Decomposability and stability of multidimensional persistence

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Chapter 0

Introduction

0.1 Introduction en français

Dans un contexte où des quantités toujours plus colossales de données sont disponibles, extraire des informations significatives et non triviales devient toujours plus difficile. Nous cherchons des solutions à des problèmes de difficulté croissante, comme la classification des types de cancer, la détection de maladie à partir de légères ombres sur des radiographies ou encore automatiser le contrôle qualité en utilisant de la reconnaissance d'image pour n'en citer que quelques-uns. Souvent, ces données sont sous la forme d'un nuage de points vivant dans $\mathbb{R}^n$. En général l'information contient beaucoup de redondance. Il est donc commun de supposer que les points sont situés, au bruit près, sur une variété de dimension d plus faible. De cette hypothèse de nombreuses méthodes permettant de réduire la dimension des données ont été étudiés. Cela comprend les méthodes linéaires comme l'analyse de composante principale, et celles non linéaires comme les machines à vecteurs (SVM) de support à noyaux. Quand la dimension est réduite, l'information initiale est simplifiée, le bruit réduit voir supprimé, et une compréhension de la répartition des données devient plus accessible. Ceci est vrai sous l'hypothèse que la méthode de réduction de dimension utilisée est adaptée à la nature des données.

Afin d'améliorer la classification, régression, ou encore l'analyse exploratoire de données, l'approche fournie par l'analyse topologique de données (TDA) est de rechercher la présence de formes dans le jeu de données. En ne conservant que les informations relativement à ces formes, on applique un processus qui peut être assimilé à une réduction de dimension. Cette information de nature topologique est résumé par des *descripteurs* de divers types : nombres, vecteurs ou encore ensemble. Dans un contexte de documents textuels, le descripteur utilisé correspond à la notion de sacs de mots (bags of words) [45]. Dans le cas de réseau de neurones ayant une structure d'auto-encodeur [36] c'est la représentation compressée de l'entrée obtenue dans le goulot d'étranglement du réseau qui joue le rôle de descripteur. Dans le domaine de l'analyse topologique de données, une approche nommée *mapper* consiste à construire un graphe qui résume l'information d'un nuage de points d'une façon visuelle : une couverture de $\mathbb{R}$ par des intervalles est choisie avec une fonction définie sur le jeu de données et à valeur dans $\mathbb{R}$. Le nerf des pré-images de la couverture par cette fonction est calculé, donnant un graphe. Une façon simple de définir une telle fonction est simplement de supprimer une colonne du jeu de données et d'utiliser ses valeurs. Par exemple si l'on considère des mesures prises dans une usine chimique [33], et une colonne représente la qualité de la substance synthétisée, on peut utiliser cette dernière.

Une autre approche qui ne requiert pas le choix d'une couverture de $\mathbb{R}$ est donnée par l'homologie persistante. Dans le cas de l'homologie persistante, l'entrée peut être un nuage de points, et la sortie une collection d'intervalles qui résument les nombreuses caractéristiques topologiques – composantes connexes, boucles, trous, ... – de la forme sous-jacente. Ces descripteurs sont calculés en utilisant l'homologie.

0.1.1 Homologie

Certaines caractéristiques de la forme d'une variété peuvent être résumées à travers un invariant appelé groupes d'homologie. Cet invariant sera introduit formellement au Chapitre 1. On présente ici l'idée principale. Dans les grandes lignes, le $n$ième groupe d'homologie encode le nombre de trous n-dimensionnels en utilisant, à translation et déformation continue le long de la variété près, des n-sphères (ou n-simplexes) non contractiles[1]. Le 0ième groupe d'homologie compte le nombre de composantes connexes. Le premier groupe d'homologie compte le nombre de cercles non contractiles, alors que le second compte le nombre de sphères non contractiles.

Pour calculer ces groupes, on peut considérer un espace topologique fait de simplexes (points, segments de ligne, triangles, tétraèdres, ...) collés ensemble. On définit alors une algèbre formée des sommes formelles de ces simplexes à coefficients dans un corps[2] $\mathbf{k}$. On définit un opérateur de bord ∂ qui envoie chaque simplex sur la somme *alternée*[3] de ses faces. On étend ensuite cette définition à toute l'algèbre par linéarité. Les groupes d'homologie

[1] Contractile signifie homotope à un point.

[2] Dans le contexte de l'homologie persistance, on considère des espaces vectoriels. Les groupes d'homologie peuvent être aussi construits avec des coefficients dans $\mathbb{Z}$. Cela donne plus d'information avec pour coût une complexité de l'objet accrue.

[3] Le mot *alterné* signifie que le signe de chacun des simplexes de cette somme alterne entre $+1$ et -1.

sont alors obtenus en prenant le quotient du noyau – moralement les boucles – de ∂ par l'image de ∂ – moralement les boucles qui sont des bords.

0.1.2 Homologie persistante

Ici, on motive et présente l'homologie persistante à travers un exemple pratique. Soit $P \subset \mathbb{R}^d$ un nuage de points. On suppose que cet ensemble est échantillonné sur une surface, comme c'est le cas de l'ensemble représenté sur la Figure 1 où les points sont pris le long d'une courbe s'enroulant autour d'un tore.

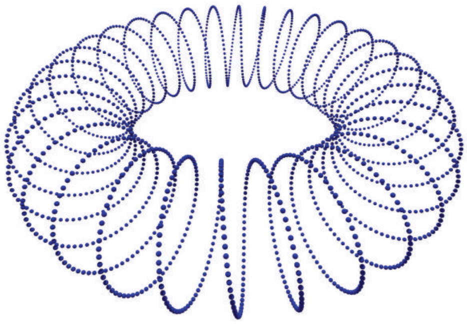


Figure 1: Points provenant d'une courbe fermée s'enroulant autour d'un tore.

Une partie de l'information topologique d'un tore $\mathbb{T}$ peut être extraite en calculant ses groupes d'homologie $H_i(\mathbb{T})$, ou pour un invariant encore plus simple on peut se contenter du rang de ces groupes $\beta_i(\mathbb{T})$, appelé nombre de Betti. Dans cette exemple nous souhaiterions retrouver, au moins partiellement, l'information que P a été échantillonné sur un tore.

Calculer $H_i(P)$ ne nous apporte rien de plus, puisque $H_0(P)$ est simplement le nombre de points dans l'ensemble et tous les $H_i(P), i \geq 1$ restants sont nuls. On peut remplacer P par l'union de boules d'un certain rayon r centrés en ses points. Notons cet ensemble $F_r = \bigcup_{p \in P} B(p, r)$. L'ensemble F_r est maintenant un volume et l'on peut calculer ses groupes d'homologie. Pour une bonne valeur de r, représentant une échelle adaptée, on peut s'attendre à retrouver les groupes d'homologie de $\mathbb{T}$. De plus, pour une plus faible valeur de r, on se retrouve à ne connecter que les points adjacents le long d'une courbe, et l'on retrouve les groupes d'homologie d'un cercle. Cela nous informe que les points ont été échantillonnée le long d'une courbe fermée.

Déterminer les valeurs de r présentant un intérêt est un problème difficile. Au lieu de chercher directement quelle est la "bonne échelle", l'homologie persistante les prend toutes en compte au sein d'un même objet. La collection de tous les volumes F_r sera plus tard appelée une filtration. L'homologie persistante calcule tous les groupes d'homologie de la collection $\{F_r, r \geq 0\}$ et les lie entre eux. L'invariant calculé, appelé un code barre, décrit quand une caractéristique topologique apparaît et combien de temps elle persiste à travers cette filtration.

0.1.3 Persistance Multidimensionnelle

Comme nous le verrons au Chapitre 3, la persistance comme décrit plus haut ne peut répondre seule à toutes les questions qui apparaissent naturellement quand l'on cherche à comprendre la topologie d'un jeu de données. Soit d_P la distance à un nuage de points $F \subseteq \mathbb{R}^d$. Considérer les boules de rayon r centrées sur les points de P est équivalent à observer les r-sous-niveaux de la fonction d_P (à valeur dans $\mathbb{R}$), c'est-à-dire les ensembles $\{x|\, d_P(x) \leq r\}$. C'est pourquoi l'homologie persistante fait sens pour n'importe quelle fonction à valeur dans $\mathbb{R}$. Une généralisation naturelle est de considérer les sous-niveaux d'une fonction à valeur dans $\mathbb{R}^n$, au lieu de $\mathbb{R}$. Cette notion est appelée multipersistance. Cela signifie que les index des groupes d'homologie ne sont plus le rayon r d'une boule, mais un élément de $\mathbb{R}^n$. Les caractéristiques topologiques évoluent maintenant le long de chacun des n différents axes, et doivent être reliées les unes aux autres par des relations de commutativité. Cette nouvelle construction accroit dramatiquement la complexité des modules de persistance, rendant invalide nombre de théorèmes disponibles dans la théorie un-dimensionnelle. Non seulement le théorème qui énonce l'existence de descripteurs ne s'étend pas, mais le nombre de candidats à ce que l'on pourrait appeler un code-barre croît tellement que les utiliser directement semble vain. Dans le Chapitre 3 on donne des exemples de sommandes complexes apparaissant en dimension 2 provenant de la littérature existante. On illustre aussi un phénomène appelé monodromie se produisant en dimension 3 via un exemple, qui signifie que l'on peut construire des carquois circulaires à l'intérieur d'un module de persistance 3-dimensionnel.

0.1.4 Contributions

Dans cette thèse nous étudions les propriétés des modules de persistance multidimensionelle dans le but d'obtenir une meilleure compréhension des sommandes et décompositions de ces derniers.

Arbres enracinés. Dans le Chapitre 3, nous introduisons un foncteur qui plonge la catégorie des représentations de carquois dont le graphe est un arbre enraciné dans la catégorie des modules de persistance indexé sur $\mathbb{R}^2$.

Cup-produit. Au Chapitre 3 nous enrichissons la structure de module de persistance provenant de l'application du foncteur homologie à une filtration. En utilisant le cup-produit des anneaux d'homologie, nous définissons un produit qui permet de construire non pas des modules de persistance mais des algèbres de persistance. Le produit de deux vecteurs du module M coïncide point à point avec le cup-produit de $H(F_t)$ où F_t est l'espace topologique de la filtration à l'instant $t \in \mathbb{R}^n$.

Persistance exacte 2 dimensionnelle point à point finie. Au Chapitre 4 nous généralisons l'approche de Crawley Beovey [26] à la multipersistance et identifions une classe de modules de persistance indexée sur $\mathbb{R}^2$ qui possède des descripteurs simples et analogues au théorème de décomposition (voir Théorème 1.4.4) existant en persistance 1-dimensionnelle. Ces descripteurs sont des collections de certains types de rectangles infinis: bandes verticales, bandes horizontales, quadrants supérieurs droits et quadrants inférieurs gauches. Les modules disposant d'une telle décomposition satisfont une certaine propriété appelée exactitude, demandant pour chaque diagramme rectangulaire traçable dans $\mathbb{R}^2$:

$$\begin{array}{ccc} M_{(s_x,t_y)} & \xrightarrow{h^t_{(s_x,t_y)}} & M_t \\ {\scriptstyle v_s^{(s_x,t_y)}}\uparrow & & \uparrow{\scriptstyle v^t_{(t_x,s_y)}} \\ M_s & \xrightarrow[h_s^{(t_x,s_y)}]{} & M_{(t_x,s_y)} \end{array} \qquad (0.1)$$

que la séquence suivante soit exacte (i.e. $\operatorname{Im}\phi = \operatorname{Ker}\psi$):

$$M_s \xrightarrow{\phi=\left(h_s^{(t_x,s_y)},\, v_s^{(s_x,t_y)}\right)} M_{(t_x,s_y)} \oplus M_{(s_x,t_y)} \xrightarrow{\psi = v^t_{(t_x,s_y)} - h^t_{(s_x,t_y)}} M_t.$$

0.1.5 Structure du document

Cette thèse est divisée en deux parties. Les deux premiers chapitres traitent de la persistance 1-dimensionnelle, mentionnent les propriétés principales et leur utilisation à travers un algorithme. Les deux derniers chapitres développent le cas de la persistance multidimensionnelle, en se focalisant essentiellement sur la dimension 2, et présentent quelques résultats dont on peut espérer qu'ils amènerons un jour à de nouveaux outils et une vision plus claire de la persistance multidimensionnelle. Le détail des chapitres est le suivant :

- Le Chapitre 1 définit l'homologie et la persistance 1-dimensionnelle. Il présente les théorèmes fondamentaux et le concept de code-barres (aussi connus sous le nom de diagrammes de persistance). Il termine par une courte présentation d'un algorithme permettant le calcul de la persistance d'une filtration.
- Le Chapter 2 suit étape par étape la démonstration de Crawley-Boevey du théorème de décomposition en des modules de persistance 1-dimensionnelle point à point de dimension finie. C'est le point de départ pour la démonstration du Chapitre 4.
- Le Chapitre 3 traite de la persistance multidimensionnelle. La définition de la persistance 1-dimensionnelle est généralisée. Nombre des théorèmes principaux ne sont plus valides et nous donnons des exemples de la complexité des sommandes indécomposables. Nous mentionnons quelques nouveaux résultats, par exemple concernant les zigzags de persistance (voir [9, 7]). Finalement, nous présentons deux contributions acessibles: un foncteur de plongement des représentations de carquois sur les arbres enracinés, et le cup-produit pour les modules de persistance.
- Le Chapitre 4 est essentiellement la démonstration de notre théorème de décomposition pour les modules 2-dimensionnelles étant exact et point à point de dimension finie. Nous terminons en mentionnant quelques applications immédiates de ce résultat.

0.2 Introduction in english

In a context where huge amounts of data are available, extracting meaningful and non trivial informations is getting harder. We are looking for solutions to problems of increasing difficulty, like classifying cancer types, detecting diseases from light shadows on radiography or automatizing quality control using image recognition to cite only a few. Often, data comes in the form of a point cloud living in $\mathbb{R}^n$. Since usually data contains redundancies, it is common to assume that the points are actually located, up to noise, on a manifold with lower dimension, say d. From this assumption various methods to reduce the dimension of the data have been studied. This includes linear methods like principal component analysis, and non linear ones like support vector machines using kernels. When the dimension gets reduced, the initial information is simplified, the noise reduced or removed, and understanding the distribution of the data becomes easier. This is true under the hypothesis that a method of reduction adapted to the nature of data has been used.

In order to improve the tasks of classification, regression, or exploratory analysis, the approach provided by topological data analysis is to look for the presence of shapes in data set. By keeping only the information relative to the shape, we apply a process which can be assimilated to dimensionality reduction. This information of a topological nature is summarized by *descriptors* of various kinds: numbers, vectors or even sets. In the context of summarizing text documents, the descriptors correspond to the notion of bag of words [45]. In the case of neural networks based on auto-encoder [36] it is the compressed representations obtained in the bottleneck of the network that play the role of a descriptor. In the area of topological data analysis, an approach named *mapper* consists in building a graph which summarizes information from the point cloud in a visual way: a covering of $\mathbb{R}$ by intervals is selected along with an $\mathbb{R}$ valued function defined on data. Then the nerve of the pre-image of the cover by this function is computed, giving a graph. An easy way to define such a function is to remove a column from the data set and use its values. For example if our data are diverse measures taken in a chemical plant [33], and one column is the quality of the product, then one can use this column.

Another approach which does not require a choice of an $\mathbb{R}$ covering is provided by persistent homology. In the case of persistence homology, the input can be a point cloud, and the output is a collection of intervals summarizing the various topological features – connected components, loops, holes, ... – of the underlying shape. These descriptors are computed using homology.

0.2.1 Homology

Properties of the shape of a manifold can be summarized through an invariant called the homology groups. This invariant will be introduced formally in Chapter 1 and we present here the underlying idea. Roughly speaking, the n-th homology group encodes the number of n-dimensional holes using, up to continuous deformations and translation along the manifold, non contractible[4] n-spheres (or n-simplices). The 0-th homology group counts the number of connected components. The 1-st homology group counts the number of non contractible circles whereas the 2-nd homology group counts the number of non contractible spheres.

To compute such groups, one can consider a topological space made of simplices (points, line segment, triangles, tetrahedra, ...) glued together. We then define an algebra made of formal sums of these simplices with coefficients in some field[5] $\mathbf{k}$. We define a boundary operator ∂ which sends a simplex to the *alternated*[6] sum of its faces and then extend its definition to the whole algebra by linearity. The homology groups are then obtained by taking the quotient of the kernel – morally the closed loops – of ∂ modulo the image of ∂ – morally the loops that are boundaries.

0.2.2 Persistent homology

We motivate and present persistent homology through a practical example. Let $P \subset \mathbb{R}^d$ be a point cloud. We assume this set of points to be sampled on a surface like the set depicted in Figure 2 where points were taken from a curve rolling around a torus.

Part of the topological information from the torus $\mathbb{T}$ can be extracted by computing its homology groups $H_i(\mathbb{T})$, or for a simpler invariant we can simply look at the rank of these groups $\beta_i(\mathbb{T})$, called the Betti numbers. In this example we would like to recover, at least partially, the information that P was sampled from a torus.

Computing $H_i(P)$ does not give much information, since $H_0(P)$ gives the number of points in the set and all the remaining $H_i(P), i \geq 1$, are zero. One could replace P by the union of balls of some radius r centered on its points. We name this set $F_r = \bigcup_{p \in P} B(p, r)$. The set F_r is now a volume and we can compute its homology groups. For a good value of r, representing a

[4]Contractile means homotopy equivalent to a point.

[5]In the setting of persistent homology, we actually consider vector spaces. Homology groups can be constructed using coefficient in the ring of integer $\mathbb{Z}$. This can give more information with the cost of a more complex object.

[6]The word *alternated* means that the sign in front of each simplex in the sum alternates between 1 and -1.

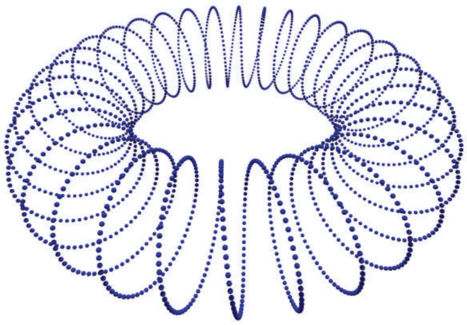


Figure 2: Points from a curve rolling on a torus

suitable geometric scale, we can expect to recover the homology groups of $\mathbb{T}$. Furthermore, for a smaller value of r we will only connect adjacent points on the curve and recover the homology groups of a circle. This is telling us that the point cloud was sampled along a closed curve.

Determining the interesting values of r is a difficult problem. Instead of finding directly the "good scales", persistent homology takes all of them together in a single object. The collection of volumes F_r will later be called a filtration. Persistent homology computes all the homology groups of the collection $\{F_r, r \geq 0\}$ and links them together. The invariant computed, called the barcode, describes when a new topological feature appears and how long it persists through the filtration.

0.2.3 Multidimensional persistence

As we will see in Chapter 3, persistence as described above cannot handle well all the questions that naturally arise when we are looking for the topology of data. Let d_P be the distance to a point cloud $P \subseteq \mathbb{R}^d$. Considering the balls of radius r centered at the points of P is equivalent to looking at the r-sublevel set of the $\mathbb{R}$ valued function d_P, namely $\{x|\, d_P(x) \leq r\}$. Thus, persistence does actually makes sense with any $\mathbb{R}$ valued function. A natural generalization of persistence is to consider sublevel sets of functions with value in $\mathbb{R}^n$ instead of $\mathbb{R}$. It is called multipersistence. This means that the index of homology groups is not the radius of a ball in $\mathbb{R}$ anymore, but an element of $\mathbb{R}^n$. Topological features can now evolve along each one of the n different axes, and should be linked to each other by a relation of commutativity. This new construction dramatically increases the complexity of the persistence modules, breaking many of the theorems available in the one-dimensional persistence theory. Not only the theorem stating the existence of descriptors fails to be extended, but the number of candidates for what we would call a descriptor increases so much in complexity that using them directly seams pointless. In Chapter 3 we gives examples of complex summands arising in dimension 2 taken from the existing literature. We illustrate by an example the phenomenon of monodromy arising in dimension 3, which means that one can build circular quivers embedded in a 3-dimensional persistence module.

0.2.4 Contributions

In this thesis we investigate the properties of multidimensional persistence modules in order to obtain a better understanding of the summands and decompositions of such modules.

Rooted trees. In Chapter 3, we introduce a functor that embeds the representations category of any quiver whose graph is a rooted tree into the category of $\mathbb{R}^2$-indexed persistence modules.

Cup product. In Chapter 3 we also enrich the structure of persistence module arising from the homology of a filtration. Using the cup product from homology rings, we describe a product that allows us to build not only persistence modules but persistence algebras. The product of two vectors of the module M coincides pointwise with the cup product of $H(F_t)$ where F_t is the topological space of a filtration at location $t \in \mathbb{R}^n$.

Exact pointwise finite dimensional 2d-persistence. In Chapter 4 we generalize the approach of Crawley Beovey [26] to multipersistence and identify a class of persistence modules indexed on $\mathbb{R}^2$ which have simple descriptor and an analog of the decomposition theorem 1.4.4 available in one dimensional persistence. This descriptor is a collection of certain kinds of infinite rectangles: horizontal bands, vertical bands, upper-right quadrants, and lower-left quadrants. The modules that have this decomposition satisfy a certain exactness condition, stating that for every rectangle diagram drawn in $\mathbb{R}^2$:

$$\begin{array}{ccc} M_{(s_x,t_y)} & \xrightarrow{h^t_{(s_x,t_y)}} & M_t \\ {\scriptstyle v_s^{(s_x,t_y)}}\uparrow & & \uparrow{\scriptstyle v^t_{(t_x,s_y)}} \\ M_s & \xrightarrow[h_s^{(t_x,s_y)}]{} & M_{(t_x,s_y)} \end{array} \tag{0.2}$$

the following sequence is exact (i.e. $\operatorname{Im}\phi = \operatorname{Ker}\psi$):

$$M_s \xrightarrow{\phi = \left(h_s^{(t_x,s_y)},\, v_s^{(s_x,t_y)}\right)} M_{(t_x,s_y)} \oplus M_{(s_x,t_y)} \xrightarrow{\psi = v^t_{(t_x,s_y)} - h^t_{(s_x,t_y)}} M_t.$$

0.2.5 Structure of the document

This thesis can be split into two parts. The first two chapters deal with one-dimensional persistence, mentioning the main properties and their use through an algorithm. The last two chapters develop the case of multidimensional persistence, focusing mostly on dimension 2, and then present some new results that we expect will lead to new tools and a clearer understanding of multidimensional persistence. The details of the chapter follow:

- Chapter 1 defines homology and one-dimensional persistence. It presents the main theorems and the concept of barcodes (also known as persistence diagrams). It ends with a short explanation of an algorithm allowing to compute the persistence of a filtration.
- Chapter 2 is a step-by-step account of Crawley-Boevey's proof of the decomposition of one-dimensional pointwise finite-dimensional persistence modules. This is the starting point of the proof from Chapter 4.
- Chapter 3 focuses on multidimensional persistence. The definition of one-dimensional persistence is generalized. Many of the main theorems from dimension one are now invalid and we give examples of the complexity of summands. We mention some known results, for example regarding zigzag-persistence (see [9, 7]). Finally, we give two accessible contributions: the embedding functor for representation of rooted trees and the cup product for persistence modules.
- Chapter 4 is for the most part the proof of our decomposition theorem for 2-dimensional modules that are both pointwise finite-dimensional and exact. We finish by mentioning some of the immediate applications of this result.

Chapter 1

1D Persistent Homology

In Chapter 4 we generalise the decomposition theorem 1.4.4, proven by Crawley-Boevey[26] whose proof is detailed in Chapter 2. In order to express why this theorem is so important in persistence theory, and to understand the applications of its generalization, we first have to introduce the objects of the theory. We start by introducing our main objects: persistence modules. We enunciate some of their properties and present an invariant of persistence modules called the barcode which summarizes the topological features contained in a persistence module. Thanks to the decomposition theorem 1.4.4 this invariant is complete.

We endow the space of persistence modules with a structure of pseudo metric space through the definition of the interleaving distance. Small perturbations in this distance can be induced by small perturbations on the objects – points clouds, and functions – from which persistence modules are computed. In parallel, we define the bottleneck distance on the space of barcode, which can be seen as an intuitive and naive way to compare two barcodes. Persistence modules and barcode having turned into two pseudo-metric spaces, we can link them between each other. We then recall the isometry theorem which states that this two spaces are indeed isometric. This result has a deep meaning. It states that small perturbations on persistence modules through the interleaving distance – induced by perturbations on the objects which gave rise to this persistence modules – are indeed the exact same thing as small perturbations on barcode through the bottleneck distance. It is the main justification of why barcode are in practice well-behaved to summarize topological information.

1.1 Homology and Cohomology

Throughout the exposition, the field of coefficients is fixed and denoted by $\mathbf{k}$. For computational purposes, it is often taken to be a finite field and most of the time for applications $\mathbf{k} = \mathbb{Z}_2$.

We will recall the definition of singular homology and cohomology. A reader unfamiliar with homology can find a valuable introduction to simplicial and singular homology, and even more, in Algebraic Topology from Allen Hatcher [34].

Let X be a topological space, and G an Abelian group. Denote by Δ^n the standard n simplex $\{x \in \mathbb{R}^{n+1} | \, x_0 + \cdots + x_n = 1, x_i \geq 0, i = 0, \ldots, n\}$.

Let σ be a n-singular simplex. We can designate σ by its ordered vertices $[v_1, \ldots, v_n] = [\sigma(e_1), \ldots, \sigma(e_n)]$ where the e_i are vertices of Δ^n. The notation $[v_0, \ldots, \hat{v_i}, \ldots, v_n]$ is the restriction of σ to the face of Δ^n whose vertices are all but v_i. The boundary operator $\partial_n : C_n(X) \to C_{n-1}(X)$ on n-simplices is defined by

$$\partial_n(\sigma) = \sum_i (-1)^i [v_0, \ldots, \hat{v_i}, \ldots, v_n].$$

The sign in the sum is alternating on the faces of Δ^n, to take the orientation into account. For example, the boundary of a simplex $[v_0, v_1]$ will be $\partial_2(\sigma) = [v_1] - [v_0]$. Two singular simplex are identified up to a minus sign if they are the same after applying a permutation of two vertices on the simplex they are defined. For example, $[v_0, v_1] = -[v_1, v_0]$. The boundary operator respects $\partial_n \circ \partial_{n+1} = 0$ and it allows us to define the degree n singular homology group $H_n = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}$. The homology group as defined above is a vector space.

Homology is a functor H_n from the category of topological spaces to the category of $\mathbf{k}$-vector spaces, which associates to a topological space X the graded $\mathbf{k}$-module $H_*(X) = \bigoplus_i H_i(X)$. Each element $\alpha \in H_i(X)$ has homology *degree* i. Similarly, we can build the dual construction: singular cohomology. The group $C^n(X; \mathbf{k})$ of *singular n-cochains with coefficients in* $\mathbf{k}$ is $Hom(C_n(X), \mathbf{k})$ the dual of the singular chain group. This collection forms a chain complex thanks to the operator δ defined by a collection of map on each chain complex $\delta^n : C^n(X; \mathbf{k}) \to C^{n+1}(X; \mathbf{k})$ given by

$$\delta^n \varphi(\sigma) = \varphi(\partial_n \sigma) = \sum_i (-1)^i \varphi([v_0, \ldots, \hat{v_i}, \ldots, v_{n+1}]),$$

for $\sigma = [v_0, \ldots, v_{n=1}]$ a $n+1$-singular simplex. The homology of this complex is called the singular cohomology, and the n-th cohomology group is $H^n(X; \mathbf{k}) = (\operatorname{Ker} \delta^n / \operatorname{Im} \delta^{n-1})$. The cohomology H^n is also a functor.

1.2 A topological construction

We are interested in algebraic objects called *persistence modules*, which encode the evolution of topological features along a notion of time. The values the time can take is called the *index set*, and the topological features are encoded by vector spaces.

We recall that a *category* $\mathcal{C}$ is a collection $Ob(\mathcal{C})$ of objects and a collection of arrows (also called morphism or maps) $Hom(\mathcal{C})$ and a low of composition of morphisms, following the axioms described bellow. Each morphism has a source object $a \in Ob(\mathcal{C})$ and a target object $b \in Ob(\mathcal{C})$. We write $Hom(a, b)$ for the collection of arrows with source a and target b and $f : a \to b$ for a such morphism. For every three objects $a, b, c \in Ob(\mathcal{C})$ we have a binary operation $Hom(a, b) \times Hom(b, c) \to Hom(a, c)$ where the composition of $f : a \to b$ and $g : b \to c$ is written $g \circ a$. The morphisms and composition low should respect the two folowing axioms :

- (identity) for every object $x \in Ob(\mathcal{C})$ there is a morphism $id_x : x \to x \in Hom(x, x)$ called the *identity morphism for* x. We require id_x to be a neutral element for the composition law. That is, for any $f : a \to x$ and $g : x \to b$, we ask $id_x \circ f = f$ and $g \circ id_x = g$.
- (associativity) we require the composition law to be associative, that is for every $f : a \to b$, $g : b \to c$ and $h : c \to d$ we should have $h \circ (g \circ f) = (h \circ g) \circ f$.

From this definition, there is exactly one identity morphism by object.

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *functor* F form $\mathcal{C}$ to $\mathcal{D}$ is a mapping that

- associate to each object x in $\mathcal{C}$ and object $F(x)$ in $\mathcal{D}$.
- associate to each morphism $f : a \to b$ in $\mathcal{C}$ a morphism $F(f) : F(x) \to F(b)$ in $\mathcal{D}$ such that
  - $F(id_x) = id_{F(x)}$ for every object $x \in \mathcal{C}$.
  - $F(g \circ f) = F(g) \circ F(f)$ for every $f : a \to b$ and $g : b \to c$ in $\mathcal{C}$.

For more detail on category theory, the Stack Project [46] is a good reference.

Definition 1.2.1. Let $(S, \leq)$ be a totally ordered set. The set S can be either finite or infinite, and will be referred as the *index set*. A 1-dimensional S-persistence module M, or simply a persistence module, is a functor from the category $(S, \leq)$ to the category of vector spaces **Vect**. The set S is called the *index set* of M. For points $s, t \in S$, the morphisms $M_s^t = M(s \leq t)$ between the two vector spaces M_s and M_t are called *structural morphisms*.

If we state it without using the language of category, a persistence module M on S is a collection of vector spaces indexed by elements of S connected with linear applications M_s^t for every $s, t \in S$, $s \leq t$. In our settings, asking M to be a functor is asking for the composition of these morphisms, that is for any $s, r, t \in S$ such that $s \leq t \leq r$, we have $M_t^r \circ M_s^t = M_s^r$.

The index sets considered most of the times are $\mathbb{N}, \mathbb{Z}$ or $\mathbb{R}$. In practical applications computer data are finite so the index set will be an interval of $\mathbb{N}$ written $[\![0, N]\!]$ for N a sufficiently large integer.

One of the most common way of constructing a persistence module is from a filtration.

Definition 1.2.2. Let $(S, \leq)$ be a totally ordered set. A 1-dimensional S-filtration $\mathcal{F}$, or simply a filtration, is a collection of topological spaces $F_s, s \in S$ such that $\forall s, t \in S$, $s \leq t \Rightarrow F_s \subseteq F_t$.

Given a filtration $\mathcal{F}$, one can obtain a persistence module $H_*(\mathcal{F})$ by applying the homology functor on the family. We then obtain a family of vector spaces $\oplus_i H_i(F_s), s \in S$. These spaces are connected by the linear application $\oplus_i H_i(F_s \subseteq F_t)$ since the homology functor sends continuous map (and therefore inclusions) to linear applications.

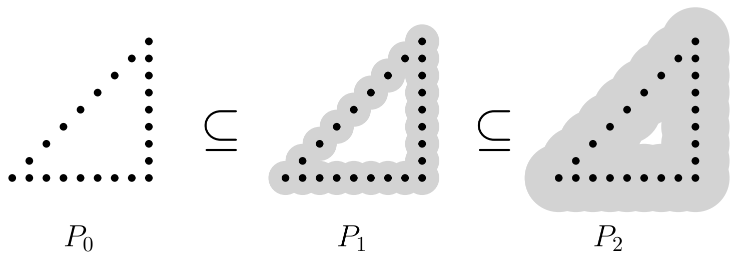


Figure 1.1: Filtration constructed from a point cloud

Given a point cloud $P \subset \mathbb{R}^m$ a filtration can be built by taking sublevel sets of the distance function to the point set $d_P : \mathbb{R}^m \to \mathbb{R}_+$ (see Figure 1.1 for a visual example) where $P_r = d_P^{-1}((-\infty, r])$.

But there is no reason to restrict ourselves to distance functions to a set. One can take any arbitrary map $f : X \to \mathbb{R}$ and look at the collection of sublevelsets $L_r = f^{-1}((-\infty, r])$. They define a filtration L^f and applying the homology functor give us a persistence module $H^*(L^f)$.

1.3 Module category

We recal that given two functors F and G from $\mathcal{C}$ to $\mathcal{D}$, a natural transformation η from F to G is a family of morphism that satisfies:

- The natural transformation associate to every object $x \in Ob(\mathcal{C})$ a morphism $\eta_x : F(x) \to G(x)$ between objects of $\mathcal{D}$.
- For every morphism $f : a \to b$ in $\mathcal{C}$ we have $\eta_b \circ F(f) = G(f) \circ \eta_a$.

See the Stack Project [46] for more details.

Since two modules M and N are defined as functors, a morphism $\varphi : M \to N$ is a natural transformation. In simple words φ is a collection of morphisms $\varphi_s : M_s \to N_s, s \in S$ such that, $\forall s, t \in S$, $s \leq t$ the diagram

$$\begin{array}{ccc} N_s & \xrightarrow{N_s^t} & N_t \\ {\scriptstyle \varphi_s}\uparrow & & \uparrow{\scriptstyle \varphi_t} \\ M_s & \xrightarrow{M_s^t} & M_t \end{array}$$

commutes. We say that φ is an *isomorphism* if each $\varphi_s, s \in S$ is an isomorphism. We denote by $\mathrm{Hom}(M, N)$ the collection of morphisms from M to N and by $\mathrm{End}(M) = \mathrm{Hom}(M, M)$ the collection of *endomorphisms*. A morphism φ of modules is said *injective morphism* if each φ_s, $s \in S$ is an injection. It is said *surjective* if each φ_s is a surjection.

A *submodule* H of a persistence module M indexed over S is a collection of subspaces $H_s \subseteq M_s$. For $s \leq t$, the morphisms H_s^t are the restriction of M_s^t to the space H_s with value in H_t. Let $\varphi : M \to N$ be a morphism. One can take for each $s \in S$ the image $\mathrm{Im}\,\varphi_s \subseteq N_s$. Since $N_s^t \circ \varphi_s = \varphi_t \circ M_s^t$ for $s \leq t \in S$, $N_s^t(\mathrm{Im}\,\varphi_s) \subseteq \mathrm{Im}\,\varphi_t$ and we get a submodule $\mathrm{Im}\,\varphi$ of the persistence module N named the image of φ. Similarly, by taking point-wise kernels, we can define $\mathrm{Ker}\,\varphi$. If we are given two modules M and N over S, we can define their *direct sum* $M \oplus N$ with spaces $(M \oplus N)_s = M_s \oplus N_s$ and morphisms $(M \oplus N)_s^t = M_s^t \oplus N_s^t$ for $s \leq t$.

Finally, we have a notion of *quotient module*. Let H be a submodule of M. We define M/H by taking as spaces the pointwise quotient $(M/H)_s = M_s/H_s$. It remains to define the morphisms between these spaces. We recall the *universal property of quotient* for vector spaces. Let G, H two vector spaces, N a subspace of G, $\pi : G \to G/N$ the projection and $f : G \to H$ a linear application with $N \subseteq \ker f$. There exists a unique linear application $\overline{f} : G/N \to H$ such that $f = \overline{f} \circ \pi$. Let $s \leq t$ be two elements of S and $\pi_t : M_t \to M_t/H_t$ be the projection of M_t on the quotient space M_t/H_t. Since H is a persistence module, $\mathrm{Ker}(\pi_t \circ M_s^t) \supseteq H_s$ and by universal property of quotient we obtain a linear application $(M/H)_s^t : M_s/H_s \to M_t/H_t$. This allows to define co-kernels. The category of persistence modules is indeed Abelian.

We also have a notion of *duality*. Given a persistence module M, one can construct its dual M^* by taking dual vector spaces $M_t^* = (M_t)^*$ of linear forms and taking transposed linear maps $(M^*)^s_t(\phi) = \phi \circ M^t_s$. We obtain a module indexed on S^{op}, where all the arrows was reversed. If S is $\mathbb{R}$ or $\mathbb{Z}$, there is a re-indexing, namely $x \mapsto -x$, that gives us a dual module with the same index set as $1M$.

Remark 1.3.1. Homology and cohomology theory are dual in the sense that under some assumption on the topological spaces – if all the topological spaces in the filtration are closed oriented manifold one can apply Poincaré duality – the dual of the persistence module obtained with homology is isomorphic to the persistence module obtained from the same filtration using cohomology. See [27].

Before going further, one can ask where does the name of persistence "modules" come from. The persistence modules are indeed graded modules over a ring, which depend on the index set.

We recall the definition of a *group ring*. Let G be a group and R a ring. The group ring $R[G]$ of G over R is essencially a R module made of the collection of finite linear combination of elements of G with coefficients in R. Sums and product are defined by linearity and distributivity. Formaly, let $R[G]$ be the set of mappings $f : G \to R$ of finite support. The scalar product αf of a scalar $\alpha \in R$ and a mapping $f \in R[G]$ is defined as the mapping $\alpha f : x \mapsto \alpha f(x)$ and the group sum of two mappings f and g is defined as the mapping $f + g : x \mapsto f(x) + g(x)$. The product of two vectors f and g is given by

$$x \mapsto \sum_{uv=x} f(u)g(v)$$

which makes sense because of the finite support hypothesis. Since G can be embedded into $R[G]$ by associating to $g \in G$ the vector $x \mapsto \mathbb{1}_{g=x}$, a vector of $R[G]$ can be written as a sum $\sum \alpha_g g$. When R is a field, $R[G]$ is an algebra over R and is also called the *group algebra* of G over R. The same construction can be done with G a semi group. Let S be an sub(semi)group of $\mathbb{R}$. We denote by $\mathbf{k}[S]$ the S-graded (semi)group algebra over $\mathbf{k}$ of the additive (semi)group S. In the case of $S = \mathbb{N}$ we get the polynomial ring $\mathbf{k}[x]$. If $S = \mathbb{Z}$ we obtain the ring of Laurent polynomials $\mathbf{k}[x, x^{-1}]$. A persistence module M over S can be endowed with a structure of $\mathbf{k}[S]$ graded module by taking the direct sum of all vector spaces

$$\mathbb{M} = \bigoplus_{s \in S} M_s$$

and defining the action of $x^t, t \in S$ on $m = \sum m_s, m_s \in M_s$ by $x^t.m = \sum M^{s+t}_s(m_s)$. The grading is with values in S and $m_s \in M_s$ is said to be of degree s.

1.4 Barcodes

From now on, we will consider modules whose index set is a subset of $\mathbb{R}$. Most of the times, S is either $\mathbb{N}, \mathbb{Z}, \mathbb{R}_+$ or $\mathbb{R}$.

When it comes to obtain a deeper understanding of a particular family of algebraic objects, a common approach is to find a decomposition of these objects, and list the smallest summands that can appear. A historical example is the unscrewing of groups known as the classification of finite groups. In the case of one dimensional persistence things go very well since under reasonable hypotheses every module can be written as a direct sum of small modules called indecomposables, the indecomposables are easy to describe and unlike group theory there is a unique way, up to isomorphisms, to reconstruct the original module from its summands.

The elementary blocks used to build one dimensional persistence are the interval modules.

Definition 1.4.1. Let I be an interval of S. An interval module is a persistence module $\mathbf{k}_I$ whose spaces are

$$(\mathbf{k}_I)_s = \begin{cases} \mathbf{k} \text{ if } s \in I, \\ 0 \text{ if } s \notin I, \end{cases}$$

and for $s \leq t$ in S, the morphisms are given by

$$(\mathbf{k}_I)_s^t = \begin{cases} \mathrm{id}_{\mathbf{k}} \text{ if } s, t \in I, \\ 0 \text{ otherwise.} \end{cases}$$

A persistence module is said to be indecomposable if it cannot be written as a direct sum of two non-trivial persistence modules. As described in [22] from Chazal et al. the endormorphism ring of an interval modules $\mathrm{End}(\mathbf{k}_I)$ is isomorphic to $\mathbf{k}$ which implies that interval modules are indecomposable.

Theorem 1.4.2. *Let I be an interval of S. The interval module $\mathbf{k}_I$ is indecomposable.*

Proof. The endormorphism ring of an interval module is indeed isomorphic to the field ($\mathrm{End}(\mathbf{k}_I) \simeq \mathbf{k}$) since once a morphism $\varphi_s : (\mathbf{k}_I)_s \to (\mathbf{k}_I)_s$ between two modules is fixed for a $s \in S$, the commutativity fix the values of the other morphisms φ_t, $t \in I$. By reductio ad absurdum we suppose that $\mathbf{k}_I$ is decomposable. Let π_1 be the endormorphism of a projection on one of the summand of this decomposition. Then π_1 is an element of the endormorphsm ring. Since $\pi_1 \circ \pi_1 = \pi_1$ and the endomorphism ring is afield, π_1 should be invertible. This is only possible if π_1 is the identity, which gives a contradiction. Therefore $\mathbf{k}_I$ is indecomposable. □

Furthermore if a decomposition in interval modules exists it is unique up to reordering.

Theorem 1.4.3 (Krull–Schmidt–Azumaya)**.** *Suppose we have an isomorphism between the two decompositions into interval modules*

$$\bigoplus_{k\in K} \mathbf{k}_{I_k} \simeq \bigoplus_{l\in L} \mathbf{k}_{J_l}.$$

Then, there exists a bijection $\sigma : K \to L$ such that $\forall k \in K$, $I_{\sigma(k)} = I_k$.

Proof. The ring of endomorphism of a summand is $\mathrm{End}(\mathbf{k}_I) \simeq \mathbf{k}$ which is a field, hence a local ring. One can apply Azumaya's Lemma [2] which give us a pairing σ of summands from each decomposition, each paired summands being isomorphic modules. An isomorphism $\mathbb{I}^I \simeq \mathbb{I}^J$ imply that the index on which $\mathbb{I}^I$ and $\mathbb{I}^J$ are non zero should be equal, so $I = J$ and this give us the expected result. □

For finitely indexed families, Gabriel's theorem (see 3.1.21) ensures the existence of a decomposition into interval summands. If the dimensions of vector spaces are finite, one can actually construct a decomposition by induction. In the case the index set S is infinite, we say that a S-persistence module M is *pointwise finite dimensional*(p.f.d.) if, $\forall s \in S$, M_s is finite dimensional. A result from Cary Webb [47] on modules over the polynomial ring $\mathbf{k}[x]$ imply, as shown in [22], that under the p.f.d. hypothesis the decomposition into intervals modules still holds when $S \simeq \mathbb{Z}$. A more general theorem, from which the previous result can be regarded as a corolary, was proven by Crawley-Boevey [26]. We state it for p.f.d. modules, although the original result holds for a wider collection of modules: the ones respecting the descending chain condition (see chapter 2). See Figure 1.4.

Theorem 1.4.4 (Crawley Boevey)**.** *Let M be a persistence module indexed over S.*

Suppose $S \subseteq \mathbb{R}$ and M p.f.d. Then M admits a decomposition in interval modules:

$$M = \bigoplus_{k\in K} \mathbb{I}^{I_k}.$$

This direct sum is locally finite: for $t \in S$ the number of summands $\mathbb{I}^{I_k}$ with $(\mathbb{I}^{I_k})_t$ non zero is finite.

The collection of intervals appearing in such a decomposition is unique and describes completely the persistence module. It is a signature of the object that gave rise to the persistence module.

Definition 1.4.5. Let M be an interval decomposable persistence module. We denote by $\mathcal{B}(M)$ the multiset of intervals – the collection of intervals

Index set	Decomposable into intervals
Finite	Yes
$\mathbb{N}$	Yes
$\mathbb{Z}$	No
$\mathbb{Z}$ and p.f.d.	Yes
$\mathbb{R}$	No
$\mathbb{R}$ and p.f.d.	Yes

Figure 1.2: Quick summary of the cases where the decomposition Theorem holds.

counted with multiplicity – appearing in the decomposition of M. We call it the *barcode* of M.

In the case the module M comes for a sublevel set filtration of $f : X \to \mathbb{R}$, we will denote the barcode by $\mathcal{B}(f)$.

1.5 Bottleneck distance

It is natural to ask whether one can compare two barcodes. This is done using the bottleneck pseudo metric. We define this distance in a relatively naive way, and state first stability result linking functions and the barcode of their sublevelset filtrations.

We start with the notion of erosion. Let I be an interval. Its *ϵ-eroded* is the interval $I^{-\epsilon} = \{x \,|\, [x-\epsilon, x+\epsilon] \subseteq I\}$. We say that an interval I is *ϵ-significant* if its ϵ-erosion is non-empty, and *ϵ-trivial* otherwise. We now define the notion of matching between two barcodes (see [9] and [22]).

Definition 1.5.1. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two barcodes. An *ϵ-matching* between $\mathcal{B}_1$ and $\mathcal{B}_2$ is a partial bijection:

$$\sigma : \mathcal{B}_1 \supseteq A \to B \subseteq \mathcal{B}_2$$

where we name $\operatorname{CoIm} \sigma = A$ the coimage and $\operatorname{Im} \sigma = B$ the image, and require the constraints:

- every interval not in the coimage is ϵ-trivial: $\forall I \in \mathcal{B}_1 - \operatorname{CoIm} \sigma, I^{-\epsilon} = \emptyset$,
- every interval not in the image is ϵ-trivial: $\forall I \in \mathcal{B}_2 - \operatorname{Im} \sigma, I^{-\epsilon} = \emptyset$,
- matched intervals are ϵ-close: $\forall I \in \operatorname{CoIm} \sigma, I^{-\epsilon} \subseteq \sigma(I), \sigma(I)^{-\epsilon} \subseteq I$.

With this definition, an ϵ-matching is simply a bijection between all the ϵ-significant and maybe some ϵ-trivial intervals of each barcode. A 0-matching is a bijection and a barcode is always 0-matched to itself. The composition of an ϵ-matching with a η-matching gives a $(\epsilon + \eta)$-matching. Hence, the barcodes form a category **Mch** where arrows are the matchings.

This notion of matchings allows us to define the distance between two barcodes as the infinimum on the ϵ of all existing ϵ-matchings.

We recall that a *pseudometric* on a set X is a non negative real valued function $d : X \times X \to \mathbb{R}_{\geq 0}$ which is symetric ($d(x, y) = d(y, x)$), is zero on the diagonal ($d(x, x) = 0$) and respect the triangular inequality ($d(x, y) + d(y, z) \geq d(x, z)$).

Proposition 1.5.2. *The* bottleneck distance *between two barcodes* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *defined as* $\mathrm{d}_{\mathbf{b}}(\mathcal{B}_1, \mathcal{B}_2) = \inf\{\epsilon \,|\, \exists\, \epsilon\text{-matching } \sigma : \mathcal{B}_1 \to \mathcal{B}_2\}$ *is a pseudometric.*

As a pseudo metric, the bottleneck distance has all the good properties of a metric but fail to differentiate some barcodes. This clear when it comes to consider singletons: singletons are 0-significant intervals, but they are ϵ-trivial for any $\epsilon > 0$, and therefore do not need to be matched with any other interval. Any barcode containing only singletons is then at distance 0 from the empty barcode.

Remark 1.5.3. It was shown in [6] that the space of barcodes with finitely many ϵ-significant intervals, for every $\epsilon > 0$, is complete and separable for the bottleneck metric.

This metric can seems arbitrary. A result from [25] (see also [31]) states that it is not. Small perturbation of the filtration induces small perturbations on the diagrams. In order to state it more formally, we introduce the notion of tame functions.

Definition 1.5.4 (Tame real valued functions)**.** Let X be a topological space, and $f : X \to \mathbb{R}$ a real valued function. A *homological regular value* $a \in \mathbb{R}$ of f is a value such that their exits a $\epsilon > 0$ such that for all pairs $x < y$ or real in the interval $[\alpha - \epsilon, \alpha + \epsilon]$, the maps $H_k(f^{-1}((-\infty, x]) \to H_k(f^{-1}((-\infty, y])$ induced by inclusion are isomorphisms for all k. A real number a that is not a homological regular value of f is called a *homological critical value* of f. A real valued function f is *tame* if it has finitely many homological critical value and $\forall a \in \mathbb{R}, H_*(f^{-1}((-\infty, a]))$ is finite dimensional.

It is pointed out in [25] that Morse functions on smooth compact manifolds are tame. The distance function to a finite point cloud is tame, and critical values correspond to a subset of times when the intersection of k balls become non empty, for $k \geq 2$. The following result state that if we perturb a little a function with respect to the uniform norm[1] $\|.\|_\infty$, the barcode do not move more with respect to the bottleneck distance.

A simplicial complex (see 1.8.1) L is a collection of simplices living in

[1] The uniform norm on real valued functions $f : X \to \mathbb{R}$ is defined by $\|f\|_\infty = \sup_{x \in X} |f(x)|$.

some $\mathbb{R}^m$ such that every simplex in L have its faces belonging to L, and the intersection of two simplices σ_1 and σ_2 of L is face of σ_1 and σ_2. We call a *triangulable topological space* a topological space X such that there exists a finite simplicial complex L and a homeomorphism $\Phi : L \to X$.

Theorem 1.5.5 (Cohen-Steiner, Edelsbrunner, Harer)**.** *Let X be a triangulable topological space and $f, g : X \to \mathbb{R}$ two tame functions. The barcodes induced by the sublevel set filtration of this functions satisfies* $\mathrm{d_b}(\mathcal{B}(f), \mathcal{B}(g)) \leq \|f - g\|_\infty$.

Through this chapter we will see stronger versions of this result. But we can already point out that stability justify the use of persistence barcode as signature for dataset in real life applications.

1.6 Persistence diagrams

Barcode was introduced by Zomorodian and Carlsson [49]. Later, Edelsbrunner, Cohen-Steiner and Harer introduced the concept of persistence diagrams in [25]. Persistence barcodes and persistence diagrams are indeed the same object observed from a different angle of view.

A decorated real $x^\circ \in \mathbb{R}^\pm$ is an element of the extended real line $x \in \mathbb{R} \cup -\infty, +\infty$ with a decoration $\circ \in \{+, -\}$. Decorated reals are ordered from the order on reals where we add that $\forall x \in \mathbb{R}$, $x^- < x^+$. Pair of decorated reals can be used to represent different types of intervals as follows:

- $[a, b]$ is written (a^-, b^+),
- $(a, b]$ is written (a^+, b^+),
- $[a, b)$ is written (a^-, b^-),
- (a, b) is written (a^+, b^-).

To represent half lines and $\mathbb{R}$ we allow ourselves to use $-\infty^-$ and $+\infty^+$ as values for a or b.

Definition 1.6.1. Let M be a persistence module. Its persistence diagram is the multiset $\mathrm{D}(M)$ of points $(x^\circ, y^\circ) \in \mathbb{R}^\pm \times \mathbb{R}^\pm$ such that $(x^\circ, y^\circ) \in \mathrm{D}(M)$ if and only if $I = (x^\circ, y^\circ) \in \mathcal{B}(M)$.

The closer a point $p \in \mathrm{D}(M)$ is to the diagonal $\Delta = \{(x^-, x^+) \,|\, x \in \mathbb{R}\}$, the less significant the interval it represents is. If the module M comes from a sublevelset filtration and since a small perturbation of the function could remove the points around the diagonal we can think of them as a form of topological noise. This is not a universal truth. When the module arise from a different process, this points can be actually an interesting feature.

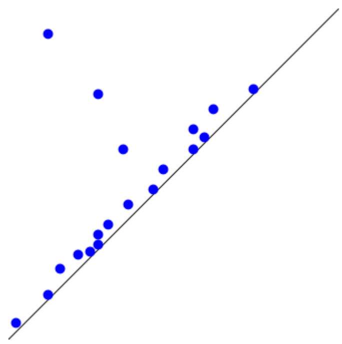


Figure 1.3: An undecorated persistence diagram

Since the bottleneck distance does not differentiate between the different types of intervals, it is common to forget the decoration. We then obtain an undecorated persistence diagram which can be represented as a subset of[2] $\overline{\mathbb{R}}^2$ (see Figure 1.3).

Barcodes endowed with the bottleneck distance allows us to use persistence as a tool for classification and dimension reduction. For other applications one requires often more structure than a metric space. This thesis focuses on the generalization of this metric space to higher dimension, but it is worth mentioning the complementary approaches that have been considered to use persistence in a broader class of applications.

- One approach in machine learning is to consider Hilbert spaces of features, allowing access to a large number of existing tools for classification, dimension reduction, clustering, etc. Different methods have been developped to send diagrams to Hilbert spaces ([1], [44], [19], [28])
- After sending diagrams into a Hilbert space, a second approach is to define kernels, allowing to compute scalar product between two diagrams ([42], [18]).
- Various approaches have been considered to allow the use of statistic and definition of mesures on diagrams, notably through landscapes ([10], [37]).
- Another approach is to look for a metric allowing a notion of barycenter. For example by replacing the bottleneck distance by the Wasserstein distance, by analogy with optimal transport theory. Comparing two persistence diagrams is in some sense equivalent to measuring the amount of mass that should be transported from one diagram to another allowing to put some on the diagonal (see [29]).

[2] We define $\overline{\mathbb{R}}$ as $\mathbb{R} \cup \{-\infty, +\infty\}$.

1.7 Interleaving and stability

For simplicity, in this part of the chapter we will assume that the index set S is either $\mathbb{R}$ or $\mathbb{R}^+$ unless stated otherwise. The definitions actually make sense for subsets of $\mathbb{R}$ stable by addition, even discrete, but we would have to be more careful while stating results while decreasing readability.

1.7.1 Interleavings

In order to obtain a better understanding of the stability properties of persistence and get rid of the triangulable space hypothesis in Theorem 1.5.5, in 2009 Chazal et al. introduced the interleaving distance in [21]. A good detailed introduction to interleavings, related properties and the interpolation lemma would be [22] by Chazal et al.

In Section 1.3 we explained how one can classify persistence modules by identifying their isomorphism classes. This notion of isomorphism is too strong when we have to deal with modules containing noise. We would like to be able to ignore small topological features, namely small intervals modules, appearing in a decomposition of a persistence module. This motivates the definition of the relation of interleaving. First, we define a shifted module and a shifted morphism.

Definition 1.7.1. Let N be a persistence module and $\delta > 0$. The δ-shift of M, written $M[\delta]$, is the persistence module with spaces $M[\delta]_t = M_{t+\delta}$ and morphisms $M[\delta]_s^t = M_{s+\delta}^{t+\delta}$. A morphism φ of degree δ between two persistence modules M and N is a morphisms $\varphi : M \to N[\delta]$. The δ-shift of a persistence modules morphism $\psi : M \to N$ is the morphism $\psi[\delta] : M[\delta] \to N[\delta]$ given by $\psi[\delta]_t = \psi_{t+\delta}$.

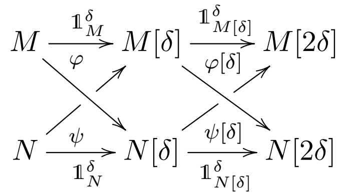
The δ-shifting is indeed a functorial operation. Given two persistence modules M and N, we write $\mathrm{Hom}^\delta(M, N)$ for the collection of morphisms from M to N of degree δ, and $\mathrm{End}^\delta(M)$ for the collection of endomorphisms. Composition of morphisms gives a map

$$\mathrm{Hom}^{\delta_2}(N, R) \times \mathrm{Hom}^{\delta_1}(M, N) \to \mathrm{Hom}^{\delta_1+\delta_2}(M, R). \tag{1.1}$$

Let M be a persistence module. A particular degree δ endomorphism is the shift map $\mathbb{1}_M^\delta$ defined by $\mathbb{1}_M^\delta : M \to M[\delta]$, $(\mathbb{1}_M^\delta)_t = M_t^{t+\delta}$. Given a persistence modules morphism $\varphi : M \to N$ and $\delta > 0$, by definition the shift map and the morphism φ commute: $\varphi \circ \mathbb{1}_M^\delta = \mathbb{1}_N^\delta \circ \varphi$. Furthermore, the collection $\{\mathbb{1}_M^\delta,\ \delta \in \mathbb{R}_{\geq 0}\}$ of shift maps on M endowed with the composition is a monoid.

Definition 1.7.2 (Interleaving)**.** Let M and N be two persistence modules indexed over S. A δ-interleaving is a pair of degree δ morphisms $\varphi : M \to N[\delta]$ and $\psi : N \to M[\delta]$ such that $\psi \circ \varphi = \mathbb{1}_M^{2\delta}$ and $\varphi \circ \psi = \mathbb{1}_N^{2\delta}$.

Given the pair (φ, ψ) as defined above, we are actually asking the commutativity of the following diagram.



Remark 1.7.3. The notion of interleaving comes naturally from the sublevelset persistence of functions. Let X be a topological space and $f, g : X \to \mathbb{R}$ two real valued functions. Suppose $\|f - g\|_\infty < \delta$. We can construct the two sublevelset persistence module $H_*(L^f)$ and $H_*(L^g)$. The inequality on their infinity distance induces on their filtrations the relations

$$L_t^f \subseteq L_{t+\delta}^g$$
$$L_t^g \subseteq L_{t+\delta}^f$$

for every $t \in S$. Applying the homology functor we get two degree δ morphisms

$$\varphi : H_*(L^f) \to H_*(L^g)[\delta]$$
$$\psi : H_*(L^g) \to H_*(L^f)[\delta]$$

which by functoriality of homology forms an interleaving pair (φ, ψ).

1.7.2 Stability

The composability of interleavings allows to define the interleaving pseudo-distance introduced in [21].

Proposition 1.7.4. *Let M and N be two persistence modules. The interleaving distance between M and N is the pseudo distance* $\mathrm{d_I}(M, N) = \inf\{\delta \in \mathbb{R}_{\geq 0} \mid M$ *and* N *are* δ*-interleaved*$\}$.

The triangle inequality is an immediate consequence of composition. Indeed, a δ_1 interleaving composed with a δ_2 interleaving gives a $\delta_1 + \delta_2$ interleaving by 1.1.

It is only a pseudo distance, since modules whose summands support is only a singleton are δ-interleaved with the zero module for all $\delta > 0$. In order to keep only the summands that are not interleaved with zero we can define the notion of radical.

Let M be a persistence module. We define its ϵ-smoothing $S^\epsilon(M)$ by $S^\epsilon(M)_t = \operatorname{Im} M_{t-\epsilon}^t \subseteq M_t$.

Definition 1.7.5 (Radical)**.** Let M be a persistence module. Its radical is the submodule

$$\operatorname{rad} M = \bigcup_{\epsilon>0} S^\epsilon(M)$$

where the union is taken pointwise since $S^\epsilon(M)_t \subseteq M_t, \forall t \in S$.

By construction $\mathrm{d_I}(M, \operatorname{rad} M) = 0$ since the radical of M is ϵ-interleaved with M for every $\epsilon > 0$. This fact is particularly useful when we study q-tame persistence modules.

Definition 1.7.6 (q-tame)**.** A persistence module M indexed over S is said to be *q-tame* if for every $s, t \in S$ we have $\operatorname{rank} M_s^t < \infty$. A function $f : X \to \mathbb{R}$ is said to be q-tame if the module $H_*(L^f)$ is q-tame.

The radical of a q-tame module is indeed interval decomposable, as proved in [23], although not always pointwise finite dimensional. It means that even if a q-tame persistence modules is not decomposable, it can be approximated by an interval decomposable module.

Example 1.7.7 (Crawley-Boevey)**.** An example from Crawley-Boevey of a q-tame module which is not interval decomposable was presented in Chapter 4 of [22]. The module

$$M = \prod_{n\in\mathbb{N}^+} \mathbb{I}^{[0,\frac{1}{n}]}$$

is uncountable dimensional at 0 (cardinality of sequences) but countable dimensional everywhere else. If it was interval dimensional, its decomposition should include infinitely many copies of $\mathbb{I}^{[0,0]}$ by cardinality of M_0. This gives a contradiction with the fact that any non-zero vector in M_0 persists for some positive time t to a space M_t.

It was noticed in [22] that the space of persistence modules endowed with the interleaving distance is actually path connected.

Lemma 1.7.8 (Interpolation)**.** *Let M and N two δ-interleaved persistence modules. There exists a 1-parameter family of persistence modules $\Gamma^x, x \in [0, \delta]$ with $\Gamma^0 = M$ and $\Gamma^\delta = N$ such that the modules Γ^x and Γ^y are $|x-y|$-interleaved for every $x, y \in [0, \delta]$.*

This result is equivalent to the existence of an extension of a certain module indexed over $\mathbb{R}^2$, a construction that will be introduced in the next chapter. This module is obtained from the embedding of two one dimensional modules along two parallel diagonal lines. An explicit construction is given in [22], but it can be obtained as an immediate corollary of the existence of Kan extensions. We show a proof later at Remark 3.1.3 in Chapter 3.

The main motivation behind the introduction of interleaving distance is the algebraic stability theorem.

Theorem 1.7.9 (Algebraic Stability)**.** *Let M and N be two q-tame persistence modules. Then* $\mathrm{d_b}(\mathcal{B}(M), \mathcal{B}(N)) \leq \mathrm{d_I}(M, N)$.

This result was proven by Chazal et al. in [22] using rectangle measures. It was later proved by Ulrich Bauer and Michael Lesnick in [4] with a simpler approach called *induced matching*. It shows that given any of the two functions forming an interleaving, one can directly construct a matching. This construction is non-functorial and the two functions forming an interleaving can give rise to two different matchings.

The converse inequality is also true, and the proof is more accessible. Given an ϵ-matching between the barcode of two decomposable modules, one can define an ϵ-interleave for each pair of intervals from this matching. Unmatched intervals being ϵ-interleaved with the zero module. The direct sum of this interleaving give an ϵ-interleaving between both modules.

In the case of q-tame modules, barcodes are defined through smoothing which allow to obtain a decomposition (for example, by considering its radical). A q-tame module is ϵ-interleaved with its radical for any $\epsilon > 0$. Two q-tame modules whose barcodes are η-matched are then $(\epsilon + \eta)$-interleaved thanks to the composition of interleavings. Since it's true for any $\epsilon > 0$, they are indeed at interleaving distance η.

Theorem 1.7.10 (Converse Stability)**.** *Let M and N be two q-tame persistence modules. Then* $\mathrm{d_I}(M, N) \leq \mathrm{d_b}(\mathcal{B}(M), \mathcal{B}(N))$.

This two results, stability and converse stability, together are called the isometry theorem. The algebraic stability theorem allows us to generalize Theorem 1.5.5 by removing the hypothesis on the space.

Let $f, g : X \to \mathbb{R}$ be two real valued functions on a topological space X. As mentioned in 1.7.3 at the end of the previous section, if the infinity norm of the two functions is smaller than δ the two sublevelset persistence modules $H_*(L^f)$ and $H_*(L^g)$ are δ-interleaved. This induces the inequality $\mathrm{d_I}(H_*(L^f), H_*(L^g)) \leq \|f - g\|_\infty$. Now suppose that both f and g are q-tame, one can apply the stability theorem. This is also the case for Morse functions on a compact manifold, whose persistence modules are p.f.d. We obtain a new expression of the stability theorem in term of infinity norm on f and g:

$$\mathrm{d_b}(\mathcal{B}(f), \mathcal{B}(g)) \leq \|f - g\|_\infty.$$

1.8 Algorithm

The one dimensional persistent homology comes with various algorithms ([49], [35], [29], [41], [3]) which allow to compute the interval decomposition of a module generated from a simplicial filtration. We start by introducing simplicial homology which allows to rephrase the computation problem in simpler terms.

1.8.1 Simplicial homology

A *simplex* is the convex hull of $k+1$ vertices[3] living in some space $\mathbb{R}^m, m \geq k$. A face is the convex hull of k of this vertices, and therefore included in the simplex.

Definition 1.8.1. A *simplicial complex* K is a set of simplicies that satisfies

- every face of a simplex of K is in K,
- the intersection of two simplices $\sigma_1, \sigma_2 \in K$ is in K

and existing as a subset of a space $\mathbb{R}^m$ for a $m \geq 1$.

Simplicial complexes are topological spaces and one can compute its singular homology. But the advantage of these spaces is that a more combinatorial way of computing homology arise. Although not obvious, this definition gives the same homology groups (See [34]).

To compute the homology of a simplicial complex, one defines a boundary operator on a simplex by taking an alternating sum of its faces. As in the case of singular homology, one consider an oriented simplex $[v_0, \ldots, v_n]$ defined by its ordered vertices. We say that two oriented simplices are equal, up to a sign, if their order differs by a transposition. For example $[v_0, v_1, v_2] = -[v_1, v_0, v_2]$. We then define a boundary operator

$$\partial_n([p_0, \ldots, p_n]) = \sum_{0 \leq i \leq n} (-1)^i [p_0, \ldots, \hat{p_i}, \ldots, p_n]$$

where $\hat{p_i}$ means the term was removed. Quotient of kernels by images of this collection of operators give us the homology of the complex.

1.8.2 Persistence

We now introduce a graded version of simplicial homology which will allow to compute persistent homology at once.

[3] The $k+1$ points should be affinely independent. If the vertices are $u_0, \ldots, u_k$, we are asking for $u_1 - u_0, \ldots, u_k - u_0$ to be linearly independent.

Definition 1.8.2. A *simplicial filtration* $\mathcal{K}$ indexed over a finite set $T = [\![1, \dots, m]\!] \subseteq \mathbb{N}$ is a collection of simplicial complex $\mathcal{K}_t, t \in T$ such that $t \leq t' \Rightarrow \mathcal{K}_t \subseteq \mathcal{K}_{t'}$ and $K_0 = \emptyset$. We ask our filtration to differ by only one simplex at each time step. In equation $\forall t \in T$, $\mathcal{K}_{t+1} \backslash \mathcal{K}_t = \{\sigma_t\}$.

A common construction of a filtration, arising from a point cloud, is the *Rips filtration.* Let P be a subset of a metric space (M, d). The diameter of a simplex $\sigma = \{p_0, \dots, p_n\}$ is defined to be $\sup\{d(x, y) \,|\, x, y \in \sigma\}$. For $s > 0$, let F_s be the set of simplices with vertices in P which have diameter at most s. The *Rips filtration* is the filtration defined by $F_s, s \in \mathbb{R}^+$.

We denote by C_n the $\mathbf{k}[x]$ module generated by taking as basis vectors the n-simplices from a filtration. Thus, C_n is the space of finite formal sums $\sum_i \alpha_i x^{k_i} \sigma_i$ with $\alpha_i \in \mathbf{k}$, $k_i \in \mathbb{N}$ and σ_i a n-simplex of the filtration. We get a structure of graded $\mathbf{k}[x]$ module by defining the *persistence degree* of a simplex, $\deg(\sigma)$, to be the first index s for which σ appears in F_s. The degree of an element $\alpha x^k \sigma$ with $\alpha \in \mathbf{k}$ is $k + \deg(\sigma)$.

For $n \geq 1$ the boundary operator ∂_n sends an oriented n-simplex to its boundary:

$$\partial_n([p_0, \dots, p_n]) = \sum_{0 \leq i \leq n} (-1)^i x^{\deg([p_0, \dots, p_n]) - \deg([p_0, \dots, \hat{p_i}, \dots, p_n])} [p_0, \dots, \hat{p_i}, \dots, p_n]$$

where $\hat{p_i}$ means the term was removed. We let ∂_0 be the null morphism. One can check that $\forall n \geq 1$, $\partial_{n-1} \circ \partial_n = 0$. Notice also the power of x is always positive since all the simplices composing the boundary have to appear in the filtration before the whole simplex. This boundary operator encodes the usual simplicial boundary operators on each F_s.

From [49], to compute the nth persistent homology of this filtration it is sufficient to find a decomposition of the homology $\operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}$. There are two approaches. First, one could could compute the homology of each space F_s and the morphism induced by inclusions $F_s \subseteq F_{s'}$ in order to finally build the $\mathbf{k}[x]$ module associated to the persistence module (see Section 1.3 for the $\mathbf{k}[x]$ structure associated to a persistence module). A simpler approach is to directly compute the homology $\operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}$.

An element of $\bigoplus C_n$ is said to be homogeneous if each of the elements on its sum have the same persistence degree. We compute a decomposition of the quotient of the cycles $Z_{n-1} = \operatorname{Ker} \partial_{n-1}$ by the boundaries $B_n = \operatorname{Im} \partial_n$ by induction on n.

Let M_n be the matrix associated to ∂_n in the standard basis of the domain formed of simplices sorted by persistence degree $\{e_i\}$ and a homogeneous basis $\{\hat{e}_j\}$ of Z_{n-1} (containing the whole image since $\partial_{n-1} \circ \partial_n = 0$). In the initial case M_1, the matrice in standard basis is already expressed in such form since ∂_0 is null and any element of the codomain of ∂_1 is a cycle. Let $\overline{M}_n$ design the Smith normal form[4] of the matrix M_n which can be obtained by Gaussian elimination on lines and columns. By definition $\overline{M}_n$ is a diagonal matrix, and computing the quotient of the kernel by the image is now trivial thanks to the following theorem.

Theorem 1.8.3 (Decomposition)**.** *Every graded module M over a graded principal ideal domain ring R decomposes uniquely into the form*[5]

$$(\bigoplus_{i=1}^{n} \Sigma^{\alpha_i} R) \oplus (\bigoplus_{j=1}^{m} \Sigma^{\gamma_j} R/d_j R)$$

where $d_j \in R$ are homogenous elements so that $d_j | d_{j+1}$, α_i, $\gamma_j \in \mathbb{Z}$, and Σ^α denotes an α-shift upward in grading.

Zero columns of $\overline{M}_n$ gives rise to a free $k[x]$ modules, whereas a degree positive coefficient a_{ii} gives rise to the torsion module $\mathbf{k}[x]/(a_{ii})$. The homology module is therefore given by the Cartesian product $\mathbf{k}[x]^q \times \mathbf{k}[x]/(a_{i_1 i_1}) \times \cdots \times \mathbf{k}[x]/(a_{i_p i_p})$ where $(a_{i_j i_j})$ are the non zero coefficients of $\overline{M}_n$ and q is the number of zero column.

Indeed, one can compute less than the Smith normal form. Let $\hat{M}_n$ be the column echelon[6] form (Figure 1.4) of M_n using only exchange of columns and the addition of an existing column multiplied by an integer, in order of reversing persistence degree.

$$\begin{bmatrix} \boxed{*} & 0 & & & 0 \\ & \boxed{*} & 0 & \dots & \\ & * & 0 & & \vdots \\ & & \boxed{*} & 0 & \\ & & * & 0 & \dots & 0 \end{bmatrix}$$

Figure 1.4: Column echelon form of a matrix. Stars are possibly non-zero values and pivots are boxed.

Actually, we can read the coefficients of the Smith normal form directly from the column echelon form. We say that a row or a column of a matrix in its normal form is a pivot row (resp. column) if it contains a pivot (Figure 1.4). For a column j with pivot on row i we can tell the basis element $\hat{e}_j$ associated contribute to a submodule $x^{\deg \hat{e}_j}.k[x]/(x^i)$ in the direct sum

[4]Smith normal form is computable for a morphism of module over a PID ring. In our case the coefficients live in the polynomial ring $k[x]$ which is PID.

[5]The direct sum is the Cartesian product on which we canonically add a group structure.

[6]We recall that in linear algebra a matrix is said to be in column echelon form if all non zero columns are on the left of all zero columns and the pivot (first non zero coefficient from the top) is strictly above the pivot of all the column on the right. See Figure 1.4.

decomposition. The two simplex e_j and e_i are said to be paired. Therefore, we obtain a $[\deg \hat{e}_j, \deg \hat{e}_j + i)$ persistence interval in our barcode. If the column j is zero and is not paired to any other simplex, the contribution is the free module $x^{\deg \hat{e}_j}.\mathbf{k}[x]$ and the interval in the barcode is $[\deg \hat{e}_j, +\infty)$. The pairing between simplices tells who creates and destroys topological features. For example a simplex e_j can create a loop and an other e_i will close it. A column associated to a simplex e_j which is zero is called a *positive simplex*. The simplex e_i associated to a non zero column is called a negative simplex.

The computation of the echelon form (see Algorithm 1.5) also give us a basis of Z_n. Indeed, the column operations conserve the homogeneous of the basis since $\deg \hat{e}_j + \deg M_n(i,j) = \deg e_i$. See [49] for details. This basis is given by the vectors associated to the 0 columns of $\hat{M}_n$. To continue our induction we need to express ∂_{n+1} in the sorted standard basis of C_n and the basis we computed for Z_n. The column operations on M_n gives row operations on M_{n+1} since the codomain of ∂_{n+1} matches the domain of M_n. Thanks to the relation between the operators all we have to do is to represent ∂_{n+1} in the matrix M_{n+1} in the standard basis of sorted simplices, and delete the rows j of M_{n+1} corresponding to pivot column j in M_n.

```
chain ComputeEchelonForm (M):
// Loop on each column of the matrix
for j = 1 to m do
  while ∃j0 < j with low(j0) = low(j) do
    add column j0 times −(M_{j0,j})^{−1} to column j
  end while
end for
```

Figure 1.5: Computing echelon form on column. We write $low(j)$ for the lowest non zero coefficient in column j. See [41].

1.8.3 Implementation

The number of simplices appearing in the computation becomes quickly large, and most of the current algorithms rely on the sparsity of the matrices to address this limitation. They do not represent all the rows and columns in memory. We present an elementary implementation taking advantage of this sparsity from [29] and computing all the homology at once. For simplification of the algorithm we consider the case where the field is $\mathbf{k} = \mathbb{Z}_2$.

The idea is to consider a square matrix whose lines and colomns correspond to each simplex of the filtration. The matrix contains only zeroes and ones. If we write $\partial[i,j]$ for the entry at the i-th line and j-th column, we have

$$\partial[i,j] = \begin{cases} 1 & \text{if } \sigma_i \text{ is a face of } \sigma_j \\ 0 & \text{otherwise.} \end{cases}$$

The ones appear in a column j correspond to the boundary[7] σ_j of the simplex σ_j. in $\partial\sigma_j$. Each column of the matrix is stored as a list and the collection of column is stored in an array.

We use an array T containing a slot for each simplex appearing in the Rips filtration. We order the simplex first by dimension, then by degree, and we break the remaining ties arbitrarily (for example by lexicographical order on the names of simplices). Each slot i is filled once a pivot is found at line i. A slot contain two things. First the index j of the column of this pivot. Second, the column corresponding of the image $\partial_{\dim \sigma^j}(\sigma^j)$ after Gaussian elimination. The second value may be used repeatedly during Gaussian elimination. This encode a pairing of the simplices σ^i and σ^j. The last thing is to look for remaining empty column which are not already paired to gather the infinite intervals.

```
chain  ComputePersistence (∂):
T ← empty array
for j = 1 to m do
  L ← ∂[j]
  while L ≠ null and T[i] with i = low(L) is not null do
    L = L + T[i]
  end while
  if  L ≠ null then
    T[low(L)] ← (j, L)
  end if
end for
return  T
```

Figure 1.6: Gaussian elimination for a column

[7] Here $\partial\sigma_j$ is the application of ∂_n to the simplex $\sigma_j \in C_n$.

Chapter 2

Proof of Crawley-Boevey's decomposition theorem

In this chapter we follow step by step the proof of Theorem 1.4.4 from Crawley-Boevey as presented in [26] and bring some intuition throughout the proof. This proof can be summarized in 3 main steps. Let M be a persistence module indexed on $\mathbb{R}$.

- First, for an interval $I \subseteq \mathbb{R}$, we define a way to count the multiplicity $\mathrm{mult}_{\mathbf{k}_I}(M)$ of the summand k_I in M. We get a functor $C_I : \mathbf{Vect}^{\mathbb{R}} \to \mathbf{Vect}$, $M \mapsto \mathbf{k}^{\mathrm{mult}_{\mathbf{k}_I}(M)}$.
- We then construct a submodule W_I of M isomorphic to $\mathbf{k}_I^{\mathrm{mult}_{\mathbf{k}_I}(M)}$, for every interval I.
- Finally, we prove that the sum of this submodules is
  - a direct sum,
  - generate the whole module M.

2.1 Subspaces

Let M be a persistence module indexed on $\mathbb{R}$. We say that M has the *descending chain* condition on kernels and images if for all $t, t_1, t_2, \dots \in \mathbb{R}$ with $t \leq t_1 < t_2 < \dots$ the chain

$$M_t \supseteq \operatorname{Ker} M_t^{t_1} \supseteq \operatorname{Ker} M_t^{t_2} \supseteq \dots$$

stabilizes and for all $t, t_1, t_2, \dots \in \mathbb{R}$ with $t \geq t_1 > t_2 > \dots$ the chain

$$M_t \supseteq \operatorname{Im} M_{t_1}^t \supseteq \operatorname{Im} M_{t_2}^t \supseteq \dots$$

stabilizes. This condition is respected when the the module is point-wise finite dimensional. We want to prove the following result.

Theorem 2.1.1. *Let M be a persistence module indexed on $\mathbb{R}$ with the descending chain condition on images and kernel. Then M has a decomposition into interval modules.*

We need a tool in order to look at the begining and end of an interval submodule. A *cut* is a partition c of $\mathbb{R}$ into two (possibly empty) sets c^-, c^+ such that $x < y$ for all $x \in c^-$ and $y \in c^+$. A pair of cuts can be used to describe a real interval. The first cut determines the beginning of the interval, and the second one the end. All types of intervals (open, closed, half open, ...) can be handled with this representation. Keeping this idea in mind, we will talk of c^- as the left end of the cut c, and c^+ at its right end. The reals in $\overline{c^+} \cap \overline{c^-}$ will be designated as the middle of the cut c.

For each real interval, described by two cuts, we are going to retrieve its multiplicity in the decomposition of a module We use cuts to select specific subspaces of the module M which will allow later to build interval submodules of M.

Let c be a cut and $t \in c^+$. We define the subspaces $\mathrm{Im}_{ct}^- \subseteq \mathrm{Im}_{ct}^+ \subseteq M_t$ as

$$\mathrm{Im}_{ct}^- = \bigcup_{s \in c^-} \mathrm{Im}\, M_s^t, \qquad \mathrm{Im}_{ct}^+ = \bigcap_{s \in c^+,\, s \leq t} \mathrm{Im}\, M_s^t.$$

Symmetrically, let c be a cut and $t \in c-$. We define the subspaces $\mathrm{Ker}_{ct}^- \subseteq \mathrm{Ker}_{ct}^+ \subseteq M_t$ by

$$\mathrm{Ker}_{ct}^- = \bigcup_{r \in c^-,\, t \leq r} \mathrm{Ker}\, M_t^r, \qquad \mathrm{Ker}_{ct}^+ = \bigcap_{r \in c^+} \mathrm{Ker}\, M_t^r.$$

By convention $\mathrm{Im}_{ct}^- = 0$ when $c^- = \emptyset$ and $\mathrm{Ker}_{ct}^+ = M_t$ when $c^+ = \emptyset$.

Im_{ct}^- contain vectors which started to come alive on the left of the cut and stay alive until time t. Im_{ct}^+ contain the vectors which are alive along the right of the cut and stay alive until time t. If we would take the quotient of Im_{ct}^+ by Im_{ct}^- we will have only the vectors that started to come to life right at the beginning of the cut. This is the main idea behind the definition of the pair $(\mathrm{Im}_{ct}^-, \mathrm{Im}_{ct}^+)$.

Similarly, Ker_{ct}^+ contains all the vectors which are mapped to zero uniformly on the right of the cut, whereas Ker_{ct}^- contain only the vectors among them which die on the left of the cut.

In other words, the $\mathrm{Im}^{\pm}$ control the left of the interval module we want to recover from M, and the $\mathrm{Ker}^{\pm}$ control the right end of this interval module. They allow to make the difference between closed or open intervals. The next lemma will allow us to manipulate the $\mathrm{Im}^{\pm}$ and $\mathrm{Ker}^{\pm}$ in a handy way. Given one of these four spaces, one can find a structural morphism whose kernel (or image) is actually equal to it.

Lemma 2.1.2 (Realization). *Let c be a cut.*

- *If $t \in c^+$, then $\exists s \in c^+$ with $s \leq t$ such that* $\mathrm{Im}^+_{ct} = \mathrm{Im}\, M^t_s$.
- *If $t \in c^-$, $c^+ \neq \emptyset$, then $\exists r \in c^+$ such that* $\mathrm{Ker}^+_{ct} = \mathrm{Ker}\, M^r_t$.

Proof. Suppose for all $s \in c^+$, $s \leq t$, $\mathrm{Im}^+_{ct} \neq \mathrm{Im}\, M^t_s$. Let $s_1 \in c^+, s_1 \leq t$. Since $\mathrm{Im}^+_{ct} \neq \mathrm{Im}\, M^t_{s_1}$ one can find $s_2 \in c^+$ such that $\mathrm{Im}\, M^t_{s_2}$ is strictly contained in $\mathrm{Im}\, M^t_{s_1}$. Applying this argument again to $\mathrm{Im}^+_{ct} \neq \mathrm{Im}\, M^t_{s_2}$ we obtain $\mathrm{Im}\, M^t_{s_3}$ strictly contained in $\mathrm{Im}\, M^t_{s_2}$. By repeating this construction we build a chain $\mathrm{Im}\, M^t_{s_1} \supset \mathrm{Im}\, M^t_{s_2} \supset \mathrm{Im}\, M^t_{s_3} \supset \dots$ that does not stabilize. Hence a contradiction. The second result is proved similarly. □

Using this tool, one can prove that the spaces $\mathrm{Im}^{\pm}_{ct}$ and $\mathrm{Ker}^{\pm}_{ct}$ are not entirely dependent on the point t. They are related with spaces defined through the same cut but different location t'.

Lemma 2.1.3 (Transportation). *Let c be a cut, and $s \leq t$ two reals.*

a) If $s, t \in c^+$, $\mathrm{Im}^{\pm}_{ct} = M^t_s(\mathrm{Im}^{\pm}_{cs})$

b) If $s, t \in c^-$, $\mathrm{Ker}^{\pm}_{cs} = (M^t_s)^{-1}(\mathrm{Ker}^{\pm}_{ct})$.

Proof. We have $M^t_s(\mathrm{Im}^+_{cs}) \subseteq \cap_u M^t_s(\mathrm{Im}\, M^t_u)) = \mathrm{Im}^+_{ct}$. By realization (Lemma 2.1.2), there exists $r \in c^+$, $r \leq s$ such that $\mathrm{Im}^+_{cs} = \mathrm{Im}\, M^s_r$. We get

$$M^t_s(\mathrm{Im}^+_{cs}) = M^t_s(\mathrm{Im}\, M^s_r) = \mathrm{Im}\, M^t_r \supseteq \mathrm{Im}^+_{ct}\,.$$

The other cases are either immediate or similar. □

2.2 Counting functor

Our next goal is to define an object that does not depend on the location $t \in \mathbb{R}$ anymore, but only on cuts. Let I be an interval. It is uniquely determined by two cuts l and u such that $l^+ \cap u^- = I$ (see Figure 2.2). They are defined by

$$l^- = \{t |\, \forall s \in I,\, t < s\}, \qquad l^+ = \{t |\, \exists s \in I,\, s \leq t\},$$
$$u^- = \{t |\, \exists s \in I,\, t \leq s\}, \qquad u^+ = \{t |\, \forall s \in I,\, s < t\}.$$

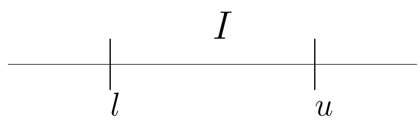


Figure 2.1: An interval determined by its two cuts.

Given an interval I and $t \in I$ we define the spaces $V_{It}^{-} \subseteq V_{It}^{+} \subseteq M_t$ by

$$\begin{aligned} V_{It}^{-} &= (\mathrm{Im}_{lt}^{-} \cap \mathrm{Ker}_{ut}^{+}) + (\mathrm{Im}_{lt}^{+} \cap \mathrm{Ker}_{ut}^{-}), \\ V_{It}^{+} &= \mathrm{Im}_{lt}^{+} \cap \mathrm{Ker}_{ut}^{+}. \end{aligned}$$

The space V_{It}^{+} keeps track of all vector that stay alive in the interval I when pushed by structural morphisms, whereas V_{It}^{-} contains the subcollection of the vectors which started to live and will die outside of the interval I. See Figure 2.2 for some examples.

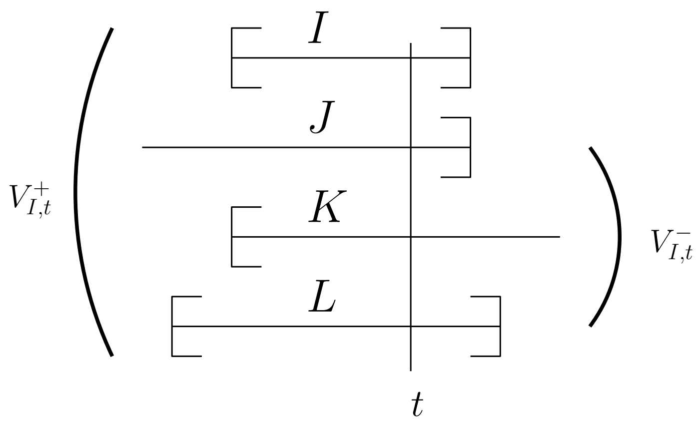


Figure 2.2: Examples of support of interval submodules whose vectors at index t will be contain in $V_{I,t}^{+}$ and $V_{I,t}^{-}$.

The following lemma shows that the definition of the quotient space V_{It}^{+}/V_{It}^{-} do not depend on the point $t \in I$. It is in some sense similar to the transportation lemma (2.1.3) and happens along the interval I.

Lemma 2.2.1. *Let I be an interval and $s, t \in I$ with $s \leq t$. We have the relation $M_s^t(V_{Is}^{\pm}) = V_{It}^{\pm}$. Furthermore the quotient map*

$$\overline{M_s^t} : V_{Is}^{+}/V_{Is}^{-} \to V_{It}^{+}/V_{It}^{-}$$

is an isomorphism.

Proof. The first relations are a consequence of the transportation lemma 2.1.3. We have $M_s^t(V_{Is}^{\pm}) \subseteq V_{It}^{\pm}$ by rules on the image of intersections and sums. Consider the case V_{Is}^-. Suppose $h \in (\mathrm{Im}_{lt}^+ \cap \mathrm{Ker}_{ut}^-)$. Then by transportation $\exists g \in \mathrm{Im}_{ls}^+$ such that $h = M_s^t(g)$. But $g \in (M_s^t)^{-1}(\{h\}) \subseteq (M_s^t)^{-1}(\mathrm{Ker}_{ut}^-) = \mathrm{Ker}_{us}^-$ by transportation. So $g \in \mathrm{Im}_{ls}^+ \cap \mathrm{Ker}_{us}^-$ and therefore $h \in M_s^t(V_{It}^+)$. Similarly we can find an inverse image to $h' \in (\mathrm{Im}_{lt}^- \cap \mathrm{Ker}_{ut}^+)$. This proves $M_s^t(V_{Is}^-) = V_{It}^-$.

The quotient map is surjective since $M_s^t(V_{Is}^+) = V_{It}^+$. To show the injectivity, it is sufficient to prove $V_{Is}^+ \cap (M_s^t)^{-1}(V_{It}^-) \subseteq V_{Is}^-$. Let $g \in V_{Is}^+$ and suppose $M_s^t(g) = h \in V_{It}^-$. Then h decomposes into the sum $h_1 + h_2 = h$ with $h_1 \in \mathrm{Im}_{lt}^- \cap \mathrm{Ker}_{ut}^+$ and $h_2 \in \mathrm{Im}_{lt}^+ \cap \mathrm{Ker}_{ut}^-$. We can find $g_1 \in \mathrm{Im}_{ls}^- \cap \mathrm{Ker}_{us}^+$ such that $M_s^t(g_1) = h_1$ by applying transportation to Im_{ls}^- and Ker_{us}^+. We deduce $M_s^t(g - g_1) = h_2 \in \mathrm{Im}_{lt}^+ \cap \mathrm{Ker}_{ut}^- \subseteq \mathrm{Ker}_{ut}^-$. This induces $g - g_1 \in \mathrm{Ker}_{us}^-$ by transportation. In the same way $g - g_1 \in \mathrm{Im}_{us}^+$. Therefore $g \in g_1 + \mathrm{Im}_{us}^+ \cap \mathrm{Ker}_{ut}^- \subseteq V_{Is}^-$. □

We now define a global object $V_I^{\pm}$ for each interval I which allow us to retrieve the spaces $V_{It}^{\pm}$, $t \in \mathbb{R}$.

Let I be an interval. The collection of spaces $V_{It}^{\pm}, t \in I$ together with the maps $M_s^t : V_{Is}^{\pm} \to V_{It}^{\pm}, s \leq t \in I$ form two projective systems[1]. This allows us to consider the inverse limits

$$V_I^{\pm} = \varprojlim_{t \in I} V_{It}^{\pm}.$$

Let $\pi_t : V_I^+ \to V_{It}^+$ be the natural projection of the inverse limit on space located at t. One can make the identification

$$V_I^- = \bigcap_{t \in I} \pi_t^{-1}(V_{It}^-) \subseteq V_I^+$$

since all projective limits $\varprojlim V_{It}^-$ are isomorphic.

It is natural to ask ourselves whether V_{It}^+/V_{It}^- is related to V_I^+/V_I^-. In general, these two objects can be different. But in our specific case they are identical. To prove this result, first we need the following lemma.

Lemma 2.2.2. *A subset $S \subseteq I$ of an interval I is said coinitial if $\forall t \in I$, $\exists s \in S, s \leq t$. Any interval I has a co-initial countable subset.*

Proof. If I has a minimum m, take $S = \{m\}$. Otherwise, take $\mathbb{Q} \cap I$. □

Now we can prove the following result.

[1] To actually obtain a projective system, we have to reverse the order on $I \subseteq \mathbb{R}$.

Lemma 2.2.3. *Let I be an interval and $t \in I$. The induced map*

$$\overline{\pi_t} : V_I^+/V_I^- \to V_{It}^+/V_{It}^-$$

is an isomorphism.

Proof. First, the inverse system V_{It}^- is Mittag-Leffler [46, Section 10.85] which means that for every t the family $M_s^t(V_{It}^-) \subseteq V_{It}^-$ for $s \leq t$ stabilizes. This is immediate since $\forall s, M_s^t(V_{It}^-) = V_{It}^-$.

We now consider the exact sequence

$$0 \to V_{It}^- \to V_{It}^+ \to V_{It}^+/V_{It}^- \to 0.$$

By Lemma 2.2.2 the hypothesis of proposition [32, Chap. 0, (13.2.2)] holds and by applying the inverse limit functor, we obtain

$$0 \to V_I^- \to V_I^+ \to \varprojlim_{t \in I} V_{It}^+/V_{It}^- \to 0$$

which is also exact. Since all the $\overline{M_s^t}$ are isomorphism by Lemma 2.2.1, we have $\forall t$, $\varprojlim_{t \in I} V_{It}^+/V_{It}^- \simeq V_{It}^+/V_{It}^-$. Looking at the morphisms between the two sequences, from this we deduce the result. □

We can define a functor $C_I : \mathbf{Vect}^{\mathbb{R}} \to \mathbf{Vect}$ which sends M to the quotient V_I^+/V_I^-. This functor "counts" the number of modules intervals of shape I that will appear in a decomposition of the module M. More precisely, the dimension $\dim C_I(M)$ will be exactly the multiplicity of the interval I in the decomposition of M. In order to find this summands as submodules of M, we first take W_I^0 to be a vector space complement of V_I^- in V_I^+. Then, we can sent it back into the module through π_t.

By Lemma 2.2.3 the restriction of π_t to W_I^0 is injective.

Lemma 2.2.4. *Let*

$$(W_I)_t = \begin{cases} \pi_t(W_I^0) & \text{if } t \in I, \\ 0 & \text{if } t \notin I. \end{cases}$$

This defines a submodule W_I of M.

Proof. For $s \leq t$ both in I, we have $M_s^t \circ \pi_s = \pi_t$ so $M_s^t(W_{Is}) = W_{It}$. If $s \in I$ and $t \notin I$ then $t \in u^+$ which implies $W_{Is} \subseteq \mathrm{Ker}_{us}^+ \subseteq \mathrm{Ker}\, M_s^t$. In case $s \notin I, s \leq t$, $W_{Is} = 0$ implies $M_s^t(W_{Is}) = 0 \subseteq W_{It}$. □

By Lemma 2.2.3 we get a pointwise decomposition $V_{It}^+ = W_{It} \oplus V_{It}^-$. We now want a direct sum decomposition of the whole module W_I into interval modules.

Lemma 2.2.5. *For each interval I, we have* $W_I \simeq \bigoplus_{b\in B} k_I$ *where* $|B| = \dim C_I(M)$.

Proof. Let B be a basis of W_I^0 and $\forall b \in B, b_t = \pi_t(b)$. For all t a basis of W_{It} is given by $\{b_t, b \in B\}$. Since $M_s^t(b_s) = b_t$ for any $s \leq t$ in I they span a submodule of W_I isomorphic to $\mathbf{k}_I$. Therefore $W_I \simeq \bigoplus_{b\in B} k_I$. □

We expect the direct sum of W_I running over all the possible intervals I to give us a decomposition of M.

2.3 Decomposition

It remains to show that

- the sum of the W_I is direct and,
- it generates the whole module M.

To prove this result we introduce the notion of a *section*[2]. We first expose properties of sections, and then latter apply them to the spaces we constructed. A section of a vector space U is a pair (F^-, F^+) of subspaces such that $F^- \subseteq F^+$. A family of sections $\{(F_\lambda^-, F_\lambda^+) |\, \lambda \in \Lambda\}$ is said to be *disjoint* if $\forall \lambda \neq \mu$ either $F_\lambda^+ \subseteq F_\mu^-$ or $F_\mu^+ \subseteq F_\lambda^-$. Disjointness is a notion of order on a family of sections. It is said to *cover* U if for all subspaces $X \subset U$, $X \neq U$ there is λ such that $X + F_\lambda^- \neq X + F_\lambda^+$. It *strongly covers* U if for all subspace $Y, Z \subseteq U$, $Z \not\subset Y$ we can find a λ such that $Y + (F_\lambda^- \cap Z) \neq Y + (F_\lambda^+ \cap Z)$. The notion of covering ask, given a proper vector subspace space X, the possibility to find an element in the family such that this pair separate X and the whole vector space.

The notion of disjointness gives the direct nature of the sum whereas the covering property will imply the generativity.

Lemma 2.3.1. *Let* $\{(F_\lambda^-, F_\lambda^+) |\, \lambda \in \Lambda\}$ *be a family disjoint of sections covering* U. *Let* W_λ *be a complement vector space* $W_\lambda \bigoplus F_\lambda^- = F_\lambda^+$ *for each* $\lambda \in \Lambda$. *Then we have the decomposition of the vector space* $U = \bigoplus_{\lambda\in\Lambda} W_\lambda$.

Proof. Let $v_{\lambda_1} + v_{\lambda_2} + \dots v_{\lambda_n} = 0$ be a linear relation of elements in $\sum W_{\lambda_i}$. The disjointness give us a total order on the pairs. By reordering one can assume that $F_{\lambda_i}^+ \subseteq F_{\lambda_1}^-$ for all $i > 1$. But since $v_{\lambda_1} = -\sum v_{\lambda_i} \in \sum F_{\lambda_i}^+ \subseteq F_{\lambda_1}^-$ we get $v_{\lambda_1} \in W_{v_{\lambda_1}} \cap F_{v_{\lambda_1}}^- \Rightarrow v_{\lambda_1} = 0$. The sum is therefore direct.

[2]This notion of section is not related to sections of a sheaf.

Let $X = \bigoplus_{\lambda \in \Lambda} W_\lambda$ and by absurd suppose $X \neq U$. By disjointness covering one can find λ such that $F_\lambda^- + X \neq F_\lambda^+ + X$. But $F_\lambda^+ + X = F_\lambda^- + W_\lambda + X = F_\lambda^- + X$ which gives a contradiction. □

The next lemma tells us how we can combine two families together.

Lemma 2.3.2. *Let $\{(F_\lambda^-, F_\lambda^+) |\, \lambda \in \Lambda\}$ and $\{(G_\gamma^-, G_\gamma^+) |\, \gamma \in \Gamma\}$ be two families of disjoint sections. The family*

$$\{(F_\lambda^- + G_\gamma^- \cap F_\lambda^+, F_\lambda^- + G_\gamma^+ \cap F_\lambda^+) |\, (\lambda, \gamma) \in \Lambda \times \Gamma\}$$

is disjoint. Let $\{(F_\lambda^-, F_\lambda^+) |\, \lambda \in \Lambda\}$ be a family of sections covering U, and $\{(G_\gamma^-, G_\gamma^+) |\, \gamma \in \Gamma\}$ be a family of sections strongly covering U. The family

$$\{(F_\lambda^- + G_\gamma^- \cap F_\lambda^+, F_\lambda^- + G_\gamma^+ \cap F_\lambda^+) |\, (\lambda, \gamma) \in \Lambda \times \Gamma\}$$

is disjoint and covering U.

Proof. Disjointness results immediately from the disjointness of Λ and Γ indexed families.

Let $X \neq U$ bet a subspace of U. By covering of U by the Λ family, we can find a $\lambda \in \lambda$ such that

$$F_\lambda^- + X \neq F_\lambda^+ + X.$$

Let $Y = F_\lambda^- + X$ and $Z = F_\lambda^+$. We have $Z \not\subset Y$ by choice of λ and by applying the strong covering property of the Γ family we get a γ such that

$$Y + (G_\gamma^- \cap Z) \neq Y + (G_\gamma^+ \cap Z).$$

Developing the expression, we obtain

$$X + F_\lambda^- + (G_\gamma^- \cap F_\lambda^+) \neq X + F_\lambda^- + (G_\gamma^+ \cap F_\lambda^+).$$

□

We now want to apply our tools to our quotient spaces. We start with a lemma telling us that the families of kernels and images we defined earlier are disjoint and strongly covering.

Lemma 2.3.3. *The families*

- $\{(\mathrm{Im}_{ct}^-, \mathrm{Im}_{ct}^+) |\, c$ *is a cut and* $t \in c^+\}$, *and*
- $\{(\mathrm{Ker}_{ct}^-, \mathrm{Ker}_{ct}^+) |\, c$ *is a cut and* $t \in c^-\}$

are disjoint and strongly cover M_t.

Proof. We treat the first case. Let c and d be two cuts with $t \in c^+ \cap d^+$. By renaming one can assume $c^+ \cap d^- \neq \emptyset$. Let $s \in c^+ \cap d^-$. By definition $s < t$ and

$$\mathrm{Im}^+_{ct} \subseteq \mathrm{Im}\, M^t_s \subseteq \mathrm{Im}^-_{dt} .$$

This gives the disjointness.

Let $Y, Z \subseteq M_t$ with $Z \not\subseteq Y$. We look at the cut c defined[3] by

$$c^- = \{s |\, M^t_s \cap Z \subseteq Y\}, \qquad c^+ = \{s |\, M^t_s \cap Z \not\subseteq Y\}.$$

It is well define since the two sets form a partition of $\mathbb{R}$: they are disjoint, any real is in one or another and they both are intervals by inclusions of the images of structural morphisms.

On one side we have

$$Y + (\mathrm{Im}^-_{ct} \cap Z) = Y + (\bigcup_{s \in c^-} \mathrm{Im}\, M^t_s \cap Z) = \bigcup_{s \in c^-} (Y + \mathrm{Im}\, M^t_s \cap Z) = Y.$$

By realization lemma (2.1.2) we can find $s < t$, $s \in c^+$ such that $\mathrm{Im}\, M^t_s = \mathrm{Im}^+_{ct}$. Hence the strong covering property

$$Y + \mathrm{Im}^+_{ct} \cap Z = Y + \mathrm{Im}\, M^t_s \cap Z \neq Y.$$

□

We can now apply the tools developed on sections to our families. We first combine them using the Lemma 2.3.2. We then prove the sum of our interval modules is direct and generate the whole module M. To do so, we prove that this is the case pointwise for every $t \in \mathbb{R}$ using Lemma 2.3.1.

Proof of Theorem 2.1.1. Let I be an interval. Let l and u be the two cuts associated to the interval I. We define

$$F^\pm_{It} = \mathrm{Im}^-_{lt} + V^\pm_{It} = \mathrm{Im}^-_{lt} + \left(\mathrm{Ker}^\pm_{ut} \cap \mathrm{Im}^+_{lt}\right) .$$

The pairs of cuts u, l with $t \in l^+ \cap u^-$ are in bijection with the real intervals containing t. By Lemma 2.3.2 and Lemma 2.3.3 the family of sections (F^-_{It}, F^+_{It}) is disjoint and cover M_t.

As mentioned earlier we have $V^+_{It} = W_{It} \oplus V^-_{It}$ (by Lemma 2.2.3). By adding Im^-_{lt} we deduce $F^+_{It} = W_{It} + F^-_{It}$. Moreover $F^-_{It} \cap W_{It} \subseteq (\mathrm{Im}^-_{lt} + V^-_{It}) \cap V^+_{It} \subseteq V^-_{It} + \mathrm{Im}^-_{lt} \cap \mathrm{Ker}^+_{ut} \subseteq V^-_{It}$ and therefore $F^-_{It} \cap W_{It} \subseteq V^-_{It} \cap W_{It} = \{0\}$ so the sum $F^+_{It} = W_{It} \oplus F^-_{It}$ is direct.

[3]In the case $s > t$, for $\mathrm{Im}\, M^t_s$ we take the whole space V_t.

By Lemma 2.3.1 we can break the space M_t in a sum of W_{It}. Indeed $M_t = \bigoplus_{I \subseteq \mathbb{R}} W_{It}$. Since this is pointwise true for every $t \in \mathbb{R}$, we obtain

$$M = \bigoplus_{I \subseteq \mathbb{R}} W_I = \bigoplus_{I \subseteq \mathbb{R}} (k_I)^{m_I}$$

where m_I is $\dim C_I(M)$. $\square$

This concludes the proof of the decomposition of p.f.d. persistence modules, and opens the way to its generalization in higher dimensions.

Chapter 3

Multipersistence

The one dimensional persistence paradigm allows us to recover topological and some geometrical information from the sampling of an object, even in high dimension, summarized as a simple collection of intervals $\mathcal{B}$ called a barcode. But one dimensional persistent homology has two major flaws. Some constructions are not robust with respect to the addition of outliers. The simple addition of a unique point can change drastically the barcode. Secondly, if we use a Rips or Cech filtration on a point cloud, it does not always take into account topological features only present at a specific density level. More generally, persistence can only take one real valued function (distance, density, ...) at a time. These two facts make one persistent homology unable to recover the topological information we are looking for when the data contain even a slight uniform noise – See Figure 3.1.

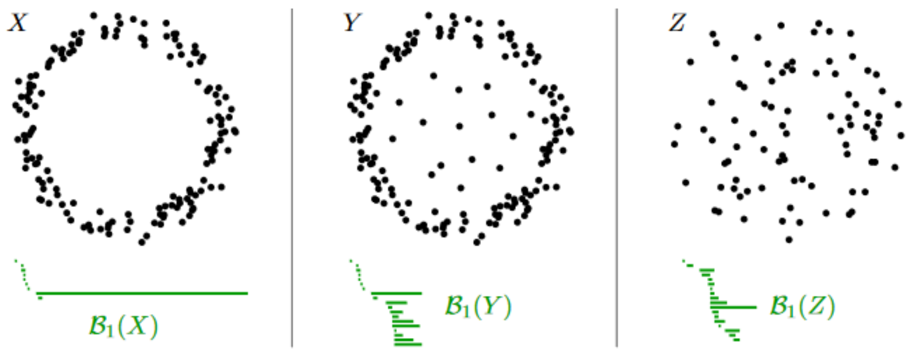


Figure 3.1: Image from [39]. Three Rips filtrations X, Y and Z. One dimensional persistence barcodes are unstable to the addition of noise and not sensitive to high density areas. Indeed the points on the circle with the addition of noise (center) is closer, according to the bottleneck metric, to pure noise (right) than to the points sampled on this circle without noise (left).

An approach to solving this problem is to consider, instead of a single $\mathbb{R}$ valued function, a pair or more of functions. For example, one could consider the distance d_P to our point cloud P, and a density function γ with high value on dense areas, and low value on outliers. We can then build a filtration indexed by these two functions, namely[1]

$$F_{x,y} = d_P^{-1}((-\infty, x]) \cap \gamma^{-1}([-y, +\infty)).$$

More generally we could take any function $f : X \to \mathbb{R}^2$ defined on a topological space X and pre-images of bidimensional sublevelsets $f^{-1}((-\infty, x] \times (-\infty, y])$.

A second approach is to filter a point cloud by density – for example, using k-neighbors, or any other density estimate – and for each slice build the Rips filtration:

$$F_{x,y} = Rips(\gamma^{-1}([-y, +\infty))_x$$

All these different constructions give a bifiltration, that is a collection of topological spaces $F_{x,y}$ such that $x' \geq x, y' \geq y \Rightarrow F_{x,y} \subseteq F_{x',y'}$. By applying the homology functor on it, we obtain a *multipersistence module.* Multipersistence does not have decomposition in interval modules as provided in one dimension by Theorem 1.4.4 and presents to us new challenges.

In this chapter, we define multidimensional persistence formally, recall some established results from the community and in the second section present new ones.

3.1 Theory

3.1.1 Multidimensional persistence modules

Multipersistence have already been studied in various way through numerous articles. Notably [17], [9] and [39].

Definition 3.1.1. Let $(S, \leq)$ be a totally ordered set and n be a positive integer. For $s, t \in S^n$ we say $s \leq t$ when $\forall 1 \leq i \leq n$, $s_i \leq t_i$. This defines a partial order on S^n. Let P be a poset (partially ordered set). A *P-filtration* $\mathcal{F}$ is a collection of topological spaces F_s with $s \in P$ such that $\forall s, t \in P$, $s \leq t \Rightarrow F_s \subseteq F_t$. An *$n$-dimensional multifiltration* $\mathcal{F}$, or simply a *multifiltration*, is an S^n-filtration.

[1] The presence of a sign in front of y is only an aesthetic trick so that the structural morphisms compose in the horizontal line from left to right. In the absence of this sign, the morphisms would be reversed along the horizontal line.

As we mentioned previously, the application of the homology functor to a filtration gives rise to a persistence module.

Definition 3.1.2. Let P be a poset. It can be identified with the category whose objects are elements of P and an arrow corresponds to a relation given by the partial order on P. A *P-persistence module* $M \in \mathbf{Vect}^P$ is a functor from the category P to the category of vector spaces $\mathbf{Vect}$. For $s, t \in S^n$, the morphism M_s^t is called a *structural morphism.* Let $(S, \leq)$ be a totaly ordered set and n be a positive integer. An *n-dimensional persistence module* M, or simply a *multipersistence module*, is an S^n-persistence module.

The special case of q 2-dimensional persistence module is referred to as *bimodules*.

Example 3.1.3. An example of a bimodule on $[\![0,1]\!]^2$:

$$\begin{array}{ccc} \mathbf{k} & \xrightarrow{id} & \mathbf{k} \\ {\scriptstyle 0}\uparrow & & \uparrow{\scriptstyle id} \\ 0 & \xrightarrow[0]{} & \mathbf{k} \end{array}$$

An S^n-persistence module M can be associated to a S^n-graded $\mathbf{k}[S^n]$ module

$$\mathbb{M} = \bigoplus_{s \in S^n} M_s.$$

For $m \in M_t$, we define the *degree* of m, written $\deg(m)$, to be this index t. Given any $t \in S$, elements which belong to M_t are called *homogeneous.* Let $t \in S^n$ and $m = \sum_s m_s, m_s \in M_s$. The multiplication by scalars is given by

$$x^t.m = \sum_s M_s^{(s_1+t_1,\ldots,s_n+t_n)}(m_s).$$

As in the case of one dimensional persistence, we can define the notions of shifting and interleaving.

Definition 3.1.4. Let N and M be two n-dimensional persistence modules and $\delta > 0$. Let $t = (t_1, \ldots, t_n)$. By $t + \delta$ we mean $(t_1 + \delta, \ldots, t_n + \delta)$. The δ-shift of M, written $M[\delta]$, is the persistence module with spaces $M[\delta]_t = M_{t+\delta}$ and morphisms $M[\delta]_s^t = M_{s+\delta}^{t+\delta}$. A morphism φ of degree δ between M and N is a morphism $\varphi : M \to N[\delta]$. The δ-shift of an n-dimensional persistence modules morphism $\psi : M \to N$ is the morphism $\psi[\delta] : M[\delta] \to N[\delta]$ given by $\psi[\delta]_t = \psi_{t+\delta}$.

Definition 3.1.5. Let M and N be two n-dimensional persistence modules. A δ-interleaving is a pair of degree δ morphisms $\varphi : M \to N[\delta]$ and $\psi : N \to M[\delta]$ such that $\psi \circ \varphi = \mathbb{1}_M^{2\delta}$ and $\varphi \circ \psi = \mathbb{1}_N^{2\delta}$.

This allows again to put a metric structure on the category of n-dimensional persistence modules.

Proposition 3.1.6. *Let M and N be two persistence modules. The* interleaving distance $\mathrm{d_I}(M,N) = \inf\{\delta \mid M \text{ and } N \text{ are } \delta\text{-interleaved}\}$ *between M and N is a pseudo distance.*

For a function $f : X \to \mathbb{R}^n$, let[2] $S(f)$ be the n-dimensional module defined by $S(f)_{(x_1,\ldots,x_n)} = H(f^{-1}((-\infty, x_1] \times \cdots \times (-\infty, x_n]))$ This implies a soft result of stability on multidimensional persistence.

Proposition 3.1.7. *Let f and g be two functions $X \to \mathbb{R}^n$ with X a topological space. Then* $\mathrm{d_I}(S(f), S(g)) \leq ||f - g||_\infty$.

3.1.2 Interval modules

We recall the definition of interval modules introduced in [5].

Definition 3.1.8. A subset $I \subseteq S^n$ is said to be an interval if:

- for all $s, r \in I$ and for all $t \in S^n$, $s \leq t \leq r \Rightarrow t \in I$,
- for all $s, r \in I$ there exists a finite sequence $t_1, t_2, \ldots, t_k \in I$ such that $s \leq t_1 \geq t_2 \leq \cdots \geq t_k \leq r$.

This second condition is referred to as *connectivity.*

Definition 3.1.9. Let I be an interval in S^n. An *interval module* is a module $\mathbb{I}^I$ whose spaces are

$$(\mathbf{k}_I)_s = \begin{cases} \mathbf{k} \text{ if } s \in I, \\ 0 \text{ if } s \notin I, \end{cases}$$

and for $s \leq t$ in S^n, the morphisms are

$$(\mathbf{k}_I)_s^t = \begin{cases} \mathrm{id}_\mathbf{k} \text{ if } s, t \in I, \\ 0 \text{ otherwise.} \end{cases}$$

Proposition 3.1.10. *Interval modules are indecomposable.*

Proof. As in the one dimensional case (1.4.3), we look at the endomorphism ring of an interval module. The connectivity and commutativity of modules implies that a morphism between two interval modules is determined by only one morphism between two non zero spaces. The endomorphism ring is then isomorphic to the field $\mathbf{k}$, hence local, and by Azumaya's Lemma [2] the module is indecomposable. □

[2]The S stands for sublevel sets.

We now introduce the category of *thin* persistence modules. Henceforth, the dimension n of the indexing poset S^n is fixed.

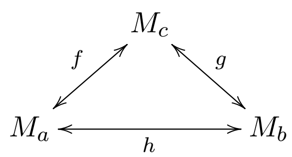
Definition 3.1.11. The category **Thin** is the category whose objects are persistence modules decomposable into direct sums of modules whose pointwise dimension is one or zero.

Remark 3.1.12. Let M be a module, pointwise one dimensional. Take its support $\mathrm{supp}(M) \subseteq S^n$ to be the set of indexes where the vector space is non-zero. We defined the relation of connectivity on the support of a module by $x \sim y \in S^n \Leftrightarrow \exists\, t_1, t_2, \ldots, t_k \in S^n$ such that $x \leq t_1 \geq t_2 \leq \cdots \geq t_k \leq y$. Each equivalence class with respect to the relation of connectivity is an interval. Furthermore, there is no non-zero morphism between two classes. Using Azumaya's Lemma, one can show that each class gives us an indecomposable module.

In dimension 2, this gives us a decomposition of M into interval modules: indeed, let N be an indecomposable submodule of M corresponding to a class of connectivity. Since M is pointwise one dimensional, any morphism between a space N_x of a thin indecomposable N and the space $\mathbb{I}_x^{\mathrm{supp}(N)}$ extends to an isomorphism between the two modules by a simple induction. It is well defined thanks to commutativity. Hence, since all the classes give submodules in direct sums, cover the whole module M, and each submodule is isomorphic to an interval module, we get the interval decomposition expected.

Remark 3.1.13. Nicolas Berkouk pointed out to me that starting with dimension 3, not all thin indecomposable modules are intervals. Here is an example from Shapira, Petit, Oudot, and Berkouk.

In $\mathbb{R}^3$ we consider three generators at coordinates $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. We move along the diagonal and take the intersection of the support of this module with a plane orthogonal to the diagonal line. While moving this plane along the diagonal $(1,1,1)$, in this intersection we see triangles growing until they touch each other as depicted in Figure 3.2 in a plane $\mathcal{P}$. We can now kill any vector belonging to the half space above the plane $\mathcal{P}$. We have the freedom of choosing any isomorphisms from our generators to the corners of the inner triangle in Figure 3.2. This leaves us with the following diagram:



with a, b and c incomparable, and all three arrows can be any isomorphism obtained by composing and inversing arrows going from and to the generators.

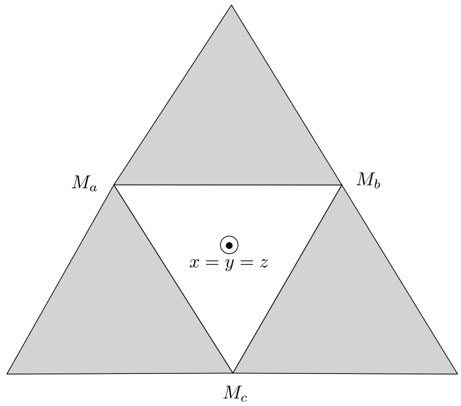
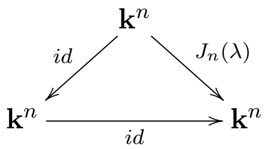


Figure 3.2: Section of a module along a plane orthogonal to the diagonal $(1, \vec{1}, 1)$.

Consider now the following triangles parametrized by the dimension n and an element $\lambda \in \mathbf{k}$:



where $J_n(\lambda)$ is the nth jordan block with coefficient λ.

Each diagram can be extended into a module thanks to our previous construction. But none of them can be decomposed into a direct sum of interval modules. The existence of such family of modules is possible thanks to the existence of a loop. It was shown by Steffen Oppermann that this do not happen with 2-dimensional persistence modules. This phenomenon, called monodromy, prevents us from defining a morphism to an interval module whose three arrows should be the identity. An other phenomenon called monodromy was introduced by Patrizio Frosini [20] but it is not clear in which way it is related to the actual monodromy of persistence modules.

3.1.3 Interpolation lemma and Kan extensions

We present a simple proof of the interpolation lemma (1.7.8) for 1-dimensional persistence modules. We need to build, given two δ interleaved persistence module M and N, a parametrized family $\Gamma^x, x \in [0, \delta]$ of one dimensional persistence modules such that $\Gamma^0 \simeq M$, $\Gamma^\delta \simeq N$ and $\mathrm{d_I}(\Gamma^x, \Gamma^y) = |x - y|, \forall x, y, \in [0, \delta]$. To do so, we construct a bimodule whose restriction along diagonal lines gives us the one dimensional modules of the parametrized family.

Let M and N be two δ-interleaved persistent modules index on $\mathbb{R}$. We start by defining a module H on Ω, the disjoint union of the lines $\Delta_M : y = x + \delta/2$ and $\Delta_N : y = x - \delta/2$.

We let $H_{(x,y)} = \begin{cases} M_{x+y} \text{ if } (x,y) \in \Delta_M, \\ N_{x+y} \text{ if } (x,y) \in \Delta_N. \end{cases}$ Morphisms between two points of Δ_M are given by structural morphisms of M. We do the same for any two points in Δ_N. Let $\varphi : M \to N[\delta]$ and $\psi : N \to N[\delta]$ be δ-interleaving morphisms. The morphism between the space at $(x,y) \in \Delta_M$ and the space at $(x+\delta, y) \in \Delta_N$ is defined by $H_{(x,y)}^{(x+\delta,y)} = \varphi_{x+y}$. Symmetrically for $(x,y) \in \Delta_N$ we define $H_{(x,y)}^{(x,y+\delta)} = \psi_{x+y}$.

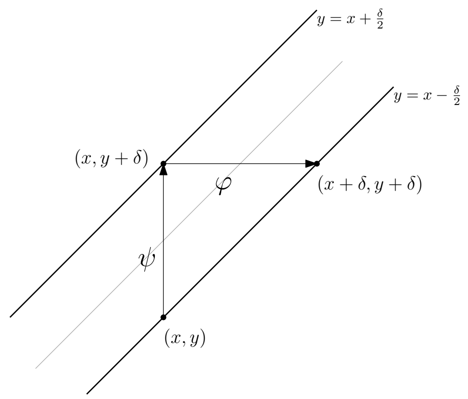


Figure 3.3: Schematic view of the persistence bimodule H.

The special case of left and right Kan extensions in the context of persistence bimodules has been described in [9]. We recall their definition and properties.

Let A, B be two posets (partially ordered sets) and a functor of posets $F : A \to B$. Given $b \in B$ let $A[F \leq b] = \{a \in A \,|\, F(a) \leq b\}$. Similarly $A[F \geq b] = \{a \in A \,|\, F(a) \geq b\}$. Let M be an A-indexed persistence module. The left Kan extension is given by

$$\mathrm{Lan}_F(M)_t = \varinjlim M_{|A[F \leq t]}$$

with the internal maps $\mathrm{Lan}_F(M)_t \to \mathrm{Lan}_F(M)_{t'}, t \leq t'$ arising from the universal property of colimits. The right Kan extension is

$$\mathrm{Ran}_F(M)_t = \varprojlim M_{|A[F \geq t]}$$

with the internal maps given by the universal property of limits.

Let N be a B-indexed persistence module, and $f : M \to N$ a morphism. The universal property of colimits induces a morphism $\mathrm{Lan}_F(f) : \mathrm{Lan}_F(M) \to \mathrm{Lan}_F(N)$. This gives us a functor $\mathrm{Lan}_F(-) : \mathbf{Vect}^A \to \mathbf{Vect}^B$.

Similarly the right Kan extension is also a functor $\mathrm{Ran}_F(-) : \mathbf{Vect}^A \to \mathbf{Vect}^B$.

An important property on left and right Kan extensions stated in [9] is that they preserve respectively product and coproducts.

Proposition 3.1.14. *The left Kan extension preserves direct sums of modules: let M_i be a familly of A-modules,*

$$\mathrm{Lan}_F(\bigoplus_i M_i) \simeq \bigoplus_i \mathrm{Lan}_F(M_i).$$

The right Kan extension preserves direct products of modules: let M_i be a family of A-modules,

$$\mathrm{Ran}_F(\prod_i M_i) \simeq \prod_i \mathrm{Ran}_F(M_i).$$

If $M = \bigoplus_i M_i$ is p.f.d. then

$$\bigoplus_i M_i = \prod_i M_i$$

and therefore the right Kan extension preserve direct sums.

With theses new tools, we get back to our construction. The poset Ω is included in the band $\mathbb{U} = \{(x, y) \in \mathbb{R}^2 | y \in [x - \frac{\delta}{2}, x + \frac{\delta}{2}]\}$. Call $F : \Omega \hookrightarrow \mathbb{U}$ the inclusion. We can compute the left Kan extension of H given by $\mathrm{Lan}_F(H)_t = \varinjlim H_{|\{s\in\Omega, s\leq t\}}$ for $t \in \mathbb{U}$ (see Figure 3.4 for an exemple).

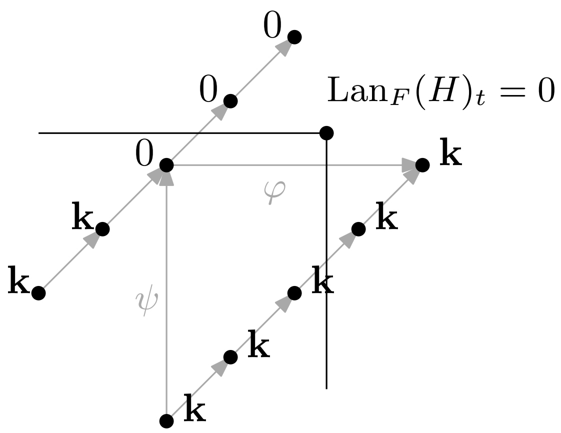


Figure 3.4: An example of left Kan extension.

The left Kan extension is the left adjoin [9] to the restriction $(-)_{|\Omega} : \mathbb{U} \to \Omega$. One can verify by hand that $(-)_{|\Delta_M} \circ \mathrm{Lan}_F = \mathrm{id}_{\mathbf{Vect}^{\Delta_M}}$ since pointwise each colimit in t gives the space M_t. Similarly $(-)_{|\Delta_N} \circ \mathrm{Lan}_F = \mathrm{id}_{\mathbf{Vect}^{\Delta_N}}$. We call $\Gamma^t, t \in [0, \delta]$, the restriction of $\mathrm{Lan}_F(H)$ to the line $\Delta_t : y = x + t - \delta/2$. We have $\Gamma^0 \simeq M$ and $\Gamma^\delta \simeq N$. For $t \leq t' \in [0, \delta]$, structural morphisms $\mathrm{Lan}_F(H)_{(x,y)}^{(x+t'-t,y)}$ along the horizontal direction gives a morphism $\varphi : \Gamma^t \to \Gamma^{t'}$. Similarly, we have a morphism $\psi : \gamma^t \to \Gamma^{t'}$ along the vertical axis. The pair (φ, ψ) forms a $|t' - t|$-interleaving between Γ^t and $\Gamma^{t'}$. The family $\Gamma_t, t \in [0, \delta]$, is therefore the collection of persistence modules announced by the interpolation lemma.

3.1.4 Quivers and their representations

Families of persistence modules defined on a finite index set are best described in the language of quivers. We introduce quiver and quiver representations (see the annex of [41] for a good reference), show how they can be turned into modules, and make the link with persistent homology. We finish this section with examples and a hint of the complexity and richness of the problem of classification of isomorphism classes.

Definition 3.1.15. A quiver Q is a set Q_0 of vertices (0-dimensional objects) of Q, a set Q_1 of edges or arrows (1-dimensional objects) of Q, and two applications $s : Q_1 \to Q_0$ returning the source of the edge and $t : Q_1 \to Q_0$ returning the target of the edge. A quiver Q is said finite if Q_0 and Q_1 are both finite sets.

This definition is identical to the one of a multigraphs. Indeed, one can draw vertices as points in the plane or 3-dimensional space and each edge by an arrow connecting the source of the edge to its target, and then obtain a directed multigraph.

Definition 3.1.16. A representation V of a quiver Q is the choice of a vector space V_p for each vertex $p \in V$, and the choice of a linear application $V_a : V_{s(a)} \to V_{t(a)}$ for each edge $a \in Q_1$. A morphism of representations $f : V \to W$ of Q is a collection of linear applications f_p, one by vertex $p \in Q_0$ going from V_p to W_p, such that for every $a \in Q_1$ the diagram

$$\begin{array}{ccc} V_{s(a)} & \xrightarrow{f_{s(a)}} & W_{s(a)} \\ \downarrow{\scriptstyle \rho_a} & & \downarrow{\scriptstyle \lambda_a} \\ V_{t(a)} & \xrightarrow{f_{t(a)}} & W_{t(a)} \end{array}$$

commutes.

- The identity morphism of a representation V is $(\mathrm{id}_V)_p = \mathrm{id}_{V_p}$, $p \in Q_0$. The zero representation is the representation with all vector spaces and morphisms equal to 0.

- A subrepresentation of V is a a representation W such that $\forall p \in Q_0, W_p \subseteq V_p$ and $\forall a \in Q_1, W_a = V_a|_{W_p}$.
- The quotient of V by a subrepresentation W is V/W defined pointwise by $(V/W)_p = V_p/W_p$ and $(V/W)_a$ is the map induced by the quotient.
- Given two representations U and V their direct sum $U \oplus V$ is the representation obtained by taking pointwise direct sums $U_p \oplus V_p, p \in Q_0$ and morphisms $U_a \oplus V_a, a \in Q_1$.
- A morphism of representations $f : U \to V$ also has a kernel $\operatorname{Ker} f$, the submodule defined pointwise by $(\operatorname{Ker} f)_p = \operatorname{Ker} f_p$, $p \in Q_0$. Similarly f also has an image and a cokernel defined analogously.

Representations of a quiver Q with their morphisms form a category $\mathbf{Rep}(Q)$, and one can show that $\mathbf{Rep}(Q)$ is Abelian.

Representation of a quiver Q over a field $\mathbf{k}$ is equivalent to a module over a ring – actually a $\mathbf{k}$-algebra – called the path algebra of Q.

A sequence of edges $\gamma = a_1 a_2 \dots a_n, a_i \in Q_1$ such that $t(a_i) = s(a_{i+1}), \forall 1 \leq i < n$ is called a path. The source of γ is the vertex $s(a_1)$ and its target is $t(a_n)$. The number n is called the length of the path. Given a vertex $p \in Q_0$ we associate a trivial path a_p whose length is 0.

Definition 3.1.17. The *path algebra* $\mathbf{k}Q$ of a quiver Q is the vector space having paths of non negative length from the quiver Q as a basis, endowed with the multiplication $\gamma.\gamma'$ equal to the concatenation of path $\gamma\gamma'$ when the sources and target matches – that is $\gamma\gamma'$ is a path – or zero otherwise.

If Q_0 is finite, $\mathbf{k}Q$ is a unitary ring with unit $\sum\limits_{p \in Q_0} a_p$.

Modules over $\mathbf{k}Q$ are naturally identified with representations of Q. Given a module M, one can recover a representation V by projecting M through the 0-length paths to obtain the spaces $V_p = a_p(M), p \in Q_0$. The morphisms are given by the 1-length paths $V_a : V_{s(a)} \to V_{t(a)}, v \mapsto a.v$. Reciprocally from a representation V one can define the module $M = \bigoplus\limits_{p \in Q_0} V_p$. The multiplication is derived by linearity from the case of a single arrow $a \in Q_1$ and vector $m \in V_p$ from the rule $a.m = V_a(m)$ if $p = s(a)$ and 0 otherwise.

Quivers and posets are not directly linked to each other. This is due to the fact that two paths from a quiver Q with the same source and target are considered different, whereas they are equal in a poset by transitivity. Some quivers Q, the ones without undirected cycles, can be regarded as a partial order on the set Q_0. The quiver (3.1) does not correspond to a partially ordered set, since ad and bc are two different paths. In a poset, we would expect the relation $ad = bc$.

$$\begin{array}{ccc} 1 & \xrightarrow{a} & 2 \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle d} \\ 3 & \xrightarrow[c]{} & 4 \end{array} \tag{3.1}$$

The two concepts can be reunified through the introduction of *quivers with relations*, or *bound quivers*.

Definition 3.1.18. A *quiver with relations* is a pair (Q, I) with Q a quiver and $I \subseteq \mathbf{k}Q$ an ideal of the path algebra. A *relation* is an element of I, that is a $\mathbf{k}$ linear combination of paths. The path algebra of (Q, I) is the quotient algebra $\mathbf{k}Q/I$.

A poset P is equivalent to a quiver with relations whose vertices are $Q_0 = P$, the arrows are given by the order on P, and paths sharing the same source and target are identified through the quotient by I.

We now focus on the problem of classifying the representation of quivers (without relations) up to isomorphisms. Let V be a representation of a quiver Q. A *decomposition* of V is a direct sum of two or more non-trivial representations that is isomorphic to V. A representation that cannot be decomposed into the direct sum of two non-trivial representations is said to be *indecomposable*. If all the spaces V_i, $i \in Q_0$ are finite dimensional – the representation V is said to be pointwise finite dimensional – one can show by recursion on the dimension of the $V_i, i \in Q_0$ that V admits a decomposition into a direct sum of indecomposable representations $\bigoplus_{\alpha \in \mathcal{A}} V^\alpha$. If Q is finite, we have the following result.

Theorem 3.1.19 (Krull, Remak, Schmidt)**.** *Let V be a representation of a finite quiver Q. The representation V decomposes as a finite sum of indecomposable representations*

$$V = \bigoplus_{\alpha \in \mathcal{A}} V^\alpha.$$

Furthermore this decomposition is unique: given a second decomposition $\bigoplus_{\beta \in \mathcal{B}} W^\beta$ there is a bijection $j : \mathcal{A} \to \mathcal{B}$ such that V^α is isomorph to $W^{j(\alpha)}$, $\forall \alpha \in \mathcal{A}$.

The existence is obtained by induction on dimensions. The uniqueness is non trivial and a consequence of Azumaya's Lemma [2].

The problem is now reduced to the classification of indecomposables up to isomorphisms. It appears that depending on the quiver Q the problem can have different complexities.

Definition 3.1.20.

- A quiver Q is said to be representation-finite if it has finitely many isomorphism classes of indecomposable representations.
- A quiver Q is said to be tame if its indecomposable classes can be described with countably finitely many one parameter families. More precisely, once fixed the dimension of each vector space of a representation, the indecomposable can be described with finitely many one parameter families.
- A quiver Q is said to be wild otherwise.

As shown in [13] quivers can encode various problems of classification of orbits induced by a group action on objects arising from linear algebra.

The quiver made of a single vertex without arrows $\bullet$ corresponds the classification of vector spaces up to basis changes. This problem corresponds to the action of $GL(V_0)$ when $Q_0 = \{0\}$ and V_0 is the unique space. This problem is solved by the notion of dimension. There is a single indecomposable, the field $\mathbf{k}$.

The quiver $\bullet \to \bullet$ encode the classification of matrices up to independent changes of basis on the source and target vector space. This is considering the action of $GL(V_0) \times GL(V_1)$ on the two vector spaces when $Q_0 = \{0, 1\}$. The matrices are classified by the introduction of the notion of rank. There are only three indecomposable classes, namely $0 \to \mathbf{k}$, $\mathbf{k} \xrightarrow{\mathrm{id}_{\mathbf{k}}} \mathbf{k}$ and $\mathbf{k} \to 0$.

The quiver $\bullet \circlearrowleft$ correspond to the classification of square matrix up to a change of basis. This problem occurs when considering the action of $GL(V_0)$ by conjugation. Assuming the field $\mathbf{k}$ is algebraically closed, the indecomposables are described by the Jordan blocks matrices. There are then countably many 1-parameter families of indecomposables. See [13] for more details on the group action and some other quivers.

A famous theorem gives conditions for quiver (without relations) is of finite type. A quiver is said to be connected if its underlying graph, i.e. the (multi)graph obtained by forgetting the orientation of arrows, is connected.

Theorem 3.1.21 (Gabriel)**.** *A connected quiver is of finite type if and only if its underlying graph is one of the Dynkin diagrams A_n, D_n, E_6, E_7, E_8 drawn in Figure 3.5.*

Donovan-Freislich and Nazarova independently classified the tame case:

Theorem 3.1.22 (Gabriel, Donovan-Freislich, Nazarova)**.** *A connected quiver is of tame type if and only if its underlying graph is one of the Euclidean diagrams $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ drawn in Figure 3.6.*

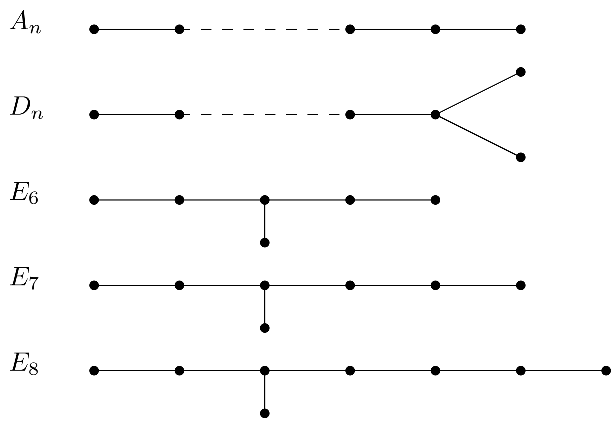


Figure 3.5: Dynkin diagrams A_n, D_n, E_6, E_7, E_8

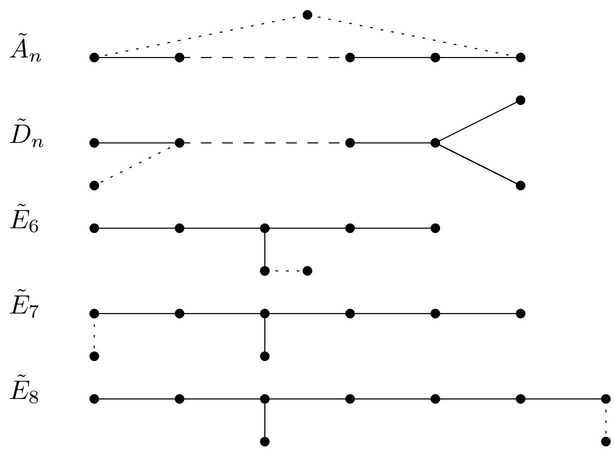


Figure 3.6: Euclidean diagrams $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$. The lines added to the Dynkin diagrams are dotted.

3.1.5 Multidimensional persistence is wild

In Section 3.1.4 we presented how simple quivers can lead to complex classification problems. Some wilde quivers can actually be "embedded" into persistence bimodules, as we will show in Section 3.2.1.

Given a poset P we say that the category of P-persistence modules is:

- representation finite, if it has finitely many classes of indecomposable representation.
- Tame, if the indecomposable classes can be indexed with countably many one parameter families.
- Wild, otherwise.

It happens that $\mathbb{Z}$-bimodules are wild, and it's also the case for most of the $[\![0,n]\!] \times [\![0,m]\!]$-modules. In the paper "The theory of Multidimensional Persistence" [17] Gunnar Carlsson and Afra Zomorodian present an explicit construction of a family of one parameter indecomposable classes. It is easy to imagine how from this construction we can build variations of this construction, leading to infinitely many one parameter indecomposable classes. We present this construction.

We can actually build this family of indecomposable by considering only a $[\![0,4]\!]^2$ grid. We set the space $M_{(0,0)}$ to be $\mathbf{k}^2$. We duplicate this space everywhere on the module and connect spaces by identities. If we take a basis for $M_{(0,0)}$ we end up with two generators a and b of the module. Let $(q_1, q_2, q_3, q_4) \in \mathbb{R}^4$ be four reals. We now look at the four submodules N_1, N_2, N_3, N_4 generated respectively at $(0,3), (1,2), (2,1), (3,0)$ by the elements $y^3.b + q_1 y^3.a,\ y^2 x^1.b + q_2 y^2 x^1.a,\ y^1 x^2.b + q_3 y^1 x^2.a,\ x^3.b + q_4 x^3.a$. The first submodule spans a line l_1 of equation $y = q_1 x$ given by the vector $y^3.b + q_1 y^3.a \in M_{(3,0)} = \mathbf{k}^2$. The three remaining modules behave similarly and we call l_2, l_3 and l_4 their respective lines.

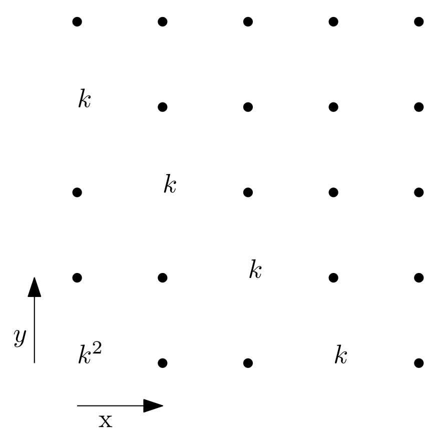


Figure 3.7: Construction of a continuous family of indecomposable

We can now consider the quotient of M by this four submodules. When we quotient by a first submodules, the remaining submodules are canonically sent to submodules of the quotient allowing to take the next quotient. Furthermore, the order in which we do the quotient doesn't mater. The obtained module $R = M/N_1/N_2/N_3/N_4$ is represented in Figure 3.7.

From one of the main results of [17] the automorphism group of R is $GL_2(\mathbf{k})$ acting as a basis change on $M_{(0,0)}$ with the effect propagated to the spaces with higher degree.

We restrict ourselves to the collection Ω of modules obtained when $l_i \neq l_j$, $\forall 1 \leq i, j \leq 4$. The other cases are treated in [17] for the curious reader.

The set of four non equal lines is stable by the action of $GL_2(\mathbf{k})$, since all matrices are invertible. The action of $GL_2(\mathbf{k})$ on lines is 3-transitive, meaning that given $l'_1, l'_2, l'_3 \in \mathbb{R}, l'_i \neq l'_j$, there exists $U \in GL_2(\mathbf{k})$ such that $U.l_1 = l'_1, U.l_2 = l'_2, U.l_3 = l'_3$. $GL_2(\mathbf{k})$ is not 4-transitive (see [13] on 3-transitivity). Let U be a matrix such that

- l_1 is sent to the horizontal axis spanned by $(1,0) \in \mathbf{k}^2$,
- l_2 is sent to the vertical axis spanned by $(0,1) \in \mathbf{k}^2$,
- l_3 is sent to the diagonal line spanned by $(1,1) \in \mathbf{k}^2$.

The remaining line $\lambda_4 = U.l_4$ identifies uniquely the isomorphism class. Therefore the isomorphism classes are in bijection with the projective line $\mathbb{P}^1(\mathbf{k})$ with the axes and diagonal removed since $l_4 \notin \{l_1, l_2, l_3\}$. This is equivalent to considering the cross-ratio $(q_1, q_2; q_3, q_4)$ (see [12] on cross-ratio). This shows that we can construct a 1-parameter family of indecomposables. We would require more to show that the grid is wild.

In [11] Mickaël Buchet and Emerson G. Escolar present some families of indecomposables that appear on small two and three dimensional grids. One of them is a family on $[\![0,4]\!] \times [\![0,1]\!]$. Let $J_d(\lambda) \in \mathcal{M}_d$ denote the Jordan block matrix of size d

$$J_d(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}.$$

The modules

$$M(d,\lambda): \begin{array}{ccccccccc} \mathbf{k}^d & \xrightarrow{\left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]} & \mathbf{k}^{2d} & \xrightarrow{\text{id}} & \mathbf{k}^{2d} & \xrightarrow{[I\ 0]} & \mathbf{k}^d & \longrightarrow & 0 \\ \uparrow & & \uparrow \left[\begin{smallmatrix} I \\ J_d(\lambda) \end{smallmatrix}\right] & & \uparrow \left[\begin{smallmatrix} I & I \\ I & J_d(\lambda) \end{smallmatrix}\right] & & \uparrow [I\ I] & & \uparrow \\ 0 & \longrightarrow & \mathbf{k}^d & \xrightarrow[\left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]]{} & \mathbf{k}^{2d} & \xrightarrow[\text{id}]{} & \mathbf{k}^{2d} & \xrightarrow[{[I\ 0]}]{} & \mathbf{k}^d \end{array}$$

are non-isomorphic indecomposables (see [11]) and imply that the $[\![0,4]\!] \times [\![0,1]\!]$-modules have infinitely many indecomposables.

3.1.6 Zigzag persistence

The theory of one-dimensional persistence with indexes over a finite set is the study of indecomposable of the quivers A_n (see Section 3.1.4) where all the arrows are pointing rightward. The $\mathbb{Z}$-indexed persistence modules are the case of the quiver A_∞ with once again all the arrows pointing rightward. A generalization of these objects, introduced in [14] is to allow the arrows to point either rightward or leftward: these objects are named persistence zigzags.

Definition 3.1.23. A finite zigzag module is a sequence of vector spaces and linear maps

$$V_1 \leftrightarrow V_2 \leftrightarrow \cdots \leftrightarrow V_n$$

where V_i, $1 \leq i \leq n$ are finite and each $\leftrightarrow$ is a morphism either pointing rightward $V_i \to V_{i+1}$ or leftward $V_i \leftarrow V_{i+1}$.

Definition 3.1.24. An infinite zigzag module, or simply a zigzag, is a sequence of finite vector spaces and linear maps index by integers

$$\cdots \leftrightarrow V_{-1} \leftrightarrow V_0 \leftrightarrow V_1 \leftrightarrow \ldots$$

where each $\leftrightarrow$ is pointing either rightward or leftward.

Gabriel's theorem ensures the existence of decompositions for any finite quivers – both the quiver and the dimension of spaces are finite – of type A_n, and therefore finite zigzags are decomposable.

By adding identity arrows and duplicating spaces, one can restrict to the modules with alternating arrows of the form

$$\cdots \leftarrow V_{-1} \leftarrow V_0 \rightarrow V_1 \leftarrow \ldots$$

where even coordinates are source and odd coordinates are sink. One can also extend finite zigzags to $\mathbb{Z}$ indexed zigzags by repeating the first and last spaces and adding identity morphisms.

Definition 3.1.25. The *interval zigzag* module $\mathbb{I}^{[a,b]}$, $a, b \in \mathbb{Z}$, is the zigzag

$$(\mathbf{k}_{[a,b]})_s = \begin{cases} \mathbf{k} \text{ if } s \in [a,b], \\ 0 \text{ if } s \notin [a,b], \end{cases}$$

whose arrows are the identity between two non zero spaces and zero otherwise.

Magnus Botnan [7] proved that like one dimensional persistence p.f.d. modules, p.f.d. zigzags over $\mathbb{Z}$ have a decomposition into interval modules. The uniqueness comes from Azumaya's Lemma [2] since the endomorphism ring of interval zigzag is isomorphic to $\mathbf{k}$ and therefore local.

Theorem 3.1.26. *Let V be a p.f.d. zigzag module. There exists a multiset of intervals $\mathcal{F}$ such that V decomposes into*

$$V = \bigoplus_{I \in \mathcal{F}} \mathbf{k}_I,$$

Furthermore, this decomposition is unique up to reordering.

Since the decomposition of a zigzag V exists, one can define the barcode $\mathcal{B}(V)$ to be the multiset of intervals occurring in the decomposition. Zigzag are usually constructed from a zigzag diagram of topological spaces

Definition 3.1.27. A (finite) *zigzag diagram* of (finitely many) topological spaces is a (finite) sequence

$$\cdots \leftrightarrow \mathbb{X}_{-1} \leftrightarrow \mathbb{X}_0 \leftrightarrow \mathbb{X}_1 \leftrightarrow \ldots$$

of topological spaces and continuous maps pointing each rightward or leftward.

Applying the homology functor gives then a zigzag module.

Given a continuous map $f : \mathbb{X} \to \mathbb{R}$ a common construction is to consider *interlevel set zigzag.* For any interval $I \subseteq \mathbb{R}$ one can define the *slice* $\mathbb{X}_I = f^{-1}(I)$. Suppose now given a *cover* $I_{2i}, i \in \mathbb{Z}$ of $\mathbb{R}$ by intervals indexed on even integers such that two consecutive intervals intersect $I_{2i} \cap I_{2(i+1)} \neq \emptyset$ but all others do not $I_{2i} \cap I_{2j} = \emptyset$, $|i - j| > 1$. We can build a zigzag diagram $\cdots \leftarrow \mathbb{X}_{-1} \to \mathbb{X}_0 \leftarrow \mathbb{X}_1 \to \ldots$ where $\mathbb{X}_{2i} = f^{-1}(I_{2i})$ and $\mathbb{X}_{2i+1} = f^{-1}(I_{2i}) \cap f^{-1}(I_{2(i+1)})$. By applying the homology functor we obtain a zigzag module.

In practical applications the map f is a real valued function on a simplicial complex. Various algorithms have been developed for computing zigzag persistence, notably in [14], [15] and [40]. One of the main advantages of zigzag persistence is that the simplicial complexes do not grow as much as in usual persistence. This leads to a lower memory footprint than the one-dimensional persistence during computation.

Zigzag can be seen as a restriction of a two dimensional persistence module. In [9] the authors give a method to reconstruct a bimodule from a zigzag. See Chapter 4 and Section 4.6.3. But interlevel set zigzag can be directly obtained as the restriction of an interlevel set bimodule M given by the filtration $F_{x,y} = f^{-1}([-x, y])$ defined on the half-plane $y > -x$. Given a cover by intervals $I_{2i}, i \in \mathbb{Z}$, of $\mathbb{R}$, we can restrict the bimodule M to the pair of endomorphisms of intervals $I_{2i}, i \in \mathbb{Z}$ and the intersections $I_{2i} \cap I_{2i+2}, i \in \mathbb{Z}$. We obtain the interlevel set zigzag associated with the cover $(I_{2i})_i$.

In [9] the algebraic stability theorem for one dimensional persistence is extended to the case of zigzags. Here is a sketch of the main idea. First, the authors define a fully faithful functor E from the category of zigzag modules to the category of bimodules on the half upper half-plane $\{(x, y), -x < y\}$. This functor allows to define an interleaving distance also denoted by $\mathrm{d_I}$ on two zigzags V and W by $\mathrm{d_I}(V, W) = \mathrm{d_I}(E(V), E(W))$ by using the interleaving distance on bimodules (see 3.1.4). The bottleneck distance considered is the usual bottleneck distance on the set of intervals. The inequality strengthens into an equality thanks to [5].

Theorem 3.1.28 (Algebraic stability for zigzags)**.** *Let V and W be two zigzag modules.*

$$\mathrm{d_b}(\mathcal{B}(V), \mathcal{B}(W)) = \mathrm{d_I}(V, W).$$

3.2 Contributions

3.2.1 Rooted trees quivers as bimodules

We are interested in the particular class of quivers whose oriented graph is a rooted tree. Under this assumption, we describe a process which allows to send representations of such a quiver to $\mathbf{Vect}^{\mathbb{R}^2}$ the category of persistence bimodules indexed by reals. We show that this operation is indeed a fully faithful functor.

Definition 3.2.1. A *tree* is a connected graph with no cycles. The vertices of the graph are called *nodes*. A tree is *countable* if the set of its nodes is countable. A *rooted tree* is a tree with one node designated as the *root*. An *oriented rooted tree* will designate a rooted tree with edges oriented away from the root[3].

An oriented rooted tree is a special case of quiver, since the vertices and edges of the rooted tree give vertices and edges of a quiver. In what follows, we will design a rooted tree by its associated quiver $Q = (Q_0, Q_1)$. In order to explicit the construction of bimodules from oriented rooted trees it will be convenient to consider the depth of a node.

Definition 3.2.2. Let G be a rooted tree, and v a node. The depth $d(v)$ of v is the length of the path – number of edges in the unique path – from the root to v. The height of G is the supremum of d over the nodes of G. A tree composed only of its root is therefore of height 0.

An oriented rooted tree will be called *bounded* when its height is finite.

Theorem 3.2.3. *Let $Q = (Q_0, Q_1)$ be a countable bounded oriented rooted tree. There exists a non-zero functor $F_Q : \mathbf{Rep}(Q) \to \mathbf{Vect}^{\mathbb{R}^2}$ sending representations of Q to $\mathbb{R}^2$-indexed bimodules.*

Proof. We show this result by induction on the maximum depth of nodes in the quiver Q. Let V be a representation of Q and $S = [0, 1)^2 \subseteq \mathbb{R}^2$ the unit square. Suppose Q is of height 0, it is composed of only the root a. To build $M = F_Q(V)$ one can set $\forall x \in S$, $M_x = V_a$ and $\forall x \notin S, M_x = 0$. The morphisms of M are set to be either zero or the identity when possible.

[3]This special case of oriented rooted tree is referred to as branching or out-tree in the literature.

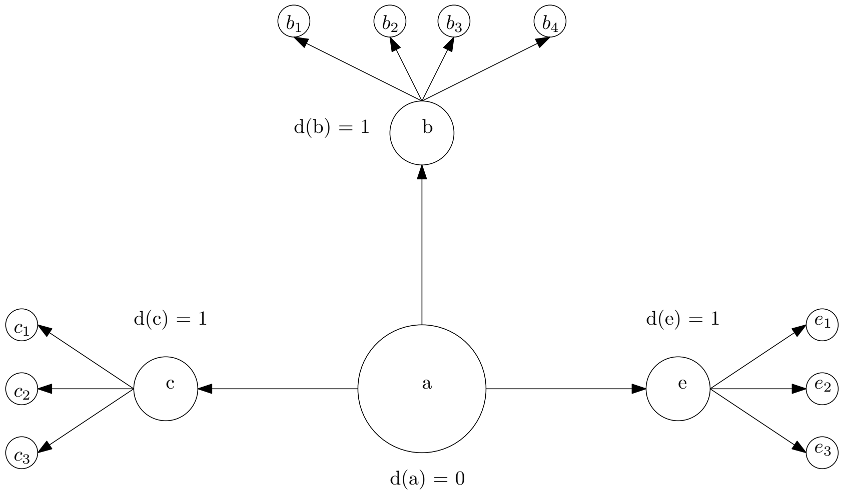


Figure 3.8: A bounded rooted tree with nodes' depths.

If the height of Q is 1, we denote by a the root of Q and by $a_1, a_2, \ldots, a_n$ the nodes of depth 1. Let $S_1, S_2, \ldots, S_n$ be squares of the form $[a, b)^2$ included in S and aligned along the anti-diagonal $\{(x, y) \in S \,|y + x = 1\}$. We ask the S_i to have bottom left corner incomparable for the partial order we defined on $\mathbb{R}^2$ and to intersect each others. We also ask the closure of the squares $\overline{S_i}$ to cover $\{(x, y) \in S \,|y+x = 1\}$. The collection of a_i can indeed be countably infinite. In a such case, we can still build a collection of squares. For example, one could take each S_i to be of size $(\frac{1}{2})^i$. See Figure 3.9 for an illustration. We will say that S_i is the square associated to a_i. This choice of squares is done only once while defining F_Q, and the same collection of squares are used for all representations of Q.

To build $M = F_Q(V)$ we set M_x to be the vector space V_{a_i} for the x belonging to the support of the ith square. In equation, $M_x = V_{a_i}, \forall x \in S_i$. Then, for all locations x in S below S_i, we set the space to be V_a. In equation $\forall x \in S, \forall i, p \in S_i,\, x < p \Rightarrow M_x = V_a$. All the remaining locations are set to zero. The morphisms are defined in a straightforward way. Arrows $M_a^{a_i}$ from V_a to V_{a_i} are set to be $V_a^{a_i}$. This is well defined since for $i \neq j$ the points in S_i and S_j are incomparable and therefore there is no arrow between them.

Let now $f : V \to W$ be a morphism between two representations of Q. As pointed earlier the choice of the partition of $\mathbb{R}^2$ depends only on Q and not on a specific representation. One can define a morphism $F_Q(f) : F_Q(V) \to F_Q(W)$ by setting $F_Q(f)_x : F_Q(V)_x \to F_Q(W)_x$ to be equal to:

- zero if $F_q(V)_x = 0$,
- f_a if $F_q(V)_x = V_a$.

The relations resulting from f being a morphism of representations give the commutativity relations required into the category of modules to allow for $F_Q(f)$ to be a morphism.

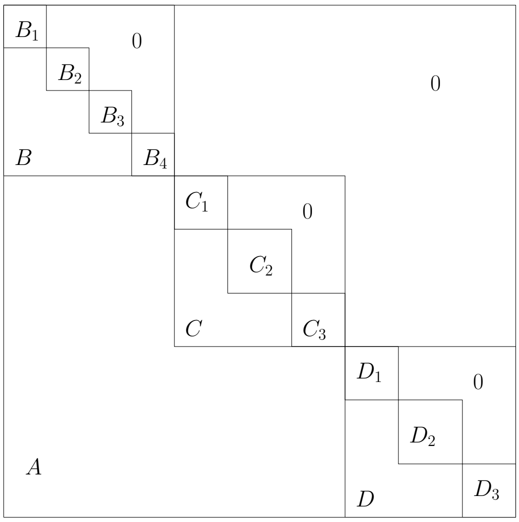


Figure 3.9: Construction of the bimodule associated to a representation[5] of a tree of height 2.

One can repeat this process by induction on the height of Q. We keep the squares already chosen for $n-1$ and subdivide each square S_q associated to a node q into as many rectangles as arrows leaving the node q. Since there is always a unique path $\gamma = e_1 \ldots e_n$ from the root to a node of depth n, the structural morphisms are well defined. See Figure 3.9 for the construction in the case of a tree of height 2.

□

The functor we constructed does not forget information about morphisms.

Proposition 3.2.4. *Let $Q = (Q_0, Q_1)$ be a countable bounded oriented rooted tree. The functor constructed above is fully faithful.*

Proof. Here is the main idea of the proof developed below. Let V be a representation of Q. The restriction to the bottom left corner of the rectangles $S_a, a \in Q_0$ constructed in the definition of F_Q is a left inverse to F_Q. Let $x_a \in S_a$ design the bottom left corner of S_a. We can retrieve each vector space V_a from $F_Q(V)_{x_a}$. The structural morphisms of the module obtained

[5] We implicitly expect the morphism from A to B_4 to be given by the composition $A \to B \to B_4$.

by this restriction are indeed the structural morphisms of the representation V. Not only the restriction is a left inverse to the functor F_Q, but it is also a right inverse on the morphisms. This is possible because sending each element $a \in Q_0$ to the point $x_a \in \mathbb{R}^2$ is a morphism of poset : $x_a \leq x_b$ if and only if there exists a path $a \to b$ in Q.

Let V and W be two representations of Q. Let f and g be two morphisms $V \to W$. Suppose $F_Q(f) = F_Q(g)$. Let a be a node and S_a be the rectangle associated. Let x be the bottom left corner of S_a. By definition $F_Q(V)_x = V_a$, $F_Q(W)_x = W_a$ and $f_a = F_Q(f)_x = F_Q(g)_x = g_a$. This gives immediately $f = g$.

Let $\phi : F_q(V) \to F_Q(W)$ be a morphism of modules. Let $a \in Q_0$ a node and $x \in S_a$ the bottom left corner of the square S_a associated to the node a. We define $f_a : F_q(V)_x = V_a \to F_q(W)_x = W_a$. Let $f : V \to W$ be the morphism given by the family over all nodes. Since on each square S_a (minus the squares included in S_a) the modules $F_Q(V)$ and $F_Q(W)$ are constant, the value of $\phi_x : F_Q(V)_x \to F_Q(W)_x$ is constant for any $x \in S_a \setminus \bigcup_{S_b \subsetneq S_a} S_b$. Therefore $F_Q(f)_x = \phi_x, \forall x \in S_a$. This applies to every squares, so $F_Q(f) = \phi$.

□

By using limits we can extend the previous result to the case of unbounded trees.

Theorem 3.2.5. *Let $Q = (Q_0, Q_1)$ be a coutable oriented rooted tree. There exists a fully faithful functor $F_Q : \mathbf{Rep}(Q) \to \mathbf{Vect}^{\mathbb{R}^2}$.*

Proof. We suppose Q unbounded since the bounded case is already treated. Let V be a representation of Q. We start with the truncation $Q^1, Q^2, \ldots$ of Q where we keep in Q_i only the nodes $a \in Q_0$ of depth $d(a) \leq i$. We can also restrict V into V^i a representation of Q^i by keeping only the spaces and morphism associated with nodes and edges present in Q^i. In the previous construction, we were careful to ensure that squares selected in the construction of F_{Q^i} are still used in the construction of $F_{Q^{i+1}}$. This allows us to consider the modules $M_i = F_{Q^i}(V^i)$.

We define a morphism $M^i \to M^{i+1}$ in the following way. Let a be a node of depth i present in Q^i with edges pointing to $a_1, a_2, \ldots$ (possibly an infinite countable number of nodes). Let $S_1, S_2, \ldots$ be the squares associated with the nodes a_j. We define f^i on the $x \in S_a$ by:

- $f^i_x = \mathrm{id}_{V_a}$ at any $x \in S_a$ such that $M^{i+1}_x = V_a$. This is the case when x is bellow the squares S_j,
- $f^i_x = V_a^{a_j}$ at any $x \in S_j$,

- $f^i_x = 0$ in the case $M^{i+1}_x = 0$ where x is above the squares S_j.

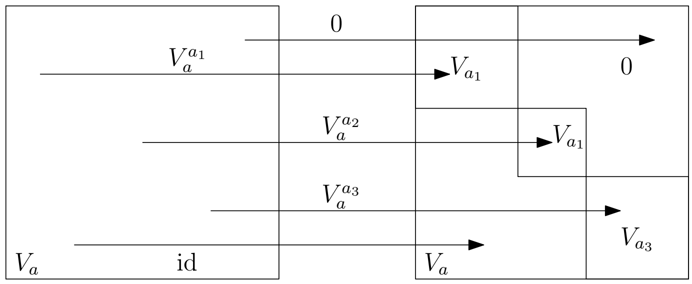


Figure 3.10: Definition of a morphism between two modules M^i on the left and M^{i+1} on the right for $x \in S_a$.

See Figure 3.2.1 for an example. This applications commute with structural morphisms of M^i and M^{i+1} inside the square S_a. The modules M^i and M^{i+1} being identical outside of the locations belonging to the rectangle associated to nodes of depth i, we put $f^i_x = \text{id}$ on the remaining locations. Since any morphism going into S_a can be split in the composition of two morphism, one going on the border of S_a and another whith domain and codomain in S_a, this ensures the commutativity with f^i. Similarly f^i commutes with morphisms leaving S_a. Therefore f^i is a morphism of modules.

This gives us a sequence $M^i, i \in \mathbb{N}$ of modules and a sequence $f^i : M^i \to M^{i+1}$ of morphisms. We can take its direct limit $M^\infty = \varinjlim M^i$. It exists since $\mathbf{Vect}^{\mathbb{R}^n}$ is a complete[6] category.

We define $F_Q(V) = M^\infty$.

It remains to show that F_Q is fully faithful. Once again, we can define a left inverse. Like in the previous case, we can look at the collection of squares S_a and the bottom left corners $x_a \in S_a$. The only difference is that the number of points is countable. The operator which sends $a \in Q_0$ on $x_a \in \mathbb{R}^2$ is a poset morphism and the same exact proof applies. □

3.2.2 Cup product and Persistence Algebras

In Section 3.1.1 we presented multipersistence modules and how they are associated to a graded module over a ring depending on the index set. We mentioned in Chapter 1 that one can use interchangeably homology or cohomology while generating (multi)persistence modules from a filtration. The

[6]Indeed one can take the pointwise limit over the index i in the category of vector spaces of ${M^i}_x, x \in \mathbb{R}^n$, and since the direct limit is functorial it and give us the structural morphism of M_∞ from the limits of the morphisms ${M^i}^y_x$.

interesting property with cohomology is that instead of homology, it comes with a notion of product named *cup product*. Their already was an attempt to link the cup product with persistence homology, notably in [48]. We now enhance the structure of persistence module by extending the cup product arising from cohomology. This turns persistence modules arising from a topological filtration into persistence algebras.

Cohomology and cup product

Let R be a ring, X a topological space and let $\varphi \in C^k(X;R)$ and $\psi \in C^l(X;R)$ be two cochains. The *cup product* $\varphi \smile \psi \in C^{k+l}(X;R)$ is given by the formula

$$\varphi \smile \psi(\sigma) = \varphi(\sigma|[v_0,\ldots,v_k])\psi(\sigma|[v_k,\ldots,v_{k+l}])$$

where the product is taken in the ring R. This product induces a product on the cohomological level, also called cup product, which can be expressed on representatives of cohomology class by the expression above.

For $\alpha \in H^k(X;R)$ and $\beta \in H^l(X;R)$, the cup product satisfies the identity $\alpha \smile \beta = (-1)^{kl}\beta \smile \alpha$.

The cup product defines a product on $H^*(X;R)$ in the obvious way, and therefore $H^*(X;R)$ is actually an R algebra. A ring respecting the commutativity relation of the cup product written above is said to be graded commutative. Indeed, the functoriality of the homology functor extends to the ring structure, and gives a functor from the category of topological spaces to the category of R-algebras. Thus, given two topological spaces $X \subseteq Y$, we have a ring morphism $H(X \subseteq Y) : H^*(Y) \to H^*(X)$. The reader can refer to [34] for the proof of the stated properties.

Construction of the ring structure

We now extend the notion of multidimensional persistent modules defined in 3.1.1 first studied in [17] to a notion of persistent algebras. This result also applies to one dimensional persistence.

Let X be a topological space. Let $F_t, t \in \mathbb{R}^n$ be an n-filtration of X, that is a collection of subsets of X such that $F_t \subset F_{t'}, \forall t \leq t'$. Let $M_t = H^*(F_t)$ be the cohomology ring $(\bigoplus_i H^i(F_t), \smile)$. Since cohomology is functorial, for two reals $s \leq t$ we have a ring morphism $M_s^t : H^*(F_s) \to H^*(F_t)$.

Let M be a persistence module indexed on $\mathbb{R}^n$ obtained from a filtration (see Definition 3.1.1). We would like to extend the ring structure of $M_t, t \in \mathbb{R}^n$ to the whole persistence module M. For $t, s \in \mathbb{R}^n$, we define $t \vee s = (\max(t_1, s_1), \ldots, \max(t_n, s_n))$.

Definition 3.2.6. Let $m, n \in M$ be two homogeneous elements with respective degrees $s = \deg m$ and $t = \deg n$. The cup product of m and n is

$$m \smile n = \left(x^{s\vee t-s}.m\right) \smile \left(x^{s\vee t-t}.n\right) \in M_{s\vee t}$$

This definition means that given two elements m and n of degree s and t, we look at the space with the smallest degree in the direct sum $\bigoplus_u M_u$ where both elements have an image. We then compute the cup product in this space. Notice that if the two elements have same degree t we end up computing the usual cup product in M_t.

We now extend this definition to any two elements of the module.

Definition 3.2.7 (Cup product)**.** Let $\alpha, \beta \in M$. They can be written in a unique way as $\alpha = \sum m_s, \beta = \sum n_t$ where $\forall s, m_s \in M_s, \forall t, m_t \in M_t$. The cup product of these two elements is

$$\alpha \smile \beta = \sum_s \sum_t \left(x^{s\vee t-s}.m_s\right) \smile \left(x^{s\vee t-t}.n_t\right)$$

Theorem 3.2.8. *$(M, \smile)$ is a graded-commutative (non-unital) ring.*

Proof. The first condition to check is the associativity. Let $\alpha = \sum_s m_s$, $\beta = \sum_t n_t$ and $\gamma = \sum_r w_r$ be three elements of M, and their decomposition as a sum of homogeneous elements. By associativity of the cup product in cohomology and of the shifts in degrees, we have :

$$\begin{aligned}
(\alpha \smile \beta) \smile \gamma &= \left(\sum_s \sum_t \left(x^{s\vee t-s}.m_s\right) \smile \left(x^{s\vee t-t}.n_t\right)\right) \smile \sum_r w_r \\
&= \sum_s \sum_t \sum_r \left(x^{s\vee t\vee r-s}.m_s\right) \smile \left(x^{s\vee t\vee r-t}.n_t\right) \smile \left(x^{s\vee t\vee r-r}.w_r\right) \\
&= \sum_s m_s \smile \left(\sum_t \sum_r \left(x^{t\vee r-t}.n_t\right) \smile \left(x^{t\vee r-r}.w_r\right)\right) \\
&= \alpha \smile (\beta \smile \gamma).
\end{aligned}$$

We now have a look at the distributivity, taking α, β and γ as above. We compute the left distributivity, and let the right one to the reader. Once again, it comes from the (left) distributivity of the cup product in cohomology and Definition 3.2.7:

$$
\begin{aligned}
\gamma \smile (\alpha+\beta) &= \left(\sum_r w_r\right) \smile (\sum_{s\in I} m_s + \sum_{t\in J} n_t) \\
&= \left(\sum_r w_r\right) \smile \left(\sum_{l\in I\cap J}(m_l+n_l) + \sum_{s\in I\setminus J} m_s + \sum_{t\in J\setminus I} n_t\right) \\
&\overset{\text{by def.}}{=} \sum_r \sum_{l\in I\cap J} x^{r\vee l - r}.w_r \smile x^{r\vee l - l}.(m_l+n_l) \\
&\quad + \sum_r \sum_{s\in I\setminus J} x^{r\vee s - r}.w_r \smile x^{r\vee s - s}.m_s \\
&\quad + \sum_r \sum_{t\in J\setminus I} x^{r\vee t - r}.w_r \smile x^{r\vee t - t}.n_t \\
&\overset{\text{by dist.}}{=} \sum_r \sum_{s\in I} x^{r\vee s - r}.w_r \smile x^{r\vee s - s}.m_s \\
&\quad + \sum_r \sum_{t\in J} x^{r\vee t - r}.w_r \smile x^{r\vee t - t}.n_t \\
&= \left(\sum_r w_r \smile \sum_{s\in I} m_s\right) + \left(\sum_r w_r \smile \sum_{t\in J} n_t\right) \\
&= \gamma \smile \alpha + \gamma \smile \beta
\end{aligned}
$$

The last property claimed is that the cup product is graded commutative. Notice that we are working with two different gradings on M. The first one is the grading given by the structure of $k[\mathbb{R}^n]$ module on M, which to an element $m \in M_t$ associate $\deg m = t$. The second one is the homological dimension. For a given i, $M^i = \bigoplus_t H^i(F_t; k)$ is actually a submodule of M. We can then rewrite M as a direct sum of submodules $\bigoplus_i M^i$. An element $m \in M^i$ can be written uniquely as a sum $m = \sum_t m_t$ where $\forall t, m_t \in H^i(F_t; k)$. The element m is said to have homology degree $\deg_{\mathrm{h}}(m) = i$. This gives us a second grading. The cup product is then graded commutative respectively to this grading. Indeed, the homology degree of $m \in M^i$ is preserved by multiplication by $x^a \in k[\mathbb{R}^n]$. Therefore, given $\alpha = \sum_s m_s$ of homology degree $\deg_{\mathrm{h}} \alpha = i$, and $\beta = \sum_t n_t$ of homology degree $\deg_{\mathrm{h}} \beta = j$, we have

$$\begin{aligned}\alpha \smile \beta &= \left(\sum_s m_s\right) \smile \left(\sum_t n_t\right)\\ &= \sum_s \sum_t \left(x^{s\vee t-s}.m_s\right) \smile \left(x^{s\vee t-t}.n_t\right)\\ &= \sum_s \sum_t (-1)^{ij} \left(x^{s\vee t-t}.n_t\right) \smile \left(x^{s\vee t-s}.m_s\right)\\ &= (-1)^{ij} \left(\beta \smile \alpha\right)\end{aligned}$$

□

Examples

A well known example of two topological spaces whose cohomology groups are the same but the ring structure given by the cup product allows to differentiate the spaces is the torus T and the wedge sum $X = S^2 \vee S^1 \vee S^1$ of two circles with a sphere. The cohomology groups of both X and T are $H^0 = \mathbb{Z}$, $H^1 = \mathbb{Z}^2, H^2 = \mathbb{Z}$ and $H^i = 0, \forall i \geq 3$. Thus, the two topological filtrations of T and X given in Figure 3.11 give rise to the same persistence module. The main point is that the cup product of the cochains given by each colored circle is zero in the case of X. In the case of the torus, this product give rise to a generator of the 2-cohomology, and therefore non zero.

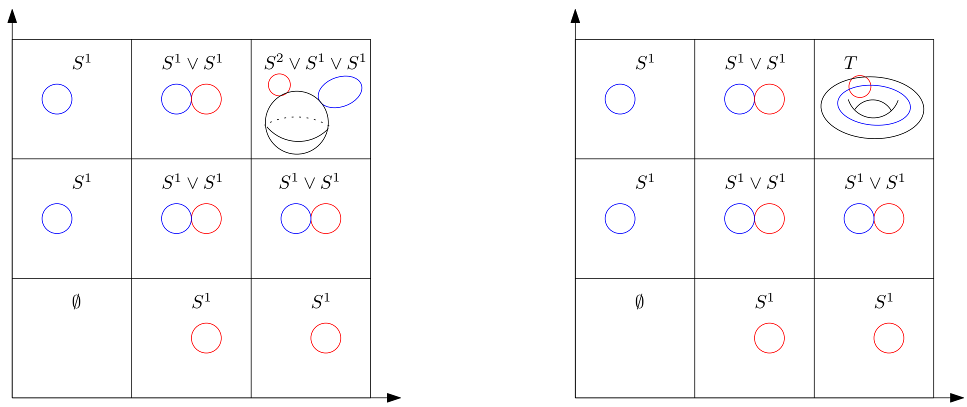


Figure 3.11: Two topological filtration wich give the same persistence bi-modules, but two different persistence algebras.

Chapter 4

Decomposition of exact p.f.d. 2-dimensional modules

As we saw in 3.1.5, we cannot expect a general classification of multidimensional persistence modules, since this configuration is wild. In this chapter based on a previously prepublished article on ArXiv [24] we characterize the class of persistence modules indexed over $\mathbb{R}^2$ that are decomposable into summands whose support has the shape of a *block*—i.e. a horizontal band, a vertical band, an upper-right quadrant, or a lower-left quadrant. Assuming the modules are pointwise finite dimensional (p.f.d.), we show that they are decomposable into block summands if and only if they satisfy a certain local property called *exactness* (see Equation 4.1 and following discussion below). This can be seen as an extension of the decomposition theorem introduced in Chapter 2 for 2-dimensional modules (also called *bimodules* for short). The proof follows the same scheme as the proof of decomposition for p.f.d. persistence modules indexed over $\mathbb{R}$. Since the product order on $\mathbb{R}^2$ is not total, the complexity of the proof is increased and some steps require more direct arguments. This work is motivated primarily by the stability theory for zigzags and interlevel-sets persistence modules, in which block-decomposable bimodules play a key part (see Chapter 3). Our results allow us to drop some of the conditions under which that theory holds, in particular the Morse-type conditions.

4.1 Main result

In a related work, Carlsson et al. [17], and Botnan [7] show that bimodules indexed on $\mathbb{Z}^2$ respecting our exactness condition can be decomposed like one dimensional persistence modules. Our result generalizes this property by considering an indexing over $\mathbb{R}^2$. Since this construction arises naturally considering functions (Section 4.6.5), it is a step further toward the understanding of multidimensional persistence.

The set on which the vector spaces of our modules will be indexed is $\mathbb{R}^2$, equipped with the usual product order:

$$\forall s, t \in \mathbb{R}^2, \quad s \leq t \quad \iff \quad s_x \leq t_x \text{ and } s_y \leq t_y.$$

Given a persistence module M over $\mathbb{R}^2$ we will sometimes write ρ_s^t, $s \leq t \in \mathbb{R}^2$ for its constituent linear maps, when it makes expressions easier to read. For clarity, ρ_s^t will be sometimes renamed v_s^t when $s_x = t_x$ ('v' for 'vertical'), and h_s^t when $s_y = t_y$ ('h' for 'horizontal').

By definition of a persistence bimodule (see Chapter 3), for any $s \leq t \in \mathbb{R}^2$ we have the following commutative diagram where the spaces and maps are taken from M:

$$\begin{array}{ccc} M_{(s_x,t_y)} & \xrightarrow{h^t_{(s_x,t_y)}} & M_t \\ {\scriptstyle v_s^{(s_x,t_y)}} \uparrow & & \uparrow {\scriptstyle v^t_{(t_x,s_y)}} \\ M_s & \xrightarrow[h_s^{(t_x,s_y)}]{} & M_{(t_x,s_y)} \end{array} \tag{4.1}$$

A module M is called *exact* if, for every $s \leq t \in \mathbb{R}^2$, the following sequence induced by (4.1) is exact (i.e. $\operatorname{Im} \phi = \operatorname{Ker} \psi$):

$$M_s \xrightarrow{\phi = \left(h_s^{(t_x,s_y)},\, v_s^{(s_x,t_y)}\right)} M_{(t_x,s_y)} \oplus M_{(s_x,t_y)} \xrightarrow{\psi = v^t_{(t_x,s_y)} - h^t_{(s_x,t_y)}} M_t.$$

Equivalently, whenever a vector in M_t has preimages in both $M_{(t_x,s_y)}$ and $M_{(s_x,t_y)}$, these preimages have a common preimage in M_s.

Remark 4.1.1. Exactness induces two properties, on images and kernels:

$$\operatorname{Im} M_s^t = \operatorname{Im} h^t_{(s_x,t_y)} \cap \operatorname{Im} v^t_{(t_x,s_y)} \tag{4.2}$$

$$\operatorname{Ker} M_s^t = \operatorname{Ker} h_s^{(t_x,s_y)} + \operatorname{Ker} v_s^{(s_x,t_y)} \tag{4.3}$$

These properties will be used repeatedly through this chapter.

Remark 4.1.2. Let M^* designate the dual module obtained by taking point-wise the dual space M_t^* of linear forms $M_t \to \mathbb{R}$ and $(M^*)_s^t$. Since $*$ is functorial, M^* is well defined as a pfd persistence bimodule. Let $\perp$ be the orthogonal operator which associates to a space V the space $V^\perp =$

$\{f \in V^* \,|\, \forall v \in V, f(v) = 0\}$. We get the relations $(\operatorname{Im} M_s^t)^\perp = \operatorname{Ker} {M^*}_t^s$ and $(\operatorname{Ker} M_s^t)^\perp = \operatorname{Im} {M^*}_t^s$. Furthermore, $(A+B)^\perp = A^\perp \cap B^\perp$ and $(A \cap B)^\perp = A^\perp + B^\perp$. This means that the two relations given by exactness can be understood as, in some weak sense, dual of each other.

Here we are interested in exact p.f.d. bimodules. Our analysis extends verbatim to exact bimodules that satisfy the *ascending and descending chain conditions* on kernels and images.

Block modules

We focus on a special type of rectangles in the plane, called *blocks*, and we use *cuts* (see Chapter 2) to parametrize them.

In Chapter 2 we presented a cut as a partition c of $\mathbb{R}$ into two (possibly empty) sets c^-, c^+ such that $x < y$ for all $x \in c^-$ and $y \in c^+$.

A non-empty rectangle R in the plane $\mathbb{R}^2$ is then uniquely defined by four cuts: two horizontal (say ${\scriptstyle|}c$ and $c{\scriptstyle|}$), and two vertical (say $\underline{c}$ and $\overline{c}$), so that $R = ({\scriptstyle|}c^+ \cap c{\scriptstyle|}^-) \times (\underline{c}^+ \cap \overline{c}^-)$. Note that R may not necessarily be open or closed, in fact the nature of each cut determines which boundaries belong to the rectangle.

A block is a certain type of rectangle where two of the cuts lie at infinity. Depending on which ones do so, we have four types of blocks:

- *birth quadrants*, for which $c{\scriptstyle|}^+ = \overline{c}^+ = \emptyset$; each such block B is then given by $\operatorname{supp}(B) = {\scriptstyle|}c^+ \times \underline{c}^+$;
- *death quadrants*, for which ${\scriptstyle|}c^- = \underline{c}^- = \emptyset$; each such block B is given by $\operatorname{supp}(B) = c{\scriptstyle|}^- \times \overline{c}^-$;
- *horizontal bands*, for which ${\scriptstyle|}c^- = c{\scriptstyle|}^+ = \emptyset$; each such block B is given by $\operatorname{supp}(B) = \mathbb{R} \times (\underline{c}^+ \cap \overline{c}^-)$;
- *vertical bands*, for which $\underline{c}^- = \overline{c}^+ = \emptyset$; each such block B is given by $\operatorname{supp}(B) = ({\scriptstyle|}c^+ \cap c{\scriptstyle|}^-) \times \mathbb{R}$.

Note that these types are not mutually exclusive, for instance $\mathbb{R}^2$ itself belongs to all four of them. In the following, we denote a block by B when its type is unspecified or irrelevant.

To any block B we associate a unique $\mathbf{k}_B$ (an interval module) having a copy of the field $\mathbf{k}$ at every point $t \in \operatorname{supp}(B)$ and zero vector spaces elsewhere, the copies of $\mathbf{k}$ being connected by identities and the rest of the maps being zero. It is immediate that any such bimodule is both p.f.d. and exact. Moreover, being exact it is invariant under taking direct sums, therefore any p.f.d. bimodule that is decomposable into blocks (or *block-decomposable* for short) is also exact. Similarily one can associate for any

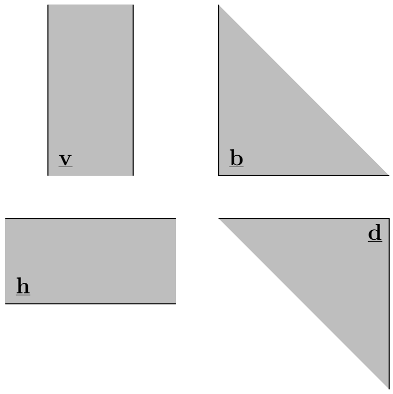


Figure 4.1: The four different type of blocks.

rectangle R in the plane a *rectangle modules* whose support is R. Such modules are p.f.d. but not necessarily exact. A module decomposable into rectangle is said *rectangle-decomposable*. Our main result states that the converse is also true:

Theorem 4.1.3 (Decomposition of exact p.f.d. bimodules)**.** *Any exact p.f.d. bimodule M decomposes as a direct sum of block modules:*

$$M \simeq \bigoplus_{B \in \mathcal{B}(M)} \mathbf{k}_B, \tag{4.4}$$

where $\mathcal{B}(M)$ is some multiset of blocks that depends on M. The decomposition is unique up to isomorphism and reordering of the terms.

We recall from Chapter 2 that a module M has the *ascending chain condition* on images if for any sequence $t \geq \cdots \geq s_2 \geq s_1$ the chain $0 \subseteq \operatorname{Im} \rho_{s_1}^t \subseteq \operatorname{Im} \rho_{s_2}^t \subseteq \ldots$ stabilizes. We now introduce the notion of descending chain: it has the *descending chain condition* on images if for any sequence $t \geq s_1 \geq s_2 \geq \ldots$ the chain $M_t \supseteq \operatorname{Im} \rho_{s_1}^t \supseteq \operatorname{Im} \rho_{s_2}^t \supseteq \ldots$ stabilizes. Ascending and descending chain conditions on kernels are defined similarly.

Remark 4.1.4. Theorem 4.1.3 holds more generally for exact bimodules that satisfy both the ascending and the descending chain conditions on kernels and images. It also holds when the modules are indexed over some open subset $\mathbb{U}$ of $\mathbb{R}^2$ that is stable under positive translations (i.e. translations by vectors with non-negative coordinates) thanks to Kan extensions as shown in Section 4.6.3.

Thus, among the p.f.d. bimodules, the ones that are block-decomposable are exactly the ones that are exact. Several applications of this result are described in Section 4.6. Among them, the study of the stability of *zigzags* in the context of *interlevel-sets persistence* (Section 4.6.5) served as the initial motivation for this work. Exactness in that setting is ensured by the Mayer-Vietoris theorem.

Proof outline

The uniqueness of the decomposition is a straightforward consequence of Azumaya's Lemma [2], the endomorphism ring of any block module being isomorphic to **k** and therefore local.

As in the one dimensional case, we proceed in three steps. Counting the number of instances of a specific interval module, realizing them as submodules of the main module, and finally proving that they are all in direct sum and cover the whole module.

Given a block decomposable persistence module M, and a specific block B, in Section 4.3 we construct a functor C_B that count the number of copies of $\mathbf{k}_B$ present in the decomposition of M. Informally, this is done by taking a vector space representing the vectors "alive" throughout B quotiented out by the vectors who already lived before entering B, or died after leaving B

We see this block modules as a submodule W_B of M. Sections 4.2 and 4.4 of this paper follow [26] for the most part. Section 4.2 provides preliminary technical material, in particular it shows some important basic properties of exact bimodules. This material is used later to prove that the submodules W_B of M constructed form an internal decomposition of the module M.

Section 4.5 shows that the submodules W_B cover the whole module M, i.e. at each point $t \in \mathbb{R}^2$ their constituent vector spaces $W_{B,t}$ generate the whole space M_t. For this we use the language of *sections*[1]. Specifically, we define sections for kernels and images independently, then we combine them into sections for the spaces $W_{B,t}$. The key idea of Section 4.5 is that when a family of sections respects a certain property called covering, some families of complement spaces constructed from this family generate the whole space. The combination turns out to be trickier to analyze than in the 1-dimensional case, requiring extra and more direct arguments. This is mainly because sections for kernels and images are less powerful in 2 dimensions, losing both the *disjointness* and *strong covering* properties that we relied on in Chapter 2 (used heavily in [26]) due to the product order on $\mathbb{R}^2$ being only a partial order. Finally, Section 4.4 shows that the W_B are in direct sum, which is equivalent to checking that their constituent spaces $W_{B,t}$ at each point $t \in \mathbb{R}^2$ are in direct sum. For this we have to use more direct arguments again because the direct sum can no longer be obtained as a byproduct of the disjointness property of sections.

[1] As in [26] the word section is used in a non-standard way, it is not a section of a sheaf.

4.2 Images and kernels

Similarly to the one-dimensional setting [26], the basic ingredients in our study are certain limits of images and kernels. We begin with restrictions of the module along the horizontal and vertical axes.

Definition 4.2.1. Let $t \in \mathbb{R}^2$. We focus on the $\mathbb{R}$-indexed modules $\{M_s,\ s_y = t_y\}$ and $\{M_u,\ u_x = t_x\}$ obtained by restricting M to the horizontal and vertical lines passing through t. Given a rectangle $R = (\lfloor c^+ \cap c\rfloor^-) \times (\underline{c}^+ \cap \overline{c}^-)$ and assuming $t \in R$, we define:

$$
\begin{array}{ll}
\mathrm{Im}^-_{\lfloor c,t} = \bigcup\limits_{x \in \lfloor c^-} \mathrm{Im}\, h^t_{(x,t_y)} & \mathrm{Im}^+_{\lfloor c,t} = \bigcap\limits_{x \in \lfloor c^+,\ x \le t_x} \mathrm{Im}\, h^t_{(x,t_y)} \\
\mathrm{Ker}^-_{c\rfloor,t} = \bigcup\limits_{x \in c\rfloor^-,\ t_x \le x} \mathrm{Ker}\, h^{(x,t_y)}_t & \mathrm{Ker}^+_{c\rfloor,t} = \bigcap\limits_{x \in c\rfloor^+} \mathrm{Ker}\, h^{(x,t_y)}_t \\
\mathrm{Im}^-_{\underline{c},t} = \bigcup\limits_{y \in \underline{c}^-} \mathrm{Im}\, v^t_{(t_x,y)} & \mathrm{Im}^+_{\underline{c},t} = \bigcap\limits_{y \in \underline{c}^+,\ y \le t_y} \mathrm{Im}\, v^t_{(t_x,y)} \\
\mathrm{Ker}^-_{\overline{c},t} = \bigcup\limits_{y \in \overline{c}^-,\ t_y \le y} \mathrm{Ker}\, v^{(t_x,y)}_t & \mathrm{Ker}^+_{\overline{c},t} = \bigcap\limits_{y \in \overline{c}^+} \mathrm{Ker}\, v^{(t_x,y)}_t
\end{array}
$$

See Figure 4.2 for an illustration. By convention we let $\mathrm{Im}^-_{c,t} = 0$ when $c^- = \emptyset$ and $\mathrm{Ker}^+_{c,t} = M_t$ when $c^+ = \emptyset$. Furthermore, we have $\mathrm{Im}^-_{c,t} \subseteq \mathrm{Im}^+_{c,t}$ and $\mathrm{Ker}^-_{c,t} \subseteq \mathrm{Ker}^+_{c,t}$ whenever the spaces are defined.

We combine the spaces in this definition together to define the notions of images and kernels[2] for a rectangle R :

$$
\begin{array}{rcl}
\mathrm{Im}^+_{R,t} & := & \mathrm{Im}^+_{\lfloor c,t} \cap \mathrm{Im}^+_{\underline{c},t}, \\
\mathrm{Im}^-_{R,t} & := & (\mathrm{Im}^-_{\lfloor c,t} + \mathrm{Im}^-_{\underline{c},t}) \cap \mathrm{Im}^+_{R,t} \\
 & = & \mathrm{Im}^-_{\lfloor c,t} \cap \mathrm{Im}^+_{\underline{c},t} + \mathrm{Im}^-_{\underline{c},t} \cap \mathrm{Im}^+_{\lfloor c,t} \\
\mathrm{Ker}^+_{R,t} & := & (\mathrm{Ker}^+_{c\rfloor,t} + \mathrm{Ker}^-_{\overline{c},t}) \cap (\mathrm{Ker}^+_{\overline{c},t} + \mathrm{Ker}^-_{c\rfloor,t}), \\
 & = & \mathrm{Ker}^+_{c\rfloor,t} \cap \mathrm{Ker}^+_{\overline{c},t} + \mathrm{Ker}^-_{c\rfloor,t} + \mathrm{Ker}^-_{\overline{c},t}, \\
\mathrm{Ker}^-_{R,t} & := & \mathrm{Ker}^-_{c\rfloor,t} + \mathrm{Ker}^-_{\overline{c},t}\,.
\end{array}
\tag{4.5}
$$

Remark finally from (4.5) that we do have the inclusions $\mathrm{Im}^-_{R,t} \subseteq \mathrm{Im}^+_{R,t}$ and $\mathrm{Ker}^-_{R,t} \subseteq \mathrm{Ker}^+_{R,t}$ for any rectangle R and point $t \in \mathrm{supp}(R)$.

Remark 4.2.2. It is worth noting that the image and kernel definitions can be related through duality. Let M^* be the vector space dual of M.

[2]Some of the equalities come from the inclusion $\mathrm{Im}^-_c \subseteq \mathrm{Im}^+_c$ and $\mathrm{Ker}^-_c \subseteq \mathrm{Ker}^+_c$. Indeed $U \subseteq U'$ and $V \subseteq V'$ implies $(U+V) \cap (U' \cap V') = U \cap V' + V \cap U'$ and $(U'+V) \cap (V'+U) = U' \cap V' + U + V$.

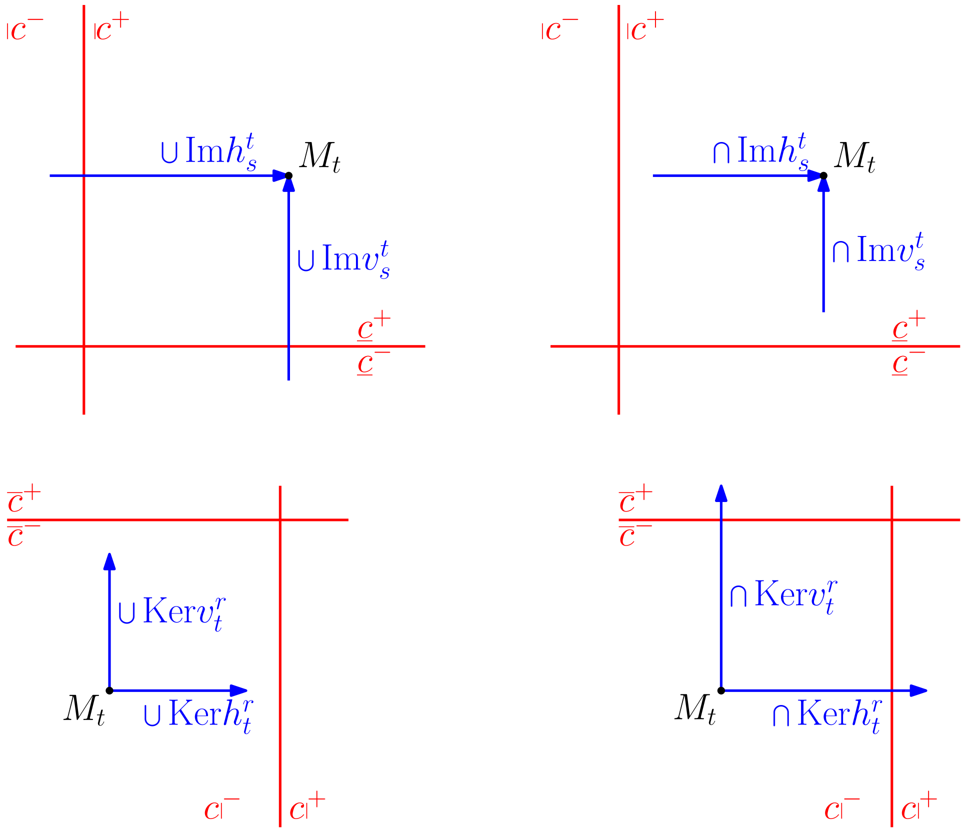


Figure 4.2: From top to bottom and from left to right: the spaces $\mathrm{Im}^-_{|\underline{c},t}$, $\mathrm{Im}^+_{|\underline{c},t}$, $\mathrm{Ker}^-_{\overline{c}|,t}$ and $\mathrm{Ker}^+_{\overline{c}|,t}$.

Since the orthogonal operator $\perp$ transform intersection in sum and sum in intersection, we have the expressions :

$$\mathrm{Im}^+_{M^*,B} = \left(\mathrm{Ker}^-_{M,B^\perp}\right)^\perp \qquad \mathrm{Im}^-_{M^*,B} = \left(\mathrm{Ker}^+_{M,B^\perp}\right)^\perp$$

$$\mathrm{Ker}^+_{M^*,B} = \left(\mathrm{Im}^-_{M,B^\perp}\right)^\perp \qquad \mathrm{Ker}^-_{M^*,B} = \left(\mathrm{Im}^+_{M,B^\perp}\right)^\perp$$

where $B^\perp$ is the block obtained by reversing the horizontal and vertical axis of B.

We recall a result from the one dimensional case introduced in Chapter 2 which states that the subspaces from Definition 4.2.1 can be realized along one dimensional restrictions of our bimodule.

Lemma 4.2.3 (Realization)**.** *Let M be an exact bimodule with ascending and descending chain condition on image and kernels. We extend M to* *a representation of the extended plane* $[-\infty, +\infty]^2$ *by letting* $M_{(\pm\infty,\cdot)} = M_{(\cdot,\pm\infty)} = 0$*. Then:*

$$
\begin{aligned}
\mathrm{Im}^{+}_{{}_{|}c,t} &= \mathrm{Im}\, h^{t}_{(x,t_y)} \textit{ for some } x \in {}_{|}c^{+} \cap (-\infty, t_x] \textit{ and any lower } x \in {}_{|}c^{+}\\
\mathrm{Im}^{-}_{{}_{|}c,t} &= \mathrm{Im}\, h^{t}_{(x,t_y)} \textit{ for some } x \in {}_{|}c^{-} \cup \{-\infty\} \textit{ and any greater } x \in {}_{|}c^{-}\\
\mathrm{Ker}^{+}_{c_{|},t} &= \mathrm{Ker}\, h^{(x,t_y)}_{t} \textit{ for some } x \in c_{|}^{+} \cup \{+\infty\} \textit{ and any lower } x \in c_{|}^{+}\\
\mathrm{Ker}^{-}_{c_{|},t} &= \mathrm{Ker}\, h^{(x,t_y)}_{t} \textit{ for some } x \in c_{|}^{-} \cap [t, +\infty) \textit{ and any greater } x \in c_{|}^{-}
\end{aligned}
$$

And similarly for the vertical cuts $\underline{c}, \overline{c}$*. Note that the spaces* $\mathrm{Im}^{\pm}_{c,t}$ *and* $\mathrm{Ker}^{\pm}_{c,t}$ *mentioned here, which are those of the extension of* M*, are the same as those of* M *since* $t \in \mathbb{R}^2$ *and the cuts considered are cuts of* $\mathbb{R}$*.*

Proof. This is Lemma 2.1 from [26], and a direct consequence of the ascending and descending chain conditions on M_t. □

4.3 The counting functor

First, we establish a result on the behavior of the spaces we defined when transported through the structural morphisms.

Lemma 4.3.1. *Let* M *be a persistence bimodule, and write* ρ^t_s, $s \leq t$ *its structural morphisms. Let* $R = ({}_{|}c^{+} \cap c_{|}^{-}) \times (\underline{c}^{+} \cap \overline{c}^{-})$ *a rectangle,* $s \leq t$ *two points in* R *and* $\triangleleft, \triangleright \in \{+, -\}$*. Then*

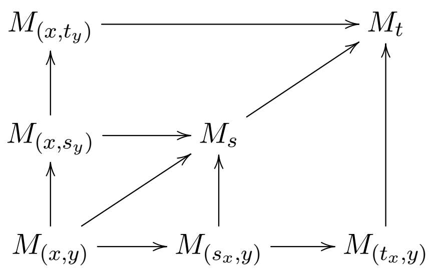
- $\rho^t_s(\mathrm{Im}^{\triangleleft}_{{}_{|}c,R,s} \cap \mathrm{Im}^{\triangleright}_{\underline{c},R,s}) = \mathrm{Im}^{\triangleleft}_{{}_{|}c,R,t} \cap \mathrm{Im}^{\triangleright}_{\underline{c},R,t}$
- ${\rho^t_s}^{-1}(\mathrm{Ker}^{\triangleleft}_{c_{|},R,t} \cap \mathrm{Ker}^{\triangleright}_{\overline{c},R,t}) = \mathrm{Ker}^{\triangleleft}_{c_{|},R,s} \cap \mathrm{Ker}^{\triangleright}_{\overline{c},R,s}$.

Proof. We extend the module M on the extended plane $[-\infty, +\infty]^2$ by $M_{(\pm\infty,.)} = M_{(.,\pm\infty)} = 0$.

First we consider images. By applying the Realization Lemma 4.2.3 we obtain $x \leq s_x \leq t_x$ and $y \leq s_y \leq t_y$ (possibly equal to $-\infty$) such that

$$
\mathrm{Im}^{\triangleleft}_{{}_{|}c,s} = \mathrm{Im}\, h^{s}_{(x,s_y)},\ \mathrm{Im}^{\triangleleft}_{{}_{|}c,t} = \mathrm{Im}\, h^{t}_{(x,t_y)},
$$
$$
\mathrm{Im}^{\triangleright}_{\underline{c},s} = \mathrm{Im}\, v^{s}_{(s_x,y)},\ \mathrm{Im}^{\triangleright}_{\underline{c},s} = \mathrm{Im}\, v^{t}_{(t_x,y)}.
$$

We then have the following commutative diagram:



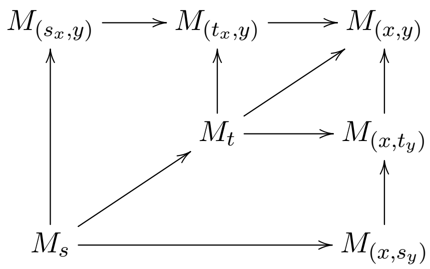
Chasing through this diagram gives:

$$\begin{aligned}\operatorname{Im} h^t_{(x,t_y)} \cap \operatorname{Im} v^t_{(t_x,y)} &\overset{\text{(Eq. 4.2)}}{=} \operatorname{Im} \rho^t_{(x,y)} = \rho^t_s(\operatorname{Im} \rho^s_{(x,y)}) \\ &\overset{\text{(Eq. 4.2)}}{=} \rho^t_s(\operatorname{Im} h^s_{(x,s_y)} \cap \operatorname{Im} v^s_{(s_x,y)}).\end{aligned}$$

We now consider kernels. The Realization Lemma 4.2.3 gives us $x \geq t_x \geq s_x$ and $y \geq t_y \leq s_y$ (possibly equal to $+\infty$) such that

$$\begin{aligned}\operatorname{Ker}^{\triangleleft}_{c|,s} = \operatorname{Ker} h_s^{(x,s_y)} \text{ and } \operatorname{Ker}^{\triangleleft}_{c|,t} = \operatorname{Ker} h_t^{(x,t_y)} \\ \operatorname{Ker}^{\triangleright}_{\overline{c},s} = \operatorname{Ker} v_s^{(s_x,y)} \text{ and } \operatorname{Ker}^{\triangleright}_{\overline{c},s} = \operatorname{Ker} v_t^{(t_x,y)}\end{aligned}$$

We then have the following commutative diagram:



Chasing through this diagram gives:

$$\begin{aligned}(\rho^t_s)^{-1}(\operatorname{Ker} h_t^{(x,t_y)} + \operatorname{Ker} v_t^{(t_x,y)}) &\overset{\text{(Eq. 4.3)}}{=} (\rho^t_s)^{-1}(\operatorname{Ker} \rho_t^{(x,y)}) = \operatorname{Ker} \rho_s^{(x,y)} \\ &\overset{\text{(Eq. 4.3)}}{=} \operatorname{Ker} h_s^{(x,s_y)} + \operatorname{Ker} v_s^{(s_x,y)}.\end{aligned}$$

□

The next lemma is an analog of the Lemma 2.1.3.

Lemma 4.3.2 (Transportation)**.** *Let $R = (\lfloor c^+ \cap c \rfloor^-) \times (\underline{c}^+ \cap \overline{c}^-)$ be a rectangle, $s \leq t$ two points in R. Then*

- $\rho_s^t(\mathrm{Im}^{\pm}_{R,s}) = \mathrm{Im}^{\pm}_{R,t}$
- ${\rho_s^t}^{-1}(\mathrm{Ker}^{\pm}_{R,t}) = \mathrm{Ker}^{\pm}_{R,s}$.

Proof. Using lemma 4.3.1 and properties $f(A + B) = f(A) + f(B)$ and $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ of structural morphisms, the result is immediate. □

We now look at specific combinations of such spaces that, intuitively, encode the generators that appear on the bottom and left boundaries of the block and die on the top and right boundaries.

Let R be a rectangle and let $t \in \mathrm{supp}(B)$. We define:

$$\begin{aligned} V^+_{R,t} &= \mathrm{Im}^+_{R,t} \cap \mathrm{Ker}^+_{R,t}, \\ V^-_{R,t} &= \mathrm{Im}^+_{R,t} \cap \mathrm{Ker}^-_{R,t} + \mathrm{Im}^-_{R,t} \cap \mathrm{Ker}^+_{R,t}\,. \end{aligned} \tag{4.6}$$

Since $\mathrm{Im}^-_{R,t} \subseteq \mathrm{Im}^+_{R,t}$ and $\mathrm{Ker}^-_{R,t} \subseteq \mathrm{Ker}^+_{R,t}$, we have $V^-_{R,t} \subseteq V^+_{R,t}$. The space $V^+_{R,t}$ contains vector from rectangle modules with support R included in M, up to some other vectors contained in $V^-_{R,t}$. So indeed, we are interested by the quotient in $V^+_{R,t}/V^-_{R,t}$ as it corresponds to vectors living in submodules of M with support R.

These spaces depend on the location t in the rectangle considered, whereas we would like to have an intrinsic object that does not have this dependency. The following results will allow us to build such an object. The following lemma is analog to lemma 2.2.1.

Lemma 4.3.3. *Let R be a rectangle. Then, for all $s \leq t \in R$ we have $\rho_s^t(V^{\pm}_{R,s}) = V^{\pm}_{R,t}$. Furthermore, the induced map $\overline{\rho_s^t} : V^+_{R,s}/V^-_{R,s} \to V^+_{R,t}/V^-_{R,t}$ is an isomorphism.*

Proof. This follows from the transportation lemma (4.3.2). First of all, we have

$$\begin{aligned} \rho_s^t(V^+_{R,s}) &= \rho_s^t(\mathrm{Im}^+_{R,s} \cap \mathrm{Ker}^+_{R,s}) \subseteq \rho_s^t(\mathrm{Im}^+_{R,s}) \cap \rho_s^t(\mathrm{Ker}^+_{R,s}) \subseteq \mathrm{Im}^+_{R,t} \cap \mathrm{Ker}^+_{R,t} = V^+_{R,t} \\ \rho_s^t(V^-_{R,s}) &= \rho_s^t(\mathrm{Im}^+_{R,s} \cap \mathrm{Ker}^-_{R,s} + \mathrm{Im}^-_{R,s} \cap \mathrm{Ker}^+_{R,s}) \\ &\subseteq \rho_s^t(\mathrm{Im}^+_{R,s}) \cap \rho_s^t(\mathrm{Ker}^-_{R,s}) + \rho_s^t(\mathrm{Im}^-_{R,s}) \cap \rho_s^t(\mathrm{Ker}^+_{R,s}) \\ &\subseteq \mathrm{Im}^+_{R,t} \cap \mathrm{Ker}^-_{R,t} + \mathrm{Im}^-_{R,t} \cap \mathrm{Ker}^+_{R,t} = V^-_{R,t} \end{aligned}$$

Thus, $\rho_s^t(V^{\pm}_{R,s}) \subseteq V^{\pm}_{R,t}$ and the induced map $\overline{\rho_s^t}$ is well-defined. We will now show that it is both injective and surjective.

Surjectivity: Take $\alpha \in V^+_{R,t} = \mathrm{Im}^+_{R,t} \cap \mathrm{Ker}^+_{R,t}$. Then, $\alpha = \rho^t_s(\beta)$ for some $\beta \in \mathrm{Im}^+_{R,s}$. But then $\beta \in {\rho^t_s}^{-1}(\alpha) \subseteq {\rho^t_s}^{-1}(\mathrm{Ker}^+_{R,t}) = \mathrm{Ker}^+_{R,s}$. Thus, $\beta \in V^+_{R,s}$ and so $\rho^t_s(V^+_{R,s}) = V^+_{R,t}$. It follows that the induced map $\overline{\rho^t_s}$ is surjective.

Injectivity: Take $\alpha \in V^+_{R,s}$ such that $\beta = \rho^t_s(\alpha) \in V^-_{R,t}$. Then, $\beta = \beta_1 + \beta_2$ with $\beta_1 \in \mathrm{Im}^-_{R,t} \cap \mathrm{Ker}^+_{R,t}$ and $\beta_2 \in \mathrm{Im}^+_{R,t} \cap \mathrm{Ker}^-_{R,t}$. Then, by the same reasoning as above, we have $\beta_1 = \rho^t_s(\alpha_1)$ for some $\alpha_1 \in \mathrm{Im}^-_{R,s} \cap \mathrm{Ker}^+_{R,s}$. Now, $\rho^t_s(\alpha - \alpha_1) = \beta_2 \in \mathrm{Ker}^-_{R,t}$ so $\alpha - \alpha_1 \in \mathrm{Ker}^-_{R,s}$. Moreover, $\alpha - \alpha_1 \in \mathrm{Im}^+_{R,s}$, so $\alpha \in V^-_{R,s}$. This implies that the induced map $\overline{\rho^t_s}$ is injective. It also implies that $\rho^t_s(V^-_{R,s}) = V^-_{R,t}$ since we already know that $\rho^t_s(V^-_{R,s}) \subseteq V^-_{R,t} \subseteq V^+_{R,t} = \rho^t_s(V^+_{R,s})$.

□

We can now define an intrinsic quotient, independent of the location of $t \in R$:

$$C_R(M) = \varprojlim_{t \in R} V^+_{R,t}/V^-_{R,t} \tag{4.7}$$

By Lemma 4.3.3, this quotient is isomorphic to $V^+_{R,t}/V^-_{R,t}$ for all $t \in R$. Moreover, its construction is functorial, since for any morphism of modules $\phi : M \to N$ there are canonically induced maps $\mathrm{Im}^\pm_{R,t}(M) \to \mathrm{Im}^\pm_{R,t}(N)$ and $\mathrm{Ker}^\pm_{R,t}(M) \to \mathrm{Ker}^\pm_{R,t}(N)$, then $V^+_{R,t}/V^-_{R,t}(M) \to V^+_{R,t}/V^-_{R,t}(N)$, and $C_R(\phi) : C_R(M) \to C_R(N)$ by universality of the limit. Thus, C_R is a functor from the category of p.f.d. bimodules satisfying the equalities of (4.2) and (4.3) to the category of finite-dimensional vector spaces. This functor is additive because the inverse limit commutes with direct products, and direct products coincide with direct sums in the category of pdf bimodules. We refer to C_R as the *counting functor* associated to the rectangle R because, as we shall see in the following, what it does is, literally, to count the multiplicity of the summand $\mathbf{k}_R$ in the direct-sum decomposition of M. In particular, we can already prove the following fact (for rectangle modules and therefore block-modules):

Lemma 4.3.4. *Assume M is p.f.d. and decomposes as a direct sum of rectangle modules. Then, for any rectangle R, the multiplicity of the summand $\mathbf{k}_R$ in the direct-sum decomposition of M is given by* $\dim C_R(M)$.

Proof. Since C_R is an additive functor, it is enough to prove the result on a single summand $\mathbf{k}_{R'}$. Let us write $R = ({}_{\shortmid}c^+ \cap c_{\shortmid}^-) \times (\underline{c}^+ \cap \overline{c}^-)$ and $R' = ({}_{\shortmid}c'^+ \cap c_{\shortmid}'^-) \times (\underline{c}'^+ \cap \overline{c}'^-)$.

Suppose first that $R' \neq R$. Then, there is a cut that differs between R and R', i.e. there is some $c \in \{{}_{\shortmid}c, \underline{c}, c_{\shortmid}, \overline{c}\}$ such that $c \neq c'$. For all $t \in R$, we then have $\bullet^+_{c,t}(\mathbf{k}_{R'}) = \bullet^-_{c,t}(\mathbf{k}_{R'})$, where $\bullet$ stands for either Im or Ker depending on whether $c \in \{{}_{|}c, \underline{c}\}$ or $c \in \{c_{|}, \overline{c}\}$. Then, by (4.5) we have $\bullet^{+}_{R,t}(\mathbf{k}_{R'}) = \bullet^{-}_{R,t}(\mathbf{k}_{R'})$, which by (4.6) implies that $V^{+}_{R,t}(\mathbf{k}_{R'}) = V^{-}_{R,t}(\mathbf{k}_{R'})$ and so $V^{+}_{R,t}(\mathbf{k}_{R'})/V^{-}_{R,t}(\mathbf{k}_{R'}) = 0$, which, taking the inverse limit as in (4.7), gives $C_R(\mathbf{k}_{R'}) = 0$.

Suppose now that $R' = R$. For any $t \in R$ and $c \in \{{}_{|}c, \underline{c}, c_{|}, \overline{c}\}$, we have $\bullet^{+}_{c,t}(\mathbf{k}_R) = (\mathbf{k}_R)_t \simeq \mathbf{k}$ and $\bullet^{-}_{c,t}(\mathbf{k}_R) = 0$, where $\bullet$ stands for either Im or Ker depending on whether $c \in \{{}_{|}c, \underline{c}\}$ or $c \in \{c_{|}, \overline{c}\}$. Then, by (4.5) we have $\mathrm{Im}^{+}_{R,t}(\mathbf{k}_R) = \mathrm{Ker}^{+}_{R,t}(\mathbf{k}_R) = (\mathbf{k}_R)_t \simeq \mathbf{k}$ while $\mathrm{Im}^{-}_{R,t}(\mathbf{k}_R) = \mathrm{Ker}^{-}_{R,t}(\mathbf{k}_R) = 0$, which by (4.6) implies that $V^{+}_{R,t}(\mathbf{k}_R) = (\mathbf{k}_R)_t \simeq \mathbf{k}$ while $V^{-}_{R,t}(\mathbf{k}_R) = 0$, and so $V^{+}_{R,t}(\mathbf{k}_R)/V^{-}_{R,t}(\mathbf{k}_R) \simeq \mathbf{k}$, which, taking the inverse limit as in (4.7), gives $C_R(\mathbf{k}_R) \simeq \mathbf{k}$ and so $\dim C_R(\mathbf{k}_R) = 1$ as desired. □

Submodules

Although $\dim C_R$ allows us to retrieve the multiplicity of a summand given a decomposition, it is not enough to prove the existence of such a decomposition. This is why we specify a submodule W_R of M for each block R and exhibit an internal direct-sum decomposition. To define this submodule we rely on the $V^{\pm}_{R,t}$.

Lemma 4.3.5. *For any rectangle R, there exists some set $S \subseteq R$ that is countable and co-initial for the partial order $\leq$.*

Proof. If there is no minimum in R, then take $S = \mathbb{Q}^2 \cap \mathrm{supp}(R)$. □

Now we can apply the construction of [26] to assign a particular submodule W_R of M to each rectangle R. The details are the same as in [26, § 4-5] but we recall them for completeness.

Consider the inverse limits:

$$V^{\pm}_R(M) = \varprojlim_{t \in R} V^{\pm}_{R,t}$$

Letting $\pi_t : V^{+}_R(M) \to V^{+}_{R,t}(M)$ denote the natural map, we can make the following identification:

$$V^{-}_R(M) = \bigcap_{t \in R} \pi_t^{-1}(V^{-}_{R,t}) \subseteq V^{+}_R(M)$$

The next lemma is the analog to Lemme 2.2.3.

Lemma 4.3.6. *For any rectangle R and any point $t \in R$, the induced map $\overline{\pi_t} : V^{+}_R(M)/V^{-}_R(M) \to V^{+}_{R,t}/V^{-}_{R,t}$ is an isomorphism.*

Proof. Recall from Lemma 4.3.3 that $\rho_s^t(V_{R,s}^-) = V_{R,t}^-$ for any $s \leq t \in R$. It follows that the Mittag-Leffler condition [32, Chap. 0, (13.1.2)] holds for the family of spaces $\{V_{R,t}^{\pm} : t \in \mathrm{supp}(R)\}$. Thanks to Lemma 4.3.5, the hypotheses of Proposition 13.2.2 from [32, Chap. 0] hold for the following system of exact sequences:

$$0 \to V_{R,t}^- \to V_{R,t}^+ \to V_{R,t}^+/V_{R,t}^- \to 0 \tag{4.8}$$

Hence the following sequence is exact:

$$0 \to V_R^-(M) \to V_R^+(M) \to C_R(M) \to 0. \tag{4.9}$$

By Lemma 4.3.3, the $\overline{\rho_s^t}$ are isomorphisms, so the inverse limit $\varprojlim_{t \in \mathrm{supp}(B)} V_{R,t}^+/V_{R,t}^-$ is isomorphic to $V_{R,t}^+/V_{R,t}^-$ for all $t \in \mathrm{supp}(R)$. By uniqueness of the limit, we obtain the result. □

From equation (4.9) we get the following corollary analog to Lemme 2.2.4.

Corollary 4.3.7. *The quotient of limits $V_R^+(M)/V_R^-(M)$ is isomorphic to $C_R(M)$.*

We now introduce a property that we will be using repeatedly. For an exact module M and B a block, kernels are included in images as follows:

Lemma 4.3.8. *Assume M is pfd and exact. Then, for any fixed $t \in \mathbb{R}^2$ and block $B = ({\shortmid}c^+ \cap c{\shortmid}^-) \times (\underline{c}^+ \cap \overline{c}^-)$ containing t:*

- $\mathrm{Ker}_{c\shortmid,t}^- \subseteq \mathrm{Im}_{\underline{c},t}^+$ *and* $\mathrm{Ker}_{\overline{c},t}^- \subseteq \mathrm{Im}_{\shortmid c,t}^+$.
- *If $c{\shortmid}^+ \neq \emptyset$ (resp. $\overline{c}^+ \neq \emptyset$) then $\mathrm{Ker}_{c\shortmid,t}^+ \subseteq \mathrm{Im}_{\underline{c},t}^+$ (resp. $\mathrm{Ker}_{\overline{c},t}^+ \subseteq \mathrm{Im}_{\shortmid c,t}^+$).*
- *If $\underline{c}^- \neq \emptyset$ (resp. ${\shortmid}c^- \neq \emptyset$) then $\mathrm{Ker}_{c\shortmid,t}^- \subseteq \mathrm{Im}_{\underline{c},t}^-$ (resp. $\mathrm{Ker}_{\overline{c},t}^- \subseteq \mathrm{Im}_{\shortmid c,t}^-$).*
- *If both $c{\shortmid}^+ \neq \emptyset \neq \underline{c}^-$ (resp. $\overline{c}^+ \neq \emptyset \neq {\shortmid}c^-$) then $\mathrm{Ker}_{c\shortmid,t}^+ \subseteq \mathrm{Im}_{\underline{c},t}^-$ (resp. $\mathrm{Ker}_{\overline{c},t}^+ \subseteq \mathrm{Im}_{\shortmid c,t}^-$).*

Proof. All four cases are proven by the same argument. We detail here the first case. The Realization Lemma 4.2.3 tells us that there exist finite values $x \geq t_x$ and $y \leq t_y$ such that $\mathrm{Ker}_{c\shortmid,t}^- = \mathrm{Ker}\, h_t^{(x,t_y)}$ and $\mathrm{Im}_{\underline{c},t}^+ = \mathrm{Im}\, v_{(t_x,y)}^t$. We then have the following exact square:

$$\begin{array}{ccc} M_t & \longrightarrow & M_{(x,t_y)} \\ \uparrow & & \uparrow \\ M_{(t_x,y)} & \longrightarrow & M_{(x,y)} \end{array}$$

The exactness of this square implies that every $\alpha \in \mathrm{Ker}\, h_t^{(x,t_y)}$ has a common antecedent $\beta \in M_{(t_x,y)}$ with $0 \in M_{(x,y)}$, meaning that $\alpha = v_{(t_x,y)}^t(\beta) \in$

$\mathrm{Im}\, v^t_{(t_x,y)}$. Therefore, $\mathrm{Ker}^-_{c|,t} \subseteq \mathrm{Im}^+_{\underline{c},t}$. The inclusion $\mathrm{Ker}^-_{\overline{c},t} \subseteq \mathrm{Im}^+_{|c,t}$ is obtained symmetrically, with the Realization Lemma 4.2.3 giving some finite $x \leq t_x$ and $y \geq t_y$ such that $\mathrm{Ker}^-_{\overline{c},t} = \mathrm{Ker}\, v_t^{(t_x,y)}$ and $\mathrm{Im}^+_{|c,t} = \mathrm{Im}\, h^t_{(x,t_y)}$. □

The next construction will be fundamental to our argument. We know how to prove it for blocks, but not for rectangle.

Proposition 4.3.9. *Assume M is exact. Then, for each block B there is a vector space W^0_B complement of $V^-_B(M)$ in $V^+_B(M)$ such that the family of subspaces defined by*

$$(W_B)_t = \begin{cases} \pi_t(W^0_B) & (t \in B) \\ 0 & (t \notin B) \end{cases}$$

defines a submodule W_B of M.

Proof. Given an arbitrary block B, let us first observe that, whatever choice of subspace W^0_B we make such that $V^+_B(M) = W^0_B \oplus V^-_B(M)$, the following properties will be satisfied:

- For any $s \leq t$ both sitting in B, $\rho^t_s((W_B)_s) \subseteq (W_B)_t$. This is because $\rho^t_s \circ \pi_s = \pi_t$ by definition of π.
- For any $s \leq t$ such that $s \notin B$ and $t \in B$, $\rho^t_s((W_B)_s) = \rho^t_s(0) = 0 \subseteq (W_B)_t$.

There only remains to show that, for a suitable choice of the subspace W^0_B, we also have $\rho^t_s(\pi_s(W^0_B)) = 0$ for all $s \leq t$ with $s \in B$ and $t \notin B$. For this we let $B = (|c^+ \cap c|^-) \times (\underline{c}^+ \cap \overline{c}^-)$ and we distinguish between the various block types:

Case B is a birth quadrant (possibly with $|c^- = \emptyset$ or $\underline{c}^- = \emptyset$). Then any choice of W^0_B works trivially, because there are no indices $s \leq t$ with $s \in B$ and $t \notin B$.

Case B is a death quadrant and not a band ($c|^+, \overline{c}^+ \neq \emptyset$). Then we will enforce $\pi_s(W^0_B) \subseteq \mathrm{Ker}^+_{c|,s} \cap \mathrm{Ker}^+_{\overline{c},s}$ for every $s \in B$, which will imply that $\rho^t_s(\pi_s(W^0_B)) \subseteq \rho^t_s(\mathrm{Ker}^+_{c|,s} \cap \mathrm{Ker}^+_{\overline{c},s}) = 0$ for every $t \geq s$ with $t \notin B$. Let then $K^+_{B,s} = \mathrm{Ker}^+_{c|,s} \cap \mathrm{Ker}^+_{\overline{c},s}$ for each $s \in B$, and consider the system formed by these vector spaces with the transition maps ρ^u_s for $s \leq u \in B$. Since $K^+_{B,s} \subseteq \mathrm{Im}^+_{B,s}$ by Lemma 4.3.8, we have $K^+_{B,s} \subseteq V^+_{B,s}$ and so the inverse limit $K^+_B(M)$ of the system can be identified as follows:

$$K^+_B(M) = \varprojlim_{s \in B} K^+_{B,s} = \bigcap_{s \in B} \pi_s^{-1}(K^+_{B,s}) \subseteq V^+_B(M)$$

We claim that $V_B^-(M) + K_B^+(M) = V_B^+(M)$. Indeed, this equality holds at every index $s \in B$ because $K_{B,s}^+ \subseteq \mathrm{Im}_{B,s}^+$:

$$V_{B,s}^+ = \mathrm{Im}_{B,s}^+ \cap (\mathrm{Ker}_{B,s}^- + K_{B,s}^+) = \mathrm{Im}_{B,s}^+ \cap \mathrm{Ker}_{B,s}^- + K_{B,s}^+ = V_{B,s}^- + K_{B,s}^+.$$

In other words, at every index $s \in B$ we have the following exact sequence:

$$0 \longrightarrow V_{B,s}^- \cap K_{B,s}^+ \xrightarrow{\alpha \mapsto (\alpha, -\alpha)} V_{B,s}^- \oplus K_{B,s}^+ \xrightarrow{(\alpha,\beta) \mapsto \alpha+\beta} V_{B,s}^+ \longrightarrow 0$$

Thus, we have an exact sequence of systems. Since every space $V_{B,s}^- \cap K_{B,s}^+$ is finite-dimensional, the Mittag-Leffler condition is satisfied and so, by Proposition 13.2.2 of [32, Chap. 0], the limit sequence is exact:

$$0 \longrightarrow V_B^-(M) \cap K_B^+(M) \longrightarrow V_B^-(M) \oplus K_B^+(M) \longrightarrow V_B^+(M) \longrightarrow 0$$

It follows that $V_B^-(M) + K_B^+(M) = V_B^+(M)$. We can then choose[3] our vector space complement $W_B^0(M)$ inside $K_B^+(M)$, which ensures that $\pi_s(W_B^0) \subseteq K_{B,s}^+$ for every $s \in B$.

Case B is a horizontal band and not a birth quadrant ($\overline{c}^+ \neq \emptyset$). Then we will enforce $\pi_s(W_B^0) \subseteq \mathrm{Im}_{\underline{c},s}^+ \cap \mathrm{Ker}_{\overline{c},s}^+$ for every $s \in B$, which will imply that $\rho_s^t(\pi_s(W_B^0)) \subseteq \rho_s^t(\mathrm{Ker}_{\overline{c},s}^+) = 0$ for every $t \geq s$ with $t \notin B$. Let then $K_{B,s}^+ = \mathrm{Im}_{\underline{c},s}^+ \cap \mathrm{Ker}_{\overline{c},s}^+$ for each $s \in B$. We have:

$$V_{B,s}^+ = \mathrm{Im}_{B,s}^+ \cap \mathrm{Ker}_{B,s}^+ = \mathrm{Im}_{|c,s}^+ \cap \mathrm{Im}_{\underline{c},s}^+ \cap (\mathrm{Ker}_{c|,s}^- + \mathrm{Ker}_{\overline{c},s}^+).$$

Since $\mathrm{Ker}_{c|,s}^- \subseteq \mathrm{Im}_{\underline{c},s}^+$ and $\mathrm{Ker}_{\overline{c},s}^- \subseteq \mathrm{Ker}_{\overline{c},s}^+ \subseteq \mathrm{Im}_{|c,s}^+$ by Lemma 4.3.8, we get:

$$V_{B,s}^+ = \mathrm{Im}_{|c,s}^+ \cap \mathrm{Ker}_{c|,s}^- + \mathrm{Im}_{\underline{c},s}^+ \cap \mathrm{Ker}_{\overline{c},s}^+ = \mathrm{Im}_{|c,s}^+ \cap \mathrm{Ker}_{c|,s}^- + K_{B,s}^+.$$

Meanwhile, we have:

$$\begin{aligned} V_{B,s}^- &= \mathrm{Im}_{B,s}^+ \cap \mathrm{Ker}_{B,s}^- + \mathrm{Im}_{B,s}^- \cap \mathrm{Ker}_{B,s}^+ \\ &= \mathrm{Im}_{|c,s}^+ \cap \mathrm{Im}_{\underline{c},s}^+ \cap (\mathrm{Ker}_{c|,s}^- + \mathrm{Ker}_{\overline{c},s}^-) + \mathrm{Im}_{B,s}^- \cap \mathrm{Ker}_{B,s}^+ \\ &= \mathrm{Im}_{|c,s}^+ \cap \mathrm{Ker}_{c|,s}^- + \mathrm{Im}_{\underline{c},s}^+ \cap \mathrm{Ker}_{\overline{c},s}^- + \mathrm{Im}_{B,s}^- \cap \mathrm{Ker}_{B,s}^+ \supseteq \mathrm{Im}_{|c,s}^+ \cap \mathrm{Ker}_{c|,s}^-. \end{aligned}$$

Hence, $V_{B,s}^- + K_{B,s}^+ = V_{B,s}^+$. By the same argument as in the previous case, we deduce that the limits satisfy $V_B^-(M) + K_B^+(M) = V_B^+(M)$. We can then choose our vector space complement $W_B^0(M)$ inside $K_B^+(M)$, which ensures that $\pi_s(W_B^0) \subseteq K_{B,s}^+$ for every $s \in B$.

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[3] This is done by a finite induction since $V_B^+(M)/V_B^-(M)$ is finite-dimensional by Lemma 4.3.6.

Case B is a vertical band and not a birth quadrant ($c|^+ \neq \emptyset$). This case is symmetric to the previous one. □

Remark 4.3.10. Since we have chosen W_B^0 such that $V_B^+(M) = V_B^-(M) \oplus W_B^0$, for every $t \in B$ we have $W_B^0 \overset{\pi_t}{\simeq} (W_B)_t$ and $V_{B,t}^+ = V_{B,t}^- \oplus (W_B)_t$ by Lemma 4.3.6.

Assuming that M decomposes as a direct sum of block modules, we have by construction and Lemma 4.3.4 that $\dim W_B^0 = \dim C_B(M)$ is equal to the multiplicity of the summand $\mathbf{k}_B$ in the decomposition of M. More generally, assuming merely that M is p.f.d. and satisfies (4.2, 4.3), we have:

Lemma 4.3.11. *W_B is isomorphic to a direct sum of copies of the block module $\mathbf{k}_B$.*

Proof. Take a basis B of W_B^0. For each $b \in B$, the element $\pi_t(b)$ is non-zero and π satisfies $\rho_s^t(\pi_s(b)) = \pi_t(b)$ for all $s \leq t$, so π spans a submodule $S(b)$ of W_B that is isomorphic to $\mathbf{k}_B$. Now, for all t the family $\{\pi_t(b) : b \in B\}$ is a basis of $W_{B,t}$, so $W_B = \bigoplus_{b \in B} S(b)$. □

4.4 Sections and direct sum

We now show that the summands W_B are in direct sum (Propositions 4.4.2 and 4.4.3), which constitute half the proof of Theorem 4.1.3. For this step we use the concept of *sections* and *disjointness* as in [26] (see Chapter 2), however we combine it with more direct arguments.

We recall the following results from [26], also mentioned in Chapter 2

Lemma 4.4.1. *Given a fixed $t \in \mathbb{R}^2$, each of the families $\{(\mathrm{Im}_{\underline{c},t}^-, \mathrm{Im}_{\underline{c},t}^+) : \underline{c}^+ \ni t_y\}$, $\{(\mathrm{Ker}_{\overline{c},t}^-, \mathrm{Ker}_{\overline{c},t}^+) : \overline{c}^- \ni t_y\}$, $\{(\mathrm{Im}_{|c,t}^-, \mathrm{Im}_{|c,t}^+) : |c^+ \ni t_x\}$, and $\{(\mathrm{Ker}_{c|,t}^-, \mathrm{Ker}_{c|,t}^+) : c|^- \ni t_x\}$ is disjoint in M_t.*

Note that this results and Lemma 2.3.2 do not allow us to conclude directly as in [26], because in our case the full family of sections is not disjoint. In fact the subfamily indexed by the birth quadrants is already not disjoint — the reader can refer to Example 4.5.5 for a counterexample. This means that we need a special treatment to establish the direct sum.

From now on, let $t \in \mathbb{R}^2$ be fixed. The proof that the family of subspaces $\{W_{B,t} : \mathrm{supp}(B) \ni t\}$ is in direct sum is divided into two parts: first, we show that, for each individual block type, the associated subfamily is in direct sum (Proposition 4.4.2); second, we show that the sum is also direct across block types (Proposition 4.4.3).

We introduce the family $\mathcal{F}_t = \{(F^-_{B,t}, F^+_{B,t}) : \mathrm{supp}(B) \ni t\}$, where the spaces $F^\pm_{B,t}$ are defined as follows:

$$\begin{aligned} F^+_{B,t} &= \mathrm{Im}^-_{B,t} + V^+_{B,t} = \mathrm{Im}^-_{B,t} + \mathrm{Ker}^+_{B,t} \cap \mathrm{Im}^+_{B,t}, \\ F^-_{B,t} &= \mathrm{Im}^-_{B,t} + V^-_{B,t} = \mathrm{Im}^-_{B,t} + \mathrm{Ker}^-_{B,t} \cap \mathrm{Im}^+_{B,t}. \end{aligned} \tag{4.10}$$

The reason is purely technical: it is somewhat easier to work with the spaces $F^\pm_{B,t}$ than with the spaces $V^\pm_{B,t}$ because their definitions are more symmetric. Since we have:

$$\begin{aligned} W_{B,t} \cap F^-_{B,t} &= W_{B,t} \cap V^+_{B,t} \cap F^-_{B,t} = W_{B,t} \cap (V^-_{B,t} + V^+_{B,t} \cap \mathrm{Im}^-_{B,t}) \\ &\subseteq W_{B,t} \cap V^-_{B,t} = 0, \end{aligned}$$

we deduce $F^+_{B,t} = W_{B,t} \oplus F^-_{B,t}$. This allows us to consider either $F^\pm_{B,t}$ or $V^\pm_{B,t}$ when it comes to prove properties of the spaces $W_{B,t}$.

Proposition 4.4.2. *For any fixed block type (horizontal band, vertical band, birth quadrant and death quadrant), the summands $W_{B,t}$ are in direct sum.*

Proof. We focus on each block type individually:

1. Horizontal bands (including ones that extend to infinity vertically, either upwards or downwards or both) Let ${\shortmid}c$ denote the trivial horizontal cut with ${\shortmid}c^- = \emptyset$. By Lemma 4.4.1, the family $\{(\mathrm{Im}^-_{\underline{c},t}, \mathrm{Im}^+_{\underline{c},t}) : \underline{c}^+ \ni t_y\}$ is disjoint. It follows, by intersecting all the spaces of this disjoint family with $\mathrm{Im}^+_{{\shortmid}c,t}$, that $\{(\mathrm{Im}^-_{\underline{c},t} \cap \mathrm{Im}^+_{{\shortmid}c,t}, \mathrm{Im}^+_{\underline{c},t} \cap \mathrm{Im}^+_{{\shortmid}c,t}) : \underline{c}^+ \ni t_y\}$ is also disjoint. By definition this is the same family as $\{(\mathrm{Im}^-_{B,t}, \mathrm{Im}^+_{B,t}) : \mathrm{supp}(B) \ni t, B \text{ a horizontal band}\}$. Then, by Lemma 2.3.2 the family $\{(F^-_{B,t}, F^+_{B,t}) : \mathrm{supp}(B) \ni t, B \text{ a horizontal band}\}$ itself is disjoint. Hence, by Lemma 2.3.1 the family of subspaces $\{W_{B,t} : \mathrm{supp}(B) \ni t\}$ is in direct sum.

2. Vertical bands (including ones that extend to infinity horizontally, either to the left or to the right or both) The treatment is symmetric.

3. Death quadrants (including ones that extend to infinity upwards or to the right or both) Take any finite family of distinct death quadrants $B_1, \cdots, B_n$ whose supports contain t. Because they are all distinct, there must be one of them (say B_1) whose support is not included in the union of the other supports. Hence there is some $r \geq t$ such that $r \in \mathrm{supp}(B_1) \setminus \bigcup_{i>1} \mathrm{supp}(B_i)$. Now, suppose there is some relation $\sum_{i=1}^n x_i = 0$ with $x_i \in W_{B_i}$ nonzero for all i. Then, by linearity of ρ^r_t we have $\sum_{i=1}^n \rho_t^r(x_i) = 0$. But each x_i with $i > 1$ is sent to zero through ρ_t^r because r lies outside the support of B_i. Hence, $\rho_t^r(x_1) = -\sum_{i=2}^n \rho_t^r(x_i) = 0$. Meanwhile, we have $\rho_t^r(x_1) \neq 0$ because the restriction of ρ_t^r to W_{B_1} is injective by Lemma 4.3.11. This raises a contradiction.

4. Birth quadrants (including ones that extend to infinity downwards or to the left or both) All we need to prove is that, for any finite family of distinct birth quadrants $B_1, \cdots, B_n$, there is at least one of them (say B_1) whose corresponding subspace $W_{B_1,t} \subseteq M_t$ is in direct sum with the ones of the other quadrants in the family. The result follows then from a simple induction on the size n of the family.

Let then $B_1, \cdots, B_n$ be such a family. Each quadrant B_i is delimited to the left by a horinzontal cut $\shortmid c_i$ and to the bottom by a vertical cut $\underline{c}_i$. Up to reordering, we can assume that B_1 has the rightmost horizontal cut and, in case of ties, it has the uppermost vertical cut among the quadrants with the same horizontal cut. Formally:

$$\shortmid c_1^+ \subseteq \bigcap_{i=2}^{n} \shortmid c_i^+$$

$$\underline{c}_1^+ \subseteq \bigcap_{\substack{i>1 \\ \shortmid c_i = \shortmid c_1}} \underline{c}_i^+$$

It follows that (the support of) B_1 contains none of the other quadrants. Those can be partitioned into two subfamilies: the ones (say $B_2, \cdots, B_k$) contain B_1 strictly, while the others $(B_{k+1}, \cdots, B_n)$ neither contain B_1 nor are contained in B_1. See Figure 4.3 for an illustration. We analyze the two subfamilies separately.

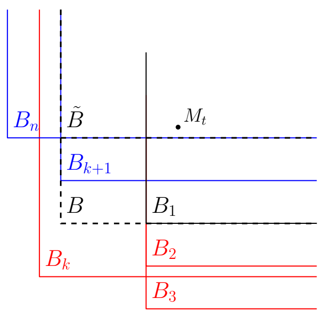

Figure 4.3: Birth quadrants partitioned into two subfamilies

For every $i \in (1, k]$, we have both $\shortmid c_i^+ \supseteq \shortmid c_1^+$ and $\underline{c}_i^+ \supseteq \underline{c}_1^+$, moreover we have either $\shortmid c_i^+ \supsetneq \shortmid c_1^+$ or $\underline{c}_i^+ \supsetneq \underline{c}_1^+$ or both. It follows that $\mathrm{Im}^+_{\shortmid c_i,t} \subseteq \mathrm{Im}^+_{\shortmid c_1,t}$ and $\mathrm{Im}^+_{\underline{c}_i,t} \subseteq \mathrm{Im}^+_{\underline{c}_1,t}$, moreover either $\mathrm{Im}^+_{\shortmid c_i,t} \subseteq \mathrm{Im}^-_{\shortmid c_1,t}$ or $\mathrm{Im}^+_{\underline{c}_i,t} \subseteq \mathrm{Im}^-_{\underline{c}_1,t}$ or both.

Hence,

$$\mathrm{Im}^+_{B_i,t} = \mathrm{Im}^+_{\shortmid c_i,t} \cap \mathrm{Im}^+_{\underline{c}_i,t} \subseteq \mathrm{Im}^+_{\shortmid c_1,t} \cap \mathrm{Im}^-_{\underline{c}_1,t} + \mathrm{Im}^-_{\shortmid c_1,t} \cap \mathrm{Im}^+_{\underline{c}_1,t} = \mathrm{Im}^-_{B_1,t}\,.$$

Summing over $i = 2, \cdots, k$ we obtain:

$$\sum_{i=2}^{k} \mathrm{Im}^+_{B_i,t} \subseteq \mathrm{Im}^-_{B_1,t}\,. \tag{4.11}$$

For every $i \in (k, r]$, we have $\shortmid c_i^+ \supsetneq \shortmid c_1^+$ and $\underline{c}_i^+ \subsetneq \underline{c}_1^+$. Let $\tilde{B}$ be the quadrant whose support is the intersection of $B_{k+1}, \cdots, B_n$. We then have $\shortmid\tilde{c}^+ \supsetneq \shortmid c_1^+$ and $\underline{\tilde{c}}^+ \subsetneq \underline{c}_1^+$, which means that $\tilde{B}$ neither contains B_1 nor is contained in B_1. Let now B be the smallest quadrant containing both B_1 and $\tilde{B}$ — this new quadrant strictly contains them. It follows from the same arguments as in the case $i \in (1, k]$ that

$$\mathrm{Im}^+_{B_1,t} \cap \left(\sum_{i=k+1}^{n} \mathrm{Im}^+_{B_i,t} \right) \subseteq \mathrm{Im}^+_{B_1,t} \cap \mathrm{Im}^+_{\tilde{B},t} = \mathrm{Im}^+_{B,t} \subseteq \mathrm{Im}^-_{B_1,t}\,. \tag{4.12}$$

Combining (4.11) and (4.12), we obtain:

$$\begin{aligned} W_{B_1,t} \cap \left(\sum_{i=2}^{k} W_{B_i,t} + \sum_{i=k+1}^{n} W_{B_i,t} \right) &\subseteq \mathrm{Im}^+_{B_1,t} \cap \left(\sum_{i=2}^{k} \mathrm{Im}^+_{B_i,t} + \sum_{i=k+1}^{n} \mathrm{Im}^+_{B_i,t} \right) \\ &= \sum_{i=2}^{k} \mathrm{Im}^+_{B_i,t} + \mathrm{Im}^+_{B_1,t} \cap \left(\sum_{i=k+1}^{n} \mathrm{Im}^+_{B_i,t} \right) \\ &\subseteq \mathrm{Im}^-_{B_1,t} \subseteq F^-_{B_1,t}, \end{aligned}$$

which is itself in direct sum with $W_{B_1,t}$. Hence the result. □

To establish the direct sum across block types, we adopt the following convention regarding blocks that belong to more than one type:

- All the blocks whose support extends to infinity both upwards and to the right are assigned to the birth quadrants.
- Among the remaining blocks, the ones whose support extends to infinity upwards are assigned to the vertical bands, while the ones whose support extends to infinity to the right are assigned to the horizontal bands.

Proposition 4.4.3. *Under the previous convention, the subspaces* $\oplus_{B:birth} W_{B:birth,t}$, $\oplus_{B:vband} W_{B:vband,t}$, $\oplus_{B:hband} W_{B:hband,t}$ *and* $\oplus_{B:death} W_{B:death,t}$ *are in direct sum.*

Proof. We order the block types as follows: birth quadrants, vertical bands, horizontal bands, death quadrants. We will prove that the summands of each block type are in direct sum with the summands of the following block types in the sequence.

1. Birth quadrants Suppose that

$$\left(\bigoplus_{B:birth} W_{B,t}\right) \cap \left(\bigoplus_{B:vband} W_{B,t} + \bigoplus_{B:hband} W_{B,t} + \bigoplus_{B:death} W_{B,t}\right) \neq 0,$$

where by our convention we treat all the blocks extending to infinity both upwards and to the right as birth quadrants. Take then a nonzero vector x in the intersection. It can be written as a linear combination of nonzero vectors $x_1, \cdots, x_n$ taken from the summands of finitely many birth quadrants $(B'_1, \cdots, B'_n)$, but also as a linear combination of nonzero vectors $y_1, \cdots, y_m$ taken from the summands of finitely many blocks of other types $(B'_1, \cdots, B'_m)$: $\sum_{i=1}^n x_i = x = \sum_{j=1}^m y_j$.

Pick a point $r \geq t \in \mathbb{R}^2$ that is large enough so that it lies outside the supports of the blocks $B'_1, \cdots, B'_m$. Such a point r exists because, by our convention, none of the blocks $B'_1, \cdots, B'_m$ extends to infinity both upwards and to the right. Meanwhile, r still lies in the supports of the birth quadrants $B_1, \cdots, B_n$. Let us then consider the image of x in M_r through the map ρ_t^r. On the one hand it is zero since $\rho_t^r(y_j) = 0$ for all j. On the other hand it is nonzero since the restriction of ρ_t^r to $\bigoplus_{i=1}^n W_{B_i,t}$ is injective by Lemma 4.3.11 and Proposition 4.4.2. This raises a contradiction.

2. Vertical bands Suppose that

$$\left(\bigoplus_{B:vband} W_B\right) \cap \left(\bigoplus_{B:hband} W_B + \bigoplus_{B:death} W_B\right) \neq 0,$$

where by our convention we treat the blocks extending to infinity upwards as vertical bands[4]. Then we can reproduce the reasoning of case 1. Take a nonzero vector x in the intersection and decompose it as a sum of nonzero vectors taken from the summands of vertical bands on the one hand, as a sum of nonzero vectors taken from the summands of horizontal bands or death quadrants on the other hand. Pick then a point $r \geq t$ with $r_x = t_x$ and with r_y large enough so that r lies outside the supports of all the horizontal bands and death quadrants involved in the decomposition of x. By looking at the image $\rho_t^r(x) \in M_r$ we can raise the same contradiction as in case 1.

3. Horizontal bands. They are treated symmetrically to the vertical bands. □

[4]The horizontal bands extending to infinity upwards have already been taken care of in case 1.

4.5 Sections and covering

In this section we show that, assuming our module M is exact, the submodules W_B, for B ranging over all blocks, cover the whole module M (Corollary 4.5.6). Notice that a rectangle module $\mathbf{k}_R$ is exact if and only if R is actually a block. Therefore, only the W_B with B a block are non zero and need to be considered. For this we use the notion of *covering* introduced in Chapter 2. From Chapter 2 we can extract the following result.

Lemma 4.5.1. *Suppose that $\{(F_\lambda^-, F_\lambda^+) : \lambda \in \Lambda\}$ is a family of sections that covers U. For each $\lambda \in \Lambda$, let W_λ be a subspace with $F_\lambda^+ = W_\lambda \oplus F_\lambda^-$. Then, $U = \sum_{\lambda\in\Lambda} W_\lambda$.*

In light of Lemma 4.5.1 and Remark 4.3.10, given a fixed $t \in \mathbb{R}^2$ we want to show that the family of sections $\mathcal{V}_t = \{(V_{B,t}^-, V_{B,t}^+) : B \ni t\}$ covers M_t. Nevertheless, we will use another family instead, the family $\mathcal{F}_t = \{(F_{B,t}^-, F_{B,t}^+) : \mathrm{supp}(B) \ni t\}$ introduced previously through Equation 4.10. Since $F_{B,t}^+ = W_{B,t} \oplus F_{B,t}^-$, we can work with either $\mathcal{V}_t$ or $\mathcal{F}_t$ to show the coverage of M by the submodules W_B.

Let us try to follow the same strategy as in the 1-dimensional setting [26] to show that $\mathcal{F}_t$ is a covering family. Consider the following 'elementary' families:

$$\mathcal{I}_t = \left\{(\mathrm{Im}_{B,t}^-, \mathrm{Im}_{B,t}^+) : B \ni t\right\},$$
$$\mathcal{K}_t = \left\{(\mathrm{Ker}_{B,t}^-, \mathrm{Ker}_{B,t}^+) : B \ni t\right\}.$$

we have:

$$\begin{aligned} W_{B,t} \cap F_{B,t}^- &= W_{B,t} \cap V_{B,t}^+ \cap F_{B,t}^- = W_{B,t} \cap (V_{B,t}^- + V_{B,t}^+ \cap \mathrm{Im}_{B,t}^-) \\ &\subseteq W_{B,t} \cap V_{B,t}^- = 0, \end{aligned}$$

therefore $F_{B,t}^+ = W_{B,t} \oplus F_{B,t}^-$.

We want to apply the following result from Chapter 2 to combine these two families into the family $\mathcal{F}_t$ and show that the latter still cover M_t:

Lemma 4.5.2. *If $\{(F_\lambda^-, F_\lambda^+) : \lambda \in \Lambda\}$ is a family of sections that covers U, and $\{G_\sigma^-, G_\sigma^+) : \sigma \in \Sigma\}$ is a family of sections that strongly covers U, then the following family covers U:*

$$\{(F_\lambda^- + G_\sigma^- \cap F_\lambda^+, F_\lambda^- + G_\sigma^+ \cap F_\lambda^+) : (\lambda, \sigma) \in \Lambda \times \Sigma\}.$$

Unfortunately, applying this lemma is not possible here, because, while the family $\mathcal{I}_t$ does cover M_t (Proposition 4.5.3), it does not strongly cover it (Example 4.5.5). Futhermore, $\mathcal{K}_t$ does not always cover M_t.

Proposition 4.5.3. *The family $\mathcal{I}_t$ covers M_t, more precisely: for any $X \subsetneq M_t$ there are two cuts $\lvert c$ and $\underline{c}$ such that $t \in \lvert c^+ \times \underline{c}^+$ and $\mathrm{Im}^-_{B,t} \subseteq X \not\supseteq \mathrm{Im}^+_{B,t}$ for any block $B = (\lvert c^+ \cap \bullet) \times (\underline{c}^+ \cap \bullet)$.*

The proof of this result relies on the strong covering properties of images and kernels induced by horizontal and vertical cuts. Indeed, by restricting the module M to the horizontal or vertical line passing through t, we have from [26] that each of the four families $\{(\mathrm{Im}^-_{\lvert c,t}, \mathrm{Im}^+_{\lvert c,t}) : \lvert c^+ \ni t_x\}$, $\{(\mathrm{Ker}^-_{c\rvert,t}, \mathrm{Ker}^+_{c\rvert,t}) : c\rvert^- \ni t_x\}$, $\{(\mathrm{Im}^-_{\underline{c},t}, \mathrm{Im}^+_{\underline{c},t}) : \underline{c}^+ \ni t_y\}$, and $\{(\mathrm{Ker}^-_{\overline{c},t}, \mathrm{Ker}^+_{\overline{c},t}) : \overline{c}^- \ni t_y\}$ strongly covers M_t, more precisely:

Lemma 4.5.4. *For any subsets $Y \subsetneq M_t$ and $Z \not\subseteq Y$, there is a horizontal cut $\lvert c$ with $t_x \in \lvert c^+$ such that $\mathrm{Im}^-_{\lvert c,t} \cap Z \subseteq Y \not\supseteq \mathrm{Im}^+_{\lvert c,t} \cap Z$. Similarly, there is a vertical cut $\underline{c}$ with $t_y \in \underline{c}^+$ such that $\mathrm{Im}^-_{\underline{c},t} \cap Z \subseteq Y \not\supseteq \mathrm{Im}^+_{\underline{c},t} \cap Z$. Same for kernels.*

Proof of Proposition 4.5.3. Let X be a subset of M_t. We apply Lemma 4.5.4 with $Z = M_t$ to get a cut $\lvert c$ such that $t_x \in \lvert c^+$ and $\mathrm{Im}^-_{\lvert c,t} \subseteq X \not\supseteq \mathrm{Im}^+_{\lvert c,t}$. We then apply this lemma again with $Z = \mathrm{Im}^+_{\lvert c,t}$ to get a cut $\underline{c}$ such that $t_y \in \underline{c}^+$ and $\mathrm{Im}^-_{\underline{c},t} \cap \mathrm{Im}^+_{\lvert c,t} \subseteq X \not\supseteq \mathrm{Im}^+_{\underline{c},t} \cap \mathrm{Im}^+_{\lvert c,t}$. Now, take any block $B = (\lvert c^+ \cap \bullet) \times (\underline{c}^+ \cap \bullet)$ containing t. Then:

$$
\begin{aligned}
\mathrm{Im}^+_{B,t} &= \mathrm{Im}^+_{\lvert c,t} \cap \mathrm{Im}^+_{\underline{c},t} \not\subseteq X \\
\mathrm{Im}^-_{B,t} &= \mathrm{Im}^-_{\lvert c,t} \cap \mathrm{Im}^+_{\underline{c},t} + \mathrm{Im}^-_{\underline{c},t} \cap \mathrm{Im}^+_{\lvert c,t} \\
&\subseteq \mathrm{Im}^-_{\lvert c,t} + \mathrm{Im}^-_{\underline{c},t} \cap \mathrm{Im}^+_{\lvert c,t} \subseteq X.
\end{aligned}
$$

Thus, $\mathcal{I}_t$ covers M_t. □

Example 4.5.5 (see Figure 4.4 for an illustration)**.** Take for M the direct sum of the modules associated with two birth quadrants B and B' whose lower-left corners are not comparable in the product order on $\mathbb{R}^2$ (see Figure 4.4 for an illustration). Take t in the intersection of the two quadrants. Call β a generator of the 1-dimensional subspace of M_t spanned by B, and call β' a counterpart for B'. Then, take $X = 0$, and for Z take the linear span of $\beta + \beta'$. Since Z is 1-dimensional, for any block C such that $\mathrm{Im}^+_{C,t} \cap Z \neq 0$, $\mathrm{Im}^+_{C,t}$ must contain at least Z, therefore C must be included in $B \cap B'$. But then $\mathrm{Im}^-_{C,t} = \mathrm{Im}^+_{C,t} = M_t \neq 0$, which means that the family $\mathcal{I}_t$ does not strongly cover M_t.

From Example 4.5.5 it follows that Lemma 4.5.2 cannot be applied with $\mathcal{I}_t$ and $\mathcal{K}_t$. This calls for a different approach to prove that $\mathcal{F}_t$ covers M_t.

Consider the submodule $L = \mathrm{Im}^+_{\mathbb{R}^2}$ of M, given at every point $t \in \mathbb{R}^2$ by $L_t = \mathrm{Im}^+_{\mathbb{R}^2,t}$. By transportation lemma this submodule has surjective interval morphism (although this property will not be used).

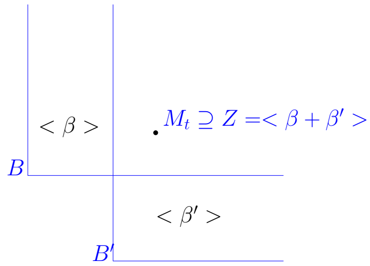

Figure 4.4: Two incomparable birth quadrants.

Consider now the submodule $N = \mathrm{Im}^+_{\mathbb{R}^2} \cap \mathrm{Ker}^-_{\mathbb{R}^2}$, given at every point $t \in \mathbb{R}^2$ by $N_t = \mathrm{Im}^+_{\mathbb{R}^2,t} \cap Ker^-_{\mathbb{R}^2,t}$. It turns out that

$$L = \mathrm{Im}^+_{\mathbb{R}^2} = V^+_{\mathbb{R}^2} = F^+_{\mathbb{R}^2}$$

and

$$N = \mathrm{Im}^+_{\mathbb{R}^2} \cap \mathrm{Ker}^-_{\mathbb{R}^2} = V^-_{\mathbb{R}^2} = F^-_{\mathbb{R}^2}.$$

Therefore, by definition of $W_{\mathbb{R}^2}$ we have $L = N \oplus W_{\mathbb{R}^2}$.

We first prove that we can generate the whole space with N and a subcollection of W_B. We designate by *strict bands* the vertical and horizontal bands with two non-trivial cuts. This are the bands which are not birth or death quadrant at the same time.

Proposition 4.5.6.

$$M = N + \left(\bigoplus_{B:strict\ band} W_B \quad \bigoplus_{B:birth\ quadrant} W_B \right).$$

Notice the direct sum includes also $W_{\mathbb{R}^2}$.

Proof. First recall that $L_t = N_t + W_{\mathbb{R}^2,t}$. Suppose that $X := L_t + \sum_{B:\text{ not death}} W_{B,t} \subsetneq M_t$.

Since the family $\mathcal{I}_t$ is covering, we can find two cuts $\overline{c}, \underline{c}$ such that

$$\mathrm{Im}^-_{\overline{c},\underline{c},t} \subseteq L_t + \sum_{B:\text{ not death}} W_{B,t},$$
$$\mathrm{Im}^+_{\overline{c},\underline{c},t} \not\subseteq L_t + \sum_{B:\text{ not death}} W_{B,t}.$$

Now, notice that $(\overline{c}, \underline{c})$ cannot be part of a death quadrant. Indeed, for $(\overline{c}, \underline{c})$ both trivial we have $\mathrm{Im}^+_{\overline{c},\underline{c},t} = \mathrm{Im}^+_{\overline{c}} \cap \mathrm{Im}^+_{\underline{c}} = L_t \subseteq X$.

We now consider the three different configurations that can arise for the cuts $(\overline{c}, \underline{c})$ and show that in each case we can find a block B such that we have $F^-_{B,t} \subseteq X \not\supseteq F^+_{B,t}$.

1 Both ${\scriptstyle|}c$ and $\underline{c}$ are nontrivial (${\scriptstyle|}c^- \neq \emptyset$ and $\underline{c}^- \neq \emptyset$). Then take $(c{\scriptstyle|}, \overline{c})$ both trivial to form a birth quadrant B, and by Lemma 4.3.8 we have:

$$
\begin{aligned}
F^+_{B,t} &= \mathrm{Im}^-_{B,t} + \mathrm{Im}^+_{B,t} \cap \mathrm{Ker}^+_{B,t} = \mathrm{Im}^-_{B,t} + \mathrm{Im}^+_{B,t} \cap M_t = \mathrm{Im}^+_{B,t} \nsubseteq X \\
F^-_{B,t} &= \mathrm{Im}^-_{B,t} + \mathrm{Im}^+_{B,t} \cap \mathrm{Ker}^-_{B,t} = \mathrm{Im}^-_{B,t} \subseteq X.
\end{aligned}
$$

2 ${\scriptstyle|}c$ is trivial while $\underline{c}$ is nontrivial. Then, $({\scriptstyle|}c, \underline{c})$ is part of an horizontal band B, possibly extending to infinity upwards. Take $c{\scriptstyle|}$ trivial. In order to determine the upper limit of the band precisely, we invoke the strong covering property of kernels on restrictions to vertical lines (Lemma 4.5.4), which gives us a pair $(\mathrm{Ker}^-_{\overline{c},t}, \mathrm{Ker}^+_{\overline{c},t})$ such that[5]:

$$
\begin{aligned}
\mathrm{Im}^-_{\underline{c},t} \cap \mathrm{Im}^+_{{\scriptstyle|}c,t} + \mathrm{Im}^+_{\underline{c},t} \cap \mathrm{Im}^+_{{\scriptstyle|}c,t} \cap \mathrm{Ker}^+_{\overline{c},t} \nsubseteq X \\
\mathrm{Im}^-_{\underline{c},t} \cap \mathrm{Im}^+_{{\scriptstyle|}c,t} + \mathrm{Im}^+_{\underline{c},t} \cap \mathrm{Im}^+_{{\scriptstyle|}c,t} \cap \mathrm{Ker}^-_{\overline{c},t} \subseteq X.
\end{aligned}
$$

Note that the cut $\overline{c}$ may be trivial ($\overline{c}^+ = \emptyset$), in which case we have $\mathrm{Ker}^+_{\overline{c},t} = M_t$. In any case, the same argument as in the proof of Lemma 4.3.8 gives $\mathrm{Ker}^-_{\overline{c},t} \subseteq \mathrm{Im}^+_{{\scriptstyle|}c,t}$ and $\mathrm{Ker}^-_{c{\scriptstyle|},t} \subseteq \mathrm{Im}^-_{\underline{c},t} \subseteq \mathrm{Im}^+_{\underline{c},t}$, where $c{\scriptstyle|}$ is the trivial horizontal cut with $c{\scriptstyle|}^+ = \emptyset$. Then:

$$
\begin{aligned}
F^+_{B,t} &= \mathrm{Im}^-_{\underline{c},t} \cap \mathrm{Im}^+_{{\scriptstyle|}c,t} + \mathrm{Im}^+_{\underline{c},t} \cap \mathrm{Im}^+_{{\scriptstyle|}c,t} \cap \mathrm{Ker}^+_{\overline{c},t} \nsubseteq X \\
F^-_{B,t} &= \mathrm{Im}^-_{\underline{c},t} \cap \mathrm{Im}^+_{{\scriptstyle|}c,t} + \mathrm{Im}^+_{\underline{c},t} \cap \mathrm{Im}^+_{{\scriptstyle|}c,t} \cap (\mathrm{Ker}^-_{\overline{c},t} + \mathrm{Ker}^-_{c{\scriptstyle|},t}) \\
&= \mathrm{Im}^-_{\underline{c},t} \cap \mathrm{Im}^+_{{\scriptstyle|}c,t} + \mathrm{Im}^+_{\underline{c},t} \cap \mathrm{Im}^+_{{\scriptstyle|}c,t} \cap \mathrm{Ker}^-_{\overline{c},t} + \mathrm{Im}^+_{\underline{c},t} \cap \mathrm{Im}^+_{{\scriptstyle|}c,t} \cap \mathrm{Ker}^-_{c{\scriptstyle|},t} \\
&= \mathrm{Im}^-_{\underline{c},t} \cap \mathrm{Im}^+_{{\scriptstyle|}c,t} + \mathrm{Im}^+_{\underline{c},t} \cap \mathrm{Im}^+_{{\scriptstyle|}c,t} \cap \mathrm{Ker}^-_{\overline{c},t} \subseteq X.
\end{aligned}
$$

3 ${\scriptstyle|}c$ is nontrivial while $\underline{c}$ is trivial. Then, $({\scriptstyle|}c, \underline{c})$ is part of a vertical band B, possibly extending to infinity rightwards. This case is symmetric to the previous one.

Now, consider the space $W_{B,t}$. We have

$$
W_{B,t} \subseteq X = L_t + \sum_{B'\text{: not death}} W_{B',t}.
$$

Therefore $F^+_{B,t} = F^-_{B,t} \oplus W_{B,t} \subseteq X \nsupseteq F^+_{B,t}$ which is a contradiction.

□

We now want to replace N in the expression of Proposition 4.5.6 by a direct sum with blocks.

We start by first breaking the summands in two groups.

[5]Recall that $\mathrm{Im}^-_{\underline{c},t} \cap \mathrm{Im}^+_{{\scriptstyle|}c,t} = \mathrm{Im}^-_{B,t} \subseteq X$ while $\mathrm{Im}^+_{\underline{c},t} \cap \mathrm{Im}^+_{{\scriptstyle|}c,t} = \mathrm{Im}^+_{B,t} \nsubseteq X$.

Proposition 4.5.7.

$$(N + \bigoplus_{B\text{:strict band}} W_B) \cap (\bigoplus_{B\text{:birth quadrant}} W_B) = 0.$$

Notice that the birth quadrants include those with lower left corner at infinity, horizontally, vertically, or both.

Proof. Suppose the contrary and let $t \in \mathbb{R}^2$ be such that the intersection at location t is not trivial. Then, there are $\alpha \neq 0 \in N_t$, $x_1 \neq 0 \in W_{B_1,t}, \ldots x_r \neq 0 \in W_{B_r,t}$ (where the blocks are strict bands), $y_1 \neq 0 \in W_{B_1,t}, \ldots y_s \neq 0 \in W_{B_s,t}$ (where the blocks are quadrants), such that $\alpha + \sum_i x_i = \sum_j y_j$.

Now, the strict bands being not birth quadrants, and α being in $\mathrm{Ker}^-_{\mathbb{R}^2,t}$, there exists $u > t$ such that $\rho_t^u(\alpha + \sum_i x_i) = 0$. However, for birth quadrants we have $\rho_t^u(\sum_j y_j) \neq 0$. This is a contradiction. □

We now look at the specific case of the submodule N, which is as we saw before closely related to L.

Proposition 4.5.8.

$$N \cap (\bigoplus_{B\text{:strict bands}} W_B) = 0$$

Proof. Since $N \subseteq L$, it is sufficient to show that $L \cap (\bigoplus_{B\text{:strict bands}} W_B) = 0$. Suppose the contrary. Let $t \in \mathbb{R}^2$ such that the intersection at location t is non trivial. Then, there is $\alpha \in L_t$
$\{0\}, x_1 \neq 0 \in B_1, \ldots, x_s \neq 0 \in B_s$ strict bands such that $\alpha = \sum_i x_i$. Assume without loss of generality that $B_1, \ldots B_r$ are horizontal bands and $B_{r+1}, \ldots B_s$ are vertical bands. We have

$$\alpha - \sum_{i=1}^{r} x_i = \sum_{i=r+1}^{s} x_i.$$

Let $u \geq t$ with $u_y = t_y$ and $u \notin B_j \, \forall j = r+1, \ldots, s$. Such a u exists since the B_j are vertical bands and not birth quadrants. Then:

$$\forall j = r+1, \ldots, s, \;\; \rho_t^u(x_j) = 0 \Rightarrow \beta := \rho_t^u(\alpha) = \sum_{i=1}^{r} \rho_t^u(x_i) \neq 0$$

since the x_i are linearly independent and belong to horizontal bands which do not go to 0 at u.

Hence:

$$L_u \cap \Big(\bigoplus_{B:\text{strict bands}} M_{B,u} \Big) \neq 0.$$

Now, $\beta \in L_u = \mathrm{Im}^+_{\mathbb{R}^2,u} = F^+_{\mathbb{R}^2,u}$ while each $\rho^u_t(x_i)$ for $1 \leq i \leq r$ belongs to $M_{B_i,u}$ such that $F^-_{B_i,u} \bigoplus W_{B_i,u} = F^+_{B_i,u}$. Since the B_i are not death quadrants, we have $F^+_{\mathbb{R}^2,u} = \mathrm{Im}^+_{\mathbb{R}^2,u} \subseteq \mathrm{Im}^-_{B_i,u} \subseteq F^-_{B_i,u}$.

Since by Lemma 2.3.2 the family

$$\{(F^-_{\mathbb{R}^2,u}, F^+_{\mathbb{R}^2,u})\} \cup \{(F^-_{B,u}, F^+_{B,u}) : B \text{ strict band}\}$$

is disjoint, $M_{\mathbb{R}^2,u}$ and $M_{B_i,u}$ are in direct sum. But since $(F^-_{\mathbb{R}^2,u}, F^+_{\mathbb{R}^2,u})$ is the minimum of the family, we have in fact that $F^+_{\mathbb{R}^2,u}$ and the $W_{B_i,u}$ are in direct sum.

Then, $\beta = 0$ and $\rho^u_t(x_i) = 0, \forall i$, which implies $x_i = 0, \forall i$, and so $\alpha = 0$ which contradicts our initial assumption.

□

We can now combine our previous results to obtain a direct sum with N generating the whole module M.

Corollary 4.5.9.

$$M = N \oplus \left(\bigoplus_{B:\ \textit{strict band}} W_B \oplus \bigoplus_{B:\ \textit{birth quadrant}} W_B \right)$$

Proof. It follows from Propositions 4.5.6, 4.5.7 and 4.5.8. □

The last remaining thing to do is to understand the structure of the sub-module N. As we can expect, it is composed of the remaining death quadrants (plus possibly some bands).

Lemma 4.5.10. *The module N is p.f.d. and exact.*

Proof. The fact that N is p.f.d. is immediate as a submodule of the p.f.d. module M.

We show the exactness. Let $s \leq t \in \mathbb{R}^2$. Let $\delta \in N_t$ such that $v^t_{(t_x,s_y)}(\beta) = \delta = h^t_{(s_x,t_y)}(\gamma)$ for some $\beta \in N_{(t_x,s_y)}$ and $\gamma \in N_{(s_x,t_y)}$. Then, by exactness of M, $\exists \alpha \in M_s$ such that $\beta \in h^{(t_x,s_y)}_s(\alpha)$ and $\gamma = v^{(s_x,t_y)}_s(\alpha)$. By transportation lemma on M, we have $\alpha \in (\rho^t_s)^{-1}(\mathrm{Ker}^-_{(M,\mathbb{R}^2,t)}) = \mathrm{Ker}^-_{(M,\mathbb{R}^2,s)}$. Let us show that α also belong to $\mathrm{Im}^+_{(M,\mathbb{R}^2,s)}$ which will conclude the proof.

$$
\begin{array}{ccc}
\gamma \in M_{(s_x,t_y)} & \longrightarrow & M_t \ni \delta \\
\uparrow & & \uparrow \\
\alpha \in M_s & \longrightarrow & M_{(t_x,s_y)} \ni \beta \\
\uparrow & & \uparrow \\
\alpha_u \in M_u & \longrightarrow & M_{(t_x,u_y)} \ni \beta_u
\end{array}
$$

Figure 4.5: Diagram with spaces considered

Let $u \leq s$ such that $u_x = s_x$ (see Figure 4.5). Since $\beta \in \operatorname{Im}^+_{M,\mathbb{R}^2,(t_x,s_y)} \subseteq \operatorname{Im}^+_{M,\underline{\infty},(t_x,s_y)}$, there exists $\beta_u \in M_{(t_x,u_y)}$ such that $v(\beta_u) = \beta$. By exactness of M, there exists $\alpha_u \in M_u$ such that $v_u^s(\alpha_u) = \alpha$ and $h_u^{(t_x,u_y)}(\alpha_u) = \beta_u$. Hence $\alpha \in \operatorname{Im} v_u^s$. Since this is true for every $u \leq s$ with $u_x = s_x$, we conclude that $\alpha \in \operatorname{Im}^+_{M,\underline{\infty},s}$. Symmetrically, $\alpha \in \operatorname{Im}^+_{M,|\infty,s}$, and so $\alpha \in \operatorname{Im}^+_{M,\mathbb{R}^2,s}$. Finally $\alpha \in \operatorname{Im}^+_{M,\mathbb{R}^2,s} \cap \operatorname{Ker}^-_{(M,\mathbb{R}^2,s)} = N_s$.

□

Proposition 4.5.11.

$$N \simeq \bigoplus_{i \in I} \mathbf{k}_{B_i}$$

where the sum is on the blocks B_i which are not a birth quadrant. This includes band starting at a finite location or at infinity, and death quadrants.

Proof. First, take N^* the dual module of N. Since the operator which associates to a vector space its linear form is functorial, N^* comes with a structure of bimodule on $(\mathbb{R}^{\mathrm{op}})^2 \simeq \mathbb{R}^2$.

Since N is exact (by Lemma 4.5.10), every rectangle

$$
\begin{array}{ccc}
C & \xrightarrow{\delta} & D \\
{\scriptstyle\beta}\uparrow & & \uparrow{\scriptstyle\gamma} \\
A & \xrightarrow{\alpha} & B
\end{array}
$$

in N is exact, i.e. the sequence

$$A \xrightarrow{\Phi=(\alpha,\beta)} B \oplus C \xrightarrow{\Psi=\gamma-\delta} D$$

is exact.

Since the dual operator $*$ is an additive and exact functor, the sequence

$$A \xleftarrow{\Phi^*} B \oplus C \xleftarrow{\Psi^*} D$$

with $\Phi* = \alpha^* - \beta^*$ and $\Psi^* = (\gamma^*, -\delta^*)$ is also exact. This implies that indeed, N^* is an exact p.f.d. module.

We consider the othogonal operator $\perp$ on subspaces of N_t. Since this operator sends sums to intersections and kernels to images, we have $\mathrm{Im}^+_{N^*,\mathbb{R}^2} = \left(\mathrm{Ker}^-_{N,\mathbb{R}^2}\right)^\perp = N^\perp = 0$.

Therefore, by Corollary 4.5.9,

$$N^* = \bigoplus_{i \in I} \mathbf{k}_{B'_i}$$

where the B'_i are blocks that are not death quadrants in $(\mathbb{R}^{\mathrm{op}})^2$.

By additivity

$$N \simeq \bigoplus_{i \in I} (k_{B'_i})^* \simeq \bigoplus_{i \in I} (k_{B_i})$$

where each B_i is the same block as B'_i with the orientation of arrows reversed. Therefore, B_i is not a birth quadrant in $\mathbb{R}^2$.

□

We finally obtain the expected decomposition:

Corollary 4.5.12.

$$M = \bigoplus_{B} W_B$$

Proof. Thanks to Corollary 4.5.9 and Proposition 4.5.11 we knows that M decomposes as a direct sum of blocks modules. Now, since the W_B are in direct sum, Lemma 4.3.4 ensures that each W_B has pointwise dimension equal to $\dim C_B(M)$, which counts each summand of the decomposition exactly once. □

This proves Theorem 4.1.3.

4.6 Extensions and Applications

4.6.1 Barcodes and stability for exact pfd bimodules

Thanks to Theorem 4.1.3, to any exact pfd persistence bimodule M we can associate the multiset of blocks $\mathcal{B}(M)$ involved in its decomposition (4.4).

This multiset is called the *barcode* of M. The following isometry result follows[6] from our Theorem 4.1.3 and from [5, 9]:

Corollary 4.6.1. *For any exact p.f.d. persistence bimodules M and N:*

$$\mathrm{d_I}(M,N) = \mathrm{d}_b(\mathcal{B}(M),\ \mathcal{B}(N)),$$

where $\mathrm{d_I}$ *and* $\mathrm{d_b}$ *denote respectively the* interleaving distance *between bimodules and the* bottleneck distance *between barcodes using the same definition as presented in [22, 38] where intervals are replaced by support of summand of a decomposition.*

4.6.2 Tame persistence bimodules

With Theorem 4.1.3 and Corollary 4.6.1 at hand, we can define barcodes for a larger class of (possibly indecomposable) persistence bimodules that are not pfd but satisfy a certain *tameness* condition, much like in the 1-dimensional setting [4, 22, 41].

Following the argument in [6], we begin by a completeness result we will require later, that is interesting in itself.

We denote by $\overline{\mathcal{B}}$ the space of "countable non-ephemeral blocks barcodes". It is the space of barcodes $\mathcal{B} \in \overline{\mathcal{B}}$ such that $\mathcal{B}$ contains only countably many closed[7] blocks, and all vertical and horizontal bands of $\mathcal{B}$ have non-zero width.

Theorem 4.6.2. *The space of countable non-ephemeral block barcodes $\overline{\mathcal{B}}$ is complete.*

Proof. Since the bottleneck distance is only a pseudo metric, any open or half open block is at distance 0 from a closed block. Therefore we choose arbitrarily to construct the limit of a Cauchy sequence as a collection of closed blocks.

First, notice that a barcode $\mathcal{B} \in \overline{\mathcal{B}}$ can be broken up to the four subsets $\mathcal{B}_{B:birth}$, $\mathcal{B}_{B:death}$, $\mathcal{B}_{B:vband}$ $\mathcal{B}_{B:hband}$. Respectively the birth shapes, death shapes, vertical band and horizontal bands of $\mathcal{B}$.

Since it is not possible to construct a matching between elements of two different collections, we can treat each type of shape independently. We will prove completeness when restricting on barcodes containing only vertical bands, the 3 other cases being almost identical, the birth and death quadrants being simpler.

[6] Strictly speaking, the result in [5, 9] is stated for bimodules indexed over the open half-plane above the minor diagonal $x+y=0$. However, a careful look at the proof reveals that the result extends easily to bimodules indexed over $\mathbb{R}^2$.

[7] We consider only the blocks that are closed subsets of $\mathbb{R}^2$ for the usual topology.

Let $(\mathcal{B}_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of barcodes containing only vertical bands. By extracting a subsequence we can suppose $\mathrm{d}_{\mathbf{b}}(\mathcal{B}_n, \mathcal{B}_{n+1}) < 2^{-(n+1)}$.

We denote by $I_1^1(i_1), i_1 \leq m_1$ (if the family is infinite, we say $m_1 = +\infty$) the family of vertical bands with width strictly greater than $1/4$ in $\mathcal{B}_1$, which is at most countable. Take a $1/4$ matching between $\mathcal{B}_1$ and $\mathcal{B}_2$. By restricting this matching to only bands of width strictly greater than $1/4$ in $\mathcal{B}_1$ we get a $1/2$ matching. Each of the $I_1^1(i_1)$ is matched to an interval $I_2^1(i_1)$ in $\mathcal{B}_2$. We then define the family of $I_2^2(i_2), i_2 \leq m_2$ to be the remaining elements in $\mathcal{B}_2$ that are not matched and of width strictly greater than $1/8$. By induction, given the families $(I_l^k(i_l))_{i_l\leq m_l}, l \leq k$ and a 2^{-k}-matching ϕ between $\mathcal{B}_k$ and $\mathcal{B}_{k+1}$ matching only bands of width greater than $2^{-(k+1)}$ from $\mathcal{B}_k$, we define $(I_{k+1}^1(i_1))_{i_1\leq m_1}, \ldots, (I_{k+1}^k(i_k))_{i_k\leq m_k}$ to be the images of the families $(I_l^k(i_l))_{i_l\leq m_l}, l \leq k$ through ϕ. Then we complete with the family $I_{k+1}^{k+1}(i_{k+1}), i_{k+1} \leq m_{k+1}$ of the bands remaining untouched in B_{k+1} with width greater than $2^{-(k+1)}$.

As previously stated, at rank k, two bands $I_k^l(i_l)$ and I_{k+1}^l are 2^{-k} matched. We write (x_k, y_k) for the horizontal coordinates of the beginning and ending of the band $I_k^l(i_l)$, and similarly (x_{k+1}, y_{k+1}) for I_{k+1}^l. It implies $|x_k - x_{k+1}| < 2^{-k}$ and $|y_k - y_{k+1}| < 2^{-k}$. Therefore, our sequence of vertical bands $(I_{l+n}^l(i_l))_{l\in\mathbb{N}}$ gives a Cauchy sequence of points in $\mathbb{R}^2$. It converges to a point with coordinates $(\lim x_n, \lim y_n)$ which can be identified with a closed band, with beginning $\lim x_n$ and ending $\lim y_n$, possibly ephemeral[8].

Let $\mathcal{B}_\infty$ be the set of all the non-ephemeral limit intervals constructed, containing vertical and horizontal bands, birth quadrants and death quadrants. By construction, $\mathcal{B}_\infty$ is countable, and contain only non-ephemeral blocks.

It remains to show that $(\mathcal{B}_n)_{n\in\mathbb{N}}$ converges to $\mathcal{B}_\infty$ for the bottleneck metric. We need to define a matching between $\mathcal{B}_n$ and $\mathcal{B}_\infty$. Given $l \leq n$ and $i_l \leq m_l$, let (x_n, y_n) be the coordinates of $I_n^l(i_l)$. We match each $I_n^l(i_l) \in \mathcal{B}_n$ with the closed bands of coordinates $(\lim x_n, \lim y_n)$ from $\mathcal{B}_\infty$. In the case the limit should be an ephemeral band, we let the interval unmatched. All bands with width strictly greater than 2^{1-n} from $\mathcal{B}_n$ are matched, by definition of the $I_n^l(i_l), l \leq n, i_l \leq m_l$. The unmatched elements from $\mathcal{B}_\infty$ are obtained as limits from interval whose width is at most 2^{1-n}. Therefore all unmatched $\mathcal{B}_\infty$ have width bounded by $2^{1-n} + 2^{-n}\sum_{j\in\mathbb{N}} 2^{-j} = 2^{1-n} + 2^{-n}$.

This gives us $\mathrm{d}_{\mathbf{b}}(\mathcal{B}_n, \mathcal{B}_\infty) \leq 2^{1-n} + 2^{-n}, \forall n$. Therefore $\mathcal{B}_\infty$ is the limit of the sequence $(\mathcal{B}_n)_{n\in\mathbb{N}}$ for the bottleneck metric.

□

[8] This is the case when $\lim x_n = \lim y_n$.

We now define the notion of tameness, a property that allows a wide variety of modules to be considered even if they are not pfd.

Definition 4.6.3. Given a fixed vector line l, a persistence bimodule M is *tame in direction* l if the map $\rho_t^{t+\epsilon}$ has finite rank for all points $t \in \mathbb{R}^2$ and for all nonzero nonnegative vectors ϵ aligned with l. M is called *tame* if it is tame in all directions.

Note that pfd modules are tame in all directions. Note also that being tame in at least two linearly independent directions implies being tame in all directions. Futhermore, being tame in at least one non-horizontal and non-vertical direction induces tameness in any other direction[9]. The properties of tame modules can be studied via their so-called *smoothings*.

Definition 4.6.4. Given a bimodule M and a nonzero nonnegative vector $\epsilon > 0$, the ϵ-*smoothing* of M is the bimodule M^ϵ with vector spaces $M_t^\epsilon = \operatorname{Im} \rho_{t-\epsilon}^t$ for $t \in \mathbb{R}^2$, connected by the linear maps induced from M.

By definition, when M is tame, its ϵ-smoothings in that direction are pfd.

Lemma 4.6.5. *If M is exact, then for any nonzero nonnegative vector ϵ, the module M^ϵ is also exact.*

Proof. We prove the result for a vertical vector ϵ for simplicity. The case of a horizontal vector ϵ is treated symmetrically. The general case follows by decomposing the vector into its horizontal and vertical components.

In the vertical case we want to show that the following commutative diagram is exact for all $s \leq t \in \mathbb{R}^2$:

$$\begin{array}{ccc} M^\epsilon_{(s_x,t_y)} & \longrightarrow & M^\epsilon_t \\ \uparrow & & \uparrow \\ M^\epsilon_s & \longrightarrow & M^\epsilon_{(t_x,s_y)} \end{array} \tag{4.13}$$

The result follows from a simple chasing argument in the diagram below,

[9] Any non-horizontal and non-vertical line is directed by a vector with two non zero coordinate. It is possible to compose a small smoothing along l and a smoothing along the vertical or horizontal line to obtain an arbitrary small smoothing in direction l'. The tameness in direction l' results immediately.

using the exactness of the upper and lower quadrangles:

$$
\begin{array}{ccc}
M_{(s_x,t_y)} & \longrightarrow & M_t \\
\uparrow & & \uparrow \\
M_s & \longrightarrow & M_{(t_x,s_y)} \\
\uparrow & & \uparrow \\
M_{s-\epsilon} & \longrightarrow & M_{(t_x,s_y)-\epsilon}
\end{array}
\tag{4.14}
$$

Take a vector $(\alpha, \beta) \in M^{\epsilon}_{(s_x,t_y)} \oplus M^{\epsilon}_{(t_x,s_y)} \subseteq M_{(s_x,t_y)} \oplus M_{(t_x,s_y)}$. Assume that it belongs to the kernel of the map $h - v$, where h and v are respectively the horizontal and vertical arrows pointing to M^{ϵ}_t in (4.13). Then, by exactness of the upper quadrangle in (4.14), there is a vector $\gamma \in M_s$ such that $\alpha = v_s^{(s_x,t_y)}(\gamma)$ and $\beta = h_s^{(t_x,s_y)}(\gamma)$. In the meantime, there is a vector $\delta \in M_{(t_x,s_y)-\epsilon}$ such that $\beta = v^{(t_x,s_y)}_{(t_x,s_y)-\epsilon}(\delta)$ since by assumption $\beta \in M^{\epsilon}_{(t_x,s_y)}$. Then, by exactness of the lower quadrangle in (4.14), there is a vector $\eta \in M_{s-\epsilon}$ such that $\gamma = v^s_{s-\epsilon}(\eta)$ and $\delta = h^{(t_x,s_y)-\epsilon}_{s-\epsilon}(\eta)$. Hence, $\gamma \in M^{\epsilon}_s$ and so (α, β) belongs to the image of (v', h'), where v' and h' are respectively the vertical and horizontal arrows originating at M^{ϵ}_s in (4.13). $\square$

Thus, when M is exact and tame, its ϵ-smoothings M^{ϵ} along any direction are exact and pfd therefore they are block decomposable by Theorem 4.1.3. Given now any sequence $(M^{\epsilon_n})_{n\in\mathbb{N}}$ of smoothings where $\|\epsilon_n\|$ goes to zero as n goes to infinity, the induced sequence of barcodes $(\mathcal{B}(M^{\epsilon_n}))_{n\in\mathbb{N}}$ is a Cauchy sequence in the bottleneck distance, by Corollary 4.6.1. Furthermore, we are going to show that $\mathcal{B}(M^{\epsilon_n})$ is countable. Let $K \subseteq \mathbb{R}^2$ be a compact. If the blocks from $\mathcal{B}(M^{\epsilon_n})$ which intersect K are not in a finite number, there would exist a point p where the dimension of M_p is unbounded. This is not possible since M is supposed p.f.d. Since $\mathbb{R}^2$ can be covered by the collection of balls $(B(0,m))_{m\in\mathbb{N}}$, and any block in $\mathcal{B}(M^{\epsilon_n})$ intersects one of these balls, we deduce that the number of blocks is countable. Since the space $\overline{\mathcal{B}}$ equipped with the bottleneck distance is complete by Theorem 4.6.2, this Cauchy sequence converges to some limit barcode $\mathcal{B}(M)$, which is independent of the choice of sequence $(\epsilon_n)_{n\in\mathbb{N}}$ and can therefore be assigned to the module M. Then, Corollary 4.6.1 and the triangle inequality induce the following isometry result:

Corollary 4.6.6. *For any exact and tame persistence bimodules M and N, the barcodes $\mathcal{B}(M)$ and $\mathcal{B}(N)$ are well-defined and we have:*

$$
\mathrm{d_I}(M, N) = \mathrm{d}_b(\mathcal{B}(M),\ \mathcal{B}(N)).
$$

4.6.3 Extensions

We stated, at Remark 4.1.4 that we can use Kan extension to obtain a more general result of our main theorem. We here detail rigorously why any exact module on an open subset $\mathbb{U} \subseteq \mathbb{R}^2$ stable by non-negative translation can be extended to an exact bimodule on $\mathbb{R}^2$. The special case of left and right Kan extension applied to bimodules have been detailed in Section 3.1.3.

Theorem 4.6.7. *Let M be an exact p.f.d. bimodule on an open subset $\mathbb{U}$ of $\mathbb{R}^2$ stable by positive translations (i.e. for all $x \in \mathbb{U}$, $v \in \mathbb{R}^2_{\geq 0}$ we have $x + v \in \mathbb{U}$). The module M can be extended to $\overline{M}$ indexed over $\mathbb{R}^2$ by taking its right Kan extension. This extension $\overline{M}$ is exact.*

Proof. For any $t \in \mathbb{R}^2$ such that $\forall u \in \mathbb{U}, t \leq u \in \mathbb{U}$ we compute the right Kan extension $Ran(M)(t) = \varprojlim M_{|\{s \geq t\} \cap \mathbb{U}}$.

This, by construction, gives us a module on $\mathbb{R}^2$ defined by $\overline{M}_t := Ran(M)(t)$ if $t \notin \mathbb{U}$ and $\overline{M}_t := M_t$ if $t \in \mathbb{U}$. It remains to prove that taking this Kan extension conserves exactness.

Consider the diagram:

$$\begin{array}{ccc} \overline{M}_x & \longrightarrow & \overline{M}_t \\ \uparrow & & \uparrow \\ \overline{M}_z & \longrightarrow & \overline{M}_y \end{array}$$

where $t \in \mathbb{U}$ and $x, y, z \notin \mathbb{U}$. Let $u \in \overline{M}_x$ and $v \in \overline{M}_y$ having same image in $\overline{M}_t$. Because $\overline{M}_x$ and $\overline{M}_y$ are cone, the projection gives us two families of vectors $\mathcal{F}_x$ and $\mathcal{F}_y$, defined respectively on $\mathbb{U} \cap Q_x$ and $\mathbb{U} \cap Q_y$, where $Q_r = \{s | \, s \geq r\}$. Since u and v have same image in $\overline{M}_t$, the commutativity with structural morphisms implies that these two families coincide on $\mathbb{U} \cap (Q_x \cap Q_y)$. To find a preimage of u and v in $\overline{M}_z$, we have to construct a family $\mathcal{F}_z$ defined on $\mathbb{U} \cap Q_z$. This will be possible thanks to the exactness and stability by positive translation of $\mathbb{U}$.

Let $(x_n)_{n \in \mathbb{N}}$ be a dense sequence of the open set $\Gamma = \mathbb{U} \cap (Q_z \setminus (Q_x \cup Q_y))$ for the usual $\mathbb{R}^2$ topology. Let $\Omega_0 = \mathbb{U} \cap (Q_x \cup Q_y)$, and start with the family $\mathcal{F}_0 = \mathcal{F}_x \cup \mathcal{F}_y$ defined on Ω_0. Since $\mathbb{U}$ is stable by positive translation and $x_0 \in \mathbb{U}$, $Q_{x_0} \subseteq \mathbb{U}$. Therefore, the border of $Q_{x_0} \setminus \Omega_0$ is a rectangle. Inductively, for each point x_n, we extend Ω_n to $\Omega_{n+1} = \Omega_n \cup Q_{x_n}$. At each step n, the border of $\Omega_n \setminus \Omega_0$ is piecewise linear, constituted only of finitely many vertical and horizontal lines. Therefore, Ω_{n+1} is the union of Ω_n and a birth quadrant. We extend the family $\mathcal{F}_n$ defined on Ω_n to a family $\mathcal{F}_{n+1}$ on Ω_{n+1} in the following way. Since the border of $\Omega_{n+1} \setminus \Omega_n$ is piecewise linear with finitely many horizontal and vertical segments, we can apply

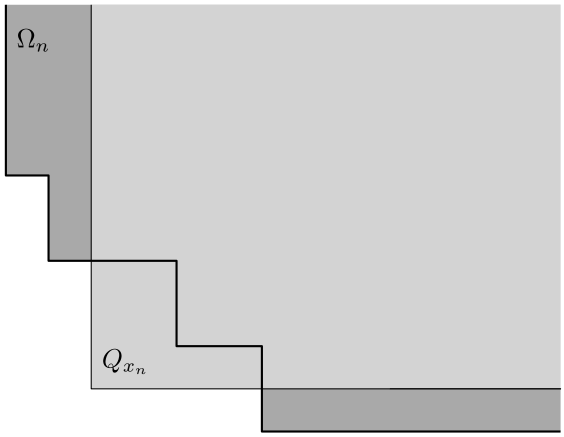

Figure 4.6: The set Ω_{n+1} as the union of Ω_n and Q_{x_n}.

finitely many time exactness on a subdivision of $\Omega_{n+1} \setminus \Omega_n$ in rectangles (see Figure 4.6). This lead to a vector in M_{x_n} generating a family defined on Q_{x_n} and coinciding with the the family previously defined on Ω_n. The family $\mathcal{F}_{n+1}$ is obtained by adding this vector and its images to $\mathcal{F}_n$. Notice that if $x_n \in \Omega_n$, then nothing have to be done and we let $\mathcal{F}_{n+1} = \mathcal{F}_n$.

By induction we obtain a family $\mathcal{F}_z$ defined on $\bigcup_{n\in\mathbb{N}} \Omega_n = Q_z \cap \mathbb{U}$. This family is an element of the limit $Ran(M)(z) = \overline{M}_z$. This element is sent to u and v through the structural morphisms of $\overline{M}$.

In the case $x \in \mathbb{U}$ or $y \in \mathbb{U}$, we can still define similar families, and the result hold. The case $t \notin \mathbb{U}$ is the easiest: the families $\mathcal{F}_z$ is simply obtained as the union $\mathcal{F}_x \cup \mathcal{F}_y$ since $(Q_x \cup Q_y) \cap \mathbb{U} = Q_z \cap \mathbb{U}$.

Therefore, the extended module is exact. □

4.6.4 Restriction principle

We now turn to another property of persistence bimodules that relates them to zigzags. We link the decomposition obtained from our result of an exact bimodule to the decomposition of a restriction. Special cases of restrictions of bi-modules frequently studied are one dimensional modules and zigzags.

Given a subposet $(\mathbb{U}, \preceq)$ of $(\mathbb{R}^2, \leq)$ (i.e. one such that $\mathbb{U}$ is a subset of $\mathbb{R}^2$ and $\preceq$ is a subset of $\leq$ as a binary relation on $\mathbb{U}$), any bimodule M on $(\mathbb{R}^2, \leq)$ restricts naturally to $(\mathbb{U}, \preceq)$ by taking the spaces M_t for $t \in \mathbb{U}$ and the maps ρ_s^t for $s \preceq t$. The resulting module is denoted by $M|_{(\mathbb{U},\preceq)}$. If M is block decomposable, then each of its summands restricts to $(\mathbb{U}, \preceq)$ in the same way, although its restriction may no longer be indecomposable.

Lemma 4.6.8. *Given a block B and a subposet $(\mathbb{U}, \preceq)$ of $(\mathbb{R}^2, \leq)$, the restriction of $\mathbf{k}_B$ to $(\mathbb{U}, \preceq)$ decomposes as follows:*

$$\mathbf{k}_B|_{(\mathbb{U},\preceq)} \simeq \bigoplus_{i\in I} \mathbf{k}_{C_B^i},$$

where the C_B^i are the equivalence classes of the symmetric closure of $\preceq$ on $\mathrm{supp}(B) \cap \mathbb{U}$*, and where $\mathbf{k}_{C_B^i}$ denotes the module having a copy of $\mathbf{k}$ at each element $t \in C_B^i$ and the identity map between any pair of points $s \preceq t \in C_B^i$, the rest of the spaces and maps being zero. Moreover, the decomposition is unique up to isomorphism and reordering of the terms, and each $M_{C_B^i}$ is indecomposable.*

Proof. The fact that the C_B^i partition $\mathrm{supp}(B) \cap \mathbb{U}$ with no order relation between them implies that their associated modules cover $\mathbf{k}_B|_{(\mathbb{U},\preceq)}$ and are in direct sum. The fact that they are themselves indecomposable comes from the fact that their endomorphism ring is isomorphic to $\mathbf{k}$ (by restriction of the one of $\mathbf{k}_B$) and therefore local. This proves also the uniqueness of the decomposition, by Azumaya's Lemma. □

By additivity, we deduce the following restriction principle:

Corollary 4.6.9 (Restriction principle)**.** *Given a subposet $(\mathbb{U}, \preceq)$ of $(\mathbb{R}^2, \leq)$, the restriction of any block decomposable bimodule M over $\mathbb{R}^2$ to $(\mathbb{U}, \preceq)$ decomposes as follows:*

$$M|_{(\mathbb{U},\preceq)} \simeq \bigoplus_{B\in\mathcal{B}(M)} \bigoplus_{i\in I_B} k_{C_B^i},$$

where the C_B^i are the equivalence classes of the symmetric closure of $\preceq$ on $\mathrm{supp}(B) \cap \mathbb{U}$*. Moreover, the decomposition is unique up to isomorphism and reordering of the terms, and each $k_{C_B^i}$ is indecomposable.*

Combined with Theorem 4.1.3, this result describes the decomposition of any module that is the restriction of some exact pfd bimodule to some subposet of $(\mathbb{R}^2, \leq)$. The special case of zigzag subposets is of particular interest:

Remark 4.6.10. Take an injective sequence of points $(s_i)_{i\in\mathbb{Z}}$ in $\mathbb{R}^2$, such that for each $i \in \mathbb{Z}$ we have either $s_i \leq s_{i+1}$ or $s_{i+1} \leq s_i$. Call $(\mathbb{U}, \preceq)$ the resulting zigzag, viewed as a subposet of $(\mathbb{R}^2, \leq)$. Then, given any exact pfd (or, more generally, block decomposable) bimodule M, the zigzag module obtained from M by restriction to $(\mathbb{U}, \preceq)$ is interval decomposable, and its summands are obtained from those of M by restriction. More precisely, each summand S_B of M gives rise to as many summands as the number of times the zigzag intersects $\mathrm{supp}(B)$.

4.6.5 Interlevel-sets persistence

We now consider a particular type of persistence bimodule that arises in the study of interlevel-sets persistence. Let $\mathbb{R}^{\mathrm{op}}$ denote the set $\mathbb{R}$ equipped with the opposite order, and let $\mathbb{U}$ be the subset of $\mathbb{R}^{\mathrm{op}} \times \mathbb{R}$ consisting of the points s such that $s_x < s_y$, equipped with the order induced from the product order on $\mathbb{R}^{\mathrm{op}} \times \mathbb{R}$. $\mathbb{U}$ is naturally identified with the set of nonempty bounded open intervals of $\mathbb{R}$, equipped with the inclusion order.

Given a topological space T and an $\mathbb{R}$-valued function $\gamma : T \to \mathbb{R}$, let $\mathcal{S}_\gamma$ denote the *interlevel-sets filtration* of γ, which assigns the space $\mathcal{S}_\gamma(s) = \gamma^{-1}((s_x, s_y))$ to any point $s \in \mathbb{U}$. $\mathcal{S}_\gamma$ can be viewed as a functor from the poset $\mathbb{U}$ to the category of topological spaces. The composition $\mathsf{H}(\mathcal{S}_\gamma)$ (where H stands for the singular homology functor with field coefficients) is then a functor from $\mathbb{U}$ to the category of vector spaces over $\mathbf{k}$. The map γ is called *pfd* (resp. *tame*) whenever $\mathsf{H}(\mathcal{S}_\gamma)$ is a pfd (resp. tame) module. Note that this module is always exact because, for any $s < t \in \mathbb{U}$, the following diagram

$$\begin{array}{ccc} \gamma^{-1}((t_x, t_y)) & \xleftarrow{\supseteq} & \gamma^{-1}((s_x, t_y)) \\ \subseteq \uparrow & & \uparrow \supseteq \\ \gamma^{-1}((t_x, s_y)) & \xleftarrow[\subseteq]{} & \gamma^{-1}((s_x, s_y)) \end{array}$$

induces an exact diagram in homology, by the Mayer-Vietoris theorem. For a pfd function whose bimoduleextend to a p.f.d. bimodule we get a decomposition by combining Theorem 4.1.3 and Theorem 4.6.7. The main decomposition theorem of [8] can be used to prove a stronger result:

Corollary 4.6.11. *For any topological space T and pfd function $\gamma : T \to \mathbb{R}$ the bimodule $\mathsf{H}(\mathcal{S}_\gamma)$ is block decomposable.*

This result answers partially a conjecture of Botnan and Lesnick [9, Conjecture 8.3]. Combined with Corollary 4.6.1, it induces a general stability result for interlevel-sets persistence, in which the functions considered do not have to be of *Morse type* [9, 30]:

Corollary 4.6.12. *For any pfd functions $\gamma, \gamma' : T \to \mathbb{R}$ whose bimodules $\mathsf{H}(\mathcal{S}_\gamma)$ and $\mathsf{H}(\mathcal{S}'_\gamma)$ extends to a p.f.d. bimodule on $\mathbb{R}^2$, the barcodes $\mathcal{B}(\mathsf{H}(\mathcal{S}_\gamma))$ and $\mathcal{B}(\mathsf{H}(\mathcal{S}_{\gamma'}))$ are well-defined and we have:*

$$\mathrm{d}_b(\mathcal{B}(\mathsf{H}(\mathcal{S}_\gamma)),\ \mathcal{B}(\mathsf{H}(\mathcal{S}_{\gamma'}))) \leq \|\gamma - \gamma'\|_\infty.$$

This result extends verbatim to tame functions with p.f.d module extension by Corollary 4.6.6. Besides, we can extend it to interlevel-sets filtrations obtained by taking preimages of bounded *closed* intervals, provided we only consider closed intervals that are not singletons[10]. The reason for this restriction is that the hypothesis of the Mayer-Vietoris theorem does not always hold for preimages of singletons, so the resulting bimodule may behave wildly on the diagonal. This can be avoided e.g. by using Steenrod-Sitnikov homology instead of singular homology. We refer the reader to [30, Section 3.2] for a more detailed treatment of this matter.

4.6.6 $\mathbb{Z}^2$-indexed modules

Let M be an exact pfd bimodule indexed over $\mathbb{Z}^2$. This means that every space M_t for $t \in \mathbb{Z}^2$ is finite-dimensional, and furthermore every instance of the quadrangle (4.1) with $s \leq t \in \mathbb{Z}^2$ commutes and is exact.

We can extend M to a piecewise-constant module $\bar{M}$ indexed over $\mathbb{R}^2$ as follows (where for $t = (t_x, t_y) \in \mathbb{R}^2$ the notation $\lfloor t \rfloor$ stands for $(\lfloor t_x \rfloor, \lfloor t_y \rfloor)$):

$$\forall t \in \mathbb{R}^2,\ \bar{M}_t = M_{\lfloor t \rfloor}$$
$$\forall s \leq t \in \mathbb{R}^2,\ \bar{M}_s^t = M_{\lfloor s \rfloor}^{\lfloor t \rfloor}$$

Obviously, since M is exact and pfd, so is $\bar{M}$. Therefore, $\bar{M}$ decomposes into block summands by Theorem 4.1.3, and so does M itself by restriction to the poset $(\mathbb{N}^2, \leq)$ (Proposition 4.6.9). The same reasoning applies to bimodules indexed over $\mathbb{N}^2$, which are extended over $\mathbb{R}^2_+$ (which is stable under positive translations). Hence:

Theorem 4.6.13. *Any exact pfd bimodule M indexed over $\mathbb{Z}^2$ (resp. $\mathbb{N}^2$) decomposes as a direct sum of block modules indexed over $\mathbb{Z}^2$ (resp. $\mathbb{N}^2$).*

Remark 4.6.14. It is interesting to point out that, due to the piecewise constant structure of the extension $\bar{M}$, the block modules involved in its decomposition are of the form $B \in \{B : \mathit{birth}, B : \mathit{death}, B : \mathit{hband}, B : \mathit{vband}\}$ with its cuts respecting $\sup c^- = \inf c^+ \in \mathbb{Z}$. The same holds for the summands in the decomposition of M. Note that a cut $(-\infty, x], (x, +\infty)$ can also be written as $(-\infty, x+1), [x+1, +\infty)$.

Suppose now that M is indexed over some finite grid, say $[\![0, n]\!] \times [\![0, m]\!]$. We can extend M to an exact pfd bimodule over $\mathbb{N}^2$ as follows. First, we extend it over $[\![0, n+1]\!] \times [\![0, m+1]\!]$ by duplicating the last row and column and by inserting identity maps. It is clear that this module is still exact and pfd. We now iterate the process in order to get an exact pfd extension of M over the entire lattice $\mathbb{N}^2$. By Theorem 4.6.13, this extension is decomposable as a direct sum of block modules. Since M is its restriction to the finite grid

[10]This means that our bimodules are once again indexed over the open half-plane above the diagonal $s_x = s_y$, whereas closed intervals are naturally parametrized by the closed half-plane above the diagonal.

$[\![0,n]\!] \times [\![0,m]\!]$, M itself decomposes as a direct sum of block modules by Corollary 4.6.9. Hence:

Corollary 4.6.15. *Any exact pfd bimodule M indexed over the finite grid $[\![0,n]\!] \times [\![0,m]\!]$ decomposes as a direct sum of block modules over the same grid.*

Remark 4.6.16. Note that in this case it is already known that M decomposes as a direct sum of indecomposables — see e.g. [43]. Our theorem identifies the summands as being blocks restricted to the finite grid. It might be possible to do so by using exactness in a more direct way though.

4.6.7 $\mathbb{Z}$-indexed zigzag modules

It is a well-known fact that pfd zigzag modules over $\mathbb{Z}$ decompose as direct sums of interval modules — see e.g. [7] for a recent treatment with a simple and elegant proof. This result is also a consequence of our main theorem. The connection happens through an extension of zigzag modules to $\mathbb{R}^2$ inspired from the one in [9].

Given a pfd zigzag module N indexed over $\mathbb{R}$, we can assume without loss of generality (by inserting in isomorphisms at the right places) that the arrow orientations in the module are alternating:

$$\cdots \longleftarrow N_{2i-2} \longrightarrow N_{2i-1} \longleftarrow N_{2i} \longrightarrow N_{2i+1} \longleftarrow N_{2i+2} \longrightarrow \cdots$$

Then we can embed the module into $\mathbb{R}^2$ by letting $M_{(i,-i)} = N_{2i}$ and $M_{(i,-i+1)} = N_{2i-1}$ for all $i \in \mathbb{Z}$, with the obvious embedding of the maps from N. The module M is indexed over the staircase $S_0 = \bigcup_{i\in\mathbb{Z}}\{(i,-i),(i,-i+1)\}$ that runs along the antidiagonal and is equipped with the partial order $\leq$ on $\mathbb{R}^2$. Now we extend[11] M to S_1, the one-step shift (upwards) of S_0, by letting $M_{(i,-i+2)}$ be the *pushout*[12] of $M_{(i-1,-i+2)} \longleftarrow M_{(i-1,-i+1)} \longrightarrow M_{(i,-i+1)}$ for all $i \in \mathbb{Z}$. We continue this process to further extend M to the two-steps shift S_2 of S_0, and so on, until M has been extended to the whole part of the integer lattice that lies above S_0. Symmetrically, we extend M to the part of the integer lattice that lies below S_0 via *pullbacks*[13].

The pushouts and pullbacks ensure that every quadrangle (4.1) with $s \in \mathbb{Z}^2$ and $t = s + (1,1) \in \mathbb{Z}^2$ commutes and is exact. Moreover, a simple

[11] We could use Kan extensions introduced earlier, but we present here a simple elementary way to do it, by hand.

[12] The pushout of $B \xleftarrow{f} A \xrightarrow{g} C$ is the space $D = (B \oplus C)/\operatorname{Im}(f,-g)$ together with the maps $b \mapsto (b,0) + \operatorname{Im}(f,-g) : B \to D$ and $c \mapsto (0,c) + \operatorname{Im}(f,-g) : C \to D$.

[13] The pullback of $B \xrightarrow{f} D \xleftarrow{g} C$ is the space $A = \operatorname{Ker}(f-g) \subseteq B \oplus C$ together with the natural projections $B \oplus C \to B$ and $B \oplus C \to C$ restricted to A.

diagram chasing argument shows that if each one of the small quadrangles commutes and is exact in the following diagram (where $s \leq t \leq u \in \mathbb{Z}^2$) then so does the big quadrangle:

$$\begin{array}{ccccc}
M_{(s_x,u_y)} & \longrightarrow & M_{(t_x,u_y)} & \longrightarrow & M_u \\
\uparrow & & \uparrow & & \uparrow \\
M_{(s_x,t_y)} & \longrightarrow & M_t & \longrightarrow & M_{(u_x,t_y)} \\
\uparrow & & \uparrow & & \uparrow \\
M_s & \longrightarrow & M_{(t_x,s_y)} & \longrightarrow & M_{(u_x,s_y)}
\end{array}$$

Thus, M is an exact $\mathbb{Z}^2$-indexed bimodule. It is also pfd because N was pfd to start with. Hence, M decomposes as a direct sum of block modules by Theorem 4.6.13. Since N is the restriction of M to the staircase S_0, N itself decomposes as a direct sum of interval modules by Corollary 4.6.9.

4.7 Conclusion

In this thesis, we first presented two results on multidimensional persistence in chapter 3. We showed a way to send a family of quiver representations into the category of bimodules. Then we defined a richer structure for both persistence and multidimensional persistence by introducing a product to the module structure, occurring naturally from the cohomology functor applied to a filtration.

In the last chapter which constitute the heart of this thesis (Chapter 4), we extended the decomposability result from [26] to a two dimensional version. This resulted in an answer to a conjecture from Botnan and Lesnick, and a deeper understanding of a subclass of bimodules.

The next steps will be to either extend this type of result to a wider class of modules, or find other classes which properties that allow to describe their indecomposables. One can hope that further research will lead to the definition of new invariants, easily computable, that would give an analog of persistence barcode for n-dimensional persistence modules, and allow to transpose the existing pipeline of persistence into higher dimensions. Some results have already been obtained notably in [16], but we are still far from the toolkit available in dimension one.

Bibliography

[1] H. Adams, T. Emerson, M. Kirby, C. Neville, R.and Peterson, P. Shipman, S. Chepushtanova, E. Hanson, F. Motta, and L. Ziegelmeier. Persistence images: A stable vector representation of persistent homology. *Journal Machine Learning Research*, 2017.

[2] G. Azumaya. Corrections and supplementaries to my paper concerning Krull-Remak-Schmidt's theorem. *Nagoya Mathematical Journal*, 1:117–124, 1950.

[3] U. Bauer, M. Kerber, and J. Reininghaus. Clear and compress: Computing persistent homology in chunks. Technical Report arXiv:1303.0477 [math.AT], March 2013.

[4] U. Bauer and M. Lesnick. Induced matchings and the algebraic stability of persistence barcodes. *Journal of Computational Geometry*, 6(2):162–191, 2015.

[5] H. Bjerkevik. Stability of higher-dimensional interval decomposable persistence modules. Research Report arXiv:1609.02086 [math.AT], April 2016.

[6] A. Blumberg, I. Gal, M. Mandell, and M. Pancia. Robust statistics, hypothesis testing, and confidence intervals for persistent homology on metric measure spaces. *Foundations of Computational Mathematics*, 14(4):745–789, 2014.

[7] M. Botnan. Interval decomposition of infinite zigzag persistence modules. *Proc. Amer. Math. Soc.*, 145, 2017.

[8] M. Botnan and W. Crawley-Boevey. Decomposition of persistence modules. Technical Report arXiv:1811.08946 [math.RT], 2018.

[9] M. Botnan and M. Lesnick. Algebraic stability of zigzag persistence modules. Research Report arXiv:1604.00655 [math.AT], April 2016.

[10] P. Bubenik. Statistical topological data analysis using persistence landscapes. *Journal of Machine Learning Research*, 16:77–102, 2015.

[11] M. Buchet and Escolar E. Realizations of indecomposable persistence modules of arbitrarily large dimension. Technical Report arXiv:1803.05722 [math.AT], March 2018.

[12] P. Caldero and J. Germoni. *Histoires hédonistes de groupes et de géométries, Tome premier.* Mathématiques en devenir. Calvage & Mounet, March 2013.

[13] P. Caldero and J. Germoni. *Histoires hédonistes de groupes et de géométries, Tome 2.* Mathématiques en devenir. Calvage & Mounet, March 2015.

[14] G. Carlsson and V. de Silva. Zigzag persistence. *Foundations of Computational Mathematics*, 10(4):367–405, Aug 2010.

[15] G Carlsson, V. de Silva, and D. Morozov. Zigzag persistent homology and real-valued functions. *Proceedings of the Annual Symposium on Computational Geometry*, pages 247–256, 2009.

[16] G. Carlsson, G. Singh, and A. Zomorodian. Computing multidimensional persistence. *Journal of Computational Geometry*, 2009.

[17] G. Carlsson and A. Zomorodian. The theory of multidimensional persistence. *Discrete and Computational Geometry*, 42(1):71–93, May 2009.

[18] M. Carrière, M. Cuturi, and S. Oudot. Sliced wasserstein kernel for persistence diagrams. *ICML*, 2017.

[19] M. Carrière, S. Oudot, and M. Ovsjanikov. Stable topological signatures for points on 3d shapes. *Proceedings 13th Symposium Geometry Processing*, 2015.

[20] A. Cerri, M. Ethier, and P. Frosini. A study of monodromy in the computation of multidimensional persistence.

[21] F. Chazal, D. Cohen-Steiner, M. Glisse, L. Guibas, and S. Oudot. Proximity of persistence modules and their diagrams. In *Proceedings of the Twenty-fifth Annual Symposium on Computational Geometry*, SCG '09, pages 237–246, New York, NY, USA, 2009. ACM.

[22] F. Chazal, V. de Silva, M. Glisse, and S. Oudot. *The structure and stability of persistence modules.* Springer Briefs in Mathematics. Springer International Publishing, 2016.

[23] W. Chazal, F. andCrawley-Boevey and V. de Silva. The observable structure of persistence modules. *Homology, Homotopy and Applications*, 18(2):247–265, 2016.

[24] J. Cochoy and S. Oudot. Decomposition of exact pfd persistence bimodules. Technical Report arXiv:1605.09726 [math.RT], 2016.

[25] D. Cohen-Steiner, H. Edelsbrunner, and J. Harer. Stability of persistence diagrams. *Discrete Comput. Geom.*, 37(1):103–120, January 2007.

[26] W. Crawley-Boevey. Decomposition of pointwise finite-dimensional persistence modules. *Journal of Algebra and Its Applications*, 14(05):1550066, 2015.

[27] V. de Silva, D. Morozov, and M. Vejdemo-Johansson. Dualities in persistent (co)homology. Research Report arXiv:1107.5665 [math.AT], July 2011.

[28] B. Di Fabio and M. Ferri. Comparing persistence diagrams through complex vectors. *CoRR*, 2015.

[29] H. Edelsbrunner and J. L. Harer. *Computational Topology, An Introduction*. Applied Mathematics. American Mathematical Society, 2010.

[30] Carlsson G., V. de Silva, S. Kališnik, and D. Morozov. Parametrized homology via zigzag persistence. Research Report arXiv:1604.03596 [math.AT], April 2016.

[31] D. Govc. On the definition of homological critical value. *ArXiv e-prints*, January 2013.

[32] A. Grothendieck. Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné) : III. étude cohomologique des faisceaux cohérents, Première partie. *Publications Mathématiques de l'IHÉS*, 11:5–167, 1961.

[33] W. Guo and A. Banerjee. Toward automated prediction of manufacturing productivity based on feature selection using topological data analysis. *IEEE International Symposium on Assembly and Manufacturing (ISAM)*, 2016.

[34] A. Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[35] G. Henselman and R. Ghrist. Matroid filtrations and computational persistent homology. Research Report arXiv:1606.00199 [math.AT], June 2016.

[36] G. E. Hinton and R. R. Salakhutdinov. Reducing the Dimensionality of Data with Neural Networks. *Science*, 313:504–507, July 2006.

[37] G. Kusano, K. Fukumizu, and Y. Hiraoka. Persistence weighted gaussian kernel for topological data analysis. *Proceedings 33rd International Conference on Machine Learning*, 2016.

[38] M. Lesnick. The theory of the interleaving distance on multidimensional persistence modules. Research Report arXiv:1106.5305 [cs.CG], June 2011.

[39] M. Lesnick and M. Wright. Interactive Visualization of 2-D Persistence Modules. *ArXiv e-prints*, December 2015.

[40] C. Maria and S. Oudot. Computing zigzag persistent cohomology. Research Report arXiv:1608.06039 [cs.CG], August 2016.

[41] S. Oudot. *Persistence Theory: From Quiver Representations to Data Analysis*. Number 209 in AMS Mathematical Surveys and Monographs. American Mathematical Society, 2015.

[42] J. Reininghaus, S. Huber, U. Bauer, and Kwitt R. A stable multi-scale kernel for topological machine learning. *Proceedings Conference Computer Vision Pattern Recognition*, 2015.

[43] C. Ringel. Introduction to representation theory of finite dimensional algebras, 2014. Izmir lectures, chapter 6 (`http://www.math.uni-bielefeld.de/~ringel/lectures/izmir/`).

[44] V. Robins and K. Turner. Principal component analysis of persistent homology rank functions with case studies of spatial point patterns, sphere packing and colloids. *Physica D: Nonlinear Phenomena*, 2016.

[45] George Soumya and Joseph Shibily. Text classification by augmenting bag of words (bow) representation with co-occurrence feature. *IOSR Journal of Com- puter Engineering*, 2014.

[46] The Stacks Project Authors. *Stacks Project*. `http://stacks.math.columbia.edu`, 2018.

[47] C. Webb. Decomposition of graded modules. *Proceedings of the American Mathematical Society*, 94(4):565–571, 1985.

[48] A. Yarmola. Persistence and computation of the cup product. Master thesis, Stanford University.

[49] A. Zomorodian and G. Carlsson. Computing persistent homology. *Discrete Comput. Geom.*, 33(2):249–274, February 2005.

Index

ÉCOLE DOCTORALE
Sciences et technologies de l'information et de la communication (STIC)

Titre : Décomposabilité et stabilité de la persistance multidimensionnelle

Mots clés : algèbre, homologie, persistance, analyse topologique de données

Résumé : Dans un contexte où des quantités toujours plus colossales de données sont disponibles, extraire des informations significatives et non triviales devient toujours plus difficile. Afin d'améliorer la classification, régression, ou encore l'analyse exploratoire de données, l'approche fournie par l'analyse topologique de données (TDA) est de rechercher la présence de formes dans le jeu de données. Dans cette thèse nous étudions les propriétés des modules de persistance multidimensionnelle dans le but d'obtenir une meilleure compréhension des sommandes et décompositions de ces derniers. Nous introduisons un foncteur qui plonge la catégorie des représentations de carquois dont le graphe est un arbre enraciné dans la catégorie des modules de persistance indexé sur $\mathbb{R}^2$. Nous enrichissons la structure de module de persistance provenant de l'application du foncteur cohomologie à une filtration en une structure d'algèbre de persistance. Enfin, nous généralisons l'approche de Crawley Beovey à la multipersistance et identifions une classe de modules de persistance indexé sur $\mathbb{R}^2$ qui possède des descripteurs simples et analogues au théorème de décomposition existant en persistance 1-dimensionnelle.

Title : Decomposability and stability of multidimensional persistence

Keywords : algebra, homology, persistence, topological data analysis

Abstract : In a context where huge amounts of data are available, extracting meaningful and non trivial information is getting harder. In order to improve the tasks of classification, regression, or exploratory analysis, the approach provided by topological data analysis is to look for the presence of shapes in data set. In this thesis, we investigate the properties of multidimensional persistence modules in order to obtain a better understanding of the summands and decompositions of such modules. We introduce a functor that embeds the representations category of any quiver whose graph is a rooted tree into the category of $\mathbb{R}^2$-indexed persistence modules. We also enrich the structure of persistence module arising from the cohomology of a filtration to a structure of persistence algebra. Finally, we generalize the approach of Crawley Beovey to multipersistence and identify a class of persistence modules indexed on $\mathbb{R}^2$ which have simple descriptor and an analog of the decomposition theorem available in one dimensional persistence.

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