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To cite this version:

Yichao Huang. Gaussian multiplicative chaos and applications to Liouville quantum gravity. Complex Variables [math.CV]. Université Pierre et Marie Curie - Paris VI, 2017. English. NNT : 2017PA066623 . tel-01993869

HAL Id: tel-01993869

https://theses.hal.science/tel-01993869v1

Submitted on 25 Jan 2019

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Université Pierre et Marie Curie

École doctorale de sciences mathématiques de Paris centre

Thèse de doctorat

Discipline : Mathématiques

présentée par

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Chaos multiplicatif Gaussien et applications à la gravité quantique de Liouville

---

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Résumé en français

Dans cette thèse, nous nous intéressons par des approches probabilistes à la gravité quantique de Liouville, introduite par Polyakov en 1981 [Pol81] sous la forme d'une intégrale de chemin sur les surfaces $2d$. Pour définir cette intégrale de chemin avec interaction exponentielle, nous partons du chaos multiplicatif Gaussien, l'outil fondamental pour définir l'exponentielle des champs Gaussiens de corrélation logarithmique.

Dans un premier temps, nous généralisons la construction de la gravité quantique de Liouville sur la sphère de Riemann par David-Kupiainen-Rhodes-Vargas à une autre géométrie avec bord, celle du disque unité. La nouveauté de ce travail réalisé en collaboration avec Rémi Rhodes et Vincent Vargas, est d'analyser avec soin le terme du bord dans l'intégrale de chemin ainsi que l'interaction entre la mesure du bord et la mesure du disque. Nous établissons rigoureusement les formules de la théorie conforme des champs en physique, telles que la covariance conforme, la formule KPZ, l'anomalie de Weyl ainsi que la borne de Seiberg. Une borne de Seiberg relaxée dans le cas de la gravité de Liouville à volume total fixé sur le disque est aussi formulée et étudiée.

Dans la seconde moitié de cette thèse, nous comparons cette construction à la Polyakov avec une autre approche de la gravité quantique de Liouville initiée par Duplantier-Miller-Sheffield. En collaboration avec deux autres jeunes chercheurs Juhan Aru et Xin Sun, nous fournissons une correspondance entre ces deux approches dans un cas simple et important, celui de la sphère de Riemann avec trois points marqués. En mélangeant les techniques de ces deux approches, nous fournissons une nouvelle procédure d'approximation qui permet de relier ces deux différentes approches.

Chapitre 1

Introduction

Nous donnons dans ce chapitre un panorama des objets étudiés dans cette thèse. Le but ici est de donner des idées qui relient ces différents objets au lieu de faire une liste exhaustive des propriétés utilisées. Notre fil conducteur sera le mouvement Brownien : on définira des objets mathématiques qui généraliseront un aspect particulier de ce dernier. Pour un traitement plus complet de chaque sujet abordé, on renvoie aux références.

Le mouvement Brownien

Le mouvement Brownien est peut-être l'objet probabiliste le plus étudié de notre époque. On peut le définir de façon intuitive avec les marches aléatoires simples. Rappelons sans entrer dans les détails le résultat suivant :

Théorème 1 (Convergence de la marche aléatoire simple sur $\mathbb{Z}$.).
Soit $(X_i)_{i\in\mathbb{N}}$ une suite de variables aléatoires indépendantes et identiquement distribuées, chacune prenant la valeur 1 ou -1 avec probabilité $1/2$. On pourra l'interpréter comme un jeu de pile ou face où l'on gagne ou perd 1 euro à chaque instant, indépendamment des autres lancers.
Notons

$$S(n) = \sum_{0\leq i\leq n} X_i. \tag{1.1}$$

$S(n)$ est alors le gain total (ou la perte totale) jusqu'à l'instant n (il ne dépend pas de la somme que possède le joueur à l'instant 0).
Par interpolation linéaire, on peut prolonger S en un processus aléatoire continu $\overline{S} : \mathbb{R}_+ \to \mathbb{R}$. Alors le processus $\overline{S}$ "proprement renormalisé" converge en loi vers un processus continu aléatoire sur $[0,1]$ que l'on appelle le mouvement Brownien.

Ce théorème admet une version plus forte due à Monroe Donsker [Don51] : la convergence ci-dessus reste vraie pour beaucoup plus de variables aléatoires.

Théorème 2 (Théorème de Donsker).
L'objet limite dans le théorème précédent reste le même si l'on remplace la suite $(X_i)_{i\in\mathbb{N}}$ par une autre suite de variables aléatoires indépendantes et identiquement distribuées, pourvu que ces variables aléatoires soient centrées et de variance 1.

Ce théorème révèle le caractère "universel" du mouvement Brownien : le mouvement Brownien est la limite d'échelle d'une famille très générale de processus discrets.

Les cascades multiplicatives

Les cascades multiplicatives, introduites par Benoit Mandelbrot, sont un analogue multiplicatif du mouvement Brownien. Nous rappelons d'abord une construction du mouvement Brownien due à Paul Lévy, puis on discutera de la construction des cascades multiplicatives par Benoit Mandelbrot.

Construction du mouvement Brownien par Paul Lévy

Commençons par une simple remarque qui sera utile plus tard :

Remarque 3.
La fonction S dans le Théorème 1 est indépendante de la somme d'argent que possède le joueur au départ, car elle code seulement la *fluctuation* au cours du temps.

Dans la suite de cette section, on suppose que $S(0) = 0$ pour fixer les idées. Dans ce cas S représente exactement la somme que possède le joueur à chaque instant.
On est en mesure de donner une version simplifiée de la construction du mouvement Brownien par Paul Lévy : l'idée de la construction est de "découvrir" le mouvement Browien sur $[0,1]$ par dichotomie de façon ponctuelle. Remarquons que la valeur du mouvement Brownien à l'instant 1 suit la loi Gaussiennne centrée réduite par le Théorème Central Limite.

Définition 4 (Construction du mouvement Brownien).
Soit Σ_I l'ensemble des fonctions réelles définies sur l'intervalle $I = [a,b]$ avec $b - a = l > 0$. On définit une opération de perturbation Φ_I sur $f \in \Sigma_I$ par

$$\Phi_I(f) = f + g_I \tag{1.2}$$

où g_I est une fonction linéaire par morceaux, obtenue en interpolant entre $g_I(a) = g_I(b) = 0$ et $g_I(\frac{a+b}{2}) = \frac{1}{2\sqrt{l}}\mathcal{N}$ avec $\mathcal{N}$ une variable de loi Gaussienne centrée réduite.

La construction du mouvement Brownien par Paul Lévy peut être formulée de la façon suivante (les variables $\mathcal{N}$ qui suivent sont indépendantes) :

1. *Prendre la fonction f constante nulle sur $[0,1]$;*
2. *Considérer la fonction f_0 linéaire $f_0 = f + t\mathcal{N}$ sur $[0,1]$;*
3. *Diviser $[0,1]$ en deux intervalles de longueur égale $I_0 = [0,1/2]$ et $I_1 = [1/2,1]$;*
4. *Dans chaque sous-intervalle, appliquer l'opération de perturbation à f_0 pour obtenir f_1 ;*
5. *Diviser chaque sous-intervalle en deux intervalles de longueur égale, ici $I_{00} = [0,1/4]$, $I_{01} = [1/4,1/2]$, $I_{10} = [1/2,3/4]$ et $I_{11} = [3/4,1]$;*
6. *Dans chaque sous-intervalle, appliquer l'opération de perturbation à f_1 pour obtenir f_2 ;*
7. *Continuer ainsi jusqu'à l'infini.*

Les opérations de divisions n'interfèrent pas entre elles si l'on travaille sur des intervalles différents : cette propriété est très forte car en effet, tout processus à accroissements indépendants et stationnaires est un processus de Lévy, qui selon la décomposition d'Itô-Lévy, est la somme d'un mouvement Brownien (avec shift) et d'un processus à sauts. Donc un processus continu centré à accroissements indépendants et stationnaires est systématiquement un mouvement Brownien.

Construction des cascades multiplicatives

Les cascades multiplicatives (parfois dites canoniques) ont été introduites par Benoit Mandelbrot dans [Man74] pour simuler la dissipation d'énergie lors d'un écoulement turbulent. Nous adaptons ici une description par Jean-Pierre Kahane et Jacques Peyrière [KP76] dans le cas binaire.

Définition 5 (Construction des cascades multiplicatives canoniques).
Prenons l'intervalle $[0,1]$. L'idée de la construction des cascades multiplicatives canoniques est analogue à celle du mouvement Brownien par Paul Lévy : on divise l'intervalle en parties de longueur égale et on introduit une perturbation aléatoire cette fois-ci multiplicative dans chaque subdivision, par une variable aléatoire positive W telle que $\mathbb{E}[W] = 1$. Concrètement, on effectue les opérations suivantes :

1. *Munissons $[0,1]$ de la mesure de Lebesgue usuelle;*
2. *Diviser $[0,1]$ en deux intervalles de longueur égale $I_0 = [0,1/2]$ et $I_1 = [1/2,1]$;*
3. *Dans chaque sous-intervalle, multiplier la mesure par une copie indépendante de W ;*
4. *Diviser chaque sous-intervalle en deux intervalles de longueur égale, ici $I_{00} = [0,1/4]$, $I_{01} = [1/4,1/2]$, $I_{10} = [1/2,3/4]$ et $I_{11} = [3/4,1]$;*
5. *Dans chaque sous-intervalle, multiplier la mesure obtenue précédemment par une copie indépendante de W (par exemple, dans l'intervalle I_{01}, la densité par rapport à la mesure de Lebesgue serait maintenant $W_0 \cdot W_{01}$ où les $W_\star$ sont indépendantes et distribuées selon W);*

Figure 1.1 – Construction des cascades multiplicatives : premières étapes. Simulation avec la loi uniforme sur $[1/2, 3/2]$.

6. *Continuer ainsi jusqu'à l'infini.*

À chaque génération n avec des intervalles de longueur 2^{-n}, on peut regarder la masse totale de l'intervalle $[0, 1]$ que l'on note Y_n. La procédure étant Markovienne (chaque opération ne dépend que de l'état actuel du système), la famille $(Y_n)_{n\in\mathbb{N}}$ est naturellement une martingale dans une filtration bien choisie. La question est de savoir si la suite $(Y_n)_{n\in\mathbb{N}}$ converge (e.g. en loi) vers une limite Y_∞, et si oui, Y_∞ est-elle une mesure non triviale ? L'étude de ces questions repose sur l'équation fonctionnelle suivante :

Lemme 6 (Équation fonctionnelle de cascades).
Les variables Y_n vérifient l'équation récurrente suivante :

$$Y_n = 2^{-1} \sum_{j=0}^{1} W(j) Y_{n-1}(j). \tag{1.3}$$

Rappelons que les variables aléatoires $W(j)$ et $Y_{n-1}(j)$ sont mutuellement indépendantes, et les $Y_{n-1}(j)$ (resp. $W(j)$) sont distribuées comme Y_{n-1} (resp. W).
Alors Y_∞ est solution de l'équation fonctionnelle suivante :

$$Z = 2^{-1} \sum_{j=0}^{1} W_j Z_j \tag{1.4}$$

où les variables aléatoires W_j et Z_j sont mutuellement indépendantes, les W_j (resp. Z_j) ayant la même distribution que W (resp. Z).

Kahane et Peyrière ont montré (entre autres) les résultats suivants sur Y_∞ :

Lemme 7 (Condition de convergence $\mathcal{L}^1$ et existence des moments).
On a les résultats suivants :

1. *Sur la non-dégénérescence de Y_∞ : LASSE (les assertions suivantes sont équivalentes).*
   a) $\mathbb{E}[Y_\infty] = 1$ *;*
   b) $\mathbb{E}[Y_\infty] > 0$ *;*
   c) Une solution Z *existe avec* $\mathbb{E}[Z] = 1$ *;*
   d) $\mathbb{E}[W \ln W] < \ln 2$.
2. *Sur l'existence des moments : on suppose que* $\mathbb{P}[W = 1] \neq 1$.
   Soit $h > 1$. *On a* $\mathbb{E}[Y_\infty^h] < +\infty$ *si et seulement si* $\mathbb{E}[W^h] < 2^{h-1}$.

On remarque que ces résultats sont aussi liés aux résultats classiques sur les marches aléatoires branchantes [Big77] : il y a en effet un lien direct entre ces deux objets. À noter cependant que la distance d'arbre n'est pas le même que la distance Euclidienne : en pratique un plongement d'arbre à la Cantor est utilisé pour passer de la géométrie de l'arbre (e.g. binaire) à la géométrie Euclidienne. En annexe A on propose un modèle de cascades multiplicatives plus homogènes en espace qui permet de trouver une corrélation mesurée directement par la distance Euclidienne.

Invariance conforme et propriété de Markov

Dans cette section, on donne une caractérisation du mouvement Brownien comme le point fixe d'une opération sur les chemins. Puis on présentera une généralisation en dimension 2, qui va nous mener à quelques propriétés importantes du champ libre Gaussien que l'on définira dans la section suivante.

Propriété de Markov du mouvement Brownien

Dans la construction de Lévy du mouvement Brownien, on voit que sur chaque sous-intervalle binaire, les opérations de perturbation créent une copie (plus petite et renormalisée) du mouvement Brownien qui relient les extrémités de cet intervalle. Cette propriété est très importante pour le mouvement Brownien : si l'on remplace une partie du mouvement Brownien standard par un autre mouvement Brownien indépendant, proprement renormalisé et qui prend les mêmes valeurs aux bords, le processus ainsi obtenu par concaténation suit encore la loi d'un mouvement Brownien standard.
Précisons ce qu'on veut dire par une copie "proprement renormalisée".

Proposition 8 (Invariance d'échelle).
Soit B un mouvement Brownien standard sur $[0, 1]$. Alors pour tout $\lambda > 0$,

$$\tilde{B}(s) = \frac{1}{\sqrt{\lambda}} B(\lambda s) \tag{1.5}$$

est encore un mouvement Brownien standard sur $[0, \frac{1}{\lambda}]$.

Quitte à translater $\tilde{B}$, on peut remplir n'importe quel intervalle de longueur positive par une copie de mouvement Brownien renormalisée ainsi.
Ensuite, précisons le sens de "prendre les mêmes valeurs aux bords". Intuitivement, on peut permuter les étapes dans la construction de Lévy : l'étape lors de laquelle on choisit la valeur du mouvement Brownien au point 1 peut être réalisée à la fin.

Proposition 9 (Conditions aux bords).
Un mouvement Brownien B sur $[0,1]$ conditionné à avoir $B(0) = a$ et $B(1) = b$ a la même loi qu'un mouvement Brownien B sur $[0,1]$ conditionné à avoir $B(0) = 0$ et $B(1) = 0$ auquel on rajoute la fonction linéaire f qui vaut a en 0 et b en 1.

Autrement dit, pour passer d'une condition aux bords à une autre, il suffit de rajouter le prolongement harmonique de la différence. Un mouvement Brownien B sur $[0,1]$ conditionné à avoir $B(0) = 0$ et $B(1) = 0$ est aussi appelé un "pont Brownien". On utilisera donc cette propriété pour changer les valeurs aux bords en rajoutant une fonction harmonique déterministe.
Enfin, énonçons la propriété de Markov sous forme de proposition :

Proposition 10 (Propriété de Markov du mouvement Brownien).
Pour tout sous-intervalle $[s,t] \subset [0,1]$, la loi de $B_{|[s,t]}$ sachant $B(s)$ et $B(t)$ est indépendante de $B_{|[0,1]\setminus[s,t]}$. En plus, cette loi est celle d'un mouvement Brownien standard sur $[s,t]$ conditionné à avoir les mêmes valeurs aux bords que B, à savoir $B(s)$ en s et $B(t)$ en t.

Un mouvement Brownien est donc une loi $\mathcal{L}$ sur l'ensemble des chemins sur $[0,1]$ invariante par l'opération de remplissage qui consiste à effacer une partie $[s,t]$ du chemin, puis remplir cette partie par une copie proprement renormalisée de chemin choisi selon la loi $\mathcal{L}$ conditionnée à avoir les bonnes valeurs aux bords.
Réciproquement, il n'est pas difficile de voir que ces propriétés caractérisent presque le mouvement Brownien : une telle loi $\mathcal{L}$ est à accroissements indépendants et stationnaires, et quelques hypothèses de plus (processus continu et centré) nous assurent qu'il s'agit bien d'un mouvement Brownien (à une constante multiplicative près).

Pré-champ libre Gaussien

Cherchons maintenant un champ Gaussien vérifiant les même propriétés énoncées au-dessus en dimension 2 :

Définition 11 (Invariance conforme intrinsèque).
On dit qu'un champ X est intrinsèquement invariant conforme si pour toute application conforme $\phi : D \to D'$ entre deux domains simplement connexes du plan, on a $\phi(X_{|D}) = X_{|D'}$ en loi.

En particulier, un tel champ X est invariant par tout automorphisme conforme de D.

Définition 12 (Conditions aux bords).
On dit qu'un champ X vérifie la règle de changement des conditions aux bords, si pour tout domaine simplement connexe $D \subset \mathbb{C}$, la loi de $X_{|D}$ moins le prolongement harmonique de $X_{|\partial D}$ dans D est la loi du champ $X_{|D}$ conditionné à être nul sur ∂D.

Autrement dit, pour passer d'une condition aux bords à une autre, il suffit de rajouter la différence au bord et son prolongement harmonique à l'intérieur.

Définition 13 (Propriété de Markov spatiale).
On dit qu'un champ X vérifie la propriété de Markov spatiale si pour tout domaine simplement connexe $D \subset \mathbb{C}$, la loi de $X_{|D}$ sachant $X_{|\partial D}$ est indépendante de $X_{\mathbb{C}\setminus D}$. De plus, cette loi est celle du champ X sur D conditionné à avoir les mêmes valeurs au bord.

On peut alors calculer explicitement la fonctions de corrélation sur le disque unité $\mathbb{D}$ d'un tel champ si l'on suppose de plus qu'il est Gaussien.

Théorème 14 (Pré-champ libre).
On appelle X un champ défini sur $D \subset \mathbb{C}$ "pré-champ libre" s'il est intrinsèquement invariant conforme, vérifie la régle de changement des conditions aux bords et la propriété de Markov spatiale. Si en plus X est Gaussien et centré, alors $X_{|\mathbb{D}}$ conditionné à être nul sur $\partial\mathbb{D}$ est de corrélation (à une constante multiplicative près)

$$G(x,y) = -\ln|x-y| + \ln|1-\overline{x}y| \tag{1.6}$$

avec $x, y \in \mathbb{D}$. On dit dans ce cas que X est un pré-champ libre Gaussien.

Idée de la preuve.
Une vraie preuve nécessite d'introduire une régularisation du champ X :

Définition 15 (Circle-average du pré-champ libre).
Soit $x \in \mathbb{D}$. Définissons le circle-average canonique du champ aléatoire X autour de x :

$$X_r(x) = \frac{1}{2\pi}\int_0^{2\pi} X(x + re^{i\theta})d\theta \tag{1.7}$$

pour tout $r \in \mathbb{R}_+^*$.

Nous donnons ici une preuve formelle et heuristique en imaginant qu'à chaque point z, $X(z)$ est une variable Gaussienne bien définie.
Tout d'abord, remarquons que $r \mapsto X_{e^{-r}}(0)$ est un processus Gaussien centré à accroissements indépendants (propriété de Markov spatiale) et stationnaires (invariance conforme). Ceci est donc un mouvement Brownien à une constante près : supposons que la constant soit égale à 1 dans la suite. Il s'ensuit que, pour tout $r \in]0,1[$,

$$\mathbb{E}[X_0(0)X_r(0)] = -\ln r \tag{1.8}$$

on voit par symétrie rotationnelle (or invariance conforme) que

$$\mathbb{E}[X(0)X(r)] = -\ln r. \tag{1.9}$$

On sait calculer alors la corrélation entre 0 et un point sur l'axe réel positif. Dans le cas général, pour n'importe quel couple (x, y), il existe une transformation conforme du disque de la forme

$$\phi(z) = \frac{z - x}{1 - \overline{x}z} e^{i\theta} \tag{1.10}$$

avec $\theta \in [0, 2\pi[$ qui envoie x sur 0 et y sur un point sur l'axe réel positif, en l'occurence le point $l = |\frac{y-x}{1-\overline{x}y}|$. Alors par invariance conforme,

$$\mathbb{E}[X(x)X(y)] = \mathbb{E}[X(0)X(l)] = -\ln|y - x| + \ln|1 - \overline{x}y|. \tag{1.11}$$

On trouve la fonction de corrélation G énoncée plus haut. □

Un champ Gaussien sur $\mathbb{D}$ avec la fonction de corrélation (1.6) est aussi appelé un champ libre Gaussien avec conditions aux bords du type Dirichlet sur $\mathbb{D}$.

Champ libre Gaussien

On définit le champ libre Gaussien avec l'intégrale de chemin. L'idée est de donner une mesure de probabilité sur l'espace des fonctions généralisées dans un domaine, qui peut être vue comme une procédure de quantification analogue à celle qu'on utilise pour définir le mouvement Brownien.

Mouvement Brownien comme quantification

L'intégrale de chemin, initiée par le physicien Richard Feynman [Fey48], est un outil mathématique puissant très utilisé en physique, en particulier dans la théorie quantique des champs. Dans le contexte du mouvement Brownien, on peut grossièrement résumer l'idée de l'intégrale de chemin de la manière suivante. Regardons l'ensemble des chemins réels définis sur l'intervalle $[0, 1]$:

$$\Sigma = \{\sigma : [0, 1] \to \mathbb{R}\} \tag{1.12}$$

et essayons de donner une énergie $S(\sigma)$ à chaque chemin. L'idée sera de donner une énergie élevée aux chemins qui font trop de "zigzags" : en effet, il semble naturel de supposer que le chemin constant nul σ_0 tel que $\sigma_0(s) = 0$ pour tout $s \in [0, 1]$ coûte moins d'énergie que

les autres.
Cette énergie est appelée "action" en physique. On définit ensuite une mesure de probabilité sur Σ en déclarant que chaque chemin σ sera tiré avec une probabilité proportionnelle à $e^{-S(\sigma)}$: c'est la distribution de Boltzmann. Autrement dit, on définit une loi de probabilité $\mathbb{P}$ sur Σ par

$$\mathbb{E}[F((B_s)_{s\in[0,1]})] = \frac{1}{Z}\int_\Sigma F(\sigma)e^{-S(\sigma)}d\sigma \tag{1.13}$$

où $d\sigma$ désigne la mesure de Lebesgue sur Σ, i.e. la mesure uniforme formelle, et Z est une constante de renormalisation pour que l'on obtienne bien une mesure de probabilité.
Il s'avère que si l'on choisit comme action

$$S(\sigma) = \frac{1}{2}\int_0^1 |\sigma'(s)|^2 ds, \tag{1.14}$$

le chemin aléatoire $(B_s)_{s\in[0,1]}$ ainsi défini est le mouvement Brownien canonique (à une constante additive près). Ceci paraît un peu paradoxal : le mouvement Brownien, rappelons-le, n'est pas dérivable, donc l'action telle qu'elle s'écrit n'a pas de sens quand il s'agit d'un chemin Brownien typique. Cependent, une façon de justifier cette définition est de passer par une discrétisation du chemin, i.e. approximer un chemin par une marche aléatoire [ADJ90].
Remarquons que l'action S ne code que la fluctuation du chemin : c'est pour cela que le mouvement Brownien est défini modulo une constante de cette façon. On pourra fixer la constante en rajoutant la condition $B(0) = 0$, puis faire varier cette valeur par une loi quelconque sur $\mathbb{R}$. Par exemple, si l'on veut que la loi de $B(0)$ soit identique à la loi de $B(1)$, il est préférable de prendre comme condition de bord $B(0) = c$ avec c une constante choisie selon la mesure de Lebesgue sur $\mathbb{R}$.
On pourrait aussi rendre la définition du mouvement Brownien invariante par homéomorphisme croissant $\xi : [0,1] \to [0,1]$:

Théorème 16 (Plongement d'un mouvement Brownien).
Soit e une fonction positive définie sur $[0,1]$. Considérons l'action

$$S(\sigma, e) = \frac{1}{2}\int_0^1 \frac{|\sigma'(s)|^2}{e(s)} ds \tag{1.15}$$

pour tout $\sigma \in \Sigma$. Alors $S(\sigma, e)$ est invariant par reparamétrisation de l'intervalle $[0,1]$, et le mouvement Brownien défini avec S l'est aussi. En effet, soit $\xi : [0,1] \to [0,1]$ un homéomorphisme croissant, alors

$$S(\sigma(s), e(s)) = S(\sigma^\xi(s), e^\xi(s)) = S(\sigma\circ\xi(s), \xi'(s)e\circ\xi(s)). \tag{1.16}$$

On interprétera

$$g(s) = e(s)^2, s \in [0,1] \tag{1.17}$$

comme une métrique sur $[0,1]$. Alors le mouvement Brownien que l'on a défini plus haut correspond au mouvement Brownien dans la métrique Euclidienne usuelle. Cette nouvelle définition nous donne une propriété d'invariance conforme du mouvement Brownien sur l'intervalle $[0,1]$ avec une métrique de fond.

Champ libre Gaussien : action et fonction de partition

En remplaçant la dérivée en dimension 1 par le gradient en dimension 2, on peut se demander si c'est possible de définir proprement une surface aléatoire de la même manière. Pour fixer les idées, plaçons dans la géométrie la plus simple dans notre cas, à savoir la sphère de Riemann $\mathbb{S} = \mathbb{C} \cup \{\infty\}$ munie d'une métrique Riemannienne g (de volume total fini), et notre ensemble de surfaces sur lequel l'action sera définie prendra la forme

$$\Sigma_{\mathbb{S}} = \{X : \mathbb{S} \to \mathbb{R}\}. \tag{1.18}$$

Nous définissons alors une mesure de probabilité à la Boltzmann sur $\Sigma_{\mathbb{S}}$:

$$\exp(-S(X,g))D_g X \tag{1.19}$$

où l'action $S(X,g)$ est remplacée par

$$S(X,g) = \frac{1}{4\pi}\int_{\mathbb{S}} |\partial^g X|^2 d\lambda_g \tag{1.20}$$

avec $D_g X$ la mesure de Lebesgue sur Σ muni de la métrique g et λ_g la forme volume de la métrique g.
Remarquons que l'action S est indépendante de la métrique g, et elle est invariante par l'opération $X \mapsto X + c$ pour n'importe quelle constante c. De manière heuristique, il n'est pas difficile de se convaincre que les champs X sont Gaussiens et qu'ils sont corrélés logarithmiquement : en faisant une intégration par parties, on a

$$S(X,g) = -\frac{1}{4\pi}\int_{\mathbb{S}} X\Delta_g X d\lambda_g = -\frac{1}{2}\int_{\mathbb{S}} \frac{1}{2\pi} X\Delta X d\lambda \tag{1.21}$$

Formellement, X est un champ Gaussien corrélé en $-\frac{1}{2\pi}\Delta^{-1}$: en dimension 2, il s'agit de la fonction logarithmique

$$G(x,y) = -\ln|x-y|. \tag{1.22}$$

Cette fonction de corrélation n'est pas définie positive car, comme pour le mouvement Brownien, on doit fixer la constante dans X. La façon dont la représentation est choisie est assez simple. D'abord, on choisit une métrique g de volume total fini sur $\mathbb{C}$ et puis on regarde le champ X_g obtenu en soustrayant la moyenne de X dans la métrique g : on obtient

ainsi un champ de moyenne nulle dans la métrique g.
Par exemple, dans [DKRV16], on considère la métrique sphérique (renormalisée)

$$\hat{g}(z) = \frac{1}{\pi(1+|z|^2)^2} \tag{1.23}$$

et on considère

$$X_{\hat{g}} = X - \int_{\mathbb{C}} X\hat{g} \tag{1.24}$$

qui est de fonction de corrélation (bien définie cette fois-ci)

$$G_{\hat{g}}(x,y) = -\ln|x-y| - \frac{1}{4}(\ln\hat{g}(x) + \ln\hat{g}(y)) + \chi \tag{1.25}$$

où χ est une constante que l'on peut calculer explicitement.

Chaos multiplicatif Gaussien

Dans cette section, nous définissons le chaos multiplicatif Gaussien introduit par Jean-Pierre Kahane en 1985 [Kah85], qui est formellement l'exponentielle d'un champ Gaussien log-corrélés. Initialement utilisé pour modéliser les phénomènes intermittents observés dans les écoulements turbulents, le chaos multiplicatif Gaussien est beaucoup utilisé aujourd'hui pour définir certains objets dans l'étude de la gravité quantique de Liouville [Kah85, RV10, RV14, Sha14, Ber17, JS15].

Martingale exponentielle et théorème de Girsanov

On sait définir l'exponentielle du mouvement Brownien depuis longtemps. Très souvent, on utilise la martingale exponentielle pour définir un shift sur le mouvement Brownien : c'est l'objet du théorème de Cameron-Martin [CM44].

Théorème 17 (Théorème de Cameron-Martin).
Soit $g : [0,1] \to \mathbb{R}$ une fonction continue de carré intégrable. Notons

$$h(t) = \int_0^t g(s)ds.$$

Soit B un mouvement Brownien sous la probabilité $\mathbb{P}$, alors sous la probabilité

$$d\mathbb{Q} = \exp\left(\int_0^1 g(s)dB(s) - \frac{1}{2}\int_0^1 g(s)^2 ds\right) d\mathbb{P} \tag{1.26}$$

le processus

$$\beta(t) = B(t) - h(t) \tag{1.27}$$

est encore un mouvement Brownien.

Ainsi, l'exponentielle d'un mouvement Brownien peut être vue comme un opérateur qui permet de "shifter" un champ Gaussien par une fonction déterministe.

Exponentielle d'un champ Gaussien corrélé logarithmiquement

Commençons par une définition :

Définition 18 (Champ Gaussien corrélé logarithmiquement).
On dit qu'un champ Gaussien X, défini sur un domaine $D \subset \mathbb{R}^d$ simplement connexe, est corrélé logarithmiquement si sa fonction de corrélation s'écrit

$$\mathbb{E}[X(x)X(y)] = -\ln|x-y| + O(1) \tag{1.28}$$

avec $x, y \in D$ et $O(1)$ une fonction continue bornée sur D.

Le but est de définir l'exponentielle de ces champs de corrélation logarithmique (donc en particulier du champ libre Gaussien). Rappelons que le théorème de Cameron-Martin permet de voir l'exponentielle du mouvement Brownien comme un opérateur de shift qui agit sur l'espace des mesures Gaussiennes. Mentionnons donc l'article [Sha14], dans lequel le chaos multiplicatif Gaussien est vu et défini comme un opérateur de shift qui agit sur l'espace des mesures Gaussiennes en dimension d, exactement de la même manière que l'exponentielle du mouvement Brownien. En particulier, une convergence en probabilité vers la mesure limite lors de la procédure de régularisation a été obtenue, indépendamment de la régularisation choisie.

Théorème 19 (Convergence en probabilité du chaos multiplicatif Gaussien).
Soit X un champ Gaussien de corrélation logarithmique sur un domaine $D \subset \mathbb{R}^d$ simplement connexe muni d'une mesure de Radon σ. Notons X_ϵ une régularisation du champ X : pour simplifier, on peut prendre le circle-average défini dans Proposition 15.
Si $\gamma^2 < 2d$, la famille des mesures aléatoires

$$\forall A \subset D, \quad M_\epsilon^\gamma(A) := \int_A e^{\gamma X_\epsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(x)^2]} d\sigma(x) \tag{1.29}$$

converge en probabilité et dans $\mathcal{L}^1$. La mesure limite

$$M^\gamma(\cdot) := \lim_{\epsilon \to 0} M_\epsilon^\gamma(\cdot) = \lim_{\epsilon \to 0} \int_\cdot e^{\gamma X_\epsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(x)^2]} d\sigma(x) \tag{1.30}$$

ainsi définie s'appelle le chaos multiplicatif Gaussien du champ X de paramètre γ.

Pour l'existence des moments de la mesure limite M, nous choisissons ici de présenter une variante un peu plus simple basée sur l'approche originale de Kahane [Kah85]. L'outil fondamental de cette approche est l'inégalité de comparaison Gaussienne due à Jean-Pierre Kahane [Kah85] :

Théorème 20 (Inégalité de convexité de Kahane).
Soient $(X_i)_{i\in I}$ et $(Y_i)_{i\in I}$ deux familles finies de variables aléatoires réelles Gaussiennes. On suppose que

$$\forall i,j \in I, \mathbb{E}[X_iX_j] \leq \mathbb{E}[Y_iY_j]. \tag{1.31}$$

Soit $F : \mathbb{R}_+ \to \mathbb{R}$ une fonction convexe avec croissance au plus polynomiale à l'infini. Alors $\forall p_i \geq 0$,

$$\mathbb{E}[F(\sum_{i\in I} p_i \exp(X_i - \frac{1}{2}\mathbb{E}[X_i^2]))] \leq \mathbb{E}[F(\sum_{i\in I} p_i \exp(Y_i - \frac{1}{2}\mathbb{E}[Y_i^2]))]. \tag{1.32}$$

Ce théorème nous permet de comparer les fonctionnelles convexes de l'exponentielle d'un vecteur Gaussien si l'on a une inégalité uniforme sur leur fonctions de corrélation. On peut étendre cette inégalité au cas continu :

Lemme 21 (Inégalité de convexité de Kahane, cas continu).
Soient $(X_t)_{t\in\mathbb{R}}$ et $(Y_t)_{t\in\mathbb{R}}$ deux processus Gaussiens centrés tels que leur noyaux de covariance f_X et f_Y soient continus et vérifient

$$\forall u,t \in \mathbb{R}, f_X(t,u) \leq f_Y(t,u). \tag{1.33}$$

Soit μ une mesure de Radon positive sur $\mathbb{R}$. Alors, pour toute fonction convexe $F : \mathbb{R}_+ \to \mathbb{R}$ avec croissance au plus polynomiale à l'infini, on a :

$$\mathbb{E}[F(\int_a^b e^{X_t - \frac{1}{2}\mathbb{E}[X_t^2]}\mu(dt))] \leq \mathbb{E}[F(\int_a^b e^{Y_t - \frac{1}{2}\mathbb{E}[Y_t^2]}\mu(dt))]. \tag{1.34}$$

On utilisera souvent ce lemme pour modifier légèrement la fonction de corrélation du champ Gaussien pour se ramener à un champ Gaussien plus simple à étudier (par exemple, un champ Gaussien qui admet certaines propriétés d'indépendance d'accroissements [ARV13, RSV14]). En effet, on a le corollaire suivant sur les moments du chaos multiplicatif Gaussien en prenant $F : x \mapsto x^p$:

Corollaire 22 (Conséquence de l'inégalité de convexité de Kahane).
Soient $(X_t)_{t\in\mathbb{R}}$ et $(Y_t)_{t\in\mathbb{R}}$ deux processus Gaussiens centrés tels que leurs noyaux de covariance f_X et f_Y soient continus et vérifient

$$\forall u,t \in \mathbb{R}, f_X(t,u) \leq f_Y(t,u) \leq f_X(t,u) + C. \tag{1.35}$$

Alors il existe $K > 0$ qui dépend seulement de C et p,

$$\begin{aligned}
&\frac{1}{K}\mathbb{E}[(\int_a^b e^{X_t - \frac{1}{2}\mathbb{E}[X_t^2]}\sigma(dt))^p] \\
\leq &\mathbb{E}[(\int_a^b e^{Y_t - \frac{1}{2}\mathbb{E}[Y_t^2]}\sigma(dt))^p] \\
\leq &K\mathbb{E}[(\int_a^b e^{X_t - \frac{1}{2}\mathbb{E}[X_t^2]}\sigma(dt))^p].
\end{aligned}$$

Ainsi, les moments du chaos multiplicatif Gaussien de l'ensemble des champs Gaussiens corrélés logarithmiquement sont comparables dans le sens du corollaire. En utilisant une généralisation du Lemme 97 en dimension d, nous en tirons le théorème suivant sur la convergence $\mathcal{L}^p$ du chaos multiplicatif Gaussien pour $p > 1$:

Théorème 23 (Sur l'existence des moments du chaos multiplicatif Gaussien).
Soit X un champ Gaussien sur un domaine borné simplement connexe $D \subset \mathbb{R}^d$ dont la corrélation s'écrit

$$\forall x, y \in D, \quad \mathbb{E}[X(x)X(y)] = -\ln|x-y| + O(1). \tag{1.36}$$

Soit M^γ la mesure du chaos multiplicatif Gaussien de paramètre $\gamma < \sqrt{2d}$ associée au champ X. Alors pour tout $p > 1$,

$$\forall A \subset D, \quad \mathbb{E}[M^\gamma(A)^p] < \infty \tag{1.37}$$

si et seulement si $p < \frac{2d}{\gamma^2}$.

Nous recommandons enfin la lecture de l'article [Ber17] dans lequel le chaos multiplicatif Gaussien dans le régime sous-critique $\gamma < \sqrt{2d}$ est défini d'une manière concise : l'idée de base est une estimation $\mathcal{L}^2$ de la martingale associée à la régularisation mais en contrôlant au préalable les valeurs extrêmes par une estimation $\mathcal{L}^1$. Voir aussi [DS11] pour le cas du champ libre Gaussien.

Analyse multifractale, moments du chaos multiplicatif Gaussien

Grâce aux outils issus de l'analyse des processus Gaussiens, nous regardons les moments du chaos multiplicatif Gaussien : nous nous intéressons d'abord à des moments d'un chaos multiplicatif Gaussien non-classique défini sur une géométrie à bord, en l'occurence le disque unité $\mathbb{D}$. Nous appliquons ensuite la même méthode pour obtenir un nouveau résultat sur la vitesse de divergence des moments dans le régime surcritique $p > \frac{2d}{\gamma^2}$.

Étude des moments pour le chaos multiplicatif Gaussien sur le disque

On étudie le chaos multiplicatif Gaussien associé au champ libre Gaussien du type Neumann sur le disque unité. Concrètement, prenons la fonction de Green du disque $\mathbb{D}$ avec condition aux bords du type Neumann dans la métrique Euclidienne :

$$G_{\mathbb{D}}(x,y) = -\ln|x-y| - \ln|1-\overline{x}y| \tag{1.38}$$

avec $x, y \in \mathbb{D}$. Considérons alors un champ Gaussien X de corrélation $G_{\mathbb{D}}$ sur $\mathbb{D}$ et la mesure du chaos multiplicatif Gaussien associé intégrée contre la mesure de Poincaré sur le disque :

$$\forall A \subset \mathbb{D}, \quad M^{\gamma}(A) := \int_A e^{\gamma X} \frac{4}{(1-(x^2+y^2))^2} d\lambda \tag{1.39}$$

Dans ce cas, l'étude des moments de cette mesure aléatoire nous permet d'établir la borne de Seiberg relaxée pour la mesure de Liouville : voir l'article [HRV15]. Remarquons que cette mesure n'est pas une mesure de chaos multiplicatif Gaussien classique car les corrélations avec un point au bord se comportent comme $-2\ln|x-y|$ au lieu de $-\ln|x-y|$.

Lemme 24 (Existence des moments du chaos multiplicatif Gaussien sur le disque).
Soit $\gamma \in]0,2[$. *Alors*

$$p < \left(\frac{1}{2} + \frac{1}{\gamma^2}\right) \vee \frac{2}{\gamma^2} \Longrightarrow \mathbb{E}[M^{\gamma}(\mathbb{D})^p] < \infty \Longrightarrow p < \frac{2}{\gamma^2}. \tag{1.40}$$

On espère obtenir une borne optimale $2/\gamma^2$ pour les deux sens : malheureusement les estimations pour $p < 1$ sont un peu plus délicates à manipuler, et on n'a pas obtenu de condition nécessaire et suffisante.
Pour établir la borne de Seiberg relaxée, on étudie de plus si cette mesure intègre les singularités logarithmiques au bord et à l'intérieur du disque. Dans ce cas, on peut appliquer une technique de [DKRV16] : les détails se trouvent [HRV15]. Le problème est essentiellement unidimensionnel et la géométrie du disque ne joue pas de rôle spécial.

Étude des moments surcritiques du chaos multiplicatif Gaussien

En appliquant la même méthode que précédemment, on parvient également à montrer un résultat sur les moments surcritiques du chaos multiplicatif Gaussien classique. On se donne un champ Gaussien X de corrélation logarithmique en dimension d, et une régularisation X_ϵ au niveau ϵ du champ : pour simplifier les idées, supposons que la régularisation est donnée par le processus circle-average.

Lemme 25 (Moments surcritique du chaos multiplicatif Gaussien).
Soient $\gamma < \sqrt{2}$ *et* $\mathbb{B}$ *le pavé unité dans* $\mathbb{R}^d$. *Notons pour tout* $p > 0$

$$\overline{\zeta}(p) = (d + \gamma^2/2)p - (\gamma^2/2)p^2 \tag{1.41}$$

la fonction de scaling.
Prenons X un champ Gaussien sur $\mathbb{B}$ de corrélation

$$G(x,y) = -\ln|x-y|.$$

Si $p < \frac{2d}{\gamma^2}$, on a

$$\mathbb{E}\left[\left(\int_{\mathbb{B}} e^{\gamma X_{1/2^n}(x) - n\gamma^2/2\ln 2} dx\right)^p\right] \leq C(p). \tag{1.42}$$

Si $p > \frac{2d}{\gamma^2}$, alors on a

$$c(p)2^{n(d-\overline{\zeta}(p))} \leq \mathbb{E}\left[\left(\int_{\mathbb{B}} e^{\gamma X_{1/2^n}(x) - n\gamma^2/2\ln 2} dx\right)^p\right] \leq C(p)2^{n(d-\overline{\zeta}(p))} \tag{1.43}$$

avec $c(p), C(p)$ des constantes qui ne dépendent que de p.

Il est facile de passer au cas général (de fonction de corrélation plus générale) en appliquant l'inégalité de convexité de Kahane (Lemme 20).

Résumé des travaux

Dans cette partie, nous présentons les travaux effectués pendant cette thèse.

[HRV15] Gravité quantique de Liouville sur le disque unité

Dans ce travail, on construit la gravité quantique de Liouville sur le disque unité $\mathbb{D}$ à partir de l'action de Liouville. Rappelons que le but est de donner une "bonne" loi de probabilité sur les fonctions définies sur le disque $\mathbb{D}$. Notons donc

$$\Sigma_{\mathbb{D}} = \{X : \mathbb{D} \to \mathbb{R}\} \tag{1.44}$$

l'ensemble des applications réelles sur le disque $\mathbb{D}$. Dans le cas du disque, l'action de Liouville est un peu différente de celle sur la sphère de Riemann étudié dans [DKRV16] : on voit notamment apparaître un second terme dans l'action lié au bord $\partial\mathbb{D}$. Cette action s'écrit, avec $\gamma \in]0,2[$:

$$\begin{aligned} S_{\mathbb{D}}(X,g) =& \frac{1}{4\pi}\int_{\mathbb{D}} \left(|\partial^g X|^2 + QR_g X + 4\pi\mu e^{\gamma X}\right) d\lambda_g \\ &+ \frac{1}{2\pi}\int_{\partial\mathbb{D}} \left(QK_g X + 2\pi\mu_\partial e^{\frac{\gamma}{2}X}\right) d\lambda_{\partial g} \end{aligned} \tag{1.45}$$

avec $d\lambda_g$ la forme volume de la métrique g sur le disque $\mathbb{D}$ et $d\lambda_{\partial g}$ l'élément de longueur de g au bord du disque $\partial\mathbb{D}$. R_g et K_g désignent respectivement la courbure de Ricci à l'intérieur de $\mathbb{D}$ et la courbure géodésique au bord $\partial\mathbb{D}$. Il y a aussi quelques constantes dans l'expression : $\mu, \mu_\partial \geq 0, \mu + \mu_\partial > 0$ sont les constantes cosmologiques à l'intérieur et au bord et $Q = 2/\gamma + \gamma/2$.
Notons qu'à la différence de l'action du champ libre Gaussien, on y rajoute notamment des termes de potentiel exponentiel du type $e^{\gamma X}$: ce terme est défini rigoureusement à l'aide de la théorie du chaos multiplicatif Gaussien.

La théorie de la gravité quantique de Liouville est une théorie conforme des champs. Ce qui nous intéresse en particulier dans une théorie conforme des champs, c'est de calculer les fonctions de corrélations. Formellement, la fonction de corrélations à $n + m$ points deux à deux distincts, avec n points à l'intérieur du disque $\mathbb{D}$ et m points au bord du disque $\partial\mathbb{D}$ est donnée par

$$\begin{aligned} &\langle \prod_{i=1,\dots,n} V_{\alpha_i}(z_i) \prod_{j=1,\dots,m} V_{\beta_j}(z'_j)\rangle_g \\ &= \int_{\Sigma_{\mathbb{D}}} \prod_{i=1,\dots,n} e^{\alpha_i X(z_i)} \prod_{j=1,\dots,m} e^{\frac{\beta_j}{2} X(z'_j)} e^{-S_{\mathbb{D}}(X,g)} D_g X \end{aligned} \tag{1.46}$$

où les α_i, β_j sont des coefficients et z_i (resp. z'_j) des points marqués l'intérieur (resp. au bord) du disque $\mathbb{D}$; $D_g X$ est la mesure uniforme formelle sur $\Sigma_{\mathbb{D}}$ muni de la métrique g.

La première chose à se demander face à cette expression est bien évidemment si la fonction de corrélation est bien définie et non-triviale. La réponse dans [HRV15] est donnée par les bornes de Seiberg [Sei90] :

Théorème 26 (Bornes de Seiberg).
Supposons que $\mu, \mu_\partial > 0$. La fonction de corrélation (1.46) *existe et est non-triviale si et seulement si les bornes suivantes, dites de Seiberg, sont vérifiées :*

$$\sum_i \alpha_i + \frac{1}{2}\sum_j \beta_j > Q, \tag{1.47}$$

$$\forall i, \quad \alpha_i < Q, \tag{1.48}$$

$$\forall j, \quad \beta_j < Q. \tag{1.49}$$

Par exemple, si $n = 0$ (aucun point z_i à l'intérieur), il faut que $m \geq 3$, i.e. au moins 3 points au bord, pour que la fonction de corrélation (1.46) existe et soit non-triviale.
Une difficulté dans la preuve est de comprendre l'interaction entre un point du bord et le volume à l'intérieur du disque $\mathbb{D}$. On peut aussi obtenir d'autres bornes de Seiberg dans le cas $\mu = 0$ et dans le cas dit "unit volume".

On s'intéresse aussi aux règles de changement de la fonction de corrélation. Par exemple, on pourrait appliquer une transformation de Möbius du disque $\mathbb{D}$ pour changer les positions des points : la fonction de corrélation se transforme selon la formule de Knizhnik-Polyakov-Zamolodchikov [KPZ88].

Théorème 27 (Relation KPZ).
Soit ϕ une transformation de Möbius $\phi : \mathbb{D} \to \mathbb{D}$. Alors on a la relation suivante, dite de KPZ :

$$\begin{aligned} &\langle \prod_{i=1,\dots,n} V_{\alpha_i}(\phi(z_i)) \prod_{j=1,\dots,m} V_{\beta_j}(\phi(z'_j))\rangle_g \\ &= \prod_{i=1,\dots,n} |\phi(z_i)|^{-2\Delta_{\alpha_i}} \prod_{j=1,\dots,m} |\phi(z'_i)|^{-\Delta_{\beta_j}} \langle \prod_{i=1,\dots,n} V_{\alpha_i}(z_i) \prod_{j=1,\dots,m} V_{\beta_j}(z'_j)\rangle_g \end{aligned} \tag{1.50}$$

où $\Delta_{\alpha_i} = \frac{\alpha_i}{2}(Q - \frac{\alpha_i}{2})$ et $\Delta_{\beta_j} = \frac{\beta_j}{2}(Q - \frac{\beta_j}{2})$.

Les Δ sont appelés "poids conformes" dans le langage de la théorie conforme des champs. Ainsi, si deux ensembles de points sont liés par une transformation conforme du disque, on peut utiliser cette relation pour relier les fonctions de corrélation associées.

On s'intéresse également aux règles de changement de la fonction de corrélation quand on modifie la métrique g par équivalence conforme. On dit que deux métriques g, g' sont conformément équivalentes quand on peut écrire $g' = e^{\varphi} g$ avec une bonne fonction φ. La règle de transformation de la fonction de corrélation dans ce cas est donnée par l'anomalie de Weyl.

Théorème 28 (Anomalie de Weyl).
Soient $g' = e^{\varphi} g$ deux métriques conformément équivalentes à la métrique Euclidienne sur le disque $\mathbb{D}$. On a la relation suivante, dite d'anomalie de Weyl :

$$\begin{aligned} &\ln \frac{\langle \prod_{i=1,\dots,n} V_{\alpha_i}(z_i) \prod_{j=1,\dots,m} V_{\beta_j}(z'_j)\rangle_{g'}}{\langle \prod_{i=1,\dots,n} V_{\alpha_i}(z_i) \prod_{j=1,\dots,m} V_{\beta_j}(z'_j)\rangle_g} \\ &= \frac{c}{96\pi} \left(\int_{\mathbb{D}} |\partial^g \varphi|^2 \, d\lambda_g + \int_{\mathbb{D}} 2R_g \varphi \, d\lambda_g + 4 \int_{\partial\mathbb{D}} K_g \varphi \, d\lambda_{\partial g} \right) \end{aligned} \tag{1.51}$$

avec $c = 1 + 6Q^2$: c'est la "charge centrale" dans le language de la théorie conforme des champs.

Pour montrer les résultats précédents, il faudrait utiliser la théorie du chaos multiplicatif Gaussien ! En effet, une régularisation est nécessaire pour donner un sens précis à la fonction de corrélation : c'est la partie technique de l'article.

Jusqu'ici, on a survolé les résultats essentiels de la théorie. Ces résultats peuvent être aussi formulés en termes de la mesure de Liouville : on se réfère à [HRV15] pour un traitement plus complet.
Nous avons défini précédemment le champ libre Gaussien avec une partie de l'action de Liouville : pour définir le champ de Liouville, il suffit de faire la même démarche avec l'expression (1.45). Considérons donc le champ de Liouville $X[g, \gamma, \mu, \mu_\partial]$ défini par

$$\mathbb{E}[F(X[g, \gamma, \mu, \mu_\partial])] = \frac{1}{Z} \int_{\Sigma_{\mathbb{D}}} F(X) e^{-S_{\mathbb{D}}(X,g)} D_g X \tag{1.52}$$

où Z est une constante de renormalisation et $D_g X$ la mesure de Lebesgue de $\Sigma_{\mathbb{D}}$ munie de la métrique g. On laissera tomber les indices $[g, \gamma, \mu, \mu_\partial]$ par abus de notation.

Il s'avère que pour que la fonction de partition soit renormalisable, il faut rajouter des "insertions" du type $e^{\alpha_i X(z_i)}$ avec $\alpha_i \in \mathbb{R}$ et $z_i \in \mathbb{D}$. Dans le langage de la théorie conforme des champs, $e^{\alpha_i X(z_i)}$ s'appelle opérateur *vertex*. On obtient alors la fonction de corrélation qu'on a vue plus haut :

$$\mathbb{E}[F(X[g, \dots])] = \frac{1}{Z} \int_{\Sigma_{\mathbb{D}}} F(X) \prod_{i=1,\dots,n} e^{\alpha_i X(z_i)} \prod_{j=1,\dots,m} e^{\frac{\beta_j}{2} X(z'_j)} e^{-S_{\mathbb{D}}(X,g)} D_g X. \tag{1.53}$$

Idem, pour simplifier on laissera tomber les indices $[g, \gamma, \mu, \mu_\partial, z_i, \alpha_i, z'_j, \beta_j]$: elles ne sont précisées que si elles changent entre les expressions.
En pratique, pour définir proprement le champ de Liouville, nous partons du terme de gradient dans l'action de Liouville : cela nous définit le champ libre Gaussien X_g de moyenne nulle dans la métrique g. Ensuite, la théorie du chaos multiplicatif Gaussien nous permet de traiter le terme du potentiel exponentiel : c'est l'exponentielle du champ Gaussien X_g corrélé logarithmiquement. Finalement, les opérateurs vertex peuvent être traités avec le théorème de Girsanov : ils introduisent un shift sur le champ X_g qui est une combinaison linéaire de singularités logarithmiques.

Un champ de Liouville X est bien défini si les insertions vérifient les bornes de Seiberg (Théorème 26). Dans ce cas, ce champ ressemble à une variante du champ libre Gaussien, et on peut définir son exponentielle par la théorie du chaos multiplicatif Gaussien : on appellera respectivement

$$\left(dM = e^{\gamma X} d\lambda_g, dM_\partial = e^{\frac{\gamma}{2} X} d\lambda_{\partial g}\right) \tag{1.54}$$

mesure de Liouville à l'intérieur du disque $\mathbb{D}$ et mesure de Liouville au bord du disque $\partial\mathbb{D}$.

On peut aussi étendre la construction de la gravité quantique de Liouville à n'importe quel domaine D conformément équivalent au disque $\mathbb{D}$ via la formule de changement conforme de domaines pour le champ de Liouville :

Lemme 29 (Changement conforme de domaines).
Soit g une métrique sur $\mathbb{D}$. Soit $\varphi : D \to \mathbb{D}$ une transformation conforme d'un domaine D simplement connexe de bord $\mathcal{C}^1$ vers le disque $\mathbb{D}$. Le champ de Liouville X_D sur D est obtenu par la relation suivante :

$$X_D[g_\varphi, z_i, z'_j] = X_{\mathbb{D}}[g, \varphi(z_i), \varphi(z'_j)] \circ \varphi + Q \ln |\varphi'| \tag{1.55}$$

où g_φ désigne la métrique $|\varphi'|^2 g(\varphi)$ sur D. Les $\lambda, \mu, \mu_\partial, \alpha_i, \beta_j$ sont identiques aux deux côtés.

Intuitivement, l'action de Liouville sur le domaine D est définie de façon à ce que la mesure de Liouville soit invariante par transformation conforme : en effet, un calcul simple permet de vérifier que cela est équivalent à la formule ci-dessus sur le champ de Liouville. La relation KPZ ci-dessus reste également valable quand on change de domaines par une transformation conforme.

Dans [HRV15], nous avons également étendu la construction de la gravité quantique de Liouville au cas critique $\gamma = 2$, en vérifiant que la fonction de corrélation peut être obtenue de façon similaire au cas $\gamma < 2$ en appliquant la renormalisation de Seneta-Heyde [DRSV14a]. Une conjecture précise sur la convergence des mesures des cartes planaires aléatoires à bord simple est aussi formulée dans [HRV15] : c'est l'analogue du théorème de Donsker pour le mouvement Brownien.

[AHS15] Sur deux constructions de la gravité quantique de Liouville

Très vaguement parlant, la gravité quantique de Liouville construit des familles de surfaces aléatoires sur une géométrie Riemannienne G (par exemple la sphère de Riemann $\mathbb{S}$) qui satisfont certaines règles de transformation "naturelles" : comment la loi d'une telle famille évolue quand on applique une transformation conforme sur G, quand on fait varier la métrique Riemannienne sur G, etc.
Les lois sur ces familles de surfaces sont données par l'action de Liouville [Pol81, Nak04] : sur la sphère $\mathbb{S}$ munie d'une métrique Riemannienne g, l'action s'écrit (avec $\gamma \in [0, 2[$)

$$S(X, g) = \frac{1}{4\pi} \int_{\mathbb{S}} \left(|\partial^g X|^2 + Q R_g X + 4\pi\mu e^{\gamma X} \right) d\lambda_g. \tag{1.56}$$

On remarque la ressemblance avec le champ libre Gaussien : on a rajouté un terme de courbure R et un terme de potentiel $\int e^{\gamma X}$ avec $\mu > 0$ une constante dite cosmologique. Enfin, $d\lambda_g$ est la forme volume de la métrique g et $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$.

Sont apparues presque en même temps deux approches différentes d'une théorie de Liouville sur la sphère [DKRV16, DMS14] :
– D'un côté, [DKRV16] part de l'action de Liouville qui définit les lois de ces familles sur chaque domaine avec au moins 3 points marqués muni d'une métrique Riemannienne : on y démontre des formules qui permettent de changer le domaine de définition de manière conforme (i.e. covariance conforme), de changer la métrique g par une équivalence conforme (i.e. anomalie de Weyl), etc. Ainsi, on démontre rigoureusement les formules fondamentales de la théorie de la gravité de Liouville ;
– De l'autre côté, [DMS14] part directement du champ libre Gaussien et utilise la décomposition orthogonale sur une géométrie fixée avec 2 points marqués (e.g. la sphère de Riemann avec les deux pôles) pour construire les surfaces aléatoires dans cette géométrie, la loi étant codée par des processus de Bessel. Ensuite, les formules telles que changement conforme de domaines (1.55) sont postulées comme définitions pour passer entre deux géométries conformes entre elles. Par exemple : les surfaces construites sur le cylindre infini avec points marqués en plus et moins infini peuvent être transportées sur la sphère de Riemann avec points marqués aux pôles par une transformation conforme.

Dans la théorie de la gravité de Liouville, il y a une façon naturelle de marquer certains points sur la surface à l'aide des opérateurs *vertex*. On se réfère à [DKRV16] pour une définition exacte, mais donnons une petite idée.
Souvenons-nous du Théorème 14 où l'on a trouvé un mouvement Brownien particulier dans le champ libre, en considérant les circle-averages autour de l'origine. On a supposé que le processus était centré, mais que se passe-t-il si l'on enlève cette hypothèse, et on restreint la propriété d'invariance conforme aux transformations conformes qui laissent invariante l'origine ? Le processus des circles-averages autour de l'origine est alors un mouvement Brownien avec drift α, et on obtient un champ X qui s'écrit au voisinage de 0 :

$$X(z) = \mathrm{GFF}_{\mathbb{D}}(z) - \alpha \log(|z|). \tag{1.57}$$

On dit que l'on a rajouté une singularité logarithmique au point 0 de poids α.
Dans le cas général, les opérateurs vertex rajoutent, via le théorème de Girsanov, des singularités de type logarithmique avec un poids quelconque autour des points prescrits. Pour que la fonction de partition de Liouville soit bien définie, il faut que les poids de ces singularités satisfassent les bornes dites de Seiberg [Sei90, DKRV16] : si des singularités sont de poids respectivement α_i aux points $z_i \in \mathbb{S}$, les bornes de Seiberg s'écrivent

$$\begin{aligned} &\sum \alpha_i > 2Q, \\ \forall i, \quad &\alpha_i < Q. \end{aligned}$$

En particulier, les bornes de Seiberg nous disent qu'au moins 3 singularités logarithmiques sont nécessaires pour donner un sens à la fonction de partition de Liouville.

Dans [DMS14], pour définir la "quantum sphère", on regarde le modèle $(\mathcal{Q}, -\infty, +\infty)$, i.e. le cylindre infini avec seulement 2 singularités logarithmiques de poids γ situées en $-\infty$ et $+\infty$. La façon la plus simple de comprendre cela est de voir ce modèle comme une classe d'équivalence de surfaces aléatoires à 3 points marqués. En effet, quelque soit la position du 3ème point $u \in \mathcal{Q}$, il existe une unique transformation conforme du cylindre infini qui envoie le triplet $(-\infty, +\infty, u)$ sur $(-\infty, +\infty, 0)$, à savoir l'application $\varphi : z \mapsto z - u$. En appliquant la règle de transformation (1.55), le champ Φ_0 défini avec $(-\infty, +\infty, 0)$ et le champ Φ_u défini avec $(-\infty, +\infty, u)$ sont reliés par

$$\Phi_0 \circ \varphi = \Phi_u. \tag{1.58}$$

La mesure de Liouville sur le cylindre infini avec 2 points marqués $(-\infty, +\infty)$ est alors la classe d'équivalence des mesures à 3 points marqués $(-\infty, +\infty, u)$ par la relation ci-dessus.

Cependant, il n'est pas agréable de travailler directement dans la classe d'équivalence des surfaces à 2 points marqués pour écrire une preuve. Notre stratégie de rédaction est alors de faire l'opération inverse : au lieu de partir de la configuration à 3 points puis passer à la classe d'équivalence, nous partons de la configuration à 2 points, puis on choisit un troisième point selon la loi à 2 points : ceci est une procédure stochastique pour marquer un point de plus u à un endroit aléatoire. Ensuite, nous appliquons la transformation $z \mapsto z - u$ pour ramener ce point en 0, puis nous montrons que c'est la même surface aléatoire définie dans [DKRV16] avec 3 points marqués $(-\infty, +\infty, 0)$, chacun ayant un poids γ.

Voici donc le théorème principal de [AHS15] :

Théorème 30 ([AHS15] : théorème principal).
Soit $\gamma \in]0, 2[$. Soit $\mu^{\gamma,\gamma,\gamma}_{DKRV}(z_1, z_2, z_3)$ la mesure de Liouville dans [DKRV16] avec trois insertions en z_1, z_2, z_3 de poids γ sur la sphère de Riemann $\mathbb{S}$.
Soit $\mu^3_{DMS}(z_1, z_2, z_3)$ la mesure construite dans [DMS14] avec trois points marqués en z_1, z_2, z_3 sur la sphère de Riemann $\mathbb{S}$: concrètement, on choisit un troisième point u selon la mesure aléatoire à deux points $(\mathcal{Q}, -\infty, +\infty)$ construite dans [DMS14], et on envoie la surface aléatoire $(\mathcal{Q}, -\infty, +\infty, u)$ sur $(\mathbb{S}, z_1, z_2, z_3)$ par l'unique application conforme entre ces deux modèles, en appliquant la règle de transformation conforme pour les familles de surfaces aléatoires.
Alors $\mu^{\gamma,\gamma,\gamma}_{DKRV}(z_1, z_2, z_3)$ et $\mu^3_{DMS}(z_1, z_2, z_3)$ définissent la même mesure aléatoire sur la sphère de Riemann $\mathbb{S}$.

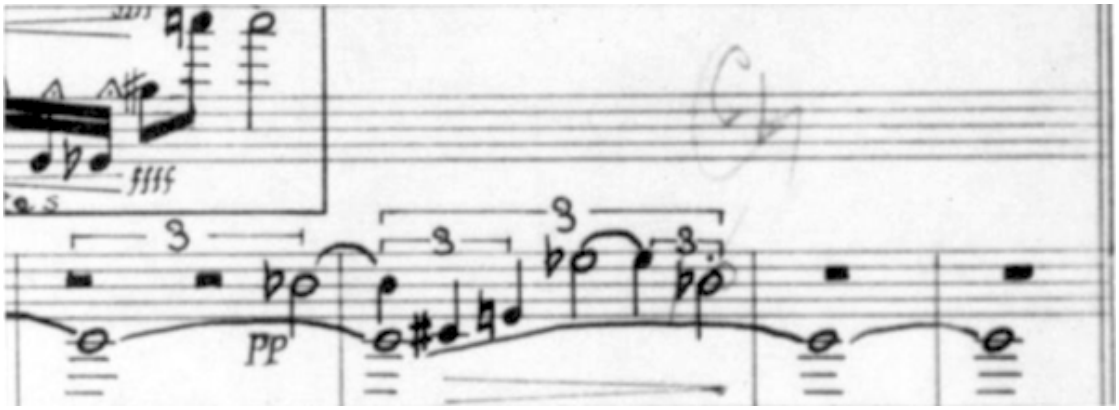

Figure 1.2 – Manuscrit original de Charles Ives, «The Unanswered Question». Ives y modifie la dernière note du thème. © Charles Ives.

Chapter 2

Liouville Quantum Gravity on the unit disk

Authors

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Abstract

Our purpose is to pursue the rigorous construction of Liouville Quantum Field Theory on Riemann surfaces initiated by F. David, A. Kupiainen and the last two authors in the context of the Riemann sphere and inspired by the 1981 seminal work by Polyakov. In this paper, we investigate the case of simply connected domains with boundary. We also make precise conjectures about the relationship of this theory to scaling limits of random planar maps with boundary conformally embedded onto the disk.

Introduction

Let us begin this introduction with a soft attempt of explanation for mathematicians of what is Liouville Quantum Field Theory (LQFT). This theory may be better understood if we first briefly recall the Feynman path integral representation of Brownian motion on $\mathbb{R}^d$. Denoting by Σ the space of paths $\sigma : [0, T] \to \mathbb{R}^d$ starting from $\sigma(0) = 0$, we define the action functional on Σ by

$$\forall \sigma \in \Sigma, \quad S_{\mathrm{BM}}(\sigma) = \frac{1}{2} \int_0^T |\dot{\sigma}(r)|^2 \, dr. \tag{2.1}$$

It is nowadays rather well understood that Brownian motion, call it B, can be understood in terms of Feynman path integrals via the relation

$$\mathbb{E}[F((B_s)_{s \leq T})] = \frac{1}{Z} \int_\Sigma F(\sigma) e^{-S_{BM}(\sigma)} D\sigma \tag{2.2}$$

where $D\sigma$ stands for a formal uniform measure on Σ and Z a renormalization constant. Brownian motion is also often said to be the canonical uniform random path in $\mathbb{R}^d$: this terminology is due to the fact the Brownian motion is the scaling limit of the simple random walk.

The reader may try to guess what could be the above picture if, instead of "canonical random path", we ask for a "canonical random Riemann surface". The answer is LQFT [1]. As in the case of Brownian motion, there are two ways to give sense to this theory: directly in the continuum in terms of Feynmann surface integrals or as scaling limit of suitable discrete models called Random Planar Maps (RPM). This picture is nowadays well understood in the physics literature since the pioneering work by Polyakov [Pol81]. The reader is referred to [HMW11, Nak04] for physics reviews, to [Pol81, Dav88, DK89, KPZ88] for founding papers in physics and to [DKRV16] for a brief introduction for mathematicians and a rigorous construction on the Riemann sphere.

In this paper, we will construct LQFT on Riemann surfaces with boundary directly in the continuum in the spirit of Feynman surface integrals. More precisely, we consider a (strict) simply connected domain D of $\mathbb{R}^2$ with a simple boundary equipped with a Riemannian metric g. Similar to the action (2.1) for Brownian motion, we must consider the

1. Liouville Quantum Field Theory (LQFT) and Liouville Quantum Gravity (LQG) are similar for the unit disk but they differ on higher genus surfaces, see [GRV16] for references and discussions.

Liouville action functional on such a Riemannian manifold. It is defined for each function $X : \overline{D} \to \mathbb{R}$ by

$$\begin{aligned} S(X,g) :=& \frac{1}{4\pi}\int_D \left(|\partial^g X|^2 + QR_g X + 4\pi\mu e^{\gamma X}\right)\lambda_g \\ &+ \frac{1}{2\pi}\int_{\partial D}\left(QK_g X + 2\pi\mu_\partial e^{\frac{\gamma}{2}X}\right)\lambda_{\partial g} \end{aligned} \tag{2.3}$$

where ∂^g, R_g, K_g, λ_g and $\lambda_{\partial g}$ respectively stand for the gradient, Ricci scalar curvature, geodesic curvature (along the boundary), volume form and line element along ∂D in the metric g: see section 2 for the definitions. The parameters $\mu, \mu_\partial \geq 0$ (with $\mu + \mu_\partial > 0$) are respectively the bulk and boundary cosmological constants and Q, γ are real parameters.

Before going into further details of the quantum field theory, let us first make a detour in Riemannian geometry to explain why the roots of LQFT are deeply connected to the theory of uniformization of Riemann surfaces. Indeed, a fundamental problem in geometry is to uniformize the surface $(\overline{D}, g)$: this means that we look for a metric g' on $\overline{D}$ conformally equivalent to g, i.e. $g' = e^u g$ for some smooth function u on $\overline{D}$, with constant Ricci scalar curvature in D and constant geodesic curvature on ∂D. Under appropriate assumptions, the unknown function u is a minimizer of the Liouville action functional (2.3). Indeed, for the particular value

$$Q = \frac{2}{\gamma}, \tag{2.4}$$

the saddle points X of this functional with Neumann boundary condition

$$\partial_{n_g}(\frac{\gamma}{2}X) + K_g = -\frac{\pi\mu_\partial\gamma^2}{2}e^{\frac{\gamma}{2}X}, \tag{2.5}$$

where ∂_{n_g} stands for the Neumann operator along ∂D, solve (if exists) the celebrated *Liouville equation*

$$\begin{aligned} -\triangle_g(\gamma X) + R_g &= -2\pi\mu\gamma^2 e^{\gamma X} \quad \text{on } D, \\ \partial_{n_g}(\frac{\gamma}{2}X) + K_g &= -\frac{\pi\mu_\partial\gamma^2}{2}e^{\frac{\gamma}{2}X} \quad \text{on } \partial D. \end{aligned} \tag{2.6}$$

Setting $u = \gamma X$ and defining a new metric $g' = e^u g$, the metric g' satisfies the relations

$$R_{g'} = -2\pi\mu\gamma^2 \quad \text{and} \quad K_{g'} = -\frac{\pi\mu_\partial\gamma^2}{2},$$

hence providing a solution to the uniformization problem of the Riemann surface $(\overline{D}, g)$. Let us further mention that, for the value of Q given by (2.4), this theory is *conformally invariant*: this means that if we choose a conformal map $\psi : \tilde{D} \mapsto D$ then the couple

(X, g) solves (2.6) on D if and only if $(X \circ \psi + Q \ln |\psi'|, g \circ \psi)$ solves (2.6) on $\tilde{D}$ [2]. These are the foundations of the theory of uniformization of surfaces with boundary in $2d$, also called *Classical Liouville field theory.*

In quantum (or probabilistic) Liouville field theory, one looks for the construction of a random field X with law given heuristically in terms of a functional integral

$$\mathbb{E}[F(X)] = Z^{-1} \int F(X) e^{-S(X,g)} DX \tag{2.7}$$

where Z is a normalization constant and DX stands for a formal uniform measure on some space of maps $X : \overline{D} \to \mathbb{R}$. This expression is in the same spirit as for the Brownian motion (2.2). This formalism describes the law of the log-conformal factor X of a formal random metric of the form $e^{\gamma X} g$ on D. Of course, this description is purely formal and giving a mathematical description of this picture is a longstanding problem since the work of Polyakov [Pol81]. It turns out that for the particular values

$$\gamma \in]0, 2], \quad Q = \frac{2}{\gamma} + \frac{\gamma}{2},$$

this field theory is expected to become a *Conformal Field Theory* (see [Gaw97] for a background on this topic). The aim of this paper is to make rigorous sense of the above heuristic picture and thereby to define a canonical random field X inspired by Feynman surface integrals. A noticeable difference with the example of Brownian motion where there is only one canonical random path (up to reparametrization) is that there is a whole family of canonical random Riemann surfaces indexed by a single parameter $\gamma \in]0, 2]$. Conformal Field Theories are characterized by their central charge $c \in \mathbb{R}$ that reflects the way the theory reacts to conformal changes of the background metric g defined on D (see Section 2). For the LQFT, we will establish that the central charge is $c = 1 + 6Q^2$: thus it can range continuously in the interval $[25, +\infty[$ and this is one of the interesting features of this theory. We will also study the conformal covariance (KPZ formula) and μ, μ_∂-dependence of this theory. Once constructed, the Liouville (random) field X allows us to define the Liouville measure, which can be thought of as the volume form associated to the random metric tensor $e^{\gamma X} g$. We will state a precise mathematical conjecture on the relationship between the Liouville measure and the scaling limit of random planar maps with a simple boundary conformally embedded onto the unit disk.

2. Let us prove this for the Neumann boundary condition; the other equation can be dealt with similarly. Since ψ is an isometry from $(\tilde{D}, g \circ \psi |\psi'|^2)$ to (D, g), we have $K_{g \circ \psi |\psi'|^2} = K_g \circ \psi$. Now applying formula (2.19) which is valid in great generality, we get that $K_{g \circ \psi} = |\psi'|(K_g \circ \psi - \frac{1}{(g \circ \psi)^{1/2} |\psi'|} \ln |\psi'|)$. Hence

$$\partial_{n_{g \circ \psi}} (\frac{\gamma}{2}(X \circ \psi + Q \ln |\psi'|)) + K_{g \circ \psi} = |\psi'|(\frac{1}{g \circ \psi} \frac{\partial(\frac{\gamma}{2} X)}{\partial n} \circ \psi + K_g \circ \psi) = -\frac{\pi \mu_\partial \gamma^2}{2} e^{\frac{\gamma}{2}(X \circ \psi + Q \ln |\psi'|)}.$$

To conclude, let us stress that the thread of the paper is inspired by [DKRV16] where the authors developped LQFT on the Riemann sphere. The main input is here to understand the phenomena related to the presence of a boundary; in particular, part of the construction relies on the theory of Gaussian multiplicative chaos (GMC) and the presence of the boundary requires to integrate against GMC measures functions that are not integrable with respect to Lebesgue measure when approaching the boundary (these technical difficulties do not appear in the case of the sphere [DKRV16] where there is no boundary): see Proposition 33 for instance.

On the difference between the David-Kupiainen-Rhodes-Vargas approach and the Duplantier-Miller-Sheffield approach

There is a conceptual difference between the approach of Duplantier-Miller-Sheffield developped in [DMS14] and the approach developed independently by David-Kupiainen and the last two authors of this paper in the work [DKRV16] where was developped LQFT on the Riemann sphere. The point of view and the objects defined in both approaches are different though one can relate both approaches in a specific case as we now describe in the case of the sphere.

The case of the Riemann sphere

In the work of David-Kupiainen-Rhodes-Vargas [DKRV16], the authors construct the correlation functions of LQFT on the Riemann sphere $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ and show that these correlation functions satisfy the axioms of a Conformal Field theory (CFT): conformal covariance (KPZ formula), Weyl anomaly, etc... The theory is indexed just like the disk by a parameter $\gamma \in]0,2]$ (with $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$) and a positive cosmological constant $\mu > 0$ (recall that in the case of the disk, the theory requires two cosmological constants: one for the bulk and one for the boundary). The theory is now based on the following action

$$S_{\mathbb{S}^2}(X,g) := \frac{1}{4\pi} \int_{\mathbb{S}^2} \left(|\partial^g X|^2 + QR_g X + 4\pi\mu e^{\gamma X} \right) \lambda_g \tag{2.8}$$

where this time ∂^g, R_g, λ_g respectively stand for the gradient, Ricci scalar curvature, volume form in the metric g. The symmetry of the theory is completely determined by γ, however the constant $\mu > 0$ is essential for the existence of the theory and in particular the correlation functions. As an output of the construction, one can define (see Les Houches lecture notes [RV16] for a simple introduction to the theory):

— Correlation functions at points $(z_i)_{1\leq i\leq n}$ and with weights $(\alpha_i)_{1\leq i\leq n}$ of "variables" $e^{\alpha_i X(z_i)}$ (called vertex operators in the physics literature) which correspond to a

rigorous definition of the following heuristic path integral formulation

$$\langle \prod_{1\leq i\leq n} e^{\alpha_i X(z_i)} \rangle := \int \mathbb{E}\left[\prod_{1\leq i\leq n} e^{\alpha_i X(z_i)} e^{-S_{\mathbb{S}^2}(X,g)}\right] DX. \tag{2.9}$$

Of course, to make sense of (2.9), one needs to regularize X and take the limit as the regularization step goes to 0. These correlation functions exist if the α_i satisfy the so-called Seiberg bounds:

$$\sum_{i=1}^{n} \alpha_i > 2Q, \quad \text{and} \quad \alpha_i < Q, \ \forall i. \tag{2.10}$$

— Random measures we will denote $Z_{(\alpha_i,z_i)_{1\leq i\leq n}}$ (when the α_i satisfy the bounds (2.10)) and unit volume random measures we will denote $Z^1_{(\alpha_i,z_i)_{1\leq i\leq n}}$ (the unit volume measures can be defined under less restrictive conditions than (2.10): see [RV16]). These are the so-called Liouville measures of the theory. The random measure $Z_{(\alpha_i,z_i)_{1\leq i\leq n}}$ is a rigorous construction of $e^{\gamma X}\lambda_g$ under the probability measure

$$F \mapsto \frac{\langle F \prod_{1\leq i\leq n} e^{\alpha_i X(z_i)}\rangle}{\langle \prod_{1\leq i\leq n} e^{\alpha_i X(z_i)}\rangle}.$$

Most of these measures can be related (conjecturally) to planar maps; however, we will only discuss the simplest measure among these measures, namely $Z^1_{(\gamma,0),(\gamma,1),(\gamma,\infty)}$, which is related to the random measure constructed in the work of Duplantier-Miller-Sheffield [DMS14].

One important aspect of the construction is that it provides very explicit expressions of the correlations and the distributions for the measures in terms of products of fractional moments of appropriate GMC measures: see Les Houches lecture notes [RV16]. Also, the moments of the measures can be expressed in terms of the correlation functions. More precisely, one has the following formula for a measurable set $B \subset \mathbb{S}^2$ and any integer $p \geq 1$:

$$\begin{aligned} &\mathbb{E}\left[\left(Z_{(\alpha_i,z_i)_{1\leq i\leq n}}(B)\right)^p\right] \\ =&\frac{\int_B \cdots \int_B \langle \prod_{1\leq j\leq p} e^{\gamma X(x_j)} \prod_{1\leq i\leq n} e^{\alpha_i X(z_i)}\rangle \lambda_g(dx_1)\cdots\lambda_g(dx_p)}{\langle \prod_{1\leq i\leq n} e^{\gamma X(z_i)}\rangle}. \end{aligned} \tag{2.11}$$

Hence, it is an essential program to compute these correlation functions since they determine for instance the moments of the measures $Z_{(\alpha_i,z_i)_{1\leq i\leq n}}$.

In the case of the sphere, the work of Duplantier-Miller-Sheffield [DMS14] constructed the so-called unit area quantum sphere we will denote $\mu_{DMS,\gamma}$ and which depends on

$\gamma \in (0,2)$. In this setting, there is no cosmological constant and therefore no correlation functions strictly speaking. The unit area quantum sphere is in fact an equivalence class of random distributions with two marked points 0 and ∞ (to be precise, one can also construct an equivalence class with one marked point but we will not discuss this case). More specifically, the authors first define an equivalence class of random surfaces with the following definition: a random distribution h_1 is equivalent to h_2 if and only if there is a (possibly random) Möbius transform ϕ that fixes 0 and ∞ such that the following holds in distribution [3]

$$h_2 = h_1 \circ \phi + Q \ln |\phi'|. \tag{2.12}$$

The unit area quantum sphere $\mu_{DMS,\gamma}$ is then defined by the equivalence class of a special distribution $h^\star$ where the radial part is sampled according to a special Bessel bridge and the non radial part is sampled according to the non radial part of a standard full plane GFF (to be precise one works in the cylinder $\mathbb{R} \times [0, 2\pi]$ which is conformally equivalent to $\mathbb{S}$): see [DMS14] for the exact definition of $h^\star$. Let us mention that $h^\star$ is such that the radial part of $h^\star$ attains its maximum on the circle of radius 1 (in the cylinder coordinates). In this setting, all the distributions h in $\mu_{DMS,\gamma}$ define random measures by the relation

$$\mu_h = \lim_{\epsilon \to 0} \epsilon^{\gamma^2/2} e^{\gamma h_\epsilon(z)} dz \tag{2.13}$$

where $h_\epsilon(z)$ is a circle average of h with center z and radius ϵ. The measures are related by

$$\mu_{h_2} = \mu_{h_1} \circ \phi \tag{2.14}$$

if h_1 and h_2 are related by (2.12). In what follows, we will identify the unit area quantum sphere with the equivalence class of the measure $\mu_{h^\star}$ with respect to relation (2.14) (rather than the equivalence class of the distribution $h^\star$ with respect to (2.12)): hence, we will say that two random measures represent the same quantum surface if they are of the form μ_{h_1} and μ_{h_2} with h_1 and h_2 related by (2.12). Therefore, in conclusion, with this slightly different definition, the unit area quantum sphere $\mu_{DMS,\gamma}$ is an equivalence class of measures such that $\mu_{h^\star} \in \mu_{DMS,\gamma}$.

The work [DMS14] is interesting because it couples measures in $\mu_{DMS,\gamma}$ to space filling variants of SLE curves: this provides an interesting framework to relate the measures in $\mu_{DMS,\gamma}$ to decorated planar maps. We now describe the relation between the two approaches in the next subsection.

Historics on the conjectured scaling limit of finite volume planar maps

In a previous paper [She16], Sheffield constructed a candidate for the scaling limit of the volume form of infinite volume planar maps (i.e. the non compact case). However,

3. The definition of the unit area quantum sphere is in fact a bit more general as one can consider other marked points than 0 and ∞ and hence more general conformal maps (which do not necessarily fix the points 0 and ∞).

he left open the construction of a candidate for the scaling limit of the volume form of finite volume planar maps (i.e. the compact case). In particular, in the case of the sphere, following the work [She16], it was clearly not expected among probabilists that there could be a rather explicit candidate to the following question:

Question: if you fix three points z_1, z_2, z_3 in the sphere $\mathbb{S}^2$, what is the scaling limit of the volume form of large finite planar maps (equipped with a natural conformal structure) embedded in the sphere where you send three points chosen at random on the map to the three fixed points z_1, z_2, z_3? [4]

Such an explicit candidate was in fact constructed in [DKRV16]: it is the measure we denote $Z^1_{(\gamma,0),(\gamma,1),(\gamma,\infty)}$ (in the case $z_1 = 0, z_2 = 1, z_3 = \infty$). More precisely, after the work [She16], the independent works [DKRV16] and [DMS14] appeared simultaneously: both works provide a description of the conjectured scaling limit of large planar maps embedded in the sphere (in [DMS14], the authors also consider the situation of the disk and the relation of these disk measures to the ones considered in this paper is expected to be similar to the relation between the measures on the sphere considered in [DKRV16] and [DMS14]), namely the measure $Z^1_{(\gamma,0),(\gamma,1),(\gamma,\infty)}$ in [DKRV16] and the equivalence class of the measure $\mu_{h^\star}$ in [DMS14], i.e. the unit area quantum sphere $\mu_{DMS,\gamma}$.

The main result of the recent work by Aru-Huang-Sun [AHS15] is to link both approaches: more precisely, they show that $\mu_{h^\star}$ and $Z_{(\gamma,0),(\gamma,1),(\gamma,\infty)}$ are equal seen as quantum surfaces with two marked points 0 and ∞. Since the two measures define the same quantum surface, one can relate both measures by a relation of the form (2.14).

On the one hand, one can perform the following procedure: choose a point z at random according to the measure $\mu_{h^\star}$ and consider the image of this measure by the conformal map ϕ_z which sends z to 1 and fixes 0 and ∞. Of course, in this setting, we have $\phi_z(x) = \frac{x}{z}$; this procedure defines a random measure we will denote $\mu^3_{DMS,\gamma}$. In more mathematical terms, the construction of $\mu^3_{DMS,\gamma}$ is:

$$\mu^3_{DMS,\gamma} = \mu_{h^\star} \circ \phi_z^{-1}, \quad z \sim \mu_{h^\star} \tag{2.15}$$

where $z \sim \mu_{h^\star}$ means you sample z along $\mu_{h^\star}$. The work [AHS15] establishes that $\mu^3_{DMS,\gamma} = Z^1_{(\gamma,0),(\gamma,1),(\gamma,\infty)}$ in distribution. The equality in distribution between these two measures is in fact non trivial to prove because the work [DKRV16] provides an explicit and tractable formula for the distribution $Z^1_{(\gamma,0),(\gamma,1),(\gamma,\infty)}$ (see expression (3.15) and (3.16) in Les Houches notes [RV16]) whereas $\mu^3_{DMS,\gamma}$ is defined by a non explicit procedure (this is because in definition (2.15) the point z is a random variable correlated to $\mu_{h^\star}$).

4. As stated, this question is not quite precise because one has to give a definition of what we mean by "natural conformal structure": we refer to the Les Houches notes [RV16] for a complete and precise exposition of the above question.

On the other hand, one can show that $Z^1_{(\gamma,0),(\gamma,1),(\gamma,\infty)}$ can be defined as a limit of the form $\lim_{\epsilon\to 0}\epsilon^{\gamma^2/2}e^{\gamma\tilde{h}_\epsilon(z)}dz$ where $\tilde{h}$ is a random distribution. Then one can consider the (random) Möbius transform ϕ which fixes 0 and ∞ such that the radial part of $\tilde{h}\circ\phi + Q\ln|\phi'|$ (mapped to the cylinder $\mathbb{R}\times[0,2\pi]$) has its maximum on the circle of radius 1. We then have in distribution

$$\mu_{h^\star} = \mu_{\tilde{h}}\circ\phi. \tag{2.16}$$

Other topologies

Finally, let us mention that both approaches can be extended to other topologies than the sphere and the disk. As a matter of fact, the approach [DMS14] in the case of the disk and sphere is an extension to the compact case of the initial approach in [She16] where was developped the theory for the half plane and the full plane. However, the approach [DMS14] has not been extended to the case of higher genus surfaces. The approach of [DKRV16] can be extended to compact Riemann surfaces of genus $\mathbf{g}\geq 1$: see [DRV16] and [GRV16]. However, since the approach of [DKRV16] is based on first defining correlation functions it seems unadapted to the non compact setting where such correlation functions do not necessarily exist.

Acknowledgements

The authors would like to thank F. David and A. Kupiainen for interesting discussions on this topic. The authors would also like to thank the anonymous referees for making comments which hopefully enabled to improve the readability of the present paper. In particular, the question raised by one of the referees motivated our writing of Section 2 which establishes the unit volume Seiberg bound.

Background and preliminary results

In order to facilitate the reading of the manuscript, we gather in this section the basics in Riemannian geometry and probability theory that we will use throughout the paper.

Metrics on the unit disk

Let us denote by $\mathbb{D}$ the unit disk in the complex plane $\mathbb{C}$ and $\partial\mathbb{D}$ its boundary. We consider the standard Laplace-Beltrami operator $\triangle$, the standard gradient ∂ and Lebesgue measure $d\lambda$ on $\mathbb{D}$, as well as the standard Neumann operator ∂_n and Lebesgue measure $d\lambda_\partial$ on $\partial\mathbb{D}$, the operators being defined with respect to the Euclidean metric if no index is given. More generally, we say that a metric $g = g(x)dx^2$ on the unit disk is conformally

equivalent to the Euclidean metric if $g(x) = e^{u(x)}$ for some function $u : \bar{\mathbb{D}} :\to \mathbb{R}$ of class $C^1(\mathbb{D}) \cap C^0(\bar{\mathbb{D}})$ such that

$$\int_{\mathbb{D}} |\partial u|^2 \, d\lambda < +\infty. \tag{2.17}$$

Notice that we use the same notation g for the metric tensor and the function which defines it but this should not lead to confusions. In that case, the Laplace-Beltrami operator $\triangle_g$ and Neumann operator ∂_{n_g} in the metric g are given by

$$\triangle_g = g^{-1}\triangle, \quad \text{and} \quad \partial_{n_g} = g^{-1/2}\partial_n.$$

We denote respectively by R_g and K_g the Ricci scalar curvature and geodesic curvature K_g in the metric g. If $g' = e^{\varphi} g$ is another metric on the unit disk conformally equivalent to the flat metric, we get the following rules for the changes of (geodesic) curvature under such a conformal change of metrics

$$R_{g'} = e^{-\varphi}(R_g - \triangle_g \varphi) \qquad \text{on } \mathbb{D}, \tag{2.18}$$

$$K_{g'} = e^{-\varphi/2}(K_g + \partial_{n_g}\varphi/2) \qquad \text{on } \partial\mathbb{D}. \tag{2.19}$$

For instance, when equipped with the Euclidean metric, the unit disk has Ricci scalar curvature 0 and geodesic curvature 1 along its boundary. Combining these data with the rules (2.18)+(2.19), one can recover the explicit expressions of R_g and K_g for any metric g conformally equivalent to the Euclidean metric. We will also consider the volume form λ_g on $\mathbb{D}$, the line element $\lambda_{\partial g}$ on $\partial\mathbb{D}$, and the gradient ∂^g associated to the metric g.

Let us further recall the Gauss-Bonnet theorem

$$\int_{\mathbb{D}} R_g \, d\lambda_g + 2\int_{\partial\mathbb{D}} K_g \, d\lambda_{\partial g} = 4\pi\chi(\mathbb{D}), \tag{2.20}$$

where $\chi(\mathbb{D})$ is the Euler characteristics of the disk (that is $\chi(\mathbb{D}) = 1$), and the Green-Riemann formula

$$\int_{\mathbb{D}} \psi \triangle_g \varphi \, d\lambda_g + \int_{\mathbb{D}} \partial^g \varphi \cdot \partial^g \psi \, d\lambda_g = \int_{\partial\mathbb{D}} \partial_{n_g} \varphi\psi \, d\lambda_{\partial g}. \tag{2.21}$$

We will denote by $m_\nu(f)$ and $m_{\partial\nu}(f)$ the mean value of f respectively in the disk $\mathbb{D}$ or the boundary $\partial\mathbb{D}$ with respect to a measure ν on $\mathbb{D}$ or $\partial\mathbb{D}$, that is

$$m_\nu(f) = \frac{1}{\nu(\mathbb{D})}\int_{\mathbb{D}} f \, d\nu, \quad m_{\partial\nu}(f) = \frac{1}{\nu(\partial\mathbb{D})}\int_{\partial\mathbb{D}} f \, d\nu.$$

If the measure ν is the volume form (or the line element on $\partial\mathbb{D}$) of some metric g, we will use the notation $m_g(f)$ (or $m_{\partial g}(f)$). When no reference to the metric g is given ($m(f)$ or $m_\partial(f)$) this means that we work with the Euclidean metric.

The Sobolev space $H^1(\mathbb{D})$ is defined as the closure of the space of smooth functions on $\bar{\mathbb{D}}$ with respect to the inner product

$$\int_{\mathbb{D}} (fh + \partial f \cdot \partial h)\, d\lambda.$$

We denote by $H^{-1}(\mathbb{D})$ its dual.

Finally, we introduce the Green function G of the Neumann problem on $\mathbb{D}$

$$G(x,y) = \ln \frac{1}{|x-y||1-x\bar{y}|}. \tag{2.22}$$

It is the unique function satisfying

1. $x \mapsto G(x,y)$ is harmonic on $\mathbb{D} \setminus \{y\}$,
2. $x \mapsto G(x,y) + \ln|y-x|$ is harmonic on $\mathbb{D}$ for all $y \in \bar{\mathbb{D}}$,
3. $\partial_n G(x,y) = -1$ for $x \in \partial\mathbb{D}, y \in \mathbb{D}$,
4. $G(x,y) = G(y,x)$ for $x, y \in \mathbb{D}$ and $x \neq y$,
5. $m_\partial G(x,\cdot) = 0$ for all $x \in \mathbb{D}$.

Recall that (2.21) combined with the properties of G implies that for all $f \in C^2(\mathbb{D}) \cap C^1(\bar{\mathbb{D}})$

$$-2\pi(f(x) - m_{\partial\mathbb{D}}(f)) = \int_{\mathbb{D}} G(x,y)\triangle f(y)\,\lambda(dy) - \int_{\partial\mathbb{D}} G(x,y)\partial_n f(y)\,\lambda_\partial(dy). \tag{2.23}$$

It is quite important to observe here that G is positive definite on $\mathbb{D}$.

Möbius transforms of the unit disk

The Möbius transforms of the unit disk are given by $\psi(x) = e^{i\alpha}\frac{x-a}{1-\bar{a}x}$ with $|a| < 1$. Recall that

$$\psi'(x) = e^{i\alpha}\frac{1-|a|^2}{(1-\bar{a}x)^2}$$

from which one gets

$$\begin{aligned} \psi(y) - \psi(x) &= (\psi'(y))^{1/2}(\psi'(x))^{1/2}(y-x), \\ 1 - \psi(x)\overline{\psi(y)} &= (\psi'(x))^{1/2}(\psi'(y))^{1/2}(1-x\bar{y}). \end{aligned} \tag{2.24}$$

The Green function for the Neumann problem defined above thus verifies

$$G(\psi(x),\psi(y)) = G(x,y) - \ln|\psi'(x)| - \ln|\psi'(y)|. \tag{2.25}$$

Gaussian Free Field with Neumann boundary conditions

We consider on $\mathbb{D}$ a Gaussian Free Field (GFF) $X_{\partial\mathbb{D}}$ with Neumann boundary conditions and vanishing mean along the boundary, namely $m_\partial(X_{\partial\mathbb{D}}) = 0$ (see [Dub09, She07] for more details about GFF). This field is a Gaussian centered distribution (in the sense of Schwartz) with covariance kernel given by the Green function of the Neumann problem with vanishing mean along the boundary

$$\mathbb{E}[X_{\partial\mathbb{D}}(x)X_{\partial\mathbb{D}}(y)] = G(x,y). \tag{2.26}$$

It can be shown that this Gaussian random distribution (in the sense of Schwartz) lives almost surely in $H^{-1}(\mathbb{D})$ (same argument as in [Dub09]).

As a distribution, the field $X_{\partial\mathbb{D}}$ cannot be understood as a fairly defined function. To remedy this problem, we will need to consider some regularizations of this field in order to deal with nice (random) functions. Thus, we introduce the regularized field $X_{\partial\mathbb{D},\epsilon}$ as follows. For $\epsilon > 0$, we let $l_\epsilon(x)$ be the length of the arc $A_\epsilon(x) = \{z \in \mathbb{D}; |z - x| = \epsilon\}$ (computed with the Euclidean line element ds on the boundary of the disk centered at x and radius ϵ). Then we set

$$X_{\partial\mathbb{D},\epsilon}(x) = \frac{1}{l_\epsilon(x)} \int_{A_\epsilon(x)} X_{\partial\mathbb{D}}(s)ds.$$

A similar regularization was considered in [DS11] and the reader can check that this field has a locally Hölder version both in the variables x and ϵ. Let us mention that we have the following two options: either $x \in \mathbb{D}$ and then for $\epsilon < \mathrm{dist}(x, \partial\mathbb{D})$ we obtain

$$X_{\partial\mathbb{D},\epsilon}(x) = \frac{1}{2\pi} \int_0^{2\pi} X_{\partial\mathbb{D}}(x + \epsilon e^{i\theta})d\theta,$$

or $x \in \partial\mathbb{D}$ and then $X_{\partial\mathbb{D},\epsilon}(x)$ is intuitively the same as above except that we integrate along the "half-circle" centered at x with radius ϵ contained in $\mathbb{D}$.

Proposition 31. *Let us denote by g_P the Poincaré metric over the unit disk*

$$g_P = \frac{1}{(1 - |x|^2)^2} dx^2. \tag{2.27}$$

We claim
1) As $\epsilon \to 0$, the convergence $\mathbb{E}[X_{\partial\mathbb{D},\epsilon}(x)^2] + \ln\epsilon \to \frac{1}{2}\ln g_P(x)$ holds uniformly over the compact subsets of $\mathbb{D}$.
2) As $\epsilon \to 0$, the convergence $\mathbb{E}[X_{\partial\mathbb{D},\epsilon}(x)^2] + 2\ln\epsilon \to -1$ holds uniformly over $\partial\mathbb{D}$.
3) Consider a Möbius transform ψ of the disk. Denote by $X_{\partial\mathbb{D}} \circ \psi_\epsilon$ the ϵ-circle average of the field $X_{\partial\mathbb{D}} \circ \psi$. Then as $\epsilon \to 0$, we have the convergence

$$\mathbb{E}[X_{\partial\mathbb{D}} \circ \psi_\epsilon(x)^2] + \ln\epsilon \to \frac{1}{2}\ln g_P(\psi(x)) - 2\ln|\psi'(x)|$$

uniformly over the compact subsets of $\mathbb{D}$ *and the convergence*

$$\mathbb{E}[X_{\partial\mathbb{D}} \circ \psi_\epsilon(x)^2] + 2\ln\epsilon \to -1 - 2\ln|\psi'(x)|$$

uniformly over $\partial\mathbb{D}$.

Proof. To prove the first statement results, apply the ϵ-circle average regularization to the Green function G in (2.22) and use the fact that the following integral vanishes

$$\int_0^{2\pi}\int_0^{2\pi} \ln\frac{1}{|e^{i\theta} - e^{i\theta'}|}\, d\theta d\theta' = 0$$

to get the uniform convergence over compact subsets of $\mathbb{E}[X_{\partial,\epsilon}(x)^2] + \ln\epsilon$ towards

$$x \mapsto \frac{1}{2}\ln g_P(x).$$

The strategy is similar for the second statement except that one gets π^{-2} times the integral $\int_0^\pi\int_0^\pi \ln\frac{1}{|e^{i\theta}-e^{i\theta'}|}\, d\theta d\theta'$, which does not vanish anymore and yields the constant -1. The third claim results from (2.25). □

Gaussian Multiplicative Chaos

Gaussian multiplicative chaos theory was introduced in [Kah85]. The reader is referred to [RV14] for a review on the topic. Here, we deal with convolution of the GFF so that as a straightforward combination of the main result in [Sha14] and Proposition 32, we claim

Proposition 32. *For* $\gamma \in [0, 2[$ *and* λ, λ_∂ *the volume form and line element on* $\mathbb{D}$, $\partial\mathbb{D}$ *of the Euclidean metric, the random measures* $e^{\gamma X_{\partial\mathbb{D}}} d\lambda$, $e^{\frac{\gamma}{2}X_{\partial\mathbb{D}}} d\lambda_\partial$ *are defined as the limits in probability*

$$e^{\gamma X_{\partial\mathbb{D}}} d\lambda = \lim_{\epsilon\to 0} \epsilon^{\frac{\gamma^2}{2}} e^{\gamma X_{\partial\mathbb{D},\epsilon}} d\lambda$$
$$e^{\frac{\gamma}{2}X_{\partial\mathbb{D}}} d\lambda_{\partial\mathbb{D}} = \lim_{\epsilon\to 0} \epsilon^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}X_{\partial\mathbb{D},\epsilon}} d\lambda_\partial$$

in the sense of weak convergence of measures over $\mathbb{D}$, $\partial\mathbb{D}$. *These limiting measures are non trivial and are two standard Gaussian Multiplicative Chaos (GMC) on* $\mathbb{D}$, $\partial\mathbb{D}$, *namely*

$$e^{\gamma X_{\partial\mathbb{D}}} d\lambda = e^{\gamma X_{\partial\mathbb{D}}(x) - \frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D}}(x)^2]} g_P(x)^{\frac{\gamma^2}{4}} \lambda(dx)$$
$$e^{\frac{\gamma}{2}X_{\partial\mathbb{D}}} d\lambda_{\partial\mathbb{D}} = e^{-\frac{\gamma^2}{8}} e^{\frac{\gamma}{2}X_{\partial\mathbb{D}}(x) - \frac{\gamma^2}{8}\mathbb{E}[X_{\partial\mathbb{D}}(x)^2]} \lambda_\partial(dx).$$

Actually, the main issue is to show that these measures give almost surely finite mass respectively to the disk and its boundary. This turns out to be obvious for the boundary measure as the expectation of the total mass of $\partial\mathbb{D}$ is finite. Concerning the bulk measure, this statement is not straightforward: observe for instance that the expectation is infinite

$$\mathbb{E}[\int_{\mathbb{D}} e^{\gamma X_{\partial\mathbb{D}}} d\lambda] = \int_{\mathbb{D}} g_P(x)^{\frac{\gamma^2}{4}} \lambda(dx)$$

as soon as $\gamma^2 \geq 2$. Yet, we show in the following proposition that the random variable $\int_{\mathbb{D}} e^{\gamma X_{\partial\mathbb{D}}} d\lambda$ is almost surely finite for all values of $\gamma \in]0,2[$.

Proposition 33. *For $\gamma \in]0,2[$, the quantities below are almost surely finite*

$$\int_{\mathbb{D}} e^{\gamma X_{\partial\mathbb{D}}} d\lambda \quad \textit{and} \quad \int_{\partial\mathbb{D}} e^{\frac{\gamma}{2} X_{\partial\mathbb{D}}} d\lambda_{\partial\mathbb{D}}.$$

Proof. As explained above, we only need to focus on the bulk measure. Observe first that its expectation is finite in the case $\gamma^2 < 2$. For $\gamma^2 \geq 2$ (in fact the argument below works for $\gamma > 1$), we prove that it has moments of small order $\alpha > 0$, which entails the a.s. finiteness of the total mass of the interior of the disk.
Recall the sub-additivity inequality for $\alpha \in]0,1[$: if $(a_j)_{1\leq j\leq n}$ are positive real numbers then

$$(a_1 + \cdots + a_n)^\alpha \leq a_1^\alpha + \cdots + a_n^\alpha.$$

Therefore we can write

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{\mathbb{D}} e^{\gamma X_{\partial\mathbb{D}}(x) - \frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D}}(x)^2]} \frac{1}{(1-|x|^2)^{\gamma^2/2}})\lambda(dx)\Big)^\alpha\Big] \\
=&\mathbb{E}\Big[\Big(\sum_{n\in\mathbb{N}} \int_{1-2^{-n}\leq|x|^2\leq 1-2^{-n-1}} e^{\gamma X_{\partial\mathbb{D}}(x) - \frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D}}(x)^2]} \frac{1}{(1-|x|^2)^{\gamma^2/2}} \lambda(dx)\Big)^\alpha\Big] \\
\leq& \sum_{n\in\mathbb{N}} 2^{n\alpha\frac{\gamma^2}{2}} \mathbb{E}\Big[\Big(\int_{1-2^{-n}\leq|x|^2\leq 1-2^{-n-1}} e^{\gamma X_{\partial\mathbb{D}}(x) - \frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D}}(x)^2]} \lambda(dx)\Big)^\alpha\Big].
\end{aligned}$$

Now we trade the GFF $X_{\partial\mathbb{D}}$ for a log-correlated field that possesses a nicer structure of correlations with the help of Kahane's convexity inequality [Kah85]. More precisely, we consider any log-correlated field on $\mathbb{R}^2$ with a white noise decomposition and invariant under rotation. For instance, let us consider a star scale invariant kernel with compact support (see [ARV13]): we choose a positive definite isotropic positive function k with compact support of class C^2 and we set

$$K_\epsilon(x) = \int_1^{\epsilon^{-1}} \frac{k(ux)}{u}\, du.$$

We consider a family of Gaussian processes $(Y_\epsilon(x))_\epsilon$ such that (see [ARV13] for the details of the construction of such fields)

$$\forall x, y \in \mathbb{R}^2, \quad \mathbb{E}[Y_\epsilon(x) Y_{\epsilon'}(y)] = K_{\max(\epsilon,\epsilon')}(x-y).$$

The reader may check that for all $r, r' \in]0,1]$ such that $1-2^{-n} \leq r^2, r'^2 \leq 1-2^{-n-1}$ and $\theta, \theta' \in [0, 2\pi]$

$$\mathbb{E}[X_{\partial\mathbb{D}}(re^{i\theta}) X_{\partial\mathbb{D}}(r'e^{i\theta'})] \geq 2\mathbb{E}[Y_{2^{-n}}(e^{i\theta}) Y_{2^{-n}}(e^{i\theta'})] - A$$

for some constant A independent of n, θ. This inequality of covariances allows us to use Kahane's convexity inequality (see [Kah85] or [RV14, Theorem 2.1]). Indeed, because the map $x \mapsto x^\alpha$ is concave, we have for some standard Gaussian random variable N independent of everything

$$\begin{aligned}
&\mathbb{E}\Big[\Big(e^{\gamma A^{1/2} N - A\gamma^2/2} \int_{1-2^{-n} \leq |x|^2 \leq 1-2^{-n-1}} e^{\gamma X_{\partial\mathbb{D}} - \frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D}}^2]} d\lambda\Big)^\alpha\Big] \\
&\leq \mathbb{E}\Big[\Big(\int_0^{2\pi} \int_{(1-2^{-n})^{1/2}}^{(1-2^{-n-1})^{1/2}} e^{\gamma\sqrt{2} Y_{2^{-n}}(e^{i\theta}) - \gamma^2 \mathbb{E}[Y_{2^{-n}}(e^{i\theta})^2]} dr d\theta\Big)^\alpha\Big] \\
&= C 2^{-n\alpha} \mathbb{E}\Big[\Big(\int_0^{2\pi} e^{\gamma\sqrt{2} Y_{2^{-n}}(e^{i\theta}) - \gamma^2 \mathbb{E}[Y_{2^{-n}}(e^{i\theta})^2]} d\theta\Big)^\alpha\Big]
\end{aligned}$$

for some constant C independent of everything. By using the comparison to Mandelbrot's multiplicative cascades as explained in [DRSV14b, Appendix B.1] to use a moment estimate in [Mad17, Proposition 2.1 and the remark just after], we have that for any $\alpha < \gamma^{-1}$ and some other constant $C > 0$

$$\sup_n \mathbb{E}\Big[\Big(n^{\frac{3\gamma}{2}} 2^{n(\gamma-1)^2} \int_0^{2\pi} e^{\gamma\sqrt{2} Y_{2^{-n}}(e^{i\theta}) - \gamma^2 \mathbb{E}[Y_{2^{-n}}(e^{i\theta})^2]} d\theta\Big)^\alpha\Big] \leq C.$$

Combining we get (up to changing the value of C to absorb the constant $\mathbb{E}[e^{\alpha\gamma A^{1/2} N - \alpha A \gamma^2/2}]$)

$$\mathbb{E}\Big[\Big(\int_{\mathbb{D}} e^{\gamma X_{\partial\mathbb{D}}(x) - \frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D}}(x)^2]} \frac{1}{(1-|x|^2)^{\gamma^2/2}}) \lambda(dx)\Big)^\alpha\Big] \leq C \sum_{n \in \mathbb{N}} 2^{n\alpha\left(\frac{\gamma^2}{2} - 1 - (\gamma-1)^2\right)} n^{-\frac{3\gamma}{2}\alpha},$$

which is finite (with $\alpha \in (0, \gamma^{-1})$) when $\gamma \in]1,2[$ because $\frac{\gamma^2}{2} - 1 - (\gamma-1)^2 < 0$ when $\gamma \in]0,2[$. □

Liouville Quantum Gravity on the disk

We are now in a position to give a precise definition of the LQFT on the disk with marked points: n points in the bulk $\mathbb{D}$ and n' points on the boundary $\partial\mathbb{D}$. In what follows,

we will first give a necessary and sufficient condition (the Seiberg bounds) on these marked points in order that the LQFT is well defined. This will allow us to give the definitions of the Liouville field and measure. Finally, we will explain how these objects behave under conformal changes of background metrics and conformal reparametrization of the domain. Basically, the approach is the same as in [DKRV16] but there are some technical differences in order to treat the interactions bulk/boundary.

Definition and existence of the partition function

LQFT on the disk will be defined in terms of three parameters $\gamma, \mu, \mu_\partial$, respectively the coupling constant and the bulk/boundary cosmological constants, together with prescribed marked points. In this section, we will assume that the parameters $\gamma, \mu, \mu_\partial$ satisfy

$$\gamma \in]0,2[, \quad \mu, \mu_\partial \geq 0 \quad \text{and} \quad \mu + \mu_\partial > 0. \tag{2.28}$$

Concerning the marked points, we fix a set of n points $(z_i)_{1\leq i\leq n}$ in the interior of $\mathbb{D}$ together with n weights $(\alpha_i)_{1\leq i\leq n} \in \mathbb{R}^n$ and n' points $(s_j)_{1\leq j\leq n'}$ on the boundary $\partial\mathbb{D}$ together with n' weights $(\beta_j)_{1\leq j\leq n'} \in \mathbb{R}^{n'}$. The family $(z_i, \alpha_i)_i$ will be called bulk marked points and the family $(s_j, \beta_j)_j$ boundary marked points.

Consider any metric $g = e^\varphi dx^2$ on the unit disk conformally equivalent to the Euclidean metric in the sense of (2.17). Our purpose is now to define the partition function $\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(\epsilon, g, F)$ of LQFT applied to a functional F. This partition function formally corresponds to the Feynmann surface integral (2.7) with action (2.3). Yet, a rigorous approach requires the regularization procedure. This is the reason why we define the regularized partition function for all $\epsilon \in]0,1]$ and bounded continuous functional F on $H^{-1}(\mathbb{D})$ by

$$\begin{aligned}
&\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(\epsilon, g, F) \qquad (2.29)\\
=&e^{\frac{1}{96\pi}\left(\int_{\mathbb{D}} |\partial \ln g|^2 \, d\lambda + \int_{\partial\mathbb{D}} 4 \ln g \, d\lambda_\partial\right)}\\
&\int_{\mathbb{R}} \mathbb{E}\Big[F(X_{\partial\mathbb{D}} + c + Q/2 \ln g) \prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i (c + X_{\partial\mathbb{D},\epsilon} + Q/2 \ln g)(z_i)} \prod_j \epsilon^{\frac{\beta_j^2}{4}} e^{\frac{\beta_j}{2}(c + X_{\partial\mathbb{D},\epsilon} + Q/2 \ln g)(s_j)}\\
&\exp\Big(-\frac{Q}{4\pi}\int_{\mathbb{D}} R_g(c + X_{\partial\mathbb{D}}) \, d\lambda_g - \mu e^{\gamma c} \epsilon^{\frac{\gamma^2}{2}} \int_{\mathbb{D}} e^{\gamma(X_{\partial\mathbb{D},\epsilon} + Q/2 \ln g)} \, d\lambda\Big)\\
&\exp\Big(-\frac{Q}{2\pi}\int_{\partial\mathbb{D}} K_g(c + X_{\partial\mathbb{D}}) \, d\lambda_{\partial g} - \mu_\partial e^{\frac{\gamma}{2} c} \epsilon^{\frac{\gamma^2}{4}} \int_{\partial\mathbb{D}} e^{\frac{\gamma}{2}(X_{\partial\mathbb{D},\epsilon} + Q/2 \ln g)} \, d\lambda_\partial\Big)\Big] \, dc.
\end{aligned}$$

The constant c which is integrated against the Lebesgue measure dc is crucial in the definition and adds extra symmetry. In particular, one has the following equality in distribution

for all Möbius transform ψ when c is distributed according to the Lebesgue measure

$$X_{\partial\mathbb{D}} \circ \psi + c \overset{(Law)}{=} X_{\partial\mathbb{D}} + c \tag{2.30}$$

To prove identity (2.30), recalll that $X_{\partial\mathbb{D}} \circ \psi - \frac{1}{2\pi}\int_{\partial\mathbb{D}} X_{\partial\mathbb{D}} \circ \psi \, d\lambda_\partial \overset{(Law)}{=} X_{\partial\mathbb{D}}$ and then use that the Lebesgue measure is invariant under translation. Identity (2.30) in distribution is essential in proving the conformal invariance properties of the theory. Now, the first natural question is to inquire whether the limit

$$\Pi^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial,\mu}(g,F) := \lim_{\epsilon\to 0} \Pi^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial,\mu}(\epsilon,g,F). \tag{2.31}$$

exists and is not trivial. Existence and non triviality will be phrased in terms of the following three conditions

$$\sum_i \alpha_i + \frac{1}{2}\sum_j \beta_j > Q, \tag{2.32}$$

$$\forall i \quad \alpha_i < Q, \tag{2.33}$$

$$\forall j \quad \beta_j < Q. \tag{2.34}$$

We claim:

Théorème 34. (Seiberg bounds) *We have the following alternatives*

1. *Assume $\mu > 0$ and $\mu_\partial \geq 0$. The partition function $\Pi^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial,\mu}(g,1)$ converges and is non trivial if and only if* (2.32)+(2.33)+(2.34) *hold.*
2. *Assume $\mu = 0$ and $\mu_\partial > 0$. The partition function $\Pi^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial,\mu}(g,1)$ converges and is non trivial if and only if* (2.32) +(2.34) *hold.*
3. *In all other cases, we have*

$$\Pi^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial,\mu}(g,1) = 0 \quad \text{or} \quad \Pi^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial,\mu}(g,1) = +\infty.$$

Along the computations involved in Theorem 34, we get the expression below for the partition function when the metric g is the Euclidean metric. Notice that considering the only Euclidean metric is not a restriction because we will see later that there is an explicit procedure to express the partition function in any background metric g in terms of that in the Euclidean metric (Weyl anomaly, Subsection 2).

Proposition 35. (Partition function) *Assume g is the Euclidean metric dx^2. Then, in each case of Theorem 34 ensuring existence and non triviality, we have*

$$\Pi^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial,\mu}(dx^2,F) = \Big(\prod_i g_P(z_i)^{-\frac{\alpha_i^2}{4}}\Big) e^{C(\mathbf{z},\mathbf{s})} \int_{\mathbb{R}} e^{\left(\sum_i \alpha_i + \sum_j \frac{\beta_j}{2} - Q\right)c} \tag{2.35}$$

$$\mathbb{E}\Big[F(X_{\partial\mathbb{D}} + H + c)\exp\Big(-\mu e^{\gamma c}\int_{\mathbb{D}} e^{\gamma H} e^{\gamma X_{\partial\mathbb{D}}}\, d\lambda - \mu_\partial e^{\frac{\gamma}{2}c}\int_{\partial\mathbb{D}} e^{\frac{\gamma}{2}H} e^{\frac{\gamma}{2}X_{\partial\mathbb{D}}}\, d\lambda_\partial\Big)\Big]\, dc,$$

where

$$H(x) = \sum_i \alpha_i G(x, z_i) + \sum_j \frac{\beta_j}{2} G(x, s_j),$$

$$C(\mathbf{z}, \mathbf{s}) = \sum_{i<i'} \alpha_i \alpha_{i'} G(z_i, z_{i'}) + \sum_{j<j'} \frac{\beta_j \beta_{j'}}{4} G(s_j, s_{j'}) + \sum_{i,j} \frac{\alpha_i \beta_j}{2} G(z_i, s_j) - \sum_j \frac{\beta_j^2}{8}.$$

Proof of Theorem 34 and Proposition 35. We begin with the Seiberg bound. Because the conformal factor φ of $g = e^{\varphi} dx^2$ is assumed to be smooth (i.e. of class C^1), we can assume without loss of generality that $\varphi = 0$. The main lines of the argument will be similar to [DKRV16, Section 3], up to a few modifications that we explain below. First observe that Propositions 32 and 33 ensure that the interaction terms

$$\lim_{\epsilon \to 0} \epsilon^{\frac{\gamma^2}{4}} \int_{\partial \mathbb{D}} e^{\frac{\gamma}{2} X_{\partial \mathbb{D}, \epsilon}} \, d\lambda_\partial \quad \text{and} \quad \lim_{\epsilon \to 0} \epsilon^{\frac{\gamma^2}{2}} \int_{\mathbb{D}} e^{\gamma X_{\partial \mathbb{D}, \epsilon}} \, d\lambda$$

are non trivial provided that $\gamma \in]0, 2[$. Hence, following [DKRV16, Section 3],

$$\Pi_{\gamma, \mu_\partial, \mu}^{(z_i, \alpha_i)_i, (s_j, \beta_j)_j}(g, 1) < +\infty$$

if and only if (2.32) holds: roughly speaking, recall that basically this amounts to claiming that the integral (A, A' are two strictly positive constants)

$$\int_{\mathbb{R}} e^{\left(\sum_i \alpha_i + \frac{1}{2} \sum_j \beta_j - Q\right) c} e^{-\mu e^{\gamma c} A - \mu_\partial e^{\frac{\gamma}{2} c} A'} \, dc$$

is converging if and only if (2.32) holds.

Recall then that the remaining part of the proof in [DKRV16, Section 3] consists in determining when a marked point causes the blowing up of the interaction measure, in which case $\Pi_{\gamma, \mu_\partial, \mu}^{(z_i, \alpha_i)_i, (s_j, \beta_j)_j}(dx^2, F) = 0$. The reason why a marked point may cause the blowing up of the interaction measure is that these marked points are handled with the Girsanov transform and this amounts to determining whether the bulk/boundary measures integrates some singularities of the type $|x - z_i|^{-\alpha_i \gamma}$ or $|x - s_j|^{-\frac{\beta_j}{2} \gamma}$. This is what we study in more details below.

Here we have two types of marked points (in the bulk or along the boundary) and two interaction measures: boundary $e^{\frac{\gamma}{2} X_{\partial \mathbb{D}}} \, d\lambda_\partial$ or bulk $e^{\gamma X_{\partial \mathbb{D}}} \, d\lambda$. A marked point (z_i, α_i) in the bulk questions whether the bulk measure integrates the singularity $x \mapsto e^{\alpha_i \gamma G(x, z_i)}$. This is exactly the same situation as in [DKRV16, Section 3]. Therefore the conclusion is the same: α_i must be strictly less than Q. The same argument settles the case of the effect of a boundary marked point (s_j, β_j) on the boundary measure: β_i must be strictly less than Q.

What is not treated in [DKRV16, Section 3] is the effect of boundary marked points on the bulk measure: namely we have to determine when the measure $e^{\gamma X_{\partial\mathbb{D}}}\,d\lambda$ integrates the singularity $x \mapsto e^{\frac{\beta_j}{2}\gamma G(x,s_j)}$ for some s_j belonging to the boundary $\partial\mathbb{D}$. Observe that the situation is more complicated as the behavior of the bulk measure is highly perturbed when approaching the boundary: recalling the expression of the bulk measure in Proposition 32, we see that on the one hand the deterministic density $g_P(x)^{\frac{\gamma^2}{4}}$ blows up along the boundary and on the other hand the field $X_{\partial\mathbb{D}}$ acquires more and more correlations, which become maximal along the boundary: as x approaches the boundary, $G(x,y)$ tends to behave like $2\ln\frac{1}{|x-y|}$ rather than $\ln\frac{1}{|x-y|}$.

Let us now analyze the situation. We want to prove that the singularity is integrable if and only if $\beta_j < Q$. Without loss of generality, we assume that $s_j = 1$. In what follows, C stands for some generic constant, which may change along the lines and does not depend on relevant quantities.

Let us first assume that the singularity is integrable, more precisely for some δ fixed small enough

$$\lim_{\epsilon\to 0}\int_{\mathbb{D}\cap B(1,\delta)} e^{\frac{\beta_j}{2}\gamma G_\epsilon(\cdot,1)}\epsilon^{\frac{\gamma^2}{2}}e^{\gamma X_{\partial\mathbb{D},\epsilon}}\,d\lambda < +\infty \tag{2.36}$$

where

$$G_\epsilon(x,y) = \mathbb{E}[X_{\partial\mathbb{D},\epsilon}(x)X_{\partial\mathbb{D},\epsilon}(y)].$$

For each $\epsilon > 0$ small enough, we denote by D_ϵ the small disk centered at $1-2\epsilon$ with radius ϵ. Notice that for ϵ small enough, this disk is contained in $B(1,\delta)\cap\mathbb{D}$. Therefore, we have the obvious relation

$$\int_{\mathbb{D}\cap B(1,\delta)} e^{\frac{\beta_j}{2}\gamma G_\epsilon(\cdot,1)}\epsilon^{\frac{\gamma^2}{2}}e^{\gamma X_{\partial\mathbb{D},\epsilon}}\,d\lambda \geq \int_{D_\epsilon} e^{\frac{\beta_j}{2}\gamma G_\epsilon(\cdot,1)}e^{\gamma X_{\partial\mathbb{D},\epsilon}-\frac{\gamma^2}{2}\mathbb{E}[X^2_{\partial\mathbb{D},\epsilon}]}e^{\frac{\gamma^2}{2}(\mathbb{E}[X^2_{\partial\mathbb{D},\epsilon}]-\ln\frac{1}{\epsilon})}\,d\lambda.$$

It is then plain to check that, for some constant C independent of ϵ and uniformly with respect to $x\in D_\epsilon$,

$$|\mathbb{E}[X_{\partial\mathbb{D},\epsilon}(x)^2] - 2\ln\frac{1}{\epsilon}| \leq C, \quad |G_\epsilon(x,1) - 2\ln\frac{1}{\epsilon}| \leq C.$$

We deduce

$$\int_{\mathbb{D}\cap B(1,\delta)} e^{\frac{\beta_j}{2}\gamma G_\epsilon(\cdot,1)}\epsilon^{\frac{\gamma^2}{2}}e^{\gamma X_{\partial\mathbb{D},\epsilon}}\,d\lambda \geq C\epsilon^{-\beta_j\gamma-\frac{\gamma^2}{2}}\int_{D_\epsilon} e^{\gamma X_{\partial\mathbb{D},\epsilon}-\frac{\gamma^2}{2}\mathbb{E}[X^2_{\partial\mathbb{D},\epsilon}]}\,d\lambda.$$

If we can establish the following estimate

$$\text{in probability,}\quad \limsup_{\epsilon\to 0}\epsilon^{-2-\gamma^2}\int_{D_\epsilon} e^{\gamma X_{\partial\mathbb{D},\epsilon}-\frac{\gamma^2}{2}\mathbb{E}[X^2_{\partial\mathbb{D},\epsilon}]}\,d\lambda = +\infty, \tag{2.37}$$

we deduce that necessarily $\beta_j < Q$ in order for (2.36) to hold.

To establish (2.37), observe (see Subsection 2) that, for some deterministic constant C independent of ϵ,

$$\sup_{\epsilon>0} \sup_{x\in D_\epsilon} |G_\epsilon(x,x) + 2\ln\epsilon| < +\infty,$$

in such a way that

$$\int_{D_\epsilon} e^{\gamma X_{\partial\mathbb{D},\epsilon} - \frac{\gamma^2}{2}\mathbb{E}[X^2_{\partial\mathbb{D},\epsilon}]}\, d\lambda \geq C\epsilon^{2+\gamma^2} e^{\gamma X_{\partial\mathbb{D},\epsilon}(1)} e^{\min_{x\in D_\epsilon} X_{\partial\mathbb{D},\epsilon}(x) - X_{\partial\mathbb{D},\epsilon}(1)}. \tag{2.38}$$

Next, we estimate the min in the above expression. Observe that ($D(2,1)$ stands for the disk centered at 2 with radius 1)

$$\min_{x\in D_\epsilon} X_{\partial\mathbb{D},\epsilon}(x) - X_{\partial\mathbb{D},\epsilon}(1) = \min_{u\in D(2,1)} Y_\epsilon(u)$$

where the Gaussian process Y_ϵ is defined by

$$Y_\epsilon(u) = X_{\partial\mathbb{D},\epsilon}(1-\epsilon u) - X_{\partial\mathbb{D},\epsilon}(1).$$

The key point is to estimate the fluctuations of the Gaussian process Y_ϵ. The reader may check (see Subsection 2) that the variance of $Y_\epsilon(2)$ is bounded independently of ϵ and that for all $z, z' \in D(2,1)$

$$\mathbb{E}[(Y_\epsilon(z) - Y_\epsilon(z'))^2] \leq C|z-z'|,$$

uniformly in $0 < \epsilon \leq 1$. Recall the Kolmogorov criterion

Théorème 36. (Kolmogorov criterion) *Let X be a continous stochastic process on $D(1,2)$. If, for some $\beta, \alpha, C > 0$:*

$$\forall x, z \in D(1,2), \quad \mathbb{E}[|X_x - X_z|^q] \leq C|x-z|^{2+\beta}.$$

For all $\delta \in]0, \frac{\beta}{q}[$, we set $L = \sup_{x\neq z} \frac{|X_x - X_z|}{|x-z|^\delta}$. Then, for all $p < q$, $\mathbb{E}[L^\beta] \leq 1 + \frac{Cp2^{\beta-q\delta}}{(q-p)(2^{\beta-q\delta}-1)}$.

One can then deduce that the family of processes $(Y_\epsilon)_\epsilon$ is tight in the space of continuous functions over $D(2,1)$ for the topology of uniform convergence. We deduce that for each subsequence, we can find R large enough such that $\min_{x\in D_\epsilon} X_{\partial\mathbb{D},\epsilon}(x) - X_{\partial\mathbb{D},\epsilon}(1) \geq -R$ with probability arbitrarily close to 1. Finally, we observe that the process $\epsilon \mapsto X_{\partial\mathbb{D},\epsilon}(1)$ behaves like a Brownian motion at time $2\ln\frac{1}{\epsilon}$ (see [DS11, section 6.1]), we can use the law of the iterated logarithm in (2.38) to complete the proof of (2.37).

Now it remains to show that the condition $\beta_j < Q$ is sufficient to have integrability, that is $\int_{\mathbb{D}\cap B(1,\delta)} e^{\gamma\frac{\beta_j}{2}G(\cdot,1)} e^{\gamma X_{\partial\mathbb{D}} - \frac{\gamma^2}{2}\mathbb{E}[X^2_{\partial\mathbb{D}}]} g_P^{\frac{\gamma^2}{4}}\, d\lambda < +\infty$. To simplify a bit the notations we will prove the following equivalent statement

$$\int_{\mathbb{H}\cap B(0,1)} e^{\gamma\frac{\beta_j}{2}G_{\mathbb{H}}(z,0)} e^{\gamma X(z) - \frac{\gamma^2}{2}\mathbb{E}[X^2(z)]} \frac{1}{\mathrm{Im}(z)^{\frac{\gamma^2}{2}}}\, \lambda(dz) < +\infty \tag{2.39}$$

where X is the GFF on $\mathbb{H}$ defined by $X = X_{\partial\mathbb{D}} \circ \psi$ where $\psi(z) = \frac{z-i}{z+i}$ is the Cayley transform mapping the upper half-plane onto the unit disk and $G_{\mathbb{H}}$ its Green function, that is $G_{\mathbb{H}}(x,y) = G(\psi(x),\psi(y))$. A simple check shows that on the ball $B(0,1)$ we have

$$G_{\mathbb{H}}(x,y) = \ln \frac{1}{|x-y||x-\bar{y}|} + g(x,y)$$

where g is a continuous bounded function. It is then also easy to see that for $x, y \in B(0,1)\cap\mathbb{H}$ and all $r \in]0,1[$

$$G_{\mathbb{H}}(rx,ry) \geq G_{\mathbb{H}}(x,y) + 2\ln\frac{1}{r} - C \tag{2.40}$$

where C is some fixed positive constant.

The same argument as in Proposition 33 shows that the quantity

$$\mathbb{E}\Big[\Big(\int_{\mathbb{H}\cap B(0,1)} e^{\gamma X(z) - \frac{\gamma^2}{2}\mathbb{E}[X^2(z)]} \frac{1}{\mathrm{Im}(z)^{\frac{\gamma^2}{2}}}\, \lambda(dz)\Big)^{\alpha}\Big]$$

is finite for $\alpha < \gamma^{-1}$. Then, for $r < 1$, we can make a change of variables $ru = z$ and then combine the relation (2.40) with Kahane's convexity inequality [Kah85] (see also [RV14, Theorem 2.1]) to deduce (for some irrelevant constant C which may change along lines)

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{\mathbb{H}\cap B(0,r)} e^{\gamma X(z) - \frac{\gamma^2}{2}\mathbb{E}[X^2(z)]} \frac{1}{\mathrm{Im}(z)^{\frac{\gamma^2}{2}}}\, \lambda(dz)\Big)^{\alpha}\Big] \\
=&r^{2\alpha}\mathbb{E}\Big[\Big(\int_{\mathbb{H}\cap B(0,1)} e^{\gamma X(ru) - \frac{\gamma^2}{2}\mathbb{E}[X^2(ru)]} \frac{1}{\mathrm{Im}(ru)^{\frac{\gamma^2}{2}}}\, \lambda(du)\Big)^{\alpha}\Big] \\
=&Cr^{(2-\frac{\gamma^2}{2})\alpha}\mathbb{E}\Big[\Big(\int_{\mathbb{H}\cap B(0,1)} e^{\gamma X(z) - \frac{\gamma^2}{2}\mathbb{E}[X^2(z)]} \frac{1}{\mathrm{Im}(z)^{\frac{\gamma^2}{2}}}\, \lambda(dz)\Big)^{\alpha}\Big]\mathbb{E}\Big[e^{\alpha\gamma Z_r - \frac{\alpha^2\gamma^2}{2}2\ln\frac{1}{r}}\Big]
\end{aligned}$$

where Z_r is a centered Gaussian random variable with variance $2\ln\frac{1}{r}$ and independent of everything. Hence, for all $r < 1$

$$\mathbb{E}\Big[\Big(\int_{\mathbb{H}\cap B(0,r)} e^{\gamma X(z) - \frac{\gamma^2}{2}\mathbb{E}[X^2(z)]} \frac{1}{\mathrm{Im}(z)^{\frac{\gamma^2}{2}}}\, \lambda(dz)\Big)^{\alpha}\Big] \leq Cr^{(2+\frac{\gamma^2}{2})\alpha - \alpha^2\gamma^2}.$$

Let $\eta > 0$. By using the Markov inequality and the above relation, we obtain

$$\begin{aligned}
&\mathbb{P}\Big(\int_{\mathbb{H}\cap B(0,r)} e^{\gamma X(z)-\frac{\gamma^2}{2}\mathbb{E}[X^2(z)]}\frac{1}{\mathrm{Im}(z)^{\frac{\gamma^2}{2}}}\lambda(dz) > r^{2+\frac{\gamma^2}{2}-\eta}\Big)\\
\leq & r^{-\alpha(2+\frac{\gamma^2}{2}-\eta)}\mathbb{E}\Big[\Big(\int_{\mathbb{H}\cap B(0,r)} e^{\gamma X(z)-\frac{\gamma^2}{2}\mathbb{E}[X^2(z)]}\frac{1}{\mathrm{Im}(z)^{\frac{\gamma^2}{2}}}\lambda(dz)\Big)^{\alpha}\Big]\\
\leq & C r^{\eta\alpha-\alpha^2\gamma^2}.
\end{aligned}$$

Choosing $\alpha > 0$ small enough so as to get $\eta\alpha - \alpha^2\gamma^2 > 0$. We can then use the Borel-Cantelli lemma to deduce that there exists a random constant R, which is finite almost surely, such that

$$\sup_{r\in]0,1]} r^{-(2+\frac{\gamma^2}{2}-\eta)}\int_{\mathbb{H}\cap B(0,r)} e^{\gamma X(z)-\frac{\gamma^2}{2}\mathbb{E}[X^2(z)]}\frac{1}{\mathrm{Im}(z)^{\frac{\gamma^2}{2}}}\lambda(dz) \leq R. \tag{2.41}$$

Now we introduce the annuli for $n \geq 0$

$$A_n = \{z \in \mathbb{H}; 2^{-n-1} \leq |z| \leq 2^{-n}\}.$$

We get from (2.41)

$$\begin{aligned}
&\int_{\mathbb{H}\cap B(0,1)} e^{\gamma\frac{\beta_j}{2}G_{\mathbb{H}}(z,0)}e^{\gamma X(z)-\frac{\gamma^2}{2}\mathbb{E}[X^2(z)]}\frac{1}{\mathrm{Im}(z)^{\frac{\gamma^2}{2}}}\lambda(dz)\\
=&\sum_{n\geq 0}\int_{A_n} e^{\gamma\frac{\beta_j}{2}G_{\mathbb{H}}(z,0)}e^{\gamma X(z)-\frac{\gamma^2}{2}\mathbb{E}[X^2(z)]}\frac{1}{\mathrm{Im}(z)^{\frac{\gamma^2}{2}}}\lambda(dz)\\
\leq & C\sum_{n\geq 0} 2^{\gamma\beta_j n}\int_{\mathbb{H}\cap B(0,2^{-n})} e^{\gamma X(z)-\frac{\gamma^2}{2}\mathbb{E}[X^2(z)]}\frac{1}{\mathrm{Im}(z)^{\frac{\gamma^2}{2}}}\lambda(dz)\\
\leq & CR\sum_{n\geq 0} 2^{\gamma\beta_j n}2^{-n(2+\frac{\gamma^2}{2}-\eta)}.
\end{aligned}$$

The proof of Theorem 34 is complete provided that we choose $0 < \eta < \gamma(Q-\beta_j)$. Once the Seiberg bounds are established, the computation of the partition function (i.e. Proposition 35) follows the same lines as in [DKRV16, Theorem 3.2]. □

Definitions of the Liouville field, Liouville measure and boundary Liouville measure

As long as one of the two conditions of Theorem 34 is satisfied, one may define the joint law of the Liouville field ϕ together with the Liouville measure $Z(\cdot)$ and boundary Liouville measure $Z_\partial(\cdot)$. In spirit, the situation is that the convergence of the partition function

entails that we get a non trivial probability law for the field $\phi = c + X_{\partial\mathbb{D}} + \frac{Q}{2}\ln g$ under the probability measure defined by the partition function. This field formally corresponds to the log-conformal factor of some random metric $e^{\gamma\phi}g$ conformally equivalent to g. Yet, observe that the field ϕ is in H^{-1} almost surely so that a rigorous description of this metric is not straightforward, at least clearly not standard. The Liouville measure that we construct below is a random measure that can be thought of as the volume form of this formal metric tensor whereas the boundary Liouville measure corresponds to the line element along the boundary. Let us mention that we could construct as well the Liouville Brownian motion by using the construction made in [GRV16, GRV14] but a rigorous construction of a distance function associated to the metric tensor $e^{\gamma\phi}g$ remains an open question.

Given a measured space E, we denote by $\mathcal{R}(E)$ the space of Radon measures on E equipped with the topology of weak convergence. The joint law of (ϕ, Z, Z_∂) is defined for all continuous bounded functional F on $H^{-1}(\bar{\mathbb{D}}) \times \mathcal{R}(\mathbb{D}) \times \mathcal{R}(\partial\mathbb{D})$ by

$$\begin{aligned}
&\mathbb{E}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial,\mu,g}\big[F(\phi, Z, Z_\partial)\big] \\
&= \frac{e^{\frac{1}{96\pi}\left(\int_{\mathbb{D}} |\partial \ln g|^2\, d\lambda + \int_{\partial\mathbb{D}} 4\ln g\, d\lambda_\partial\right)}}{\Pi^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial,\mu}(g,1)} \lim_{\epsilon\to 0}\int_{\mathbb{R}} \prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i(X_{\partial\mathbb{D},\epsilon}+Q/2\ln g)(z_i)} \prod_j \epsilon^{\frac{\beta_j^2}{4}} e^{\frac{\beta_j}{2}(X_{\partial\mathbb{D},\epsilon}+Q/2\ln g)(s_j)} \\
&E\Big[F\Big(X_{\partial\mathbb{D}} + c + Q/2\ln g, e^{\gamma c}\epsilon^{\frac{\gamma^2}{2}} e^{\gamma(X_{\partial\mathbb{D},\epsilon}+Q/2\ln g)}\, d\lambda, e^{\frac{\gamma}{2}c}\epsilon^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}(X_{\partial\mathbb{D},\epsilon}+Q/2\ln g)}\, d\lambda_\partial\Big) \\
&\exp\Big(-\frac{Q}{4\pi}\int_{\mathbb{D}} R_g(c + X_{\partial\mathbb{D}})\, d\lambda_g - \mu e^{\gamma c}\epsilon^{\frac{\gamma^2}{2}}\int_{\mathbb{D}} e^{\gamma(X_{\partial\mathbb{D},\epsilon}+Q/2\ln g)}\, d\lambda\Big) \\
&\exp\Big(-\frac{Q}{2\pi}\int_{\partial\mathbb{D}} K_g(c + X_{\partial\mathbb{D}})\, d\lambda_{\partial g} - \mu_\partial e^{\frac{\gamma}{2}c}\epsilon^{\frac{\gamma^2}{4}}\int_{\partial\mathbb{D}} e^{\frac{\gamma}{2}(X_{\partial\mathbb{D},\epsilon}+Q/2\ln g)}\, d\lambda_\partial\Big)\Big]\, dc.
\end{aligned}$$

We denote by $\mathbb{P}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial,\mu,g}$ the associated probability measure. In the following subsections, we will mention several interesting properties satisfied by these objects.

Conformal changes of metric and Weyl anomaly

Here we want to determine the dependence of the partition function (2.31) (as well as the Liouville field/measures) on the metric g conformally equivalent to the Euclidean metric. In fact, this dependence enables to determine the central charge of the theory:

Théorème 37. (Weyl anomaly)

1. *Given two metrics g, g' conformally equivalent to the flat metric and $g' = e^{\varphi}g$, we have*

$$\ln\frac{\Pi^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial,\mu}(g',F)}{\Pi^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial,\mu}(g,F)} = \frac{1+6Q^2}{96\pi}\Big(\int_{\mathbb{D}} |\partial^g\varphi|^2\, d\lambda_g + \int_{\mathbb{D}} 2R_g\varphi\, d\lambda_g + 4\int_{\partial\mathbb{D}} K_g\varphi\, d\lambda_{\partial_g}\Big).$$

2. *The law of the triple* (ϕ, Z, Z_∂) *under* $\mathbb{P}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial,\mu,g}$ *does not depend on the metric* g *in the conformal equivalence class of the Euclidean metric.*

In the language of CFT, the above theorem states that the central charge of LQFT is $1+6Q^2$: see the lecture notes [Gaw97].

Proof. In (2.31), we use the Girsanov transform to the exponential term

$$\exp\Big(-\frac{Q}{4\pi}\int_{\mathbb{D}} R_g X_{\partial\mathbb{D}}\, d\lambda_g - \frac{Q}{2\pi}\int_{\partial\mathbb{D}} K_g X_{\partial\mathbb{D}}\, d\lambda_{\partial g}\Big),$$

which has the effect of shifting the field X by

$$-\frac{Q}{4\pi}\int_{\mathbb{D}} R_g G_{\partial\mathbb{D}}(\cdot,z)\,\lambda_g(dz) - \frac{Q}{2\pi}\int_{\partial\mathbb{D}} G_{\partial\mathbb{D}}(\cdot,z) K_g\, d\lambda_{\partial g}.$$

Then we use the rules (2.18)+(2.19)+(2.23) to see that this shift is equal to

$$-\frac{Q}{2}(\ln g - m_\partial(\ln g)).$$

Due to the Girsanov renormalization, the whole partition function will be multiplied by the exponential of the variance of the field $\frac{Q}{4\pi}\int_{\mathbb{D}} R_g X_{\partial\mathbb{D}}\, d\lambda_g + \frac{Q}{2\pi}\int_{\partial\mathbb{D}} K_g X_{\partial\mathbb{D}}\, d\lambda_{\partial g}$, which can be computed with (2.18)+(2.19)+(2.23) and is given by

$$\frac{Q^2}{16\pi}\int_{\mathbb{D}} |\partial \ln g|^2\, d\lambda.$$

Hence, by making the changes of variables $v = c + \frac{Q}{2} m_\partial(\ln g)$, we get

$$\begin{aligned}
&\Pi^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial,\mu}(g,F)\\
&\quad= e^{\frac{6Q^2}{96\pi}\int_{\mathbb{D}} |\partial \ln g|^2\, d\lambda + \frac{Q^2}{2} m_\partial(\ln g)} e^{\frac{1}{96\pi}\left(\int_{\mathbb{R}^2} |\partial \ln g|^2\, d\lambda + \int_\partial 4 \ln g\, d\lambda_\partial\right)}\\
&\qquad \lim_{\epsilon\to 0}\int_{\mathbb{R}} e^{\left(\sum_i \alpha_i + \frac{1}{2}\sum_j \beta_j - Q\right)v}\mathbb{E}\Big[F(X_{\partial\mathbb{D}} + v, e^{\gamma v} e^{\gamma X_{\partial\mathbb{D}}}\, d\lambda, e^{\frac{\gamma}{2}v} e^{\frac{\gamma}{2}X_{\partial\mathbb{D}}}\, d\lambda_\partial)\\
&\qquad \prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i X_{\partial\mathbb{D},\epsilon}(z_i)} \prod_j \epsilon^{\frac{\beta_j^2}{4}} e^{\frac{\beta_j}{2} X_{\partial\mathbb{D},\epsilon}(s_j)}\\
&\qquad \exp\Big(-\mu e^{\gamma v}\epsilon^{\frac{\gamma^2}{2}}\int_{\mathbb{D}} e^{\gamma X_{\partial\mathbb{D},\epsilon}}\, d\lambda - \mu_\partial e^{\frac{\gamma}{2}v}\epsilon^{\frac{\gamma^2}{4}}\int_{\partial\mathbb{D}} e^{\frac{\gamma}{2}X_{\partial\mathbb{D},\epsilon}}\, d\lambda_\partial\Big)\Big]\, dc\\
&\quad= e^{\frac{1+6Q^2}{96\pi}\left(\int_{\mathbb{D}} |\partial \ln g|^2\, d\lambda + \int_{\partial\mathbb{D}} 4\ln g\, d\lambda_\partial\right)} \Pi^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial,\mu}(dx^2, F).
\end{aligned}$$

To complete the proof for two metrics g, g' conformally equivalent to the Euclidean metric, say $g' = e^{\varphi} g$, we apply twice the above result to get

$$\begin{aligned}
&\ln \frac{\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(g',F)}{\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(g,F)} \\
&= \frac{1+6Q^2}{96\pi}\Big(\int_{\mathbb{D}} |\partial \ln g'|^2 \, d\lambda + \int_{\partial\mathbb{D}} 4 \ln g' \, d\lambda_\partial - \int_{\mathbb{D}} |\partial \ln g|^2 \, d\lambda - \int_{\partial\mathbb{D}} 4 \ln g \, d\lambda_\partial \Big) \\
&= \frac{1+6Q^2}{96\pi}\Big(\int_{\mathbb{D}} |\partial \varphi|^2 \, d\lambda + 2 \int_{\mathbb{D}} \partial\varphi \cdot \partial \ln g \, d\lambda + \int_{\partial\mathbb{D}} 4\varphi \, d\lambda_\partial \Big).
\end{aligned}$$

Now we use (2.21)+(2.18) to get

$$\ln \frac{\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(g',F)}{\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(g,F)} = \frac{1+6Q^2}{96\pi}\Big(\int_{\mathbb{D}} |\partial\varphi|^2 \, d\lambda + 2 \int_{\mathbb{D}} \varphi R_g \, d\lambda_g + 4 \int_{\partial\mathbb{D}} \varphi (1 + \frac{1}{2}\partial_n \ln g) \, d\lambda_\partial \Big).$$

We complete the proof with (2.19). □

Conformal covariance and KPZ formula

Now we want to establish the conformal covariance of the partition function, i.e. to determine its behavior under the action of Möbius transforms on the marked points. We focus here on the case when the background metric is the Euclidean one: as shown by the Weyl anomaly (Theorem 37), this is not a restriction. One thus looks at

$$\begin{aligned}
&\Pi_{\gamma,\mu_\partial,\mu}^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}(dx^2,F) \qquad (2.42) \\
&= \lim_{\epsilon\to 0} \int_{\mathbb{R}} e^{\left(\sum_i \alpha_i + \frac{1}{2}\sum_j \beta_j - Q\right)c} \mathbb{E}\Big[F(X_{\partial\mathbb{D},\epsilon} + c) \prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i X_{\partial\mathbb{D},\epsilon}(\psi(z_i))} \prod_j \epsilon^{\frac{\beta_j^2}{4}} e^{\frac{\beta_j}{2} X_{\partial\mathbb{D},\epsilon}(\psi(s_j))} \\
&\quad \exp\Big(-\mu e^{\gamma c} \epsilon^{\frac{\gamma^2}{2}} \int_{\mathbb{D}} e^{\gamma X_{\partial\mathbb{D},\epsilon}} \, d\lambda - \mu_\partial e^{\frac{\gamma}{2}c} \epsilon^{\frac{\gamma^2}{4}} \int_{\partial\mathbb{D}} e^{\frac{\gamma}{2} X_{\partial\mathbb{D},\epsilon}} \, d\lambda_\partial \Big) \Big] \, dc.
\end{aligned}$$

where ψ is a Möbius transform of the unit disk.

We use the following convention for the rest of this section. If M is a measure on a measurable space E and $\psi : E \to E$ is a bi-measurable bijection then the measure $M \circ \psi$ is defined by the relation $M \circ \psi(A) = M(\psi(A))$ for all measurable set $A \subset E$.

Théorème 38. *Let ψ be a Möbius transform of the disk. Then*

$$\Pi_{\gamma,\mu_\partial,\mu}^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}(dx^2,1) = \prod_i |\psi'(z_i)|^{-2\Delta_{\alpha_i}} \prod_j |\psi'(s_j)|^{-\Delta_{\beta_j}} \Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(dx^2,1)$$

where the conformal weights $\triangle_\alpha$ are defined by

$$\triangle_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2}).$$

Furthermore the law of the triple (ϕ, Z, Z_∂) under $\mathbb{P}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial,\mu,dx^2}$ is the same as that of the triple $\big(\phi \circ \psi + Q \ln|\psi'|, Z \circ \psi, Z_\partial \circ \psi\big)$ under $\mathbb{P}^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}_{\gamma,\mu_\partial,\mu,dx^2}$.

Proof. To facilitate the comprehension, we take only into consideration the law of the Liouville field and we leave to the reader the details of the whole proof for the triple (ϕ, Z, Z_∂). We first study the behavior of the measure under the Möbius transform ϕ:

Lemme 39. *For any $f \in C^2(\overline{\mathbb{D}})$, we have*

$$\begin{aligned}
&(X_{\partial\mathbb{D}} \circ \psi, \lim_{\epsilon\to 0}\int_{\mathbb{D}} f\epsilon^{\frac{\gamma^2}{2}} e^{\gamma X_{\partial\mathbb{D},\epsilon}} d\lambda, \lim_{\epsilon\to 0}\int_{\mathbb{D}} f\epsilon^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2} X_{\partial\mathbb{D},\epsilon}} d\lambda_\partial)\\
\overset{law}{=}\ &(X_{\partial\mathbb{D}} + m_\partial(X_{\partial\mathbb{D}} \circ \psi), \lim_{\epsilon\to 0}\int_{\mathbb{D}} f \circ \psi e^{\gamma(X_{\partial\mathbb{D},\epsilon} + m_\partial(X_{\partial\mathbb{D}}\circ\psi))}|\psi'|^{Q\gamma} d\lambda,\\
&\lim_{\epsilon\to 0}\int_{\mathbb{D}} f \circ \psi e^{\frac{\gamma}{2}(X_{\partial\mathbb{D},\epsilon} + m_\partial(X_{\partial\mathbb{D}}\circ\psi))}|\psi'|^{\frac{Q\gamma}{2}} d\lambda_\partial)
\end{aligned}$$

Proof of Lemma 39. Using Proposition 31, we have that

$$\lim_{\epsilon\to 0}\mathbb{E}[X_{\partial\mathbb{D},\epsilon}(\psi(x))^2] - \mathbb{E}[(X_{\partial\mathbb{D}} \circ \psi)_{\frac{\epsilon}{|\phi'(x)|}}(x)^2] = 0$$

on $\mathbb{D}$ and on $\partial\mathbb{D}$.
As $|\phi'(x)|$ is always larger than a constant that is strictly positive, we can use the result in [Sha14] to show that the measures

$$(\frac{\epsilon}{|\phi'|})^{\frac{\gamma^2}{2}} e^{\gamma X_{\partial\mathbb{D},\epsilon}\circ\psi} d\lambda$$

and

$$\epsilon^{\frac{\gamma^2}{2}} e^{\gamma(X_{\partial\mathbb{D}}\circ\psi)_\epsilon} d\lambda$$

converge in probability to the same random measure on $\mathbb{D}$.
Similarly,

$$(\frac{\epsilon}{|\phi'|})^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2} X_{\partial\mathbb{D},\epsilon}\circ\psi} d\lambda_\partial$$

and

$$\epsilon^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}(X_{\partial\mathbb{D}}\circ\psi)_\epsilon} d\lambda_\partial$$

converge in probability to the same limit measure on $\partial\mathbb{D}$.
We also have, by change of variables in the integrand

$$\int_{\mathbb{D}} f\,\epsilon^{\frac{\gamma^2}{2}} e^{\gamma X_{\partial\mathbb{D},\epsilon}} d\lambda = \int_{\mathbb{D}} f\circ\psi\,\epsilon^{\frac{\gamma^2}{2}} e^{\gamma X_{\partial\mathbb{D},\epsilon}\circ\psi}|\psi'|^2 d\lambda = \int_{\mathbb{D}} f\circ\psi\,(\frac{\epsilon}{|\psi'|})^{\frac{\gamma^2}{2}} e^{\gamma X_{\partial\mathbb{D},\epsilon}\circ\psi}|\psi'|^{Q\gamma} d\lambda$$

and similarly

$$\int_{\mathbb{D}} f\,\epsilon^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2} X_{\partial\mathbb{D},\epsilon}} d\lambda_\partial = \int_{\mathbb{D}} f\circ\psi\,\epsilon^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2} X_{\partial\mathbb{D},\epsilon}\circ\psi}|\psi'| d\lambda_\partial = \int_{\mathbb{D}} f\circ\psi\,(\frac{\epsilon}{|\psi'|})^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2} X_{\partial\mathbb{D},\epsilon}\circ\psi}|\psi'|^{\frac{Q\gamma}{2}} d\lambda_\partial$$

Combining the above arguments, we conclude the proof by recalling the change of metric formula

$$X_{\partial\mathbb{D}}\circ\psi - m_\partial(X_{\partial\mathbb{D}}\circ\psi) \overset{law}{=} X_{\partial\mathbb{D}}, \tag{2.43}$$

which can be verified using the definition of m_∂ and the Green function. □

Anticipating the formula (2.43), we use the change of variables $\overline{c} = c + m_\partial(X_\partial\circ\psi)$ to write

$$\begin{aligned}
&\Pi^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}_{\gamma,\mu_\partial,\mu}(dx^2, F)\\
&= \lim_{\epsilon\to 0}\int_{\mathbb{R}} e^{\left(\sum_i \alpha_i + \frac{1}{2}\sum_j \beta_j - Q\right)(\overline{c} - m_\partial(X_{\partial\mathbb{D}}\circ\psi))}\\
&\mathbb{E}\Big[F(X_{\partial\mathbb{D},\epsilon} + \overline{c} - m_\partial(X_{\partial\mathbb{D}}\circ\psi))\prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i X_{\partial\mathbb{D},\epsilon}(\psi(z_i))}\prod_j \epsilon^{\frac{\beta_j^2}{4}} e^{\frac{\beta_j}{2} X_{\partial\mathbb{D},\epsilon}(\psi(s_j))}\\
&\exp\Big(-\mu e^{\gamma\overline{c}}\epsilon^{\frac{\gamma^2}{2}}\int_{\mathbb{D}} e^{\gamma(X_{\partial\mathbb{D},\epsilon} - m_\partial(X_{\partial\mathbb{D}}\circ\psi))}\,d\lambda - \mu_\partial e^{\frac{\gamma}{2}\overline{c}}\epsilon^{\frac{\gamma^2}{4}}\int_{\mathbb{D}} e^{\frac{\gamma}{2}(X_{\partial\mathbb{D},\epsilon} - m_\partial(X_{\partial\mathbb{D}}\circ\psi))}\,d\lambda_{\partial g}\Big)\Big]\,dc.
\end{aligned}$$

We now apply the Girsanov transform to the factor $e^{Qm_\partial(X_{\partial\mathbb{D}}\circ\psi)}$. This will shift the law of the field $X_{\partial\mathbb{D}}$, which becomes

$$X_{\partial\mathbb{D}} + \frac{Q}{2\pi}\int_{\partial\mathbb{D}} G(\cdot,\psi(z))\lambda_\partial(dz)$$

We now introduce a useful constant in the following calculation

$$D_\psi = \int_{\partial\mathbb{D}}\int_{\partial\mathbb{D}} G(\psi(y),\psi(z))\lambda_\partial(dy)\lambda_\partial(dz) = 4\pi^2\mathbb{E}[m_\partial(X_{\partial\mathbb{D}}\circ\psi)^2].$$

We also introduce the function

$$H(y) = \int_{\partial\mathbb{D}} G(\psi(y),\psi(z))\lambda_\partial(dz)$$

so that $D_\psi = \int_{\partial\mathbb{D}} H(y)\lambda_\partial(dy)$. Recall that $\int_{\partial\mathbb{D}} G(y,z)\lambda_\partial(dz) = 0$ for all y.

Under the Girsanov transform $X_{\partial\mathbb{D}}(x) - m_\partial(X_{\partial\mathbb{D}} \circ \psi)$ becomes $X_{\partial\mathbb{D}}(x) - m_\partial(X_{\partial\mathbb{D}} \circ \psi) + \frac{Q}{2\pi}H(\psi^{-1}(x)) - \frac{Q}{4\pi^2}D_\psi$ and we get

$$\begin{aligned}
&\Pi_{\gamma,\mu_\partial,\mu}^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}(dx^2, F) \\
&= \lim_{\epsilon\to 0} e^{\frac{Q^2}{8\pi}D_\psi} \int_{\mathbb{R}} e^{\left(\sum_i \alpha_i + \frac{1}{2}\sum_j \beta_j - Q\right)\bar{c}} \\
&\mathbb{E}\Big[F(X_{\partial\mathbb{D},\epsilon} + \bar{c} - m_\partial(X_{\partial\mathbb{D}} \circ \psi) + \frac{Q}{2\pi}H(\psi^{-1}(\cdot)) - \frac{Q}{4\pi^2}D_\psi) \\
&\prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i(X_{\partial\mathbb{D},\epsilon}(\psi(z_i)) - m_\partial(X_{\partial\mathbb{D}}\circ\psi) + \frac{Q}{2\pi}H(z_i) - \frac{Q}{4\pi^2}D_\psi)} \\
&\prod_j \epsilon^{\frac{\beta_j^2}{4}} e^{\frac{\beta_j}{2}(X_{\partial\mathbb{D},\epsilon}(\psi(s_j)) - m_\partial(X_{\partial\mathbb{D}}\circ\psi) + \frac{Q}{2\pi}H(s_j) - \frac{Q}{4\pi^2}D_\psi)} \\
&\exp\Big(-\mu e^{\gamma\bar{c}}\epsilon^{\frac{\gamma^2}{2}} \int_{\mathbb{D}} e^{\gamma(X_{\partial\mathbb{D}}(x) - m_\partial(X_{\partial\mathbb{D}}\circ\psi) + \frac{Q}{2\pi}H(\psi^{-1}(x)) - \frac{Q}{4\pi^2}D_\psi)}\, d\lambda \\
&- \mu_\partial e^{\frac{\gamma}{2}\bar{c}}\epsilon^{\frac{\gamma^2}{4}} \int_{\mathbb{D}} e^{\frac{\gamma}{2}(X_{\partial\mathbb{D}}(x) - m_\partial(X_{\partial\mathbb{D}}\circ\psi) + \frac{Q}{2\pi}H(\psi^{-1}(x)) - \frac{Q}{4\pi^2}D_\psi)}\, d\lambda_{\partial g}\Big)\Big]\, dc.
\end{aligned}$$

Notice the relation (consequence of (2.25))

$$\frac{Q}{2\pi}H(x) = Q\ln\frac{1}{|\psi'(x)|} + \frac{Q}{8\pi^2}D_\psi$$

the D_ψ part with cancel out the first exponential term in the above expression when we do the change of variables $c = \bar{c} - \frac{Q}{8\pi^2}D_\psi$.
Now using (2.43), (32) and Lemma 39, we finally have

$$\begin{aligned}
&\Pi_{\gamma,\mu_\partial,\mu}^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}(dx^2, F) \\
&= \prod_i |\psi'(z_i)|^{\alpha^2/2 - \alpha Q} \prod_j |\psi'(s_j)|^{\beta^2/4 - \beta Q/2} \\
&\lim_{\epsilon\to 0} \int_{\mathbb{R}} e^{\left(\sum_i \alpha_i + \frac{1}{2}\sum_j \beta_j - Q\right)c} \mathbb{E}\Big[F(X_{\partial\mathbb{D},\epsilon}\circ\psi^{-1} - Q\ln|\psi'(\psi^{-1}(\cdot))| + c)\prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i(X_{\partial\mathbb{D},\epsilon}(z_i))} \\
&\prod_j \epsilon^{\frac{\beta_j^2}{4}} e^{\frac{\beta_j}{2}(X_{\partial\mathbb{D},\epsilon}(s_j))} \exp\Big(-\mu e^{\gamma c}\epsilon^{\frac{\gamma^2}{2}} \int_{\mathbb{D}} e^{\gamma X_{\partial\mathbb{D},\epsilon}}\, d\lambda - \mu_\partial e^{\frac{\gamma}{2}c}\epsilon^{\frac{\gamma^2}{4}} \int_{\mathbb{D}} e^{\frac{\gamma}{2}X_{\partial\mathbb{D},\epsilon}}\, d\lambda_{\partial g}\Big)\Big]\, dc.
\end{aligned}$$

This completes the proof of the theorem. □

Conformal changes of domains

In this section, we explain how to construct the LQFT on domains that are conformally equivalent to the unit disk. Basically, the idea is to find a conformal map sending this

domain to the unit disk and to use the conformal covariance property of the LQFT.

Let D be a simply connected (strict) domain of $\mathbb{C}$, say with a C^1 Jordan boundary. From the Riemann mapping theorem, we can consider a conformal map $\psi : D \to \mathbb{D}$. If we further consider marked points (z_i, α_i) in D and boundary marked points $(s_j, \beta_j)_j$ in ∂D, they will be sent respectively to $(\psi(z_i), \alpha_i)$ in $\mathbb{D}$ and to the boundary marked points $(\psi(s_j), \beta_j)_j$ in $\partial\mathbb{D}$. Finally, the uniformization theorem tells us that there is no restriction if we assume that D is equipped with a metric of the type $g_\psi = |\psi'|^2 g(\psi)$ for some metric g on $\mathbb{D}$.

The Liouville partition function on (D, g_ψ) applied to a functional F reads

$$\begin{aligned}
&\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(D, g_\psi, F) \qquad (2.44)\\
=&e^{\frac{1}{96\pi}\left(\int_{\mathbb{D}} |\partial \ln g|^2 \, d\lambda + \int_{\partial\mathbb{D}} 4 \ln g \, d\lambda_\partial\right)}\\
&\lim_{\epsilon\to 0} \int_{\mathbb{R}} \mathbb{E}\Big[F(X_\nu + c + Q/2 \ln g_\psi) \prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i (c + X_{\nu,\epsilon} + Q/2 \ln g_\psi)(z_i)} \prod_j \epsilon^{\frac{\beta_j^2}{4}} e^{\frac{\beta_j}{2}(c + X_{\nu,\epsilon} + Q/2 \ln g_\psi)(s_j)}\\
&\exp\Big(-\frac{Q}{4\pi}\int_D R_{g_\psi}(c + X_\nu)\, d\lambda_{g_\psi} - \mu e^{\gamma c} \epsilon^{\frac{\gamma^2}{2}} \int_D e^{\gamma(X_{\nu,\epsilon} + Q/2 \ln g_\psi)}\, d\lambda\Big)\\
&\exp\Big(-\frac{Q}{2\pi}\int_{\partial D} K_{\partial g_\psi}(c + X_\nu)\, d\lambda_{\partial g_\psi} - \mu_\partial\, e^{\frac{\gamma}{2}c} \epsilon^{\frac{\gamma^2}{4}} \int_{\partial D} e^{\frac{\gamma}{2}(X_{\nu,\epsilon} + Q/2 \ln g_\psi)}\, d\lambda_\partial\Big)\Big]\, dc,
\end{aligned}$$

where X_ν is a GFF on D with Neumann boundary condition and vanishing ν-mean. By shift invariance of the Lebesgue measure, the choice of ν is irrelevant and it will be convenient to take $X_\nu = X_{\partial\mathbb{D}} \circ \psi$, which is free boundary GFF with vanishing mean for the line element on ∂D in the metric $|\psi'|^2 dx^2$ on D.

Proposition 40. *Let D be a simply connected (strict) domain of $\mathbb{C}$ with a C^1 Jordan boundary. Then we have the relation*

$$\begin{aligned}
&\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(D, g_\psi, F(\phi, Z, Z_\partial))\\
=&\prod_i |\psi'(z_i)|^{2\triangle_{\alpha_i}} \prod_j |\psi'(s_j)|^{\triangle_{\beta_j}} \Pi_{\gamma,\mu_\partial,\mu}^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}(\mathbb{D}, g, F(\phi \circ \psi + Q \ln|\psi'|, Z \circ \psi, Z_\partial \circ \psi)).
\end{aligned}$$

In particular:

1. *we have the following relation between the partition functions ($F = 1$)*

$$\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(D, g_\psi, 1) = \prod_i |\psi'(z_i)|^{2\triangle_{\alpha_i}} \prod_j |\psi'(s_j)|^{\triangle_{\beta_j}} \Pi_{\gamma,\mu_\partial,\mu}^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}(\mathbb{D}, g, 1).$$

2. *The law of the triple (ϕ, Z, Z_∂) under $\mathbb{P}_{\gamma,\mu_\partial,\mu,(D,g_\psi)}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}$ is the same as $(\phi \circ \psi + Q \ln|\psi'|, Z, Z_\partial)$ under $\mathbb{P}_{\gamma,\mu_\partial,\mu,(\mathbb{D},g)}^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}$.*

Proof. Again we only treat a functional F depending only on the Liouville field for simplicity. Applying Lemma 39 and using that $R_{g_\psi}(x) = R_g(\psi(x))$ and $K_{g_\psi}(x) = K_g(\psi(x))$ (because ψ is a conformal map), (2.44) is equal to

$$\begin{aligned}
&\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(D, g_\psi, F) \qquad (2.45)\\
=&e^{\frac{1}{96\pi}\left(\int_{\mathbb{D}} |\partial \ln g|^2\, d\lambda + \int_{\partial\mathbb{D}} 4\ln g\, d\lambda_\partial\right)} \prod_i |\psi'(z_i)|^{Q\alpha_i - \frac{\alpha_i^2}{2}} \prod_j |\psi'(s_j)|^{Q\frac{\beta_j}{2} - \frac{\beta_j^2}{4}}\\
&\lim_{\epsilon\to 0} \int_{\mathbb{R}} dc\, \mathbb{E}\Big[F((X_{\partial\mathbb{D}} + c + Q/2\ln g)\circ\psi + Q\ln|\psi'|)\\
&\prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i(c + X_{\partial\mathbb{D},\epsilon} + Q/2\ln g)(\psi(z_i))} \prod_j \epsilon^{\frac{\beta_j^2}{4}} e^{\frac{\beta_j}{2}(c + X_{\partial\mathbb{D},\epsilon} + Q/2\ln g)(\psi(s_j))}\\
&\exp\Big(-\frac{Q}{4\pi}\int_D R_g(\psi)(c + X_{\partial\mathbb{D}}\circ\psi) g(\psi)|\psi'|^2\, d\lambda - \mu e^{\gamma c}\epsilon^{\frac{\gamma^2}{2}} \int_{\mathbb{D}} e^{\gamma(X_{\partial\mathbb{D},\epsilon} + Q/2\ln g)}\, d\lambda\Big)\\
&\exp\Big(-\frac{Q}{2\pi}\int_{\partial D} K_g(\psi)(c + X_{\partial\mathbb{D}}\circ\psi)|\psi'| g^{1/2}(\psi)\, d\lambda_\partial - \mu_\partial e^{\frac{\gamma}{2}c}\epsilon^{\frac{\gamma^2}{4}} \int_{\partial\mathbb{D}} e^{\frac{\gamma}{2}(X_{\partial\mathbb{D},\epsilon} + Q/2\ln g)}\, d\lambda_\partial\Big)\Big]\\
=&\prod_i |\psi'(z_i)|^{2\triangle_{\alpha_i}} \prod_j |\psi'(s_j)|^{\triangle_{\beta_j}} \Pi_{\gamma,\mu_\partial,\mu}^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}(\mathbb{D}, g, F(\phi\circ\psi + Q\ln|\psi'|)).
\end{aligned}$$

This completes the proof. □

Law of the volume of space/boundary

We want to express here the (joint) law of the volume of bulk/boundary on the unit disk equipped with the Euclidean metric. It will be convenient to express this law in terms of the couple of random measures (Z_0, Z_0^∂) under $\mathbb{P}$ respectively defined on $\mathbb{D}$ and $\partial\mathbb{D}$ by (recall Proposition 32)

$$Z_0 = e^{\gamma H} e^{\gamma X_{\partial\mathbb{D}}}\, d\lambda, \quad Z_0^\partial = e^{\frac{\gamma}{2}H} e^{\frac{\gamma}{2}X_{\partial\mathbb{D}}}\, d\lambda_\partial \qquad (2.46)$$

with

$$H(x) = \sum_i \alpha_i G(x, z_i) + \sum_j \frac{\beta_j}{2} G(x, s_j). \qquad (2.47)$$

We further introduce the ratio

$$R = \frac{Z_0(\mathbb{D})}{Z_0^\partial(\partial\mathbb{D})^2}.$$

By definition of the law of the bulk/boundary Liouville measures, we have

$$\begin{aligned}
&\mathbb{E}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial,\mu,dx^2}\left[F(Z,Z_\partial)\right]\\
=&(\Pi^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial,\mu}(dx^2,1))^{-1}\int_{\mathbb{R}} e^{\left(\sum_i \alpha_i+\frac{1}{2}\sum_j \beta_j-Q\right)c}\mathbb{E}\Big[F(e^{\gamma c}Z_0,e^{\frac{\gamma}{2}c}Z_0^\partial)\\
&\exp\Big(-\mu e^{\gamma c}Z_0(\mathbb{D})-\mu_\partial e^{\frac{\gamma}{2}c}Z_0^\partial(\partial\mathbb{D})\Big)\Big]\,dc\\
=&\frac{2}{\gamma}(\Pi^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial,\mu}(dx^2,1))^{-1}\int_0^\infty y^{\frac{2}{\gamma}(\sum_i \alpha_i+\frac{1}{2}\sum_j \beta_j-Q)-1}\\
&\mathbb{E}\Big[F\Big(y^2R\frac{Z_0}{Z_0(\mathbb{D})},y\frac{Z_0^\partial}{Z_0^\partial(\partial\mathbb{D})}\Big)\exp\Big(-\mu y^2R-\mu_\partial y\Big)Z_0^\partial(\partial\mathbb{D})^{-\frac{2}{\gamma}(\sum_i \alpha_i+\frac{1}{2}\sum_j \beta_j-Q)}\Big]\,dy.
\end{aligned}$$

This is the general formula. It may be useful to state as a particular example the case $\mu_\partial = 0$ as it often arises in the study of random planar maps with a boundary.

Corollaire 41. *Assume $\mu_\partial = 0$ and $\mu > 0$. The joint law of the bulk/boundary Liouville measures is given by*

$$\begin{aligned}
&\mathbb{E}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial=0,\mu,dx^2}\left[F(Z,Z_\partial)\right]\\
=&\ \mathcal{Z}^{-1}\int_{\mathbb{R}} u^{\frac{1}{\gamma}\left(\sum_i \alpha_i+\frac{1}{2}\sum_j \beta_j-Q\right)-1}\mathbb{E}\Big[F\Big(u\frac{Z_0}{Z_0(\mathbb{D})},u^{\frac{1}{2}}\frac{Z_0^\partial}{Z_0(\mathbb{D})^{\frac{1}{2}}}\Big)Z_0(\mathbb{D})^{-\frac{1}{\gamma}(\sum_i \alpha_i+\frac{1}{2}\sum_j \beta_j-Q)}\Big]e^{-\mu u}\,du.
\end{aligned}$$

where $\mathcal{Z}$ is a renormalization constant to get a probability measure. In particular, the law of the volume of space follows a Gamma law with parameters

$$\Big(\frac{\sum_i \alpha_i+\frac{1}{2}\sum_j \beta_j-Q}{\gamma},\mu\Big)$$

and the random variable $Z(\mathbb{D})$ is independent of the random measures

$$\Big(\frac{Z}{Z(\mathbb{D})},\frac{Z_\partial}{Z(\mathbb{D})^{\frac{1}{2}}}\Big).$$

If we further condition the total bulk measure to be 1, the unit volume Liouville measure on the disk as in Corollary 41 can be defined when the following three conditions are satisfied

$$\forall i,\quad \alpha_i < Q, \tag{2.48}$$

$$\forall j,\quad \beta_j < Q, \tag{2.49}$$

$$Q-\sum_i \alpha_i-\frac{1}{2}\sum_j \beta_j < \frac{2}{\gamma}\wedge 2\min_i(Q-\alpha_i)\wedge \min_j(Q-\beta_j), \tag{2.50}$$

see Corollary 60 for a precise statement. This shows that the Seiberg bounds (2.32)+(2.33)+(2.34) can be relaxed when conditioning on finite total volume.

Remarque 42. We can also look at what happens when we set $\mu = 0$ and condition the total boundary Liouville length measure to be 1. In this case, one can treat bulk insertions and boundary insertions seperately as in [DKRV16, Section 3.4] to obtain similar relaxed Seiberg bounds without additional technical difficulties. For completeness we state the result in the following corollary.

Corollaire 43. *Assume $\mu = 0$ and $\mu_\partial > 0$. The joint law of the bulk/boundary Liouville measures is given by*

$$\begin{aligned}
&\mathbb{E}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial,\mu=0,dx^2}\left[F(Z,Z_\partial)\right] \\
&= \mathcal{Z}^{-1}\int_{\mathbb{R}} y^{\frac{2}{\gamma}\left(\sum_i \alpha_i+\frac{1}{2}\sum_j \beta_j-Q\right)-1} \\
&\quad \mathbb{E}\Big[F\Big(y^2\frac{Z_0}{Z_0^\partial(\partial\mathbb{D})^2}, y\frac{Z_0^\partial}{Z_0^\partial(\partial\mathbb{D})}\Big) Z_0^\partial(\partial\mathbb{D})^{-\frac{2}{\gamma}(\sum_i \alpha_i+\frac{1}{2}\sum_j \beta_j-Q)}\Big] e^{-\mu u}\, dy.
\end{aligned}$$

where $\mathcal{Z}$ is a renormalization constant to get a probability measure. In particular, the law of the total length of the boundary follows a Gamma law with parameters

$$\big(2\gamma^{-1}\big(\sum_i \alpha_i + \frac{1}{2}\sum_j \beta_j - Q\big), \mu\big)$$

and the random variable $Z_\partial(\partial\mathbb{D})$ is independent of the random measures

$$\big(\frac{Z}{Z_\partial(\partial\mathbb{D})^2}, \frac{Z_\partial}{Z_\partial(\partial\mathbb{D})}\big).$$

The unit boundary length Liouville measure on the disk can be defined under the following conditions:

$$\forall j, \quad \beta_j < Q, \tag{2.51}$$

$$Q - \sum_i \alpha_i - \frac{1}{2}\sum_j \beta_j < \frac{2}{\gamma} \wedge \min_j (Q - \beta_j), \tag{2.52}$$

Remarque 44. Since the geometrical KPZ formula established in [RV11] has been established almost surely with respect to the GFF expectation, it holds for the Liouville measure in our context almost surely too.

Liouville QFT at $\gamma = 2$

Here we explain how to construct LQFT on the unit disk in the important case $\gamma = 2$. The reason why this case is so specific is that it is no more superrenormalizable at small

scales. In other words the interaction terms $e^{2X_{\partial\mathbb{D}}}d\lambda$ or $e^{X_{\partial\mathbb{D}}}d\lambda_\partial$ can no more be obtained as a Wick ordering, i.e. a subcritical Gaussian multiplicative chaos: it corresponds to the phase transition in Gaussian multiplicative chaos theory. Indeed, the standard renormalizations

$$\epsilon^2\, e^{2X_{\partial\mathbb{D},\epsilon}}d\lambda \quad \text{and} \quad \epsilon\, e^{X_{\partial\mathbb{D},\epsilon}}d\lambda_\partial$$

yield vanishing limiting measures. To get a non trivial limit, an extra push $\sqrt{\ln\frac{1}{\epsilon}}$ is necessary, which is called the Seneta-Heyde norming. For Gaussian multiplicative chaos, this has been investigated in [DRSV14a] for a white noise decomposition of the GFF, which does not exactly correspond to our framework as we work with convolution cutoff approximations. So, we explain in this section how to generalize the results in [DRSV14a] to convolutions.

We first claim

Théorème 45. *The family of boundary approximation measures on* $\partial\mathbb{D}$

$$\sqrt{\ln\frac{1}{\epsilon}}\,\epsilon\, e^{X_{\partial\mathbb{D},\epsilon}}d\lambda_\partial$$

converges in probability as ϵ *goes to* 0 *towards a non trivial limiting measure, which has moments of order* q *for all* $q < 1$.

Théorème 46. *The family of bulk approximation measures on* $\mathbb{D}$

$$\sqrt{\ln\frac{1}{\epsilon}}\,\epsilon^2\, e^{2X_{\partial\mathbb{D},\epsilon}}d\lambda$$

converges in probability as ϵ *goes to* 0 *towards a non trivial limiting measure, which has moments of order* q *for all* $q < 1$.

Remarque 47. Actually, our proof for the two above theorems establishes convergence in probability for a large class of cutoff approximations with mollifying family, not only the circle average family.

Proof of Theorem 45. The strategy is the following: first we show the convergence in probability of a specific family of white noise cutoff approximations. Then we will show that this entails the convergence in probability for a whole class of convolution approximations, including circle averages.

Recall that if we consider a centered Gaussian distribution X on the boundary of the unit disk with the following covariance structure

$$\mathbb{E}[X(e^{i\theta})X(e^{i\theta})] = 2\ln\frac{1}{|e^{i\theta}-e^{i\theta'}|},$$

then the law on the boundary of the GFF $X_{\partial\mathbb{D}}$ is given by

$$X_{\partial\mathbb{D}} = X - \frac{1}{2\pi}\int_0^{2\pi} X(e^{i\theta})d\theta.$$

Our first step is to construct $X_{\partial\mathbb{D}}$ as a function of some white noise W and of a smooth Gaussian process Y. This decomposition will be convenient to establish convergence in probability of the approximating measures based on martingale techniques. We will recover the situation of approximations based on convolution of $X_{\partial\mathbb{D}}$ after that.

Recall the following decomposition (see [RV10])

$$\forall x \in \mathbb{R}^2, \quad \ln_+ \frac{1}{|x|} = 2\int_0^1 (t - |x|^{\frac{1}{2}})_+ \frac{dt}{t^2} + 2(1 - |x|^{\frac{1}{2}})_+ .$$

Now we construct two Gaussian distributions: the first one $\bar{X}$ will have the covariance structure of the first term in the above right-hand side and the second one Y the second term.

Lemme 48. *There exists a white noise W on $[1, +\infty[\times\partial\mathbb{D}$ and a family of centered Gaussian processes $(\bar{X}_\epsilon)_{\epsilon\in]0,1]}$ on $\partial\mathbb{D}$, which are measurable functions of this white noise, such that*

$$\forall 0 < \epsilon < \epsilon' \leq 1, \quad \bar{X}_\epsilon - \bar{X}_{\epsilon'} \text{ is independent of } \sigma\{X_u(e^{i\theta}); \epsilon' \leq u \leq 1, \theta \in [0, 2\pi]\} \tag{2.53}$$

and

$$\mathbb{E}[\bar{X}_\epsilon(e^{i\theta})\bar{X}_\epsilon(e^{i\theta'})] = 2\int_{\sqrt{\epsilon}}^1 (t - |e^{i\theta} - e^{i\theta'}|^{\frac{1}{2}})_+ \frac{dt}{t^2} = \int_1^{\frac{1}{\epsilon}} (1 - |v(e^{i\theta} - e^{i\theta'})|^{\frac{1}{2}})_+ \frac{dv}{v}. \tag{2.54}$$

The limiting distribution $\bar{X} = \lim_{\epsilon\to 0} \bar{X}_\epsilon$ is a centered Gaussian distribution with covariance structure

$$\mathbb{E}[\bar{X}(e^{i\theta})\bar{X}(e^{i\theta'})] = 2\int_0^1 (t - |e^{i\theta} - e^{i\theta'}|^{\frac{1}{2}})_+ \frac{dt}{t^2}.$$

Finally, for any smooth function R on $[1, +\infty[\times\partial\mathbb{D}$ with compact support, the function

$$z \in \partial\mathbb{D} \mapsto T_\epsilon(R)(z) := \mathbb{E}[\bar{X}_\epsilon(z)W(R)]$$

is a continuous function which converges uniformly as $\epsilon \to 0$ towards

$$z \in \partial\mathbb{D} \mapsto T(R)(z) := \mathbb{E}[\bar{X}(z)W(R)].$$

This lemma is proved in Appendix 2. Then we consider a centered Gaussian field Y independent of $(\bar{X}_\epsilon)_{\epsilon\in]0,1]}$ with covariance given by

$$\mathbb{E}[Y(e^{i\theta})Y(e^{i\theta'})] = (1 - |e^{i\theta} - e^{i\theta'}|^{\frac{1}{2}})_+ .$$

Recall that such a kernel is indeed positive definite [Gne01].

Now we can set

$$X_{\partial\mathbb{D}} = \bar{X} + Y - \frac{1}{2\pi}\int_0^{2\pi}(\bar{X}(e^{i\theta}) + Y(e^{i\theta}))d\theta.$$

This is a construction of $X_{\partial\mathbb{D}}$ as a function of (W, Y). Now we would like to use [DRSV14a] to show that the random measures

$$\sqrt{\ln\frac{1}{\epsilon}}\,\epsilon\, e^{\bar{X}_\epsilon} d\lambda_\partial \tag{2.55}$$

converges in probability to a non trivial limiting random measure as $\epsilon \to 0$. To this purpose, observe that the covariance $(k_\epsilon)_{\epsilon\in]0,1]}$ kernels of the family $(\bar{X}_\epsilon)_{\epsilon\in]0,1]}$ can be written as

$$k_\epsilon(e^{i\theta}, e^{i\theta'}) = \int_1^{\frac{1}{\epsilon}} \frac{k(v, e^{i\theta}, e^{i\theta'})}{v}\, dv \quad \text{with} \quad k(v, e^{i\theta}, e^{i\theta'}) = (1 - |v(e^{i\theta} - e^{i\theta'})|^{\frac{1}{2}})_+.$$

Such a kernel k satisfies the properties

A.1 k is nonnegative, continuous.

A.2 k is Hölder on the diagonal, more precisely $\forall\theta, \theta', \forall v \geq 1$,

$$|k(v, e^{i\theta}, e^{i\theta}) - k(v, e^{i\theta}, e^{i\theta'})| \leq v^{1/2}|e^{i\theta} - e^{i\theta'}|^{1/2}$$

A.3 k satisfies the integrability condition

$$\sup_{\theta,\theta'} \int_{\frac{1}{|e^{i\theta}-e^{i\theta'}|}}^{\infty} \frac{k(v, e^{i\theta}, e^{i\theta'})}{v}\, dv < +\infty.$$

A.4 for all $\epsilon \in]0, 1]$, $\int_1^{\frac{1}{\epsilon}} \frac{k(v,e^{i\theta},e^{i\theta})}{v}\, dv = \ln\frac{1}{\epsilon}$,

A.5 $k(v, e^{i\theta}, e^{i\theta'}) = 0$ for $|e^{i\theta} - e^{i\theta'}| \geq v^{-1}$.

Observe in particular that [A.2] implies that

$$|\ln\frac{1}{\epsilon} - k_\epsilon(e^{i\theta}, e^{i\theta'})| \leq \int_1^{1/\epsilon} \frac{|e^{i\theta} - e^{i\theta'}|^{1/2}}{v^{1/2}} dv \leq C(|e^{i\theta} - e^{i\theta'}|/\epsilon)^{1/2}$$

for some constant C (independent of ϵ). In particular we have the property

$$|e^{i\theta} - e^{i\theta'}| \leq \epsilon \quad \Rightarrow \quad |\ln\frac{1}{\epsilon} - k_\epsilon(e^{i\theta}, e^{i\theta'})| \leq C. \tag{2.56}$$

These properties are the only assumptions used in [RV15] to construct the derivative martingale and in [DRSV14a] to prove the Seneta-Heyde norming. Therefore the family of random measures (2.55) converges in probability towards a non trivial random measures, which has moments of order q for all $q < 1$ (see [DRSV14a]).

Hence, if $X_\epsilon = \bar{X}_\epsilon + Y - \frac{1}{2\pi}\int_0^{2\pi}(\bar{X}_\epsilon(e^{i\theta}) + Y(e^{i\theta}))d\theta$, then

$$M_\epsilon = \sqrt{\ln\frac{1}{\epsilon}}\,\epsilon\, e^{X_\epsilon(e^{i\theta})}d\lambda_\partial$$

converges in probability to a random measure M' which is a measurable function of the white noise W and the process Y, call it $F(W, Y)$.

Now, we show convergence in probability of $\sqrt{\ln\frac{1}{\epsilon}}\,\epsilon\, e^{X_{\partial\mathbb{D},\epsilon}}d\lambda_\partial$, where $X_{\partial\mathbb{D},\epsilon}$ is the circle average approximation of $X_{\partial\mathbb{D}}$ towards the same limit M'. The ideas in the following stem from the techniques developed in [RV14] along with some variant of Lemma 49 in [Sha14] (we will not recall Lemma 49 as our proof will be self contained).

For this, we introduce $X^1_{\partial\mathbb{D}}$, an independent copy of $X_{\partial\mathbb{D}}$, and $X^1_{\partial\mathbb{D},\epsilon}$ its circle average approximation. Let us define for $t \in [0,1]$ and $\theta \in [0, 2\pi]$

$$Z_\epsilon(t, e^{i\theta}) = \sqrt{t}X^1_{\partial\mathbb{D},\epsilon}(e^{i\theta}) + \sqrt{1-t}X_\epsilon(e^{i\theta}).$$

Now, we set

$$M^1_\epsilon = \sqrt{\ln\frac{1}{\epsilon}}\,\epsilon\, e^{X^1_{\partial\mathbb{D},\epsilon}(e^{i\theta})}d\lambda_\partial.$$

We first show that M^1_ϵ converges in distribution to $M' = F(W, Y)$. From [RV10, Proof of Theorem 2.1], one gets that for all $\alpha < 1$

$$\begin{aligned}
&\overline{\lim_{\epsilon\to 0}}\left|\mathbb{E}[M^1_\epsilon(B)^\alpha] - \mathbb{E}[M_\epsilon(B)^\alpha]\right| \\
&\leq c\frac{\alpha(1-\alpha)}{2}C_A\overline{\lim_{\epsilon\to 0}}\int_0^1 \mathbb{E}\Big[\Big(\sqrt{\ln\frac{1}{\epsilon}}\int_{\partial\mathbb{D}} e^{Z_\epsilon(t,\cdot)-\frac{1}{2}\mathbb{E}[Z_\epsilon(t,\cdot)^2]}\,d\lambda_\partial\Big)^\alpha\Big]\,dt \\
&+ c\,\overline{C}_A\overline{\lim_{\epsilon\to 0}}\int_0^1 \mathbb{E}\Big[\Big(\sup_{0\leq i<\frac{1}{A\epsilon}}\sqrt{\ln\frac{1}{\epsilon}}\int_{2iA\epsilon}^{2(i+1)A\epsilon} e^{Z_\epsilon(t,e^{i\theta})-\frac{1}{2}\mathbb{E}[Z_\epsilon(t,e^{i\theta})^2]}d\theta\Big)^\alpha\Big]dt,
\end{aligned}$$

where

$$C_A = \overline{\lim_{\epsilon\to 0}}\sup_{|e^{i\theta}-e^{i\theta'}|\geq A\epsilon}|\mathbb{E}[X^1_{\partial\mathbb{D},\epsilon}(e^{i\theta})X^1_{\partial\mathbb{D},\epsilon}(e^{i\theta'})] - \mathbb{E}[X_\epsilon(e^{i\theta})X_\epsilon(e^{i\theta'})]|$$

and

$$\overline{C}_A = \overline{\lim_{\epsilon\to 0}}\sup_{|e^{i\theta}-e^{i\theta'}|\leq A\epsilon}|\mathbb{E}[X^1_{\partial\mathbb{D},\epsilon}(e^{i\theta})X^1_{\partial\mathbb{D},\epsilon}(e^{i\theta'})] - \mathbb{E}[X_\epsilon(e^{i\theta})X_\epsilon(e^{i\theta'})]|.$$

The reader can check that $\overline{C}_A$ is bounded independently of A and $\lim_{A\to\infty} C_A = 0$. Since $\mathbb{E}\big[\Big(\sqrt{\ln\frac{1}{\epsilon}}\int_0^1 e^{Z_\epsilon(t,u)-\frac{1}{2}\mathbb{E}[Z_\epsilon(t,u)^2]}\,du\Big)^\alpha\big]$ is also bounded independently of everything (by comparison with Mandelbrot's multiplicative cascades as explained in the [DRSV14b, Appendix] and [DRSV14a, Appendix B.4]), we are done if we can show that for all $t \in [0,1]$

$$\overline{\lim_{\epsilon\to 0}}\ \mathbb{E}\Big[\Big(\sup_{0\le i<\frac{1}{A\epsilon}}\sqrt{\ln\frac{1}{\epsilon}}\int_{2iA\epsilon}^{2(i+1)A\epsilon} e^{Z_\epsilon(t,e^{i\theta})-\frac{1}{2}\mathbb{E}[Z_\epsilon(t,e^{i\theta})^2]}d\theta\Big)^\alpha\Big] = 0. \tag{2.57}$$

Notice that this quantity is less than

$$\big(\ln\frac{1}{\epsilon}\big)^{\alpha/2}\epsilon^\alpha\mathbb{E}\Big[\Big(e^{\sup_{\theta\in[0,2\pi]} Z_\epsilon(t,e^{i\theta})-\frac{1}{2}\mathbb{E}[Z_\epsilon(t,e^{i\theta})^2]}\Big)^\alpha\Big]. \tag{2.58}$$

To estimate this quantity, we use the main result of [Aco14]: more precisely, setting

$$m_\epsilon = 2\ln\frac{1}{\epsilon} - \frac{3}{2}\ln\ln\frac{1}{\epsilon},$$

we claim that there exist two constants $C, c > 0$ such that for ϵ small enough

$$\forall x \ge 0,\quad \mathbb{P}\Big(\Big|\max_{\theta\in[0,2\pi]} Z_\epsilon(t,e^{i\theta}) - m_\epsilon\Big| \ge x\Big) \le Ce^{-cx}.$$

In particular we get that for $\alpha < c$

$$\sup_\epsilon \mathbb{E}\Big[\Big(e^{\sup_{\theta\in[0,2\pi]} Z_\epsilon(t,e^{i\theta})}\Big)^\alpha\Big] < \infty.$$

Plugging this estimate into (2.58), we see that the quantity (2.58) is less than

$$C'\big(\ln\frac{1}{\epsilon}\big)^{\alpha/2}\epsilon^{2\alpha}e^{\alpha m_\epsilon} = C'\big(\ln\frac{1}{\epsilon}\big)^{-\alpha}.$$

for some constant $C' > 0$. This proves the claim (2.57), hence the convergence in law of the random measure M^1_ϵ towards $M' = F(W,Y)$.

Now we deduce that the family $(W, Y, M^1_\epsilon)_\epsilon$ converges in law. Take any smooth function R on $[1,+\infty[\times\partial\mathbb{D}$ with compact support, any continuous function g on $\partial\mathbb{D}$, any bounded continuous function G on $\mathbb{R}$ and $u \in \mathbb{R}$. We have by using the Girsanov transform

$$\mathbb{E}[e^{W(R)+uY}G(M^1_\epsilon(g))] = e^{\frac{1}{2}\mathrm{Var}[W(R)+uY]}\mathbb{E}[G(M^1_\epsilon(e^{T_\epsilon(R)}g)]$$

where $T_\epsilon(R)$ is defined in Lemma 48. The quantity in the right-hand side converges as $\epsilon \to 0$ towards

$$e^{\frac{1}{2}\mathrm{Var}[W(R)+uY]}\mathbb{E}[G(M'(e^{T(R)}g)] = \mathbb{E}[e^{W(R)+uY}G(M'(g))].$$

Hence our claim about the convergence in law of the triple $(W, Y, M^1_\epsilon)_\epsilon$ towards $(W, Y, M' = F(W, Y))$ is proved.

Now we consider the family $(W, Y, M^1_\epsilon, F(W, Y))_\epsilon$, which is tight. Even if it means extracting a subsequence, it converges in law towards some $(\mathcal{W}, \mathcal{Y}, \mathcal{M}, \bar{\mathcal{M}})$. We have just shown that the law of $(\mathcal{W}, \mathcal{Y}, \mathcal{M})$ is that of $(\mathcal{W}, \mathcal{Y}, F(\mathcal{W}, \mathcal{Y}))$, i.e. the same as the law of $(\mathcal{W}, \mathcal{Y}, \bar{\mathcal{M}})$. Hence $\mathcal{M} = \bar{\mathcal{M}}$ almost surely. Therefore $M^1_\epsilon - F(W, Y)$ converges in law towards 0, hence in probability. Since the convergence in probability of the family $(M^1_\epsilon)_\epsilon$ implies the convergence of probability of every family $(\widehat{M}_\epsilon)_\epsilon$ that has the same law as $(M^1_\epsilon)_\epsilon$, the proof of Theorem 45 is complete.

Finally, one can notice that instead of $X_{\partial\mathbb{D},\epsilon}$ we could have considered any smooth convolution approximation of X. □

Proof of Theorem 46. Let us consider the Poisson kernel on the unit disk

$$\forall 0 \leq r < 1, \forall \theta \in [0, 2\pi], \quad P_r(\theta) = \sum_{n \in \mathbb{Z}} r^{|n|} e^{in\theta}.$$

We can then consider the harmonic extension inside the unit disk of the trace of the GFF $X_{\partial\mathbb{D}}$ along the boundary

$$P_X(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t) X_{\partial\mathbb{D}}(e^{it})\, dt.$$

It is plain to see that P_X is a continuous Gaussian process inside the unit disk. If we set

$$X^{\mathrm{Dir}} = X_{\partial\mathbb{D}} - P_X,$$

one can check that we get a GFF with Dirichlet boundary condition in the unit disk. Therefore, by continuity of P_X inside $\mathbb{D}$, the convergence in probability of the random measures $(\epsilon^2\, e^{2X_{\partial\mathbb{D},\epsilon}(x)} d\lambda)_\epsilon$ boils down to showing the convergence in probability for the random measures

$$(\epsilon^2\, e^{2X^{\mathrm{Dir}}_\epsilon(x)} d\lambda)_\epsilon$$

where $(X^{\mathrm{Dir}}_\epsilon)_\epsilon$ stands for the circle average approximations of the GFF X^{Dir}. Given the fact that the Seneta-Heyde norming has been proved in [DRSV14a] for a white noise decomposition of X^{Dir}, we can use the same argument as in the proof of Theorem 45 to show that convergence for the white noise approximation family entails the convergence in probability for the circle average approximations. □

From now on, the construction of the Liouville LQG on the unit disk for $\gamma = 2$ follows

the same lines as for $\gamma < 2$ by taking the limit as $\epsilon \to 0$ of the quantity

$$\Pi_{2,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(g,F) \tag{2.59}$$

$$= e^{\frac{1}{96\pi}\left(\int_{\mathbb{D}} |\partial \ln g|^2 \, d\lambda + \int_{\partial\mathbb{D}} 4 \ln g \, d\lambda_\partial\right)} \lim_{\epsilon\to 0} \int_{\mathbb{R}} \mathbb{E}\Big[F(X_{\partial\mathbb{D}} + c + \ln g) \prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i (c + X_{\partial\mathbb{D},\epsilon} + \ln g)(z_i)}$$

$$\prod_j \epsilon^{\frac{\beta_j^2}{4}} e^{\frac{\beta_j}{2}(c + X_{\partial\mathbb{D},\epsilon} + \ln g)(s_j)} \exp\Big(-\frac{2}{4\pi}\int_{\mathbb{D}} R_g(c + X_{\partial\mathbb{D}}) \, d\lambda_g - \mu e^{2c}\sqrt{-\ln\epsilon}\,\epsilon^2 \int_{\mathbb{D}} e^{2(X_{\partial\mathbb{D},\epsilon} + \ln g)} \, d\lambda\Big)$$

$$\exp\Big(-\frac{2}{2\pi}\int_{\partial\mathbb{D}} K_g(c + X_{\partial\mathbb{D}}) \, d\lambda_{\partial g} - \mu_\partial e^{c}\sqrt{-\ln\epsilon}\,\epsilon \int_{\partial\mathbb{D}} e^{(X_{\partial\mathbb{D},\epsilon} + \ln g)} \, d\lambda_\partial\Big)\Big] \, dc.$$

defined for all continuous and bounded functional F on $H^{-1}(\mathbb{D})$. From now on, the properties of LQG (and their proofs) on the disk for $\gamma = 2$ are the same as for $\gamma < 2$ except Proposition 33, which needs some extra care that we treat now.

Proposition 49. *The quantities below are almost surely finite*

$$\int_{\mathbb{D}} e^{2X_{\partial\mathbb{D}}} d\lambda \quad \textit{and} \quad \int_{\partial\mathbb{D}} e^{X_{\partial\mathbb{D}}} d\lambda_{\partial\mathbb{D}}.$$

Proof. Recall the sub-additivity inequality for $\alpha \in]0,1[$: if $(a_j)_{1\le j\le n}$ are positive real numbers then

$$(a_1 + \cdots + a_n)^\alpha \le a_1^\alpha + \cdots + a_n^\alpha.$$

Now we use Kahane's convexity inequality [RV14, Theorem 2.1] to compare the Gaussian multiplicative chaos with standard dyadic lognormal cascade (once again we refer to [DRSV14b, Appendix B.1] for full details). We consider the dyadic tree with i.i.d. weights with Gaussian law $\mathcal{N}(0, \ln 2)$ on the edges of the tree and denote by Y_j^n the sum of these weights starting from the root up to the dyadic indexed by j at generation n. We denote by $(Z_j)_j$ an i.i.d sequence (independent of everything) standing for the mass of the dyadic cascade at criticality rooted at the dyadic j at generation n. From [Mad17] these random variables have distribution tail $\mathbb{P}(Z_j > x) \le \frac{C}{x}$ for some constant $C > 0$, and $\mathbb{E}[Z_j^q] < \infty$ for $q < 1$. Hence we get

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{\mathbb{D}} e^{2X_{\partial\mathbb{D}}(x) - 2\mathbb{E}[X_{\partial\mathbb{D}}(x)^2]} \frac{1}{(1-|x|^2)^2})\lambda(dx)\Big)^\alpha\Big] \\
&\le \sum_{n\in\mathbb{N}} 2^{2n\alpha} \mathbb{E}\Big[\Big(\int_{1-2^{-n}\le|x|^2\le 1-2^{-n-1}} e^{2X_{\partial\mathbb{D}}(x) - 2\mathbb{E}[X_{\partial\mathbb{D}}(x)^2]} \lambda(dx)\Big)^\alpha\Big] \\
&\le \sum_{n\in\mathbb{N}} \mathbb{E}\Big[\Big(\sum_{j=1}^{2^n} Z_j e^{2\sqrt{2}(Y_j^n - \sqrt{2}\ln 2\, n)}\Big)^\alpha\Big] \\
&= \sum_{n\in\mathbb{N}} \frac{1}{n^{3\alpha}} \mathbb{E}\Big[\Big(\sum_{j=1}^{2^n} Z_j e^{2\sqrt{2}(Y_j^n - \sqrt{2}\ln 2\, n + \frac{3}{2\sqrt{2}}\ln n)}\Big)^\alpha\Big].
\end{aligned}$$

Let $\eta \in]0,1[$. By Jensen and for some constant $B > 0$

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\sum_{j=1}^{2^n} Z_j e^{2\sqrt{2}(Y_j^n - \sqrt{2}\ln 2\, n + \frac{3}{2\sqrt{2}}\ln n)}\Big)^{\alpha}\Big] \\
=&\mathbb{E}\Big[\Big(\sum_{j=1}^{2^n} \mathbb{E}[Z_j^{1-\eta}] e^{2\sqrt{2}(1-\eta)(Y_j^n - \sqrt{2}\ln 2\, n + \frac{3}{2\sqrt{2}}\ln n)}\Big)^{\frac{\alpha}{1-\eta}}\Big] \\
\leq& B\mathbb{E}\Big[\Big(\sum_{j=1}^{2^n} e^{2\sqrt{2}(1-\eta)(Y_j^n - \sqrt{2}\ln 2\, n + \frac{3}{2\sqrt{2}}\ln n)}\Big)^{\frac{\alpha}{1-\eta}}\Big]
\end{aligned}$$

From [Mad17] again, this last expectation is bounded independently of n provided that we choose $2\alpha(1-\eta) < 1$. In that case, up to changing the value of B, we get

$$\mathbb{E}\Big[\Big(\int_{\mathbb{D}} e^{2X_{\partial\mathbb{D}}(x) - 2\mathbb{E}[X_{\partial\mathbb{D}}(x)^2]} \frac{1}{(1-|x|^2)^2})\lambda(dx)\Big)^{\alpha}\Big] \leq B \sum_{n\in\mathbb{N}} \frac{1}{n^{3\alpha}},$$

which can be obviously made finite provided that $\alpha > 1/3$. □

Conjectures related to planar quadrangulations with boundary

We consider $\overline{\mathcal{Q}}_{n,p}$ the set of quandrangulations of size n, i.e. with n inner faces and a simple boundary of length $2p$ with one marked edge on the boundary and one marked face inside. Now to each quadrangulation Q with a marked point inside and a marked point on the boundary (we choose at random a point in the marked face at a point on the marked edge), we associate a standard conformal structure (by gluing Euclidean squares along their edges as prescribed by the quadrangulation) and map it to the disk such that the interior point gets mapped to 0 and the frontier point to 1. We give volume a^2 to each quadrilateral and length a to each edge on the boundary: we denote $\nu_{Q,a}$ the corresponding volume measure and $\nu^{\partial}_{Q,a}$ the corresponding boundary length measure. Recall that we have the following asymptotics as $n, p \to \infty$ with $\frac{n}{p^2}$ tending to some value (see appendix):

$$|\overline{\mathcal{Q}}_{n,p}| \sim e^{n\ln 12} e^{2p\ln\frac{3}{\sqrt{2}}} n^{-3/2} \frac{\sqrt{3p}}{2\pi} e^{-\frac{9(2p)^2}{16n}}$$

and we set $\mu^c = \ln 12$, $\mu^c_{\partial} = \ln\frac{3}{\sqrt{2}}$ (these two constants are not universal as they depend on the class of map one considers, i.e. are different for triangulations, etc...). Now, we consider the measures $(\nu_a, \nu^{\partial}_a)$ defined by the following expression for all F

$$\mathbb{E}^a[F(\nu_a,\nu_a^\partial)] = \frac{1}{Z_a}\sum_{n,p} e^{-\bar\mu n} e^{-\bar\mu_\partial 2p} \sum_{Q\in\mathcal{Q}_{n,p}} F(\nu_{Q,a},\nu_{Q,a}^\partial),$$

where the constants $\bar\mu, \bar\mu_\partial$ are functions of $a > 0$ defined by $\bar\mu = \mu_c + a^2\mu$, $\bar\mu_\partial = \mu_\partial^c + a\mu_\partial$ and Z_a is a normalization constant. We can now state a precise mathematical conjecture:

Conjecture 1. *The limit in law* $\lim_{a\to 0}(\nu_a,\nu_a^\partial)$ *exists in the product space of Radon measures equipped with the topology of weak convergence and is given (up to deterministic constants) by the Liouville measure of LQG with parameter* $\gamma = \sqrt{\frac{8}{3}}$*, appropriate cosmological constants and* $\alpha_1 = \gamma, \beta_1 = \gamma$ *and points* $z_1 = 0, s_1 = 1$.

Here we give a few more details on the above conjecture. It states the existence of constants $\bar C, \bar c > 0$ such that

$$\lim_{a\to 0}\mathbb{E}^a[F(\nu_a,\nu_a^\partial)] = \mathbb{E}^{(0,\gamma),(1,\gamma)}_{\gamma,\bar C\mu,\bar c\mu_\partial}[F(\bar C Z,\bar c Z_\partial)] \tag{2.60}$$

with $\gamma = \sqrt{\frac{8}{3}}$. Looking at Section 2, recall that we have for all $\gamma \in]\sqrt{2},2[$

$$\begin{aligned}&\mathbb{E}^{(0,\gamma),(1,\gamma)}_{\gamma,\bar C\mu,\bar c\mu_\partial}[F(\bar C Z,\bar c Z_\partial)]\\ &= \frac{1}{C_{\mu,\mu_\partial,\gamma}}\int_{\mathbb{R}} e^{(\gamma-\frac{2}{\gamma})c}\mathbb{E}\Big[F(\bar C e^{\gamma c}Z_0,\bar c e^{\frac{\gamma}{2}c}Z_0^\partial)\exp\Big(-\bar C\mu e^{\gamma c}Z_0(\mathbb{D}) - \bar c\mu_\partial e^{\frac{\gamma}{2}c}Z_0^\partial(\partial\mathbb{D})\Big)\Big]\,dc\end{aligned} \tag{2.61}$$

where the couple (Z_0, Z_0^∂) is defined by

$$Z_0 = e^{\gamma H}e^{\gamma X_{\partial\mathbb{D}}}\,d\lambda, \quad Z_0^\partial = e^{\frac{\gamma}{2}H}e^{\frac{\gamma}{2}X_{\partial\mathbb{D}}}\,d\lambda_\partial$$

where the couple $(e^{\gamma X_{\partial\mathbb{D}}}\,d\lambda, e^{\frac{\gamma}{2}X_{\partial\mathbb{D}}}\,d\lambda_\partial)$ is a standard couple of Gaussian chaos measures defined by a limiting procedure in Proposition 32 and

$$H(x) = \gamma G(x,0) + \frac{\gamma}{2}G(x,1).$$

with G the standard Green function in the disk (see (2.22) for the definition). The constants $\bar C, \bar c$ are non universal in the sense that they depend on the class of planar map you consider. For instance, the constants $\bar C, \bar c$ will be different if you consider triangulations instead of quadrangulations. It would be interesting to know these constants (in the case of quadrangulations say); however, we do not know how to compute them as it requires information on the joint law of $(Z_0(\mathbb{D}), Z_0^\partial(\partial\mathbb{D}))$.

It is known that the joint law of the total volume and the total boundary length of (ν_a, ν_a^∂) is given by the following density within the regime of conjecture 1 (see Appendix 2)

$$\frac{1}{D_{\mu,\mu_\partial}} V^{-3/2} l^{1/2} e^{-\mu V} e^{-\mu_\partial l} e^{-\frac{9l^2}{16V}} dl dV. \tag{2.62}$$

In fact, the above distribution should be universal, i.e. should not depend on the planar map model, except for the $\frac{9}{16}$ constant in $e^{-\frac{9l^2}{16V}}$ which is specific to quadrangulations and in the case of triangulations (for instance) one should get a different constant than $\frac{9}{16}$. One can in fact read on relations (2.61) and (2.62) where the relation $\gamma = \sqrt{\frac{8}{3}}$ comes from. Indeed, for any function G, by making a simple change of variables $V = \bar{C} e^{\gamma c} Z_0(\mathbb{D})$ in (2.61) we get that

$$\begin{aligned} &\mathbb{E}^{(0,\gamma),(1,\gamma)}_{\gamma,\bar{C}\mu,\bar{c}\mu_\partial}[G(\bar{C}Z(\mathbb{D}))e^{\bar{c}\mu_\partial Z_\partial(\partial\mathbb{D})}] \\ &= \frac{1}{\gamma C_{\mu,\mu_\partial,\gamma}} \mathbb{E}[\frac{1}{(\bar{C}Z_0(\mathbb{D}))^{1-2/\gamma^2}}] \int_0^\infty G(V) V^{-\frac{2}{\gamma^2}} e^{-\mu V} dV. \end{aligned}$$

Similarly, one has

$$\begin{aligned} &\frac{1}{D_{\mu,\mu_\partial}} \int_0^\infty \int_0^\infty \left(G(V) e^{\mu_\partial l}\right) V^{-3/2} l^{1/2} e^{-\mu V} e^{-\mu_\partial l} e^{-\frac{9l^2}{16V}} dl dV \\ &= \frac{\int_0^\infty \tilde{l}^{-1/4} e^{-\frac{9}{16}\tilde{l}^2} d\tilde{l}}{2D_{\mu,\mu_\partial}} \int_0^\infty G(V) V^{-\frac{3}{4}} e^{-\mu V} dV, \end{aligned}$$

by using the change of variable $\tilde{l} = \frac{l^2}{V}$. This shows that the only possible choice for (2.60) to hold is γ such that $\frac{2}{\gamma^2} = \frac{4}{3}$, i.e. $\gamma = \sqrt{\frac{8}{3}}$.

One could also state similar conjectures with three distinct marked points on the boundary (instead of one interior marked point and one marked point on the boundary) or/and by conditioning on the measures to have fixed volume (instead of the Boltzmann weight setting of conjecture 1). One could also state similar conjectures where the quadrangulation is chosen according to the partition function of a model of statistical physics (at critical temperature): in that case, the value of γ in conjecture 1 will depend on the model and can be read on the asymptotics of the partition function of the quadrangulation (in a way similar to the way we derived the relation $\gamma = \sqrt{\frac{8}{3}}$ for uniform quadrangulations).

Finally, let us mention that variants of the measures defined by (2.61) (where you fix three points on the boundary and condition on the volume of the bulk measure or the boundary) should be related (in a similar way as the sphere case) to the unit area quantum disk and the unit boundary length quantum disk which appear in [DMS14].

Appendix

Asymptotics of quadrangulations with a boundary

Here we take material from [BG09] (see also [CM15]). Let $\mathcal{Q}_{n,p}$ denote quandrangulations of size n with a simple boundary of length $2p$ and a marked point on the frontier. Then we have

$$|\mathcal{Q}_{n,p}| = \frac{1}{3^p}\frac{(3p)!}{p!(2p-1)!}3^n\frac{(2n+p-1)!}{(n-p+1)!(n+2p)!}.$$

We are interested in the asymptotics of $|\mathcal{Q}_{n,p}|$ as $n, p \to \infty$ with $\frac{p^2}{n}$ fixed. Notice that we have within this asymptotic:

$$\begin{aligned}(2n+p-1)! &\sim \sqrt{2\pi}2^{2n+p-1}e^{(2n+p-1)\ln n+p-1+\frac{p^2}{4n}-(2n+p-1)}\sqrt{2n},\\ (n-p+1)! &\sim \sqrt{2\pi}e^{(n-p+1)\ln n-p+1+\frac{p^2}{2n}-(n-p+1)}\sqrt{n},\\ (n+2p)! &\sim \sqrt{2\pi}e^{(n+2p)\ln n+2p+\frac{2p^2}{n}-(n+2p)}\sqrt{n}.\end{aligned}$$

Hence, we get that

$$\frac{(2n+p-1)!}{(n-p+1)!(n+2p)!} \sim \sqrt{\frac{1}{\pi}}n^{-5/2}2^{2n+p-1}e^{-\frac{9p^2}{4n}}.$$

Also,

$$\frac{(3p)!}{p!(2p-1)!} \sim \frac{\sqrt{3}}{\sqrt{\pi}}\sqrt{p}(\frac{27}{4})^p$$

in such way that we get

$$|\mathcal{Q}_{n,p}| \sim 12^n(\frac{9}{2})^p n^{-5/2}\frac{\sqrt{3p}}{2\pi}e^{-\frac{9p^2}{4n}}.$$

Finally, if $\overline{\mathcal{Q}}_{n,p}$ denotes the set of quandrangulations of size n with a simple boundary of length $2p$ with one marked point on the frontier and one marked point inside then we get

$$|\overline{\mathcal{Q}}_{n,p}| \sim e^{n\ln 12}e^{2p\ln\frac{3}{\sqrt{2}}}n^{-3/2}\frac{\sqrt{3p}}{2\pi}e^{-\frac{9(2p)^2}{16n}}.$$

Some auxiliary estimates

Here we give hints for some estimates used in the proof of Theorem 34 and Proposition 35. We stick to the notations used in this proof.

Lemme 50. *On boundary behavior of the regularized Green function G_ϵ: remember that D_ϵ is the disk of radius ϵ centered at $1-2\epsilon$, we claim that*

$$\sup_{\epsilon>0}\sup_{x\in D_\epsilon}|G_\epsilon(x,x)+2\ln\epsilon|<+\infty.$$

As a consequence, one sees that if $x\in D_\epsilon$,

$$|\mathbb{E}[X_{\partial\mathbb{D},\epsilon}(x)^2]-2\ln\frac{1}{\epsilon}|\leq C,\quad |G_\epsilon(x,1)-2\ln\frac{1}{\epsilon}|\leq C.$$

Proof. Let us calculate $G_\epsilon(x,x)$ for $\epsilon>0$ small enough. Recall that the non-regularized Green function $G(x,y)$ is the sum of $\ln\frac{1}{|x-y|}$ and $\ln\frac{1}{|1-x\overline{y}|}$. We have already seen that the ϵ-regularization of $\ln\frac{1}{|x-y|}$ part of $G_\epsilon(x,x)$ will simply be $-\ln\epsilon$ as in the proof of proposition 31. Now for the $\ln\frac{1}{|1-x\overline{y}|}$ part, we remark a scaling relation: we can compare what is happening at ϵ with that at $\epsilon/2$ via the following observation (with $a,b>0$ both small of order ϵ)

$$\ln\frac{|a/2+b/2-ab/4|}{|a+b-ab|}-\ln\frac{1}{2}=\ln\frac{|a+b-ab/2|}{|a+b-ab|}\asymp\frac{|ab/2|}{|a+b-ab|}\leq|a|$$

By taking $a=1-(x+\epsilon e^{i\theta})$ and $b=1-(\overline{x}-\epsilon e^{i\theta'})$ we can establish

$$\sup_{\epsilon>0}\sup_{x\in D_\epsilon}|\frac{1}{4\pi^2}\int_{\mathbb{S}_1}\int_{\mathbb{S}_1}\ln\frac{1}{|1-(x+\epsilon e^{i\theta})(\overline{x}+e^{i\theta'})|}d\theta d\theta'+\ln\epsilon|<+\infty$$

Together we get the first part of the lemma.
The first inequality in the second part of the lemma comes as a direct consequence. The second inequality can be proved using a similar scaling relation as in the above proof. □

Now we establish another estimate concerning the process Y_ϵ. Recall that Y_ϵ is the Gaussian process defined as $Y_\epsilon(u)=X_{\partial\mathbb{D},\epsilon}(1-\epsilon u)-X_{\partial\mathbb{D},\epsilon}(1)$ and $D(2,1)$ is the disk centered at 2 with radius 1.

Lemme 51. *For all $z,z'\in D(2,1)$,*

$$\mathbb{E}[(Y_\epsilon(z)-Y_\epsilon(z'))^2]\leq C|z-z'|$$

uniformly in $0<\epsilon\leq 1$.

Proof. It suffices to prove that uniformly in ϵ,

$$|G_\epsilon(1-\epsilon z,1-\epsilon z)-G_\epsilon(1-\epsilon z,1-\epsilon z')|\leq C|z-z'|.$$

For the $\ln \frac{1}{|x-y|}$ part of G, it suffices to prove that the following function is Lipschitz in r for $r \in [0,2]$

$$f(r) = \frac{1}{4\pi^2} \int_{\mathbb{S}_1} \int_{\mathbb{S}_1} \ln \frac{1}{|e^{i\theta} - re^{i\theta'}|} d\theta d\theta'$$

notice that $f(1) = 0$. But we have already seen that $f(r) = 0$ when $r \leq 1$ and this implies that $f(r) = \ln r$ when $r > 1$.
As of the $\ln \frac{1}{|1-x\overline{y}|}$ part, we will write the difference as

$$\frac{1}{4\pi^2} \int_{\mathbb{S}_1} \int_{\mathbb{S}_1} \ln \frac{|1-(x+\epsilon e^{i\theta})((\overline{y}-\overline{x})+\overline{x}+\epsilon e^{i\theta'})|}{|1-(x+\epsilon e^{i\theta})(\overline{x}+\epsilon e^{i\theta'})|} d\theta d\theta'$$

where $x = 1 - \epsilon z$ and $y = 1 - \epsilon z'$. Then we note $t = \frac{\overline{y-x}}{\epsilon}$ and this becomes

$$\frac{1}{4\pi^2} \int_{\mathbb{S}_1} \int_{\mathbb{S}_1} \ln \frac{|1/\epsilon^2 - (x/\epsilon + e^{i\theta})(t + \overline{x}/\epsilon + e^{i\theta'})|}{|1/\epsilon^2 - (x/\epsilon + e^{i\theta})(\overline{x}/\epsilon + e^{i\theta'})|} d\theta d\theta'$$

As the derivative with respect to t is continuous and uniformly bounded in ϵ for $|t| \leq 2$, our proof is complete. □

Moment estimates on Gaussian multiplicative chaos

Moment estimate for an insertion on the boundary

To define properly unit volume Liouville measures in Corollary 41, we need several moment estimates on the GMC measure on the unit disk that we study in this section.
The techniques of proof for different values γ are not the same: we seperate the regimes $\gamma \in]0, \sqrt{2}[$ (Lemma 52) and $\gamma \in [\sqrt{2}, 2[$ (Lemma 58). The main difference from the unit volume Liouville measure on the Riemann sphere (see [DKRV16, Section 3.4]) is that we have to give moment bounds of same kind of non classical GMC measure near the boundary (with or without insertions at the boundary), where, as we will explain after, the $2d$-GMC measure collapses and behaves like a $1d$-GMC measure: analysing this transition carefully is the goal of Lemma 52 and Lemma 58.

We first look at the regime $\gamma \in]0, \sqrt{2}[$ and give a necessary and sufficient condition for the finiteness of the GMC measure on the unit disk.

Lemme 52. *Let $\gamma \in]0, \sqrt{2}[$. Consider $\beta < Q$ and $s \in \partial\mathbb{D}$. Then for all $p > 0, \delta > 0$,*

$$\mathbb{E}\left[\left(\int_{\mathbb{D}\cap B(s,\delta)} e^{\gamma\frac{\beta}{2}G(\cdot,s)} e^{\gamma X_{\partial\mathbb{D}} - \frac{\gamma^2}{2}\mathbb{E}[X^2_{\partial\mathbb{D}}]} g_P^{\frac{\gamma^2}{4}} d\lambda\right)^p\right] < +\infty \tag{2.63}$$

if and only if $p < \frac{2}{\gamma^2} \wedge \frac{1}{\gamma}(Q-\beta)$.

Remarque 53. In the following of this section, we will work with an exact scale-invariant Gaussian process on $\mathbb{H}$ with covariance

$$G_{\mathbb{H}}(x,y) = \ln \frac{R}{|x-y||x-\bar{y}|}.$$

This is not exactly the same kernel as the process we will consider in the half-plane geometry: they differ by at most a finite constant near 0. However, by Kahane's inequality, they possess the same moment bound for Lemma 52 and Lemma 58. Furthermore, restricted to a small compact near 0, $G_{\mathbb{H}}$ is indeed positive definite: one can first sample a log-correlated Gaussian field on $B(0,\epsilon)\cap\mathbb{C}$ with correlation $\ln^+ \frac{1}{|x-y|}$ (this is indeed a definite-positive kernel, see [RV10]) for a **fixed** small ϵ, then define the Gaussian field on $B(0,\epsilon)\cap\mathbb{H}$ by assigning $\frac{X(z)+X(\bar{z})}{\sqrt{2}}$ at point z: this new field has correlation $G_{\mathbb{H}}$ on $B(0,\epsilon)\cap\mathbb{H}$. This is a standard procedure, see for example [She16, Section 3.2].

Proof. By rotational invariance of the problem, we can suppose that $s=-1$. Let $\psi(z) = \frac{z-i}{z+i}$ be the Cayley transform which maps the upper half-plane onto the unit disk and sends 0 to -1. We set $G_{\mathbb{H}}(x,y) = G(\psi(x),\psi(y))$ and $X = X_{\partial\mathbb{D}}\circ\psi$ which has covariance $G_{\mathbb{H}}$.
In light of the above Remark 53, we will work with the following form of $G_{\mathbb{H}}$:

$$G_{\mathbb{H}}(x,y) = \ln \frac{1}{|x-y||x-\bar{y}|},$$

we have that for $x,y\in\mathbb{H}$ and all $r\in]0,1[$,

$$G_{\mathbb{H}}(rx,ry) = G_{\mathbb{H}}(x,y) + 2\ln\frac{1}{r}, \tag{2.64}$$

which is an exact scaling relation.
With these notations, by conformal invariance, it is equivalent to study whether

$$\mathbb{E}\left[\left(\int_S e^{\gamma\frac{\beta}{2}G_{\mathbb{H}}(z,0)} e^{\gamma X(z)-\frac{\gamma^2}{2}\mathbb{E}[X^2(z)]}\frac{1}{\mathrm{Im}(z)^{\frac{\gamma^2}{2}}}\lambda(dz)\right)^p\right] < +\infty, \tag{2.65}$$

where S denotes some small square $[-\epsilon,\epsilon]\times[0,2\epsilon]$ in $\mathbb{H}$ such that $G_{\mathbb{H}}$ is definite positive on S.
ϵ will be fixed in the rest of the section, it is solely chosen to ensure that $G_{\mathbb{H}}$ is well-defined.
Let $X_{1/2^n}$ denotes a 2^{-n}-regularisation of X by convolution with a mollifier of compact support (for instance the 2^{-n}-circle average) and note

$$J_n(B) = \int_B e^{\gamma\frac{\beta}{2}G_{\mathbb{H}}(z,0)} e^{\gamma X_{1/2^n}(z)-\frac{\gamma^2}{2}\mathbb{E}[X^2_{1/2^n}(z)]}\frac{1}{\mathrm{Im}(z)^{\frac{\gamma^2}{2}}}\lambda(dz) \tag{2.66}$$

for all borelians B in $\mathbb{H}$. We prove that:

$$\limsup_n \mathbb{E}\left[J_n(S)^p\right] < +\infty, \tag{2.67}$$

if and only if $p < \frac{2}{\gamma^2} \wedge \frac{1}{\gamma}(Q-\beta)$. One then follows [DKRV16, Section A.] to finish off the proof.

In the following we denote by

$$\overline{\zeta}(p) = (2+\gamma^2/2)p - \gamma^2 p^2 \tag{2.68}$$

the scaling exponent for the measure J_n.
We first study the quantity in equation (2.67) in the case $\beta = 0$ (i.e. no insertion at the boundary), then we pass to the case of general β by slightly modifying an argument in [DKRV16, Section A.], for which we briefly recall the details in the following.

The case $\beta = 0$:
Remark that $\frac{2}{\gamma^2} < \frac{Q}{\gamma}$ such that in the case $\beta = 0$, we only need to prove that :

$$\limsup_n \mathbb{E}\left[J_n(S)^p\right] < +\infty \quad \text{if and only if} \quad p < \frac{2}{\gamma^2}. \tag{2.69}$$

One checks by Fubini that with $\gamma \in]0, \sqrt{2}[$, $J_n(S)$ possesses finite first moment

$$\mathbb{E}\left[\int_S e^{\gamma X(z) - \frac{\gamma^2}{2}\mathbb{E}[X^2(z)]} \frac{1}{\mathrm{Im}(z)^{\frac{\gamma^2}{2}}} \lambda(dz)\right] < +\infty, \tag{2.70}$$

so we will only focus on values of $p > 1$ in the regime $\gamma \in]0, \sqrt{2}[$.
We will decompose S into three parts and use scaling relations to explore the multifractal structure of X and the associated mesure J_n. Let

$$S_1 = [-\epsilon, 0] \times [0, \epsilon], S_2 = [0, \epsilon] \times [0, \epsilon], S_3 = [-\epsilon, \epsilon] \times [\epsilon, 2\epsilon]$$

and remark that, by scaling

$$\mathbb{E}\left[J_{n+1}(S_1)^p\right] = 2^{-\overline{\zeta}(p)}\mathbb{E}\left[J_n(S)^p\right] \tag{2.71}$$

where $\overline{\zeta}(p)$ is the half-plane scaling exponent for the kernel $G_{\mathbb{H}}$ as defined in equation (2.68).

Before entering the proof, we first provide three estimations on the correlations of J_n. The motivation behind these estimations is that we want to have some kind of "decorrelation"

inequalities in order to compare some cross-terms with leading terms in polynomial expansions.
Let $p, q > 0$ and $A = [a_1^l, a_1^r] \times [a_2^l, a_2^r]$, $B = [b_1^l, b_1^r] \times [b_2^l, b_2^r]$ two rectangles in S. We call A and B adjacent if $a_1^r = b_1^l$ (the right boundary of A is at the same level as the left boundary of B).

Lemme 54 (Cutoff estimation: non-adjacent case).
Let $p, q > 0$. Suppose that $a_1^r < b_1^l$, so that $dist(A, B) = \delta > 0$. Then

$$\mathbb{E}[J_n(A)^p]\mathbb{E}[J_n(B)^q] \leq \mathbb{E}[J_n(A)^p J_n(B)^q] \leq \delta^{-2pq\gamma^2}\mathbb{E}[J_n(A)^p]\mathbb{E}[J_n(B)^q]. \quad (2.72)$$

Lemme 55 (Cutoff estimation: adjacent to non-adjacent case).
Let $p, q > 0$. Suppose that $a_1^r = b_1^l$. Let $B_t = [b_1^l + t, b_1^r + t] \times [b_2^l, b_2^r]$ be a translation of B on the first coordinate with $t > 0$ (so that we are moving away from A). Then there exists some constant C that does not depend on n such that

$$\mathbb{E}[J_n(A)^p J_n(B_t)^q] \leq C\mathbb{E}[J_n(A)^p J_n(B)^q]. \quad (2.73)$$

Lemme 56 (Cutoff estimation: symmetric case).
Let $p, q > 0$. Let $H_L = \{x \in \mathbb{R}^2; x_1 = L\}$ the hyperplan of $\mathbb{R}^2$ with the first coordiante equal to L. Let $\overline{B}$ be the symmetry of B with respect to $H = H_{a_1^r} = H_{b_1^l}$. Then

$$\mathbb{E}[J_n(A)^p J_n(B)^q] \leq \mathbb{E}[J_n(A)^p J_n(\overline{B})^q]. \quad (2.74)$$

Lemme 57 (Cutoff estimation: adjacent case).
Let $p, q > 0$. Suppose that $a_1^r = b_1^l$. Let $B_t = [b_1^l + t, b_1^r + t] \times [b_2^l, b_2^r]$ a translation of B on the first coordinate with $t < 0$ (so that we are moving towards A) such that $[b_1^l + t, b_1^r + t] \subset [a_1^l, a_1^r]$. Then there exists some constant C that does not depend on n such that

$$\mathbb{E}[J_n(A)^p J_n(B)^q] \leq C\mathbb{E}[J_n(A)^p J_n(B_t)^q]. \quad (2.75)$$

For continuity of lecture, we postpone the proofs of these lemmas until the end of this section. We now explain how to finish the $\beta = 0$ case using these estimations.

• Suppose $\limsup_n \mathbb{E}\left[J_n(S)^p\right] < +\infty$ and let us prove that $p < 2/\gamma^2$.
If $p \geq 4/\gamma^2$, we already know since Kahane that the p-th moment of a $2d$-GMC measure explodes. Thus, when $p \geq 4/\gamma^2$, $\mathbb{E}\left[J_n(S)^p\right]$ explodes in the limit since $\mathbb{E}\left[J_n(S_3)^p\right]$ does. Let $1 \leq p < 4/\gamma^2$ so that $\mathbb{E}\left[J_n(S_3)^p\right]$ converges to some finite quantity by Kahane. By super-additivity and scale invariance (equation (2.68)), we have

$$\begin{aligned}
&\mathbb{E}\left[J_n(S)^p\right] \\
\geq &\mathbb{E}\left[J_n(S_1)^p\right] + \mathbb{E}\left[J_n(S_2)^p\right] + \mathbb{E}\left[J_n(S_3)^p\right] \\
\geq &2^{1-\overline{\zeta}(p)}\mathbb{E}\left[J_{n-1}(S)^p\right] + C
\end{aligned}$$

for some constant $C > 0$ independent of n. Since $1 - \overline{\zeta}(p) \geq 0$ when $p \geq 2/\gamma^2$, necessarily $p < 2/\gamma^2$ if $\mathbb{E}\left[J_n(S)^p\right]$ is uniformly bounded in n.

• Suppose $p < 2/\gamma^2$ and prove $\limsup_n \mathbb{E}\left[J_n(S)^p\right] < +\infty$.

We will show this by induction on p: suppose $\limsup_n \mathbb{E}\left[J_n(S)^k\right] < +\infty$ for some integer k and we prove $\limsup_n \mathbb{E}\left[J_n(S)^p\right] < +\infty$ for all $p < 2/\gamma^2$ such that $k < p \leq k+1$.

We want to point out that this is a refinement of the arguments in [KP76].

First we use Muirhead's inequality (see Corollary 62) to write

$$\begin{aligned}
&\mathbb{E}\left[J_n(S_1+S_2)^p\right] \\
\leq &\mathbb{E}\left[J_n(S_1)^p\right] + \mathbb{E}\left[J_n(S_2)^p\right] + C(\mathbb{E}[J_n(S_1)^k J_n(S_2)^{p-k}] + \mathbb{E}[J_n(S_1)^{p-k} J_n(S_2)^k]) \\
= &\, 2\mathbb{E}\left[J_n(S_1)^p\right] + C\mathbb{E}[J_n(S_1)^k J_n(S_2)^{p-k}]
\end{aligned}$$

where we used the symmetry of J_n on S_1 and S_2. Now consider some large integer that we denote by δ^{-1} (so that δ is small), and cut S_1 into δ^{-1} equal parts of the form

$$S_1^i = [-i\delta\epsilon, -(i-1)\delta\epsilon] \times [0, \epsilon], \quad i = 1, \ldots, \delta^{-1}.$$

We also cut S_2 into two parts

$$S_2^l = [0, \delta\epsilon] \times [0, 1], \quad S_2^r = [\delta\epsilon, \epsilon] \times [0, \epsilon].$$

Notice that S_2^l and S_1 are adjacent in the sense of Lemma 57. It follows from the same lemma, since S_1^i is a horizontal translation of S_2^l, that for all $i = 1, \ldots, \delta^{-1}$,

$$\mathbb{E}[J_n(S_1)^k J_n(S_2^l)^{p-k}] \leq C\mathbb{E}[J_n(S_1)^k J_n(S_1^i)^{p-k}]$$

for some C independent of n.

Since $0 < p - k \leq 1$, using Jensen's inequlity for the concave function $x \mapsto x^{p-k}$,

$$\begin{aligned}
&\mathbb{E}[J_n(S_1)^p] \\
\geq &\mathbb{E}[J_n(S_1)^k J_n(S_1)^{p-k}] \\
\geq &\delta^{1-(p-k)} \sum_i \mathbb{E}[J_n(S_1)^k J_n(S_1^i)^{p-k}] \\
\geq &C\delta^{-(p-k)} \mathbb{E}[J_n(S_1)^k J_n(S_2^l)^{p-k}].
\end{aligned}$$

On the other hand, we can compare $\mathbb{E}[J_n(S_1)^k J_n(S_2^r)^{p-k}]$ and $\mathbb{E}[J_n(S_1)^p]$ via Lemma 54: applying this lemma to S_2^r and S_1,

$$\begin{aligned}
&\mathbb{E}[J_n(S_1)^k J_n(S_2^r)^{p-k}] \\
\leq &C\delta^{-2p(p-k)\gamma^2} \mathbb{E}[J_n(S_1)^k] \mathbb{E}[J_n(S_2^r)^{p-k}] \\
\leq &C\delta^{-2p(p-k)\gamma^2}.
\end{aligned}$$

where we used the finiteness for the k-th moment and the $(p-k)$-th moment thanks to induction hypothesis, and C always independent of n.
Gathering all pieces of information above, we get by sub-additivity,

$$\begin{aligned}
&\mathbb{E}[J_n(S_1+S_2)^p] \\
\leq& 2\mathbb{E}\left[J_n(S_1)^p\right] + C\mathbb{E}[J_n(S_1)^k J_n(S_2)^{p-k}] \\
\leq& 2\mathbb{E}\left[J_n(S_1)^p\right] + C\mathbb{E}[J_n(S_1)^k J_n(S_2^l)^{p-k}] + C\mathbb{E}[J_n(S_1)^k J_n(S_2^r)^{p-k}] \\
\leq& (2+C\delta^{p-k})\mathbb{E}\left[J_n(S_1)^p\right] + C\delta^{-2p(p-k)\gamma^2}
\end{aligned}$$

where C changes from line to line but is always independent of n.
Since $1 \leq p < 2/\gamma^2$, one checks that it is possible to choose (independently of n) a δ such that $\alpha = 2^{-\overline{\zeta}(p)}(2+C\delta^{p-k}) < 1$. With this choice, we get

$$\mathbb{E}[J_n(S_1+S_2)^p] \leq \alpha\mathbb{E}[J_{n-1}(S)^p] + C. \tag{2.76}$$

Finally, applying Minkowski's inequality to $J_n(S)$ with the cutting S_1, S_2, S_3, we get, for $1 \leq p < 2/\gamma^2$,

$$\begin{aligned}
&\mathbb{E}[J_n(S)^p]^{1/p} \\
\leq& (\mathbb{E}\left[J_n(S_1+S_2)^p\right])^{1/p} + \mathbb{E}[J_n(S_3)^p]^{1/p} \\
\leq& \left(\alpha\mathbb{E}\left[J_{n-1}(S)^p\right] + C\right)^{1/p} + C
\end{aligned}$$

again, the estimate on S_3 is classical since Kahane, and $C > 0$ independent of n.
By sub-additivity,

$$\begin{aligned}
&\mathbb{E}[J_n(S)^p]^{1/p} \\
\leq& \alpha^{1/p}\mathbb{E}\left[J_{n-1}(S)^p\right]^{1/p} + C^{1/p} + C.
\end{aligned}$$

With $\alpha < 1$, this ensures that the sequence $\mathbb{E}[J_n(S)^p]^{1/p}$ is uniformly bounded in n (and so is $\mathbb{E}[J_n(S)^p]$), thus completes our proof for the $\beta = 0$ case.

The case β general: We now add an insertion of weight β at the boundary of $\mathbb{H}$ at 0. We study directly the field X and exploit its exact scaling relation as in [DKRV16, Section A.]. Let

$$I(B) = \mathbb{E}\left[\left(\int_B e^{\gamma\frac{\beta}{2}G_{\mathbb{H}}(\cdot,s)} e^{\gamma X_{\partial\mathbb{D}} - \frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D}}^2]} g_P^{\frac{\gamma^2}{4}}\, d\lambda\right)^p\right] \tag{2.77}$$

for borelians B in $\mathbb{H}$.
Let us fix a small ϵ such that $G_{\mathbb{H}}$ on $B(0,\epsilon)$ is well-defined.
Let $A = \mathbb{H} \cap B(0,\epsilon)$ and $A_i = \mathbb{H} \cap (B(0,2^{-i}\epsilon) \backslash B(0,2^{-n-1}\epsilon))$ for all $i = 0, 1, \ldots,$ a

partition of A into half-annulis. We know by the case $\beta = 0$ that for all i, $I(A_i) < \infty$ iff $p < \frac{2}{\gamma^2}$. Furthermore, by exact-scaling,

$$I(A_i) = 2^{p\gamma\beta} 2^{-\overline{\zeta}(p)} I(A_{i-1})$$

and remark that $f(p) = p\gamma\beta - \overline{\zeta}(p)$ changes sign at $p = \frac{Q-\beta}{\gamma}$.
Suppose that $p > 1$, then by super-additivity,

$$I(A) \geq \sum_i I(A_i), \tag{2.78}$$

from which one deduces that $I(A) < \infty$ implies $p < \frac{Q-\beta}{\gamma}$. On the other hand, by Minkowski's inequality,

$$I(A)^{1/p} \leq \sum_i I(A_i)^{1/p}, \tag{2.79}$$

from which one deduces that $p < \frac{Q-\beta}{\gamma}$ implies $I(A) < \infty$. □

We now turn our attention to the regime $\gamma \in [\sqrt{2}, 2[$. Here we deal with moments of order $0 < p < 1$, which is something quite unusual in the GMC theory: see the remark below the following lemma.

Lemme 58. *Let $\gamma \in [\sqrt{2}, 2[$. Consider $\beta < Q$ and $s \in \partial\mathbb{D}$. Then:*
i)) For all $p > 0, \delta > 0$,

$$\mathbb{E}\left[\left(\int_{\mathbb{D}\cap B(s,\delta)} e^{\gamma\frac{\beta}{2}G(\cdot,s)} e^{\gamma X_{\partial\mathbb{D}} - \frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D}}^2]} g_P^{\frac{\gamma^2}{4}}\, d\lambda\right)^p\right] < +\infty$$

if $p < \frac{2}{\gamma^2} \wedge \frac{1}{\gamma}(Q-\beta)$.
ii) Conversely, if

$$\mathbb{E}\left[\left(\int_{\mathbb{D}\cap B(s,\delta)} e^{\gamma\frac{\beta}{2}G(\cdot,s)} e^{\gamma X_{\partial\mathbb{D}} - \frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D}}^2]} g_P^{\frac{\gamma^2}{4}}\, d\lambda\right)^p\right] < +\infty$$

then $p \leq (\frac{1}{2} + \frac{1}{\gamma^2}) \wedge \frac{1}{\gamma}(Q-\beta)$.

Remarque 59. It is plausible to think that we can improve the bound in ii) to obtain the same moment bound as in the $\gamma \in [0, \sqrt{2}[$ regime. This would require some sharper estimations that are not studied in the classical GMC theory – because we are dealing with moments smaller than 1, which always exist for classical GMC measures (since the first moment is either finite (in sub-critical phase) or zero if the field degenerates (in critical or super-critical case) for classical GMC measures).

Proof. We put ourselves in the same settings as in the proof of the previous lemma. In particular we work with the half-plane representation. We also follow the same notations.
• We first look at assertion i).
The case $\beta = 0$: the bound in the $\beta = 0$ is reduced to $p < \frac{2}{\gamma^2} < 1$.
First suppose that $p < \frac{2}{\gamma^2}$ and prove that the p-th moment is finite. Since $p < \frac{2}{\gamma^2} \leq 1$, we can make use the sub-addivitiy inequality and exact scale-invariance to write

$$\begin{aligned}
&\mathbb{E}[J_{n+1}(S)^p] \\
\leq &\mathbb{E}[J_{n+1}(S_1 + S_2 + S_3)^p] \\
\leq &\mathbb{E}[J_{n+1}(S_1)^p] + \mathbb{E}[J_{n+1}(S_2)^p] + \mathbb{E}[J_{n+1}(S_3)^p] \\
\leq &2^{1-\overline{\zeta}(p)}\mathbb{E}[J_n(S)^p] + C
\end{aligned}$$

where C is some finite constant independent of n.
With the assumption that $\gamma < 2$ we have $\frac{1}{2} < \frac{2}{\gamma^2}$. Choose some $p' > p$ such that $\frac{1}{2} < p' < \frac{2}{\gamma^2}$ and observe that $1 - \overline{\zeta}(p') < 0$, thus from the above relation, the p'-th moment of the measure $J(S)$ exists and is finite (because we would have $2^{1-\overline{\zeta}(p')} < 1$). Since $p < p'$, the p-th moment of the measure $J(S)$ exists and is finite.
The case β general:
For completeness, we sketch a proof here although it is essentially the same as in [DKRV16, Section A.] and as the proof for general boundary β-insertions in the $\gamma \in (0, \sqrt{2})$ regime above.
Let $p < 1$ and

$$I(B) = \mathbb{E}\left[\left(\int_B e^{\gamma\frac{\beta}{2}G_{\mathbb{H}}(\cdot,s)} e^{\gamma X_{\partial\mathbb{D}} - \frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D}}^2]} g_P^{\frac{\gamma^2}{4}}\, d\lambda\right)^p\right] \tag{2.80}$$

for borelians B in $\mathbb{H}$.
Let us fix a small ϵ such that $G_{\mathbb{H}}$ on $B(0,\epsilon)$ is well-defined.
Let $A = \mathbb{H} \cap B(0,\epsilon)$ and $A_i = \mathbb{H} \cap (B(0, 2^{-i}\epsilon) \backslash B(0, 2^{-n-1}\epsilon))$ for all $i = 0, 1, \ldots,$ a partition of A into half-annulis. We know by the case $\beta = 0$ that for all i, $I(A_i) < \infty$ if $p < \frac{2}{\gamma^2}$. Furthermore, by exact-scaling,

$$I(A_i) = 2^{p\gamma\beta} 2^{-\overline{\zeta}(p)} I(A_{i-1})$$

and remark that $f(p) = p\gamma\beta - \overline{\zeta}(p)$ changes sign at $p = \frac{Q-\beta}{\gamma}$.
Suppose that $p < 1$, then by sub-additivity,

$$I(A) \leq \sum_i I(A_i), \tag{2.81}$$

from which one deduces that $p < \frac{Q-\beta}{\gamma}$ implies $I(A) < \infty$.

• We now turn our attention to ii).
The case $\beta = 0$:
Since $p < \frac{2}{\gamma^2} \leq 1$, we can make use the sub-addivitiy inequality, exact scale-invariance, then Jensen's inequality to write

$$\begin{aligned}
&\mathbb{E}[J_{n+1}(S)^p] \\
\geq &\mathbb{E}[J_{n+1}(S_1+S_2)^p] \\
\geq &2^{p-1}(\mathbb{E}[J_{n+1}(S_1)^p] + \mathbb{E}[J_{n+1}(S_2)^p]) \\
\geq &2^p 2^{-\overline{\zeta}(p)}\mathbb{E}[J_n(S)^p]
\end{aligned}$$

Studying the sign of $p - \overline{\zeta}(p)$ gives us the bound $p \leq \frac{1}{2} + \frac{1}{\gamma^2}$ if the sequence $\mathbb{E}[J_n(S)^p]$ is uniformly bounded in n.
The case β general:
One can check that the same argument above as in assertion i) goes through by replacing sub-additivity by Minkowski's inequality for $p < 1$. □

We now gather the sufficient conditions in Lemma 52 and Lemma 58 to define the unit volume measure of the disk.

Corollaire 60. *Suppose $\alpha_i, \beta_j < Q$ for all i, j. The random variable $Z_0(\mathbb{D})$ has a moment of order $\frac{Q-\sum_i \alpha_i - \frac{1}{2}\sum_j \beta_j}{\gamma}$ if*

$$Q - \sum_i \alpha_i - \frac{1}{2}\sum_j \beta_j < \frac{2}{\gamma} \wedge 2\min_i(Q-\alpha_i) \wedge \min_j(Q-\beta_j). \tag{2.82}$$

In particular, under these conditions, the unit volume measure given by Corollary 41 is well defined.

Proof. Let

$$p = \frac{Q - \sum_i \alpha_i - \frac{1}{2}\sum_j \beta_j}{\gamma}.$$

From Lemma 52 and Lemma 58, the random variable $Z_0(S)$ has a moment of order p if S is a small open neighborhood of any boundary point, not containing any bulk insertions, under the condition

$$p < \frac{2}{\gamma^2} \wedge \frac{1}{\gamma}\min_j(Q-\beta_j).$$

Since the boundary of $\mathbb{D}$ is finite, one can find a covering of a small open neighborhood of $\partial\mathbb{D}$ that does not contain any bulk insertions using a finite number of such S. On the

complementary of this neighborhood, from the results of unit volume Liouville measure on the Riemann sphere ([DKRV16, Section 3.4]), we know that Z_0 has a moment of order p under the condition

$$p < \frac{4}{\gamma^2} \wedge \frac{2}{\gamma} \min_i (Q - \alpha_i).$$

Combining these two considerations we get that $Z_0(\mathbb{D})$ has a moment of order p if

$$p < \frac{2}{\gamma^2} \wedge \frac{2}{\gamma} \min_i (Q - \alpha_i) \wedge \frac{1}{\gamma} \min_j (Q - \beta_j).$$

Replacing p by $\frac{Q - \sum_i \alpha_i - \frac{1}{2}\sum_j \beta_j}{\gamma}$ yields the corollary. □

We now give proofs of the cutoff estimations (Lemma 54 to Lemma 57). The idea is to construct auxilary Gaussian processes and use a continous version of Slepian's lemma (Proposition 63 below, see [Zei16, Theorem 3] for a general formulation) to compare their functionals.

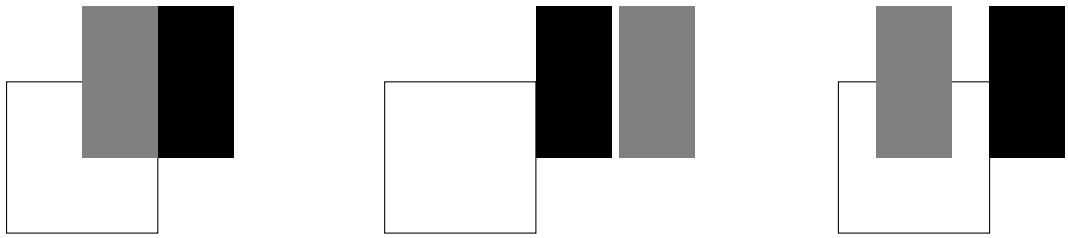

Figure 2.1 – Cut-off estimations: A is illustrated by a white box, B by a black box. The grey box denotes the box B after some linear transformation.
Left: the symmetry case. Middle: B is moving away from A. Right: B is moving inside of A.

Proof of Lemma 54.
Let $p, q > 0$. Let $(\mathbf{x}, \mathbf{y})$ be a Gaussian vector and define

$$f(x_1, \ldots, x_k, y_1 \ldots, y_l) = \left(\sum_{1 \le i \le k} p_i e^{\gamma x_i} \right)^p \left(\sum_{1 \le j \le l} q_j e^{\gamma y_j} \right)^q$$

with p_i, q_j some non-negative weights. One verifies that

$$\partial_{x_i, y_j} f \ge 0$$

for all couples (x_i, y_j).
Let $\overline{X}$ be a Gaussian field defined on the same set with the same kernel as X, independent

of X. Let $\overline{M}_n$ be the regularised GMC measure associated with $\overline{X}$. Let (α_i) (resp. (β_j)) be a discret net approximating A (resp. B).
Now consider two sets of Gaussian vectors:
– $(\mathbf{x}, \mathbf{y})$ with $x_i = X(\alpha_i), y_j = X(\beta_j)$;
– $(\overline{\mathbf{x}}, \overline{\mathbf{y}})$ with $\overline{x}_i = X(\alpha_i), \overline{y}_j = \overline{X}(\beta_j)$.
The difference between these two vectors is that by changing the y component, the A part and the B part of the second vector becomes independent. It is clear that these two vectors possess the same correlation structure except that for all couples (x_i, y_j) (resp. $(\overline{x}_i, \overline{y}_j)$), and we have

$$\mathbb{E}[\overline{x}_i \overline{y}_j] \leq \mathbb{E}[x_i y_j].$$

This allows us to apply Slepian's lemma (see Proposition 63) to conclude that

$$\mathbb{E}[f(\overline{x}_i, \overline{y}_j)] \leq \mathbb{E}[f(x_i, y_j)].$$

Taking the Riemann integral limit, we obtain the first inequality in Lemma 54.

For the other inequality, we apply the same idea: choose three mutually independent standard normal Gaussian distributions $N, N', \overline{N}$, independent of X and $\overline{X}$ and consider two sets of Gaussian vectors:
– $(\mathbf{x}, \mathbf{y})$ with $x_i = X(\alpha_i) + rN, y_j = X(\beta_j) + rN'$;
– $(\overline{\mathbf{x}}, \overline{\mathbf{y}})$ with $\overline{x}_i = X(\alpha_i) + r\overline{N}, \overline{y}_j = \overline{X}(\beta_j) + r\overline{N}$;
where r is a constant we choose later. Observe that if $r^2 \geq -2\ln\delta$, then these two vectors possess the same correlation structure except that for all couples (x_i, y_j) (resp. $(\overline{x}_i, \overline{y}_j)$), we have

$$\mathbb{E}[x_i y_j] \leq \mathbb{E}[\overline{x}_i \overline{y}_j].$$

Now we apply Slepian's lemma (see Proposition 63) to conclude that

$$\mathbb{E}[f(x_i, y_j)] \leq \mathbb{E}[f(\overline{x}_i, \overline{y}_j)].$$

Taking the Riemann integral limit, we obtain

$$\mathbb{E}[e^{pr\gamma N} e^{qr\gamma N'} M_n(A)^p M_n(B)^q] \leq \mathbb{E}[e^{(p+q)r\gamma \overline{N}} M_n(A)^p \overline{M}_n(B)^q].$$

Choosing $r^2 = -2\ln\delta$, we get

$$\mathbb{E}[M_n(A)^p M_n(B)^q] \leq \delta^{-2pq\gamma^2} \mathbb{E}[M_n(A)^p] \mathbb{E}[M_n(B)^q].$$

This gives the second inequality in Lemma 54. □

Proof of Lemma 55.
We apply the same idea as in the proof of Lemma 54 with the same function f. We only

give hints about the Gaussian vectors we consider: the rest follows the same lines as in the above proof.
Suppose (α_i) approximate A and (β_j) approximate B. Let $\beta_j^t = \beta_j + te_1$ where e_1 is the first coordinate. Consider two sets of Gaussian vectors:
– $(\mathbf{x}, \mathbf{y})$ with $x_i = X(\alpha_i), y_j = X(\beta_j)$;
– $(\overline{\mathbf{x}}, \overline{\mathbf{y}})$ with $\overline{x}_i = X(\alpha_i), \overline{y}_j = X(\beta_j^t)$.
These two vectors possess the same correlation structure except that for all couples (x_i, y_j) (resp. $(\overline{x}_i, \overline{y}_j)$), and we have

$$\mathbb{E}[\overline{x}_i \overline{y}_j] \leq \mathbb{E}[x_i y_j].$$

In particular, the constant can be chosen to be $C = 1$ for all $t > 0$. □

Proof of Lemma 56.
We apply the same idea as in the proof of Lemma 54 with the same function f. We only give hints about the Gaussian vectors we consider: the rest follows the same lines as in the above proof.
Suppose (α_i) approximate A and (β_j) approximate B. Let $\overline{\beta}_j$ be the symmetry of β_j with respect to the hyperplan $H = H_{a_1^r} = H_{b_1^l}$. Consider two sets of Gaussian vectors:
– $(\mathbf{x}, \mathbf{y})$ with $x_i = X(\alpha_i), y_j = X(\beta_j)$;
– $(\overline{\mathbf{x}}, \overline{\mathbf{y}})$ with $\overline{x}_i = X(\alpha_i), \overline{y}_j = X(\overline{\beta}_j)$.
These two vectors possess the same correlation structure except that for all couples (x_i, y_j) (resp. $(\overline{x}_i, \overline{y}_j)$), and we have

$$\mathbb{E}[\overline{x}_i \overline{y}_j] \geq \mathbb{E}[x_i y_j].$$

Applying Slepian's lemma gives us the desired inequality. □

Proof of Lemma 57.
Suppose $t < 0$ and such that $[b_1^l + t, b_1^r + t] \subset [a_1^l, a_1^r]$. For simplicity we left out the second coordinate (i.e. a_2, b_2) in the following calculations: for example, the rectangle $A = [a_1^l, a_1^r] \times [a_2^l, a_2^r]$ will be simply denoted by $[a_1^l, a_1^r]$. We do not change the second cooordinate of A (resp. B) in the following calculation where all linear shifts and symmetries preserve the second coordinate.
Because for all $x, y \geq 0$, $x^p + y^p$ and $(x+y)^p$ only differ by some positive multiplicative constant (which only depends on p), we have, with C denoting some constant independent

of n which might change from line to line:

$$
\begin{aligned}
&\mathbb{E}[M_n(A)^p M_n(B_t)^q] \\
&\overset{ⓐ}{\geq} C(\mathbb{E}[M_n([a^l, b^l+t])^p M_n([b^l+t, b^r+t])^q] + \mathbb{E}[M_n([b^l+t, a^r])^p M_n([b^l+t, b^r+t])^q]) \\
&\overset{ⓑ}{=} C(\mathbb{E}[M_n([a^l-t, b^l])^p M_n([b^l, b^r])^q] + \mathbb{E}[M_n([b^l+t, a^r])^p M_n([a^r+b^l-b^r, a^r])^q]) \\
&\overset{ⓒ}{\geq} C(\mathbb{E}[M_n([a^l-t, a^r])^p M_n([b^l, b^r])^q] + \mathbb{E}[M_n([b^l+t, a^r])^p M_n([b^l, b^r])^q]) \\
&\overset{ⓓ}{\geq} C(\mathbb{E}[M_n([a^l-t, a^r])^p M_n([b^l, b^r])^q] + \mathbb{E}[M_n([a^l, a^l-t])^p M_n([b^l, b^r])^q]) \\
&\overset{ⓔ}{\geq} C\mathbb{E}[M_n([a^l, a^r])^p M_n([b^l, b^r])^q] \\
&= C\mathbb{E}[M_n(A)^p M_n(B)^q]
\end{aligned}
$$

where
– ⓐ comes from the comparaison between $x^p + y^p$ and $(x+y)^p$ above;
– ⓑ comes from the horizontal translation invariance of X (for the first term) and symmetry of X (for the second term);
– ⓒ comes from the symmetry case Lemma 56 (applied to the second term);
– ⓓ comes from the adjacent-to-non-adjacent case Lemma 55 (applied to the second term);
– ⓔ comes from (again) the comparaison between $x^p + y^p$ and $(x+y)^p$ above.
This concludes Lemma 57.
One can find a pictural representation of each inequality above in Figure 2.2 below. □

Muirhead's inequality

Proposition 61 (Simple case of Muirhead's inequality).
Let $0 < p_1 \leq p_2 \leq \frac{p}{2}$. *Then for* $a, b \geq 0$,

$$a^{p_1} b^{p-p_1} + a^{p-p_1} b^{p_1} \geq a^{p_2} b^{p-p_2} + a^{p-p_2} b^{p_2}.$$

Proof of Proposition 61.
We can rewrite the inequality as

$$a^{p_1} b^{p_1} (a^{p_2-p_1} - b^{p_2-p_1})(a^{p-p_2-p_1} - b^{p-p_2-p_1}) \geq 0,$$

and this is true because by assumption, $p_2 \geq p_1$, $p \geq p_1 + p_2$. □

Inspired by [KP76], we recall a consequence of this inequality:

Corollaire 62 (Expansion inequality).
Let $k \in \mathbb{N}^*$ *and* $k \leq p \leq k+1$. *Then for all* $x, y \geq 0$, *we have*

$$(x+y)^p \leq x^p + y^p + C(x^k y^{p-k} + x^{p-k} y^k) \tag{2.83}$$

where C *is a constant depending only on* p.

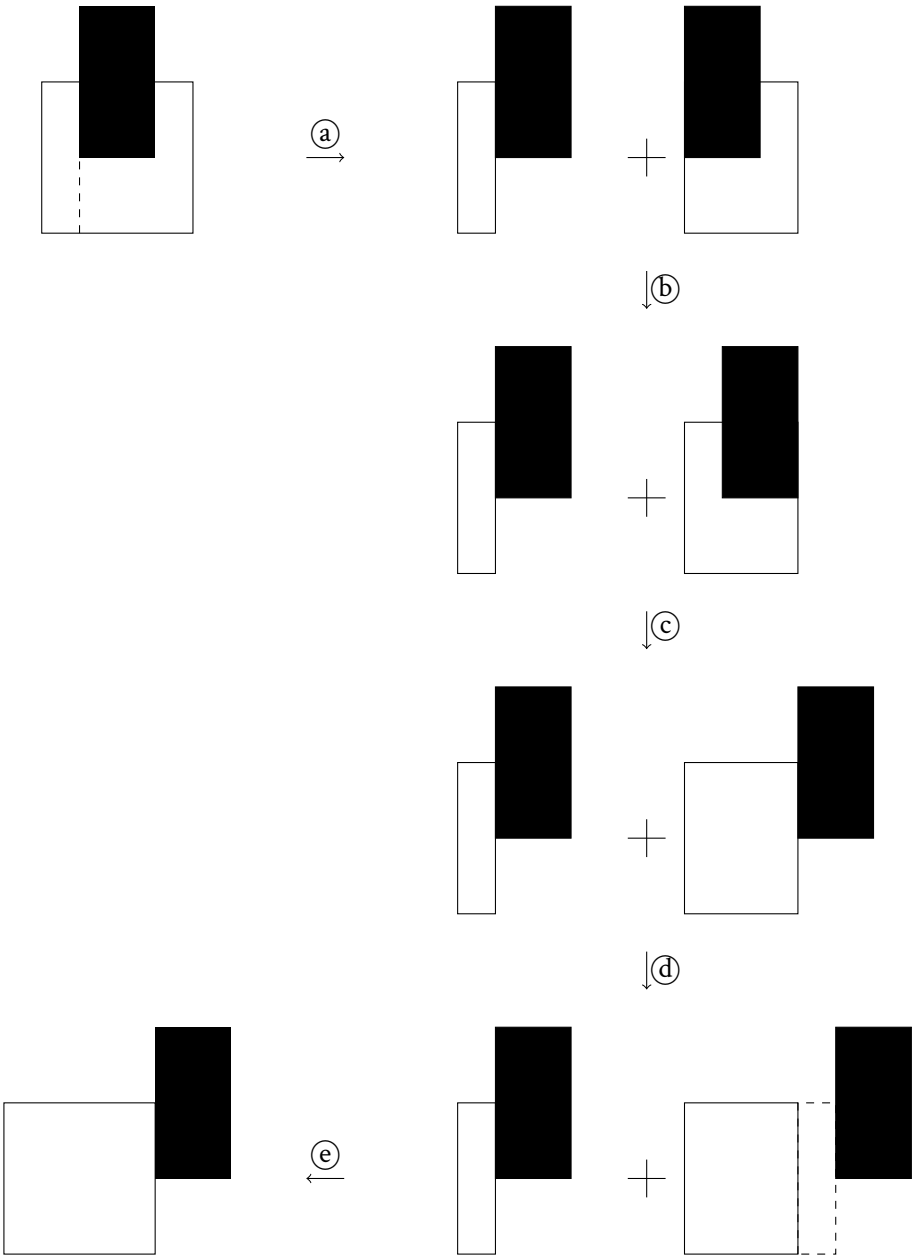

Figure 2.2 – Proof of Lemma 57, a graphical representation.

Proof of Corollary 62.
We have, by sub-additivity of $x \mapsto x^{\frac{p}{k+1}}$,

$$\begin{aligned}
&(x+y)^p \\
=&((x+y)^{p/k+1})^{k+1} \\
\leq&(x^{p/k+1}+y^{p/k+1})^{k+1} \\
\leq&x^p+y^p+\sum_{i=1}^{k} C_{k+1}^{i} x^{\frac{ip}{k+1}} y^{\frac{(k+1-i)p}{k+1}}.
\end{aligned}$$

By Proposition 63, for each pair of crossterms with index $(i, k+1-i)$, we have (with $1 \leq i \leq k$)

$$x^{\frac{ip}{k+1}} y^{\frac{(k+1-i)p}{k+1}} + x^{\frac{(k+1-i)p}{k+1}} y^{\frac{ip}{k+1}} \leq x^k y^{p-k} + x^{p-k} y^k.$$

Since the number of cross-terms is finite, one can choose C big enough (say $C = 2^{k+1}$) to conclude. □

Slepian's lemma

Proposition 63 (Slepian's lemma). *Let $T = \{1, \ldots, n\}$. Let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional centered Gaussian vectors. Assume the existence of some subset $\Omega \subset T \times T$ such that:*

$$\begin{aligned}
\mathbb{E}[X_i X_j] &\leq \mathbb{E}[Y_i Y_j], \quad (i,j) \in \Omega. \\
\mathbb{E}[X_i X_j] &= \mathbb{E}[Y_i Y_j], \quad (i,j) \notin \Omega.
\end{aligned}$$

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is smooth, with appropriate growth at infinity (exponential growth is fine) as well as its first and second derivative, and

$$\partial_{ij} f \geq 0 \quad (\textit{resp.} \leq 0), \quad (i,j) \in \Omega.$$

Then $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ (resp. $\mathbb{E}[f(X)] \geq \mathbb{E}[f(Y)]$).

Proof of Proposition 63.
See [Zei16, Theorem 3] for a slightly stronger form. □

Proof of Lemma 48

We introduce the Fourier coefficients $\alpha_v(n) \geq 0$ for $n \in \mathbb{Z}, v \in [1, \infty[$ given by

$$|\alpha_v(n)|^2 = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} (1 - |v(e^{i\theta} - 1)|^{\frac{1}{2}})_+ d\theta.$$

We consider a standard white noise W on $[1,\infty[\times\partial\mathbb{D}$ and we set

$$\bar{X}_\epsilon(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \alpha_v(n)e^{in\theta}\frac{1}{\sqrt{2\pi}}\int_1^{\frac{1}{\epsilon}}\int_0^{2\pi}\frac{e^{-inu}}{\sqrt{2\pi v}}W(dv,du)$$

Observe that $\alpha_v(n) = \alpha_v(-n)$ for $n \geq 0$ in such a way that $\bar{X}_\epsilon$ is real-valued. Then we can check that

$$\mathbb{E}[\bar{X}_\epsilon(e^{i\theta})\overline{\bar{X}_{\epsilon'}(e^{i\theta})}] = \sum_{n\in\mathbb{Z}} e^{in(\theta-\theta')}\int_1^{\frac{1}{\epsilon}}|\alpha_v(n)|^2\frac{dv}{v} = \int_1^{\frac{1}{\epsilon}}(1-|v(e^{i\theta}-e^{i\theta'})|^{\frac{1}{2}})_+\frac{dv}{v}.$$

Also, notice that we have

$$\bar{X}(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \alpha_v(n)e^{in\theta}\int_1^{\infty}\int_0^{2\pi}\frac{e^{-inu}}{\sqrt{2\pi v}}W(dv,du).$$

Now we compute the correlations between the family $(\bar{X}_\epsilon)_\epsilon$ and the white noise W. We consider a smooth function $H : [1,+\infty[\to \mathbb{R}$ with compact support and a smooth function f on $\partial\mathbb{D}$: we set $F = H\otimes f$ and

$$W(F) = \frac{1}{\sqrt{2\pi}}\int_{[1,+\infty[\times[0,2\pi]} H(v)f(e^{iu})W(dv,du).$$

Therefore, by considering the Fourier coefficients $(c_n(f))_n$ of f, we obtain

$$\begin{aligned}
T_\epsilon(F)(e^{i\theta}) =& \mathbb{E}[\bar{X}_\epsilon(e^{i\theta})W(H\otimes f)]\\
=& \sum_n \frac{1}{2\pi}\int_{[1,1/\epsilon[\times[0,2\pi]}\frac{\alpha_v(n)}{\sqrt{v}}e^{in\theta}f(e^{iu})e^{-inu}H(v)\,dvdu\\
=& \sum_n c_n(f)e^{in\theta}\int_{[1,1/\epsilon[}\frac{\alpha_v(n)}{\sqrt{v}}H(v)\,dv.
\end{aligned}$$

Because H has compact support, it is readily seen that this series defines a continuous function of θ, which converges uniformly as $\epsilon\to 0$ towards a continuous function given by

$$T(F)(e^{i\theta}) = \sum_n c_n(f)e^{in\theta}\int_{[1,\infty[}\frac{\alpha_v(n)}{\sqrt{v}}H(v)\,dv.$$

Backgrounds on fractional Brownian sheet

We look at the main theorem in [Aco14] and we slightly modify the hypothesis (1.2). Let $\{(Y_\epsilon^x : x\in[0,1]^d\}_{\epsilon>0}$ be a family of centerd Gaussian fields indexed by $[0,1]^d$ where d is the dimension of the space. We suppose that for some constant $0 < C_Y < \infty$,

$$\forall x,y\in[0,1]^d, \forall\epsilon>0, |Cov(Y_\epsilon^x, Y_\epsilon^y) + \log(\max\{\epsilon,|x-y|\})| \leq C_Y \tag{2.84}$$

$$\mathbb{E}[(Y_\epsilon^x - Y_\epsilon^y)^2] \leq C_Y \epsilon^{-1/2}|x-y|^{1/2} \text{ if } |x-y| \leq \epsilon \tag{2.85}$$

where $|\cdot|$ is the Euclidean distance.
We claim that

Théorème 64. *There exist constants $0 < c, C < \infty$ and a small $\epsilon_0 > 0$ (all depending of C_Y and d) such that for all $0 < \epsilon \leq \epsilon_0$ and all $\lambda \geq 0$,*

$$\mathbb{P}[|\max_{x\in[0,1]^d} Y_\epsilon^x - m_\epsilon| \geq \lambda] \leq Ce^{-c\lambda}$$

We adapt the proof by introducing the fractional Brownian sheet. Recall that a (one-dimensional) fractional Brownian sheet $B_0^H = \{B_0^H, t \in \mathbb{R}^N\}$ with Hurst index $H = (H_1, \ldots, H_N), 0 < H_j < 1$ is a real-valued centered Gaussian field with covariance structure

$$\mathbb{E}[B_0^H(s)B_0^H(t)] = \prod_{j=1}^N \frac{1}{2}(|s_j|^{2H_j} + |t_j|^{2H_j} - |s_j - t_j|^{2H_j}), s, t \in \mathbb{R}^N.$$

In particular B_0^H is self-similar, i.e. for all constants $c > 0$,

$$\left\{B_0^H(ct), t \in \mathbb{R}^N\right\} = \left\{c^{\sum_{j=1}^N H_j} B_0^H(t), t \in \mathbb{R}^N\right\}$$

in distribution.
In view of comparing with equation (2.85), we will choose a d-dimensional vector H with all H_j equal to $1/4$. Let us denote this particular fractional Brownian sheet by Φ.
We now define the field Φ_ϵ on $[0, \epsilon[^d$ by linearly shrinking the region $[p, 2p[^d$ of Φ onto $[0, \epsilon[^d$, that is, $(\Phi_\epsilon(x), x \in [0, \epsilon[^d) = (\Phi(l(x)), l(x) \in [p, 2p[^d)$ where l is the affine map from $[0, \epsilon[^d$ to $[p, 2p[^d$. Notice that Φ_ϵ depends on the choice of p, and p can be chosen as large as desired.
Let us recall two estimations that are useful for the proof (compare with equations (2.7) and (2.8) in [Aco14]):
Following the definition of fractional Brownian sheet:

$$p^{d/2} \leq Var(\Phi_\epsilon(x)) \leq (2p)^{d/2} \tag{2.86}$$

Combine self-similarity of Φ with lemma 3.4 from [AX05] we deduce that there exist $c, C > 0$ such that

$$cp^{d/2}\epsilon^{-1/2}|x-y|_2^{1/2} \leq \mathbb{E}[(\Phi_\epsilon(x) - \Phi_\epsilon(y))^2] \leq Cp^{d/2}\epsilon^{-1/2}|x-y|_2^{1/2} \tag{2.87}$$

where $|\cdot|_2$ is the 2-norm, which is equivalent to the Euclidean norm.
New following [Aco14] we will divide $[0, 1[^d$ into boxes of side length $\epsilon > 0$ and assign values to each box using independent copies of Φ_ϵ.
We first recover Lemma 2.2 in [Aco14]. We claim that

Lemme 65. *There exist constants* $0 < c, C < \infty$ *(depending on* p *and* d*) such that*

$$\sup_{v \in V_\epsilon} \mathbb{P}(\sup_{x \in \square_\epsilon^v} \Phi_\epsilon(x) \geq \lambda) \leq Ce^{-c\lambda^2} \tag{2.88}$$

To prove this lemma we use Fernique's majorizing measure argument. Notice that

$$B(x,r) := \{y \in \square_\epsilon^v : \mathbb{E}[(\Phi_\epsilon(x) - \Phi_\epsilon(y))^2] \leq r^2\} \supset \{y \in \square_\epsilon^v : Cp^{d/2}\epsilon^{-1/2}|y-x|_2^{1/2} \leq r^2\}$$

so that

$$\mu(B(x,r)) \geq Cr^{4d}$$

for some $C > 0$ depending on p and d.
Applying the majorizing measure technique we obtain

$$\mathbb{E}[\sup_{x \in \square_\epsilon^v} \Phi_\epsilon(x)] \leq C \int_0^\infty \sqrt{-\log(cr^{4d})}dr \leq C < \infty$$

then we complete the proof of Lemma 2.2 by invoking Borell's inequality:

$$\mathbb{E}[\sup_{x \in \square_\epsilon^v} \Phi_\epsilon(x) \geq C + \lambda] \leq e^{-\lambda^2/2(2p)^{d/2}}$$

the quantity on the right results from (2.86).
We then follow exactly the same steps as in [Aco14] (the only difference is to replace some d's by $d/2$'s because of (2.86)) to recover the main theorem.

Chapter 3

Two perspectives of the 2D unit area quantum sphere and their equivalence

Authors

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Abstract

2D Liouville quantum gravity (LQG) is used as a toy model for 4D quantum gravity and is the theory of world-sheet in string theory. Recently there has been growing interest in studying LQG in the realm of probability theory: David, Kupiainen, Rhodes, Vargas (2014) and Duplantier, Miller, Sheffield (2014) both provide a probabilistic perspective of the LQG on the 2D sphere. In particular, in each of them one may find a definition of the so-called unit area quantum sphere. We examine these two perspectives and prove their equivalence by showing that the respective unit area quantum spheres are the same. This is done by considering a unified limiting procedure for defining both objects.

Introduction

2D Liouville quantum gravity (LQG) was first introduced by Polyakov in [Pol81] as a framework for integrating over surfaces. On the one hand, this integration over surfaces can be used to describe the time-evolution of bosonic strings and on the other hand, it provides a model for a random metric with a fixed topology, i.e. for quantum gravity [Sei90]. Whereas the theory for surfaces of non-trivial moduli remains to be understood, the basic theory and the constructions for the 2D sphere are well-known in the physics literature [Nak04]. In the realm of rigorous probability theory, however, even the understanding of the simplest case, i.e. the theory of the random metric on the 2D sphere is relatively recent.

Liouville quantum gravity on the sphere is the limiting object of natural discrete random planar map models on the sphere. This provides one way to define the limiting object as a random measure endowed metric space [LG13, Mie13], however the conformal structure is lacking in this framework. Indeed, as in the case of 2D Riemannian manifolds, this metric, although highly non-smooth, should nevertheless determine a conformal structure such that both the metric and the measure are described using a 2D Gaussian free field defined on this conformal structure [Pol81, Sei90]. For now (except for the recent progress in the case of $\gamma = \sqrt{8/3}$, see [MS15a, MS16a, MS16b]) one can only define the corresponding volume form, that takes roughly the form '$e^{\gamma h}dz$' where h is a 2D Gaussian free field type of distribution and dz the volume element. In fact, two different constructions of the volume form corresponding to the unit area quantum sphere have recently appeared in the probability literature.

[DKRV16] rigorously constructed the Liouville quantum field theory on the sphere by proving that the partition function of the theory is indeed well-defined. As a consequence, [DKRV16] also provides an exact formulation of the Liouville measure on the sphere conditioned to have unit volume. Later [HRV15, DRV16] generalized the construction to the disk and to the torus and in [DKRV15], the authors constructed the Liouville quantum gravity on the sphere in a certain critical situation – when the so-called Seiberg bound is saturated. In [KRV15] the authors further verify that the conformal Ward and BPZ identities for Liouville quantum field theory can be derived in the probabilistic framework. This is an important step in their project of unifying the path integral approach and the conformal bootstrap approach of Liouville conformal field theory. An important feature of this approach is the use of Gaussian multiplicative chaos theory and its strong link to the original physics literature.

Another approach stems from [She16], where the author suggested a limiting procedure involving the Gaussian free field. Following up this work, in [DMS14], the authors

provided a more concrete construction via Bessel processes and showed that it is equivalent to the limiting procedure suggested in [She16]. [DMS14] also rigorously constructed objects like quantum disks, quantum wedges and quantum cones. In [MS15b], the authors provide further constructions of the quantum sphere. In particular, they show that the unit area quantum sphere can be obtained from mating a pair of correlated continuum trees of finite diameter, which are given by space filling SLE curves. This exemplifies an important aspect of their approach: the interplay between Gaussian free field and Schramm-Loewner evolution, already known to be linked to conformal field theories [Car05].

The [DKRV16] approach considers the so-called Liouville measure directly in the space of random measures whereas in the [DMS14] approach one defines the quantum sphere as an equivalence classes of random measures. On one hand, the [DKRV16] approach is more explicit. On the other hand, the notion of equivalence class enables [DMS14] to work with one or two marked points whereas the framework of [DKRV16] is restricted to 3 or more points.

Both approaches provide evidence that they give the correct scaling limit of the random planar maps weighted by critical statistical mechanics models with 3 uniformly chosen marked points, (see [DKRV16, Section 5.3] and [MS15b, Section 1.2],[GS15]). Following the notions preferred by the original authors, the candidate via the [DKRV16] approach is called the *unit volume Liouville measure with three insertion points of weight* γ. The candidate via the [DMS14] approach is called the *unit area quantum sphere with three marked points*. A natural question is whether the two constructions actually agree. The main aim of this article is to give an affirmative answer to this question (Theorem 66). On the way we also revisit the two perspectives in some detail and provide a unified framework to work with both of them.

We finally remark that there is yet a third mathematical approach to defining a unit area quantum sphere in the case when $\gamma = \sqrt{8/3}$ (the so-called pure gravity). In this case, the unit area quantum sphere can be defined as the scaling limit of a large class of random planar maps including uniform triangulations, uniform quadrangulations etc. Indeed, it has been shown that these random planar maps converge as metric spaces to the Brownian map in the Gromov-Hausdorff topology [LG13, Mie13]. Moreover, [MS16c, MS15a] recently announced that there is a canonical way of putting a metric on the unit area quantum sphere with $\gamma = \sqrt{8/3}$ via the quantum Loewner evolution to obtain the Brownian map, thus proving the equivalence between the Brownian map and the unit area quantum sphere defined in [DMS14]. Our result relates [DKRV16] to this body of works, by showing that the unit area quantum sphere can be constructed also via their approach.

Acknowledgements We are grateful to Scott Sheffield for suggesting this problem and helpful discussions. We are also thankful to François David, Bertrand Duplantier, Jason Miller, Rémi Rhodes and Vincent Vargas for useful discussions. We thank the anonymous referee and Bertrand Duplantier for their careful reading and many helpful comments on the manuscript. The authors would like to thank the Isaac Newton Institute for Mathemat-

ical Sciences, Cambridge, for support and hospitality during the program *Random Geometry* where most of the work on this paper was undertaken. J. Aru acknowledges support of the SNF grant SNF-155922, Y. Huang would like to thank the support from the Cambridge-PSL French embassy fund and X. Sun was partially supported by NSF grant DMS-1209044.

Outline and main results

We will revisit the two constructions of the unit area quantum sphere in detail in Section 3. Here we first give a brief description in order to state our main theorem.

In [DKRV16], given an integer $k \geq 3$, $z_1, \cdots, z_k \in \mathbb{C} \cup \{\infty\}$ and $\alpha_1 \cdots, \alpha_k \in \mathbb{R}$ with $\alpha_1, \cdots, \alpha_k$ satisfying certain bounds, one can use the Gaussian multiplicative chaos theory to construct the so-called Liouville measure on the sphere with log singularity α_i at z_i. Then by conditioning on the quantum area to be 1, one obtains a Liouville measure with unit volume. We denote the law of unit volume Liouville measures obtained in this way by $\mu_{\mathrm{DKRV}}^{\alpha_1,\cdots,\alpha_k}$. The detailed construction will be provided in Section 3.

In [DMS14], the construction of the unit area quantum sphere is based on the notion of quantum surfaces — an equivalence class of random distributions with a number of marked points. In Section 3 we revisit the Bessel process construction of the unit area quantum sphere with 2 marked points, whose law is denoted by μ_{DMS}^2. The unit area quantum sphere with $k \geq 3$ marked points (denoted by μ_{DMS}^k) can be obtained by first sampling according to μ_{DMS}^2 and then sampling $k-2$ points according to the quantum area independently of each other.

The following is the main theorem of the paper:

Théorème 66. *For $\gamma \in (0,2)$,*

$$\mu_{\mathrm{DMS}}^3 = \mu_{\mathrm{DKRV}}^{\gamma,\gamma,\gamma}.$$

More precisely, if we embed μ_{DMS}^3 such that the three marked points are fixed at $z_1, z_2, z_3 \in \mathbb{C} \cup \{\infty\}$, then it has the same distribution as $\mu_{\mathrm{DKRV}}^{\gamma,\gamma,\gamma}$ with marked points of weight γ, γ, γ at z_1, z_2, z_3.

Remarque 67. Whereas we state the equivalence in terms of measures, it also holds in terms of underlying fields and thus underlying quantum surfaces. Indeed, in [BSS14], it is shown that a Gaussian free field type field and the Liouville measure it induces determine each other. One can check that this result also applies in the current context.

Remarque 68. Notice that $\mu_{\mathrm{DMS}}^4 \neq \mu_{\mathrm{DKRV}}^{\gamma,\gamma,\gamma,\gamma}$. In fact, according to the definition, the cross ratio of the 4 marked points of μ_{DMS}^4 is a non-trivial random variable. On the side of $\mu_{\mathrm{DKRV}}^{\gamma,\gamma,\gamma,\gamma}$, however, the cross ratio is simply the cross ratio of the four given points z_1, z_2, z_3, z_4. For the same reason Theorem 66 is not true for any $k > 3$. The value $k = 3$ is special because on the one hand it fixes a conformal structure on the sphere, but on the other hand all spheres with three marked points remain conformally equivalent.

On the one hand, the area measure of random planar quadrangulations with n faces, with each face carrying area $1/n$ and with four marked points chosen independently from the area measure, should conjecturally converge to μ^4_{DMS} in the limit $n \to \infty$. On the other hand, if one first embeds random planar quadrangulations with n faces, with each face carrying area $1/n$ and with three marked points, to the Riemann sphere with three marked points z_1, z_2, z_3 using say circle-packing embedding, and then adds a suitable singularity at a fixed point z_4 (see [DKRV16] section 5.3), one expects to obtain in the $n \to \infty$ limit the random measure $\mu^{\gamma,\gamma,\gamma,\gamma}_{\mathrm{DKRV}}$ with four marked points at z_1, z_2, z_3, z_4.

The idea of the proof is to approximate $\mu^{\gamma,\gamma,\gamma}_{\mathrm{DKRV}}$ and μ^3_{DMS} using essentially the same limiting procedure. In Section 3 we will provide an approximation scheme for $\mu^{\gamma,\gamma,\gamma}_{\mathrm{DKRV}}$ which is very close to the one in [DMS14, Section 5.3]. In Section 3 we will show that a slight perturbation of the scheme in Section 3 leads to both $\mu^{\gamma,\gamma,\gamma}_{\mathrm{DKRV}}$ and μ^3_{DMS}, thus concluding the proof.

Two perspectives of the unit area quantum sphere

In this section we first provide necessary background on the whole-plane Gaussian free field and then describe the two perspectives in some more detail.

The whole plane Gaussian free field

As in [She07], the whole plane GFF h is random distribution module an additive constant satisfying

$$\mathrm{Var}[(h,\phi)] = -\int \phi(x)\log|x-y|\phi(y)dxdy, \quad \forall \phi \in C_0^\infty(\mathbb{R}^2) \text{ s.t. } \int \phi(x)dx = 0.$$

In other words, it is a Gaussian process only indexed by zero mean test functions.

One can also make h into an honest random distribution (i.e. a scalar field) by pinning it: namely, we choose some finite measure $\rho(z)dz$ on $\mathbb{C} \cup \{\infty\}$ satisfying

$$\int\int |\rho(x)\log|x-y|\rho(y)|dxdy < \infty$$

and set the average

$$(h,\rho) = \int h(z)\rho(z)dz$$

to be zero, i.e. to pin the field using ρ. We sometimes call $\rho(z)dz$ the background measure. Then all the other averages (h,ρ') are defined by choosing $c > 0$ such that $c\rho$ has the same mass as ρ' and setting

$$(h,\rho') := (h,\rho' - c\rho).$$

If ρ has unit mass, then the Green's function of such a field is given by [DKRV16]:

$$G_\rho(z,w) = \log\frac{1}{|z-w|} - m_\rho(\log\frac{1}{|w-\cdot|}) - m_\rho(\log\frac{1}{|z-\cdot|}) + \theta_\rho, \tag{3.1}$$

where we set

$$m_\rho(f) := \int_{\mathbb{R}^2} f(z)\rho(z)dz$$

and

$$\theta_\rho = -\int\int_{\mathbb{R}^2\times\mathbb{R}^2} \rho(z)\log|z-w|\rho(w)dwdz.$$

We denote the whole-plane GFF pinned using ρ by h_ρ, its probability measure by $\mathbb{P}_\rho$ and the corresponding expectation operator by $\mathbb{E}^{\mathbb{P}_\rho}$. For a different background measure the scalar whole plane GFF differs by a random constant. No choice of background measure makes the field truly conformal invariant – conformal transformations change the pinning of the field. [DKRV16] circumvents this by tensoring the whole plane GFF with the Lebesgue measure on $\mathbb{R}$ to obtain a Mobius invariant, but infinite measure. We will see the precise meaning in the next subsection.

One of the natural choices for the background measure is the normalized area measure of the spherical metric

$$\hat{g}(x) = \pi^{-1}(1+|x|^2)^{-2}$$

of total mass 1. More explicitly, we define $h_{\hat{g}}$ to be the random distribution with covariance given by [DKRV16, Equation (2.12)]:

$$G_{\hat{g}}(x,y) := \log\frac{1}{|x-y|} - \frac{1}{4}(\log\hat{g}(x) + \log\hat{g}(y)) - \frac{1}{2}.$$

We refer to [MS13] for more information on whole plane Gaussian free field. For later purpose we record [MS13, Proposition 2.10] stating that one can approximate the whole-plane GFF using Dirichlet free fields on a sequence of growing domains:

Proposition 69. *Suppose D_n is a sequence of growing domains with harmonically non-trivial boundary containing 0 s.t. $\mathrm{dist}(0,\partial D_n) \to \infty$. In each D_n we define a Dirichlet GFF h_n with uniformly bounded boundary values. Then on any fixed bounded domain D, the restrictions of h_n to D converge in total variational distance to the restriction of the whole plane GFF to D, seen as a distribution modulo additive constant.*

David-Kupiainen-Rhodes-Vargas' approach

In this section we provide a detailed description of the construction of the unit volume Liouville measure in [DKRV16]. We identify the sphere $\mathbb{S}^2$ with $\mathbb{C}\cup\{\infty\}$ via the stereographic projection map. Before going into details of the construction, we remark that

in order to understand the Definition 70 of the unit volume Liouville measure, one only needs (3.2), (3.4), (3.7) and the definition of the Gaussian multiplicative chaos measure, as explained between (3.3) and (3.4).

For the spherical metric $\hat{g}$, the first step in constructing the Liouville field on $\mathbb{C} \cup \{\infty\}$ with marked points at $z_1, \ldots, z_k$ and log-singularities α_i at z_i is to consider the whole-plane GFF with a shift term corresponding to the curvature of the metric and additional singularities corresponding to the marked points:

$$h_L(z) = h_{\hat{g}}(z) + \frac{Q}{2} \log \hat{g}(z) + \sum_{i=1}^{k} \alpha_i G_{\hat{g}}(z, z_i). \tag{3.2}$$

Here $Q = 2/\gamma + \gamma/2$ with $\gamma \in (0, 2)$ a fixed parameter. Now, given the so-called cosmological constant $\hat{\mu} > 0$ and the parameter $\gamma \in (0, 2)$, we define the partition function for the Liouville field, motivated by the physics literature. For any bounded continuous functional F acting on $H^{-1}(\mathbb{C})$ we set:

$$\Pi_{\gamma,\hat{\mu}}^{(z_i,\alpha_i)_i}(F) = e^{C(z)} \prod_i \hat{g}(z_i)^{\Delta_{\alpha_i}} \int_{\mathbb{R}} e^{sc} \mathbb{E}^{\mathbb{P}_{\hat{g}}}[F(c + h_L) \exp(-\hat{\mu} e^{\gamma c} \mu_{h_L}(\mathbb{C}))] dc. \tag{3.3}$$

Here, for a fixed parameter γ, $\mu_{h_L}(\cdot)$ denotes the Gaussian multiplicative chaos measure constructed from the log-correlated field $h_L(z)$. Shortly, it is the limit in probability of the measures $\mu_{h_L}^{\varepsilon}(\cdot)$ defined by setting for any Borel $A \subset \mathbb{C}$,

$$\mu_{h_L}^{\varepsilon}(A) = \int_A \varepsilon^{\frac{\gamma^2}{2}} \exp(\gamma h_L^{\varepsilon}(z)) dz,$$

where h_L^{ε} is, for example, the circle-average approximation of h_L. See e.g. [DKRV16]) for more details. Moreover, we set

$$\begin{aligned} s &= \sum_{i=1}^{k} \alpha_i - 2Q, \\ C(z) &= \frac{1}{2} \sum_{i \neq j} \alpha_i \alpha_j G_{\hat{g}}(z_i, z_j) + \frac{\theta_{\hat{g}} + \log 2}{2} \sum_i \alpha_i^2, \\ \Delta_\alpha &= \frac{\alpha}{2}(Q - \frac{\alpha}{2}). \end{aligned} \tag{3.4}$$

Finally, the integration over c corresponds to the tensoring of the free field with the Lebesgue measure mentioned above.

In [DKRV16, Theorem 3.2] it is proved that the partition function (3.3) is non-trivial, finite and can be approximated by a circle-average regularization procedure if and only if

$\alpha_1, \cdots, \alpha_k \in \mathbb{R}$ satisfy the following Seiberg bounds:

$$\sum_{i=1}^{k} \alpha_i > 2Q \quad \text{(first Seiberg bound);}$$
$$\alpha_i < Q, \ \forall i \quad \text{(second Seiberg bound).}$$

We remark that the two Seiberg bounds yield $k \geq 3$. This suits well with the need of three marked points for fixing the conformal structure of the sphere.

Under these constraints the partition function gives rise to a probability measure on $H^{-1}(\mathbb{C})$, called the *Liouville field*. Its law is denoted by $\mathbb{P}^{\gamma,\hat{\mu}}_{(z_i,\alpha_i,\hat{g})}$ and it is described by setting:

$$\mathbb{E}^{\gamma,\hat{\mu}}_{(z_i,\alpha_i,\hat{g})}[F] = \frac{\Pi^{(z_i,\alpha_i)_i}_{\gamma,\hat{\mu}}(F)}{\Pi^{(z_i,\alpha_i)_i}_{\gamma,\hat{\mu}}(1)}. \tag{3.5}$$

for all bounded continuous functionals F on $H^{-1}(\mathbb{C})$.

Via this partition function and again using the theory of Gaussian multiplicative chaos, the Liouville field also induces a corresponding Liouville measure $M(\cdot)$ on the sphere by specifying for all bounded continuous functional F on $\mathbb{R}^n_+$ and all Borel sets $A_1, \cdots, A_n$ in $\mathbb{C}$:

$$\begin{aligned}
&\mathbb{E}^{\gamma,\hat{\mu}}_{(z_i,\alpha_i,\hat{g})}[F(M(A_1), \cdots, M(A_n)] \\
=&\frac{\int_{\mathbb{R}} e^{sc}\mathbb{E}^{\mathbb{P}_{\hat{g}}}[F(e^{\gamma c}\mu_{h_L}(A_1), \cdots, e^{\gamma c}\mu_{h_L}(A_n)) \exp(-\hat{\mu}e^{\gamma c}\mu_{h_L}(\mathbb{C}))]dc}{\int_{\mathbb{R}} e^{sc}\mathbb{E}^{\mathbb{P}_{\hat{g}}}[\exp(-\hat{\mu}e^{\gamma c}\mu_{h_L}(\mathbb{C}))]dc}.
\end{aligned}$$

Using a change of variables $e^{\gamma c}\mu_{h_L}(\mathbb{C}) = y$ we can rewrite this as follows:

$$\begin{aligned}
&\mathbb{E}^{\gamma,\hat{\mu}}_{(z_i,\alpha_i,\hat{g})}[F(M(A_1), \cdots, M(A_n))] \\
=&\frac{\int_0^\infty \mathbb{E}^{\mathbb{P}_{\hat{g}}}\left[F(y\frac{\mu_{h_L}(A_1)}{\mu_{h_L}(\mathbb{C})}, \cdots, y\frac{\mu_{h_L}(A_n)}{\mu_{h_L}(\mathbb{C})})\mu_{h_L}(\mathbb{C})^{-\frac{s}{\gamma}}\right] e^{-\hat{\mu}y}y^{\frac{s}{\gamma}-1}dy}{\hat{\mu}^{\frac{s}{\gamma}}\Gamma(\frac{s}{\gamma})\mathbb{E}^{\mathbb{P}_{\hat{g}}}\left[\mu_{h_L}(\mathbb{C})^{-\frac{s}{\gamma}}\right]}.
\end{aligned}$$

Thus the Liouville measure on the sphere can be in fact described as follows:

1. $M(\mathbb{C})$ is independent of the normalized measure $\bar{M}(\cdot) := M(\cdot)/M(\mathbb{C})$;
2. $M(\mathbb{C})$ is distributed as Gamma distribution $\Gamma(\frac{s}{\gamma}, \hat{\mu})$;
3. The normalized measure $\bar{M}(\cdot)$ can be sampled by the normalized measure of μ_{h_L} re-weighted by $\mu_{h_L}(\mathbb{C})^{-\frac{s}{\gamma}}$. Namely, write $\bar{\mu}_{h_L}(\cdot) := \mu_{h_L}(\cdot)/\mu_{h_L}(\mathbb{C})$ then

$$\mathbb{E}^{\gamma,\hat{\mu}}_{(z_i,\alpha_i,\hat{g})}[F(\bar{M}(\cdot))] = \frac{\mathbb{E}^{\mathbb{P}_{\hat{g}}}\left[F(\bar{\mu}_{h_L}(\cdot))\mu_{h_L}(\mathbb{C})^{-\frac{s}{\gamma}}\right]}{\mathbb{E}^{\mathbb{P}_{\hat{g}}}\left[\mu_{h_L}(\mathbb{C})^{-\frac{s}{\gamma}}\right]}. \tag{3.6}$$

Notice that from this explicit structural description of the Liouville field, we see that the cosmological constant $\hat{\mu}$ only effects the total volume of the Liouville measure of the sphere, while the normalized measure is independent of $\hat{\mu}$. It is natural to then take the reweighted normalized measure $\bar{\mu}_{h_L}(\cdot)$ as the definition of the unit volume Liouville measure with specific log-singularities. As stated in [DKRV16, Lemma 3.10], for (3.6) to make sense, we only need (see Lemma 73):

$$Q - \frac{\sum_{i=1}^{k} \alpha_i}{2} < \frac{2}{\gamma} \wedge \min_i (Q - \alpha_i) \quad \text{and} \quad \alpha_i < Q, \forall i. \tag{3.7}$$

Now we are ready to give a formal definition of $\mu_{\mathrm{DKRV}}^{\alpha_1,\cdot,\alpha_k}$. The expression for the Liouville field comes from the same change of variables as above:

Définition 70. *Fix $\gamma \in (0,2)$. Let $k \geq 3$, $z_1, \cdots, z_k$ be distinct points in $\mathbb{C} \cup \{\infty\}$ and $\alpha_1, \cdots, \alpha_k \in \mathbb{R}$ satisfying (3.7). Let h_L be defined as in (3.2), s as in (3.4) and $\mu_{h_L}(\cdot)$ be the Gaussian multiplicative chaos measure corresponding to h_L.*

Then the unit volume Liouville field on the sphere with marked points $(z_1, \cdots, z_k)$ and log-singularities $(\alpha_1, \cdots, \alpha_k)$ is a random field distributed as

$$h_L - \gamma^{-1} \log \mu_{h_L}(\mathbb{C})$$

under the re-weighted measure

$$d\widehat{\mathbb{P}}^{(\alpha_i,z_i)_i} := \mathbb{E}^{\mathbb{P}_{\hat{g}}} \left[\mu_{h_L}(\mathbb{C})^{\frac{-s}{\gamma}} \right]^{-1} \mu_{h_L}(\mathbb{C})^{\frac{-s}{\gamma}} d\mathbb{P}_{\hat{g}}.$$

Correspondingly, the unit volume Liouville measure, denoted hereafter by $\mu_{\mathrm{DKRV}}^{\alpha_1,\cdot,\alpha_k}$, is the measure distributed as $\bar{\mu}_{h_L}(\cdot)$ under $\widehat{\mathbb{P}}^{(\alpha_i,z_i)_i}$.

Notice that we did not include $\hat{g}$ in the notation of $\widehat{\mathbb{P}}^{(\alpha_i,z_i)_i}$. This is due to Theorem 71 which proves that the unit volume Liouville measure does not depend on the choice of normalization. We will be interested in the special case of $k = 3$ and $\alpha_i = \gamma$ for all $i = 1, 2, 3$. Note that in this case (3.7) is satisfied for all $\gamma \in (0, 2)$.

Properties of the Liouville measure: moments, covariance under Mobius transforms and independence of the background measure

In this section we record two invariance properties for the unit volume Liouville measure $\mu_{\mathrm{DKRV}}^{\alpha_1,\cdots,\alpha_k}$, and a criterion for the existence of moments of the Liouville measure.

First, we claim Definition 70 is independent of the background measure, justifying our notation $\widehat{\mathbb{P}}^{(\alpha_i,z_i)_i}$ in that definition. Indeed, in Definition 70 the only role of the background measure is to fix the whole-plane GFF. We will show in Theorem 71 below that this choice does not matter. This corresponds to the Weyl anomaly described in [DKRV16, Lemma 3.11.2)].

Consider a probability measure $\rho(z)dz$ and let h_ρ be the whole-plane GFF normalized such that $(h,\rho)=0$. As above, denote by $m_\rho(f) := \int_{\mathbb{R}^2} f(z)\rho(z)dz$ and set

$$h_{L(\rho)}(z) = h_\rho(z) + 2Qm_\rho(\log\frac{1}{|z-\cdot|}) + \sum_{i=1}^{k}\alpha_i G_\rho(z,z_i), \tag{3.8}$$

where z_i are marked points with log-singularities α_i, and as in [DKRV16], G_ρ is the Green function associated with the background measure ρ as defined in (3.1).

Notice that for the normalized spherical metric $\hat{g}$, the field $h_{L(\hat{g})}$ is the same as h_L defined above. Finally, write $\mu_{h_{L(\rho)}}$ as the volume measure corresponding to the field $h_{L(\rho)}$.

Théorème 71. *Take any ρ_1,ρ_2 as above and let $s = \sum_{i=1}^{k}\alpha_i - 2Q$. The fields*

$$h_{L(\rho_1)} - \gamma^{-1}\log\mu_{h_{L(\rho_1)}}(\mathbb{C}) \quad \textit{under the measure} \quad c\mu_{h_{L(\rho_1)}}(\mathbb{C})^{-\frac{s}{\gamma}}d\mathbb{P}_{\rho_1}$$

and

$$h_{L(\rho_2)} - \gamma^{-1}\log\mu_{h_{L(\rho_2)}}(\mathbb{C}) \quad \textit{under the measure} \quad c\mu_{h_{L(\rho_2)}}(\mathbb{C})^{-\frac{s}{\gamma}}d\mathbb{P}_{\rho_2}$$

are equal in law. Here the constants c are chosen to make the above measures probability measures.

Proof. To begin with, recall that there is an identification between the whole-plane GFF pinned using some background measure ρ and the whole-plane GFF seen as a modulo-constant distribution: the passage from the latter to the former was described in Subsection 3 and to pass from former to the latter, we just test the GFF against zero mean test functions. Using this identification, it suffices to prove the theorem using the probability measure $d\mathbb{P}$ induced by the whole-plane GFF on the modulo-constant distributions. We write (h,ϕ) for this modulo constant field acting on zero mean distributions ϕ. Now, observe that under the measure $d\mathbb{P}$, we have that almost surely $h_{\rho_1} = h_{\rho_2} - (h,\rho_1-\rho_2)$. Write also

$$g_{\rho_i} = h_{L(\rho_i)} - h_{\rho_i}, \quad i=1,2.$$

Then in particular

$$h_{L(\rho_1)} = h_{\rho_2} + g_{\rho_1} - (h,\rho_1-\rho_2).$$

As for $i=1,2$ the quantity $h_{L(\rho_i)} - \gamma^{-1}\log\mu_{h_{L(\rho_i)}}(\mathbb{C})$ remains unchanged after adding a constant to the field, we can write:

$$h_{L(\rho_1)} - \gamma^{-1}\log\mu_{h_{L(\rho_1)}}(\mathbb{C}) = h_{\rho_2} + g_{\rho_1} - \gamma^{-1}\log\mu_{h_{\rho_2}+g_{\rho_1}}(\mathbb{C}).$$

In addition, we also have that

$$\mu_{h_L(\rho_1)}(\mathbb{C})^{-\frac{s}{\gamma}} = \mu_{h_{\rho_2}+g_{\rho_1}}(\mathbb{C})^{-\frac{s}{\gamma}}\exp(s(h,\rho_1-\rho_2)).$$

By the Cameron-Martin theorem for the modulo-additive constant Gaussian free field (see e.g. [Jan97]), reweighing $d\mathbb{P}$ by $\exp(s(h, \rho_1 - \rho_2))$ induces for any zero mean test function ϕ a drift equal to

$$-s \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \phi(z) \log |z - w| (\rho_1 - \rho_2)(w) dz dw$$

on (h, ϕ). Now as the difference of Green's functions $G_{\rho_1}(z, z_i) - G_{\rho_2}(z, z_i)$ satisfies

$$\Delta(G_{\rho_1}(z, z_i) - G_{\rho_2}(z, z_i)) = \rho_1 - \rho_2, \quad \text{for all } 1 \leq i \leq k.$$

Finally, one can verify that

$$-s \int_{\mathbb{R}^2} \phi(z) \log |z - w| (\rho_1 - \rho_2)(w) dw = g_{\rho_2} - g_{\rho_1} + C$$

for some absolute constant C. Now Theorem 71 follows. □

Second, the following theorem says that the unit volume Liouville measure transforms covariantly under Mobius transforms:

Théorème 72. *[DKRV16, Theorem 3.7]*

Let ψ be a Mobius transform of the sphere. The law of the unit volume Liouville field ϕ under $\widehat{\mathbb{P}}^{(\alpha_i, z_i)_i}$ is the same as that of $\phi \circ \psi + Q \log |\psi'|$ under $\widehat{\mathbb{P}}^{(\alpha_i, \psi(z_i))_i}$. The area measure under $\widehat{\mathbb{P}}^{(\alpha_i, \psi(z_i))_i}$ has the same law as the pushforward of the area measure under $\widehat{\mathbb{P}}^{(\alpha_i, z_i)_i}$.

Finally, for the moment estimate that we use later on, we first recall a slightly modified formulation of Lemma 3.10 in [DKRV16]:

Lemme 73. *Fix $\gamma \in (0, 2)$ and let $\rho = \hat{g}$, i.e. take the spherical measure as the pinning measure. Consider the field $h_{L(\rho)}$ as in (3.8) under the law $\mathbb{P}_\rho$. Then for any $q \in \mathbb{R}$ with $q < \frac{4}{\gamma^2} \wedge \min_i \frac{2}{\gamma}(Q - \alpha_i)$ we have that $\mathbb{E}^{\mathbb{P}_\rho}[\mu_{h_{L(\rho)}}(\mathbb{C})^q] < \infty$.*

We will need the same estimate in the case where $\rho = \mathfrak{c}$, where we denote by $\mathfrak{c}$ the uniform probability measure on the unit circle.

Corollaire 74. *The lemma above holds holds for $\rho = \mathfrak{c}$.*

Proof. Notice that

$$\left| m_{\hat{g}}(\log \frac{1}{|z - \cdot|}) - m_{\mathfrak{c}}(\log \frac{1}{|z - \cdot|}) \right| = \left| \frac{1}{4} \log \hat{g}(z) + \log(|z| \vee 1) \right| < C \tag{3.9}$$

where C is some finite constant independent of z.

It follows from the definition of the Green's functions (3.1) that $G_{\hat{g}}$ and $G_{\mathfrak{c}}$ differ at most by some finite constant. Moreover, as remarked above, the difference between $h_{\hat{g}}$ and $h_{\mathfrak{c}}$ is a Gaussian of finite variance. Thus from (3.8) we see that the difference between $h_{L(\hat{g})}$ and $h_{L(\mathfrak{c})}$ is bounded by some Gaussian with bounded mean and variance and we conclude. □

In fact, the same moment bound holds for a much larger class of pinning measures ρ. However, for the more general case we do not have such a short and simple proof.

Duplantier-Miller-Sheffield's approach

We now briefly describe the approach to quantum surfaces initiated in [She16, DMS14]. In particular we will show how to make sense of the unit area quantum sphere even in the case of two marked points.

The underlying idea is to think of quantum surfaces as abstract surfaces, and of different conformal parametrizations as of different embeddings of these surfaces. One way to do this is to define an equivalence class of random measures with a transformation rule under conformal mappings.

Définition 75. *Let $\gamma \in (0,2)$ and $Q = 2/\gamma + \gamma/2$. Suppose we have a random distribution h on a domain D. Then a random surface with $k \geq 0$ marked points is an equivalence class of $(k+2)$-tuples $(D, h, x_1, \ldots, x_k)$ (with $x_i \in \overline{D}$) under the following equivalence relation: two embeddings $(D_i, h_i, x_1^i, \ldots, x_k^i)$ for $i = 1, 2$ are considered equivalent if there is a conformal map $\phi : D_2 \to D_1$ such that*

1. $h_2 = h_1 \circ \phi + Q \log |\phi'|$;
2. *it induces a correspondence between marked points on D_1 and marked points on D_2:*

$$x_j^1 = \phi(x_j^2), j = 1, \ldots, k.$$

In each fixed embedding, one can define the corresponding Liouville measure μ_{h_1} (resp. μ_{h_2}) as the Gaussian multiplicative chaos measure of parameter γ corresponding to the field h_1 (resp. h_2). Here the transformation rule in the definition is chosen so that this γ-Liouville measure is defined intrinsically for any equivalence class: $\mu_{h_1} = \phi_* \mu_{h_2}$. The marked points in this framework correspond to the so called typical points of this Liouville measure, i.e. they have γ-singularities.

Remarque 76. If we have too few marked points to fix the automorphisms of D, then the sigma-algebra of h does not coincide with that of the random surface – i.e. only events that are invariant under the remaining automorphisms are measurable w.r.t the random surface.

Remarque 77. Given a random surfaces with fewer marked points than are necessary to fix an embedding, we can still find well-defined embeddings of this surface: for example, we can just choose the missing number of points in a measurable way w.r.t. the random surface constructed with given marked points and then use all these points to choose a well-defined embedding.

Encoding surfaces using Bessel processes

In [DMS14], the authors introduce a way to encode random surfaces using Bessel processes and give a definition of the quantum unit sphere with two marked points.

To motivate the construction, consider the infinite cylinder $\mathcal{Q} = \mathbb{R} \times [0, 2\pi]$ with $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{2\pi\}$ identified. Here two marked points are given by the ends of the cylinder $\{-\infty, +\infty\}$ and the remaining degrees of Mobius freedom are: 1) rotations around the axis of the cylinder; 2) horizontal shifts.

These degrees of freedom pair well with the decomposition of the GFF on $\mathcal{Q}$ into radial and angular parts [DMS14, Lemma 4.2]: the radial part is invariant under rotations and transforms under horizontal shifts while the angular part is invariant under horizontal shifts and transforms under rotations. This comes directly from the orthogonal decomposition of the Dirichlet space $\mathcal{H}(\mathcal{Q})$ on the cylinder, that is given by the closure of smooth functions f on $\mathcal{Q}$ under the Dirichlet norm $\|\nabla f\|$ [DMS14, Lemma 4.2]:

Lemme 78. *Let $\mathcal{H}_1(\mathcal{Q}) \subset \mathcal{H}(\mathcal{Q})$ be the subspace of of functions which are constant on circles of the form $u \times [0, 2\pi]$ with endpoints 0 and 2π identified and $u \in \mathbb{R}$. Also, let $\mathcal{H}_2(\mathcal{Q}) \subset \mathcal{H}(\mathcal{Q})$ be the subspace of functions on $\mathcal{Q}$ which are of mean zero on all such circles. Then $\mathcal{H}_1(\mathcal{Q})$ and $\mathcal{H}_2(\mathcal{Q})$ form an orthogonal decomposition of the space $\mathcal{H}(\mathcal{Q})$.*

Remarque 79. We call the $\mathcal{H}_1(\mathcal{Q})$ and $\mathcal{H}_2(\mathcal{Q})$ components the radial and angular components respectively.

Let $\delta = 4 - 8/\gamma^2$ and let ν_δ^{BES} be Bessel excursion measure of dimension δ. In order to define μ^2_{DMS}, we first introduce an infinite measure on quantum surfaces with two marked points as follows:

1. Parameterize quantum surfaces by $(h, \mathcal{Q}, \infty, -\infty)$.
2. Sample an excursion e according to ν_δ^{BES}.
3. The radial $\mathcal{H}_1(\mathcal{Q})$ component of the quantum sphere is given by reparametrizing $2\gamma^{-1} \log e$ to have quadratic variation du.
4. The angular $\mathcal{H}_2(\mathcal{Q})$ component is sampled independently from the law of the $\mathcal{H}_2(\mathcal{Q})$ component of a whole-plane GFF on $\mathcal{Q}$.

Following [MS15b], we denote this measure by M_{BES}.

Définition 80. *The unit area quantum sphere with two marked points is M_{BES} conditioned on the surface having unit quantum area. We denote its law by μ^2_{DMS}. Given an integer $k \geq 3$ and if we first sample of instance of μ^2_{DMS} then conditionally independently sample $k-2$ points on the surface according to the quantum area, the resulting quantum surface with k marked points is called the unit area quantum sphere with k marked points, whose law is denoted by μ^k_{DMS}. By forgetting one marked point in μ^2_{DMS}, we obtain the unit area quantum sphere with 1 marked point μ^1_{DMS}.*

Remarque 81. Although M_{BES} is an infinite measure, the restriction of M_{BES} to the event that the surface has quantum area larger than a positive number is finite. One can check

that it is also similarly possible to make sense of M_{BES} conditioned on have quantum area 1 to obtain a probability measure.

Notice that a priori μ^2_{DMS} having only two marked points, comes without a canonical embedding. Choosing an embedding amounts to fixing the rotation and the horizontal translation in some way. The rotation can be fixed arbitrarily. For the radial part, we find it convenient to use the so called *maxima embedding*, where we fix the horizontal shift by requiring the location of the radial maxima to be at zero.

Limiting procedure for the 2-point sphere

In [DMS14, Proposition 5.13], a limiting procedure is described to obtain μ^2_{DMS}. It roughly goes as follows: take a Dirichlet GFF h on the unit disk $\mathbb{D}$ with zero boundary conditions, and condition its Gaussian multiplicative chaos measure μ_h to have mass

$$e^C \leq \mu_h(\mathbb{D}) \leq e^C(1+\delta)$$

for $C > 0$ large and $\delta > 0$ small. Denote the conditional law by $h_{C,\delta}$. Let w be a point in $\mathbb{D}$ sampled from $\bar{\mu}_{h_{C,\delta}}$. It is then shown that the quantum surfaces

$$(h_{C,\delta} - \gamma^{-1}C, \mathbb{D}, w, \infty)$$

converge in law to μ^2_{DMS}, when we first let $C \to \infty$ and then let $\delta \to 0$.

Here, convergence means that in some fixed embedding of these quantum surfaces the conditioned γ-Liouville measures converge weakly in law to the measure corresponding to μ^2_{DMS} in this embedding. For example, one can

- map $\mathbb{C}$ to the cylinder $\mathcal{Q}$ via the logarithmic map, sending ∞ to $-\infty$ and w to ∞;
- use the change-of-coordinates formula of Definition 75 to transform the field;
- horizontally shift the resulting field so that the maxima of its radial part is achieved at $\mathrm{Re}z = 0$;
- consider the γ-Liouville measure of the resulting field and, in order to obtain a measure on $\mathcal{Q}$, extend this measure by zero outside of the shifted image of $\mathbb{D}$.

Convergence of $(h_{C,\delta} - \gamma^{-1}C, \mathbb{D}, w, \infty)$ then means convergence in law of these measures on the cylinder $\mathcal{Q}$ under weak topology. The two marked points w and ∞ become respectively ∞ and $-\infty$ on $\mathcal{Q}$.

By examining the corresponding proof in [DMS14], we lift the following statement:

Proposition 82. *Let*

$$h = h_0 - \gamma \log|z| - C$$

where h_0 is a zero boundary GFF on $\mathbb{D}$ and C is a constant. The quantum surface

$$(h, \mathbb{C}, 0, \infty)$$

conditioned on $e^{-\gamma\delta} \leq \mu_h(\mathbb{D}) \leq e^{\gamma\delta}$ *converges to* μ^2_{DMS}.
More precisely, if we embed the quantum surface $(h, \mathbb{C}, 0, \infty)$ *into* $(\mathcal{Q}, +\infty, -\infty)$ *and fix the horizontal shift such that the maxima of the radial part is achieved at* $\mathrm{Re} z = 0$*, then as* $C \to \infty$ *and then* $\delta \to 0$ *the conditional law of the quantum surface converges to the maxima embedding of* μ^2_{DMS} *described in Section 3 in the sense the corresponding* γ*-Liouville measures converge weakly in law.*

A limiting procedure for $\mu_{\mathrm{DKRV}}^{\gamma,\gamma,\gamma}$

This section is devoted to proving an approximation scheme for $\mu_{\mathrm{DKRV}}^{\gamma,\gamma,\gamma}$. Let $D^\varepsilon = \varepsilon^{-1}\mathbb{D}$ be a large disk and h_0^ε be the zero boundary GFF on D^ε. Let further $z_1, z_2 \in \mathbb{D}$ and consider a GFF with two interior γ singularities at z_1, z_2, and boundary values chosen such that in the limit they give rise to a third γ singularity at infinity, i.e. we set:

$$h^\varepsilon(z) = h_0^\varepsilon(z) + (2Q - \gamma)\log\varepsilon + \gamma G_{D^\varepsilon}(z, z_1) + \gamma G_{D^\varepsilon}(z, z_2). \tag{3.10}$$

Here G_{D^ε} is the Dirichlet Green function of D^ε.

Let $\mathbb{P}^\varepsilon$ be the corresponding probability measure and μ_{h^ε} be the associated Liouville measure of h^ε. Furthermore, let

$$A_\varepsilon = (h_0^\varepsilon, \xi_\varepsilon)_\nabla$$

where

$$\xi_\varepsilon = -\log(|z| \vee 1) - \log\varepsilon.$$

Then A_ε is the circle average of h_0^ε around $\partial\mathbb{D}$.

Théorème 83. *In the above setup, suppose* $\delta > 0$*,*

$$E_\delta^\varepsilon = \{\mu_{h^\varepsilon}(\mathbb{C}) \in [e^{-\gamma\delta}, e^{\gamma\delta}]\}$$

and

$$H^\varepsilon = \{A_\varepsilon + (2Q - 3\gamma)\log\varepsilon \geq -|\log\varepsilon|^{2/3}\}.$$

By first letting $\varepsilon \to 0$ *then* $\delta \to 0$*,* μ_{h^ε} *conditioned on* $E_\delta^\varepsilon \cap H^\varepsilon$ *converges in law to* $\mu_{\mathrm{DKRV}}^{\gamma,\gamma,\gamma}$ *of Definition 70 and with singularities at* z_1, z_2, ∞*. Here the topology of convergence is the weak topology of measures on* $\mathbb{C} \cup \{\infty\}$*.*

In the case $\gamma \geq \sqrt{2}$, the proof of Lemma 84 below implies that one can omit conditioning on H^ε and the theorem still holds. See Remark 87 for details. The proof of Proposition 90 in Section 3 also suggests that the same is true when $\gamma < \sqrt{2}$. However, we cannot confirm this via a short argument for now.

Preliminary calculations, notations and heuristics

Using conformal invariance of the Green's function, we can write

$$h^\varepsilon = h_0^\varepsilon + (2Q - 3\gamma)\log\varepsilon - \gamma\log|z - z_1| - \gamma\log|z - z_2| + r_\varepsilon,$$

where $r_\varepsilon = \gamma\log|1 - \varepsilon^2\bar{z}_1 z| + \gamma\log|1 - \varepsilon^2\bar{z}_2 z|$.

Recall that A_ε is the circle average of h_0^ε around $\partial\mathbb{D}$ and

$$\mathrm{Var}[A_\varepsilon] = (\xi_\varepsilon, \xi_\varepsilon)_\nabla = -\log\varepsilon.$$

By the orthogonal decomposition of the GFF (see e.g. [She07, Section 2.4]), we can write

$$h_0^\varepsilon = h_{\mathfrak{c}}^\varepsilon + \frac{A_\varepsilon}{\mathrm{Var}[A_\varepsilon]}\xi_\varepsilon,$$

where $h_{\mathfrak{c}}^\varepsilon$ is distributed as h_0^ε conditioned to have circle-average zero on $\partial\mathbb{D}$ and is independent of A_ε.

Let $h_{\mathfrak{c}}$ be the whole-plane GFF normalized such that the circle average around $\partial\mathbb{D}$ is zero, i.e. in other words we consider the scalar whole-plane GFF with the background measure equal to the uniform measure on the unit circle. For consistence of notation we denote this unit measure by $\mathfrak{c}$. Then using the notations of Section 2.2, we write

$$h_L^\varepsilon = h_{\mathfrak{c}}^\varepsilon - (2Q - 3\gamma)\log(|z| \vee 1) - \gamma\log|z - z_1| - \gamma\log|z - z_2|, \tag{3.11}$$

$$h_L = h_{\mathfrak{c}} - (2Q - 3\gamma)\log(|z| \vee 1) - \gamma\log|z - z_1| - \gamma\log|z - z_2|. \tag{3.12}$$

The corresponding probability laws are denoted by $\mathbb{P}_{\mathfrak{c}}^\varepsilon, \mathbb{P}_{\mathfrak{c}}$. The field $h_{\mathfrak{c}}$ now corresponds to the field of (3.8), in the case where the background measure is the unit mass distributed uniformly on the unit circle. By Proposition 69, $\mathbb{P}_{\mathfrak{c}}$ is the limiting measure of $\mathbb{P}_{\mathfrak{c}}^\varepsilon$, where the topology is given by convergence in total variation on any bounded domain.

We can now write

$$h^\varepsilon = h_L^\varepsilon + \left(\frac{A_\varepsilon}{\mathrm{Var}[A_\varepsilon]} - (2Q - 3\gamma)\right)\xi_\varepsilon + r_\varepsilon \tag{3.13}$$

and by the orthogonal decomposition we have $d\mathbb{P}_\varepsilon = dA_\varepsilon \otimes d\mathbb{P}_{\mathfrak{c}}^\varepsilon$. The key observation is that if we now consider the probability measure

$$d\mathbb{Q}^\varepsilon \propto \exp\{(2Q - 3\gamma)A_\varepsilon\}d\mathbb{P}^\varepsilon,$$

then under $\mathbb{Q}^\varepsilon$:

1. by Girsanov theorem, the law of $\widetilde{A}_\varepsilon := A_\varepsilon + (2Q - 3\gamma)\log\varepsilon$ is a centered Gaussian with variance $|\log\varepsilon|$;
2. h_L^ε has the same law as under $\mathbb{P}^\varepsilon$ and is independent of $\widetilde{A}_\varepsilon$.

Now write

$$h^\varepsilon = h_L^\varepsilon + \widetilde{A}_\varepsilon + \widetilde{A}_\varepsilon g_\varepsilon(z) + r_\varepsilon, \tag{3.14}$$

where $g_\varepsilon(z) = \log(|z| \vee 1)/\log \varepsilon$.

Writing h^ε in this way is useful, as under $\mathbb{Q}^\varepsilon$, $\widetilde{A}_\varepsilon g_\varepsilon(z) + r_\varepsilon$ vanish as ε goes to 0 and therefore h^ε is roughly speaking an independent sum of h_L^ε and $\widetilde{A}_\varepsilon$. Since h_L^ε tends to h_L in law and the law of $\widetilde{A}_\varepsilon$ vaguely tends to Lebesgue measure, this should remind the reader of the whole-plane GFF tensorized with Lebesgue measure, as discussed in Section 3 and used in [DKRV16]. Making this connection precise is the main content of the rest of this section.

Statement of the main lemma and proof of Theorem 83

In the rest of this section we omit r_ε, as it is a harmonic function with $\max_z |r_\varepsilon(z)| = o_\varepsilon(1)$ and plays no role in the later argument. Given a Liouville measure μ_h with almost surely finite total mass, let $\bar{\mu}_h(\cdot) = \mu_h(\cdot)/\mu_h(\mathbb{C})$. Recall the events $\mathbb{E}_\delta^\varepsilon$ and H^ε from Theorem 83:

$$E_\delta^\varepsilon = \{\mu_{h^\varepsilon}(\mathbb{C}) \in [e^{-\gamma\delta}, e^{\gamma\delta}]\}.$$

In light of (3.14), under $\mathbb{Q}^\varepsilon$ it is instrumental to write

$$H^\varepsilon = \{\widetilde{A}_\varepsilon \geq -|\log \varepsilon|^{2/3}\}.$$

Now Theorem 83 is a consequence of the following lemma:

Lemme 84. *Suppose F is a nonnegative bounded continuous functional on the space of probability measures on $\mathbb{C} \cup \{\infty\}$ with the topology of weak convergence and $a = \gamma^{-1}(2Q - 3\gamma)$. Then for all $\delta > 0$*

$$\lim_{\varepsilon\to 0} \sqrt{2\pi \operatorname{Var}[\widetilde{A}_\varepsilon]}\, \mathbb{E}^{\mathbb{Q}^\varepsilon}[F(\bar{\mu}_{h_L^\varepsilon + \widetilde{A}_\varepsilon g_\varepsilon(z)}) \mu^a_{h_L^\varepsilon + \widetilde{A}_\varepsilon g_\varepsilon}(\mathbb{C}) \mathbf{1}_{E_\delta^\varepsilon} \mathbf{1}_{H^\varepsilon}] = 2\delta \mathbb{E}^{\mathbb{P}_{\mathfrak{c}}}[F(\bar{\mu}_{h_L}) \mu^a_{h_L}(\mathbb{C})]. \tag{3.15}$$

We postpone the proof of Lemma 84 in Section 3 and proceed to the proof of Theorem 83.

Proof of Theorem 83. Let $\mathbb{P}'_{\mathfrak{c}}$ be the probability measure we obtained via re-weighting $\mathbb{P}_{\mathfrak{c}}$ by $\mu^a_{h_L}(\mathbb{C})$. Then by Theorem 71 and Definition 70, the law of $\mu_{\mathrm{DKRV}}^{\gamma,\gamma,\gamma}$ is given by $\bar{\mu}_{h_L}$ under $\mathbb{P}'_{\mathfrak{c}}$.

To prove Theorem 83, we first notice that since we are conditioning on μ_{h^ε} having total mass between $[e^{-\gamma\delta}, e^{\gamma\delta}]$ and we are letting $\delta \to 0$ eventually, it suffices to show that $\bar{\mu}_{h^\varepsilon}$ converges to $\bar{\mu}_{h_L}$ under $\mathbb{P}'_{\mathfrak{c}}$. In other words it suffices to show that for all F as in Lemma 84 we have

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\mathbb{E}^{\mathbb{P}^\varepsilon}[F(\bar{\mu}_{h^\varepsilon})|E^\varepsilon_\delta\cap H^\varepsilon] = \mathbb{E}^{\mathbb{P}'_{\mathfrak{c}}}[F(\bar{\mu}_{h_L})].$$

Recall that $a = \gamma^{-1}(2Q-3\gamma)$. We have

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{\mathbb{E}^{\mathbb{P}^\varepsilon}[F(\bar{\mu}_{h^\varepsilon})\mathbf{1}_{E^\varepsilon_\delta}\mathbf{1}_{H^\varepsilon}]}{\mathbb{E}^{\mathbb{P}^\varepsilon}[E^\varepsilon_\delta\cap H^\varepsilon]} = \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{\mathbb{E}^{\mathbb{Q}^\varepsilon}[F(\bar{\mu}_{h^\varepsilon})e^{-(2Q-3\gamma)A_\varepsilon}\mathbf{1}_{E^\varepsilon_\delta}\mathbf{1}_{H^\varepsilon}]}{\mathbb{E}^{\mathbb{Q}^\varepsilon}[e^{-(2Q-3\gamma)A_\varepsilon}\mathbf{1}_{E^\varepsilon_\delta}\mathbf{1}_{H^\varepsilon}]}. \tag{3.16}$$

Now under the event E^ε_δ we have

$$e^{-(2Q-3\gamma)A_\varepsilon}\propto e^{-(2Q-3\gamma)\widetilde{A}_\varepsilon} = (1+O(\delta))\mu^a_{h^\varepsilon_L+\widetilde{A}_\varepsilon g_\varepsilon(z)}(\mathbb{C}).$$

Therefore the right hand side of (3.16) equals

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{\mathbb{E}^{\mathbb{Q}^\varepsilon}[F(\bar{\mu}_{h^\varepsilon_L+\widetilde{A}_\varepsilon g_\varepsilon(z)})\mu^a_{h^\varepsilon_L+\widetilde{A}_\varepsilon g_\varepsilon(z)}(\mathbb{C})\mathbf{1}_{E^\varepsilon_\delta}\mathbf{1}_{H^\varepsilon}]}{\mathbb{E}^{\mathbb{Q}^\varepsilon}[\mu^a_{h^\varepsilon_L+\widetilde{A}_\varepsilon g_\varepsilon(z)}(\mathbb{C})\mathbf{1}_{E^\varepsilon_\delta}\mathbf{1}_{H^\varepsilon}]}.$$

By Lemma 84, it is equal to

$$\frac{\mathbb{E}^{\mathbb{P}_{\mathfrak{c}}}[F(\bar{\mu}_{h_L})\mu^a_{h_L}(\mathbb{C})]}{\mathbb{E}^{\mathbb{P}_{\mathfrak{c}}}[\mu^a_{h_L}(\mathbb{C})]} = \mathbb{E}^{\mathbb{P}'_{\mathfrak{c}}}[F(\bar{\mu}_{h_L})],$$

from which Theorem 83 follows. □

Proof of Lemma 84

The proof of Lemma 84 relies on two simple Fubini identities and a continuity result.

Lemme 85. *Let X, Y, Z be random variables where $\mathbb{E}[|Z|]<\infty$. For fixed $\delta>0$, let*

$$E^1_\delta(x) = \{X\in[-x-\delta,\delta-x]\}, E^2_\delta(x) = \{X\le -x+\delta, Y\ge -x-\delta\}.$$

Then

$$\int_{-\infty}^{\infty}\mathbb{E}[Z1_{E^1_\delta(x)}]dx = 2\delta\mathbb{E}[Z], \tag{3.17}$$

$$\int_{-\infty}^{\infty}\mathbb{E}[Z1_{E^2_\delta(x)}]dx = \mathbb{E}[Z((Y-X+2\delta)\vee 0)]. \tag{3.18}$$

Proof. (3.17) is the special case of (3.18) when $X=Y$. So we only prove (3.18). By Fubini Theorem the left hand side of (3.18) equals

$$\mathbb{E}\left[\int_{-\infty}^{\infty}Z1_{\{-Y-\delta\le x\le -X+\delta\}}dx\right] = \mathbb{E}[Z((Y-X+2\delta)\vee 0)].$$ □

Lemme 86. *Suppose a_ε is a sequence of deterministic numbers tending to 0 as $\varepsilon \to 0$. Let h_L^ε, h_L be defined as in Section 3. Then $\bar{\mu}_{h_L^\varepsilon + a_\varepsilon \log(|z|\vee 1)}$ weakly converges to $\bar{\mu}_{h_L}$ in the space probability measure on $\mathbb{C} \cup \{\infty\}$ endowed with the weak convergence topology.*

Proof. By tightness, $\bar{\mu}_{h_L^\varepsilon + a_\varepsilon \log(|z|\vee 1)}$ has subsequential limits. We only need to show the uniqueness of the limiting object. By Proposition 69, h_L^ε restricted to $R\mathbb{D}$ converges to h_L in total variation for all $R > 0$. Moreover, $a_\varepsilon \log(|z| \vee 1)$ converges to 0 locally uniformly in $\mathbb{C}$. Therefore on $R\mathbb{D}$, $\mu_{h_L^\varepsilon + a_\varepsilon \log(|z|\vee 1)}$ converges in total variation distance to μ_{h_L}. Since we are considering a normalized measure, the only thing left to show is that $\mu_{h_L^\varepsilon + a_\varepsilon \log(|z|\vee 1)}(\mathbb{C})$ is tight.
Take $0 < c < Q - \gamma$. Then $\mu_{h_L^\varepsilon + a_\varepsilon \log(|z|\vee 1)}(\mathbb{C}) < \mu_{h_L^\varepsilon + c \log(|z|\vee 1)}(\mathbb{C})$ for small enough ε. On the other hand the field $h_L + c\log(|z| \vee 1)$ has at $0, 1, \infty$ singularities $(\gamma, \gamma, \gamma + c)$. These singularities satisfy the Seiberg bounds, and so by Corollary 74, $\mu_{h_L + c\log(|z|\vee 1)}(\mathbb{C})$ has some finite positive moment. Hence $\mu_{h_L + c\log(|z|\vee 1)}(\mathbb{C}) < \infty$ almost surely. But now, by considering the radial decomposition of h_L^ε, $\mu_{h_L^\varepsilon + c\log(|z|\vee 1)}(\mathbb{C})$ weakly converges to $\mu_{h_L + c\log(|z|\vee 1)}(\mathbb{C})$ and tightness follows. □

Proof of Lemma 84. Denote the left hand side of (3.15) by L_ε and let

$$E_\delta^\varepsilon(x) = \{\gamma^{-1} \log \mu_{h_L^\varepsilon + x g_\varepsilon(z)}(\mathbb{C}) \in [-x - \delta, -x + \delta]\}.$$

Then by conditioning on $\widetilde{A}_\varepsilon = x$, we have

$$L_\varepsilon = \int_{-|\log \varepsilon|^{\frac{2}{3}}}^{\infty} \mathbb{E}^{\mathbb{P}_{\mathfrak{c}}^\varepsilon}[F(\bar{\mu}_{h_L^\varepsilon + x g_\varepsilon(z)}) \mu^a_{h_L^\varepsilon + x g_\varepsilon(z)}(\mathbb{C}) \mathbf{1}_{E_\delta^\varepsilon(x)}] \exp\left\{-\frac{x^2}{2|\log \varepsilon|}\right\} dx. \tag{3.19}$$

Let

$$\Sigma_\varepsilon = \int 1_{x \in [-|\log \varepsilon|^{\frac{2}{3}}, |\log \varepsilon|^{\frac{2}{3}}]} \mathbb{E}^{\mathbb{P}_{\mathfrak{c}}^\varepsilon}[F(\bar{\mu}_{h_L^\varepsilon + x g_\varepsilon(z)}) \mu^a_{h_L^\varepsilon + x g_\varepsilon(z)}(\mathbb{C}) \mathbf{1}_{E_\delta^\varepsilon(x)}] \exp\left\{-\frac{x^2}{2|\log \varepsilon|}\right\} dx$$

and $\Gamma_\varepsilon = L_\varepsilon - \Sigma_\varepsilon$. If $a \le 0$ (i.e. $\gamma \ge \sqrt{2}$), then

$$\mu_{h_L^\varepsilon + x g_\varepsilon(z)}(\mathbb{C}) \ge \mu_{h_L^\varepsilon + x g_\varepsilon(z)}(\mathbb{D}) = \mu_{h_L^\varepsilon}(\mathbb{D})$$

and we have

$$\Gamma_\varepsilon \le O_F(1) \int_{x > |\log \varepsilon|^{\frac{2}{3}}} \mathbb{E}^{\mathbb{P}_{\mathfrak{c}}^\varepsilon}[\mu^a_{h_L^\varepsilon}(\mathbb{D})] \exp\left\{-\frac{x^2}{2|\log \varepsilon|}\right\} dx = o_\varepsilon(1). \tag{3.20}$$

If $a > 0$ (i.e. $\gamma < \sqrt{2}$), we get

$$\Gamma_\varepsilon \le O_F(1) \int_{x > |\log \varepsilon|^{\frac{2}{3}}} e^{-\gamma a x} \exp\left\{-\frac{x^2}{2|\log \varepsilon|}\right\} \mathbb{P}_{\mathfrak{c}}^\varepsilon[E_\delta^\varepsilon(x)] dx = o_\varepsilon(1).$$

Here $e^{-\gamma a x}$ comes from the fact that if $E^\varepsilon_\delta(x)$ occurs and $\widetilde{A}_\varepsilon = x$, then $\mu_{h^\varepsilon_L + x g_\varepsilon(z)} e^{\gamma x} = 1 + o_\delta(1)$.

It remains to deal with Σ_ε. From the argument in Lemma 86 and Skorokhod's representation theorem, we can couple h^ε_L, h_L so that almost surely for all $x \in [-|\log \varepsilon|^{\frac{2}{3}}, |\log \varepsilon|^{\frac{2}{3}}]$, we have $\bar{\mu}_{h^\varepsilon_L + x g_\varepsilon(z)}$ converges to $\bar{\mu}_{h_L}$ and $\mu_{h^\varepsilon_L + x g_\varepsilon(z)}(\mathbb{C})$ converges to $\mu_{h_L}(\mathbb{C})$ almost surely.

Now let $E_\delta(x) = \{\gamma^{-1} \log \mu_{h_L}(\mathbb{C}) \in [-x-\delta, -x+\delta]\}$. Observe that by Lemma 85

$$\int_{-\infty}^{\infty} \mathbb{E}^{\mathbb{P}_c}[F(\bar{\mu}_{h_L}) \mu^a_{h_L}(\mathbb{C}) \mathbf{1}_{E_\delta(x)}]\, dx = 2\delta \mathbb{E}^{\mathbb{P}_c}[F(\bar{\mu}_{h_L}) \mu^a_{h_L}(\mathbb{C})] = \text{RHS of (3.15)}.$$

Therefore to finish the proof we only need to show that the dominated convergence theorem can be applied to evaluate $\lim_{\varepsilon \to 0} \Sigma_\varepsilon$.

When $a \le 0$, since $g_\varepsilon(z) = 0$ on $\mathbb{D}$, the integrand inside Σ_ε is controlled by

$$O_F(1) \mathbb{E}^{\mathbb{P}^\varepsilon_c}[\mu^a_{h^\varepsilon_L}(\mathbb{D}) \mathbf{1}_{\hat{E}^\varepsilon_\delta(x)}]. \tag{3.21}$$

By Lemma 85

$$\int_{-\infty}^{\infty} \mathbb{E}^{\mathbb{P}^\varepsilon_c}[\mu^a_{h^\varepsilon_L}(\mathbb{D}) \mathbf{1}_{\hat{E}^\varepsilon_\delta(x)}]\, dx = \mathbb{E}^{\mathbb{P}^\varepsilon_c}[\mu^a_{h^\varepsilon_L}(\mathbb{D})((Y_\varepsilon - X_\varepsilon + 2\delta) \vee 0)].$$

where

$$X_\varepsilon = \gamma^{-1} \log \mu_{h^\varepsilon_L + |\log \varepsilon|^{\frac{2}{3}} g_\varepsilon(z)}(\mathbb{C}) \text{ and } Y_\varepsilon = \gamma^{-1} \log \mu_{h^\varepsilon_L - |\log \varepsilon|^{\frac{2}{3}} g_\varepsilon(z)}(\mathbb{C}),$$
$$\hat{E}^\varepsilon_\delta(x) = \{X_\varepsilon \le -x + \delta, Y_\varepsilon \ge -x - \delta\}.$$

By Corollary 74, for $c > 0$ small enough, we have that $\mathbb{E}^{\mathbb{P}_c}[\mu^{a+c}_{h_L + c \log(|z| \vee 1)}] < \infty$. Thus by the dominated convergence theorem,

$$\lim_{\varepsilon \to 0} \mathbb{E}^{\mathbb{P}^\varepsilon_c}[\mu^a_{h^\varepsilon_L}(\mathbb{D})((Y_\varepsilon - X_\varepsilon + 2\delta) \vee 0)] = 2\delta \mathbb{E}^{\mathbb{P}_c}[\mu^a_{h^L}(\mathbb{D})].$$

Since

$$\int_{-\infty}^{\infty} \mathbb{E}^{\mathbb{P}_c}[\mu^a_{h_L}(\mathbb{D}) \mathbf{1}_{E_\delta(x)}]\, dx = 2\delta \mathbb{E}^{\mathbb{P}_c}[\mu^a_{h_L}(\mathbb{D})],$$

we have

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \mathbb{E}^{\mathbb{P}^\varepsilon_c}[\mu^a_{h^\varepsilon_L}(\mathbb{D}) \mathbf{1}_{\hat{E}^\varepsilon_\delta(x)}]\, dx = \int_{-\infty}^{\infty} \mathbb{E}^{\mathbb{P}_c}[\lim_{\varepsilon \to 0} \mu^a_{h^\varepsilon_L}(\mathbb{D}) \mathbf{1}_{\hat{E}^\varepsilon_\delta(x)}]\, dx.$$

Now we can apply the dominated convergence theorem to evaluate $\lim_{\varepsilon \to 0} \Sigma_\varepsilon$.

When $a > 0$, (i.e. $\gamma < \sqrt{2}$), we replace the term in (3.21) by

$$O_F(1) \mathbb{E}^{\mathbb{P}^\varepsilon_c}[\mu^a_{h^\varepsilon_L + \log |\varepsilon|^{\frac{2}{3}} g_\varepsilon(z)}(\mathbb{C}) \mathbf{1}_{\hat{E}^\varepsilon_\delta(x)}].$$

The rest of the argument works line by line. □

Remarque 87. Notice that when $a \leq 0$ (i.e. $\gamma \geq \sqrt{2}$) the estimate in (3.20) still holds if we integrate against the whole $\mathbb{R}$ instead of $[-|\log \varepsilon|^{2/3}, \infty]$. This means that in Lemma 84 and hence also in Theorem 83, we can replace $E^\varepsilon_\delta \cap H^\varepsilon$ by E^ε_δ if $\gamma \in [\sqrt{2}, 2)$. This does not directly work if $\gamma < \sqrt{2}$, since a is now positive and the integral over $[-\infty, -|\log \varepsilon|^{2/3}]$ is not as easily controlled.

Equivalence of 3-point spheres

In this section we finish the proof of Theorem 66. By the Mobius invariance of $\mu^{\gamma,\gamma,\gamma}_{\mathrm{DKRV}}$ in Theorem 72 and the relation between μ^2_{DMS} and μ^3_{DMS} in Definition 80, it suffices to show that

Théorème 88. *$\mu^{\gamma,\gamma,\gamma}_{\mathrm{DKRV}}$ with marked points at $0, 1, \infty$ can be obtained by first sampling a third point w from μ^2_{DMS} with marked points at $0, \infty$ and then applying the Mobius transformation that maps $(0, \infty, w)$ to $(0, \infty, 1)$.*

We prove Theorem 88 by constructing two very close limiting procedures. The first one gives μ^3_{DMS} embedded such that the marked points are $0, 1, \infty$. The second one gives $\mu^{\gamma,\gamma,\gamma}_{\mathrm{DKRV}}$ with marked points $0, 1, \infty$.

Let $D^\varepsilon, h^\varepsilon_0, G_{D^\varepsilon}$ be defined as in Section 3 and

$$h^\varepsilon(z) = h^\varepsilon_0(z) + (2Q - \gamma)\log \varepsilon + \gamma G_{D^\varepsilon}(z, 0). \tag{3.22}$$

Notice that h^ε here is *not* the same as the one defined in Section 3, indeed here h^ε only has a single log singularity inside D^ε.

Denote

$$E^\varepsilon_\delta = \{\mu_{h^\varepsilon}(\mathbb{C}) \in [e^{-\gamma\delta}, e^{\gamma\delta}]\},$$

let $\mathbb{P}^\varepsilon_\delta$ be the law of h^ε conditioning on E^ε_δ and $d\hat{\mathbb{P}}^\varepsilon_\delta = c\mu_{h^\varepsilon(\mathbb{C})} d\mathbb{P}^\varepsilon_\delta$ where c is chosen to make $\hat{\mathbb{P}}^\varepsilon_\delta$ a probability measure. Given an instance of h^ε under $\mathbb{P}^\varepsilon_\delta$, let w^ε be a point sampled according to the quantum area. Let $\hat{w}^\varepsilon$ be sampled in the same way as w^ε with $\hat{\mathbb{P}}^\varepsilon_\delta$ in place of $\mathbb{P}^\varepsilon_\delta$. Now we claim that

Proposition 89. *Suppose $(w^\varepsilon, h^\varepsilon)$ is sampled from $\mathbb{P}^\varepsilon_\delta$ as above. Apply the Mobius transformation that maps w^ε to 1 and fixes 0 and ∞. Let h^ε transform according to the coordinate change rule as in Definition 75. If we first let $\varepsilon \to 0$ then $\delta \to 0$, then the resulting Liouville area measure converges weakly in law to that of μ^3_{DMS} with the embedding chosen such that the marked points are $0, 1, \infty$.*

Proof. Since $G_{D^\varepsilon}(z, 0) = -\log|\varepsilon z|$, the field in (3.22) can be written as

$$h^\varepsilon = h^\varepsilon_0 - \gamma \log|z| + (2Q - 2\gamma)\log \varepsilon.$$

We now set the constant C in Proposition 82 to be $(2\gamma - 3Q)\log\varepsilon$ and apply the coordinate change $z \mapsto \varepsilon z$, then Proposition 82 yields that $(h^\varepsilon, \mathbb{C}, 0, \infty)$ converges to μ^2_{DMS} in the sense that in the maxima embedding, the area measures converge weakly. Therefore we have a coupling of $(h^\varepsilon, \mathbb{C}, 0, \infty)$, μ^2_{DMS}, and points w^ε and w sampled from μ_{h^ε} and μ^2_{DMS} respectively, such that under maximal embedding μ_{h^ε} a.s. converges to μ^2_{DMS} and $\lim_{\varepsilon\to 0}|w^\varepsilon - w| = 0$ in probability. Now Proposition 89 follows. □

We also have an analogous statement for $\mu^{\gamma,\gamma,\gamma}_{\mathrm{DKRV}}$:

Proposition 90. *Suppose $(\hat{w}^\varepsilon, h^\varepsilon)$ is sampled from $\hat{\mathbb{P}}^\varepsilon_\delta$ defined above. Apply the Mobius transformation that maps $\hat{w}^\varepsilon$ to 1 and fixes 0 and ∞. Let h^ε transform according to the coordinate change rule. If we first let $\varepsilon \to 0$ then $\delta \to 0$, then the resulting Liouville area measure converges weakly in law to that of $\mu^{\gamma,\gamma,\gamma}_{\mathrm{DKRV}}$ with marked points at $0, 1, \infty$.*

Let us see how Theorem 88 now follows.

Proof of Theorem 88. Since $\mu_{h^\varepsilon}(\mathbb{C}) \in [e^{-\gamma\delta}, e^{\gamma\delta}]$ under $\mathbb{P}^\varepsilon_\delta$, the total variational distance between $\mathbb{P}^\varepsilon_\delta$ and $\hat{\mathbb{P}}^\varepsilon_\delta$ is $o_\delta(1)$ and so is the total variational distance between $(w^\varepsilon, h^\varepsilon)$ and $(\hat{w}^\varepsilon, h^\varepsilon)$. Therefore the two limiting objects in Proposition 89 and Proposition 90 are the same, which implies Theorem 88. □

The rest of the section is devoted to proving Proposition 90.

The location of the third point sampled from μ^2_{DMS}

We start by controlling the location of the sampled points w^ε and $\hat{w}^\varepsilon$ defined above.

Lemme 91. *Let w^ε be defined in Proposition 89. We have*

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\log|w^\varepsilon|/|\log\varepsilon|^{2/3} = 0 \quad \textit{in law under } \mathbb{P}^\varepsilon_\delta.$$

The same statement is true for $\hat{w}^\varepsilon$ defined in Proposition 90.

Proof of Lemma 91. As the total variational distance between $(w^\varepsilon, h^\varepsilon)$ and $(\hat{w}^\varepsilon, h^\varepsilon)$ is $o_\delta(1)$, it suffices to prove the result for w^ε.

We use the map $z \mapsto e^{-z}$ to pull $(h^\varepsilon, \mathbb{C})$ back to $\mathcal{Q} = \mathbb{R}\times[0, 2\pi]$. Then D^ε is mapped to the half cylinder $(\log\varepsilon, +\infty)\times[0, 2\pi]$ and $0, \infty$, w^ε are mapped to $+\infty, -\infty, -\log w^\varepsilon$ respectively. The radial component of h^ε can be written as $X_t = B_t - (Q-\gamma)t$ where B_t evolves as a standard Brownian motion with $B_{\log\varepsilon} = 2(Q-\gamma)\log\varepsilon$.

Let L^ε be the location where $B_t - (Q-\gamma)t$ achieves the maxima. By the proof of Proposition 89, $L^\varepsilon + \log|w^\varepsilon|$ is tight, in particular

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}(\log|w^\varepsilon| + L^\varepsilon)/|\log\varepsilon|^{2/3} = 0 \quad \text{in law.} \tag{3.23}$$

Thus we only need to show that

$$\lim_{\varepsilon\to 0} L^\varepsilon / |\log \varepsilon|^{2/3} = 0 \quad \text{in law.} \tag{3.24}$$

In fact it suffices to show this under a certain conditioning. Indeed, let F_C^ε be the event that the maxima of $B_t - (Q-\gamma)t$ is bigger than $-C$. Then by the argument in [DMS14, Lemma 5.5 and Proposition 5.7],

$$\begin{aligned} &\mathbb{P}[F_C^\varepsilon | E_\delta^\varepsilon] \to 1 && \text{as } C \to \infty \text{ uniformly in } \varepsilon \text{ for fixed } \delta \text{ and} \\ &\mathbb{P}[E_\delta^\varepsilon | F_C^\varepsilon] > 0 && \text{uniformly in } \varepsilon \text{ for fixed } \delta, C. \end{aligned} \tag{3.25}$$

Therefore we only need to show that (3.24) holds for fixed δ, C, conditioning on F_C^ε.

To prove this, let $T^\varepsilon = \inf\{t \geq \log\varepsilon : B_t - (Q-\gamma)t \geq -C\}$. By the Markov property of Brownian motion, conditioning on F_C^ε, $L^\varepsilon - T^\varepsilon$ does not depend on ε, C, δ, and is a distribution associated with standard drifted Brownian motion. Thus it remains to show that for fixed C, conditioning on F_C^ε,

$$\lim_{\varepsilon\to 0} T^\varepsilon / |\log\varepsilon|^{2/3} = 0 \quad \text{in law,}$$

which follows from the classical Lemma 92 below by setting $a = Q - \gamma$ and properly translating the picture. □

Lemme 92. *Consider the drifted Brownian motion $B_t - at$ where B_t is a standard Brownian motion with $B_t = 0$ and $a > 0$. Let $T_A = \inf\{t \in [0,\infty[: B_t - at = A\}$ with $A > 0$. Then conditional on $T_A < \infty$,*

$$\lim_{A\to\infty} \frac{T_A - a^{-1}A}{A^{2/3}} = 0 \quad \textit{in law.} \tag{3.26}$$

Proof. Recall [DMS14, Lemma 3.6], conditioning on $T_A < \infty$, $B_t - at$ can be sampled as follows:

1. Sample a standard Brownian motion X^1 with linear drift a until it hits A and let T_A be the first time that X^1 hits A.
2. Sample standard Brownian motion X^2 with linear drift $-a$.
3. Concatenate $X^1[0, T_A]$ with $X^2(\cdot - T_A)$.

It is well known (e.g. in [KS91], also used in [DS11]) that the law of T_A is the inverse Gaussian distribution with parameters $(a^{-1}A, a^{-2}A^2)$. Since $E[T_A] = a^{-1}A$ and $\mathrm{Var}[T_A] = aA$, Lemma 92 follows from Markov inequality. □

Rooted measure and $\mu_{\mathrm{DKRV}}^{\gamma,\gamma,\gamma}$

In this section we prove Proposition 90. We will argue that the setting obtained by sampling a third point and mapping it to 1 is close to that of Theorem 83 with $z_1 = 0, z_2 = 1$, and that thus we can apply the arguments of Section 3 to obtain convergence to $\mu_{\mathrm{DKRV}}^{\gamma,\gamma,\gamma}$.

Recall that $(\hat{w}^\varepsilon, h^\varepsilon)$ under $\hat{\mathbb{P}}_\varepsilon^\delta$ can be sampled from the following probability measure:

$$ce^{\gamma h^\varepsilon(z)} 1_{E_\delta^\varepsilon} dh^\varepsilon dz$$

where dh^ε is the law of h^ε as in (3.22) and c is a normalizing constant. We can perform this sampling in two steps:

1. sample $(\hat{w}^\varepsilon, h^\varepsilon)$ from $ce^{\gamma h^\varepsilon(z)} dh^\varepsilon dz$, which is the so-called rooted measure of h^ε;
2. condition on the event E_δ^ε.

Under the rooted measure $ce^{\gamma h^\varepsilon(z)} dh^\varepsilon dz$, h^ε can be written as ([DS11, Section 3.3])

$$h^\varepsilon = \tilde{h}_0^\varepsilon + (2Q - \gamma)\log\varepsilon + \gamma G_{D^\varepsilon}(\cdot, \hat{w}^\varepsilon) + \gamma G_{D^\varepsilon}(\cdot, 0) \tag{3.27}$$

where $\hat{w}^\varepsilon$ is sampled from its marginal law under the rooted measure [1] and $\tilde{h}_0^\varepsilon$ is an zero boundary GFF on D^ε that is independent of $\hat{w}^\varepsilon$.

As stated in Proposition 90, we apply the Mobius transform $z \mapsto (\hat{w}^\varepsilon)^{-1} z$ to pull $(0, \hat{w}^\varepsilon, \infty)$ back to $(0, 1, \infty)$. Let $\hat{D}^\varepsilon = |\hat{w}^\varepsilon|^{-1} D^\varepsilon$, then the resulting field after coordinate change is

$$\hat{h}^\varepsilon = \hat{h}_0^\varepsilon + (2Q - \gamma)\log\varepsilon + \gamma G_{\hat{D}^\varepsilon}(\cdot, 1) + \gamma G_{\hat{D}^\varepsilon}(\cdot, 0) + Q\log|\hat{w}^\varepsilon| \tag{3.28}$$

where $\hat{h}_0^\varepsilon$ denotes the zero boundary GFF on $\hat{D}^\varepsilon$. Moreover, the event E_δ^ε becomes $\mu_{\hat{h}^\varepsilon}(\mathbb{C}) \in [e^{-\gamma\delta}, e^{\gamma\delta}]$, and thus by conditioning $\hat{h}_\varepsilon$ on the event E_δ^ε, we obtain the field in the statement in Proposition 90.

In other words, if we first sample $(\hat{w}^\varepsilon, h^\varepsilon)$ according to $\hat{\mathbb{P}}_\delta^\varepsilon$, then map $(0, \hat{w}^\varepsilon, \infty)$ to $(0, 1, \infty)$, then the resulting field after coordinate change of h^ε is exactly $\hat{h}^\varepsilon$ on $\hat{D}^\varepsilon$ as in (3.28) conditioning on $\mu_{\hat{h}^\varepsilon}(\mathbb{C}) \in [e^{-\gamma\delta}, e^{\gamma\delta}]$.

Proof of Proposition 90. By Lemma 91, $\log|\hat{w}^\varepsilon| = o_\varepsilon(|\log\varepsilon|^{\frac{2}{3}})$ with probability $1 - o_{\varepsilon,\delta}(1)$ under $\hat{\mathbb{P}}_\delta^\varepsilon$. Here and henceforth we use the notation $o_{\varepsilon,\delta}(1)$ to denote a quantity such that $\lim_{\delta\to 0}\lim_{\varepsilon\to 0} o_{\varepsilon,\delta}(1) = 0$.

Let $|\hat{D}^\varepsilon|$ be the radius of $\hat{D}^\varepsilon$. By Lemma 91, we can write

$$\hat{h}^\varepsilon = \hat{h}_0^\varepsilon - (2Q - \gamma + q_\varepsilon)\log|\hat{D}^\varepsilon| + \gamma G_{\hat{D}^\varepsilon}(\cdot, 1) + \gamma G_{\hat{D}^\varepsilon}(\cdot, 0) \tag{3.29}$$

1. As explained in [DS11], the marginal law of $\hat{w}^\varepsilon$ has density $\propto \int e^{\gamma h^\varepsilon(z)} dh^\varepsilon$ where the integration is over the law of h^ε.

where under $\hat{\mathbb{P}}^{\varepsilon}_{\delta}$,

$$|\hat{D}^{\varepsilon}| \to \infty \text{ and } \lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{q_{\varepsilon}}{|\log\varepsilon|^{-1/3}} = 0 \quad \text{in law.} \tag{3.30}$$

Now let $\hat{A}_{\varepsilon}$ be the circle average of $\hat{h}^{\varepsilon}_0$ along $\partial\mathbb{D}$ and

$$\hat{H}^{\varepsilon} = \{\hat{A}_{\varepsilon} + (2Q - 3\gamma + q_{\varepsilon})\log\varepsilon \geq -|\log\varepsilon|^{2/3}\}.$$

We claim that

$$\hat{\mathbb{P}}^{\varepsilon}_{\delta}[\hat{H}^{\varepsilon}] = 1 - o_{\varepsilon,\delta}(1). \tag{3.31}$$

We will prove (3.31) as a corollary of Lemma 93 below. Let us now see how it implies Proposition 90. Given (3.31), to study the limiting measure in Proposition 90 as $\varepsilon \to 0$ and then $\delta \to 0$, it suffices to study the limiting measure obtained by the following procedure:

Step 1 sample a pair of random variables $(\hat{D}^{\varepsilon}, q_{\varepsilon})$ such that as $\varepsilon \to 0, \delta \to 0$, it holds that $|\hat{D}^{\varepsilon}| \to \infty$ and $q_{\varepsilon} = o\left(\left|\log|\hat{D}^{\varepsilon}|\right|^{-1/3}\right)$ in law. (Notice that in particular the pair obtained from the marginal law on $\hat{w}^{\varepsilon}$ as above satisfies these conditions);

Step 2 given $(\hat{D}^{\varepsilon}, q_{\varepsilon})$, sample a zero boundary GFF on $\hat{D}^{\varepsilon}$ (denoted by $\hat{h}^{\varepsilon}_0$) and construct $\hat{h}^{\varepsilon}$ by (3.29). Condition on both $\mu_{\hat{h}^{\varepsilon}(\mathbb{C})} \in [e^{-\gamma\delta}, e^{\gamma\delta}]$ and $\hat{H}^{\varepsilon}$ occuring.

In fact, although the measure in Proposition 90 before taking the limit is not exactly as the one sampled from the above two steps, by (3.30) and (3.31), the limits of these two procedures coincide in law.

Now we are ready to use the proof of Theorem 83 for the case $z_1 = 0, z_2 = 1$. First, notice that the sampling of the Step 2 is independent of Step 1 given $\hat{D}^{\varepsilon}$. Thus to obtain the limiting law in Proposition 90 we can simply assume that q_{ε} is a deterministic vanishing sequence and $\hat{D}^{\varepsilon}$ is a sequence of deterministic growing disks. Now suppose that we replace $(2Q - \gamma)$ by $(2Q - \gamma + b_{\varepsilon})$ in (3.10) where b_{ε} is a deterministic sequence converging to 0. Then one can step-by-step examine that in Section 3, if whenever encountering $(2Q - 3\gamma)$ we replace it by $(2Q - 3\gamma + b_{\varepsilon})$, the same arguments still work. In particular, the conclusion of Theorem 83 still holds, thus Proposition 90 follows. □

To finish off, we prove the claim (3.31). By examining (3.28), (3.29) along with (3.30), we see that with probability $1 - o_{\varepsilon,\delta}(1)$ the difference between $\hat{A}_{\varepsilon} + (2Q - 3\gamma)\log\varepsilon$ and the circle average of $\hat{h}^{\varepsilon}$ along $\partial\mathbb{D}$ is $o(|\log\varepsilon|^{2/3})$. Moreover, the circle average of $\hat{h}^{\varepsilon}$ along $\partial\mathbb{D}$ and the circle average of h^{ε} along $|\hat{w}^{\varepsilon}|\partial\mathbb{D}$ differ by $Q\log|\hat{w}^{\varepsilon}| = o(|\log\varepsilon|^{2/3})$ again with probability $1 - o_{\varepsilon,\delta}(1)$. Finally, since the total variational distance of $\hat{\mathbb{P}}^{\varepsilon}_{\delta}$ and $\mathbb{P}^{\varepsilon}_{\delta}$ is $o_{\delta}(1)$, we have that $\hat{\mathbb{P}}^{\varepsilon}_{\delta}[\hat{H}^{\varepsilon}] \geq \mathbb{P}^{\varepsilon}_{\delta}[\widetilde{H}^{\varepsilon}] - o_{\varepsilon,\delta}(1)$, where $\widetilde{H}^{\varepsilon}$ is the event that the circle average of h^{ε} along $|w^{\varepsilon}|\partial\mathbb{D}$ is larger than $-|\log\varepsilon|^{2/3}$. Thus (3.31) follows from:

Lemme 93. $\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\mathbb{P}^{\varepsilon}_{\delta}[\widetilde{H}^{\varepsilon}] = 1$.

Proof. Recall the setting of the proof of Lemma 91. In the cylindrical coordinates, the radial component of h^ε can be written as $X_t^\varepsilon = B_t - (Q-\gamma)t$ such that B_t evolves as a standard Brownian motion and $B_{\log\varepsilon} = 2(Q-\gamma)\log\varepsilon$. By the coordinate change formula and Lemma 91,

$$\mathbb{P}_\delta^\varepsilon[\widetilde{H}^\varepsilon] \geq \mathbb{P}_\delta^\varepsilon\left[X^\varepsilon_{-\log|w^\varepsilon|} \geq -|\log\varepsilon|^{2/3}/2\right] - o_{\varepsilon,\delta}(1). \tag{3.32}$$

Let L^ε be the location where X_t^ε achieves its maxima and F^ε be the event that $X^\varepsilon_{L^\varepsilon} \geq -|\log\varepsilon|^{2/3}/4$. Then (3.25) in Lemma 91 implies that:

$$\mathbb{P}[F^\varepsilon|E_\delta^\varepsilon] = 1 - o_\varepsilon(1) \quad \text{for fixed } \delta. \tag{3.33}$$

But now the proof of Proposition 89 implies that $X^\varepsilon_{\log|w^\varepsilon|} - X^\varepsilon_{L^\varepsilon}$ is a tight sequence of random variables. Hence (3.32) and (3.33) together yield Lemma 93. □

Annexe A

Vers une construction des cascades multiplicatives plus homogène en espace?

Comparons la construction du mouvement Brownien à la Paul Lévy avec la construction des cascades multiplicatives ci-dessus : on remarque une différence entre les opérations qui admet une conséquence majeure. D'un côté, pour le mouvement Brownien, l'opération de recollement introduit une contrainte forte sur les bords, à savoir que au point où les deux mouvements Browniens se rencontrent, ils doivent prendre la même valeur, ce qui se traduit ensuite par une corrélation entre ces deux chemins. De l'autre côté, pour les cascades multiplicatives, l'opération de division agit indépendemment sur les intervalles, ainsi, la structure de corrélation n'est pas homogène et dépend de la manière dont on divise les intervalles.
Dans la suite, on verra un modèle qui pourrait être vu comme la limite des cascades multiplicatives : le champ libre Gaussien en dimension 2. Contrairement aux cascades, le champ libre Gaussien admet une corrélation quasi-homogène en espace. Une question naturelle serait de se demander :

Question 1. *En rajoutant un conditionnement sur les valeurs aux bords lors de l'opération de division, peut-on obtenir, à la limite, une variante des cascades multiplicatives dont la corrélation est homogène en espace?*

On essaie de donner une réponse à cette question, en nous donnant plus de liberté sur la façon de diviser les intervalles pour avoir plus de symétrie.
Déjà, il est plus naturel de supprimer la distinction entre les extrémités 0 et 1 lors de la construction des cascades multiplicatives. On est amené alors à travailler sur le tore $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.
Comme il n'y a aucun point privilégié comme "milieu du tore", pour diviser le tore en $\mathbb{T}$ en deux parties égales (bien sûr, on pourra imaginer d'autres façons de diviser, mais ceci

nous semble être la plus simple), on choisit un point x uniformément selon la mesure de Lebesgue sur $[0,1]$ plus on divise le tore $\mathbb{T}$ en

$$\mathbb{T} = [x, x+1/2[\cup [x+1/2, x+1[\tag{A.1}$$

(avec la convention $x+1 = x$).
Plus généralement, définissons une opération de division sur n'importe quel intervalle avec les extrémités identifiées :

Définition 94 (Opérations de division).
Soit $I_{s,t} = [s,t[\subset \mathbb{R}$ un intervalle de $\mathbb{R}$ de longueur $l = t - s > 0$. Nous définissons une opération de division de l'intervalle $I = I_{s,t}$ comme suit :

1. *Choisisson un point $x \in [s,t[$ selon la mesure uniforme sur $I_{s,t}$;*
2. *Diviser l'intervalle $I_{s,t}$ en deux parties de longueur (au sens de Lebesgue) $l/2$: une partie sera un intervalle $I^1_{s,t}$ de longueur $l/2$ qui admet x comme l'une des extrémités, et l'autre partie $I^2_{s,t}$ sera le complémentaire de $I^1_{s,t}$ dans $I_{s,t}$.*

Il n'est pas difficile de voir que cette division est unique une fois x est choisi.

Autrement dit, si l'on identifie les extrémités, on diviser l'intervalle (ou le tore) $I_{s,t}$ en deux intervalles de longueur égale $l/2$ avec x l'une des extrémités.
On définit alors une variante des cascades multiplicatives, en rajoutant ce "shift" aléatoire lors de la division des intervalles.

Définition 95 (Cascades multiplicatives homogènes).
On construit un modèle de cascades multiplicatives homogènes sur $I = [0,1[$.
Choisissons une variable aléatoire W fixée, $(W^i_j)_{i\in\mathbb{N}, 0\leq j\leq 2^i-1}$ des copies indépendantes de W.
Munissons $I = [0,1]$ de la mesure de Lebesgue usuelle et effectuons successivement les opérations de division ci-dessus de la manière suivante :

1. *Notons Y_0 la mesure totale de $I = [0,1[$ muni de la mesure de Lebesgue $d\lambda^0$: $Y_0 = \lambda^0([0,1[) = 1$;*
2. *Effectuer l'opération de division sur I pour obtenir deux parties I^1_0 et I^1_1 (de longueur usuelle 2^{-1});*
3. *Dans chaque sous-partie I^1_j pour $0 \leq j \leq 1$, multiplifier la mesure $d\lambda^0$ par W^1_j (qui est une copie indépendante de W) et notons cette nouvelle mesure $d\lambda^1$;*
4. *Notons $Y_1 = \lambda^1([0,1[)$ la mesure totale de $I = [0,1[$ muni de la nouvelle mesure $d\lambda^1$;*
5. *Diviser chaque sous parties I^1_j par l'opération de division pour obtenir I^2_j avec $0 \leq j \leq 3$ (chacune de longueur usuelle 2^{-2});*
6. *Dans chaque sous-partie I^1_j pour $0 \leq j \leq 1$, multiplifier la mesure $d\lambda^1$ par W^2_j (qui est une copie indépendante de W) et notons cette nouvelle mesure $d\lambda^2$;*

7. *Notons* $Y_2 = \lambda^2([0,1[)$ *la mesure totale de* $I = [0,1[$ *muni de la nouvelle mesure* $d\lambda^2$*;*
8. *Ainsi de suite jusqu'à l'infini.*

Pour que la question soit intéressante, supposons que $\mathbb{E}[W] = 1$. Un choix pour W qui est directement lié à notre sujet de thèse sera de prendre

$$W = e^{C\mathcal{N} - \frac{C^2}{2}} \tag{A.2}$$

où $C \in \mathbb{R}_+$ et $\mathcal{N}$ est une variable aléatoire qui suit la loi normale standard $\mathcal{N}(0,1)$. Alors pour chaque $i \in \mathbb{N}$, la densité $\lambda^i(x)$ avec $x \in [0,1[$ est le produit de i copies indépendantes de W. Il s'ensuit alors, en prenant le logarithme, que la densité de λ^i est l'exponentielle d'un processus Gaussien $(X^i(x))_{x\in[0,1[}$ indexé par $I = [0,1[$.
On pourrait s'intéresser à la fonction de corrélation de ce processus Gaussien X^i. Un calcul montre que

Proposition 96 (Corrélation logarithmique).
La fonction de corrélation du champ Gaussien X^i *converge, quand* i *tend vers l'infini, vers une fonction de corrélation du type logarithme. Plus précisément, si*

$$K^i(x,y) = \mathbb{E}[X^i(x)X^i(y)] \tag{A.3}$$

alors

$$\lim_{i\to\infty} K^i(x,y) = -\gamma \log |x-y|_{\mathbb{T}} + O(1) \tag{A.4}$$

où $\gamma = C\ln 2 > 0$ *et* $O(1)$ *est une fonction (symmétrique) continue bornée sur* $I^2 = [0,1[^2$.
Ici, $|\cdot|_{\mathbb{T}}$ *désigne la distance du tore* $\mathbb{T}$*, i.e.*

$$|x-y|_{\mathbb{T}} = \min\{|x-y|, |1-|x-y||\}. \tag{A.5}$$

Nous vérifions que ces cascades multiplicatives homogènes satisfont encore l'équation fonctionnelle (1.4) du Lemme 6, et les résultats de Kahane et Peyrière dans le Lemme 7 sont encore valides dans ce cas et s'écrivent :

Lemme 97 (Condition de convergence $\mathcal{L}^1$ et existence des moments).
On a les résultats suivants :
• *1) Sur la non-dégénérescence :*
Les assertions suivantes sont équivalentes :
a) $\mathbb{E}[Y_\infty] = 1$*;*
b) $\mathbb{E}[Y_\infty] > 0$*;*
c) Une solution Z *existe avec* $\mathbb{E}[Z] = 1$*;*
d) $\gamma < \sqrt{2}$.
• *2) Sur l'existence des moments :*
Soit $h > 1$. *On a* $\mathbb{E}[Y_\infty^h] < +\infty$ *si et seulement si* $h < \frac{2}{\gamma^2}$.

Ce résultat s'étend naturellement au cas multidimensionnel. Les champs Gaussiens corrélés logarithmiquement seront étudiés plus loin et plus en détails : voir aussi [BZ12] pour un modèle de processus de branchement (dit "modifié") qui interpole entre le processus de branchement classique et le champ libre Gaussien sur un tore.

Bibliographie

[Aco14] Javier Acosta. Tightness of the recentered maximum of log-correlated Gaussian fields. *Electron. J. Probab.*, 19 :25 pp., 2014.

[ADJ90] Jan Ambjørn, Bergfinnur Jøgvan Durhuus, and T. Jonsson. A random walk representation of the dirac propagator. *Nuclear Physics, Section B*, 330(2-3) :509–522, 1 1990.

[AHS15] Juhan Aru, Yichao Huang, and Xin Sun. Two perspectives of the 2D unit area quantum sphere and their equivalence. *arXiv preprint arXiv :1512.06190*, 2015.

[ARV13] Romain Allez, Rémi Rhodes, and Vincent Vargas. Lognormal star-scale invariant random measures. *Probability Theory and Related Fields*, pages 1–38, 2013.

[AX05] Antoine Ayache and Yimin Xiao. Asymptotic properties and hausdorff dimensions of fractional brownian sheets. *Journal of Fourier Analysis and Applications*, 11(4) :407–439, 2005.

[Ber17] Nathanaël Berestycki. An elementary approach to Gaussian multiplicative chaos. *Electron. Commun. Probab.*, 22 :12 pp., 2017.

[BG09] Jérémie Bouttier and Emmanuel Guitter. Distance statistics in quadrangulations with a boundary, or with a self-avoiding loop. *Journal of Physics A : Mathematical and Theoretical*, 42(46) :465208, 2009.

[Big77] J. D. Biggins. Martingale convergence in the branching random walk. *Journal of Applied Probability*, 14(1) :25–37, 1977.

[BSS14] N. Berestycki, S. Sheffield, and X. Sun. Equivalence of Liouville measure and Gaussian free field. *ArXiv e-prints*, October 2014.

[BZ12] Maury Bramson and Ofer Zeitouni. Tightness of the recentered maximum of the two-dimensional discrete gaussian free field. *Communications on Pure and Applied Mathematics*, 65(1) :1–20, 2012.

[Car05] J. Cardy. SLE for theoretical physicists. *Annals of Physics*, 318(1) :81 – 118, 2005. Special Issue.

[CM44] R. H. Cameron and W. T. Martin. Transformations of weiner integrals under translations. *Annals of Mathematics*, 45(2) :386–396, 1944.

[CM15] Nicolas Curien and Grégory Miermont. Uniform infinite planar quadrangulations with a boundary. *Random Structures & Algorithms*, 47(1) :30–58, 2015.

[Dav88] François David. Conformal field theories coupled to 2d gravity in the conformal gauge. *Modern Physics Letters A*, 03(17) :1651–1656, 1988.

[DK89] J. Distler and H. Kawai. Conformal field theory and 2-d quantum gravity or who's afraid of Joseph Liouville ? *Nucl. Phys.*, B321 :509–517, 1989.

[DKRV15] F. David, A. Kupiainen, R. Rhodes, and V. Vargas. Renormalizability of Liouville Quantum Gravity at the Seiberg bound. *ArXiv e-prints*, June 2015.

[DKRV16] François David, Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. Liouville quantum gravity on the riemann sphere. *Communications in Mathematical Physics*, 342(3) :869–907, 2016.

[DMS14] B. Duplantier, J. Miller, and S. Sheffield. Liouville quantum gravity as a mating of trees. *ArXiv e-prints*, September 2014.

[Don51] Monroe D. Donsker. An invariance principle for certain probability limit theorems. *Memoirs of the American Mathematical Society*, 6, 1951.

[DRSV14a] Bertrand Duplantier, Rémi Rhodes, Scott Sheffield, and Vincent Vargas. Renormalization of critical gaussian multiplicative chaos and kpz relation. *Communications in Mathematical Physics*, 330(1) :283–330, 2014.

[DRSV14b] Bertrand Duplantier, Rémi Rhodes, Scott Sheffield, and Vincent Vargas. Critical gaussian multiplicative chaos : Convergence of the derivative martingale. *Ann. Probab.*, 42(5) :1769–1808, 09 2014.

[DRV16] François David, Rémi Rhodes, and Vincent Vargas. Liouville quantum gravity on complex tori. *Journal of Mathematical Physics*, 57(2) :022302, 2016.

[DS11] Bertrand Duplantier and Scott Sheffield. Liouville quantum gravity and KPZ. *Inventiones mathematicae*, 185(2) :333–393, 2011.

[Dub09] Julien Dubédat. Sle and the free field : Partition functions and couplings. *Journal of the American Mathematical Society*, 22(4) :995–1054, 2009.

[Fey48] R. P. Feynman. Space-time approach to non-relativistic quantum mechanics. *Reviews of Modern Physics*, 20 :367–387, April 1948.

[Gaw97] Krzysztof Gawedzki. Lectures on conformal field theory. *Quantum fields and strings : A course for mathematicians*, 1,2 :727–805, 1996/1997.

[Gne01] Tilmann Gneiting. Criteria of pólya type for radial positive definite functions. *Proceedings of the American Mathematical Society*, 129(8) :2309–2318, 2001.

[GRV14] Christophe Garban, Rémi Rhodes, and Vincent Vargas. On the heat kernel and the Dirichlet form of Liouville Brownian motion. *Electron. J. Probab.*, 19 :no. 96, 25, 2014.

[GRV16] Colin Guillarmou, Rémi Rhodes, and Vincent Vargas. Polyakov's formulation of $2d$ bosonic string theory. *arXiv preprint arXiv :1607.08467*, 2016.

[GS15] E. Gwynne and X. Sun. Scaling limits for the critical Fortuin-Kastelyn model on a random planar map III : finite volume case. *ArXiv e-prints*, October 2015.

[HMW11] Daniel Harlow, Jonathan Maltz, and Edward Witten. Analytic continuation of liouville theory. *Journal of High Energy Physics*, 2011(12) :71, 2011.

[HRV15] Y. Huang, R. Rhodes, and V. Vargas. Liouville Quantum Gravity on the unit disk. *ArXiv e-prints*, February 2015.

[Jan97] S. Janson. *Gaussian Hilbert Spaces.* Cambridge Tracts in Mathematics. Cambridge University Press, 1997.

[JS15] Janne Junnila and Eero Saksman. The uniqueness of the Gaussian multiplicative chaos revisited. *arXiv preprint arXiv :1506.05099*, 2015.

[Kah85] J-P Kahane. Sur le chaos multiplicatif. *Ann. Sci. Math. Québec*, 1985.

[KP76] Jean-Pierre Kahane and Jacques Peyrière. Sur certaines martingales de benoit mandelbrot. *Advances in Mathematics*, 22(2) :131 – 145, 1976.

[KPZ88] V.G. Knizhnik, A.M. Polyakov, and A.B. Zamolodchikov. Fractal structure of 2d—quantum gravity. *Modern Physics Letters A*, 03(08) :819–826, 1988.

[KRV15] A. Kupiainen, R. Rhodes, and V. Vargas. Conformal Ward and BPZ Identities for Liouville quantum field theory. *ArXiv e-prints*, December 2015.

[KS91] I. Karatzas and S.E. Shreve. *Brownian Motion and Stochastic Calculus.* Graduate Texts in Mathematics. Springer New York, 1991.

[LG13] Jean-François Le Gall. Uniqueness and universality of the Brownian map. *Ann. Probab.*, 41(4) :2880–2960, 2013.

[Mad17] Thomas Madaule. Convergence in law for the branching random walk seen from its tip. *Journal of Theoretical Probability*, 30(1) :27–63, 2017.

[Man74] Benoit B. Mandelbrot. Intermittent turbulence in self-similar cascades : divergence of high moments and dimension of the carrier. *Journal of Fluid Mechanics*, 62(2) :331–358, 1974.

[Mie13] Grégory Miermont. The Brownian map is the scaling limit of uniform random plane quadrangulations. *Acta Math.*, 210(2) :319–401, 2013.

[MS13] J. Miller and S. Sheffield. Imaginary geometry IV : interior rays, whole-plane reversibility, and space-filling trees. *ArXiv e-prints*, February 2013.

[MS15a] J. Miller and S. Sheffield. Liouville quantum gravity and the Brownian map I : The QLE(8/3,0) metric. *ArXiv e-prints*, July 2015.

[MS15b] J. Miller and S. Sheffield. Liouville quantum gravity spheres as matings of finite-diameter trees. *ArXiv e-prints*, June 2015.

[MS16a] J. Miller and S. Sheffield. Liouville quantum gravity and the Brownian map II : geodesics and continuity of the embedding. *ArXiv e-prints*, May 2016.

[MS16b] J. Miller and S. Sheffield. Liouville quantum gravity and the Brownian map III : the conformal structure is determined. *ArXiv e-prints*, August 2016.

[MS16c] Jason Miller and Scott Sheffield. Quantum loewner evolution. *Duke Math. J.*, 165(17) :3241–3378, 11 2016.

[Nak04] Y. Nakayama. Liouville field theory – a decade after the revolution. *Int. J. Mod. Phy.*, A19(2771), 2004.

[Pol81] A.M. Polyakov. Quantum geometry of bosonic strings. *Physics Letters B*, 103(3) :207 – 210, 1981.

[RSV14] Rémi Rhodes, Julien Sohier, and Vincent Vargas. Levy multiplicative chaos and star scale invariant random measures. *Ann. Probab.*, 42(2) :689–724, 03 2014.

[RV10] Raoul Robert and Vincent Vargas. Gaussian multiplicative chaos revisited. *The Annals of Probability*, 38(2) :605–631, 2010.

[RV11] Rémi Rhodes and Vincent Vargas. KPZ formula for log-infinitely divisible multifractal random measures. *ESAIM Probability and Statistics*, 15 :358, 2011.

[RV14] Rémi Rhodes and Vincent Vargas. Gaussian multiplicative chaos and applications : A review. *Probab. Surveys*, 11 :315–392, 2014.

[RV15] Rémi Rhodes and Vincent Vargas. Liouville brownian motion at criticality. *Potential Analysis*, 43(2) :149–197, 2015.

[RV16] Rémi Rhodes and Vincent Vargas. Lecture notes on Gaussian multiplicative chaos and Liouville Quantum Gravity. *arXiv preprint arXiv :1602.07323*, 2016.

[Sei90] N. Seiberg. Notes on quantum liouville theory and quantum gravity. *Progress of Theoretical Physics Supplement*, 102 :319–349, 1990.

[Sha14] A. Shamov. On Gaussian multiplicative chaos. *arXiv preprint arXiv :1407.4418*, July 2014.

[She07] Scott Sheffield. Gaussian free fields for mathematicians. *Probability theory and related fields*, 139(3-4) :521–541, 2007.

[She16] Scott Sheffield. Conformal weldings of random surfaces : SLE and the quantum gravity zipper. *Ann. Probab.*, 44(5) :3474–3545, 09 2016.

[Zei16] Ofer Zeitouni. Gaussian Fields : Notes for Lectures. *Personal homepage*, 2016.