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Singular elliptic problems in perforated and two-component domains

Federica Raimondi

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Spécialité Mathématiques

Préparée au sein de Université de Rouen Normandie
et de Università della Campania Luigi Vanvitelli

Problèmes elliptiques singuliers dans des domaines perforés et à deux composants

Singular elliptic problems in perforated and two-component domains

**Présentée et soutenue par
Federica RAIMONDI**

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Abstract

This thesis is mainly devoted to the study of some singular elliptic problems posed in perforated domains. Denoting by Ω_ε^* a domain perforated by ε -periodic holes of ε -size, we prove existence and uniqueness of the solution, for fixed ε , as well as homogenization and correctors results for the following singular problem:

$$\begin{cases} -\operatorname{div}\left(A\left(\frac{x}{\varepsilon}, u_\varepsilon\right)\nabla u_\varepsilon\right) = f\zeta(u_\varepsilon) & \text{in } \Omega_\varepsilon^*, \\ u_\varepsilon = 0 & \text{on } \Gamma_0^\varepsilon, \\ \left(A\left(\frac{x}{\varepsilon}, u_\varepsilon\right)\nabla u_\varepsilon\right)\nu + \varepsilon^\gamma \rho\left(\frac{x}{\varepsilon}\right)h(u_\varepsilon) = \varepsilon g\left(\frac{x}{\varepsilon}\right) & \text{on } \Gamma_1^\varepsilon, \end{cases}$$

where homogeneous Dirichlet and nonlinear Robin conditions are prescribed on the exterior boundary Γ_0^ε and on the boundary of the holes Γ_1^ε , respectively. The quasilinear matrix field A is elliptic, bounded, periodic in the first variable and Carathéodory. The nonlinear singular lower order term is the product of a continuous function ζ (singular in zero) and f whose summability depends on the growth of ζ near its singularity. The nonlinear boundary term h is a C^1 increasing function, ρ and g are periodic nonnegative functions with prescribed summabilities. To investigate the asymptotic behaviour of the problem, as $\varepsilon \rightarrow 0$, we apply the Periodic Unfolding Method by D. Cioranescu-A. Dalmlamian-G. Griso, adapted to perforated domains by D. Cioranescu-A. Dalmlamian-P. Donato-G. Griso-R. Zaki.

Finally, we show existence and uniqueness of a weak solution of the same equation in a two-component domain $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$, being Γ the interface between the connected component Ω_1 and the inclusions Ω_2 . More precisely we consider

$$\begin{cases} -\operatorname{div}(A(x, u)\nabla u) + \lambda u = f\zeta(u) & \text{in } \Omega \setminus \Gamma, \\ u = 0 & \text{on } \partial\Omega, \\ (A(x, u_1)\nabla u_1)\nu_1 = (A(x, u_2)\nabla u_2)\nu_1 & \text{on } \Gamma, \\ (A(x, u_1)\nabla u_1)\nu_1 = -h(u_1 - u_2) & \text{on } \Gamma, \end{cases}$$

where ν_1 is the unit external normal vector to Ω_1 and λ a nonnegative real number. Here h represents the proportionality coefficient between the continuous heat flux and the jump of the solution and it is assumed to be bounded and nonnegative on Γ .

Abstract

Questa tesi tratta principalmente di alcuni problemi ellittici singolari posti in domini perforati. Denotando con Ω_ε^* un dominio periodicamente perforato da buchi di misura ε , mostriamo esistenza e unicità della soluzione, ad ε fissato, così come risultati di omogeneizzazione e correttori per il seguente problema singolare:

$$\begin{cases} -\operatorname{div}\left(A\left(\frac{x}{\varepsilon}, u_\varepsilon\right)\nabla u_\varepsilon\right) = f\zeta(u_\varepsilon) & \text{in } \Omega_\varepsilon^*, \\ u_\varepsilon = 0 & \text{su } \Gamma_0^\varepsilon, \\ \left(A\left(\frac{x}{\varepsilon}, u_\varepsilon\right)\nabla u_\varepsilon\right)\nu + \varepsilon^\gamma \rho\left(\frac{x}{\varepsilon}\right)h(u_\varepsilon) = \varepsilon g\left(\frac{x}{\varepsilon}\right) & \text{su } \Gamma_1^\varepsilon, \end{cases}$$

dove sono prescritte condizioni di Dirichlet omogenee sulla frontiera esterna Γ_0^ε e di Robin non lineari su quella dei buchi interni Γ_1^ε . Il campo matriciale quasi lineare A è ellittico, limitato, periodico nella prima variabile e di Carathéodory. Il termine singolare non lineare è prodotto di una funzione continua ζ (singolare nello zero) e di f , la cui sommabilità dipende dalla crescita di ζ vicino alla sua singolarità. Il termine di bordo non lineare h è una funzione crescente e derivabile con continuità, mentre ρ e g sono funzioni periodiche non negative con assegnate sommabilità. Per studiare il comportamento asintotico del problema, al tendere di ε a zero, applichiamo il "periodic unfolding method" (cfr. D. Cioranescu-A. Dalmlamian-G. Griso e D. Cioranescu-A. Dalmlamian-P. Donato-G. Griso-R. Zaki per domini perforati).

Infine, mostriamo esistenza e unicità della soluzione debole per la stessa equazione in un dominio a due componenti $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$, essendo Γ l'interfaccia tra la componente connessa Ω_1 e le inclusioni Ω_2 . Più precisamente consideriamo

$$\begin{cases} -\operatorname{div}(A(x, u)\nabla u) + \lambda u = f\zeta(u) & \text{in } \Omega \setminus \Gamma, \\ u = 0 & \text{su } \partial\Omega, \\ (A(x, u_1)\nabla u_1)\nu_1 = (A(x, u_2)\nabla u_2)\nu_1 & \text{su } \Gamma, \\ (A(x, u_1)\nabla u_1)\nu_1 = -h(u_1 - u_2) & \text{su } \Gamma, \end{cases}$$

dove λ è un numero reale non negativo e h rappresenta il coefficiente di proporzionalità tra il flusso continuo del calore e il salto della soluzione, ed è assunto essere limitato e non negativo su Γ .

Résumé

Cette thèse est consacrée principalement à l'étude de quelques problèmes elliptiques singuliers dans un domaine Ω_ε^* , périodiquement perforé par des trous de taille ε . On montre l'existence et l'unicité d'une solution, pour tout ε fixé, ainsi que des résultats de homogénéisation et correcteurs pour le problème singulier suivant:

$$\begin{cases} -\operatorname{div}\left(A\left(\frac{x}{\varepsilon}, u_\varepsilon\right)\nabla u_\varepsilon\right) = f\zeta(u_\varepsilon) & \text{dans } \Omega_\varepsilon^*, \\ u_\varepsilon = 0 & \text{sur } \Gamma_0^\varepsilon, \\ \left(A\left(\frac{x}{\varepsilon}, u_\varepsilon\right)\nabla u_\varepsilon\right)\nu + \varepsilon^\gamma \rho\left(\frac{x}{\varepsilon}\right)h(u_\varepsilon) = \varepsilon g\left(\frac{x}{\varepsilon}\right) & \text{sur } \Gamma_1^\varepsilon, \end{cases}$$

où l'on prescrit des conditions de Dirichlet homogènes sur la frontière extérieure Γ_0^ε et des conditions de Robin non linéaires sur la frontière des trous Γ_1^ε . Le champ matriciel quasi linéaire A est elliptique, borné, périodique dans la première variable et de Carathéodory. Le terme singulier non linéaire est le produit d'une fonction continue ζ (singulier en zéro) et de f , dont la sommabilité dépend de la croissance de ζ près de sa singularité. Le terme de bord non linéaire h est une fonction croissante de classe C^1 , ρ et g sont des fonctions périodiques non négatives avec sommabilité convenables. Pour étudier le comportement asymptotique du problème quand $\varepsilon \rightarrow 0$, on applique la méthode de l'éclatement périodique due à D. Cioranescu-A. Dalambian-G. Griso (cf. D. Cioranescu-A. Dalambian-P. Donato-G. Griso-R. Zaki pour les domaines perforés).

Enfin, on montre l'existence et l'unicité de la solution faible pour la même équation, dans un domaine à deux composants $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$, étant Γ l'interface entre la composant connecté Ω_1 et les inclusions Ω_2 . Plus précisément on considère

$$\begin{cases} -\operatorname{div}(A(x, u)\nabla u) + \lambda u = f\zeta(u) & \text{dans } \Omega \setminus \Gamma, \\ u = 0 & \text{sur } \partial\Omega, \\ (A(x, u_1)\nabla u_1)\nu_1 = (A(x, u_2)\nabla u_2)\nu_1 & \text{sur } \Gamma, \\ (A(x, u_1)\nabla u_1)\nu_1 = -h(u_1 - u_2) & \text{sur } \Gamma, \end{cases}$$

où λ est un réel non négatif et h représente le coefficient de proportionnalité entre le flux de chaleur et le saut de la solution, et il est supposé être borné et non négatif sur Γ .

Contents

Introduction	6
Existence for singular problems	7
A general presentation	8
Our contribution	9
Homogenization of singular problems	11
A general presentation	11
The periodic unfolding method	14
Our contribution	17
1 Existence and uniqueness results for a class of singular elliptic problems in perforated domains	19
1.1 Introduction	20
1.2 Setting of the problem	22
1.3 A priori estimates	27
1.4 An approximate problem	31
1.5 An existence result for the singular problem	36
1.6 A uniqueness result	40
1.7 A regularity result	44
2 Homogenization of a class of singular elliptic problems in perforated domains	46
2.1 Introduction	47
2.2 Setting of the problem and main results	49
2.3 The periodic unfolding method	56
2.4 A priori estimates	60
2.5 A crucial auxiliary result	64
2.6 Proof of Theorem 2.8	75

3 Existence and uniqueness results for a class of singular elliptic problems in two-component domains	82
3.1 Introduction	82
3.2 Setting of the problem	84
3.3 A priori estimates	86
3.4 Main results	89
Perspectives	93
Bibliography	94

Introduction

The aim of this thesis is to study the mathematical model of the stationary heat diffusion in heterogeneous materials. We analyze two different frameworks: the case of perforated materials and the one of materials with inclusions. One can find several examples of these composite materials in the world around us: let bear in mind the aluminium foam, the fiberglass (employed, for instance, to build swimming pools and aircrafts), the carbon fibers, the mudbricks (made of mud and straw), mix of aggregates like wood reinforced with cellulose fibers, aluminium conductors or composite artificial limbs.

In our study, the mathematical model of the stationary heat diffusion is the following quasilinear elliptic equation with a singular lower order term:

$$-div(B(x, u)\nabla u) + \lambda u = f\zeta(u).$$

The quasilinear diffusion term consists in an uniformly elliptic and bounded Carathéodory matrix field $B(x, t)$, λ is a nonnegative real number, $\zeta(s)$ is a nonnegative real function singular at $s = 0$ and f is a nonnegative datum whose summability depends on the growth of ζ near its singularity.

From the physical point of view, quasilinear diffusion matrix fields describe the behavior of materials, like glass or wood, in which the heat diffusion depends on the range of the temperature (see [50] for more details). A physical phenomenon described by a source term that depends on the solution itself and becomes infinite when the solution vanishes is the one of an electrical conductor where each point becomes a source of heat as a current flows in it. Indeed, in this framework, the solution $u(x)$ represents the temperature at the point x , the function $f(x)$ is the square of local voltage drop in the material and $\zeta(u)$ denotes the electrical conductivity. Then, generation of heat occurs with a rate given by $f(x)\zeta(u)$. We refer to [42, Section 3] for details on the physical meaning of this problem.

The materials in which the heat diffusion takes place are modeled by perforated and two-component domains. According to the domain we study different kinds of boundary conditions. First we consider a perforated domain, which is obtained by removing a bounded closed set (representing one or more holes) from a bounded connected open set. We prescribe a homogeneous Dirichlet condition on the exterior boundary, while we impose a nonlinear

Robin condition on the boundary of the holes. Such nonlinear boundary conditions model certain chemical reactions at the boundaries of perforations (e.g. [37]). For this model, we investigate the existence and uniqueness of the solution as well as the homogenization and related correctors results.

Also we consider the case where the previous holes are replaced by inclusions of a different material and that an imperfect contact exists on the interface separating the two components. For this domain we prescribe a Dirichlet condition on the exterior boundary and a jump of the solution proportional to the heat flux through the interface. This kind of boundary condition models a rough interface between the two components and we refer to [21] for a physical justification of this model (see also [42] and [59]). By similar techniques as those used for the previous model, we prove existence and uniqueness results for this problem.

The plan of the thesis is the following one:

- In Chapter 1 we study existence, uniqueness and regularity results for a weak solution of the singular problem posed in a perforated domain with nonlinear Robin condition.
- In Chapter 2 we deal with the homogenization of the previous class of problems assuming that they are posed in a periodically perforated domain.
- In Chapter 3 we prove existence, uniqueness and regularity results for a weak solution of the singular problem posed in a two-component domain with a jump of the solution on the interface.

Before going into details on the study presented in the next chapters, we address the reader to the following overview of the content of the thesis, organized into two parts.

Existence for singular problems

The first part of this work concerns some existence, uniqueness and regularity results for two classes of singular problems.

In Chapter 1, appeared in [46], we look for a weak solution of the following problem:

$$\begin{cases} -\operatorname{div}(B(x, u)\nabla u) + \lambda u = f\zeta(u) & \text{in } \mathcal{O}, \\ u = 0 & \text{on } \Gamma_0, \\ (B(x, u)\nabla u)\nu + \rho h(u) = g & \text{on } \Gamma_1, \end{cases} \quad (0.1)$$

where \mathcal{O} denotes a perforated domain of \mathbb{R}^N ($N \geq 2$), Γ_0 the exterior boundary, Γ_1 the boundary of the holes and ν the unit external normal vector to \mathcal{O} . Concerning the Robin boundary condition, the function ρ is assumed to be bounded and nonnegative on Γ_1 , the

nonlinear boundary term h is an increasing and continuously differentiable function whose derivative satisfies suitable growth assumptions, and g is a nonnegative function with opportune summability.

In Chapter 3, appeared in [49], we consider a two-component domain $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ of \mathbb{R}^N ($N \geq 2$), where Γ denotes the interface between the connected component Ω_1 and the inclusions Ω_2 . We focus our attention on the following problem:

$$\begin{cases} -\operatorname{div}(B(x, u)\nabla u) + \lambda u = f\zeta(u) & \text{in } \Omega \setminus \Gamma, \\ u = 0 & \text{on } \partial\Omega, \\ (B(x, u_1)\nabla u_1)\nu_1 = (B(x, u_2)\nabla u_2)\nu_1 & \text{on } \Gamma, \\ (B(x, u_1)\nabla u_1)\nu_1 = -h(u_1 - u_2) & \text{on } \Gamma, \end{cases} \quad (0.2)$$

where ν_1 denotes the unit external normal vector to Ω_1 . Here, the proportionality coefficient h between the continuous heat flux and the jump of the solution is assumed to be bounded and nonnegative on Γ .

A general presentation

Elliptic problems with singular nonlinear terms have been widely studied in the last years. We refer here to the works presented in [4], [17], [18], [24], [42], [51], [52], [53] and [54]. More precisely, in [18], [51] and [42] the authors deal with mild singularities as long as in [52] with a strong one. In [51] they are interested in finding a function u which satisfies the following problem:

$$\begin{cases} u \geq 0 & \text{in } \Omega, \\ -\operatorname{div}(A(x)\nabla u) = F(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded set of \mathbb{R}^N , $A \in L^\infty(\Omega)^{N \times N}$ is a coercive matrix and F is a Carathéodory function such that

$$0 \leq F(x, u) \leq l(x) \left(\frac{1}{u^\gamma} + 1 \right) \quad \text{a.e. } x \in \Omega, \forall s > 0,$$

with $0 < \gamma \leq 1$ and some $l(x)$ belonging to a suitable Lebesgue space. In [52] they consider the case of strong singularities, the more general one where $F(x, u)$ can have any singular behavior as u approaches to zero.

In [4], [24], [53] and [54] a different lower order term appears: $\frac{|\nabla u|^2}{u^\gamma}$, having a quadratic dependence with respect to the gradient and a singular dependence with respect to the solution. Moreover, nonlinear Robin boundary conditions have been treated in the papers

[34], [19] and [24] for the quasilinear case. For the linear equation with this kind of boundary condition, we refer to [25], [34] and [37]. Whereas jump of the solution on the interface is considered in [42].

What is usually done in literature to prove the existence of at least a solution of a singular problem, is first to give some a priori estimates for its solution. Then, the problem is approximated by a sequence of nonsingular problems, whose existence of solutions (that we denote by u_m) have to be showed. The a priori estimates, that also apply to u_m , allow to extract a subsequence converging (at the limit on m) to a solution of the singular problem. The keystone in these proofs is the analysis of the singular term near the singularity. More precisely, one estimates the integral of the singular term on the singular set $\{0 \leq u_m \leq \delta\}$ for $\delta > 0$ sufficiently small. Then, one can pass to the limit as $m \rightarrow +\infty$ and as $\delta \rightarrow 0$, by using some classical results from Stampacchia, as well as the Lebesgue dominated convergence and Vitali theorems and inequalities like Young's and Hölder's.

Our contribution

The singularity we deal with in the whole thesis is the same of the papers mentioned above. In particular we take the same hypothesis as in [42], which inspired our work. Namely, we assume that the singular lower order term $f\zeta$ is as follows:

$$\left\{ \begin{array}{l} \text{i) } \zeta : [0, +\infty[\rightarrow [0, +\infty] \text{ is a function such that} \\ \zeta \in \mathcal{C}^0([0, +\infty[), 0 \leq \zeta(s) \leq \frac{1}{s^\theta} \text{ for every } s \in]0, +\infty[, \text{ with } 0 < \theta \leq 1; \\ \text{ii) } f \geq 0 \text{ a.e. in } \Omega, \text{ with } f \in L^l(\Omega), \text{ for } l \geq \frac{2}{1+\theta} (\geq 1). \end{array} \right.$$

Nevertheless, in [42] the problem is posed through rough surface, while here we have a perforated domain as well as a two-component domain together with quasilinear diffusion terms. When we start looking for solutions of problems (0.1) and (0.2), the difficulties arise in dealing simultaneously with the quasilinear matrix field, the singular datum and the boundary conditions. The techniques used to prove existence, uniqueness and regularity results apply to both perforated and two-component model cases under opportune modifications concerning the boundary terms. The huge difference between the two models consists in the proofs of the a priori estimates. They are well detailed in Sections 1.3 - 3.3.

As usual in the search of solutions, we first derive some a priori estimates. Among all we point out the third a priori estimate (see Propositions 1.9 - 3.5) that we get following some ideas of [42], [51] and [52]. This bound is crucial in treating the singular term since it gives an estimate of the integral of this term on the singular set $\{0 \leq u \leq \delta\}$, for $\delta > 0$ sufficiently small. Indeed the main tool used to handle the singular term is to split its integral into the

sum of two integrals, the one on the set where the solution is close to the singularity and the one where it is far from it, which results not singular.

Then, we take a sequence of nonsingular problems with a bounded nonlinearity in the equation that approximates our problem, namely we consider the following equation:

$$-\operatorname{div}(B(x, u_m)\nabla u_m) + \lambda u_m = T_m(f\zeta(|u_m|)),$$

where, for every $m \in \mathbb{N}$, T_m is the truncation function at level m and u_m its solution. To show the existence of a solution of this approximating problem, due to the presence of the quasilinear term, we use the Schauder fixed-point theorem, arguing similarly to [4], [19], [24], [51] and [53].

Finally, the previous a priori estimates also apply to u_m , uniformly with respect to m . So that, via accurate measure arguments, we pass to the limits, as m goes to infinity and δ goes to zero, obtaining that the sequence of solutions of the approximating problems converges to a solution of our initial problem.

To ensure the uniqueness of such a solution, as usual in literature, some additional hypotheses on the matrix field B and on the nonlinear function ζ are required. Concerning the quasilinear term, we make the following assumption:

$$\left\{ \begin{array}{l} \text{There exists a real function } \omega : \mathbb{R} \rightarrow \mathbb{R} \text{ satisfying the following conditions:} \\ \text{i) } \omega \text{ is continuous and non decreasing, with } \omega(t) > 0 \forall t > 0; \\ \text{ii) } |B(x, t_1) - B(x, t_2)| \leq \omega(|t_1 - t_2|) \text{ for a.e. } x \in \Omega, \forall t_1 \neq t_2; \\ \text{iii) } \forall y > 0, \lim_{x \rightarrow 0^+} \int_x^y \frac{dt}{\omega(t)} = +\infty, \end{array} \right.$$

which was originally introduced in [23] by M.Chipot (see also [3]) for quasilinear nonsingular problem with Dirichlet conditions. We also refer to [12] and [13] for linear Robin conditions and to [19] for nonlinear Robin conditions. Let us remark that if B is uniformly Lipschitz-continuous in t with constant L , then $w(t) \doteq Lt$ satisfies the required assumption. Concerning the function ζ a monotonicity hypothesis is needed, as done in [42] and [52], namely ζ is non increasing in $[0, +\infty[$. We explicitly observe that we do not assume any monotonicity property for ζ except for this result.

Also, under stronger hypotheses on the summabilities of the data f and g , we are able to prove a regularity result i.e. the boundedness of the solution. To do this, we adapt to our needs an argument of [24], making use of classical results of Stampacchia [69], as in [24], [42], [51] and [54].

Homogenization of singular problems

The second part of the thesis entirely develops in the second chapter, published in [47]. We study there the homogenization of a class of quasilinear elliptic problems with singular lower order terms posed in periodically perforated domains.

Let us denote by Ω_ε^* the periodically perforated domain we are going to consider. It is obtained by removing from a given bounded open set, Ω , the set of the ε -periodic holes of the same size as the period. The boundary of Ω_ε^* is decomposed into the union of Γ_1^ε and Γ_0^ε , which denote the boundary of the holes well contained in Ω and the remaining exterior boundary, respectively. Thus, in Chapter 2 we focus our interest on studying the asymptotic behavior, as ε goes to zero, of the following problem:

$$\begin{cases} -\operatorname{div}(A^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon) = f\zeta(u_\varepsilon) & \text{in } \Omega_\varepsilon^*, \\ u_\varepsilon = 0 & \text{on } \Gamma_0^\varepsilon, \\ (A^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon)\nu + \varepsilon^\gamma \rho_\varepsilon(x)h(u_\varepsilon) = g_\varepsilon & \text{on } \Gamma_1^\varepsilon, \end{cases} \quad (0.3)$$

where ν is the unit outward normal to the holes.

The oscillating coefficients' matrix field A^ε in the quasilinear diffusion term is defined by $A^\varepsilon(x, t) = A\left(\frac{x}{\varepsilon}, t\right)$, where the matrix field A is uniformly elliptic, bounded, periodic in the first variable and Carathéodory. The functions ζ and f are the same of problems (0.1) and (0.2). Concerning the Robin boundary condition, $\rho_\varepsilon(x) = \rho\left(\frac{x}{\varepsilon}\right)$ where the function ρ is assumed to be periodic, nonnegative and bounded on the boundary of the holes. The nonlinear boundary term h is an increasing and continuously differentiable function whose derivative satisfies suitable growth assumptions, and $g_\varepsilon(x) = \varepsilon g\left(\frac{x}{\varepsilon}\right)$, where g is a periodic nonnegative function with prescribed summability.

Let us now briefly introduce the reader to the homogenization theory, and then analyze the homogenization process for our problem.

A general presentation

The mathematical theory of the homogenization developed in the last fifty years. It focuses its interest on modeling materials with periodic or heterogeneous structures. It has attracted the attention of many scientists in various fields since several physical and engineering applications present problems with these types of structures.

Among all these problems one finds the modeling of composite materials. They are materials containing two or more finely mixed constituents. These composite materials are widely used nowadays since their macroscopic behavior is better than the ones of their individual constituents. Indeed the physical properties of the composite are not simply an average of those ones of the components.

The aim of the homogenization is to describe the macroscopic behavior of composite materials taking into account their microscopic properties. To describe these composites, we use two different kinds of scales: the microscopic one, which describes the heterogeneities of the sample, and the macroscopic one, which describes its global behavior. The phenomena we want to describe are related to composite materials presenting small heterogeneities. Smaller are the heterogeneities, better is the mixture that appears as a homogeneous material at the macroscopic level.

The goal of this theory is to remove the heterogeneities by homogenizing the mixture. In this way we obtain a new material, the so called 'homogeneous' material, which replaces the heterogeneous one.

The motivation is the following one. The modeling of the phenomena taking place in these composite media, think for instance to the heat transfer, leads to partial differential equations whose coefficients (describing the properties of the material) turn out to be highly oscillating. This represents a big difficulty in solving numerically these equations.

The homogenization theory allows us to examine the composite material by looking over the corresponding homogeneous one. It results easier to deal with, due to the presence of constant coefficients in the corresponding mathematical model, and gives a good approximation of the material.

In our work we consider materials in which the heterogeneities are very small compared with the global dimension of the sample. Also we suppose they are evenly distributed, actually this is a realistic assumption for a wide range of applications. This kind of distribution of the heterogeneities can be modeled by periodicity, which is characterized by a small parameter ε , representing the period. Letting ε go to zero, we find a limit problem (the 'homogenized' problem) whose study corresponds to consider the global behavior of the homogeneous material, at a physical level. Here the periodicity comes from both the heterogeneous media and the high oscillating coefficients.

The first results about the homogenization theory date back to the French, Russian and Italian schools of the 1970's. They have been presented by Bensoussan, Lions and Papanicolaou in [16], Sanchez-Palencia in [65], Bakhvalov and Panasenko in [9]. Since then, many authors have examined in depth this topic. We address the reader to the historical books [8], [16], [31], [63], [66], [56] as well as [9], [36] and [22] for the case of perforated domains and oscillating boundaries, and the references quoted therein.

We refer to [5], [6] for the first results on the periodic homogenization of quasilinear elliptic problems in the case of fixed domains, and to [14], [15] for quadratic nonlinear elliptic problems. The homogenization of elliptic problems in perforated domains was originally studied in [36], [35] with Neumann conditions. While the homogenization of quasilinear problems in

periodically perforated domains with nonlinear Robin boundary conditions has been considered in [20], [25] and [26], where is also considered a nonlinear term which is quadratic with respect to the gradient. The homogenization of various singular elliptic problems has been studied, for instance, in [17], [42], [43] and [51].

Quite a few methods can be applied for studying the homogenization of periodic structures. The following ones are the main methods introduced in the last decades:

- ***The multiple-scales method***

It is a classical method used in mechanics which applies when there are some small parameters describing different scales. It is also called *the asymptotic expansion method* and it looks for a formal asymptotic expansion of the form

$$u^\varepsilon(x) = u^0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u^1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u^2\left(x, \frac{x}{\varepsilon}\right) + \dots, \quad (0.4)$$

where, for $i = 0, 1, 2, \dots$, $u^i = u^i(x, y)$ are Y -periodic with respect to the second variable, being Y the reference cell. The initial problem gives rise to an infinite system of equations, having a particular structure. The first equation contains only u^0 as unknown function. The second equation contains u^1 and u^0 as unknown functions, and so on. Once it is known a solution of the previous equation, it is possible to determinate a solution of the next one. By this method, it turns out that u^0 depends only on x even if, in (0.4), it is a priori an oscillating function, since it depends on $\frac{x}{\varepsilon}$. This is why u^0 is expected to be the homogenized solution. So that, if there is an equation satisfied by u^0 , it will be the homogenized equation one was looking for.

The multiple-scales method has been introduced by E. Sanchez-Palencia in [64] and [65], by J.-L. Lions in [57], by A. Bensoussan, J.-L. Lions and G. Papanicolaou in [16]. See also the books [31], [63] and [56].

- ***The Tartar's oscillating test function method***

It is a rigorous method proposed by L. Tartar in [70] and [71]. Its main idea consists in the construction of suitable oscillating test functions obtained by periodizing the solution of a problem set in the reference cell. The main feature of the method is the presence of the adjoint operator in the previous reference problem, which allows to cancel all the terms containing two weakly convergent sequences.

- ***The two-scale convergence method***

It has been introduced in an abstract framework by G. Nguetseng in [62] and soon after developed in applied frameworks by G. Allaire in [1] and [2]. It deals with the convergence of integrals of the form

$$\int_{\Omega} v^\varepsilon(x) \Psi\left(x, \frac{x}{\varepsilon}\right) dx,$$

where $\{v^\varepsilon\}$ is a bounded sequence in $L^2(\Omega)$ and Ψ is a smooth function, periodic with respect to the second variable. This method gives a new approach for studying the homogenization problems. Let us mention that the two-scale convergence method justifies a posteriori the asymptotic expansion developed in the multi-scales method: any sequence $\{u^\varepsilon\}$ given by (0.4) two-scale converges to the first term of the expansion, $u^0(x, y)$.

- ***The periodic unfolding method***

It was originally introduced by D. Cioranescu, A. Dalambian and G. Griso in [28] and [29] for the study of classical homogenization problems in the periodic case for fixed domains. It was extended to the case of perforated domains in [32] and [33] by the same authors together with P. Donato and R. Zaki. The main feature of this method is the unfolding operator that doubles the dimension of the space. It transforms any integral over Ω in an integral over $\Omega \times Y$, being Ω and Y the domain and the reference cell, respectively. This permits to transfer all the oscillations on the second variable and gives a simple approach in the periodic homogenization studies (see the next section for details).

Finally, let us point out that the homogenization theory does not restrict itself to the analysis of periodic structures. There exist several methods in this direction such as the G-convergence introduced by S. Spagnolo in [67] (see also [68]) to study problems with symmetric coefficients and the H-convergence introduced by L. Tartar in [70] and developed by F. Murat and L. Tartar in [60] and [61] for the case of non symmetric coefficients. For a large class of nonlinear problems, E. De Giorgi, S. Spagnolo and T. Franzoni introduced in [38], [39] and [40] a general mathematical theory called Γ -convergence.

The periodic unfolding method

All the homogenization results of Chapter 2 are proved by means of the periodic unfolding method. We now give a short presentation of this method applied to perforated domains (that is our framework) and we refer to the papers quoted above for the case of fixed ones. We first construct the domain. Let $N \in \mathbb{N}, N \geq 2$ and Ω be a connected bounded open set in \mathbb{R}^N whose boundary $\partial\Omega$ is Lipschitz-continuous. Let $b = \{b_1, \dots, b_N\}$ be a basis of \mathbb{R}^N and define by Y the following reference cell:

$$Y \doteq \left\{ y \in \mathbb{R}^N : y = \sum_{i=1}^N y_i b_i, (y_1, \dots, y_N) \in (0,1)^N \right\}.$$

Also, T denotes the reference hole, which is a (nonempty) open subset of \mathbb{R}^N such that $\overline{T} \subset Y$, and ∂T is Lipschitz-continuous with a finite number of connected components.

The perforated cell, that is the part of the cell occupied by the material, is defined by $Y^* \doteq Y \setminus \bar{T}$.

Let $\{\varepsilon\}_{\varepsilon>0}$ be a positive parameter taking values in a sequence converging to zero and set

$$G \doteq \left\{ \xi \in \mathbb{R}^N : \xi = \sum_{i=1}^N k_i b_i, (k_1, \dots, k_N) \in \mathbb{Z}^N \right\} \quad T_\varepsilon \doteq \bigcup_{\xi \in G} \varepsilon(\xi + T),$$

where $\varepsilon(\xi + T)$ are disjoint translated sets of T rescaled by ε .

The perforated domain Ω_ε^* is defined by removing from Ω the closure of holes T_ε , that is $\Omega_\varepsilon^* \doteq \Omega \setminus \bar{T}_\varepsilon$. Moreover, we define by $\widehat{\Omega}_\varepsilon$ the interior of the largest union of cells $\varepsilon(\xi + \bar{Y})$ completely included in Ω and by Λ_ε its complement with respect to Ω that contains the cells $\varepsilon(\xi + \bar{Y})$ intersecting $\partial\Omega$, i.e.

$$\widehat{\Omega}_\varepsilon \doteq \text{interior} \left\{ \bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \bar{Y}) \right\} \quad \text{and} \quad \Lambda_\varepsilon \doteq \Omega \setminus \widehat{\Omega}_\varepsilon,$$

where $\Xi_\varepsilon \doteq \{\xi \in G, \varepsilon(\xi + Y) \subset \Omega\}$. The corresponding perforated sets are

$$\widehat{\Omega}_\varepsilon^* \doteq \widehat{\Omega}_\varepsilon \setminus \bar{T}_\varepsilon \quad \text{and} \quad \Lambda_\varepsilon^* \doteq \Omega_\varepsilon^* \setminus \widehat{\Omega}_\varepsilon^*.$$

We decompose the boundary of the perforated domain Ω_ε^* as done, for instance, in [33],[34], [20] and [25], that is

$$\partial\Omega_\varepsilon^* = \Gamma_0^\varepsilon \cup \Gamma_1^\varepsilon \quad \text{where} \quad \Gamma_1^\varepsilon \doteq \partial\widehat{\Omega}_\varepsilon^* \cap \partial T_\varepsilon \quad \text{and} \quad \Gamma_0^\varepsilon \doteq \partial\Omega_\varepsilon^* \setminus \Gamma_1^\varepsilon.$$

Now, let $z \in \mathbb{R}^N$, we denote by $[z]_Y$ its integer part such that $z - [z]_Y$ belongs to Y and set $\{z\}_Y \doteq z - [z]_Y$.

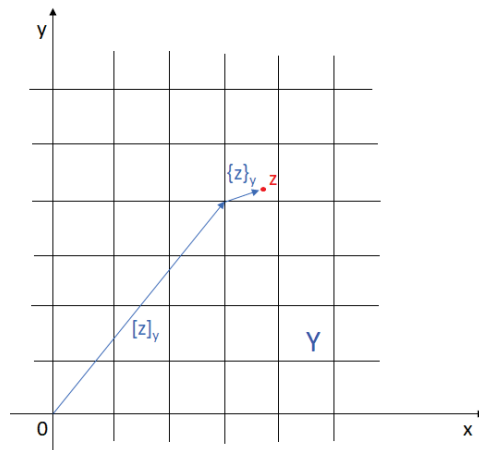


Figure 1. Definition of $[z]_Y$ and $\{z\}_Y$

Then, for every positive ε , we write

$$x = \varepsilon \left(\left[\frac{x}{\varepsilon} \right]_Y + \left\{ \frac{x}{\varepsilon} \right\}_Y \right) \quad \forall x \in \mathbb{R}^N.$$

For any Lebesgue-measurable function ϕ on Ω_ε^* , the unfolding operator $\mathcal{T}_\varepsilon^*$ is defined as follows:

$$\mathcal{T}_\varepsilon^*(\phi)(x, y) \doteq \begin{cases} \phi \left(\varepsilon \left[\frac{x}{\varepsilon} \right]_Y + \varepsilon y \right) & \text{a.e. for } (x, y) \in \widehat{\Omega}_\varepsilon \times Y^*, \\ 0 & \text{a.e. for } (x, y) \in \Lambda_\varepsilon \times Y^*. \end{cases}$$

Keeping in mind the decomposition in Figure 1, observe that there exist two scales: the macroscopic scale x giving the position of the point in the domain, and the microscopic scale $y = \frac{x}{\varepsilon}$ giving the position of the point in the reference cell Y^* and capturing the oscillations. Therefore the unfolding operator maps functions defined on oscillating domains into functions defined on fixed ones. This is the reason why this method does not need to use any extension operator. Moreover, if $\{v_\varepsilon\}$ is a bounded sequence in $L^p(\Omega_\varepsilon^*)$, with $p \in (1, +\infty)$, then the weak convergence of the unfolded sequence $\{\mathcal{T}_\varepsilon^*(v_\varepsilon)\}$ in $L^p(\Omega \times Y^*)$ is equivalent to the two-scale convergence of $\{v_\varepsilon\}$ in $L^p(\Omega_\varepsilon^*)$. Whence the unfolding method allows to choose a wider range of admissible test functions, compared to those eligible to check the two-scale convergence.

The operator $\mathcal{T}_\varepsilon^*$ is a linear and continuous operator from $L^p(\Omega_\varepsilon^*)$ to $L^p(\Omega \times Y^*)$ such that, for $p \in [1, +\infty)$,

- $\mathcal{T}_\varepsilon^*(\phi\psi) = \mathcal{T}_\varepsilon^*(\phi)\mathcal{T}_\varepsilon^*(\psi)$ for every $\phi, \psi \in L^p(\Omega_\varepsilon^*)$,
- $\|\mathcal{T}_\varepsilon^*(\phi)\|_{L^p(\Omega \times Y^*)} \leq |Y|^{\frac{1}{p}} \|\phi\|_{L^p(\Omega_\varepsilon^*)}$ for every $\phi \in L^p(\Omega_\varepsilon^*)$.

One of its main features is the validity of the following **integration formula**, for all $\phi \in L^1(\Omega_\varepsilon^*)$:

$$\int_{\widehat{\Omega}_\varepsilon^*} \phi(x) dx = \int_{\Omega_\varepsilon^*} \phi(x) dx - \int_{\Lambda_\varepsilon^*} \phi(x) dx = \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(\phi)(x, y) dx dy.$$

As a consequence, one obtains the **unfolding criterion for integrals (u.c.i.)** stating that if ϕ_ε is in $L^1(\Omega_\varepsilon^*)$ and

$$\int_{\Lambda_\varepsilon^*} |\phi_\varepsilon| dx \rightarrow 0,$$

then

$$\int_{\Omega_\varepsilon^*} \phi_\varepsilon dx - \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(\phi_\varepsilon) dx dy \rightarrow 0.$$

This means that, at the limit for ε going to zero, the integral of ϕ_ε on Ω_ε^* is equivalent to its unfolded on the fixed domain $\Omega \times Y^*$.

We conclude by defining the boundary unfolding operator, introduced to handle the nonlinear

Robin condition on the boundaries of the holes.

For any Lebesgue-measurable function ϕ on $\partial\widehat{\Omega}_\varepsilon^* \cap \partial T_\varepsilon$, the boundary unfolding operator $\mathcal{T}_\varepsilon^b$ is defined as follows:

$$\mathcal{T}_\varepsilon^b(\phi)(x, y) \doteq \begin{cases} \phi\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{a.e. for } (x, y) \in \widehat{\Omega}_\varepsilon \times \partial T, \\ 0 & \text{a.e. for } (x, y) \in \Lambda_\varepsilon \times \partial T. \end{cases}$$

Properties analogous to those listed above hold for the boundary operator. We refer to Section 2.3 for more details on the properties of these operators.

Our contribution

Coming back to problem (0.3), we apply the results of Chapter 1 to a ε -periodically perforated domain to get the existence and the uniqueness of its weak solution, for every fixed ε . By adapting to our case some arguments introduced in Section 1.3 we obtain similar uniform a priori estimates of the solution u_ε . Let us point out that the crucial third bound (cf. Proposition 2.22) provides an estimate of the integral of the singular term close to the singularity in terms of the quasilinear one.

As for the existence results, also in the homogenization process we have to deal with the quasilinear term, the singular one and the nonlinear boundary condition. In order to study the quasilinear term, we prove a crucial convergence, given by Theorem 2.27, which is the main tool when proving our results. It shows that the gradient of u_ε behaves like that of the solution of a suitable linear problem associated with a weak cluster point of the sequence $\{u_\varepsilon\}$, as $\varepsilon \rightarrow 0$. This idea was originally introduced in [14] (see also [15]) where some nonlinear problems with quadratic growth are considered. The main difference with respect to [14] consists in the choice of the test functions used in the proof. Indeed, in the quadratic growth case the authors take exponential test functions, while here we have to use appropriate test functions that allow us to treat the singular term. To construct these functions, taking into account the homogenization results of [25], we adapt to our needs some techniques from [42]. As already mentioned, we split the singular term into the sum of two integrals: one on the set where the solution is close to the singularity and one where it is far from it, which results not singular. Near the singularity, we make use of the estimate given by Proposition 2.22. In this way, we shift the study from the singular term to the quasilinear one, for which we can use the previous result.

Concerning the nonlinear Robin boundary condition, the arguments used in [25] to study the boundary terms still apply to our case, with appropriate modifications, the hypotheses being slightly different.

We also prove a corrector result, which states that the correctors for the nonsingular quasilinear problem in perforated domains studied in [25] are also correctors for our problem.

Let us point out that for $h \equiv 0$ we obtain, as a particular case, homogenization and corrector results for the problem when a (homogeneous or not) Neumann boundary condition on Γ_1^ε is prescribed, which are already new in the literature.

The main novelty of this work is that, for the first time, the unfolding method is used to treat a singular term as $f\zeta(u_\varepsilon)$. This is essential in order to get the almost everywhere convergences needed to study the asymptotic behavior of the problem. Although in [42] the problem presents the same singular term, it is posed through rough surface. Whence, since the measure of the interface is zero, a compactness result in $L^2(\Omega)$ from [48] gives the almost everywhere convergences. In our case, due to the presence of the holes, this argument does not apply, and the almost everywhere convergences can be obtained only by using the unfolding method.

Chapter 1

Existence and uniqueness results for a class of singular elliptic problems in perforated domains

P. Donato, S. Monsurrò and F. Raimondi, *Existence and uniqueness results for a class of singular elliptic problems in perforated domains*, Ric. Mat., **66**, pp. 333–360, doi.org/10.1007/s11587-016-0303-y (2017).

In this chapter we prove some existence, uniqueness and regularity results for a weak solution of a quasilinear elliptic nonlinear problem posed in a domain perforated by one or more holes. The nonlinear term is of the form $f\zeta(u)$, where $\zeta(s)$ is singular at $s = 0$. On the boundary of the holes we impose a nonlinear Robin condition, while on the exterior boundary we prescribe a homogeneous Dirichlet condition. The difficulty here arises in dealing simultaneously with the quasilinear matrix field, the singular datum and the nonlinear Robin condition. To show the existence of a solution we approximate the problem with a sequence of nonsingular problems for which the existence of a solution is proved via the Schauder fixed-point theorem. The main tool when passing to the limit in the approximate problem is to split the integral of the singular term into the sum of two integrals, one on the set where the solution is very close to the singularity and one where it is far from it. To obtain the uniqueness of the solution we need to require additional assumptions on the quasilinear term and a monotonicity property for the singular one. Under suitable stronger hypotheses on the data, we also prove the boundedness of the solution.

1.1 Introduction

We study the existence, the uniqueness and the regularity for a weak solution u of a quasilinear elliptic singular problem posed in a perforated domain \mathcal{O} of R^N , $N \geq 2$.

The domain \mathcal{O} is obtained by removing a bounded closed set $\overline{\mathcal{O}}_1$, representing one or more holes in \mathcal{O}_0 , from a bounded connected open set \mathcal{O}_0 . Denoted by Γ_0 and Γ_1 the boundaries of \mathcal{O}_0 and \mathcal{O}_1 respectively, we have that $\partial\mathcal{O} = \Gamma_0 \cup \Gamma_1$. On the exterior boundary Γ_0 we prescribe a homogeneous Dirichlet condition, while on the boundary of the holes Γ_1 we impose a nonlinear Robin condition. More precisely, we focus on the following problem:

$$\begin{cases} -\operatorname{div}(B(x, u)\nabla u) + \lambda u = f\zeta(u) & \text{in } \mathcal{O}, \\ u = 0 & \text{on } \Gamma_0, \\ (B(x, u)\nabla u)\nu + \rho h(u) = g & \text{on } \Gamma_1, \end{cases}$$

where ν denotes the unit external normal vector to \mathcal{O} . The quasilinear term consists in a uniformly elliptic and bounded Carathéodory matrix field $B(x, t)$, λ is a nonnegative real number, $\zeta(s)$ is a nonnegative real function singular at $s = 0$ and f is a nonnegative datum whose summability depends on the growth of ζ near the singularity. Concerning the Robin boundary condition, the function ρ is assumed to be bounded and nonnegative on Γ_1 , the nonlinear boundary term h is an increasing and continuously differentiable function whose derivative satisfies suitable growth assumptions, and g is a nonnegative function with prescribed summability.

The difficulty arises in dealing simultaneously with the quasilinear matrix field, the singular datum and the nonlinear Robin condition.

Quasilinear diffusion matrix fields describe the behavior of materials, like glass or wood, in which the heat diffusion depends on the range of the temperature (see for instance [50]). Nonlinear boundary conditions model certain chemical reactions at the boundaries of perforations (e.g. [37]). A physical phenomenon described by a source term that depends on the solution itself and becomes infinite when the solution vanishes is the one of an electrical conductor where each point becomes a source of heat as a current flows in it. Indeed, in this framework, in the non-stationary case, the solution $u(x, t)$ represents the temperature at the point x and at the time t , the function $f(x, t)$ is the square of local voltage drop in \mathcal{O} and $\zeta(u)$ denotes the electrical conductivity. Then generation of heat occurs with a rate given by $f(x, t)\zeta(u)$. So that in the stationary case one obtains $f(x)\zeta(u)$. We refer to [42, Section 3] for details on the physical meaning of this problem.

As usual in the literature, in order to prove the existence of a solution for our problem, we approximate it with a sequence of nonsingular problems with a bounded nonlinearity in the equation. To show the existence of a solution of the approximating problem, due to the

presence of the quasilinear term, we use the Schauder fixed-point theorem, arguing similarly to [19] and [24].

To prove the existence result, we follow some ideas of [42], [51] and [52], which present a similar singular term. Indeed we split the integral of the singular term into the sum of two integrals, the one on the set where the solution is close to the singularity and the one where it is far from it. Then suitable a priori estimates, together with accurate measure arguments, allow us to treat each term and pass to the limit.

As far as it concerns the uniqueness of the solution, we need to require additional assumptions on the matrix field B and on the function ζ . For the quasilinear term we make the same assumptions of [3] (see also [19] and the references quoted therein), while for ζ we prescribe that it is a non increasing function. We explicitly observe that we do not assume any monotonicity property for ζ except for this result.

We also prove a regularity result (boundedness of the solution) under stronger hypotheses on the summabilities of the data f and g . To prove this, we adapt to our needs an argument of [24], making use of classical results of Stampacchia [69], as in [24], [42], [51] and [54].

Elliptic problems with singular nonlinear terms have been studied in [4], [24], [26], [42], [51], [52], [53] and [54]. More precisely, [42] and [51] present a mild singularity as long as [52] a strong one. The problem we deal with in the present paper has the same singularity of the one in [42], where the problem is posed in a two-component domain with a rough interface. Here the problem is quasilinear and posed in a perforated domain with both Dirichlet and Robin boundary conditions. In [4], [24], [26], [53] and [54] the nonlinear term is singular with respect to the solution and has a quadratic growth with respect to its gradient.

Let us mention that in the papers [19], [24] and [26], quoted above, the authors consider nonlinear Robin boundary condition. For the linear equation with this kind of boundary condition, we refer to [25], [34] and [37].

The paper is organized as follows:

In Section 2 we present the setting of the problem and we recall some preliminary results.

In Section 3 we prove some a priori estimates, including an estimate on the integral of the singular term close to the singular set $\{u = 0\}$.

In Section 4 we show the existence of a solution of the approximate problem.

In Section 5 we present the existence result for our problem.

In Section 6 we deal with the uniqueness result of the solution.

In Section 7 we discuss the boundedness of a solution of our problem.

1.2 Setting of the problem

The problem studied in this paper is posed in a perforated domain \mathcal{O} , obtained by removing from a bounded connected open set of \mathbb{R}^N a bounded closed set, representing one or more holes in the open set.

More precisely, we suppose $N \geq 2$ and denote by

- \mathcal{O}_0 a connected bounded open set in \mathbb{R}^N and Γ_0 its boundary,
- \mathcal{O}_1 a bounded open set such that $\overline{\mathcal{O}_1} \subset \mathcal{O}_0$ and its boundary Γ_1 is Lipschitz continuous,
- $\mathcal{O} := \mathcal{O}_0 \setminus \overline{\mathcal{O}_1}$ the perforated domain,
- ν the unit outward normal to \mathcal{O} ,
- χ_E the characteristic function of any subset E of \mathbb{R}^N ,
- $\mathcal{M}(\alpha, \beta, E)$ the set of matrix fields $A = (a_{i,j})_{1 \leq i,j \leq N} \in (L^\infty(E))^{N \times N}$ such that $(A(x)\lambda, \lambda) \geq \alpha|\lambda|^2$ and $|A(x)\lambda| \leq \beta|\lambda|$, $\forall \lambda \in \mathbb{R}^N$ and a.e. on E .

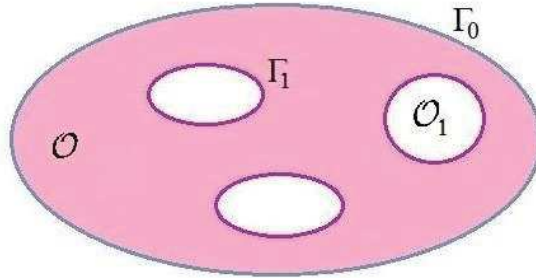


Figure 1.1. The perforated domain \mathcal{O}

In this paper we are interested in the study of the following problem:

$$\begin{cases} -\operatorname{div}(B(x, u)\nabla u) + \lambda u = f\zeta(u) & \text{in } \mathcal{O}, \\ u = 0 & \text{on } \Gamma_0, \\ (B(x, u)\nabla u)\nu + \rho h(u) = g & \text{on } \Gamma_1, \end{cases} \quad (1.1)$$

where we prescribe a Dirichlet condition on the exterior boundary of \mathcal{O} and a nonlinear Robin condition on the boundary of the holes.

In the sequel we make the following hypotheses:

H₁) The real $N \times N$ matrix field $B : (x, t) \in \mathcal{O} \times \mathbb{R} \mapsto B(x, t) = (b_{i,j}(x, t))_{i,j=1,\dots,N} \in \mathbb{R}^{N^2}$ satisfies the following conditions:

$$\left\{ \begin{array}{l} \text{i) } B \text{ is a Carathéodory function,} \\ \text{i.e. } B(x, \cdot) \text{ is continuous for a.e. } x \in \mathcal{O} \\ \text{and } B(\cdot, t) \text{ is measurable for every } t \in \mathbb{R}; \\ \text{ii) } B(\cdot, t) \in M(\alpha, \beta, \mathcal{O}), \text{ for every } t \in \mathbb{R}. \end{array} \right.$$

H₂) $\lambda \geq 0.$

H₃) $\left\{ \begin{array}{l} \text{i) } \zeta : [0, +\infty[\rightarrow [0, +\infty] \text{ is a function such that} \\ \zeta \in \mathcal{C}^0([0, +\infty[), 0 \leq \zeta(s) \leq \frac{1}{s^\theta} \text{ for every } s \in]0, +\infty[, \text{ with } 0 < \theta \leq 1; \\ \text{ii) } f \geq 0 \text{ a.e. in } \mathcal{O}_0, \text{ with } f \in L^l(\mathcal{O}_0), \text{ for } l \geq \frac{2}{1+\theta} (\geq 1). \end{array} \right.$

H₄) The functions g and ρ are such that

$$\left\{ \begin{array}{l} \text{i) } 0 \leq g \in L^s(\Gamma_1), \text{ with } \begin{cases} s \geq \frac{2(N-1)}{N} & \text{if } N > 2, \\ s > 1 & \text{if } N = 2; \end{cases} \\ \text{ii) } 0 \leq \rho \in L^\infty(\Gamma_1). \end{array} \right.$$

H₅) Either

$$f \not\equiv 0$$

or

$$\partial H_{N-1}(\Gamma_1) > 0 \quad \text{and} \quad g \not\equiv 0,$$

where ∂H_{N-1} is the $(N-1)$ -Hausdorff measure in \mathbb{R}^N .

H₆) The function h is an increasing and continuously differentiable function such that for some positive constant C and an exponent q one has

$$\left\{ \begin{array}{l} h(0) = 0, \\ |h'(s)| \leq C(1 + |s|^{q-1}), \forall s \in \mathbb{R}, \\ \text{with } 1 \leq q < \infty \text{ if } N = 2, \text{ and } 1 \leq q \leq \frac{N}{N-2} \text{ if } N > 2. \end{array} \right.$$

REMARK 1.1. Since h is an increasing function and $h(0) = 0$,

$$(h(s) - h(s'))(s - s') \geq 0, \quad \forall s, s' \in \mathbb{R}, \quad (1.2)$$

which implies

$$h(s)s \geq 0, \quad \forall s \in \mathbb{R}. \quad (1.3)$$

Moreover, from the boundedness of the derivative, we have

$$|h(s)| \leq C(1 + |s|^q), \quad \forall s \in \mathbb{R}, \quad (1.4)$$

with q as in H_6).

REMARK 1.2. In [24] the function g is supposed with higher summability, that is $s > N - 1$, which is necessary there since the authors look for bounded solutions. Here we will do the same assumption only for our last result about regularity, but for the existence and uniqueness results this is not necessary and we can suppose the minimal assumption on the summability of g , given in H_4).

We introduce now the natural framework for the study of problem (1.1) and its variational formulation.

Let V be the space defined by $V = \{v \in H^1(\mathcal{O}) : v = 0 \text{ on } \Gamma_0\}$. For the Poincaré and Sobolev inequalities, there exists a positive constant $C_t = C_t(\mathcal{O})$ such that

$$\|u\|_{L^t(\mathcal{O})} \leq C_t \|\nabla u\|_{L^2(\mathcal{O})}, \quad \forall u \in V,$$

for all $t \in [2, +\infty)$ if $N = 2$, and for all $t \in [2, 2^*]$ if $N > 2$, where $2^* = \frac{2N}{N-2}$. In the sequel, we will use $C_{\mathcal{O}}$ instead of $C_2(\mathcal{O})$.

Consequently, V can be equipped by the norm

$$\|u\|_V := \|\nabla u\|_{L^2(\mathcal{O})}, \quad \forall u \in V, \quad (1.5)$$

which is equivalent to the H^1 -norm.

REMARK 1.3. From the Sobolev and the trace inclusions it follows that (see [34]):

- if $N = 2$, the map $u \in V \mapsto u \in L^r(\Gamma_1)$ is continuous for all $r \in [1, +\infty)$;
- if $N > 2$, the map $u \in V \mapsto u \in L^{\frac{2(N-1)}{N-2}}(\Gamma_1)$ is continuous.

That is, there exists a positive constant $\gamma_r = \gamma_r(\mathcal{O})$ such that

$$\|u\|_{L^r(\Gamma_1)} \leq \gamma_r \|u\|_V, \quad \forall u \in V,$$

for all $r \in [1, +\infty)$ if $N = 2$, and $r = \frac{2(N-1)}{N-2}$ if $N > 2$.

We recall also the following results proved in [34] (see also [19]):

PROPOSITION 1.4. [19][34] *Let h be a function satisfying assumption H_6). Suppose that r and r' are such that*

$$\begin{aligned} & i) \text{ if } N = 2, \ r' \in [1, +\infty) \text{ and } r \geq qr'; \\ & ii) \text{ if } N > 2, \ r = \frac{2(N-1)}{N-2} \text{ and } r' = \frac{2(N-1)}{N} \left(\text{i.e. } \frac{1}{r} + \frac{1}{r'} = 1 \right). \end{aligned} \quad (1.6)$$

Then

1) The map $u \in V \mapsto h(u) \in L^{r'}(\Gamma_1)$ is weakly continuous; that is, it maps a weakly convergent sequence in V into weakly convergent functions in $L^{r'}(\Gamma_1)$.

2) The map $u \in H^1(\mathcal{O}) \mapsto u \in L^r(\Gamma_1)$ is continuous, and for every $u, v \in H^1(\mathcal{O})$, we have $vh(u) \in L^1(\Gamma_1)$.

3) The map $u \in L^r(\Gamma_1) \mapsto h(u) \in L^{r'}(\Gamma_1)$ is continuous and bounded and there exists a positive constant c such that

$$\|h(u)\|_{L^{r'}(\Gamma_1)} \leq c(1 + \|u\|_{L^r(\Gamma_1)}^{\frac{r}{r'}}).$$

REMARK 1.5. Property 2) in Proposition 1.4 allows us to state that under hypothesis $H_4)_{ii}$,

$$\int_{\Gamma_1} \rho h(u) \varphi d\sigma < +\infty, \quad \forall \varphi \in V \text{ and } \forall u \in V.$$

We also have under hypothesis $H_4)_i$,

$$\int_{\Gamma_1} g \varphi d\sigma < +\infty, \quad \forall \varphi \in V.$$

Indeed, when $N > 2$, Proposition 1.4 implies that

$$\varphi \in L^{\frac{2(N-1)}{N-2}}(\Gamma_1) \quad \text{and} \quad g \in L^s(\Gamma_1) \subset L^{\frac{2(N-1)}{N}}(\Gamma_1) = L^{\left(\frac{2(N-1)}{N-2}\right)' }(\Gamma_1),$$

since $s > \frac{2(N-1)}{N} = \left(\frac{2(N-1)}{N-2}\right)'$. When $N = 2$, $g \in L^s(\Gamma_1)$ and, by Remark 1.3, $\varphi \in L^{s'}(\Gamma_1)$.

PROPOSITION 1.6. [19] *Let h be a function satisfying assumption H_6). Let $t_1 \geq 1$ such that*

$$\begin{aligned} & i) \text{ if } N = 2, \ t_1 \in [1, 2[; \\ & ii) \text{ if } N > 2, \ t_1 = \frac{2 \cdot 2^*}{2^* + 2q} = \frac{2N}{q(N-2) + N}. \end{aligned}$$

If u and v are in $H^1(\mathcal{O})$, then $h(u)v \in W^{1,t_1}(\mathcal{O})$.

Furthermore let us recall the classical decomposition $u = u^+ - u^-$ for every function $u \in V$, where u^+ and u^- are the nonnegative functions defined by

$$u^+ := \max\{u, 0\}, \quad u^- := -\min\{u, 0\} \quad \text{a.e. in } \mathcal{O}.$$

In the sequel we will also use the two real functions introduced below.

For every $k > 0$, T_k denotes the usual truncation function at level k given by (see Figure 1.2)

$$T_k(s) = \begin{cases} -k, & \text{if } s < -k, \\ s, & \text{if } |s| \leq k, \\ k, & \text{if } s > k; \end{cases}$$

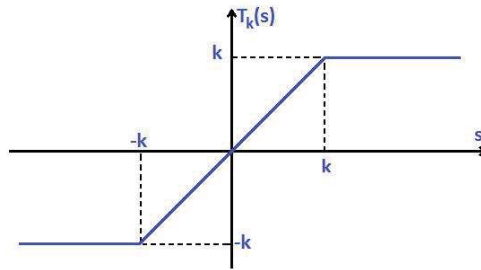


Figure 1.2. The truncation function

and for every $\delta > 0$, the auxiliary function Z_δ is defined by (see Figure 1.3)

$$Z_\delta(s) = \begin{cases} 1, & \text{if } 0 \leq s \leq \delta, \\ -\frac{s}{\delta} + 2, & \text{if } \delta \leq s \leq 2\delta, \\ 0, & \text{if } s \geq 2\delta. \end{cases} \quad (1.7)$$

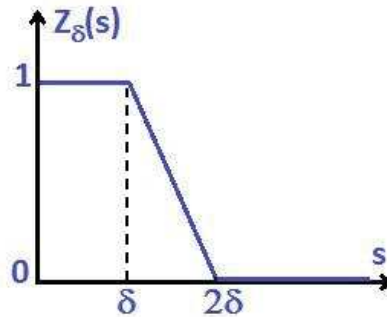


Figure 1.3. The auxiliary function

Observe that, if $\varphi \in V$, then $T_k(\varphi)$ converges to φ almost everywhere in \mathcal{O} so that, via the Lebesgue dominated convergence theorem,

$$T_k(\varphi) \rightarrow \varphi \text{ strongly in } V, \text{ as } k \rightarrow \infty. \quad (1.8)$$

Moreover $0 \leq Z_\delta(s) \leq 1$ for any positive real s and

$$\nabla Z_\delta(s) = \chi_{\{\delta \leq s \leq 2\delta\}} \left(-\frac{1}{\delta} \nabla s \right). \quad (1.9)$$

Our aim is to prove an existence result for problem (1.1), whose variational formulation is the following:

$$\left\{ \begin{array}{l} \text{Find } u \in V \text{ such that } u > 0 \text{ a.e. in } \mathcal{O}, \\ \int_{\mathcal{O}} f\zeta(u)\varphi dx < +\infty \text{ and} \\ \int_{\mathcal{O}} B(x, u)\nabla u \nabla \varphi dx + \int_{\Gamma_1} \rho h(u)\varphi d\sigma + \int_{\mathcal{O}} \lambda u \varphi dx \\ = \int_{\mathcal{O}} f\zeta(u)\varphi dx + \int_{\Gamma_1} g\varphi d\sigma, \quad \forall \varphi \in V. \end{array} \right. \quad (1.10)$$

A function u satisfying (1.10) is called a solution of problem (1.1).

1.3 A priori estimates

In this section, we prove some a priori estimates for a solution u of problem (1.1). These estimates also apply to the solution u_m of the approximate problem (4.1), used in the next section; in this case they are uniform with respect to m , which will play an important role when passing to the limit in the problem, as $m \rightarrow +\infty$. We adapt here some ideas from [42]. Set

$$s_1 := \begin{cases} \frac{2(N-1)}{N-2} & \text{if } N > 2, \\ s' & \text{if } N = 2, \end{cases} \quad (1.11)$$

where s is given by $H_4)_i$.

PROPOSITION 1.7. *Under assumptions $H_1)$ - $H_6)$, let $u \in V$ be a solution of problem (1.1). The following a priori estimate holds:*

$$\|u\|_V \leq c, \quad (1.12)$$

where c depends on α , $C_{\mathcal{O}}$, γ_{s_1} , $\|f\|_{L^1(\mathcal{O})}$ and $\|g\|_{L^s(\Gamma_1)}$.

Proof. Let $u \in V$ be a solution of problem (1.1) and let us choose u as test function in its variational formulation (1.10). Using H_1 , H_2 , H_3 , H_4 ii, (1.3), and (1.5), we get

$$\begin{aligned} \alpha \|\nabla u\|_{L^2(\mathcal{O})}^2 &\leq \int_{\mathcal{O}} B(x, u) \nabla u \nabla u dx + \int_{\mathcal{O}} \lambda u^2 dx + \int_{\Gamma_1} \rho h(u) u d\sigma \\ &= \int_{\mathcal{O}} f \zeta(u) u dx + \int_{\Gamma_1} g u d\sigma \\ &\leq \int_{\mathcal{O}} f u^{1-\theta} dx + \int_{\Gamma_1} g u d\sigma. \end{aligned}$$

Applying Young, Hölder and Poincaré inequalities, using Remark 1.3, Remark 1.5 and (1.11), we obtain

$$\begin{aligned} \alpha \|\nabla u\|_{L^2(\mathcal{O})}^2 &\leq \eta \|u\|_{L^2(\mathcal{O})}^2 + c(\eta) \|f\|_{L^{\frac{2}{1+\theta}}(\mathcal{O})}^{\frac{2}{1+\theta}} + c_1 \|g\|_{L^s(\Gamma_1)} \|u\|_{L^{s_1}(\Gamma_1)} \\ &\leq \eta \|u\|_{L^2(\mathcal{O})}^2 + c(\eta) \|f\|_{L^{\frac{2}{1+\theta}}(\mathcal{O})}^{\frac{2}{1+\theta}} + c_1 \gamma_{s_1} \|g\|_{L^s(\Gamma_1)} \|u\|_V \\ &\leq \eta C_{\mathcal{O}}^2 \|u\|_V^2 + c(\eta) c_2 \|f\|_{L^{\frac{2}{1+\theta}}(\mathcal{O})}^{\frac{2}{1+\theta}} \\ &\quad + \eta c_1 \gamma_{s_1} \|g\|_{L^s(\Gamma_1)} \|u\|_V^2 + c(\eta) c_1 \gamma_{s_1} \|g\|_{L^s(\Gamma_1)}. \end{aligned} \tag{1.13}$$

We get

$$\begin{aligned} \|u\|_V^2 &\leq \eta \|u\|_V^2 (C_{\mathcal{O}}^2 + c_1 \gamma_{s_1} \|g\|_{L^s(\Gamma_1)}) \\ &\quad + c(\eta) \left(c_2 \|f\|_{L^{\frac{2}{1+\theta}}(\mathcal{O})}^{\frac{2}{1+\theta}} + c_1 \gamma_{s_1} \|g\|_{L^s(\Gamma_1)} \right). \end{aligned}$$

Whence, reminding that $\|f\|_{L^{\frac{2}{1+\theta}}(\mathcal{O})}$ and $\|g\|_{L^s(\Gamma_1)}$ are finite quantities (due to hypotheses H_3 ii and H_4 i) and taking η sufficiently small so that $\alpha - \eta(C_{\mathcal{O}}^2 + c_1 \gamma_{s_1} \|g\|_{L^s(\Gamma_1)}) \geq 0$, we deduce that there exists a constant c such that

$$\|u\|_V \leq c,$$

where the constant c depends on α , $C_{\mathcal{O}}$, γ_{s_1} , $\|f\|_{L^{\frac{2}{1+\theta}}(\mathcal{O})}$ and $\|g\|_{L^s(\Gamma_1)}$. \square

PROPOSITION 1.8. *Under assumptions H_1)- H_6), let $u \in V$ be a solution of problem (1.1). Then*

$$\|f \zeta(u) \varphi\|_{L^1(\mathcal{O})} \leq c, \tag{1.14}$$

for every positive $\varphi \in V$ with c depending on α , β , $C_{\mathcal{O}}$, λ , r' , γ_r , γ_{s_1} , $\|\varphi\|_V$, $\|f\|_{L^1(\mathcal{O})}$, $\|\rho\|_{L^\infty(\Gamma_1)}$ and $\|g\|_{L^s(\Gamma_1)}$, where r and r' are as in (1.6).

Proof. Let $u \in V$ be a solution of problem (1.1) and let us choose a nonnegative $\varphi \in V$ as test function in its variational formulation (1.10). Since f , ζ and φ are nonnegative, we have

$$\begin{aligned} 0 &\leq \int_{\mathcal{O}} f\zeta(u)\varphi dx = \int_{\mathcal{O}} B(x, u)\nabla u\nabla\varphi dx \\ &+ \int_{\Gamma_1} \rho h(u)\varphi d\sigma + \int_{\mathcal{O}} \lambda u\varphi dx - \int_{\Gamma_1} g\varphi d\sigma. \end{aligned} \quad (1.15)$$

We will use the Hölder inequality for each term. For $H_1)_{ii}$ we get

$$\int_{\mathcal{O}} B(x, u)\nabla u\nabla\varphi dx \leq \beta\|\nabla u\|_{L^2(\mathcal{O})}\|\nabla\varphi\|_{L^2(\mathcal{O})}.$$

Taking r and r' verifying assumptions (1.6), from $H_4)_{ii}$, Proposition 1.4 and Remark 1.5, we have

$$\begin{aligned} \int_{\Gamma_1} \rho h(u)\varphi d\sigma &\leq \|\rho\|_{L^\infty(\Gamma_1)}\|h(u)\varphi\|_{L^1(\Gamma_1)} \\ &\leq \|\rho\|_{L^\infty(\Gamma_1)}\|h(u)\|_{L^{r'}(\Gamma_1)}\|\varphi\|_{L^r(\Gamma_1)} \\ &\leq \|\rho\|_{L^\infty(\Gamma_1)}c' \left(1 + \|u\|_{L^r(\Gamma_1)}^{\frac{r}{r'}}\right) \|\varphi\|_{L^r(\Gamma_1)} \\ &\leq \|\rho\|_{L^\infty(\Gamma_1)}c' \left(1 + \gamma_r^{\frac{r}{r'}}\|u\|_V^{\frac{r}{r'}}\right) \gamma_r\|\varphi\|_V. \end{aligned}$$

For the Poincaré inequality we get

$$\int_{\mathcal{O}} \lambda u\varphi dx \leq \lambda\|u\|_{L^2(\mathcal{O})}\|\varphi\|_{L^2(\mathcal{O})} \leq \lambda C_{\mathcal{O}}^2\|\nabla u\|_{L^2(\mathcal{O})}\|\nabla\varphi\|_{L^2(\mathcal{O})}.$$

Since g and φ are nonnegative, the last term in (1.15) is lower or equal to zero. Now using the a priori estimate (1.12) of Proposition 1.7, we obtain

$$0 \leq \int_{\mathcal{O}} f\zeta(u)\varphi dx \leq c,$$

where the constant c depends on α , β , $C_{\mathcal{O}}$, λ , r' , γ_r , γ_{s_1} , $\|\varphi\|_V$, $\|f\|_{L^1(\mathcal{O})}$, $\|\rho\|_{L^\infty(\Gamma_1)}$ and $\|g\|_{L^s(\Gamma_1)}$. \square

The following proposition gives an estimate of the integral of the singular term close to the singular set $\{u = 0\}$, which is crucial for the proofs of next results.

PROPOSITION 1.9. *Under assumptions $H_1)$ - $H_6)$, let $u \in V$ be a solution of problem (1.1) and δ a fixed positive real number. Then,*

$$\begin{aligned} \int_{\{0 \leq u \leq \delta\}} f\zeta(u)\varphi dx &\leq \int_{\mathcal{O}} B(x, u)\nabla u\nabla\varphi Z_\delta(u) dx \\ &+ h(2\delta)\|\rho\|_{L^\infty(\Gamma_1)}\|\varphi\|_{L^1(\Gamma_1)} + 2\delta\lambda\|\varphi\|_{L^1(\mathcal{O})}, \end{aligned} \quad (1.16)$$

for every $\varphi \in V$, $\varphi \geq 0$, where Z_δ is defined by (1.7).

Proof. Let $u \in V$ be a solution of problem (1.1) and $\varphi \in V$ a nonnegative function. Let us choose, for $k > 0$, $Z_\delta(u)T_k(\varphi) \in V$ as test function in the variational formulation (1.10).

Using (1.9), we have

$$\begin{aligned} & \int_{\mathcal{O}} B(x, u) \nabla u \nabla T_k(\varphi) Z_\delta(u) dx - \frac{1}{\delta} \int_{\mathcal{O} \cap \{\delta < u < 2\delta\}} B(x, u) \nabla u \nabla T_k(\varphi) dx \\ & + \int_{\Gamma_1} \rho h(u) Z_\delta(u) T_k(\varphi) d\sigma + \lambda \int_{\mathcal{O}} u Z_\delta(u) T_k(\varphi) dx \\ & - \int_{\Gamma_1} g Z_\delta(u) T_k(\varphi) d\sigma = \int_{\mathcal{O}} f \zeta(u) Z_\delta(u) T_k(\varphi) dx. \end{aligned}$$

From H_1) and since g, φ, δ and α are nonnegative, the previous equality implies

$$\begin{aligned} & \int_{\{0 \leq u \leq \delta\}} f \zeta(u) T_k(\varphi) dx = \int_{\{0 \leq u \leq \delta\}} f \zeta(u) Z_\delta(u) T_k(\varphi) dx \leq \int_{\mathcal{O}} f \zeta(u) Z_\delta(u) T_k(\varphi) dx \\ & \leq \int_{\mathcal{O}} B(x, u) \nabla u \nabla T_k(\varphi) Z_\delta(u) dx + \int_{\Gamma_1} \rho h(u) Z_\delta(u) T_k(\varphi) d\sigma + \lambda \int_{\mathcal{O}} u Z_\delta(u) T_k(\varphi) dx. \end{aligned}$$

From the definition of Z_δ and the growth of h , we have $h(u) \leq h(2\delta)$ on the set $\{0 \leq u \leq 2\delta\}$, so that

$$\begin{aligned} & \int_{\Gamma_1} \rho h(u) Z_\delta(u) T_k(\varphi) d\sigma \leq \int_{\Gamma_1} \rho h(u) T_k(\varphi) \chi_{\{0 \leq u \leq 2\delta\}} d\sigma \\ & \leq h(2\delta) \int_{\Gamma_1} \rho T_k(\varphi) d\sigma \leq h(2\delta) \|\rho\|_{L^\infty(\Gamma_1)} \|T_k(\varphi)\|_{L^1(\Gamma_1)}, \end{aligned}$$

where we used again Proposition 1.4. Also, we obtain

$$\begin{aligned} & \lambda \int_{\mathcal{O}} u Z_\delta(u) T_k(\varphi) dx \leq \lambda \int_{\mathcal{O}} u T_k(\varphi) \chi_{\{0 \leq u \leq 2\delta\}} dx \\ & \leq 2\delta \lambda \int_{\mathcal{O}} T_k(\varphi) dx = 2\delta \lambda \|T_k(\varphi)\|_{L^1(\mathcal{O})}. \end{aligned}$$

For every $k > 0$, we get

$$\begin{aligned} & \int_{\{0 \leq u \leq \delta\}} f \zeta(u) T_k(\varphi) dx \leq \int_{\mathcal{O}} B(x, u) \nabla u \nabla T_k(\varphi) Z_\delta(u) dx \\ & + h(2\delta) \|\rho\|_{L^\infty(\Gamma_1)} \|T_k(\varphi)\|_{L^1(\Gamma_1)} + 2\delta \lambda \|T_k(\varphi)\|_{L^1(\mathcal{O})}. \end{aligned}$$

We can pass to the limit as k tends to infinity in this inequality. Convergence (1.8) and Fatou's lemma applied on the left-hand side give us

$$\begin{aligned} & \int_{\{0 \leq u \leq \delta\}} f \zeta(u) \varphi dx \leq \int_{\mathcal{O}} B(x, u) \nabla u \nabla \varphi Z_\delta(u) dx \\ & + h(2\delta) \|\rho\|_{L^\infty(\Gamma_1)} \|\varphi\|_{L^1(\Gamma_1)} + 2\delta \lambda \|\varphi\|_{L^1(\mathcal{O})}, \end{aligned}$$

that ends the proof. \square

1.4 An approximate problem

As usual for this kind of problem (see for instance [4], [19], [24], [34], [42], [51], [52] and [53]), in order to study the existence of a solution of problem (1.1), we approximate our problem by a sequence of nonsingular problems with bounded nonlinearity in the equation. First we show here that these approximate problems admit at least a solution. Due to the presence of the quasilinear term, as in the articles quoted above, we make use of the Schauder fixed-point theorem stated below. Then in the next section, using the a priori estimates stated in the previous one, we show that the sequence of solutions of these problems converges to a solution of our initial problem.

Therefore, we define the following sequence of nonsingular problems, which approximate problem (1.1):

$$\begin{cases} -\operatorname{div}(B(x, u_m)\nabla u_m) + \lambda u_m = T_m(f\zeta(|u_m|)) & \text{in } \mathcal{O}, \\ u_m = 0 & \text{on } \Gamma_0, \\ (B(x, u_m)\nabla u_m)\nu + \rho h(u_m) = g & \text{on } \Gamma_1, \end{cases} \quad (1.17)$$

where, for every $m \in \mathbb{N}$, the function T_m is the truncation function at level m . Then, the variational formulation associated with the approximate problem (1.17) reads

$$\begin{cases} \text{Find } u_m \in V \text{ such that} \\ \int_{\mathcal{O}} B(x, u_m)\nabla u_m \nabla \varphi dx + \int_{\Gamma_1} \rho h(u_m)\varphi d\sigma + \int_{\mathcal{O}} \lambda u_m \varphi dx \\ = \int_{\mathcal{O}} T_m(f\zeta(|u_m|))\varphi dx + \int_{\Gamma_1} g\varphi d\sigma, \quad \forall \varphi \in V. \end{cases} \quad (1.18)$$

Thanks to this approximation, we avoid the singularity at 0.

Now we prove a theorem that provides the existence of a solution of this approximate problem for every fixed $m \in \mathbb{N}$. Some existence results for the linear and quasilinear nonsingular cases with a right-hand side $f \in L^p$, have been proved in [19] and [34], respectively.

For the next result, we follow some ideas from [24], where the problem present a singular lower-order term depending on the gradient of the solution with a quadratic growth. Here, we have a different singular nonlinearity; hence, in the next proof we only treat in details the term containing our singularity.

Let us first recall the Schauder fixed-point theorem:

THEOREM 1.10. (Schauder fixed-point theorem) *Let K be a convex closed set of a Banach space B . Let $T : K \subset B \rightarrow B$ be a continuous map such that $T(K) \subset K$ and $T(K)$ is relatively compact in K . Then T has a fixed point in K .*

Let us prove the following existence result for problem (1.17):

THEOREM 1.11. *Under assumptions $H_1)$ - $H_6)$, for every fixed $m \in \mathbb{N}$, the approximate problem (1.17) admits at least a solution.*

Proof. For simplicity we denote here u_m by u , omitting its dependence of m .

Fix $w \in V$ and consider the problem

$$\begin{cases} \text{Find } u \in V \text{ such that } \forall v \in V, \\ \int_{\mathcal{O}} B(x, w) \nabla u \nabla v dx + \int_{\Gamma_1} \rho h(u) v d\sigma + \int_{\mathcal{O}} \lambda u v dx \\ = \int_{\mathcal{O}} T_m(f\zeta(|w|)) v dx + \int_{\Gamma_1} g v d\sigma. \end{cases} \quad (1.19)$$

The existence and uniqueness of the solution of this problem has been proved in [34] for $\lambda = 0$, but the case $\lambda > 0$ does not present any substantial difference. In order to prove the existence of a solution of problem (1.18), we apply the Schauder fixed-point theorem.

Let us choose a suitable convex closed set of V . To do that, we take u as test function in (1.19). Using $H_1)$, $H_2)$, $H_3)$, $H_4)_{ii}$, (1.3), (1.5) and the Young inequality $|u| \leq \eta u^2 + c(\eta)$, $\forall \eta > 0$, we get

$$\begin{aligned} \alpha \|u\|_V^2 &\leq \int_{\mathcal{O}} B(x, w) \nabla u \nabla u dx + \int_{\mathcal{O}} \lambda u^2 dx + \int_{\Gamma_1} \rho h(u) u d\sigma \\ &\leq \int_{\mathcal{O}} |T_m(f\zeta(|w|)) u| dx + \int_{\Gamma_1} |g u| d\sigma \\ &\leq \eta \int_{\mathcal{O}} T_m(f\zeta(|w|)) u^2 dx + c(\eta) \int_{\mathcal{O}} T_m(f\zeta(|w|)) dx + \int_{\Gamma_1} |g u| d\sigma \\ &\leq \eta \|T_m(f\zeta(|w|))\|_{L^\infty(\mathcal{O})} \|u\|_{L^2(\mathcal{O})}^2 + c(\eta) \|T_m(f\zeta(|w|))\|_{L^\infty(\mathcal{O})} |\mathcal{O}| + \int_{\Gamma_1} |g u| d\sigma, \end{aligned}$$

where $|\mathcal{O}|$ denotes the measure of \mathcal{O} . Now, using the Poincaré inequality and the same arguments as in (1.13), keeping in mind the definition of s_1 , i.e. (1.11), we get

$$\begin{aligned} \alpha \|u\|_V^2 &\leq \eta m C_{\mathcal{O}}^2 \|u\|_V^2 + c(\eta) m |\mathcal{O}| \\ &\quad + \eta c_1 \gamma_{s_1} \|g\|_{L^s(\Gamma_1)} \|u\|_V^2 + c(\eta) c_1 \gamma_{s_1} \|g\|_{L^s(\Gamma_1)}, \end{aligned}$$

to obtain

$$\alpha \|u\|_V^2 \leq \eta \|u\|_V^2 (m C_{\mathcal{O}}^2 + c_1 \gamma_{s_1} \|g\|_{L^s(\Gamma_1)}) + c(\eta) (m |\mathcal{O}| + c_1 \gamma_{s_1} \|g\|_{L^s(\Gamma_1)}).$$

Whence, reminding that $\|g\|_{L^s(\Gamma_1)}$ is a finite quantity (due to hypothesis $H_4)_{i}$) and taking η sufficiently small so that $\alpha - \eta (m C_{\mathcal{O}}^2 + c_1 \gamma_{s_1} \|g\|_{L^s(\Gamma_1)}) \geq 0$, we deduce that there exists a constant C_0 such that the solution u of (1.19) satisfies

$$\|u\|_V \leq C_0. \quad (1.20)$$

Let us consider the map $T : w \in V \mapsto T(w) = u \in V$, where u is the solution of problem (1.19). In order to apply the Schauder fixed-point theorem, set

$$K := \{v \in V \mid \|v\|_V \leq C_0\},$$

which is a closed convex subset of V . Observe that, thanks to (1.20),

$$T(K) \subset K.$$

First, let us prove that $T(K)$ is relatively compact in K . Let $\{w_n\}_{n \in \mathbb{N}}$ be a sequence in K and let $u_n = T(w_n)$ be a solution of (1.19) corresponding to w_n . Since (1.20) holds true for every u_n , we have that $\{u_n\}_{n \in \mathbb{N}} \subset B(0, C_0)$. This allows us to extract a subsequence $\{u_{n_k}\}_{k \in \mathbb{N}}$ from $\{u_n\}_{n \in \mathbb{N}}$ and, for the boundedness of $\{w_n\}_{n \in \mathbb{N}}$ in V (by definition of K) and the compactness of $H^1(\mathcal{O})$ in $L^2(\mathcal{O})$, a subsequence $\{w_{n_k}\}_{k \in \mathbb{N}}$ from $\{w_n\}_{n \in \mathbb{N}}$ such that

$$\begin{cases} \text{i) } w_{n_k} \rightharpoonup w \text{ weakly in } V, \\ \text{ii) } w_{n_k} \rightarrow w \text{ strongly in } L^2(\mathcal{O}), \\ \text{iii) } u_{n_k} \rightharpoonup u^* \text{ weakly in } V. \end{cases} \quad (1.21)$$

From (1.21)_{ii} we have that

$$w_{n_k} \rightarrow w \text{ a.e. in } \mathcal{O}, \quad (1.22)$$

up to a subsequence.

The same computation done in [19] (see also [24]) gives, for every function $v \in V$,

$$(B(x, w) - B(x, w_{n_k})) \nabla v \rightarrow 0 \text{ strongly in } L^2(\mathcal{O}), \quad (1.23)$$

and

$$\begin{aligned} & \lim_{k \rightarrow +\infty} \int_{\mathcal{O}} B(x, w_{n_k}) \nabla u_{n_k} \nabla v dx + \int_{\Gamma_1} \rho h(u_{n_k}) v d\sigma + \lambda \int_{\mathcal{O}} u_{n_k} v dx \\ &= \int_{\mathcal{O}} B(x, w) \nabla u^* \nabla v dx + \int_{\Gamma_1} \rho h(u^*) v d\sigma + \lambda \int_{\mathcal{O}} u^* v dx. \end{aligned} \quad (1.24)$$

From (1.22) and for the properties of T_m and ζ , we can apply again the Lebesgue dominated convergence theorem to have

$$T_m(f\zeta(|w_{n_k}|)) \rightarrow T_m(f\zeta(|w|)) \text{ strongly in } L^2(\mathcal{O}), \quad (1.25)$$

and

$$\lim_{k \rightarrow +\infty} \int_{\mathcal{O}} T_m(f\zeta(|w_{n_k}|)) v dx = \int_{\mathcal{O}} T_m(f\zeta(|w|)) v dx. \quad (1.26)$$

Accordingly u^* satisfies

$$\begin{aligned} & \int_{\mathcal{O}} B(x, w) \nabla u^* \nabla v dx + \int_{\Gamma_1} \rho h(u^*) v d\sigma + \int_{\mathcal{O}} \lambda u^* v dx \\ &= \int_{\mathcal{O}} T_m(f\zeta(|w|)) v dx + \int_{\Gamma_1} g v d\sigma, \quad \forall v \in V, \end{aligned} \quad (1.27)$$

which implies that $u^* = T(w)$.

It remains to prove that

$$T(w_{n_k}) = u_{n_k} \rightarrow u^* \text{ strongly in } V. \quad (1.28)$$

To do that, we take $v = (u_{n_k} - u^*)$ as test function in (1.19) (written for the subsequence n_k) and in (1.27). After subtraction of the two identities, using H_1) and (1.2), we obtain

$$\begin{aligned} \alpha \|u_{n_k} - u^*\|_V^2 &\leq \int_{\mathcal{O}} B(x, w_{n_k}) \nabla(u_{n_k} - u^*) \nabla(u_{n_k} - u^*) dx \\ &+ \int_{\mathcal{O}} \lambda(u_{n_k} - u^*)(u_{n_k} - u^*) dx + \int_{\Gamma_1} \rho(h(u_{n_k}) - h(u^*))(u_{n_k} - u^*) d\sigma \\ &= \int_{\mathcal{O}} [T_m(f\zeta(|w_{n_k}|)) - T_m(f\zeta(|w|))](u_{n_k} - u^*) dx \\ &+ \int_{\mathcal{O}} (B(x, w) - B(x, w_{n_k})) \nabla u^* \nabla(u_{n_k} - u^*) dx. \end{aligned} \quad (1.29)$$

From (1.25) and (1.21)_{iii}, we deduce that

$$\lim_{k \rightarrow +\infty} \int_{\mathcal{O}} [T_m(f\zeta(|w_{n_k}|)) - T_m(f\zeta(|w|))](u_{n_k} - u^*) dx = 0.$$

On the other hand, from (1.21)_{iii} and (1.23), the second term in right-hand side of (1.29) goes to 0 as k goes to infinity. Whence, passing to the limit in (1.29), we obtain (1.28), which proves that $T(\{w_n\})$ is relatively compact.

Next, we show the continuity of T . Let $\{w_n\}_{n \in \mathbb{N}}$ be a sequence in K such that

$$w_n \rightarrow w \text{ strongly in } V, \quad (1.30)$$

and let $u_n = T(w_n)$ be the solution of (1.19) corresponding to w_n .

Using (1.20) and (1.30), we can extract a subsequence $\{u_{n_k}\}_{k \in \mathbb{N}}$ from $\{u_n\}_{n \in \mathbb{N}}$ and a subsequence $\{w_{n_k}\}_{k \in \mathbb{N}}$ from $\{w_n\}_{n \in \mathbb{N}}$ such that

$$\begin{cases} \text{i) } u_{n_k} \rightharpoonup u \text{ weakly in } V, \\ \text{ii) } u_{n_k} \rightarrow u \text{ strongly in } L^2(\mathcal{O}), \\ \text{iii) } w_{n_k} \rightarrow w \text{ a.e. in } \mathcal{O}. \end{cases} \quad (1.31)$$

We need first to show that

$$u = T(w). \quad (1.32)$$

To do that, we pass to the limit in (1.19) (written for the subsequence n_k).

For (1.31)_{iii} and for all $v \in V$, the same arguments used to deduce (1.26) give

$$\lim_{k \rightarrow +\infty} \int_{\mathcal{O}} T_m(f\zeta(|w_{n_k}|)) v dx = \int_{\mathcal{O}} T_m(f\zeta(|w|)) v dx.$$

We also have (see (1.24))

$$\begin{aligned} & \lim_{k \rightarrow +\infty} \int_{\mathcal{O}} B(x, w_{n_k}) \nabla u_{n_k} \nabla v dx + \int_{\Gamma_1} \rho h(u_{n_k}) v d\sigma + \lambda \int_{\mathcal{O}} u_{n_k} v dx \\ &= \int_{\mathcal{O}} B(x, w) \nabla u \nabla v dx + \int_{\Gamma_1} \rho h(u) v d\sigma + \lambda \int_{\mathcal{O}} u v dx. \end{aligned} \quad (1.33)$$

Thus, for all $v \in V$,

$$\begin{aligned} & \int_{\mathcal{O}} B(x, w) \nabla u \nabla v dx + \int_{\Gamma_1} \rho h(u) v d\sigma + \int_{\mathcal{O}} \lambda u v dx \\ &= \int_{\mathcal{O}} T_m(f\zeta(|w|)) v dx + \int_{\Gamma_1} g v d\sigma. \end{aligned}$$

The uniqueness of the solution of problem (1.19) proves (1.32) and the fact that the whole sequence u_n converges weakly to u in V .

To prove the continuity, it remains to show that

$$u_n \rightarrow u \text{ strongly in } V. \quad (1.34)$$

This is done as in the proof of (1.28) observing that, under assumptions (1.30)–(1.31), inequality (1.29) still holds true with $u^* = u$. Hence, T satisfies the assumptions of the Schauder fixed-point theorem, so that problem (1.17) has at least a solution in V . \square

We also have:

PROPOSITION 1.12. *Let u_m be a solution of problem (1.17). Then*

$$u_m \geq 0 \text{ a. e. in } \mathcal{O}.$$

Proof. Taking $\varphi = -u_m^-$ as test function in (1.18), we get

$$\begin{aligned} & - \int_{\mathcal{O}} B(x, u_m) \nabla u_m \nabla u_m^- dx - \int_{\Gamma_1} \rho h(u_m) u_m^- d\sigma - \int_{\mathcal{O}} \lambda u_m u_m^- dx \\ &= - \int_{\mathcal{O}} T_m(f\zeta(|u_m|)) u_m^- dx - \int_{\Gamma_1} g u_m^- d\sigma. \end{aligned}$$

Let observe that

$$-\nabla u_m \nabla u_m^- = (\nabla u_m^-)^2 \text{ and } -u_m u_m^- = (u_m^-)^2$$

and due to H_6)

$$h(u_m) u_m^- \leq 0 \text{ a.e. in } \mathcal{O}.$$

Consequently, applying the trace theorem in $H^1(\mathcal{O})$,

$$h(u_m) u_m^- \leq 0 \text{ a.e. on } \Gamma_1.$$

Then, using the ellipticity of B and H_2), we obtain

$$\alpha \int_{\mathcal{O}} |\nabla u_m^-|^2 dx \leq - \int_{\mathcal{O}} T_m(f\zeta(|u_m|)) u_m^- dx - \int_{\Gamma_1} g u_m^- d\sigma.$$

Since f , ζ , u_m^- and g are nonnegative, the integrals on the right-hand side are lower or equal to zero, so that

$$0 \leq \alpha \int_{\mathcal{O}} |\nabla u_m^-|^2 dx \leq 0.$$

This means that $|\nabla u_m^-| = 0$, which implies that u_m^- is equal to a constant c in \mathcal{O} . Since $u_m = 0$ on Γ_0 , also $u_m^- = 0$ on Γ_0 , so that $c = 0$. This shows that $u_m = u_m^+$ almost everywhere in \mathcal{O} and concludes the proof. \square

1.5 An existence result for the singular problem

In this section we show the existence of a solution of our problem, following [24], [42], [51], [52] and [53]. More precisely, [24] and [53] deal with an elliptic problem with a different singularity under nonlinear Robin and Dirichlet boundary conditions, respectively. In [51] and [52], the semilinear elliptic equations have more general singularities together with Dirichlet boundary conditions. In [42], the problem is posed through rough surfaces and presents the same singularity we treat in the present paper. Here the difficulty arises in dealing simultaneously with the quasilinear matrix field, the singular datum and the nonlinear Robin condition. We will also use the following particular case of the strong maximum principle given by [55, Theorem 8.19]:

THEOREM 1.13. [55] *Suppose that the matrix field A belongs to $M(\alpha, \beta, \mathcal{O})$ and the function d_1 is in $L^\infty(\mathcal{O})$ with $d_1 \leq 0$. Let $v \in H_0^1(\mathcal{O})$ such that*

$$\operatorname{div}(A(x)\nabla v) + d_1(x)v \geq 0 \text{ in } \mathcal{O}. \quad (1.35)$$

Then, if for some ball $B \subset\subset \mathcal{O}$ we have $\sup_B v = \sup_{\mathcal{O}} v \geq 0$, the function v must be constant in \mathcal{O} .

THEOREM 1.14. *Under assumptions H_1)- H_6), problem (1.10) admits at least a solution.*

Proof. Fix $m \in \mathbb{N}$, let $u_m \in V$ be a solution of the approximate problem (1.17). Observe that, since $T_m(f\zeta(|u_m|))$ satisfies the same assumptions as the function $f\zeta(|u_m|)$ for every fixed $m \in \mathbb{N}$, Proposition 1.7, Proposition 1.8 and Proposition 1.9 apply to u_m , as mentioned in Section 3. Moreover Proposition 1.12 implies that

$$u_m \geq 0 \quad \text{a.e. in } \mathcal{O}. \quad (1.36)$$

Hence we can write $\zeta(u_m)$ instead of $\zeta(|u_m|)$ in the problem.

Consequently, from Proposition 1.7, there exists $u \in V$ such that, up to a subsequence,

$$\begin{cases} \text{i) } u_m \rightharpoonup u \text{ weakly in } V \\ \text{ii) } u_m \rightarrow u \text{ strongly in } L^2(\mathcal{O}) \\ \text{iii) } u_m \rightarrow u \text{ a.e. in } \mathcal{O}. \end{cases} \quad (1.37)$$

Convergence (1.36) and (1.37)_{iii} imply that

$$u \geq 0 \quad \text{a.e. in } \mathcal{O}. \quad (1.38)$$

Let us now take $\varphi \in V$, $\varphi \geq 0$ and use $T_l(\varphi) \in V \cap L^\infty(\mathcal{O})$ as test function in (1.18), for $l > 0$ fixed. Then

$$\begin{aligned} & \int_{\mathcal{O}} B(x, u_m) \nabla u_m \nabla T_l(\varphi) dx + \int_{\Gamma_1} \rho h(u_m) T_l(\varphi) d\sigma + \int_{\mathcal{O}} \lambda u_m T_l(\varphi) dx \\ &= \int_{\mathcal{O}} T_m(f\zeta(u_m)) T_l(\varphi) dx + \int_{\Gamma_1} g T_l(\varphi) d\sigma. \end{aligned} \quad (1.39)$$

Also, using the fact that $T_l(\varphi) \geq 0$, from Proposition 1.8 (written with $T_l(\varphi)$ instead of φ) we have the uniform estimate

$$\|T_m(f\zeta(u_m)) T_l(\varphi)\|_{L^1(\mathcal{O})} \leq c$$

with c independent of m . From (1.37)_{iii} and the properties of ζ and T_m , we have that

$$\lim_{m \rightarrow \infty} T_m(f\zeta(u_m)) T_l(\varphi) = f\zeta(u) T_l(\varphi) \quad \text{a.e. in } \mathcal{O},$$

for every fixed $l > 0$. Since f, ζ and φ are nonnegative, we can apply Fatou's lemma to obtain

$$\int_{\mathcal{O}} f\zeta(u) T_l(\varphi) dx < +\infty. \quad (1.40)$$

Now we want to pass to the limit in (1.39) for $\varphi \in V$, $\varphi \geq 0$ and $l > 0$ fixed, as $m \rightarrow +\infty$. But, concerning the integral $\int_{\mathcal{O}} T_m(f\zeta(u_m)) T_l(\varphi) dx$, we can apply the Lebesgue dominated convergence theorem only far from the singularity. To avoid this problem, as in [42],[51] and [52], for every positive δ we divide the integral in two terms, that is

$$\begin{aligned} & \int_{\mathcal{O}} T_m(f\zeta(u_m)) T_l(\varphi) dx = \int_{\{0 \leq u_m \leq \delta\}} T_m(f\zeta(u_m)) T_l(\varphi) dx \\ & + \int_{\{u_m > \delta\}} T_m(f\zeta(u_m)) T_l(\varphi) dx \doteq I_m + J_m. \end{aligned} \quad (1.41)$$

From Proposition 1.9 written for u_m and $T_l(\varphi)$, we have the following estimate for the integral I_m :

$$\begin{aligned} I_m &\leq \int_{\mathcal{O}} B(x, u_m) \nabla u_m \nabla T_l(\varphi) Z_\delta(u_m) dx \\ &\quad + h(2\delta) \|\rho\|_{L^\infty(\Gamma_1)} \|T_l(\varphi)\|_{L^1(\Gamma_1)} + 2\delta\lambda \|T_l(\varphi)\|_{L^1(\mathcal{O})}, \end{aligned}$$

which implies that

$$\begin{aligned} 0 &\leq \limsup_{m \rightarrow \infty} I_m \leq \limsup_{m \rightarrow \infty} \int_{\mathcal{O}} B(x, u_m) \nabla u_m \nabla T_l(\varphi) Z_\delta(u_m) dx \\ &\quad + h(2\delta) \|\rho\|_{L^\infty(\Gamma_1)} \|T_l(\varphi)\|_{L^1(\Gamma_1)} + 2\delta\lambda \|T_l(\varphi)\|_{L^1(\mathcal{O})}. \end{aligned} \tag{1.42}$$

From H_1), (1.37)_{iii} and the continuity of Z_δ (see (1.7)), it follows that

$$\lim_{m \rightarrow \infty} {}^t B(x, u_m) \nabla T_l(\varphi) Z_\delta(u_m) = {}^t B(x, u) \nabla T_l(\varphi) Z_\delta(u) \text{ a.e. in } \mathcal{O}$$

and the estimate

$$|{}^t B(x, u_m) \nabla T_l(\varphi) Z_\delta(u_m)| \leq \beta |\nabla T_l(\varphi) Z_\delta(u_m)| \leq \beta |\nabla T_l(\varphi)| \in L^2(\mathcal{O}) \text{ a.e. in } \mathcal{O},$$

for all $m \in \mathbb{N}$. Applying the Lebesgue dominated convergence theorem, we obtain

$${}^t B(x, u_m) \nabla T_l(\varphi) Z_\delta(u_m) \rightarrow {}^t B(x, u) \nabla T_l(\varphi) Z_\delta(u) \text{ strongly in } L^2(\mathcal{O}),$$

as $m \rightarrow +\infty$. Consequently, using (1.42) we get

$$\begin{aligned} 0 &\leq \limsup_{m \rightarrow \infty} I_m \leq \int_{\mathcal{O}} B(x, u) \nabla u \nabla T_l(\varphi) Z_\delta(u) dx \\ &\quad + h(2\delta) \|\rho\|_{L^\infty(\Gamma_1)} \|T_l(\varphi)\|_{L^1(\Gamma_1)} + 2\delta\lambda \|T_l(\varphi)\|_{L^1(\mathcal{O})}. \end{aligned}$$

To pass to the limit as δ goes to 0 in the last inequality, we observe that, by dominated convergence,

$$\lim_{\delta \rightarrow 0} \int_{\mathcal{O}} B(x, u) \nabla u \nabla T_l(\varphi) Z_\delta(u) dx = \int_{\mathcal{O}} B(x, u) \nabla u \nabla T_l(\varphi) \chi_{\{u=0\}} dx = 0,$$

since ∇u vanishes where u is equal to zero. Using the hypothesis H_6), we have

$$\lim_{\delta \rightarrow 0} \limsup_{m \rightarrow \infty} I_m = 0. \tag{1.43}$$

On the other hand, by the same arguments used in [42], [51] and [52] to treat a term similar to J_m , we have

$$\lim_{\delta \rightarrow 0} \lim_{m \rightarrow \infty} J_m = \int_{\mathcal{O}} f\zeta(u) T_l(\varphi) \chi_{\{u>0\}} dx, \quad \delta \in \mathbb{R}^+ \setminus C, \tag{1.44}$$

where C is a countable set of values.

Finally, passing to the limit in (1.41), combining (1.43) and (1.44), we get

$$\limsup_{m \rightarrow \infty} \int_{\mathcal{O}} T_m(f\zeta(u_m))T_l(\varphi)dx = \int_{\mathcal{O}} f\zeta(u)T_l(\varphi)\chi_{\{u>0\}}dx, \quad (1.45)$$

for every $\varphi \in V$, $\varphi \geq 0$. But this remains true for every $\varphi \in V$ with any sign, using the fact that $\varphi = \varphi^+ - \varphi^-$ with φ^+ , φ^- nonnegative and that $T_l(\varphi^+ - \varphi^-) = T_l(\varphi^+) - T_l(\varphi^-)$.

Now, arguing as done in the proof of the Theorem 1.11 in order to obtain (1.24), we can pass to the limit in the left-hand side of (1.39), as $m \rightarrow +\infty$. Then, using (1.38), (1.40) and (1.45), we obtain that the limit function u satisfies the following problem:

$$\left\{ \begin{array}{l} \text{Find } u \in V \text{ such that } u \geq 0 \text{ a.e. in } \mathcal{O}, \\ \int_{\mathcal{O}} f\zeta(u)T_l(\varphi)dx < +\infty \text{ and} \\ \int_{\mathcal{O}} B(x, u)\nabla u\nabla T_l(\varphi)dx + \int_{\Gamma_1} \rho h(u)T_l(\varphi)d\sigma + \int_{\mathcal{O}} \lambda u T_l(\varphi)dx \\ = \int_{\mathcal{O}} f\zeta(u)T_l(\varphi)\chi_{\{u>0\}}dx + \int_{\Gamma_1} gT_l(\varphi)d\sigma, \quad \forall \varphi \in V. \end{array} \right. \quad (1.46)$$

Let us pass to the limit in (1.46) as l goes to infinity.

Thanks to (1.8), the only term to check is the singular one. To do this, we observe that

$$\liminf_{l \rightarrow \infty} f\zeta(u)T_l(\varphi) = f\zeta(u)\varphi \quad \text{a.e. in } \mathcal{O}$$

and, since $f\zeta(u)T_l(\varphi) \in L^1(\mathcal{O})$, we have

$$f\zeta(u)\varphi \in L^1(\mathcal{O}),$$

via Fatou's lemma. Whence we get

$$f\zeta(u)T_l(\varphi) \leq f\zeta(u)\varphi \in L^1(\mathcal{O}), \quad \forall l \in \mathbb{N} \text{ and a.e. in } \mathcal{O},$$

so, applying the Lebesgue dominated convergence theorem, we obtain

$$f\zeta(u)T_l(\varphi) \rightarrow f\zeta(u)\varphi \text{ strongly in } L^1(\mathcal{O}).$$

Hence we deduce that the limit function u satisfies the following problem:

$$\left\{ \begin{array}{l} \text{Find } u \in V \text{ such that } u \geq 0 \text{ a.e. in } \mathcal{O}, \\ \int_{\mathcal{O}} f\zeta(u)\varphi dx < +\infty \text{ and} \\ \int_{\mathcal{O}} B(x, u)\nabla u\nabla \varphi dx + \int_{\Gamma_1} \rho h(u)\varphi d\sigma + \int_{\mathcal{O}} \lambda u \varphi dx \\ = \int_{\mathcal{O}} f\zeta(u)\varphi\chi_{\{u>0\}}dx + \int_{\Gamma_1} g\varphi d\sigma, \quad \forall \varphi \in V. \end{array} \right. \quad (1.47)$$

In order to show that $u > 0$ almost everywhere in \mathcal{O} , we observe that u is such that

$$-\operatorname{div}(B(x, u)\nabla u) + \lambda u \geq 0.$$

Therefore setting $v := -u$, it results

$$\operatorname{div}(B(x, u)\nabla v) - \lambda v \geq 0$$

and v satisfies (1.35) written for $A(x) = B(x, u(x))$ and $d_1(x) = -\lambda$. Moreover $v \leq 0$ in \mathcal{O} and $v = 0$ on Γ_0 , so that $\sup_{\mathcal{O}} v = 0$.

Suppose now there exists a ball B well contained in \mathcal{O} such that $\sup_B v = 0$. Then from the strong maximum principle stated in Proposition 1.13, using again the boundary condition on Γ_0 , we derive $v \equiv 0$ in \mathcal{O} , that is $u \equiv 0$ in \mathcal{O} .

Since u is a solution of problem (1.47), the hypothesis $h(0) = 0$ and the positivity of f , ζ and g imply that each integral on the right-hand side of (1.47) is zero, for every $\varphi \in \mathcal{D}(\mathcal{O})$. This means that $f \equiv 0$ on \mathcal{O} and $g \equiv 0$ on Γ_1 , which contradicts assumption H_5). Consequently $u > 0$ almost everywhere in \mathcal{O} and $\chi_{\{u>0\}} \equiv 1$. Then, u satisfies problem (1.10). \square

1.6 A uniqueness result

The next result deals with the uniqueness of the solution of problem (1.10). As usual in the literature, to ensure the uniqueness of the solution, we need additional hypotheses on the function ζ and the matrix field B . Concerning the quasilinear term, we make the same assumption (see $H_{1'}$) below) introduced in [23] by Chipot (see also [3]) for quasilinear nonsingular problem with Dirichlet conditions (we refer to [12] and [13] for linear Robin conditions and to [19] for nonlinear Robin conditions). For the nonlinearity ζ we require a monotonicity hypothesis as in [42] and [52]. Let us mention that, while in [52] the uniqueness result is a consequence of the comparison principle, here in the proof we follow some arguments from [19]. More precisely, we suppose the following assumptions:

$H_{1'}$) There exists a real function $\omega : \mathbb{R} \rightarrow \mathbb{R}$ satisfying the following conditions:

$$\left\{ \begin{array}{l} \text{i) } \omega \text{ is continuous and non decreasing, with } \omega(t) > 0 \ \forall t > 0; \\ \text{ii) } |B(x, t_1) - B(x, t_2)| \leq \omega(|t_1 - t_2|) \text{ for a.e. } x \in \mathcal{O}, \forall t_1 \neq t_2; \\ \text{iii) } \forall y > 0, \lim_{x \rightarrow 0^+} \int_x^y \frac{dt}{\omega(t)} = +\infty. \end{array} \right.$$

$H_{3'}$) The function ζ is non increasing in $[0, +\infty[$.

REMARK 1.15. If B is uniformly Lipschitz-continuous in t , i.e. there exists a constant L such that

$$|B(x, t_1) - B(x, t_2)| \leq L|t_1 - t_2| \text{ for almost every } x \in \mathcal{O} \text{ and for every } t_1 \neq t_2,$$

then $w(t) := Lt$ satisfies assumptions $H_{1'}$).

THEOREM 1.16. *Under assumptions H_1 - H_6) and the additional ones $H_{1'}$) and $H_{3'}$), problem (1.10) admits a unique solution.*

Proof. Let us assume that u_1 and u_2 are two solutions of problem (1.10). Taking the difference between the two variational formulations, we get

$$\begin{aligned} & \int_{\mathcal{O}} [B(x, u_1) \nabla u_1 - B(x, u_2) \nabla u_2] \nabla v dx + \int_{\Gamma_1} \rho [h(u_1) - h(u_2)] v d\sigma \\ & + \lambda \int_{\mathcal{O}} (u_1 - u_2) v dx = \int_{\mathcal{O}} f[\zeta(u_1) - \zeta(u_2)] v dx, \end{aligned} \quad (1.48)$$

for every function v in V . We want to prove that $u_1 = u_2$.

Proceeding as in [19] (see also [3], [23]), we define:

$$F_\mu(s) := \begin{cases} \int_\mu^s \frac{dt}{\omega^2(t)} & \text{if } s \geq \mu, \\ 0 & \text{otherwise;} \end{cases} \quad (1.49)$$

where μ is a positive parameter converging to 0 and ω is given by $H_{1'}$). By known results (see for instance [23, Corollary 2.15]), $F_\mu(u_1 - u_2) \in V$ and

$$\nabla F_\mu(u_1 - u_2) = \frac{1}{\omega^2(u_1 - u_2)} \nabla(u_1 - u_2), \quad (1.50)$$

for every fixed $\mu > 0$. Let us set

$$E := \{x \in \mathcal{O} \mid (u_1 - u_2)(x) > \mu\}$$

and take $F_\mu(u_1 - u_2) \in V$ as test function in (1.48).

Adding and subtracting the term $\int_{\mathcal{O}} B(x, u_1) \nabla u_2 \nabla F_\mu(u_1 - u_2) dx$, we obtain

$$\begin{aligned} & \int_{\mathcal{O}} B(x, u_1) \nabla(u_1 - u_2) \nabla F_\mu(u_1 - u_2) dx + \int_{\mathcal{O}} [B(x, u_1) - B(x, u_2)] \nabla u_2 \nabla F_\mu(u_1 - u_2) dx \\ & + \int_{\Gamma_1} \rho [h(u_1) - h(u_2)] F_\mu(u_1 - u_2) d\sigma + \lambda \int_{\mathcal{O}} (u_1 - u_2) F_\mu(u_1 - u_2) dx \\ & = \int_{\mathcal{O}} f[\zeta(u_1) - \zeta(u_2)] F_\mu(u_1 - u_2) dx. \end{aligned}$$

From the definition of F_μ (see (1.49)), using (1.50) and the fact that $\nabla F_\mu(u_1 - u_2) = 0$

outside E , we get

$$\begin{aligned}
 & \int_E B(x, u_1) \frac{1}{\omega^2(u_1 - u_2)} \nabla(u_1 - u_2) \nabla(u_1 - u_2) dx \\
 & + \int_E [B(x, u_1) - B(x, u_2)] \frac{1}{\omega^2(u_1 - u_2)} \nabla u_2 \nabla(u_1 - u_2) dx \\
 & + \int_{\Gamma_1} \rho[h(u_1) - h(u_2)] F_\mu(u_1 - u_2) d\sigma + \lambda \int_{\mathcal{O}} (u_1 - u_2) F_\mu(u_1 - u_2) dx \\
 & = \int_{\mathcal{O}} f[\zeta(u_1) - \zeta(u_2)] F_\mu(u_1 - u_2) dx.
 \end{aligned}$$

So, from H_1) we have:

$$\begin{aligned}
 \alpha \left\| \frac{\nabla(u_1 - u_2)}{\omega(u_1 - u_2)} \right\|_{L^2(E)}^2 & \leq \int_E B(x, u_1) \frac{1}{\omega^2(u_1 - u_2)} \nabla(u_1 - u_2) \nabla(u_1 - u_2) dx \\
 & = - \int_E [B(x, u_1) - B(x, u_2)] \frac{1}{\omega^2(u_1 - u_2)} \nabla u_2 \nabla(u_1 - u_2) dx \\
 & - \int_{\Gamma_1} \rho[h(u_1) - h(u_2)] F_\mu(u_1 - u_2) d\sigma - \lambda \int_{\mathcal{O}} (u_1 - u_2) F_\mu(u_1 - u_2) dx \\
 & + \int_{\mathcal{O}} f[\zeta(u_1) - \zeta(u_2)] F_\mu(u_1 - u_2) dx.
 \end{aligned} \tag{1.51}$$

Now, from $H_4)_{ii}$, H_6) and (1.49), we obtain

$$\rho[h(u_1) - h(u_2)] F_\mu(u_1 - u_2) \geq 0 \quad \text{a.e. in } \mathcal{O}. \tag{1.52}$$

Since $F_\mu(u_1 - u_2) \in V$, Proposition 1.6 (written for $t_1 = 1$) gives $\rho[h(u_1) - h(u_2)] F_\mu(u_1 - u_2) \in W^{1,1}(\mathcal{O})$, which implies

$$\rho[h(u_1) - h(u_2)] F_\mu(u_1 - u_2) \geq 0 \quad \text{a.e. on } \Gamma_1, \tag{1.53}$$

thanks to (1.52) and the trace theorem. Again from (1.49) and H_2), we have also

$$\lambda(u_1 - u_2) F_\mu(u_1 - u_2) \geq 0 \quad \text{a.e. in } \mathcal{O}. \tag{1.54}$$

On the other hand, in view of hypotheses $H_3)_{ii}$ and $H_3')$, we get

$$f[\zeta(u_1) - \zeta(u_2)] F_\mu(u_1 - u_2) \leq 0 \quad \text{a.e. in } \mathcal{O}. \tag{1.55}$$

Collecting (1.53), (1.54) and (1.55), we obtain by (1.51)

$$\begin{aligned}
 \alpha \left\| \frac{\nabla(u_1 - u_2)}{\omega(u_1 - u_2)} \right\|_{L^2(E)}^2 & \leq \int_E B(x, u_1) \frac{1}{\omega^2(u_1 - u_2)} \nabla(u_1 - u_2) \nabla(u_1 - u_2) dx \\
 & \leq \int_E \frac{|[B(x, u_1) - B(x, u_2)] \nabla u_2 \nabla(u_1 - u_2)|}{\omega^2(u_1 - u_2)} dx.
 \end{aligned}$$

Now, using hypothesis $H_{1'})_{ii}$ and the Cauchy-Schwarz inequality, we have

$$\alpha \left\| \frac{\nabla(u_1 - u_2)}{\omega(u_1 - u_2)} \right\|_{L^2(E)}^2 \leq \int_E \frac{|\nabla u_2| |\nabla(u_1 - u_2)|}{\omega(u_1 - u_2)} dx \leq \|\nabla u_2\|_{L^2(E)} \left\| \frac{\nabla(u_1 - u_2)}{\omega(u_1 - u_2)} \right\|_{L^2(E)},$$

which implies

$$\left\| \frac{\nabla(u_1 - u_2)}{\omega(u_1 - u_2)} \right\|_{L^2(E)} \leq \frac{1}{\alpha} \|\nabla u_2\|_{L^2(E)} \leq C \|\nabla u_2\|_{L^2(\mathcal{O})} \leq C', \quad (1.56)$$

with C and C' independent of μ .

Currently we set

$$G_\mu(s) := \begin{cases} \int_\mu^s \frac{dt}{\omega(t)} & \text{if } s \geq \mu, \\ 0 & \text{otherwise;} \end{cases} \quad (1.57)$$

where μ is given as in (1.49).

As before, we have $G_\mu(u_1 - u_2) \in V$ and $\nabla G_\mu(u_1 - u_2) = \frac{1}{\omega(u_1 - u_2)} \nabla(u_1 - u_2)$. Whence, using the fact that $G_\mu(u_1 - u_2)$ is equal to zero on $\mathcal{O} \setminus E$, (1.56) becomes

$$\|G_\mu(u_1 - u_2)\|_V = \|\nabla G_\mu(u_1 - u_2)\|_{L^2(\mathcal{O})} \leq C', \quad \text{for every } \mu.$$

Hence, there exists a sequence $\{\mu_k\}_{k \in \mathbb{N}}$ converging to 0, as $k \rightarrow \infty$, and a function $G \in V$ such that, up to a subsequence,

$$\begin{cases} G_{\mu_k}(u_1 - u_2) \rightharpoonup G \text{ weakly in } V, \\ G_{\mu_k}(u_1 - u_2) \rightarrow G \text{ strongly in } L^2(\mathcal{O}) \text{ and a.e. in } \mathcal{O}. \end{cases}$$

This implies that

$$\lim_{\mu_k \rightarrow 0} G_{\mu_k}(u_1 - u_2)(x) = G(x) < +\infty \quad \text{a.e. in } \mathcal{O}, \quad \text{as } k \rightarrow \infty.$$

But from definition (1.57) and hypothesis $H_{1'})_{iii}$, we have also

$$\lim_{\mu_k \rightarrow 0} G_{\mu_k}(u_1 - u_2)(x) = \lim_{\mu_k \rightarrow 0} \int_{\mu_k}^{(u_1 - u_2)(x)} \frac{dt}{\omega(t)} = +\infty \quad \text{a.e. in } E.$$

Consequently, as $\mu \rightarrow 0$,

$$|E| = |\{x \in \mathcal{O} \mid (u_1 - u_2)(x) > 0\}| = 0,$$

so that $u_1 \leq u_2$ almost everywhere in \mathcal{O} . Interchanging the roles of u_1 and u_2 , we obtain $u_1 = u_2$ almost everywhere in \mathcal{O} . \square

1.7 A regularity result

In this last section we discuss the boundedness of a solution of problem (1.10). As mentioned in Remark 1.2, in order to get bounded solutions we need to require a greater summability of the data f and g , as done in [24], [25], [26], [42], [51] and [54].

Hence, instead of $H_3)_{ii}$ and $H_4)_i$, we make the following hypotheses:

$$H_{3''}) \quad 0 \leq f \in L^l(\mathcal{O}_0), \text{ with } l > \frac{N}{2}.$$

$$H_{4'}) \quad 0 \leq g \in L^s(\Gamma_1), \text{ with } s > N - 1.$$

REMARK 1.17. Let us mention that, as showed in [42], assumption $H_{3''})$ is stronger than $H_3)_{ii}$. Furthermore, observe that if $N = 2$, $H_{4'})$ and $H_4)_i$ are the same hypothesis; if $N > 2$, $H_{4'})$ implies $H_4)_i$ since $\frac{2(N-1)}{N} < 2 \leq N - 1$.

As usual when proving the boundedness of a solution (see for instance [24], [42] and [51]), to show the next theorem we use some classical ideas from Stampacchia [69].

THEOREM 1.18. *Under assumptions $H_1)–H_6)$, suppose further $H_{3''})$ and $H_{4'})$ hold true. Then, any solution of (1.10) is bounded.*

Proof. Let us set, for every real number $h \geq 1$,

$$A_h := \{x \in \mathcal{O} : u(x) > h\} \quad \text{and} \quad D_h := \{x \in \Gamma_1 : u(x) > h\}.$$

Since $u \in V$, denoting by $|A_h|$ the measure of A_h , we get

$$h^2 |A_h| < \int_{A_h} u^2 dx \leq c,$$

and consequently $|A_h| \leq \frac{c}{h^2}$. Then

$$\lim_{h \rightarrow +\infty} |A_h| = 0.$$

From [24, Lemma 3.4], thanks to Proposition 1.7 we have also

$$\lim_{h \rightarrow +\infty} \partial H_{N-1}(D_h) = 0,$$

where ∂H_{N-1} is the $(N - 1)$ -Hausdorff measure in \mathbb{R}^N .

Now, let us choose $G_h(u) := (u - h)^+ \in V$ as test function in (1.10), with $h \geq 1$. Since

$$\nabla u = \nabla G_h(u) - \nabla[(u - h)^-] \quad \text{and}$$

$$\int_{\mathcal{O}} B(x, u) \nabla[(u - h)^-] \nabla G_h(u) dx = 0,$$

we get

$$\begin{aligned} & \int_{\mathcal{O}} B(x, u) \nabla G_h(u) \nabla G_h(u) dx + \int_{\Gamma_1} \rho h(u) G_h(u) d\sigma + \int_{\mathcal{O}} \lambda u G_h(u) dx \\ &= \int_{\mathcal{O}} f \zeta(u) G_h(u) dx + \int_{\Gamma_1} g G_h(u) d\sigma. \end{aligned}$$

In view of H_1) and taking into account that $G_h(u)$, ρ , λ and $h(u)$ are nonnegative (since $u > 0$), the previous equality yields

$$\alpha \int_{\mathcal{O}} [\nabla G_h(u)]^2 dx \leq \int_{\mathcal{O}} f \zeta(u) G_h(u) dx + \int_{\Gamma_1} g G_h(u) d\sigma.$$

Since $H_3)_i$ and $h \geq 1$, we obtain $\zeta(u) \leq \frac{1}{u^\theta} < \frac{1}{h^\theta} \leq 1$ on the set where $G_h(u) \neq 0$ which gives

$$\alpha \int_{\mathcal{O}} [\nabla G_h(u)]^2 dx \leq \int_{\mathcal{O}} f G_h(u) dx + \int_{\Gamma_1} g G_h(u) d\sigma.$$

This implies that u belongs to $L^\infty(\mathcal{O})$. Indeed, from this point, we can use exactly the same arguments of the proof of Theorem 3.2 in [24] (written for $\delta = 0$), based on classical results from Stampacchia [69]). \square

Chapter 2

Homogenization of a class of singular elliptic problems in perforated domains

P. Donato, S. Monsurrò and F. Raimondi, *Homogenization of a class of singular elliptic problems in perforated domains*, *Nonlinear Analysis*, **173**, pp. 180–208, (2018).

In this chapter we study the asymptotic behavior of a class of quasilinear elliptic problems posed in a domain perforated by ε -periodic holes of size ε . The quasilinear equations present a nonlinear singular lower order term $f\zeta(u_\varepsilon)$, where u_ε is the solution of the problem at ε -level, ζ is a continuous function singular in zero and f a function whose summability depends on the growth of ζ near its singularity. We prescribe a nonlinear Robin condition on the boundary of the holes contained in Ω and a homogeneous Dirichlet condition on the exterior boundary. The particular case of a Neumann boundary condition on the holes is already new.

The main tool in the homogenization process consists in proving a suitable convergence result, which shows that the gradient of u_ε behaves like that of the solution of a suitable linear problem associated with a weak cluster point of the sequence $\{u_\varepsilon\}$, as $\varepsilon \rightarrow 0$. This allows us not only to pass to the limit in the quasilinear term, but also to study the singular term near its singularity, via an accurate a priori estimate. We also get a corrector result for our problem.

The main novelty of this work is that for the first time the unfolding method is used to treat a singular term as $f\zeta(u_\varepsilon)$. This plays an essential role in order to get an almost everywhere convergence of the solution u_ε , needed in the study the asymptotic behavior of the problem.

2.1 Introduction

We deal with the homogenization of a class of quasilinear elliptic problems with singular lower order terms posed in periodically perforated domains. In our study, we use the periodic unfolding method, originally introduced in [28] and [29]. The perforated domain Ω_ε^* is obtained by removing from a connected bounded open set Ω of \mathbb{R}^N , $N \geq 2$, a set of ε -periodic holes of size ε . The boundary of Ω_ε^* is decomposed into Γ_1^ε and Γ_0^ε , which denote the boundary of the holes well contained in Ω and the remaining exterior boundary, respectively (see Section 2.2 for details). We prescribe a nonlinear Robin condition on Γ_1^ε and a homogeneous Dirichlet condition on Γ_0^ε .

More precisely, we study the asymptotic behavior, as ε goes to zero, of the following problem:

$$\begin{cases} -\operatorname{div}(A^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon) = f\zeta(u_\varepsilon) & \text{in } \Omega_\varepsilon^*, \\ u_\varepsilon = 0 & \text{on } \Gamma_0^\varepsilon, \\ (A^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon)\nu + \varepsilon^\gamma \rho_\varepsilon(x)h(u_\varepsilon) = g_\varepsilon & \text{on } \Gamma_1^\varepsilon, \end{cases}$$

where ν is the unit outward normal to the holes.

The oscillating coefficients' matrix field A^ε in the quasilinear diffusion term is defined by $A^\varepsilon(x, t) = A\left(\frac{x}{\varepsilon}, t\right)$, where the matrix field A is uniformly elliptic, bounded, periodic in the first variable and Carathéodory. The nonlinear real function $\zeta(s)$ is nonnegative and singular at $s = 0$, while f is a nonnegative datum whose summability depends on the growth of ζ near its singularity. Concerning the Robin boundary condition, $\rho_\varepsilon(x) = \rho\left(\frac{x}{\varepsilon}\right)$ where the function ρ is assumed to be periodic, nonnegative and bounded on ∂T , the nonlinear boundary term h is an increasing and continuously differentiable function whose derivative satisfies suitable growth assumptions, and $g_\varepsilon(x) = \varepsilon g\left(\frac{x}{\varepsilon}\right)$, where g is a periodic nonnegative function with prescribed summability.

From the physical point of view, the quasilinear diffusion term describes the behavior of some materials, like glass or wood, in which the heat diffusion depends on the temperature (see [50] for more details). A source term depending on the solution itself and becoming infinite when the solution vanishes may model an electrical conductor, where each point becomes a source of heat as a current flows in it (see [42, Section 3]). Moreover, nonlinear Robin boundary conditions describe certain chemical reactions at the boundaries of perforations (e.g. [37]).

The existence and uniqueness of the weak solution of the problem, for every fixed ε , have already been proved by the authors in [46]. Uniform a priori estimates of the solution u_ε are obtained by adapting to our case some arguments introduced in [46]. Let us mention that the third bound (cf. Proposition 2.22) provides an estimate of the integral of the singular term close to the singular set $\{u_\varepsilon = 0\}$, in terms of the quasilinear one.

In the homogenization process we have to pass to the limit in the quasilinear term, in the singular one and in the nonlinear Robin condition. In order to study the quasilinear term, we prove a crucial convergence, given by Theorem 2.27, which is the main tool when proving our results. It shows that the gradient of u_ε behaves like that of the solution of a suitable linear problem associated with a weak cluster point of the sequence $\{u_\varepsilon\}$, as $\varepsilon \rightarrow 0$. This idea was originally introduced in [14] (see also [15]) where some nonlinear problems with quadratic growth are considered. The main difference with respect to [14] consists in the choice of the test functions used in the proof. Indeed, in the quadratic growth case the authors take exponential test functions, while here we have to use appropriate test functions that allow us to treat the singular term. To construct these functions, taking into account the homogenization results of [25], we adapt to our needs some techniques from [42].

As far as it concerns the singular term, as done in [42] and [46], we split it into the sum of two integrals: one on the set where the solution is close to the singularity and one where it is far from it, which results not singular. Near the singularity, we make use of the estimate given by Proposition 2.22. In this way, we shift the study of the singular term to that of the quasilinear one, for which we can use the previous result.

Let us point out that for $h \equiv 0$ we obtain, as a particular case, homogenization and corrector results for the problem when a (homogeneous or not) Neumann boundary condition on Γ_1^ε is prescribed, which are already new in the literature.

Concerning the nonlinear Robin boundary condition, the arguments used in [25] to study the boundary terms still apply to our case, with appropriate considerations, the hypotheses being slightly different.

Finally, Theorem 2.9 states that the correctors for the nonsingular quasilinear problem in perforated domains studied in [25] are also correctors for our problem.

As already mentioned, all the results of this paper are proved by means of the periodic unfolding method. It was introduced by D. Cioranescu, A. Dalambian and G. Griso in [28] and [29] for fixed domains and extended to perforated ones in [32] and [33].

The main novelty of this work is that, for the first time, the unfolding method is used to treat a singular term as $f\zeta(u_\varepsilon)$. This is essential in order to get the almost everywhere convergences needed to study the asymptotic behavior of the problem. Although in [42] the problem presents the same singular term, it is posed through rough surface. Whence, since the measure of the interface is zero, a compactness result in $L^2(\Omega)$ from [48] gives the almost everywhere convergences. In our case, due to the presence of the holes, this argument does not apply, and the almost everywhere convergences can be obtained only by using the unfolding method.

We refer to [15], [14] for the first results on the periodic homogenization of quadratic nonlinear elliptic problems, and to [5], [6] for that of quasilinear problems, in the case of fixed

domains. The homogenization of quasilinear problems in periodically perforated domains with nonlinear Robin boundary conditions has been considered in [20], [25], and in [26], where is also considered a nonlinear term which is quadratic with respect to the gradient. The homogenization of various singular elliptic problems has been studied, for instance, in [17], [42] and [51].

The paper is organized as follows:

In Section 2 we present the setting of the problem and we state the main results.

In Section 3 we give a short presentation of the periodic unfolding operator for perforated domains.

In Section 4 we prove two a priori estimates uniform with respect to ε and we give an estimate of the integral of the singular term close to the singular set $\{u_\varepsilon = 0\}$.

In Section 5 we state and prove the crucial auxiliary convergence result given by Theorem 2.27.

In Section 6 we prove the homogenization theorem.

2.2 Setting of the problem and main results

Throughout the paper, we use the notation introduced in [27] and [29] for the periodic unfolding method in perforated domains.

For $N \in \mathbb{N}, N \geq 2$, we suppose that Ω is a connected bounded open set in \mathbb{R}^N whose boundary $\partial\Omega$ is Lipschitz-continuous. Let $b = \{b_1, \dots, b_N\}$ be a basis of \mathbb{R}^N and define by Y the following reference cell:

$$Y \doteq \left\{ y \in \mathbb{R}^N : y = \sum_{i=1}^N y_i b_i, (y_1, \dots, y_N) \in (0,1)^N \right\}. \quad (2.1)$$

Also, T denotes the reference hole, which is a (nonempty) open subset of \mathbb{R}^N such that $\bar{T} \subset Y$, and ∂T is Lipschitz-continuous with a finite number of connected components.

The perforated cell, that is the part of the cell occupied by the material, is defined by $Y^* \doteq Y \setminus \bar{T}$.

Let $\{\varepsilon\}_{\varepsilon>0}$ be a positive parameter taking values in a sequence converging to zero and set

$$G \doteq \left\{ \xi \in \mathbb{R}^N : \xi = \sum_{i=1}^N k_i b_i, (k_1, \dots, k_N) \in \mathbb{Z}^N \right\} \quad T_\varepsilon \doteq \bigcup_{\xi \in G} \varepsilon(\xi + T),$$

where $\varepsilon(\xi + T)$ are disjoint translated sets of T rescaled by ε .

The perforated domain Ω_ε^* is defined by removing from Ω the closure of holes T_ε , that is (see Figure 2.1)

$$\Omega_\varepsilon^* \doteq \Omega \setminus \bar{T}_\varepsilon.$$

Moreover, we define by $\widehat{\Omega}_\varepsilon$ the interior of the largest union of cells $\varepsilon(\xi + \overline{Y})$ completely included in Ω and by Λ_ε its complement with respect to Ω that contains the cells $\varepsilon(\xi + \overline{Y})$ intersecting $\partial\Omega$, i.e.

$$\widehat{\Omega}_\varepsilon \doteq \text{interior} \left\{ \bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \overline{Y}) \right\} \quad \text{and} \quad \Lambda_\varepsilon \doteq \Omega \setminus \widehat{\Omega}_\varepsilon,$$

where

$$\Xi_\varepsilon \doteq \{ \xi \in G, \varepsilon(\xi + Y) \subset \Omega \}.$$

The corresponding perforated sets are

$$\widehat{\Omega}_\varepsilon^* \doteq \widehat{\Omega}_\varepsilon \setminus \overline{T}_\varepsilon \quad \text{and} \quad \Lambda_\varepsilon^* \doteq \Omega_\varepsilon^* \setminus \widehat{\Omega}_\varepsilon^*.$$

We decompose the boundary of the perforated domain Ω_ε^* as done, for instance, in [33],[34], [20] and [25], that is

$$\partial\Omega_\varepsilon^* = \Gamma_0^\varepsilon \cup \Gamma_1^\varepsilon \quad \text{where} \quad \Gamma_1^\varepsilon \doteq \partial\widehat{\Omega}_\varepsilon^* \cap \partial T_\varepsilon \quad \text{and} \quad \Gamma_0^\varepsilon \doteq \partial\Omega_\varepsilon^* \setminus \Gamma_1^\varepsilon.$$

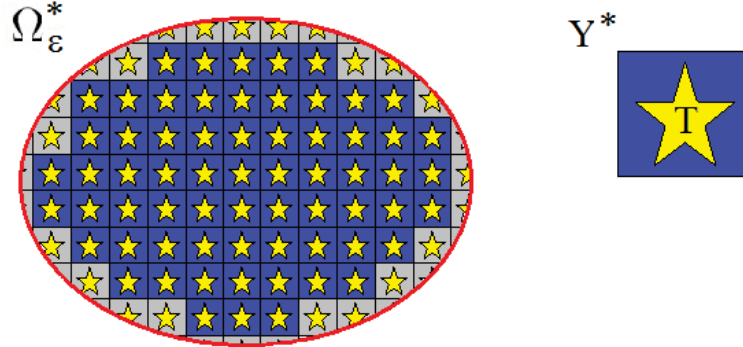


Figure 2.1. The perforated domain Ω_ε^* , $\widehat{\Omega}_\varepsilon^*$, Λ_ε^* and the reference cell Y^*

In the sequel, we also denote by:

- $\mathcal{M}_{Y^*}(v) \doteq \frac{1}{|Y^*|} \int_{Y^*} v(y) dy$ the average of any function $v \in L^1(Y^*)$,
- $\mathcal{M}_{\partial T}(v) \doteq \frac{1}{|\partial T|} \int_{\partial T} v(y) d\sigma_y$ the average of any function $v \in L^1(\partial T)$,
- \sim the zero extension to the whole of Ω of functions defined on Ω_ε^* ,
- ν the unit outward normal to Ω or Y^* ,
- $\theta \doteq \frac{|Y^*|}{|Y|}$ the proportion of the material,

- c different positive constants independent of ε ,
- $\mathcal{M}(\alpha, \beta, Y)$ the set of matrix fields $A = (a_{i,j})_{1 \leq i,j \leq N} \in (L^\infty(Y))^{N \times N}$ such that $(A(x)\lambda, \lambda) \geq \alpha|\lambda|^2$ and $|A(x)\lambda| \leq \beta|\lambda|$, $\forall \lambda \in \mathbb{R}^N$ and a.e. in Y , with $\alpha, \beta \in \mathbb{R}$, $0 < \alpha < \beta$,
- $\{e_1, \dots, e_N\}$ the canonical basis of \mathbb{R}^N .

Furthermore let us recall the classical decomposition for every real function u

$$u = u^+ - u^-, \quad u^+ \doteq \max\{u, 0\} \quad \text{and} \quad u^- \doteq -\min\{u, 0\} \quad \text{a.e. in } \Omega, \quad (2.2)$$

where u^+ and u^- are both nonnegative. We also recall that

$$\chi_{\Omega_\varepsilon^*} \rightharpoonup \theta \quad \text{weakly* in } L^\infty(\Omega), \quad \text{as } \varepsilon \rightarrow 0.$$

DEFINITION 2.1. Let f be a function defined almost everywhere in \mathbb{R}^N and Y given by (2.1). The function f is called Y – *periodic* if and only if

$$f(x + ky_i b_i) = f(x) \quad \text{a.e. in } \mathbb{R}^N, \quad \forall k \in \mathbb{Z} \quad \text{and} \quad \forall i = 1, \dots, N.$$

The problem

Our aim is to study the asymptotic behavior, as ε goes to zero, of the following problem:

$$\begin{cases} -\operatorname{div}(A^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon) = f\zeta(u_\varepsilon) & \text{in } \Omega_\varepsilon^*, \\ u_\varepsilon = 0 & \text{on } \Gamma_0^\varepsilon, \\ (A^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon)\nu + \varepsilon^\gamma \rho_\varepsilon(x)h(u_\varepsilon) = g_\varepsilon & \text{on } \Gamma_1^\varepsilon, \end{cases} \quad (2.3)$$

where a nonlinear Robin condition on the boundary of the holes Γ_1^ε and a homogeneous Dirichlet condition on the exterior boundary Γ_0^ε are prescribed. The matrix field A^ε is defined in (2.4), and the functions ρ_ε and g_ε in (2.5).

Assumptions on the data

Throughout this paper, we make the following hypotheses, where the Y -periodicity is taken in the sense of Definition 2.1:

H₁) The real $N \times N$ matrix field $A : (y, t) \in Y \times \mathbb{R} \mapsto A(y, t) = (a_{i,j}(y, t))_{i,j=1,\dots,N} \in \mathbb{R}^{N^2}$

satisfies the following conditions:

- $$\left\{ \begin{array}{l} \text{i) } A(\cdot, t) \text{ is } Y\text{-periodic for every } t; \\ \text{ii) } A \text{ is a Carathéodory function, i.e.} \\ \quad - A(y, \cdot) \text{ is continuous for a.e. } y \in Y, \\ \quad - A(\cdot, t) \text{ is measurable for every } t \in \mathbb{R}; \\ \text{iii) } A(\cdot, t) \in \mathcal{M}(\alpha, \beta, Y), \text{ for every } t \in \mathbb{R}; \\ \text{iv) there exists a real function } \omega : \mathbb{R} \rightarrow \mathbb{R} \text{ satisfying the following conditions:} \\ \quad - \omega \text{ is continuous and non decreasing, with } \omega(t) > 0 \forall t > 0, \\ \quad - |A(y, t_1) - A(y, t_2)| \leq \omega(|t_1 - t_2|) \text{ for a.e. } y \in Y, \forall t_1 \neq t_2, \\ \quad - \forall s > 0, \lim_{y \rightarrow 0^+} \int_y^s \frac{dt}{\omega(t)} = +\infty. \end{array} \right.$$

H₂) The functions ζ and f verify

- $$\left\{ \begin{array}{l} \text{i) } \zeta : [0, +\infty[\rightarrow [0, +\infty] \text{ is a function such that} \\ \quad \zeta \in \mathcal{C}^0([0, +\infty[), 0 \leq \zeta(s) \leq \frac{1}{s^k} \text{ for every } s \in]0, +\infty[, \text{ with } 0 < k \leq 1; \\ \text{ii) } \zeta \text{ is non increasing;} \\ \text{iii) } f \geq 0 \text{ a.e. in } \Omega, \text{ with } f \in L^l(\Omega), \text{ for } l \geq \frac{2}{1+k} (\geq 1). \end{array} \right.$$

H₃) Either

- $$\text{i) } 0 \leq g \in L^s(\partial T), \text{ with } \mathcal{M}_{\partial T}(g) \neq 0 \text{ and } \begin{cases} s \geq \frac{2(N-1)}{N} & \text{if } N > 2, \\ s > 1 & \text{if } N = 2; \end{cases}$$

or

- $$\text{ii) } g \equiv 0.$$

H₄) Either $f \not\equiv 0$ or $g \not\equiv 0$.

H₅) $\gamma \geq 1$, and ρ is a nonnegative Y -periodic function in $L^\infty(\partial T)$.

H₆) The function h is an increasing and continuously differentiable function such that for some positive constant C and an exponent q one has

$$\left\{ \begin{array}{l} h(0) = 0, \\ |h'(s)| \leq C(1 + |s|^{q-1}), \forall s \in \mathbb{R}, \\ \text{with } 1 \leq q < \infty \text{ if } N = 2, \text{ and } 1 \leq q \leq \frac{N}{N-2} \text{ if } N > 2. \end{array} \right.$$

◇

Under the above assumptions we set, for almost every $x \in \Omega$ and every $t \in \mathbb{R}$,

$$A^\varepsilon(x, t) \doteq A\left(\frac{x}{\varepsilon}, t\right), \quad (2.4)$$

and, for almost every $x \in \Gamma_1^\varepsilon$,

$$g_\varepsilon(x) \doteq \varepsilon g\left(\frac{x}{\varepsilon}\right), \quad \rho_\varepsilon(x) \doteq \rho\left(\frac{x}{\varepsilon}\right). \quad (2.5)$$

Now we introduce the natural framework for the study of problem (2.3), that is the space

$$V_\varepsilon \doteq \{v \in H^1(\Omega_\varepsilon^*) : v = 0 \text{ on } \Gamma_0^\varepsilon\}.$$

REMARK 2.2. It is known (see for instance [35, Lemma 1], [36]) that the Poincaré inequality in V_ε holds with a constant c_P independent of ε , that is

$$\|v\|_{L^2(\Omega_\varepsilon^*)} \leq c_P \|\nabla v\|_{L^2(\Omega_\varepsilon^*)} \quad \forall v \in V_\varepsilon. \quad (2.6)$$

Consequently, the space V_ε can be equipped by the norm

$$\|v\|_{V_\varepsilon} \doteq \|\nabla v\|_{L^2(\Omega_\varepsilon^*)} \quad \forall v \in V_\varepsilon,$$

which is equivalent to the H^1 -norm via a constant independent on ε .

Moreover, the Sobolev embedding theorems also apply to V_ε independently on ε .

The variational formulation associated with problem (2.3) reads

$$\left\{ \begin{array}{l} \text{Find } u_\varepsilon \in V_\varepsilon \text{ such that } u_\varepsilon > 0 \text{ a.e. in } \Omega_\varepsilon^*, \\ \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)\varphi dx < +\infty \text{ and} \\ \int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon \nabla \varphi dx + \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon h(u_\varepsilon)\varphi d\sigma \\ = \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)\varphi dx + \int_{\Gamma_1^\varepsilon} g_\varepsilon \varphi d\sigma, \quad \forall \varphi \in V_\varepsilon. \end{array} \right. \quad (2.7)$$

Let us point out that the Lipschitz-continuous assumption on ∂T is necessary in order to write here the surface integrals on the boundary of the holes.

We recall the following result:

THEOREM 2.3 ([46]). *Under assumptions H_1)- H_6), problem (2.7) admits a unique solution.*

REMARK 2.4. It can be seen from [46, proofs of Theorems 5.2 and 6.2]), that hypotheses $H_1)_{i,ii,iii}$, $H_2)_{i,iii}$, H_3 - H_6) provide the existence of at least a solution of our problem. To ensure the uniqueness of the solution, as usual in the literature, some additional hypotheses on the matrix field A and on the nonlinear function ζ are required. This is why in [46] the uniqueness is proved assuming that the quasilinear term verifies hypothesis $H_1)_{iv}$, which was originally introduced in [23] by M. Chipot for quasilinear nonsingular problems (see also [19]). Concerning the function ζ the monotonicity hypothesis $H_2)_{ii}$ is assumed, as done in [42] and [52].

Let us mention that if A is uniformly Lipschitz-continuous in t with constant L , then $w(t) \doteq Lt$ satisfies the assumption $H_1)_{iv}$.

Let us introduce here, for every fixed $t \in \mathbb{R}$, the homogenized matrix field $A^0(t)$, defined by

$$A^0(t)\lambda \doteq \frac{1}{|Y^*|} \int_{Y^*} A(y, t) \nabla_y \widehat{\omega}_\lambda(y, t) dy \quad \forall \lambda \in \mathbb{R}^N, \quad (2.8)$$

where, for every $t \in \mathbb{R}$ and $\lambda \in \mathbb{R}^N$,

$$\widehat{\omega}_\lambda(y, t) = \lambda \cdot y - \widehat{\chi}_\lambda(y, t)$$

and $\widehat{\chi}_\lambda(y, t)$ is solution of the following problem:

$$\left\{ \begin{array}{ll} -\operatorname{div}(A(\cdot, t) \nabla_y \widehat{\chi}_\lambda(\cdot, t)) = -\operatorname{div}(A(\cdot, t)\lambda) & \text{in } Y^*, \\ A(\cdot, t)(\lambda - \nabla_y \widehat{\chi}_\lambda(\cdot, t))\nu = 0 & \text{on } \partial T, \\ \widehat{\chi}_\lambda(\cdot, t) \text{ Y-periodic,} \\ \frac{1}{|Y^*|} \int_{Y^*} \widehat{\chi}_\lambda(y, t) dy = 0. \end{array} \right. \quad (2.9)$$

The homogenized matrix A^0 is that originally introduced in [35] for linear problems with Neumann conditions in perforated domain, successively extended to quasilinear ones in [5], [6].

We recall (see [20] and [35]) that the matrix A^0 satisfies the following properties:

- i) A^0 is continuous and $A^0(t) \in M\left(\alpha, \frac{\beta^2}{\alpha}, \Omega\right)$ for every $t \in \mathbb{R}$;
- ii) there exists a positive constant C , depending only on α, β, Y and T such that

$$|A^0(t_1) - A^0(t_2)| \leq C\omega(|t_1 - t_2|) \quad (2.10)$$

for every $t_1, t_2 \in \mathbb{R}$, with $t_1 \neq t_2$, where ω is the function given in $H_1)$.

Now we state the main results of this paper. The following one is proved in Section 4:

PROPOSITION 2.5. *Under assumptions H_1)- H_6), let $u_\varepsilon \in V_\varepsilon$ be the unique solution of problem (2.7). Then, there exist a subsequence (still denoted by ε) and two functions $u_0 \in H_0^1(\Omega)$ and $\widehat{u} \in L^2(\Omega; H_{per}^1(Y^*))$ with $\mathcal{M}_{Y^*}(\widehat{u}) = 0$, such that*

$$\left\{ \begin{array}{ll} i) \mathcal{T}_\varepsilon^*(u_\varepsilon) \rightarrow u_0 & \text{strongly in } L^2(\Omega; H^1(Y^*)) \text{ and a.e. in } \Omega \times Y^*, \\ ii) \mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon)) \rightarrow \zeta(u_0) & \text{a.e. in } \Omega \times Y^*, \\ iii) \widetilde{u}_\varepsilon \rightharpoonup \theta u_0 & \text{weakly in } L^2(\Omega), \\ iv) \mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) \rightharpoonup \nabla u_0 + \nabla_y \widehat{u} & \text{weakly in } L^2(\Omega \times Y^*), \\ v) \mathcal{T}_\varepsilon^b(h(u_\varepsilon)) \rightarrow h(u_0) & \text{strongly in } L^t(\Omega; W^{1-\frac{1}{t}, t}(\partial T)), \end{array} \right. \quad (2.11)$$

where

$$t \doteq \begin{cases} \in (1; 2) & \text{if } N = 2 \text{ and } q > 1, \\ \frac{2N}{q(N-2)+2} & \text{otherwise;} \end{cases} \quad (2.12)$$

and q is given by H_6). Moreover,

$$u_0 \geq 0 \text{ a.e. in } \Omega \quad \text{and} \quad \int_\Omega f\zeta(u_0)\varphi dx < +\infty, \quad \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega). \quad (2.13)$$

We also have the result below, which is proved at the end of Section 5.

PROPOSITION 2.6. *Under assumptions H_1)- H_6), let (u_0, \widehat{u}) be given by Proposition 2.5. Then*

$$\widehat{u}(y, x) = - \sum_{i=1}^N \widehat{\chi}_{e_i}(y, u_0(x)) \frac{\partial u_0}{\partial x_i}(x) \in L^2(\Omega; H_{per}^1(Y^*)).$$

REMARK 2.7. The proof of Proposition 2.6 makes use of some results from [25] and [26], where is also considered the case $g \not\equiv 0$ with $\mathcal{M}_{\partial T}(g) = 0$. Here, since we need to assume $g \geq 0$, we can only have the cases $\mathcal{M}_{\partial T}(g) \neq 0$ or $g \equiv 0$.

The homogenization result for problem (2.7) is given by the following theorem, whose proof is presented in Section 6:

THEOREM 2.8. *Under assumptions H_1)- H_6), let $u_\varepsilon \in V_\varepsilon$ be the unique solution of problem (2.7) and (u_0, \widehat{u}) be given by Propositions 2.5-2.6. Then the pair (u_0, \widehat{u}) is the unique solution of the limit equation*

$$\left\{ \begin{array}{l} \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega) \text{ and } \forall \psi \in L^2(\Omega; H_{per}^1(Y^*)) \\ \int_{\Omega \times Y^*} A(y, u_0)(\nabla u_0 + \nabla_y \widehat{u})(\nabla \varphi + \nabla_y \psi) dx dy + |Y|c_\gamma \int_\Omega h(u_0)\varphi dx \\ = |Y^*| \int_\Omega f\zeta(u_0)\varphi dx + |\partial T|\mathcal{M}_{\partial T}(g) \int_\Omega \varphi dx, \end{array} \right. \quad (2.14)$$

where c_γ is defined by

$$c_\gamma \doteq \begin{cases} \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(\rho) & \text{if } \gamma = 1, \\ 0 & \text{if } \gamma > 1. \end{cases} \quad (2.15)$$

Finally, $u_0 > 0$ almost everywhere in Ω and u_0 is the unique solution of the following singular limit problem:

$$\begin{cases} -\operatorname{div}(A^0(u_0)\nabla u_0) + c_\gamma h(u_0) = \theta f\zeta(u_0) + \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g) & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.16)$$

where the homogenized matrix $A^0(t)$ is given by (2.8) and verifies

$$A^0(u_0)\nabla u_0 = \frac{1}{|Y|} \int_{Y^*} A(y, u_0)(\nabla u_0 + \nabla_y \hat{u}) dy. \quad (2.17)$$

Consequently, convergences (2.11) hold for the whole sequence.

Also let us introduce the usual corrector matrix field for perforated domains C^ε (see [41]), defined by

$$C^\varepsilon(\cdot, t) = (C_{i,j}^\varepsilon(\cdot, t))_{i,j=1,\dots,N} \in \mathbb{R}^{N^2}, \quad \text{for every } t \in \mathbb{R},$$

where

$$\begin{cases} C^\varepsilon(x, t) = C\left(\frac{x}{\varepsilon}, t\right) & \text{a.e. in } \Omega_\varepsilon^*, \\ C_{i,j}(y, t) = \frac{\partial \hat{\omega}_j}{\partial y_i}(y, t), \quad i, j = 1, \dots, N & \text{a.e. on } Y^*. \end{cases} \quad (2.18)$$

Finally, we prove the following corrector result, which shows that the corrector of a suitable associated non-singular problem introduced in Section 5 (see (2.33)) is also a corrector for our problem:

THEOREM 2.9. *Under assumptions of Theorem 2.8, let C^ε be defined by (2.18). Then*

$$\lim_{\varepsilon \rightarrow 0} \|\nabla u_\varepsilon - C^\varepsilon(\cdot, u_\varepsilon)\nabla u_0\|_{L^1(\Omega_\varepsilon^*)} = 0.$$

Proof. This result is a direct consequence of Theorem 2.27 in Section 5 and the corrector result for the non-singular case given by [25, Theorem 4.11]. \square

2.3 The periodic unfolding method

As already mentioned in the introduction, the periodic unfolding method was introduced by D. Cioranescu, A. Dalmaian and G. Griso in [28] and [29] for fixed domains and extended to perforated ones in [33]-[32]. In this section, we give a short presentation of the periodic

unfolding operator for perforated domains. We refer to [27] and [29] for detailed proofs of the results presented here.

Let $z \in \mathbb{R}^N$, we denote by $[z]_Y$ its integer part such that $z - [z]_Y$ belongs to Y and set $\{z\}_Y \doteq z - [z]_Y$. Then, for every positive ε ,

$$x = \varepsilon \left(\left[\frac{x}{\varepsilon} \right]_Y + \left\{ \frac{x}{\varepsilon} \right\}_Y \right) \quad \forall x \in \mathbb{R}^N.$$

DEFINITION 2.10. For any Lebesgue-measurable function ϕ on Ω_ε^* , the unfolding operator $\mathcal{T}_\varepsilon^*$ is defined as follows:

$$\mathcal{T}_\varepsilon^*(\phi)(x, y) \doteq \begin{cases} \phi \left(\varepsilon \left[\frac{x}{\varepsilon} \right]_Y + \varepsilon y \right) & \text{a.e. for } (x, y) \in \widehat{\Omega}_\varepsilon \times Y^*, \\ 0 & \text{a.e. for } (x, y) \in \Lambda_\varepsilon \times Y^*. \end{cases} \quad (2.19)$$

This definition makes sense even if ϕ is Lebesgue-measurable on $\widehat{\Omega}_\varepsilon^*$, since we extend it by zero in Λ_ε^* . Observe that, by definition, one has

$$\mathcal{T}_\varepsilon^*(\zeta(\phi))(x, y) = \begin{cases} \zeta(\mathcal{T}_\varepsilon^*(\phi)(x, y)) & \text{a.e. in } \widehat{\Omega}_\varepsilon \times Y^*, \\ 0 & \text{a.e. in } \Lambda_\varepsilon \times Y^*, \end{cases} \quad (2.20)$$

for all measurable functions ϕ on Ω_ε^* and any function ζ satisfying $H_2)_i$. Moreover, the unfolding operator $\mathcal{T}_\varepsilon^*$ has the properties listed below:

PROPOSITION 2.11 ([27][29][33]). *Let $p \in [1, +\infty)$.*

1. $\mathcal{T}_\varepsilon^*$ is a linear and continuous operator from $L^p(\Omega_\varepsilon^*)$ to $L^p(\Omega \times Y^*)$.
2. $\mathcal{T}_\varepsilon^*(\phi\psi) = \mathcal{T}_\varepsilon^*(\phi)\mathcal{T}_\varepsilon^*(\psi)$ for every $\phi, \psi \in L^p(\Omega_\varepsilon^*)$.
3. Let $\phi \in L^p(Y^*)$ be a Y -periodic function and set $\phi_\varepsilon(x) = \phi\left(\frac{x}{\varepsilon}\right)$. Then

$$\mathcal{T}_\varepsilon^*(\phi_\varepsilon)(x, y) = \phi(y) \quad \text{a.e. in } \widehat{\Omega}_\varepsilon \times Y^*.$$

4. For all $\phi \in L^1(\Omega_\varepsilon^*)$, the following **integration formula** holds:

$$\int_{\widehat{\Omega}_\varepsilon^*} \phi(x) dx = \int_{\Omega_\varepsilon^*} \phi(x) dx - \int_{\Lambda_\varepsilon^*} \phi(x) dx = \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(\phi)(x, y) dx dy.$$

5. $\|\mathcal{T}_\varepsilon^*(\phi)\|_{L^p(\Omega \times Y^*)} \leq |Y|^{\frac{1}{p}} \|\phi\|_{L^p(\Omega_\varepsilon^*)}$ for every $\phi \in L^p(\Omega_\varepsilon^*)$.
6. For $\phi \in L^p(\Omega)$,

$$\mathcal{T}_\varepsilon^*(\phi) \rightarrow \phi \quad \text{strongly in } L^p(\Omega \times Y^*).$$

7. Let $\{\phi_\varepsilon\}$ be a sequence in $L^p(\Omega)$ such that $\phi_\varepsilon \rightarrow \phi$ strongly in $L^p(\Omega)$. Then

$$\mathcal{T}_\varepsilon^*(\phi_\varepsilon) \rightarrow \phi \quad \text{strongly in } L^p(\Omega \times Y^*).$$

COROLLARY 2.12 ([29]) (**Unfolding criterion for integrals (u.c.i.)**). Let ϕ_ε be in $L^1(\Omega_\varepsilon^*)$. If

$$\int_{\Lambda_\varepsilon^*} |\phi_\varepsilon| dx \rightarrow 0,$$

then

$$\int_{\Omega_\varepsilon^*} \phi_\varepsilon dx - \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(\phi_\varepsilon) dx dy \rightarrow 0.$$

In the sequel, if $\{\phi_\varepsilon\}$ is a sequence satisfying u.c.i., we write

$$\int_{\Omega_\varepsilon^*} \phi_\varepsilon dx \simeq \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(\phi_\varepsilon) dx dy.$$

PROPOSITION 2.13 ([29]). *The following properties hold:*

- Let $\{\phi_\varepsilon\} \subset L^p(\Omega_\varepsilon^*)$ such that $\|\phi_\varepsilon\|_{L^p(\Omega_\varepsilon^*)} \leq c$, with $p \in (1, +\infty)$, and $\psi \in L^p(\Omega_\varepsilon^*)$, then

$$\int_{\Omega_\varepsilon^*} \phi_\varepsilon \psi dx \simeq \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(\phi_\varepsilon) \mathcal{T}_\varepsilon^*(\psi) dx dy.$$

- Let $\{\phi_\varepsilon\} \subset L^p(\Omega_\varepsilon^*)$ such that $\|\phi_\varepsilon\|_{L^p(\Omega_\varepsilon^*)} \leq c$, with $p \in (1, +\infty)$, and $\{\psi_\varepsilon\}$ a bounded sequence in $L^{p_0}(\Omega_\varepsilon^*)$ with $\frac{1}{p} + \frac{1}{p_0} < 1$. Then

$$\int_{\Omega_\varepsilon^*} \phi_\varepsilon \psi_\varepsilon dx \simeq \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(\phi_\varepsilon) \mathcal{T}_\varepsilon^*(\psi_\varepsilon) dx dy.$$

As a consequence of the previous proposition, the following convergence result holds:

PROPOSITION 2.14 ([27][29][33]). Let $\{\phi_\varepsilon\} \subset L^p(\Omega_\varepsilon^*)$ such that $\|\phi_\varepsilon\|_{L^p(\Omega_\varepsilon^*)} \leq c$, with $p \in [1, +\infty)$. If

$$\mathcal{T}_\varepsilon^*(\phi_\varepsilon) \rightharpoonup \phi \quad \text{weakly in } L^p(\Omega \times Y^*),$$

then

$$\tilde{\phi}_\varepsilon \rightharpoonup \theta \mathcal{M}_{Y^*}(\phi) \quad \text{weakly in } L^p(\Omega).$$

We also recall the definition and some properties of the boundary unfolding operator.

DEFINITION 2.15. For any Lebesgue-measurable function ϕ on $\partial\widehat{\Omega}_\varepsilon^* \cap \partial T_\varepsilon$, the boundary unfolding operator $\mathcal{T}_\varepsilon^b$ is defined as follows:

$$\mathcal{T}_\varepsilon^b(\phi)(x, y) \doteq \begin{cases} \phi\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{a.e. for } (x, y) \in \widehat{\Omega}_\varepsilon \times \partial T, \\ 0 & \text{a.e. for } (x, y) \in \Lambda_\varepsilon \times \partial T. \end{cases} \quad (2.21)$$

PROPOSITION 2.16 ([27][29][33]). *Let $p \in (1, +\infty)$.*

1. $\mathcal{T}_\varepsilon^b$ is a linear and continuous operator from $L^p(\partial\widehat{\Omega}_\varepsilon^* \cap \partial T_\varepsilon)$ to $L^p(\Omega \times \partial T)$.
2. $\mathcal{T}_\varepsilon^b(\phi\psi) = \mathcal{T}_\varepsilon^b(\phi)\mathcal{T}_\varepsilon^b(\psi)$ for every $\phi, \psi \in L^p(\partial\widehat{\Omega}_\varepsilon^* \cap \partial T_\varepsilon)$.
3. Let $\phi \in L^p(\partial T)$ be a Y -periodic function and set $\phi_\varepsilon(x) = \phi(\frac{x}{\varepsilon})$. Then

$$\mathcal{T}_\varepsilon^b(\phi_\varepsilon)(x, y) = \phi(y) \quad \text{a.e. in } \widehat{\Omega}_\varepsilon \times \partial T.$$

4. For all $\phi \in L^1(\partial\widehat{\Omega}_\varepsilon^* \cap \partial T_\varepsilon)$, the **integration formula** is given by

$$\int_{\Gamma_\varepsilon^1} \phi(x) d\sigma_x = \frac{1}{\varepsilon|Y|} \int_{\Omega \times \partial T} \mathcal{T}_\varepsilon^b(\phi)(x, y) dx d\sigma_y.$$

5. $\|\mathcal{T}_\varepsilon^b(\phi)\|_{L^p(\Omega \times \partial T)} \leq \varepsilon^{\frac{1}{p}}|Y|^{\frac{1}{p}}\|\phi\|_{L^p(\partial T_\varepsilon)}$ for every $\phi \in L^p(\partial\widehat{\Omega}_\varepsilon^* \cap \partial T_\varepsilon)$.
6. For $\phi \in L^p(\partial\widehat{\Omega}_\varepsilon^* \cap \partial T_\varepsilon)$,

$$\mathcal{T}_\varepsilon^b(\phi) \rightarrow \phi \quad \text{strongly in } L^p(\Omega \times \partial T).$$

We summarize in the proposition below some results that have been obtained in the literature (we refer to [32], [33] as well to [27] for convergences $i) - iv)$, and to [20] for $v)$).

PROPOSITION 2.17. *Let $\{w_\varepsilon\}$ be a sequence such that $w_\varepsilon \in V_\varepsilon$ and $\|w_\varepsilon\|_{V_\varepsilon} \leq c$. Then, there exist a subsequence (still denoted by ε) and two functions $(w_0, \widehat{w}) \in H_0^1(\Omega) \times L^2(\Omega; H_{per}^1(Y^*))$ with $\mathcal{M}_{Y^*}(\widehat{w}) = 0$, such that*

$$\left\{ \begin{array}{ll} i) \mathcal{T}_\varepsilon^*(w_\varepsilon) \rightarrow w_0 & \text{strongly in } L^2(\Omega; H^1(Y^*)), \\ ii) \mathcal{T}_\varepsilon^*(\nabla w_\varepsilon) \rightharpoonup \nabla w_0 + \nabla_y \widehat{w} & \text{weakly in } L^2(\Omega \times Y^*), \\ iii) \widehat{w}_\varepsilon \rightharpoonup \theta w_0 & \text{weakly in } L^2(\Omega), \\ iv) \mathcal{T}_\varepsilon^b(h(w_\varepsilon)) \rightarrow h(w_0) & \text{strongly in } L^t(\Omega; W^{1-\frac{1}{t}, t}(\partial T)), \\ v) \mathcal{T}_\varepsilon^*(A^\varepsilon(x, w_\varepsilon)) \rightarrow A(y, w_0) & \text{a.e. in } \Omega \times Y^*, \end{array} \right. \quad (2.22)$$

where

$$t \doteq \begin{cases} \in (1; 2) & \text{if } N = 2 \text{ and } q > 1, \\ \frac{2N}{q(N-2)+2} & \text{otherwise;} \end{cases} \quad (2.23)$$

and q given by H_6).

Here, let us just point out that

$$\mathcal{T}_\varepsilon^*(A^\varepsilon(x, u_\varepsilon(x))) = \begin{cases} A(y, \mathcal{T}_\varepsilon^*(u_\varepsilon)(x, y)) & \text{a.e. for } (x, y) \in \widehat{\Omega}_\varepsilon \times Y^*, \\ 0 & \text{a.e. for } (x, y) \in \Lambda_\varepsilon \times Y^*. \end{cases} \quad (2.24)$$

We recall now an important result, which plays an essential role when treating surface integrals containing periodic functions. It was originally proved in [30], then revisited and improved by unfolding in [27] and [33] (see also [25, Propositions 3.7-3.8] for the case $g \in L^s$).

PROPOSITION 2.18 ([25]). *Let g be a function satisfying hypothesis $H_3)_i$. Then, for every $\varphi \in V_\varepsilon$, the following inequality holds:*

$$\left| \int_{\Gamma_1^\varepsilon} g\left(\frac{x}{\varepsilon}\right) \varphi(x) d\sigma \right| \leq \frac{c}{\varepsilon} (|\mathcal{M}_{\partial T}(g)| + \varepsilon) \|\nabla \varphi\|_{L^2(\Omega_\varepsilon^*)}.$$

Furthermore, if $\mathcal{M}_{\partial T}(g) = 0$, then for every $\varphi \in V_\varepsilon$, one has

$$\lim_{\varepsilon \rightarrow 0} \int_{\Gamma_1^\varepsilon} g\left(\frac{x}{\varepsilon}\right) \varphi(x) d\sigma = 0.$$

Moreover, if $\{w_\varepsilon\} \subset V_\varepsilon$ is a sequence such that (2.22)_i holds, then

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Gamma_1^\varepsilon} g\left(\frac{x}{\varepsilon}\right) w_\varepsilon d\sigma = \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g) \int_{\Omega} w_0 dx.$$

2.4 A priori estimates

In the following propositions we establish two a priori estimates for a solution of problem (2.7), which are uniform with respect to ε . Let us point out that in the sequel we can suppose $\varepsilon < 1$ without loss of generality, since ε will tend to zero.

PROPOSITION 2.19. *Under assumptions $H_1)$ - $H_6)$, let $u_\varepsilon \in V_\varepsilon$ be the solution of problem (2.7). The following a priori estimate holds:*

$$\|u_\varepsilon\|_{V_\varepsilon} \leq c, \quad (2.25)$$

where c depends on α , c_P , $\|f\|_{L^1(\Omega)}$ and $\mathcal{M}_{\partial T}(g)$.

Proof. Let $u_\varepsilon \in V_\varepsilon$ be the solution of problem (2.3) and let us choose u_ε as test function in its variational formulation (2.7). The same computation made in the proof of the [46, Proposition 3.1], together with the nonnegativity of ε and γ , gives

$$\begin{aligned} \alpha \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon^*)}^2 &\leq \int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla u_\varepsilon \nabla u_\varepsilon dx + \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon h(u_\varepsilon) u_\varepsilon d\sigma \\ &\leq \int_{\Omega_\varepsilon^*} f u_\varepsilon^{1-k} dx + \int_{\Gamma_1^\varepsilon} g_\varepsilon u_\varepsilon d\sigma. \end{aligned}$$

Applying the Young inequality with exponents $\frac{2}{1-k}$ and $\frac{2}{1+k}$, we have for every $\eta_1 > 0$,

$$f u_\varepsilon^{1-k} \leq c(\eta_1) f^{\frac{2}{1+k}} + \eta_1 u_\varepsilon^2.$$

This implies, in view of the Poincaré inequality (2.6), that

$$\alpha \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon^*)}^2 \leq \eta_1 c_P^2 \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon^*)}^2 + c(\eta_1) \|f\|_{L^{\frac{2}{1+k}}(\Omega)}^{\frac{2}{1+k}} + \int_{\Gamma_1^\varepsilon} g_\varepsilon u_\varepsilon d\sigma,$$

where c_P is the Poincaré constant in V_ε . Moreover, in view of the definition of g_ε and Proposition 2.18, for the Young inequality we have

$$\int_{\Gamma_1^\varepsilon} g_\varepsilon u_\varepsilon d\sigma \leq c(|\mathcal{M}_{\partial T}(g)| + \varepsilon) \|u_\varepsilon\|_{V_\varepsilon} \leq \eta_2 \|u_\varepsilon\|_{V_\varepsilon}^2 + c(\eta_2),$$

independently on ε . Whence, choosing η_1 and η_2 sufficiently small so that $\alpha - \eta_1 c_P^2 - \eta_2 > 0$, we deduce that there exists a constant c , independent on ε , such that

$$\|u_\varepsilon\|_{V_\varepsilon} \leq c,$$

where c depends on α , c_P , $\|f\|_{L^1(\Omega)}$ and $\mathcal{M}_{\partial T}(g)$. \square

In the next a priori estimate, we will use the following property that we prove for the reader's convenience:

LEMMA 2.20. *Let Ω_1 be a bounded open set in \mathbb{R}^N with $\partial\Omega_1$ Lipschitz-continuous. If u belongs to $H^1(\Omega_1) \cap L^\infty(\Omega_1)$, then $u \in L^\infty(\partial\Omega_1)$.*

Proof. Set $k := \|u\|_{L^\infty(\Omega_1)}$ and take $u_n \in \mathcal{D}(\Omega_1)$ such that

$$u_n \rightarrow u \quad \text{strongly in } H^1(\Omega_1).$$

Then, by known results, if T_k denotes the truncation function at level k , we have

$$T_k(u_n) \rightarrow T_k(u) = u \quad \text{strongly in } H^1(\Omega_1),$$

as $n \rightarrow +\infty$. Hence

$$T_k(u_n) \rightarrow u \quad \text{strongly in } L^2(\partial\Omega_1),$$

and, up to a subsequence, almost everywhere on $\partial\Omega_1$. Since $|T_k(u_n)| \leq k$, this gives

$$u \leq k \quad \text{a.e. on } \partial\Omega_1,$$

which implies the result. \square

PROPOSITION 2.21. *Under assumptions $H_1)$ - $H_6)$, let $u_\varepsilon \in V_\varepsilon$ be the solution of problem (2.7). Then, up to a subsequence,*

$$\|f\zeta(u_\varepsilon)\varphi\|_{L^1(\Omega_\varepsilon^*)} \leq c, \quad (2.26)$$

for every nonnegative $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with c depending on $\alpha, \beta, c_P, |Y|, \|\rho\|_{L^\infty(\partial T)}, \|\nabla\varphi\|_{L^2(\Omega)}, \|f\|_{L^1(\Omega)}$ and $\mathcal{M}_{\partial T}(g)$.

Proof. Let $u_\varepsilon \in V_\varepsilon$ be the solution of problem (2.7) and let us choose a nonnegative function $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function. Since f, ζ, g and φ are nonnegative, using $H_1)$ together with the Hölder inequality, we have

$$\begin{aligned} 0 &\leq \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)\varphi dx = \int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon\nabla\varphi dx + \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon h(u_\varepsilon)\varphi d\sigma - \int_{\Gamma_1^\varepsilon} g_\varepsilon\varphi d\sigma \\ &\leq \beta\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon^*)}\|\nabla\varphi\|_{L^2(\Omega_\varepsilon^*)} + \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon h(u_\varepsilon)\varphi d\sigma. \end{aligned}$$

As far as it concerns the surface integral, thanks to the properties of the boundary unfolding operator (see Proposition 2.16) and $H_5)$, we get

$$\begin{aligned} \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon h(u_\varepsilon)\varphi d\sigma &= \frac{\varepsilon^{\gamma-1}}{|Y|} \int_{\Omega \times \partial T} \mathcal{T}_\varepsilon^b(\rho_\varepsilon)\mathcal{T}_\varepsilon^b(h(u_\varepsilon))\mathcal{T}_\varepsilon^b(\varphi) dx d\sigma_y \\ &= \frac{\varepsilon^{\gamma-1}}{|Y|} \int_{\Omega \times \partial T} \rho(y)\mathcal{T}_\varepsilon^b(h(u_\varepsilon))\mathcal{T}_\varepsilon^b(\varphi) dx d\sigma_y \\ &\leq \frac{\varepsilon^{\gamma-1}}{|Y|} \|\rho\|_{L^\infty(\partial T)} \int_{\Omega \times \partial T} \mathcal{T}_\varepsilon^b(h(u_\varepsilon))\mathcal{T}_\varepsilon^b(\varphi) dx d\sigma_y. \end{aligned} \quad (2.27)$$

In order to estimate the right-hand side of (2.27), let t be a real number verifying (2.23). Thanks to a priori estimate (2.25), Proposition 2.17 holds true; so that, up to a subsequence, $\mathcal{T}_\varepsilon^b(h(u_\varepsilon))$ is bounded in $L^t(\Omega \times \partial T)$. In addition, by using the assumption on φ , Lemma 2.20 and Proposition 2.16₆ one has the boundedness of $\mathcal{T}_\varepsilon^b(\varphi)$ in $L^{t'}(\Omega \times \partial T)$, where t' is the conjugate of t . Applying the Hölder inequality we obtain

$$\begin{aligned} &\frac{\varepsilon^{\gamma-1}}{|Y|} \|\rho\|_{L^\infty(\partial T)} \int_{\Omega \times \partial T} \mathcal{T}_\varepsilon^b(h(u_\varepsilon))\mathcal{T}_\varepsilon^b(\varphi) dx d\sigma_y \\ &\leq \frac{\varepsilon^{\gamma-1}}{|Y|} \|\rho\|_{L^\infty(\partial T)} \|\mathcal{T}_\varepsilon^b(h(u_\varepsilon))\|_{L^t(\Omega \times \partial T)} \|\mathcal{T}_\varepsilon^b(\varphi)\|_{L^{t'}(\Omega \times \partial T)}. \end{aligned}$$

This, together with (2.25) and hypothesis $H_5)$, allows to state that

$$0 \leq \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)\varphi dx \leq c,$$

where the constant c depends on α , β , c_P , $|Y|$, $\|\rho\|_{L^\infty(\partial T)}$, $\|\nabla\varphi\|_{L^2(\Omega)}$, $\|f\|_{L^1(\Omega)}$ and $\mathcal{M}_{\partial T}(g)$. \square

The following result gives an estimate of the integral of the singular term close to the singular set $\{u_\varepsilon = 0\}$. It is inspired by arguments used in [51], [52] and [42], which present a similar singular term. The estimate follows immediately from the a priori estimate [46, Proposition 3.3] in view of [59, Lemma 2.7].

PROPOSITION 2.22 ([46]). *Under assumptions $H_1)$ - $H_6)$, let $u_\varepsilon \in V_\varepsilon$ be the solution of problem (2.7) and δ a fixed positive real number. Then,*

$$\int_{\{0 < u_\varepsilon \leq \delta\}} f\zeta(u_\varepsilon)\varphi dx \leq \int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon\nabla\varphi Z_\delta(u_\varepsilon)dx + c\varepsilon^{\gamma-\frac{1}{2}}h(2\delta)\|\rho\|_{L^\infty(\partial T)}\|\varphi\|_{V_\varepsilon},$$

for every $\varphi \in V_\varepsilon$, $\varphi \geq 0$ and c independent on ε , where Z_δ is an auxiliary function defined by

$$Z_\delta(s) = \begin{cases} 1, & \text{if } 0 \leq s \leq \delta, \\ -\frac{s}{\delta} + 2, & \text{if } \delta \leq s \leq 2\delta, \\ 0, & \text{if } s \geq 2\delta. \end{cases}$$

The estimate given in Proposition 2.21 allows us to prove Proposition 2.5.

Proof of Proposition 2.5. Let u_ε be the solution of problem (2.7). Proposition 2.19 allows us to apply Proposition 2.17 which provides the existence of $u_0 \in H_0^1(\Omega)$ and $\widehat{u} \in L^2(\Omega; H_{per}^1(Y^*))$ with $\mathcal{M}_{Y^*}(\widehat{u}) = 0$ such that, up to a subsequence, one has convergences (2.11)_i, (2.11)_{iii}-(2.11)_v, where $\mathcal{T}_\varepsilon^*$ is the unfolding operator defined by (2.19).

To prove (2.11)_{ii}, observe that, by construction, for every $x \in \Omega$ there exists $\varepsilon_x > 0$ such that

$$x \in \widehat{\Omega}_\varepsilon, \quad \forall \varepsilon \leq \varepsilon_x.$$

Consequently, since (2.20) holds true, for almost every $(x, y) \in \Omega \times Y^*$, there exists $\varepsilon_x > 0$ such that

$$\mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon)) = \zeta(\mathcal{T}_\varepsilon^*(u_\varepsilon)), \quad \forall \varepsilon \leq \varepsilon_x. \quad (2.28)$$

On the other hand, using the continuity of ζ and (2.11)_i, we have

$$\zeta(\mathcal{T}_\varepsilon^*(u_\varepsilon)) \rightarrow \zeta(u_0) \quad \text{a.e. in } \Omega \times Y^*. \quad (2.29)$$

This, together with (2.28), gives (2.11)_{ii}.

To show that u_0 is nonnegative almost everywhere in Ω , we note that every solution u_ε is positive almost everywhere in Ω_ε^* , by Theorem 2.3. Then, the definition of the unfolding operator implies $\mathcal{T}_\varepsilon^*(u_\varepsilon) \geq 0$ almost everywhere in $\Omega \times Y^*$ so that, in view of (2.11)_i,

$$u_0 \geq 0 \quad \text{a.e. in } \Omega.$$

It remains to prove the second condition in (2.13). Let us choose first a nonnegative $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Propositions 2.11_{2,4} and 2.21, for the subsequence mentioned before, lead to

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon)) \mathcal{T}_\varepsilon^*(\varphi) dx dy \leq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon^*} f \zeta(u_\varepsilon) \varphi dx < +\infty. \quad (2.30)$$

Now, from Proposition 2.11₆, $\mathcal{T}_\varepsilon^*(f)$ and $\mathcal{T}_\varepsilon^*(\varphi)$ converge to f and φ , respectively, almost everywhere in $\Omega \times Y^*$, up to a subsequence. Thus, by (2.11)_{ii},

$$\mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon)) \mathcal{T}_\varepsilon^*(\varphi) \rightarrow f \zeta(u_0) \varphi \quad \text{a.e. in } \Omega \times Y^*.$$

Since $\mathcal{T}_\varepsilon^*(f)$, $\mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon))$ and $\mathcal{T}_\varepsilon^*(\varphi)$ are nonnegative functions, we can use Fatou's lemma and (2.30) to obtain

$$\frac{1}{|Y|} \int_{\Omega \times Y^*} f \zeta(u_0) \varphi dx dy \leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon)) \mathcal{T}_\varepsilon^*(\varphi) dx dy < +\infty.$$

Being the functions f and u_0 independent on y , this implies in particular that

$$\theta \int_{\Omega} f \zeta(u_0) \varphi dx < +\infty$$

and ends the proof for $\varphi \geq 0$. For φ with any sign, it suffices to decompose it as in (2.2). \square

2.5 A crucial auxiliary result

In this section we state and prove the convergence result given in Theorem 2.27, which is our main tool when proving Theorem 2.8. Let u_0 a weak cluster point of the sequence $\{u_\varepsilon\}$. Then, Theorem 2.27 shows that the gradient of our solution u_ε is equivalent to the gradient of the solution v_ε of a suitable linear problem, (2.35), associated with u_0 . This idea was originally introduced in [14] (see also [15]) in the homogenization of some nonlinear problem with quadratic growth. We refer to [41] and [26] for the case of perforated domains. We also refer to [42] for the case of a singular nonlinearity verifying $H_2)_i$, in a two-component domain with a oscillating interface. Here we adapt some techniques from [26] and [42], taking into account the homogenization results proved in [25].

In order to prove Theorem 2.27, the following preliminary result is needed (see [25] and [26, Remark 3.1]).

PROPOSITION 2.23 ([26]). *There exists a linear operator $\mathcal{L}_\varepsilon : H^{-1}(\Omega) \rightarrow V'_\varepsilon$ satisfying the following condition:*

If $\{\varphi_\varepsilon\}$ is a sequence such that

$$\|\varphi_\varepsilon\|_{V_\varepsilon} \leq c \quad \text{and} \quad \widetilde{\varphi}_\varepsilon \rightharpoonup \theta \varphi_0 \quad \text{weakly in } L^2(\Omega), \quad (2.31)$$

then

$$\lim_{\varepsilon \rightarrow 0} \langle \mathcal{L}_\varepsilon(Z), \varphi_\varepsilon \rangle_{V'_\varepsilon, V_\varepsilon} = \langle Z, \varphi_0 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \quad (2.32)$$

Notice that, see [26, Remark 3.1], if (2.31) holds, one has $\varphi_0 \in H_0^1(\Omega)$.

REMARK 2.24. Let us point out that our sequence $\{u_\varepsilon\}$ satisfies (2.31), thanks to Proposition 2.19 and (2.11)_{iii}.

Here, as in [26], the suitable linear problem associated with problem (2.3) is

$$\begin{cases} -\operatorname{div}(A^\varepsilon(x, u_\varepsilon) \nabla v_\varepsilon) = \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_0) \nabla u_0) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g)) & \text{in } \Omega_\varepsilon^*, \\ v_\varepsilon = 0 & \text{on } \Gamma_0^\varepsilon, \\ (A^\varepsilon(x, u_\varepsilon) \nabla v_\varepsilon) \nu + \varepsilon^\gamma \rho_\varepsilon(x) h(u_\varepsilon) = g_\varepsilon & \text{on } \Gamma_1^\varepsilon, \end{cases} \quad (2.33)$$

where c_γ is defined by (2.15) and whose homogenization has been studied in [25]. One can easily check (see [26]), using this homogenization result, that

$$\begin{cases} -\operatorname{div}(A^0(u_0) \nabla v_0) = -\operatorname{div}(A^0(u_0) \nabla u_0) & \text{in } \Omega, \\ v_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, the uniqueness of this problem gives the following result stated in [26]:

LEMMA 2.25 ([26]). *Under the assumptions of Proposition 2.5, let v_ε be the solution of problem (2.35). Then, v_ε satisfies*

$$\|v_\varepsilon\|_{V_\varepsilon} \leq c \quad \text{and} \quad \tilde{v}_\varepsilon \rightharpoonup \theta u_0 \quad \text{weakly in } L^2(\Omega), \quad (2.34)$$

with u_0 given by Proposition 2.5.

Hence, the gradient of the solution v_ε of problem (2.33) appears to be a natural candidate in order to show its equivalence (in the L^2 -norm) to the gradient of our solution u_ε .

The variational formulation of problem (2.33) is

$$\begin{aligned} & \int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla v_\varepsilon \nabla \varphi dx + \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon h(u_\varepsilon) \varphi d\sigma \\ &= \int_{\Gamma_1^\varepsilon} g_\varepsilon \varphi d\sigma + \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_0) \nabla u_0) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g)), \varphi \rangle_{V'_\varepsilon, V_\varepsilon}, \quad \forall \varphi \in V_\varepsilon. \end{aligned} \quad (2.35)$$

Unlike the problem treated in [26], under our assumptions, the functions v_ε are not necessarily bounded. Hence, as in [42], we define the following auxiliary functions u_m :

$$\forall m \in \mathbb{N}, m \geq 1, \quad u_m \doteq T_m(u_0),$$

where T_m is the usual truncation function at level m , so that

$$0 \leq u_m \leq u_0 \quad \text{and} \quad u_m \rightarrow u_0 \quad \text{strongly in } H_0^1(\Omega), \quad \text{as } m \rightarrow +\infty. \quad (2.36)$$

Then, we define by v_ε^m the solution of the following problem:

$$\begin{cases} -\operatorname{div}(A^\varepsilon(x, u_\varepsilon)\nabla v_\varepsilon^m) = \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_0)\nabla u_m) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)) & \text{in } \Omega_\varepsilon^*, \\ v_\varepsilon^m = 0 & \text{on } \Gamma_0^\varepsilon, \\ (A^\varepsilon(x, u_\varepsilon)\nabla v_\varepsilon^m)\nu + \varepsilon^\gamma \rho_\varepsilon(x)h(u_\varepsilon) = g_\varepsilon & \text{on } \Gamma_1^\varepsilon, \end{cases}$$

whose variational formulation is

$$\begin{aligned} & \int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon)\nabla v_\varepsilon^m\nabla\varphi dx + \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon h(u_\varepsilon)\varphi d\sigma \\ &= \int_{\Gamma_1^\varepsilon} g_\varepsilon\varphi d\sigma + \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_0)\nabla u_m) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)), \varphi \rangle_{V_\varepsilon', V_\varepsilon}, \quad \forall \varphi \in V_\varepsilon. \end{aligned} \quad (2.37)$$

The existence and the uniqueness of such a solution $v_\varepsilon^m \in V_\varepsilon$ is straightforward proved by using the Lax-Milgram theorem. Also, from [25, Corollary 4.6] written for $z_\varepsilon = u_\varepsilon$ and for $Z = -\operatorname{div}(A^0(u_0)\nabla u_m) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)$, one has, for every fixed m ,

$$\|v_\varepsilon^m\|_{V_\varepsilon} \leq c \quad \text{and} \quad \widetilde{v_\varepsilon^m} \rightharpoonup \theta u_m \quad \text{weakly in } L^2(\Omega), \quad (2.38)$$

with c independent on m and ε . Hence, v_ε^m satisfies (2.31) and, from Proposition 2.17, the following convergence holds true, up to a subsequence:

$$\mathcal{T}_\varepsilon^*(v_\varepsilon^m) \rightarrow u_m \quad \text{strongly in } L^2(\Omega; H^1(Y^*)). \quad (2.39)$$

Moreover, by classical results from [69], we have that for every fixed m

$$\|v_\varepsilon^m\|_{L^\infty(\Omega_\varepsilon^*)} \leq c_m, \quad \text{for every } \varepsilon. \quad (2.40)$$

LEMMA 2.26. *The sequence $\{(v_\varepsilon^m)^-\}$ satisfies conditions (2.31). Moreover*

$$\mathcal{T}_\varepsilon^b((v_\varepsilon^m)^-) \rightharpoonup 0 \quad \text{weakly in } L^r(\Omega \times \partial T), \quad \forall r \geq 1. \quad (2.41)$$

Proof. In view of the estimate in (2.38), one has

$$\|(v_\varepsilon^m)^-\|_{V_\varepsilon} \leq \|v_\varepsilon^m\|_{V_\varepsilon} \leq c.$$

Hence, Proposition 2.17 implies

$$\mathcal{T}_\varepsilon^*((v_\varepsilon^m)^-) \rightarrow u_m^- = 0 \quad \text{strongly in } L^2(\Omega; H^1(Y^*)), \quad (2.42)$$

since $u_m \geq 0$ and (2.39) implies convergence (2.42) in $L^2(\Omega \times Y^*)$. Also, by using Proposition 2.14, one has

$$\widetilde{(v_\varepsilon^m)^-} \rightharpoonup \theta \mathcal{M}_{Y^*}(u_m^-) = 0 \quad \text{weakly in } L^2(\Omega).$$

Moreover, from (2.42) we get in particular

$$\mathcal{T}_\varepsilon^*((v_\varepsilon^m)^-) \rightharpoonup u_m^- = 0 \quad \text{weakly in } L^2(\Omega \times \partial T),$$

and the convergence (2.41) derives from (2.40) and Lemma 2.20. \square

We can now state the main result of this section.

THEOREM 2.27. *Under assumptions $H_1)$ - $H_6)$, let u_ε and v_ε be solutions of problems (2.7) and (2.35), respectively. Then, for the subsequence verifying (2.11),*

$$\lim_{\varepsilon \rightarrow 0} \|\nabla u_\varepsilon - \nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon^*)} = 0.$$

Proof. The proof is done in 3 steps. It makes use of some ideas from [42] concerning the singular term, split into two terms: one near the singularity and one far from it which results not singular.

Step 1. Let us prove that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon^*} [(v_\varepsilon^m)^-]^2 dx = 0. \quad (2.43)$$

Observe that, from the Poincaré inequality (2.6),

$$0 \leq \int_{\Omega_\varepsilon^*} [(v_\varepsilon^m)^-]^2 dx \leq c_P^2 \int_{\Omega_\varepsilon^*} |\nabla (v_\varepsilon^m)^-|^2 dx, \quad (2.44)$$

for any fixed $m \geq 1$. Then it suffices to prove that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon^*} |\nabla (v_\varepsilon^m)^-|^2 dx = 0 \quad \forall m \geq 1. \quad (2.45)$$

Let us choose $-(v_\varepsilon^m)^- \in V_\varepsilon$ as test function in (2.37) getting

$$\begin{aligned} & - \int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla v_\varepsilon^m \nabla (v_\varepsilon^m)^- dx - \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon h(u_\varepsilon) (v_\varepsilon^m)^- d\sigma \\ & = - \int_{\Gamma_1^\varepsilon} g_\varepsilon (v_\varepsilon^m)^- d\sigma + \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_0) \nabla u_m) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g)), -(v_\varepsilon^m)^- \rangle_{V_\varepsilon', V_\varepsilon}. \end{aligned}$$

From the ellipticity of A and observing that $\nabla v_\varepsilon^m = \nabla (v_\varepsilon^m)^+ - \nabla (v_\varepsilon^m)^-$ and

$$\int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla (v_\varepsilon^m)^+ \nabla (v_\varepsilon^m)^- dx = 0,$$

we have

$$\begin{aligned}
 \alpha \|\nabla(v_\varepsilon^m)^-\|_{L^2(\Omega_\varepsilon^*)}^2 &\leq \int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla(v_\varepsilon^m)^- \nabla(v_\varepsilon^m)^- dx + \int_{\Gamma_1^\varepsilon} g_\varepsilon(v_\varepsilon^m)^- d\sigma \\
 &= \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon h(u_\varepsilon)(v_\varepsilon^m)^- d\sigma + \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_0)\nabla u_m) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g)), -(v_\varepsilon^m)^- \rangle_{V_\varepsilon', V_\varepsilon}.
 \end{aligned} \tag{2.46}$$

From Lemma 2.26 and Proposition 2.23, we get

$$\begin{aligned}
 \lim_{\varepsilon \rightarrow 0} \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_0)\nabla u_m) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g)), -(v_\varepsilon^m)^- \rangle_{V_\varepsilon', V_\varepsilon} \\
 = \langle (-\operatorname{div}(A^0(u_0)\nabla u_m) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g)), 0 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0.
 \end{aligned} \tag{2.47}$$

Also, by using Proposition 2.16_{2,3,4}, (2.11)_v and (2.41) of Lemma 2.26, we get

$$\begin{aligned}
 0 &\leq \lim_{\varepsilon \rightarrow 0} \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon h(u_\varepsilon)(v_\varepsilon^m)^- d\sigma = \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^{\gamma-1}}{|Y|} \int_{\Omega \times \partial T} \rho(y) \mathcal{T}_\varepsilon^b(h(u_\varepsilon)) \mathcal{T}_\varepsilon^b((v_\varepsilon^m)^-) dx d\sigma_y \\
 &\leq \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^{\gamma-1}}{|Y|} \|\rho\|_{L^\infty(\partial T)} \int_{\Omega \times \partial T} \mathcal{T}_\varepsilon^b(h(u_\varepsilon)) \mathcal{T}_\varepsilon^b((v_\varepsilon^m)^-) dx d\sigma_y = 0,
 \end{aligned}$$

which, together with (2.46)-(2.47), gives (2.45).

Step 2. In this step we show that

$$\lim_{m \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon^*} |\nabla(u_\varepsilon - v_\varepsilon^m)|^2 dx = 0. \tag{2.48}$$

To do that, let us choose $u_\varepsilon - v_\varepsilon^m \in V_\varepsilon$ as test function in (2.7) and (2.37). By subtraction and $H_1)_{iii}$, one has

$$\begin{aligned}
 \alpha \|\nabla(u_\varepsilon - v_\varepsilon^m)\|_{L^2(\Omega_\varepsilon^*)}^2 &\leq \int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla(u_\varepsilon - v_\varepsilon^m) \nabla(u_\varepsilon - v_\varepsilon^m) dx = \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)(u_\varepsilon - v_\varepsilon^m) dx \\
 &\quad - \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_0)\nabla u_m) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g)), u_\varepsilon - v_\varepsilon^m \rangle_{V_\varepsilon', V_\varepsilon}.
 \end{aligned} \tag{2.49}$$

First we pass to the limit as $\varepsilon \rightarrow 0$, and then as $m \rightarrow +\infty$. Let us prove that, as $\varepsilon \rightarrow 0$, we have

$$\begin{aligned}
 \alpha \limsup_{\varepsilon \rightarrow 0} \|\nabla(u_\varepsilon - v_\varepsilon^m)\|_{L^2(\Omega_\varepsilon^*)}^2 &\leq \theta \int_{\Omega} f\zeta(u_0)(u_0 - u_m) \chi_{\{u_0 > 0\}} dx \\
 &\quad - \langle (-\operatorname{div}(A^0(u_0)\nabla u_m) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g)), u_0 - u_m \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.
 \end{aligned} \tag{2.50}$$

Concerning the second term in the right-hand side of (2.49), let us observe that $u_\varepsilon - v_\varepsilon^m$ satisfies (2.31), from Remark 2.24 and (2.38). So that, by Proposition 2.23

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_0)\nabla u_m) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g)), u_\varepsilon - v_\varepsilon^m \rangle_{V'_\varepsilon, V_\varepsilon} \\ = \langle (-\operatorname{div}(A^0(u_0)\nabla u_m) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g)), u_0 - u_m \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \end{aligned} \quad (2.51)$$

Let $\delta > 0$. We split the integral of the singular term in (2.49) into two terms and write

$$\begin{aligned} \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)(u_\varepsilon - v_\varepsilon^m) dx &= \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)(u_\varepsilon - (v_\varepsilon^m)^+) dx + \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)(v_\varepsilon^m)^- dx \\ &\leq I_\varepsilon^\delta + J_\varepsilon^\delta + K_\varepsilon, \end{aligned} \quad (2.52)$$

where

$$\begin{aligned} I_\varepsilon^\delta &\doteq \int_{\Omega_\varepsilon^* \cap \{0 < u_\varepsilon \leq \delta\}} f\zeta(u_\varepsilon)u_\varepsilon dx, & J_\varepsilon^\delta &\doteq \int_{\Omega_\varepsilon^* \cap \{u_\varepsilon > \delta\}} f\zeta(u_\varepsilon)(u_\varepsilon - (v_\varepsilon^m)^+) dx \\ \text{and } K_\varepsilon &\doteq \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)(v_\varepsilon^m)^- dx. \end{aligned}$$

We will compute separately these three limits.

Limit of I_ε^δ : When $k < 1$ we get

$$0 \leq I_\varepsilon^\delta \leq \int_{\Omega_\varepsilon^* \cap \{0 < u_\varepsilon \leq \delta\}} f u_\varepsilon^{1-k} dx \leq \delta^{1-k} \int_{\Omega} f \chi_{\{0 < u_\varepsilon \leq \delta\}} dx \leq c\delta^{1-k},$$

since the functions involved are all nonnegative, $f \in L^1(\Omega)$ and for the growth conditions on ζ near its singularity (see H_2);). Consequently,

$$\lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} I_\varepsilon^\delta = 0. \quad (2.53)$$

While in the case $k = 1$, one has

$$0 \leq I_\varepsilon^\delta \leq \int_{\Omega_\varepsilon^* \cap \{0 < u_\varepsilon \leq \delta\}} f \frac{1}{u_\varepsilon} u_\varepsilon dx = \int_{\Omega_\varepsilon^*} f \chi_{\{0 < u_\varepsilon \leq \delta\}} \chi_{\{u_0 \neq \delta\}} dx + \int_{\Omega_\varepsilon^*} f \chi_{\{0 < u_\varepsilon \leq \delta\}} \chi_{\{u_0 = \delta\}} dx. \quad (2.54)$$

Observe that

$$\int_{\Omega_\varepsilon^*} f \chi_{\{0 < u_\varepsilon \leq \delta\}} \chi_{\{u_0 = \delta\}} dx = 0, \quad (2.55)$$

for every $\delta \in \mathbb{R}^+ \setminus D$, where D is the countable set given by

$$D = \{\delta \in \mathbb{R}^+ : |\{(x, y) \in \Omega \times Y^* : u_0(x) = \delta\}| > 0\} \quad (2.56)$$

(see for instance [53], [42] and [46]).

In what follows, we take $\delta \in \mathbb{R}^+ \setminus D$. In order to study the first term in the right-hand side, we make use of the periodic unfolding method. From Proposition 2.11_{2,4} one has

$$\begin{aligned}
 & \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon^*} f \chi_{\{0 < u_\varepsilon \leq \delta\}} \chi_{\{u_0 \neq \delta\}} dx \\
 &= \lim_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(\chi_{\{0 < u_\varepsilon \leq \delta\}}) \mathcal{T}_\varepsilon^*(\chi_{\{u_0 \neq \delta\}}) dx dy + \lim_{\varepsilon \rightarrow 0} \int_{\Lambda_\varepsilon^*} f \chi_{\{0 < u_\varepsilon \leq \delta\}} \chi_{\{u_0 \neq \delta\}} dx \\
 &= \lim_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(\chi_{\{0 < u_\varepsilon \leq \delta\}}) \mathcal{T}_\varepsilon^*(\chi_{\{u_0 \neq \delta\}}) dx dy,
 \end{aligned} \tag{2.57}$$

since $f \in L^1(\Omega)$ and $|\Lambda_\varepsilon| = 0$ when $\varepsilon \rightarrow 0$. Moreover, by definition of the unfolding operator, one has

$$\mathcal{T}_\varepsilon^*(\chi_{\{0 < u_\varepsilon \leq \delta\}})(x, y) = \begin{cases} \chi_{\{0 < \mathcal{T}_\varepsilon^*(u_\varepsilon)(x, y) \leq \delta\}} & \text{a.e. in } \widehat{\Omega}_\varepsilon \times Y^*, \\ 0 & \text{a.e. in } \Lambda_\varepsilon \times Y^*, \end{cases}$$

and, in view of (2.11)_i, up to a subsequence,

$$\mathcal{T}_\varepsilon^*(\chi_{\{0 < u_\varepsilon \leq \delta\}}) \rightarrow \chi_{\{0 < u_0 \leq \delta\}} \quad \text{a.e. in } \Omega \times Y^*. \tag{2.58}$$

Consequently, by Proposition 2.11₆, up to a subsequence,

$$\mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(\chi_{\{0 < u_\varepsilon \leq \delta\}}) \mathcal{T}_\varepsilon^*(\chi_{\{u_0 \neq \delta\}}) \rightarrow f \chi_{\{0 < u_0 \leq \delta\}} \chi_{\{u_0 \neq \delta\}} = f \chi_{\{0 < u_0 < \delta\}} \quad \text{a.e. in } \Omega \times Y^*. \tag{2.59}$$

Moreover from Proposition 2.11₅ we have

$$\int_{E \times Y^*} \mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(\chi_{\{0 < u_\varepsilon \leq \delta\}}) \mathcal{T}_\varepsilon^*(\chi_{\{u_0 \neq \delta\}}) dx dy \leq \int_{E \times Y^*} \mathcal{T}_\varepsilon^*(f) dx dy \leq |Y| \|f\|_{L^1(E)},$$

for any measurable set $E \times Y^* \subset \Omega \times Y^*$. Via the absolute continuity of the Lebesgue integral and applying the Vitali theorem, from (2.59) we obtain

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(\chi_{\{0 < u_\varepsilon \leq \delta\}}) \mathcal{T}_\varepsilon^*(\chi_{\{u_0 \neq \delta\}}) dx dy = \frac{1}{|Y|} \int_{\Omega \times Y^*} f \chi_{\{0 < u_0 < \delta\}} dx dy.$$

Then, since $\delta \in \mathbb{R}^+ \setminus D$, by (2.54), (2.55) and (2.57),

$$0 \leq \lim_{\varepsilon \rightarrow 0} I_\varepsilon^\delta \leq \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon^*} f \chi_{\{0 < u_\varepsilon \leq \delta\}} \chi_{\{u_0 \neq \delta\}} dx = \theta \int_{\Omega} f \chi_{\{0 < u_0 < \delta\}} dx.$$

Using here the Lebesgue theorem,

$$0 \leq \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} I_\varepsilon^\delta \leq \lim_{\delta \rightarrow 0} \theta \int_{\Omega} f \chi_{\{0 < u_0 < \delta\}} dx = \theta \int_{\Omega} f \chi_{\{u_0 = 0\}} dx.$$

Now observe that, since $\zeta(u_0) = +\infty$ when $u_0 = 0$, (2.13) implies that

$$\text{meas}(\{x \in \Omega \mid u_0 = 0 \text{ and } f > 0\}) = 0.$$

Hence, $f\chi_{\{u_0=0\}} = 0$ a.e. in Ω and also

$$\lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} I_\varepsilon^\delta = 0. \quad (2.60)$$

Limit of J_ε^δ : We write

$$J_\varepsilon^\delta = \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)(u_\varepsilon - (v_\varepsilon^m)^+) \chi_{\{u_\varepsilon > \delta\}} \chi_{\{u_0 \neq \delta\}} dx + \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)(u_\varepsilon - (v_\varepsilon^m)^+) \chi_{\{u_\varepsilon > \delta\}} \chi_{\{u_0 = \delta\}} dx, \quad (2.61)$$

where, as before, $\int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)(u_\varepsilon - (v_\varepsilon^m)^+) \chi_{\{u_\varepsilon > \delta\}} \chi_{\{u_0 = \delta\}} dx = 0$ except for a countable set of values of δ which will be excluded. Hence, by unfolding

$$\begin{aligned} J_\varepsilon^\delta &= \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)(u_\varepsilon - (v_\varepsilon^m)^+) \chi_{\{u_\varepsilon > \delta\}} \chi_{\{u_0 \neq \delta\}} dx \\ &= \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon)) \mathcal{T}_\varepsilon^*(u_\varepsilon - (v_\varepsilon^m)^+) \mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon > \delta\}}) \mathcal{T}_\varepsilon^*(\chi_{\{u_0 \neq \delta\}}) dx dy \\ &\quad + \int_{\Lambda_\varepsilon^*} f\zeta(u_\varepsilon)(u_\varepsilon - (v_\varepsilon^m)^+) \chi_{\{u_\varepsilon > \delta\}} \chi_{\{u_0 \neq \delta\}} dx. \end{aligned} \quad (2.62)$$

By using the fact that $u_\varepsilon - (v_\varepsilon^m)^+ \leq u_\varepsilon$, the Hölder inequality with exponents $\frac{2}{1+k}$ and $\frac{2}{1-k}$, Remark 2.24 and H_2), since $l \geq \frac{2}{1+k}$, we have, as $\varepsilon \rightarrow 0$,

$$\begin{aligned} \int_{\Lambda_\varepsilon^*} f\zeta(u_\varepsilon)(u_\varepsilon - (v_\varepsilon^m)^+) \chi_{\{u_\varepsilon > \delta\}} \chi_{\{u_0 \neq \delta\}} dx &\leq \int_{\Lambda_\varepsilon^*} f\zeta(u_\varepsilon) u_\varepsilon dx \\ &\leq \int_{\Lambda_\varepsilon^*} f u_\varepsilon^{1-k} dx \leq \|f\|_{L^{\frac{2}{1+k}}(\Lambda_\varepsilon^*)} \|u_\varepsilon\|_{L^2(\Omega_\varepsilon^*)}^{1-k} \leq \|f\|_{L^1(\Lambda_\varepsilon^*)} \|u_\varepsilon\|_{L^2(\Omega_\varepsilon^*)}^{1-k} \rightarrow 0. \end{aligned} \quad (2.63)$$

On the other hand, in order to apply the Vitali theorem to the first integral on the right-hand side of (2.62), let us first observe that (2.11)_i and (2.39) yield, up to a subsequence,

$$\mathcal{T}_\varepsilon^*(u_\varepsilon - (v_\varepsilon^m)^+) \rightarrow u_0 - u_m \quad \text{a.e. in } \Omega \times Y^*.$$

Hence, by (2.11)_{ii} and the same arguments used to prove (2.59), we have that, almost everywhere in $\Omega \times Y^*$,

$$\mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon)) \mathcal{T}_\varepsilon^*(u_\varepsilon - (v_\varepsilon^m)^+) \mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon > \delta\}}) \mathcal{T}_\varepsilon^*(\chi_{\{u_0 \neq \delta\}}) \rightarrow f\zeta(u_0)(u_0 - u_m) \chi_{\{u_0 > \delta\}}. \quad (2.64)$$

Moreover, from the growth condition on ζ one has

$$\mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon)) < \frac{1}{\delta^k} \quad \text{on the set } \{u_\varepsilon > \delta\}, \quad (2.65)$$

so that

$$\begin{aligned} & \int_{E \times Y^*} |\mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon)) \mathcal{T}_\varepsilon^*(u_\varepsilon - (v_\varepsilon^m)^+) \mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon > \delta\}}) \mathcal{T}_\varepsilon^*(\chi_{\{u_0 \neq \delta\}})| dx dy \\ & \leq \int_{E \times Y^*} \mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon)) \mathcal{T}_\varepsilon^*(u_\varepsilon) dx dy + \frac{1}{\delta^k} \int_{E \times Y^*} \mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*((v_\varepsilon^m)^+) dx dy, \end{aligned} \quad (2.66)$$

for any measurable set $E \times Y^* \subset \Omega \times Y^*$. By using (2.20), $H_2)_i$, the Hölder inequality with exponents $\frac{2}{1+k}$ and $\frac{2}{1-k}$, Proposition 2.11₅, (2.6) and estimate (2.25), we obtain

$$\int_{E \times Y^*} \mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon)) \mathcal{T}_\varepsilon^*(u_\varepsilon) dx dy \leq \int_{E \times Y^*} \mathcal{T}_\varepsilon^*(f) [\mathcal{T}_\varepsilon^*(u_\varepsilon)]^{1-k} dx dy \leq c \|f\|_{L^{\frac{2}{1+k}}(E)}. \quad (2.67)$$

Next, in view of Proposition 2.11₅ and (2.40),

$$\frac{1}{\delta^k} \int_{E \times Y^*} \mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*((v_\varepsilon^m)^+) dx dy \leq \frac{|Y|}{\delta^k} c_m \|f\|_{L^1(E)}. \quad (2.68)$$

Hence, from (2.62), using (2.63) and the Vitali theorem (in view of (2.64) and (2.66)-(2.68)), we obtain

$$\lim_{\varepsilon \rightarrow 0} J_\varepsilon^\delta = \theta \int_{\Omega} f \zeta(u_0) (u_0 - u_m) \chi_{\{u_0 > \delta\}} dx.$$

On the other hand, assumption $H_2)$ and the Hölder inequality imply that

$$\int_{\Omega} f u_0^{1-k} dx \leq \|f\|_{L^{\frac{2}{1+k}}(\Omega)} \|u_0^{1-k}\|_{L^{\frac{2}{1-k}}(\Omega)} \leq \|f\|_{L^1(\Omega)} \|u_0\|_{L^2(\Omega)}^{1-k} < +\infty. \quad (2.69)$$

Consequently, from $H_2)_i$, (2.69), the boundedness of u_m and (2.13) of Proposition 2.5,

$$0 \leq f \zeta(u_0) (u_0 - u_m) \chi_{\{u_0 > \delta\}} \leq f u_0^{1-k} + f \zeta(u_0) u_m \in L^1(\Omega),$$

which, applying the Lebesgue dominated convergence theorem, provides

$$\lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} J_\varepsilon^\delta = \theta \int_{\Omega} f \zeta(u_0) (u_0 - u_m) \chi_{\{u_0 > 0\}} dx. \quad (2.70)$$

Limit of K_ε : Using similar arguments as in [42], we prove now that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon^*} f \zeta(u_\varepsilon) (v_\varepsilon^m)^- dx = 0. \quad (2.71)$$

Let us observe that for every $\delta \in \mathbb{R}^+ \setminus D$, where D is given by (2.56), we have

$$\int_{\Omega_\varepsilon^* \cap \{u_\varepsilon > \delta\}} f \zeta(u_\varepsilon) (v_\varepsilon^m)^- \chi_{\{u_0 = \delta\}} dx = 0.$$

Hence, for $\delta_0 \in \mathbb{R}^+ \setminus D$, we can write

$$\begin{aligned} \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)(v_\varepsilon^m)^- dx &= \int_{\Omega_\varepsilon^* \cap \{0 < u_\varepsilon \leq \delta_0\}} f\zeta(u_\varepsilon)(v_\varepsilon^m)^- dx \\ &+ \int_{\Omega_\varepsilon^* \cap \{u_\varepsilon > \delta_0\}} f\zeta(u_\varepsilon)(v_\varepsilon^m)^- \chi_{\{u_0 \neq \delta_0\}} dx \doteq A_\varepsilon + B_\varepsilon. \end{aligned} \quad (2.72)$$

From Proposition 2.22 written for $\delta = \delta_0$ we get

$$0 \leq A_\varepsilon \leq \int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla u_\varepsilon \nabla (v_\varepsilon^m)^- Z_{\delta_0}(u_\varepsilon) dx + c\varepsilon^{\gamma-\frac{1}{2}} h(2\delta_0) \|\rho\|_{L^\infty(\partial T)} \|(v_\varepsilon^m)^-\|_{V_\varepsilon}.$$

This implies, by using $H_1)_{iii}$, H_5), H_6), the Hölder inequality and (2.25),

$$\begin{aligned} 0 \leq A_\varepsilon &\leq \beta \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon^*)} \|Z_{\delta_0}(u_\varepsilon) \nabla (v_\varepsilon^m)^-\|_{L^2(\Omega_\varepsilon^*)} + c\varepsilon^{\gamma-\frac{1}{2}} h(2\delta_0) \|\rho\|_{L^\infty(\partial T)} \|(v_\varepsilon^m)^-\|_{V_\varepsilon} \\ &\leq (c + \varepsilon^{\gamma-\frac{1}{2}} c_1) \|\nabla (v_\varepsilon^m)^-\|_{L^2(\Omega_\varepsilon^*)}, \end{aligned}$$

which gives, via (2.45) and H_5),

$$\lim_{\varepsilon \rightarrow 0} A_\varepsilon = 0. \quad (2.73)$$

In order to prove that also

$$\lim_{\varepsilon \rightarrow 0} B_\varepsilon = 0, \quad (2.74)$$

by using the integration formula (see Proposition 2.11), we write B_ε as follows:

$$\begin{aligned} B_\varepsilon &= \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon)) \mathcal{T}_\varepsilon^*((v_\varepsilon^m)^-) \mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon > \delta_0\}}) \mathcal{T}_\varepsilon^*(\chi_{\{u_0 \neq \delta_0\}}) dx dy \\ &+ \int_{\Lambda_\varepsilon^*} f\zeta(u_\varepsilon)(v_\varepsilon^m)^- \chi_{\{u_\varepsilon > \delta_0\}} \chi_{\{u_0 \neq \delta_0\}} dx. \end{aligned}$$

Concerning the second term in the right-hand side of the previous equation, from the growth condition on ζ on the set $\{u_\varepsilon > \delta_0\}$ and (2.40) it results

$$\begin{aligned} \int_{\Lambda_\varepsilon^*} f\zeta(u_\varepsilon)(v_\varepsilon^m)^- \chi_{\{u_\varepsilon > \delta_0\}} \chi_{\{u_0 \neq \delta_0\}} dx &\leq \frac{1}{\delta_0^k} \int_{\Lambda_\varepsilon^*} f(v_\varepsilon^m)^- dx \\ &\leq \frac{1}{\delta_0^k} \|(v_\varepsilon^m)^-\|_{L^\infty(\Omega_\varepsilon^*)} \|f\|_{L^1(\Lambda_\varepsilon^*)} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0; \end{aligned}$$

while for treating the first one we can use the Vitali theorem. Indeed, from (2.65), Proposition 2.11₅, the Hölder inequality and (2.40), we have

$$\int_{E \times Y^*} \mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon)) \mathcal{T}_\varepsilon^*((v_\varepsilon^m)^-) \mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon > \delta_0\}}) \mathcal{T}_\varepsilon^*(\chi_{\{u_0 \neq \delta_0\}}) dx dy$$

$$\leq |Y| \frac{1}{\delta_0^k} \int_{\Omega_\varepsilon^* \cap E} f(v_\varepsilon^m)^- dx \leq \frac{|Y|}{\delta_0^k} c_m \|f\|_{L^1(E)},$$

for any measurable set $E \times Y^* \subset \Omega \times Y^*$. Also, by Lemma 2.26 and the same arguments used before to prove (2.64), we get, up to a subsequence,

$$\mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon)) \mathcal{T}_\varepsilon^*((v_\varepsilon^m)^-) \mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon > \delta_0\}}) \mathcal{T}_\varepsilon^*(\chi_{\{u_0 \neq \delta_0\}}) \rightarrow f\zeta(u_0) u_m^- \chi_{\{u_0 > \delta_0\}} = 0 \text{ a.e. in } \Omega \times Y^*,$$

since u_m is nonnegative by definition. Hence, putting together (2.72)-(2.74), we get (2.71).

At present, in order to prove (2.48), we collect (2.49), (2.51)-(2.53), (2.60), (2.70)-(2.71), getting (2.50).

It remains now to pass to the limit as $m \rightarrow +\infty$ in (2.50). In this inequality the second term in the right-hand side goes to zero in view of (2.36). The first one also goes to zero via the Lebesgue theorem, since

$$0 \leq f\zeta(u_0)(u_0 - u_m) \chi_{\{u_0 > 0\}} \leq f\zeta(u_0) u_0 \chi_{\{u_0 > 0\}} \leq f u_0^{1-k} \in L^1(\Omega),$$

due to (2.69). Thus we obtain (2.48).

Step 3. The final step is to show that

$$\lim_{m \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon^*} |\nabla(v_\varepsilon^m - v_\varepsilon)|^2 dx = 0. \quad (2.75)$$

We take $v_\varepsilon^m - v_\varepsilon$ as test function in the variational formulations (2.37) and (2.35). By subtraction, we have

$$\int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla(v_\varepsilon^m - v_\varepsilon) \nabla(v_\varepsilon^m - v_\varepsilon) dx = \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_0) \nabla(u_m - u_0))), v_\varepsilon^m - v_\varepsilon \rangle_{V_\varepsilon', V_\varepsilon}.$$

Passing to the limit on ε , for H_1) and Proposition 2.23 (whose assumptions are satisfied both by v_ε and v_ε^m thanks to (2.34) and (2.38)), we get

$$\begin{aligned} 0 &\leq \alpha \lim_{\varepsilon \rightarrow 0} \|\nabla(v_\varepsilon^m - v_\varepsilon)\|_{L^2(\Omega_\varepsilon^*)}^2 \leq \lim_{\varepsilon \rightarrow 0} \langle \mathcal{L}_\varepsilon(-\operatorname{div}(A^0(u_0) \nabla(u_m - u_0))), v_\varepsilon^m - v_\varepsilon \rangle_{V_\varepsilon', V_\varepsilon} \\ &= \langle -\operatorname{div}(A^0(u_0) \nabla(u_m - u_0)), u_m - u_0 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \end{aligned}$$

This together with (2.36) gives

$$0 \leq \lim_{m \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \alpha \|\nabla(v_\varepsilon^m - v_\varepsilon)\|_{L^2(\Omega_\varepsilon^*)}^2 \leq \lim_{m \rightarrow +\infty} \int_{\Omega} A^0(u_0) \nabla(u_m - u_0) \nabla(u_m - u_0) dx = 0.$$

Finally combining convergences (2.48) and (2.75), we obtain the desired result. \square

As an important consequence of Theorem 2.27, we are able now to prove Proposition 2.6.

Proof of Proposition 2.6. Let v_ε the solution of problem (2.35). The homogenization result given in [25, Theorem 4.5] for $v_0 = u_0$ gives, for the subsequence verifying (2.11),

$$\mathcal{T}_\varepsilon^*(\nabla v_\varepsilon) \rightharpoonup \nabla u_0 + \nabla_y \widehat{v} \quad \text{weakly in } L^2(\Omega \times Y^*), \quad (2.76)$$

where, in our case $\mathcal{M}_{\partial\Gamma}(g) \neq 0$ or $g = 0$ (see [25, (5.22)]), \widehat{v} is a function in $L^2(\Omega; H_{per}^1(Y^*))$ with $\mathcal{M}_{Y^*}(\widehat{v}) = 0$, such that

$$\widehat{v}(x, y) = - \sum_{i=1}^N \widehat{\chi}_{e_i}(y, u_0(x)) \frac{\partial u_0}{\partial x_i}(x),$$

with $\widehat{\chi}_{e_i}$ solution of (2.9), written for $\lambda = e_i$. Now let us observe that, from Theorem 2.27 and Proposition 2.117, we get

$$\mathcal{T}_\varepsilon^*(\nabla u_\varepsilon - \nabla v_\varepsilon) \rightarrow 0 \quad \text{strongly in } L^2(\Omega \times Y^*).$$

This, together with (2.11)_{iv} and (2.76), leads to

$$\nabla_y \widehat{v} = \nabla_y \widehat{u} \quad \text{a.e. in } \Omega \times Y^*, \quad (2.77)$$

which implies $\widehat{v} = \widehat{u} + w(x)$, for some function w only depending on x . Since $\mathcal{M}_{Y^*}(\widehat{v}) = \mathcal{M}_{Y^*}(\widehat{u}) = 0$ and

$$\mathcal{M}_{Y^*}(\widehat{v}) = \mathcal{M}_{Y^*}(\widehat{u}) + \mathcal{M}_{Y^*}(w),$$

we derive $w = 0$ and

$$\widehat{v} = \widehat{u}.$$

Whence we obtain the convergence

$$\mathcal{T}_\varepsilon^*(\nabla v_\varepsilon) \rightharpoonup \nabla u_0 + \nabla_y \widehat{u} \quad \text{weakly in } L^2(\Omega \times Y^*), \quad (2.78)$$

and the claimed expression of \widehat{u} . □

2.6 Proof of Theorem 2.8

First, let us observe that, under our assumptions, convergences (2.11) hold true for a subsequence of u_ε (still denoted by ε), where $u_0 \in H_0^1(\Omega)$ is a nonnegative function and $\widehat{u} \in L^2(\Omega; H_{per}^1(Y^*))$ is given by Proposition 2.6. Also we have the validity of (2.13).

Then, we now identify the limit problem satisfied by (u_0, \widehat{u}) . To do that, we take $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\phi \in \mathcal{D}(\Omega)$ and $\xi \in \mathcal{C}_{per}^1(Y^*)$, and use

$$\psi_\varepsilon(x) = \varphi(x) + \varepsilon \phi(x) \xi\left(\frac{x}{\varepsilon}\right) \in V_\varepsilon$$

as test function in (2.7), obtaining

$$\int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla u_\varepsilon \nabla \psi_\varepsilon dx + \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon h(u_\varepsilon) \psi_\varepsilon d\sigma = \int_{\Omega_\varepsilon^*} f \zeta(u_\varepsilon) \psi_\varepsilon dx + \int_{\Gamma_1^\varepsilon} g_\varepsilon \psi_\varepsilon d\sigma. \quad (2.79)$$

Let us consider the solution v_ε of problem (2.35) and write

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla u_\varepsilon \nabla \psi_\varepsilon dx &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla v_\varepsilon \nabla \psi_\varepsilon dx \\ &+ \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla (u_\varepsilon - v_\varepsilon) \nabla \psi_\varepsilon dx. \end{aligned} \quad (2.80)$$

By using assumption H_1), the Hölder inequality and Theorem 2.27, one has

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla (u_\varepsilon - v_\varepsilon) \nabla \psi_\varepsilon dx = 0, \quad (2.81)$$

taking into account that the norm of ψ_ε is bounded in V_ε .

Moreover, by the same arguments used in the proof of Theorem 4.5 of [25] when showing (5.5) and (5.16) (written for $z_\varepsilon = u_\varepsilon$ and $Z = -\operatorname{div}(A^0(u_0) \nabla u_0) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g)$), we obtain

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla v_\varepsilon \nabla \psi_\varepsilon dx = \frac{1}{|Y|} \int_{\Omega \times Y^*} A(y, u_0) (\nabla u_0 + \nabla_y \hat{u}) (\nabla \varphi + \phi \nabla_y \xi) dx dy, \quad (2.82)$$

in view of (2.78). Let us notice that in [25] the function φ is taken in $\mathcal{D}(\Omega)$ and then a density argument is used. Here this is not possible due to the presence of the singular term, so that we need to choose $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. The only difference with respect to [25] is that now we have to compute the following additional limit:

$$\int_{\Lambda_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla v_\varepsilon \nabla \varphi dx \leq \beta \|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon^*)} \|\nabla \varphi\|_{L^2(\Lambda_\varepsilon^*)} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

The main difficulty is then to pass to the limit in the singular term. Let us show that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon^*} f \zeta(u_\varepsilon) \psi_\varepsilon dx = \theta \int_{\Omega} f \zeta(u_0) \varphi \chi_{\{u_0 > 0\}} dx. \quad (2.83)$$

Let us define

$$\mu_\varepsilon(x) \doteq \varepsilon \phi(x) \xi \left(\frac{x}{\varepsilon} \right), \quad \text{that is } \psi_\varepsilon = \varphi + \mu_\varepsilon. \quad (2.84)$$

We remark that, from Propositions 2.11_{2,3,6}-2.16_{2,3}, one has

$$\mathcal{T}_\varepsilon^*(\mu_\varepsilon) = \varepsilon \mathcal{T}_\varepsilon^*(\phi) \xi, \quad \mathcal{T}_\varepsilon^b(\mu_\varepsilon) = \varepsilon \mathcal{T}_\varepsilon^b(\phi) \xi \quad \text{and} \quad \nabla \mu_\varepsilon = \varepsilon \nabla \phi \xi \left(\frac{\cdot}{\varepsilon} \right) + \phi \nabla_y \xi \left(\frac{\cdot}{\varepsilon} \right),$$

and

$$\begin{cases} \text{i) } \mathcal{T}_\varepsilon^*(\mu_\varepsilon) \rightarrow 0 & \text{strongly in } L^2(\Omega \times Y^*), \\ \text{ii) } \mathcal{T}_\varepsilon^*(\nabla \mu_\varepsilon) \rightarrow \phi \nabla_y \xi & \text{strongly in } L^2(\Omega \times Y^*). \end{cases} \quad (2.85)$$

From now on, without loss of generality we can assume $\varphi \geq 0$ and $\mu_\varepsilon \geq 0$ in (2.84). Indeed we can decompose the functions in their positive and negative parts as in (2.2). As in the proof of Theorem 2.27, we split the singular integral into two terms: one near the singularity and one far from it. Then, for every positive δ we write

$$0 \leq \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)\psi_\varepsilon dx = \int_{\{0 < u_\varepsilon \leq \delta\}} f\zeta(u_\varepsilon)\psi_\varepsilon dx + \int_{\{u_\varepsilon > \delta\}} f\zeta(u_\varepsilon)\psi_\varepsilon dx \doteq I_\varepsilon^\delta + J_\varepsilon^\delta. \quad (2.86)$$

In view of Proposition 2.22 written for $\varphi, \mu_\varepsilon \geq 0$ we have

$$0 \leq \limsup_{\varepsilon \rightarrow 0} I_\varepsilon^\delta \leq \limsup_{\varepsilon \rightarrow 0} \left[\int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla u_\varepsilon \nabla \psi_\varepsilon Z_\delta(u_\varepsilon) dx + c\varepsilon^{\gamma-\frac{1}{2}} h(2\delta) \|\rho\|_{L^\infty(\partial T)} \|\psi_\varepsilon\|_{V_\varepsilon} \right].$$

From H₅), H₆) and the boundedness of the norm of ψ_ε in V_ε , the second term in the right-hand side of the above inequality vanishes as ε approaches to zero, for every positive δ .

Then, by Proposition 2.11_{2,4} we have

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} I_\varepsilon^\delta &\leq \limsup_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(A^\varepsilon(x, u_\varepsilon)) \mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) \mathcal{T}_\varepsilon^*(\nabla \psi_\varepsilon Z_\delta(u_\varepsilon)) dx dy \\ &+ \limsup_{\varepsilon \rightarrow 0} \int_{\Lambda_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla u_\varepsilon \nabla \psi_\varepsilon Z_\delta(u_\varepsilon) dx. \end{aligned}$$

In view of assumption H₁), the Hölder inequality, Proposition 2.19 and the construction of ψ_ε we get

$$\limsup_{\varepsilon \rightarrow 0} \int_{\Lambda_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla u_\varepsilon \nabla \psi_\varepsilon Z_\delta(u_\varepsilon) dx = 0.$$

Then, if ${}^t A$ denotes the transposed matrix field of A ,

$$\begin{aligned} 0 \leq \limsup_{\varepsilon \rightarrow 0} I_\varepsilon^\delta &\leq \limsup_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(A^\varepsilon(x, u_\varepsilon)) \mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) \mathcal{T}_\varepsilon^*(\nabla \psi_\varepsilon Z_\delta(u_\varepsilon)) dx dy \\ &= \limsup_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*({}^t A^\varepsilon(x, u_\varepsilon)) \mathcal{T}_\varepsilon^*(\nabla \psi_\varepsilon Z_\delta(u_\varepsilon)) \mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) dx dy. \end{aligned} \quad (2.87)$$

We now apply the Vitali theorem to prove that

$$\mathcal{T}_\varepsilon^*({}^t A^\varepsilon(x, u_\varepsilon)) \mathcal{T}_\varepsilon^*(\nabla \psi_\varepsilon Z_\delta(u_\varepsilon)) \rightarrow {}^t A(y, u_0) (\nabla \varphi + \phi \nabla_y \xi) Z_\delta(u_0) \quad \text{strongly in } L^2(\Omega \times Y^*). \quad (2.88)$$

By the same arguments used to prove (2.11)_{ii} (see proof of Proposition 2.5), we obtain, up to a subsequence,

$$\mathcal{T}_\varepsilon^*(Z_\delta(u_\varepsilon)) \rightarrow Z_\delta(u_0) \quad \text{a.e. in } \Omega \times Y^*. \quad (2.89)$$

From (2.22)_v, Proposition 2.11₆, (2.85)_{ii} and (2.89) it follows that, up to a subsequence,

$$\mathcal{T}_\varepsilon^*({}^tA^\varepsilon(x, u_\varepsilon))\mathcal{T}_\varepsilon^*(\nabla\psi_\varepsilon)\mathcal{T}_\varepsilon^*(Z_\delta(u_\varepsilon)) \rightarrow {}^tA(y, u_0)(\nabla\varphi + \phi\nabla_y\xi)Z_\delta(u_0) \quad \text{a.e. in } \Omega \times Y^*. \quad (2.90)$$

Moreover, from (2.24), H₁), Proposition 2.11_{2,5} and the definitions of ψ_ε and Z_δ , we get

$$\begin{aligned} & \int_{E \times Y^*} |\mathcal{T}_\varepsilon^*({}^tA^\varepsilon(x, u_\varepsilon))\mathcal{T}_\varepsilon^*(\nabla\psi_\varepsilon)\mathcal{T}_\varepsilon^*(Z_\delta(u_\varepsilon))|^2 dx dy \leq \beta^2 \int_{E \times Y^*} |\mathcal{T}_\varepsilon^*(\nabla\psi_\varepsilon)|^2 dx dy \\ & \leq \beta^2 |Y| \int_{E \cap \Omega_\varepsilon^*} |\nabla\psi_\varepsilon|^2 dx \leq c \int_E |\nabla\varphi|^2 dx + c \int_E |\nabla\Phi\xi|^2 dx + c \int_E |\Phi\nabla\xi|^2 dx, \end{aligned}$$

for any measurable set $E \times Y^* \subset \Omega \times Y^*$. This, together with (2.90), allows us to apply the Vitali theorem to get (2.88). Using this convergence in (2.87), together with (2.11)_{iv}, we have

$$0 \leq \limsup_{\varepsilon \rightarrow 0} I_\varepsilon^\delta \leq \frac{1}{|Y|} \int_{\Omega \times Y^*} A(y, u_0)(\nabla u_0 + \nabla_y \hat{u})(\nabla\varphi + \phi\nabla_y\xi)Z_\delta(u_0) dx dy.$$

In order to pass to the limit as δ goes to zero in this inequality, let us observe that

$$Z_\delta(u_0) \rightarrow \chi_{\{u_0=0\}} \quad \text{a.e. in } \Omega \quad \text{and} \quad Z_\delta(u_0) \leq 1.$$

Hence, using the Lebesgue dominated convergence theorem,

$$0 \leq \lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} I_\varepsilon^\delta \leq \frac{1}{|Y|} \int_{\Omega \times Y^*} A(y, u_0)(\nabla u_0 + \nabla_y \hat{u})(\nabla\varphi + \phi\nabla_y\xi)\chi_{\{u_0=0\}} dx dy = 0, \quad (2.91)$$

since, due to the expression of \hat{u} given by Proposition 2.6, the functions ∇u_0 and $\nabla_y \hat{u}$ vanish where u_0 is equal to 0.

On the other hand, in order to study the limit behavior of the term J_ε^δ defined in (2.86), observe first that

$$\int_{\Lambda_\varepsilon^*} f\zeta(u_\varepsilon)\psi_\varepsilon\chi_{\{u_\varepsilon>\delta\}} dx \leq \frac{1}{\delta^k} \int_{\Lambda_\varepsilon^*} f\psi_\varepsilon dx \leq \frac{c}{\delta^k} \|f\|_{L^1(\Lambda_\varepsilon^*)} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0,$$

since ψ_ε is, in particular, in $L^\infty(\Omega)$. Hence, arguing as in (2.61), and using the integration formula from Proposition 2.11, we can write:

$$\begin{aligned} 0 & \leq \lim_{\varepsilon \rightarrow 0} J_\varepsilon^\delta = \lim_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(f)\mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon))\mathcal{T}_\varepsilon^*(\psi_\varepsilon)\mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon>\delta\}}) dx dy \\ & = \lim_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(f)\mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon))\mathcal{T}_\varepsilon^*(\psi_\varepsilon)\mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon>\delta\}})\mathcal{T}_\varepsilon^*(\chi_{\{u_0 \neq \delta\}}) dx dy \\ & \quad + \lim_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(f)\mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon))\mathcal{T}_\varepsilon^*(\psi_\varepsilon)\mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon>\delta\}})\mathcal{T}_\varepsilon^*(\chi_{\{u_0=\delta\}}) dx dy. \end{aligned} \quad (2.92)$$

As in the proof of (2.88), we apply the Vitali theorem to the first term in the right-hand side of (2.92). From (2.65), Proposition 2.11_{2,5} and the fact that $\psi_\varepsilon \in L^\infty(\Omega)$, one has

$$\begin{aligned} \int_{E \times Y^*} |\mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon)) \mathcal{T}_\varepsilon^*(\psi_\varepsilon) \mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon > \delta\}}) \mathcal{T}_\varepsilon^*(\chi_{\{u_0 \neq \delta\}})| dx dy \\ \leq \frac{|Y|}{\delta^k} \int_{\Omega_\varepsilon^* \cap E} |f \psi_\varepsilon| dx \leq \frac{c}{\delta^k} \|f\|_{L^1(E)}, \end{aligned}$$

for any measurable set $E \times Y^* \subset \Omega \times Y^*$. Since, by Proposition 2.11₆ and (2.85)_i, up to a subsequence,

$$\mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon)) \mathcal{T}_\varepsilon^*(\psi_\varepsilon) \mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon > \delta\}}) \mathcal{T}_\varepsilon^*(\chi_{\{u_0 \neq \delta\}}) \rightarrow f \zeta(u_0) \varphi \chi_{\{u_0 > \delta\}} \quad \text{a.e. in } \Omega \times Y^*,$$

the Vitali theorem gives

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon)) \mathcal{T}_\varepsilon^*(\psi_\varepsilon) \mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon > \delta\}}) \mathcal{T}_\varepsilon^*(\chi_{\{u_0 \neq \delta\}}) dx dy \\ = \frac{1}{|Y|} \int_{\Omega \times Y^*} f \zeta(u_0) \varphi \chi_{\{u_0 > \delta\}} dx dy = \theta \int_{\Omega} f \zeta(u_0) \varphi \chi_{\{u_0 > \delta\}} dx. \end{aligned}$$

Finally, in order to pass to the limit in the last equality, as δ goes to zero, we use again the Lebesgue theorem. Since

$$f \zeta(u_0) \varphi \chi_{\{u_0 > \delta\}} \rightarrow f \zeta(u_0) \varphi \chi_{\{u_0 > 0\}} \quad \text{a.e. in } \Omega$$

and by the estimate in (2.13)

$$0 \leq f \zeta(u_0) \varphi \chi_{\{u_0 > \delta\}} \leq f \zeta(u_0) \varphi \in L^1(\Omega),$$

we obtain

$$\begin{aligned} \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon)) \mathcal{T}_\varepsilon^*(\psi_\varepsilon) \mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon > \delta\}}) \mathcal{T}_\varepsilon^*(\chi_{\{u_0 \neq \delta\}}) dx dy \\ = \theta \int_{\Omega} f \zeta(u_0) \varphi \chi_{\{u_0 > 0\}} dx. \end{aligned} \tag{2.93}$$

As for the second term in the right-hand side of (2.92), for every $\delta \in \mathbb{R}^+ \setminus D$, with D given by (2.56), it results

$$\frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon)) \mathcal{T}_\varepsilon^*(\psi_\varepsilon) \mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon > \delta\}}) \mathcal{T}_\varepsilon^*(\chi_{\{u_0 = \delta\}}) dx dy = 0. \tag{2.94}$$

Consequently, we collect (2.92)-(2.94) and get

$$\lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} J_\varepsilon^\delta = \theta \int_{\Omega} f \zeta(u_0) \varphi \chi_{\{u_0 > 0\}} dx, \tag{2.95}$$

which implies, by (2.86) and (2.91), that (2.83) holds true.

Observe now that the arguments used in the proof of [25, Theorem 4.5], when $\varphi \in \mathcal{D}(\Omega)$, still apply to the case $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, giving

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon h(u_\varepsilon) \psi_\varepsilon d\sigma = c_\gamma \int_{\Omega} h(u_0) \varphi dx, \quad (2.96)$$

with c_γ defined by (2.15). On the other hand, as a consequence of convergences (2.85),

$$\mathcal{T}_\varepsilon^*(\mu_\varepsilon) \rightarrow 0 \quad \text{strongly in } L^2(\Omega; H^1(Y^*)),$$

so that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Gamma_1^\varepsilon} g_\varepsilon \psi_\varepsilon d\sigma = \lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Gamma_1^\varepsilon} g\left(\frac{x}{\varepsilon}\right) (\varphi + \mu_\varepsilon) d\sigma = \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g) \int_{\Omega} \varphi dx, \quad (2.97)$$

by using Proposition 2.18 written first for $\omega_\varepsilon = \varphi$ and then for $\omega_\varepsilon = \mu_\varepsilon$.

Then, when passing to the limit in (2.79), we combine (2.80)-(2.83), (2.96)-(2.97) and have

$$\begin{aligned} & \int_{\Omega \times Y^*} A(y, u_0) (\nabla u_0 + \nabla_y \widehat{u}) (\nabla \varphi + \phi \nabla_y \xi) dx dy + |Y| c_\gamma \int_{\Omega} h(u_0) \varphi dx \\ &= |Y^*| \int_{\Omega} f \zeta(u_0) \varphi \chi_{\{u_0 > 0\}} dx + |\partial T| \mathcal{M}_{\partial T}(g) \int_{\Omega} \varphi dx, \end{aligned}$$

for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\phi \in \mathcal{D}(\Omega)$ and $\xi \in \mathcal{C}_{per}^1(Y^*)$. By density we obtain

$$\begin{aligned} & \int_{\Omega \times Y^*} A(y, u_0) (\nabla u_0 + \nabla_y \widehat{u}) (\nabla \varphi + \nabla_y \psi) dx dy + |Y| c_\gamma \int_{\Omega} h(u_0) \varphi dx \\ &= |Y^*| \int_{\Omega} f \zeta(u_0) \varphi \chi_{\{u_0 > 0\}} dx + |\partial T| \mathcal{M}_{\partial T}(g) \int_{\Omega} \varphi dx, \end{aligned} \quad (2.98)$$

for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $\psi \in L^2(\Omega; H_{per}^1(Y^*))$.

Let us now notice that formula (2.17) is a known consequence of the expression of \widehat{u} given in Proposition 2.6. Moreover, in view of (2.17), a standard computation shows that u_0 is a solution of the following problem:

$$\begin{cases} -\operatorname{div}(A^0(u_0) \nabla u_0) + c_\gamma h(u_0) = \theta f \zeta(u_0) \chi_{\{u_0 > 0\}} + \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g) & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.99)$$

Also, u_0 results the unique solution of this problem, since [46, Theorem 6.2] holds under assumptions $H_2)$ - $H_6)$ and the conditions satisfied by A^0 given in (2.10). This implies the uniqueness of \widehat{u} under the condition $\mathcal{M}_{Y^*}(\widehat{u}) = 0$, in view of Proposition 2.6. The uniqueness of u_0 provides the validity of convergence (2.11) for the whole sequence.

It remains to show that $u_0 > 0$ almost everywhere in Ω . To do that, we prove that $v \doteq -u_0 <$

0 a.e. in Ω . Observe that, since (2.13) holds, $v \leq 0$ in Ω and $v = 0$ on $\partial\Omega$, so that $\sup_{\Omega} v = 0$. Now, if we suppose there exists a ball B well contained in Ω such that $\sup_B v = 0 = \sup_{\Omega} v$, then, from the strong maximum principle, by using the boundary condition on $\partial\Omega$, we derive $v \equiv 0$ in Ω , that is $u_0 \equiv 0$ in Ω .

Since u_0 is a solution of problem (2.99), whose variational formulation is

$$\int_{\Omega} A^0(u_0) \nabla u_0 \nabla \varphi dx + c_{\gamma} \int_{\Omega} h(u_0) \varphi dx = \theta \int_{\Omega} f \zeta(u_0) \chi_{\{u_0 > 0\}} \varphi dx + \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g) \int_{\Omega} \varphi dx,$$

$\forall \varphi \in H_0^1(\Omega) \cap L^{\infty}(\Omega)$. The hypothesis $h(0) = 0$, the positivity of θ , f , ζ and g imply that each integral on the right-hand side above is zero, for every $\varphi \in \mathcal{D}(\Omega)$. This means that $f \equiv 0$ on Ω and $g \equiv 0$ on ∂T , which contradicts assumption H_4). Consequently $u_0 > 0$ almost everywhere in Ω and $\chi_{\{u_0 > 0\}} \equiv 1$. Then u_0 satisfies the limit equation (2.16). \square

Chapter 3

Existence and uniqueness results for a class of singular elliptic problems in two-component domains

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In this chapter we give an application of the results used in Chapter 1. We prove some existence, uniqueness and regularity results for a weak solution of a quasilinear elliptic singular problem posed in a two-component domain. On the interface between the two components the heat flux is proportional to the jump of the temperature. Even here the difficulty arises in dealing simultaneously with the quasilinear matrix field, the singular datum and the interface condition.

3.1 Introduction

We study the existence, the uniqueness and the regularity for a weak solution u of a quasilinear elliptic singular problem posed in a two-component domain $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ of R^N , $N \geq 2$, with an imperfect contact on the interface Γ .

More precisely, we focus on the following problem described in Section 3.2:

$$\begin{cases} -\operatorname{div}(B(x, u)\nabla u) + \lambda u = f\zeta(u) & \text{in } \Omega \setminus \Gamma, \\ u = 0 & \text{on } \partial\Omega, \\ (B(x, u_1)\nabla u_1)\nu_1 = (B(x, u_2)\nabla u_2)\nu_1 & \text{on } \Gamma, \\ (B(x, u_1)\nabla u_1)\nu_1 = -h(u_1 - u_2) & \text{on } \Gamma, \end{cases} \quad (3.1)$$

where ν_1 is the unit external normal vector to Ω_1 , and we prescribe a Dirichlet condition on the exterior boundary and a jump of the solution on the interface. The quasilinear term consists in a uniformly elliptic and bounded Carathéodory matrix field $B(x, t)$, λ is a nonnegative real number, $\zeta(s)$ is a nonnegative real function singular at $s = 0$ and f is a nonnegative datum whose summability depends on the growth of ζ near the singularity. As for the boundary condition, the term h appearing in the proportionally coefficient between the continuous heat flux and the jump of the solution is assumed to be bounded and non-negative on Γ .

The difficulty arises in dealing simultaneously with the quasilinear matrix field, the singular datum and the interface boundary condition.

Quasilinear diffusion matrix fields describe the behavior of materials like glass or wood, in which the heat diffusion depends on the range of the temperature (see for instance [42]). The boundary condition on Γ models a jump of the solution through a rough interface and we refer to [21] for a physical justification of this model (see also [42] and [59]). A source term that depends on the solution itself and becomes infinite when the solution vanishes can model, for instance, an electrical conductor where each point becomes a source of heat as a current flows in it. We refer to [42, Section 3] for details on the physical meaning of this problem.

The main results are stated in Section 3.4.

In order to prove the existence of a solution for our problem, we approximate it with a sequence of nonsingular problems with a bounded nonlinearity in the equation. Then, suitable a priori estimates, together with accurate measure arguments, allow us to pass to the limit in the approximate problem, as done in [42], [51] and [52], which present a similar singular term.

For the uniqueness of the solution, we need to require additional assumptions on the matrix field B and on the function ζ . In particular we prescribe that ζ is a nonincreasing function, observing that (as in [46]) we do not assume any monotonicity property for it except for this result.

We also state the boundedness of a solution assuming stronger hypothesis on the summability of the datum f .

Let us point out that the main novelty with respect to the case studied in [46] concerns the

proofs of the a priori estimates, given here in detail in Section 3.3. Then, we describe the approximate problem and, for the sake of brevity, merely sketch the remaining proofs. They are obtained by adapting to the present case the corresponding ones from [46], with obvious modifications concerning the boundary term.

3.2 Setting of the problem

To describe the two-component domain, we suppose $N \geq 2$ and denote by

- Ω a connected bounded open set in \mathbb{R}^N and $\partial\Omega$ its boundary,
- $\Omega := \Omega_1 \cup \Omega_2 \cup \Gamma$ the two-component domain, where Ω_2 is an open set such that $\overline{\Omega_2} \subset \Omega$ with a Lipschitz continuous boundary Γ and $\Omega_1 := \Omega \setminus \overline{\Omega_2}$,
- ν_i the unit outward normal to Ω_i for $i = 1, 2$,
- $u_i = u|_{\Omega_i}$ the restriction to Ω_i of a function u defined in Ω ,
- \sim the zero extension to the whole of Ω of functions defined on Ω_1 and Ω_2 ,
- χ_E the characteristic function of any subset E of \mathbb{R}^N ,
- $\mathcal{M}(\alpha, \beta, E)$ the set of matrix fields $A = (a_{i,j})_{1 \leq i,j \leq N} \in (L^\infty(E))^{N \times N}$ such that

$$(A(x)\lambda, \lambda) \geq \alpha|\lambda|^2 \text{ and } |A(x)\lambda| \leq \beta|\lambda|, \quad \forall \lambda \in \mathbb{R}^N \text{ and a.e. on } E.$$

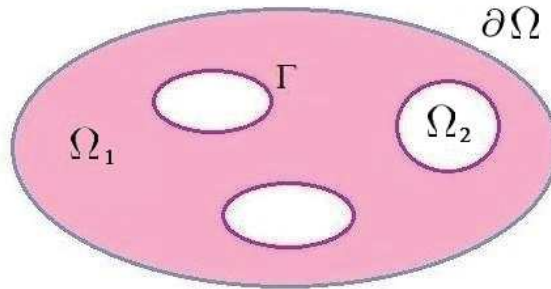


Figure 3.1. The two-component domain Ω

In this paper we study problem (3.1) under the following hypotheses:

- H₁)** The real $N \times N$ matrix field $B : (x, t) \in \Omega \times \mathbb{R} \mapsto B(x, t) = (b_{i,j}(x, t))_{i,j=1}^N \in \mathbb{R}^{N^2}$ is a Carathéodory function such that $B(\cdot, t) \in \mathcal{M}(\alpha, \beta, \Omega)$, for every $t \in \mathbb{R}$;

H₂) $\zeta \in C^0([0, +\infty[)$, $0 \leq \zeta(s) \leq \frac{1}{s^\theta}$ for every $s \in]0, +\infty[$, with $0 < \theta \leq 1$;

H₃) $\lambda \geq 0$, $0 \leq f \not\equiv 0$ a.e. in Ω , with $f \in L^l(\Omega)$, for $l \geq \frac{2}{1+\theta} (\geq 1)$;

H₄) $h \in L^\infty(\Gamma)$ and there exists $h_0 \in \mathbb{R}$ such that $0 < h_0 < h(y)$ a.e. on Γ .

We introduce now the natural framework for the study of problem (3.1) and its variational formulation. Let V_1 be the space defined by

$$V_1 = \{v \in H^1(\Omega_1) : v = 0 \text{ on } \partial\Omega\} \quad \text{with} \quad \|v\|_{V_1} := \|\nabla v\|_{L^2(\Omega_1)}.$$

Let us define $V := \{v \equiv (v_1, v_2) : v_1 \in V_1 \text{ and } v_2 \in H^1(\Omega_2)\}$, equipped with the norm

$$\|v\|_V^2 := \|\nabla v_1\|_{L^2(\Omega_1)}^2 + \|\nabla v_2\|_{L^2(\Omega_2)}^2 + \|v_1 - v_2\|_{L^2(\Gamma)}^2. \quad (3.2)$$

We identify $\nabla v := \widetilde{\nabla}v_1 + \widetilde{\nabla}v_2$, so that $\|v\|_V^2 = \|\nabla v\|_{L^2(\Omega \setminus \Gamma)}^2 + \|v_1 - v_2\|_{L^2(\Gamma)}^2$.

REMARK 3.1. A Poincaré inequality holds true in V_1 , i.e. there exists a positive constant C_p such that

$$\|v_1\|_{L^2(\Omega_1)} \leq C_p \|\nabla v_1\|_{L^2(\Omega_1)}, \quad \text{for every } v_1 \in V_1.$$

We recall also the following equivalence of the norms proved in [59]:

PROPOSITION 3.2. *The norm given in (3.2) is equivalent to the norm of $V_1 \times H^1(\Omega_2)$, i.e. there exist two positive constants c_1 and c_2 such that*

$$c_1 \|v\|_V \leq \|v\|_{V_1 \times H^1(\Omega_2)} \leq c_2 \|v\|_V, \quad \forall v \in V.$$

Furthermore let us recall the classical decomposition $u = u^+ - u^-$ for every function $u \in V$, where u^+ and u^- are the nonnegative functions defined by

$$u^+ := \max\{u, 0\}, \quad u^- := -\min\{u, 0\} \quad \text{a.e. in } \Omega.$$

In particular, for every $v \in V$ and also for their traces on Γ one has

$$\begin{aligned} (v_1 - v_2)(v_1^- - v_2^-) &= (v_1^+ - v_2^+)(v_1^- - v_2^-) - (v_1^- - v_2^-)^2 \\ &= -v_1^+ v_2^- - v_1^+ v_1^- - (v_1^- - v_2^-)^2 \leq 0. \end{aligned}$$

In the sequel we will also use the two real functions introduced below.

For every $k > 0$, T_k denotes the usual truncation function at level k . Observe that, if $\varphi \in V$, then $T_k(\varphi)$ converges to φ almost everywhere in Ω so that, via the Lebesgue dominated convergence theorem,

$$T_k(\varphi) \rightarrow \varphi \text{ strongly in } V, \text{ as } k \rightarrow \infty. \quad (3.3)$$

Let also Z_δ be, for every $\delta > 0$, the auxiliary function defined by

$$Z_\delta(s) = \begin{cases} 1, & \text{if } 0 \leq s \leq \delta, \\ -\frac{s}{\delta} + 2, & \text{if } \delta \leq s \leq 2\delta, \\ 0, & \text{if } s \geq 2\delta. \end{cases} \quad (3.4)$$

Observe that $0 \leq Z_\delta(s) \leq 1$ for any positive real s and

$$\nabla Z_\delta(s) = \chi_{\{\delta \leq s \leq 2\delta\}} \left(-\frac{1}{\delta} \nabla s \right). \quad (3.5)$$

Our aim is to prove an existence result for problem (3.1), whose variational formulation is the following:

$$\left\{ \begin{array}{l} \text{Find } u \in V \text{ such that } u > 0 \text{ a.e. in } \Omega, \\ \int_{\Omega} f\zeta(u)\varphi dx < +\infty \text{ and} \\ \int_{\Omega \setminus \Gamma} B(x, u) \nabla u \nabla \varphi dx + \int_{\Omega} \lambda u \varphi dx + \int_{\Gamma} h(u_1 - u_2)(\varphi_1 - \varphi_2) d\sigma \\ = \int_{\Omega} f\zeta(u)\varphi dx, \quad \forall \varphi = (\varphi_1, \varphi_2) \in V. \end{array} \right. \quad (3.6)$$

A function u satisfying (3.6) is called a solution to problem (3.1).

3.3 A priori estimates

In this section, we prove some a priori estimates for a solution u to problem (3.1). As usual in the literature, these estimates are the main tool when proving the existence of u . Indeed, since they also apply to a solution u_m of the approximate problem (4.1), where they are uniform with respect to m , they allow to pass to the limit.

PROPOSITION 3.3. *Under assumptions H_1)- H_4), let $u \in V$ be a solution to problem (3.1). The following a priori estimates hold true:*

$$\|\nabla u\|_{L^2(\Omega \setminus \Gamma)} \leq C_1 \|f\|_{L^{\frac{2}{1+\theta}}(\Omega)}, \quad (3.7)$$

where C_1 depends on α and C_p .

Also

$$\|u_1 - u_2\|_{L^2(\Gamma)} \leq C_2 \|f\|_{L^{\frac{2}{1+\theta}}(\Omega)}, \quad (3.8)$$

where C_2 depends on α , h_0 and C_p .

Proof. Let $u \in V$ be a solution of problem (3.1). Choosing u as test function in (3.6), using H_1)- H_4) and applying the Young inequality we get

$$\begin{aligned} & \alpha \|\nabla u\|_{L^2(\Omega \setminus \Gamma)}^2 + h_o \int_{\Gamma} (u_1 - u_2)^2 d\sigma \\ & \leq \int_{\Omega \setminus \Gamma} B(x, u) \nabla u \nabla u dx + \int_{\Omega} \lambda u^2 dx + \int_{\Gamma} h (u_1 - u_2)^2 d\sigma \\ & = \int_{\Omega} f \zeta(u) u dx \leq \int_{\Omega} f u^{1-\theta} dx \leq \eta \|u\|_{L^2(\Omega)}^2 + c(\eta) \|f\|_{L^{\frac{2}{1+\theta}}(\Omega)}^{\frac{2}{1+\theta}}. \end{aligned}$$

From Remark 3.1 we obtain

$$\alpha \|\nabla u\|_{L^2(\Omega \setminus \Gamma)}^2 + h_o \int_{\Gamma} (u_1 - u_2)^2 d\sigma \leq c(\eta, C_p) \|u\|_{V_1 \times H^1(\Omega_2)}^2 + c(\eta) \|f\|_{L^{\frac{2}{1+\theta}}(\Omega)}^{\frac{2}{1+\theta}}.$$

Thanks to Proposition 3.2 we get

$$\alpha \|\nabla u\|_{L^2(\Omega \setminus \Gamma)}^2 + h_o \|u_1 - u_2\|_{L^2(\Gamma)}^2 - c(\eta, C_p) c_2 \|u\|_V^2 \leq c(\eta) \|f\|_{L^{\frac{2}{1+\theta}}(\Omega)}^{\frac{2}{1+\theta}}. \quad (3.9)$$

Taking η sufficiently small so that $\alpha - c(\eta, C_p) c_2 \geq 0$ and $h_o - c(\eta, C_p) c_2 \geq 0$, firstly we can neglect the nonnegative boundary term in (3.9) and deduce (3.7). Neglecting now the terms outside Γ in the left-hand side of (3.9), we deduce (3.8). \square

PROPOSITION 3.4. *Under assumptions H_1)- H_4), let $u \in V$ be a solution of problem (3.1). Then*

$$\|f \zeta(u) \varphi\|_{L^1(\Omega)} \leq c,$$

for every positive $\varphi \in V$ with c depending on $\alpha, \beta, \lambda, h_o, C_p, c_2, C_1, C_2, \|\varphi\|_{H_0^1(\Omega)}$ and $\|f\|_{L^1(\Omega)}$.

Proof. Let $u \in V$ be a solution of problem (3.1) and let us choose a nonnegative $\varphi \in H_0^1(\Omega)$ as test function in (3.6). Since f, ζ and φ are nonnegative and the boundary term vanishes, using the Hölder inequality for each term, we have

$$\begin{aligned} 0 & \leq \int_{\Omega} f \zeta(u) \varphi dx = \int_{\Omega \setminus \Gamma} B(x, u) \nabla u \nabla \varphi dx + \int_{\Omega} \lambda u \varphi dx \\ & \leq \beta \|\nabla u\|_{L^2(\Omega \setminus \Gamma)} \|\nabla \varphi\|_{L^2(\Omega \setminus \Gamma)} + \lambda \|u\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)}. \end{aligned}$$

From Remark 3.1 and Proposition 3.2, thanks to estimates (3.7) and (3.8), we have

$$\begin{aligned} & \int_{\Omega \setminus \Gamma} B(x, u) \nabla u \nabla \varphi dx + \int_{\Omega} \lambda u \varphi dx \\ & \leq \beta C_1 \|f\|_{L^{\frac{1}{1+\theta}}(\Omega)}^{\frac{1}{1+\theta}} \|\nabla \varphi\|_{L^2(\Omega \setminus \Gamma)} + C(\lambda, C_p, c_2, C_1, C_2) \|f\|_{L^{\frac{1}{1+\theta}}(\Omega)}^{\frac{1}{1+\theta}} \|\varphi\|_{L^2(\Omega)}. \end{aligned}$$

The above inequalities imply the result for $\varphi \in H_0^1(\Omega)$. If now $\varphi = (\varphi_1, \varphi_2) \in V$ is nonnegative, since Γ is Lipschitz continuous, there exist nonnegative ψ_1 and ψ_2 in $H_0^1(\Omega)$ such that $\varphi = (\varphi_1, \varphi_2) = (\psi_{1|\Omega_1}, \psi_{2|\Omega_2})$. Then

$$0 \leq \int_{\Omega_1} f\zeta(u)\varphi_1 dx + \int_{\Omega_2} f\zeta(u)\varphi_2 dx \leq \int_{\Omega} f\zeta(u)\psi_1 dx + \int_{\Omega} f\zeta(u)\psi_2 dx \leq c.$$

□

The following proposition estimates the integral of the singular term close to the singular set $\{u = 0\}$ and is crucial for the proofs of next results.

PROPOSITION 3.5. *Under assumptions $H_1)$ - $H_4)$, let $u \in V$ be a solution of problem (3.1) and δ a fixed positive real number. Then,*

$$\begin{aligned} \int_{\{0 \leq u \leq \delta\}} f\zeta(u)\varphi dx &\leq \int_{\Omega \setminus \Gamma} B(x, u)\nabla u \nabla \varphi Z_\delta(u) dx \\ &+ 2\delta\lambda\|\varphi\|_{L^1(\Omega)} + 2\delta\|h\|_{L^\infty(\Gamma)}\|\varphi_1 + \varphi_2\|_{L^1(\Gamma)}, \end{aligned}$$

for every $\varphi \in V$, $\varphi \geq 0$, where Z_δ is defined by (3.4).

Proof. Let $u \in V$ be a solution of problem (3.1) and $\varphi \in V$ a nonnegative function. Choose, for $k > 0$, $Z_\delta(u)T_k(\varphi) \in V$ as test function in (3.6). Using (3.5), we have

$$\begin{aligned} &\int_{\Omega \setminus \Gamma} B(x, u)\nabla u \nabla T_k(\varphi)Z_\delta(u) dx - \frac{1}{\delta} \int_{\{\Omega \setminus \Gamma\} \cap \{\delta < u < 2\delta\}} B(x, u)\nabla u \nabla T_k(\varphi) dx \\ &+ \lambda \int_{\Omega} uZ_\delta(u)T_k(\varphi) dx + \int_{\Gamma} h(u_1 - u_2)[Z_\delta(u_1)T_k(\varphi_1) - Z_\delta(u_2)T_k(\varphi_2)] d\sigma \\ &= \int_{\Omega} f\zeta(u)Z_\delta(u)T_k(\varphi) dx. \end{aligned}$$

From $H_1)$ and since φ, δ and α are nonnegative, the previous equality implies

$$\begin{aligned} &\int_{\{0 \leq u \leq \delta\}} f\zeta(u)T_k(\varphi) dx \\ &= \int_{\{0 \leq u \leq \delta\}} f\zeta(u)Z_\delta(u)T_k(\varphi) dx \leq \int_{\Omega} f\zeta(u)Z_\delta(u)T_k(\varphi) dx \\ &\leq \int_{\Omega \setminus \Gamma} B(x, u)\nabla u \nabla T_k(\varphi)Z_\delta(u) dx + \lambda \int_{\Omega} uZ_\delta(u)T_k(\varphi) dx \\ &+ \int_{\Gamma} h(u_1 - u_2)[Z_\delta(u_1)T_k(\varphi_1) - Z_\delta(u_2)T_k(\varphi_2)] d\sigma. \end{aligned}$$

Since u , Z_δ and T_k are nonnegative, using the Hölder inequality we also have

$$\begin{aligned}
 & \lambda \int_{\Omega} u Z_\delta(u) T_k(\varphi) dx + \int_{\Gamma} h(u_1 - u_2) [Z_\delta(u_1) T_k(\varphi_1) - Z_\delta(u_2) T_k(\varphi_2)] d\sigma \\
 & \leq \lambda \int_{\Omega} u T_k(\varphi) \chi_{\{0 \leq u \leq 2\delta\}} dx + \int_{\Gamma} h [u_1 Z_\delta(u_1) T_k(\varphi_1) + u_2 Z_\delta(u_2) T_k(\varphi_2)] d\sigma \\
 & \leq 2\delta \lambda \int_{\Omega} T_k(\varphi) dx + \int_{\Gamma} h [u_1 \chi_{\{0 \leq u_1 \leq 2\delta\}} T_k(\varphi_1) + u_2 \chi_{\{0 \leq u_2 \leq 2\delta\}} T_k(\varphi_2)] d\sigma \\
 & \leq 2\delta \lambda \|T_k(\varphi)\|_{L^1(\Omega)} + 2\delta \|h\|_{L^\infty(\Gamma)} \|T_k(\varphi_1) + T_k(\varphi_2)\|_{L^1(\Gamma)}.
 \end{aligned}$$

For every $k > 0$, we get

$$\begin{aligned}
 & \int_{\{0 \leq u \leq \delta\}} f\zeta(u) T_k(\varphi) dx \leq \int_{\Omega \setminus \Gamma} B(x, u) \nabla u \nabla T_k(\varphi) Z_\delta(u) dx \\
 & + 2\delta \lambda \|T_k(\varphi)\|_{L^1(\Omega)} + 2\delta \|h\|_{L^\infty(\Gamma)} \|T_k(\varphi_1) + T_k(\varphi_2)\|_{L^1(\Gamma)}.
 \end{aligned}$$

We can pass to the limit as k tends to infinity in this inequality. Convergence (3.3) and Fatou's lemma applied on the left-hand side give the result. \square

3.4 Main results

As usual for this kind of problem (see, for instance, [4], [19], [24], [42], [51], [52] and [53]), in order to study the existence of a solution of problem (3.1), we approximate our problem by a sequence of nonsingular problems and using the a priori estimates stated in the previous section, we pass to the limit.

Here, we define the approximate problem by

$$\begin{cases}
 -\operatorname{div}(B(x, u_m) \nabla u_m) + \lambda u_m = T_m(f\zeta(|u_m|)) & \text{in } \Omega \setminus \Gamma, \\
 u_m = 0 & \text{on } \partial\Omega, \\
 (B(x, u_{m1}) \nabla u_{m1}) \nu_1 = (B(x, u_{m2}) \nabla u_{m2}) \nu_1 & \text{on } \Gamma, \\
 (B(x, u_{m1}) \nabla u_{m1}) \nu_1 = -h(u_{m1} - u_{m2}) & \text{on } \Gamma,
 \end{cases} \quad (3.10)$$

where, for every $m \in \mathbb{N}$, the function T_m is the truncation function at level m .

The variational formulation associated with the approximate problem (3.10) reads

$$\begin{cases}
 \text{Find } u_m \in V \text{ such that} \\
 \int_{\Omega \setminus \Gamma} B(x, u_m) \nabla u_m \nabla \varphi dx + \int_{\Omega} \lambda u_m \varphi dx + \int_{\Gamma} h(u_{m1} - u_{m2}) (\varphi_1 - \varphi_2) d\sigma \\
 = \int_{\Omega} T_m(f\zeta(|u_m|)) \varphi dx, \quad \forall \varphi \in V.
 \end{cases}$$

Thanks to this approximation, we avoid the singularity at 0.

The existence of a solution of this problem makes use of arguments similar to [10] and [11]. Moreover, as in [46] and [42] one can prove that $u_m \geq 0$ a. e. in Ω , so that we can write $\zeta(u_m)$ instead of $\zeta(|u_m|)$ in the problem.

THEOREM 3.6. *Under assumptions $H_1)$ - $H_4)$, problem (3.6) admits at least a solution.*

Proof. For $m \in \mathbb{N}$, let $u_m \in V$ be a solution of the approximate problem (3.10). Observe that, since $T_m(f\zeta(|u_m|))$ satisfies the same assumptions as the function $f\zeta(|u_m|)$ for every fixed $m \in \mathbb{N}$, the a priori estimates of Section 3.3 apply uniformly with respect to m .

This implies that there exists $u \in V$ such that, up to a subsequence,

$$\left\{ \begin{array}{l} \text{i) } u_m \rightharpoonup u \text{ weakly in } V, \\ \text{ii) } u_m \rightarrow u \text{ strongly in } L^2(\Omega) \text{ and a.e. in } \Omega, \\ \text{iii) } u_{m_1} - u_{m_2} \rightarrow u_1 - u_2 \text{ strongly in } L^2(\Gamma). \end{array} \right.$$

From now on, the proof follows that of Theorem 5.2 from [46]. □

We also state the two results below, which can be proved following the proofs of the corresponding results from [46].

The first one deals with uniqueness and it is stated under the assumption introduced in [23] for a quasilinear problem with Dirichlet boundary conditions.

THEOREM 3.7. *Under assumptions $H_1)$ - $H_4)$, suppose further that ζ is nonincreasing in $[0, +\infty[$ and there exists a continuous and nondecreasing function $\omega : \mathbb{R} \rightarrow \mathbb{R}$, with $\omega(t) > 0 \forall t > 0$, satisfying the following conditions:*

$$\left\{ \begin{array}{l} \text{i) } |B(x, t_1) - B(x, t_2)| \leq \omega(|t_1 - t_2|) \text{ for a.e. } x \in \Omega, \forall t_1 \neq t_2; \\ \text{ii) } \forall y > 0, \lim_{x \rightarrow 0^+} \int_x^y \frac{dt}{\omega(t)} = +\infty. \end{array} \right.$$

Then problem (3.6) admits a unique solution.

Proof. We adapt to our case the ideas of [23] (see also [19]). To do that, let u and w be two solutions of problem (3.6).

For any $\mu > 0$, set $E := \{x \in \Omega \setminus \Gamma \mid (u - w)(x) > \mu\}$ and define

$$F_\mu(s) := \begin{cases} \int_\mu^s \frac{dt}{\omega^2(t)} & \text{if } s \geq \mu, \\ 0 & \text{otherwise.} \end{cases}$$

Subtracting the variational formulations of u and v , by similar computations as those done in [23], one can show that

$$\left\| \frac{\nabla(u-w)}{\omega(u-w)} \right\|_{L^2(E)} \leq C,$$

with C independent on μ . This inequality can be written as

$$\|G_\mu(u-w)\|_V = \|\nabla G_\mu(u-w)\|_{L^2(\Omega)} \leq C,$$

where

$$G_\mu(s) := \begin{cases} \int_\mu^s \frac{dt}{\omega(t)} & \text{if } s \geq \mu, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that there exists a sequence $\{\mu_k\}_{k \in \mathbb{N}}$ converging to 0 such that, up to a subsequence,

$$\begin{cases} G_{\mu_k}(u-w) \rightharpoonup G \text{ weakly in } V, \\ G_{\mu_k}(u-w) \rightarrow G \text{ strongly in } L^2(\Omega) \text{ and a.e. in } \Omega. \end{cases}$$

Accordingly

$$\lim_{\mu_k \rightarrow 0} G_{\mu_k}(u-w)(x) = G(x) < +\infty \quad \text{a.e. in } \Omega \setminus \Gamma, \quad \text{as } k \rightarrow \infty;$$

but from the definition of G_μ we have also

$$\lim_{\mu_k \rightarrow 0} G_{\mu_k}(u-w)(x) = \lim_{\mu_k \rightarrow 0} \int_{\mu_k}^{(u-w)(x)} \frac{dt}{\omega(t)} = +\infty \quad \text{a.e. in } E.$$

This involves that $|E| = 0$, as μ goes to zero, so that $u \leq w$ almost everywhere in $\Omega \setminus \Gamma$. Interchanging the roles of u and w , we get $u = w$ almost everywhere in Ω . \square

We state now a boundedness result for a solution of problem (3.6), whose proof makes use of classical ideas from Stampacchia [69]. In order to get bounded solutions, we need to require a greater summability of the datum f , as done in [24], [42], [51] and [54].

Hence, instead of H_3), we make the following hypothesis:

$$H_{3'}) \quad 0 \leq f \in L^l(\Omega), \text{ with } l > \frac{N}{2}.$$

Let us mention that, as showed in [42], assumption $H_{3'})$ is stronger than H_3).

THEOREM 3.8. *Under assumptions $H_1)$ – $H_4)$, suppose further $H_{3'})$ holds true. Then, any solution of (3.6) is bounded.*

Proof. For any real number $h \geq 1$, set

$$A_h := \{x \in \Omega \setminus \Gamma : u(x) > h\} \quad \text{and} \quad D_h := \{x \in \Gamma : u(x) > h\}.$$

Since $u \in V$,

$$\lim_{h \rightarrow +\infty} |A_h| = 0 \quad \text{and} \quad \lim_{h \rightarrow +\infty} \partial H_{N-1}(D_h) = 0.$$

Now, let us choose $G_h(u) := (u-h)^+ \in V$ as test function in (3.6). Since G_h is nondecreasing and

$$\zeta(u) \leq \frac{1}{u^\theta} < \frac{1}{h^\theta} \leq 1 \quad \text{where } G_h(u) \neq 0,$$

we obtain

$$\alpha \int_{\Omega \setminus \Gamma} [\nabla G_h(u)]^2 dx \leq \int_{\Omega} f G_h(u) dx.$$

From now on, we can use exactly the same arguments of the proof of Theorem 3.2 in [24] (written for $\delta = 0$, $g = 0$ and with obvious modifications concerning the boundary term), based on classical results from Stampacchia [69], to obtain that u belongs to $L^\infty(\Omega)$. \square

Perspectives

In this thesis we have proved some existence and uniqueness results for singular elliptic problems posed in perforated and two-component domains. Also we have obtained homogenization and related correctors for the first class of problems in periodically perforated domains.

With this in mind, a first future work might concern the study of the asymptotic behavior of the second class of problems, namely to study at the limit on $\varepsilon \rightarrow 0$ the following problem:

$$\left\{ \begin{array}{ll} -\operatorname{div}(A^\varepsilon(x, u^\varepsilon)\nabla u^\varepsilon) = f\zeta(u^\varepsilon) & \text{in } \Omega^\varepsilon \setminus \Gamma^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \\ (A^\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1^\varepsilon = (A^\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon)\nu_1^\varepsilon & \text{on } \Gamma^\varepsilon, \\ (A^\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1^\varepsilon = -\varepsilon^\gamma h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon) & \text{on } \Gamma^\varepsilon, \end{array} \right.$$

where $\Omega^\varepsilon = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Gamma^\varepsilon$ denotes the two-component domain, Γ^ε the periodic interface and ν_1^ε is the unit normal vector to the connected component Ω_1^ε .

We assume the quasilinear diffusion term is uniformly elliptic, bounded, periodic in the first variable and Carathéodory. The singular lower order term $f\zeta$ is as in the whole thesis. Concerning the boundary condition, we suppose $\gamma \geq 1$, $h \in L^\infty(\Gamma)$ and that there exists $h_0 \in \mathbb{R}$ such that $0 < h_0 < h(y)$ almost everywhere on Γ . We set $h^\varepsilon(x) = h(\frac{x}{\varepsilon})$ almost everywhere on Γ^ε .

Homogenization of a composite with imperfect interface has been studied, for instance, in [7] [10], [44], [45] and [59]. We expect to adapt to our needs the techniques used in Chapter 2 in order to obtain the homogenized problem and the corrector result, by means of the periodic unfolding method adjusted to the case of two-component domains.

Next, it would be interesting to consider the existence and the homogenization of these two classes of problems in the time-dependent setting, extending the presented results to the parabolic case.

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