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Sahar Albosaily

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Normandie Université

THÈSE

Pour obtenir le diplôme de doctorat

Spécialité (Mathématiques Appliquées)

Préparée au sein de « l'Université de Rouen Normandie »

Optimal investment and consumption strategies for spread financial markets

**Stratégies optimales pour la consommation et l'investissement
dans les marchés financiers de "spread"**

**Présentée et soutenue par
Sahar ALBOSAILY**

**Thèse soutenue publiquement le (7 Décembre 2018)
devant le jury composé de**

Monsieur / Mikhail KAMENSKIY	Professeur/ Université d'État de Voronej, Voronej, Russie	Rapporteur
Monsieur / Emmanuel LÉPINETTE	MdC, HDR / Université de Paris-Dauphin	Rapporteur
Madame / Laurence CARASSUS	Professeur / Léonard de Vinci Pôle universitaire, Research Centre et URCA, Courbevoie	Examineur
Monsieur / Paul LESCOT	Professeur / Université de Rouen	Examineur, Président du jury
Monsieur / Serguei PERGAMENHTCHIKOV	Professeur / Université de Rouen	Directeur de thèse

Thèse dirigée par Serguei PERGAMENHTCHIKOV, laboratoire LMRS



To my life-coaches, my loving parents Mohammad and Noura Albusaili: because I owe it all to you. Many Thanks!

لوالدي محمد و نورة البصيلي، اهديكم هذا الجهد الذي لولاكم بعد الله لما كان ...

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements.

Sahar ALBOSAILY
January 2019

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Abstract

This thesis studies the consumption/investment problem for the spread financial market defined by the Ornstein–Uhlenbeck (OU) process. Recently, the OU process has been used as a proper financial model to reflect underlying prices of assets. The thesis consists of 8 Chapters.

Chapter 1 presents a general literature review and a short view of the main results obtained in this work where different utility functions have been considered.

The optimal consumption/investment strategy are studied in Chapter 2 for the power utility functions for small time interval, that $0 < t < T < T_0$. Main theorems have been stated and the existence and uniqueness of the solution has been proven. Numeric approximation for the solution of the HJB equation has been studied and the convergence rate has been established. In this case, the convergence rate for the numerical scheme is super geometrical, i.e., more rapid than any geometrical ones. A special *verification theorem* for this case has been shown. In this chapter, we have studied the Hamilton–Jacobi–Bellman (HJB) equation through the Feynman–Kac (FK) method. The existence and uniqueness theorem for the classical solution for the HJB equation has been shown.

Chapter 3 extended our approach from the previous chapter of the optimal consumption/investment strategies for the power utility functions for any time interval where the power utility coefficient γ should be less than $1/4$.

Chapter 4 addressed the optimal consumption/investment problem for logarithmic utility functions for multivariate OU process in the base of the stochastic dynamical programming method. As well it has been shown a special *verification theorem* for this case. It has been demonstrated the existence and uniqueness theorem for the classical solution for the HJB equation in explicit form. As a consequence the optimal financial strategies were constructed. Some examples have been stated for a scalar case and for a multivariate case with diagonal volatility.

Stochastic volatility markets has been considered in Chapter 5 as an extension for the previous chapter of optimization problem for the logarithmic utility functions.

Chapter 6 proposed some auxiliary results and theorems that are necessary for the work.

Numerical simulations has been provided in Chapter 7 for power and logarithmic utility functions. The fixed point value h for power utility has been presented. We study the constructed strategies by numerical simulations for different parameters. The value function for the logarithmic utilities has been shown too.

Finally, Chapter 8 reflected the results and possible limitations or solutions.

Keywords Financial spread markets · Ornstein–Uhlenbeck processes · Optimal consumption/investment problem · Stochastic control · Dynamical programming · Hamilton–Jacobi–Bellman equation · Feynman–Kac mapping · Numerical schemes.

Mathematics Subject Classification (2010) primary MSC 60P05 · secondary MSC 60G05

Résumé

Dans cette thèse, on étudie le problème de la consommation et de l'investissement pour le marché financier de "spread" (différence entre deux actifs) défini par le processus Ornstein - Uhlenbeck (OU). Ce manuscrit se compose de sept chapitres. Le chapitre 1 présente une revue générale de la littérature et un bref résumé des principaux résultats obtenus dans ce travail où différentes fonctions d'utilité sont considérées.

Dans le chapitre 2, on étudie la stratégie optimale de consommation/investissement pour les fonctions puissances d'utilité pour un intervalle de temps réduit à $0 < t < T < T_0$. Dans ce chapitre, nous étudions l'équation de Hamilton-Jacobi-Bellman (HJB) par la méthode de Feynman - Kac (FK). L'approximation numérique de la solution de l'équation de HJB est étudiée et le taux de convergence est établi. Il s'avère que dans ce cas, le taux de convergence du schéma numérique est super-géométrique, c'est-à-dire plus rapide que tous ceux géométriques. Les principaux théorèmes sont énoncés et des preuves de l'existence et de l'unicité de la solution sont données. Un théorème de vérification spécial pour ce cas des fonctions puissances est montré.

Le chapitre 3 étend notre approche au chapitre précédent à la stratégie de consommation/investissement optimale pour tout intervalle de temps pour les fonctions puissances d'utilité où l'exposant γ doit être inférieur à $1/4$.

Dans le chapitre 4, on résout le problème optimal de consommation/investissement pour les fonctions logarithmiques d'utilité dans le cadre du processus OU multidimensionnel en se basant sur la méthode de programmation dynamique stochastique. En outre, on montre un théorème de vérification spécial pour ce cas. Le théorème d'existence et d'unicité pour la solution classique de l'équation de HJB sous forme explicite est également démontré. En conséquence, les stratégies financières optimales sont construites. Quelques exemples sont donnés pour les cas scalaires et pour les cas multivariés à volatilité diagonale.

Le modèle de volatilité stochastique est considérée dans Chapter 5 comme une extension du chapitre précédent des fonctions logarithmique d'utilité.

Le chapitre 6 propose des résultats et des théorèmes auxiliaires nécessaires au travail.

Le chapitre 7 fournit des simulations numériques pour les fonctions puissances et logarithmiques d'utilité. La valeur du point fixé h de l'application de FK pour les fonctions

puissances d'utilité est présentée. Nous comparons les stratégies optimales pour différents paramètres à travers des simulations numériques. La valeur du portefeuille pour les fonctions logarithmiques d'utilité est également obtenue.

Enfin, nous concluons nos travaux et présentons nos perspectives dans le chapitre 8.

Mots-clés Marché financier de "spread" · Le processus d'Ornstein–Uhlenbeck · Problème optimal d'investissement et de consommation · Contrôle stochastique · Programmation dynamique · L'équation de Hamilton–Jacobi–Bellman · L'application de Feynman–Kac · Schémas numériques.

Classification par sujet de mathématiques (2010) primaire MSC 60P05 · secondaire MSC 60G05

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Chapter 1

Introduction

One of the fundamental problems in financial mathematics is distributing the endowment between assets to obtain a high return at the end of the time contract. Robert Merton has been the first to investigate a consumption/investment problem for Black–Scholes (Bl–Sch) markets with constant coefficients (Merton, [64]). In his paper, he formulated the optimization problem and maximized utilities. Thereafter, the interest in the consumption/investment problem has increased among researchers. Therefore, Merton’s classical problem has been generalized and extended by considering more complicated forms such as including transaction costs and bankruptcy. This thesis studies the optimal consumption/investment problem during a fixed time interval $[0, T]$ for a financial market generated by risky spread assets defined through the Ornstein–Uhlenbeck (OU) processes. The interest in the subject came from a paper that studied a one dimensional pure investment problem with one spread of risky assets for a finite time interval (Boguslavsky and Boguslavskaya, [13]). It considered a power utility over the final wealth X_T^γ for $-\infty < \gamma < 1$ by using OU process to model the spread for risky assets.

In this thesis, we develop the problem proposed in (Boguslavsky and Boguslavskaya [13]) by adding a riskless asset and a consumption to the model. Therefore, from financial perspectives, the investor has more options to choose from during a pre-specified time interval, either investing or consuming. So, in this research our goal is to optimize this problem to get the optimal solution in order to help the trader to take a good decision.

We suppose that a spread (difference) of risky assets follows an OU process which is given by

$$dS_t = -\kappa(S_t - \theta)dt + \sigma dW_t,$$

where $\kappa > 0$ is the speed of mean reverting, $\sigma > 0$ is a noise component and W_t is a Wiener process. So, this process will revert to its long term mean θ . For simplifying, it is been

assumed that the long term mean θ is zero. The process S_t is normally distributed and is given by

$$S_t = \theta + (S_s - \theta)e^{-\kappa(t-s)} + \sigma \int_s^t e^{-\kappa(t-u)} dW_u,$$

with the following parameters

$$\mathbf{E}(S_{t+s}|S_t) = \theta + (S_t - \theta)e^{-\kappa s} \quad \text{and} \quad \text{Var}(S_{t+s}|S_t) = \left(\frac{1 - e^{-2\kappa s}}{2\kappa} \right) \sigma^2.$$

As seen in Fig. 1.1 below, the Ornstein–Uhlenbeck process is presented for a single path in Fig. 1.1a and for 10 paths in Fig. 1.1b with mean reverting equals to zero and parameters σ and κ equal to one in the single path and $\kappa = 0.5$ in multipath. The trader's position α_t in

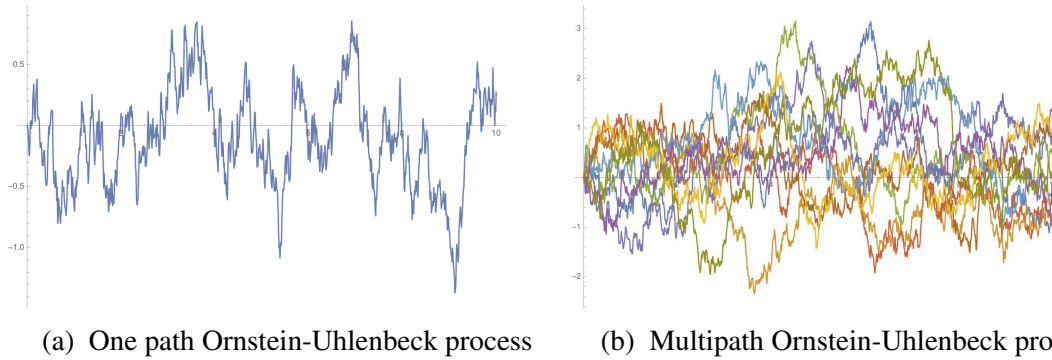


Fig. 1.1 Ornstein-Uhlenbeck process

the risky asset at time t and the consumption c_t until time t are the control parameters in our optimization problem with the assumption that short selling is allowed. Noting that, there is no conditions on wealth. The wealth dynamics for the control parameters is given by

$$dX_t^v = (rX_t^v - \kappa_1 \alpha_t S_t - c_t)dt + \alpha_t \sigma dW_t. \quad (1.0.1)$$

Our goal is to maximize the utility for all admissible strategies $v = (\alpha, c) \in \mathcal{V}(\zeta)$, i.e., for $\zeta = (x, s)$,

$$J^*(\zeta) = \sup_{v \in \mathcal{V}} J(\zeta, v) := \sup_{v \in \mathcal{V}} \mathbf{E}_\zeta \left(\int_0^T \mathbf{f}(c_u) du + \mathbf{h}(X_T^v) \right), \quad (1.0.2)$$

where \mathbf{E}_ζ is the conditional expectation under the condition that $\zeta_0 = (X_0, S_0) = \zeta$ and X_T^v is the terminal wealth over the strategy v and the functions \mathbf{f} and \mathbf{h} are representing the utility functions.

In this thesis, we use power and logarithmic utility functions.

1.1 Spread markets

This thesis focuses on the spread of risky assets, also called the relative spread trading, which can be explained as pairs trading in a financial market. The idea of spread trading is not new, it has been used for nearly three decades in Wall Street market. Among the pioneers of this idea was Nunzio Tartaglia's quantitative group at Morgan Stanley investment bank and financial services company.

Several studies have used the notion of spread to examine the behavior of the financial markets. For example, for the precious metals markets, studies have examined the spread of gold futures and the Treasury bill futures (Monroe and Cohn [69]). In addition, spread has been studied for oil markets. For example, Grima and Paulson [32] investigated the long term price relationship between futures prices of crude oil, unleaded gasoline, and heating oil. Such problems are of prime interest for practical investors such as those in the electricity and gas markets, and also in other sectors like microstructure level within the airline industry. However, although the idea of pairs trading is widely used, the academic research about it is still small (Gatev et al. [29]).

In this thesis, we are concerned with the time-series approach of pairs trading. Moreover, these problems for BI-Sch markets and stochastic utility markets are considered in many papers (see, for example, Karatzas and Shreve [44], Klüppelberg and Pergamenchtchikov [48], Duffie et al. [17] and Berdjane and Pergamenchtchikov [9]). The affine processes proposed in Duffie et al. [17] and Kraft et al. [45] to be used in the financial markets in the general framework, however, unfortunately we can not use these methods due to the additional variable in the HJB equation corresponding to the risky asset.

The aim of the spread (or pairs trading) strategy is to gain profits from mispricing of two assets. Therefore, the actual price of an asset is not of high importance, but rather the anomalies of this asset (Gatev et al. [29]). In order to break down the concept of spread in more detail, we have to explain first the notion of cointegration. In this dissertation, we are concerned with certain types of assets. They have to be cointegrated, which means that the assets should be correlated and have certain behavior together over time. Therefore, they diverge from each other for a certain volatility and they return back to their mean-reverting point. This gives the opportunity to get a profit when the price of the two assets diverges from the standard value.

This thesis is not concerned with the idea of how to choose the assets as it is beyond our focus. However, there are several papers that have studied the choice of pairs by using some tests such as the Minimum Distance method (MDM), Augmented Dickey-Fuller (ADF) Test

and Granger Causality (GC) (see, for example, Raghava and Bharadwaj[75]). Therefore, in order to know how this mechanism works, let us assume that we have two assets and these two assets are cointegrated. Thus, they have to converge over time to their standard position. The speculator can profit by going short or long between these two assets as they divert from the mean-reverting point. More precisely, going long for the asset that is below the mean-reverting line and going short in the other asset that is above the mean-reverting line. Therefore, this strategy is quite useful in any market case. It enables the investor to hedge the exposure of the market. In our model development, firstly we use the OU process to model the spread markets. It is a mean-reverting and stationary process (see, for example, Boguslavsky and Boguslavskaya [13] and the references therein).

1.2 Dynamic programming method

We recall what means the dynamic programming method for deterministic systems.

To this end, we fix a terminal time T , the controlled dynamics ODE

$$\begin{cases} \dot{x}(s) = f(x(s), v(s)), t \leq s \leq T, \\ x(t) = x. \end{cases}$$

with the associated payoff functional

$$J_{x,t}[v(\cdot)] = \int_t^T r(x(s), v(s)) ds + g(x(T)).$$

Definition 1.2.1. For $x \in \mathbb{R}^n$, $0 \leq t \leq T$, define the value function $V(x, t)$ to be the greatest payoff possible if we start at $x \in \mathbb{R}^n$ at time t . In other words,

$$V(s, t) := \sup_{v \in \mathcal{V}} J_{x,t}[v(\cdot)] \quad x \in \mathbb{R}^n, 0 \leq t \leq T.$$

Notice then that

$$V(x, T) = g(x) \quad x \in \mathbb{R}^n.$$

We want to show that the value function V satisfies a certain nonlinear partial differential equation.

Theorem 1.2.1 (Hamilton-Jacobi-Bellman equation). *Assume that the value function V is a C^1 function of the variables (x, t) . Then V solves the nonlinear partial differential equation*

$$V_t(x, t) + \max_{v \in \mathcal{V}} \{ \mathbf{f}(x, v) \cdot \nabla_x V(x, t) + r(x, v) \} = 0, \quad x \in \mathbb{R}^n, 0 \leq t \leq T,$$

with the terminal condition

$$V(x, T) = g(x) \quad (x \in \mathbb{R}^n).$$

Remark 1.2.1. We can write the latter HJB equation as

$$V_t(x, t) + H(x, \nabla_x V) = 0 \quad (x \in \mathbb{R}^n, 0 \leq t \leq T),$$

and for the PDE Hamiltonian

$$H(x, p) := \max_{v \in \mathcal{V}} H(x, p, v) = \max_{v \in \mathcal{V}} \{ \mathbf{f}(x, v) \cdot p + r(x, v) \}$$

where $x, p \in \mathbb{R}^n$ and $x \cdot y$ is the scalar product.

Proof. The proof is stated in (Lawrence [56], Theorem 5.1) □

Dynamic programming method is used to design optimal control by firstly solving the HJB equation and computing the value function V . Then by using the obtained value function and the HJB equation to find the optimal control $\alpha^*(\cdot)$. In order to do this we define a parameter value $\alpha(x, t) \in A$ where the maximum in HJB is attained for each $x \in \mathbb{R}^n$ and each $0 \leq t \leq T$, i.e., select $\alpha(x, t)$ such that

$$V_t(x, t) + \mathbf{f}(x, \alpha(x, t)) \cdot \nabla_x V(x, t) + r(x, \alpha(x, t)) = 0.$$

Then by Solving the following ODE

$$\begin{cases} \dot{x}^*(s) = \mathbf{f}(x^*(s), \alpha(x^*(s), s)) & t \leq s \leq T, \\ x(t) = x, \end{cases}$$

we define the optimal control

$$\alpha^*(s) := \alpha(x^*(s), s).$$

In the stochastic dynamic programming, we use the Feynman-Kac (FK) formula given in this theorem.

Theorem 1.2.2. *Let $K(x)$ be a nonnegative, continuous function, and let $f(x)$ be bounded and continuous. Suppose that $u(x,t)$ is a bounded function that satisfies the following partial differential equation*

$$u_t = \frac{1}{2}u_{xx} - K(x)u,$$

and the initial condition

$$u(x,0) = f(x).$$

Then

$$u(x,t) = \mathbf{E}_x \exp \left\{ - \int_0^t K(W_s) ds \right\} f(W_t),$$

where under the probability measure \mathbf{P}^x , the process $(W_t)_{t \geq 0}$ is Wiener process started at x .

In the stochastic dynamic programming, usually through the FK formula, one obtains that the value in Eq. (1.0.2) satisfies the HJB equation which is parabolic PDE of the second order.

1.3 Problems

Generally, in this thesis, we consider the following market model.

$$\begin{cases} d\check{S}_t = r\check{S}_t dt, & \check{S}_0 = 1, \\ dS_t = -\kappa S_t dt + \sigma dW_t, & S_0 > 0, \end{cases}$$

where $\kappa > 0$ is the market mean-reverting parameter from \mathbb{R} and $\sigma > 0$ is the market volatility. We assume that the bond's interest rate $r \leq \kappa$. Let now $\check{\alpha}_t$ be the number of riskless assets (bonds) denoted by \check{S} and let α_t be the investment position in risky assets (stocks) at the moment $0 \leq t \leq T$, and the consumption rate is given by a non negative integrated function $(c_t)_{0 \leq t \leq T}$ (Karatzas and Shreve [44]). Thus, the wealth process is given by

$$X_t = \check{\alpha}_t \check{S}_t + \alpha_t S_t,$$

and therefore,

$$dX_t = \check{\alpha}_t d\check{S}_t + \alpha_t dS_t - c_t dt.$$

We define the financial strategy as

$$v = (v_t)_{0 \leq t \leq T} = (\alpha_t, c_t)_{0 \leq t \leq T}.$$

Then the differential equation for the wealth process for the financial strategy can be written as

$$dX_t^v = (rX_t^v - \kappa_1 \alpha_t S_t - c_t)dt + \alpha_t \sigma dW_t,$$

where $\kappa_1 = \kappa + r > 0$. Our main goal is to maximize the value function Eq. (1.0.2) for different utility functions \mathbf{f} and \mathbf{h} .

In Chapter 2 and Chapter 3, we consider the power utility functions, i.e., $\mathbf{f}(x) = \mathbf{h}(x) = x^\gamma$, where $0 < \gamma < 1$. According to the dynamical programming method, we need to study the corresponding Hamilton–Jacobi–Bellman (HJB) equation. In order to find the HJB solution, we will use probabilistic representation for the parabolic partial differential equation (PDE) on the basis of the Feynman–Kac (FK) mapping. Therefore, we develop the fixed-point method for the FK mapping by constructing a special completed metrical space in $C^{1,0}(\mathbb{R} \times [0, T])$ and we show that in the introduced metrical space the FK mapping is contracted. In addition, through the fixed-point solution for the FK mapping we show the existence and uniqueness theorem for the HJB equation. Then we represent the HJB solution through the fixed-point of the FK’s mapping. Moreover, we develop the verification theorem method for this problem by applying the general verification theorem for a general optimal stochastic control problem for the positive utility functions. We also study the moment properties for the optimal wealth process to provide the uniform integrability property for the HJB solution calculated for the optimal strategy. Therefore, we show a new special verification theorem for the spread markets by checking all conditions stated in the general optimal stochastic control problem. Lastly, we do the following numerical analysis for the application of the constructed optimal strategies. Using the contracted properties of the FK mapping we find the upper bound in the explicit form for the approximation accuracy of the iterative scheme. Minimizing the obtained upper bound, we highlight that the convergence rate is super geometrical, i.e. more rapid than any geometrical one.

In Chapter 4 and Chapter 5, we consider the logarithmic utility functions, i.e., $\mathbf{f}(x) = \mathbf{h}(x) = \ln x$, for the problem Eq. (1.0.2). We find the HJB solution in the explicit form and we construct the optimal strategies. In this thesis the optimal strategies are found by the following plan.

- We consider the optimization problem in a framework of the optimal stochastic control and to resolve this problem we use the dynamical programming method.
- According to the dynamical programming method, we need to study the corresponding Hamilton–Jacobi–Bellman (HJB) equation.

- To find the HJB solution in the power utility case, we will use the probabilistic representation for the parabolic partial differential equation (PDE) on the basis of the Feynman–Kac (FK) mapping.
- We develop the fixed-point method for the FK mapping:
 1. we construct a special completed metrical space in $C^{1,0}(\mathbb{R} \times [0, T])$;
 2. we show that in the introduced metrical space the FK mapping is contracted;
 3. through the fixed-point solution for the FK mapping we show the existence and uniqueness theorem for the HJB equation;
 4. we represent the HJB equation through the fixed-point of the FK's mapping.
- We develop the verification theorem method:
 1. we apply the general verification theorem for a general optimal stochastic control problem for the positive utilities functions;
 2. we study the moment properties for the optimal wealth process to check the uniform integrability property for the HJB solution calculated for the optimal strategy;
 3. we show the verification theorem for the spread markets by checking all conditions stated in the general optimal stochastic control problem.
- We do the following numerical analysis for the application of the constructed optimal strategies for power utility case:
 1. using the contracted properties of the FK mapping we find the upper bound in the explicit form for the approximation accuracy of the iterative scheme;
 2. minimizing the obtained upper bound, we highlight that the convergence rate is super geometrical, i.e. more rapid than any geometrical one.

This thesis is organised as follows: In Chapter 2, we consider the optimization for power utility function in a scalar case where the time interval is quite small. We proof a modified *verification theorem* for spread markets. In addition the Cauchy problem is been studied for this problem in order to proof the existence of its the solution.

In Chapter 3, the time interval issue is no longer a problem. So, the optimization for power utility function is considered for any time interval.

In Chapter 4, we study the optimization problem for logarithmic utility functions for multi-dimensional case. This problem is different than the previous two problems that been discussed in the first two chapters. In this problem we have not a nonlinear term in the HJB equation that we obtain, however, the methods that been used before for the power utility function case do not apply for this case. In this chapter, the solution for the HJB equation has been obtained explicitly. The verification theorem in this case is shown as well. In addition, some examples have been demonstrated for scalar case and for diagonal volatility in multivariate case.

In Chapter 5, we continue the study of the optimization problem for stochastic volatility markets.

In Chapter 6, the auxiliary lemmas and theorems have been stated such as Cauchy problem and special verification theorems. Finally, numerical analysis has been shown in Chapter 7 by using different parameters to show the effect of these parameters on the strategies and wealth.

Chapter 2

Optimisation for power utility function on small time interval

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This Chapter deals with an optimal consumption/investment problem during a fixed time interval $[0, T]$ for a financial market generated by risky spread assets defined through the Ornstein–Uhlenbeck (OU) processes. We consider a small time interval where the maturity time $T < T_0$. The investor will make decisions regarding to investing and consuming for a portfolio based on a power utility function of the form x^γ for $\gamma \in (0, 1)$. Through this chapter, we develop a new method for the probabilistic analysis of the parabolic PDE. Similarly

to (Berdjane and Pergamenchtchikov [9]), we study the Hamilton–Jacobi–Bellman (HJB) equation through the Feynman–Kac (FK) representation.

2.1 Market model

Let $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbf{P})$ be a standard filtered probability space with the Wiener process $(W_t)_{0 \leq t \leq T}$ and $\mathcal{F}_t = \sigma\{W_u, u \leq t\}$. Our financial market consists of one *riskless asset* $(\check{S}_t)_{0 \leq t \leq T}$ and *risky spread asset* $(S_t)_{0 \leq t \leq T}$, and is governed by the following equations:

$$\begin{cases} d\check{S}_t = r\check{S}_t dt, & \check{S}_0 = 1, \\ dS_t = -\kappa S_t dt + \sigma dW_t, & S_0 > 0. \end{cases} \quad (2.1.1)$$

Here the constant $\kappa > 0$ is the market mean-reverting parameter from \mathbb{R} and $\sigma > 0$ is the market volatility. We assume that the interest rate r of riskless asset (bond) \check{S} should be less than κ . Let now $\check{\alpha}_t$ be the number of shares in the riskless asset \check{S} and α_t be the investment position in risky assets (stocks) at the moment $0 \leq t \leq T$, and the consumption rate is given by a non negative integrated function $(c_t)_{0 \leq t \leq T}$ [44]. Thus, the wealth process is given by

$$X_t = \check{\alpha}_t \check{S}_t + \alpha_t S_t.$$

Using the self financial principle from [44] then the wealth process X_t can be written as

$$dX_t = \check{\alpha}_t d\check{S}_t + \alpha_t dS_t - c_t dt. \quad (2.1.2)$$

We define the financial strategy as

$$v = (v_t)_{0 \leq t \leq T} = (\alpha_t, c_t)_{0 \leq t \leq T}.$$

So, replacing now in (2.1.2) the differentials $d\check{S}_t$ and dS_t by their definitions in (2.1.1), we obtain the differential equation for the wealth process corresponding to the financial strategy v :

$$dX_t^v = (rX_t^v - \kappa_1 \alpha_t S_t - c_t) dt + \alpha_t \sigma dW_t, \quad (2.1.3)$$

where $\kappa_1 = \kappa + r > 0$.

Definition 2.1.1. The financial strategy $v = (v_t)_{0 \leq t \leq T}$ is called admissible if this stochastic control process is adapted and the equation (2.1.3) has a unique strong non negative solution

and the following conditions hold.

$$\int_0^T \alpha_t^2 dt < \infty \quad \text{and} \quad \int_0^T c_t dt < \infty.$$

We denote by \mathcal{V} the set of all admissible financial strategies. For initial endowment $x > 0$, admissible strategy $v \in \mathcal{V}$, and state process $\zeta_t = (X_t^v, S_t)$, we introduce the following objective function for $0 < \gamma < 1$

$$J(\zeta, t, v) := \mathbf{E}_{\zeta, t} \left(\int_t^T c_u^\gamma du + \varpi (X_T^v)^\gamma \right), \quad (2.1.4)$$

where $\varpi > 0$ is some fixed constant, $\mathbf{E}_{\zeta, t}$ is the conditional expectation with respect to $\zeta_t = \zeta = (x, s)$. We set $J(\zeta, v) = J(\zeta, 0, v)$. Our goal is to maximize the objective function (2.1.4), i.e.

$$\sup_{v \in \mathcal{V}} J(\zeta, v).$$

To do this we use the dynamical programming method. Therefore, we need to study the optimization problem for the objective function (2.1.4), i.e., for any $0 \leq t \leq T$,

$$J^*(\zeta, t) = \sup_{v \in \mathcal{V}_t} J(\zeta, t, v), \quad (2.1.5)$$

where \mathcal{V}_t is the set of all admissible financial strategies $v \in \mathcal{V}$ such that $(v_u)_{t \leq u \leq T}$ is $\mathcal{F}_{t, u}$ adapted, $\mathcal{F}_{t, u} = \sigma\{W_s - W_t, t \leq s \leq u\}$.

Remark 2.1.1. The coefficient $0 < \varpi < \infty$ explains the investor's preference between consumption and pure investment problem. Therefore, we did not consider the case where $\varpi = 0$, as in reality the trader is more interested in the terminal wealth than consumption.

2.2 Main parameters

First we introduce the following ordinary differential equation

$$\dot{g}(t) - 2\gamma_2 g(t) + \gamma_1 g^2(t) + \gamma_3 = 0 \quad \text{and} \quad g(T) = 0,$$

where \dot{g} is the derivative of g ,

$$\gamma_1 = \frac{\sigma^2}{1-\gamma}, \quad \gamma_2 = \frac{\gamma \kappa_1}{1-\gamma} + \kappa, \quad \kappa_1 = \kappa + r \quad \text{and} \quad \gamma_3 = \frac{\gamma \kappa_1^2}{(1-\gamma)\sigma^2}.$$

One can check directly that

$$g(t) = \check{\gamma}_2 - \vartheta - \frac{2\vartheta(\check{\gamma}_2 - \vartheta)}{e^{\omega(T-t)}(\check{\gamma}_2 + \vartheta) - \check{\gamma}_2 + \vartheta}, \quad (2.2.1)$$

where $\check{\gamma}_2 = \gamma_2/\gamma_1$, $\check{\gamma}_3 = \gamma_3/\gamma_1$, $\omega = 2\vartheta\gamma_1$ and $\vartheta = \sqrt{\check{\gamma}_2^2 - \check{\gamma}_3}$.

Note that, as seen in Fig. 2.1, the function $g(t)$ is decreasing, i.e., $\max_{0 \leq t \leq T} g(t) = g(0)$. One can check directly that

$$g(0) \leq \frac{\sqrt{\gamma}\kappa_1}{\sigma^2}. \quad (2.2.2)$$

From Fig. 2.1, we set $r = 0.1$ and $\gamma = 0.2$. We see that the function $g(t)$ is aggressively decreasing in Fig. 2.1c at time $t = 0.9$ when we choose $\kappa = 5$ and $\sigma = 1$. However for Fig. 2.1a and (Fig. 2.1b) the function curve is less aggressive when we consider respectively $\kappa = 1$ and $\sigma = 1$ ($\kappa = 1$ and $\sigma = 0.5$). Taking into account that $r \leq \kappa$, we get that $\check{\gamma}_2^2 \geq \check{\gamma}_3$.

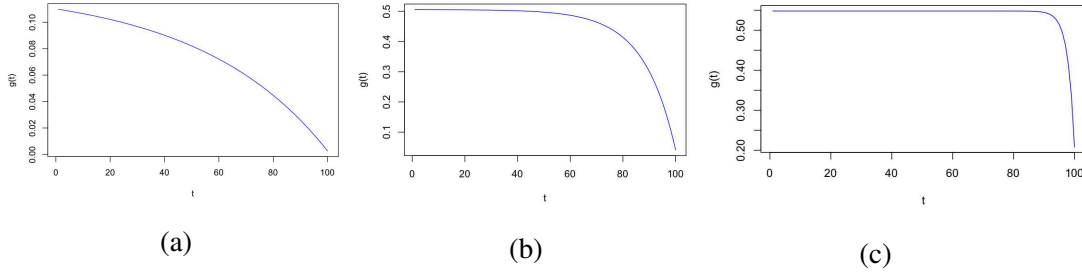


Fig. 2.1 The function $g(t)$ with different parameters κ and σ .

Furthermore, we set

$$\mathbf{B}_1 = \frac{1}{\gamma_1} \left(\sqrt{\frac{\pi}{2T}} + \sqrt{\frac{|\pi - 4T\sigma^2\omega^{\frac{1}{\gamma-1}}|}{2T}} \right) \quad \text{and} \quad \mathbf{B}_0 = \check{q}_1 T, \quad (2.2.3)$$

where $\check{q}_1 = \frac{\sqrt{\gamma}\kappa_1}{2} + r\gamma + (1 - \gamma)\omega^{\frac{1}{\gamma-1}} + \frac{\gamma_1}{2}\mathbf{B}_1^2$. We denote by $C_+^{1,0}(\mathbb{R} \times [0, T])$, the set of all positive functions from $C^{1,0}(\mathbb{R} \times [0, T])$, i.e. the set of all $\mathbb{R} \times [0, T] \rightarrow \mathbb{R}_+$ continuous partial derivatives with respect to the first variable s and continuous functions in the second variable t . Now we introduce the following set

$$\mathcal{X} = \left\{ h \in C_+^{1,0}(\mathbb{R} \times [0, T]) : \sup_{s,t} h(s,t) \leq \mathbf{B}_0, \quad \sup_{s,t} |h_s(s,t)| \leq \mathbf{B}_1 \right\}. \quad (2.2.4)$$

For some $\varkappa > 1$, which we will precise later, we introduce the metric in this space

$$\rho(f, h) = \sup_{s \in \mathbb{R}, 0 \leq t \leq T} e^{-\varkappa(T-t)} \Upsilon_{f,h}(s, t), \quad (2.2.5)$$

where $\Upsilon_{f,h}(s, t) = |h(s, t) - f(s, t)| + |h_s(s, t) - f_s(s, t)|$. Now, for any $0 \leq t \leq T$ and $s \in \mathbb{R}$, we introduce the process $(\eta_u^{s,t})_{t \leq u \leq T}$ as the solution of the following stochastic differential equation

$$d\eta_u^{s,t} = g_1(u) \eta_u^{s,t} du + \sigma d\check{W}_u, \quad \eta_t^{s,t} = s, \quad (2.2.6)$$

where $g_1(t) = \gamma_1 g(t) - \gamma_2$ and $(\check{W}_u)_{u \geq 0}$ is a standard Brownian motion. It is clear that $\eta_u^{s,t} \sim \mathcal{N}(s\mu(u, t), \sigma_1^2(u, t))$, with

$$\mu(u, t) = \exp \left\{ \int_t^u g_1(v) dv \right\} \quad \text{and} \quad \sigma_1^2(u, t) = \sigma^2 \int_t^u \mu^2(u, z) dz.$$

Now, for any $h \in \mathcal{X}$, we define the FK mapping as

$$\mathcal{L}_h(s, t) = \int_t^T \mathbf{E} \Psi_h(\eta_u^{s,t}, u) du, \quad (2.2.7)$$

where $\Psi_h(s, t) = \Gamma_0(s, t, h(s, t), h_s(s, t))$ and

$$\Gamma_0(s, t, y_1, y_2) = \frac{\sigma^2 y_2^2}{2(1-\gamma)} + \frac{\sigma^2 g(t)}{2} + r\gamma + (1-\gamma)\varpi_1 G(s, t, y_1). \quad (2.2.8)$$

Here, the coefficient $\varpi_1 = \varpi^{-1/(1-\gamma)}$ and

$$G(s, t, y) = \exp \left\{ -\frac{1}{1-\gamma} \left(\frac{s^2}{2} g(t) + y \right) \right\}. \quad (2.2.9)$$

We assume that $T < T_0$ and

$$T_0 = \min(P_1, P_2, P_3), \quad (2.2.10)$$

where

$$P_1 = \frac{\kappa(1-\gamma)}{2(3+\gamma)\kappa_2}, \quad P_2 = \frac{\gamma(1-\gamma)}{(3+\gamma)(\gamma+1)\sigma^2 g(0)}, \quad P_3 = \frac{\pi}{4\sigma^2}$$

and $\kappa_2 = \kappa_1^2 \left(1/\sigma^2 + 1/2 + g(0)/\kappa_1 \right)$.

As seen in Table 2.1, the value of T_0 has been shown for different parameters r , κ , σ and γ .

Condition for maturity time T_0						
r	κ	σ	γ	P_1	P_2	P_3
0.01	5	5	0.2	0.04463311	0.01859671	0.03141593
0.01	5	5	0.02	0.05923305	0.008980411	0.03141593
0.01	0.1	5	0.02	2.457447	0.4090169	0.03141593
0.01	0.1	5	0.2	1.851728	0.8469954	0.03141593
0.01	0.1	0.3	0.2	0.0623069	0.8469954	8.726646
0.01	5	0.3	0.2	0.001501814	0.01859671	8.726646
0.01	5	0.3	0.9	0.0001152886	0.002555438	8.726646
0.9	5	0.3	0.9	8.312999E-05	0.002169957	8.726646

Table 2.1 Time limit T_0 with differencnt parameters r , κ , σ and γ

Remark 2.2.1. Note that we will use the FK mapping (2.2.7) to study the HJB equation which will be defined in the next section.

2.3 Hamilton–Jacobi–Bellman (HJB) equation

Denoting by $\zeta_t = (X_t, S_t)$, we can rewrite equations (2.1.1) and (2.1.3) as,

$$d\zeta_t = a(\zeta_t, v_t)dt + b(\zeta_t, v_t)dW_t, \quad (2.3.1)$$

where

$$a(\zeta, \mathbf{u}) = \begin{pmatrix} rx - \kappa_1 \alpha s - c \\ -\kappa s \end{pmatrix}, \quad b(\zeta, \mathbf{u}) = \begin{pmatrix} \alpha \sigma \\ \sigma \end{pmatrix} \quad \text{and} \quad \mathbf{u} = (\alpha, c).$$

Now, for the Eq. (2.1.3), we introduce the Hamilton function. For any

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

we set,

$$H(\zeta, q, M) := \sup_{\mathbf{u} \in \Theta} H_0(\zeta, q, M, \mathbf{u}) \quad \text{and} \quad (2.3.2)$$

$$H_0(\zeta, q, M, \mathbf{u}) := a'(\zeta, \mathbf{u})q + \frac{1}{2} \text{tr}[bb'(\zeta, \mathbf{u})M] + c^\gamma,$$

where $\Theta = \mathbb{R} \times \mathbb{R}_+$ and the prime "′" here denotes the transposition. In order to find the solution to the value function (2.1.5), we need to solve the HJB equation which is given by

$$\begin{cases} z_t(\zeta, t) + H(\zeta, t, \partial z(\zeta, t), \partial^2 z(\zeta, t)) = 0, & t \in [0, T], \\ z(\zeta, T) = \bar{\omega}x^\gamma, & \zeta \in \mathbb{R}_+ \times \mathbb{R}, \end{cases} \quad (2.3.3)$$

where

$$\partial z(\zeta, t) = \begin{pmatrix} z_x \\ z_s \end{pmatrix} \quad \text{and} \quad \partial^2 z(\zeta, t) = \begin{pmatrix} z_{xx} & z_{xs} \\ z_{sx} & z_{ss} \end{pmatrix}.$$

Moreover, here

$$\begin{aligned} H_0(\zeta, t, q, M, \mathbf{u}) &= \frac{\alpha^2 \sigma^2}{2} M_{11} + (\sigma^2 M_{12} - \kappa_1 s q_1) \alpha + \frac{1}{2} \sigma^2 M_{22} \\ &\quad + r x q_1 - \kappa s q_2 - c q_1 + c^\gamma. \end{aligned}$$

Note that, in view of the definition (2.3.2), the Hamilton function $H(\zeta, t, q, M) = \infty$ if $M_{11} \geq 0$ or $q_1 \leq 0$, and for $M_{11} < 0$ and $q_1 > 0$,

$$H(\zeta, t, q, M) = H_0(\zeta, t, q, M, \mathbf{u}^0),$$

where the optimal value $\mathbf{u}^0 = (\alpha^0, c^0)$ is defined as

$$\alpha^0 = \alpha^0(s, q, M) = \frac{\kappa_1 s q_1}{\sigma^2 M_{11}} - \frac{M_{21}}{M_{11}} \quad \text{and} \quad c^0 = c^0(s, q, M) = \left(\frac{q_1}{\gamma}\right)^{\frac{1}{\gamma-1}}. \quad (2.3.4)$$

Taking these into account, then by (2.3.3), we obtain the following form for the HJB equation

$$\begin{aligned} z_t(\zeta, t) + \frac{1}{2} \frac{(\sigma^2 z_{xs} - \kappa_1 s z_x)^2}{\sigma^2 |z_{xx}|} + \frac{\sigma^2 z_{ss}}{2} + r x z_x - \kappa s z_s \\ + (1 - \gamma) \left(\frac{z_x}{\gamma}\right)^{\frac{\gamma}{\gamma-1}} = 0, \end{aligned} \quad (2.3.5)$$

where $z(\zeta, T) = \bar{\omega}x^\gamma$. To study this equation we use the following representation

$$z(x, s, t) = \bar{\omega}x^\gamma \exp \left\{ \frac{s^2}{2} g(t) + Y(s, t) \right\}. \quad (2.3.6)$$

The function $g(\cdot)$ is defined in (2.2.1), and

$$\begin{cases} Y_t(s, t) + \frac{1}{2} \sigma^2 Y_{ss}(s, t) + sg_1(t) Y_s(s, t) + \Psi_Y(s, t) = 0, \\ Y(s, T) = 0, \end{cases} \quad (2.3.7)$$

where $\Psi_Y(s, t)$ is given in (2.2.8) and the function $g_1(\cdot)$ is defined in (2.2.6). As we will see later that equation (2.3.7) has a solution in $\mathbf{C}^{2,0}(\mathbb{R} \times [0, T])$ which can be represented as a fixed point for the FK mapping

$$h(s, t) = \mathbf{E} \int_t^T \Psi_h(\eta_u^{s,t}, u) du = \mathcal{L}_h(s, t). \quad (2.3.8)$$

To construct the optimal strategies we use the optimal value of the Hamilton function (2.3.4) and the solution given by equation (2.3.6). We set

$$\check{\alpha}^0(\zeta, t) = \alpha^0(s, t, \partial z(\zeta, t), \partial^2 z(\zeta, t)) \quad \text{and} \quad \check{c}^0(\zeta, t) = c^0(s, t, \partial z(\zeta, t), \partial^2 z(\zeta, t)).$$

It is easy to see that in this case, these functions can be represented as

$$\begin{aligned} \check{\alpha}^0(\zeta, t) &= \frac{\kappa_1 s z_x(\zeta, t)}{\sigma^2 z_{xx}(\zeta, t)} - \frac{z_{xs}(\zeta, t)}{z_{xx}(\zeta, t)} = \check{\beta}(s, t)x, \\ \check{c}^0(\zeta, t) &= \left(\frac{z_x(\zeta, t)}{\gamma} \right)^{\frac{1}{\gamma-1}} = \check{G}(s, t)x, \end{aligned} \quad (2.3.9)$$

where

$$\check{\beta}(s, t) = \frac{1}{1-\gamma} \left(sg(t) + h_s(s, t) - \frac{\kappa_1}{\sigma^2} s \right) \quad \text{and} \quad \check{G}(s, t) = \varpi^{\frac{1}{\gamma-1}} G(s, t, h(s, t)).$$

Now we set the following stochastic equation to define the optimal wealth process, i.e., we set

$$dX_t^* = a^*(t) X_t^* dt + b^*(t) X_t^* dW_t, \quad (2.3.10)$$

where $a^*(t) = A^*(S_t, t)$, $b^*(t) = B^*(S_t, t)$,

$$A^*(s, t) = r - \kappa_1 s \check{\beta}(s, t) - \check{G}(s, t) \quad \text{and} \quad B^*(s, t) = \sigma \check{\beta}(s, t). \quad (2.3.11)$$

By Itô formula we can obtain that

$$X_t^* = x \exp \left\{ \int_0^t a^*(u) du \right\} \mathcal{E}_{0,t}(b^*) \quad (2.3.12)$$

and

$$\mathcal{E}_{0,t}(b^*) = \exp \left\{ \int_0^t b^*(u) dW_u - \frac{1}{2} \int_0^t (b^*(u))^2 du \right\}.$$

Using the stochastic differential equation (2.3.10) we define the optimal strategies:

$$\alpha_t^* = \check{\alpha}^0(\zeta_t^*, t) \quad \text{and} \quad c_t^* = \check{c}^0(\zeta_t^*, t), \quad (2.3.13)$$

where $\zeta_t^* = (X_t^*, S_t^*)'$ and X_t^* is defined in (2.3.10). The prime "′" denotes the transposition.

Remark 2.3.1. Note, the main difference in the HJB equation (2.3.5) from the one in (Boguslavsky and Boguslavskaya [13]) is the last nonlinear term, as we see, we can not use the solution method from (Boguslavsky and Boguslavskaya[13]). One can check that the solution for pure investment problem from (Boguslavsky and Boguslavskaya[13]) can be obtained in Eq. (2.3.13) as $\varpi \rightarrow \infty$.

2.4 Main results

First we study the HJB equation.

Theorem 2.4.1. *Assume that $0 < T < T_0$, with T_0 is given in Eq. (2.2.10), then equation Eq. (2.3.5) has the solution defined by Eq. (2.3.6), where Y is the unique solution of Eq. (2.3.7) in \mathcal{X} and is the fixed point for the FK mapping, i.e., $Y = h$, and $h = \mathcal{L}_h$.*

Theorem 2.4.2. *Assume that $0 < T < T_0$, then the optimal value of $J(t, \zeta, v)$ is given by*

$$\max_{v \in \mathcal{V}} J(\zeta, t, v) = J(\zeta, t, v^*) = \varpi x^\gamma \exp \left\{ \frac{s^2}{2} g(t) + h(s, t) \right\},$$

where the optimal control $v^* = (\alpha^*, c^*)$ for all $0 \leq t \leq T$ is given in Eq. (2.3.13) with the function Y defined in Eq. (2.3.8). The optimal wealth process $(X_t^*)_{0 \leq t \leq T}$ is the solution to Eq. (2.3.10).

Let us now define the approximation sequence $(h_n)_{n \geq 1}$ for h as $h_0 = 0$, and for $n \geq 1$, as

$$h_n = \mathcal{L}_{h_{n-1}}. \quad (2.4.1)$$

In the following theorems we show that the approximation sequence goes to the fixed function h , i.e. $h = \mathcal{L}_h$.

Theorem 2.4.3. *For any $0 < \delta < 1/2$, the approximation*

$$\|h - h_n\| = O(n^{-\delta n}) \quad \text{as } n \rightarrow \infty,$$

where $\|f\| = \sup_{s,t} (|f(s,t)| + |f_s(s,t)|)$.

Remark 2.4.1. Note that the convergence rate for the fixed-point solution is super geometrical.

Now we define the approximation. We set

$$\alpha_n^*(\zeta, t) = \check{\beta}_n(s, t)x \quad \text{and} \quad c_n^*(\zeta, t) = \check{G}_n(s, t)x$$

where

$$\check{\beta}_n(s, t) = \frac{1}{1-\gamma} \left(sg(t) + \frac{\partial h_n(s, t)}{\partial s} - \frac{\kappa_1}{\sigma} s \right) \quad \text{and} \quad \check{G}_n(s, t) = \varpi^{\frac{1}{\gamma-1}} G(s, t, h_n(s, t)).$$

Theorem 2.4.4. *For any $0 < \delta < 1/2$*

$$\sup_{\substack{\zeta \\ 0 \leq t \leq T}} \left(|\alpha^*(\zeta, t) - \alpha_n^*(\zeta, t)| + |c^*(\zeta, t) - c_n^*(\zeta, t)| \right) = O(n^{-\delta n}), \quad \text{as } n \rightarrow \infty.$$

Remark 2.4.2. As it is seen from Theorem 2.4.1 the approximation scheme for the HJB equation implies the approximation for the optimal strategy with super geometrical rate. i.e. more rapid than any geometrical ones.

2.5 Properties of the Feynman–Kac (FK) mapping

We need to study the properties of the mapping (2.2.7).

Proposition 2.5.1. *The space (\mathcal{X}, ρ) is the completed metrical space.*

Proposition 2.5.2. *Assume that $T \leq \pi/4\sigma^2$. Then $\mathcal{L}_h \in \mathcal{X}$ for any $h \in \mathcal{X}$, i.e. $\mathcal{L}_h: \mathcal{X} \rightarrow \mathcal{X}$.*

Proof. The function $\mathcal{L}_h(s, t)$ is given in Eq. (2.2.7) and can be written as

$$\begin{aligned} \mathcal{L}_h(s, t) &= \frac{\sigma^2}{2} \int_t^T g(u) du + \frac{\sigma^2}{2(1-\gamma)} \mathbf{E} \int_t^T h_s^2(\eta_u^{s,t}, u) du + r\gamma(T-t) \\ &\quad + (1-\gamma)\varpi^{\frac{1}{\gamma-1}} \mathbf{E} \int_t^T G(\eta_u^{s,t}, u, h(\eta_u^{s,t}, u)) du, \end{aligned}$$

with $G(s, t, y)$ is given in Eq. (2.2.9).

Therefore,

$$\begin{aligned} |\mathcal{L}_h(s, t)| &\leq \frac{\sigma^2}{2} g(0)(T-t) + \frac{\sigma^2}{2(1-\gamma)} \mathbf{B}_1^2(T-t) + r\gamma(T-t) \\ &\quad + (1-\gamma)\varpi^{\frac{1}{\gamma-1}}(T-t) \leq \mathbf{B}_0, \end{aligned} \quad (2.5.1)$$

where \mathbf{B}_0 and \mathbf{B}_1 are given in Eq. (2.2.3). Then by taking the derivative with respect to s , we get

$$\begin{aligned} \frac{\partial}{\partial s} \mathcal{L}_h(s, t) &= \frac{\sigma^2}{2(1-\gamma)} \frac{\partial}{\partial s} \mathbf{E} \int_t^T h_s^2(\eta_u^{s,t}, u) du \\ &\quad + (1-\gamma)\varpi^{\frac{1}{\gamma-1}} \frac{\partial}{\partial s} \mathbf{E} \int_t^T G(\eta_u^{s,t}, u, h(\eta_u^{s,t}, u)) du. \end{aligned}$$

From Lemma 6.5.1 and as $\|G(\eta_u^{s,t}, u, h(\eta_u^{s,t}, u))\|_{t, \infty} \leq 1$, we have

$$\left| \frac{\partial}{\partial s} \mathcal{L}_h(s, t) \right| \leq \frac{\sigma^2}{2(1-\gamma)} \sqrt{\frac{2(T-t)}{\pi}} \mathbf{B}_1^2 + (1-\gamma)\varpi^{\frac{1}{\gamma-1}} \sqrt{\frac{2(T-t)}{\pi}}.$$

Then by taking into account the definition of \mathbf{B}_1 in Eq. (2.2.4) we obtain,

$$\left| \frac{\partial}{\partial s} \mathcal{L}_h(s, t) \right| \leq \frac{\sigma^2}{2(1-\gamma)} \sqrt{\frac{2T}{\pi}} \mathbf{B}_1^2 + (1-\gamma)\varpi^{\frac{1}{\gamma-1}} \sqrt{\frac{2T}{\pi}} \leq \mathbf{B}_1.$$

So, we get that $\mathcal{L}_h \in \mathcal{X}$. Hence Proposition 2.5.2. \square

Proposition 2.5.3. For all $f \in \mathcal{X}$, for all s , and $0 \leq t \leq T$,

$$\frac{\partial}{\partial s} \mathcal{L}_f(s, t) = \int_t^T \left(\int_{\mathbb{R}} \Gamma_0(z, t, f(z, u), f_s(z, u)) \check{\rho}(s, t, z, u) dz \right) du,$$

where Γ_0 is as in (2.2.8) and

$$\check{\rho}(s, t, z, u) = \frac{\partial}{\partial s} \varphi(s, t, z, u) = K \frac{\mu(u, t)}{\sigma_1(u, t)} \varphi(s, t, z, u), \quad (2.5.2)$$

where

$$\varphi(s, z, u) = \frac{e^{-\frac{\kappa^2}{2}}}{\sqrt{2\pi}\sigma_1(u, t)} \quad \text{and} \quad K(s, z, u) = \frac{z - s\mu(u, t)}{\sigma_1(u, t)}. \quad (2.5.3)$$

Proposition 2.5.4. *The mapping \mathcal{L} is contraction in \mathcal{X} , i.e. for any $0 < \lambda < 1$, there exists $\varkappa \geq 1$ in the metric (2.2.5) such that for any h and $f \in \mathcal{X}$,*

$$\rho(\mathcal{L}_h, \mathcal{L}_f) \leq \lambda \rho(h, f). \quad (2.5.4)$$

Proof. Using the definition of the mapping \mathcal{L}_h given in (2.2.7), we obtain that for any h and f from \mathcal{X} ,

$$\begin{aligned} \mathcal{L}_h - \mathcal{L}_f &= \frac{\sigma^2}{2(1-\gamma)} \mathbf{E} \int_t^T \left(h_s^2(\eta_u^{s,t}, u) - f_s^2(\eta_u^{s,t}, u) \right) du \\ &\quad + (1-\gamma) \varpi^{\frac{1}{\gamma-1}} \mathbf{E} \int_t^T \left(G\left(\eta_u^{s,t}, u, h(\eta_u^{s,t}, u)\right) - G\left(\eta_u^{s,t}, u, f(\eta_u^{s,t}, u)\right) \right) du. \end{aligned}$$

Taking into account that the function G is lipschitzian, i.e. for any $y_1 \geq 0$ and $y_2 \geq 0$

$$|G(s, t, y_1) - G(s, t, y_2)| \leq \frac{1}{1-\gamma} |y_1 - y_2|,$$

we obtain that

$$\begin{aligned} |\mathcal{L}_h - \mathcal{L}_f| &\leq \frac{\sigma^2}{2(1-\gamma)} \left| \int_t^T \mathbf{E} \left(h_s^2(\eta_u^{s,t}, u) - f_s^2(\eta_u^{s,t}, u) \right) du \right| \\ &\quad + \varpi^{\frac{1}{\gamma-1}} \mathbf{E} \int_t^T \left| h(\eta_u^{s,t}, u) - f(\eta_u^{s,t}, u) \right| du. \end{aligned} \quad (2.5.5)$$

Recall that f and h belong to \mathcal{X} , i.e. the difference for the squares of their derivatives can be estimated as $|h_s^2(z, u) - f_s^2(z, u)| \leq 2\mathbf{B}_1 |h_s(z, u) - f_s(z, u)|$. Therefore,

$$|\mathcal{L}_h(s, t) - \mathcal{L}_f(s, t)| \leq \left(\frac{\sigma^2 \mathbf{B}_1}{(1-\gamma)} + \varpi^{\frac{1}{\gamma-1}} \right) \int_t^T \Upsilon_{h,f}^*(u) e^{-\varkappa(T-u)} e^{\varkappa(T-u)} du,$$

where $\Upsilon_{h,f}^*(t) = \sup_{y \in \mathbb{R}^p} \Upsilon_{h,f}(y,t)$. In view of definition (2.2.4),

$$\begin{aligned} |\mathcal{L}_h(s,t) - \mathcal{L}_f(s,t)| &\leq \left(\frac{\sigma^2 \mathbf{B}_1}{(1-\gamma)} + \varpi^{\frac{1}{\gamma-1}} \right) \rho(h,f) \int_t^T e^{\varkappa(T-u)} du \\ &\leq \left(\frac{\sigma^2 \mathbf{B}_1}{(1-\gamma)} + \varpi^{\frac{1}{\gamma-1}} \right) \frac{\rho(h,f)}{\varkappa} e^{\varkappa(T-t)}. \end{aligned}$$

Therefore, for all $0 \leq t \leq T$,

$$\sup_{s \in \mathbb{R}} |\mathcal{L}_h(s,t) - \mathcal{L}_f(s,t)| \leq \frac{\tilde{\mathbf{B}}_1}{\varkappa} \rho(h,f) e^{\varkappa(T-t)} \quad \text{and} \quad \tilde{\mathbf{B}}_1 = \frac{\sigma^2 \mathbf{B}_1}{(1-\gamma)} + \varpi^{\frac{1}{\gamma-1}}.$$

The partial derivative of $\mathcal{L}(s,t)$ with respect to s is given by

$$\begin{aligned} \frac{\partial}{\partial s} \mathcal{L}_h(s,t) &= \frac{\sigma^2}{2(1-\gamma)} \mathbf{E} \frac{\partial}{\partial s} \int_t^T h_s^2(\eta_u^{s,t}, u) du \\ &\quad + (1-\gamma) \varpi^{\frac{1}{\gamma-1}} \mathbf{E} \frac{\partial}{\partial s} \int_t^T G(\eta_u^{s,t}, u, h(\eta_u^{s,t})) du. \end{aligned}$$

By taking the expectation we obtain

$$\begin{aligned} \frac{\partial}{\partial s} \mathcal{L}_h(s,t) &= \frac{\sigma^2}{2(1-\gamma)} \int_t^T \int_{\mathbb{R}} h_s^2(z,u) \frac{\partial}{\partial s} \varphi(z,u) dz du \\ &\quad + (1-\gamma) \varpi^{\frac{1}{\gamma-1}} \int_t^T \int_{\mathbb{R}} G(z,u, h(z,u)) \frac{\partial}{\partial s} \varphi(z,u) dz du, \end{aligned}$$

where $\check{\rho}(s,t,z,u) = \partial \varphi(s,t,z,u) / \partial s$ and $\varphi(s,t,z,u)$ is given in (2.5.3). Therefore, for $u > t$ and for some constant $\mathbf{c}^* \geq 0$

$$\sup_{s \in \mathbb{R}} \int_{\mathbb{R}} |\check{\rho}(s,t,z,u)| dz \leq \frac{\mathbf{c}^*}{\sqrt{u-t}}. \quad (2.5.6)$$

Putting now $\hat{\alpha}_1 = \sigma^2(2-2\gamma)^{-1}$ and $\hat{\alpha}_2 = (1-\gamma)\varpi^{\frac{1}{\gamma-1}}$, we obtain that

$$\begin{aligned} \left| \frac{\partial}{\partial s} \mathcal{L}_h(s,t) - \frac{\partial}{\partial s} \mathcal{L}_f(s,t) \right| &= \left| \int_t^T \int_{\mathbb{R}} \left(\hat{\alpha}_1 (h_s^2(z,u) - f_s^2(z,u)) \right. \right. \\ &\quad \left. \left. + \hat{\alpha}_2 (G(z,u, h(z,u)) - G(z,u, f(z,u))) \right) \check{\rho}(s,t,z,u) dz du \right|. \end{aligned}$$

Here, note that

$$\left| \hat{\alpha}_1(h_s^2(z, u) - f_s^2(z, u)) + \hat{\alpha}_2(G(z, u, h(z, u)) - G(z, u, f(z, u))) \right| \leq \mathbf{B}_2 \Upsilon_{f,h}^*(u),$$

where $\mathbf{B}_2 = (2\hat{\alpha}_1 \tilde{\mathbf{B}}_1 + \hat{\alpha}_2(1 - \gamma))$. Thus,

$$\left| \frac{\partial}{\partial s} \mathcal{L}_h(s, t) - \frac{\partial}{\partial s} \mathcal{L}_f(s, t) \right| \leq \mathbf{B}_2 \int_t^T \Upsilon_{f,h}^*(u) \left(\int_{\mathbb{R}} |\check{\rho}(s, t, z, u)| dz \right) du.$$

Using here the bound (2.5.6), we obtain that

$$\left| \frac{\partial}{\partial s} \mathcal{L}_h(s, t) - \frac{\partial}{\partial s} \mathcal{L}_f(s, t) \right| \leq \mathbf{B}_2 \sqrt{\frac{2}{\pi}} \int_t^T \frac{1}{\sqrt{u-t}} \Upsilon_{f,h}^*(u) e^{-\varkappa(T-u)} e^{\varkappa(T-u)} du.$$

Using again here the definition (2.2.4) we get

$$\begin{aligned} \left| \frac{\partial}{\partial s} \mathcal{L}_h(s, t) - \frac{\partial}{\partial s} \mathcal{L}_f(s, t) \right| &\leq \sqrt{\frac{2}{\pi}} \mathbf{B}_2 \rho(f, h) \int_t^T \frac{e^{\varkappa(T-u)}}{\sqrt{u-t}} du \\ &\leq \frac{2}{\pi} \mathbf{B}_2 \rho(f, h) e^{\varkappa(T-t)} \int_t^T \frac{e^{-\varkappa(u-t)}}{\sqrt{u-t}} du \leq \mathbf{B}_2 \rho(f, h) \frac{e^{\varkappa(T-t)}}{\sqrt{\varkappa}}. \end{aligned}$$

Therefore,

$$\left| \frac{\partial}{\partial s} \mathcal{L}_h(s, t) - \frac{\partial}{\partial s} \mathcal{L}_f(s, t) \right| \leq \mathbf{B}_2 \rho(f, h) \frac{e^{\varkappa(T-t)}}{\sqrt{\varkappa}}.$$

Thus

$$\begin{aligned} \left| \mathcal{L}_h(s, t) - \mathcal{L}_f(s, t) \right| + \left| \frac{\partial}{\partial s} \mathcal{L}_h(s, t) - \frac{\partial}{\partial s} \mathcal{L}_f(s, t) \right| \\ \leq \left(\frac{\tilde{\mathbf{B}}_1}{\varkappa} e^{\varkappa(T-t)} + \mathbf{B}_2 \frac{e^{\varkappa(T-t)}}{\sqrt{\varkappa}} \right) \rho(f, h). \end{aligned}$$

So, taking into account that $\varkappa > 1$, we get

$$\rho(\mathcal{L}_h, \mathcal{L}_f) \leq \frac{\tilde{\mathbf{B}}_2}{\sqrt{\varkappa}} \rho(f, h) \quad \text{where} \quad \tilde{\mathbf{B}}_2 = \tilde{\mathbf{B}}_1 + \mathbf{B}_2. \quad (2.5.7)$$

Choosing here $\varkappa = (\tilde{\mathbf{B}}_2)^2 / \lambda^2$, we obtain the inequality (2.5.4). Hence Proposition 2.5.4.

□

□

Proposition 2.5.5. *For the mapping \mathcal{L} there exists a unique fixed point h from \mathcal{X} , i.e. $\mathcal{L}_h = h$, such that for any $n \geq 1$ and for any $\varkappa > (\tilde{\mathbf{B}}_2)^2$*

$$\rho(h, h_n) \leq \mathbf{B}^* \lambda^n, \quad \lambda = \frac{\tilde{\mathbf{B}}_2}{\sqrt{\varkappa}}, \quad (2.5.8)$$

where $\mathbf{B}^* = (\mathbf{B}_0 + \mathbf{B}_1)/(1 - \lambda)$, with \mathbf{B}_0 and \mathbf{B}_1 are defined in Eq. (2.2.3).

Proof. We want to show that the approximation sequence $(h_n)_{n \geq 1}$ converge to a fixed point h , where $h_0 = 0$ and $h_n = \mathcal{L}_{h_{n-1}}$ for $n \geq 1$. Using here Proposition 2.5.3, we obtain that $\rho(h_n, h_{n+1}) = \rho(\mathcal{L}_{h_{n-1}}, \mathcal{L}_{h_n}) \leq \lambda \rho(h_{n-1}, h_n)$. Therefore,

$$\rho(h_n, h_{n+1}) \leq \lambda \rho(\mathcal{L}_{h_{n-1}}, \mathcal{L}_{h_n}) \leq \lambda^2 \rho(h_{n-2}, h_{n-1}) \leq \dots \leq \lambda^n \rho(h_0, h_1).$$

Note that Eq. (2.2.4) implies directly that $\rho(h_0, h_1) \leq \mathbf{B}_0 + \mathbf{B}_1$. So, for $m > n$,

$$\rho(h_n, h_m) \leq (\lambda^n + \lambda^{n+1} + \dots + \lambda^{m-1})(\mathbf{B}_0 + \mathbf{B}_1) \leq \sum_{i=n}^{\infty} \lambda^i (\mathbf{B}_0 + \mathbf{B}_1).$$

Therefore, there exists h , such that $\rho(h_n, h) \rightarrow 0$, i.e., for all n , we obtain Eq. (2.2.5). Hence Proposition 2.5.5. \square

2.6 Properties of the fixed-point function h

In this section we study some regularity properties for the function h . First we study the smoothness with respect to the variable s .

Proposition 2.6.1. *If $h \in \mathcal{X}$ is a fixed point for \mathcal{L} i.e. $h = \mathcal{L}_h$, then for any $0 < \beta < 1$,*

$$\sup_{0 \leq t \leq T} \sup_{s_1, s_2} \frac{|h_s(s_1, t) - h_s(s_2, t)|}{|s_1 - s_2|^\beta} < \infty.$$

Proof. As

$$\frac{\partial}{\partial s} h(s, t) = \int_t^T \int_{\mathbb{R}} \Psi_h(z, u) \check{\rho}(s, t, z, u) dz du,$$

where $\Psi_h(z, u)$ and $\check{\rho}(s, t, z, u)$ are given in (2.2.8) and (2.5.2) respectively. Therefore,

$$\left| \frac{\partial}{\partial s} h(s_1, t) - \frac{\partial}{\partial s} h(s_2, t) \right| = \left| \int_t^T \Psi_h(z, u) \left(\int_{\mathbb{R}} (\check{\rho}(s_1, t, z, u) - \check{\rho}(s_2, t, z, u)) dz \right) du \right|.$$

If $\Delta = |s_1 - s_2| > 1$ then,

$$\begin{aligned} \frac{1}{\Delta^\beta} \left| \frac{\partial}{\partial s} h(s_1, t) - \frac{\partial}{\partial s} h(s_2, t) \right| &\leq \int_t^T \left(\int_{\mathbb{R}} |\check{\rho}(s_1, t, z, u)| dz \right) du \\ &\quad + \int_t^T \left(\int_{\mathbb{R}} |\check{\rho}(s_2, t, z, u)| dz \right) du < \infty. \end{aligned}$$

For $0 < \Delta < 1$, then,

$$\frac{1}{\Delta^\beta} \left| \frac{\partial}{\partial s} h(s_1, t) - \frac{\partial}{\partial s} h(s_2, t) \right| \leq \mathbf{B}_0 \int_t^T \left(\int_{\mathbb{R}} \frac{|\check{\rho}(s_1, t, z, u) - \check{\rho}(s_2, t, z, u)|}{\Delta^\beta} dz \right) du,$$

where from (2.5.1), $\int_t^T \Psi_h(z, u) du \leq \mathbf{B}_0$, and \mathbf{B}_0 is given in (2.2.3). Let

$$I(\Delta) = \int_t^T \left(\int_{\mathbb{R}} \frac{|\check{\rho}(s_1, t, z, u) - \check{\rho}(s_2, t, z, u)|}{\Delta^\beta} dz \right) du.$$

Then we can rewrite it as

$$\begin{aligned} I(\Delta) &= \int_t^{t+\Delta_1} \int_{\mathbb{R}} \frac{|\check{\rho}(s_1, t, z, u) - \check{\rho}(s_2, t, z, u)|}{\Delta^\beta} dz du \\ &\quad + \int_{t+\Delta_1}^T \int_{\mathbb{R}} \frac{|\check{\rho}(s_1, t, z, u) - \check{\rho}(s_2, t, z, u)|}{\Delta^\beta} dz du. \end{aligned}$$

Putting $\Delta_1 = \Delta^{2\beta}$ we obtain that

$$\begin{aligned} I(\Delta) &\leq \frac{1}{\Delta^\beta} \int_t^{t+\Delta_1} \left(\int_{\mathbb{R}} |\check{\rho}(s_1, t, z, u)| dz + \int_{\mathbb{R}} |\check{\rho}(s_2, t, z, u)| dz \right) du \\ &\quad + \frac{1}{\Delta^\beta} \int_{t+\Delta_1}^T \int_{\mathbb{R}} |\check{\rho}(s_1, t, z, u) - \check{\rho}(s_2, t, z, u)| dz du. \end{aligned}$$

Taking into account the bound (2.5.5), we estimate the integral $I(\Delta)$ as

$$I(\Delta) \leq \frac{\sqrt{2}}{\sqrt{\pi}} + \frac{1}{\Delta^\beta} \int_{t+\Delta_1}^T \int_{\mathbb{R}} |\check{\rho}(s_1, t, z, u) - \check{\rho}(s_2, t, z, u)| dz du.$$

Then

$$I(\Delta) \leq \frac{\sqrt{2}}{\sqrt{\pi}} + \frac{1}{\Delta^\beta} \int_{t+\Delta_1}^T \int_{s_1}^{s_2} \left(\int_{\mathbb{R}} |\check{\rho}_s(v, t, z, u)| \, dz \right) \, dv \, du,$$

where

$$\check{\rho}_s = \frac{\partial}{\partial s} \check{\rho}(v, t, z, u) = \frac{\mu^2}{\sqrt{2\pi}\sigma_1^3} e^{-\frac{K^2}{2}} (K^2 - 1)$$

and $K = K(s, z, u)$ is given in (2.5.3). Thus

$$|\check{\rho}_s| \leq \frac{1}{\sqrt{2\pi}\sigma_1^3} e^{-\frac{K^2}{2}} (K^2 + 1),$$

and

$$\int_{\mathbb{R}} |\check{\rho}_s(v, t, z, u)| \, dz = \frac{1}{\sigma_1^2} \int_{\mathbb{R}} |K^2 + 1| e^{-\frac{K^2}{2}} \, dK \leq \frac{\mathbf{c}^*}{\sigma_1^2}.$$

Taking into account that $\sigma_1^{-2} \leq \mathbf{c}^*(u-t)^{-1}$ for some $\mathbf{c}^* > 0$, we get

$$I(\Delta) \leq \frac{2\sqrt{2}}{\sqrt{\pi}} + \mathbf{c}^* \Delta^{1-\beta} \int_{t+\Delta_1}^T \frac{1}{u-t} \, du \leq \mathbf{c}^* + \Delta^{1-\beta} (|\ln \Delta_1| + |\ln T|).$$

Hence Proposition 2.6.1. □

Now we need to study the smoothness property with respect to t . We show now that the function f and its derivatives are Höldarians.

Proposition 2.6.2. *Let $h = \mathcal{L}_h$, with $h \in \mathcal{X}$. Therefore, for all t , for all $N \geq 1$, and $0 < \beta < 1/2$,*

$$\sup_{\substack{0 \leq t_1 \leq T \\ 0 \leq t_2 \leq T}} \sup_{|s| \leq N} \left(\frac{|h(s, t_1) - h(s, t_2)| + |h_s(s, t_1) - h_s(s, t_2)|}{|t_1 - t_2|^\beta} \right) < \infty.$$

Proof. Firstly, note that

$$h(s, t) = \int_t^T \Gamma(s, t, u) \, du \quad \text{and} \quad \Gamma(s, t, u) = \int_{\mathbb{R}} \Psi_h(z, u) \varphi(s, t, z, u) \, dz.$$

Therefore, for any $0 \leq t_1 \leq t_2 \leq T$,

$$h(s, t_2) - h(s, t_1) = \int_{t_2}^T \left(\Gamma(s, t_2, u) du - \Gamma(s, t_1, u) \right) du - \int_{t_1}^{t_2} \Gamma(s, t_1, u) du .$$

Let now $\Delta = t_2 - t_1$ and $\Delta_1 = \Delta^{2\beta}$ for some $0 < \beta < 1/2$. Taking into account that Γ is bounded, we obtain that for some $\mathbf{c}^* > 0$,

$$\frac{1}{\Delta^\beta} |h(s, t_2) - h(s, t_1)| \leq \frac{\mathbf{c}^*}{\Delta^\beta} I(\Delta) + c \Delta^{1-\beta} ,$$

where $I(\Delta) = \int_{t_2}^T \int_{\mathbb{R}} |\Omega(z, u)| dz du$ and $\Omega(z, u) = \varphi(s, t_2, z, u) - \varphi(s, t_1, z, u)$. We represent this term as $I(\Delta) = I_1(\Delta) + I_2(\Delta)$, where

$$I_1(\Delta) = \int_{t_2}^{t_2+\Delta_1} \int_{\mathbb{R}} |\Omega(z, u)| dz du \quad \text{and} \quad I_2(\Delta) = \int_{t_2+\Delta_1}^T \int_{\mathbb{R}} |\Omega(z, u)| dz du .$$

It is clear that $I_1(\Delta) \leq 2\Delta_1$. To estimate the term $I_2(\Delta)$ note that

$$|\Omega(z, u)| = |\varphi(s, t_2, z, u) - \varphi(s, t_1, z, u)| \leq \int_{t_1}^{t_2} |\varphi_t(s, \theta, z, u)| d\theta ,$$

where

$$\varphi_t(s, \theta, z, u) = \frac{\partial}{\partial t} \varphi(s, t, z, u) = \left(\frac{\sigma_2(u, t)}{2 \sqrt{2\pi} \sigma_1^3(u, t)} - \frac{K\dot{K}}{\sqrt{2\pi}\sigma_1} \right) e^{-\frac{K^2}{2}} .$$

The dot "·" here is the derivative with respect to t and $\sigma_2 = \dot{\sigma}_1$. Denoting by $\mu_1 = \dot{\mu}$, we obtain that

$$\dot{K} = \frac{\partial}{\partial t} \left(\frac{z - s\mu}{\sigma_1} \right) = \frac{s\mu_1}{\sigma_1} - \frac{z - s\mu}{\sigma_1^2} \dot{\sigma}_1 = -\frac{s\mu_1}{\sigma_1} - \frac{1}{2} K \frac{\sigma_2}{2\sigma_1^2} .$$

Taking into account that μ_1 is bounded, we obtain that for some $\mathbf{c}^* > 0$,

$$\left| \frac{\partial}{\partial t} \varphi(s, t, z, u) \right| \leq \mathbf{c}^* (1 + |s|) e^{-\frac{K^2}{2}} \frac{(K^2 + |K| + 1)}{\sigma_1^3} .$$

Therefore, for some $\mathbf{c}^* > 0$ and $u > t$

$$\int_{\mathbb{R}} \left| \frac{\partial}{\partial t} \varphi(s, t, z, u) \right| dz \leq \frac{\mathbf{c}^* (1 + |s|)}{u - t} ,$$

and we get

$$\begin{aligned} |I_2(\Delta)| &\leq \mathbf{c}^* (1 + |s|) \int_{t_1}^{t_2} \left(\int_{t_2+\Delta_1}^T \frac{1}{u-\theta} du \right) d\theta \\ &\leq \mathbf{c}^* (1 + |s|) \Delta \int_{t_2+\Delta_1}^T \frac{du}{u-t_2} \leq \mathbf{c}^* (1 + |s|) \Delta |\ln \Delta_1|. \end{aligned}$$

Therefore, for some $\mathbf{c}^* > 0$

$$\limsup_{\Delta \rightarrow 0} \frac{1}{\Delta^\beta} |h(s, t_2) - h(s, t_1)| \leq \mathbf{c}^* (1 + |s|).$$

Now to prove the second part we firstly take the partial derivative of the function h which may be represented by

$$\frac{\partial}{\partial s} h(s, t) = \frac{1}{\sqrt{2\pi}} \int_t^T \frac{\mu(u, t)}{\sigma_1^2(u, t)} \left(\int_{\mathbb{R}} \Psi_h(u, z) K e^{-\frac{K^2}{2}} dz \right) du. \quad (2.6.1)$$

Then,

$$\begin{aligned} \frac{\partial}{\partial s} h(s, t) &= \int_t^T \frac{\mu(u, t)}{\sigma_1(u, t)} \left(\int_{\mathbb{R}} \Psi_h(s\mu + \sigma_1 K, u) K \frac{e^{-\frac{K^2}{2}}}{\sqrt{2\pi}} dK \right) du \\ &= \int_t^T \frac{\mu(u, t)}{\sigma_1(u, t)} \left(\mathbf{E} \Psi_h(s\mu(u, t) + \sigma_1(u, t) \xi, u) \xi \right) du, \end{aligned}$$

where $\xi \sim \mathcal{N}(0, 1)$. So, we can represent the derivative (2.6.1) as

$$\frac{\partial}{\partial s} h(s, t) = \int_t^T \mathbf{q}(t, u) du \quad \text{and} \quad \mathbf{q}(t, u) = \mathbf{q}_1(t, u) \mathbf{q}_2(t, u),$$

where $\mathbf{q}_1(t, u) = \mathbf{E} \xi \Psi_h(s\mu(u, t) + \sigma_1(u, t) \xi, u)$ and $\mathbf{q}_2(t, u) = \mu(u, t) / \sigma_1(u, t)$. Setting now $\mathbf{q}_3(u) = \mathbf{q}(t_2, u) - \mathbf{q}(t_1, u)$, we obtain that

$$\frac{\partial}{\partial s} h(s, t_2) - \frac{\partial}{\partial s} h(s, t_1) = \int_{t_2}^T \mathbf{q}_3(u) du - \int_{t_1}^{t_2} \mathbf{q}(t_1, u) du.$$

Now we recall that the function Ψ_h is bounded, i.e. $|\mathbf{q}(t, u)| \leq \mathbf{c}^*/\sqrt{u-t}$ for some $\mathbf{c}^* > 0$. Therefore,

$$\begin{aligned} \left| \frac{\partial}{\partial s} h(s, t_2) - \frac{\partial}{\partial s} h(s, t_1) \right| &\leq \int_{t_2}^T |\mathbf{q}_3(u)| du + \int_{t_1}^{t_2} \frac{\mathbf{c}^*}{\sqrt{u-t_1}} du \\ &\leq I_1^*(\Delta) + I_2^*(\Delta) + 2\mathbf{c}^* \sqrt{\Delta}, \end{aligned}$$

where

$$I_1^*(\Delta) = \int_{t_2}^{t_2+\Delta_1} |\mathbf{q}_3(u)| du \quad \text{and} \quad I_2^*(\Delta) = \int_{t_2+\Delta_1}^T |\mathbf{q}_3(u)| du.$$

Similarly, for $0 < t_1 < t_2$

$$I_1^*(\Delta) \leq \mathbf{c}^* \int_{t_2}^{t_2+\delta_1} \left(\frac{1}{\sqrt{u-t_2}} + \frac{1}{\sqrt{u-t_1}} \right) du \leq 4\mathbf{c}^* \sqrt{\Delta_1}.$$

To estimate $I_2^*(\Delta)$, note that

$$\begin{aligned} |\mathbf{q}_3(u)| &= |\mathbf{q}_1(u, t_2) \mathbf{q}_2(u, t_2) - \mathbf{q}_1(u, t_1) \mathbf{q}_2(u, t_1)| \\ &\leq \left| \mathbf{q}_2(u, t_2) \left(\mathbf{q}_1(u, t_2) - \mathbf{q}_1(u, t_1) \right) \right| + \left| \mathbf{q}_1(u, t_1) \left(\mathbf{q}_2(u, t_2) - \mathbf{q}_2(u, t_1) \right) \right|. \end{aligned}$$

Moreover, noting that

$$\mathbf{q}_2(u, t) = \frac{\mu(u, t)}{\sigma_1(u, t)} \leq \frac{1}{\sigma_1(u, t)} \leq \frac{\mathbf{c}^*}{\sqrt{u-t}},$$

we obtain that for $u > t$,

$$|\mathbf{q}_3(u)| \leq c \left(\left| \mathbf{q}_2(u, t_2) - \mathbf{q}_2(u, t_1) \right| + \frac{1}{\sqrt{u-t_1}} \left| \mathbf{q}_1(u, t_2) - \mathbf{q}_1(u, t_1) \right| \right).$$

From the definition of q_1 , we can obtain that for some $\mathbf{c}^* > 0$,

$$\left| \mathbf{q}_2(u, t_2) - \mathbf{q}_2(u, t_1) \right| \leq \int_{t_1}^{t_2} \frac{1}{(u-\theta)^{\frac{3}{2}}} d\theta \leq \frac{\Delta}{(u-t_2)^{\frac{3}{2}}},$$

and

$$|\mathbf{q}_3(u)| \leq \frac{\Delta}{(u-t_2)^{\frac{3}{2}}} + \frac{\mathbf{c}^*}{\sqrt{u-t_1}} \left| \mathbf{q}_1(u, t_2) - \mathbf{q}_1(u, t_1) \right|.$$

It should be noted that Proposition 2.6.1 implies that for any $0 < \beta < 1$ and for some $\mathbf{c}^* > 0$,

$$\left| \Psi_h(s_2, t) - \Psi_h(s_1, t) \right| \leq \mathbf{c}^* |s_2 - s_1|^\beta.$$

So,

$$|\mathbf{q}_1(u, t_2) - \mathbf{q}_1(u, t_1)| \leq \mathbf{c}^* (1 + |s|^\beta) \left(|\mu(u, t_2) - \mu(u, t_1)|^\beta + |\sigma_1(u, t_2) - \sigma_1(u, t_1)|^\beta \right).$$

We recall that $|\sigma_1(u, t_2) - \sigma_1(u, t_1)| \leq \delta / \sqrt{u - t_2}$. Therefore,

$$|\mathbf{q}_1(u, t_2) - \mathbf{q}_1(u, t_1)| \leq (1 + |s|^\beta) \left(\frac{\Delta^\beta}{(\sqrt{u - t_2})^\beta} \right) \leq (1 + |s|^\beta) \left(\frac{\Delta^\beta}{(\sqrt{u - t_2})^\beta} \right).$$

Thus,

$$|\mathbf{q}_3(u)| \leq \frac{\Delta}{(u - t_2)^{\frac{3}{2}}} + \frac{(1 + |s|^\beta)}{\sqrt{u - t_2}} \frac{\Delta^\beta}{(\sqrt{u - t_2})^\beta}.$$

As a result,

$$\begin{aligned} I_2^*(\Delta) &\leq (1 + |s|^\beta) \int_{t_2 + \Delta_1}^T \left(\frac{\Delta}{(u - t_2)^{\frac{3}{2}}} + \frac{\Delta^\beta}{(u - t_2)^{\frac{\beta}{2} + \frac{1}{2}}} \right) du \\ &\leq \left(\frac{\Delta}{\sqrt{\Delta_1}} + \Delta^\beta (\Delta_1)^{\frac{1-\beta}{2}} \right) (1 + |s|^\beta). \end{aligned}$$

Therefore, for any $0 < \beta < 1/2$,

$$\overline{\lim}_{\Delta \rightarrow 0} \frac{I_1^*(\Delta) + I_2^*(\Delta)}{\Delta^\beta} < \infty.$$

Hence, Proposition 2.6.2. □

2.7 Proofs

2.7.1 Proof of Theorem 2.4.1

Let $h \in \mathcal{X}$ be the fixed point for the mapping \mathcal{L} , i.e. $h = \mathcal{L}_h$. Consider now the following equation

$$Y_t(s, t) + \frac{\sigma^2 Y_{ss}(s, t)}{2} + sg_1(t)Y_s(s, t) + \Psi_h(s, t) = 0, \quad Y(s, T) = 0, \quad (2.7.1)$$

where $g_1(t) = \gamma_1 g(t) - \gamma_2 > 0$ and $\Psi_h(s, t)$ is given in (2.2.7). Then we change the variables as $u(s, t) = Y(s, T - t)$. So we get

$$u_t(s, t) - \frac{\sigma^2 u_{ss}(s, t)}{2} - sg_1(T - t)u_s(s, t) - \Psi_h(s, T - t) = 0, \quad u(s, 0) = 0. \quad (2.7.2)$$

We can rewrite the previous equation as

$$u_t(s, t) - \frac{\sigma^2 u_{ss}(s, t)}{2} + a(s, t, u, u_s) = 0, \quad u(s, 0) = 0,$$

where $a(s, t, u, p) = -sg_1(T - t)p - \Psi_h(s, T - t)$. Taking into account that

$$\Psi_{max} = \sup_{s \in \mathbb{R}} \sup_{0 \leq t \leq T} \Psi_h(s, T - t) < \infty,$$

we obtain that $a(s, t, u, 0)u = -\Psi_h(s, T - t)|u| \geq -\Psi_{max}|u|$, i.e. the condition in (6.2.4) holds with $\Phi(r) = \Psi_{max}$ and $b = 0$. In view of Propositions 2.6.1 and 2.6.2, the function Ψ_h satisfies the Hölder condition C_5) for any $0 < \beta < 1/2$. By using Theorem 6.2.1 we obtain that equation (2.7.2) has a bounded solution. Therefore, there exists a solution of equation (2.7.1). In order to prove this theorem we use the probabilistic representation. Now, we define a stopping time τ_n

$$\tau_n = \inf \{n \geq t : |\eta_u^{s,t}| \geq n\} \wedge T,$$

where the process $(\eta_u^{s,t})_{u \geq t}$ is defined in (2.2.6). By Itô formula we obtain that

$$\begin{aligned} Y(s, t) = & - \int_t^{\tau_n} \left((Y_t(\eta_u^{s,t}, u) + g_1(u)\eta_u^{s,t}Y_s(\eta_u^{s,t}, u) + \frac{\sigma^2}{2}Y_{ss}(\eta_u^{s,t}, u)) \right) du \\ & - \int_t^{\tau_n} Y_s(\eta_u^{s,t}, u) d\check{W}_u + Y(\eta_{\tau_n}^{s,t}, \tau_n). \end{aligned}$$

Taking into account equation (2.3.7), we obtain that

$$Y(s, t) = \int_t^{\tau_n} \Psi_h(\eta_u^{s,t}, u) du - \int_t^{\tau_n} Y_s(\eta_u^{s,t}, u) d\check{W}_u + Y(\tau_n, \eta_{\tau_n}^{s,t}).$$

As $\mathbf{E} \int_t^{\tau_n} Y_s(\eta_u^{s,t}, u) d\check{W}_u = 0$, we obtain

$$Y(s, t) = \mathbf{E} \int_t^{\tau_n} \Psi_h(\eta_u^{s,t}, u) du + \mathbf{E}Y(\tau_n, \eta_{\tau_n}^{s,t}).$$

Note here that the solution of equation (2.7.1) is bounded. So, by Dominated Convergence theorem and in view of the boundary condition in (2.7.1) we obtain that

$$\lim_{n \rightarrow \infty} \mathbf{E}Y(\eta_{\tau_n}^{s,t}, \tau_n) = \mathbf{E} \lim_{n \rightarrow \infty} Y(\eta_{\tau_n}^{s,t}, \tau_n) = \mathbf{E}Y(\eta_T^{s,t}, T) = 0.$$

Moreover, taking into account that $\Psi_h \geq 0$, by the Monotone Convergence theorem we obtain

$$Y(s, t) = \mathbf{E} \lim_{n \rightarrow \infty} \int_t^{\tau_n} \Psi_h(\eta_u^{s,t}, u) du = \mathbf{E} \int_t^T \Psi_h(\eta_u^{s,t}, u) du,$$

i.e. $Y(s, t) = \mathcal{L}_h(s, t) = h$. Hence Theorem 2.4.1. \square

2.7.2 Proof of Theorem 2.4.2

To proof this theorem we use the verification theorem 6.3.1 and find the solution to the HJB equation(2.3.5) using FK mapping with h a fixed point for the mapping \mathcal{L} . Therefore, the function

$$z(\zeta, t) = \varpi x^\gamma \exp \left\{ \frac{s^2}{2} g(t) + h(s, t) \right\}, \quad (2.7.3)$$

is the solution of the HJB equation (2.3.5). By using this function we calculate the optimal control variables in (2.3.9) and we obtain the strategies (2.3.10) - (2.3.13). Hence \mathbf{H}_3). Now we want to check condition \mathbf{H}_4). First note that the equation

$$d\zeta_t^* = a(\zeta_t^*, t)dt + b(\zeta_t^*, t)dW_t, \quad t \geq 0, \quad \text{and} \quad \zeta_0^* = x,$$

is identical to the linear equation (2.3.10). By the assumptions on the market parameters, the coefficients $a(t)$ and $b(t)$ are continuous. So, it has the positive solution Eq. (2.3.12) and therefore, we obtain \mathbf{H}_4). To check the condition \mathbf{H}_5), we need the following Lemma.

Lemma 2.7.1. *If $0 < T < T_0$, then there exists some $\check{\delta} > 1$ such that*

$$\sup_{\tau \in \mathcal{M}_t} \mathbf{E} \left(z^{\check{\delta}}(\zeta_\tau^*, \tau) \mid \zeta_t = \zeta \right) < \infty, \quad (2.7.4)$$

where \mathcal{M}_t is the set of all stopping times with $t \leq \tau \leq T$ and the function z is given in (2.7.3).

Lemma 2.7.1 yields condition \mathbf{H}_5), where $z(\zeta, t) = \bar{\omega} x^\gamma \exp\{s^2 g(t)/2 + h(s, t)\}$ and h is the fixed point (2.3.8). Now, the verification Theorem 6.3.1 implies Theorem 2.4.2. \square

2.7.3 Proof of Theorem 2.4.3

We set $\Delta_n(y, t) = h(y, t) - h_n(y, t)$. So, from Eq. (2.5.8), we obtain that

$$\begin{aligned} \Upsilon_{h, h_n}^*(y, t) &= \sup_{(y, t) \in \mathcal{X}} (|\Delta_n(y, t)| + |D_y \Delta_n(y, t)|) \leq e^{\varkappa T} \rho(h, h_n) \\ &\leq (\mathbf{B}_0 + \mathbf{B}_1) \frac{\lambda^n}{1 - \lambda} e^{\varkappa T} = (\mathbf{B}_0 + \mathbf{B}_1) \exp\{H(\lambda, \varkappa)\}, \end{aligned}$$

where $H(\lambda, \varkappa) = \varkappa T + n \ln \lambda - \ln(1 - \lambda)$ and \mathbf{B}_0 and \mathbf{B}_1 are defined in (2.2.3).

So, if we take $\varkappa = n(\tilde{\mathbf{B}}_2)^2$ with \mathbf{B}_2 defined in Eq. (2.5.7), then we obtain, for $\lambda = 1/\sqrt{n}$, that

$$\Upsilon_{h, h_n}^*(y, t) = O(n^{-\delta n}),$$

for any $0 < \delta < 1/2$. Hence Theorem 2.4.3. \square

2.7.4 Proof of Proposition 2.5.1

One can check directly that the set \mathcal{X} is closed in the set $\mathbf{C}^{1,0}(\mathbb{R} \times [0, T])$ which is complete.

So, the space (\mathcal{X}, ρ) is complete also. Hence Theorem 2.5.1. \square

2.7.5 Proof of Proposition 2.5.3

Firstly, note that from the definition of the mapping in (2.2.7) we get that for any $\delta > 0$,

$$\frac{\mathcal{L}_h(s + \delta, t) - \mathcal{L}_h(s, t)}{\delta} = \int_t^T \int_{\mathbb{R}} \left(\Psi_h(z, u) \left(\frac{\varphi(s + \delta, t, z, u) - \varphi(s, t, z, u)}{\delta} \right) dz \right) du.$$

Taking into account that the function $\check{\rho}$ is continuously differentiable, we can rewrite

$$\frac{\varphi(s + \delta, t, z, u) - \varphi(s, t, z, u)}{\delta} = \frac{1}{\delta} \int_s^{s+\delta} \check{\rho}(v, t, z, u) dv = \check{\rho}(s, t, z, u) + \mathbf{D}_\delta(s, t, z, u),$$

where $\check{\rho}(s, t, z, u) = \partial \varphi(s, t, z, u) / \partial s$ and

$$\mathbf{D}_\delta(s, t, z, u) = \frac{1}{\delta} \int_s^{s+\delta} (\check{\rho}(v, t, z, u) - \check{\rho}(s, t, z, u)) dv.$$

So,

$$\frac{\mathcal{L}_h(s + \delta, t) - \mathcal{L}_h(s, t)}{\delta} = \int_t^T \left(\int_{\mathbb{R}} \Psi_h(z, u) \check{\rho}(s, t, z, u) dz \right) du + \mathbf{G}_\delta,$$

where $\mathbf{G}_\delta = \int_t^T \left(\int_{\mathbb{R}} \Psi_h(z, u) \mathbf{D}_\delta(s, t, z, u) dz \right) du$. Now we have to prove that the term \mathbf{G}_δ goes to zero as $\delta \rightarrow 0$.

As the function $\Psi_h(s, t)$ is bounded for $h \in \mathcal{X}$, therefore,

$$|\mathbf{G}_\delta| \leq \Psi^* \int_t^T \frac{1}{\delta} \left(\int_s^{s+\delta} \mathbf{L}(v, u) dv \right) du \leq \Psi^* \int_t^T \mathbf{L}_\delta^*(u) du,$$

where $\Psi^* = \sup_{z \in \mathbb{R}, 0 \leq t \leq T} |\Psi_h(z, u)|$, the function $\mathbf{L}_\delta^*(u) = \max_{s \leq v \leq s+\delta} \mathbf{L}(v, u)$ and

$$\mathbf{L}(v, u) = \int_{\mathbb{R}} |\check{\rho}(v, t, z, u) - \check{\rho}(s, t, z, u)| dz.$$

We can check directly that for some $\mathbf{c}^* > 0$

$$\sup_{0 < \delta < 1} \mathbf{L}_\delta^*(u) \leq \frac{\mathbf{c}^*}{\sqrt{u-t}}.$$

Moreover, note that for some $N > 1$

$$\mathbf{L}(v, u) \leq \int_{|z| \leq N} |\check{\rho}(v, t, z, u) - \check{\rho}(s, t, z, u)| dz + \int_{|z| > N} |\check{\rho}(v, t, z, u) - \check{\rho}(s, t, z, u)| dz.$$

The first part approaches zero when $N \rightarrow 0$, and

$$\begin{aligned} \int_{|z|>N} |\check{\rho}(v, t, z, u)| dz &= \frac{\mu(u, t)}{\sqrt{2\pi}\sigma_1(u, t)} \int_{|\sigma_1 y + v\mu| > N} |y| e^{-\frac{y^2}{2}} dy \\ &\leq \frac{\mu(u, t)}{\sqrt{2\pi}\sigma_1(u, t)} \int_{|y| > N_1} |y| e^{-\frac{y^2}{2}} dy \rightarrow 0 \quad \text{as } N \rightarrow \infty, \end{aligned}$$

where $N_1 = (N - (|s| + \delta)|\mu|)/\sigma_1$, and s, μ, σ_1 are fixed.

Thus, for any s, t , and u ,

$$\lim_{\delta \rightarrow 0} \mathbf{L}_\delta^*(u) = 0. \quad (2.7.5)$$

So, by Lebesgue dominated convergence theorem,

$$\int_t^T \mathbf{L}_\delta^*(u) du \rightarrow 0.$$

Hence Theorem 2.5.3. \square

2.7.6 Proof of Lemma 2.7.1

Proof. From the optimal wealth process given in Eq. (2.3.10) through Itô formula we have that

$$X_t^* = x \exp \left\{ \int_0^t a^*(u) du \right\} \mathcal{E}_{0,t}(b^*),$$

where the function a^* and b^* are defined in Eq. (2.3.10) and

$$\mathcal{E}_{0,t}(b^*) = \exp \left\{ \int_0^t b^*(u) dW_u - \frac{1}{2} \int_0^t (b^*(u))^2 du \right\}.$$

We will show Lemma 2.7.1 for $\check{\delta} = 1 + (1 - \gamma)/2\gamma$, taking into account that $z(\zeta, t) \leq \mathbf{c}^* x^\gamma \exp\{s^2 g(0)/2\}$. To this end it is sufficient to show that

$$\sup_{\tau \in \mathcal{M}_t} \mathbf{E} \left((X_\tau^*)^{\delta_1} \exp \left\{ \frac{\check{\delta}_1}{2} S_\tau^2 \right\} \middle| X_t = x, S_t = s \right) < \infty,$$

where $\delta_1 = \gamma\check{\delta} = (1 + \gamma)/2 < 1$ and $\check{\delta}_1 = g(0)\check{\delta}$. Note that $(\mathcal{E}_{t,u})_{u \geq t}$ is supermartingale and $\mathbf{E}\mathcal{E}_{t,\tau}(b^*) \leq 1$ for any stopping time $\tau \in \mathcal{M}$. Moreover, note that

$$S_\tau = e^{-\varkappa(T-t)}s + \xi_{t,\tau} \quad \text{and} \quad \xi_{t,\tau} = \sigma e^{-\varkappa\tau} \int_t^\tau e^{\varkappa u} dW_u.$$

Since $|S_\tau| \leq |s| + |\xi_{t,\tau}|$, one needs to check that

$$\sup_{\tau \in \mathcal{M}_t} \mathbf{E} \left((X_\tau^*)^{\delta_1} \exp \{ \check{\delta}_1 \xi_{t,\tau}^2 \} \right) < \infty.$$

By Hölder inequality we obtain that for $p = (1 + \delta_1)/2\delta_1$ and $q = (1 + \delta_1)/(1 - \delta_1)$

$$\mathbf{E}_{\zeta,t} (X_\tau^*)^{\delta_1} \exp \{ \check{\delta}_1 \xi_{t,\tau}^2 \} \leq \left(\mathbf{E}_{\zeta,t} (X_\tau^*)^{\delta_2} \right)^{\frac{1}{p}} \left(\mathbf{E}_{\zeta,t} \exp \{ q \check{\delta}_1 \xi_{t,\tau}^2 \} \right)^{\frac{1}{q}},$$

where $\delta_2 = p\delta_1 = (1 + \delta_1)/2 < 1$. Note that

$$\mathbf{E}_{\zeta,t} (X_\tau^*)^{\delta_2} = x^{\delta_2} \mathbf{E} \exp \left\{ \delta_2 \int_t^\tau a^*(u) du \right\} \left(\mathcal{E}_{t,\tau}(b^*) \right)^{\delta_2}.$$

By Hölder inequality, for $r = 1/\delta_2$ and $q_1 = 1/(1 - \delta_2)$

$$\begin{aligned} \mathbf{E}_{\zeta,t} (X_\tau^*)^{\delta_2} &\leq x^{\delta_2} \left(\mathbf{E} \exp \left\{ \frac{\delta_2}{1 - \delta_2} \int_t^\tau a^*(u) du \right\} \right)^{1 - \delta_2} \left(\mathbf{E}_{t,\tau} \mathcal{E}_{t,\tau}(b^*) \right)^{\delta_2} \\ &\leq x^{\delta_2} \left(\mathbf{E} \exp \left\{ \frac{\delta_2}{1 - \delta_2} \int_t^T |a^*(u)| du \right\} \right)^{1 - \delta_2}. \end{aligned} \tag{2.7.6}$$

Moreover, note that

$$|a^*(t)| \leq \left(g(0) + \frac{\kappa_1}{\sigma^2} \right) \kappa_1 s^2 + \kappa_1 |s| \mathbf{B}_1 + 1 + r \leq \kappa_2 s^2 + \mathbf{c}^*,$$

where \mathbf{c}^* is some constant and $\kappa_2 = \kappa_1^2(1/\sigma^2 + 1/2) + g(0)\kappa_1$. So, for some $\mathbf{c}^* > 0$

$$\int_t^T |a^*(u)| du \leq 2\kappa_2 \int_t^T \xi_{t,u}^2 du + \mathbf{c}^*.$$

Let us show now that

$$\mathbf{E} \exp \left\{ \tilde{\kappa}_2 \int_t^T \xi_{t,u}^2 du \right\} < \infty,$$

where $\tilde{\kappa}_2 = 2\delta_2\kappa_2/(1 - \delta_2) = \left(2(3 + \gamma)\kappa_1^2(1/\sigma^2 + 1/2 + g(0)/\kappa_1)\right)/(1 - \gamma)$.

$$\mathbf{E} \exp \left\{ \tilde{\kappa}_2 \left(\int_t^T \xi_{t,u}^2 du \right) \right\} = \sum_{m=0}^{\infty} \frac{\tilde{\kappa}_2^m}{m!} \mathbf{E} \left(\int_t^T \xi_{t,u}^2 du \right)^m < \infty.$$

Moreover, note that in view of the Hölder inequality

$$\mathbf{E} \left(\int_t^T \xi_{t,u}^2 du \right)^m \leq (T - t)^{m-1} \int_t^T \mathbf{E} \xi_{t,u}^{2m} du.$$

Taking into account that $\xi_{t,u} \sim \mathcal{N}(0, \int_t^u e^{-2\kappa(u-v)} dv)$, we obtain that

$$\mathbf{E} \xi_{t,u}^{2m} = (2m - 1)!! \left(\int_t^u e^{-2\kappa(u-v)} dv \right)^m \leq \frac{m!}{\kappa^m}$$

and

$$\mathbf{E} \left(\int_t^T \xi_{t,u}^2 du \right)^m \leq m! \frac{T^m}{\kappa^m}.$$

Therefore,

$$\mathbf{E} \exp \left\{ \tilde{\kappa}_2 \int_t^T \xi_{t,u}^2 du \right\} = \sum_{m=0}^{\infty} \frac{\tilde{\kappa}_2^m}{m!} \mathbf{E} \left(\int_t^T \xi_{t,u}^2 du \right)^m \leq \sum_{m=0}^{\infty} \left(\frac{\tilde{\kappa}_2 T}{\kappa} \right)^m.$$

In view of the definition of T_0 in Eq. (2.2.10) we obtain that the condition $T < T_0$ implies that $T < \kappa/\tilde{\kappa}_2$, i.e. this series is finite. Moreover, by Proposition 6.5.3, we have that $\mathbf{E} \xi_{t,\tau} \leq m!(2\sigma^2 T)^m$ for all $m \geq 1$, and for any $\tau \in \mathcal{M}_t$. So

$$\begin{aligned} \mathbf{E}_{\zeta,t} \exp \{ q \check{\delta}_1 \xi_{t,\tau}^2 \} &= 1 + \sum_{m=1}^{\infty} \frac{(q \check{\delta}_1)^m}{m!} \mathbf{E} \xi_{t,\tau}^m \\ &\leq 1 + \sum_{m=1}^{\infty} \frac{(q \check{\delta}_1)^m}{m!} m! (2\sigma^2 T)^m \leq 1 + \sum_{m=1}^{\infty} (2q \check{\delta}_1 \sigma^2 T)^m < \infty, \end{aligned}$$

for $T < 1/2q\sigma^2\check{\delta}_1 = \gamma(1 - \gamma)/(3 + \gamma)(1 + \gamma)\sigma^2g(0)$, which is true due to the condition $T < T_0$. Hence Lemma 2.7.1 \square

Chapter 3

Optimisation for power utility functions for any time interval

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In this chapter we consider the problem Eq. (2.1.5) for any time interval. Therefore, This can be considered to be a generalisation to the previous chapter.

3.1 Main results

First we study the HJB equation Eq. (2.3.3).

Theorem 3.1.1. *Assume that $\bar{\omega} \geq (16T/\pi)^{1-\gamma}$, then equation (2.3.3) has the solution defined by (2.3.6), where Y is the unique solution of (2.3.7) in \mathcal{X} and is the fixed point for the FK mapping, i.e., $Y = h$, and $h = \mathcal{L}_h$.*

Theorem 3.1.2. *Assume that $\bar{\omega} \geq (16T/\pi)^{1-\gamma}$ and $0 < \gamma < 1/4$, then the optimal value of the objective function (2.1.5) is given by*

$$J^*(\zeta, t) = J(\zeta, t, \mathbf{v}^*) = \bar{\omega} x^\gamma \exp \left\{ \frac{s^2}{2} g(t) + h(s, t) \right\},$$

where h is a fixed point solution and the optimal control $\mathbf{v}_t^* = (\alpha_t^*, c_t^*)$ for all $0 \leq t \leq T$ is given in (2.3.13) with the function $h \in \mathcal{X}$ defined in (2.3.8), i.e. $h = \mathcal{L}_h$. The optimal wealth process $(X_t^*)_{0 \leq t \leq T}$ is the solution to (2.3.10).

Proof. The proofs of main theorems are proven by making use of verification theorem 6.3.1 and Lemma 3.2.2 similarly to the proofs of Theorem 2.4.1 and Theorem 2.4.2 respectively. \square

Remark 3.1.1. It should be noted that in Theorem 3.1.1 and Theorem 3.1.2, we impose only some conditions on the parameter ϖ as a function of the time interval $[0, T]$. There is no other conditions on ϖ and T . This means that for any time interval $T > 0$, we treat the optimization problem Eq. (2.1.5) with some fixed coefficients ϖ more than some threshold depending on T . In this sense, we find the optimal strategy for Eq. (2.1.5) for any time interval $[0, T]$. Different from the previous chapter, we have no additional conditions on T . We only impose that $0 < \gamma < 1/4$.

3.2 Properties of optimal strategies

We need to study the properties of the mapping (2.2.7).

Proposition 3.2.1. *Assume that $\varpi \geq (16T/\pi)^{1-\gamma}$. Then $\mathcal{L}_h \in \mathcal{X}$ for any $h \in \mathcal{X}$, i.e. $\mathcal{L}_h: \mathcal{X} \rightarrow \mathcal{X}$.*

Proof. The proof follows from Proposition 2.5.2. \square

Lemma 3.2.2. *If $0 < \gamma < 1/4$, then there exists some $\check{\delta} > 1$ such that*

$$\sup_{\tau \in \mathcal{M}_t} \mathbf{E} \left(z^{\check{\delta}}(\varsigma_\tau^*, \tau) | \varsigma_t^* = \varsigma \right) < \infty, \quad (3.2.1)$$

where \mathcal{M}_t is the set of all stopping times with $t \leq \tau \leq T$ and the function z is given in (2.7.3).

Proof. From the optimal wealth process given in (2.3.10) and through Itô formula we have that

$$X_t^* = x \exp \left\{ \int_0^t a^*(u) du \right\} \mathcal{E}_{0,t}(b^*),$$

where the functions a^* and b^* are defined in (2.3.10) and

$$\mathcal{E}_{0,t}(b^*) = \exp \left\{ \int_0^t b^*(u) dW_u - \frac{1}{2} \int_0^t (b^*(u))^2 du \right\}.$$

Therefore,

$$X_t^* = x \exp \left\{ \int_0^t k^*(u) du + \int_0^t b^*(u) dW_u \right\} \quad \text{and} \quad k^*(u) = K^*(S_u, u),$$

where $K^*(s, t) = A^*(s, t) - (B^*(s, t))^2/2$ and the functions $A^*(s, t)$ and $B^*(s, t)$ are defined in (2.3.11). Taking into account the bound in inequality (2.2.2), we obtain that

$$z(\zeta, t) \leq \mathbf{c}^* x^\gamma \exp \left\{ \frac{s^2}{2} g(0) \right\} \leq \exp \left\{ \frac{s^2 \sqrt{\gamma} \kappa_1}{2\sigma^2} \right\}.$$

Therefore, it is sufficient to show that

$$\sup_{\tau \in \mathcal{M}_t} \mathbf{E} \left(\Upsilon_\tau \middle| X_t = x, S_t = s \right) < \infty, \quad (3.2.2)$$

where $\Upsilon_\nu = (X_\nu^*)^{\delta_1} \exp \left\{ \delta_2 \frac{S_\nu^2}{\sigma^2} \right\}$. It is clear that

$$\mathbf{E} \left(\Upsilon_\tau \middle| X_t = x, S_t = s \right) = x^{\delta_1} \mathbf{E} \exp \left\{ \delta_1 J_{t,\tau} + \delta_2 \frac{S_{t,\tau}^2}{\sigma^2} \right\}, \quad (3.2.3)$$

where $J_{t,v} = \int_t^v k_t^*(u) du + \int_t^v b_t^*(u) dW_u$, $k_t^*(u) = K^*(S_{t,u}, u)$, $b_t^*(u) = B^*(S_{t,u}, u)$ and for $u > t$

$$S_{t,u} = e^{-\kappa(u-t)} s + \sigma \xi_{t,u} \quad \text{and} \quad \xi_{t,u} = e^{-\kappa u} \int_t^u e^{\kappa v} dW_v.$$

Note now, that the upper bound from Eq. (6.5.5) in Proposition 6.5.4 implies that for any $N > 0$

$$\sup_{\tau \in \mathcal{M}_t} \mathbf{E} e^{N|\xi_{t,\tau}|} < \infty. \quad (3.2.4)$$

Therefore, we can write that

$$\frac{S_{t,u}^2}{\sigma^2} = \xi_{t,u}^2 + \check{S}_{t,u},$$

where the term $\check{S}_{t,u}$ satisfies the property (3.2.4). Moreover, taking into account that

$$d\xi_{t,u}^2 = -2\kappa \xi_{t,u}^2 du + du + 2\xi_{t,u} dW_u, \quad \xi_{t,t} = 0,$$

So, the power in the exponential in the left side of the equality (3.2.3) can be represented as

$$\delta_1 J_{t,v} + \delta_2 \frac{S_{t,v}^2}{\sigma^2} = L_{t,v} + \check{S}_{t,u}, \quad \text{and} \quad L_{t,v} = \int_t^v \zeta_1(t, z) dW_z + \int_t^v \zeta_2(t, z) dz,$$

where $\zeta_1(t, z) = \delta_1 b_t^*(z) + 2\check{\delta}_1 \xi_{t,z}$ and $\zeta_2(t, z) = \delta_1 k_t^*(z) - 2\check{\delta}_1 \kappa \xi_{t,z}^2 + \check{\delta}_1$. Therefore, to show (3.2.2) it is sufficient to check that

$$\sup_{\tau \in \mathcal{M}_t} \mathbf{E} \exp \{L_{t,\tau} + \check{S}_{t,\tau}\} < \infty. \quad (3.2.5)$$

Taking into account here that

$$\sup_{\tau \in \mathcal{M}_t} \mathbf{E} \exp \left\{ 2 \int_t^\tau \zeta_1(t, z) dW_z - 2 \int_t^\tau \zeta_1^2(t, z) dz \right\} \leq 1,$$

we obtain through the Cauchy–Schwarz inequality, that for any $\tau \in \mathcal{M}_t$

$$\mathbf{E} \exp \{L_{t,\tau}\} \leq \sqrt{\mathbf{E} \exp \left\{ 2 \int_t^\tau D_t(v) dv + 2\check{S}_{t,\tau} \right\}},$$

where $D_t(v) = \zeta_2(t, v) + \zeta_1^2(t, v)$.

Note now that, from the definitions of $A^*(s, t)$ and $B^*(s, t)$ in (2.3.10), we can represent these functions as

$$a_t^*(v) = \frac{\kappa_1^2}{1-\gamma} \tilde{g}(v) \xi_{t,v}^2 + \check{a}_t^*(v) \quad \text{and} \quad b_t^*(v) = -\frac{\kappa_1}{1-\gamma} \tilde{g}(v) \xi_{t,v} + \check{b}_t^*(v),$$

where $\tilde{g}(v) = 1 - \sigma^2 g(v)/\kappa_1$ and the functions $\check{a}_t^*(v)$ and $\check{b}_t^*(v)$ are such that for any $N > 0$,

$$\mathbf{E} \exp \left\{ N \int_t^T |\check{a}_t^*(v)| dv \right\} < \infty \quad (3.2.6)$$

and $\check{b}_t^*(v)$ is bounded i.e., $\sup_{t \leq v \leq T} |\check{b}_t^*(v)| < \infty$. Note, that the function $D_t(v)$ can be represented as

$$D_t(v) = G(v) \xi_{t,v}^2 + \check{D}_t(v),$$

where the function $\check{D}_t(v)$ satisfies the property (3.2.6). The function $\check{g}(v)$ can be represented as

$$G(v) = -t_1 \check{g}^2(v) + t_2 \check{g}(v) - t_3. \quad (3.2.7)$$

Here,

$$t_1 = \left(\frac{1}{2} - \delta_1 \right) \frac{\delta_1 \kappa_1^2}{(1-\gamma)^2}, \quad t_2 = \frac{\delta_1 \kappa_1}{1-\gamma} (\kappa_1 - 4\delta_2) \quad \text{and} \quad t_3 = 2\delta_2 (\kappa - 2\delta_2).$$

If we take here

$$\check{\delta} = \frac{1}{2\sqrt{\gamma}}, \quad \delta_1 = \gamma\check{\delta} = \frac{\sqrt{\gamma}}{2} \quad \text{and} \quad \delta_2 = \frac{\check{\delta}\sqrt{\gamma}\kappa_1}{2} = \frac{\kappa_1}{4},$$

then we obtain that

$$G(v) = -\frac{(1-\sqrt{\gamma})\sqrt{\gamma}\kappa_1^2}{4(1-\gamma)^2}\tilde{g}^2(v) - \frac{\kappa_1(2\kappa-\kappa_1)}{4} \leq 0.$$

So, this implies the upper bound (3.2.1) and, taking into account that $\check{\delta} > 1$ for $\gamma < 1/4$ we come to Lemma 3.2.2. \square

Chapter 4

Optimisation for logarithmic utility functions

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This chapter deals with an optimal investment/consumption problem during a fixed time interval $[0, T]$ for a financial market generated by one non-risky \check{S} asset and risky spread (difference) assets S_t defined through the general multivariate Ornstein–Uhlenbeck (OU) processes with constant volatility for logarithmic utility functions.

4.1 Market model

Let $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbf{P})$ be a standard filtered probability space with Wiener processes $W = (W_t)_{0 \leq t \leq T} \in \mathbb{R}^m$ and $\mathcal{F}_t = \sigma\{W_u, u \leq t\}$. Our financial market consists of one *riskless bond* $(\check{S}_t)_{0 \leq t \leq T}$ and *risky spread stocks* $(S_t)_{0 \leq t \leq T}$ governed by the following equations:

$$\begin{cases} d\check{S}_t = r\check{S}_t dt, & \check{S}_0 = 1, \\ dS_t = AS_t dt + \sigma dW_t, & S_0 > 0, \end{cases} \quad (4.1.1)$$

where $r \geq 0$ is the interest rate for riskless asset, the d vector risky assets $S_t = (S_1(t), S_2(t), S_3(t), \dots, S_d(t))$, the standard Brownian motion $(W_t)_{0 \leq t \leq T}$ with values in \mathbb{R}^m , the volatility σ is a $d \times m$ matrix such that $(\sigma\sigma')^{-1}$ exists, and the $d \times d$ mean reverting matrix A is given by

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1d} \\ a_{21} & a_{22} & \dots & a_{2d} \\ \vdots & & \ddots & \\ a_{d1} & a_{d2} & \dots & a_{dd} \end{pmatrix}, \quad (4.1.2)$$

We assume that

$$\sup_{x \in \mathbb{R}^d} \frac{x'Ax}{|x|^2} < 0, \quad (4.1.3)$$

$\text{Re}\lambda_i(A) < 0$. where $'$ denotes the transposition. Note that this condition implies that the real parts of the eigenvalues are negative i.e. $\text{Re}\lambda_i(A) < 0$. Let now $\check{\alpha}_t$ be the number of riskless assets \check{S} and $\alpha_t = (\alpha_1(t), \alpha_2(t), \dots, \alpha_d(t)) \in \mathbb{R}^d$ be the number of risky assets at the moment $0 \leq t \leq T$, and the consumption rate is given by a non negative integrated function $(c_t)_{0 \leq t \leq T}$ [44]. Thus the wealth process for $X_t = \check{\alpha}_t \check{S}_t + \alpha_t' S_t$ is given by

$$dX_t = \check{\alpha}_t d\check{S}_t + \alpha_t' dS_t - c_t dt,$$

which can be written as

$$dX_t = (rX_t - \alpha_t' \widehat{S}_t - c_t) dt + \alpha_t' \sigma dW_t, \quad (4.1.4)$$

where $\widehat{S}_t = A_1 S_t = (\widehat{S}_1(t), \dots, \widehat{S}_d(t))' \in \mathbb{R}^d$ and $A_1 = rI_d - A$, the prime $'$ denotes the transposition. Note that in this case the matrix A_1 is invertible, i.e. there exists A_1^{-1} . In the sequel we denote the financial strategy by $v_t = (\alpha_t, c_t)' \in \Theta = \mathbb{R}^d \times \mathbb{R}_+$ and the wealth process (4.1.4) corresponding to this strategy by X_t^v . Moreover, we set $\zeta_t^v = (X_t^v, S_t)' \in \mathbb{R}^n$, where $n = d + 1$.

In this dissertation, we use the logarithmic utility functions, i.e., we need the following definition for the admissible strategies.

Definition 4.1.1. The strategy $v = (v_t)_{0 \leq t \leq T}$ is called admissible if it is adapted, equation (4.1.4) has a unique nonnegative strong solution and the following conditions hold.

$$\mathbf{E} \left(\int_0^T (\ln c_t)_- dt \right) < \infty, \quad \mathbf{E} \sup_{0 \leq t \leq T} (\ln(X_t^v))_- < \infty, \quad (4.1.5)$$

and

$$\int_0^T |\alpha_t|^2 dt < +\infty.$$

We denote by \mathcal{V} the set of all admissible financial strategies.

Now for any $v \in \mathcal{V}$ and $\zeta = (x, s)'$ from $\Xi = \mathbb{R}_+ \times \mathbb{R}^d$, we define the objective function as

$$\mathbf{J}(\zeta, v) := \mathbf{E}_\zeta \left(\int_0^T (\ln c_u) du + \varpi \ln(X_T^v) \right),$$

where \mathbf{E}_ζ is the expectation under condition $\zeta_0^v = \zeta$. Our goal in this thesis is to maximize this function, i.e.

$$\mathbf{J}^*(\zeta) := \sup_{v \in \mathcal{V}} \mathbf{J}(\zeta, v). \quad (4.1.6)$$

To study this problem we use the stochastic dynamic programming method. To this end we need to consider the optimization problems on the interval $[t, T]$ for any $0 \leq t < T$. For the problem on the interval $[t, T]$ we use the strategies v from \mathcal{V} such that the process $(v_u)_{t \leq u \leq T}$ is adapted to the field family $(\mathcal{F}_{t,u})_{t \leq u \leq T}$, where $\mathcal{F}_{t,u} = \sigma\{W_v - W_t, t \leq v \leq u\}$. The class of such strategies we denote by \mathcal{V}_t . Now we need to study the value functions $(\mathbf{J}^*(\zeta, t))_{0 \leq t \leq T}$ defined as

$$\mathbf{J}^*(\zeta, t) = \sup_{v \in \mathcal{V}_t} \mathbf{E}_{\zeta, t} \left(\int_t^T (\ln c_u) du + \varpi \ln(X_T^v) \right),$$

where $\varpi > 0$ and $\mathbf{E}_{\zeta, t}$ is the expectation under condition $\zeta_t^v = \zeta = (x, s) \in \Xi$. Thus we need to study the HJB equation which is given in the following section.

4.2 Hamilton–Jacobi–Bellman equation

Using the process ζ_t^v , we can rewrite the wealth and stock equations given in Eq. (4.1.1) and (4.1.4) respectively in the following form

$$d\zeta_t^v = \check{a}(\zeta_t^v, v_t) dt + \check{b}(\zeta_t^v, v_t) dW_t, \quad \zeta_0 = \zeta, \quad (4.2.1)$$

where $\check{a} \in \mathbb{R}^n$ and \check{b} is the matrix of $n \times m$ functions such that for any $\zeta = (x, s) \in \Xi$

$$\check{a}(\zeta, \mathbf{u}) = \begin{pmatrix} rx - \alpha' \hat{s} - c \\ \tilde{s} \end{pmatrix} \quad \text{and} \quad \check{b}(\zeta, \mathbf{u}) = \begin{pmatrix} \alpha' \sigma \\ \sigma \end{pmatrix},$$

where $\hat{s} = A_1 s$, and we denote by $\tilde{s} = A s = (\tilde{s}_1, \dots, \tilde{s}_d)' \in \mathbb{R}^d$, the control variable $\mathbf{u} = (\alpha, c)$ with $\alpha \in \mathbb{R}^d$ and $c \geq 0$. Now, for any $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_n)' \in \mathbb{R}^n$ and $n \times n$ symmetric matrix $\mathbf{M} = (\mathbf{M}_{ij})_{1 \leq i, j \leq n}$, we set the Hamilton function as

$$H(\zeta, \mathbf{q}, \mathbf{M}) := \sup_{\mathbf{u} \in \Theta} H_0(\zeta, \mathbf{q}, \mathbf{M}, \mathbf{u}), \quad \Theta = \mathbb{R}^d \times \mathbb{R}_+, \quad (4.2.2)$$

where

$$H_0(\zeta, \mathbf{q}, \mathbf{M}, \mathbf{u}) := \check{\alpha}'(\zeta, \mathbf{u})\mathbf{q} + \frac{1}{2} \text{tr}[\check{b}\check{b}'(\zeta, \mathbf{u})\mathbf{M}] + \ln c.$$

In order to study problem (4.1.6), we need to solve the HJB equation which is given by

$$\begin{cases} z_t(\zeta, t) + H(\zeta, \partial z(\zeta, t), \partial^2 z(\zeta, t)) = 0, & t \in [0, T], \\ z(\zeta, T) = \varpi \ln x, & \zeta \in \mathbb{R}^n, \end{cases} \quad (4.2.3)$$

where $\partial z(\zeta, t) = (z_x, z_{s_1}, \dots, z_{s_d})' \in \mathbb{R}^n$ and

$$\partial^2 z(\zeta, t) = \begin{pmatrix} z_{xx} & z_{xs_1} & z_{xs_2} & \cdots & z_{xs_d} \\ z_{xs_1} & z_{s_1 s_1} & z_{s_1 s_2} & \cdots & z_{s_1 s_d} \\ \vdots & & & \ddots & \\ z_{xs_d} & z_{s_d s_1} & z_{s_d s_2} & \cdots & z_{s_d s_d} \end{pmatrix}_{n \times n}.$$

To calculate the Hamilton function (4.2.2), note that

$$\begin{aligned} H_0(\zeta, \mathbf{q}, \mathbf{M}, v) &= (rx - \alpha' \hat{s} - c)\mathbf{q}_1 + \sum_{i=1}^d \tilde{s}_i \mathbf{q}_{1+i} \\ &+ \frac{1}{2} \left(\alpha' \sigma \sigma' \alpha \mathbf{M}_{11} + 2 \sum_{i=1}^d \langle \sigma \sigma' \alpha \rangle_i \mathbf{M}_{1,1+i} \right. \\ &\left. + \sum_{k,i=1}^d \langle \sigma \sigma' \rangle_{ki} \mathbf{M}_{1+k,1+i} \right) + \ln c. \end{aligned}$$

The symbol $\langle X \rangle_i$ denotes the i^{th} element of the vector X and $\langle Y \rangle_{ij}$ denotes the $(i, j)^{\text{th}}$ element of the matrix Y . Note that due to (4.2.2), if $\mathbf{M}_{11} \geq 0$ or $\mathbf{q}_1 \leq 0$ then the Hamilton function $H(\zeta, \mathbf{q}, \mathbf{M}) = \infty$. So, we maximize the function $H_0(\zeta, \mathbf{q}, \mathbf{M}, v)$ over α and c under condition that $\mathbf{M}_{11} < 0$ and $\mathbf{q}_1 > 0$. We obtain that optimal values for this maximization

problem are given by

$$\alpha^0(s, \mathbf{q}, \mathbf{M}) = \frac{(\sigma\sigma')^{-1}\tau}{\mathbf{M}_{11}} \quad \text{and} \quad c^0(s, \mathbf{q}, \mathbf{M}) = \frac{1}{\mathbf{q}_1}, \quad (4.2.4)$$

where $\tau = \tau(s, \mathbf{q}_1, \mu) = \mathbf{q}_1 \widehat{s} - \sigma\sigma'\mu$ and $\mu = (\mathbf{M}_{1,1+1}, \dots, \mathbf{M}_{1,1+d})'$. Now we replace α_i^0 and c^0 into H_0 to obtain the Hamilton function, so we get

$$\begin{aligned} H(\zeta, \mathbf{q}, \mathbf{M}) &= rx\mathbf{q}_1 - \ln\mathbf{q}_1 + \frac{\tau'(\sigma\sigma')^{-1}\tau}{2|\mathbf{M}_{11}|} + \sum_{i=1}^d \tilde{s}_i \mathbf{q}_{1+i} \\ &\quad + \frac{1}{2} \sum_{k,i=1}^d \langle \sigma\sigma' \rangle_{ki} \mathbf{M}_{1+i,1+k} - 1. \end{aligned}$$

From the preceding Hamilton function and the HJB equation (4.2.3), we obtain

$$\begin{cases} z_t + rxz_x + \frac{\tau'_0(\sigma\sigma')^{-1}\tau_0}{2|z_{xx}|} - 1 - \ln z_x + \sum_{i=1}^d \tilde{s}_i z_{s_i} + \frac{1}{2} \sum_{k,i=1}^d \langle \sigma\sigma' \rangle_{ki} z_{s_i s_k} = 0, \\ z(\zeta, T) = \ln x, \quad \text{for any } \zeta \in \Xi, \end{cases} \quad (4.2.5)$$

where $\tau_0 = \tau(s, \partial z/\partial x, \partial^2 z/\partial x \partial s)$. To write the solution for this equation, we need to introduce the $d \times d$ matrix $g = (g_{ij})_{1 \leq i, j \leq d}$ which is the solution of the following differentiable equation

$$\dot{g} + \frac{1}{2}\rho(t)A'_1(\sigma\sigma')^{-1}A_1 + (g + g')A = 0, \quad g(T) = 0. \quad (4.2.6)$$

Here, the dot “ \cdot ” denotes the derivative. Moreover, we set

$$f(t) = \sum_{k,i=1}^d \langle \sigma\sigma' \rangle_{ki} (\tilde{g}_{ki}(v) + \tilde{g}_{ik}(v)) + f_0(t), \quad (4.2.7)$$

where $\tilde{g}(t) = \int_t^T g(v)dv$,

$$f_0(t) = \frac{1}{2}r(t^2 - 2t(T+1) + T(T+2)) - \rho(t)\ln\rho(t) \quad \text{and} \quad \rho(t) = T - t + 1.$$

We show that the equation (4.2.5) has the following solution

$$z(x, s, t) = \rho(t)\ln x + s'g(t)s + f(t). \quad (4.2.8)$$

Remark 4.2.1. As we see in the HJB equation, the additional variable $s \in \mathbb{R}^d$ is the main difference from the BI-Sch market.

4.3 Main results

First of all we have to study the HJB equation (4.2.5) to calculate the value function (4.1.6).

Theorem 4.3.1. *The function (4.2.8) satisfies the HJB equation (4.2.5).*

Furthermore, to construct the optimal strategies we set

$$\check{\alpha}^0(\zeta, t) = \alpha^0(\zeta, \partial z, \partial^2 z) = -(\sigma\sigma')^{-1}\widehat{s}x \quad \text{and} \quad \check{c}^0(\zeta, t) = c^0(\zeta, \partial z, \partial^2 z) = \frac{x}{\rho(t)}. \quad (4.3.1)$$

Recall that $\widehat{s} = A_1 s = (\widehat{s}_1, \dots, \widehat{s}_d)' \in \mathbb{R}^d$. Using these functions we define the optimal strategies $v^* = (\alpha^*, c^*)$ as

$$\alpha^*(t) = \check{\alpha}^0(\zeta_t^*, t) = -(\sigma\sigma')^{-1}\widehat{S}_t X_t^* \quad \text{and} \quad c^*(t) = \check{c}^0(\zeta_t^*, t) = \frac{X_t^*}{\rho(t)}. \quad (4.3.2)$$

Here $\zeta_t^* = (X_t^*, S_t)$ and X_t^* is the optimal wealth process defined by the following stochastic differential equation

$$dX_t^* = X_t^* a^*(t) dt + X_t^* (b^*(t))' dW_t, \quad X_0^* = x, \quad (4.3.3)$$

where

$$a^*(t) = r - \widehat{S}_t' (\sigma\sigma')^{-1} \widehat{S}_t - \frac{1}{\rho(t)} \quad \text{and} \quad b^*(t) = \sigma' (\sigma\sigma')^{-1} \widehat{S}_t.$$

Now we show that these processes are optimal solutions for the problem (4.1.6).

Theorem 4.3.2. *The processes (4.3.2) and (4.3.3) are the optimal strategies for the problem (4.1.6) and*

$$J^*(x, s, t) = z(x, s, t) = \rho(t) \ln x + s' g(t) s + f(t), \quad (4.3.4)$$

where ρ, g and f are given in (4.2.6).

Example 1. *For one dimensional case where a riskless and risky assets are given respectively by*

$$\begin{cases} d\check{S}_t = r\check{S}_t dt, & \check{S}_0 = 1, \\ dS_t = -\kappa S_t dt + \sigma dW_t, & S_0 > 0, \end{cases} \quad (4.3.5)$$

where $r \geq 0$ is the interest rate of the riskless asset, $\kappa > 0$ and σ are respectively the mean reverting speed and the volatility for risky assets. Therefore, for $\kappa_1 = \kappa + r > 0$, the optimal

strategies and the HJB equation are given by

$$\alpha^*(t) = \check{\alpha}^0(\zeta_t^*, t) = -\frac{\kappa_1 S_t X_t^*}{\sigma^2} \quad \text{and} \quad c^*(t) = \check{c}^0(\zeta_t^*, t) = \frac{X_t^*}{\rho(t)},$$

where $\check{\alpha}^0$ and \check{c}^0 are defined in Eq. (5.3.5). Moreover, the differential wealth process for this example is given by

$$dX_t^* = X_t^* a^*(t) dt + X_t^* b^*(t) dW_t, \quad (4.3.6)$$

where

$$a^*(t) = r + \kappa_1^2 S_t^2 / \sigma^2 - 1/\rho(t) \quad \text{and} \quad b^*(t) = \kappa_1 S_t / \sigma.$$

Example 2. For multidimensional case where the market assets are given by

$$\begin{cases} d\check{S}_t = r\check{S}_t dt, & \check{S}_0 = 1, \\ dS_t = AS_t dt + \sigma dW_t, & S_0 > 0, \end{cases} \quad (4.3.7)$$

where r is the interest rate for riskless asset \check{S} and $S_t = (S_1(t), S_2(t), S_3(t), \dots, S_d(t)) \in \mathbb{R}^d$ is a d -dimensional vector of risky assets, (W_t) is a standard Brownian motion with values in \mathbb{R}^d , the market volatility matrix $\sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_d)$, and the mean reverting matrix A is given by

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1d} \\ a_{21} & a_{22} & \dots & a_{2d} \\ \vdots & & \ddots & \\ a_{d1} & a_{d2} & \dots & a_{dd} \end{pmatrix},$$

with negative real eigenvalues i.e. $\text{Re}\lambda_i(A) < 0$. The optimal wealth process $(X_t^*)_{0 \leq t \leq T}$ is defined by the following stochastic equation

$$dX_t^* = X_t^* a^*(t) dt + X_t^* (b^*(t))' dW_t, \quad X_0^* = x,$$

where

$$a^*(t) = r + \sum_{i=1}^d \frac{\widehat{S}_i^2(t)}{\sigma_i^2} - \frac{1}{\rho(t)}, \quad b^*(t) = (b_1^*(t), \dots, b_d^*(t))' \quad \text{and} \quad b_i^*(t) = \widehat{S}_i(t) / \sigma_i.$$

Using the preceding stochastic differential equation, the optimal strategies

$v^* = (\alpha^*, c^*)$ for all $0 \leq t \leq T$ is of the form:

$$\alpha_i^*(t) = \check{\alpha}_i^0(\zeta_t^*, t) = -\frac{\widehat{S}_i(t)X_t^*}{\sigma_i^2} \quad \text{and} \quad c^*(t) = \check{c}^0(\zeta_t^*, t) = \frac{X_t^*}{\rho(t)},$$

where $\check{\alpha}_i^0 = (\check{\alpha}_1^0, \dots, \check{\alpha}_d^0) \in \mathbb{R}^d$.

Remark 4.3.1. It should be noted that the behaviour of these optimal strategies are described by the transformed spread process $\widehat{S}_t = A_1 S_t'$. In the scalar case this is the same as S_t . However, in the general multidimensional case we need to take into account all components of the spread processes.

4.4 Proofs

4.4.1 Proof of Theorem 4.3.1

Proof. Now, by taking the derivatives of $z(\zeta, t)$ defined in (4.2.8) with respect to t and s and apply them into equation (4.2.5) we obtain

$$\begin{aligned} s' \dot{g}(t)s + \dot{f}(t) + r\rho(t) + \sum_{k,i=1}^d (\langle \sigma \sigma' \rangle_{ki} (g_{ki} + g_{ik})) - \ln \rho(t) - 1 \\ + \sum_{j=1}^d \sum_{l=1}^d A_{jl} s_l \langle (g + g')s \rangle_j + \frac{\rho(t) \widehat{s}' (\sigma \sigma')^{-1} \widehat{s}}{2} = 0, \end{aligned}$$

where the dot “ $\dot{\cdot}$ ” denotes the first derivative and g is a $d \times d$ matrix defined in (4.2.6). Then this can be written as

$$\begin{aligned} s' \left(\dot{g}(t) + \frac{1}{2} \rho(t) A_1' (\sigma \sigma')^{-1} A_1 - A'(g + g') \right) s + \dot{f}(t) + \sum_{k,i=1}^d \langle \sigma \sigma' \rangle_{ki} (g_{ki} + g_{ik}) - 1 \\ - \ln \rho(t) + r\rho(t) = 0. \end{aligned}$$

After calculation we get that for $s \in \mathbb{R}^d$,

$$f(t) = \sum_{k,i=1}^d \langle \sigma \sigma' \rangle_{ki} (\tilde{g}_{ki}(v) + \tilde{g}_{ik}(v)) + f_0(t),$$

and

$$s'(\dot{g}(t) + \frac{1}{2}\rho(t)A'_1(\sigma\sigma')^{-1}A_1 - A'(g + g'))s = 0, \quad g(T) = 0,$$

where

$$\tilde{g}(t) = \int_t^T g(v)dv \quad \text{and} \quad f_0(t) = \frac{1}{2}r(t^2 - 2t(T+1) + T(T+2)) + \rho(t)\ln\rho(t).$$

The last term in the preceding equation can be written as

$$\begin{aligned} \langle A'(g + g') \rangle_{ij} &= \sum_{l=1}^d \langle A' \rangle_{il} (g_{lj} + g_{jl}), \\ &= \sum_{l=1}^d \left(\langle A \rangle_{li} g_{lj} + \langle A \rangle_{li} g_{jl} \right). \end{aligned}$$

Let us denote by $H = (h_{ij})_{1 \leq i, j \leq d}$, where $h = \text{vect}(H)$ a vector in \mathbb{R}^m such that $h = (h_1, h_2, \dots, h_m)$, with $h_{(j-1)d+i} = \langle H \rangle_{i,j}$ and $Z(t) = \text{vect}(g(t))$ where $Z_{(j-1)d+i} = g_{ij}$. Therefore, the last equation becomes

$$\begin{aligned} \langle A'(g + g') \rangle_{ij} &= \sum_{l=1}^d \langle A \rangle_{li} \left(Z_{(j-1)d+l} + Z_{(l-1)d+j} \right), \\ &= \sum_{l=1}^d \langle A \rangle_{li} Z_{(j-1)d+l} + \sum_{l=1}^d \langle A \rangle_{li} Z_{(l-1)d+j}, \\ &= \sum_{l=1}^d \sum_{k=1}^d \left(\langle A \rangle_{li} \mathbf{1}_{\{k=j\}} Z_{(k-1)d+l} + \langle A \rangle_{ki} \mathbf{1}_{\{l=j\}} Z_{(k-1)d+l} \right). \end{aligned}$$

This can be written in the following form

$$\text{Vect}(A'(g + g')) = \Gamma Z,$$

where $\Gamma = (\check{\gamma}_{s,t})$ and $\hat{\gamma}_{s,t} = \langle A \rangle_{li} \mathbf{1}_{\{k=j\}} + \langle A \rangle_{ki} \mathbf{1}_{\{l=j\}}$, with $s = (j-1)d + i$ and $t = (k-1)d + l$. Therefore, for all $m \times m$ matrix Γ , with $m = d^2$,

$$\langle A'(g + g') \rangle_{ij} = \langle \Gamma Z \rangle_{ij},$$

where $\Gamma = (\hat{\gamma}_{s,t})_{1 \leq s, t \leq m}$. Thus, equation (4.2.6) can be written as

$$\dot{Z} - \Gamma Z + \frac{1}{2}\rho(t)\tilde{b} = 0, \quad Z(T) = 0,$$

where $\tilde{b} = \text{vect}(A_1'(\sigma\sigma')^{-1}A_1) \in \mathbb{R}^m$. Therefore, the solution of $Z(t)$ is given by

$$Z(t) = \tilde{b} \frac{1}{2} \int_t^T \rho(v) e^{\Gamma(v-t)} dv. \quad (4.4.1)$$

This proves Theorem 4.3.1. \square

4.4.2 Proof of Theorem 4.3.2

Proof. We apply the verification Theorem 6.4.1 to the Problem (4.1.6) for the stochastic control differential equation (4.2.1). First note, that from the definition of the risk asset in (4.1.1) it follows that for $t < u < T$ and $s \in \mathbb{R}^d$

$$S_u = e^{A(u-t)} s + \xi_{t,u} \quad \text{and} \quad \xi_{t,u} = \int_t^u e^{A(u-v)} \sigma dW_v. \quad (4.4.2)$$

It should be noted that the upper bound (1.1.11) of Proposition 1.1.2 in [39] implies directly that

$$\mathbf{E} \left(\sup_{t \leq u \leq T} |\xi_{t,u}|^2 \right) < \infty. \quad (4.4.3)$$

Therefore, Theorem 4.3.1 and the last inequality in (4.1.5) imply the condition \overline{H}_1 .

Moreover, note that the linear equation (4.3.3) has the strong unique solution X_t^* given as

$$X_t^* = x \exp \left\{ \int_0^t \left(a^*(u) - |\check{b}^*(u)|^2 / 2 \right) du + \int_0^t (\check{b}^*(u))' dW_u \right\}. \quad (4.4.4)$$

Therefore, the strategy $v^* = (v_t^*)_{0 \leq t \leq T}$ with $v_t^* = (\alpha_t^*, c_t^*)$ defined in (4.3.2) and (4.3.3) belongs to the class \mathcal{V} and satisfies the condition \overline{H}_2). To check the condition \overline{H}_3) we have to show the upper bound (6.4.5), i.e.

$$\mathbf{E}_{\zeta,t} \sup_{t \leq u \leq T} |z(\zeta_u^*, u)| < \infty \quad (4.4.5)$$

for any $\zeta \in \mathbb{R}_+ \times \mathbb{R}^n$. Taking into account that in the HJB solution (4.2.8) the functions g and f are bounded it suffices to check that

$$\mathbf{E}_{\zeta,t} \sup_{t \leq u \leq T} (|\ln(X_u^*)| + |S_u|^2) < \infty.$$

So, in view of (4.4.2) and (4.4.3) one needs to check that

$$\mathbf{E}_{\zeta,t} \sup_{t \leq u \leq T} |\ln(X_u^*)| < \infty. \quad (4.4.6)$$

For this, taking into account the representation (4.4.4), it suffices to check that

$$\mathbf{E}_{\zeta,t} \int_t^T \left(|\check{a}^*(u)| + |\check{b}^*(u)|^2 \right) du + \sqrt{\mathbf{E}_{\zeta,t} \sup_{t \leq u \leq T} \left(\int_t^u (b^*(v))' dW_v \right)^2} < \infty.$$

Note now, that from the definition of the functions $\check{a}^*(t)$ and $\check{b}^*(t)$ in (4.3.3) and the conditions of this theorem, we have that

$$|\check{a}^*(t)| \leq \mathbf{c}_1(1 + |S_t|^2) \quad \text{and} \quad |\check{b}^*(t)| \leq \mathbf{c}_2 |S_t| \quad (4.4.7)$$

for some constant $\mathbf{c}_1 > 0$ and $\mathbf{c}_2 > 0$. Moreover, using Doob's martingale inequality, the equality (4.4.3) and the bound (4.4.2) we obtain that

$$\mathbf{E}_{\zeta,t} \sup_{t \leq u \leq T} \left(\int_t^u (b^*(v))' dW_v \right)^2 \leq 4\mathbf{E}_{\zeta,t} \left(\int_t^T (b^*(u))' dW_u \right)^2 = 4\mathbf{E}_{\zeta,t} \int_t^T |b^*(u)|^2 du < \infty.$$

The last inequality follows immediately from Eq. (4.4.5) and the second bound in Eq. (4.4.7). This proves Theorem 4.3.2. \square

Chapter 5

Stochastic volatility model

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In this Chapter, we consider the stochastic volatility markets which are popular in mathematical finance (see, for example, [22], [25], [34], etc.).

5.1 Market model

Let $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbf{P})$ be a standard filtered probability space with $(\mathcal{F}_t)_{0 \leq t \leq T}$ adapted independent Wiener processes $(W_t^{(1)})_{0 \leq t \leq T}$ and $(W_t^{(2)})_{0 \leq t \leq T}$. Our financial market consists of one *riskless bond* $(\check{S}_t)_{0 \leq t \leq T}$ and *risky spread stocks* $(S_t)_{0 \leq t \leq T}$ governed by the following equations:

$$\begin{cases} dS_t = -\kappa S_t dt + \sigma(y_t) dU_t, & S_0 > 0, \\ dy_t = \Lambda(y_t) dt + \beta dW_t^{(2)}, & \beta > 0, \end{cases} \quad (5.1.1)$$

where y_0 is a fixed nonrandom initial value, $U_t = \sqrt{1 - \check{\beta}^2} W_t^{(1)} + \check{\beta} W_t^{(2)}$ and $0 < \check{\beta} < 1$. Here the constant $\kappa > 0$ is the market mean-reverting parameter from \mathbb{R} and $\sigma(y) > 0$ is the market volatility. We assume that the interest rate $r \leq \kappa$. Let now $\check{\alpha}_t$ and α_t be the number of

riskless assets \check{S} and risky assets S respectively in the moment $0 \leq t \leq T$, the consumption rate is given by a non negative integrated function $(c_t)_{0 \leq t \leq T}$ [44]. Then the wealth process $X_t = \check{\alpha}_t \check{S}_t + \alpha_t S_t$ satisfies the following equation

$$dX_t = (rX_t - \alpha_t \kappa_1 S_t - c_t)dt + \alpha_t \sigma(y_t) dU_t, \quad (5.1.2)$$

where the initial endowment $X_0 > 0$ is a fixed nonrandom constant and $\kappa_1 = \kappa + r$. We denote by \mathcal{V} of all adapted processes $v_t = (\alpha_t, c_t)$ such that almost sure

$$\int_0^T \alpha_t^2 dt < \infty, \quad \int_0^T c_t dt < \infty$$

and the equation (5.1.2) has an unique strong non negative solution. In the sequel we denote

$$\zeta_t = (X_t, S_t, y_t)', \quad (5.1.3)$$

where "′" denotes the transposition. We denote by \mathcal{V} the class of all admissible financial strategies defined in Definition 4.1.1 for the Eq. (5.1.2).

Then for any $v \in \mathcal{V}$ and $\zeta = (x, s, y) \in \mathbb{R}^3$, we define the objective function as

$$J(\zeta, v) := \mathbf{E}_\zeta \left(\int_0^T (\ln c_u) du + \varpi \ln(X_T^v) \right),$$

where X^v is the wealth process (5.1.2) corresponding to the strategy $v \in \mathcal{V}$ and \mathbf{E}_ζ is the expectation under condition $\zeta_0 = \zeta = (x, s, y)$. The coefficient $\varpi > 0$ is a some parameter giving the preferences between consumptions and investments. Our goal, in this Chapter, is to maximize this function, i.e.

$$J^*(\zeta) := \sup_{v \in \mathcal{V}} J(\zeta, v). \quad (5.1.4)$$

To study this problem we use the stochastic dynamic programming method. To this end we need to study the value functions $(J^*(\zeta, t))_{0 \leq t \leq T}$ defined as

$$J^*(\zeta, t) = \sup_{v \in \mathcal{V}_t} \mathbf{E}_{\zeta, t} \left(\int_t^T (\ln c_u) du + \varpi \ln(X_T^v) \right),$$

where $\mathbf{E}_{\zeta, t}$ is the expectation under condition $\zeta_0 = \zeta = (x, s, y)$ and \mathcal{V}_t is the set of all admissible financial strategies $v \in \mathcal{V}$ which are adapted to $\mathcal{F}_{t, u} = \sigma\{W_s^{(1)} - W_t^{(1)}, W_s^{(2)} -$

$W_t^{(2)}, t \leq s \leq u$ on the interval $t \leq u \leq T$. Thus we need to study the HJB equation which is given in the following section.

5.2 Hamilton–Jacobi–Bellman equation

First, we rewrite the stock and wealth equations in the following form

$$d\zeta_t = \check{a}(\zeta_t, v_t)dt + \check{b}(\zeta_t, v_t)dW_t, \quad W_t = \begin{pmatrix} W_t^{(1)} \\ W_t^{(2)} \end{pmatrix}, \quad (5.2.1)$$

where $\check{a} \in \mathbb{R}^3$ and \check{b} is the matrix of 3×2 functions such that for any $\zeta = (x, s, y)$ and $v = (\alpha, c)$

$$\check{a}(\zeta, \mathbf{u}) = \begin{pmatrix} rx - \alpha\kappa_1s - c \\ -\kappa s \\ \Lambda(y) \end{pmatrix} \quad \text{and} \quad \check{b}(\zeta, \mathbf{u}) = \begin{pmatrix} \alpha\sigma(y)\sqrt{1-\check{\beta}^2}; & \alpha\sigma(y)\check{\beta} \\ \sigma(y)\sqrt{1-\check{\beta}^2}; & \sigma(y)\check{\beta} \\ 0; & \beta \end{pmatrix}. \quad (5.2.2)$$

Now, for any $\mathbf{q} = (q_1, q_2, q_3)' \in \mathbb{R}^3$ and 3×3 symmetric matrix $\mathbf{M} = (\mathbf{M}_{ij})_{1 \leq i, j \leq 3}$, we set the Hamilton function as

$$H(\zeta, \mathbf{q}, \mathbf{M}) := \sup_{v \in \Theta} H_0(\zeta, \mathbf{q}, \mathbf{M}, v), \quad \Theta \in \mathbb{R} \times \mathbb{R}_+, \quad (5.2.3)$$

where

$$H_0(\zeta, \mathbf{q}, \mathbf{M}, v) := \check{a}'(\zeta, v)\mathbf{q} + \frac{1}{2}\text{tr}[\check{b}\check{b}'(\zeta, v)\mathbf{M}] + \ln c.$$

In order to study problem (5.1.4), we need to solve the HJB equation which is given by

$$\begin{cases} z_t(\zeta, t) + H(\zeta, \partial z(x, t), \partial^2 z(x, t)) = 0, & t \in [0, T], \\ z(x, t) = \bar{w} \ln x, & \zeta \in \mathbb{R}^3, \end{cases} \quad (5.2.4)$$

where $\partial z(x, t) = (z_x, z_s, z_y)' \in \mathbb{R}^3$ and

$$\partial^2 z(x, t) = \begin{pmatrix} z_{xx} & z_{xs} & z_{xy} \\ z_{xs} & z_{ss} & z_{sy} \\ z_{xy} & z_{ys} & z_{yy} \end{pmatrix}_{3 \times 3}.$$

To calculate the Hamilton function (5.2.3), note that

$$\begin{aligned} H_0(\zeta, \mathbf{q}, \mathbf{M}, v) = & (rx - \alpha\kappa_1s - c)\mathbf{q}_1 + \kappa s\mathbf{q}_2 + \Lambda(y)\mathbf{q}_3 + \frac{1}{2}\alpha^2\sigma^2(y)\mathbf{M}_{11} + \alpha\sigma^2(y)\mathbf{M}_{12} \\ & + \alpha\sigma(y)\beta\check{\beta}\mathbf{M}_{13} + \frac{1}{2}\sigma^2(y)\mathbf{M}_{22} + \sigma(y)\beta\check{\beta}\mathbf{M}_{23} + \frac{1}{2}\beta^2\mathbf{M}_{33} + \ln c. \end{aligned}$$

We can represent this function as

$$H_0(\zeta, \mathbf{q}, \mathbf{M}, v) = \frac{1}{2}\alpha^2\sigma^2(y)\mathbf{M}_{11} + \alpha Q(\zeta, \mathbf{q}, \mathbf{M}) - c\mathbf{q}_1 + \ln c + Q_0(\zeta, \mathbf{q}, \mathbf{M}), \quad (5.2.5)$$

where $Q(\zeta, \mathbf{q}, \mathbf{M}) = \sigma^2(y)\mathbf{M}_{12} + \sigma(y)\beta\check{\beta}\mathbf{M}_{13} - \kappa_1s\mathbf{q}_1$ and

$$Q_0(\zeta, \mathbf{q}, \mathbf{M}) = rx\mathbf{q}_1 + \kappa s\mathbf{q}_2 + \Lambda(y)\mathbf{q}_3 + \frac{1}{2}\sigma^2(y)\mathbf{M}_{22} + \sigma(y)\beta\check{\beta}\mathbf{M}_{23} + \frac{1}{2}\beta^2\mathbf{M}_{33}$$

Note that due to (5.2.3), if $\mathbf{M}_{11} \geq 0$ or $\mathbf{q}_1 \leq 0$ then the Hamilton function $H(\zeta, \mathbf{q}, \mathbf{M}) = \infty$. So, we maximize the function $H_0(\zeta, \mathbf{q}, \mathbf{M}, v)$ over α and c under condition that $\mathbf{M}_{11} < 0$ and $\mathbf{q}_1 > 0$. We obtain that optimal values for this maximization problem are given by

$$\alpha^0(\zeta, \mathbf{q}, \mathbf{M}) = \frac{Q(\zeta, \mathbf{q}, \mathbf{M})}{\sigma^2(y)|\mathbf{M}_{11}|} \quad \text{and} \quad c^0(\zeta, \mathbf{q}, \mathbf{M}) = \frac{1}{\mathbf{q}_1}. \quad (5.2.6)$$

Now we replace α_i^0 and c^0 into H_0 to obtain the Hamilton function, so we get

$$H(\zeta, \mathbf{q}, \mathbf{M}) = \frac{Q^2(\zeta, \mathbf{q}, \mathbf{M})}{2\sigma^2(y)|\mathbf{M}_{11}|} - 1 - \ln \mathbf{q}_1 + Q_0(\zeta, \mathbf{q}, \mathbf{M}). \quad (5.2.7)$$

Setting $\tilde{Q}(\zeta, t) = Q(\zeta, \partial z, \partial^2 z, t)$ and $\tilde{Q}_0(\zeta, t) = Q_0(\zeta, \partial z, \partial^2 z, t)$, we can represent the equation (5.2.4) as

$$z_t + \frac{\tilde{Q}^2(\zeta, t)}{2\sigma^2(y)|z_{xx}|} - 1 - \ln z_x + \tilde{Q}_0(\zeta, t) = 0, \quad (5.2.8)$$

where $z(x, t) = \ln x$ for any $\zeta \in \mathbb{R}_+ \times \mathbb{R}^2$.

5.3 Main results

First of all we have to study the HJB equation (5.2.8) to calculate the value function (5.1.4). To this end we define the following linear partial derivative equations for the functions $g(y, t)$,

$h(y,t)$ and $f(y,t)$.

$$g_t + \frac{\beta^2}{2} g_{yy} + 2\kappa g + \Lambda(y)g_y = -\frac{\kappa_1^2 \rho(t)}{\sigma^2(y)}, \quad g(y,T) = 0. \quad (5.3.1)$$

Then

$$h_t + \frac{\beta^2}{2} h_{yy} + \kappa h + \Lambda(y)h_y = -\sigma(y)\beta\check{\beta}g_y, \quad h(y,T) = 0. \quad (5.3.2)$$

Finally,

$$f_t + \frac{\beta^2 f_{yy}}{2} + \Lambda(y)f_y = \Upsilon(y,t), \quad f(y,T) = 0, \quad (5.3.3)$$

where $\Upsilon(y,t) = 1 + \ln \rho(t) - r\rho(t) - \sigma^2(y)g/2 - \sigma(y)\beta\check{\beta}h_y$.

Theorem 5.3.1. *Assume that the functions $\sigma(\cdot)$ and $\Lambda(\cdot)$ are bounded, continuously derivable with bounded derivatives and $\inf_{y \in \mathbb{R}} \sigma(y) > 0$. Then the HJB solution is*

$$z(x,s,y,t) = \rho(t) \ln x + \frac{s^2}{2} g(y,t) + sh(y,t) + f(y,t), \quad \rho(t) = T - t + 1, \quad (5.3.4)$$

where the functions g , h and f are bounded and defined by the partial equations (5.3.1), (5.3.2) and (5.3.3).

Now, using the form from (5.3.4) or $z(x,s,y,t)$ we construct the optimal strategies. We set

$$\check{\alpha}^0(\varsigma,t) = \alpha^0(\varsigma, \partial z, \partial^2 z) = -\frac{\kappa_1 s}{\sigma^2(y)} x \quad \text{and} \quad \check{c}^0(\varsigma,t) = c^0(\varsigma, \partial z, \partial^2 z) = \frac{x}{\rho(t)}. \quad (5.3.5)$$

Using these functions, we define the optimal strategies $v^* = (\alpha^*, c^*)$ as

$$\alpha^*(t) = \check{\alpha}^0(\varsigma_t^*, t) = -\frac{\kappa_1 S_t}{\sigma^2(y_t)} X_t^* \quad \text{and} \quad c^*(t) = \check{c}^0(\varsigma_t^*, t) = \frac{X_t^*}{\rho(t)}. \quad (5.3.6)$$

Here $\varsigma_t^* = (X_t^*, S_t, y_t)'$ and X_t^* is the optimal wealth process defined by the following stochastic differential equation

$$dX_t^* = X_t^* a^*(t) dt + X_t^* b^*(t) dU_t, \quad X_0^* = x, \quad (5.3.7)$$

where

$$a^*(t) = r + \frac{\kappa_1 S_t^2}{\sigma^2(y_t)} - \frac{1}{\rho(t)} \quad \text{and} \quad b^*(t) = -\frac{\kappa_1 S_t}{\sigma(y_t)}.$$

Theorem 5.3.2. *Assume that the functions $\sigma(\cdot)$ and $\Lambda(\cdot)$ are bounded, continuously derivable with bounded derivatives and $\inf_{y \in \mathbb{R}} \sigma(y) > 0$. Then the process $\mathbf{v}_t^* = (\alpha_t^*, c_t^*)$ defined in (5.3.6) and (5.3.7) is the solution for the problem (5.1.4).*

5.4 Proofs

5.4.1 Proof of Theorem 5.3.1

Proof. Let us use the following form for the HJB equation Eq. (5.2.8)

In this case $\tilde{Q}(\zeta, t) = -\kappa_1 s \rho(t)/x$ and

$$\tilde{Q}_0(\zeta, t) = r\rho(t) + \kappa s z_s + \Lambda(y)z_y + \frac{\sigma^2(y)z_{ss}}{2} + \sigma(y)\beta\check{\beta}z_{sy} + \frac{\beta^2 z_{yy}}{2}.$$

Moreover, the last function can be represented as

$$\tilde{Q}_0(\zeta, t) = \frac{s^2}{2}K_2(y, t) + sK_1(y, t) + K_0(y, t), \quad (5.4.1)$$

where

$$K_2(y, t) = \frac{\beta^2}{2}g_{yy} + 2\kappa g + \Lambda(y)g_y, \quad K_1(y, t) = \kappa h + \Lambda(y)h_y + \sigma(y)\beta\check{\beta}g_y + \frac{\beta^2}{2}h_{yy}$$

and

$$K_0(y, t) = r\rho(t) + \frac{\sigma^2(y)g}{2} + \Lambda(y)f_y + \sigma(y)\beta\check{\beta}h_y + \frac{\beta^2 f_{yy}}{2}.$$

Therefore, from the equation (5.2.8) we obtain that

$$\frac{s^2}{2} \left(g_t + \frac{\kappa_1^2 \rho(t)}{\sigma^2(y)} + K_2(y, t) \right) + s(h_t + K_1(y, t)) + f_t - 1 - \ln \rho(t) + K_0(y, t) = 0.$$

From here we obtain the following linear partial equations (5.3.1), (5.3.2) and (5.3.3). By changing the variable $t \rightarrow T - t$ in (5.3.1) and using the condition of this theorem, we can apply Theorem 6.1.1 for any $0 < l < 1$, i.e. this equation has the unique bounded solution with the bounded derivatives. So, we can do the same for the equations (5.3.2) and (5.3.3). This proves Theorem 5.3.1. \square

5.4.2 Proof of the verification Theorem 5.3.2

Proof. We apply the verification Theorem 6.4.1 to the Problem (5.1.4) for the stochastic control differential equation (5.2.1). First note, that Theorem 5.3.1 implies the condition \bar{H}_1) with the function (5.3.4). Moreover, note that the linear equation (5.3.7) has the strong unique solution X_t^* given as

$$X_t^* = x \exp \left\{ \int_0^t \left(\check{a}^*(v) - (\check{b}^*(v))^2 / 2 \right) dv + \int_0^t \check{b}^*(v) dU_v \right\}. \quad (5.4.2)$$

Therefore, the strategy $v^* = (v_t^*)_{0 \leq t \leq T}$ with $v_t^* = (\alpha_t^*, c_t^*)$ defined in (5.3.6) and (5.3.7) belongs to the class \mathcal{V} and satisfies the condition \bar{H}_2). To check the condition \bar{H}_4) we have to show the upper bound (6.4.5), i.e.

$$\mathbf{E}_{\zeta, t} \sup_{t \leq u \leq T} |z(\zeta_u^*, u)| < \infty \quad (5.4.3)$$

for any $\zeta \in \mathbb{R}_+ \times \mathbb{R}^2$. Taking into account that in the HJB solution (5.3.4) the functions g, h and f are bounded it suffices to check that

$$\mathbf{E}_{\zeta, t} \sup_{t \leq u \leq T} (|\ln(X_u^*)| + S_u^2) < \infty.$$

Moreover, note that for $t < u < T$

$$S_u = e^{-\kappa(u-t)} S_t + \xi_{t,u} \quad \text{and} \quad \xi_{t,u} = \int_t^u e^{-\kappa(u-v)} \sigma(y_v) dW_v.$$

So, one needs to check that

$$\mathbf{E} \left(\sup_{t \leq u \leq T} \xi_u^2 \middle| y_t = y \right) < \infty \quad \text{and} \quad \mathbf{E}_{\zeta, t} \sup_{t \leq u \leq T} |\ln(X_u^*)| < \infty. \quad (5.4.4)$$

It should be noted that the first inequality follows immediately from the upper bound Eq. (6.5.4). As to the last inequality in (5.4.4) in view of the representation (5.4.2) it suffices to check that

$$\mathbf{E}_{\zeta, t} \int_t^T (|\check{a}^*(u)| + (\check{b}^*(u))^2) du + \sqrt{\mathbf{E}_{\zeta, t} \sup_{t \leq u \leq T} \left(\int_t^u \check{b}^*(v) dW_v \right)^2} < \infty.$$

Note now, that from the definition of the functions $\check{a}^*(t)$ and $\check{b}^*(t)$ in (5.3.7) and the conditions of this theorem, we have that $|\check{a}^*(u)| \leq \mathbf{c}(1 + S_u^2)$ and $|\check{b}^*(u)| \leq \mathbf{c}|S_u|$ for some constant $\mathbf{c} > 0$. Moreover, using Doob's martingale inequality we obtain that

$$\mathbf{E}_{\zeta,t} \sup_{t \leq u \leq T} \left(\int_t^u \check{b}^*(v) dW_v \right)^2 \leq 4\mathbf{E}_{\zeta,t} \left(\int_0^T \check{b}^*(u) dW_u \right)^2 = 4\mathbf{E}_{\zeta,t} \int_0^T |\check{b}^*(u)|^2 du.$$

Therefore, to obtain the last inequality in (5.4.4) we need to check that

$$\mathbf{E} \left(\int_t^T \xi_u^2 du \mid y_t = y \right) < \infty.$$

This bound follows immediately from the first inequality (5.4.4). This proves Theorem 5.3.2. \square

Chapter 6

Toolbox

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In this chapter we state a toolbox

6.1 Cauchy Problem for linear parabolic equations

Suppose $u(y, t)$ is the classical solution of the following nonlinear problem

$$\begin{cases} u_t - \sum_{i,j=1}^n a_{ij}(y, t) u_{y_i y_j} + \sum_{j=1}^n a_j(y, t) u_{y_j} + a(y, t) u = \Upsilon(y, t), \\ u(y, 0) = \varphi(y), \quad y \in \mathbb{R}^n. \end{cases} \quad (6.1.1)$$

We need the following condition.

CL) Assume that there exist the constants $0 < \mathbf{c}_{min} \leq \mathbf{C}_{max} < \infty$ such that for any $y \in \mathbb{R}^n$, $t \in [0, T]$ and $(z_1, \dots, z_n) \in \mathbb{R}^n$

$$\mathbf{c}_{min} \leq \sum_{i,j=1}^n z_i z_j a_{ij}(y, t) \leq \mathbf{C}_{max}.$$

Now we recall some notations from [54]. Let now be $l > 0$ non integer, i.e. $l = [l] + \{l\}$, where $[l]$ is the integer part and l is the fractional and $0 < \{l\} < 1$. We denote by $\mathcal{H}^l(\mathbb{R}^n)$ the Banach space of the $\mathbb{R}^n \rightarrow \mathbb{R}$ functions ϕ having $m = [l]$ derivatives which satisfy the Hölder conditions with the exponent $\alpha = \{l\}$ and bounded the Hölder constants. The norm is defined as

$$|\phi|^{(l)} = \sum_{j=0}^m \sum_{i_1+\dots+i_n=j} \sup_{y \in \mathbb{R}^n} \left| \frac{\partial^j}{(\partial y_1)^{i_1} \dots (\partial y_n)^{i_n}} \phi(y) \right| + \sum_{i_1+\dots+i_n=m} \left\langle \frac{\partial^j}{(\partial y_1)^{i_1} \dots (\partial y_n)^{i_n}} \phi \right\rangle^\alpha,$$

where

$$\langle u \rangle^\alpha = \sup_{x, y \in \mathbb{R}^n} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

Moreover, the space $\mathcal{H}^{l, l/2}(\mathbb{R}^n \times [0, T])$ is the Banach space of all $\mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$ functions $u(y, t)$ which for any $p + 2r \leq m$ have the continuous bounded derivatives

$$\partial_{i_1, \dots, i_n}^p \partial^r u = \frac{\partial^p \partial^r}{(\partial y_1)^{i_1} \dots (\partial y_n)^{i_n} (\partial t)^r} u, \quad i_1 + \dots + i_n = p.$$

The norm in this case is defined as the following:

$$|u|^{(l)} = \sum_{j=0}^m \sum_{p+2r=j} \sum_{i_1+\dots+i_n=p} \sup_{y \in \mathbb{R}^n, t \in [0, T]} |\partial_{i_1, \dots, i_n}^p \partial^r u(y, t)| + |u|_y^{(l)} + |u|_t^{(l/2)},$$

where

$$|u|_y^{(l)} = \sum_{p+2r=m} \sum_{i_1+\dots+i_n=p} \left\langle \partial_{i_1, \dots, i_n}^p \partial^r u \right\rangle_y^\alpha$$

and

$$|u|_t^{(l/2)} = \sum_{p+2r=m-1} \sum_{i_1+\dots+i_n=p} \left\langle \partial_{i_1, \dots, i_n}^p \partial^r u \right\rangle_t^{(\alpha+1)/2} + \sum_{p+2r=m} \sum_{i_1+\dots+i_n=p} \left\langle \partial_{i_1, \dots, i_n}^p \partial^r u \right\rangle_t^{\alpha/2}.$$

Here the Hölder constants are defined as

$$\langle u \rangle_y^\alpha = \sup_{x, y \in \mathbb{R}^n, t \in [0, T]} \frac{|u(x, t) - u(y, t)|}{|x - y|^\alpha} \quad \text{and} \quad \langle u \rangle_t^\alpha = \sup_{x \in \mathbb{R}^n, t, u \in [0, T]} \frac{|u(x, t) - u(x, u)|}{|t - u|^\alpha}.$$

Now we give the uniqueness and existence theorem for the Cauchy problem (6.1.1).

Theorem 6.1.1 (Theorem 5.1, Chapter 4, section 5, p.320 of [54]). *Assume that the condition (CL) holds. Moreover, let $l > 0$ be a non integer number for which the functions $a_{ij}(y, t)$,*

$a_j(y, t)$ and $a(y, t)$ belong to the class $\mathcal{H}^{l, l/2}(\mathbb{R}^n \times [0, T])$. Then for any $\Upsilon \in \mathcal{H}^{l, l/2}(\mathbb{R}^n \times [0, T])$ and $\varphi \in \mathcal{H}^{l+2}(\mathbb{R}^n)$, the equation (6.1.1) has a unique solution from the space $\mathcal{H}^{l+2, l/2+1}(\mathbb{R}^n \times [0, T])$ such that for some constant $c > 0$

$$|u|^{(l+2)} \leq c \left(|f|^{(l)} + |\varphi|^{(l+2)} \right). \quad (6.1.2)$$

6.2 Cauchy Problem for quasilinear parabolic equations

Suppose $u(x, t)$ is the classical solution of the following nonlinear problem

$$\begin{cases} \mathcal{L}u \equiv u_t - \sum_{1 \leq i, j \leq n} a_{ij}(x, t, u, u_x) u_{x_i x_j} + a(x, t, u, u_x) = 0, \\ u|_{t=0} = u(x, 0) = \psi_0(x). \end{cases} \quad (6.2.1)$$

We assume that there exists some functions (a_1, a_2, \dots, a_n) , such that

$$a_{ij}(x, t, u, p) \equiv \frac{\partial a_i(x, t, u, p)}{\partial p_j} \quad \text{and} \quad (6.2.2)$$

$$A(x, t, u, p) \equiv a(x, t, u, p) - \sum_{i=1}^n \frac{\partial a_i}{\partial u} p_i - \sum_{i=1}^n \frac{\partial a_i}{\partial x_i}. \quad (6.2.3)$$

Now for any $N \geq 1$,

$$\Gamma_N = \{(x, t) : |x| \leq N, 0 \leq t \leq T\}.$$

We introduce the following conditions for ensuring the existence of the solution $u(x, t)$ of Cauchy problem.

Suppose that the following conditions hold.

C₁) For all $N \geq 1$,

$$\psi_0(x) \in \mathcal{H}^{2+\beta}(\Gamma_N) \quad \text{and} \quad \max_{E_n} |\psi_0(x)| < \infty.$$

C₂) There exists $b \geq 0$ and some $\mathbb{R}_+ \rightarrow \mathbb{R}_+$ function Φ , such that for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and for all $0 \leq t \leq T$,

$$A(x, t, u, 0)u \geq -\Phi(|u|)|u| - b, \quad (6.2.4)$$

and

$$\int_0^\infty \frac{d\tau}{\Phi(\tau)} = \infty.$$

C₃) For $t \in (0, T]$ for arbitrary $x, u, p \in \mathbb{R}^n$, and any $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$, there exists $0 < \nu < \mu$ such that

$$\sum_{1 \leq i, j \leq n} a_{ij}(x, t, u, p) \xi_i \xi_j \geq 0 \quad \text{and} \quad \nu |\xi|^2 \leq a_{ij}(x, t, u, p) \xi_i \xi_j \leq \mu |\xi|^2.$$

C₄) The functions $a_i(x, t, u, p)$ and $a(x, t, u, p)$ are continuous, the functions $(a_i)_{1 \leq i \leq n}$ are differentiable with respect to x, u and $p \in \mathbb{R}^n$, and for any $N \geq 1$ there exists $\mu_1 = \mu_1(N)$ such that

$$\sup_{\substack{(x, t) \in \Gamma_N \\ |u| \leq N \\ p \in \mathbb{R}^n}} \frac{\sum_{i=1}^n (|a_i| + |\frac{\partial a_i}{\partial u}|)(1 + |p|) + \sum_{i, j=1}^n |\frac{\partial a_i}{\partial x_j}| + |a|}{(1 + |p|)^2} \leq \mu_1(N).$$

C₅) For all $N \geq 1$, and for all $|x| \leq N, 0 \leq t \leq T, |u| \leq N$ and $|p| \leq N$, the functions $a_i, a, \partial a_i / \partial p_j, \partial a_i / \partial u$, and $\partial a_i / \partial x_i$ are continuous functions satisfying a Hölder condition in x, t, u and p with exponents $\beta, \beta/2, \beta$ and β respectively for some $\beta > 0$.

Theorem 6.2.1 (See Theorem 8.1, Chapter 5, Section 8, p.495 of [54]). Assume that the conditions **C₁)–C₅)** hold. Then there exists at least one solution $u(x, t)$ of Cauchy problem (6.2.1) that is bounded in $\mathbb{R}^n \times [0, T]$ which belongs to $\mathcal{H}^{2+\beta, 1+\beta/2}(\Gamma_N)$ for any $N \geq 1$.

6.3 Verification theorem for positive utility functions

Now we give the verification theorem from [9]. Consider on the interval $[0, T]$, the stochastic control process is given by the n –dimensional Itô process with its values in $\mathfrak{E} \subseteq \mathbb{R}^n$.

$$\begin{cases} d\zeta_t^v &= a(\zeta_t^v, t, v)dt + b(t, \zeta_t^v, v)dW_t, \quad t \geq 0, \\ \zeta_0^v &= x \in \mathfrak{E} \subseteq \mathbb{R}^n, \end{cases} \quad (6.3.1)$$

where $(W_t)_{0 \leq t \leq T}$ is a standard m –dimensional Brownian motion. We set

$\mathcal{F}_t = \sigma\{W_u, 0 \leq u \leq t\}$ for any $0 < t \leq T$. We assume that the control process v_t takes values in some set $\Theta \subseteq \mathbb{R}^q$ for some integer $q \geq 1$. Note that, a takes its values in \mathbb{R}^n and b is the $n \times m$ matrix. Moreover, we assume that the coefficients a and b satisfy the following conditions:

V_1) For all $t \in [0, T]$ the functions $a(., t, .)$ and $b(., t, .)$ are continuous on $\Xi \times \Theta$.

V_2) For every deterministic vector $v \in \Theta$, the stochastic differential equation

$$d\zeta_t^v = a(\zeta_t^v, t, v)dt + b(\zeta_t^v, t, v)dW_t,$$

has a unique strong solution.

Now we introduce an admissible control process for the equation (6.3.1). The stochastic control process $v = (v_t)_{t \geq 0} = (\alpha_t, c_t)_{t \geq 0}$ is called admissible on $[0, T]$ if it is $(\mathcal{F}_t)_{0 \leq t \leq T}$ progressively measurable with values in Θ , and equation (6.3.1) has a unique strong a.s. continuous solution $\zeta_t^v \in \Xi$ such that

$$\int_0^T (|a(\zeta_t^v, t, v_t)| + |b(\zeta_t^v, t, v_t)|^2) dt < \infty \quad \text{a.s.}$$

We denote by \mathcal{V} the set of all admissible control processes with respect to equation (6.3.1). Moreover, let $\mathbf{f} : \Xi \times [0, T] \times \Theta \rightarrow [0, \infty)$ and $\mathbf{h} : \Xi \rightarrow [0, \infty)$ be continuous utility functions. Now we consider the cost function

$$\mathbf{J}(x, t, v) = \mathbf{E}_{x,t} \left(\int_t^T \mathbf{f}(\zeta_u, u, v_u) du + \mathbf{h}(\zeta_T^v) \right), \quad 0 \leq t \leq T, \quad (6.3.2)$$

where $\mathbf{E}_{x,t}$ is the expectation under condition on $\zeta_t = x$. We consider for this function, the following optimal control problem,

$$\sup_{v \in \mathcal{V}} \mathbf{J}(x, 0, v).$$

To this end, we used dynamical programming method, according to which one needs to study the optimization problems on the interval $[t, T]$ for any $0 \leq t \leq T$, i.e., to solve the optimization problem

$$\mathbf{J}^*(x, t) := \sup_{v \in \mathcal{V}_t} \mathbf{J}(x, t, v), \quad (6.3.3)$$

where \mathcal{V}_t is the set of all admissible financial strategies $v \in \mathcal{V}$ which are adapted to $\mathcal{F}_{t,u} = \sigma\{W_s - W_t, t \leq s \leq u\}$ on the interval $t \leq u \leq T$.

To this end we introduce the Hamilton function, i.e. for any ζ and $0 \leq t \leq T$, with $\mathbf{q} \in \mathbb{R}^n$

and symmetric $n \times n$ matrix \mathbf{M} we set

$$H(x, t, q, \mathbf{M}) := \sup_{\theta \in \Theta} H_0(x, t, \mathbf{q}, \mathbf{M}, \theta),$$

where

$$H_0(x, t, \mathbf{q}, \mathbf{M}, \theta) := a'(x, t, \theta)\mathbf{q} + \frac{1}{2}tr[bb'(x, t, \theta)\mathbf{M}] + \mathbf{f}(x, t, \theta).$$

In order to find the solution to (6.3.3) we investigate the HJB equation

$$\begin{cases} z_t(x, t) + H(x, t, z_x(x, t), z_{xx}(x, t)) = 0, & t \in [0, T], \\ z(x, T) = \mathbf{h}(x), & x \in \mathbb{R}^n. \end{cases} \quad (6.3.4)$$

Here, z_t denotes the partial derivative of z with respect to t , $z_x(x, t)$ the gradient vector with respect to x in Ξ and $z_{xx}(x, t)$ denotes the symmetric hessian matrix, that is the matrix of the second order partial derivatives with respect to x .

We assume the following conditions hold:

H₁) *The functions \mathbf{f} and \mathbf{h} are non negative.*

H₂) *There exists a function $z(x, t)$ from $\mathbf{C}^{2,1}(\mathbb{R}^n \times [0, T])$ from $\mathbb{R}^n \times [0, T] \rightarrow (0, \infty)$ which satisfies the HJB equation .*

H₃) *There exists a measurable function $\theta^* : \mathbb{R}^n \times [0, T] \rightarrow \Theta$ such that for all $x \in \mathbb{R}^n$ and $0 \leq t \leq T$,*

$$H(x, t, z_x(x, t), z_{xx}(x, t)) = H_0(x, t, z_x(x, t), z_{xx}(x, t), \theta^*(x, t)).$$

H₄) *For any $x \in \Xi$ and $0 \leq t \leq T$, there exists a unique strong solution to the Itô equation*

$$d\zeta_u^* = a(\zeta_u^*, u, v_u^*)du + b(\zeta_u^*, u, v_u^*)dW_u, \quad \zeta_t^* = x, \quad t \leq u \leq T,$$

where $v_u^* = \theta^*(\zeta_u^*, u)$, the optimal control process $v^* = (v_s^*)_{t \leq s \leq T} \in \mathcal{V}_t$.

H₅) *There exists some $\check{\delta} > 1$ such that for all $x \in \mathbb{R}^n$ and $0 \leq t \leq T$*

$$\sup_{\tau \in \mathcal{M}_t} \mathbf{E}_{x,t} (z(\zeta_\tau^*, \tau))^{\check{\delta}} < \infty,$$

where \mathcal{M}_t is the set of all stopping times in $[t, T]$.

Theorem 6.3.1. *Assume that conditions \mathbf{H}_1 - \mathbf{H}_5 hold, then for any $0 \leq t \leq T$ the process $(v_s^*)_{t \leq s \leq T}$ defined in H_5) is a solution for the problem (6.3.3) and $z(x, t) = \mathbf{J}^*(x, t)$.*

6.4 Verification theorem for any utility functions

In this section, we consider the optimization problem Eq. (6.3.3) for any utility functions \mathbf{f} and \mathbf{h} . To this end we need to modify the definition of admissible strategies.

Definition 6.4.1. The stochastic control process $v = (v_t)_{t \geq 0} = (\alpha_t, c_t)_{t \geq 0}$ is called admissible on $[0, T]$ with respect to Eq. (6.3.1) if it is $(\mathcal{F}_t)_{0 \leq t \leq T}$ progressively measurable with values in Θ , and Eq. (6.3.1) has a unique strong a.s. continuous solution $(\zeta_t^v)_{0 \leq t \leq T}$ such that

$$\mathbf{E} \int_0^T (\mathbf{f}(\zeta_u^v, u, v_u))_- dt < \infty, \quad \mathbf{E} \sup_{0 \leq t \leq T} (\mathbf{h}(\zeta_t^v))_- < \infty, \quad (6.4.1)$$

and

$$\int_0^T (|\check{a}(\zeta_u^v, u, v_u)| + |\check{b}(\zeta_u^v, u, v_u)|^2) dt + \int_0^T |\mathbf{f}(\zeta_u^v, u, v_u)| du < \infty \quad \text{a.s.} \quad (6.4.2)$$

We denote by \mathcal{V} the set of all admissible control processes with respect to equation Eq. (6.3.1).

Our goal is to study the problem Eq. (6.3.3) for the admissible strategies defined in the Definition 6.4.1. To this end we introduce the Hamilton function, i.e. for any x and $0 \leq t \leq T$, with $\mathbf{q} \in \mathbb{R}^n$ and symmetric $n \times n$ matrix \mathbf{M} we set

$$H(x, t, \mathbf{q}, \mathbf{M}) := \sup_{\theta \in \Theta} H_0(x, t, \mathbf{q}, \mathbf{M}, \theta), \quad (6.4.3)$$

where

$$H_0(x, t, \mathbf{q}, \mathbf{M}, \theta) := \check{a}'(x, t, \theta) \mathbf{q} + \frac{1}{2} \text{tr}[\check{b} \check{b}'(x, t, \theta) \mathbf{M}] + \mathbf{f}(x, t, \theta).$$

In order to find the solution to Eq. (4.1.6), we investigate the HJB equation

$$\begin{cases} z_t(x, t) + H(x, t, z_x(x, t), z_{xx}(x, t)) = 0, & t \in [0, T], \\ z(x, T) = \mathbf{h}(x), & x \in \mathbb{R}^n. \end{cases} \quad (6.4.4)$$

Here, z_t denotes the partial derivative of z with respect to t , $z_x(x, t)$ the gradient vector with respect to x in \mathbb{R}^n and $z_{xx}(x, t)$ denotes the symmetric hessian matrix, that is the matrix of the second order partial derivatives with respect to x .

We assume the following conditions hold:

$\bar{\mathbf{H}}_1$) There exists a $\Xi \times [0, T] \rightarrow \mathbb{R}$ function $z(x, t)$ from $\mathbf{C}^{2,1}(\Xi \times [0, T])$ which satisfies the HJB equation.

$\bar{\mathbf{H}}_2$) There exists a measurable function $\theta^* : \mathbb{R}^n \times [0, T] \rightarrow \Theta$, such that for all $x \in \mathbb{R}^n$ and $0 \leq t \leq T$,

$$H(x, t, z_x(x, t), z_{xx}(x, t)) = H_0(x, t, z_x(x, t), z_{xx}(x, t), \theta^*(x, t)).$$

$\bar{\mathbf{H}}_3$) Assume that for any $v \in \mathcal{V}$, any $0 \leq t \leq T$ and x ,

$$\mathbf{E}_{x,t} \sup_{t \leq u \leq T} (z(\zeta_u^v, u))_- < \infty.$$

$\bar{\mathbf{H}}_4$) For any $x \in \Xi$ and $0 \leq t \leq T$, there exists a unique strong solution with values in Ξ to the Itô equation

$$d\zeta_u^* = \check{a}(\zeta_u^*, u, v_u^*) du + \check{b}(\zeta_u^*, u, v_u^*) dW_u, \quad \zeta_t^* = x, \quad t \leq u \leq T,$$

where $v_u^* = \theta^*(\zeta_u, u)$, the optimal control process $v^* = (v_s^*)_{t \leq s \leq T} \in \mathcal{V}_t$ and

$$\mathbf{E}_{x,t} \sup_{t \leq u \leq T} |z(\zeta_u^*, u)| < \infty. \quad (6.4.5)$$

Theorem 6.4.1. Assume that conditions $\bar{\mathbf{H}}_1$)- $\bar{\mathbf{H}}_4$) hold, then for any $0 \leq t \leq T$, the process $(v_s^*)_{t \leq s \leq T}$ is a solution to the problem Eq. (6.3.3).

Proof. For $v \in \mathcal{V}$, let X^v be the associated wealth process with initial value $X_0^v = x$. Now, for any fixed $L > 0$ define a stopping time

$$\tau_L = \inf \left\{ s \geq t : \int_t^s |\check{b}'(\zeta_u^v, u) \partial_\zeta z(\zeta_u^v, u)|^2 du \geq L \right\} \wedge T.$$

Note that condition (6.4.2) implies that $\tau_L \rightarrow T$ as $L \rightarrow \infty$ a.s.. By continuity of $z(\cdot, \cdot)$ and of $(\zeta_t^v)_{0 \leq t \leq T}$ we obtain

$$\lim_{L \rightarrow \infty} z(\zeta_{\tau_L}^v, \tau_L) = z(\zeta_T^v, T) = \mathbf{h}(\zeta_T^v) \quad \text{a.s..} \quad (6.4.6)$$

To simplify we use the notation $\check{a}_t = \check{a}(\zeta_t, v_t, t)$ and $\check{b}_t = \check{b}(\zeta_t, v_t, t)$. Then by Itô formula

$$dz(\zeta_t, t) = z_t(\zeta_t, t) dt + \sum_{i=1}^n \frac{\partial}{\partial \zeta_i} z(\zeta_t, t) d\zeta_i + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2}{\partial \zeta_i \partial \zeta_j} z(\zeta_t, t) d \langle \zeta_i, \zeta_j \rangle_t. \quad (6.4.7)$$

By using Eq. (6.3.1), the preceding equation becomes

$$\begin{aligned} dz(\zeta_t, t) = & z_t(\zeta_t, t) + (\partial z(\zeta_t, t))' \check{\alpha}_t^v dt + \frac{1}{2} \text{tr}(\check{b}_t^v (\check{b}_t^v)' \partial^2 z(\zeta_t, t)) dt \\ & + (\partial z(\zeta_t, t))' \check{b}_t^v dW_t. \end{aligned}$$

Taking the integration for both sides we get

$$z(\zeta_{\tau_L}, \tau_L) - z(\zeta_t, t) = \int_t^{\tau_L} A_u du + \int_t^{\tau_L} (\partial z(\zeta_u, u))' \check{b}_u^v dW_u,$$

where $A_u = z_u(\zeta_u, u) + (\partial z(\zeta_u, u))' \check{\alpha}_u^v + \text{tr}(\check{b}_u^v (\check{b}_u^v)' \partial^2 z(\zeta_u, u)) / 2$. Taking into account that $\mathbf{E}_{x,t} z(\zeta_t, t) = z(x, t)$ and

$$\mathbf{E}_{x,t} \int_t^{\tau_L} (\partial z(\zeta_u, u))' \check{b}_u^v dW_u = 0,$$

we obtain that

$$z(x, t) = \mathbf{E}_{x,t} z(\zeta_{\tau_L}, \tau_L) - \mathbf{E}_{x,t} \int_t^{\tau_L} A_u du.$$

Moreover, noting that $z_u(x, u) = -H(x, z_x, z_{xx})$, we can represent the processes $-A_u$ as:

$$-A_u = H(\zeta_u, \partial z(\zeta_u, u), \partial^2 z(\zeta_u, u)) - H_0(\zeta_u, \partial z(\zeta_u, u), \partial^2 z(\zeta_u, u), v_u) + \mathbf{f}(\zeta_u, u, v_u).$$

So, for $v \in \mathcal{V}$,

$$-A_u \geq \mathbf{f}(\zeta_u, u, v_u).$$

As to this term, note that,

$$\int_t^{\tau_L} \mathbf{f}(\zeta_u, u, v_u) du = \int_t^{\tau_L} (\mathbf{f}(\zeta_u, u, v_u))_+ du - \int_t^{\tau_L} (\mathbf{f}(\zeta_u, u, v_u))_- du.$$

We recall that

$$\mathbf{E} \int_0^T (\mathbf{f}(\zeta_u, u, v_u))_- du < \infty.$$

Therefore, we obtain by the Monotone Convergence Theorem that

$$\lim_{L \rightarrow \infty} \mathbf{E} \int_0^{\tau_L} \mathbf{f}(\zeta_u, u, v_u) du = \mathbf{E} \int_t^T \mathbf{f}(\zeta_u, u, v_u) du.$$

Note also, that in view of the condition $\overline{\mathbf{H}}_3$)

$$\mathbf{E}_{x,t} \sup_{L \geq 1} (z(\zeta_{\tau_L}^v, \tau_L))_- \leq \mathbf{E}_{x,t} \sup_{t \leq u \leq T} (z(\zeta_u^v, u))_- < \infty.$$

We thereby Fatou's Lemma obtain that

$$\lim_{L \rightarrow \infty} \mathbf{E}_{x,t} z(\zeta_{\tau_L}^v, \tau_L) \geq \mathbf{E}_{x,t} \lim_{L \rightarrow \infty} z(\zeta_{\tau_L}^v, \tau_L) = \mathbf{E}_{x,t} z(\zeta_T^v, T) = \mathbf{E}_{x,t} \mathbf{h}(\zeta_T).$$

Finally, we obtain that

$$z(x, t) \geq \mathbf{E}_{x,t} \left(\int_t^T \mathbf{f}(\zeta_u^v, u, v_u) du + \mathbf{h}(\zeta_T) \right) = J(x, t, v).$$

Therefore, $z(x, t) \geq J^*(x, t)$ for all $0 \leq t \leq T$. Similarly, replacing the strategies v given by the optimal strategies v^* as defined in $\overline{\mathbf{H}}_2$) and $\overline{\mathbf{H}}_4$) we obtain

$$z(x, t) = \mathbf{E}_{x,t} \int_t^{\tau_L} \mathbf{f}(\zeta_u^v, u, v_u) du + \mathbf{E}_{x,t} z(\zeta_{\tau_L}^*, \tau_L).$$

The upper bound (6.4.5) implies that the family $\{z(\zeta_{\tau_L}^*, \tau_L)\}_{L \geq 1}$ is uniformly integrable. Therefore, the limit equation (6.4.6) yields

$$\lim_{L \rightarrow \infty} \mathbf{E}_{x,t} z(\zeta_{\tau_L}^*, \tau_L) = \mathbf{E}_{x,t} \lim_{L \rightarrow \infty} z(\zeta_{\tau_L}^*, \tau_L) = \mathbf{E} z(\zeta_T, T) = \mathbf{E}_{x,t} \mathbf{h}(\zeta_T^*),$$

and we obtain

$$\begin{aligned} z(x, t) &= \lim_{L \rightarrow \infty} \mathbf{E}_{x,t} \int_t^{\tau_L} \mathbf{f}(\zeta_u^*, u, v_u^*) du + \lim_{L \rightarrow \infty} \mathbf{E}_{x,t} z(\zeta_{\tau_L}^*, \tau_L) \\ &= \mathbf{E}_{x,t} \left(\int_t^T \mathbf{f}(\zeta_u^*, u, v_u^*) du + \mathbf{h}(\zeta_T^*) \right) = J(x, t, v^*). \end{aligned}$$

We arrive at $z(x, t) = J^*(x, t)$. This proves Theorem 6.4.1. □

6.5 Technical Lemmas

6.5.1 The smoothness properties of the process in Eq. (2.2.6)

Lemma 6.5.1. *For any bounded function Q in \mathcal{X} and for $u > t$,*

$$\left| \frac{\partial}{\partial s} \int_t^T \mathbf{E}Q(\eta_u^{s,t}, u) du \right| \leq Q_t^* \frac{2}{\sigma} \sqrt{\frac{2(T-t)}{\pi}},$$

where

$$Q_t^* = \sup_{\substack{s \in \mathbb{R} \\ t \leq u \leq T}} |Q(s, u)|.$$

Proof. By Fubini theorem, if a function $Q > 0$, then

$$\frac{\partial}{\partial s} \mathbf{E} \int_t^T Q(\eta_u^{s,t}, u) du = \frac{\partial}{\partial s} \int_t^T \mathbf{E}Q(\eta_u^{s,t}, u) du.$$

As

$$\mathbf{E}Q(\eta_u^{s,t}, u) = \frac{1}{\sigma_1(u, t)} \int_{\mathbb{R}} Q(y, u) \varphi\left(\frac{y - s\mu(u, t)}{\sigma_1(u, t)}\right) dy,$$

where

$$\varphi(\theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\theta^2}{2}} \quad \text{and} \quad \theta = \frac{z - x\mu(u, t)}{\sigma_1(u, t)}.$$

Then we have that,

$$\mathbf{E}Q(\eta_u^{s,t}, u) = \frac{1}{\sqrt{2\pi}\sigma_1(u, t)} \int_{\mathbb{R}} Q(y, u) \exp\left\{-\frac{(y - s\mu(u, t))^2}{2\sigma_1^2(u, t)}\right\} dy.$$

Thus by deriving the last expression with respect to s we get

$$\frac{\partial}{\partial s} \mathbf{E}Q(\eta_u^{s,t}, u) = \frac{\mu(u, t)}{\sqrt{2\pi}\sigma_1^3(u, t)} \int_{\mathbb{R}} Q(y, u) (y - s\mu(u, t)) \exp\left\{-\frac{(y - s\mu(u, t))^2}{2\sigma_1^2(u, t)}\right\} dy.$$

Then by letting

$$v = \frac{y - s\mu(u, t)}{\sigma_1(u, t)},$$

the preceding equation becomes

$$\frac{\partial}{\partial s} \mathbf{E}Q(\eta_u^{s,t}, u) = \frac{\mu(u, t)}{\sqrt{2\pi}\sigma_1(u, t)} \int_{\mathbb{R}} Q(s\mu(u, t) + v\sigma_1(u, t), u) v e^{-v^2/2} dv.$$

By taking the absolute value for both sides we get

$$\begin{aligned}
\left| \frac{\partial}{\partial s} \mathbf{E}Q(\eta_u^{s,t}, u) \right| &\leq \frac{\mu(u,t)}{\sqrt{2\pi}\sigma_1(u,t)} \int_{\mathbb{R}} |Q(s\mu(u,t) + v\sigma_1(u,t), u)| |v| e^{-v^2/2} dv, \\
&\leq Q_t^* \frac{\mu(u,t)}{\sqrt{2\pi}\sigma_1(u,t)} \int_{\mathbb{R}} |v| e^{-v^2/2} dv, \\
&= Q_t^* \sqrt{\frac{2}{\pi}} \frac{\mu(u,t)}{\sigma(u,t) \left(\int_t^u \mu^2(u,z) dz \right)^{1/2}},
\end{aligned}$$

where $Q_t^* = \sup_{y \in \mathbb{R}, u \geq t} |Q(y, u)|$. Therefore,

$$\left| \frac{\partial}{\partial s} \mathbf{E}Q(\eta_u^{s,t}, u) \right| \leq Q_t^* \sqrt{\frac{2}{\pi}} \frac{\mu(u,t)}{\sigma_1(u,t)}.$$

Since the integral

$$\begin{aligned}
\int_t^T \frac{\mu(u,t)}{\sigma_1(u,t)} du &= \int_t^T \frac{e^{-\int_t^u g_1(v) dv}}{\sigma \sqrt{\int_t^u e^{-2\int_z^u g_1(v) dv} dz}} du, \\
&\leq \int_t^T \frac{1}{\sigma \sqrt{u-t}} du = \frac{2}{\sigma} \sqrt{T-t}.
\end{aligned}$$

Therefore,

$$\left| \frac{\partial}{\partial s} \mathbf{E}Q(\eta_u^{s,t}, u) \right| \leq Q_t^* \sqrt{\frac{2}{\pi}} \frac{1}{\sigma \sqrt{u-t}}. \quad (6.5.1)$$

Then by taking the integral from t to T ,

$$\left| \frac{\partial}{\partial s} \int_t^T \mathbf{E}Q(\eta_u^{s,t}, u) du \right| \leq Q_t^* \sqrt{\frac{2}{\pi}} \int_t^T \frac{1}{\sigma \sqrt{u-t}} du \leq Q_t^* \frac{2}{\sigma} \sqrt{\frac{2(T-t)}{\pi}}. \quad (6.5.2)$$

Hence Lemma 6.5.1. □

In the sequel we need the following condition:

\underline{H} : There exists $\gamma > 0$ such that

$$y^* f(t, y) \leq -\gamma |y|^2 \quad \forall t \in \mathbb{R}_+, \quad y \in \mathbb{R}^q.$$

Lemma 6.5.2 (Lemma 1.1.1 in [39]). *Let y be the solution of the following equation*

$$dy_t = f(t, y_t)dt + G_t dW_t, \quad y_0 = 0 \quad (6.5.3)$$

with f satisfying (\underline{H}) and $\|G\|_T \leq M$ where $T \in \mathbb{R}_+$ and $M = M_G$ is a constant.

Then for every integer $m \geq 1$ the following inequalities hold:

$$\mathbf{E}|y_t|^{2m} \leq k_m(t) \quad \forall t \in \mathcal{R}_+,$$

$$\mathbf{E}|y_\tau|^{2m} \leq c_m(T) \quad \forall \tau \in \mathcal{T}_T,$$

where

$$k_m(t) := (2m-1)!! M^{2m} \left(\frac{1 - e^{-2\gamma t}}{2\gamma} \right)^m,$$

$$c_m(T) := m(2m-1)!! M^2 \int_0^T k_{m-1}(u) du.$$

Proposition 6.5.3 (Proposition 1.1.2 in [39]). *Let y be the solution of Eq. (6.5.3). Under the assumptions of Lemma 6.5.2*

$$\mathbf{E}|y_t|^{2m} \leq m!(2M^2 t)^m,$$

$$\mathbf{E}|y_\tau|^{2m} \leq m!(2M^2 T)^m \quad \forall \tau \in \mathcal{T}_T,$$

$$\mathbf{E} \max_{0 \leq t \leq T} |y_t|^{2m} \leq (3 + m \ln m) m! (2M^2 T)^m. \quad (6.5.4)$$

Proposition 6.5.4 (Proposition 1.1.5 in [39]). *Let y be the solution of Eq. (6.5.3). Under the assumptions of Lemma 6.5.2*

$$\begin{aligned} \mathbf{E}|y_t|^{2m} &\leq m!(M^2/\gamma)^m, \\ \mathbf{E}|y_\tau|^{2m} &\leq 2mm!(M^2/\gamma)^m\gamma T \quad \forall \tau \in \mathcal{T}_T, \\ \mathbf{E} \max_{0 \leq t \leq T} |y_t|^{2m} &\leq (3 + m \ln m)2mm!(M^2/\gamma)^m(1 \vee (\gamma T)). \end{aligned} \tag{6.5.5}$$

Chapter 7

Numerical Simulation

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7.1 Numerical Simulation for power utility functions

For optimisation problem with power utility function, we simulate the function represented by the fixed-point $h(s, t)$ given by

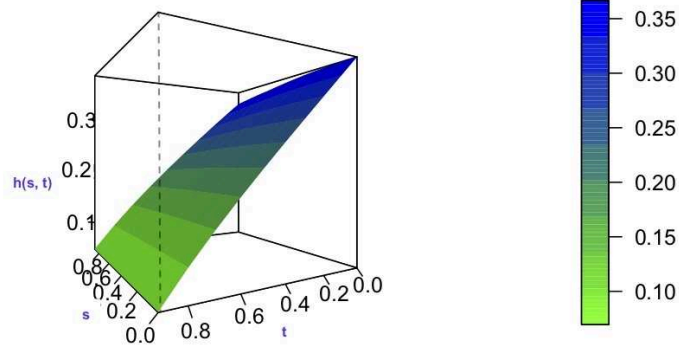
$$\mathcal{L}_h(s, t) = \int_t^T \mathbf{E} \Psi_h(\eta_u^{s,t}, u) du, \quad (7.1.1)$$

where $\Psi_h(\eta_u^{s,t}, u)$ is given in Eq. (2.2.7). We set $h_0 = 0$ and

$$h_n(s, t) = \mathcal{L}_{h_{n-1}}(s, t) \quad \text{for } n \geq 1.$$

The main goal is to implement the fixed point scheme. We use Monte Carlo method to calculate the expectation in Eq. (7.1.1) we generate a large number N of random paths of the process $(\eta_u^{s,t})_{u \geq t}$ then we replace the integral by the empirical mean using Monte Carlo method. We consider the following parameter values with $\sigma = 1$, $\gamma = 0.5$, $\kappa = 0.5$ and $r = 0.05$ for the time interval $[0, 1]$ and 1000 times of iteration.

To simulate the strategies for the problem considering power utility functions in Chapter 2,

Fig. 7.1 The Limit function $h(s, t)$

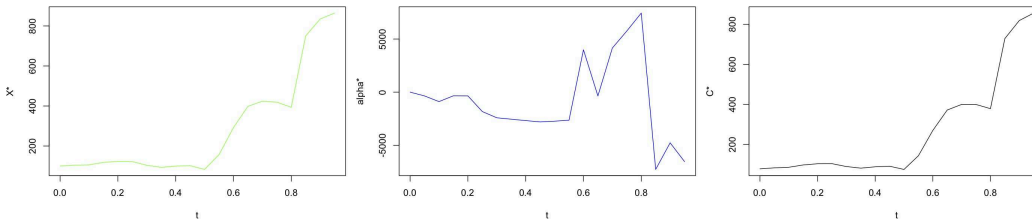
we use the following equations for optimal strategies given in Eq. (2.3.13).

$$\alpha_t^* = \check{\alpha}^0(\zeta_t^*, t) = \check{\beta}(s, t)x^* \quad \text{and} \quad c_t^* = \check{c}^0(\zeta_t^*, t) = \check{G}(s, t)x^*,$$

where

$$\check{\beta}(s, t) = \frac{1}{1-\gamma} \left(sg(t) + h_s(s, t) - \frac{\kappa_1}{\sigma^2} s \right) \quad \text{and} \quad \check{G}(s, t) = \varpi^{\frac{1}{\gamma-1}} G(s, t, h(s, t)).$$

As seen in Fig. 7.2 that the graph for both the wealth and the consumption is nearly identical. This is due to the function G that in this case, it almost equals to one. In addition, we assumed that $\varpi = 1$.



(a) The wealth process X_t^* . (b) Optimal investment α_t^* . (c) Optimal consumption c_t^* .

Fig. 7.2 The wealth process x^* with the parameters α^* and c^* when $\sigma = 0.1$, $r = 0.05$ and $\kappa = 5$.

7.2 Numerical Simulation for logarithmic utility functions

For 1–dimensional case. Fig. 7.3 shows the value function $z(\zeta, t)$ given in Eq. (4.2.8) by

$$z(x, s, t) = \rho(t) \ln x + s'g(t)s + f(t).$$

The optimal strategies for 1– dimensional case as in Example 1 are given by

$$\alpha^*(t) = \check{\alpha}^0(\zeta_t^*, t) = -\frac{\kappa_1 S_t X_t^*}{\sigma^2} \quad \text{and} \quad c^*(t) = \check{c}^0(\zeta_t^*, t) = \frac{X_t^*}{\rho(t)}.$$

The differential wealth process for this example is given by

$$dX_t^* = X_t^* a^*(t) dt + X_t^* b^*(t) dW_t,$$

where $a^*(t)$ and $b^*(t)$ are given in Eq. (4.3.6). The following parameters have been used: $T = 1$, $r = 0.01$, $\kappa = 0.1$, $\sigma = 0.5$ with the initial endowment $x = 100$.

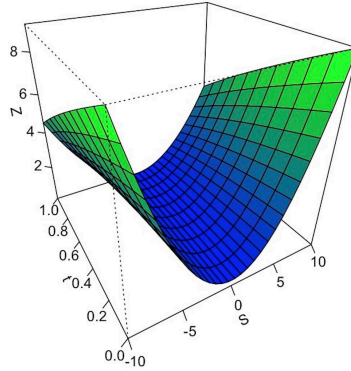
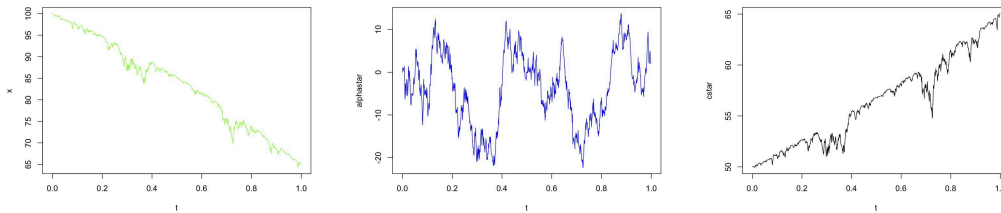


Fig. 7.3 The value function.

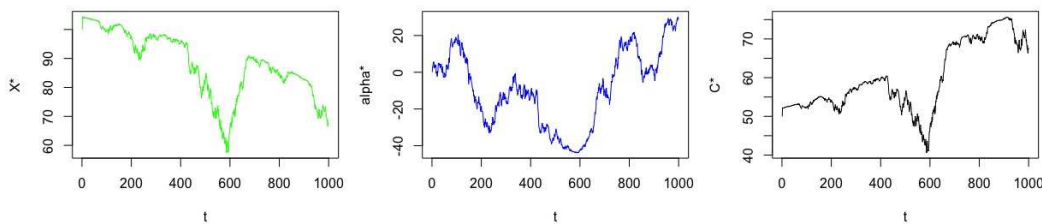
Now, we simulate the optimal strategies α_t^* and c_t^* given in Eq. (2.3.13) with the optimal wealth process x_t^* . In the following figures, we used different parameters to show the behaviour of the strategies with different values of r, κ and σ . As seen in the figures below, the behaviour of the wealth process is increasing constantly when κ has large values (see Fig. 7.8a and Fig. 7.12a). However, it is clear that the wealth process is decreasing when κ has a quite small value as seen in Fig. 7.4a and Fig. 7.10a. In addition we see that the volatility in the investment process increases and decreases depending on the fraction κ_1/σ^2 . Thus the range of volatility in figures (Fig. 7.4b - Fig. 7.11b) is less than the ones in figures (Fig. 7.12b - Fig. 7.15b) which jumps to 4000 points or even more aggressively in Fig. 7.16b. This is due to the higher number we get from the fraction which is nearly 50 and 70 for

Fig. 7.16b where the volatility reaches 10^5 .



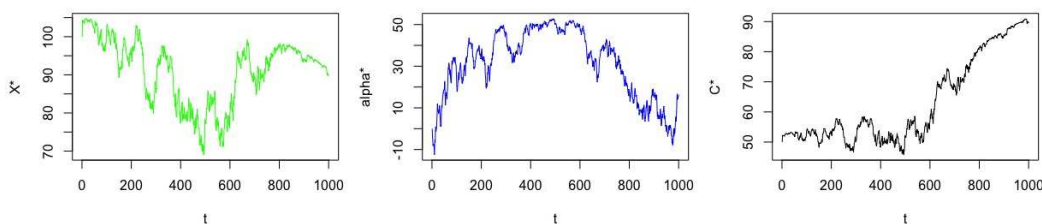
(a) The wealth process X_t^* . (b) Optimal investment α_t^* . (c) Optimal consumption c_t^* .

Fig. 7.4 The wealth process with the parameters α and c when $\sigma = 1$, $r = 0.01$ and $\kappa = 0.5$.



(a) The wealth process X_t^* . (b) Optimal investment α_t^* . (c) Optimal consumption c_t^* .

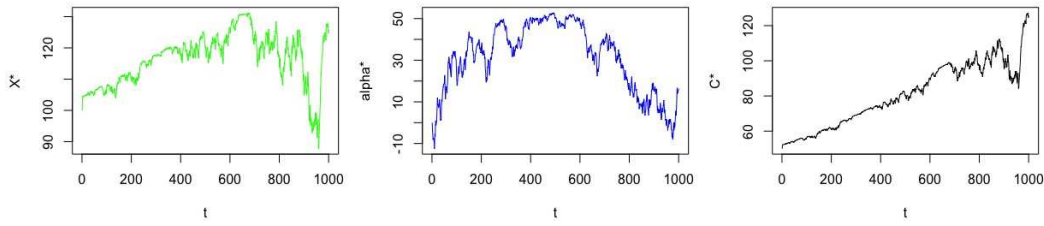
Fig. 7.5 The wealth process X with the parameters α and c when $\sigma = 1$, $r = 0.1$ and $\kappa = 0.5$.



(a) The wealth process X_t^* . (b) Optimal investment α_t^* . (c) Optimal consumption c_t^* .

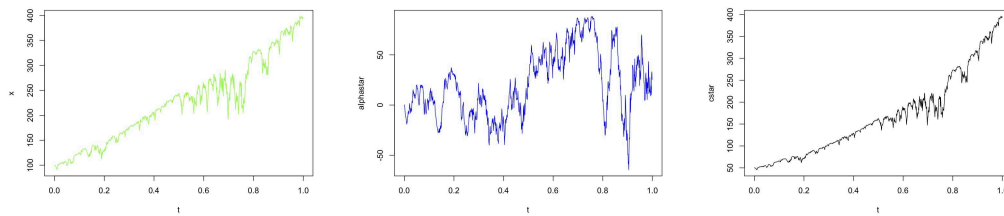
Fig. 7.6 The wealth process X with the parameters α and c when $\sigma = 1$, $r = 0.2$ and $\kappa = 0.5$.

Remark 7.2.1. As seen in the previous simulations, that we illustrate the optimal calculations



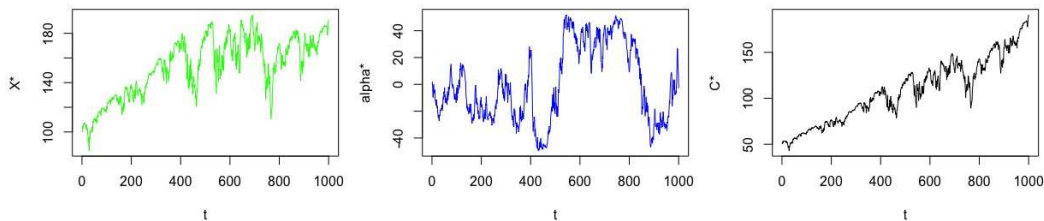
(a) The wealth process X_t^* . (b) Optimal investment α_t^* . (c) Optimal consumption c_t^* .

Fig. 7.7 The wealth process X with the parameters α and c when $\sigma = 1$, $r = 0.5$ and $\kappa = 0.5$.



(a) The wealth process X_t^* . (b) Optimal investment α_t^* . (c) Optimal consumption c_t^* .

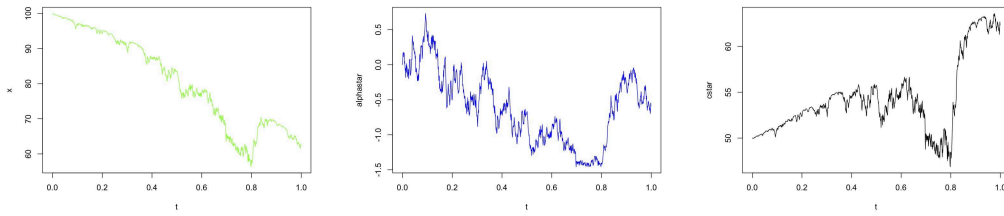
Fig. 7.8 The wealth process with the parameters α and c when $\sigma = 5$, $r = 4$ and $\kappa = 5$.



(a) The wealth process X_t^* . (b) Optimal investment α_t^* . (c) Optimal consumption c_t^* .

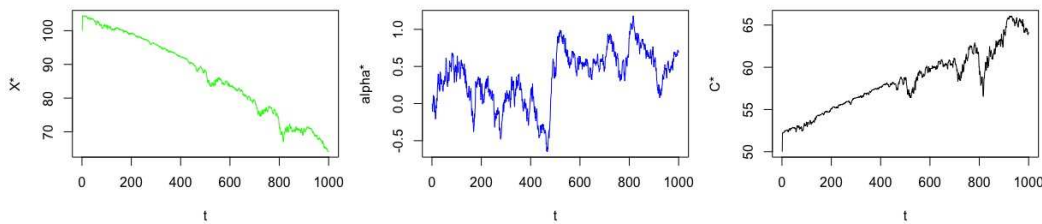
Fig. 7.9 The wealth process X with the parameters α and c when $\sigma = 5$, $r = 0$ and $\kappa = 5$.

Remark 7.2.2. Numerical simulation is a technique for practice and learning that can be applied to many different disciplines such as financial markets. It is a technique to amplify real experiences with guided ones that evoke or replicate substantial aspects of the real market in a fully interactive fashion. It provides a valuable tool in learning practical dilemmas.



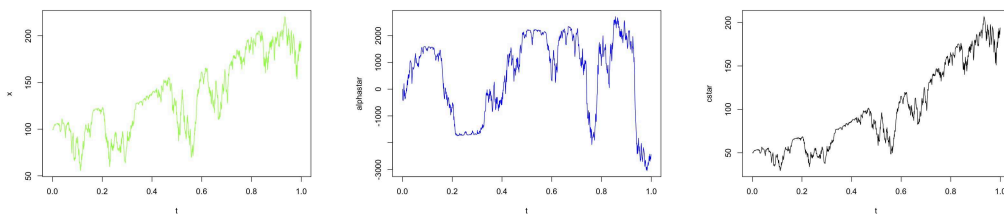
(a) Wealth process X_t^* (b) Optimal investment α_t^* . (c) Optimal consumption c_t^* .

Fig. 7.10 The wealth process with the parameters α and c when $\sigma = 20$, $r = 0.01$ and $\kappa = 0.5$ with $n = 1000$.



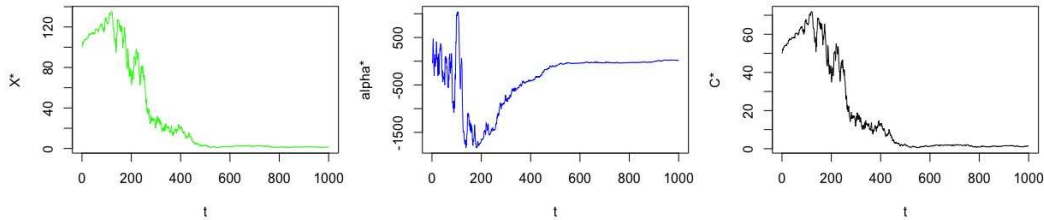
(a) The wealth process X_t^* . (b) Optimal investment α_t^* . (c) Optimal consumption c_t^* .

Fig. 7.11 The wealth process X with the parameters α and c when $\sigma = 20$, $r = 0$ and $\kappa = 0.5$.



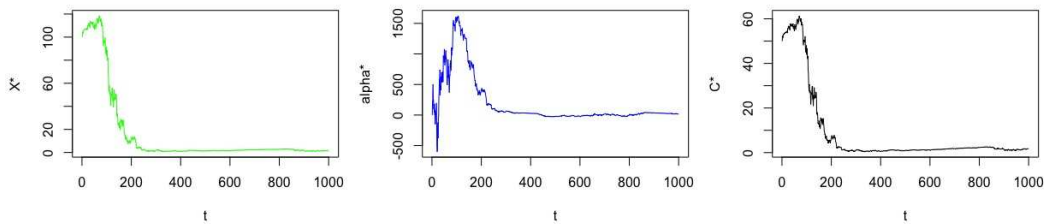
(a) The wealth process X_t^* . (b) Optimal investment α_t^* . (c) Optimal consumption c_t^* .

Fig. 7.12 The wealth process with the parameters α and c when $\sigma = 0.1$, $r = 0.01$ and $\kappa = 5$ with $n = 1000$.



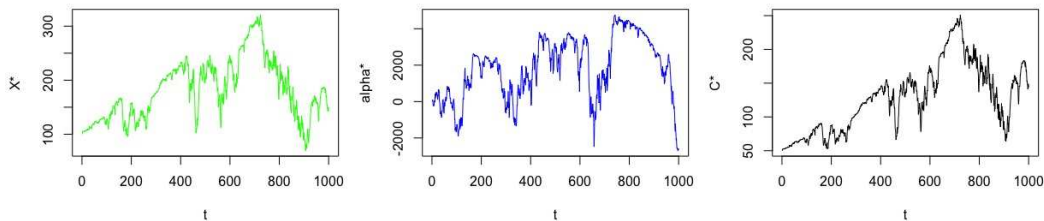
(a) The wealth process X_t^* . (b) Optimal investment α_t^* . (c) Optimal consumption c_t^* .

Fig. 7.13 The wealth process X with the parameters α and c when $\sigma = 0.1$, $r = 0.1$ and $\kappa = 5$.



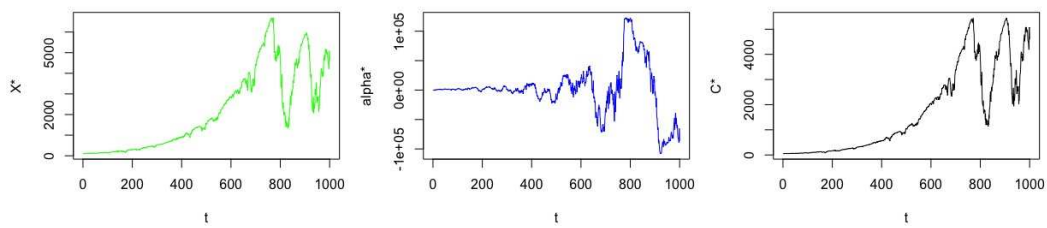
(a) The wealth process X_t^* . (b) Optimal investment α_t^* . (c) Optimal consumption c_t^* .

Fig. 7.14 The wealth process X with the parameters α and c when $\sigma = 0.1$, $r = 0.2$ and $\kappa = 5$.



(a) The wealth process X_t^* . (b) Optimal investment α_t^* . (c) Optimal consumption c_t^* .

Fig. 7.15 The wealth process X with the parameters α and c when $\sigma = 0.1$, $r = 1$ and $\kappa = 5$.



(a) The wealth process X_t^* . (b) Optimal investment α_t^* . (c) Optimal consumption c_t^* .

Fig. 7.16 The wealth process X with the parameters α and c when $\sigma = 0.1$, $r = 3$ and $\kappa = 5$.

Chapter 8

Conclusion

In conclusion, this thesis considered an optimisation problem for consumption and investment in two cases power and logarithmic utility functions by choosing Ornstein-Uhlenbeck process to model the spread between risky assets. For power utility functions, two cases had been discussed for small time interval and for any time interval $[0, T]$. For the first case, the choice of time is small and the maturity time should be less than T_0 where it depends on the mean reverting κ , the market volatility σ and the utility coefficient γ . In the second case we have no conditions on time interval, however the coefficient ϖ that describes the investor's preference between pure investing or consuming is conditioned to be greater than or equal to the fraction $(16T/\pi)^{1-\gamma}$.

For logarithmic utility functions it had been used the dynamic programming method to solve the Hamilton-Jacobi-Bellman equation and two examples had been stated to show the problem in scalar case and in multivariate case where the volatility σ was considered to be diagonal matrix. Moreover, explicit solutions had been stated for the HJB equation. In addition, the verification theorem had been applied to show that there exists a unique strong solution for the Itô equation given by the strategy proposed. Nonetheless, numerical simulations had been stated for the scalar case of logarithmic utility case to demonstrate the behaviour of the strategies and the wealth process by considering different values of the mean-reverting and volatility. As shown that the wealth process is decreasing relatively with κ . The optimal consumption/investment strategy for logarithmic utility functions had been explicitly calculated. In addition, we extended the work by considering stochastic volatility of the same problem.

This dissertation had discussed the optimisation strategies for both power and logarithmic utilities. However, it is been assumed that the market has no constraints and transaction costs. As a result, it is recommended for further work considering these conditions. In addition, in this thesis, it had been considered the scalar case for power utility functions. Prof.

Pergamenshchikov and me are working now on multivariate case of this problem. Therefore, further researches are recommended to develop these problems to fulfil the real market, such as taking into account the transaction costs.

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Appendix A

The R simulation codes

A.1 Ornstein-Uhlenbeck process

```
1 #***** Mathematica program *****
2 #***** For single path *****
3 proc[k_, \[Sigma]_] :=
4   ItoProcess[\[DifferentialD]s[
5     t] == -k s[t] \[DifferentialD]t + \[Sigma] \[DifferentialD]
6     ]w[t],
7   s[t], {s, 0}, t, w \[Distributed] WienerProcess[]]
8   proc[1, 1]
9   paths = RandomFunction[proc[1, 1], {0, 10, 0.01}, 1]
10  ListLinePlot[paths]
11 #***** For multipath *****
12 proc[k_, \[Sigma]_] :=
13   ItoProcess[\[DifferentialD]s[
14     t] == -k s[t] \[DifferentialD]t + \[Sigma] \[DifferentialD]
15     ]w[t],
16   s[t], {s, 0}, t, w \[Distributed] WienerProcess[]]
17   paths = RandomFunction[proc[0.5, 1], {0, 10, 0.01}, 10]
18   ListLinePlot[paths]
```


A.2 Simulation for power utility

```

1  #*****Wiener process*****
2  n ← 10^5          # Number of increments that we made (steps).
3  T ← 1            # Terminal time (usually we use it =1).
4  t0 ← 0           # Initial time
5  dt ← (T- t0)/n  # Increments size (delta)
6  t ← seq(t0, T, length=n+1)
7  W ← c(t0, cumsum(sqrt(dt)*rnorm(n)))
8  plot(t, W, type="l", main="Wiener process", ylim=c(-1,1))
9  #----- Another method for Wiener process-----
10
11 T ← 1; t0 ← 0; N ← 1000
12 xi ← rnorm(n=1000, 0, 1) #Create normal distributed random
13 variable with mean=0
14 # and standard deviation=1 also with n=1000 random
15 variables to chose from.
16 plot(xi, type="l") #type="l" to connect point.
17 Wt ← sqrt((T-t0)/N)*cumsum(xi) #To add continuously
18 # xi[i+1]=sqrt((T-t0)/N) *(xi[i]+rnorm(1) )
19 plot(Wt, type="l")
20 length(Wt)
21 #***For the value function in Power Utility ***
22 #----- Defining the Constantes -----
23 sigma ← 1; gamma ← 0.5; kapp ← 0.5; r ← 0.05
24 gamma1 ← (sigma^2)/(1-gamma)
25 gamma2 ← (gamma*(kapp+r)/(1-gamma))+kapp
26 c1 ← 1; c ← 0.5; T ← 1; M ← 1000
27 #-----For the Integral function-----
28 integ ← function(t,u,m){
29   v ← t+(u-t)*(0:(m-1))/m
30   g1 ← gamma1*(c1-exp(-c*(T-v)))-gamma2
31   return(sum(g1)*(u-t)/m)
32   }
33 integ(0,0.4,100000) ### quel m prendre pour
34 avoir une bonne approximation de l'integrale ?
35 Sigma2 ← function(t,u,m){

```

```

36   v ← t+(u-t)*(0:(m-1))/m
37   mu2 ← integ(v,u,m)**2
38   return(sum(mu2)*(u-t)/m
39           }
40 Sigma2(0,0.4,100000)
41 z ← T*(0:(M-1))/M
42 xi ← rnorm(1000,0,1)
43 #----- Eta function -----
44 eta ← function(s,t,u,z,m){
45   return(sum((z≤u)*integ(z,u,m)*xi)*((u-t)/m)
46   *sqrt(Sigma2(t,u,m))+s*exp(integ(t,u,m)))
47 }
48 eta(1,0,0.4,z,100000)
49 #----- G function -----
50 G ← function(s,t,y)
51 {
52   gt ← c1-exp(-c*(T-t))
53   return(exp(-(1/(1-gamma))*(gt*s*s/2+y)))
54 }
55 G(0.44,0.4,1)
56 #----- Gamma0 function -----
57 Gamma0 ← function(s,t,y1,y2)
58 {
59   gt ← c1-exp(-c*(T-t))
60   return((gamma1/2)*y2**2+(sigma**2)*gt/2
61   +r*gamma+(1-gamma)*1*G(s,t,y1))
62 }
63 Gamma0(1,4,1,0)
64 #----- Expectation function -----
65 EPsi ← function(s,t,u,etaa,h,hs,N)
66 { e ← 0
67   for (i in 1:N){
68     va ← rnorm(1000,0,1)
69     eta(s,t,u,z,10000)
70     e ← e+sum(Gamma0(etaa,u,h,hs))
71   }
72   return(e/N)

```

```

73 }
74
75 EPsi(0.44,0.4,.6,eta(1,0,0.4,z,100000),0,0,1000)
76
77 #----- L function -----
78 L ← function(s,t,h,hs,T,m)
79 { l ← (T-t)/m
80   N ← 1000
81   u ← t+(0:(m-1))*l
82   return(l*sum(EPsi(s,t,u,eta(s,t,u,z,10000),h,hs,N)))
83 }
84
85 L(0.44,0.4,0,0,1,10000)
86
87
88 #***** Second try *****
89 m ← 100; K ← 6; I ← 5; J ← 5
90 s1 ← (0:(I-1))/I
91 t1 ← T*(0:(J-1))/J
92 h ← array(rep(0,I*J*K),dim=c(I,J,K))
93 hs ← array(rep(1,I*J*K),dim=c(I,J,K))
94 for (k in 1:K-1)
95 {
96   for (j in 1:J)
97   {
98     h[1,j,k+1]=L(s1[1],t1[j],h[1,j,k],1,T,m)
99     h[2,j,k+1]=L(s1[2],t1[j],h[2,j,k],1,T,m)
100    hs[1,j,k+1]=(h[2,j,k+1]-h[1,j,k+1])/(s1[2]-s1[1])
101    for (i in 2:I-1)
102    {
103      h[i-1,j,k+1]=L(s1[i-1],t1[j],h[i-1,j,k],1,T,m)
104      ## pas forcement utile
105      h[i,j,k+1]=L(s1[i],t1[j],h[i,j,k],1,T,m)
106      h[i+1,j,k+1]=L(s1[i+1],t1[j],h[i+1,j,k],1,T,m)
107      #hs[i,j,k+1]=(h[i+1,j,k+1]-h[i-1,j,k+1])/(s1[i+1]-s1[i-1
108      ])

```

```

109     }
110     }
111     #h[I, j, k+1]=L(s1[I], t1[j], h[I, j, k], 1, T, m)
112     hs[I, j, k+1]=(h[I-1, j, k+1]-h[I, j, k+1]) / (s1[I-1]-s1[I])
113     }
114 }
115 h[2, , ]
116 hs[1, , ]
117 s1
118 #-----
119 #----- To plot the strategies -----
120 A1 ← r(T-0) -
121 x ← exp(A1)
122 alpha11 ← (s1*4+hs+s1*(kappa+r))*x
123 plot(alpha11)
124 #second try
125 T ← 1
126 r ← 0.01
127 n ← 10
128 gamma ← 0.2
129 kappa ← 5
130 sigma ← 0.1
131 kappal ← kappa+r
132 dt ← T/n
133 dw ← rnorm(n, 0, sqrt(T/n))
134 s0 ← 0
135 x0 ← 100
136 s ← c(s0)
137 x ← c(x0)
138 s ← array(rep(0, n))
139 I ← 5
140 J ← 5
141 s1 ← (0:(I-1))/I
142 x ← array(rep(0, n))
143 astar ← array(rep(0, n))
144 bstar ← array(rep(0, n))
145 t ← T*(0:(n-1))/n

```

```

146 rho ← T-t+1
147 astar[1] ← r+kappa1^2*s0^2/sigma^2-1/rho[1]
148 bstar[1] ← -kappa1*s0/sigma
149 for (i in 2:(n)) {
150   s[1] ← s0
151   x[1] ← x0
152   #astar[1] ← 0
153   s[i] ← s[i-1] - kappa*s[i-1]*dt + sigma*dw[i-1]
154   astar[i] ← r-kappa1*s[i]*(s[i]+ hs[i] - kappa1*s[i]/sigma^
155     2)
156   / (1- gamma)
157   bstar[i] ← sigma*(s[i]+hs[i]- kappa1*s[i]/sigma^2)
158   / (1- gamma)
159   x[i] ← x[i-1] + astar[i-1]*x[i-1]*dt + bstar[i-1]*(x[i-1])
160   *dw[i-1]
161 }
162 alphastar ← x*(s +hs -(kappa1*s)/sigma^2)/(1 - gamma)
163 cstar ← x*exp(-(s^2/2 +h)/(1 - gamma))
164 #-----
165 #----- Plot -----
166 nbcoll ← 100
167 jet.colors ← colorRampPalette( c("blue", "green") )
168 color ← jet.colors(nbcoll)
169 facetcol ← cut(zfacet, nbcoll)
170 persp(s1, t1, h, col = color[facetcol])
171
172 #persp(s1, t1, h, col = color[facetcol], phi = 30, theta = -30,
173 xlab = "S", ylab = "t", zlab = "Z", ticktype = "detailed")
174
175 # Strategies
176 T ← 1
177 r ← 0.05
178 #n ← 10
179 gamma ← 0.20
180 kappa ← 5
181 sigma ← 0.1

```

```
182 kappal ← kappa+r
183 #dt ← T/n
184 dt ← T/J
185 #dw ← rnorm(n,0,sqrt(T/n))
186 dw ← rnorm(n,0,sqrt(T/J))
187 #s0 ← 0
188 x0 ← 100
189 #s ← c(s0)
190 #X ← c(x0)
191 #s ← array(rep(0,n))
192 S ← array(rep(0,I))
193 #I ← 10 #it was 5
194 #J ← 10 # it was 5
195 #s1 ← (0:(I-1))/I
196 X ← array(rep(0,J))
197 astar ← array(rep(0,J))
198 bstar ← array(rep(0,J))
199 cstar ← array(rep(0,J))
200 t ← T*(0:(J-1))/J
201 rho ← T-t+1
202 gt ← c1 - ((2*theta*c1)/(exp(omega*(T-t))*c-c))
203
204
205 for (j in 2:J) {
206   #s[1] ← s0
207
208   #astar[1] ← 0
209   S[j] ← S[j-1] - kappa*S[j-1]*dt + sigma*dw[j-1]
210 }
211
212 max(S)
213 min(S)
214
215 ds=(max(S)-min(S))/I
216 ds
217 indice=rep(1,J)
218 compteur=0
```

```

219 for (j in 1:J) {
220     compteur=0
221     while (S[j] ≥ min(S)+compteur*ds) {
222         compteur=compteur+1
223         #print("YY")
224     }
225     indice[j]=compteur
226 }
227
228 astar[1] ← r-kappal*S[1]*(S[1]*gt[1]+ hs[indice[1],1,6] -
          kappal*S[1]/sigma^2)/(1- gamma)+G(S[1],1,h[indice[1],1,6])
229
230 bstar[1] ← sigma*(S[1]*gt[1]+hs[indice[1],1,6]- kappal*S[1]/
          sigma^2)/(1- gamma)
231 X[1] ← x0
232 for (j in 2:J) {
233     #s[1] ← s0
234
235     #astar[1] ← 0
236     #S[j] ← S[j-1] - kappa*S[j-1]*dt + sigma*dw[j-1]
237     astar[j] ← r-kappal*S[j]*(S[j]*gt[j]+ hs[indice[1],j,6] -
          kappal*S[j]/sigma^2)/(1- gamma)+G(S[j],1,h[indice[1],j,6])
238     bstar[j] ← sigma*(S[j]*gt[j]+hs[indice[1],j,6]- kappal*S[j]/
          sigma^2)/(1- gamma)
239     X[j] ← X[j-1] + astar[j-1]*X[j-1]*dt +bstar[j-1]*(X[j-1])*dw
          [j-1]
240 }
241 alphastar ← X*(S*gt +hs[indice[1],,6] -(kappal*S)/sigma^2)/(1
          - gamma)
242 for (j in 1:J) {
243     cstar[j] ← X[j]*G(S[j],t[j],h[indice[1],j,6])
244 }
245
246 plot(t,cstar,type="l", xlab="t", ylab="C*")
247 plot(t,alphastar,type="l", xlab="t", ylab="alpha*", col="blue")
248 plot(t,X,type="l", xlab="t", ylab="X*", col="green")

```

A.3 Simulation for log utility

```

1  ### defining the variables ###
2
3  T=1
4  r=0.01
5  kappa=0.1
6  kappal=kappa+r
7  sigma=0.5
8  x=100
9  s ← seq(-10,10, length=30)
10 t ← seq(0,1, by=.1)
11
12 ## The function g
13 g ← function(t) (-kappal^2/2*sigma^2 )
14     * ((-2*kappa*exp(2*kappa*(t-T))
15     + t-T-1 + exp(2*kappa*(t-T)) - 1 )/(4* kappa^2))
16
17 ## The function f
18 f ← function(t) sigma^2*g(t) + T-t-(T-t+1)* log(T-t+1)
19
20 ## The function Z
21 Zvarsigma ← function(s,t) log(T-t+1) +s^2*g(t)+f(t)
22
23 ## The plot ###
24
25 z ← outer(s,t, Zvarsigma)
26 jet.colors ← colorRampPalette( c("blue", "green") )
27 nbcol ← 100
28 color ← jet.colors(nbcol)
29 nrz ← nrow(z)
30 ncz ← ncol(z)
31 zfacet ← z[-1, -1] + z[-1, -ncz] + z[-nrz, -1] + z[-nrz, -ncz]
32 facetcol ← cut(zfacet, nbcol)
33
34 persp(s, t, z, col = color[facetcol], phi = 30, theta = -30,
      xlab = "S",

```



```
35         ylab = "t", zlab = "Z", ticktype = "detailed")
36 # ticktype -- to give details in the numbers or values of each
   variable
37
38 ## The strategies ##
39 rho ← T-t+1
40 astar ← r+(kappal*s^2/sigma^2)-1/rho
41 bstar ← -kappal*s/sigma
42 xstar ← seq(0.1,100)
43 alphastar← s*xstar/sigma^2
44 persp(s, xstar,alphastar, xlab = "S", ylab = "xstar", zlab = "
   alpha^*")
45 cstar ← xstar/(T-t+1)
46 persp(xstar, t, cstar, xlab = "xstar", ylab = "t", zlab = "c^*"
   )
```

Listing A.1 The value function