# Quantisation of moduli spaces and connections

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par

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Composition du Jury :

| | | | |
|---|---|---|---|
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A mia madre: irriducibile;
A mia sorella, sempre un passo davanti a me;
a mio padre, e come a lui agli altri, per quel che vale.

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These people improved my recent life in ways for which there were no substitutes.

I dedicate my last paragraph to my family and my friends closer to "home". These few words cannot convey the importance of their distant, yet constant presence; neither could many other more.

*"[...] at least now it should be clear that you're not quantising the curve..."*
*Anonymous*

# Synthèse de la thèse en français

Dans cette thèse on étudie la quantification de :

1. Systèmes Hamiltoniens dépendants du temps. Leur flot décrit des équations différentielles non-linéaires qui gèrent les déformations isomonodromiques de certaines connexions méromorphes sur la sphère de Riemann. Il s'agit ici de quantification par déformation.

2. Espaces de modules, qui sont réalisés par des variétés hyperKähler. Ils paramétrisent $\mathrm{SL}(2,\mathbb{C})$-connexions holomorphes sur un tore dans une structure complexe, et $\mathrm{SL}(2,\mathbb{C})$-fibrés de Higgs dans une autre. Il s'agit ici de quantification géométrique.

Pour introduire ces deux projets, nous commençons par rappeler les constructions que nous voulons généraliser.

## Quantification par déformation de connexions de isomonodromie

Soient $m, n$ deux entiers positifs. On définit

$$\mathfrak{g} \coloneqq \mathfrak{gl}(n,\mathbb{C}), \qquad \mathbf{B} \coloneqq \mathbb{C}^m \setminus \{\text{diags}\} = \Big\{(t_1,\dots,t_m) \in \mathbb{C}^m \mid t_i \neq t_j \text{ si } i \neq j\Big\},$$

et on fixe un produit scalaire invariant sur $\mathfrak{g}$, ce qui permet d'identifier $\mathfrak{g} \cong \mathfrak{g}^*$. Considérons le fibré vectoriel holomorphe trivial de rang $n$ sur la sphère, avec la connexion

$$\nabla = d - \sum_i \frac{R_i}{z-t_i} dz, \qquad \text{où} \qquad R_i \in \mathfrak{g}, \quad (t_1,\dots,t_m) \in \mathbf{B}, \tag{1}$$

et où $z$ est une coordonnée qui identifie $\mathbb{C}P^1 \cong \mathbb{C} \cup \{\infty\}$. Les équations différentielles afin qu'une section locale $\psi$ soit horizontale pour la connexion s'écrivent

$$\frac{d\psi}{dz} = \left(\sum_i \frac{R_i}{z-t_i}\right)\psi. \tag{2}$$

Les solutions de (2) sont des fonctions multivaluées. La façon dans laquelle elle se transforment par prolongement analytique le long d'un boucle autour d'un pole (simple) est dictée par la monodromie du système. La monodromie exprime l'action du groupe fondamental de la sphère — où l'on a enlevé les pôles — sur l'espace des solutions fondamentales de (2). Si les positions des pôles et les résidus varient, la monodromie en général change. Une classe importante de déformations est définie précisément en demandant que ceci n'arrive pas : celles-ci sont les déformations isomonodromiques des systèmes Fuchsiens (sur la sphère).
Schlesinger [Sch05] a dérivé un système d'équations différentielles pour les résidus $R_i$ — en tant que fonctions des positions $t_i$ — qui code de telles déformations isomonodromiques.

Résoudre ces équations d'isomonodromie revient à calculer le flot du système Hamiltonien dépendant du temps $H_i^{\mathrm{Sch}} \colon \mathfrak{g}^m \times \mathbf{B} \longrightarrow \mathbb{C}$ défini par:

$$H_i^{\mathrm{Sch}}(R_1, \ldots, R_m, t_1, \ldots, t_m) = \sum_{i \neq j} \frac{\mathrm{Tr}(R_i R_j)}{t_i - t_j}, \qquad \text{pour} \qquad 1 \leq i \leq m. \tag{3}$$

Les équations d'isomonodromie s'écrivent alors $\{H_i^{\mathrm{Sch}}, R_j\} = \frac{\partial R_j}{\partial t_i}$, où l'on utilise le crochet de Poisson de $\mathfrak{g}^m \cong (\mathfrak{g}^*)^m$. Géométriquement, elles sont équivalentes à une connexion de Ehresmann intégrable sur la fibration de Poisson triviale $\mathfrak{g}^m \times \mathbf{B} \longrightarrow \mathbf{B}$. Quantifier cette connexion d'isomonodromie signifie quantifier les Hamiltoniennes $H_i^{\mathrm{Sch}}$, et il y a un formalisme algébrique standard pour cela. À savoir, la restriction fibre-à-fibre des $H_i^{\mathrm{Sch}}$ définit des fonctions polynomiales sur $\mathfrak{g}^m$, c'est-à-dire des éléments de l'algèbre symétrique $\mathrm{Sym}(\mathfrak{g}^*)^{\otimes m} \cong \mathrm{Sym}(\mathfrak{g})^{\otimes m}$, et le théorème de Poincaré–Birkhoff–Witt donne une application $\mathcal{Q}_{\mathrm{PBW}} \colon \mathrm{Sym}(\mathfrak{g})^{\otimes m} \longrightarrow U(\mathfrak{g})^{\otimes m}$. Comme l'algèbre enveloppante universelle est une quantification de l'algèbre symétrique, la quantification fibre-à-fibre

$$\mathcal{Q}_{\mathrm{PBW}}(H_i^{\mathrm{Sch}}) \colon U(\mathfrak{g})^{\otimes m} \times \mathbf{B} \longrightarrow \mathbf{B}$$

définit un système Hamiltonien quantique dépendant du temps qui quantifie (3). De plus, [Res92; Har96] ont montré que les équations dynamiques qui lui sont associés correspondent aux équations de KZ universelles, c'est-à-dire

$$\frac{\partial \psi}{\partial t_i} = \left( \sum_{j \neq i} \frac{\Omega_{ij}}{t_i - t_j} \right) \psi. \tag{4}$$

Celles-ci sont des équations pour une section locale $\psi$ du fibré vectoriel trivial $U(\mathfrak{g})^{\otimes m} \times \mathbf{B} \longrightarrow \mathbf{B}$ (cf. [EFK98]). Du point de vue mathématiques, les éléments $\Omega_{ij}$ dénotent l'action d'un tenseur $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ par multiplication gauche sur l'$i$-ème et le $j$-ème facteur du produit tensoriel $U(\mathfrak{g})^{\otimes m}$, et la preuve que KZ quantifie le système de Schlesinger revient à la remarque que la fonction $\mathrm{Tr}(R_i R_j)$ est quantifiée par l'opérateur $\Omega_{ij}$.
Tout comme pour les équations de Schlesinger, les équations de KZ (4) sont équivalentes à une connexion plate — dans le fibré vectoriel trivial $U(\mathfrak{g})^{\otimes m} \times \mathbf{B} \longrightarrow \mathbf{B}$ —, et toute cette discussion peut se résumer en disant que la connexion de KZ est une quantification de la connexion de Schlesinger.

Ceci est le programme que l'on veut généraliser, en admettant des singularités irrégulières dans les connexions méromorphes. La première étape dans cette direction a été la dérivation de [Boa02] de la connexion de DMT à partir d'un problème d'isomonodromie irrégulier. Nous considérons ici les déformations isomonodromiques de connexions méromorphes sur la sphère de Riemann ayants un pôle d'ordre deux à l'infini et un pôle simple en zéro :

$$\nabla = d - \left( T + \frac{R}{z} \right) dz, \qquad \text{où} \qquad T, R \in \mathfrak{g}, \tag{5}$$

et où $T = \mathrm{diag}(t_1, \ldots, t_n)$ est diagonale avec spectre simple. Maintenant nous déformons $T$ dans la partie régulière $\mathfrak{h}_{\mathrm{reg}}$ de l'algèbre de Cartan $\mathfrak{h} \subseteq \mathfrak{g}$ des matrices diagonales, et

nous écrivons accordement des équations d'isomonodromie pour le résidu $R$. Les Hamiltoniennes d'isomonodromie s'écrivent alors

$$H_i^{\mathrm{dSch}} = \sum_{i \neq j} \frac{R_{ij} R_{ji}}{t_i - t_j}. \tag{6}$$

Elles sont définies sur la fibration triviale $\mathfrak{g} \times \mathfrak{h}_{\mathrm{reg}} \longrightarrow \mathfrak{h}_{\mathrm{reg}}$, et leur quantification à là PBW produit les Hamiltoniennes quantiques (dépendants du temps) de la connexion de DMT :

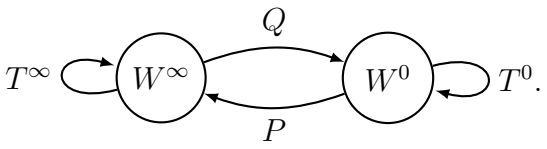
$$\widehat{H}_i^{\mathrm{DMT}} = \frac{1}{2} \sum_{j \neq i} \frac{\widehat{e}_{ij} \cdot \widehat{e}_{ji} + \widehat{e}_{ji} \cdot \widehat{e}_{ij}}{t_i - t_j} : \mathfrak{h}_{\mathrm{reg}} \longrightarrow U(\mathfrak{g}),$$

où $e_{ij}$ est la base canonique de $\mathfrak{g}$.

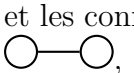
Le système d'isomonodromie classique $H_i^{\mathrm{dSch}}$ est une version duale de celui de Schlesinger, dans le sens suivant. L'espace des temps pour (3) est l'espace des configurations de $m$ pôles sur la sphère où aucune pair de pôle ne se réunit, tandis que dans (6) on peut varier le spectre de la matrice diagonale $T$ — qui code le type irrégulier $T dz$ à l'infini — de façon que les valeurs propres restent distinctes. Ces derniers correspondent aux temps d'isomonodromie irréguliers considérés par Jimbo–Miwa–Môri–Sato (JMMS) [Jim+80]. Harnad [Har94] a montré que les deux familles de temps dans le système de JMMS peuvent être renversées, et ceci n'est qu'une partie de la dualité d'opérateurs différentiels rationnels suivante :

$$\frac{d}{dz} - \left( T^0 + Q(z - T^\infty)^{-1} P \right) \longmapsto \frac{d}{dz} + \left( T^\infty + P(z - T^0)^{-1} Q \right), \tag{7}$$

où $W^\infty$, $W^0$ sont des espaces vectoriels complexes de dimension finie, et $T^0, T^\infty, Q, P$ sont des fonctions linéaires allant dans les directions indiquées dans ce graphe orienté :




Donc le côté gauche de (7) (resp. le côté droit) agit sur des sections locales du fibré vectoriel holomorphe trivial de fibre $W^0$ sur la sphère (resp. de fibre $W^\infty$). Cette dualité induit en particulier une correspondance entre les connexions (1) et (5). Cette dernière sous-tend la dualité de Howe/quantique de [Bau99] utilisé dans [Tol02] pour relier les connexions de KZ et de DMT pour $\mathfrak{g} = \mathfrak{gl}_n$.

Nous pouvons ainsi dire qu'il existe une correspondance entre les problèmes d'isomonodromie et les connexions quantiques intégrables qui sont associées aux représentations du graphe  , c'est-à-dire du graphe complet sur deux nœuds. À ces deux nœuds nous associons un espace de temps qui correspondent aux variations admissibles des spectres des endomorphismes semi-simple $T^\infty$ et $T^0$ : ces déformations sont définies comme celles qui ne changent pas les décompositions de $W^\infty$ et $W^0$ en espaces propres. À son tour, ces déformations sont les déformations admissibles d'une surface de Riemann sauvage équipée

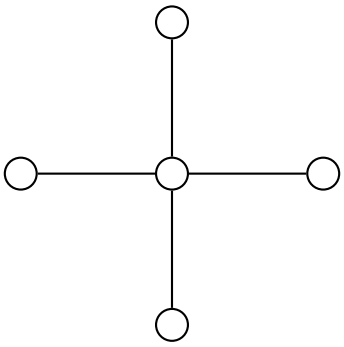


Figure 1: L'étoile est le graphe complet biparti associé à KZ et DMT.

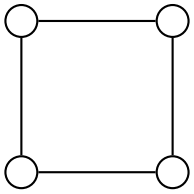
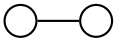


Figure 2: Un graphe biparti complet générique correspond à la connexion de FMTV.

avec un type irrégulier qui transporte un pôle d'ordre deux à l'infini, plus un nombre fini de pôles simples. Pour coder ces déformations admissibles dans le graphe, on peut éclater les nœuds de  suivant les décompositions en sommes directes de $W^{\infty}$ et $W^{0}$ en obtenant un graphe *biparti* complet (cf. § 3.1).

En [Boa12b], Boalch a montré qu'on peut généraliser ce point de vue à des graphes $k$-partis complets arbitraires. Ceci étend la théorie Hamiltonienne des déformations d'isomonodromie de connexions méromorphes sur la sphère de façon à inclure un cas particulier de [JMU81]. Les nouveaux systèmes Hamiltoniens intégrables associés à des graphes $k$-partis complets simplement lacés sont appelés "système d'isomonodromie simplement lacés" (*simply-laced isomonodromy systems* en anglais). Comme nous pouvons nous y attendre, ils impliquent $k$ familles de temps qui correspondent toujours à des déformations de sphères sauvages en généralisant les deux familles de temps du système de JMMS. De plus, à ce niveau-ci, il y a un groupe de $\mathrm{SL}(2,\mathbb{C})$-symétries évident qui inclue la transformée de Fourier–Laplace. Cette transformée définit (7) pour un graphe biparti complet. Le groupe de symétrie conduit au groupe de Weyl su système de racines du graphes.

Suite à cette discussion, nous avons l'impression qu'il devrait y avoir une théorie quantique sous-jacente à cette extension en mécanique classique. À savoir, on conjecture l'existence d'une famille de connexions plates associées à des graphes $k$-partis complets qui quantifie par déformation les systèmes d'isomonodromie simplement lacés, et qui contient KZ, DMT et FMTV comme cas particuliers (Fig. 1 et 2).
Le premier nouvel exemple est associé à un triangle car dans le langage de [Boa12b] ce graphe correspond à un système d'isomonodromie pour des connexions ayants un pôle d'ordre trois à l'infini et qui ne peuvent pas être ramenées à des connexions avec rang de Poincaré inférieur en utilisant les $\mathrm{SL}(2,\mathbb{C})$-symétries ci-dessus (Fig. 3).

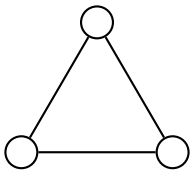


Figure 3: Quelle est la connexion quantique plate du triangle ?

La première question qu'on se pose est alors la suivante :

1. *Peut-on construire une quantification par déformation des systèmes d'isomonodromie simplement lacés ?*

Si cela avait été possible, on obtiendrait une nouvelle famille de "connexions quantiques simplement lacés" (*simply-laced quantum connections* en anglais : SLQC) à partir de la quantification par déformation de connexions d'isomonodromie. Toutefois, il faut remarquer qu'une quantification arbitraire n'a aucune raison de préserver la propriété cruciale d'intégrabilité. Nous devons alors renforcer la question précédente.

2. *Peut-on construire un système Hamiltonien quantique intégrable qui quantifie les systèmes d'isomonodromie simplement lacés ?*

Une telle construction serait plus intéressante si elle était une généralisation de KZ [KZ84], DMT [MT05] et FMTV [Fel+00], qui sont toutes obtenues à partir de la quantification de cas particuliers des systèmes d'isomonodromie simplement lacés. Plus précisément, Schlesinger et son dual sont associés aux graphes étoilés, et JMMS au graphe biparti complet arbitraire.

3. *Peut-on construire les connexions quantiques simplement lacés de façon qu'elles se réduisent à KZ, DMT et FMTV dans les cas pertinents ?*

Pour résumer, répondre aux trois questions précédentes signifie remplir la dernière colonne de la table suivante :

| Système d'isomonodromie | Schlesinger | Schlesinger dual | JMMS | SLIMS |
|---|---|---|---|---|
| Espace des temps | $\mathbb{C}^m \setminus \{\text{diags}\}$ | $\mathbb{C}^n \setminus \{\text{diags}\}$ | $\mathbb{C}^m \setminus \{\text{diags}\} \times \mathbb{C}^n \setminus \{\text{diags}\}$ | $\prod_1^k \mathbb{C}^{d_i} \setminus \{\text{diags}\}$ |
| Connexion quantique | KZ | DMT | FMTV | SLQC |

Ceci va être traité dans le Chap. 3.

# Quantification Kählerienne de l'espace de module de connexions plates

Soit $\Sigma$ une surface réelle, fermée, orientable de genre $g \geq 2$. Posons $K \coloneqq \mathrm{SU}(2)$. Nous pouvons considérer l'espace de modules $\mathcal{M}_{\mathrm{fl}}(\Sigma, K)$ de classes de $S$-équivalence de $K$-connexions plates semi-stables sur le $K$-fibré principal trivial au-dessus de $\Sigma$. Les connexions irréductibles (pour lesquelles la $S$-équivalence se réduit à la relation ordinaire d'isomorphisme) définissent un sous-espace lisse $\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}} \subseteq \mathcal{M}_{\mathrm{fl}}$.
Atiyah & Bott [AB83] et Goldman [Gol84] ont prouvé qu'il existe une structure symplectique entière naturelle $\omega$ sur $\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}$. Nous pouvons alors définir la préquantification de $(\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}, \omega)$. La définition de données de préquantification pour $(\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}, \omega)$ est suggérée par la théorie des champs de Chern–Simons [CS74] pour la variété $X \coloneqq \Sigma \times [0, 1]$. Chern–Simons est une théorie des champs topologique en dimension 3 (cf. [Fre95]). En ce sens-ci, l'espace de modules $\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}$ est l'espace des conditions au bord pour la théorie de Chern–Simons sur la 3-variété $X$. Du coup, une des motivations principales derrière la quantification de $\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}$ est de définir une quantification de la théorie de Chern–Simons (compacte).

La préquantification de la variété symplectique lisse $(\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}, \omega)$ est, par la suite, perfectionnée en une quantification par rapport à des polarisations Kähleriennes. À savoir, pour toute structure complexe $\tau$ sur la surface $\Sigma$ on trouve une structure Kähler $I_\tau$ sur $(\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}, \omega)$ et on considère l'espace de Hilbert quantique de sections $I_\tau$-holomorphe à carré intégrable du fibré de préquantification $L$ — au niveau $k \geq 1$. Pour faire cela, on équipe $L$ avec l'opérateur $\overline{\partial}$ défini par la partie $(0, 1)$ de la connexion de préquantification et on utilise la norme $L^2$ naturelle sur les sections lisses de $L$ induite par la métrique de préquantification et la mesure de Liouville :

$$\mathcal{H}_{\mathbb{C},\tau}^{(k)} \coloneqq H^0\left(\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}, L^{\otimes k}\right) \cap L^2(\mathcal{M}_{\mathrm{fl}}, \mu). \tag{8}$$

La structure complexe $I_\tau$ dépend de $\tau$ à isotopie près, et donc l'espace de Teichmüller $\mathcal{T}_\Sigma$ de $\Sigma$ paramétrise une famille d'espaces vectoriels dont les projectifiés sont des modèles pour la quantification de $(\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}, \omega)$. Toutefois, en physique, la quantification est par principe indépendante du choix des polarisations.
Du point de vue mathématique, on souheterait des identifications canoniques entre les projectifiés des espaces de Hilbert (8) pour des choix différents de $\tau \in \mathcal{T}_\Sigma$. La solution proposée par Axelrod & Della Pietra & Witten [ADW91] et Hitchin [Hit90] est de construire une connexion projectivement plate dans le fibré vectoriel $\mathcal{H}^{(k)} \longrightarrow \mathcal{T}_\Sigma$ dont la fibre au-dessus de $\tau$ est l'espace (8) (Witten suggère déjà cette idée en [Wit89] et trace son origine dans la théorie conforme des champs). Cette connexion a été, par la suite, appelée la connexion de Hitchin dans la communauté mathématique ; Hitchin a établi la platitude projective en utilisant les Hamiltoniennes Poisson-commutatives du système intégrable introduit par lui même en [Hit87b]. Une nouvelle construction de la connexion de Hitchin par opérateurs différentiels globaux agissant sur des sections lisses du fibré de préquantification a été donnée après par Andersen [And12], dans un contexte plus général (cf. aussi [And06] pour une application de ce point de vue pour démontrer la fidélité asymptotique des représentations quantiques du *mapping class group* de la surface, ainsi que le plus récent [AR16]).

Ceci accompli la quantification Kählerienne de l'espace de modules de connexions unitaire plate au-dessus d'une surface de Riemann il y a environ 25 ans.

La prochaine étape naturelle dans ce programme est la quantification de la théorie de Chern–Simons complexe, déjà considérée par Witten en [Wit91]. Du point de vue des espaces de modules, on complexifie le groupe $K$ pour obtenir un groupe réductif complexe $G$ — l'archétype étant $G = \mathrm{SL}(2,\mathbb{C})$, la complexification de $\mathrm{SU}(2)$ — et on considère l'espace de module $\mathcal{M}_{\mathrm{dR}}(\Sigma, G)$ de $G$-connexions plates au-dessus de $\Sigma$.
La situation classique est davantage plus riche que celle du cas compact puisque maintenant la structure complexe $J$ du groupe $G$ se combine avec toute structure complexe $I_\tau$ qui provient d'une structure de surface de Riemann sur $\Sigma$ pour définir un triplet hyperKähler $(I_\tau, J, K_\tau = I_\tau J)$. La construction de Atiyah–Bott produit cette fois-ci une structure symplectique $J$-holomorphe $\omega_{\mathbb{C}}$ sur l'espace de modules. Ceci a été montré par Hitchin [Hit87a] en introduisant un espace de module hyperKähler $\mathfrak{M}$ de solutions aux équations d'auto-dualité sur $\Sigma$ (maintenant appelées équations de Hitchin) qui a été relié à $\mathcal{M}_{\mathrm{dR}}$ Donaldson en [Don87] — compagnon de [Hit87a]) — et plus généralement par Corlette en [Cor88].
Plus précisément, l'espace de modules de Hitchin est une variété hyperKähler qui sous-tend trois variétés algébriques non-isomorphes, une desquelles est $\mathcal{M}_{\mathrm{dR}}$ — dans la structure complexe $J$ —, et une autre étant l'espace de modules $\mathcal{M}_{\mathrm{Dol}}(\Sigma, G)$ de $G$-fibrés de Higgs au-dessus de $\Sigma$ — dans la structure complexe $I_\tau$. La troisième description algébrique est celle de l'espace de modules de Betti $\mathcal{M}_{\mathrm{B}}(\Sigma, G)$ des $G$-représentations du groupe fondamental de $\Sigma$, à conjugaison près.

La correspondance lisse entre $G$-fibré de Higgs et $G$-connexions plates/$G$-systèmes locaux a été établie d'après les travaux précédemment cités de Hitchin, Donaldson, Corlette et finalement Simpson [Sim92]. Simpson a également suggéré en [Sim94; Sim97] de considérer cette correspondance comme une correspondance de Hodge pour le premier groupe de cohomologie non-Abelienne de la variété Kähler compacte $\Sigma$ — à coefficients dans $G$ —, parallèlement aux théories cohomologiques usuelles de de Rham, Dolbeault et Betti — à coefficients en $\mathbb{C}^* \cong \mathrm{GL}_1(\mathbb{C})$.

Une question naturelle se pose alors : construire la quantification géométrique de l'espace de modules $\mathfrak{M}$ par rapport à des polarisations Kähleriennes extraites de la sphère de structures Kähler. Plus précisément, on remplace le niveau quantique de la théorie par un paramètre complexe $t = k + i\sigma \in \mathbb{C}^*$, et on l'utilise pour définir une structure symplectique réelle sur $\mathfrak{M}$ à partir de la forme complexe de Atiyah–Bott $\omega_{\mathbb{C}}$ :

$$\omega_t := \frac{t\omega_{\mathbb{C}} + \bar{t}\overline{\omega_{\mathbb{C}}}}{2} = \Re(t\omega_{\mathbb{C}}).$$

La construction de données de préquantification $(L^{(t)}, \nabla, h)$ au niveau $t$ est adaptée du cas compact. Pour tout point $\tau$ dans l'espace de Teichmüller, nous avons une combinaison linéaire distinguée $I_\tau^{(t)}$ du triplet hyperKähler qui donne une structure complexe et qui dépend du niveau quantique à homothétie près. Géométriquement, varier le nombre complexe unitaire $t/|t| \in U(1)$ donne une paramétrisation du cercle de structures complexes $I_\tau^{(t)} = k' I_\tau - \sigma' K_\tau$ dans la sphère Kähler, où $k' + i\sigma' = \frac{k+i\sigma}{\sqrt{k^2+\sigma^2}}$ est la normalisation de $t$. Ce cercle est l'équateur de la sphère si l'on prend $\pm J$ comme étant les pôles.

Nous avons donc une famille de polarisations Kähleriennes parametrisée par l'espace de Teichmüller qui, en conséquence, définit le fibré quantique $\mathcal{H}^{(t)}_{\mathbb{C}}$ de sections holomorphes de $L^{(t)}$, comme c'était dans le cas d'un groupe compact.

La question qu'on se pose est alors la suivante :

1. *Peut-on construire une connexion projectivement plate dans $\mathcal{H}^{(t)}_{\mathbb{C}}$, pour tout choix de $t \in \mathbb{C}^*$ ?*

Ceci permettrait d'établir la quantification Kählerienne de l'espace de modules de connexions plates pour le groupe de jauge complexe $G$. En cette généralité, cette question est toujours ouverte, accordement avec le fait que la quantification de la théorie de Chern–Simons complexe reste un sujet de recherche active, au moins du point de vue mathématique.

Ce que Witten a fait en [Wit91] est différent. Il a considéré des polarisations réelles, toujours dépendantes d'une structure complexe sur $\Sigma$, et il a par la suite définit une connexion projectivement plate dans le fibré quantique $\mathcal{H}^{(t)}_{\mathbb{R}}$ de sections polarisées. Les détails mathématiques ont été donnés par Andersen & Gammelgaard [AG14], qui ont appelé cette connexion la connexion de Hitchin–Witten.
Comme on assume que la construction de Witten est la plus pertinente du point de vue de la théorie quantique des champs, on devrait relier la connexion pour la quantification Kählerienne de $\mathcal{M}^{\mathrm{st}}_{\mathrm{dR}}$ et la connexion de Hitchin–Witten. Une manière naturelle est de chercher à construire un isomorphisme $\mathcal{B}^{(t)} \colon \mathcal{H}^{(t)}_{\mathbb{R}} \longrightarrow \mathcal{H}^{(t)}_{\mathbb{C}}$ entre les fibrés quantiques et conjuguer la connexion de Hitchin–Witten à travers cet isomorphisme. Le candidat naturel est — une petite variation de — la transformée de Bargmann [Bar61].

2. *Peut-on construire une identification entre les espaces des sections polarisées par polarisations réelles et les sections holomorphes, pour tout point de l'espace de Teichmüller ?*

3. *Peut-on calculer une formule pour la conjugaison de la connexion de Hitchin–Witten dans le fibré vectoriel $\mathcal{H}^{(t)}_{\mathbb{C}}$ ?*

Nous pouvons nous attendre à ce que la connexion de Hitchin dans $\mathcal{H}^{(t)}_{\mathbb{C}}$ soit unique. Plus précisément, la construction de la connexion de Hitchin par opérateurs différentiels globaux (d'ordre deux) se base sur une Ansatz qui peut être généralisée en ce contexte non-compact. Ceci est notamment important si l'on veut tenir compte d'un élément important de l'espace de modules pour la théorie complexe : l'action de dilation naturelle de $\mathbb{C}^*$ sur les champs de Higgs.
Nous voudrions utiliser cette action pour induire une action du groupe $U(1)$ sur l'espace de Hilbert quantique des sections holomorphes en définissant ainsi une gradation sur $\mathbb{Z}$ qui donnerait une décomposition en composantes isotypiques :

$$\mathcal{H}^{(t)}_{\mathbb{C},\tau} = \bigoplus_{m\in\mathbb{Z}} \mathcal{H}^{(t)}_{\mathbb{C},\tau,m}, \qquad \text{où} \qquad \mathcal{H}^{(t)}_{\mathbb{C},\tau,m} = \left\{ s \in \mathcal{H}^{(t)}_{\mathbb{C},\tau} \,\middle|\, \lambda.s = \lambda^m s \right\}.$$

Ceci est une des façons pour traiter le fait que les espaces des états de la théorie de Chern–Simons complexe sont de dimension infinie.

Actuellement, il existe un thème de recherche dédié à calculer des formules pour la dimension des composantes isotypiques, avec le caveat que l'action de $U(1)$ soit bien définie sur les espaces de sections holomorphes. Si par exemple on suppose que $\Im(t) = \sigma = 0$, alors $I_\tau^{(t)} = I_\tau$ est fixée par l'action du cercle, et [AGP16] établie une formule à la Verlinde dans ce contexte-ci.
Si l'on admet une partie imaginaire non nulle dans le niveau quantique, au contraire, alors le diffeomorphisme $\lambda \in U(1)$ va changer $I_\tau^{(t)}$ en $\lambda^* I_\tau^{(t)}$. Géométriquement, le tiré-en-arrière correspond à une rotation par $\lambda$ dans la sphère Kähler ; plus précisément, comme la structure complexe de Dolbeault $I_\tau$ est fixée par l'action, la rotation est le long de l'équateur de la sphère, si cette fois-ci on fixe $\pm I_\tau$ comme étant les pôles. Donc la rotation est autour d'un axe orthogonal à celui de la rotation induite par une variation du niveau quantique $t$.
En tout cas, pour définir une action dans le fibré des sections holomorphes nous avons besoin d'une identification canonique entre l'espace des sections $I_\tau^{(t)}$-holomorphes de $L^{(t)}$ et celui des sections $\lambda^* I_\tau^{(t)}$-holomorphes.

La première idée qu nous avons poursuivi pour résoudre ce problème est une extension de la configuration de Hitchin et Witten. À savoir, nous pouvons montrer que les espaces des sections holomorphes pour les structures Kähleriennes $\lambda^* I_\tau^{(t)}$ se situent dans un sous-fibré vectoriel du fibré vectoriel trivial de préquantification, au-dessus d'une base étendue

$$C^\infty(\mathfrak{M}, L^{(t)}) \times \mathcal{T}_\Sigma \times U(1) \longrightarrow \mathcal{T}_\Sigma \times U(1).$$

Par la suite, nous pouvons essayer de démontrer que ce fibré peut être muni d'une connexion projectivement plate. La variété $\mathcal{T}_\Sigma \times U(1)$ devrait être considérée comme un "espace de Teichmüller étendu" pour $\Sigma$, ce qui est une des motivations principales de cette thèse.

Cependant, une première difficulté technique se présente : l'action de $U(1)$ n'est pas symplectique. On doit donc trouver une bonne définition pour les données de préquantification pour la famille de variétés symplectiques $(\mathfrak{M}, \lambda^*\omega_t)_{\lambda \in U(1)}$ parametrisées au-dessus d'un cercle.

4. *Peut-on construire des données de préquantification sur l'espace de modules qui sont compatibles avec l'action de Hitchin ?*

La situation change les règles usuelles de la quantification géométrique qui demandent de quantifier une variété symplectique *fixée* par rapport à des polarisations qui varient.
Si les données de préquantification sont définies, alors le fibré trivial de préquantification l'est aussi. Nous pouvons désormais montrer que les espaces de sections $\lambda^* I_\tau^{(t)}$-holomorphes définissent un sous-fibré vectoriel à l'intérieur. Une des manières est de répondre à la question suivante.

5. *Existe-t-il une connexion dans le fibré de préquantification au-dessus de l'espace de Teichmüller étendu qui préserve l'holomorphicité fibre-à-fibre ?*

Si nous avions une connexion projectivement plate avec monodromie triviale autour du cercle, alors nous aurions une action du cercle sur les espaces $\mathcal{H}^{(t)}_{\mathbb{C},\tau}$ de sections holomorphes. Nous pourrions en effet composer l'isomorphisme défini par $\lambda$ avec le transport parallèle le long du chemin qui relie $\lambda^* I^{(t)}_\tau$ à $I^{(t)}_\tau$, c'est-à-dire le chemin de $(\tau, \lambda)$ à $(\tau, 1)$ dans l'espace de Teichmüller étendu.

Une autre possibilité en genre un est d'utiliser une généralisation de la transformée de Bargmann de la question 2, liée au couplage générique entre des polarisations linéaires transverses sur un espace vectoriel. Plus précisément, si $\lambda \in U(1)$ et $\tau \in \mathcal{T}_\Sigma$ ont été choisis, alors on peut faire la chose suivante.
Premièrement, l'inverse de la transformée de Bargmann donne une application $\mathcal{H}_{\mathbb{C},\tau} \longrightarrow \mathcal{H}_{\mathbb{R},\tau}$. Deuxièmement,$\lambda$ agit par tiré-en-arrière en envoyant $\mathcal{H}_{\mathbb{R},\tau}$ dans l'espace $\mathcal{H}_{\mathbb{R},\lambda^*\tau}$ de sections qui sont horizontales pour la polarisation réelle tournée par $\lambda$. Enfin, le couplage non-dégénéré entre $\mathcal{H}_{\mathbb{R},\lambda^*\tau}$ et $\mathcal{H}_{\mathbb{C},\tau}$ ferme le triangle, et on peut définir cette composition comme étant l'action du cercle sur $\mathcal{H}_{\mathbb{C},\tau}$.

6. *Peut-on calculer une formule pour cette action sur des sections holomorphes du fibré de préquantification ?*

Dans le Chap. 4 nous considérons le cas d'une surface de genre un et le groupe $G = \mathrm{SL}(2,\mathbb{C})$, et nous répondons affirmativement à toutes ces questions.
En particulier, nous montrons qu'il existe un fibré quantique (étendu) non trivial bien défini de sections holomorphes au-dessus de $\mathcal{T}_\Sigma \times U(1)$. Nous n'avons cependant pas de preuve du fait que ce fibré supporte une connexion projectivement plate.

# Résumé des résultats principaux

Dans le Chap. 3 on répond aux questions 1, 2 et 3, dans la direction de la quantification par déformation. Plus précisément, soit $\mathcal{G}$ le graphe $k$-parti complet sur les nœuds $I = \coprod_{j\in J} I^j$, décoré par un *reading* $a\colon I \longrightarrow \mathbb{P}(\mathbb{C}^2)$ et des espaces vectoriels de dimension finie $\{V_i\}_{i\in I}$. Soit $\{H_i\}_{i\in I}$ le système d'isomonodromie simplement lacé associé à ces données.

On prouve alors les proposition suivantes.

**Theorem.** *Il existe un système Hamiltonian quantique naturel dépendant du temps $\{\widehat{H}_i\}_{i\in I}$ qui quantifie par déformation $\{H_i\}_i$.*

La définition des systèmes d'isomonodromie simplement lacés est rappelée en § 3.1. Les Hamiltoniennes quantiques $\widehat{H}_i$ sont définies en § 3.4 en utilisant l'idée de potentiels quantiques introduite en § 3.3. Ces systèmes constituent les connexions quantiques simplement lacés associées au graphe décoré $\mathcal{G}$.

**Theorem.** *Toutes les connexions quantiques simplement lacées sont fortement plates.*

La preuve de ce résultat est le contenu de § 3.5 et § 3.6.

Après avoir construit cette nouvelle famille de connexions plates, on la compare avec d'autres connexions quantiques déjà apparues dans la littérature.

**Theorem.** *Les systèmes d'isomonodromie simplement lacés se réduisent au système de Schlesinger, dans le cas particulier du reading dégénéré d'un graphe étoilé sans temps irréguliers. La connexion quantique simplement lacée associée se réduit au système de KZ, et donc elle quantifie Schlesinger.*

La première affirmation est démontrée en § 3.7.2 et § 3.7.3 ; la seconde en § 3.7.4 et § 3.7.5.

**Theorem.** *Les systèmes d'isomonodromie simplement lacés se réduisent au système de Schlesinger dual, dans le cas particulier du reading dégénéré d'un graphe étoilé sans temps réguliers.*

Cette affirmation autour de systèmes classiques est démontrée en § 3.8.4.

**Theorem.** *Une correction naturelle de la connexion quantique simplement lacée associée se réduit au système de DMT, et sa différence avec la connexion quantique simplement lacée est nulle à la limite semi-classique.*

**Corollary.** *Une réduction de la connexion quantique simplement lacée quantifie le système de Schlesinger dual.*

Ces affirmations autour de systèmes quantiques sont démontrées en § 3.8.5.

**Theorem.** *Une restriction de la connexion de FMTV peut être corrigée pour obtenir une quantification du système de JMMS, et la différence est nulle à la limite semi-classique. Donc le système de FMTV quantifie le système de JMMS.*

Ceci est démontré en § 3.9.3.

**Theorem.** *Une correction naturelle de la connexion quantique simplement lacée se réduit au système de FMTV, dans la cas particulier du reading dégénéré d'un graphe biparti complet. Sa différence avec la connexion quantique simplement lacée est nulle à la limite semi-classique.*

**Corollary.** *Une réduction de la connexion quantique simplement lacée quantifie le système de JMMS.*

Les deux dernières affirmations sont démontrées en § 3.9.4.

Finalement, on compare la connexion quantique simplement lacée avec la quantification de [NS11] en § 3.10.

Dans le Chap. 4 on répond aux questions 1, 2, 3, 4, 5, dans la direction de la quantification géométrique. Plus précisément, soit $G = \mathrm{SL}(2, \mathbb{C})$, $\Sigma$ une surface réelle fermée orientable de genre un, $t \in \mathbb{C}^*$ un niveau quantique. Considérons l'espace de modules de Hitchin $\mathfrak{M} = \mathfrak{M}(\Sigma, G)$ associé à ces données.

On démontre alors les propositions suivantes.

**Theorem.** *Il existe une connexion de Hitchin plate pour la quantification géométrique de $\mathfrak{M}$ au niveau $t$, par rapport à une famille distinguée de structures complexes dans la sphère Kähler de $\mathfrak{M}$.*

L'espace de module est étudié de manière détaillée en § 4.2 ; en particulier, la famille des polarisations Kähleriennes parametrisée par l'espace de Teichmüller de $\Sigma$ est introduite en § 4.2.5. La formule explicite d'une connexion qui préserve l'holomorphicité est donnée en § 4.3.2. La preuve de platitude est le contenu de § 4.3.3.

Après avoir construit cette connexion, on considère l'action de Hitchin.

**Theorem.** *Des données de préquantification qui sont compatibles avec l'action de Hitchin sur les champs de Higgs existent pour $\mathfrak{M}$.*

Ceci est discuté en § 4.4.4. D'autres aspects de l'action de $U(1)$ sont explorés en § 4.4.3, où l'on décrit aussi la façon dans laquelle les polarisations Kähleriennes sont tournées.

**Theorem.** *Il existe une connexion dans le fibré de préquantification au-dessus de l'espace de Teichmüller étendu $\mathcal{T}_\Sigma \times U(1)$ de $\Sigma$ qui préserve les sections holomorphes, au niveau $t$.*

Ceci est décrit en § 4.5. Une formule explicite pour une connexion qui préserve l'holomorphicité est donnée en Thm. 4.3.

Enfin, on considère la transformée de Bargmann.

**Theorem.** *Soit $\tau$ un point dans l'espace de Teichmüller de $\Sigma$, et soient $P_{\mathbb{R}}, P$ la polarisation réelle et Kählerienne associée sur $\mathfrak{M}$, respectivement. Il existe un isomorphisme canonique défini entre les espaces de Hilbert des sections $P_{\mathbb{R}}$-polarisées et $P$-polarisées, en utilisant la transformée de Bargmann.*

Ceci est discuté en § 4.6. La formule explicite pour la transformée est donnée en Thm. 4.4. De plus, en § 4.6.4 on calcule une formule explicite pour la conjugaison de la connexion de Hitchin–Witten du côté des polarisations Kähleriennes, en utilisant Thm. 4.6 : ce théorème constitue une généralisation des relations de commutations usuelles entre la représentation de Fock et de Schrödinger de l'algèbre de Weyl. La connexion de Hitchin–Witten est introduite en § 4.6.4.

**Theorem.** *L'action de Hitchin est transformée par la transformée de Bargmann en une transformée intégrale qui agit sur l'espace de Segal–Bargmann de dimension 2. Le noyau de cette transformée généralise le reproducing kernel associé aux états cohérents.*

Ceci est discuté en § 4.6.5 et une formule pour l'action est donnée en Thm. 4.7. Enfin, dans la même section on explique comment conjuguer dérivées et multiplications par des fonctions par rapport à l'action du cercle qu'on vient de calculer, ce qui correspond à la première étape pour conjuguer les deux connexions (plates) construites précédemment.

# Contents

# Chapter 1

# Introduction

The main results of this thesis are the following:

1. We construct a new family of flat quantum connections generalising the Knizhnik–Zamolodchikov (KZ) connection [KZ84], the De Concini–Millson–Toledano-Laredo (DMT) connection [MT05] and the Felder–Markov–Tarasov–Varchenko (FMTV) connection [Fel+00]. These systems are obtained via deformation quantisation of Hamiltonian systems controlling the isomonodromic deformations of meromorphic connections on the Riemann sphere.

2. We construct the geometric quantisation of the Hitchin moduli space for the group $\mathrm{SL}(2,\mathbb{C})$, on a compact Riemann surface of genus one — with respect to Kähler polarisations — as well as an explicit mapping class group invariant Hitchin connection. We also consider the circle action on the Hitchin moduli space, and construct a connection that is simultaneously compatible with the action and preserves holomorphicity. Finally, we explain how to relate our constructions with the geometric quantisation of Witten [Wit91], using the Bargmann transform [Seg63; Bar61], and we also turn Hitchin's action on the moduli space into an action on the space of holomorphic sections using the pairing with coherent states.

The first project is discussed in Chap. 3, expanding the preprint [Rem17]; the second one in Chap. 4. The necessary mathematical background is recalled in Chap. 2, whereas § A provides some motivational material rooted in quantum mechanics, and explains the origin of much of the terminology we use.

Before introducing the two projects in further details we wish to underline one important driving motivation.

## Extended Teichmüller spaces

One essential feature of the quantisation of symplectic moduli spaces of flat smooth unitary connections on a closed real surface $\Sigma$ of genus $g$ is that every complex structure on

$\Sigma$ yields a Kähler polarisation on the moduli space. Since the polarisation only depends on the isomorphism class of the Riemann surface, the moduli space $\mathcal{M}_g$ of smooth projective curves of genus $g$ — or rather the moduli stack — parametrises polarisations on the moduli space of connections, and thus becomes the base of a bundle fitting together the quantisations corresponding to different choices. Equivalently, we can work over the Teichmüller space $\mathcal{T}_g \longrightarrow \mathcal{M}_g$ in a mapping class group equivariant way — with the advantage that $\mathcal{T}_g$ is a manifold.
In the mathematical formalism of geometric quantisation, the fibres of the quantum bundle are Hilbert spaces, whose projectivisation are the phase-spaces of the quantum-mechanical theory; in the deformation quantisation approach, the fibres are noncommutative algebras that quantise the algebra of functions — i.e. classical observables — on a complex symplectic moduli space of connections.

One important point of the thesis is that one may construct quantum bundles over bases that extend the Riemann moduli space $\mathcal{M}_g$, at least in the deformation approach. More precisely, the extension we propose in Chap. 3 is motivated by the observation that the moduli spaces of *meromorphic* connections provide far-reaching generalisations of those for nonsingular ones. Whereas this is well-established for the moduli spaces themselves — which we view as symplectic phase-spaces for classical-mechanical systems —, their quantisation is pretty much an open problem, and quite a broad one in this generality.

In the case of connections having simple poles, one can define (hyper-)Kähler moduli spaces of such connections over *pointed* Riemann surfaces, with the marked points parametrising the positions of the poles. The moduli space of pointed Riemann surfaces then becomes the base of a family of symplectic manifolds, which one may quantise fibrewise.
The fact that this bundle is projectively flat, which is the mathematical way of phrasing the independence of the quantisation of the polarisation, is understood as follows in genus zero: from the viewpoint of geometric quantisation one has the Hitchin connection, on the bundle of holomorphic sections of the Chern–Simons prequantum bundle, and from the viewpoint of conformal field theory one has the KZ connection [KZ84], on the Verlinde bundle of conformal blocks for the Wess–Zumino–Witten model. The interesting point for us is that the KZ connection can be derived via a simple deformation quantisation of the isomonodromy equations for the logarithmic connections on the sphere (the Schlesinger system), whereas the construction of the Hitchin connection is arguably more complex.
In Chap. 3 we achieve the next natural step in this program, allowing for irregular singularities in the connections that go beyond previously studied examples.

To define symplectic moduli spaces of connections with irregular singularities one has to fix local normal forms for the principal parts at the poles, involving the irregular type and the residue. The crucial remark is that the extra moduli of the irregular types behave in the quantisation just as the moduli of the underlying pointed surface, and thus the good setup becomes that of pointed Riemann surfaces with choices of irregular types at the marked points, i.e. "wild Riemann surfaces".
This means that the moduli space of wild Riemann surfaces is naturally the base of a symplectic bundle of moduli spaces of meromorphic connections, generalising the moduli of

pointed Riemann surfaces for regular singularities, and even further the classical Riemann moduli space $\mathcal{M}_g$ for no singularities at all. We consider again the case of genus zero, and we show that one can indeed construct quantum bundles over spaces of restricted deformations of wild Riemann spheres, via fibrewise deformation quantisation of the associated moduli spaces of meromorphic connection. Moreover, we quantise the associated isomonodromy connection to construct (new) quantum connections in the quantum bundle, generalising the relation between Schlesinger and KZ.
The construction of flat connections out of deformation quantisation of irregular isomonodromy problems was first considered in [Boa02], where the Harnad-dual version of the Schlesinger system — in the sense of [Har94] — was quantised to yield the DMT connection of [MT05]. The simply-laced quantum connections of Chap. 3 subsume this construction, as well as that of the FMTV connection of [Fel+00] as a quantisation of the combined isomonodromy system of [Jim+80].

In Chap. 4, instead, we extend the Teichmüller space of surfaces of genus one to contain a further real direction, taking into account Hitchin's circle action on Higgs fields. This results in a larger space parametrising Kähler polarisations on Hitchin's moduli space, and the construction of a connection preserving the subspaces of holomorphic sections proves that there is indeed a well defined quantum bundle over the extended base.
The question whether the extended quantum bundle carries a projectively flat connection remains open, but an interesting new feature has already appeared: one no longer quantises a fixed symplectic manifold with respect to varying polarisations, but rather has to allow for the symplectic structure to vary as well — along the circle action in our case. This extends the usual setup of geometric quantisation, and forces one to look for new Ansatz for the construction of Hitchin connections.

We now introduce the two quantisation projects in more details, recalling the constructions that they aim to generalise.

## Deformation quantisation of isomonodromy connections

Let $m, n$ be positive integers, set

$$\mathfrak{g} \coloneqq \mathfrak{gl}(n, \mathbb{C}), \qquad \mathbf{B} \coloneqq \mathbb{C}^m \setminus \{\text{diags}\} = \Big\{(t_1, \dots, t_m) \in \mathbb{C}^m \mid t_i \neq t_j \text{ if } i \neq j\Big\},$$

and fix an invariant scalar product on $\mathfrak{g}$ so to identify $\mathfrak{g} \cong \mathfrak{g}^*$. Consider the trivial rank $n$ holomorphic vector bundle over the sphere, equipped with the connection

$$\nabla = d - \sum_i \frac{R_i}{z - t_i} dz, \qquad \text{where} \qquad R_i \in \mathfrak{g}, \quad (t_1, \dots, t_m) \in \mathbf{B}, \tag{1.1}$$

where $z$ is a coordinate that identifies $\mathbb{C}P^1 \cong \mathbb{C} \cup \{\infty\}$. The differential equations for a local section $\psi$ to be covariantly constant then read

$$\frac{d\psi}{dz} = \left(\sum_i \frac{R_i}{z - t_i}\right) \psi. \tag{1.2}$$

Solutions to (1.2) are multi-valued, and the way in which they transform when analytically continued along a loop that encircles a (simple) pole is dictated by the monodromy of the system. The monodromy expresses the action of the fundamental group of the punctured sphere — with the poles removed — on the space of fundamental solutions of (1.2). If the position of the poles and the residues are varied, the monodromy will change in general. An important class of deformations is defined precisely by requiring that this does not happen: these are the isomonodromic deformations of Fuchsian systems (on the sphere).
Schlesinger [Sch05] derived a system of differential equations for the residues $R_i$ — as functions of the positions $t_i$ — which codes such isomonodromic deformations. Solving these isomonodromy equations amounts to compute the flow of the time-dependent Hamiltonian system $H_i^{\mathrm{Sch}}\colon \mathfrak{g}^m \times \mathbf{B} \longrightarrow \mathbb{C}$ defined by:

$$H_i^{\mathrm{Sch}}(R_1, \dots, R_m, t_1, \dots, t_m) = \sum_{i \neq j} \frac{\mathrm{Tr}(R_i R_j)}{t_i - t_j}, \qquad \text{for} \qquad 1 \leq i \leq m. \tag{1.3}$$

The isomonodromy equations then read $\{H_i^{\mathrm{Sch}}, R_j\} = \frac{\partial R_j}{\partial t_i}$, where one uses the linear Poisson bracket of $\mathfrak{g}^m \cong (\mathfrak{g}^*)^m$. Geometrically, they are the same as an integrable Ehresmann connection on the trivial Poisson fibration $\mathfrak{g}^m \times \mathbf{B} \longrightarrow \mathbf{B}$. To quantise this isomonodromy connection means to quantise the Hamiltonians $H_i^{\mathrm{Sch}}$, and there is a standard algebraic machinery for that. The fibrewise restriction of the $H_i^{\mathrm{Sch}}$ defines polynomial functions on $\mathfrak{g}^m$, i.e. elements of the symmetric algebra $\mathrm{Sym}(\mathfrak{g}^*)^{\otimes m} \cong \mathrm{Sym}(\mathfrak{g})^{\otimes m}$, and the Poincaré–Birkhoff–Witt theorem results in a map $\mathcal{Q}_{\mathrm{PBW}}\colon \mathrm{Sym}(\mathfrak{g})^{\otimes m} \longrightarrow U(\mathfrak{g})^{\otimes m}$. Since the universal enveloping algebra is a quantisation of the symmetric one, the fibrewise quantisation

$$\mathcal{Q}_{\mathrm{PBW}}(H_i^{\mathrm{Sch}})\colon U(\mathfrak{g})^{\otimes m} \times \mathbf{B} \longrightarrow \mathbf{B}$$

defines a time-dependent quantum Hamiltonian system which quantises (1.3). Moreover, [Res92; Har96] showed that the dynamical equations associated to it are the (universal) KZ equations, i.e.

$$\frac{\partial \psi}{\partial t_i} = \left( \sum_{j \neq i} \frac{\Omega_{ij}}{t_i - t_j} \right) \psi. \tag{1.4}$$

These are equations for a local section $\psi$ of the trivial vector bundle $U(\mathfrak{g})^{\otimes m} \times \mathbf{B} \longrightarrow \mathbf{B}$ (see [EFK98]). From the mathematical viewpoint, the element $\Omega_{ij}$ denotes the action of a tensor $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ by left multiplication on the $i$th and $j$th slot of the tensor product $U(\mathfrak{g})^{\otimes m}$, and the proof that KZ quantises the Schlesinger system then boils down to the observation that the function $\mathrm{Tr}(R_i R_j)$ quantises to the operator $\Omega_{ij}$.
Just as it was for Schlesinger equations, the KZ equations (1.4) amount to a flat connection — in the trivial vector bundle $U(\mathfrak{g})^{\otimes m} \times \mathbf{B} \longrightarrow \mathbf{B}$ —, and the whole discussion is then resumed by saying that the KZ connection is a quantisation of the Schlesinger connection.

This is the program that one wants to generalise, allowing for irregular singularity in the meromorphic connections. The first step in this direction was the derivation of [Boa02] of the DMT connection from an irregular isomonodromy problem. One considers the isomonodromic deformations of meromorphic connections on the Riemann sphere

having one pole of order two at infinity and a simple pole at zero:

$$\nabla = d - \left(T + \frac{R}{z}\right) dz, \qquad \text{where} \qquad T, R \in \mathfrak{g}, \tag{1.5}$$

and where $T = \mathrm{diag}(t_1, \ldots, t_n)$ is diagonal with simple spectrum. Now one deforms $T$ inside the regular part $\mathfrak{h}_{\mathrm{reg}}$ of the Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ of diagonal matrices, and accordingly writes isomonodromy equations for the residue $R$. The isomonodromy Hamiltonians read

$$H_i^{\mathrm{dSch}} = \sum_{i \neq j} \frac{R_{ij} R_{ji}}{t_i - t_j}. \tag{1.6}$$

They are defined on the trivial fibration $\mathfrak{g} \times \mathfrak{h}_{\mathrm{reg}} \longrightarrow \mathfrak{h}_{\mathrm{reg}}$, and their PBW-quantisation yields the time-dependent quantum Hamiltonians defining the DMT connection:

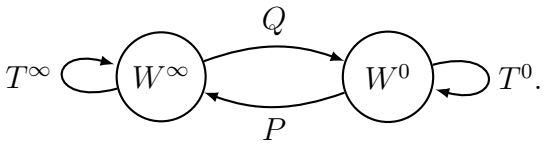
$$\widehat{H}_i^{\mathrm{DMT}} = \frac{1}{2} \sum_{j \neq i} \frac{\widehat{e}_{ij} \cdot \widehat{e}_{ji} + \widehat{e}_{ji} \cdot \widehat{e}_{ij}}{t_i - t_j} : \mathfrak{h}_{\mathrm{reg}} \longrightarrow U(\mathfrak{g}),$$

where $e_{ij}$ is the canonical basis of $\mathfrak{g}$.

The classical isomonodromy system $H_i^{\mathrm{dSch}}$ is a dual version of Schlesinger, in the following sense. The space of times for (1.3) is the space of configurations of $m$ poles on the sphere where no poles coalesce, whereas in (1.6) one may vary the spectrum of the diagonal matrix $T$ — coding the irregular type $T dz$ at infinity — so that no eigenvalues $t_i$ coalesce. The latter correspond to the irregular isomonodromy times considered by Jimbo–Miwa–Môri–Sato (JMMS) [Jim+80]. Harnad [Har94] has shown that the two collections of times in the JMMS system may be swapped, entering into a larger duality of rational differential operators:

$$\frac{d}{dz} - \left(T^0 + Q(z - T^\infty)^{-1} P\right) \longmapsto \frac{d}{dz} + \left(T^\infty + P(z - T^0)^{-1} Q\right), \tag{1.7}$$

where $W^\infty$, $W^0$ are finite-dimensional complex vector spaces, and $T^0, T^\infty, Q, P$ are linear functions mapping as in this quiver:




Thus the left-hand side of (1.7) (resp. the right-hand side) acts on local sections of the trivial holomorphic vector bundle with fibre $W^0$ over the sphere (resp. with fibre $W^\infty$). This duality induces in particular a correspondence between the connections (1.1) and (1.5). This underlies the quantum/Howe duality of [Bau99] used in [Tol02] to relate the KZ and DMT connections for $\mathfrak{g} = \mathfrak{gl}_n$.

Thus, in brief, there is a correspondence between isomonodromy problems and flat quantum connections which is attached to representations of the graph  , i.e. the complete graph on two nodes. Each node carries a set of times associated to variations

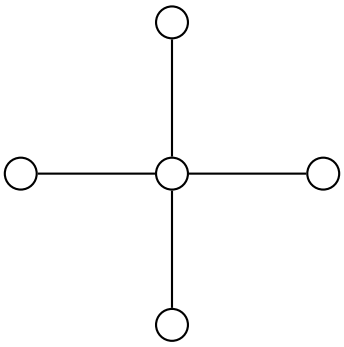
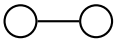


Figure 1.1: The star is the complete bipartite graph associated to KZ and DMT.

of the spectra of the semisimple endomorphisms $T^\infty$ and $T^0$ that do not change the eigenspace decomposition of $W^\infty$ and $W^0$. In turn, these are precisely admissible deformations of a wild Riemann sphere equipped with an irregular type that carries a pole of order two at infinity, plus a finite number of simple poles. To encode this admissible deformations in the graph, one may also splay the nodes of  according to the decomposition in direct sum of $W^\infty$ and $W^0$, getting as result a complete *bipartite* graph (see § 3.1).

The natural question whether this graph-theoretic viewpoint could be generalised to arbitrary complete $k$-partite graphs was answered positively in [Boa12b], extending the Hamiltonian theory of isomonodromic deformations of meromorphic connections over the sphere to include an extension of a particular case of [JMU81]. The new integrable Hamiltonian systems attached to simply-laced, complete $k$-partite graphs are called "simply-laced isomonodromy systems". As expected, they involve $k$ collections of times that still correspond to deformations of wild spheres, generalising the two collections of times in the JMMS system. Moreover, at this level there is now an apparent group of $\mathrm{SL}(2,\mathbb{C})$-symmetries, which includes the Fourier–Laplace transform defining (1.7) for a complete bipartite graph, and which leads to the Weyl group of the root system of the graph.

From this discussion, one gets the impression that there should be a quantum theory underlying this classical-mechanical extension, i.e. a family of flat quantum connections attached to complete $k$-partite graphs which deformation-quantise the simple-laced isomonodromy systems, and which contains KZ, DMT and FMTV as particular cases (Fig. 1.1 and 1.2).
The first new example is attached to a triangle, because in the language of [Boa12b] this graph corresponds to an isomonodromy systems for connections having a pole of order three at infinity, which cannot be carried to connections with lower Poincaré rank using the aforementioned $\mathrm{SL}(2,\mathbb{C})$-symmetries (Fig. 1.3).

The first question we ask, then, is the following:

1. *Can one construct a deformation quantisation of the simply-laced isomonodromy systems?*

If this were possible, one would get a new family of "simply-laced quantum connections" (SLQC) out of the deformation quantisation of isomonodromy connections. Notice

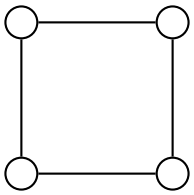


Figure 1.2: A generic complete bipartite graph yields the FMTV connection.

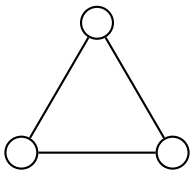


Figure 1.3: What is the flat quantum connection of the triangle?

however that an arbitrary quantisation procedure need not preserve the crucial property of integrability. One should thus reinforce the previous question.

2. *Can one construct an integrable quantum Hamiltonian system that quantises the simply-laced isomonodromy systems?*

Such a construction would be more interesting if it were a generalisation of KZ [KZ84], DMT [MT05] and FMTV [Fel+00], which are all obtained from the quantisation of particular cases of the simply-laced isomonodromy systems. More precisely, both Schlesinger and its dual are attached to a star-shaped graph, and JMMS to a generic complete bipartite one.

3. *Can one construct the simply-laced quantum connections so that they reduce to KZ, DMT and FMTV, in the relevant setup?*

In brief, answering to the previous three questions means filling-in the last column of the following table:

| Isomonodromy system | Schlesinger | Dual Schlesinger | JMMS | SLIMS |
|---|---|---|---|---|
| Space of times | $\mathbb{C}^m \setminus \{\text{diags}\}$ | $\mathbb{C}^n \setminus \{\text{diags}\}$ | $\mathbb{C}^m \setminus \{\text{diags}\} \times \mathbb{C}^n \setminus \{\text{diags}\}$ | $\prod_1^k \mathbb{C}^{d_i} \setminus \{\text{diags}\}$ |
| Quantum connection | KZ | DMT | FMTV | SLQC |

This is what we do in Chap. 3.

## Kähler quantisation of moduli spaces of flat connections

Let $\Sigma$ be a closed, orientable, real surface of genus $g \geq 2$, and $K := \mathrm{SU}(2)$. One considers the moduli space $\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}(\Sigma, K)$ of $S$-equivalence classes of semistable, flat $K$-connections on the trivial principal $K$-bundle on $\Sigma$. Irreducible connections (on which $S$-equivalence reduces to the ordinary relation of isomorphism) define an open smooth subset $\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}} \subseteq \mathcal{M}_{\mathrm{fl}}$. Atiyah & Bott [AB83] and Goldman [Gol84] proved that there is a natural integral symplectic structure $\omega$ on $\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}$, and thus one can define the prequantisation of $(\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}, \omega)$. The definition of prequantum data for $(\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}, \omega)$ is suggested by Chern–Simons field theory [CS74] for the manifold $X := \Sigma \times [0, 1]$. Chern–Simons is a 3-dimensional, topological field theory (see [Fre95]). In this sense, the moduli space $\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}$ is the space of boundary conditions for the theory on the 3-manifold $X$, and one motivation behind the quantisation of $\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}$ is to define a quantisation of (compact) Chern–Simons theory.

The prequantisation of the smooth symplectic locus $(\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}, \omega)$ is then upgraded to geometric quantisation with respect to Kähler polarisations. Namely, for each complex structure $\tau$ on the surface $\Sigma$ one finds a Kähler structure $I_\tau$ on $(\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}, \omega)$, and then considers the quantum Hilbert space of square-summable $I_\tau$-holomorphic sections of the prequantum bundle $L$ — at level $k \geq 1$ —, equipping $L$ with the $\overline{\partial}$-operator defined by the $(0, 1)$-part of the prequantum connection, and using the natural $L^2$-norm on smooth sections of $L$ which is induced by the prequantum Hermitian metric and the Liouville measure:

$$\mathcal{H}_{\mathbb{C},\tau}^{(k)} := H^0\left(\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}, L^{\otimes k}\right) \cap L^2(\mathcal{M}_{\mathrm{fl}}, \mu). \tag{1.8}$$

The complex structure $I_\tau$ only depends on $\tau$ up to isotopy, and thus the Teichmüller space $\mathcal{T}_\Sigma$ to $\Sigma$ parametrises a family of vector spaces, whose projectivisations are models for the quantisation of $(\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}, \omega)$. This is however not the full story, since in physics quantisation is assumed to be independent of the choice of polarisations.
From the mathematical viewpoint, one would like to have canonical identifications among the projectivisations of the Hilbert spaces (1.8) for different choices of $\tau \in \mathcal{T}_\Sigma$. The solution proposed by Axelrod & Della Pietra & Witten [ADW91] and Hitchin [Hit90] is to construct a projectively flat connection in the vector bundle $\mathcal{H}^{(k)} \longrightarrow \mathcal{T}_\Sigma$, whose fibre over $\tau$ is the space (1.8) (Witten already suggests this idea in [Wit89], and traces its origins in conformal field theory). This connection has subsequently been called the Hitchin connection in the mathematical community; Hitchin established projective flatness by exploiting the Poisson-commuting Hamiltonians of the integrable system he introduced in [Hit87b]. A new construction of the Hitchin connection via global differential operators acting on smooth sections of the prequantum bundle was later given by Andersen [And12], in a more general context (see also [And06] for an application of this viewpoint to proving the asymptotic faithfulness of the quantum representation of the mapping class group of the surface, as well as the more recent [AR16]).
This achieved the Kähler quantisation of the moduli space of flat unitary connections on a Riemann surface, some 25 years ago.

The next natural step in this program is the quantisation of complex Chern–Simons theory, already considered by Witten in [Wit91]. From the viewpoint of moduli spaces,

one complexifies the group $K$ to obtain a complex reductive group $G$ — the archetype being $G = \mathrm{SL}(2,\mathbb{C})$, the complexification of $\mathrm{SU}(2)$ — and considers the moduli space $\mathcal{M}_{\mathrm{dR}}(\Sigma, G)$ of flat $G$-connections on $\Sigma$.
The classical situation is much richer than in the compact case, since now the complex structure $J$ of the group $G$ combines with any complex structure $I_\tau$ arising from a a Riemann surface structure on $\Sigma$ to yield a hyperKähler triple $(I_\tau, J, K_\tau = I_\tau J)$. The construction of Atiyah–Bott now yields a $J$-holomorphic symplectic structure $\omega_{\mathbb{C}}$ on the moduli space. This was shown by Hitchin [Hit87a], who introduced a hyperKähler moduli space $\mathfrak{M}$ of solutions to self-duality equations on $\Sigma$ (now Hitchin equations), which was related to $\mathcal{M}_{\mathrm{dR}}$ by Donaldson in [Don87] — companion to [Hit87a] — and more generally by Corlette in [Cor88].
More precisely, the Hitchin moduli space is an hyperKähler manifold that underlies three nonisomorphic complex algebraic varieties, one of which is $\mathcal{M}_{\mathrm{dR}}$ — in the complex structure $J$—, and another one being the moduli space $\mathcal{M}_{\mathrm{Dol}}(\Sigma, G)$ of $G$-Higgs bundles on $\Sigma$ — in the complex structure $I_\tau$. The third algebraic description is that of the Betti moduli space $\mathcal{M}_{\mathrm{B}}(\Sigma, G)$ of $G$-representations of the fundamental group of $\Sigma$, up to conjugation.

The smooth correspondence between $G$-Higgs bundles and flat $G$-connections/local $G$-systems was established by the aforementioned works of Hitchin, Donaldson, Corlette, and finally Simpson [Sim92]. Simpson also suggested in [Sim94; Sim97] that one should think of this correspondence as a Hodge correspondence for the first non-Abelian cohomology of the compact Kähler manifold $\Sigma$ — with coefficients in $G$ —, in analogy to the usual cohomological theories of de Rham, Dolbeault and Betti — with coefficients in $\mathbb{C}^* \cong \mathrm{GL}_1(\mathbb{C})$.

A natural project now arises: construct the geometric quantisation of the moduli space $\mathfrak{M}$ with respect to distinguished Kähler polarisations extracted from the sphere of Kähler structures. More precisely, one replaces the quantum level of the theory with a complex parameter $t = k + i\sigma \in \mathbb{C}^*$, and use it to define a real symplectic structure on $\mathfrak{M}$ out of the complex Atiyah–Bott form $\omega_{\mathbb{C}}$:

$$\omega_t := \frac{t\omega_{\mathbb{C}} + \bar{t}\overline{\omega_{\mathbb{C}}}}{2} = \Re(t\omega_{\mathbb{C}}).$$

The construction of prequantum data $(L^{(t)}, \nabla, h)$ at level $t$ is adapted from the compact case. For any point $\tau$ in Teichmüller space there is now a distinguished linear combination $I_\tau^{(t)}$ of the hyperKähler triple that yields a complex structure, and which depends on the quantum level up to dilations. Geometrically, moving the unitary complex number $t/|t| \in U(1)$ parametrises the circle of complex structures $I_\tau^{(t)} = k' I_\tau - \sigma' K_\tau$ in the Kähler sphere, where $k' + i\sigma' = \frac{k+i\sigma}{\sqrt{k^2+\sigma^2}}$ is the normalisation of $t$. This circle is the equator of the sphere, if one takes $\pm J$ to be the poles.
One thus has a family of Kähler polarisations parametrised by Teichmüller space, which in turn defines the quantum bundle $\mathcal{H}_{\mathbb{C}}^{(t)}$ of holomorphic sections of $L^{(t)}$, as it was for a compact gauge group.

The question we ask is then:

1. *Can one construct a projectively flat connection on $\mathcal{H}_{\mathbb{C}}^{(t)}$, for all choice of $t \in \mathbb{C}^*$?*

This would achieve the Kähler quantisation of the moduli space of flat connections for the complex gauge group $G$. In this generality, this question remains open to these days, reflecting the fact that the quantisation of complex Chern–Simons theory is still a current research topic, at least from a mathematician's viewpoint.

What Witten did in [Wit91] was different. He considered real polarisations on the moduli space, still depending on a complex structure on $\Sigma$, and accordingly defined a projectively flat connection in the resulting quantum bundle $\mathcal{H}^{(t)}_{\mathbb{R}}$ of real-polarised sections. The mathematical details have been laid out by Andersen & Gammelgaard [AG14], who called this connection the Hitchin–Witten connection.
Since it is assumed that Witten's construction is the most sensible one from the standpoint of quantum field theory, one should relate the Hitchin connection for the Kähler quantisation of $\mathcal{M}^{\mathrm{st}}_{\mathrm{dR}}$ and the Hitchin–Witten connection. A natural way to do this is looking for an isomorphism $\mathcal{B}^{(t)}\colon \mathcal{H}^{(t)}_{\mathbb{R}} \longrightarrow \mathcal{H}^{(t)}_{\mathbb{C}}$ of the quantum bundles, and conjugate the Hitchin–Witten connection through it. The natural candidate is — a slight variation of — the Bargmann transform [Bar61].

2. *Can one construct an identification between the spaces of real-polarised and holomorphic sections for all points in Teichmüller space?*

3. *Can one compute a formula for the conjugation of the Hitchin–Witten connection on the vector bundle $\mathcal{H}^{(t)}_{\mathbb{C}}$?*

Note that one does not expect the Hitchin connection in $\mathcal{H}^{(t)}_{\mathbb{C}}$ to be unique. More precisely, the construction of the Hitchin connection via global differential operators (of order two) relies on an Ansatz which may be relaxed in this new noncompact setting. This is particularly important if one wants to take into account an important feature of the moduli space for the complex theory: the natural dilation action of $\mathbb{C}^*$ on Higgs fields. One would like to use this action to induce a $U(1)$-action on the quantum Hilbert spaces of holomorphic sections, thereby defining a $\mathbb{Z}$-grading which will give a decomposition into isotypical components:

$$\mathcal{H}^{(t)}_{\mathbb{C},\tau} = \bigoplus_{m\in\mathbb{Z}} \mathcal{H}^{(t)}_{\mathbb{C},\tau,m}, \qquad \text{where} \qquad \mathcal{H}^{(t)}_{\mathbb{C},\tau,m} = \left\{ s \in \mathcal{H}^{(t)}_{\mathbb{C},\tau} \,\middle|\, \lambda.s = \lambda^m s \right\}.$$

This is one way around the fact that the spaces of states of complex Chern–Simons theory are infinite-dimensional.

There is a current research trend dedicated to computing formulae for the dimension of the isotipical components, with the caveat that the $U(1)$-action be well defined on the spaces of holomorphic sections. If for example one assumes that $\Im(t) = \sigma = 0$, then $I^{(t)}_\tau = I_\tau$ will be fixed by the circle action, and [AGP16] established a Verlinde-type formula in this context.
If one allows for a nonvanishing imaginary part in the quantum level, instead, then the diffeomorphism $\lambda \in U(1)$ will move $I^{(t)}_\tau$ to $\lambda^* I^{(t)}_\tau$. Geometrically, the pull-back amounts to a rotation by $\lambda$ inside the Kähler sphere; more precisely, since the Dolbeault complex structure $I_\tau$ is fixed by this action, the rotation happens along the equator of the sphere,

if this time one fixes $\pm I_\tau$ to be the poles. Hence the rotation happens around an axis which is orthogonal with respect to that of the rotation induced by varying the quantum level $t$.
Nonetheless, to define an action on the quantum bundle of holomorphic sections one now needs to find a canonical identification between the spaces of $I_\tau^{(t)}$-holomorphic sections of $L^{(t)}$ and the $\lambda^* I_\tau^{(t)}$-holomorphic ones.

The first idea we pursued to solve this problem is an extension of the original setup of Hitchin and Witten. Namely, one may try to show that the the spaces of holomorphic sections for the Kähler structures $\lambda^* I_\tau^{(t)}$ fit into a vector subbundle of the trivial prequantum bundle over an extended base

$$C^\infty(\mathfrak{M}, L^{(t)}) \times \mathcal{T}_\Sigma \times U(1) \longrightarrow \mathcal{T}_\Sigma \times U(1).$$

Second, we try to prove that this bundle carries a projectively flat connection. The manifold $\mathcal{T}_\Sigma \times U(1)$ should be thought as an "extended Teichmüller" space for $\Sigma$, which is one of our motivating viewpoint throughout the thesis.

Anyway, a first technical difficulty arises immediately: the $U(1)$-action is not symplectic, and thus one has to find a proper definition of prequantum data for the family of symplectic manifolds $(\mathfrak{M}, \lambda^*\omega_t)_{\lambda \in U(1)}$ parametrised over the circle.

4. *Can one construct prequantum data on the moduli space which are compatible with the Hitchin action?*

This situation changes the usual rules of geometric quantisation, which prescribe to quantise a *fixed* symplectic manifold with respect to varying polarisations.
If the prequantum data are defined, then the trivial prequantum bundle is defined too. One now has to prove that the spaces of $\lambda^* I_\tau^{(t)}$-holomorphic sections define a subbundle inside of it. One way to show this is to answer the following question.

5. *Is there a connection in the prequantum bundle over the extended Teichmüller space that preserves holomorphicity fibrewise?*

If one had a projectively flat connection with trivial monodromy around the circle, then one might indeed define a circle action on the spaces $\mathcal{H}_{\mathbb{C},\tau}^{(t)}$ of holomorphic sections. One would compose the isomorphism defined by $\lambda$ with the parallel transport along any path that connects $\lambda^* I_\tau^{(t)}$ back to $I_\tau^{(t)}$, i.e. from $(\tau, \lambda)$ back to $(\tau, 1)$ inside the extended Teichmüller space.

Another possibility in genus one is to use a generalisation of the Bargmann transform of question 2, related to the generic pairing among transverse linear polarisations on a vector space. More precisely, if $\lambda \in U(1)$ and $\tau \in \mathcal{T}_\Sigma$ are chosen, then one might do the following.
First, the inverse Bargmann transform provides a map $\mathcal{H}_{\mathbb{C},\tau} \longrightarrow \mathcal{H}_{\mathbb{R},\tau}$, and then $\lambda$ acts by pull-back sending $\mathcal{H}_{\mathbb{R},\tau}$ to the space of sections $\mathcal{H}_{\mathbb{R},\lambda^*\tau}$ which are horizontal along the real polarisation rotated by $\lambda$. Finally, the nondegenerate pairing between $\mathcal{H}_{\mathbb{R},\lambda^*\tau}$ and $\mathcal{H}_{\mathbb{C},\tau}$ closes the triangle, and we define this composition to be the circle action on $\mathcal{H}_{\mathbb{C},\tau}$.

6. *Can one compute a formula for this action on holomorphic sections of the prequantum bundle?*

In Chap. 4 we consider the case of a genus one surface and the group $G = \mathrm{SL}(2, \mathbb{C})$, and we answer positively to the all these questions.
In particular we show that there is a well defined (extended) nontrivial quantum bundle of holomorphic sections over $\mathcal{T}_\Sigma \times U(1)$. We do not however have a proof that this bundle carries a projectively flat connection.

## Summary of main results

In Chap. 3 we answer to questions 1, 2 and 3, in the direction of deformation quantisation. More precisely, let $\mathcal{G}$ be the complete $k$-partite graph on nodes $I = \coprod_{j\in J} I^j$, decorated with a reading $a\colon I \longrightarrow \mathbb{P}(\mathbb{C}^2)$ and finite-dimensional vector spaces $\{V_i\}_{i\in I}$, and define $\{H_i\}_{i\in I}$ to be the simply-laced isomonodromy system associated to these data.

Then we prove the following statements.

**Theorem.** *There exists a natural time-dependent quantum Hamiltonian system $\{\widehat{H}_i\}_{i\in I}$ that deformation-quantises $\{H_i\}_i$.*

The definition of the simply-laced isomonodromy systems is recalled in § 3.1. The quantum Hamiltonians $\widehat{H}_i$ are defined in § 3.4, using the idea of quantum potentials introduced in § 3.3. They constitute the simply-laced quantum connection attached to the decorated graph $\mathcal{G}$.

**Theorem.** *All simply-laced quantum connections are strongly flat.*

The proof of this result is the content of § 3.5 and § 3.6.

After constructing this new family of flat connections, we compare it with other quantum connections which already appeared in the literature.

**Theorem.** *The simply-laced isomonodromy systems reduce to the Schlesinger system, in the special case of the degenerate reading of a star-shaped graph with no irregular times. The associated simply-laced quantum connection reduces to the KZ system, and thus it quantises Schlesinger.*

The former statement is proven in § 3.7.2 and § 3.7.3; the latter in § 3.7.4 and § 3.7.5.

**Theorem.** *The simply-laced isomonodromy systems reduce to the dual Schlesinger system, in the special case of the degenerate reading of a star-shaped graph with no regular times.*

This statement about classical systems is proven in § 3.8.4.

**Theorem.** *A natural correction of the associated simply-laced quantum connection reduces to the DMT system, and its difference with the simply-laced quantum connection vanishes in the semiclassical limit. In particular,*

**Corollary.** *A reduction of the simply-laced quantum connection quantises the dual Schlesinger system.*

These statements about quantum systems is proven in § 3.8.5.

**Theorem.** *A restriction of the FMTV connection can be corrected to obtain the quantisation of the JMMS system, and the difference vanishes semiclassically. Hence the FMTV system quantises the JMMS system.*

This is proven in § 3.9.3.

**Theorem.** *A natural correction of the simply-laced quantum connections reduce to the FMTV system, in the special case of the degenerate reading of a complete bipartite graph. The difference with the SLQC vanishes semiclassically.*

**Corollary.** *A reduction of the simply-laced quantum connection is a quantisation of JMMS.*

The last two claims are proven in § 3.9.4.

Finally, we compare the simply-laced quantum connections with the quantisation of [NS11] in § 3.10.

In Chap. 4 we answer to questions 1, 2, 3, 4, 5, in the direction of geometric quantisation. More precisely, set $G = \mathrm{SL}(2,\mathbb{C})$, let $\Sigma$ be a real closed orientable surface of genus one, $t \in \mathbb{C}^*$ a quantum level, and consider the Hitchin moduli space $\mathfrak{M} = \mathfrak{M}(\Sigma, G)$ attached to this data.

Then we prove the following statements.

**Theorem.** *There exists a flat Hitchin connection for the geometric quantisation of $\mathfrak{M}$ at level $t$, with respect to a distinguished family of complex structures inside the Kähler sphere of $\mathfrak{M}$.*

The moduli space itself is studied in detail in § 4.2; in particular, the family of Kähler polarisations parametrised by the Teichmüller space to $\Sigma$ is introduced in § 4.2.5. The explicit formula of a connection preserving holomorphicity is given in § 4.3.2. The proof of flatness is the content of § 4.3.3.

After constructing this flat connection, we turn to the Hitchin action.

**Theorem.** *There exist prequantum data on $\mathfrak{M}$ which are compatible with the Hitchin action on Higgs fields.*

This is detailed in § 4.4.4. More features of the $U(1)$-action are explored in § 4.4.3, where we also describe how the Kähler polarisations are rotated.

**Theorem.** *There exists a connection in the prequantum bundle over the extended Teichmüller space $\mathcal{T}_\Sigma \times U(1)$ to $\Sigma$ which preserves holomorphic sections, at level $t$.*

This is described in § 4.5. An explicit formula for a connection that preserves holomorphicity is given in Thm. 4.3.

Finally we turn to the Bargmann transform.

**Theorem.** *Let $\tau$ be a point in the Teichmüller space to $\Sigma$, and let $P_{\mathbb{R}}, P$ be the associated real and Kähler polarisation on $\mathfrak{M}$, respectively. There is a canonical isomorphism between the Hilbert spaces of $P_{\mathbb{R}}$-polarised sections and $P$-polarised sections, defined via the Bargmann transform.*

This is discussed in § 4.6. The explicit formula for the transform is given in Thm. 4.4. Moreover, in § 4.6.4 we compute an explicit formula for the conjugate of the Hitchin–Witten connection onto the Kähler-polarised side, using Thm. 4.6: this theorem is a generalisation of the usual commutation relations between the Schrödinger and the Fock representation of the Weyl algebra. The Hitchin–Witten connection is introduced in § 4.6.4.

**Theorem.** *The Hitchin action gets turned by the Bargmann transform into an integral transform, acting on the 2-dimensional Segal–Bargmann space. The kernel of the transform generalises the reproducing kernel associated to coherent states.*

This is discussed in § 4.6.5, with a formula for the action given in Thm. 4.7. Finally, in the same section we explain how to conjugate derivatives and function multiplications with respect to the circle action just computed, which is the first step to conjugate the two (flat) connections previously constructed.

# Chapter 2

# Background

In this chapter we will introduce two mathematical frameworks used to tackle the problem of quantisation introduced in § A.4: *deformation quantisation* in § 2.1 and *geometric quantisation* in § 2.2. The purpose of the exposition is to be able to discuss the original content of the thesis (chapters 3 and 4) in an — almost — self-contained fashion.

The origins of the deformation approach to quantisation can be traced back to the papers [Bay+78a; Bay+78b]. An overview on deformation quantisation can be read in [Wei95]. Our particular viewpoint follows the first chapters of [ES98], as well as the notes [Eti07].
The origins of the geometric approach to quantisation can instead be traced back to [Kir76; Kos70; Sou70], which used the orbit method to classify unitary representations of Lie groups. A standard reference for geometric quantisation is the book [Woo80]; the second chapter of [AN01], written by Kirillov, provides a sharp and shorter introduction.

After discussing quantisation abstractly, in § 2.3 we introduce the classical phase-spaces whose quantisation has been the primary concern of this project: moduli spaces of connections. The relevant citations in this case are scattered along the exposition.

All the quantum-mechanical terminology used in the forthcoming sections is motivated by the material of the appendix A.

## 2.1 Deformation quantisation

Throughout this section on deformation quantisation, all vector spaces and algebras will be taken over $\mathbb{C}$. The layout of this section is the following.

In § 2.1.1 we define formal deformation quantisation.
In § 2.1.2 we discuss gradings and filtrations, and in § 2.1.3 we use this to define filtered quantisation. This construction is seen to be a particular case of formal quantisation, as explained in § 2.1.4. § 2.1.2 also contains the definitions of the *Weyl algebra* (2.5) of a symplectic vector space and the *universal enveloping algebra* (2.6) of a Lie algebra, which will be the quantum algebras used in Chap. 3.

In § 2.1.5 we define two quantisation maps associated with Weyl algebras and universal enveloping algebras: the *PBW quantisation* and the *Weyl quantisation*, respectively (2.8) and (2.9).
Finally, in § 2.1.6 we review classical and quantum Hamiltonian reduction from an algebraic viewpoint, which will be relevant for Chap. 3. This is likely to be less familiar to the reader than the usual Marsden–Weinstein reduction.

### 2.1.1 Formal deformation quantisation

Let $\hbar$ be a complex variable, and set $K \coloneqq \mathbb{C}[\![\hbar]\!]$. This is the ring of formal power series in $\hbar$, with complex coefficients. It has a natural topology, induced by the *$\hbar$-norm*, which is defined by

$$\left|\sum_{k\geq n} c_k \hbar^k\right| \coloneqq C^{-n} \in \mathbb{R}_{\geq 0}, \tag{2.1}$$

if $c_n \neq 0$, and where $C > 1$ is a real constant. Moreover, $K$ is a complete metric space for the metric induced by this norm.

One now works in the category of topological $K$-modules $V$, which are topological vector spaces together with with an algebraic $K$-module structure $K \longrightarrow \mathrm{End}(V)$. A natural way to construct them is the following. If $V_0$ is a vector space, set

$$V_0[\![\hbar]\!] \coloneqq \left\{ \sum_{k\geq 0} v_k \hbar^k \,\middle|\, v_k \in V_0 \text{ for all } k \geq 0 \right\}.$$

This has a structure of a $K$-module, by letting $\hbar \in K$ act via the natural multiplication. Moreover, it carries a natural $\hbar$-norm defined as in Eq. (2.38), which defines a structure of topological $K$-module.

**Remark 2.1.** One has an embedding

$$\varphi\colon V_0 \otimes_{\mathbb{C}} K \hookrightarrow V_0[\![\hbar]\!], \qquad \varphi\colon v \otimes \sum_{k\geq 0} c_k \hbar^k \longmapsto \sum_{k\geq 0} c_k v \cdot \hbar^k,$$

which is an isomorphism as long as $V_0$ is finite-dimensional over $\mathbb{C}$. In general, the image of $\varphi$ corresponds to the space of power series $\sum_{k\geq 0} w_k \hbar^k$ such that the dimension of the complex span of the family $\{w_k\}_{k\geq 0}$ is finite.

**Definition 2.1.** A $K$-module $V$ is topologically free if there exists a vector space $V_0$ together with an isomorphism of $K$-modules $V \cong V_0[\![\hbar]\!]$.

If $V$ is a topologically free $K$-module, then one can uniquely recover the space $V_0$ from $V$ up to isomorphism, via the quotient

$$V_0 \cong V \big/ \hbar V,$$

where, $\hbar V \coloneqq \{\hbar v \mid v \in V\}$.

**Definition 2.2.** A topologically free $K$-algebra $A$ is a topologically free $K$-module with an associative algebra structure.

If $A$ is a topologically free $K$-algebra, the set $\hbar A \subseteq A$ is a two-sided ideal. Since $A$ is topologically free, one may write $A \cong A_0[\![\hbar]\!]$ for some vector space $A_0$, and since $A_0 \cong A\big/\hbar A$ is a quotient with respect to a two-sided ideal then $A_0$ is itself an algebra.

One may now do the opposite, that is consider an algebra $A_0$ and look for $K$-algebras $A$ such that $A_0$ is the ring of coefficients for the power series in $A$.

**Definition 2.3.** A formal, one-parameter deformation of an algebra $A_0$ consists of a topologically free $K$-algebra $A$ and an isomorphism $A\big/\hbar A \cong A_0$.

Deformations always exist: if $A_0$ is an algebra, take $A \coloneqq A_0[\![\hbar]\!]$.

The crucial observation of the theory of deformation quantisation is that every deformation $A$ of $A_0$ induces a Poisson structure $\{\cdot,\cdot\}_0$ on $A_0$, *provided that* $A_0$ *is commutative*. To see this, pick elements $f_0, g_0 \in A_0$, and lift them arbitrarily to elements $f, g \in A$. Now consider the commutator $\Big[f, g\Big] \in A$, whose class in $A_0$ must vanish because of the commutativity of $A_0$. This means that $\Big[f, g\Big] \in \hbar A$, and thus the element $\frac{1}{\hbar}\Big[f, g\Big]$ exists in $A$. Moreover, the class of this element in the quotient $A_0$ does not depend on the lifts $f, g$ chosen: if $f' = f + \hbar x$, $g' = g + \hbar y \in A$ are two other lifts of $f_0, g_0$, then the difference $\frac{1}{\hbar}\Big[f', g'\Big] - \frac{1}{\hbar}\Big[f, g\Big]$ expands as a sum of commutators, and it vanishes in the commutative algebra $A_0$.

It makes thus sense to define $\{f_0, g_0\}_0 \in A_0$ to be the class of the corrected commutator:

$$\{f_0, g_0\}_0 \coloneqq \frac{1}{\hbar}\Big[f, g\Big] + \hbar A. \tag{2.2}$$

This is the sound mathematical way of expressing the relaxation of the Dirac axiom 4 that is discussed at the end of § A.4 (cf. Eq. (A.12), up to a multiplication by the opposite of the imaginary unit $i$).

Start again from a commutative Poisson algebra $(A_0, \{\cdot,\cdot\})$.

**Definition 2.4.** A formal, one-parameter deformation quantisation of $(A_0, \{,\})$ is a formal deformation $A$ of $A_0$ such that the induced Poisson structure $\{\cdot,\cdot\}_0$ on $A_0$ coincides with $\{\cdot,\cdot\}$. In this case, $A_0$ is called the semiclassical limit of $A$, and the canonical projection $\sigma\colon A \longrightarrow A_0$ is called the semiclassical limit too.

Notice that neither existence nor uniqueness of a quantisation $A$ of a given commutative Poisson algebra $A_0$ are trivial problems. Conversely, the semiclassical limit of a topologically free $K$-algebra is always uniquely determined.

Now, pick a Poisson manifold $(M, \{\cdot,\cdot\})$, i.e. a classical phase space (see § A.2).

**Definition 2.5.** A (formal, one parameter) deformation quantisation of $M$ is a deformation quantisation of some distinguished Poisson algebra of functions $A_0 \subseteq C^\infty(M, \mathbb{C})$.

One can now make sense of the quantisation of a time-dependent Hamiltonian system (see Def. A.5 and Def A.7).

**Definition 2.6.** Let $H_i \colon \mathbf{B} \longrightarrow A_0 \subseteq C^\infty(M, \mathbb{C})$ be a time-dependent Hamiltonian system on the phase-space $M$, with space of times $\mathbf{B}$, and assume $A$ to be a deformation quantisation of $A_0$, with semiclassical limit $\sigma \colon A \longrightarrow A_0$. A time-dependent quantum Hamiltonian system $\widehat{H}_i \colon \mathbf{B} \longrightarrow A$ is a quantisation of $\{H_i\}_i$ if the identity $\sigma\Big(\widehat{H}_i\Big) = H_i$ holds for all times $t \in \mathbf{B}$.

In general, one would like to take $A_0$ to be as big as possible, in order to be able to quantise many classical observables. One allows for subalgebras in the definition because of the *no-go* theorems that we discuss in § A.4. A concrete example of this phenomenon is the following.

**Remark 2.2** (Quantisation breaks symmetries)**.**
Assume the phase-space $(M, \omega)$ to be symplectic. The group of symplectomorphisms $\mathrm{Symp}(M, \omega)$ of $M$ will act faithfully on $C^\infty(M, \mathbb{C})$, so that one has an injective morphism

$$\varphi \colon \mathrm{Symp}(X, \omega) \hookrightarrow \mathrm{Aut}\Big(C^\infty(M, \mathbb{C})\Big).$$

**Theorem 2.1** (Groenewold–van Howe)**.**
*There exists no pair $(A, \Phi)$ where $A$ is a quantisation of the classical algebra $C^\infty(M, \mathbb{C})$ and $\Phi \colon \mathrm{Symp}(M, \mathbb{C}) \longrightarrow \mathrm{Aut}\Big(A\Big)$ a group morphism lifting $\varphi$.*

### 2.1.2 Gradings and filtrations

The actual type of deformation quantisation we will deal with is somewhat simpler, and it is called *filtered quantisations*. To define it, one first needs to introduce the notions of gradings and filtrations, which we do in this section.

Consider a vector space $B$.

**Definition 2.7.** A $\mathbb{Z}$-grading on $B$ is a decomposition $B = \bigoplus_{k \in \mathbb{Z}} B_k$ in vector subspaces. Elements of $B_k$ are said to be homogeneous elements of degree $k$. The grading is said to be positive if $B_{-k} = (0)$ for all $k > 0$. A positively $\mathbb{Z}$-graded vector space is a vector space equipped with a positive $\mathbb{Z}$-grading. A morphism of such spaces is a linear map preserving the $\mathbb{Z}$-grading.

We will hereafter drop "$\mathbb{Z}$" from the notation, since this is the only grading monoid we will ever consider.

**Example 2.1.** If $A, B$ are graded vector space, there exists a natural grading on the tensor product $A \otimes B$, suggested by the distributivity of tensor products in direct sums. It is defined by

$$A \otimes B \cong \bigoplus_{k \in \mathbb{Z}} (A \otimes B)_k, \qquad \text{where} \qquad (A \otimes B)_k \coloneqq \bigoplus_{l+m=k} A_l \otimes B_m.$$

Let us now define what we mean for a graded space $B$ to have a compatible associative algebra structure.

**Definition 2.8.** Let $B \cong \bigoplus_{k\in\mathbb{Z}} B_k$ be a graded vector space. A graded associative algebra structure on $B$ is an associative product $\mu\colon B \otimes B \longrightarrow B$ such that $B_l \cdot B_m \subseteq B_{l+m}$ for $l, m \in \mathbb{Z}$.[1] The grading is positive if it is a positive grading of vector spaces. A positively graded associative algebra is a vector space $B$ provided with all this structure. We will refer to such objects as graded algebras.

If one has a graded algebra $B \cong \bigoplus_{k\geq 0} B_k$, then one can construct new ones by taking appropriate quotients. Consider a complex subspace $I \subseteq B$.

**Definition 2.9.** $I$ is a homogeneous space if $I = \bigoplus_{k\geq 0} I \cap B_k$.

If $I$ is in addition a two-sided ideal, then the quotient algebra $B\big/I$ has a natural positive grading:

$$B/I \cong \bigoplus_{k\geq 0} \Big( B_k \big/ (I \cap B_k) \Big).$$

**Example 2.2.** The chief example of this construction, as far as this document is concerned, is the quotient,

$$\mathrm{Sym}(V) \cong \mathrm{Tens}(V) \big/ I_0$$

where $V$ is a vector space. Here $\mathrm{Tens}(V)$ is the tensor algebra

$$\mathrm{Tens}(V) = \bigoplus_{k\geq 0} V^{\otimes k},$$

and $I_0$ is the two-sided ideal generated by the commutators:

$$v \otimes w - w \otimes v, \qquad v, w \in V.$$

It is clear that an ideal generated by homogeneous elements is homogeneous. The algebra $\mathrm{Sym}(V^*)$ has a natural interpretation as the algebra of polynomial functions $f\colon V \longrightarrow \mathbb{C}$, with the usual notions of homogeneous elements and degree. This is also the algebra of regular functions $\mathbb{C}[V] = \mathscr{O}_V(V)$ of $V$, thinking of the vector space $V$ as an affine complex space $\mathbb{A}_{\mathbb{C}}$ of the same dimension, with structural sheaf $\mathscr{O}_V$.

To be more explicit, choose a basis $\mathcal{B} \coloneqq \{e_1, \ldots, e_n\} \subseteq V$, with associated dual basis $\{de_1, \ldots, de_n\} \subseteq V^*$. Then an explicit identification $\mathrm{Sym}(V^*) \cong \mathscr{O}_V(V)$ is given on homogeneous elements of degree $k$ by

$$\prod_{1\leq j\leq k} de_{i_j} \longmapsto f, \qquad \text{where} \qquad f(X_1, \ldots, X_n) \coloneqq \prod_{1\leq j\leq k} X_{i_j},$$

for indices $i_1, \ldots, i_k \in \{1, \ldots, n\}$. The notation for the leftmost product indicates that one has a symmetric tensor products $de_{i_1} \otimes \cdots \otimes de_{i_k}$. The complex variables $X_1, \ldots, X_n$ are precisely the global coordinates on $V \cong \mathbb{C}^n$ associated to the basis $\mathcal{B} \subseteq V$. If one

[1] This is a morphism of graded vector spaces, if the tensor product is endowed with the natural grading.

thinks of them as (linear) functions $X_i \colon V \longrightarrow \mathbb{C}$, then actually $X_i = de_i$, and the above identification is tautological. Notice that, in this identification, the embedding $V^* \cong \mathrm{Sym}(V^*)_1 \hookrightarrow \mathrm{Sym}(V^*)$ is simply the inclusion of homogeneous polynomials of degree one in the space of all polynomials.

To define graded Poisson algebras, one must also introduce shifts.

**Definition 2.10.** Let $B = \bigoplus_{k\in\mathbb{Z}} B_k$ be a graded vector space, and $n$ an integer. The $n$-shift of $B$ is the graded vector space $B[n]$ having the same underlying complex vector space, endowed with the grading

$$B = \bigoplus_{k\in\mathbb{Z}} B[n]_k, \qquad \text{where} \qquad B[n]_k \coloneqq B_{k+n}.$$

**Definition 2.11.** Let $B$ be a graded associative algebra, and $n$ an integer. A graded, $n$-shifted Poisson structure on $B$ is a grading preserving Poisson bracket

$$\{\cdot,\cdot\} \colon B \wedge B \longrightarrow B[-n].$$

**Example 2.3** (Symplectic vector spaces)**.**
The symmetric algebra $\mathrm{Sym}(V^*) \cong \mathscr{O}_V(V)$ admits a 2-shifted Poisson structure, provided that $(V,\omega)$ is a symplectic vector space. Indeed, there must be a natural Poisson bracket on $\mathscr{O}_V(V)$, since symplectic manifolds are Poisson (see Rem. A.1). To explicitly describe it, fix a Darboux basis $\mathcal{B} \coloneqq \{q_1,\dots,q_n,p_1,\dots,p_n\} \subseteq V$ of $V$, so that

$$\omega = \sum_i dq_i \wedge dp_i \in V^* \wedge V^*.$$

The coordinate $q_i$ are called the *generalised positions*, and $p_i$ the *generalised momenta*. The choice of such coordinates determines a Lagrangian splitting $(V,\omega) \cong (T^*Q, \omega_{\mathrm{can}})$, where one sets $Q \coloneqq \mathrm{span}_{\mathbb{C}}\{q_i\}_i$, and $\omega_{\mathrm{can}}$ is the canonical symplectic structure of the cotangent bundle. Now let us compute the Hamiltonian vector fields of the coordinate functions $dq_i, dp_i \colon V \longrightarrow \mathbb{C}$. To this hand, notice that one has

$$\omega(\partial_{q_i},\cdot) = \sum_j dq_j \wedge dp_j(q_i,\cdot) = \sum_j \delta_{ij} dp_j = dp_i,$$

and

$$\omega(\partial_{p_i},\cdot) = \sum_j dq_j \wedge dp_j(\partial_{p_i},\cdot) = -\sum_j \delta_{ij} dq_j = -dq_i.$$[2]

These identities can be rewritten as

$$\iota_{\partial_{q_i}}\omega - dp_i = 0, \qquad \iota_{\partial_{p_i}}\omega + dq_i = 0.$$

According to the convention (A.3), this means that $-\partial_{q_i}$ and $\partial_{p_i}$ are the Hamiltonian vector fields of $dp_i$ and $dq_i$, respectively.

[2] In these computation one thinks of $\partial_{q_i}$ and $\partial_{p_i}$ as the derivatives in the direction of the coordinates $q_i, p_i$, which is an elementary differential-geometric notion. However, using the canonical identification $TV \cong V \times V$, these translation invariant vector fields correspond to the vectors $q_i, p_i \in \mathcal{B}$ of the chosen Darboux basis of $V$.

To check that the Poisson bracket shifts the degree down by 2, it is enough to consider homogeneous elements of degree 1, since they generate the symmetric algebra. One must thus see that $\{V^*, V^*\} \subseteq \mathbb{C}$, which follows from:

$$\begin{aligned} \{dq_i, dq_j\} = \omega(\partial_{p_i}, \partial_{p_j}) = 0 = \omega(\partial_{q_i}, \partial_{q_j}) = \{dp_i, dp_j\}, \\ \{dq_i, dp_j\} = -\omega(\partial_{p_i}, \partial_{q_j}) = \omega(\partial_{q_j}, \partial_{p_i}) = \delta_{ij}. \end{aligned} \tag{2.3}$$

These relations are collectively called the *canonical commutation relations*. This particular 2-shifted graded Poisson algebra is the model for the classical observables quantised in Chap. 3.

**Example 2.4** (Dual Lie algebras)**.**
The algebras of functions on the dual of every Lie algebra is a 1-shifted graded Poisson algebra.

Indeed, let $\left(\mathfrak{g}, \left[\cdot, \cdot\right]\right)$ be a Lie algebra, and let us define a Poisson bracket

$$\{\cdot, \cdot\} \colon \mathscr{O}_{\mathfrak{g}^*}(\mathfrak{g}^*) \wedge \mathscr{O}_{\mathfrak{g}^*}(\mathfrak{g}^*) \longrightarrow \mathscr{O}_{\mathfrak{g}^*}(\mathfrak{g}^*).$$

Again, one thinks of $\mathfrak{g}^*$ as a complex affine space, with structural sheaf $\mathscr{O}_{\mathfrak{g}^*}$. The global sections of this sheaf are just polynomial functions $\mathfrak{g}^* \longrightarrow \mathbb{C}$, which one can identify with $\mathrm{Sym}(\mathfrak{g})$, as done in Ex. 2.2. Moreover, in this identification $\mathfrak{g} \subseteq \mathrm{Sym}(\mathfrak{g})$ corresponds to the space of linear functions on $\mathfrak{g}^*$, because of the canonical isomorphism $\mathfrak{g}^{**} \cong \mathfrak{g}$. This being said, it is then enough to define the restriction of the Poisson bracket to $\mathfrak{g} \wedge \mathfrak{g} \subseteq \mathrm{Sym}(\mathfrak{g}) \wedge \mathrm{Sym}(\mathfrak{g})$, and its extension to the whole of the symmetric algebra will be uniquely determined by the Leibnitz identity, because $\mathfrak{g}$ generates $\mathrm{Sym}(\mathfrak{g})$ as associative algebra. Let us then set

$$\{x, y\}(\xi) \coloneqq \langle \xi, \left[x, y\right] \rangle, \tag{2.4}$$

where $x, y \in \mathfrak{g}$, $\xi \in \mathfrak{g}^*$, and $\langle \cdot, \cdot \rangle \colon \mathfrak{g}^* \otimes \mathfrak{g} \longrightarrow \mathbb{C}$ is the canonical duality. One can show that this indeed defines a Poisson bracket on elements of degree one. Moreover, the formula shows that $\{\mathfrak{g}, \mathfrak{g}\} \subseteq \mathfrak{g} \cong \mathrm{Sym}(\mathfrak{g})_1$, because the bracket of two linear functions is still a linear function. This shows that the full Poisson bracket is 1-shifted. The Poisson bracket defined by Eq. (2.4) is called the Lie–Berezin–Kirillov bracket.

Let us now define filtrations, which are a weaker version of gradings. Consider again a vector space $B$.

**Definition 2.12.** An increasing, exhaustive $\mathbb{Z}$-filtration on $B$ is a family of subspaces $\{B_{\leq k}\}_{k \in \mathbb{Z}}$ of $B$ such that $B_l \subseteq B_m$ if $l \leq m$, and $\bigcup_{k \geq 0} B_{\leq k} = B$. The filtration is positive if $B_{\leq -k} = (0)$ for all $k > 0$. In this case, the order of $b \in B$ is the nonnegative integer

$$\mathrm{ord}(b) \coloneqq \min\{k \in \mathbb{Z}_{\geq 0} \mid b \in B_{\geq k}\}.$$

A positively, exhaustively $\mathbb{Z}$-filtered vector space is a vector space equipped with an increasing, exhaustive, positive $\mathbb{Z}$-filtration. A morphism of such spaces is a linear map preserving the filtrations.

The notions of a filtered associative algebra is a perfect analogue of Def. 2.8.

**Definition 2.13.** Let $B = \bigcup_{k\in\mathbb{Z}}$ be a filtered vector space. A filtered associative algebra structure on $B$ is an associative product $\mu\colon B\otimes B \longrightarrow B$ such that $\mu\Big(B_{\leq l}, B_{\leq m}\Big) \subseteq B_{\leq l+m}$, for all $l, m \in \mathbb{Z}$. The filtration is positive if it is positive as a filtration of vector spaces. A positively filtered associative algebra is a vector space $B$ with all this structure.

There is no need to define generic $n$-shifted filtered Poisson algebras, since the only examples of such structures that we will consider are those arising from commutators $\Big[\cdot,\cdot\Big]\colon B\wedge B \longrightarrow B$ of a filtered associative product.

**Remark 2.3** (Difference between gradings and filtrations)**.**
It is clear that filtrations are a weaker notion then gradings, since if $B \cong \oplus_{k\in\mathbb{Z}} B_k$ is a graded vector space then $B_{\leq k} \coloneqq \bigoplus_{m\leq k} B_m$ defines a filtration on $B$. The main difference between graded algebras and filtered algebras is that for the latter there is no notion of homogeneous elements.

More precisely, one might consider the elements of the quotient $B_{\leq k}\Big/B_{\leq k-1}$, which is the same as looking for a splitting $B_{\leq k} = C_k \oplus B_{\leq k-1}$, and declare $C_k$ to be the space of homogeneous elements of degree $k$. The difference between a filtration and a grading is that there is in general no canonical choice for such complement spaces $\{C_k\}_{k\in\mathbb{Z}}$.

In the filtered context, one can take quotients without additional constraints, meaning that if $B$ is a filtered algebra and $B \longrightarrow B' \longrightarrow 0$ is an exact sequence of algebras, then there is a natural filtration on $B'$: one just takes the images of the subspaces defining the filtration on $B$. The product is defined as for ordinary quotients of ring.

**Example 2.5** (Weyl algebra)**.**
This example is central for the results of Chap. 3. Consider again a symplectic vector space $(V,\omega)$, and let $\mathrm{Tens}(V)$ be the tensor algebra of $V$. In Ex. 2.2 we defined the symmetric algebra as a quotient of $\mathrm{Tens}(V)$ with respect to a homogeneous ideal. Now we consider the two-sided ideal $I_\omega \subseteq \mathrm{Tens}(V)$ generated by the nonhomogeneous elements

$$x\otimes y - y\otimes x - \omega(x,y), \qquad \text{where} \qquad x, y \in V.$$

The quotient $W(V,\omega) \coloneqq \mathrm{Tens}(V)\Big/I_\omega$ has the structure of a filtered associative algebra, with the quotient filtration. It is called the *Weyl algebra* of the symplectic vector space $V$.

To describe the Weyl algebra by means of generators and relations, let us pick again a Darboux basis $\mathcal{B} \coloneqq \{q_1,\dots,q_n,p_1,\dots,p_n\} \subseteq V$, as in Ex. 2.3. Now, there is a natural inclusion $V \hookrightarrow W(V,\omega)$, given by applying the canonical projection $\pi\colon TV \longrightarrow W(V,\omega)$ after the inclusion $\iota\colon V \hookrightarrow TV$. The overall composition is injective, since $\iota(V) \subseteq TV$ does not intersect the ideal $I_\omega \subseteq TV$. We will denote $\widehat{v} \coloneqq \pi\circ\iota(v) \in W(V,\omega)$ the image of an element $v \in V$ in the Weyl algebra for this composition. The notation is chosen to indicate that this is the linear part of a quantisation map (see § 2.1.5).

In particular, the elements $\widehat{q_i}, \widehat{p_i} \in W(V,\omega)$ generate the Weyl algebra. This is because $TV$ is generated in degree one, and $\pi$ is a surjective morphism of associative algebras. Moreover, the elements $q_i, p_i$ satisfy no nontrivial relations in $TV$, since $TV$ is by definition

the free associative algebra on the vector space $V$. Hence the only nontrivial relations that their classes $\widehat{q}_i, \widehat{p}_i$ satisfy in the Weyl algebra are those modulo $I_\omega$, namely:

$$\Big[x, y\Big] = \omega(x, y) \mod I_\omega, \qquad \text{for} \qquad x, y \in V.$$

Hence

$$\begin{aligned} \Big[\widehat{q}_i, \widehat{q}_j\Big] &= 0 = \Big[\widehat{p}_i, \widehat{p}_j\Big], \\ \Big[\widehat{q}_i, \widehat{p}_j\Big] &= \delta_{ij}, \end{aligned} \tag{2.5}$$

for all $1 \leq i, j \leq n$. These are the quantum canonical commutation relations (cf. Eq. (2.3)). The Weyl algebra (of dimension $n$) can thus be presented as the algebra on $2n$ generators satisfying the above commutation relations.

Notice that there is a dual realisation of the Weyl algebra. Consider the canonical linear isomorphism $\omega^\sharp \colon V^* \longrightarrow V$, induced by the nondegenerate, alternating symplectic pairing $\omega \colon V \wedge V \longrightarrow \mathbb{C}$ (cf. Rem. A.1). There is now a unique symplectic structure on $V^*$ such that $\omega^\sharp$ is a symplectomorphism — which we may denote $\omega^*$ — obtained by pull-back:

$$\omega^*(f, g) \coloneqq \omega\Big(\omega^\sharp(f), \omega^\sharp(g)\Big), \qquad \text{for} \qquad f, g \in V^*.$$

The Weyl algebra $W(V^*, \omega^*)$ of the dual symplectic space is then canonically isomorphic to $W(V, \omega)$.[3]

The intrinsic way of thinking about this is the following. The symplectic vector space $(V, \omega)$, considered as an affine complex space, comes equipped with a 2-shifted Poisson bracket $\{\cdot, \cdot\} \colon \mathscr{O}_V(V) \wedge \mathscr{O}_V(V) \longrightarrow \mathscr{O}_V(V)$, according to Ex. 2.3. We have seen that this implies that its restriction to linear functions yields an alternating bilinear form on the dual space $\{\cdot, \cdot\} \colon V^* \wedge V^* \longrightarrow \mathbb{C}$. This is exactly the dual structure $\omega^*$ we're talking about in the previous paragraph, because $\omega^\sharp(f) = X_f$ is the Hamiltonian vector field associated to $f \in V^*$ (which one may also think as a translation invariant 1-form on $V$). Thus:

$$\omega^*(f, g) = \omega\Big(\omega^\sharp(f), \omega^\sharp(g)\Big) = \omega(X_f, X_g) = \{f, g\},$$

for $f, g \in V^*$. Hence, in essence, one may say that the Weyl algebra $W(V^*, \omega^*)$ is obtained from the tensor algebra $\mathrm{Tens}(V^*)$ by modding out the Poisson structure. This is analogous what happens for the universal enveloping algebras $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$, discussed just below in Ex. 2.6.

Finally, if one is interested in explicit generators and relations, notice that the dual basis $\{dq_1, \ldots, dq_n, dp_1, \ldots, dp_n\}$ provides a presentation of $W(V^*, \omega^*)$ which is the same as that for $W(V, \omega)$, since

$$\{dq_i, dq_j\} = 0 = \{dp_i, dp_j\}, \qquad \{dq_i, dp_j\} = \delta_{ij}$$

[3] The canonical isomorphism is given by the universal property of the quotient applied to the composition $\pi \circ \mathrm{Tens}(\omega^\sharp) \colon \mathrm{Tens}(V^*) \longrightarrow W(V, \omega)$, where $\mathrm{Tens}(\omega^\sharp) \colon \mathrm{Tens}(V^*) \longrightarrow \mathrm{Tens}(V)$ is the image of the morphism $\omega^\sharp$ under the functor Tens, and again $\pi \colon \mathrm{Tens}(V) \longrightarrow W(V, \omega)$ is the canonical projection.

for all $1 \leq i, j \leq n$, reading again the classical canonical commutation relations (2.3). Hence

$$\left[\widehat{dq_i}, \widehat{dq_j}\right] = 0 = \left[\widehat{dp_i}, \widehat{dp_j}\right],$$
$$\left[\widehat{dq_i}, \widehat{dp_j}\right] = \delta_{ij}.$$

**Example 2.6** (Universal enveloping algebras)**.**
Let $\left(\mathfrak{g}, \left[\cdot, \cdot\right]\right)$ be a Lie algebra, and consider its tensor algebra $\mathrm{Tens}(\mathfrak{g})$. This is a filtered algebra, and one considers its quotient with respect to the two-sided ideal $I_1 \subseteq \mathrm{Tens}(\mathfrak{g})$ generated by the nonhomogeneous elements

$$x \otimes y - y \otimes x - \left[x, y\right], \qquad \text{for} \qquad x, y \in \mathfrak{g}.$$

One sets $U(\mathfrak{g}) \coloneqq \mathrm{Tens}(\mathfrak{g})\big/I_1$, and calls this the *universal enveloping algebra* of $\left(\mathfrak{g}, \left[\cdot, \cdot\right]\right)$.

The algebra $U(\mathfrak{g})$ is an associative filtered algebra, with the quotient filtration. Notice that the tensor algebra $\mathrm{Tens}(\mathfrak{g})$ is naturally Poisson, if one extends the Lie bracket of $\mathfrak{g}$ by demanding the Leibnitz rule to be satisfied, i.e. generalising the following equality to arbitrary tensor products:

$$\left[x \otimes y, z\right] = x \otimes \left[y, z\right] + \left[y, z\right] \otimes x, \qquad \text{for} \qquad x, y, z \in \mathfrak{g},$$

and similarly for the other slot. Hence one may say that the enveloping algebra is obtained from the tensor algebra by modding out its Poisson structure, just as for the Weyl algebra introduced in 2.5. Nonetheless, the associative algebra $U(\mathfrak{g})$, endowed with the Poisson structure defined by its commutator (cf. Ex. A.1), is a filtered Poisson algebra. The shift follows from that of $\mathrm{Sym}(\mathfrak{g}) \cong \mathscr{O}_{\mathfrak{g}^*}(\mathfrak{g}^*)$.

Let us describe in some more detail the case of $\mathfrak{g} = \mathfrak{gl}(V)$, where $V$ is a vector space of dimension $n$, because this is relevant to the results of § 3.7, § 3.8 and § 3.9. Picking a basis $\mathcal{B} \coloneqq \{e_i\}_i \subseteq V$ of $V$, one may define a basis for $\mathfrak{g}$ as follows: denote $e_{ij} \colon V \longrightarrow V$ the element $e_{ij} \coloneqq e_i \otimes de_j \in V \otimes V^* \cong \mathrm{End}(V)$. These elements satisfy the commutation relations

$$\left[e_{ij}, e_{kl}\right] = \delta_{jk} e_{il} - \delta_{il} e_{kj}.$$

There is now an inclusion $\mathfrak{g} \hookrightarrow U(\mathfrak{g})$, provided by the composing the canonical injection $\iota \colon \mathfrak{g} \hookrightarrow \mathrm{Tens}(\mathfrak{g})$ with the canonical projection $\pi \colon \mathrm{Tens}(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$. Let us denote $\widehat{x} \coloneqq \pi \circ \iota(x)$, for $x \in \mathfrak{g}$, underlying the fact that this is the restriction of a quantisation map (see § 2.1.5). Then, reasoning as in Ex. 2.5, the enveloping algebra $U(\mathfrak{g})$ is the algebra on generators $\{\widehat{e_{ij}}\}_{i,j}$, subjected to the relations

$$\left[\widehat{e}_{ij}, \widehat{e}_{kl}\right] = \delta_{ik} \widehat{e}_{il} - \delta_{il} \widehat{e}_{kj},$$

for all $i, j, k, l$.

### 2.1.3 Filtered deformations and filtered quantisations

We now want to use the material of the previous section to give a new definition of deformations of associative algebras, and quantisation of commutative associative algebras. The main idea is to replace the general notion of a semiclassical limit by the grading map.

**Definition 2.14.** Let $B = \bigcup_{k\in\mathbb{Z}} B_{\leq k}$ be a filtered associative algebra. The associated graded of $B$ is the graded algebra $\mathrm{gr}(B)$ defined as follows. One sets $\mathrm{gr}(B)_k \coloneqq B_{\leq k}\big/B_{\leq k-1}$, and then

$$\mathrm{gr}(B) \coloneqq \bigoplus_{k\in\mathbb{Z}} \mathrm{gr}(B)_k,$$

as graded vector space. The product of two classes $a + B_{\leq l-1}, b + B_{\leq m-1}$, having representatives $a \in B_{\leq l}, b \in B_{\leq m}$ is (well) defined to be the class of the product $a \cdot b \in B_{\leq l+m}$ in $\mathrm{gr}(B)_{l+m}$. One writes $\sigma_k \colon B_{\leq k} \longrightarrow \mathrm{gr}(B)_k$ for the canonical projection, where $k \in \mathbb{Z}$, and $\sigma \colon B \longrightarrow \mathrm{gr}(B)$ for the map defined by $\sigma_k$ on $B_{\leq k} \setminus B_{\leq k-1}$.

Notice that $B$ and $\mathrm{gr}(B)$ are isomorphic as vector spaces. The examples of the previous sections are related like this.

**Example 2.7.** Consider a symplectic vector space $(V, \omega)$. We have a commutative, graded, 2-shifted Poisson algebra $A_0 \coloneqq \mathrm{Sym}(V^*) \cong \mathscr{O}_V(V)$ associated to this space, as well as a filtered, associative $A \coloneqq W(V^*, \omega^*)$. One can show that $\mathrm{gr}(A) \cong A_0$.

To do this, consider the composition $\sigma\circ\pi \colon \mathrm{Tens}(V^*) \longrightarrow \mathrm{gr}(A)$, where $\pi \colon \mathrm{Tens}(V^*) \longrightarrow A$ is the canonical projection, and $\sigma \colon A \longrightarrow \mathrm{gr}(A)$ the grading map. One can now show that the kernel of this map contains the ideal $I_0 \subseteq \mathrm{Tens}(V^*)$ defining the symmetric algebra (see Ex. 2.2). This is the two-sided ideal generated by the commutators $f \otimes g - g \otimes f$, for $f, g \in V^* \subseteq \mathrm{Sym}(V^*)$. Now, one has

$$\pi\Big(f \otimes g - g \otimes f\Big) = \Big[\pi(f), \pi(g)\Big] = \Big[\widehat{f}, \widehat{g}\Big] \in A.$$

This element vanishes in the associated graded algebra, because one has $\Big[\widehat{f}, \widehat{g}\Big] = \{f, g\}$ in the Weyl algebra, and the left-hand side a priori lives in $A_{\leq 2}$, whereas the right-hand side lives in $A_{\leq 0} \cong \mathbb{C}$, so that $\sigma\Big(\Big[\widehat{f}, \widehat{g}\Big]\Big) = \sigma_2\Big(\Big[\widehat{f}, \widehat{g}\Big]\Big) = \sigma_2\Big(\{f, g\}\Big) = 0$.

The universal property of the quotient provides a surjective morphism

$$A_0 \cong \mathrm{Tens}(V^*)\big/I_0 \longrightarrow \mathrm{gr}(A),$$

and one can show that it is injective. Up to identifying $A_0$ with $\mathrm{gr}(A)$, one then has the following relation between the two Poisson structures

$$\sigma\Big[\widehat{f}, \widehat{g}\Big] = \{f, g\} = \Big\{\sigma\Big(\widehat{f}\Big), \sigma\Big(\widehat{g}\Big)\Big\},$$

for $f, g \in A_0$, and for arbitrary lifts $\widehat{f}, \widehat{g} \in A$. This equality has been shown for elements of degree one in $A_0$, comparing the canonical commutation relations of Ex. 2.5 and Ex. 2.3. The Leibnitz identity then yields the general formula. This will be relevant for Prop. 2.1.

**Example 2.8.** Reasoning exactly as in the above example 2.7, one can show that the associated graded algebra $\mathrm{gr}\Big(U(\mathfrak{g})\Big)$ of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\Big(\mathfrak{g}, \Big[\cdot,\cdot\Big]\Big)$ is isomorphic to the symmetric algebra $\mathrm{Sym}(\mathfrak{g})$ of $\mathfrak{g}$. Moreover, up to identifying $\mathrm{Sym}(\mathfrak{g}) \cong \mathscr{O}_{\mathfrak{g}^*}(\mathfrak{g}^*)$ with $\mathrm{gr}\Big(U(\mathfrak{g})\Big)$, one has the same relation for the Poisson brackets as in the previous example 2.7. In this case the identity on elements of degree one follows directly from the definition of the Poisson bracket of $\mathfrak{g}^*$ as dual to the Lie bracket $\Big[\cdot,\cdot\Big]$. This will be relevant for Prop. 2.1, and it also one way of expressing the theorem of Poincaré–Birkhoff–Witt.

These two examples are instances of *filtered deformations*.

**Definition 2.15.** Let $A_0$ be a graded algebra. A filtered deformation $A$ of $A_0$ is a filtered algebra such that $\mathrm{gr}(A) \cong A_0$.

The crucial remark is the following: if $A$ is a filtered algebra whose associated graded is commutative, then there is a canonical 1-shifted Poisson bracket $\{\cdot,\cdot\}_A$ on $\mathrm{gr}(A)$, defined by the condition that

$$\sigma_{k+l-1}\Big[x, y\Big] = \{\sigma_k(x), \sigma_l(y)\}_A, \qquad \text{for} \qquad x \in A_{\leq k}, y \in A_{\leq l}.$$

Just as we did in the case of deformation quantisation, we use this fact to define *filtered quantisations*. Let then $(A_0, \{\cdot,\cdot\})$ be a commutative, graded Poisson algebra.

**Definition 2.16.** A filtered quantisation of $A_0$ is a filtered deformation $A$ of $A_0$, such that moreover

$$\sigma_{k+l-1}\Big[x, y\Big] = \{\sigma_k(x), \sigma_l(y)\}, \qquad \text{for} \qquad x \in A_{\leq k}, y \in A_{\leq l},$$

where one identifies $A_0 \cong \mathrm{gr}(A)$.

The discussions of 2.3, 2.7, 2.6 and 2.8 then immediately yield the following standard result.

**Proposition 2.1.** *Let $\mathfrak{g}$ be a Lie algebra, and $(V, \omega)$ be a symplectic vector space.*

- *The Weyl algebra $W(V, \omega)$ of $(V, \omega)$ is a filtered quantisation of the algebra of polynomial functions $\mathrm{Sym}(V^*)$ on $V$.*
- *The universal enveloping algebra $U(\mathfrak{g})$ is a filtered quantisation of the algebra of polynomial function $\mathrm{Sym}(\mathfrak{g})$ on $\mathfrak{g}^*$.*

What is left to do is to relate this notion to that of a formal deformation quantisation of § 2.1. The main idea is that the former is a particular case of the latter. Moreover, given a filtered quantisation, there is a universal construction to get a formal deformation quantisation out of it.

### 2.1.4 Rees construction

The notions of formal deformation quantisation (§ 2.1) and filtered quantisation (§ 2.1.3) can be related by the so-called *Rees construction*.

**Definition 2.17.** Let $A = \bigoplus_{k\in\mathbb{Z}} A_{\leq k}$ be a filtered algebra. The Rees algebra of $A$ at level $\hbar$ is the algebra

$$\mathcal{R}_\hbar(A) \coloneqq \bigoplus_{k\in\mathbb{Z}} A_{\leq k}\hbar^k \subseteq A[\hbar].$$

To get back to topologically free $\mathbb{C}[\![\hbar]\!]$-modules, one may consider a completion of this Rees algebra:

$$\widehat{A} \coloneqq \left\{ \sum_{k\geq 0} a_k \hbar^k \,\middle|\, a_k \in A_{\leq k} \text{ for all } k \geq 0, \lim_{k\longrightarrow+\infty} \Big(k - \operatorname{ord}(a_k)\Big) = +\infty \right\} \subseteq A[\![\hbar]\!].$$

Now assume that $A$ is a filtered deformation of $A_0 \cong \operatorname{gr}(A)$.

**Proposition 2.2.** *The algebra $\widehat{A}$ is a formal, one-parameter deformation of $A_0$.*

*Proof.* Consider the morphism $\varphi\colon \widehat{A} \longrightarrow A_0$ defined by

$$\varphi\left(\sum_{k\geq 0} a_k \hbar^k\right) = \sum_{k\geq 0} \sigma_k(a_k), \tag{2.6}$$

identifying $\operatorname{gr}(A) = \sigma(A)$ with $A_0$. The sum on the right-hand side is finite, since by definition $\operatorname{ord}(a_k) = k$ only for finitely many indices $k$, so that $\sigma_k(a_k) = 0$ in all but finitely many cases.

The map $\varphi$ is surjective, and its kernel is precisely $\hbar\widehat{A}$, so that $A_0 \cong \widehat{A}\big/\hbar\widehat{A}$. □

In a sense, the morphism $\varphi$ define by Eq. (2.6) is the true semiclassical limit.

Now assume that $\Big(A, \big[\cdot,\cdot\big]\Big)$ is a filtered quantisation of $(A_0, \{\cdot,\cdot\})$.

**Proposition 2.3.** *The algebra $\widehat{A}$ is a formal, one-parameter deformation quantisation of $A_0$.*

*Proof.* All that is left to show is that the Poisson bracket on $A_0$ comes from the commutator of $\widehat{A}$, by means of (2.2). This follows precisely from the condition of filtered quantisation in Def. 2.16. □

Because of Prop. 2.3, we shall consider filtered quantisation as a good notion of deformation quantisation for commutative, graded Poisson algebras, and keep the notation $A$ for the quantum algebras.

### 2.1.5 Quantisation maps

Let us conclude this section on deformation quantisation by adding something to the quantisations of Prop. 2.1. Namely, in both cases one can define a "quantisation" map, i.e. a distinguished right-inverse of the semiclassical limit.[4] The idea is the same both for enveloping algebras and Weyl algebras, and relies on symmetrisation.

Let $\Big(\mathfrak{g}, \big[\cdot,\cdot\big]\Big)$ be a Lie algebra, and $(V,\omega)$ a symplectic vector space. Denote $A \coloneqq U(\mathfrak{g})$, $A_0 \coloneqq \operatorname{Sym}(\mathfrak{g}) \cong \operatorname{gr}(A)$, $B \coloneqq W(V^*,\omega^*)$ and $B_0 \coloneqq \operatorname{Sym}(V^*) \cong \operatorname{gr}(B)$. One has natural

[4] We use quotation marks, because these maps are a weaker notion than that presented in § A.4.

inclusions $A_0 \hookrightarrow \mathrm{Tens}(\mathfrak{g})$, $B_0 \hookrightarrow \mathrm{Tens}(V^*)$, given on monomials of degree $k$ by the same formula:

$$\prod_{1\leq i\leq k} X_i \longmapsto \frac{1}{k!}\sum_{\tau\in\Sigma_k} X_{\tau_1}\otimes\cdots\otimes X_{\tau_k}, \tag{2.7}$$

where $X_i \in A_0$ or $B_0$, and where $\Sigma_k$ is the symmetric group on $k$ objects. This is just the standard fact that — in characteristic zero — symmetric algebras can be realised as subalgebras of the tensor algebras, whereas a priori they are defined as quotients.

One can now compose this map with the canonical projections $\pi_W\colon \mathrm{Tens}(V^*) \longrightarrow B$ and $\pi\colon \mathrm{Tens}(\mathfrak{g}) \longrightarrow A$. This defines maps $\mathcal{Q}_W\colon B_0 \longrightarrow B$ and $\mathcal{Q}_{\mathrm{PBW}}\colon A_0 \longrightarrow A$ which are sections of the respective semiclassical limits, because of the normalisation taken in (2.7).

**Definition 2.18.** The map $\mathcal{Q}_W$ is called the Weyl quantisation. The map $\mathcal{Q}_{\mathrm{PBW}}$ is called the PBW quantisation.

Notice that the explicit formula for the quantisation maps reads just as the symmetrisation (2.7), if one replaces the element $X_i$ in the relevant symmetric algebra with its class $\widehat{X}_i := \pi\iota(X_i)$, as explained in Ex. 2.5 and Ex. 2.6. Namely, if $k$ is a nonnegative integer, and if $X_i \in \mathfrak{g}$ for $1 \leq i \leq k$:

$$\mathcal{Q}_{\mathrm{PBW}}\left(\prod_{i=1}^{k} X_i\right) = \frac{1}{k!}\sum_{\tau\in\Sigma_k} \widehat{X}_{\tau_1}\cdot\ldots\cdot\widehat{X}_{\tau_k}. \tag{2.8}$$

Similarly, if $Y_i \in V^*$ for all $1 \leq i \leq k$:

$$\mathcal{Q}_W\left(\prod_{i=1}^{k} Y_i\right) = \frac{1}{k!}\sum_{\tau\in\Sigma_k} \widehat{Y}_{\tau_1}\cdot\ldots\cdot\widehat{Y}_{\tau_k}. \tag{2.9}$$

We will see in Chap. 3 how to use a variation of these maps in order to quantise interesting time-dependent Hamiltonian systems.

### 2.1.6 Hamiltonian reduction: an algebraic viewpoint

Here we briefly review the algebraic viewpoint on Hamiltonian actions and Hamiltonian reductions, which is suited to deformation quantisation (see [Eti07]). This material will be used in Chap. 3, more precisely in § 3.7, 3.8 and 3.9.

Let $(A_0, \{\cdot,\cdot\})$ be a commutative Poisson algebra, and $\mathfrak{g}$ a Lie algebra. Denote by $\mathrm{Der}(A_0)$ the Lie algebra of Poisson derivations of $A_0$, i.e. of $\mathbb{C}$-linear maps $\psi\colon A_0 \longrightarrow A_0$ satisfying

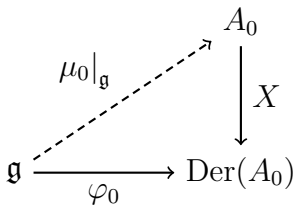
$$\psi(fg) = \psi(f)g + f\psi(g), \qquad \psi\Big(\{f,g\}\Big) = \{\psi(f),g\} + \{f,\psi(g)\}, \qquad \text{for} \qquad f,g \in A_0,$$

with the Lie bracket provided by the commutator.

**Definition 2.19.** A $\mathfrak{g}$-action on $A_0$ is a morphism of Lie algebras $\varphi_0\colon \mathfrak{g} \longrightarrow \mathrm{Der}(A_0)$.

Now, one has a natural map $X\colon A_0 \longrightarrow \mathrm{Der}(A_0)$, associating to an element $f \in A_0$ the bracket $X_f = \{f, \cdot\}\colon A_0 \longrightarrow A_0$. This is a Poisson derivation of $A_0$, because of the Leibnitz and Jacobi identities. The notation mimic that for the Hamiltonian vector field of a function $f$ defined on a Poisson manifold (see A.2).
From an algebraic perspective, $X$ is just the morphism defining the *adjoint* action of $A_0$ on itself, considered as a Lie algebra.

**Definition 2.20.** A (classical) moment map for the $\mathfrak{g}$-action $\varphi_0\colon \mathfrak{g} \longrightarrow \mathrm{Der}(A_0)$ is a morphism $\mu_0\colon \mathrm{Sym}(\mathfrak{g}) \longrightarrow A_0$ of Poisson algebras whose restriction to $\mathfrak{g}$ lifts $\varphi_0$ through $X$:




Recall that $\mathrm{Sym}(\mathfrak{g}) \cong \mathscr{O}_{\mathfrak{g}^*}(\mathfrak{g}^*)$ is Poisson, as explained in Ex. 2.4. The lifting condition means that

$$X\Big(\mu_0(x)\Big).a = \{\mu_0(x), a\} = \varphi_0(x).a \qquad \text{for } x \in \mathrm{Sym}(\mathfrak{g}), a \in A_0.$$

In particular the $\mathfrak{g}$-action is completely determined by $\mu_0$. Moreover, $\mu_0$ is completely determined by its restriction to $\mathfrak{g}$.

**Definition 2.21.** Let $\varphi_0\colon \mathfrak{g} \longrightarrow A_0$ be a $\mathfrak{g}$-action on $A_0$. The subring of $\mathfrak{g}$-invariants of $A_0$ is:

$$A_0^{\mathfrak{g}} \coloneqq \Big\{a \in A_0 \mid \varphi_0(\mathfrak{g}).a = 0\Big\}.$$

**Remark 2.4.** The subring $A_0^{\mathfrak{g}}$ is a $\mathbb{C}$-subalgebra. Moreover, it is a Poisson subalgebra, because $\varphi_0$ is a Poisson derivation by definition. Notice that if $\mu_0$ is a moment for $\varphi_0$, then the invariant algebra equals

$$A_0^{\mathfrak{g}} = \Big\{a \in A_0 \mid \{\mu_0(\mathfrak{g}), a\} = 0\Big\},$$

which means that it is the centraliser of $\mu(\mathfrak{g}) \subseteq A_0$ in $A_0$.

These notions are introduced to define reductions of $A_0$ with respect to the moment map $\mu_0$ and the choice of an ideal $I \subseteq \mathrm{Sym}(\mathfrak{g})$. More precisely, one wants to define the *Hamiltonian reduction* $R(A_0, \mu_0, I)$ as a Poisson algebra. To this end, let $J \subseteq A_0$ be the ideal generated by $\mu_0(I)$.

**Lemma 2.1.** *One has* $\{A_0^{\mathfrak{g}}, J\} \subseteq J$.

*Proof.* Pick $c \in A_0^{\mathfrak{g}}, b \in J$. One may write $b = \sum_i b_i \mu_0(a_i)$ for suitable $b_i \in A_0, a_i \in I$. One has:

$$\{c, b\} = \sum_i \{c, b_i\mu_0(a_i)\} = \sum_i b_i\{c, \mu_0(a_i)\} + \mu_0(a_i)\{c, b_i\} = \sum_i \mu_0(a_i)\{c, b_i\} \in J,$$

since $\{\mu_0(a_i), c\} = 0$ by $\mathfrak{g}$-invariance. □

An immediate corollary is that the invariant part $J^{\mathfrak{g}} \subseteq A_0^{\mathfrak{g}}$ of $J$ is a Poisson ideal in $A_0^{\mathfrak{g}}$.

**Definition 2.22.** The (classical) Hamiltonian reduction of $A_0$ with respect to the moment $\mu_0 \colon \mathrm{Sym}(\mathfrak{g}) \longrightarrow A_0$ and the ideal $I \subseteq \mathrm{Sym}(\mathfrak{g})$ is the quotient

$$R(A_0, \mu_0, I) \coloneqq A_0^{\mathfrak{g}} \Big/ J^{\mathfrak{g}}.$$

This quotient is a Poisson algebra, thanks to the above remark.

**Remark 2.5.** This is indeed the good algebraic notion that corresponds to the reduction of a symplectic manifold with respect to the Hamiltonian action of a Lie group. To explicitly state the correspondence, let $(M, \omega)$ be a symplectic manifold, equipped with a symplectic action of a Lie group $G$, with Lie algebra $\mathfrak{g}$.

The $G$-action preserves the Poisson structure of $A_0 \coloneqq C^\infty(M, \mathbb{C})$, in the sense that the action of $G$ on $A_0$ defined by pull-back satisfies

$$g.\{f, h\} = \{g.f, g.h\}, \qquad g.(fh) = (g.f) \cdot (g.h),$$

where $g \in G$, $f, h \in A_0$. Hence the differential of this $G$-action $\rho \colon G \longrightarrow \mathrm{Aut}(A_0)$, at the identity $e \in G$, provides a morphism $d_e\rho \colon \mathfrak{g} \longrightarrow \mathrm{Der}(A_0)$ of Lie algebras, which is a $\mathfrak{g}$-action on $A_0$ by Def. 2.19.
If one thinks of the geometric action on points of $M$, this $\mathfrak{g}$-action is the standard morphism that associates to $x \in \mathfrak{g}$ the vector field $\widehat{x} \in C^\infty(M, TM)$ on $M$ defined by

$$\widehat{x}(p) = \frac{d}{dt}\Big( \exp^{tx} .p \Big)\Big|_{t=0}$$

where $p \in M$, and $\exp \colon \mathfrak{g} \longrightarrow G$ is the exponential map of $G$. There is now a standard identification between derivatives of $A_0$ and vector fields on $M$.

Now assume that the $G$-action is Hamiltonian, equipped with a ($G$-equivariant) moment map $\mu \colon M \longrightarrow \mathfrak{g}^*$. The dual viewpoint provides a map $\mu^* \colon \mathfrak{g} \longrightarrow A_0$, usually called a *comoment*, which admits a unique extension to a morphism of Poisson algebras $\mu^* \colon \mathrm{Sym}(\mathfrak{g}) \longrightarrow A_0$. Moreover, by definition, this map sends an element $x \in \mathfrak{g}$ to the function $\mu^*(x) \in A_0$ whose Hamiltonian vector field is $\widehat{x} = d_e\rho(x) \in \mathrm{Der}(A_0)$, and thus $\mu^*$ is a moment map for the $\mathfrak{g}$-action, according to Def. 2.20.

Next there is the question of the reduction. Let $\mathcal{O} \subseteq \mathfrak{g}^*$ be a closed $G$-invariant set, for example the closure of a coadjoint orbit. Then one defines the Hamiltonian reduction of $M$ with respect to the moment $\mu$ and the set $\mathcal{O}$ as the quotient

$$M /\!\!/_{\mathcal{O}} G \coloneqq \mu^{-1}(\mathcal{O}) \big/ G,$$

which inherits a natural symplectic structure from $\omega$ (and which may fail to be smooth). The natural algebraic counterpart of this construction would be to consider the algebra of functions on this Hamiltonian reduction. The way to match this up with Def. 2.22 is the following.

Consider the ideal $I \subseteq \mathrm{Sym}(\mathfrak{g}) \cong \mathscr{O}_{\mathfrak{g}^*}(\mathfrak{g}^*)$ of functions $\mathfrak{g}^* \longrightarrow \mathbb{C}$ that vanish on the closed set $\mathcal{O} \subseteq \mathfrak{g}^*$. Then the ideal $J \subseteq A_0$ generated by $\mu^*(I)$ is the ideal of functions that vanish on $\mu^{-1}(\mathcal{O}) \subseteq M$, and the quotient $A_0 \big/ J$ is canonically identified with the ring of functions on the closed set $\mu^{-1}(\mathcal{O})$. Finally, taking the $G$-invariant part of the quotient ring would provide the ring of functions on the quotient $M /\!\!/_{\mathcal{O}} G$. Recall that by definition one sets

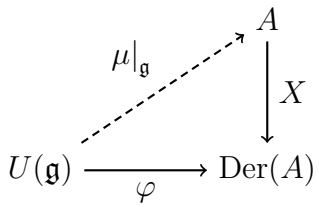
$$A_0^G \coloneqq \{f \in A_0 \mid G.f = f\} \subseteq A_0,$$

and differentiating $g.f = f$ at the identity $e \in G$ shows that this is the same as $A_0^{\mathfrak{g}}$ of 2.21. Hence, a priori, the ring of functions of $M /\!\!/_{\mathcal{O}} G$ is the quotient $\left(A_0 \big/ J\right)^{\mathfrak{g}}$. One can show that invariant ring is isomorphic to the quotient $A_0^{\mathfrak{g}} \big/ J^{\mathfrak{g}}$ in some fortunate cases, e.g. when $G$ and $\mathfrak{g}$ are reductive.

On the whole, the construction we described above is the natural algebraic and contravariant counterpart of the classical Marsden–Weinstein reduction. If one were to work with with affine Poisson varieties $X$ over $\mathbb{C}$, provided with algebraic actions of reductive Lie group $G$ that satisfy all of the above, then one would have $\mathrm{Spec}\left(\mathscr{O}_X(X)\right) \cong X$, and one could define $X /\!\!/_{\mathcal{O}} G \coloneqq \mathrm{Spec}\left(R(\mathscr{O}_X(X), \mu, I)\right)$, using the same notations as above (and taking care that $\mathcal{O} \subseteq \mathfrak{g}^*$ is now Zariski-closed). This produces an affine Poisson scheme that may be singular.

In the same algebraic fashion as above, and exploiting the standard PBW quantisation $\mathcal{Q}_{PBW} \colon \mathfrak{g} \longrightarrow U(\mathfrak{g})$ for Lie algebras as in § 2.1.5, one defines quantum moments. To this end, let us denote $\sigma \colon U(\mathfrak{g}) \longrightarrow \mathrm{Sym}(\mathfrak{g})$ the semiclassical limit, and let $A$ be a possibly noncommutative associative algebra over $\mathbb{C}$, equipped with a $\mathfrak{g}$-action $\varphi \colon \mathfrak{g} \longrightarrow \mathrm{Der}(A)$.[5]

**Definition 2.23.** A quantum moment for $\varphi$ is an associative morphism $\mu \colon U(\mathfrak{g}) \longrightarrow A$ such that its restriction to $\mathfrak{g}$ lifts $\varphi$ through the natural morphism $X \colon A \longrightarrow \mathrm{Der}(A)$:




Just as in the classical case, the action is determined by $\mu$, which is in turn completely determined by its restriction to $\mathfrak{g}$.

Now suppose to be in the context of filtered quantisation, as in § 2.1.3. Assume hence that $A$ be filtered, and that $A_0 = \mathrm{gr}(A)$ is commutative. Call $\sigma' \colon A \longrightarrow A_0$ the semiclassical limit, and consider two $\mathfrak{g}$-actions $\varphi_0 \colon \mathfrak{g} \longrightarrow \mathrm{Der}(A_0)$, $\varphi \colon \mathfrak{g} \longrightarrow \mathrm{Der}(A)$ which admit a classical moment $\mu_0 \colon \mathrm{Sym}(\mathfrak{g}) \longrightarrow A_0$ and a quantum moment $\mu \colon U(\mathfrak{g}) \longrightarrow A$, respectively.

[5] If $A$ is associative, a $\mathbb{C}$-linear map $f \colon A \longrightarrow A$ is a derivation if $f(ab) = f(a)b + af(b)$ for all $a, b \in A$. This is however the same as asking for it to be a Poisson derivation with respect to the Poisson bracket defined by the commutator of $A$ (see A.1).

**Definition 2.24.** The quantum moment $\mu\colon U(\mathfrak{g}) \longrightarrow A$ is quantisation of the classical moment $\mu_0\colon \mathrm{Sym}(\mathfrak{g}) \longrightarrow A_0$ if they are intertwined by the semiclassical limits:

$$\begin{array}{ccc} U(\mathfrak{g}) & \xrightarrow{\mu} & A \\ \sigma \downarrow & \circlearrowleft & \downarrow \sigma' \\ \mathrm{Sym}(\mathfrak{g}) & \xrightarrow[\mu_0]{} & A_0 \end{array}$$

If one were to pick the completion of the Rees algebra of $A$, thereby getting a genuine $\hbar$-deformation quantisation, as explained in § 2.1.4, then one might say that $\mu \equiv \mu_0(mod(\hbar))$.

**Remark 2.6.** Notice that definition Def. 2.24 implies in particular that $\varphi_0 \circ \sigma = \sigma' \circ \varphi$ as well, since

$$\varphi_0\Big(\sigma\widehat{f}\Big).g = \{\mu_0(\sigma\widehat{f}), g\} = \{\mu_0(\sigma\widehat{f}), \sigma'\widehat{g}\} = \{\sigma'\mu(\widehat{f}), \sigma'(\widehat{g})\} = \sigma'\Big[\mu(\widehat{f}), \widehat{g}\Big] = \sigma'\varphi(\widehat{f}).\widehat{g},$$

for all $\widehat{f} \in U(\mathfrak{g})$, $g \in A_0$, and for all lift $\widehat{g} \in A$.

One now defines the quantum Hamiltonian reduction $R_q(A, \mu, I)$, where $(A, \mu, I)$ is a triple consisting of an associative algebra $A$, a quantum moment $\mu\colon U(\mathfrak{g}) \longrightarrow A$, and a two-sided ideal $I \subseteq U(\mathfrak{g})$. One considers the left-ideal $J \coloneqq A.\mu(I) \subseteq A$ generated by $\mu(I)$, which need not be two-sided.

**Lemma 2.2.** *The invariant part $J^{\mathfrak{g}} \subseteq A^{\mathfrak{g}}$ is a two-sided ideal.*

Here again one sets $A^{\mathfrak{g}}$ to be the algebra of $\mathfrak{g}$-invariants:

$$A^{\mathfrak{g}} \coloneqq \Big\{b \in A \Big| \Big[\mu(\mathfrak{g}), b\Big] = 0\Big\}.$$

*Proof.* It is clear that $J^{\mathfrak{g}} = J \cap A^{\mathfrak{g}}$ is an ideal in $A^{\mathfrak{g}}$. To see that it is bilateral, pick $c \in A^{\mathfrak{g}}$, $b \in J^{\mathfrak{g}}$. One may write $b = \sum_i b_i\mu(a_i)$ for suitable $b_i \in A$, $a_i \in I$. Then:

$$bc = \sum_i b_i\mu(a_i)c = \sum_i b_ic\mu(a_i) + b_i\Big[\mu(a_i), c\Big] = \sum_i b_ic\mu(a_i) \in J^{\mathfrak{g}},$$

where $\Big[\mu(a_i), c\Big] = 0$ since $c$ is $\mathfrak{g}$-invariant. □

**Definition 2.25.** The quantum Hamiltonian reduction of $A$ with respect to the quantum moment $\mu\colon U(\mathfrak{g}) \longrightarrow A$ and the two-sided ideal $I \subseteq U(\mathfrak{g})$ is the quotient:

$$R_q(A, \mathfrak{g}, I) \coloneqq A^{\mathfrak{g}}\Big/J^{\mathfrak{g}}.$$

Notice that this quotient is an associative algebra, because of the above lemma.

## 2.2 Geometric quantisation

The geometric approach to quantisation starts with a smooth symplectic manifold $(M, \omega)$ as phase-space, which moreover satisfies a certain integrality condition.

In § 2.2.1 we define the prequantisation of $(M, \omega)$, which results in a Hilbert space $\mathcal{H}$. Next, in § 2.2.2 we introduce *polarisations* $P \subseteq T_{\mathbb{C}}M$ on $(M, \omega)$, and in § 2.2.3 we construct the true quantum phase-space $\mathcal{H}_P$ of the theory out this data.
In § 2.2.4 we define the Hitchin connection, which provides a canonical way to identify the spaces that correspond to different choices of polarisations, solving the problem of the geometric quantisation of $(M, \omega)$.
Finally, in § 2.2.5 we explain how to introduce the deformation parameter $\hbar$ into the geometric picture, thereby providing a direct link with deformation quantisation.

### 2.2.1 Prequantisation

The main idea of geometric quantisation is to realise the quantum phase-space $\mathbb{P}(\mathcal{H})$ underlying $(M, \omega)$ out of smooth sections of a complex line bundle $\pi\colon L \longrightarrow M$ over $M$. Given such a line bundle, one has a priori no way to define an Hermitian product on $C^\infty(M, L)$. However, if $h$ is a Hermitian metric on $L$, then it makes sense to consider the space of sections whose $h$-norm squared is integrable with respect to some measure on $M$. This is better, but there is still a priori no way to let a classical observable $f\colon M \longrightarrow \mathbb{R}$ act on the space of sections — apart from fibrewise multiplication — when the line bundle is not trivial. The next good idea is then to ask for a connection $\nabla$ on $L$, so that the first-order operators $\nabla_{X_f}\colon C^\infty(M, L) \longrightarrow C^\infty(M, L)$ of covariant derivative may act on the space of smooth sections, where $X_f$ is the Hamiltonian vector field of $f$.

This quick overview motivates the following few definitions and remarks.

**Definition 2.26.** A set of prequantum data on the symplectic manifold $(M, \omega)$ is a triple $(L, \nabla, h)$, where

1. $\pi\colon L \longrightarrow M$ is a smooth, complex line bundle over $B$.
2. $h \in C^\infty(M, L^* \otimes \overline{L}^*)$ is a smooth Hermitian metric on $L$.
3. $\nabla$ is a $h$-parallel connection in $L$ satisfying $F_\nabla = -i\omega$.

One also refers to $(L, \nabla, h)$ as a *prequantum triple*, to $\pi\colon L \longrightarrow M$ as the prequantum line bundle and to $\nabla$ as the prequantum connection. Here $F_\nabla \in A^2\big(M, \mathrm{End}(L)\big) \cong A^2(M, \mathbb{C})$ is the curvature of $\nabla$. The constant $i$ is necessary because $\omega$ is a real form, whereas $F_\nabla$ is purely imaginary. This is conceptually due to the fact that the local 1-forms defining the connection take values in the Lie algebra $\mathfrak{u}(1) \cong i\mathbb{R}$ of the unitary group $U(1)$, since $\nabla$ is parallel for the Hermitian metric $h$ on $L$:

$$\partial_X h(s, t) = h(\nabla_X s, t) + h(s, \nabla_X t), \qquad \text{for all } s, t \in C^\infty(M, L), X \in C^\infty(M, TM).$$

Moreover, the normalised curvature form $\frac{i}{2\pi}F_\nabla \in A^2(M,\mathbb{R})$ has integer periods, and thus $\frac{\omega}{2\pi}$ must be an *integer* form in order for a set of prequantum data to exist.

**Definition 2.27.** A differential form $\alpha \in \Omega^k(M,\mathbb{R})$ is integer if its de Rham cohomology class $[\alpha] \in H^k_{\mathrm{dR}}(M,\mathbb{R})$ lies in the image of the natural map

$$H^k(M,\mathbb{Z}) \longrightarrow H^k(M,\mathbb{R}) \cong H^k_{\mathrm{dR}}(M,\mathbb{R}),$$

defined for the ordinary singular cohomology.

We thus immediately see an obstruction to the geometric quantisation of $(M,\omega)$.

**Definition 2.28.** The symplectic manifold $(M,\omega)$ is prequantisable if $\frac{\omega}{2\pi}$ is an integer form.[6]

A set prequantum data may exist only if $(M,\omega)$ is prequantisable. The nice starting result of the theory is the fact that the converse is true.

**Theorem 2.2** (Konstant–Souriau–Weyl)**.**
*Let $M$ be a smooth manifold, and $\alpha \in A^2(M,\mathbb{R})$ a closed 2-form. If $\alpha$ is integer then there exists a triple $(L,\nabla,h)$ as in Def. 2.26, such that $\frac{1}{2\pi}F_\nabla = -i\alpha$.*

The Chern class of the prequantum line bundle $\pi\colon L \longrightarrow M$ is fixed. In particular, the line bundle $L$ is uniquely determined as a smooth complex line bundle. We now consider the vector space $C^\infty(M,L)$ of smooth sections of $L$, and try to construct a Hilbert space out of it. To this end, consider the *Liouville* measure on $(M,\omega)$, that is the measure associated to the volume form

$$\mu := \frac{\omega^{\wedge n}}{n!},$$

if $M$ is $2n$-dimensional. Then, if $s \in C^\infty(M,L)$ is a section, one can consider the real-valued function $h(s,s)\colon M \longrightarrow \mathbb{R}_{\geq 0}$, and consider the space of square-summable sections:

$$\left\{ s \in C^\infty(M,L) \middle| \int_M h(s,s)\mu < +\infty \right\}.$$

This space is pre-Hilbert, endowed with the inner product

$$\langle s \mid t \rangle := \int_M h(s,t)\mu \in \mathbb{C}. \tag{2.10}$$

**Definition 2.29.** The space $\mathcal{H}$ of the prequantisation of $(M,\omega)$ with respect to the prequantum data $(L,\nabla,h)$ is the Hilbert completion of the space of square-summable sections of $L$.

Hence $\mathcal{H}$ generalises the space $C^\infty(\mathbb{R}^{2n},\mathbb{C}) \cap L^2(\mathbb{R}^{2n}, dx_1 \dots dx_{2n})$ of square-integrable complex-valued functions for the $2n$-dimensional Lebesgue measure, which would correspond to a trivial line bundle over flat $2n$-dimensional space (cf. Rem. 2.9).

As noted above, one now has two basic types of differential operators acting on $\mathcal{H}$, associated to a smooth function $f \in C^\infty(M,\mathbb{C})$:

[6] One may also find in the literature that the prequantum condition is expressed as $F_\nabla = -2\pi i\omega$. In this case $[\omega]$ itself must be an integer class.

- The function multiplication $\mu_f \colon s \longmapsto fs$, an operator of order zero.
- The covariant derivative $\nabla_{X_f} \colon s \longmapsto \nabla_{X_f} s$, where $X_f = \{f, \cdot\}$ is the Hamiltonian vector field of $f$, an operator of order one.

We are admittedly ignoring some analytical issues here. Namely, these operators are a priori defined on $C^\infty(M, L)$. One should check whether they act on $\widetilde{\mathcal{H}}$, and then extend them to $\mathcal{H}$ by density. Even after doing that, it is not clear what their domain $\mathcal{D} \subseteq \mathcal{H}$ will be. At any rate, the procedure of quantisation that we are describing must be corrected in order to work: the space $\mathcal{H}$ is in a sense too big (see Rem. 2.9).

**Definition 2.30.** The prequantum operator attached to $f \in C^\infty(M, \mathbb{C})$ is

$$\widehat{f} \coloneqq \mu_f - i\nabla_{X_f} \in \mathrm{End}(\mathcal{H}).$$

If we think of the map $f \longmapsto \widehat{f}$ as a quantisation map $\mathcal{Q} \colon C^\infty(M, \mathbb{C}) \longrightarrow \mathrm{End}(\mathcal{H})$, in the sense of § A.4, then we might try to show that is satisfies some of the Dirac axioms listed there, up to a multiplication by Planck constant. For instance, it is clear that the map is $\mathbb{C}$-linear, and the function constant to 1 acts as the identity. Moreover, the strong version 4 of the compatibility of the Poisson structures holds.

**Proposition 2.4.** *One has* $\mathcal{Q}\big(\{f, g\}\big) = i\big[\widehat{f}, \widehat{g}\big]$ *for all* $f, g \in C^\infty(M, \mathbb{C})$.

*Proof.* Pick functions $f, g \in C^\infty(M, \mathbb{C})$. One has

$$\mathcal{Q}\big(\{f, g\}\big) = \mu_{\{f,g\}} - i\nabla_{X_{\{f,g\}}} = \{f, g\} - i\nabla_{\big[X_f, X_g\big]},$$

because $f \longmapsto X_f$ is a morphism of Lie algebras with the chosen convention (see (A.4)). Now, one has

$$\nabla_{\big[X_f, X_g\big]} = \big[\nabla_{X_f}, \nabla_{X_g}\big] - F_\nabla(X_f, X_g) = \big[\nabla_{X_f}, \nabla_{X_g}\big] + i\omega(X_f, X_g) = \big[\nabla_{X_f}, \nabla_{X_g}\big] + i\{f, g\}.$$

Hence, on the whole:

$$\mathcal{Q}\big(\{f, g\}\big) = 2\{f, g\} - i\big[\nabla_{X_f}, \nabla_{X_g}\big]$$

Let us now compute the commutator:

$$i\big[\widehat{f}, \widehat{g}\big] = i\big[\mu_f - i\nabla_{X_f}, \mu_g - i\nabla_{X_g}\big] = i\big[\mu_f, \mu_g\big] + \big[\mu_f, \nabla_{X_g}\big] + \big[\nabla_{X_f}, \mu_g\big] - i\big[\nabla_{X_f}, \nabla_{X_g}\big].$$

The first addend vanishes, since function multiplication is commutative. Hence all that is left is show is that

$$\big[\mu_f, \nabla_{X_g}\big] - \big[\mu_g, \nabla_{X_f}\big] = 2\{f, g\}.$$

Now, if $s \in C^\infty(M, L)$:

$$\Big[\mu_f, \nabla_{X_g}\Big] s = f\nabla_{X_g} s - \nabla_{X_g}(fs) = -df(X_g)s.$$

Hence

$$\Big[\mu_f, \nabla_{X_g}\Big] = -X_g(f) = -\{g, f\} = \{f, g\},$$

and similarly

$$\Big[\mu_g, \nabla_{X_f}\Big] = \{g, f\}.$$

□

One can also verify that the algebra of classical observables is preserved, according to axiom 2.

**Proposition 2.5.** *Let $f \in C^\infty(M, \mathbb{R})$ be a real-valued function, and $s, t \in C^\infty(M, L)$ two sections. One has*

$$\langle \mathcal{Q}(f)s \mid t\rangle = \langle s \mid \mathcal{Q}(f)t\rangle.$$

*Proof.* Let us fix $f, s, t$ as in the statement.

It is clear that $\langle fs \mid t\rangle = \langle s \mid ft\rangle$, because $\langle \cdot \mid \cdot \rangle$ is sesquilinear. We must then show that $\langle -i\nabla_{X_f} s \mid t\rangle = \langle s \mid -i\nabla_{X_f} t\rangle$, i.e.

$$\langle \nabla_{X_f} s \mid t\rangle + \langle s \mid \nabla_{X_f} t\rangle = 0.$$

In terms of integral, this means showing that

$$0 = \int_M h\Big(\nabla_{X_f} s, t\Big)\mu + \int_M h\Big(s, \nabla_{X_f} t\Big)\mu = \int_M X_f\Big(h(s,t)\Big)\mu,$$

Using the fact that $\nabla$ is $h$-parallel:

$$X_f\Big(h(s,t)\Big) = h\Big(\nabla_{X_f} s, t\Big) + h\Big(s, \nabla_{X_f} t\Big).$$

Introduce the Lie derivative $\mathcal{L}_{X_f}$ along the vector field $X_f$, which allows for computing derivatives of arbitrary differential forms. Using the generalised Leibnitz identity:[7]

$$\mathcal{L}_{X_f}\Big(h(s,t)\mu\Big) = \mathcal{L}_{X_f}\Big(h(s,t) \wedge \mu\Big) = \mathcal{L}_{X_f} \wedge \Big(h(s,t)\Big)\mu + h(s,t) \wedge \mathcal{L}_{X_f}(\mu).$$

The first addend is just the product $X_f\Big(h(s,t)\Big)\mu$ appearing in the integral. Indeed, $\mathcal{L}_{X_f}\Big(h(s,t)\Big)$ is the ordinary derivative $X_f\Big(h(s,t)\Big)$ of the nonnegative, smooth function $h(s,t)\colon M \longrightarrow \mathbb{R}_{\geq 0}$. The second term vanishes, because of the Leibnitz identity, and the fact that $\mathcal{L}_{X_f}(\omega) = 0$, $n!\mu = \omega^{\wedge n}$. Recall that $\mathcal{L}_{X_f}(\omega) = 0$ follows from the Cartan formula:

$$\mathcal{L}_{X_f}(\omega) = (d \circ \iota_{X_f} + \iota_{X_f} \circ d)(\omega) = d(\iota_{X_f}\omega) + \iota_{X_f}(d\omega) = -d^2 f = 0.$$

[7] That is $\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X(\alpha) \wedge \mathcal{L}_X(\beta)$ for all $X \in C^\infty(M, TM)$ and $\alpha, \beta \in A^\bullet(M, \mathbb{C})$.

Now, using again the Cartan formula

$$\mathcal{L}_{X_f}\Big(h(s,t)\mu\Big) = \Big(d\circ\iota_{X_f} + \iota_{X_f}\circ d\Big)\Big(h(s,t)\mu\Big) = d\Big(\iota_{X_f}\Big(h(s,t)\mu\Big)\Big),$$

since $h(s,t)\mu \in \Omega^{2n}(M,\mathbb{R})$ is closed, being a form of top degree.

One can finally conclude, using Stokes theorem:

$$\int_M X_f\Big(h(s,t)\Big)\mu = \int_M d\Big(\iota_{X_f}\Big(h(s,t)\mu\Big)\Big) = \int_{\partial M}\iota_{X_f}\Big(h(s,t)\mu\Big) = 0.$$

□

The constructions described so far has made some significant steps forward in the search for a solution to the problem of quantisation of § A.4. Unfortunately, there are major drawbacks, which explain the prefix "pre" in "prequantisation". From the mathematical viewpoint, the aberration can already be seen in the failure of the Dirac axiom 5 on irreducibility. From the viewpoint of quantum mechanics, the present construction violates Heisenberg's indeterminacy principle. To see why, let us apply the material of this section to a simple classical system: a free particle in three-dimensional space.

**Example 2.9** (Dirac prequantisation)**.**
The classical phase-space of a free point particle in three-dimensional space is $(T^*\mathbb{R}^3, \omega_{\text{can}})$, using the standard Hamiltonian viewpoint for which the momenta variables live in cotangent spaces.[8] The canonical symplectic form $\omega_{\text{can}} = \sum_i dq_i \wedge dp_i$ is exact. One has $\omega = d\lambda$, where $\lambda \coloneqq -\sum_i p_i dq_i$ is the so-called *Liouville potential*. In particular one has $[\omega] = 0$ in $H^2_{\text{dR}}(T^*\mathbb{R}^3,\mathbb{R})$, and thus our phase-space is prequantisable.

One can take the prequantum data $(L,\nabla,h)$ consisting of the trivial complex line bundle $L \coloneqq \mathbb{C}\times T^*\mathbb{R}^3 \longrightarrow T^*\mathbb{R}^3$, the translation invariant metric $h$ which is everywhere equal to the standard inner product $h(s,t) \coloneqq s\bar{t}$, and the prequantum connection

$$\nabla \coloneqq d - i\lambda.$$

In this context one has $C^\infty(T^*\mathbb{R}^3, L) \cong C^\infty(T^*\mathbb{R}^3,\mathbb{C})$, and

$$\mathcal{H} \cong L^2(\mathbb{R}^6, dqdp),$$

where $dqdp$ is the standard *Lebesgue measure* on $\mathbb{R}^6 \cong T^*\mathbb{R}^3$.

We now see why this is not acceptable from the quantum mechanical viewpoint. The classical observables $q_i, p_i$ do not Poisson commute: $\{q_i, p_i\} = 1$. This means that their quantisation $\widehat{q_i}, \widehat{p_i}$ correspond to quantum observables whose values cannot be measured simultaneously. In particular, it makes little sense for a wave function $\psi \in \mathcal{H}$ to depend on both positions and momenta. One would rather like to have a quantum phase-space like $\mathcal{H}_1 \cong L^2(Q, dq)$, where $Q \cong \mathbb{R}^3$ is the configuration space spanned by the positions, or $\mathcal{H}_2 \cong L^2(P, dp)$, where $P \cong \mathbb{R}^3$ is the space of momenta. These spaces correspond to the usual position and momentum representations for a wave function $\psi$, and they illustrate the general idea that one should only have functional dependence on a Lagrangian submanifold of the classical phase-space.

[8] This is in essence the dual viewpoint to Lagrangian mechanics. The two are related by the Legendre transform: [Arn89].

This remark shows that the prequantisation just proposed fails to recover the canonical quantisation of the free particle in flat space, which is an essential request for any viable quantisation procedure, as expressed in § A.4. We now turn to *polarisations*, in order to find a way to correct the space $\mathcal{H}$ by "cutting out" some directions.

### 2.2.2 Polarisations

We will now restrict the prequantum phase-space $\mathcal{H}$ by considering sections of the prequantum line bundle which are covariantly constant along chosen Lagrangian submanifolds of the classical phase-space, i.e. along submanifolds having maximally isotropic subspaces.

Let again $(M, \omega)$ be a symplectic manifold.

**Definition 2.31.** A polarisation in $(M, \omega)$ is a smooth, integrable, Lagrangian distribution $P \subseteq T_{\mathbb{C}}M$ inside the complexified tangent bundle to $M$, such that the dimension of $P \cap \overline{P} \subseteq T_{\mathbb{C}}M$ is constant.

The complexified tangent bundle is the tensor product $T_{\mathbb{C}}M \coloneqq TM \otimes_{\mathbb{R}} \mathbb{C}$ of vector bundles over $M$, where $\mathbb{C}$ denotes the trivial complex line bundle. One tacitly extends $\omega$ to the complexification by imposing $\mathbb{C}$-linearity, getting a new complex, close, nondegenerate 2-form $\omega \in C^\infty\big(M, \Lambda^2 T^*_{\mathbb{C}}M\big)$. A Lagrangian subspace $L \subseteq (T_{\mathbb{C}}M)_p$ is then a vector subspace such that $L = L^{\perp\omega_p}$, where $p \in M$. Its real dimension is equal to the dimension of $M$, or half the real dimension of $(T_{\mathbb{C}}M)_p$. Since the distribution has constant rank, the Frobenius theorem applies to show that $P$ is integrable if and only if it is involutive, i.e. closed for the Lie bracket of vector fields.

It follows from the definition 2.31 that the conjugate $\overline{P} \subseteq T_{\mathbb{C}}M$ of a polarisation is still a polarisation. There is a priori no condition on the dimension of the integrable subbundle $P \cap \overline{P} \subseteq T_{\mathbb{C}}M$, but the two extreme cases are so important that one introduces further terminology for them.

**Definition 2.32.** Let $P \subseteq T_{\mathbb{C}}M$ be a polarisation on the symplectic manifold $(M, \omega)$.

- If $P = \overline{P}$ then $P$ is said to be real.
- If $P \cap \overline{P} = (0)$ then $P$ is said to be pseudo-Kähler. If moreover the nondegenerate Hermitian form $i\omega \colon P \otimes \overline{P} \longrightarrow \mathbb{C}$ on $P$ is positive definite, then the polarisation is said to be Kähler.

The Hermitian form in the statement is defined at $p \in M$ by $(u, v) \longmapsto i\omega_p(u, \overline{v})$, for $u, v \in P_x \subseteq (T_{\mathbb{C}}M)_x$. Its radical is precisely $P \cap \overline{P}$. Hence a polarisation is real if and only if $i\omega = 0$, and pseudo-Kähler if and only if $i\omega$ is nondegenerate.

Notice that a real polarisation can be obtained from a smooth, integrable, Lagrangian distribution $R \subseteq TM$, by means of $P \coloneqq R \otimes_{\mathbb{R}} \mathbb{C}$. Conversely, if $P$ is a real polarisation, then $P$ is the complexification of $R = P \cap TM$, and thus considering distributions in the complexified tangent bundle allows for more general objects.

**Remark 2.7.** Geometrically, real polarisations are the same as Lagrangian foliations on $M$, up to integrating the real subbundle of $TM$ to Lagrangian submanifolds of $M$ (and one sometimes requires this foliation to be a smooth fibration).

**Remark 2.8.** The existence of global polarisations has topological implications which we will not delve into. It is also to deal with such issues that one works in the complexified tangent bundle.
For the sake of an example, there are no real polarisation on the symplectic sphere $(\mathbb{S}^2, \omega)$, where $\omega$ is the standard volume form. This is because a real polarisation in this case is just a smooth distributions of lines inside $T\mathbb{S}^2$, which can be given an orientation up to lifting it to a two-sheeted cover of $\mathbb{S}^2$. This cover is diffeomorphic to the 2-sphere, and it now carries a nowhere vanishing vector field: impossible.

**Remark 2.9.** In general, one calls a polarisation $P \subseteq T_{\mathbb{C}}M$ *admissible* if the subbundle $P + \overline{P} \subseteq T_{\mathbb{C}}M$ is integrable. This condition is automatically satisfied for real and Kähler polarisations, since in the former case one has $P + \overline{P} = P$, and in the latter $P + \overline{P} = T_{\mathbb{C}}M$. Still speaking in full generality, one says that a polarisation $P \subseteq T_{\mathbb{C}}M$ is *positive* if the Hermitian form $i\omega \colon P \otimes \overline{P} \longrightarrow \mathbb{C}$ is positive semidefinite. This is the case of Kähler polarisations by definition, and of real polarisations because in that case $i\omega = 0$, as remarked above.

The terminology for Kähler polarisations is justified as follows. If $I$ is a complex structure on $M$ that makes $(M, \omega, I)$ into a Kähler manifold, then one may consider the antiholomorphic tangent bundle $P \coloneqq T_{0,1}M \subseteq T_{\mathbb{C}}M$ as polarisation. Indeed, this distribution is smooth, and it is integrable if and only if $I$ is integrable — by the theorem of Newlander–Nirenberg — which is assumed. Moreover, the compatibility with $\omega$ makes it isotropic, and thus Lagrangian for dimension reasons. Then $P$ is pseudo-Kähler because

$$P \oplus \overline{P} = T_{0,1}M \oplus T_{1,0}M = T_{\mathbb{C}}M,$$

and finally the positive definiteness of $i\omega \colon T_{0,1}M \otimes T_{1,0}M \longrightarrow \mathbb{C}$ follows from the compatibility of $I$ with $\omega$, i.e. from the positivity of the Kähler metric $g = \omega\Big(\cdot, I(\cdot)\Big)$.

The converse is true.

**Proposition 2.6.** *If $P$ is a Kähler polarisation on $(M, \omega)$, then there exists a complex structure $I$ on $M$ such that $(M, \omega, I)$ is Kähler and $P = T_{0,1}M$.*

*Proof.* One defines the complex structure as the multiplication by $-i$ on $P$ and by $i$ on $\overline{P}$.
More precisely, let $x \in M$ be a point. Since $P \oplus \overline{P} = T_{\mathbb{C}}M$, we can decompose each tangent vector $v \in T_xM \subseteq (T_{\mathbb{C}}M)_x$ as $v = v' + v''$, with $v' \in P_x$ and $v'' \in \overline{P}_x$ uniquely determined. Since $v = \overline{v}$, one has $\overline{v'} + \overline{v''} = v' + v''$. Because $\overline{v'} \in \overline{P}_x$ and $\overline{v''} \in P_x$, it must be that $\overline{v'} = v''$ and $\overline{v''} = v'$, by the uniqueness of the decomposition of $v \in P_x \oplus \overline{P}_x$. Then we define $I_x \colon T_xM \longrightarrow T_xM$ via

$$I_x(v) \coloneqq i(v'' - v').$$

This is indeed an endomorphism of $T_xM$, since

$$\overline{I_x(v)} = -i(\overline{v''} - \overline{v'}) = i(\overline{v'} - \overline{v''}) = i(v'' - v') = I_x(v).$$

By definition, the complexification of $I$ yields automorphisms of the complexified tangent spaces for which $P$ is the eigenbundle of eigenvalue $-i$, and $\overline{P}$ the eigenbundle of eigenvalue $i$. This almost-complex structure is then integrable, since $P = T_{0,1}M$ is integrable by hypothesis.

Finally, $I$ is compatible with $\omega$; indeed, if $v = v' + v''$ and $w = w' + w''$ are tangent vectors at some point that we omit from the notation, then one has

$$\begin{aligned}
&\omega\Big(I(v), I(w)\Big) = \\
&\quad = \omega\Big( - iv' + iv'', -iw' + iw''\Big) = -\omega(v', w') + \omega(v', w'') + \omega(v'', w') - \omega(v'', w'') = \\
&\quad = \omega(v', w'') + \omega(v'', w'),
\end{aligned}$$

where in the last passage we used the fact that $P$ and $\overline{P}$ are $\omega$-isotropic. For the same reason, the last term is equal to $\omega(v' + v'', w' + w'') = \omega(v, w)$, as it was to be shown. The positivity of the bilinear symmetric form $g = \omega\Big(\cdot, I(\cdot)\Big)$ follows from the positivity of the Hermitian form $i\omega$. □

Before leaving the general theory of polarisations, exploiting them in § 2.2.3 to correct the prequantisation procedure of § 2.2.1, let us briefly discuss the case of flat space. This is relevant for the discussion of § 4.6.

**Example 2.10** (Linear polarisations on flat space)**.**
Let $M = T^*\mathbb{R}^n$, equipped with the standard symplectic structure $\omega_{\text{can}}$. There is in this case an important, distinguished class of polarisations on $M$: the constant ones. This makes sense because the tangent bundle $TM \cong M \times M$ is trivial, and thus a vector subspace $V \subseteq M \otimes \mathbb{C} \cong \mathbb{C}^{2n}$ defines the subbundle $P_V \subseteq T_{\mathbb{C}}M$ such that $P_V(x) = V$ for all $x \in M$. Such polarisations are said to be *linear*. The space of linear polarisations is then naturally identified with the Lagrange–Graßmann manifold $\Lambda(\mathbb{C}^{2n})$ of $\mathbb{C}^{2n}$, i.e. the compact manifold of complex dimension $\frac{1}{2}n(n+1)$ parametrising Lagrangian subspaces of $(\mathbb{C}^{2n}, \omega_{\text{can}})$, where we abusively denote $\omega_{\text{can}}$ the complexification of the canonical cotangent symplectic form.
We now set $\Lambda_+(\mathbb{C}^{2n})$ to be the subspace of positive linear polarisation. The group $\text{Sp}(2n, \mathbb{R})$ of linear symplectic transformations of $M$ acts on $\Lambda(\mathbb{C}^{2n})$, preserving $\Lambda_+(\mathbb{C}^{2n})$, and splitting the latter space into $n+1$ orbits, according to the rank of $i\omega_{\text{can}}$:

$$\Lambda_+(\mathbb{C}^{2n}) = \bigcup_{k=0}^{n} \Lambda_+^k(\mathbb{C}^{2n}).$$

The subset $\Lambda_+^0(\mathbb{C}^{2n})$ is closed, and consists of the real, linear polarisations. Indeed, if $i\omega_{\text{can}}$ vanishes on $P_V \otimes \overline{P_V}$ then one has $\overline{P_V} \subseteq P_V^{\perp\omega}$; but one also has $P_V^{\perp\omega} = P_V$, since $P_V$ is Lagrangian, and thus $\overline{P_V} = P_V$ by the equality of dimensions: $P_V$ is real. Conversely, if $P_V$ is real then $i\omega_{\text{can}}$ vanishes on $P_V \otimes \overline{P_V} = P_V \otimes P_V$.
On the opposite side, if $i\omega_{\text{can}} > 0$ then one has $P_V \cap \overline{P_V} = (0)$, since all vectors in the intersections are $\omega_{\text{can}}$-isotropic. Hence the polarisation is Kähler, by Def. 2.32, and the converse is essentially one way of rewriting the definition.

### 2.2.3 Quantisation

Consider a prequantisable symplectic manifold $(M, \omega)$, together with prequantum data $(L, \nabla, h)$ (as in § 2.2.1) and a polarisation $P \subseteq T_{\mathbb{C}}M$ (as in 2.2.2).

We now correct the prequantum phase-space $\mathcal{H}$, by only considering sections which are covariantly constant along the polarisation.

**Definition 2.33.** A section $s\colon M \longrightarrow L$ of the prequantum bundle is $P$-polarised if

$$\nabla_X s = 0 \qquad \text{for all} \qquad X \in C^\infty(M, P). \tag{2.11}$$

We denote $C^\infty_P(M, L)$ the space of smooth $P$-polarised sections of $L$.

**Remark 2.10.** The notation $X \in C^\infty(M, P)$ indicates that $X$ is a smooth section of the polarisation $P \subseteq T_{\mathbb{C}}M$. Such vector fields are said to be *tangent* or *adapted* to the polarisation. In particular, $X$ is a complex vector field on $M$, and the covariant derivative $\nabla_X$ is well defined.

We can now repeat the same construction as in § 2.2.1, but for polarised sections.

**Definition 2.34.** The space of the geometric quantisation of $(M, \omega)$ with respect to the polarisation $P$ is the Hilbert completion $\mathcal{H}_P$ of the pre-Hilbert space of square-integrable, $P$-polarised sections of $L$:

$$\left\{ s \in C^\infty_P(M, L) \,\middle|\, \int_M h(s, s) < +\infty \right\}.$$

The completion is taken with respect to the restriction of the Hermitian product (2.10) to $P$-polarised sections.

Notice that $\mathcal{H}_P \subseteq \mathcal{H}$ is a closed Hilbert subspace. We will speak of the *Kähler quantisation* of $(M, \omega)$ to refer to the geometric quantisation with respect to Kähler polarisations.

**Remark 2.11.** There are further issues to discuss at this point.

First, assume that the polarisation $P$ is real, and let $\mathcal{F}$ be a leaf of the associated Lagrangian foliation on $M$. Then one can consider the holonomy of any section of $L$ along a loop in $\mathcal{F}$, with respect to the prequantum connection $\nabla$. The restriction of $\nabla$ to the line bundle $L|_{\mathcal{F}}$ is flat, since the curvature is a multiple of the symplectic form, and $\mathcal{F}$ is a (maximal) isotropic submanifold. Hence the holonomy along loops based at $p \in \mathcal{F}$ defines a 1-dimensional representation $\rho_\nabla\colon \pi_1(\mathcal{F}, p) \longrightarrow \mathbb{C}^* \cong \mathrm{GL}_{\mathbb{C}}(L_p)$, i.e. a character of the fundamental group. If this character is not trivial on a loop class $[\gamma] \in \pi_1(\mathcal{F}, p)$, then any solution of (2.11) vanishes on $\mathcal{F}$, because of the equality $s = \rho_\nabla\big([\gamma]\big)s$.
One way out is to let $M_0 \subseteq M$ be the union of all leaves having trivial holonomies. This is the *Bohr–Sommerfeld* submanifold of $(M, \omega, P, L, \nabla)$, and then one may consider solutions of (2.11) which are concentrated on $M_0$. In the case we will consider in Chap. 4, however, the leaves of our real polarisations will be simply connected, in which case $M_0 = M$.

A second issue for real polarisations is that of integrability. Namely, the $h$-norm of a $P$-polarised section is constant along the leaves of $P$, since $\nabla$ is $h$-parallel:

$$X.h(s,s) = h\Big(\nabla_X s, s\Big) + h\Big(s, \nabla_X s\Big) = 0,$$

if $X \in C^\infty(M,P)$, and if $s$ is a solution of (2.11). In particular, the function $h(s,s)$ fails to be integrable on any non-compact leaf of $P$, and $s$ is not an $L^2$-section.
One way out of this is the introduction of $L$-valued half-forms on $M$. In the case we will consider in Chap. 4, however, there will be a canonical global transverse to the Lagrangian foliation defined by the real polarisation. In this case, the set of leaves is in natural correspondence to the transverse, which has a natural measure. One can thus replace the integral of a $P$-polarised over $M$ with an integral over the transverse.

A third issue is that the prequantum operators of Def. 2.30 need not preserve a given polarisation $P$, be it real of Kähler. This is one of the fundamental issues of geometric quantisation: one would like to have projective identifications among the Hilbert spaces which correspond to different choices of polarisations. We will present in § 2.2.4 a viable solution to this problem.

Let us discuss in more details the case of Kähler polarisations, which will be central in Chap. 4.

**Example 2.11** (Kähler quantisation)**.**
Consider a prequantisable Kähler manifold $(M,\omega,I)$, together with prequantum data $(L,\nabla,h)$. The complex structure $I$ on $M$ implicitly defines a Kähler polarisation via the antiholomorphic tangent bundle $P \coloneqq T_{0,1}M \subseteq T_{\mathbb{C}}M$ to $M$ (see 2.32), and we can define $\mathcal{H}_P$ as in Def. 2.34. Let us understand this space in more details.

First, every quasi-complex structure $I \in C^\infty(M, TM \otimes T^*M) \cong C^\infty\Big(M, \mathrm{End}(TM)\Big)$ on $M$ provides a splitting $A^1(M,\mathbb{C}) = A^{1,0}(M,\mathbb{C}) \oplus A^{0,1}(M,\mathbb{C})$, according to the direct sum $T^*_{\mathbb{C}}M \cong T^{1,0}M \oplus T^{0,1}M$.[9] Namely, one sets $A^{1,0}(M,\mathbb{C}) \coloneqq C^\infty(M, T^{1,0}M)$, and $A^{0,1}(M,\mathbb{C}) \coloneqq C^\infty(M, T^{0,1}M)$. This decomposition can now be extended to the prequantum connection $\nabla$, which one should now think as a differential operator

$$\nabla \colon C^\infty(M,L) \longrightarrow A^1(M,L) \cong C^\infty(M, T^*_{\mathbb{C}}M \otimes L)$$

of order one. One thus has $\nabla = \nabla^{1,0} + \nabla^{0,1}$, where in particular

$$\nabla^{0,1} \colon C^\infty(M,L) \longrightarrow A^{0,1}(M,L) \cong C^\infty(M, T^{0,1}M \otimes L).$$

The explicit formula for the $(0,1)$-part of the connection is

$$\nabla^{0,1} = \frac{1}{2}\Big(\mathrm{Id} + iI\Big)\nabla. \tag{2.12}$$

One can now upgrade $\pi \colon L \longrightarrow M$ to a *holomorphic* line bundle $(L,\overline{\partial})$, using $\overline{\partial} \coloneqq \nabla^{0,1}$ as Dolbeault operator. The integrability of this operator follows from the prequantum condition that $F_\nabla = -i\omega \in A^{1,1}(M,\mathbb{C})$, since

$$\overline{\partial}^2 = \Big(\nabla^{0,1}\Big)^2 = \Big(\nabla^2\Big)^{0,2} = (F_\nabla)^{0,2} = -i\omega^{0,2} = 0.$$

[9] Recall that $T^{1,0}M$ (resp. $T^{0,1}M$) is the eigenbundle of eigenvalue $i$ (resp. $-i$) for the $\mathbb{C}$-linear extension of the dual structure $I^* \in C^\infty\big(M, \mathrm{End}(T^*M)\big)$ to $T^*_{\mathbb{C}}M$.

Notice that now the connection $\nabla$ is tautologically upgraded to the Chern connection of the Hermitian line bundle $(L, \overline{\partial}, h)$. Moreover, by the very definition of $\nabla^{0,1}$, a section $s\colon M \longrightarrow L$ is $P$-polarised if and only if it lies in the kernel of $\overline{\partial}$, i.e. if and only if it is a holomorphic section. Hence in this case one has

$$C^\infty_P(M,L) \cong H^0(M,L) \coloneqq \mathrm{Ker}\left(\overline{\partial}\colon C^\infty(M,L) \longrightarrow A^{0,1}(M,L)\right).$$

Here $H^0(M,L)$ is the standard, cohomological-theoretic notation for the space of holomorphic sections of $(L,\overline{\partial}) \longrightarrow M$, which depends on the Kähler structure $I$. Then $\mathcal{H}_P$ is the completion of the space of square-summable holomorphic sections of $L$.

This construction is a much more satisfactory quantisation procedure, as the following example indicate.

**Example 2.12** (Dirac quantisation)**.**
Let us improve on Ex. 2.9 by adding a polarisation into account.

Consider again the exact symplectic manifold $(T^*\mathbb{R}^3, \omega_{\mathrm{can}} = d\lambda)$, with prequantum data $(L = \mathbb{C}\times T^*\mathbb{R}^3, \nabla = d - i\lambda, h = dz\otimes d\overline{z})$. The prequantum phase-space of the theory is $\mathcal{H} \cong L^2(\mathbb{R}^6, dqdp)$, where $(q,p)$ are the natural Darboux coordinates for $\omega_{\mathrm{can}}$. We now want to cut the number of variables in half, by means of a polarisation. To get the position representation, for which pure states are elements $\psi \in L^2(\mathbb{R}^3, dq)$, one must then ask for all sections to be constant along the vertical fibres of the cotangent bundle. It is a nice fact that this amounts to the choice of a real polarisation.

Set then $R_{(q,p)} \coloneqq T^*_q\mathbb{R}^3$, and check that this defines a real polarisation $P \coloneqq R\otimes_{\mathbb{R}} \mathbb{C}$ inside $T_{\mathbb{C}}(T^*\mathbb{R}^3)$. It is clear that the distribution is smooth and isotropic. Since $\mathrm{Dim}_{\mathbb{R}}\left(R_{(q,p)}\right) = 3$ for all $(q,p)\in T^*\mathbb{R}^3$, it is then Lagrangian. It is simple to show that it is integrable, because the vector sub-bundle $R\subseteq T(T^*\mathbb{R}^3)$ is the tangent distribution to a foliation in the phase-space, namely to the vertical foliation defined by the fibres of the canonical projection $\pi\colon T^*\mathbb{R}^3 \longrightarrow \mathbb{R}^3$. There is little to distinguish between the distribution and the foliation, in this case. We then define $\mathcal{H}_P \subseteq \mathcal{H}$ to be the completion of the space of smooth, square-integrable, $P$-polarised sections, according to Def. 2.34.
To be more explicit, let us look in more detail to the space of $P$-polarised sections, which is the same as that of sections which are covariantly constant along the directions of $R$.

Pick then a section $s\colon T^*\mathbb{R}^3 \longrightarrow L$, and consider the vector fields $\{\partial_{p_i}\}_i \subseteq C^\infty(T^*\mathbb{R}^3, R)$, which provide a trivialising frame of $R$. One has, for $k\in\{1,2,3\}$:

$$\nabla_{\partial_{p_k}} s = \partial_{p_k} s - i\lambda(\partial_{p_k})s = \partial_{p_k} s,$$

since $\partial_{p_k}$ is in the kernel of $\lambda = -\sum_i p_i dq_i$. Hence a $P$-polarised section is one satisfying the following first-order, linear homogeneous system of PDEs:

$$\partial_{p_k} s = 0, \qquad \text{for} \qquad k\in\{1,2,3\},$$

which is just saying that they are constant along the vertical directions. For example the constant section $r = 1$ would be a solution.

It is not always the case that covariantly constant sections are just sections which are constant along the polarisation. It happened here because we took care to choose a symplectic potential $\lambda$ that vanishes along the polarisation. Such a symplectic potential is said to be *adapted* to the polarisation. If we had taken $\lambda' \coloneqq \sum_i q_i dp_i$, thereby changing the prequantum connection to $\nabla' = d - i\lambda'$, then the condition for polarised sections would have read

$$\partial_{p_k} s = iq_k s, \qquad k = 1, 2, 3,$$

which admits e.g. the solution

$$r(q,p) = \exp\left(i \sum_j q_j p_j\right) = e^{iq\cdot p} \in \mathbb{C},$$

where $\cdot$ denotes the ordinary scalar product for the Euclidean space $\mathbb{R}^3$.

Regardless of the choice of symplectic potential, one can find a $h$-unitary, nowhere vanishing section $r\colon M \longrightarrow L$ which is $P$-polarised.[10] This is in particular a frame of $L$, and every other section $s \in C^\infty(M, L)$ can now be written $s = fr$ for a suitable function $f \in C^\infty(M, \mathbb{C})$. Now it really makes sense to identify sections of the trivial prequantum bundle with complex-valued functions on $M$. Moreover, a $P$-polarised section now correspond to functions which are plainly constant along the polarisation, because of the Leibnitz rule:

$$\nabla_X(fr) = \partial_X(f)r + \nabla_X r = \partial_X(f)r,$$

for all $X \in C^\infty(M, TM)$. Under this identification, one can finally write

$$\mathcal{H}_P \cong L^2(Q, \mathbb{C}, dq) \cap C^\infty(Q, \mathbb{C}),$$

recovering the good quantum phase-space.

It is now a small step to get to the operators of the so-called *canonical quantisation*, or *Dirac quantisation*, of flat space. Introduce back Planck constant $\hbar$, and consider the following operators, acting on $\mathcal{H}_P$:

$$\widehat{q_k} \coloneqq \mu_{q_k}, \qquad \widehat{p_k} \coloneqq -i\hbar\partial_{q_k}.$$

In this way, one recovers the $\hbar$-deformed canonical commutation relations:

$$\left[\widehat{q_k}, \widehat{q_l}\right] = 0 = \left[\widehat{p_k}, \widehat{p_l}\right],$$
$$\left[\widehat{q_k}, \widehat{p_l}\right] = \delta_{kl} i\hbar,$$

for $1 \leq k, l \leq 3$. The nontrivial commutator for the position and the momentum relative to the same direction are the mathematical way of expressing Heisenberg uncertainty principle. The operators $\widehat{q_i}, \widehat{p_i}$ have *continuous spectrum* equal to the whole of $\mathbb{R}$, and one can measure them in the state $\psi \in \mathcal{H}_P$ in order to get a real number lying in their spectrum, as far as the mathematical formalisation is concerned (see § A).

[10] Since $h$ is just the fibrewise standard Hermitian product on $\mathbb{C}$, being $h$-unitary is equivalent to take values in $U(1) \subseteq \mathbb{C}$.

To do some explicit dynamics, one could consider the classical observable

$$H(q,p) := \frac{1}{2}|p|^2 = \frac{1}{2}\sum_{j=1}^{3} p_j^2,$$

a.k.a. kinetic energy (for a particle of mass 1). The canonical quantisation of this function is then the quantum Hamiltonian

$$\widehat{H} := -\frac{\hbar^2}{2}\sum_{j=1}^{3}\partial_{q_j}^2 = -\frac{\hbar^2}{2}\Delta,$$

acting on $\mathcal{H}_P$ via a multiple of the standard Laplacian operator for the flat Riemannian metric.

This is not the end of the general theory of geometric quantisation. For example, one might introduce *metaplectic corrections* in order to correct for the spectra of the operators defined at the end of this procedure. This direction has not been pursued in the thesis. Rather, the original results of Chap. 4 consist in new constructions of the Hitchin connection, which we shall now introduce.

### 2.2.4 The Hitchin connection

In the previous sections § 2.2.1, § 2.2.2 and § 2.2.3 we explained how to attach a quantum phase-space $\mathcal{H}_P$ to the data of a prequantisable smooth manifolds $(M,\omega)$, a prequantum triple $(L,\nabla,h)$ and a polarisation $P \subseteq T_{\mathbb{C}}M$. The problem we face now is the following: the quantum theory should be independent of the chosen polarisation, quantum mechanics tells us. This means that one would like to have canonical isomorphisms $\mathbb{P}(\mathcal{H}_P) \cong \mathbb{P}(\mathcal{H}_{P'})$ for the projective spaces of two geometric quantisations of $(M,\omega)$ associated to different polarisations $P, P' \subseteq T_{\mathbb{C}}M$. This is because these projective spaces are the true quantum phase-spaces of the theory, according to Def. A.6.

To solve this problem in the context of the quantisation of compact Chern–Simons theory, [Hit90] and [ADW91] independently proposed the following solution.

Suppose there to exist a family $\{P_\tau\}_{\tau\in\mathcal{T}}$ of polarisations on the classical phase-space $(M,\omega)$ at hand, parametrised by some smooth manifold $\mathcal{T}$. Consider the trivial vector bundle $\pi\colon \mathcal{H}\times\mathcal{T} \longrightarrow \mathcal{T}$, and notice that the subspace $\mathcal{H}_{P_\tau} \subseteq \mathcal{H}$ lives in the fibre $\pi^{-1}(P_\tau)$. The program is then the following:

1. Show that those subspaces can be glued into a vector sub-bundle $\mathcal{V} \longrightarrow \mathcal{T}$, so that $\mathcal{V}_\tau = \mathcal{H}_{P_\tau}$ for all $P \in \mathcal{T}$.
2. Define a connection $\widehat{\nabla}$ on the vector bundle $\mathcal{V} \longrightarrow \mathcal{T}$.
3. Use the parallel transport $\mathrm{PT}_\gamma^{\widehat{\nabla}}\colon \mathcal{V}_{\gamma(0)} \longrightarrow \mathcal{V}_{\gamma(1)}$ of $\widehat{\nabla}$ along paths $\gamma\colon [0,1] \longrightarrow \mathcal{T}$, in order to provide isomorphisms $\mathcal{H}_{P_\tau} \cong \mathcal{H}_{P_{\tau'}}$ between different fibres of $\mathcal{V}$, where $\tau = \gamma(0)$, $\tau' = \gamma(1)$.[11]

[11] See Def. 2.45.

4. Show that $\widehat{\nabla}$ is projectively flat.

5. Use some topological property of the base $\mathcal{T}$ to conclude that the induced isomorphisms are canonical.

**Definition 2.35.** The connection $\widehat{\nabla}$ is called the Hitchin connection for the geometric quantisation of $(M, \omega)$ with respect to the family of polarisations $\{P_\tau\}_{\tau \in \mathcal{T}}$.

The first two points 1 and 2 are actually dealt with simultaneously, as follows. One looks for a connection $\widehat{\nabla}$ in the trivial bundle $\mathcal{H} \times \mathcal{T} \longrightarrow \mathcal{T}$ that preserves the subspaces $\mathcal{H}_{P_\tau} \subseteq \mathcal{H}$. If such a connection exists, then $\mathcal{V}$ is a smooth vector sub-bundle, and the restriction of $\widehat{\nabla}$ is what one is after. Since the prequantum bundle is trivial, one has the following formula:

$$\widehat{\nabla} = \nabla^T + u,$$

where $\nabla^T$ is the trivial connection, and $u$ is a one-form on the base taking values in endomorphisms of $\mathcal{H}$. The *Hitchin Ansatz* is to take $u \in A^1\Big(\mathcal{T}, \mathcal{D}(M, L)\Big)$: the one-form takes values in differential operators acting on smooth sections of the prequantum line bundle. The problem is now to choose $u$ so that the subspaces $\mathcal{H}_{P_\tau}$ are preserved.

One now has isomorphisms $\mathrm{PT}_\gamma^{\widehat{\nabla}} \colon \mathcal{H}_{P_{\gamma(0)}} \longrightarrow \mathcal{H}_{P_{\gamma(1)}}$, according to 3, but they a priori depend on the choice of the path $\gamma \colon [0,1] \longrightarrow \mathcal{T}$. If however $\widehat{\nabla}$ is projectively flat (see Def. A.9), as indicated by 4, then the induced isomorphisms

$$\mathbb{P}\Big(\mathrm{PT}_\gamma^{\widehat{\nabla}}\Big) \colon \mathbb{P}(\mathcal{V})_{P_{\gamma(0)}} \longrightarrow \mathbb{P}(\mathcal{V})_{P_{\gamma(1)}}$$

on the projectivisation $\mathbb{P}(\mathcal{V}) \longrightarrow \mathcal{T}$ of $\mathcal{V}$ will only depend on the homotopy class $[\gamma] \in \Pi_1(\mathcal{T})$ of the path $\gamma$. Finally, following the lead of the last item 5, if $\mathcal{T}$ is for example simply connected, then these isomorphisms will be totally canonical. One may then finally define the Hilbert space of the geometric quantisation $\mathcal{H}_M$ of $(M, \omega)$ as the isomorphism class of $\mathcal{H}_{P_\tau}$, for any $\tau \in \mathcal{T}$. This is the same a declaring that $\mathcal{H}_M$ is the space of $\widehat{\nabla}$-covariant constant sections of $\mathcal{V}$.

This is the final output of the geometric approach to quantisation, as far as this thesis is concerned.

### 2.2.5 The quantum level

Before leaving geometric quantisation, a last important remark is at hand. Namely, one may introduce the deformation parameter $\hbar$ into the theory, and consider semiclassical asymptotic expansions for $\hbar \longrightarrow 0$. This will also provide a bridge towards the deformation quantisation, as introduced in § 2.1.

Pick again a prequantisable smooth manifold $(M, \omega)$, prequantised by the triple $(L, \nabla, h)$. Let $k$ be a nonnegative integer, call it the *quantum level* of the theory, and replace the prequantum data by the new $(L^{\otimes k}, \nabla^{\otimes k}, h^{\otimes k})$, where $\pi \colon L^{\otimes k} \longrightarrow M$ is the $k$-fold tensor

power of $L$, and the remaining data are the naturally induced metric and connection in the line bundle $L^{\otimes k}$. One now has the new prequantisation condition

$$F_{\nabla^{\otimes k}} = kF_{\nabla} = -ik\omega.$$

It would have been the same to consider $k$-depending prequantum data $(L^{(k)}, \nabla^{(k)}, h^{(k)})$ on $(M, \omega)$ such that $F_{\nabla^{(k)}} = -i\omega_k$, where $\omega_k$ is a $k$-deformation of $\omega$, in this case just equal to $k\omega$. This is the viewpoint of § 4.2.4.

All the above material can be now be recast in those $k$-dependent terms, to define prequantum Hilbert spaces $\mathcal{H}^{(k)}$, and quantum Hilbert spaces $\mathcal{H}_P^{(k)}$.[12] There is one technical reason for doing this, which is allowing for prequantum bundles with more $P$-polarised sections, as the level $k$ grows. However, the most deep reason consists in considering asymptotic properties of the geometric quantisation $\mathcal{H}_P^{(k)}$, as $k \longrightarrow +\infty$. The link with deformation theory comes by introducing $\hbar \coloneqq k^{-1}$, so that the above limit is actually the semiclassical one (see § A.4). The following example will make this more precise, in the context of Kähler quantisation (2.11).

**Example 2.13** (Berezin–Toeplitz quantisation)**.**
Let $(M, \omega, I)$ be a prequantisable, compact Kähler manifold, $k \in \mathbb{Z}_{\geq 0}$ a quantum level, and $(L, \nabla, h)$ prequantum data. According to Ex. 2.11, the Hilbert space $\mathcal{H}_P^{(k)}$ of the Kähler quantisation of $(M, \omega, I)$ at level $k$ is the metric completion of the subspace of holomorphic sections of $L^{\otimes k}$,

$$C_P^{\infty}(M, L^{\otimes k}) = H^0\Big(M, L^{\otimes k}\Big),$$

within the space of square-integrable ones, where $P \coloneqq T_{0,1}M \subseteq T_{\mathbb{C}}M$. Similarly, the prequantum Hilbert space $\mathcal{H}^{(k)}$ is the completion of the full space of square-integrable section of $L^{\otimes k}$, so that one may write $\mathcal{H}^{(k)} \cong L^2(M, L^{\otimes k}, \mu)$, where $\mu \coloneqq \frac{\omega^{\wedge n}}{n!}$, and $\mathrm{Dim}_{\mathbb{R}}(M) = 2n$. One can now try to define a $k$-dependent quantisation map

$$\mathcal{Q}_k \coloneqq C^{\infty}(M, \mathbb{C}) \longrightarrow \mathrm{End}\left(\mathcal{H}_P^{(k)}\right),$$

by improving on the definition of the prequantum operators 2.30. One way of doing this is provided by *Toeplitz operators*.

To introduce them, denote by $\Pi_k \colon \mathcal{H}^{(k)} \longrightarrow \mathcal{H}_P^{(k)}$ the orthogonal projection onto the closed quantum subspace. Now, if $f \in C^{\infty}(M, \mathbb{C})$ is a smooth function, the operator $\mu_f$ will act on $\mathcal{H}^{(k)}$, but it need not preserve the space of holomorphic sections. However, the composition $T^{(k)}f \coloneqq \Pi_k \circ \mu_f \colon \mathcal{H}_P^{(k)} \longrightarrow \mathcal{H}_P^{(k)}$ is forcefully an operator acting on the quantum Hilbert space $\mathcal{H}_P^{(k)}$.

**Definition 2.36.** The operator $T^{(k)}f \in \mathrm{End}\left(\mathcal{H}_P^{(k)}\right)$ is the Toeplitz operator of level $k$ associated to the function $f \colon M \longrightarrow \mathbb{C}$. The assignment $T^{(k)} \colon C^{\infty}(M, \mathbb{C}) \longrightarrow \mathrm{End}\left(\mathcal{H}_P^{(k)}\right)$ is the Berezin–Toeplitz quantisation.

[12] It is important to note that here one does not necessarily have tensor or direct products, whence the heavier the notation.

One can show that these operators are a hidden version of (a $k$-deformed version of) the prequantum operators $\mathcal{Q}^{(k)}(f)$ of Def. 2.30. Namely, one can relate $T_g^{(k)}$ to $\mathcal{Q}^{(k)}(f)$, where $g \coloneqq f - \frac{1}{2k}\Delta f$ (see [Tuy87]). Moreover, some of the desired Dirac axioms of § A.4 are recovered, thanks to the following results.

**Theorem 2.3.** *One has* $\|T_f^{(k)}\| \leq \|f\|_\infty$ *for all* $f \in C^\infty(M,\mathbb{C})$, *where* $\|\cdot\|_\infty$ *is the* $\infty$*-norm, and* $\|\cdot\|$ *the operator norm with respect to the scalar product* (2.10). *Moreover*

$$\lim_{k\longrightarrow+\infty} \|T_f^{(k)}\| \longrightarrow \|f\|_\infty.$$

This guarantees that the Toeplitz operators are bounded, hence continuous. The limit indicates that the norm of of the quantum operator is semiclassically equal to that of the classical observable, which is coherent with Bohr correspondence principle.

The next result shows even more strikingly that the weakening of axiom 4 holds in this geometric context, thereby essentially providing a deformation quantisation of the classical Poisson algebra $A_0 \coloneqq C^\infty(M,\mathbb{C})$. The aim of this result is to compare the $\hbar$-deformed commutator $\frac{i}{\hbar}\left[T_f^{(k)}, T_g^{(k)}\right] = ik\left[T_f^{(k)}, T_g^{(k)}\right]$ with the Berezin–Toeplitz quantisation $T_{\{f,g\}}^{(k)}$ of the Poisson bracket of $f, g \in A_0$. One expects the difference to be infinitesimal for $\hbar \longrightarrow 0$, and this is precisely what happens for the operator norms.

**Theorem 2.4.** *Let* $f, g \in A_0$. *One has:*

$$\left\| ik\left[T_f^{(k)}, T_g^{(k)}\right] - T_{\{f,g\}}^{(k)} \right\| = O\left(\frac{1}{k}\right), \qquad k \longrightarrow +\infty.$$

These two theorems are quoted from [BMS94].

## 2.3 Moduli spaces of connections

Here we introduce the classical phase-spaces we are interested in: moduli spaces of flat connections on Riemann surfaces. We will neither delve into the general categorical definition of coarse and fine moduli spaces as representable functors, nor into the general question of actually constructing moduli space for a given classification problem. Rather, we point to the results in the literature that guarantee that the moduli spaces we consider exist, and that they have the correct geometric structure in order to try to quantise them.

Throughout this section $K$ is a connected compact Lie group, and $G = K_{\mathbb{C}}$ its complexification: it is a complex reductive group containing $K$ as maximal compact subgroup. The symbol $\Sigma$ will denote a smooth complex projective curve, a compact Riemann surface or a compact orientable smooth surface, depending on the context.

In § 2.3.1 we introduce the moduli space of $\mathcal{M}_{\mathrm{dR}}(\Sigma, G)$ flat $G$-connections on $\Sigma$, which is the classical-mechanical setup for the geometric quantisation results of Chap. 4. For the sake of a self-contained exposition, we also briefly introduce the moduli space $\mathcal{M}_{\mathrm{Dol}}(\Sigma, G)$ of $G$-Higgs bundles on $\Sigma$, as well as the character variety $\mathcal{M}_{\mathrm{B}}(\Sigma, G)$ of $\Sigma$, in the broader context of the non-Abelian Hodge correspondence on curves.

In § 2.3.2 we consider the particular case of unitary connections, i.e. connections for the compact group $K \subseteq G$. In § 2.3.3 we then recall the successful geometric quantisation of the moduli space of $C^\infty$, flat SU(2)-connections: this is a direct application of the setup of § 2.2 to a particular, interesting classical phase-space. The complexification of this construction is what we look at in Chap. 4.
In § 2.3.4 we introduce the moduli space $\mathcal{M}^*$ of meromorphic connections on holomorphically trivial vector bundles over $\Sigma = \mathbb{C}P^1$, which is the classical-mechanical setup for the deformation quantisation results of Chap. 3.
Finally, in § 2.3.5 we discuss the Schlesinger system after introducing the notion of holonomy/monodromy for flat connections. We also apply the material of § 2.1 to deformation-quantise the Schlesinger system, getting the Knizhnik–Zamolodchikov connection. The project detailed in Chap. 3 is a generalisation of this procedure.

### 2.3.1 Nonsingular connections

Let us fix a positive integer $n$, and consider the group $G \coloneqq \mathrm{GL}_n(\mathbb{C})$. The category of principal $G$-bundles on the nonsingular, complex projective curve $\Sigma$ is then equivalent to that of rank $n$, complex vector bundles on $\Sigma$. All the discussion of this section can be generalised to a generic complex reductive group by consistently replacing vector bundles with $G$-principal bundles.[13] Nonetheless, one must pay attention that there are a priori three types of principal $G$-bundles, or complex vector bundles over $\Sigma$: algebraic, holomorphic and smooth ones.

To clarify the distinction, let $\mathscr{O}_\Sigma$ be the sheaf of regular function on $\Sigma$ as a complex algebraic variety, $\mathscr{O}_\Sigma^{\mathrm{an}}$ its sheaf of holomorphic function as a complex manifold, and $C^\infty_\Sigma$ its sheaf of complex-valued smooth functions as a smooth manifold. One has inclusions $\mathscr{O}_\Sigma(U) \subseteq C^\infty_\Sigma(U)$ for all open sets $U \subseteq \Sigma$, making $C^\infty_\Sigma$ into an $\mathscr{O}_\Sigma$-module.
Now, because of the fact that $\Sigma$ is projective, the first two sheaves are related by the classical GAGA correspondence of [Ser56]. This means that the classification problem for algebraic vector bundles and connections on $\Sigma$ provides a moduli space which is in natural bijection to that for the same type of holomorphic objects on the underlying Riemann surface. More precisely, the category of coherent $\mathscr{O}_\Sigma$-modules on the smooth projective curve (e.g. algebraic vector bundles of finite rank) is equivalent to that of coherent $\mathscr{O}_\Sigma^{\mathrm{an}}$-modules (e.g. holomorphic vector bundles of finite rank) on the underlying Riemann surface. The equivalence is an essentially surjective, fully-faithful functor; this means precisely that the map induced on the moduli spaces parametrising isomorphism classes is bijective.
With this correspondence clarified, we will work in the holomorphic category, but abusively omit the superscript "an" from the notation.

On the whole, we will consider holomorphic vector bundles on $\Sigma$, i.e. locally-free sheaves of $\mathscr{O}_\Sigma$-modules, and smooth complex vector bundle on $\Sigma$, i.e. locally-free sheaves of $C^\infty_\Sigma$-modules. We will use the same notation for a vector bundle and its associated sheaf of sections.

**Definition 2.37.** A holomorphic $G$-connection on $\Sigma$ is a pair $(V, \nabla)$ consisting of a

[13] And forms taking values in endomorphism bundles with forms taking values in adjoint bundles.

holomorphic, rank $n$ vector bundle on $\Sigma$ and a first-order differential operator

$$\nabla\colon V \longrightarrow \Omega^1_\Sigma \otimes V,$$

satisfying the Leibnitz identity:

$$\nabla(fs) = df \otimes s + f\nabla s,$$

for any local section $s$ of $V$ and a holomorphic function $f$ on $\Sigma$.

The symbol $\Omega^p_\Sigma$ denotes the sheaf of holomorphic $p$-forms on $\Sigma$, i.e. differential forms of type $(p,0)$ with holomorphic coefficients. In a local holomorphic trivialisation $V|_U \cong \mathbb{C}^n \times U$ of $V$ over the open set $U \subseteq \Sigma$, one will have a formula

$$\nabla|_U = d - A,$$

where $A \in \mathrm{End}(\mathbb{C}^n) \otimes \Omega^1_\Sigma$ is the connection form. The curvature of the connection $\nabla$ is by definition

$$F_\nabla = \nabla^2 \in \Omega^2_\Sigma \otimes \mathrm{End}(V).$$

This differential form vanishes, for dimension reasons: there are no holomorphic 2-forms on $\Sigma$. Hence all holomorphic connections on $\Sigma$ are flat.

**Remark 2.12.** Before discussing stability conditions, thereby defining the moduli space we want to quantise, let us consider the equivalent smooth picture. First, if $V$ is a holomorphic vector bundle on $\Sigma$, then one may consider the underlying $C^\infty$ bundle

$$V^\infty := V \otimes_{\mathscr{O}_\Sigma} C^\infty_\Sigma$$

on $\Sigma$. By definition, a local section of $V^\infty$ on the open set $U \subseteq \Sigma$ can be written as a linear combination $t = \sum_i f_i s_i$, where $s_i\colon U \longrightarrow V$ are holomorphic sections, and $f_i\colon U \longrightarrow \mathbb{C}$ smooth functions. One can then extend $\nabla$ to a connection on the underlying smooth vector bundle by imposing the Leibnitz identity; namely, all terms of the equality

$$\nabla\left(\sum_i f_i s_i\right) = \sum_i df_i \otimes s_i + f_i \nabla s_i$$

are well defined. The result is a differential operator

$$\nabla\colon V^\infty \longrightarrow A^1_\Sigma \otimes V^\infty,$$

where $A^1_\Sigma$ is the sheaf of smooth, differential one-forms on $\Sigma$. This smooth connection is still flat, because the curvature $F_\nabla \in \mathrm{End}(V) \otimes A^2_\Sigma$ vanishes on holomorphic sections, and $\nabla$ is just defined by extending scalars.

Conversely, let $(V^\infty, \nabla)$ be a flat, smooth connection on $\Sigma$, i.e. a smooth complex vector bundle $\pi\colon V^\infty \longrightarrow \Sigma$ equipped with a flat connection $\nabla\colon V^\infty \longrightarrow V^\infty \otimes A^1_\Sigma$. Then in particular one has $\left(\nabla^{0,1}\right)^2 = \nabla^{0,2} = 0$, and so the differential operator

$$\overline{\partial}_{V^\infty} := \nabla^{0,1}\colon V^\infty \longrightarrow V^\infty \otimes A^{0,1}_\Sigma$$

defines a holomorphic structure on $V^\infty$, where $A^{p,q}_\Sigma$ denotes the sheaf of smooth differential forms of bidegree $(p,q)$ on $\Sigma$. One then defines $V := \mathrm{Ker}\left(\overline{\partial}_{V^\infty}\right) \subseteq V^\infty$ to be the holomorphic vector bundle defined by the Dolbeault operator $\overline{\partial}_{V^\infty}$ within $V^\infty$. Now, the restriction of $\nabla$ to $V$ (i.e. the $(1,0)$-part of $\nabla$) is by definition a holomorphic connection.

Now that the equivalence is clear, let us define stability conditions for holomorphic connections.

**Definition 2.38.** Let $\pi\colon V \longrightarrow \Sigma$ be a vector bundle over $\Sigma$. The slope of $V$ is the ratio

$$\mu(V) \coloneqq \frac{\mathrm{Deg}(V)}{\mathrm{Rk}(V)},$$

where $\mathrm{Deg}(V)$ is the degree of $V$, and $\mathrm{Rk}(V)$ the rank of $V$.

Recall that the degree of the vector bundle $V$ only depends on its smooth structure, and it is defined as follows. One has a short exact sequence of sheaves on $\Sigma$:

$$(0) \longrightarrow \mathbb{Z} \longrightarrow \mathscr{O}_\Sigma \longrightarrow \mathscr{O}_\Sigma^* \longrightarrow (1).$$

Here $\mathbb{Z}$ is the constant sheaf associated to the ring $\mathbb{Z}$, and $\mathscr{O}_\Sigma^*$ is the sheaf of nowhere vanishing holomorphic functions on $\Sigma$. The first nontrivial arrow is the natural embedding of $\mathbb{Z}$ into the sheaf of holomorphic functions on $\Sigma$, and the surjective morphism of sheaves $\mathscr{O}_\Sigma \longrightarrow \mathscr{O}_\Sigma^*$ is given by the exponentiation $f \longmapsto e^{2\pi i f}$ of local holomorphic functions $f$ on $\Sigma$. For general cohomological-algebraic reasons, this short sequence defines a long exact sequence in Čech cohomology, and one in particular has a coboundary map

$$\delta\colon \check{\mathrm{H}}^1(\Sigma, \mathscr{O}_\Sigma^*) \longrightarrow \check{\mathrm{H}}^2(\Sigma, \mathbb{Z}) \cong \mathbb{Z}.$$

Now one recalls that the first cohomology group $\check{\mathrm{H}}^1(\Sigma, \mathscr{O}_\Sigma^*)$ classifies line bundles over $\Sigma$. Namely, a Čech 1-cocycle $\{\varphi_{ij}\colon U_i \cap U_j \longrightarrow \mathbb{C}^* \cong \mathrm{GL}(1,\mathbb{C})\}_{i,j\in I}$ associated to an open cover $\{U_i\}_{i\in I}$ of $\Sigma$ will define transition functions for a line bundle $\pi\colon L \longrightarrow \Sigma$ which is trivial over the open sets $U_i$, and its isomorphism class will be provided by adding all possible Čech 1-coboundaries. Then, by definition, the degree of $L$ is the integer $\mathrm{Deg}(L)$ associated to its cohomology class under the map $\delta$ above. Finally, the degree of a generic vector bundle $V$ is the degree $\mathrm{Deg}\Big(\det(V)\Big) \in \mathbb{Z}$ of its determinant line bundle.

**Definition 2.39.** Let $(V, \nabla)$ be a holomorphic $G$-connection on $\Sigma$. A subconnection of $(V, \nabla)$ is a vector subbundle $W \subseteq V$ such that $\nabla(W) \subseteq W \otimes \Omega^1_\Sigma$, endowed with the restricted connection $\nabla|_W$. The connection $(V, \nabla)$ is said to be:

- semistable, if one has $\mu(W) \leq \mu(V)$ for all subconnections such that $W \neq (0)$.
- stable, if one has $\mu(W) < \mu(V)$ for all subconnections such that $(0) \neq W \neq V$.
- polystable, if it is a direct sum of stable connections of one and the same slope.

**Remark 2.13.** This definition is given in this full generality in order to match it up with that for $G$-Higgs bundles and $G$-bundles. In this context, however, there are many simplifications.

The basic remark is that the degree of a vector bundle carrying a flat connection vanishes. Hence a holomorphic connection $(V, \nabla)$ is carried by a vector bundle of slope zero, as well as for all its nontrivial subconnections. Hence all holomorphic connections are semistable, and the only stable ones are those which are irreducible, i.e. those without nontrivial subconnections. Similarly, a connection is polystable if and only if it is semisimple. The same is true if one replaces "holomorphic" with "smooth and flat".

We now consider the moduli spaces $\mathcal{M}^{\mathrm{st}}_{\mathrm{dR}}(\Sigma, G)$ (resp. $\mathcal{M}_{\mathrm{dR}}(\Sigma, G)$) of isomorphism classes of stable (resp. polystable), holomorphic $G$-connections on $\Sigma$. A stream of result relates this moduli space — or rather the underlying moduli stack — to two other classification problems associated to the pair $(\Sigma, G)$, and show that it has the structure of a hyper-Kähler manifold. Let us now briefly review these results, starting from the definition of Higgs bundles.

**Definition 2.40.** A Higgs bundle on $\Sigma$ is a pair $(V, \Phi)$, consisting of a holomorphic vector bundle $V$ on $\Sigma$ and a Higgs field $\Phi$ on $V$. In turn, a Higgs field on $V$ is an $\mathrm{End}(V)$-valued holomorphic 1-form on $\Sigma$:

$$\Phi \in H^0\Big(\Sigma, \Omega^1_\Sigma \otimes \mathrm{End}(V)\Big).$$

A Higgs subbundle of $(V, \Phi)$ is a vector subbundle $W \subseteq V$ such that $\Phi(W) \subseteq W \otimes \Omega^1_\Sigma$. The Higgs bundle $(V, \Phi)$ is said to be

- semistable, if one has $\mu(W) \leq \mu(V)$ for all Higgs subbundles such that $W \neq (0)$.
- stable, if one has $\mu(W) < \mu(V)$ for all Higgs subbundles such that $(0) \neq W \neq V$.
- polystable, if it is a direct sum of stable Higgs bundles of one and the same slope.

We denote $\mathcal{M}^{\mathrm{st}}_{\mathrm{Dol}}(\Sigma, G)$ and $\mathcal{M}_{\mathrm{Dol}}(\Sigma, G)$ the moduli spaces of isomorphism classes of stable and polystable Higgs bundles of slope zero, respectively.

**Remark 2.14.** For the sake of generality, we shall also express the definition in terms of principal $G$-bundles. In this language, a $G$-Higgs bundle on $\Sigma$ is a pair $(P, \Phi)$, where $P$ is a principal $G$-bundle and $\Phi$ a section of $\mathrm{Ad}(P) \otimes K$, where $K$ is the canonical line bundle of the Riemann surface. Since $G$ is a classical reductive group, such pairs $(P, \Phi)$ correspond to triples $(V, \theta, \Phi)$, where $V$ is a holomorphic vector bundle, $\theta$ a reduction of the structure group of $V$ from $\mathrm{GL}_n(\mathbb{C})$ to $G$ (vacuous in the context of this section) and $\Phi$ is a $K$-twisted endomorphism of $V$ which is compatible with $\theta$.

In Chap. 4 we will pursue the viewpoint of vector bundles, for the structure group $\mathrm{SL}(2, \mathbb{C})$: a $\mathrm{SL}(2, \mathbb{C})$-Higgs bundle is a rank 2 holomorphic vector bundle $V$ with trivial determinant, together with a holomorphic 1-form $\Phi$ that takes values in traceless endomorphisms $\mathrm{End}_0(V) \subseteq \mathrm{End}(V)$.

Higgs fields were defined in [Hit87a]. The existence of the moduli space $\mathcal{M}_{\mathrm{Dol}}(\Sigma, G)$ was proved by Hitchin in the case of rank two, and by Simpson [Sim92; Nit91] in the general case. Simpson extended these result to an arbitrary reductive complex group in [Sim94]. The upshot is the following.

**Theorem 2.5** (Nitsure, Simpson)**.** *The moduli space $\mathcal{M}_{\mathrm{Dol}}$ is a quasi-projective algebraic variety, containing $\mathcal{M}^{\mathrm{st}}_{\mathrm{Dol}}$ as smooth locus.*

In [Hit87a] it is also shown that $\mathcal{M}_{\mathrm{Dol}}$ is in natural correspondence with the space of (equivalence classes of) solutions of a certain system of differential equations on $\Sigma$.

Namely, one considers a smooth complex vector bundle $\pi\colon V^\infty \longrightarrow \Sigma$, together with a smooth Hermitian metric $h$. The *Hitchin equations* read

$$\begin{cases} F_A + \left[\Phi, \Phi^*\right] & = 0 \\ \overline{\partial}_A \Phi = 0, \end{cases} \tag{2.13}$$

where $A$ is a $h$-unitary connection on $V^\infty$, $\overline{\partial}_A \coloneqq A^{0,1}$ the holomorphic structure on $V^\infty$ associated to $A$, and $\Phi$ a smooth, $\mathrm{End}(V^\infty)$-valued $(1,0)$-form on $\Sigma$. One accordingly introduces a moduli space $\mathfrak{M} = \mathfrak{M}(\Sigma, G)$ of pairs $(A, \Phi)$ that solve (2.13), up to gauge-equivalence.

**Theorem 2.6** (Hitchin, Simpson)**.**
*The moduli space $\mathfrak{M}$ of solutions to Hitchin equations (2.13), up to equivalence, is the hyper-Kähler quotient of an infinite dimensional vector space, with (2.13) expressing the vanishing of the hyper-Kähler moment map. Moreover, this hyper-Kähler manifold is isomorphic to $\mathcal{M}_{\mathrm{Dol}}$ in one complex structure coming from the Kähler sphere of $\mathfrak{M}$.*

The natural map $\mathfrak{M} \longrightarrow \mathcal{M}_{\mathrm{Dol}}$ is obtained as follows. To a solution $(A, \Phi)$ of (2.13) on the smooth vector bundle $V^\infty$, one associates the holomorphic vector bundle $V \coloneqq \mathrm{Ker}\left(\overline{\partial}_A\right) \subseteq V^\infty$, so that now $\Phi$ is tautologically a holomorphic section of $\Omega^1_\Sigma \otimes \mathrm{End}(V)$, i.e. a $G$-Higgs field. The theorem is there to prove the nontrivial statements that this correspondence is well defined on isomorphism classes, and that it induces a bijection with the polystable part of the Dolbeault moduli space.

Similarly, one can relate the hyper-Kähler moduli space $\mathfrak{M}$ to that of polystable (i.e. completely reducible) algebraic/holomorphic connections on $\Sigma$ (see [Cor88], as well as [Don87], which is a companion paper to [Hit87a]).

**Theorem 2.7** (Corlette, Donaldson)**.**
*The hyper-Kähler moduli space $\mathfrak{M}$ is isomorphic to $\mathcal{M}_{\mathrm{dR}}$ in a different complex structure coming from the Kähler sphere of $\mathfrak{M}$.*

A more precise statement is that a flat connection $(V^\infty, \nabla)$ on $\Sigma$ is polystable, then $V$ supports a *harmonic metric*. The equations for such metrics are then related to (2.13).

In this case the natural bijection $\mathfrak{M} \cong \mathcal{M}_{\mathrm{dR}}$ is constructed as follows. To a solution $(A, \Phi)$ of (2.13) on a smooth vector bundle $V^\infty$ one associates the smooth $G$-connection $\nabla = A + \Phi + \Phi^*\colon V^\infty \longrightarrow V^\infty \otimes A^1_\Sigma$. The first equation of (2.13) then assures that $\nabla$ is flat, and thus its gauge class is a point of $\mathcal{M}_{\mathrm{dR}}$. Then [Don87] shows that any irreducible, flat connection on $\Sigma$ is gauge-equivalent to one of the form $\nabla = A + \Phi + \Phi^*$, where $(A, \Phi)$ solves (2.13).

One gets by composition a diffeomorphism $\mathcal{M}_{\mathrm{Dol}} \cong \mathcal{M}_{\mathrm{dR}}$, which is called the non-Abelian Hodge correspondence. This terminology was introduced by Simpson, and it is motivated by cohomological reasons (see [Sim92; Sim94; Sim97]). In brief, on the compact Kähler manifold $\Sigma$ one has three different cohomological theories named after Betti, de Rham and Dolbeault, defined for the Abelian group $\mathbb{C}^* = \mathrm{GL}_1(\mathbb{C})$; the moduli spaces $\mathcal{M}_{\mathrm{dR}}, \mathcal{M}_{\mathrm{Dol}}$ just introduced then correspond to the non-Abelian analogue of the latter two.

What is left is the Betti viewpoint, which provides a third, different complex algebraic avatar of the (coarse moduli space underlying the) first cohomology group $\check{\mathrm{H}}^1(\Sigma, G)$. This is obtained by considering the moduli space of $G$-representations of the fundamental group of $\Sigma$, up to conjugation. We thus set

$$\mathcal{M}_{\mathrm{B}}(\Sigma, G) \coloneqq \mathrm{Hom}\left(\pi_1(\Sigma), G\right)\big/G, \tag{2.14}$$

which is also called the *$G$-character variety* of $\Sigma$. Notice that we omit the base point from the fundamental group, for the following reason. If $p, q \in \Sigma$ are two points, then there is an isomorphism $\varphi_\mu \colon \pi_1(\Sigma, p) \longrightarrow \pi_1(\Sigma, q)$ for all choice of continuous path $\mu \colon [0,1] \longrightarrow \Sigma$ that starts at $p$ and ends at $q$. Namely, if $[\gamma]$ is the homotopy class of a loop $\gamma$ based at $p$, then one sets

$$\varphi_\mu([\gamma]) \coloneqq [\mu^{-1} * \gamma * \mu],$$

where $*$ denotes path concatenation (written from left to right), and $\mu^{-1}$ denotes the path $\mu$ traced backwards. If one chooses a second path $\mu'$ that joins $p$ to $q$, then the composition $\varphi_{\mu'}^{-1} \circ \varphi_\mu \colon \pi_1(\Sigma, p) \longrightarrow \pi_1(\Sigma, p)$ results in an inner automorphism of the fundamental group at $p$, namely the conjugation by the loop $\mu^{-1} * \mu'$. Such automorphism are turned into inner automorphisms of $G$ by any representation $\rho \colon \pi_1(\Sigma, p) \longrightarrow G$, and thus the class $[\rho] \in \mathcal{M}_B$ does not depend on them.

The variety $\mathcal{M}_B$ is constructed as the GIT quotient of a complex affine variety, with respect to the action of the reductive group $G$ (see [MFK94]). Namely, if one let $g$ be the genus of $\Sigma$, then one has a presentation

$$\mathrm{Hom}\left(\pi_1(\Sigma), G\right) \cong \left\{ (A_i, B_i)_{1 \leq i \leq g} \,\middle|\, \prod_i \left[A_i, B_i\right] - I = 0 \right\} \subseteq G^{2g},$$

where $I \in G$ is the identity matrix. This is an affine variety within $G^{2g}$, cut out by algebraic equations, and its quotient with respect to the $G$-action

$$g.(A_i, B_i)_i \coloneqq (gA_ig^{-1}, gB_ig^{-1})_i$$

is what one is after. One can show that the stable points for this action correspond to irreducible representations, that the polystable ones correspond to semisimple representations, and that all points are semistable (see [Ric88]). This is in full accordance with the terminology for flat connections, as it should, because of the following standard result.

**Theorem 2.8.** *The map $\nu \colon \mathcal{M}_{\mathrm{dR}} \longrightarrow \mathcal{M}_{\mathrm{B}}$ that sends a flat $G$-connections to its monodromy data is a bijection that respects polystable and stable points. It induces a non-algebraic biholomorphism $\mathcal{M}_{\mathrm{dR}}^{\mathrm{st}} \cong \mathcal{M}_{\mathrm{B}}^{\mathrm{st}}$.*

The application $\nu$ may be called the $C^\infty$ Riemann–Hilbert correspondence, of which we will see an explicit instance in § 4.2.1 for a curve $\Sigma$ of genus one and $G = \mathrm{SL}_2(\mathbb{C})$. Let us just recall here that to a flat, nonsingular $G$-connection $\nabla$ one associates its holonomy representation $\nu(\nabla) \colon \pi_1(\Sigma) \longrightarrow G$, and that this data reconstructs the connection entirely (see § 2.3.5 for more details on holonomy, where we will also introduce Deligne's Riemann–Hilbert correspondence for logarithmic connections on the sphere).

Conversely, to an $n$-dimensional complex representation $\rho\colon \pi_1(\Sigma) \longrightarrow \mathrm{GL}(\mathbb{C}^n)$ one can associate a flat connection $(V_\rho, \nabla_\rho)$ on $\Sigma$, as follows. The *universal cover* $\widetilde{\Sigma} \longrightarrow \Sigma$ is a principal $\pi_1(\Sigma)$-bundle, with the fundamental group acting freely on the right via deck transformations; one then considers the vector bundle on $\Sigma$ associated to the representation $\rho$, whose total space is the quotient

$$V_\rho \coloneqq \widetilde{\Sigma} \times_\rho \mathbb{C}^n = \widetilde{\Sigma} \times \mathbb{C}^n \Big/ \pi_1(\Sigma),$$

where $[\gamma] \in \pi_1(\Sigma)$ acts diagonally on the right as $[\gamma].(p, v) \coloneqq \Big(p.[\gamma], \rho([\gamma])^{-1}.v\Big)$. The clutching maps on this vector bundle are actually constant: it is a *local system* on $\Sigma$. Hence it makes sense to define a connection $\nabla_\rho\colon V_\rho \longrightarrow A^1_\Sigma \otimes V_\rho$ on $V_\rho$ by declaring that $\nabla|_{U_i} = d$ for local trivialisations $V_\rho|_{U_i} \cong \mathbb{C}^n \times U_i$ of $V_\rho$ on an open cover $\{U_i\}_{i\in I}$ of $\Sigma$ such that the transition maps $g_{ij}\colon U_i \cap U_j \longrightarrow G$ are constant for all $i, j \in I$. These locally trivial connections glue to a global connection because the terms $g_{ij}^{-1} dg_{ij}$ of the local gauge transformations vanish all. This connection is tautologically flat, because its connection forms vanish in the given trivialisation. Hence one can associate a holomorphic connection to all representation of $\pi_1(\Sigma)$ (the Narasimhan–Seshadri theorem that we recall in 2.10 exploits this fact).

**Remark 2.15.** The Betti description $\mathcal{M}_\mathrm{B}$ of the hyper-Kähler moduli space provides the most explicit presentation of its underlying smooth manifold. It also shows that the moduli space is trivial if the genus $g$ of $\Sigma$ is zero, i.e. when $\Sigma \cong \mathbb{C}P^1$. In that case one can still get nontrivial moduli spaces by allowing for singular objects, that is for *meromorphic* connections and Higgs fields: see § 2.3.4. This is precisely the setup for the deformation quantisation results of Chap. 3.
Still speaking of lower genus, the case of genus $g = 1$ leaves no stable representations, because in that case one deals with commuting pair of matrices; it is then better to consider polystable objects, i.e. semisimple ones. This is the situation which is relevant to the geometric quantisation results of Chap. 4. The classification of polystable objects up to isomorphism is the same as the classification of semistable ones up to a weaker equivalence relation, called $S$-equivalence, where "S" stays for "semisimple". This is because in any $S$-equivalence class there exists a polystable object, unique up to isomorphism, and because the notion of $S$-equivalence is the same as that of isomorphism on polystable objects.
To clarify what $S$-equivalence is about, we consider holomorphic vector bundles. One can show that any semistable holomorphic vector bundle $V$ on $\Sigma$ admits a Jordan–Hölder filtration

$$(0) \subseteq W_1 \subsetneq \cdots \subsetneq W_k = V,$$

where every quotient $W_l \big/ W_{l-1}$ is stable of slope $\mu(W_l \big/ W_{l-1}) = \mu(V)$.[14] Thus $V$ is a filtered vector space in the sense of § 2.1.2, and the associated graded $\mathrm{gr}(V) = \bigoplus_i W_i \big/ W_{i-1}$ is defined as a holomorphic vector bundle on $\Sigma$. One can show that the isomorphism class of $\mathrm{gr}(V)$ is uniquely determined by $V$, although the Jordan–Hölder filtration need not be unique. Two semistable vector bundles on $\Sigma$ are said to be $S$-equivalent if their associated graded are isomorphic.

[14] This is a variation of the Harder–Narasimhan filtration of an arbitrary vector bundle, introduced in [HN75].

If now $V$ is polystable, then it is by definition a semistable vector bundle provided with a grading in stable vector bundles. Up to picking an order, the associated filtration satisfies the above conditions, and thus the isomorphism class of the associated graded $\mathrm{gr}(V)$ uniquely identifies the isomorphism classes of the direct addends of $V$. This fixes the isomorphism class of $V$ among the polystable vector bundles.

### 2.3.2 Nonsingular unitary connections

We now replace the complex reductive group $G$ by the compact maximal subgroup $K$ that it complexifies. The chief case we consider here is that of $K = U(n)$, whose complexification is the group $G = \mathrm{GL}_n(\mathbb{C})$ of the previous section § 2.3.1. A principal $K$-bundle is then the same as a vector bundle equipped with an Hermitian metric. Assume moreover in this sections that the genus $g$ of $\Sigma$ is at least two.

In this case there are only two different moduli spaces to consider, as far as algebraic structure is concerned. First, there is the moduli space of *unitary* representations of the fundamental group of $\Sigma$, up to equivalence. This is the quotient

$$\mathrm{Char}(\Sigma, K) \coloneqq \mathrm{Hom}\left(\pi_1(\Sigma), K\right)\Big/K.$$

The $C^\infty$ Riemann–Hilbert correspondence identifies this space with that of flat $K$-connections on $\Sigma$, and it makes little sense to distinguish the two. One can still show that the irreducible part is a smooth manifold. In this case there is however no natural way to define a complex algebraic structure on the quotient, since $K$ is not a complex variety.[15]
The way around this comes from the second moduli space we consider: it is $\mathrm{Bun}_G(\Sigma, 0)$, parametrising equivalence classes of degree zero, algebraic/holomorphic principal $G$-bundles on $\Sigma$, i.e. holomorphic, degree zero, rank $n$ vector bundles $\pi\colon V \longrightarrow \Sigma$. One defines the slope of such a bundle as in Def. 2.38, which is always zero in this context of vanishing degree. Next, the definition of stability is given in accordance to Def. 2.39 and 2.40, only that one now considers *all* vector subbundles, and not just the ones fixed by a $G$-connection or a $G$-Higgs field. One then has a moduli space $\mathrm{Bun}_G^{\mathrm{st}}(\Sigma, 0)$ of isomorphism classes of stable, holomorphic $G$-bundles on $\Sigma$ of degree zero, as well as the following result (see [Mum63]).

**Theorem 2.9** (Mumford)**.**
*The moduli space* $\mathrm{Bun}_G^{\mathrm{st}}(\Sigma, 0)$ *has the structure of a nonsingular, quasiprojective variety over* $\mathbb{C}$*.*

The link between this moduli space and that of irreducible, flat $K$-connections is provided by the theorem of Narasimhan and Seshadri (see [NS65]). Recall from the previous section that a representation $\rho$ of $\pi_1(\Sigma)$ with values in $G = K_{\mathbb{C}}$ provides a flat connection $(V_\rho, \nabla_\rho)$ on $\Sigma$. Hence a unitary representation $\rho\colon \pi_1(\Sigma) \longrightarrow K \subseteq G$ does the same, and one may just consider the associated holomorphic vector bundle $V_\rho$ on $\Sigma$, forgetting about the fact that it carries a holomorphic connection $\nabla_\rho$ (but noticing that this forces $\mathrm{Deg}(V_\rho) = 0$).

[15] However the complex structures of $\Sigma$ does provide a complex structure on the moduli space of flat connections: see § 2.3.3.

**Theorem 2.10** (Narasimhan–Seshadri)**.**
*A degree zero, holomorphic vector bundle on $\Sigma$ is stable if and only if it comes from an irreducible, unitary representation of $\pi_1(\Sigma)$.*

This theorem fills in the nontrivial details of showing that the association $\rho \longmapsto V_\rho$ induces a well defined map on the moduli spaces, and it sends $K$-conjugacy classes of irreducible connections bijectively on the stable locus of $\mathrm{Bun}_G(\Sigma, 0)$. A different, later proof consists in showing that the stability condition on a vector bundles $V$ amounts to that for the existence of a metric $h$ such that the associated Chern connection is flat, and then the Narasimhan–Seshadri correspondence is obtained by taking the local system defined on $\Sigma$ by that flat connection (see [Don83]). This is all nowadays subsumed by the results of the previous sections, where one considers $\mathrm{Bun}^{\mathrm{st}}(\Sigma, 0)$ as the subvariety of $\mathcal{M}_{\mathrm{Dol}}(\Sigma, G)$ where the $G$-Higgs fields vanish, noticing that the stability conditions in this case match up. Then the theorem of Narasimhan–Seshadri is a corollary of the non-Abelian Hodge correspondence, even though saying so reverses the history of the development of this vast subject.

After the theorem of Narasimhan–Seshadri, one expected that the space of all unitary representation of the fundamental group should provide a compactification of $\mathrm{Bun}_G^{\mathrm{st}}(\Sigma, 0)$, resulting in a (singular) projective variety. This was shown two years later in [Ses67].

**Remark 2.16.** The tangent space to a principal $G$-bundle $V$ on $\Sigma$ is naturally identified with the first cohomology space $H^1\big(\Sigma, \mathrm{End}(V)\big)$. Hence a $G$-Higgs field $\Phi \in H^0\big(\Sigma, \mathrm{End}(V) \otimes \Omega^1_\Sigma\big)$ is naturally a cotangent vector to $\mathrm{Bun}_G^{\mathrm{st}}$ at $V$. This provides an embedding

$$T^* \mathrm{Bun}_G^{\mathrm{st}} \subseteq \mathcal{M}_{\mathrm{Dol}}^{\mathrm{st}},$$

which is however not an isomorphism. The point is that there do exist stable $G$-Higgs bundles defined on non-stable principal $G$-bundles over $\Sigma$. The correct statement is that $\mathcal{M}_{\mathrm{Dol}}^{\mathrm{st}}$ is a partial compactification of the cotangent bundle $T^* \mathrm{Bun}_G^{\mathrm{st}}$. The same exact phenomenon occurs for the polystable locus.

The next section is devoted to recalling the geometric quantisation of the moduli space of flat SU(2)-connections.

### 2.3.3 Kähler quantisation of the moduli space of flat unitary connections

We first recall how the moduli space of irreducible, unitary flat connections on a nonsingular, complex projective curve $\Sigma$ of genus $g \geq 2$ has a natural symplectic structure. The mathematical details we leave out are to be found in [AB83] (and [Gol84] for a different, finite-dimensional approach). Let us agree to denote $\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}} = \mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}(\Sigma)$ the moduli space. This is the same as the irreducible locus of the character variety considered in the previous section.

Let us then switch to the compact group $K \coloneqq \mathrm{SU}(n)$ in this section, which is both simple and simply connected. In the language of vector bundles of rank $n$, intersecting the

structure group with $\mathrm{SL}(\mathbb{C}^n)$ amounts to having a trivial determinant bundle. Consider the trivial principal $K$-bundle $P_T \coloneqq K \times \Sigma \longrightarrow \Sigma$ on $\Sigma$, and let $\mathcal{A} = \mathcal{A}(\Sigma, K)$ be the infinite-dimensional space of unitary $K$-connections on it. In general, fixing the topological type of the bundle is a genuine restriction, without which one cannot even hope to have a connected moduli space. In the previous sections this has been done by fixing the degree, which in the case of principal $\mathrm{GL}_n(\mathbb{C})$-bundle is an integer (i.e. an element of the fundamental group of the structure group). Since $K$ is simply connected, however, all $K$-principal bundles over $\Sigma$ surface are trivialisable, and thus the choice made is just that of a global section.[16]

The space $\mathcal{A}$ of $K$-connections on $P_T$ is an affine space having $A^1(\Sigma, \mathfrak{k}) \cong A^1\Big(\Sigma, \mathrm{Ad}(P_T)\Big)$ as vector space of translations, where $\mathrm{Ad}(P_T)$ denotes the vector bundle associated to $P_T$ by the *Adjoint representation* of $K$ on its Lie algebra $\mathfrak{k} \coloneqq \mathfrak{su}(n)$. This is because the difference of any two $K$-connections on $P_T$ is given by a global $\mathfrak{k}$-valued one-form $A \in A^1(\Sigma, \mathfrak{k})$. In particular, the tangent spaces can be everywhere identified with the space of translations:

$$T_\nabla \mathcal{A} \cong A^1(\Sigma, \mathfrak{k}), \qquad \text{for all} \qquad \nabla \in \mathcal{A}.$$

The moduli space $\mathcal{M}_{\mathrm{fl}}$ is then the set of gauge-orbits of flat connections on $P_T$, i.e. the quotient of $\mathcal{A}$ for the action of the gauge group $\mathcal{K}$ of $P_T$, i.e. the group of bundle automorphisms of $P_T$ over the identity. Its elements are $K$-equivariant diffeomorphisms $g\colon P_T \longrightarrow P_T$ such that one has a commutative triangle

$$\begin{array}{ccc} P_T & \xrightarrow{\ g\ } & P_T \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \Sigma & \end{array}$$

where $\pi$ is the projection of the principal $K$-bundle. Because of the $K$-equivariance, such maps are coded by smooth functions $g\colon \Sigma \longrightarrow K$, and thus $\mathcal{K} \cong C^\infty(\Sigma, K)$. Writing the gauge group like this, one also has an identification $\mathrm{Lie}(\mathcal{K}) \cong C^\infty(\Sigma, \mathfrak{k})$. The action of $g \in \mathcal{K}$ on a connection $\nabla = d - A$ is given by:

$$g.\nabla = d - \Big(\mathrm{Ad}_g(A) + g^*\theta\Big),$$

where $\theta \in A^1(K, \mathfrak{k})$ is the Maurer–Cartan 1-form of $K$. More concretely, if $p \in \Sigma$ is a point on the surface, then $g^*\theta_p$ is the composition of $dg_p\colon T_p\Sigma \longrightarrow T_{g(p)}K$ with the left translation $g^{-1}\colon T_{g(p)}K \longrightarrow T_eK \cong \mathfrak{k}$.

Now, there exists a canonical symplectic structure $\omega$ on $\mathcal{A}$, defined at a connection $\nabla \in \mathcal{A}$ by:

$$\omega_\nabla\colon A^1(\Sigma, \mathfrak{k}) \wedge A^1(\Sigma, \mathfrak{k}) \longrightarrow \mathbb{R}, \qquad \omega_\nabla(A, B) \coloneqq -\frac{1}{2\pi}\int_\Sigma \mathrm{Tr}(A \wedge B). \tag{2.15}$$

[16] Principal $K$-bundles on $\Sigma$ are classified by homotopy classes of maps $\Sigma \longrightarrow BK$, where $BK$ is the basis of the universal principal $K$-bundle $EK \longrightarrow BK$: see [Aud04], § VI.1. If $K$ is 1-connected, then $BG$ is 3-connected, and all maps from a surface to $BG$ are homotopic to a constant.

In words, one uses an Ad-invariant scalar product on $\mathfrak{k}$ to construct a real-valued 2-form out of two $\mathfrak{k}$-valued 1-forms.[17] Since $\mathcal{A}$ is an affine space, it makes sense to consider translation-invariant tensors on it, and $\omega$ is exactly one of such.

Next, one can show that the $\mathcal{K}$-action preserves $\omega$, i.e. $g^*\omega = \omega$ for all $g \in \mathcal{K}$. This is due to the invariance of the bilinear form chosen on $\mathfrak{k}$. Moreover, there exists a moment map $\mu\colon \mathcal{A} \longrightarrow \mathrm{Lie}(\mathcal{K})^* \cong C^\infty(\Sigma, \mathfrak{k})^*$ for the gauge-action. Namely, one can show that the formula

$$\Big(\mu(A)\Big)(f) = \int_\Sigma \mathrm{Tr}\left(F_A \wedge f\right)$$

works, where $F_A = -dA + A \wedge A \in A^2(\Sigma, \mathfrak{k})$ is the curvature of $\nabla = d - A$ (see also [Aud04], Chap. 5). By definition the zero level-set $\mathcal{A}^0 \coloneqq \mu^{-1}(0) \subseteq \mathcal{A}$ is then the space of flat $K$-connections on $\Sigma$. The usual Marsden–Weinstein symplectic reduction goes through, and thus the quotient

$$\mathcal{A} /\!\!/_0 \mathcal{K} \coloneqq \mathcal{A}^0 \big/ \mathcal{K}$$

is symplectic, with symplectic form abusively written $\omega$. This description is clearly purely formal, and one needs nontrivial analytic arguments to make this infinite-dimensional quotient into a manifold. Nevertheless, if one now only considers irreducible connections $\mathcal{A}^{\mathrm{irr}} \subseteq \mathcal{A}$, then the quotient is precisely the stable part $\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}$ of the moduli space of flat connections.

One should turn to the construction of prequantum data for the symplectic manifold $(\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}, \omega)$, at level $k \geq 1$. To this end, consider the trivial line bundle $\widetilde{L} \coloneqq \mathcal{A} \times \mathbb{C}$, equipped with the tautological Hermitian metric $\widetilde{h}$. Since this bundle is trivial, any global, complex 1-form $\alpha$ on $\mathcal{A}$ will define a connection in it. Set then

$$\alpha_A(B) = -\frac{1}{4\pi} \int_\Sigma \mathrm{Tr}(A \wedge B),$$

for $A \cong d - A \in \mathcal{A}$, $B \in T_\nabla \mathcal{A}$. The connection $\widetilde{\nabla}^{(k)} = d - ik\alpha$ on $\widetilde{L}^{\otimes k}$ is then $\widetilde{h}$-unitary, and it is a prequantum connection at level $k$ since $F_{\widetilde{\nabla}^{(k)}} = -ikd\alpha = -ik\omega$. What is left is to find a way to reduce this prequantum data by the gauge action. To this end, one looks for multipliers

$$\Theta^{(k)}\colon \mathcal{K} \times \mathcal{A} \longrightarrow U(1),$$

i.e. functions satisfying

$$\Theta^{(k)}\Big(h, g.A\Big) \cdot \Theta^{(k)}(h, A) = \Theta^{(k)}(gh, A), \qquad \text{for} \qquad g, h \in \mathcal{K}, A \in \mathcal{A}.$$

Such choices provides a linearisation of the $\mathcal{K}$-action from $\mathcal{A}$ to $\widetilde{L}$, by means of

$$g.(A, \eta) \coloneqq \Big(g.A, \Theta^{(k)}(g, A)\eta\Big), \qquad \text{for} \qquad g \in \mathcal{K}, A \in \mathcal{A}, \eta \in \mathbb{C}.$$

Their abstract definition is provided by Chern–Simons theoretical arguments (see [And+17], section 3.1). Consider the oriented 3-manifold $X \coloneqq \Sigma \times [0, 1]$. Choose a connection $A \in \mathcal{A}$ and a gauge-group element $g \in \mathcal{K}$, and set $\widetilde{A} \coloneqq \pi^* A$, where $\pi\colon X \longrightarrow \Sigma$ is the natural

[17] The normalisation of the trace pairing is tuned according to [AG14]; anyhow, since $\mathfrak{k}$ is simple, every $\mathrm{Ad}_K$-invariant scalar product is a multiple of the Killing form of $\mathfrak{k}$, i.e. $B(X, Y) = 4\,\mathrm{Tr}(XY)$.

projection; this is a connection on the trivial, principal $K$-bundle over $X$. Moreover, choose an isotopy $\widetilde{g}\colon X \longrightarrow K$ from the trivial gauge-transformation to $g$. The transform $\widetilde{g}.\widetilde{A} \in A^1(X, \mathfrak{k})$ is defined for all $t \in [0,1]$, and one sets

$$\Theta^{(k)}(g, A) \coloneqq \exp\Big( - 2\pi i k S_{\mathrm{CS}}(\widetilde{g}.\widetilde{A})\Big) \in U(1),$$

where $S_{\mathrm{CS}}\colon A^1(X, \mathfrak{k}) \longrightarrow \mathbb{R}$ denotes the Chern–Simons action on $X$, i.e.

$$S_{\mathrm{CS}}(A) \coloneqq \frac{1}{8\pi^2}\int_X \mathrm{Tr}\Big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\Big).$$

The constant is chosen so that the cohomology class of $\frac{1}{6}\Big(\theta \wedge [\theta \wedge \theta]\Big)$ is integral inside $H^3_{\mathrm{dR}}(K, \mathfrak{k})$. This assures that a gauge-transformation changes the action by an integer (see [Fre95]).
Now one can restrict all prequantum data to the space of irreducible flat connections, and notice that the lifted actions fixes the symplectic potential and the metric, thereby defining prequantum data $(L^{\otimes k}, \nabla^{(k)}, h)$ at level $k$ on the symplectic moduli space $(\mathcal{M}^{\mathrm{st}}_{\mathrm{fl}}, \omega)$: the prequantisation has been achieved.

Now let us move on to the Kähler quantisation of $(\mathcal{M}^{\mathrm{st}}_{\mathrm{fl}}, \omega)$. One should use the full data of the problem, and recall that $\Sigma$ carries a complex structure. Using some elementary Hodge theory, this induces a complex structure $I$ on the moduli space.
More precisely, considering $\Sigma$ as a compact Kähler manifold, hence in particular a Riemannian manifold, it comes equipped with *Hodge star* operators

$$*_l\colon A^l(\Sigma, \mathbb{C}) \longrightarrow A^{2-l}(\Sigma, \mathbb{C}),$$

for $l \in \{0,1,2\}$, satisfying $*_l \circ *_l = (-1)^l\,\mathrm{Id}$. One can then extend them naturally to $\mathfrak{k}$-valued differential forms, and the case $l = 1$ provides a complex structures on the vector space $A^1(\Sigma, \mathfrak{k})$, which can be upgraded to a translation-invariant almost-complex structure on $\mathcal{A}$, since $T\mathcal{A} \cong \mathcal{A} \times A^1(\Sigma, \mathfrak{k})$, as argued above. Such almost-complex structures are integrable, precisely because they are translation invariant.
Next, one recalls that the tangent space $T_{[A]}\mathcal{M}_{\mathrm{fl}}$ of the moduli space at the gauge-equivalence class $[A]$ of a unitary connection $A \in A^1(\Sigma, \mathfrak{k})$ is identified with the first $A$-twisted cohomology group $H^1(\Sigma, d_A)$, which means the first cohomology group of the complex

$$A^0\Big(\Sigma, \mathrm{Ad}(P_T)\Big) \xrightarrow{d_A} A^1\Big(\Sigma, \mathrm{Ad}(P_T)\Big) \xrightarrow{d_A} A^2\Big(\Sigma, \mathrm{Ad}(P_T)\Big).$$

The fact that this is actually a complex means precisely that $d_A$ is a flat connection.

Finally, one recalls that the Hodge star preserves harmonic forms, and that such forms are representatives for de Rham cohomology classes. More precisely, one has an identification

$$H^1(\Sigma, d_A) \cong \mathrm{Ker}(\Delta_A),$$

where $\Delta_A = d_A d_A^* + d_A^* d_A$ is the Hodge Laplacian attached to the first-order differential operator $d_A$, whose kernel is invariant for $*_1$ (this explicitly uses the compactness of $\Sigma$). The automorphism $I \coloneqq -*_1$ is by definition the complex structure on $\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}$ associated to a complex structure on the curve. The sign is there to assure that $\omega\Big(\cdot, I(\cdot)\Big) > 0$.

The triple $\Big(\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}, \omega, I\Big)$ is then a smooth, compact Kähler manifold. Using the material of § 2.2, and particularly the example 2.11, one can try define its Kähler quantisation $\mathcal{H}_P$, where $P \coloneqq T_{0,1}\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}$. This is done with respect to the prequantum data described above.

**Remark 2.17.** Once a complex structure is chosen on $\Sigma$, the theorem of Narasimhan–Seshadri 2.10 provides an identification of $\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}$ with the moduli space $\mathrm{Bun}_G^{\mathrm{st}}(\Sigma, 0)$ of stable holomorphic $G$-bundle on $\Sigma$ of degree zero, where $G = \mathrm{SL}(2, \mathbb{C})$ is the complexification of $K$. This smooth projective variety admits a compactification $\mathrm{Bun}_G(\Sigma, 0)$, which is the moduli space of semistable holomorphic $G$-bundles of degree zero up to $S$-equivalence. One can show that the Picard group $\mathrm{Pic}\Big(\mathrm{Bun}_G(\Sigma, 0)\Big)$ — i.e. the group of isomorphisms classes of holomorphic line bundles over the moduli space — is isomorphic to $\mathbb{Z}$ (see [DN89]). Finally, the Chern–Simons line bundle at level $k$ is (the smooth line bundle underlying) $\mathcal{L}^{\otimes k}$, where $\mathcal{L}$ is an ample generator of the Picard group.

This discussion indicates that there is a natural family of Kähler polarisations on the prequantisable symplectic manifold $\Big(\mathcal{M}_{\mathrm{fl}}^{\mathrm{st}}, \omega\Big)$, parametrised by complex structures on $\Sigma$. However, the notion of holomorphicity on the moduli space only depends on the biholomorphism class of the complex manifold $\Sigma$, and thus one has a Kähler polarisation for every point in the Riemann moduli space $\mathcal{M}_g$ for closed oriented surfaces of genus $g$. Since this space is singular, one prefers working with the Teichmüller space $\mathcal{T}_g$ of closed orientable surfaces of genus $g$, which is a contractible, finite-dimensional space parametrising marked Riemann surfaces structures on $\Sigma$ (see the beginning of § 4.2 for more details on this important point).

Following the program presented in § 2.2.4, one gets a vector bundle $\pi\colon \mathcal{V}^{(k)} \longrightarrow \mathcal{T}_g$ of holomorphic section, sitting inside the trivial bundle $\pi\colon C^\infty\Big(\mathcal{M}_{\mathrm{fl}}, L^{\otimes k}\Big) \times \mathcal{T}_g \longrightarrow \mathcal{T}_g$ of smooth section of $L^{\otimes k}$. Then [Hit90] and [ADW91] independently defined a projectively flat connection $\widehat{\nabla}$ inside $\mathcal{V}^{(k)}$. Since the base is contractible, this produces the desired canonical identifications among Hilbert spaces arising from different choices of Kähler polarisations.

**Remark 2.18.** The mapping class group $\Gamma_g$ of closed oriented surfaces of genus $g$ — defined in § 4.3 — acts on all the data of the geometric quantisation. One can also show that the Hitchin connection is invariant under this action, and thus the projectively flat connection descends on a bundle over the Riemann moduli space $\mathcal{M}_g$, which is the actual parameter space for the Kähler polarisations.[18]

The advantage of considering $\Gamma_g$-invariant connections on $\mathcal{T}_g$ instead of genuine connections on $\mathcal{M}_g$ is that the former space is smooth, whereas the latter is a (Deligne–Mumford) orbifold. One should really consider the moduli stack of Riemann surfaces to construct the good objects.

[18] Incidentally, the action of $\Gamma_g$ on the fibres of $\mathcal{V}^k$ is precisely the origin of the quantum mapping class group representation in the geometric quantisation setting (see e.g. [Mas03]).

### 2.3.4 Meromorphic connections

Let us consider again a compact Riemann surface $\Sigma$, and set $G \coloneqq \mathrm{GL}_n(\mathbb{C})$, $\mathfrak{g} \coloneqq \mathrm{Lie}(G)$. We have seen how to attach three moduli spaces to the first non-Abelian cohomology group of $\Sigma$, taking values in $G$: holomorphic Higgs bundles, holomorphic connections and complex representation of $\pi_1(\Sigma)$ of dimension $n$. We now consider one natural extension of this program, where one replace nonsingular objects by meromorphic ones. To this end, fix a positive divisor $D$ on $\Sigma$, and let $\Omega^1_\Sigma(D)$ be the sheaf of meromorphic 1-forms on $\Sigma$ with poles bounded by $D$.

**Definition 2.41.** A meromorphic connection on $\Sigma$ bounded by the divisor $D$ is a pair $(V, \nabla)$, where $V$ is a holomorphic vector bundle on $\Sigma$ and $\nabla$ is a first order differential operator

$$\nabla \colon V \longrightarrow \Omega^1_\Sigma(D) \otimes V,$$

satisfying the Leibnitz identity:

$$\nabla(fs) = f\nabla s + df \otimes s,$$

for any local holomorphic function $f$, and local holomorphic section $s$.

Notice the strict analogy with Def. 2.37. More concretely, let $z$ be a local holomorphic coordinate on $\Sigma$ on the open set $U \subseteq \Sigma$, and let $\nabla$ be a meromorphic connection on $\pi \colon V \longrightarrow \Sigma$. Then the restriction of $\nabla$ to $U$ may be written

$$\nabla|_U = d - \mathcal{B}dz,$$

where $\mathcal{B}$ is a meromorphic, $\mathrm{End}(V)$-valued function on $U$ bounded by $D$. If one writes $D = \sum_i k_i(p_i)$, with $k_i > 0$ for all $i$, and if $z$ is a local coordinate vanishing at $p_i$, then one will have

$$\mathcal{B} = \sum_{j=1}^{k_i} \frac{A^{(i)}_j}{z^j} + \mathcal{B}_0, \qquad \text{where} \qquad A^{(i)}_j \in \mathfrak{g} \quad \text{for all} \quad 1 \leq j \leq k_i, \tag{2.16}$$

and where $\mathcal{B}_0$ is a matrix of holomorphic functions. As it was before, one can replace holomorphic, rank $n$ vector bundles with principal $G$-bundles, and $\mathrm{End}(V)$-valued forms with $\mathfrak{g}$-valued ones.

One can now introduce a moduli space for meromorphic connections on rank $n$, holomorphic vector bundles over $\Sigma$ with poles bounded by a positive divisor $D$. It turns out however that one also needs to fix further data at the poles $p_i$, related to (unramified) normal forms, as well as a parabolic structure on the curve $\Sigma$ with marked points defined by the support of $D$. Moreover, one has to introduce stability conditions totally analogous to Def. 2.39, replacing the degree by the parabolic degree.
The general theory may be read in [Boa01; Nit93; Sim94], while [Boa12a] provides a quick review. These constructions will not not be needed in full generality, as far as the results presented in Chap. 3 are concerned. We thus immediately restrict our attention to the case of a curve $\Sigma$ of genus zero, i.e. $\Sigma \cong \mathbb{C}P^1$, and we introduce the remaining piece of data needed to the define the moduli spaces.

**Definition 2.42.** Let $p \in \mathbb{C}P^1$ be a point, and $z$ a local holomorphic coordinate on $\mathbb{C}P^1$ that vanishes at $p$. An (unramified) irregular type for the group $G$, at the point $p \in \mathbb{C}P^1$, is an element

$$Q = \sum_{i=1}^{k-1} \frac{A_i}{z^i} \in \mathfrak{t}(\!(z)\!)\Big/\mathfrak{t}[\![z]\!],$$

where $\mathfrak{t} \subseteq \mathfrak{g}$ is the subalgebra of diagonal matrices, and $k$ a positive integer. The centraliser of $Q$ is the subgroup

$$H \coloneqq \{g \in G \mid gA_ig^{-1} = A_i \text{ for all } i\} \subseteq G,$$

and we denote $\mathfrak{h} \subseteq \mathfrak{g}$ its Lie algebra.

The following definition is instead given for all genera.

**Definition 2.43.** Let $\Sigma$ be a Riemann surface. A structure of wild Riemann surface $\mathbf{\Sigma} = \big(\Sigma, \{p_i\}_i, \{Q_i\}_i\big)$ on $\Sigma$ is a decoration consisting of

- Distinct marked points $p_i \in \Sigma$.
- Irregular types $Q_i$ at the points $p_i$.

These wild Riemann surfaces are the good base spaces for holomorphic vector bundles carrying meromorphic connections. The marked points code the positions of the poles, and the irregular types provide distinguished local normal forms around the poles.

**Definition 2.44.** Let $\mathbf{\Sigma} = \big(\mathbb{C}P^1, \{p_i\}_i, \{Q_i\}_i\big)$ be a wild Riemann surface that underlies the Riemann sphere, and let $\mathcal{O}_i \subseteq \mathfrak{h}_i$ be a choice of adjoint $H_i$-orbit for all $i$, where $H_i$ is the centraliser of the irregular type $Q_i$. We denote by $\mathcal{M}^* = \mathcal{M}^*\big(\mathbf{\Sigma}, G, \{\mathcal{O}_i\}_i\big)$ the moduli space of gauge-equivalence classes of meromorphic connections $\nabla$ living on globally holomorphically trivial vector bundle over $\mathbb{C}P^1$, such that

1. $\nabla$ has poles at the marked points, and only there.
2. $\nabla$ is locally holomorphically isomorphic near $p_i$ to a connection of the form

$$d - \left(dQ_i + \Lambda_i \frac{dz}{z} + \mathcal{B}_0^i dz\right), \tag{2.17}$$

   where $\Lambda_i \in \mathcal{O}_i$ and $\mathcal{B}_0^i$ is a local $\mathfrak{g}$-valued holomorphic function.

3. $\nabla$ has no nontrivial subconnections.

The last requirement is what the more general stability condition boils down in this context. The choice of adjoint orbits instead amounts to fixing a symplectic leaf inside a bigger Poisson moduli space.

**Remark 2.19.** The moduli space $\mathcal{M}^*$ is a refined geometric viewpoint on differential-analytic objects, i.e. gauge-equivalence classes of systems of linear, first order, ordinary differential equations with rational coefficients. Indeed, a meromorphic connection $(V, \nabla)$ on the trivial, rank $n$ vector bundle over $\mathbb{C}P^1$ is defined by a global section $\mathcal{A}$ of the sheaf $\Omega^1_{\mathbb{C}P^1}(D) \otimes \mathrm{End}(V)$, in the sense that one has $\nabla = d - \mathcal{A}$. Choose a local holomorphic coordinate $z$ that identifies $\mathbb{C}P^1 \cong \mathbb{C} \coprod \{\infty\}$, so that the support of $D$ lies in $\mathbb{C}$, and let $v \colon U \longrightarrow V$ be a local holomorphic section of $V$ on the open set $U \subseteq \mathbb{C}$. Then the equation for the flatness of $v$ reads

$$dv = \mathcal{A}v, \qquad \text{i.e.} \qquad \frac{dv}{dz} = \mathcal{A}(dz_z)v.$$

This is precisely a system of linear ODEs with rational coefficients for the components of $v$.

Before picking the actual irregular types that will be relevant to the discussion of 3, let us briefly recall the finite-dimensional presentation of $\mathcal{M}^*$ as a smooth, complex symplectic quotient. To this end, consider the group

$$G_k \coloneqq \mathrm{GL}_n\left(\mathbb{C}[z]/z^k\right),$$

where $k$ is a positive integer. One has $G_1 = G$, whereas a generic element of $G_k$ can be written as a polynomial

$$g = \sum_{i=0}^{k-1} g_i z^i, \qquad \text{with} \quad g_i \in G \quad \text{for all } i.$$

This means that the Lie algebra $\mathfrak{g}_k$ of $G_k$ consists of elements

$$X = \sum_{i=0}^{k-1} X_i z^i, \qquad \text{with} \quad X_i \in \mathfrak{g} \quad \text{for all } i,$$

where the Lie bracket is defined expanding that of $\mathfrak{g} = \mathfrak{g}_1$ by bilinearity, and deleting terms that carry a power $z^l$ with $l \geq k$. There exists a canonical identification of the dual Lie algebra $\mathfrak{g}_k^*$ with the space of polar parts of meromorphic connections having one pole of order $k$ on the trivial, holomorphic rank $n$ vector bundle over a disc. If one has a local coordinate $z$ on the disc, such a polar part is a sum

$$A = \sum_{i=1}^{k} A_i \frac{dz}{z^i}, \qquad \text{where} \quad A_i \in \mathfrak{g} \quad \text{for all } i,$$

and the identification with $\mathfrak{g}_k^*$ is provided by the nondegenerate pairing

$$\langle A, X \rangle \coloneqq \mathrm{Res}_0\left(\mathrm{Tr}(A \cdot X)\right) = \sum_{i=1}^{k} \mathrm{Tr}(A_i X_{i-1}).$$

Now one fixes a coadjoint orbit $\mathcal{O}_i \subseteq \mathfrak{g}_{k_i}^*$ for all $1 \leq i \leq m$, according to the irregular type $Q_i$ and the residue $\Lambda_i \in \mathfrak{h}_i$, chosen as above. Namely, one lets $\mathcal{O}_i$ be the orbit through the point

$$dQ_i + \Lambda_i \frac{dz}{z} \in \mathfrak{g}_{k_i}^*,$$

where $z$ is as usual a coordinate on $\mathbb{C}P^1$ vanishing at the marked point $p_i$.

**Theorem 2.11.** *Ignoring stability, the moduli space $\mathcal{M}^*$ is isomorphic to the complex symplectic quotient*

$$\mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!\!/_0 G,$$

*where the group $G \subseteq \bigcap_{k\geq 1} G_k$ acts diagonally on the product of coadjoint orbits $\prod_i \mathcal{O}_i \subseteq \prod_i \mathfrak{g}_{k_i}^*$, if one has $m$ marked points on the wild sphere.*

Notice that for $m = 0$ the moduli space is trivial. This is in accordance with the fact that the moduli space for nonsingular flat connections on the sphere is identified with the representation of its (trivial) fundamental group.

**Remark 2.20.** This is one of the results of [Boa01], where one considers the naïf symplectic quotient of the product of orbits at the zero level of the moment map, without a restriction to irreducible connections (which are the stable objects in our context). However, for a generic choice of the orbits $\mathcal{O}_i$ one can assure the absence of nontrivial subconnections, and thus $\mathcal{M}^*$ consists of stable objects. In particular, it is smooth.

**Example 2.14.** The case with $m$ simple poles yields

$$\mathcal{M}^* \cong \left\{ (R_1, \ldots, R_m) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_m \,\middle|\, \sum_i R_i = 0 \right\} \Big/ G,$$

with $\mathcal{O}_i \subseteq \mathfrak{g}^*$ for all $i$. Since $\mathfrak{g}$ is reductive, one has an isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ obtained by fixing an $\mathrm{Ad}_G$-invariant scalar product.
In this case one point of $\mathcal{M}^*$ corresponds to a *Fuchsian system* of the form

$$d - \sum_{i=1}^{m} \frac{R_i}{z - t_i} dz,$$

on the trivial, rank $n$ vector bundle $V \cong \mathbb{C}^n \times \Sigma \longrightarrow \Sigma$, where $t_i \in \mathbb{C}$ are distinct marked points. The moment map $\mu\colon \prod_i \mathcal{O}_i \longrightarrow \mathfrak{g}^*$ for the diagonal, (co)adjoint $G$-action is the sum of inclusions $\mathcal{O}_i \hookrightarrow \mathfrak{g}^*$. Its vanishing provides the above condition that the sum of the residues $R_i$ on $\mathbb{C} \subseteq \mathbb{C}P^1$ is zero; this in particular assures that there is no singularity at $\infty$.

This example has several other interesting features that will be explored in § 2.3.5.

**Remark 2.21.** Finally, the particular moduli space $\mathcal{M}^*$ that is relevant to the results of Chap. 3 is obtained by considering the following type of polar divisor on $\mathbb{C}P^1$. Choose points $t_1, \ldots t_m \in \mathbb{C}$, and considers meromorphic connections $\nabla$ bounded by the effective divisor

$$D := \sum_{i=1}^{m} (p_i) + 3(\infty).$$

Such connections have by definitions at most simple poles at the points $p_i$, plus a pole of order at most three at $z = \infty$. To fix the normal form at infinity, let $A, T \in \mathfrak{t}$ be diagonal matrices, and consider the irregular type $Q := \frac{Az^2}{2} + Tz$ at $\infty$. One can now fix arbitrary adjoint orbits $\mathcal{O}_i \subseteq \mathfrak{g}$ for the Fuchsian singularities, plus an orbit $\mathcal{O} \subseteq \mathfrak{h}$, where $\mathfrak{h} \subseteq \mathfrak{g}$ is the algebra of matrices commuting with $A$ and $T$.

According to the discussion above, we consider connections $\nabla = d - \mathcal{A}$ on the trivial vector bundle $\mathbb{C}^n \times \mathbb{C}P^1$, which are of the form

$$\mathcal{A} = \left( Az + C + \sum_{i=1}^{m} \frac{R_i}{z - t_i} \right) dz,$$

where $R_i \in \mathcal{O}_i$, and $C \in \mathfrak{g}$. Moreover, in accordance with Def. 2.44, we ask that $\nabla$ is locally holomorphically isomorphic to a connection of the form

$$\left( Az + T + \frac{\Lambda}{z} + \mathcal{B}_0 \right) dz$$

at $z = \infty$, where $\mathcal{B}_0$ is a holomorphic term and $\Lambda \in \mathcal{O}$ a residue lying in the correct orbit. It is a nontrivial problem to find criteria that assure the existence of such connections which are moreover stable. This is the (unramified), additive Deligne–Simpson problem, in which we will not delve (see [Boa12b], § 10). The last thing we wish to recall from [Boa12b] is that in this case $\mathcal{M}^*$ can be realised as a quiver variety. Namely, one can find a suitable graph $\mathcal{G}$ on nodes $I$, together with vector spaces $\{V_i\}_{i \in I}$, such that the following holds.

**Theorem 2.12.** *One has an isomorphism*

$$\mathcal{M}^* \cong T^* \operatorname{Rep}\left(\mathcal{G}, \{V_i\}_i\right) /\!\!/_{\mathcal{O}} \widehat{H},$$

*where* $\widehat{H} := \prod_{i \in I} \mathrm{GL}(V_i)$ *acts on representations by simultaneous conjugations, and where* $\mathcal{O}$ *is a suitable coadjoint orbit.*

The representation space can also be introduced as a space of presentations of modules for the one-dimensional Weyl algebra (i.e. the algebra 2.5 with only two generators); this notwithstanding, the moduli space is the complex symplectic quotient of a finite-dimensional symplectic vector space. See the beginning of § 3.1 for more details.

**Remark 2.22.** In a parallel fashion to § 2.3.1, one can also attach Poisson moduli spaces $\mathcal{M}_{\mathrm{Dol}}$ and $\mathcal{M}_{\mathrm{B}}$ to any wild Riemann surface $\mathbf{\Sigma}$, in addition to $\mathcal{M}_{\mathrm{dR}}$, generalising the non-Abelian Hodge theoretic viewpoints of Dolbeault and Hitchin. The Dolbeault moduli space parametrises meromorphic $G$-Higgs fields on holomorphic vector bundles over $\Sigma$, whereas the Betti one is a more far-fetched generalisation of the character variety of $G$-representations of the curve.
If further choice are made to pick symplectic leaves, then [BB04] and [Sab99] realize the triple of moduli spaces as an hyper-Kähler manifold having three different, distinguished complex algebraic structures, and one does have a wild, non-Abelian Hodge correspondence as soon as certain parameters are matched up (coadjoint orbits signatures for $\mathcal{M}_{\mathrm{Dol}}$ and $\mathcal{M}_{\mathrm{dR}}$, and conjugacy classes for $\mathcal{M}_{\mathrm{B}}$). In the case of simple poles, this was first set up in [Sim90].

The idea behind the definition of the Betti moduli space is that it should still be a repository for generalised monodromy data of connections, even when their holonomy representation is no longer enough to code them uniquely. The good notion is that

of *Stokes data*, which we will not need. Nonetheless, one still has a wild Riemann–Hilbert correspondence $\nu\colon \mathcal{M}_{\mathrm{dR}} \longrightarrow \mathcal{M}_{\mathrm{B}}$ that generalises Thm. 2.8. The proof that it is symplectic can be read in [Boa01].
Getting back to the setting of trivial vector bundles over a wild sphere $\boldsymbol{\Sigma}$, one gets an exponential-like map $\nu\colon \mathcal{M}^* \longrightarrow \mathcal{M}_{\mathrm{B}}$, that relates the "additive" moduli space $\mathcal{M}^*$ to the "multiplicative" Betti side.
In the next section 2.3.5 we will consider this map in the case of simple poles on the sphere. The correspondence is essentially the same as the nonsingular one, if one takes care to replace the Riemann sphere with the punctured sphere $\mathbb{C}P^1 \setminus \Delta$, removing the support of the polar divisor (this is due to [Del70]).

Finally, notice that it follows from Def. 2.44 that if one varies the wild curve $\boldsymbol{\Sigma}$ over a family $\mathbf{B}$ then one finds symplectic bundles $\boldsymbol{\mathcal{M}}^* \longrightarrow \mathbf{B}$, $\boldsymbol{\mathcal{M}}_{\mathbf{B}} \longrightarrow \mathbf{B}$, whose fibres at a wild curve $\boldsymbol{\Sigma} \in \mathbf{B}$ is the relevant moduli space of connections or Stokes data.

### 2.3.5 Schlesinger and KZ

In this section we expand example 2.14, which introduces the moduli space of Fuchsian systems on the sphere, in two directions: first, we discuss the monodromy data of a logarithmic connection on a vector bundle over the sphere, thereby defining its isomonodromic deformations, which yield the Schlesinger system of [Sch05]; second, we give a proof of the fact that the PBW quantisation of the Schlesinger Hamiltonians, defined according to § 2.1.5, yields the Knizhnik–Zamolodchikov system of [KZ84], as was first remarked in [Res92]. Extending this deformation quantisation procedure is precisely the content of the next chapter 3.

Let us then temporarily start from an abstract, nonsingular viewpoint. Let $M$ be a smooth manifold, $V$ a smooth, complex vector bundle on $M$, and $\nabla\colon V \longrightarrow V \otimes A^1_M$ a smooth connection on $V$. Let $p, q \in M$ be two points which are connected by a smooth path $\gamma\colon [0,1] \longrightarrow M$.

**Definition 2.45.** The parallel transport of $\nabla$ along $\gamma$ is the $\mathbb{C}$-linear isomorphism

$$\mathrm{PT}^{\nabla}_{\gamma}\colon V_p \longrightarrow V_q$$

that associates to $v \in V_p$ the value at $t = 1$ of the solution of the initial value problem

$$\begin{cases} \gamma^*\nabla s &= 0 \\ s(0) &= v \end{cases} \tag{2.18}$$

This is a differential equation for a smooth section $s$ of the pull-back vector bundle $\gamma^*V$ over $[0,1]$, equipped with the pull-back connection $\gamma^*\nabla$.

**Remark 2.23.** Concretely, one considers the vector field $\partial_t$ that trivialises $T[0,1] \cong [0,1] \times \mathbb{R}$, and then one considers the linear, first-order ODE

$$\nabla_{\dot\gamma(t)} s = 0,$$

where $\dot\gamma(t)$ is the image of $\partial_t$ for the tangent map $d\gamma\colon T[0,1] \longrightarrow TM$ (the velocity of $\gamma$). If one has a local trivialisation $V|_U \cong U \times \mathbb{C}^n$ for $V$ over an open set $U \subseteq M$ that contains $\gamma\Big([0,1]\Big)$, then one can write $\nabla = d - \alpha$ for a suitable connection form $\alpha \in A^1\Big(U, \mathrm{End}(\mathbb{C}^n)\Big)$, and the system becomes

$$\partial_t s(t) = \alpha\Big(\dot\gamma(t)\Big)s(t),$$

for a vector-valued function $s\colon [0,1] \longrightarrow \mathbb{C}^n$. If $v \in \mathbb{C}^n \cong V_p$, then (2.18) admits a unique solution $s$, because of the Cauchy–Lipschitz theorem, and one sets

$$\mathrm{PT}^\nabla_\gamma(v) \coloneqq s(1) \in \mathbb{C}^n \cong V_q.$$

In particular, if $p = q \in M$, then one gets an automorphism $PT^\nabla_\gamma \in \mathrm{GL}_{\mathbb{C}}(V_p)$. Moreover, one can show that if $\nabla$ is flat then $\mathrm{PT}^\nabla_\gamma$ only depends on the homotopy class $[\gamma] \in \Pi_1(M)$ of $\gamma$ in the fundamental grupoid of $M$, and that path-concatenation gets turned into composition of linear maps. Hence a flat connection produces representations $\nu(\nabla)_p\colon \pi_1(M,p) \longrightarrow \mathrm{GL}_{\mathbb{C}}(V_p)$ for all $p \in M$.

**Definition 2.46.** The representation $\nu(\nabla)_p$ is called the holonomy representation of $\nabla$ at $p \in M$. The image $\nu(\nabla)_p\Big(\pi_1(M,p)\Big) \subseteq \mathrm{GL}(V_p)$ is the holonomy group of the flat connection $(V,\nabla)$ at $p \in M$.

The symbol $\nu$ is chosen so to match it up with the Riemann–Hilbert correspondence of Thm. 2.8. Indeed, if $G \coloneqq \mathrm{GL}(\mathbb{C}^n)$, then the monodromy data of a holomorphic $G$-connection on a compact Riemann surface $\Sigma$ of genus $g \geq 0$ amounts to its holonomy around loops that generate $\pi_1(\Sigma,p)$, where $p \in \Sigma$ (the base point is immaterial when these data are considered only up to conjugation). Notice that a priori the notion of monodromy is related to differential equations in complex domain, and to the analytic continuation of their (holomorphic) solutions along differentiable paths. We will only momentarily discuss this viewpoint in the context of Ex. 2.14.

To express the nonsingular Riemann–Hilbert correspondence more concretely, one can choose the classes of $2g$ loops $a_1, b_1, \ldots, a_g, b_g \in \pi_1(\Sigma,p)$, together with an implicit ordering, such that one has a presentation

$$\pi_1(\Sigma,p) \cong \left\langle (a_i,b_i)_{1\leq i\leq g} \,\middle|\, \prod_i [a_i,b_i] = e \right\rangle.$$

In this case the correspondence $\nu\colon \mathcal{M}_{\mathrm{dR}} \longrightarrow \mathcal{M}_{\mathrm{B}}$ is precisely obtained by associating to a gauge-equivalence class of holomorphic connections $[\nabla]$ the $G$-orbit of matrices $(M_1, N_1, \cdots, M_g, N_g) \in \mathrm{GL}(V_p)^{2g}$ defined by

$$M_i \coloneqq \mathrm{PT}^\nabla_{a_i}, \qquad N_i \coloneqq \mathrm{PT}^\nabla_{b_i},$$

where $G$ acts by diagonal conjugation. This will be used in § 4.1 and § 4.2.1.

Let us now move on to the case of $\Sigma = \mathbb{C}P^1$ with marked points, i.e. to the setup of Ex. 2.14. Let $m$ be a positive integer, and consider the marked curve $\boldsymbol{\Sigma} =$

$\left(\mathbb{C}P^1, \{t_i\}_{i\leq i\leq m}\right)$, where $t_i \in \mathbb{C}P^1$. Here irregular types vanish: it is a particular case of Def. 2.43. This pointed curve carries meromorphic connections

$$\nabla = d - \sum_i \frac{R_i}{z - t_i}, \tag{2.19}$$

over the trivial rank $n$ holomorphic vector bundle $V \cong \mathbb{C}^n \times \mathbb{C}P^1$ on the sphere, where $A_i \in \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ are residues, and $z$ is a local coordinate that identifies $\mathbb{C}P^1 \cong \mathbb{C}\coprod\{\infty\}$. We assume that $\sum_i R_i = 0$, so that the $m$-tuple $(R_1, \ldots, R_m)$ defines a point of the moduli space $\mathcal{M}^*$ of Def. 2.44, as soon as one restrict the residues to live in prescribed adjoint orbits $\mathcal{O}_i \in \mathfrak{g} \cong \mathfrak{g}^*$.

The Fuchsian system (2.19) admits global solutions which are multi-valued functions on $\mathbb{C}P^1 \setminus \Delta$, where $\Delta = \{t_i\}_i$ is the support of the polar divisor. A multi-valued function is a holomorphic function $s\colon \widetilde{\mathbb{C}P^1 \setminus \Delta} \longrightarrow \mathbb{C}^n$ defined on the universal cover of the punctured sphere. These solutions form a complex vector space of dimension $n$, on which the fundamental group of $\mathbb{C}P^1 \setminus \Delta$ acts, and this representation is by definition the monodromy of $\nabla$ (see [Bea93]). Looking for a system (2.19) with a prescribed monodromy is precisely one instance of the Riemann–Hilbert problem (which does not always admit a solution: [Bol89] constructs a counterexample).
This notion of monodromy is however equivalent to considering the holonomy representation of the holomorphic connection on $\mathbb{C}P^1\setminus\Delta$ obtained by restricting $(V, \nabla)$ outside of the singular locus. One may pick a point $p \in \mathbb{C}P^1\setminus\Delta$, choose classes $c_1, \ldots c_m \in \pi_1(\mathbb{C}P^1\setminus\Delta, p)$ of loops that run once around the punctures, and then find the presentation

$$\pi_1(\mathbb{C}P^1 \setminus \Delta, p) \cong \left\langle (c_1, \ldots, c_m) \middle| \prod_i c_i = e \right\rangle .$$

Hence the monodromy data of a connection as (2.19) is the data of $m$ invertible matrices $M_1, \ldots, M_m$ of size $n$ that factorise the identity matrix in the given order. The restriction of the residues $R_i$ to adjoint orbits $\mathcal{O}_i$ on the "additive" de Rham side amounts to restricting the monodromy matrices $M_i$ to conjugacy orbits $\mathcal{C}_i = \exp(2\pi\sqrt{-1}\mathcal{O}_i) \subseteq G$ on the "multiplicative" Betti side. Then the map $\nu\colon \mathcal{M}^*_{\mathrm{dR}}\left(\mathbf{\Sigma}, G, \{\mathcal{O}_i\}_i\right) \longrightarrow \mathcal{M}^{\mathrm{irr}}_{\mathrm{B}}\left(\mathbf{\Sigma}, G, \{\mathcal{C}\}_i\right)$ is well defined. The proof that $\nu$ is a holomorphic symplectic maps in this case is due to Hitchin and Iwasaki: [Hit97; Iwa91; Iwa92] (this was extended in [Boa01]). Notice incidentally that the multiplicative Deligne–Simpson problem consists precisely in finding conjugacy classes $\{\mathcal{C}_i\}_i$ such that the Betti moduli space is non-empty.

Now one can consider the following question. If one deforms the irregular curve $\mathbf{\Sigma}$ ever so slightly, by a small variation of the positions of the poles $p_i$ inside the configuration space $\mathbf{B} \coloneqq \mathrm{Conf}_m(\mathbb{C})$ of $m$-tuples of complex points, then the monodromy map $\nu$ will take a different value in the same moduli space as before.[19] This deformation will vary the connection (2.19), and one may ask the natural question: how should one modify the

[19] If the classes of the loops $c_i$ are chosen to bound a small disc around each puncture, then one may move every pole within its disc and keep the same presentation for the fundamental group of the punctured sphere. Notice also that we tacitly restricted deformations to a subclass of admissible ones, where coalescence of singularities is forbidden.

residues $R_i$ so that the monodromy of the new meromorphic connection is the same as the starting one? Such a deformation of $\nabla$ is thus called *isomonodromic*.
The answer to this question, due to [Sch05], has been known for more then a century now. If one thinks of the residues $R_i$ as the dependent variables, which are expressed as functions of the independent time-like variables $t_i$, then one finds the following system of nonlinear, first-order differential equations for the matrices $R_i$:

$$dR_i = \sum_{j \neq i} \frac{\left[R_i, R_j\right]}{t_i - t_j}(dt_i - dt_j). \tag{2.20}$$

This is the *Schlesinger system*. Geometrically, it amounts to a flat Ehresmann connection on the trivial symplectic fibration $\mathcal{M}^* \times \mathbf{B} \longrightarrow \mathbf{B}$, as explained in § A.2. Namely, one can show that the system (2.20) admits an Hamiltonian formulation, meaning that there is a time-dependent Hamiltonian system $H_i \colon \mathcal{M}^* \times \mathbf{B} \longrightarrow \mathbb{C}$ with space of (complex) times $\mathbf{B}$ such that one has

$$\frac{\partial (R_i)_{kl}}{\partial t_j} = \{H_j, (R_i)_{kl}\},$$

where one thinks of the components $(R_i)_{kl}$ of the residue $R_i$ as functions on $\mathcal{M}^*$. One finds the explicit formula

$$H_i^{\mathrm{Sch}} = H_i^{\mathrm{Sch}}(R_1, \dots, R_m, t_1, \dots t_m) = \sum_{j \neq i} \frac{\mathrm{Tr}(R_i R_j)}{t_i - t_j} \in \mathbb{C}. \tag{2.21}$$

This functions are called the *Schlesinger Hamiltonians*. They constitute a time-dependent, classical Hamiltonian system, in the sense of Def. A.5. To match up with that definition, one should define the horizontal 1-form

$$\varpi^{\mathrm{Sch}} \coloneqq \sum_i H_i dp_i,$$

defined on the trivial symplectic fibration at hand.

This time-dependent system is strongly flat, in the sense of Def. A.4. Now, as explained in § A.2, one can also consider the time-dependent Hamiltonians as sections of the "universal" vector bundle $A_0 \times \mathbf{B} \longrightarrow \mathbf{B}$, where $A_0 \coloneqq \mathscr{O}_{\mathfrak{g}}^{\otimes m}$ is the algebra of regular functions on the Poisson variety $\mathfrak{g}^m \cong \left(\mathfrak{g}^m\right)^*$. Recall that the dual of any Lie algebra is Poisson, via the Lie–Berezin–Kirillov bracket presented in Ex. 2.4. It is an important fact that the functions $H_i$ are invariant under the diagonal, adjoint $G$-action on $\mathfrak{g}^m$, which is just the statement that the trace is a class function. In particular, the formula (2.21) defines functions that can be harmlessly restricted to the product of adjoint orbits $\prod_i \mathcal{O}_i \subseteq \mathfrak{g}^m$ where one chooses the residue, and that moreover define functions on the quotient

$$\mathcal{M}^* \cong \mathcal{O}_1 \cdots \mathcal{O}_m /\!\!/_0 G,$$

where one uses Thm. 2.11. The advantage of working at the level of the trivial Poisson fibration $\mathfrak{g}^m \times \mathbf{B} \longrightarrow \mathbf{B}$ is that one can now use the discussion of § 2.1.3 in order to quantise the Schlesinger system (in the sense of Def. 2.6). Namely, according to Prop. 2.1, and using the universal property of tensor products, the quantum algebra $A \coloneqq U(\mathfrak{g})^{\otimes m}$

provides a filtered quantisation of $A_0$, and 2.18 defines a quantisation map $\mathcal{Q}\colon A_0 \longrightarrow A$, whose explicit formula is (2.8): it is the PBW quantisation of the symmetric algebra $\mathrm{Sym}\left(\mathfrak{g}^m\right) \cong \mathrm{Sym}(\mathfrak{g})^{\otimes m}$.

The rest of this section is devoted to proving the following result.

**Theorem 2.13.** *The PBW quantisation of the Schlesinger system (2.21) yields the (universal) Knizhnik–Zamolodchikov connection.*

Recall that the Knizhnik–Zamolodchikov connection (to be referred to as "KZ" hereafter) of [KZ84] is a connection $\widehat{\nabla}_{\mathrm{KZ}}$ in the trivial vector bundle

$$U(\mathfrak{g})^{\otimes m} \times \mathbf{B} \longrightarrow \mathbf{B}.$$

Its origins lie in certain two-dimensional models for conformal field theory, named after Wess, Zumino and Witten. The KZ connection can be written

$$\widehat{\nabla}_{\mathrm{KZ}} = d_{\mathbf{B}} - \sum_{1\leq i\leq m} \left( \sum_{i\neq j} \frac{\Omega_{ij}}{t_i - t_j} \right) dt_i, \tag{2.22}$$

for certain operators $\Omega_{ij}\colon U(\mathfrak{g})^{\otimes m} \longrightarrow U(\mathfrak{g})^{\otimes m}$. To define them, one considers the standard invariant $\mathbb{C}$-bilinear form

$$(\cdot,\cdot)\colon \mathfrak{g}\otimes\mathfrak{g} \longrightarrow \mathbb{C}, \qquad (A,B) \coloneqq \mathrm{Tr}(AB),$$

which provides an isomorphism $\mathfrak{g}^* \longrightarrow \mathfrak{g}$, as it is nondegenerate. Next, one considers the identity $\mathrm{Id}_{\mathfrak{g}} \in \mathfrak{g}\otimes\mathfrak{g}^*$, and uses the trace-duality on the second factor to turn it into an element $\Omega \in \mathfrak{g}\otimes\mathfrak{g}$. This does not depend on a choice of basis, but if one does fix a $(\cdot,\cdot)$-orthonormal basis $\{x_i\}_i$ of $\mathfrak{g}$, then

$$\mathrm{Id}_{\mathfrak{g}} = \sum_i x_i \otimes dx_i \in \mathfrak{g}\otimes\mathfrak{g}^*, \qquad \Omega = \sum_i x_i \otimes x_i \in \mathfrak{g}\otimes\mathfrak{g},$$

since $x_i$ is then dual to itself with respect to the trace-duality.
Now one uses the natural embedding $\iota_U\colon \mathfrak{g} \hookrightarrow U(\mathfrak{g})$ to consider the element $\iota_U(\Omega) \in U(\mathfrak{g})\otimes U(\mathfrak{g})$, and defines $\Omega_{ij}$ as its left multiplication on the $i$th and $j$th slot of the product $U(\mathfrak{g})^{\otimes m}$, and the identity elsewhere (for $i \neq j$).

Let us now set

$$\widehat{H}_i^{\mathrm{KZ}} \coloneqq \sum_{j\neq i} \frac{\Omega_{ij}}{t_i - t_j}, \tag{2.23}$$

Those are the *KZ Hamiltonians*. They constitute a time-dependent, strongly flat, quantum Hamiltonian system, in the sense of Def. A.7 and A.8. To precisely match it up with A.7, one should construct the 1-form

$$\widehat{\varpi}_{\mathrm{KZ}} \coloneqq \sum_{i\neq j} \Omega_{ij} d\log(t_i - t_j) \in A^1\Big(\mathbf{B}, \mathrm{End}\left(U(\mathfrak{g})^{\otimes m}\right)\Big).$$

Notice that every connection of the form $\widehat{\nabla}^{\hbar}_{\mathrm{KZ}} = d_{\mathbf{B}} - \hbar\widehat{\varpi}_{\mathrm{KZ}}$ would be strongly flat as well, for every complex parameter $\hbar \in \mathbb{C}$. Using the construction recalled in § 2.1.4 one can

add such a quantum parameter in the KZ connection, to match the above expression with what is usually found in the literature (one normally considers the dual Coxeter number of $\mathfrak{g}$: see [EFK98]).

Now, the Schlesinger Hamiltonians are smooth sections

$$H_i^{\mathrm{Sch}} \colon \mathbf{B} \longrightarrow \mathrm{Sym}(\mathfrak{g}^*)^{\otimes m},$$

since to every point $\{t_i\}_i \in \mathbf{B}$ in the base they associate a polynomial function on $\mathfrak{g}^m$. Thanks to the trace-duality $\mathfrak{g}^* \cong \mathfrak{g}$, one may as consider them as elements of $\mathrm{Sym}(\mathfrak{g})^{\otimes m}$, abusively written $H_i^{\mathrm{Sch}}$ as well. Now it makes sense to compare the fibrewise quantisation $\mathcal{Q}_{\mathrm{PBW}}(H_i^{\mathrm{Sch}}) \in U(\mathfrak{g})^{\otimes m}$ with the KZ Hamiltonian (2.23). The proof of Thm. 2.13 follows from the following remark, by appealing to the $\mathbb{C}$-linearity of the PBW quantisation.

**Proposition 2.7.** *One has* $\mathcal{Q}_{\mathrm{PBW}}\Big(\mathrm{Tr}(R_iR_j)\Big) = \Omega_{ij}$ *for all* $i \neq j \in \{1, \dots, m\}$.

*Proof.* The function $(R_1, \dots, R_m) \longmapsto \mathrm{Tr}(R_iR_j)$ is precisely the invariant scalar product $(\cdot,\cdot) \in \mathfrak{g}^* \otimes \mathfrak{g}^*$ on the $i$th and $j$th slot of the product $\mathfrak{g}^m$, and it does not depend on the other variables. Hence it is enough to show that the PBW-quantisation of the trace-dual $K \in \mathfrak{g} \otimes \mathfrak{g} \subseteq \mathrm{Sym}(\mathfrak{g}) \otimes \mathrm{Sym}(\mathfrak{g})$ of $(\cdot,\cdot)$ equals $\iota_U(\Omega) \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$.

Now, in a $(\cdot,\cdot)$-orthonormal basis one has $(\cdot,\cdot) = \sum_i dx_i \otimes dx_i$, and thus actually

$$K = \sum_i x_i \otimes x_i = \Omega.$$

The conclusion then follows from the fact that the PBW-quantisation on elements of degree one corresponds to the tautological embedding $\iota_U$ on $\mathrm{Sym}(\mathfrak{g})_1 \cong \mathfrak{g}$. □

It is in this sense that one says that the KZ connection is a quantisation of the Schlesinger system.

**Remark 2.24.** Finally, we recall here an intrinsic, geometric approach to the isomonodromy connection. The base space $\mathbf{B}$ of admissible deformation of pointed curves carries the symplectic bundles $\boldsymbol{\mathcal{M}}^* \longrightarrow \mathbf{B}$ and $\boldsymbol{\mathcal{M}}_{\mathbf{B}} \longrightarrow \mathbf{B}$ that have the relevant moduli spaces as fibres. In the Betti bundle the notion of isomonodromy is simpler: one may locally keep Stokes matrices constant (see [Boa01]). In particular, this bundle is equipped with a flat connection, the isomonodromy connection. The pull-back of this connection along the fibrewise Riemann–Hilbert correspondence $\nu \colon \mathcal{M}^* \longrightarrow \mathcal{M}_{\mathrm{B}}$ then defines a nontrivial, flat Ehresmann connection inside the de Rham bundle, and this is precisely the connection coded by the Schlesinger system. It is also shown in [Boa01] that the isomonodromy connection is symplectic, meaning that integrating it yields symplectic isomorphism among different fibres of $\boldsymbol{\mathcal{M}}^*$.
Alternatively, one can provide a cohomological interpretation of the isomonodromy connection, following Simpson [Sim94], and see the isomonodromy connection as the Gauß–Manin connection in non-Abelian cohomology for the group $G$. In general, if $\pi \colon Y \longrightarrow X$ is a fibration, one can replace the fibres $\pi^{-1}(t) = Y_t$ by their cohomology $H^\bullet(Y_t, \mathbb{C})$, where $t \in X$. This vector bundle carries a natural flat connection, which is the Gauß–Manin

connection. Abstractly, this connections arises from the existence of canonical identifications $H^\bullet(Y_t, \mathbb{C}) \cong H^\bullet(Y_s, \mathbb{C})$, obtained from the homotopy invariance of the cohomology: the restriction $Y|_D$ to a ball $D \subseteq X$ has the homotopy type of any fibre $Y_t$, for $t \in D$. It turns out that this extends to the first $G$-valued cohomology group of the punctured curve. Since the moduli spaces are avatars for this non-Abelian cohomology, this abstractly defines a flat connection in the symplectic bundles de Rham, which is the isomonodromy connection.

# Chapter 3

# Simply-laced quantum connections

In this chapter we describe the original results obtained in the direction of deformation quantisation. The aim was a generalisation of the construction outlined in § 2.3.5, more precisely Thm. 2.13, stating that the KZ connection (2.22) can be obtained as a deformation quantisation of the Schlesinger system (2.21). These results have already been presented in [Rem17].

Let us start by summarising the content of the following sections.

In § 3.1 we recall the definition of the simply-laced isomonodromy systems (SLIMS). This involves a trivial fibration $\mathbb{F}_a = \mathbb{M} \times \mathbf{B} \longrightarrow \mathbf{B}$, and a collection of time-dependent Hamiltonian functions $H_i \colon \mathbb{F}_a \longrightarrow \mathbb{C}$. These systems are attached to simply-laced, complete $k$ partite graphs.
In § 3.2 we realise the Hamiltonians as traces of cycles on the graph (classical potentials), and we study the Poisson bracket of such traces.
In § 3.3 we define quantum potentials, which are related to the quantum algebra $A$ similarly to how classical potentials are related to the functions on $\mathbb{M}$.
In § 3.4 we explain how to quantise the Hamiltonians $H_i$ to elements $\widehat{H}_i$, thereby defining the simply-laced quantum connection (SLQC).
In § 3.5 and § 3.6 we prove the main theorem (Theorem 3.2) that the simply-laced quantum connection is strongly flat.
In § 3.7 we show that a reduction of the simply-laced quantum connection yields the KZ connection in the special case of a star-shaped graph, i.e. a complete bipartite graph where one part has only one node. This means that this SLQC quantises the Schlesinger system, because so does KZ.
In § 3.8 we consider the Harnad-dual data of the previous section, and we show that a natural (strongly flat) correction of the SLQC reduces to the DMT connection. This correction amounts to a reordering within the quantum Hamiltonians, and it does not tamper with the classical dynamics, as it vanishes in the semiclassical limit. This means that this SLQC quantises the dual Schlesinger system, because so does DMT.
In § 3.9 we put together the results of the previous two sections to show that an analogous natural correction of the SLQC reduces to the FMTV connection. Together with the fact that FMTV quantises the JMMS system, this proves that this SLQC quantises the JMMS

system.
Finally, in § 3.10 we compare the SLQC with the quantisation [NS11] of a particular case of the JMU system. We argue that our cycle-theoretic formulation recovers part of the explicit Hamiltonian formulation for the systems that lie in the intersection of the isomonodromy equations of [Boa12b] and [JMU81], as well as its quantisation. This is also the occasion for writing down the simply-laced quantum connection explicitly in rank 3, via differential operators acting on polynomial functions on $\mathbb{C}^3$.

## 3.1 Simply-laced isomonodromy systems

One of the results of [Boa12b] is an explicit Hamiltonian formulation of isomonodromic deformations for certain families $\mathcal{M}^* \times \mathbf{B} \longrightarrow \mathbf{B}$ of moduli spaces of meromorphic connections (on holomorphically trivial vector bundles) on the Riemann sphere. The general setup for the definition of the moduli spaces $\mathcal{M}^*$ has been recalled in § 2.3.4, culminating in Def. 2.44. We then discussed isomonodromy in the case of simple poles, in § 2.3.5.

The theory of isomonodromy equations for meromorphic connections on the Riemann sphere has been greatly extended in [JMU81], passing from Fuchsian systems to almost generic rational matrices; the restriction is that all highest irregular terms be regular semisimple. In this chapter we relax the requirement of regularity, admitting nonsimple eigenvalues, but only consider a particular choice of irregular types, as explained in Rem. 2.21: one allows for at most one singularity of Poincaré rank 2 at infinity, together with simple poles. With these choices, one has two alternative realisations of the moduli space:

- $\mathcal{M}^*$ is isomorphic to the symplectic quotient of a symplectic space $\mathbb{M}$ of presentations of modules for the one-dimensional Weyl algebra.
- $\mathcal{M}^*$ is isomorphic to the Nakajima variety of a suitably defined graph.

We will start by discussing the former viewpoint, since it is the most natural, and very soon turn to the latter one. The upshot is the definition of a symplectic vector space $(\mathbb{M}, \omega_a)$ that surjects onto the moduli space $\mathcal{M}^*$. We will also introduce a base space $\mathbf{B}$ of admissible deformations which will play the role of the space of (complex) times for the isomonodromy systems, as explained in § A.2. Finally, and more importantly, we will define the simply-laced isomonodromy systems as a strongly-flat, time-dependent (classical) Hamiltonian systems on the trivial symplectic fibration $\mathbb{M} \times \mathbf{B} \longrightarrow \mathbb{B}$.

Let us then get started by considering three complex square matrices $\alpha, \beta, \gamma$ of size $n$. The $n \times n$ matrix

$$M \coloneqq \alpha\partial + \beta z - \gamma$$

takes its coefficients from the unitary $\mathbb{C}$-algebra $A_1(\mathbb{C})$ generated by the two symbols $z$, $\partial = \partial_z$, submitted to the commutation relation $\left[\partial, z\right] = 1$. This is precisely the Weyl algebra $W\left(T^*\mathbb{C}, \omega_{\text{can}}\right)$, in the sense of Ex. 2.5, where $\omega_{\text{can}}$ is the canonical symplectic form of the cotangent bundle. We call it the one-dimensional Weyl algebra because of

its tautological representation by differential operators acting on holomorphic functions on one complex variable. One then lets $M$ act on holomorphic functions $v\colon U \longrightarrow \mathbb{C}^n$ defined on open sets $U \subseteq \mathbb{C}$, and the equation $Mv = 0$ is a local system of differential equations for $v$.
Elementary $\mathcal{D}$-modules' theory relates the space of solutions of this system to the quotient $\mathcal{N} \coloneqq A_1^n(\mathbb{C})\Big/A_1^n(\mathbb{C})\cdot M$ (the ring $\mathcal{D}$ of differential operators being exactly the Weyl algebra in this case). This quotient is a (left) module for the one-dimensional Weyl algebra, and thus the matrices $\alpha, \beta, \gamma$ parametrise such objects. More precisely, one has a presentation of $\mathcal{N}$ defined by the exact sequence of $A_1(\mathbb{C})$-modules

$$A_1^n(\mathbb{C}) \xrightarrow{(\cdot M)} A_1^n(\mathbb{C}) \longrightarrow (0)$$

Instead of taking the full family of presentations of $A_1(\mathbb{C})$-modules defined by triples $\alpha, \beta, \gamma \in M_n(\mathbb{C})$, one restricts $\alpha, \beta$ to be commuting, diagonalisable matrices whose kernels have trivial intersection. If one lists the eigenvalues $\alpha_i, \beta_j$ of $\alpha$ and $\beta$ respectively, in some given order, then one gets points

$$a_i \coloneqq \Big[ -\beta_i \colon \alpha_i \Big]$$

of the complex projective line $\mathbb{P}(\mathbb{C}^2)$. The point at infinity would correspond to the (possibly trivial) kernel of $\alpha$ in the affine chart $\mathbb{P}(\mathbb{C}^2) \setminus \Big\{\Big[1\colon 0\Big]\Big\} \longrightarrow \mathbb{C}$ that sends $\Big[z_1 \colon z_2\Big]$ to the ratio $z_1\Big/z_2$. We identify $\mathbb{P}(\mathbb{C}^2)$ with $\mathbb{C} \coprod \{\infty\}$ in this way, but we wish to distinguish this copy of the Riemann sphere from the Riemann surface $\Sigma = \mathbb{C}P^1$ that carries the meromorphic connections: this is why we use an alternative notation.
Let now $\{a_j\}_{j\in J} \subseteq \mathbb{P}(\mathbb{C}^2)$ be the set of points thus obtained, and set $k \coloneqq |J|$ to be the cardinality of the index set. We agree to write $\infty \in J$ for the index that corresponds to the point at infinity, if it exists. For each $j \in J$ consider the vector subspace $W^j \subseteq V \coloneqq \mathbb{C}^n$ defined as the joint eigenspace for the endomorphisms $\alpha, \beta$ corresponding to pairs $(\alpha_i, \beta_i)$ of eigenvalues lying over $a_j$:

$$W^j \coloneqq \Big\{ v \in V \Big| \alpha v = \alpha_i v, \quad \beta v = \beta_i v, \quad \Big[\alpha_i \colon \beta_i\Big] = a_j \Big\}.$$

Since $\alpha$ and $\beta$ are simultaneously diagonalisable, one has $V = \bigoplus_{j\in J} W^j$. One further requires that the restriction $\gamma|_{W^j}$ be semisimple for all $j \in J$.

Let us now write $U^\infty = \bigoplus_{j\in J\setminus\{\infty\}} W^j$. If $\alpha$ is nonsingular then $V = U^\infty$, otherwise $U^\infty$ equals the whole of $V$ "minus" the kernel of $\alpha$. One can multiply $M$ on the left so to put in the following normalised form

$$M = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \partial + \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -A \end{pmatrix} z - \gamma \in \mathrm{End}(V) \otimes A_1(\mathbb{C}),$$

where the block diagonal decomposition is taken with respect to the direct sum $V = W^\infty \oplus U^\infty$; also, one has necessarily $A = \sum_{j\in J\setminus\{\infty\}} a_j I^j$, where $I^j\colon U^\infty \longrightarrow U^\infty$ is the

idempotent for $W^j$.
The endomorphism $\gamma \in \mathrm{End}(W^\infty \oplus U^\infty)$ admits a similar block decomposition

$$\gamma = \begin{pmatrix} T^\infty & P \\ Q & T+B \end{pmatrix},$$

where $T^\infty \in \mathrm{End}(W^\infty)$, $T, B \in \mathrm{End}(U^\infty)$ and

$$W^\infty \underset{P}{\overset{Q}{\rightleftarrows}} U^\infty.$$

The diagonal part $\delta(\gamma)$ of $\gamma$ in this decomposition is then

$$\delta(\gamma) =: \widehat{T} = \begin{pmatrix} T^\infty & 0 \\ 0 & T \end{pmatrix},$$

whereas the off-diagonal part $\gamma^o$ is

$$\gamma^o =: \Gamma = \begin{pmatrix} 0 & P \\ Q & B \end{pmatrix}.$$

By hypothesis, $T^\infty$ and $T$ are both semisimple.

One may now write explicitly the system of ODEs $Mv = 0$ as

$$\partial v = \Big(Az + B + T + Q(z - T^\infty)^{-1}P\Big)v,$$

for a local, $U^\infty$-valued holomorphic function $v$ defined on open sets of $\mathbb{C}$.[1] As explained in Rem. 2.19, this system of first order ODEs with rational coefficients is the same as the meromorphic connection

$$\nabla = d - \mathcal{A}, \qquad \text{where} \qquad \mathcal{A} \coloneqq \Big(Az + B + T + Q(z - T^\infty)^{-1}P\Big)dz, \tag{3.1}$$

living in the trivial vector bundle $U^\infty \times \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ over the sphere. As stated before, such connections have a pole of order at most three at infinity, plus Fuchsian singularities at the spectrum of $T^\infty$. The elements of the diagonal matrices $T^\infty$ and $T$ will be the time variables of the simply laced isomonodromy systems; we call them the *regular* and *irregular* times, respectively.

**Remark 3.1.** One of the main advantages of working at this higher level is that many symmetries become apparent. More precisely, if $N = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$, then the one can act on $M = \alpha\partial + \beta z - \gamma$ by

$$N.M \coloneqq \alpha(a\partial + bz) + \beta(c\partial + dz) - \gamma.$$

[1] We abusively keep the same notation as before for the local section. If $\infty \in \{a_j\}_{j\in J}$ then $U^\infty \neq \mathbb{C}^n$.

This action comes from the fact that $\mathrm{SL}_2(\mathbb{C})$ is the group of linear symplectomorphisms of $(T^*\mathbb{C}, \omega_{\mathrm{can}})$, whose action can be lifted to the Weyl algebra $A_1(\mathbb{C})$, and then to the tensor product $\mathrm{End}(V) \otimes A_1(\mathbb{C})$ where $M$ lives.[2] Doing this changes the meromorphic connections, as well as the underlying vector bundle on which it is defined; to see this, just remark that $N$ acts on the finite set $\{a_j\}_{j\in J} \subseteq \mathbb{P}(\mathbb{C}^2)$ via the restriction of a Möbius transformation, and thus the rank of the vector bundle need not stay fixed.

A particular case of such transformation is obtained by taking $N = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which results in the transformation $\partial \longmapsto -z$, $z \longmapsto \partial$, i.e. the Fourier–Laplace transform (at the level of differential operators).

The important fact is that the isomonodromic deformations of such connections are all governed by the same system of nonlinear differential equations, when one varies the time-like variables $T^\infty$ and $T$.

We can now define the symplectic vector space that will work as classical phase-space for the system we wish to quantise. This space must contain the dependent variables of the isomonodromy equations for the connections (3.1), which are the irregular terms $Q, P$ and $B$, i.e. the off-diagonal part $\Gamma$ of $\gamma$. With this idea in mind, one sets

$$\mathbb{M} \coloneqq \mathrm{End}(V)^o = \bigoplus_{i\neq j\in J} \mathrm{Hom}(W^i, W^j),$$

equipped with the symplectic form

$$\omega_a \coloneqq \sum_{i\neq j\in J\setminus\{\infty\}} \frac{\mathrm{Tr}\left(dB^{ij} \wedge dB^{ji}\right)}{2(a_i - a_j)} + \sum_{j\in J\setminus\{\infty\}} \mathrm{Tr}\left(dB^{j\infty} \wedge dB^{\infty j}\right),$$

where $B^{ij}$ is the linear map from $W^j$ to $W^i$ for all $i \neq j \in J$. In matrix notation, this is precisely the corresponding block component of the matrix $B$ that appears above. The subscript "$a$" stresses the dependence of this symplectic form on the weights $a_j \in \mathbb{C}$, for $j \neq \infty$.

To introduce the independent time-variables of the isomonodromy equations, that is the diagonal part $\widehat{T}$ of $\gamma$, one now looks to a more refined decomposition of $V$; namely, consider the decomposition of the direct addends $W^j$ in eigenspaces for the semisimple matrices $T^j \coloneqq \gamma|_{W^j}$. One may accordingly write

$$W^j = \bigoplus_{i\in I^j} V_i,$$

where $I^j$ is the set of eigenspaces of $T^j$. This decomposition defines an open space

$$\mathbf{B} \coloneqq \prod_{j\in J} \mathbb{C}^{|I^j|} \setminus \{\text{diags}\} \subseteq \mathbb{C}^{|I|}, \tag{3.2}$$

where $I \coloneqq \coprod_{j\in J} I^j$.

[2] One must also check that if $\alpha, \beta$ are commuting semisimple matrices whose kernels intersect trivially, then the same holds for $a\alpha + c\beta$ and $b\alpha + d\beta$, provided that $ad - bc \neq 0$.

**Remark 3.2.** One may think of $\mathbf{B}$ as a space of deformations of the diagonal matrices $T^j = \sum_{i\in I^j} \mathrm{Id}_i\, t_i$, where $\mathrm{Id}_i\colon W^j \longrightarrow W^j$ is the idempotent for $V_i$ in $W^J$, and where $\{t_i\}_{i\in I^j}$ is the spectrum of $T^j$. Such deformations preserve the decomposition of $W^j$ into eigenspaces for $T^j$, since removing the "diagonals" from $\mathbb{C}^{|I^j|}$ precisely avoids that different eigenvalues will coalesce. This must be thought of as a generalisation of the base space $\mathrm{Conf}_m(\mathbb{C})$ for the trivial fibration on which the Schlesinger system is defined (in the case of $m$ simple poles; see § 2.3.5).

We shall refer to these as the *admissible deformations*.

Finally, one defines $\mathbb{F}_a \coloneqq \mathbb{M} \times \mathbf{B}$ as a trivial fibration in symplectic vector spaces, and proceeds to define a strongly flat Ehresmann connection inside $\mathbb{F}_a$ that codes the isomonodromic deformations of the connections (3.1), with respect to variations of $T^\infty$ and $T$, but keeping the highest irregular term fixed.[3] The sense in which this Ehresmann connection codes isomonodromy equations is totally analogous to what happens in the situation of § 2.3.5 for the Schlesinger system.
Before doing this, however, we turn to the graph-theoretic presentation of the theory, turning around the one just made from the viewpoint of $\mathcal{D}$-modules. To this end, notice that the construction of the trivial symplectic fibration $\mathbb{F}_a$ only relies on the choices of:

- A finite set $J$.
- A second finite set $I$, together with a partition $I = \coprod_{j\in J} I^j$ indexed by $J$.
- An embedding $a\colon J \hookrightarrow \mathbb{P}(\mathbb{C}^2)$, written $a\colon j \longmapsto a_j$ (where we still just write $\infty \in J$ for the index $a^{-1}(\infty)$, if it exists).

Starting from all this data, let $\widetilde{\mathcal{G}}$ be the complete graph on nodes $J$, and $\mathcal{G}$ the complete $k$-partite graph on nodes $I$. The latter means that two nodes of $\mathcal{G}$ are adjacent if and only if they lie in different parts of $I$. Both $\widetilde{\mathcal{G}}$ and $\mathcal{G}$ are by definition simply-laced, i.e. without edge loops or repeated edges.

**Definition 3.1.** The embedding $a\colon J \hookrightarrow \mathbb{P}(\mathbb{C}^2)$ is called a reading of $\widetilde{\mathcal{G}}$. The reading is generic if $\infty \notin a(J)$, and degenerate otherwise. We extend the reading to a map $a\colon I \longrightarrow \mathbb{P}(\mathbb{C}^2)$ by declaring it to be constant to $a_j$ on each part $I^j$ of $I$, and we call it a reading for $\mathcal{G}$.

Let us abusively denote by $\widetilde{\mathcal{G}}$ the quiver associated to the graph $\widetilde{\mathcal{G}}$: it is the quiver on nodes $I$ having a pair of antiparallel arrows for each edge of $\widetilde{\mathcal{G}}$. The same abuse of notation will be taken for $\mathcal{G}$.
Finally, we attach complex vector spaces $V_i$ to the nodes of $\mathcal{G}$, and we set $W^j \coloneqq \bigoplus_{i\in I^j} V_i$.

These data determine the same symplectic fibration $\mathbb{F}_a = (\mathbb{M}, \omega_a) \times \mathbf{B}$ as above, but now one thinks of $\mathbb{M}$ as the space of representations of the quiver $\widetilde{\mathcal{G}}$ with respect to the

[3] Indeed, the diagonal term $A$ only appear in the definition of the symplectic structure, but not in the space $\mathbb{M}$ itself. If one allows for $A$ to vary too, then one would no longer have a trivial symplectic fibration.

vector spaces $\{W^j\}_{j\in J}$:

$$\mathbb{M} \cong \mathrm{Rep}\left(\widetilde{\mathcal{G}}, \{W_j\}_{j\in J}\right) = \bigoplus_{i\neq j\in J} \mathrm{Hom}(W^i, W^j).$$

Notice that this is the same as the representation space of the finer $k$-partite graph, if one just decomposes the linear maps $B^{ij}\colon W^j \longrightarrow W^i$ in block components with respect to the direct sums $W_j = \bigoplus_{i\in I^j} V_i$.

As a last piece of notation, define $X^{ij}\colon W^j \longrightarrow W^i$ to be the scalar multiplication of $B^{ij}$ by the weight $\phi_{ij} \in \mathbb{C}$, where

$$\phi_{ij} = -\phi_{ji} \coloneqq \begin{cases} (a_i - a_j)^{-1}, & a_i, a_j \neq \infty \\ 1, & a_i = \infty \end{cases}.$$

With this notation introduced, the formula for the symplectic form simplifies to

$$\omega_a \coloneqq \frac{1}{2} \sum_{i\neq j\in J} \mathrm{Tr}\left(dX^{ij} \wedge dB^{ji}\right).$$

**Remark 3.3.** Notice that we did not choose an orientation for the arrows of neither graphs. Such a choice is necessary in order to characterise the representation space as a cotangent bundle, which leads to the usual definition of the Nakajima varieties of a quiver ([Nak94]): a representation of a graph is the same thing as a representation of the double of any quiver obtained by choosing an orientation of the graph. The symplectic form $\omega_a$ instead relies on the reading $a\colon J \hookrightarrow \mathbb{P}(\mathbb{C}^2)$.
This notwithstanding, it is shown in [Boa12b] that $(\mathbb{M}, \omega_a)$ and the cotangent bundle

$$T^*\left(\bigoplus_{e\in\mathcal{G}_1} \mathrm{Hom}\left(V_{t(e)}, V_{h(e)}\right)\right)$$

are isomorphic as Hamiltonian $\widehat{H}$-spaces, where $t, h\colon \mathcal{G}_1 \longrightarrow I$ denote any choice of tail and head functions, assigning to each edge of $\mathcal{G}$ its source and target, respectively.

The symplectic reduction of $(\mathbb{M}, \omega_a)$ with respect to $\widehat{H}$ is then isomorphic to a Nakajima quiver variety, for all choice of orbits in the dual Lie algebra of $\widehat{H}$. This symplectic reduction is also isomorphic to the moduli space $\mathcal{M}^*$ described in Rem. 2.21: see again [Boa12b], from which we recall one last result.

**Theorem 3.1.** *The isomonodromy deformation (isomonodromy) equations for the meromorphic connections (3.1) admit an Hamiltonian formulation via a time-dependent Hamiltonian system $H_i\colon \mathbb{F}_a \longrightarrow \mathbb{C}$. Moreover, the Hamiltonian system is strongly flat, in the sense of Def. A.4:*

$$\{H_i, H_j\} = 0 = \frac{\partial H_i}{\partial t_j} - \frac{\partial H_j}{\partial t_i}, \qquad \text{for } i \neq j.$$

As stated above, the isomonodromy equations are nonlinear first order PDEs for $\Gamma = \begin{pmatrix} 0 & P \\ Q & B \end{pmatrix}$, as a function of $\widehat{T} = \begin{pmatrix} T^\infty & 0 \\ 0 & T \end{pmatrix}$. The fact that this problem admits an Hamiltonian formulation means that the time-dependent Hamiltonian system $H_i\colon \mathbb{F}_a \longrightarrow \mathbb{C}$ in

the statement is such that the isomonodromy equations can be written

$$\frac{\partial \Gamma_j}{\partial t_i} = \{H_i, \Gamma_j\},$$

for all components $\Gamma_i$ of a local section $\Gamma$ of the fibration.

The definition of the Hamiltonians is implicitly given by defining the following horizontal 1-form $\varpi = \sum_{i\in I} H_i dt_i$:

$$\varpi \coloneqq \frac{1}{2}\operatorname{Tr}\left(\widetilde{\Xi\Gamma}\delta(\Xi\Gamma)\right) - \operatorname{Tr}\left(\Xi\gamma\Xi d\widehat{T}\right) + \operatorname{Tr}\left(X^2 T dT\right) + \operatorname{Tr}\left(PAQT^\infty dT^\infty\right). \tag{3.3}$$

One will have $H_i = \langle \varpi, \partial_{t_i}\rangle$.

In the above formula we define $\Xi \coloneqq \phi(\Gamma)$ and $X \coloneqq \phi(B)$, applying the alternating weights $\phi_{ij} \in \mathbb{C}$ componentwise. Also, $\delta(\Xi\Gamma)$ denotes the diagonal part of $\Xi\Gamma$ in the direct sum decomposition $V = \bigoplus_{j\in J} W^j$, and one defines

$$\widetilde{\Xi\Gamma} \coloneqq \operatorname{ad}_{\widehat{T}}^{-1}\left[d\widehat{T}, \Xi\Gamma\right].$$

As explained in § A.2, the horizontal 1-form $\varpi$ is in itself a time-dependent Hamiltonian system, expressed in a coordinate-independent fashion. Its coefficients $H_i$ can be thought of as global sections of the (universal) vector bundle

$$A_0 \times \mathbf{B} \longrightarrow \mathbf{B},$$

where $A_0 \coloneqq \mathscr{O}(\mathbb{M}) \cong \operatorname{Sym}(\mathbb{M}^*)$ is the algebra of regular function on the affine complex space $\mathbb{M}$.

**Definition 3.2.** The simply-laced isomonodromy system (SLIMS) attached to the complete $k$-partite graph $\mathcal{G}$ on nodes $I$, to the vector spaces $\{V_i\}_{i\in I}$ and to the reading $a\colon I \longrightarrow \mathbb{C}\cup\{\infty\}$ is the time-dependent Hamiltonian system (3.3).

This is the classical system we wish to quantise, in the sense explained in § 2.1. To this end, we first express the algebraic functions $H_i$ as traces of potentials on the $k$-partite quiver.

## 3.2 Classical potentials

Consider again the complete $k$-partite quiver $\mathcal{G}$ on nodes $I = \coprod_{j\in J} I^j$.

**Definition 3.3.** A potential $W$ on $\mathcal{G}$ is a $\mathbb{C}$-linear combination of oriented cycles in $\mathcal{G}$, defined up to cyclic permutations of their arrows. The space of potentials is denoted $\mathbb{C}\mathcal{G}_{\mathrm{cycl}}$.

Every potential $W \in \mathbb{C}\mathcal{G}_{\mathrm{cycl}}$ defines a regular function on $\mathbb{M}$, by taking the traces of its cycles in every given representation. Thus a time-dependent potential $W\colon \mathbf{B} \longrightarrow \mathbb{C}\mathcal{G}_{\mathrm{cycl}}$ will define a global section $\operatorname{Tr}(W)\colon \mathbf{B} \longrightarrow A_0$.

Introducing the natural notation $I^i := \pi^{-1}(\pi(i)) \subseteq I$ for the part of $I$ containing the node $i \in I$, consider the following potentials on $\mathcal{G}$:

$$\begin{aligned} W_i(2) &:= \sum_{j \in I \setminus I^i} (t_i - t_j)\alpha_{ij}\alpha_{ji}, \\ W_i(3) &:= \sum_{j,l \in I \setminus I^i \,:\, I^j \neq I^l} (a_j - a_l)\alpha_{il}\alpha_{lj}\alpha_{ji}, \\ W_i(4) &:= \sum_{m \in I^i \setminus \{i\}, j,l \in I \setminus I^i} (a_i - a_j)(a_i - a_l)\frac{\alpha_{ij}\alpha_{jm}\alpha_{ml}\alpha_{li}}{t_i - t_m}. \end{aligned} \tag{3.4}$$

Here $\alpha_{ij}$ denotes the arrow from $j$ to $i$ in $\mathcal{G}$, and one writes a cycle in $\mathcal{G}$ as the sequence of its arrows, reading from right to left. We will speak of an $m$-cycle to refer to a cycle $C = \alpha_m \dots \alpha_1$ having $m$ arrows, and write $l(C) = m$ in that case.

**Proposition 3.1.** *The Hamiltonian $H_i$ of the simply-laced isomonodromy system (3.3) is the sum of the traces of the potentials (3.4), for a generic reading of $\mathcal{G}$:*

$$H_i = \operatorname{Tr}\Big(W_i(4)\Big) + \operatorname{Tr}\Big(W_i(3)\Big) + \operatorname{Tr}\Big(W_i(2)\Big).$$

*Moreover, in a degenerate reading one only needs to change the weights of the same cycles that appear in the potentials (3.4).*

Notice that the trace $\operatorname{Tr}\Big(W_i(d)\Big)$ defines an homogeneous element of degree $d$ in $A_0$, for $d \in \{2, 3, 4\}$.

*Proof.* This follows from an explicit expansion of the formula for $\varpi$. Let us provide some detail.

One may write $\varpi = \varpi(4) + \varpi(3) + \varpi(2)$, taking $\varpi(d)$ to be the term of order $d$ of the time-dependent system, i.e. the one satisfying $\langle \varpi(d), \partial_{t_i} \rangle = \operatorname{Tr}\Big(W_i(d)\Big)$. The highest-order term expands as:

$$\begin{aligned} \varpi(4) &= \frac{1}{2}\operatorname{Tr}\Big(\widetilde{\Xi\Gamma}\delta(\Xi\Gamma)\Big) = \frac{1}{2}\operatorname{Tr}\Big(\widetilde{XB}\delta(XB)\Big) = \\ &= \frac{1}{2}\sum_{i,j,k,l,m,n,p} (a_m - a_n)(a_m - a_p)\operatorname{Tr}(X_{ij}^{mn}X_{jk}^{nm}X_{kl}^{mp}X_{li}^{pm})d\log(t_i^m - t_k^m) \end{aligned}$$

In the above sum, one has $n \neq m \neq p \in J, i \neq k \in I^m, j \in I^n, l \in I^p$.

This means that the order-four part of the Hamiltonian for the time $t_i$ will be

$$\langle \varpi(4), \partial_{t_i} \rangle = \sum_{j,k \neq i,l,n,p} (a_m - a_n)(a_m - a_p)\frac{\operatorname{Tr}(X_{ij}^{mn}X_{jk}^{nm}X_{kl}^{mp}X_{li}^{pm})}{t_i - t_k}.$$

The crucial remark is that one is summing over all 4-cycles of the form

$$V_i \longrightarrow V_l \longrightarrow V_k \longrightarrow V_j \longrightarrow V_i,$$

each one with the weighted by the (time-dependent) complex coefficient $\frac{(a_m-a_n)(a_m-a_p)}{t_i-t_k}$. The numerator of this weight only depends on the coarse decomposition $V=\bigoplus_{j\in J}W^j$, whereas the denominator depends on the finer $V=\bigoplus_{j\in J,i\in I^j}V_i^j$. In particular, this shows that the potential $W_i(4)$ is chosen correctly in (3.4).

Similarly, going down with the order:

$$\varpi(3)=-\operatorname{Tr}(XBXdT)=-\sum_{m,n,p,i,j,l}(a_n-a_p)\operatorname{Tr}(X_{ij}^{mn}X_{jl}^{np}X_{li}^{pm})dt_i$$

where $m\neq n\neq p\in J, i\in I^m, j\in I^n, l\in I^p$. Hence one is now summing all 3-cycles:

$$V_i\longrightarrow V_l\longrightarrow V_j\longrightarrow V_i,$$

each one with weight $(a_p-a_n)\in\mathbb{C}$ (a constant function on $\mathbf{B}$ in this case).

Finally, one has:

$$\varpi(2)=\operatorname{Tr}(X[X,T]dT)=\sum_{m,n,i,j}\operatorname{Tr}(X_{ij}^{mn}X_{ji}^{nm})(t_i-t_j)dt_i$$

where $m\neq n\in J, i\in I^m, j\in I^n$. This is a sum over the '2-cycles

$$V_i\longrightarrow V_j\longrightarrow V_i,$$

with weights $(t_i-t_j)\in C^\infty(\mathbf{B},\mathbb{C})$. □

We made the choice to express all algebraic functions on $\mathbb{M}$ as polynomials in the corrected variables $X^{ij}=\phi_{ij}B^{ij}$. This is done in order to achieve simpler-looking formulae for the potentials.

**Corollary 3.1.** *The simply-laced Hamiltonians $H_i$ are invariant for the natural $\widehat{H}$-action on $\mathbb{M}$.*

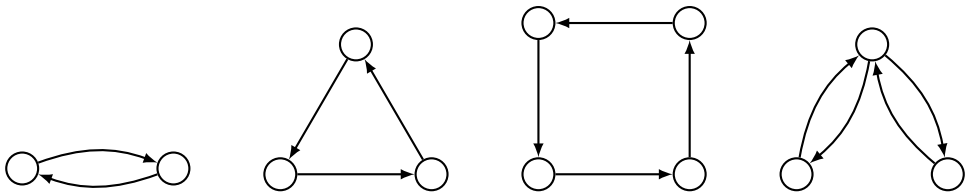
This just follows from the fact that the trace is a class function, and $\widehat{H}=\prod_{i\in I}\mathrm{GL}_{\mathbb{C}}(V_i)$ acts by simultaneous conjugations. In particular, the Hamiltonians $H_i$ descend to a time-dependent Hamiltonian system on the symplectic reduction of $(\mathbb{M},\omega_a)$ at any coadjoint orbit, i.e. on the trivial symplectic fibration $\mathcal{M}^*\times\mathbf{B}\longrightarrow\mathbf{B}$. One can thus work on the upper level of $\mathbb{M}$ to quantise a time-dependent classical system on the moduli space itself.

Let us now introduce some terminology, for further use.

**Definition 3.4.** The potentials $W_i(n)$ of (3.4) are called the (classical) isomonodromy potentials, for $i\in I, 2\leq n\leq 4$. The isomonodromy cycles are the cycles that appear in their addends. The isomonodromy 4-cycles can be further divided in two families:

1. nondegenerate, if they touch 4 distinct nodes of $\mathcal{G}$.
2. degenerate, if they touch 3 distinct nodes of $\mathcal{G}$.

A picture will help clarifying; the isomonodromy cycles are the following type of cycles in $\mathcal{G}$:

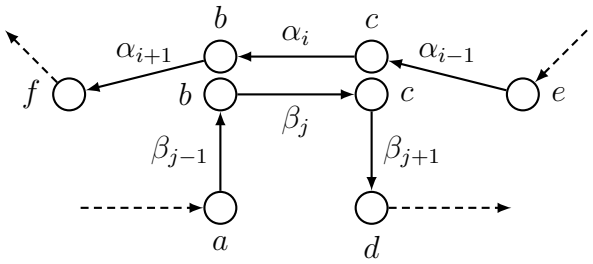


In order from left to right, one has 2-cycles, 3-cycles, nondegenerate 4-cycles and degenerate 4-cycles. Notice that the degenerate 4-cycle are the glueing of two 2-cycles at a well defined node, which will be called their *centre*.[4]

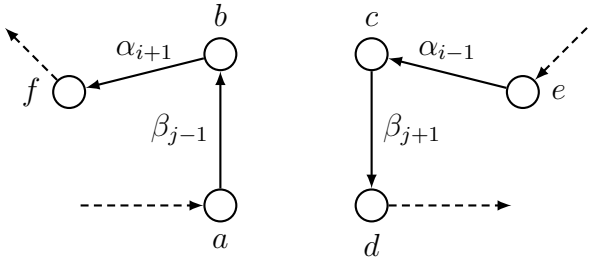
As a last remark, there is a natural (positively) graded Lie structure $\{,\}$ on $\mathbb{C}\mathcal{G}_{\text{cycl}}$, where the gradation is given by cycle length, called the necklace Lie algebra structure (see e.g. [BL02; Eti07]). If $\alpha$ is an arrow in $\mathcal{G}$, we write $\alpha^*$ for its (unique) antiparallel one.

**Definition 3.5.** Pick two oriented cycles $C_1 = \alpha_n \cdots \alpha_1$ and $C_2 = \beta_m \cdots \beta_1$ in $\mathcal{G}$. The Lie bracket $\{C_1, C_2\}$ is a weighted sum of $(n+m-2)$-cycles obtained as follows. For all pairs of antiparallel arrows $\alpha_i, \beta_j = \alpha_i^*$, one deletes that pair and glues together the two remaining cycles. The weights are determined by the defining relation of the Poisson bracket of $A_0$.

To see this graphically, fix a pair $i, j$ such that $\alpha_i = \beta_j^*$, and introduce the notation $t(\alpha), h(\alpha) \in I$ for the tail and the head of an arrow $\alpha$ in $\mathcal{G}$, respectively; these are the starting node of $\alpha$ and the end node of $\alpha$, respectively. Set then $a = t(\beta_{j-1}), b = h(\beta_{j-1}) = h(\alpha_i), c = h(\beta_j) = h(\alpha_{i-1}), d = h(\beta_{j+1}), e = t(\alpha_{i-1}), f = h(\alpha_{i+1}) \in I$. Then the local picture before deleting arrows looks like this:




Afterwards, one will have:




[4]The other two nodes must lie in one and the same part of $I$, looking at the indices of $W_i(4)$ in (3.4).

Now, the nice fact is that this bracket comes from the Poisson structure $\{\cdot,\cdot\}_{A_0}$ of $A_0$.

**Proposition 3.2.** *One has*

$$\mathrm{Tr}\left\{C_1, C_2\right\} = \left\{\mathrm{Tr}(C_1), \mathrm{Tr}(C_2)\right\}_{A_0} \in A_0$$

*for all cycles* $C_1, C_2 \in \mathbb{C}\mathcal{G}_{\mathrm{cycl}}$.

*Proof.* The proof consists of a direct expansion of the Poisson bracket

$$\left\{\mathrm{Tr}(C_1), \mathrm{Tr}(C_2)\right\}_{A_0} = \left\{\mathrm{Tr}(X^{\alpha_n}\cdots X^{\alpha_1}), \mathrm{Tr}(X^{\beta_m}\cdots X^{\beta_1})\right\}_{A_0}.$$

To simplify the notation, introduce multi-indices $K \coloneqq (k_n, \dots, k_1), L \coloneqq (l_m, \dots, l_1)$ that vary respectively in the products of sets

$$D_{C_1} \coloneqq \{1, \dots, d_{h(\alpha_n)}\}\times\cdots\times\{1, \dots, d_{h(\alpha_1)}\} \quad \text{and} \quad D_{C_2} \coloneqq \{1, \dots, d_{h(\beta_m)}\}\times\cdots\times\{1, \dots, d_{h(\beta_1)}\},$$

where $d_i \coloneqq \dim(V_i)$ for all $i \in I$.

One Has:

$$\left\{\mathrm{Tr}(X^{\alpha_n}\cdots X^{\alpha_1}), \mathrm{Tr}(X^{\beta_m}\cdots X^{\beta_1})\right\}_{A_0} = \sum_{K\in D_{C_1}, L\in D_{C_2}} \left\{X^{\alpha_n}_{k_n,k_{n-1}}\cdots X^{\alpha_1}_{k_1,k_n}, X^{\beta_m}_{l_m,l_{m-1}}\cdots X^{\beta_1}_{l_1,l_m}\right\}_{A_0} =$$

$$= \sum_{K,L,1\le i\le n,1\le j\le m} \left\{X^{\alpha_i}_{k_i,k_{i-1}}, X^{\beta_j}_{l_j,l_{j-1}}\right\}_{A_0} X^{\alpha_n}_{k_n,k_{n-1}}\cdots \widehat{X^{\alpha_i}_{k_i,k_{i-1}}} \cdots X^{\alpha_1}_{k_1,k_n}\cdot X^{\beta_m}_{l_m,l_{m-1}}\cdots \widehat{X^{\beta_j}_{l_j,l_{j-1}}}\cdots X^{\beta_1}_{l_1,l_m} =$$

$$= \sum_{K,L,i,j:\alpha_i=\beta_j^*} c_{ij}\delta_{k_i,l_{j-1}}\delta_{k_{i-1},l_j} X^{\alpha_n}_{k_n,k_{n-1}}\cdots \widehat{X^{\alpha_i}_{k_i,k_{i-1}}} \cdots X^{\alpha_1}_{k_1,k_n}\cdot X^{\beta_m}_{l_m,l_{m-1}}\cdots \widehat{X^{\beta_j}_{l_j,l_{j-1}}}\cdots X^{\beta_1}_{l_1,l_m} =$$

$$= \sum_{K,L,i,j:\alpha_i=\beta_j^*} c_{ij} X^{\alpha_{i-1}}_{k_{i-1},i-2}\cdots X^{\alpha_1}_{k_1,k_n} X^{\alpha_n}_{k_n,k_{n-1}}\cdots X^{\alpha_{i+1}}_{k_{i+1},k_i}\cdot X^{\beta_{j-1}}_{k_i,l_{j-2}}\cdots X^{\beta_1}_{l_1,l_m} X^{\beta_m}_{l_m,l_{m-1}}\cdots X^{\beta_{j+1}}_{l_{j+1},k_{i-1}} =$$

$$= \sum_{i,j:\alpha_i=\beta_j^*} c_{ij}\,\mathrm{Tr}(X^{\beta_m}\cdots X^{\beta_{j+1}}\cdot X^{\alpha_{i-1}}\cdots X^{\alpha_1}X^{\alpha_n}\cdots X^{\alpha_{i+1}}\cdot X^{\beta_{j-1}}\cdots X^{\beta_1}).$$

Here a hat denotes an arrow that has been left out, and one used

$$\left\{X^{\alpha_i}_{k_i,k_{i-1}}, X^{\beta_j}_{l_j,l_{j-1}}\right\}_{A_0} = c_{ij}\delta_{k_i,l_{j-1}}\delta_{k_{i-1},l_j} \in \mathbb{C}.$$

Notice that $\alpha_i = \beta_j^*$ implies $t(\alpha_{i+1}) = h(\beta_{j-1})$ and $h(\alpha_{i-1}) = t(\beta_{j+1})$. Hence the compositions $X^{\alpha_{i+1}}\cdot X^{\beta_{j-1}}$ and $X^{\beta_{j+1}}\cdot X^{\alpha_{i-1}}$ make sense. □

Conceptually, what happens is the following. The invariant regular functions on $\mathbb{M}$ for the action of $\widehat{H}$ consist of the $\mathbb{C}$-algebra $A_0^{\widehat{H}} \subseteq A_0$ generated by traces of cycles. Hence we have an injective map $\mathrm{Tr}\colon \mathbb{C}\mathcal{G}_{\mathrm{cycl}} \hookrightarrow A_0^{\widehat{H}}$, and the above discussion shows that this is a Lie algebras' morphism: the necklace Lie bracket is the pull-back of the Poisson bracket on $A_0$.

Last, notice that it is not possible to upgrade $\mathbb{C}\mathcal{G}_{\mathrm{cycl}}$ to a Poisson algebra using the natural concatenation product, since $\mathrm{Tr}(AB) \neq \mathrm{Tr}(A)\,\mathrm{Tr}(B)$ is clearly generally false for endomorphisms $A, B$ of a vector space. Rather, one should define a formal product of cycles that satisfies the same rules as the product of their traces, i.e. be commutative. This is well expressed by the following elementary algebraic fact.

**Proposition 3.3.** *Let $V$ be a complex vector space, and $\iota\colon V \hookrightarrow B$ a $\mathbb{C}$-linear embedding into a $\mathbb{C}$-algebra. Then there is a natural tensor map* $\mathrm{Tens}(\iota)\colon \mathrm{Tens}(V) \longrightarrow B$, *defined on pure tensors as*

$$\mathrm{Tens}(\iota)(v_1 \otimes \cdots \otimes v_n) \coloneqq \iota_1(v) \cdots \iota_n(v) \in B.$$

*This map is surjective on the subalgebra $B' \subseteq B$ generated by the vector space $\iota(V) \subseteq B$, and it induces an isomorphism of $\mathbb{C}$-algebras*

$$\mathrm{Tens}(V) \Big/ \mathrm{Ker}(\mathrm{Tens}(\iota)) \cong B'.$$

This is just an application of the universal properties of tensor products and quotients. In the case at hand, one finds an isomorphism $A_0^{\widehat{H}} \cong \mathrm{Sym}(\mathbb{C}\mathcal{G}_{\mathrm{cycl}})$. This identification is just saying that all $\widehat{H}$-invariant regular functions on $\mathbb{M}$ are monomials of (traces of) cycles. Notice that the Lie bracket of $\mathbb{C}\mathcal{G}_{\mathrm{cycl}}$ is now tautologically upgraded to a Poisson bracket on $\mathrm{Sym}(\mathbb{C}\mathcal{G}_{\mathrm{cycl}})$, and this symmetric algebra is isomorphic to $A_0^G$ as a graded commutative Poisson algebra, because of Prop. 3.2. We will present a quantum counterpart of this, in § 3.3.

**Remark 3.4.** As a first application of the cycle-theoretic viewpoint, one can provide a direct proof of "half" of the strong flatness of the SLIMS. More precisely, remark that one has

$$\mathrm{Tr}(\partial_{t_i} W_j) = \partial_{t_i} \mathrm{Tr}(W_j)$$

for all $i, j \in I$, where $W_j$ is an isomonodromy potential. This is because the derivative does not modify the cycles that make up the potentials, but only their weights. Hence showing that $\partial_{t_i} H_j - \partial_{t_j} H_i = 0$ is equivalent to showing that $\partial_{t_i} W_j - \partial_{t_j} W_i = 0$, because of the injectivity of $\mathrm{Tr}\colon \mathbb{C}\mathcal{G}_{\mathrm{cycl}} \hookrightarrow A_0$.

**Proposition 3.4.** *One has $\partial_{t_i} W_j = \partial_{t_j} W_i$ for all $i, j \in I$.*

*Proof.* One can clearly assume $i \neq j \in I$. Then one has

$$\partial_{t_j} W_i(2) = \begin{cases} -\alpha_{ij}\alpha_{ji}, & I^i \neq I^j \\ 0, & \text{else} \end{cases} \qquad \text{and} \qquad \partial_{t_i} W_j(2) = \begin{cases} -\alpha_{ji}\alpha_{ij}, & I^i \neq I^j \\ 0, & \text{else} \end{cases}.$$

Also

$$\partial_{t_j} W_i(3) = 0 = \partial_{t_i} W_j(3),$$

since all 3-cycles are actually time-independent in our setting. Finally,

$$\partial_{t_j} W_i(4) = \begin{cases} \sum_{m,l \in I \setminus I^i} (a_i - a_m)(a_i - a_l) \frac{\alpha_{im}\alpha_{mj}\alpha_{jl}\alpha_{li}}{(t_i - t_j)^2}, & I^i = I^j \\ 0, & \text{else} \end{cases}$$

and

$$\partial_{t_i} W_j(4) = \begin{cases} \sum_{m,l \in I \setminus I^j} (a_j - a_m)(a_j - a_l) \frac{\alpha_{jl}\alpha_{li}\alpha_{im}\alpha_{mj}}{(t_j - t_i)^2}, & I^i = I^j \\ 0, & \text{else} \end{cases}.$$

This is seen explicitly on the formulae (3.4), and proves the claim, because

$$\alpha_{ij}\alpha_{ji} = \alpha_{ji}\alpha_{ij} \in \mathbb{C}\mathcal{G}_{\mathrm{cycl}} \qquad \text{and} \qquad \alpha_{jl}\alpha_{li}\alpha_{im}\alpha_{mj} = \alpha_{im}\alpha_{mj}\alpha_{jl}\alpha_{li} \in \mathbb{C}\mathcal{G}_{\mathrm{cycl}}.$$

For the case of 4-cycles, one must also recall that $a_i = a_j$ if $I^i = I^j$, because the reading only depends on the parts of $I$. □

## 3.3 Quantisation of potentials

Let now $A \coloneqq W(\mathbb{M}, \omega_a)$ be the Weyl algebra of $(\mathbb{M}, \omega_a)$, defined as in 2.5. Recall that there is a natural quantisation map $\mathcal{Q}_W \colon A_0 \longrightarrow A$, described in § 2.1.5.
The problem we face is that if one naively uses this canonical quantisation to quantise the SLIMS, in the sense of Def. 2.6, then one gets a time-dependent quantum system which fails to be flat. The problem lies in the necessity of symmetrising all monomials in $A_0$ to have a well defined map in the quantum algebra. Namely, recall that one can associate $n!$ different quantum operators to all monomial $f = X_1 \cdots X_n$ of degree $n$. One has elements

$$\widehat{X}_{\sigma(1)} \cdots \widehat{X}_{\sigma(n)} \in A_{\leq n},$$

for all permutation $\sigma \in \Sigma_n$ on $n$ objects, where $\widehat{X}_i \in A$ is the class of $X_i \in \mathrm{Tens}(\mathbb{M}^*)$ in $A$ (see § 2.1 for all the background details). The lack of a canonical way to pick one of them is one of the main issues of the theory, i.e. extending the quantisation $X \longmapsto \widehat{X}$ of linear functions to a full quantisation $f \longmapsto \widehat{f} \colon A_0 \longrightarrow A$.

This notwithstanding, the idea we propose is to code quantum operators via decorated cycles, just as we coded ($\widehat{H}$-invariant) functions on $\mathbb{M}$ via cycles.
Consider again the complete $k$-partite quiver $\mathcal{G}$.

**Definition 3.6.** An anchored cycle $\widehat{C}$ is an oriented cycle in $\mathcal{G}$ with a starting arrow fixed, to be called the anchor of $\widehat{C}$. We will denote this by underlining the anchor:

$$\widehat{C} = \alpha_n \cdots \underline{\alpha_1},$$

where $\alpha_n, \ldots, \alpha_1$ are arrows in $\mathcal{G}$.

Now suppose that $f \in A_0$ is a monomial coming from the trace of a cycle $C = \alpha_n \cdots \alpha_1 \in \mathbb{C}\mathcal{G}_{\mathrm{cycl}}$. This means that $f$ is one monomial of the sum

$$\mathrm{Tr}(C) = \sum_K X^{\alpha_n}_{k_n,k_{n-1}} \cdots X^{\alpha_1}_{k_1,k_n} \in A_0,$$

where $K = (k_n, \ldots, k_1)$ is an appropriate multi-index. Now, if one picks an anchor for $C$, say that $\widehat{C} \coloneqq \alpha_n \cdots \underline{\alpha_1}$, then the quantum operator

$$\widehat{f} \coloneqq \sum_{K \in D_C} \widehat{X}^{\alpha_n}_{k_n,k_{n-1}} \cdots \widehat{X}^{\alpha_1}_{k_1,k_n} \in A_{\leq n}$$

is uniquely defined, and one can in turn define $\mathrm{Tr}(\widehat{C}) \in A$ to be that operator. In hindsight, and more intrinsically, one could just consider the operator-valued matrix

$$\widehat{X}^{\alpha_n} \cdots \widehat{X}^{\alpha_1} \in A \otimes \mathrm{End}(V_i),$$

where $i \coloneqq t(\alpha_1) \in I$ is the starting node of $\widehat{C}$. Taking the trace then amounts again to contracting $V_i$ against $V_i^*$.

As a final remark, notice that two different anchored cycles $\widehat{C}_1, \widehat{C}_2$ may define the same quantum operator. This happens when their two underlying cycles coincide under an "admissible" permutation of their arrows: no arrow $\alpha$ may pass over its antiparallel $\alpha^*$. This is because the entries of matrices $\widehat{X}^{\alpha}, \widehat{X}^{\beta}$ commute if and only if $\alpha \neq \beta^*$, according to the defining relations of $A$. This motivates the next definitions.

**Definition 3.7.** Consider an anchored cycle $\widehat{C} = \alpha_n \cdots \underline{\alpha_1}$ on $\mathcal{G}$. An admissible permutation of its arrows consists in dividing the word $\alpha_n \cdots \alpha_1$ in two subwords

$$A = \alpha_n \cdots \alpha_{n-i}, \qquad B = \alpha_{n-i-1} \cdots \alpha_1$$

such that no arrow in $A$ has its antiparallel in $B$, and in swapping $A$ and $B$. This yields a new anchored cycle $\widehat{C}' = \alpha_{n-i-1} \cdots \alpha_1 \alpha_n \cdots \underline{\alpha_i}$.

**Definition 3.8.** Let $\widehat{\mathbb{C}\mathcal{G}_{\mathrm{cycl}}}$ be the complex vector space spanned by anchored cycles in $\mathcal{G}$, defined up to admissible permutations of their arrows. Its elements will be called quantum potentials, its generators quantum cycles.
One denotes by $\sigma\colon \widehat{\mathbb{C}\mathcal{G}_{\mathrm{cycl}}} \longrightarrow \mathbb{C}\mathcal{G}_{\mathrm{cycl}}$ the map that forgets the anchor, which we call the semiclassical limit. A quantum potential $\widehat{W}$ is a quantisation of the potential $W$ if $\sigma(\widehat{W}) = W$.

There exists now a well defined $\mathbb{C}$-linear injective map $\mathrm{Tr}\colon \widehat{\mathbb{C}\mathcal{G}_{\mathrm{cycl}}} \hookrightarrow A$, together with a commutative square where the quantum and classical traces intertwine the semiclassical limit: $\sigma\Big(\mathrm{Tr}(\widehat{C})\Big) = \mathrm{Tr}(C) \in A_0$ for all quantum cycles $\widehat{C} \in \widehat{\mathbb{C}\mathcal{G}_{\mathrm{cycl}}}$ that quantises the classical cycle $C \in \mathbb{C}\mathcal{G}_{\mathrm{cycl}}$. Moreover, one can use Prop. 3.3 to produce a cycle-theoretic analogue of the Weyl algebra. Namely, one considers the tensor map

$$\mathrm{Tens}(\mathrm{Tr})\colon\ \mathrm{Tens}(\widehat{\mathbb{C}\mathcal{G}_{\mathrm{cycl}}}) \longrightarrow A,$$

which is surjective on the subalgebra of $A' \subseteq A$ generated by traces of quantum cycles. One thus has an isomorphism of associative algebras

$$\mathrm{Tens}(\widehat{\mathbb{C}\mathcal{G}_{\mathrm{cycl}}}) \Big/ \mathrm{Ker}(\mathrm{Tens}(\mathrm{Tr})) \cong A'.$$

The quotient on the left-hand side is thus an associative (quantum) algebra that has an analogous relation with $A' \subseteq A$ as $\mathrm{Sym}(\mathbb{C}\mathcal{G}_{\mathrm{cycl}})$ has with $A_0^{\widehat{H}}$. Notice that this is abstract, as we do not have a nice description of the kernel of the quantum trace map. However, one has an identification

$$\mathbb{C} \oplus \widehat{\mathbb{C}\mathcal{G}_{\mathrm{cycl}}} \cong A'_{\leq 1},$$

with respect to the quotient filtration on $A'$. Indeed, this happens because $\mathrm{Tens}(\mathrm{Tr})$ is by definition the identity on $\mathbb{C}$ (trace of empty cycles, if one will), and it is injective on the vector space generated by quantum cycles.

Finally, $A'$ now has a well defined product, defined on quantum cycles by

$$\mathrm{Tr}\left(\widehat{C}_1 \widehat{C}_2\right) = \mathrm{Tr}(\widehat{C}_1)\,\mathrm{Tr}(\widehat{C}_2) \in A'.$$

This makes $A'$ into a filtered associative algebra, provided with a semiclassical limit

$$\sigma\colon A' \longrightarrow A_0^{\widehat{H}},$$

which is defined on monomials by forgetting anchors $\sigma\colon \widehat{C}_1 \cdots \widehat{C}_n \longmapsto \sigma(\widehat{C}_1) \cdots \sigma(\widehat{C}_n)$.

In this noncommutative context it is even more important to allow for formal products of cycles, in order to keep track of the anchoring, as exemplified by the next proposition.

**Proposition 3.5.** *Pick a quantum cycle $\widehat{C} = \alpha_n \cdots \underline{\alpha_1}$, and let $\widehat{C}' = \alpha_j \cdots \alpha_1 \alpha_n \cdots \underline{\alpha_{j+1}}$ be a change of anchoring. One has*

$$\widehat{C} - \widehat{C}' = \sum_{1 \leq l \leq j} \sum_{\alpha_m = \alpha_l^*} c_{lm} \widehat{C}_{lm} \widehat{D}_{lm}$$

*where $\widehat{C}_{lm}, \widehat{D}_{lm}$ are two quantum cycle obtained by deleting the arrows $\alpha_k, \alpha_l = \alpha_k^*$, and where $c_{lm} \in \mathbb{C}$ is the usual structure constant coming from $\omega_a$.*

*Proof.* First, notice that one has a telescopic sum:

$$\widehat{C} - \widehat{C}' = \sum_{0 \leq l \leq j-1} \Big( \alpha_l \cdots \alpha_1 \alpha_n \cdots \underline{\alpha_{l+1}} - \alpha_{l+1} \cdots \alpha_1 \alpha_n \cdots \underline{\alpha_{l+2}} \Big),$$

so that the problem reduces to make one arrow commute.
Now pick traces and compute, introducing again the multi-index $K$ associated to $C = \sigma(\widehat{C}) = \sigma(\widehat{C}')$:

$$\begin{aligned}
&\operatorname{Tr}\Big( \alpha_l \cdots \alpha_1 \alpha_n \cdots \underline{\alpha_{l+1}} - \alpha_{l+1} \cdots \alpha_1 \alpha_n \cdots \underline{\alpha_{l+2}} \Big) = \\
&= \sum_{K \in D_C} \widehat{X}^{\alpha_l}_{k_l,k_{l-1}} \cdots \widehat{X}^{\alpha_1}_{k_1,k_n} \widehat{X}^{\alpha_n}_{k_n,k_{n-1}} \cdots \widehat{X}^{\alpha_{l+1}}_{k_{l+1},k_l} - \widehat{X}^{\alpha_{l+1}}_{k_{l+1},k_l} \cdots \widehat{X}^{\alpha_1}_{k_1,k_n} \widehat{X}^{\alpha_n}_{k_n,k_{n-1}} \cdots \widehat{X}^{\alpha_{l+2}}_{k_{l+2},k_{l+1}} = \\
&= \sum_{K \in D_C} \Big[ \widehat{X}^{\alpha_l}_{k_l,k_{l-1}} \cdots \widehat{X}^{\alpha_1}_{k_1,k_n} \widehat{X}^{\alpha_n}_{k_n,k_{n-1}} \cdots \widehat{X}^{\alpha_{l+2}}_{k_{l+2},k_{l+1}}, \widehat{X}^{\alpha_{l+1}}_{k_{l+1},k_{l+2}} \Big] = \\
&= \sum_{K \in D_C} \sum_{\alpha_m = \alpha_{l+1}^*} \widehat{X}^{\alpha_l}_{k_l,k_{l-1}} \cdots \widehat{X}^{\alpha_{m+1}}_{k_{m+1},k_m} \Big[ \widehat{X}^{\alpha_m}_{k_m,k_{m-1}}, \widehat{X}^{\alpha_{l+1}}_{k_{l+1},k_l} \Big] \widehat{X}^{\alpha_{m-1}}_{k_{m-1},k_{m-2}} \cdots \widehat{X}^{\alpha_{l+2}}_{k_{l+2},k_{l+1}} = \\
&= \sum_{K \in D_C} \sum_{\alpha_m = \alpha_{l+1}^*} \Big( c_{m,l+1} \delta_{k_l,k_m} \delta_{k_{m-1},k_{l+1}} \Big) \widehat{X}^{\alpha_l}_{k_l,k_{l-1}} \cdots \widehat{X}^{\alpha_{m+1}}_{k_{m+1},k_m} \widehat{X}^{\alpha_{m-1}}_{k_{m-1},k_{m-2}} \cdots \widehat{X}^{\alpha_{l+2}}_{k_{l+2},k_{l+1}} = \\
&= \sum_{K'} \sum_{\alpha_m = \alpha_{l+1}^*} c_{m,l+1} \widehat{X}^{\alpha_l}_{k_m,k_{l-1}} \cdots \widehat{X}^{\alpha_{m+1}}_{k_{m+1},k_m} \cdot \widehat{X}^{\alpha_{m-1}}_{k_{m-1},k_{m-2}} \cdots \widehat{X}^{\alpha_{l+2}}_{k_{l+2},k_{m-1}} = \\
&= \sum_{\alpha_m = \alpha_{l+1}^*} c_{m,l+1} \operatorname{Tr}(\alpha_l \cdots \underline{\alpha_{m+1}}) \operatorname{Tr}(\alpha_{m-1} \cdots \underline{\alpha_{l+2}}) =: \sum_{\alpha_m = \alpha_{l+1}^*} c_{m,l+1} \operatorname{Tr}(\widehat{C}_{m,l+1}) \operatorname{Tr}(\widehat{D}_{m,l+1}).
\end{aligned}$$

The last identity defines the cycles $\widehat{C}_{m,l+1}$ and $\widehat{D}_{m,l+1}$. Also, we used

$$c_{m,l+1} \delta_{k_l,k_m} \delta_{k_{m-1},k_{l+1}} = \Big[ \widehat{X}^{\alpha_m}_{k_m,k_{m-1}}, \widehat{X}^{\alpha_{l+1}}_{k_{l+1},k_l} \Big] \in \mathbb{C}.$$

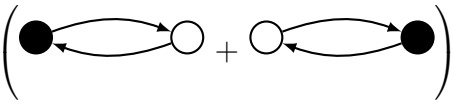
Now, by definition, the above computation yields

$$\alpha_l \cdots \alpha_1 \alpha_n \cdots \underline{\alpha_{l+1}} - \alpha_{l+1} \cdots \alpha_1 \alpha_n \cdots \underline{\alpha_{l+2}} = \sum_{\alpha_m = \alpha_{l+1}^*} c_{m,l+1} \widehat{C}_{m,l+1} \widehat{D}_{m,l+1},$$

and summing on $l \in \{0, \ldots, j-1\}$ gives the conclusion. □

The following few definitions now come naturally. Consider again the classical isomonodromy cycles of § 3.2. The 3-cycles and the nondegenerate 4-cycles do not contain pairs of antiparallel arrows, so that one can quantise such a cycle $C$ by choosing any anchor: all of them are equivalent. As for 2-cycles and degenerate 4-cycles, we make the following choices.

**Definition 3.9.** The quantisation of a degenerate 4-cycles is the quantum cycle having the same underlying classical cycle, anchored at any arrow coming out of its centre. The quantisation of a two cycle $C =$  is by definition the quantum potential

$\widehat{C} = \frac{1}{2}\Bigg($  $\Bigg)$.

In this picture and in all that follow, the black nodes are the tails of the anchors.

**Remark 3.5.** In the case of degenerate 4-cycles, a priori specifying a starting arrow is more than specifying a starting node, but in this case there is no ambiguity: changing the order of the arrows coming out of the central node amounts to an admissible permutation of the arrows of the degenerate 4-cycles. This is because such a cycle can be written as a word $C = \beta^*\beta\alpha^*\alpha$, where $\alpha, \beta$ are the two distinct arrows of $\mathcal{G}$ coming out of the centre. Now, the two possible anchors at the centre correspond to the quantisations $\widehat{C}_1 = \beta^*\beta\alpha^*\underline{\alpha}$ and $\widehat{C}_2 = \alpha^*\alpha\beta^*\underline{\beta}$. These two are equivalent, using the admissible permutation that swaps the two 2-cycles: one can move $\beta^*\beta$ to the right of $\alpha^*\alpha$ without changing the relative order of the antiparallel pairs $(\alpha, \alpha^*), (\beta, \beta^*)$.

This is totally canonical, and does not rely on a full quantisation $\mathcal{Q}\colon A_0 \longrightarrow A$. One can however show that it amounts to correcting the standard Weyl quantisation.

## 3.4 Definition of the simply-laced quantum connection

Consider now the isomonodromy potentials $W_i = W_i(4) + W_i(3) + W_i(2) \in \mathbb{C}\mathcal{G}_{\mathrm{cycl}}$ of (3.4).

**Definition 3.10.** The quantum isomonodromy potential $\widehat{W}_i \in \widehat{\mathbb{C}\mathcal{G}_{\mathrm{cycl}}}$ at the node $i$ is the sum of the quantisations of its isomonodromy cycles. The quantum isomonodromy Hamiltonian $\widehat{H}_i\colon \mathbf{B} \longrightarrow A$ is the trace of the quantum isomonodromy potential at the node $i$, i.e. $\widehat{H}_i := \mathrm{Tr}(\widehat{W}_i)$.

This is a quantisation of the classical isomonodromy Hamiltonian $H_i\colon \mathbf{B} \longrightarrow A_0$, in the sense that the identity $\sigma(\widehat{H}_i) = H_i$ is true everywhere on $\mathbf{B}$. Consider now the trivial "universal" bundle $\mathbb{E}_a := A \times \mathbf{B} \longrightarrow \mathbf{B}$.

**Definition 3.11.** The (universal) simply-laced quantum connection $\widehat{\nabla}$ is the connection on $\mathbb{E}_a$ defined by

$$\widehat{\nabla} := d_{\mathbf{B}} - \widehat{\varpi}, \qquad \text{where } \widehat{\varpi} := \sum_{i\in I} \widehat{H}_i dt_i.$$

Note that $\Omega^1(\mathbf{B}, A) \subseteq \Omega^1\Big(\mathbf{B}, \mathrm{End}(\mathbb{E}_a)\Big)$, where one lets $\widehat{H}_i$ act linearly on the fibre $A$ of $\mathbb{E}_a$ by left multiplication. If one thinks of $\widehat{\varpi}$ as an element of $A^1(\mathbf{B}, A)$, then this is the same as a time-dependent quantum Hamiltonian system (see § A.3).

The main result is the following.

**Theorem 3.2.** *$\widehat{\nabla}$ is strongly flat, in the sense of Def. A.8:*

$$\left[\widehat{H}_i, \widehat{H}_j\right] = 0 = \frac{\partial \widehat{H}_i}{\partial t_j} - \frac{\partial \widehat{H}_j}{\partial t_i}, \qquad \text{for } i, j \in I.$$

To the proof of Thm. 3.2 we dedicate the next two sections.

## 3.5 Proof of strong flatness: I

Let us start by showing that

$$\partial_{t_i} \widehat{H}_j - \partial_{t_j} \widehat{H}_i = 0$$

for all $i, j \in I$. This follows from a lemma.

**Lemma 3.1.** *Let $W_i \colon \mathbf{B} \longrightarrow \mathbb{C}\mathcal{G}$ be a classical isomonodromy potential. Then*

$$\partial_{t_j} \widehat{W}_i = \widehat{\partial_{t_j} W_i}, \qquad \partial_{t_j} \operatorname{Tr}(\widehat{W}_i) = \operatorname{Tr}(\partial_{t_j} \widehat{W}_i),$$

*for all $j \in I$.*

*Proof.* The first set of identities are due to the fact that the quantisation does not depend on $\mathbf{B}$. Moreover, as already mentioned at the end of § 3.2, tacking a derivative does not change the type of cycles that make up the potential, but only modifies their weights. This means that the quantisation $\widehat{\partial_{t_j} W_i}$ is well defined, and that taking traces (both of classical and quantum potentials) commutes with picking derivatives. □

Using the second set of identities of the lemma, it is thus enough to verify that one has $\partial_{t_i} \widehat{W_j} - \partial_{t_j} \widehat{W_i} = 0$ for all $i, j \in I$, because the trace of the left-hand side is precisely the difference $\partial_{t_i} \widehat{H}_j - \partial_{t_j} \widehat{H}_i$. Finally, to prove this, one exploits Thm. 3.1, borrowing the statement

$$\partial_{t_i} W_j = \partial_{t_j} W_i.$$

This is precisely Prop. 3.4, which implies that

$$\widehat{\partial_{t_i} W_j} = \widehat{\partial_{t_j} W_i}.$$

Then the first set of identity of the above lemma permits to conclude. Notice that crucial fact that the quantisation is "symmetric on $\mathcal{G}$", in the sense that the quantisation $\widehat{H}_i$ of the Hamiltonian $H_i$ does not depend on the base node $i \in I$.

## 3.6 Proof of strong flatness: II

One is left to show that the quantum isomonodromy Hamiltonians commute. By bilinearity, this reduces to the problem of computing commutators of the form

$$\left[\operatorname{Tr}(\widehat{C}_1), \operatorname{Tr}(\widehat{C}_2)\right] \in A$$

where $\widehat{C}_1, \widehat{C}_2$ are quantum isomonodromy cycles.

### 3.6.1 Commutators of quantum cycles

The first thing to do is to see whether one can still write this element in terms of traces of quantum cycles. Let us get back to our quantum algebra $A'$ whose commutator is defined by $\mathrm{Tr}\left([\widehat{C}_1, \widehat{C}_2]\right) = \left[\mathrm{Tr}(\widehat{C}_1), \mathrm{Tr}(\widehat{C}_2)\right]$.

We would like to be able to give a characterization of $[\widehat{C}_1, \widehat{C}_2]$ along the lines of Prop. 3.2, but unfortunately that used the commutativity of the product on $A_0$. This means that we cannot a priori hope that the commutator of quantum cycles be a quantum potential: one must a priori allow for higher-order elements.
The crucial point of our proof is that one can show that the desired property holds for the cycles we have to consider. Set $\widehat{\mathrm{IMD}} \subseteq \widehat{\mathbb{C}\mathcal{G}_{\mathrm{cycl}}}$ to be the vector space spanned by the quantum isomonodromy cycles.

**Proposition 3.6.** *The restriction* $[\cdot,\cdot]\colon \widehat{\mathrm{IMD}} \wedge \widehat{\mathrm{IMD}} \longrightarrow A'$ *takes values into* $\widehat{\mathbb{C}\mathcal{G}_{\mathrm{cycl}}}$. *Moreover, if* $\widehat{C}_1, \widehat{C}_2 \in \widehat{\mathrm{IMD}}$ *then the commutator* $\left[\widehat{C}_1, \widehat{C}_2\right]$ *is obtained by a suitable anchoring of the Poisson bracket* $\{C_1, C_2\}$, *where* $C_i \coloneqq \sigma(\widehat{C}_i)$.

The proof of Prop. 3.6 relies on a lemma, plus two separate verifications.

**Lemma 3.2.** *Pick two quantum cycles* $\widehat{C}_1, \widehat{C}_2$, *with semiclassical limit* $C_1, C_2$. *Assume that one of* $\widehat{C}_1, \widehat{C}_2$ *is a 2-cycle, or that one of them does not contain pairs of antiparallel arrows. Then Prop. 3.6 holds for* $\left[\widehat{C}_1, \widehat{C}_2\right] \in \widehat{\mathbb{C}\mathcal{G}_{\mathrm{cycl}}}$.

*Proof.* Write $\widehat{C}_1 = \alpha_n \cdots \underline{\alpha_1}, \widehat{C}_2 = \beta_m \cdots \underline{\beta_1}$, for $\alpha_i, \beta_j \in \mathcal{G}_1$, and introduce again multi-indices $K \coloneqq (k_n, \ldots, k_1), L \coloneqq (l_m, \ldots, l_1)$ varying respectively in

$$D_{C_1} \coloneqq \{1, \ldots, d_{h(\alpha_n)}\} \times \cdots \times \{1, \ldots, d_{h(\alpha_1)}\} \quad \text{and} \quad D_{C_2} \coloneqq \{1, \ldots, d_{h(\beta_m)}\} \times \cdots \times \{1, \ldots, d_{h(\beta_1)}\}.$$

Now, one wants to expand

$$\left[\mathrm{Tr}(\widehat{C}_1), \mathrm{Tr}(\widehat{C}_2)\right] = \sum_{K \in D_{C_1}, L \in D_{C_2}} \left[\widehat{X}^{\alpha_n}_{k_n,k_{n-1}} \cdots \widehat{X}^{\alpha_1}_{k_1,k_n}, \widehat{X}^{\beta_m}_{l_m,l_{m-1}} \cdots \widehat{X}^{\beta_1}_{l_1,l_m}\right].$$

By applying Leibnitz rule recursively, one finds two possible developments:

$$\begin{aligned}\left[\mathrm{Tr}(\widehat{C}_1), \mathrm{Tr}(\widehat{C}_2)\right] = &\sum_{K \in D_{C_1}, L \in D_{C_2}} \sum_{i,j:\alpha_i = \beta_j^*} [\widehat{X}^{\alpha_i}_{k_i,k_{i-1}}, \widehat{X}^{\beta_j}_{l_j,l_{j-1}}] \widehat{X}^{\beta_m}_{l_m,l_{m-1}} \cdots \widehat{X}^{\beta_{j+1}}_{l_{j+1},l_j} \cdot \\ &\cdot \widehat{X}^{\alpha_n}_{k_n,k_{n-1}} \cdots \widehat{X}^{\alpha_{i+1}}_{k_{i+1},k_i} \widehat{X}^{\alpha_{i-1}}_{k_{i-1},k_{i-2}} \cdots \widehat{X}^{\alpha_1}_{k_1,k_n} \cdot \widehat{X}^{\beta_{j-1}}_{l_{j-1},l_{j-2}} \cdots \widehat{X}^{\beta_1}_{l_1,l_m} = \\ = &\sum_{K \in D_{C_1}, L \in D_{C_2}} \sum_{i,j:\alpha_i = \beta_j^*} [\widehat{X}^{\alpha_i}_{k_i,k_{i-1}}, \widehat{X}^{\beta_j}_{l_j,l_{j-1}}] \widehat{X}^{\alpha_n}_{k_m,k_{n-1}} \cdots \widehat{X}^{\alpha_{i+1}}_{k_{i+1},k_i} \cdot \\ &\cdot \widehat{X}^{\beta_m}_{l_m,l_{m-1}} \cdots \widehat{X}^{\beta_{j+1}}_{l_{j+1},l_j} \widehat{X}^{\beta_{j-1}}_{l_{j-1},l_{j-2}} \cdots \widehat{X}^{\beta_1}_{l_1,l_m} \cdot \widehat{X}^{\alpha_{i-1}}_{k_{i-1},k_{i-2}} \cdots \widehat{X}^{\alpha_1}_{k_1,k_n}\end{aligned}$$

Here we could displace all commutators of variables on the left, since they're just constants lying in the centre of $A$.

Now, one would like to write all the words of length $n+m-2$ in the sum as traces of quantum cycles. One consistent way to ensure this is precisely the condition given in the statement. Indeed, if — say — $\widehat{C}_1$ satisfies the hypothesis, then it has no pair of antiparallel arrows, as soon as one of its arrows is removed. Then one is free to write

$$\widehat{X}^{\alpha_n}_{k_n,k_{n-1}} \cdots \widehat{X}^{\alpha_{i+1}}_{k_{i+1},k_i} \widehat{X}^{\alpha_{i-1}}_{k_{i-1},k_{i-2}} \cdots \widehat{X}^{\alpha_1}_{k_1,k_n} = \widehat{X}^{\alpha_{i-1}}_{k_{i-1},k_{i-2}} \cdots \widehat{X}^{\alpha_1}_{k_1,k_n} \widehat{X}^{\alpha_n}_{k_n,k_{n-1}} \cdots \widehat{X}^{\alpha_{i+1}}_{k_{i+1},k_i}$$

and thus all addends get the desired form:

$$\begin{aligned}\Big[\operatorname{Tr}(\widehat{C}_1), \operatorname{Tr}(\widehat{C}_2)\Big] = \sum_{K\in D_{C_1}, L\in D_{C_2}} \sum_{i,j:\alpha_i=\beta_j^*} [\widehat{X}^{\alpha_i}_{k_i,k_{i-1}}, \widehat{X}^{\beta_j}_{l_j,l_{j-1}}] \widehat{X}^{\beta_m}_{l_m,l_{m-1}} \cdots \widehat{X}^{\beta_{j+1}}_{l_{j+1},l_j} \cdot \\ \cdot \widehat{X}^{\alpha_{i-1}}_{k_{i-1},k_{i-2}} \cdots \widehat{X}^{\alpha_1}_{k_1,k_n} \widehat{X}^{\alpha_n}_{k_n,k_{n-1}} \cdots \widehat{X}^{\alpha_{i+1}}_{k_{i+1},k_i} \cdot \widehat{X}^{\beta_{j-1}}_{l_{j-1},l_{j-2}} \cdots \widehat{X}^{\beta_1}_{l_1,l_m} = \\ = \sum_{i,j:\alpha_i=\beta_j^*} c_{ij} \operatorname{Tr}\left(\widehat{X}^{\beta_m} \cdots \widehat{X}^{\beta_{j+1}} \widehat{X}^{\alpha_{i-1}} \cdots \widehat{X}^{\alpha_1} \widehat{X}^{\alpha_n} \cdots \widehat{X}^{\alpha_{i+1}} \widehat{X}^{\beta_{j-1}} \cdots \widehat{X}^{\beta_1}\right),\end{aligned}$$

where we used $[\widehat{X}^{\alpha_i}_{k_i,k_{i-1}}, \widehat{X}^{\beta_j}_{l_j,l_{j-1}}] = c_{ij}\delta_{k_i,l_{j-1}}\delta_{k_{i-1},l_j} \in \mathbb{C}$. This is the same structure constant as in the classical case, precisely because $A$ is a quantisation of $A_0$. Notice that the second development would work if $\widehat{C}_2$ were the quantum cycle satisfying the hypothesis.

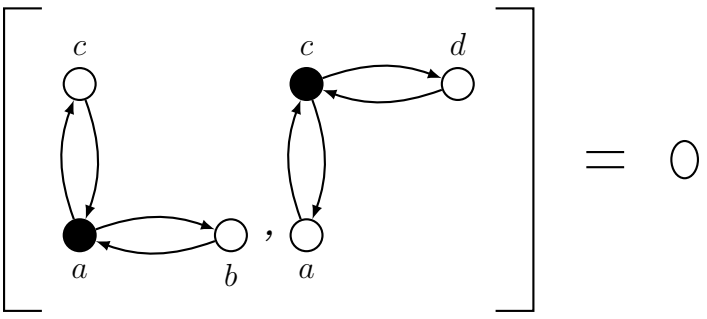
Comparing the above formula with that of $\{C_1, C_2\} = \sigma\Big(\big[\widehat{C}_1, \widehat{C}_2\big]\Big)$ shows that the commutator is a lift of the Poisson bracket. □

The only IMD cycles that do not satisfy the hypothesis of Lem. 3.2 are the degenerate 4-cycles. Hence one must still show that the commutator of two such cycles follows the same rule. This leads us to check the possible *intersections* of cycles in $\mathcal{G}$.

**Definition 3.12.** Two (classical or quantum) cycles are said to intersect if there exists an arrow of the first with its antiparallel in the second. The intersection is said to be nontrivial if the two cycles are different.

Notice that two classical cycles (resp. quantum cycles) may have a nonvanishing Poisson bracket (resp. vanishing commutator) only if they intersect nontrivially. Now, two degenerate 4-cycles have only two possible nontrivial intersections: either they have the centre in common, or they do not.

**Proposition 3.7.** *Pick nodes $a, b, c, d \in I$ such that the sequences of nodes $(a, b, a, c)$ and $(a, c, d, c)$ define two degenerate 4-cycles. Then the following commutator vanishes:*



Here we sketched quantum cycle by drawing a black node where their anchor starts.

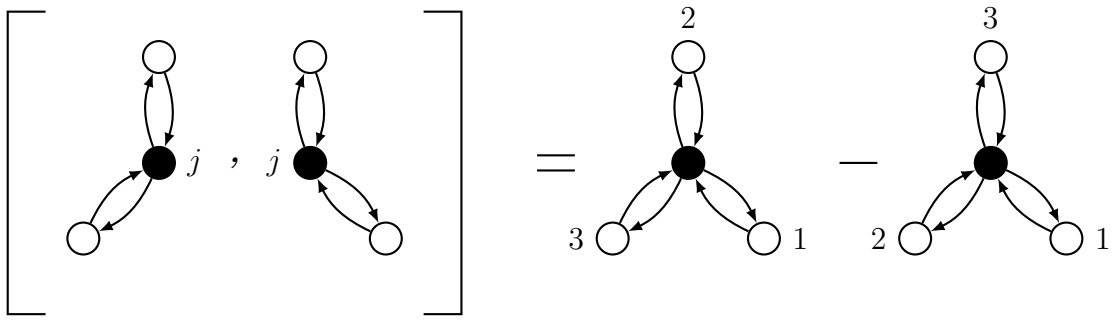
*Proof.* Let's set $\alpha \coloneqq (a,b), \beta \coloneqq (b,c), \gamma \coloneqq (c,d)$. Then one has:

$$\begin{aligned}
&\Big[Tr(\widehat{X}^{\beta^*}\widehat{X}^{\beta}\widehat{X}^{\alpha^*}\widehat{X}^{\alpha}, \widehat{X}^{\beta}\widehat{X}^{\beta^*}\widehat{X}^{\gamma^*}\widehat{X}^{\gamma})\Big] = \sum_{i,j,k,l,m,n,o,p} \Big[\widehat{X}^{\beta^*}_{ij}\widehat{X}^{\beta}_{jk}\widehat{X}^{\alpha^*}_{kl}\widehat{X}^{\alpha}_{li}, \widehat{X}^{\beta}_{mn}\widehat{X}^{\beta^*}_{no}\widehat{X}^{\gamma^*}_{op}\widehat{X}^{\gamma}_{pm}\Big] = \\
&= \sum_{i,j,k,l,m,n,o,p} \Big[\widehat{X}^{\beta^*}_{ij}\widehat{X}^{\beta}_{jk}, \widehat{X}^{\beta}_{mn}\widehat{X}^{\beta^*}_{no}\Big]\widehat{X}^{\alpha^*}_{kl}\widehat{X}^{\alpha}_{li}\widehat{X}^{\gamma^*}_{op}\widehat{X}^{\gamma}_{pm} = \\
&= \sum_{i,j,k,l,m,n,o,p} \widehat{X}^{\beta^*}_{ij}\widehat{X}^{\beta}_{mn}\Big[\widehat{X}^{\beta}_{jk}, \widehat{X}^{\beta^*}_{no}\Big]\widehat{X}^{\alpha^*}_{kl}\widehat{X}^{\alpha}_{li}\widehat{X}^{\gamma^*}_{op}\widehat{X}^{\gamma}_{pm} + \Big[\widehat{X}^{\beta^*}_{ij}, \widehat{X}^{\beta}_{mn}\Big]\widehat{X}^{\beta^*}_{no}\widehat{X}^{\beta}_{jk}\widehat{X}^{\alpha^*}_{kl}\widehat{X}^{\alpha}_{li}\widehat{X}^{\gamma^*}_{op}\widehat{X}^{\gamma}_{pm} = \\
&= \sum_{i,j,k,l,m,n,o,p} c_{\beta\beta^*}\delta_{jo}\delta_{kn}\widehat{X}^{\beta^*}_{ij}\widehat{X}^{\beta}_{mn}\widehat{X}^{\alpha^*}_{kl}\widehat{X}^{\alpha}_{li}\widehat{X}^{\gamma^*}_{op}\widehat{X}^{\gamma}_{pm} + c_{\beta^*\beta}\delta_{jm}\delta_{in}\widehat{X}^{\beta^*}_{no}\widehat{X}^{\beta}_{jk}\widehat{X}^{\alpha^*}_{kl}\widehat{X}^{\alpha}_{li}\widehat{X}^{\gamma^*}_{op}\widehat{X}^{\gamma}_{pm} = \\
&= c_{\beta\beta^*}\Big(\operatorname{Tr}(\widehat{X}^{\beta^*}\widehat{X}^{\gamma^*}\widehat{X}^{\gamma}\widehat{X}^{\beta}\widehat{X}^{\alpha^*}\widehat{X}^{\alpha} - \operatorname{Tr}(\widehat{X}^{\beta^*}\widehat{X}^{\gamma^*}\widehat{X}^{\gamma}\widehat{X}^{\beta}\widehat{X}^{\alpha^*}\widehat{X}^{\alpha})\Big) = 0.
\end{aligned}$$

One uses that $c_{\beta\beta^*} \in \mathbb{C}$ is alternating in the arrows of $\mathcal{G}$. □

The next intersection asks instead to show that the following picture is true.

**Proposition.** *Pick two degenerate 4-cycles with the central node $j \in I$ in common. If they intersect nontrivially, then one has, ignoring structure constants:*




The number at the peripheral nodes indicates the order in which one must touch them, starting from the centre (the tail of the anchor).

*Proof.* Denote $\beta, \beta^*$ the arrows that the two cycle have in common, with $t(\beta) = j$. Set then $\alpha, \alpha^*$ to be the remaining arrows of the leftmost cycle, with $t(\alpha) = j$, and similarly

for $\gamma, \gamma^*$, where $t(\gamma) = j$. Then one has:

$$\Big[\mathrm{Tr}(\widehat{X}^{\beta^*}\widehat{X}^{\beta}\widehat{X}^{\alpha^*}\widehat{X}^{\alpha}), \mathrm{Tr}(\widehat{X}^{\beta^*}\widehat{X}^{\beta}\widehat{X}^{\gamma^*}\widehat{X}^{\gamma})\Big] = \sum_{i,j,k,l,m,n,o,p} \Big[\widehat{X}^{\beta^*}_{ij}\widehat{X}^{\beta}_{jk}\widehat{X}^{\alpha^*}_{kl}\widehat{X}^{\alpha}_{li}, \widehat{X}^{\beta^*}_{mn}\widehat{X}^{\beta}_{no}\widehat{X}^{\gamma^*}_{op}\widehat{X}^{\gamma}_{pm}\Big] =$$

$$= \sum_{i,j,k,l,m,n,o,p} \Big[\widehat{X}^{\beta^*}_{ij}\widehat{X}^{\beta}_{jk}, \widehat{X}^{\beta^*}_{mn}\widehat{X}^{\beta}_{no}\Big]\widehat{X}^{\alpha^*}_{kl}\widehat{X}^{\alpha}_{li}\widehat{X}^{\gamma^*}_{op}\widehat{X}^{\gamma}_{pm} =$$

$$= \sum_{i,j,k,l,m,n,o,p} \widehat{X}^{\beta^*}_{ij}\Big[\widehat{X}^{\beta}_{jk}, \widehat{X}^{\beta^*}_{mn}\Big]\widehat{X}^{\beta}_{no}\widehat{X}^{\alpha^*}_{kl}\widehat{X}^{\alpha}_{li}\widehat{X}^{\gamma^*}_{op}\widehat{X}^{\gamma}_{pm} + \widehat{X}^{\beta^*}_{mn}\Big[\widehat{X}^{\beta^*}_{ij}, \widehat{X}^{\beta}_{no}\Big]\widehat{X}^{\beta}_{jk}\widehat{X}^{\alpha^*}_{kl}\widehat{X}^{\alpha}_{li}\widehat{X}^{\gamma^*}_{op}\widehat{X}^{\gamma}_{pm} =$$

$$= \sum_{i,j,k,l,m,n,o,p} c_{\beta\beta^*}\delta_{jn}\delta_{km}\widehat{X}^{\beta^*}_{ij}\widehat{X}^{\beta}_{no}\widehat{X}^{\alpha^*}_{kl}\widehat{X}^{\alpha}_{li}\widehat{X}^{\gamma^*}_{op}\widehat{X}^{\gamma}_{pm} + c_{\beta^*\beta}\delta_{io}\delta_{jk}\widehat{X}^{\beta^*}_{mn}\widehat{X}^{\beta}_{jk}\widehat{X}^{\alpha^*}_{kl}\widehat{X}^{\alpha}_{li}\widehat{X}^{\gamma^*}_{op}\widehat{X}^{\gamma}_{pm} =$$

$$= c_{\beta\beta^*}\Big(\mathrm{Tr}(\widehat{X}^{\alpha^*}\widehat{X}^{\alpha}\widehat{X}^{\beta^*}\widehat{X}^{\beta}\widehat{X}^{\gamma^*}\widehat{X}^{\gamma}) - \mathrm{Tr}(\widehat{X}^{\beta^*}\widehat{X}^{\beta}\widehat{X}^{\alpha^*}\widehat{X}^{\alpha}\widehat{X}^{\gamma^*}\widehat{X}^{\gamma})\Big),$$

□

### 3.6.2 Anchors

Let us decompose the classical IMD potentials $W_i, W_j$ into a sum of classical IMD cycles: $W_i = \sum_k c_k C_k, W_j = \sum_l d_l D_l$. After expanding their vanishing Poisson bracket by bilinearity, one will find itself with a sum of potentials:

$$0 = \{W_i, W_j\} = \sum_{k,l} c_{kl}\{C_k, D_l\}. \tag{3.5}$$

Putting together all the cycles that coincide as elements of $\mathbb{C}\mathcal{G}_{\mathrm{cycl}}$, one will get to a finer decomposition

$$0 = \{W_i, W_j\} = \sum_m e_m E_m \in \mathbb{C}\mathcal{G}_{\mathrm{cycl}}.$$

Now, since we're assuming that $E_m \neq E_{m'}$ for $m \neq m'$ in this sum, one has necessarily $e_m = 0$ for all $m$: any finite family of distinct cycles in $\mathcal{G}$ is free inside $\mathbb{C}\mathcal{G}_{\mathrm{cycl}}$, by definition.

Now, thanks to Prop. 3.6, one will find a similar development:

$$\Big[\widehat{W}_i, \widehat{W}_j\Big] = \sum_{k,l} c_{kl}\Big[\widehat{C}_k, \widehat{D}_l\Big],$$

with $\widehat{C}_k, \widehat{D}$ being the quantisation of $C_k, D_l$, and more importantly with $\Big[\widehat{C}_k, \widehat{D}_l\Big]$ being a quantisation of $\{C_k, D_l\}$. One would now like to have

$$\Big[\widehat{W}_i, \widehat{W}_j\Big] = \sum_m e_m \widehat{E}_m \in \widehat{\mathbb{C}\mathcal{G}_{\mathrm{cycl}}},$$

with the same constants $e_m \in \mathbb{C}$, for some lift $\widehat{E}_m$ of $E_m$. This happens if and only if every time that one has $\{C_k, D_l\} = \{C_{k'}, D_{l'}\}$ in (3.5), then one also has $\Big[\widehat{C}_k, \widehat{D}_l\Big] = \Big[\widehat{C}_{k'}, \widehat{D}_{l'}\Big]$

in $\widehat{\mathbb{C}\mathcal{G}_{\mathrm{cycl}}}$. Since these two commutators have the same underlying classical cycle, this happens if and only if their anchors are equivalent, which can clearly be achieved whenever both $C_k$ and $D_l$ are not degenerate 4-cycles. Indeed, for 3-cycles and nondegenerate 4-cycles the choice of anchor is totally immaterial, and one can change the anchor of a 2-cycle at the cost of adding a term of order zero, i.e. a central additive constant which will not tamper with the commutator.
Hence one is led to consider the nontrivial intersections of nondegenerate 4-cycles with other types of cycles. The only ones which yield a nonvanishing Poisson brackets are described in plain words as:

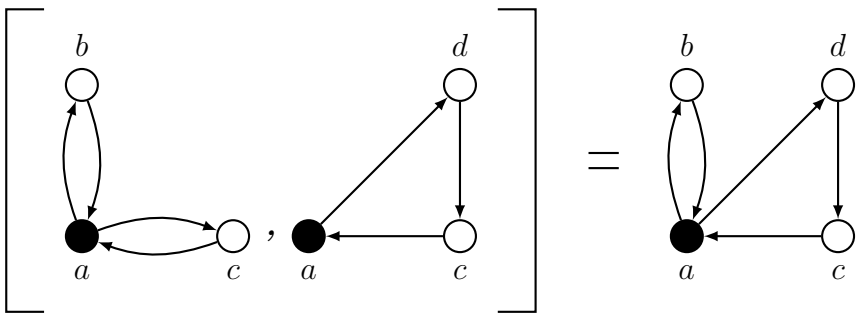
1. a 3-cycle and a degenerate 4-cycle with one pair of antiparallel arrows in common[5]
2. two degenerate 4-cycles with the centre in common
3. a nondegenerate 4-cycle and a degenerate one, with one pair of antiparallel arrows in common
4. same as the one just above, with two pairs in common

All these intersections give cycles which are classically distinguishable. Also, n ° 2 has already been dealt with above.

We will now show that these nontrivial intersections yield equivalent quantum potentials, as needed in order to conclude the proof of Thm. 3.2. We will thus sketch three commutators of quantum cycles, with the logic behind the pictures always being to summarise longer computations in noncommutative variables, exploiting Prop. 3.6. We also explicitly write down these computations in Darboux coordinates, so to simplify the formulae.

Let us start from n°1. One can verify that this nontrivial intersection produces as Poisson brackets 5-cycles which are built from glueing a 2-cycle to a 3-cycle, with the two having no antiparallel arrows in common. It would then be enough to choose anchors so that one always follows the 3-cycle first, and this can indeed be done.

**Proposition 3.8.** *Pick nodes $a, b, c, d \in I$ so that $(a, d, c)$ defines a 3-cycle. Assume also that $a$ and $b$ are adjacent. Then one may choose Darboux coordinates so that:*




[5]It is impossible that they have two pairs of antiparallel arrows in common, since the three nodes of a IMD 3-cycle all lie in three different parts of $I$, whereas the nodes of a degenerate 4-cycle lie in two different parts.

*Proof.* Set $\alpha := (a,d), \beta := (d,c), \gamma := (c,a)$ and $\varepsilon := (a,b)$. Then one has:

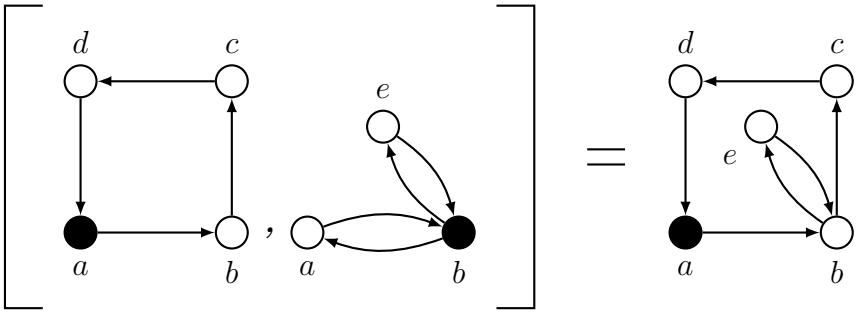
$$\Big[\mathrm{Tr}(\widehat{X}^{\varepsilon^*}\widehat{X}^{\varepsilon}\widehat{X}^{\gamma}\widehat{X}^{\gamma^*}), \mathrm{Tr}(\widehat{X}^{\gamma}\widehat{X}^{\beta}\widehat{X}^{\alpha})\Big] = \sum_{i,j,k,l,m,n,o} \Big[\widehat{X}^{\varepsilon^*}_{ij}\widehat{X}^{\varepsilon}_{jk}\widehat{X}^{\gamma}_{kl}\widehat{X}^{\gamma^*}_{li}, \widehat{X}^{\gamma}_{mn}\widehat{X}^{\beta}_{no}\widehat{X}^{\alpha}_{om}\Big] =$$
$$= \sum_{i,j,k,l,m,n,o} \widehat{X}^{\varepsilon^*}_{ij}\widehat{X}^{\varepsilon}_{jk}\widehat{X}^{\gamma}_{kl}\Big[\widehat{X}^{\gamma^*}_{li}, \widehat{X}^{\gamma}_{mn}\Big]\widehat{X}^{\beta}_{no}\widehat{X}^{\alpha}_{om} =$$
$$= \sum_{i,j,k,l,m,n,o} \delta_{ln}\delta_{im}\widehat{X}^{\varepsilon^*}_{ij}\widehat{X}^{\varepsilon}_{jk}\widehat{X}^{\gamma}_{kl}\widehat{X}^{\beta}_{no}\widehat{X}^{\alpha}_{om} = \mathrm{Tr}(\widehat{X}^{\varepsilon^*}\widehat{X}^{\varepsilon}\widehat{X}^{\gamma}\widehat{X}^{\beta}\widehat{X}^{\alpha}).$$

□

Now, every time that such a nontrivial intersection arises, one can base the 3-cycle as in the above figures without loss of generality, and the resulting 5-cycle will start at its 3-subcycle. In particular, two such commutators will be equal if and only if their associated classical brackets are, which is the result one is after.

One may next consider n °3. This nontrivial intersection produces a 6-cycle as Poisson bracket, built from glueing a nondegenerate 4-cycle and a 2-cycle, with the two having no antiparallel arrows in common. It would then be enough to choose anchors so that one always follows the 4-cycle first, and this can indeed be done.

**Proposition 3.9.** *Pick nodes $a, b, c, d, e \in I$ such that $(a, b, c, d)$ defines a 4-cycle. Assume that $b$ and $e$ are adjacent. Then one can choose Darboux coordinates so that:*



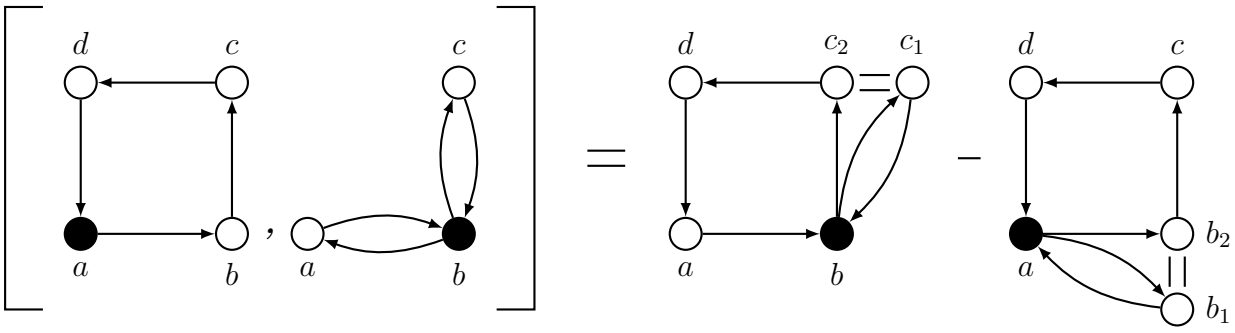

*Proof.* Set $\alpha := (a,b), \beta := (b,c), \gamma := (c,d), \varepsilon := (d,a)$ and $\zeta := (b,e)$. Then one has:

$$\Big[\mathrm{Tr}(\widehat{X}^{\varepsilon}\widehat{X}^{\gamma}\widehat{X}^{\beta}\widehat{X}^{\alpha}), \mathrm{Tr}(\widehat{X}^{\zeta^*}\widehat{X}^{\zeta}\widehat{X}^{\alpha}\widehat{X}^{\alpha^*})\Big] = \sum_{i,j,k,l,m,n,o,p} \Big[\widehat{X}^{\varepsilon}_{ij}\widehat{X}^{\gamma}_{jk}\widehat{X}^{\beta}_{kl}\widehat{X}^{\alpha}_{li}, \widehat{X}^{\zeta^*}_{mn}\widehat{X}^{\zeta}_{no}\widehat{X}^{\alpha}_{op}\widehat{X}^{\alpha^*}_{pm}\Big] =$$
$$= \sum_{i,j,k,l,m,n,o,p} \widehat{X}^{\varepsilon}_{ij}\widehat{X}^{\gamma}_{jk}\widehat{X}^{\beta}_{kl}\widehat{X}^{\zeta^*}_{mn}\widehat{X}^{\zeta}_{no}\widehat{X}^{\alpha}_{op}\Big[\widehat{X}^{\alpha}_{li}, \widehat{X}^{\alpha^*}_{pm}\Big] = \sum_{i,j,k,l,m,n,o,p} \delta_{ip}\delta_{lm}\widehat{X}^{\varepsilon}_{ij}\widehat{X}^{\gamma}_{jk}\widehat{X}^{\beta}_{kl}\widehat{X}^{\zeta^*}_{mn}\widehat{X}^{\zeta}_{no}\widehat{X}^{\alpha}_{op} =$$
$$= \sum_{i,j,k,l,m,n} \widehat{X}^{\varepsilon}_{ij}\widehat{X}^{\gamma}_{jk}\widehat{X}^{\beta}_{kl}\widehat{X}^{\zeta^*}_{lm}\widehat{X}^{\zeta}_{mn}\widehat{X}^{\alpha}_{ni} = \mathrm{Tr}(\widehat{X}^{\varepsilon}\widehat{X}^{\gamma}\widehat{X}^{\beta}\widehat{X}^{\zeta^*}\widehat{X}^{\zeta}\widehat{X}^{\alpha}).$$

□

Finally, one should check n °4.

**Proposition 3.10.** *Pick nodes $a, b, c, d \in I$ defining a 4-cycle. One may choose Darboux coordinates so that:*




On the right-hand side one has split the nodes $c = c_1 = c_2$ and $b = b_1 = b_2$, which are both touched twice. The point of this proposition is the same as before: up to changing the anchor of the nondegenerate 4-cycle, all 6-cycles that appear as a result of this type of nontrivial intersection will have equivalent anchors (one follows the 2-cycle first).

*Proof.* Let's denote $\alpha := (a,b), \beta := (b,c), \gamma := (c,d)$, and $\varepsilon = (d,a)$. Then one has:

$$\Big[\mathrm{Tr}(\widehat{X}^{\varepsilon}\widehat{X}^{\gamma}\widehat{X}^{\beta}\widehat{X}^{\alpha}), \mathrm{Tr}(\widehat{X}^{\alpha}\widehat{X}^{\alpha^*}\widehat{X}^{\beta^*}\widehat{X}^{\beta})\Big] = \sum_{i,j,k,l,m,n,o,p} \Big[\widehat{X}^{\varepsilon}_{ij}\widehat{X}^{\gamma}_{jk}\widehat{X}^{\beta}_{kl}\widehat{X}^{\alpha}_{li}, \widehat{X}^{\alpha}_{mn}\widehat{X}^{\alpha^*}_{no}\widehat{X}^{\beta^*}_{op}\widehat{X}^{\beta}_{pm}\Big] =$$

$$= \sum_{i,j,k,l,m,n,o,p} \widehat{X}^{\varepsilon}_{ij}\widehat{X}^{\gamma}_{jk}\widehat{X}^{\beta}_{kl}\Big[\widehat{X}^{\alpha}_{li}, \widehat{X}^{\alpha}_{mn}\widehat{X}^{\alpha^*}_{no}\widehat{X}^{\beta^*}_{op}\widehat{X}^{\beta}_{pm}\Big] + \widehat{X}^{\varepsilon}_{ij}\widehat{X}^{\gamma}_{jk}\Big[\widehat{X}^{\beta}_{kl}, \widehat{X}^{\alpha}_{mn}\widehat{X}^{\alpha^*}_{no}\widehat{X}^{\beta^*}_{op}\widehat{X}^{\beta}_{pm}\Big]\widehat{X}^{\alpha}_{li} =$$

$$= \sum_{i,j,k,l,m,n,o,p} \widehat{X}^{\varepsilon}_{ij}\widehat{X}^{\gamma}_{jk}\widehat{X}^{\beta}_{kl}\widehat{X}^{\alpha}_{mn}\Big[\widehat{X}^{\alpha}_{li}, \widehat{X}^{\alpha^*}_{no}\Big]\widehat{X}^{\beta^*}_{op}\widehat{X}^{\beta}_{pm} + \widehat{X}^{\varepsilon}_{ij}\widehat{X}^{\gamma}_{jk}\widehat{X}^{\alpha}_{mn}\widehat{X}^{\alpha^*}_{no}\Big[\widehat{X}^{\beta}_{kl}, \widehat{X}^{\beta^*}_{op}\Big]\widehat{X}^{\beta}_{pm}\widehat{X}^{\alpha}_{li} =$$

$$= \sum_{i,j,k,l,m,n,o,p} \delta_{lo}\delta_{in}\widehat{X}^{\varepsilon}_{ij}\widehat{X}^{\gamma}_{jk}\widehat{X}^{\beta}_{kl}\widehat{X}^{\alpha}_{mn}\widehat{X}^{\beta^*}_{op}\widehat{X}^{\beta}_{pm} - \delta_{lo}\delta_{kp}\widehat{X}^{\varepsilon}_{ij}\widehat{X}^{\gamma}_{jk}\widehat{X}^{\alpha}_{mn}\widehat{X}^{\alpha^*}_{no}\widehat{X}^{\beta}_{pm}\widehat{X}^{\alpha}_{li} =$$

$$= \sum_{i,j,k,l,m,n,o,p} \delta_{lo}\delta_{in}\widehat{X}^{\alpha}_{mn}\widehat{X}^{\varepsilon}_{ij}\widehat{X}^{\gamma}_{jk}\widehat{X}^{\beta}_{kl}\widehat{X}^{\beta^*}_{op}\widehat{X}^{\beta}_{pm} - \delta_{lo}\delta_{kp}\widehat{X}^{\varepsilon}_{ij}\widehat{X}^{\gamma}_{jk}\widehat{X}^{\beta}_{pm}\widehat{X}^{\alpha}_{mn}\widehat{X}^{\alpha^*}_{no}\widehat{X}^{\alpha}_{li} =$$

$$= \sum_{i,j,k,l,m,n} \widehat{X}^{\alpha}_{ij}\widehat{X}^{\varepsilon}_{jk}\widehat{X}^{\gamma}_{kl}\widehat{X}^{\beta}_{lm}\widehat{X}^{\beta^*}_{mn}\widehat{X}^{\beta}_{ni} - \widehat{X}^{\varepsilon}_{ij}\widehat{X}^{\gamma}_{jk}\widehat{X}^{\beta}_{kl}\widehat{X}^{\alpha}_{lm}\widehat{X}^{\alpha^*}_{mn}\widehat{X}^{\alpha}_{ni} =$$

$$= \mathrm{Tr}(\widehat{X}^{\alpha}\widehat{X}^{\varepsilon}\widehat{X}^{\gamma}\widehat{X}^{\beta}\widehat{X}^{\beta^*}\widehat{X}^{\beta}) - \mathrm{Tr}(\widehat{X}^{\varepsilon}\widehat{X}^{\gamma}\widehat{X}^{\beta}\widehat{X}^{\alpha}\widehat{X}^{\alpha^*}\widehat{X}^{\alpha}).$$

□

This computation concludes the proof of Thm. 3.2.

## 3.7 KZ and the star

Here we show that the KZ connection (2.22) of § 2.3.5 is a reduction of the simply-laced quantum connection for the degenerate reading of a star with no irregular times. Recall that a star is a complete bipartite graph having one part with a single node. The statements we will prove are the following.

**Theorem 3.3.** *The Schlesinger connection quantises to the KZ connection via the standard PBW isomorphism.*

This is proven in § 2.3.5.

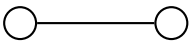
**Theorem 3.4.** *The SLIMS reduce to the Schlesinger system, in the special case of the degenerate reading of a star with no irregular times.*

This is proven in § 3.7.2 and § 3.7.3.

**Theorem 3.5.** *The SLQC reduces to the KZ system, in the same case as above.*

This is proven in § 3.7.4 and § 3.7.5.

### 3.7.1 Simply-laced quantum connection of a star

The general construction of § 3.1 must be reduced to the following data: the set $J$ has cardinality $k = 2$ the reading is $a(J) = \{+\infty, 0\}$, and $T^0 = 0$. One considers the complete graph  on nodes $J$. The splayed graph $\mathcal{G}$ will be a star on nodes $I = I^0 \coprod I^\infty = \{0\} \coprod I^\infty$. It will be centred at 0, and have $m := |I^\infty|$ legs.
The base space of times is then $\mathbf{B} = \mathbb{C}^m \setminus \{\text{diags}\}$, the vector phase-space is

$$\mathbb{M} = \mathrm{Hom}(W^\infty, W^0) \oplus \mathrm{Hom}(W^0, W^\infty),$$

equipped with the symplectic form $\omega_a = \mathrm{Tr}(dQ \wedge dP)$, where one considers linear maps $Q\colon W^\infty \longrightarrow W^0$, $P\colon W^0 \longrightarrow W^\infty$. If $W^\infty = \bigoplus_{i\in I^\infty} V_i$, then one will write $Q_i$ for the component of $Q$ in $V_i^* \otimes W^0$, and $P_i$ for the component of $P$ in $(W^0)^* \otimes V_i$. Notice that we've basically chosen an orientation of $\widetilde{\mathcal{G}}$, which wasn't necessary. However, this provides Darboux coordinates:

$$\{(Q_i)_{kl}, (P_j)_{mn}\} = \delta_{ij}\delta_{kn}\delta_{lm}, \qquad \{(Q_i)_{kl}, (Q_j)_{mn}\} = 0 = \{(P_i)_{kl}, (P_j)_{mn}\}.$$

Those data code a space of meromorphic connections of the form

$$\nabla = d - \sum_{i\in I^\infty} \frac{Q_i P_i}{z - t_i} dz,$$

on the trivial vector bundle $W^0 \times \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$. Here $\{t_i\}_{i\in I^\infty} \in \mathbf{B}$, and $(Q, P) \in \mathbb{M}$.

The isomonodromic deformations of those connections are coded by the Hamiltonian system

$$\varpi = \frac{1}{2}\,\mathrm{Tr}\left(\widetilde{PQ}PQ\right) = \sum_{i\in I^\infty} H_i dt_i \in \Omega^0(\mathbb{F}_a, \pi^* T^*\mathbf{B}),$$

where $\pi\colon \mathbb{F}_a = \mathbb{M} \times \mathbf{B} \longrightarrow \mathbf{B}$ is the trivial symplectic fibration of § 3.1. This system spells out as

$$H_i(Q, P, T^\infty) = \sum_{i\neq j\in I^\infty} \frac{\mathrm{Tr}(P_i Q_j P_j Q_i)}{t_i - t_j} \in \mathrm{Sym}(\mathbb{M}^*). \tag{3.6}$$

Indeed, $H_i$ is the sum of the traces of all (necessarily degenerate) 4-cycles at the node $i$, whereas 3-cycles cannot appear in a bipartite context, and all 2-cycles are not there because of $A = T^0 = dT^0 = 0$.

The SLQC at hand is then

$$\widehat{\nabla} = d_{\mathbf{B}} - \widehat{\varpi} = d_{\mathbf{B}} - \sum_{i \in I^\infty} \left( \sum_{i \neq j \in I^\infty} \frac{\mathrm{Tr}(\widehat{Q}_j \widehat{P}_j \widehat{Q}_i \widehat{P}_i)}{t_i - t_j} \right) dt_i. \tag{3.7}$$

This is a connection on the trivial (quantum) vector bundle $\mathbb{E}_a = A \times \mathbf{B} \longrightarrow \mathbf{B}$, where $A := W(\mathbb{M}^*, \omega_a)$. Here again one denotes by $\omega_a$ the symplectic form induced on $\mathbb{M}^*$ by the linear isomorphism induced by the symplectic pairing $\mathbb{M} \wedge \mathbb{M} \longrightarrow \mathbb{C}$. As explained when introducing the Weyl algebra, this bilinear alternating map $\mathbb{M}^* \wedge \mathbb{M}^* \longrightarrow \mathbb{C}$ is the restriction of the Poisson bracket of $\mathscr{O}(\mathbb{M})$ to linear function.

The main theorem 3.2 assures that $\widehat{\nabla}$ is strongly flat. Let us set

$$\widehat{H}_i := \sum_{i \neq j} \frac{\mathrm{Tr}(\widehat{Q}_j \widehat{P}_j \widehat{Q}_i \widehat{P}_i)}{t_i - t_j},$$

for the quantum Hamiltonians defining the simply-laced quantum connection. Let us prove explicitly that the connection is strongly flat. To this end, the only nontrivial verification is that for the commutators.

**Proposition 3.11.** *One has*

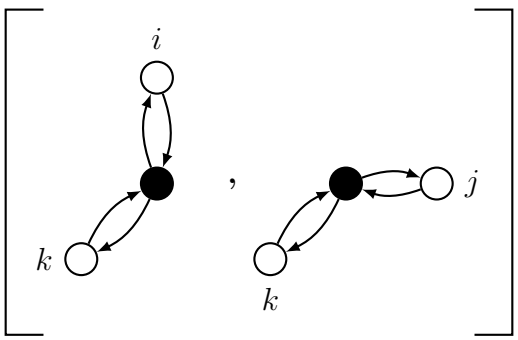
$$\left[\widehat{H}_i, \widehat{H}_j\right] = 0$$

*for all* $i, j \in I^\infty$.

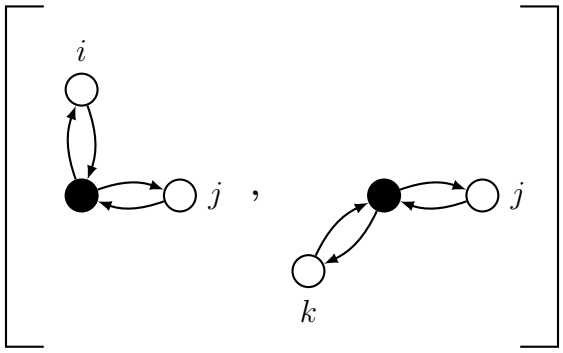
*Proof.* Pick $i \neq j \in I^\infty$. The trick is to decompose the commutator in the following sum:

$$\begin{aligned}\left[\widehat{H}_i, \widehat{H}_j\right] = &\sum_{k \in I^\infty \setminus \{i,j\}} \frac{1}{(t_i - t_k)(t_j - t_k)} \left[ \mathrm{Tr}(\widehat{Q}_k \widehat{P}_k \widehat{Q}_i \widehat{P}_i), \mathrm{Tr}(\widehat{Q}_k \widehat{P}_k \widehat{Q}_j \widehat{P}_j) \right] + \\ &+ \frac{1}{(t_i - t_j)(t_j - t_k)} \left[ \mathrm{Tr}(\widehat{Q}_j \widehat{P}_j \widehat{Q}_i \widehat{P}_i), \mathrm{Tr}(\widehat{Q}_j \widehat{P}_j \widehat{Q}_k \widehat{P}_k) \right] + \\ &+ \frac{1}{(t_i - t_k)(t_j - t_i)} \left[ \mathrm{Tr}(\widehat{Q}_i \widehat{P}_i \widehat{Q}_k \widehat{P}_k), \mathrm{Tr}(\widehat{Q}_i \widehat{P}_i \widehat{Q}_j \widehat{P}_j) \right]\end{aligned}$$

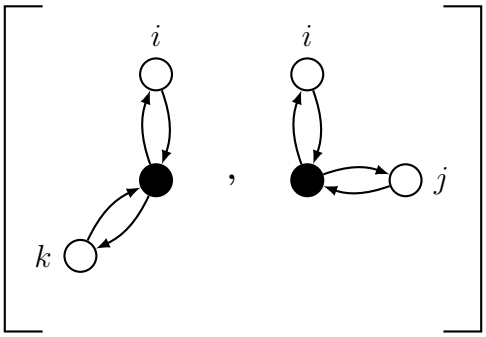
This decomposition is suggested by looking at the degenerate 4-cycles at the nodes $i, j$ with 2-cycles in common, which leaves the following nontrivial intersections:




and




and




All those cycles are based at the centre $0 \in I$ of the star. Those intersections give precisely the terms above. Now, using Prop. 3.6.1 for the commutators, one finds:

$$\begin{aligned}\left[\widehat{H}_i, \widehat{H}_j\right] &= \sum_{k\neq i,j} \frac{1}{(t_i-t_k)(t_j-t_k)} \operatorname{Tr}\left([\widehat{Q}_i\widehat{P}_i, \widehat{Q}_k\widehat{P}_k]\widehat{Q}_j\widehat{P}_j\right) + \\ &+ \frac{1}{(t_i-t_j)(t_j-t_k)} \operatorname{Tr}\left([\widehat{Q}_i\widehat{P}_i, \widehat{Q}_j\widehat{P}_j]\widehat{Q}_k\widehat{P}_k\right) + \frac{1}{(t_i-t_k)(t_j-t_i)} \operatorname{Tr}\left([\widehat{Q}_k\widehat{P}_k, \widehat{Q}_i\widehat{P}_i]\widehat{Q}_j\widehat{P}_j\right) = \\ &= \sum_{k\neq i,j} \left[\frac{1}{(t_i-t_k)(t_j-t_k)} - \frac{1}{(t_i-t_j)(t_j-t_k)} - \frac{1}{(t_i-t_k)(t_j-t_i)}\right] \operatorname{Tr}\left([\widehat{Q}_j\widehat{P}_j, \widehat{Q}_i\widehat{P}_i]\widehat{Q}_k\widehat{P}_k\right).\end{aligned}$$

Finally, this vanishes thanks to the identity

$$\frac{1}{(t_i-t_k)(t_j-t_k)} - \frac{1}{(t_i-t_j)(t_j-t_k)} - \frac{1}{(t_i-t_k)(t_j-t_i)} = 0.$$

□

**Remark 3.6.** It is precisely this type of computation that motivated the introduction of (traces of) quantum potentials. Moreover, cyclic identities of the type above for the functions $(t_i - t_k)^{-1}(t_j - t_k)^{-1}$ are also used in the proof of flatness of the KZ connection. Indeed, we view the commutation relations among degenerate 4-cycles as a lift of the so-called Kohno relations for the operators $\Omega_{ij}$ of the KZ connection (2.22).

### 3.7.2 The classical reduction

The main idea is to replace the product $Q_iP_i \in \mathrm{End}(W^0)$ that appears in (3.6) with the residue $R_i \in \mathrm{End}(W^0)$ that appears in (2.21). This transforms the former into the latter, as one sees directly on the explicit formulae.

To make this rigorous, get back to the vector spaces $W^0, W^\infty = \bigoplus_{i\in I^\infty} V_i$, and set $\mathfrak{g}^0 := \mathfrak{gl}(W^0)$. Thanks to the trace-pairing, one has a canonical identification

$$L_i := W^0 \otimes V_i^* = \mathrm{Hom}(V_i, W^0) \cong \mathrm{Hom}(W^0, V_i) = (W^0)^* \otimes V_i,$$

and this for all $i \in I^\infty$. One now considers the map given by the composition of linear functions, that is

$$\mu_i \colon L_i \oplus L_i^* \longrightarrow \mathfrak{g}^0, \qquad (A, B) \longmapsto AB,$$

This best expressed by choosing bases. If $\{e_j\}_j$ is a basis of $W^0$, then one has a canonical basis of $\mathfrak{g}^0$, provided by $\{e_{jk} := e_j \otimes de_k\}_{jk}$: $e_{jk}$ is the endomorphism that maps $e_k$ into $e_j$. If one next chooses a basis $\{f_j^{(i)}\}_j$ of $V_i$, then one has a basis $\{(ef^{(i)})_{jk} := e_j \otimes df_k^{(i)}\}_{jk}$ of $L_i$: $(ef^{(i)})_{jk}$ is the linear function that maps $f_k^{(i)}$ into $e_j$. The trace-duality sends this to a basis $\{(f^{(i)}e)_{jk} := f_j^{(i)} \otimes de_k\}_{jk}$ of $L_i^*$, with $(f^{(i)}e)_{jk} \colon W^0 \longrightarrow V_i$ sending $e_k$ to $f_j^{(i)}$.
With those choices made, the product $\mu_i \colon T^*L_i \longrightarrow \mathfrak{g}^0$ reads

$$\mu_i((ef^{(i)})_{jk}, (f^{(i)}e)_{lm}) = \mu_i\Big((e_j \otimes df_k^{(i)}) \oplus (f_l^{(i)} \otimes de_m)\Big) = \delta_{kl} e_j \otimes de_m = \delta_{kl} e_{jm}.$$

Then one expands by bilinearity, to get the usual matrix product $L_i \otimes L_i^* \longrightarrow \mathfrak{g}^0$. Notice that $\mu_i$ is nothing but a restriction of the moment map for the standard action of $\mathrm{GL}(W^0)$ on $\mathbb{M}$. Equivalently, it is the Poisson map for the restricted action of $\mathrm{GL}(W^0)$ on the invariant symplectic subspace $T^*L_i \subseteq \mathbb{M}$:

$$g.(Q_j, P_j)_{j\in I^\infty} = (Q'_j, P'_j)_{j\in I^\infty},$$

with

$$(Q'_j, P'_j) = \begin{cases} (Q_j, P_j), & j \neq i, \\ (gQ_i, P_i g^{-1}), & \text{else.} \end{cases}$$

In particular, $\mu_i$ is a Poisson map.

This was the situation for a single leg of the star-shaped graph $\mathcal{G}$. One can now glue the maps $\mu_i$ to the full moment map

$$\mu \colon \mathbb{M} = \bigoplus_{i\in I^\infty} T^*L_i \longrightarrow \mathfrak{g}^0, \qquad \mu \colon (Q_i, P_i)_{i\in I^\infty} \longmapsto \sum_{i\in I^\infty} \mu_i(Q, P) = \sum_{i\in I^\infty} Q_iP_i,$$

which will satisfy the same as above. We shall also abusively denote $\mu \colon \mathbb{M} \longrightarrow (\mathfrak{g}^0)^m$ the map $(Q_i, P_i)_{i\in I^\infty} \longmapsto (Q_iP_i)_{i\in I^\infty}$ that separates the components.

**Proposition 3.12.** *One has $\mu^*(H_i^{\mathrm{Sch}}) = H_i$ for all $i \in I^\infty$.*

*Proof.* By linearity, it is enough to check that

$$\mu^* \operatorname{Tr}(R_i R_j) = \operatorname{Tr}(Q_i P_i Q_j P_j),$$

for $i \neq j$. This follows from the fact that

$$(R_i)_{kl} = de_{kl}(R_i), \qquad (Q_i)_{kl} = d(ef^{(i)})_{kl}(Q_i), \qquad (P_i)_{kl} = d(f^{(i)}e)_{kl}(P_i),$$

with the same notation as above, by the very definition of the component maps with respect to the bases $\{e_j\}_j$ of $W^0$ and $\{f_j^{(i)}\}_j$ of $V_i$. One may write $de_{kl}^{(i)}$ for the coordinate function $R \longmapsto (R_i)_{kl} = de_{kl}(R_i)$.

Having understood this notation, one has

$$\mu_i^* de_{jk}^{(i)}\Big((ef^{(i)})_{lm} \oplus (f^{(i)}e)_{no}\Big) = de_{jk}^{(i)}\mu_i\Big((ef^{(i)})_{lm} \oplus (f^{(i)}e)_{no}\Big) = de_{jk}^{(i)}(\delta_{mn}e_{lo}^{(i)}) = \delta_{mn}\delta_{jl}\delta_{ko},$$

which yields the formula

$$\mu_i^* de_{jk}^{(i)} = \sum_m d(ef^{(i)})_{jm} \otimes d(f^{(i)}e)_{mk} \in \mathscr{O}(T^*L_i).$$

Hence the following computation yields the result:

$$\begin{aligned}
&\mu^* \operatorname{Tr}(R_i R_j) = \mu^* \sum_{k,l} (R_i)_{kl}(R_j)_{lk} = \sum_{k,l} \Big(\mu_i^*(R_i)_{kl}\Big)\Big(\mu_j^*(R_j)_{lk}\Big) = \\
&= \sum_{k,l} \Big(\mu_i^* de_{kl}^{(i)}\Big) \otimes \Big(\mu_j^* de_{lk}^{(j)}\Big) = \sum_{k,l,m,n} d(ef^{(i)})_{km} \otimes d(f^{(i)}e)_{ml} \otimes d(ef^{(j)})_{ln} \otimes d(f^{(j)}e)_{nk} = \\
&= \sum_{k,l,m,n} (Q_i)_{km}(P_i)_{ml}(Q_j)_{ln}(P_j)_{nk} = \operatorname{Tr}(Q_i P_i Q_j P_j).
\end{aligned}$$

□

This proposition means that there is a consistent way to pass from the star-shaped SLIMS to the Schlesinger ones, by the change of variable $R_i \coloneqq Q_i P_i$. We view this as a classical reduction, in the following sense.

### 3.7.3 Dual symplectic pairs and classical Hamiltonian reduction

Consider two smooth affine complex Poisson varieties $P_1, P_2$.

**Definition 3.13.** A smooth affine symplectic variety $(M, \omega)$ over $\mathbb{C}$, together with a couple of Poisson maps $\mu_1, \mu_2 \colon M \longrightarrow P_i$, is called a symplectic dual pair if one has

$$\{\mu_1^*(\mathscr{O}(P_1)), \mu_2^*(\mathscr{O}(P_2))\}_\omega = 0,$$

where $\mathscr{O}(P_i)$ are the global sections of the structural sheaf of $P_i$, and with $\{\cdot,\cdot\}_\omega$ being the Poisson bracket on $\mathscr{O}(M)$ defined by $\omega$.

This notion was introduced in [Kar89].[6] Such a situation arises in particular for the moment maps with respect to commuting Hamiltonian actions. To prove this, pick two complex algebraic groups $G_1, G_2$ with Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$, and a symplectic manifold $(M, \omega)$.

**Lemma 3.3.** *Assume that $G_1, G_2$ act on $(M, \omega)$ with momenta $\mu_i \colon M \longrightarrow \mathfrak{g}_i^*$. If the two actions commute, then one has a dual symplectic pair.*

*Proof.* One can consider the natural action of $G \coloneqq G_1 \times G_2$ on $(M, \omega)$. It admits the moment

$$\mu = \mu_1 \oplus \mu_2 \colon M \longrightarrow \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

Now one has, for $x_i \in \mathfrak{g}_i$:

$$\begin{aligned}\{\mu_1^*(x_1), \mu_2^*(x_2)\}_\omega = \{\mu^*(x_1, 0), \mu^*(0, x_2)\}_\omega = \mu^*\Big([(x_1, 0), (0, x_2)]\Big) = \\ = \mu^*\Big([x_1, 0], [0, x_2]\Big) = \mu^*(0, 0) = 0,\end{aligned}$$

where one used the fact that $\mu^* \colon \mathfrak{g}_1 \oplus \mathfrak{g}_2 \longrightarrow \mathscr{O}(M)$ is a morphism of Lie algebras, plus the definition of the direct sum of Lie algebras. □

To apply this to the case at hand, set $G^0 \coloneqq \mathrm{GL}(W^0), G^\infty \coloneqq \prod_{i \in I^\infty} \mathrm{GL}(V_i)$, with Lie algebras $\mathfrak{g}^0, \mathfrak{g}^\infty$. The two groups act on $(\mathbb{M}, \omega_a)$ with commutative Hamiltonian actions. Introduce the notation $\mu_0 \colon \mathbb{M} \longrightarrow \mathfrak{g}^0 \cong (\mathfrak{g}^0)^*$ and $\mu_\infty \colon \mathbb{M} \longrightarrow \mathfrak{g}^\infty \cong (\mathfrak{g}^\infty)^*$ for the momenta. One would now like to relate the subalgebra $\mu_0^*(\mathscr{O}(\mathfrak{g}^0)) \subseteq \mathscr{O}(\mathbb{M})$ to that of regular functions on the reduction $\mathbb{M}\big/G^\infty$. Recall that this is by definition the affine Poisson scheme defined as

$$\mathbb{M}\big/G^\infty \coloneqq \mathrm{Spec}\left(\mathscr{O}(\mathbb{M})^{G^\infty}\right),$$

using elementary affine GIT theory. This is well defined, because one is acting via a reductive group on affine space, but it can have geometric issues: the space may be nonreduced and/or singular. Here we are however only interested in the dual functional viewpoint, i.e. the algebraic one, which is more suited to deformation quantisation.

Recall from § 2.1.6 that one defines the (classical) Hamiltonian reduction $R\Big(\mathscr{O}(\mathbb{M}), \mathfrak{g}^\infty, I\Big)$ of $\mathscr{O}(\mathbb{M})$ with respect to the Lie algebra $\mathfrak{g}^\infty$ and an ideal $I \subseteq \mathrm{Sym}(\mathfrak{g}^\infty)$, assuming there to be a (co)moment $\mu_\infty^* \colon \mathrm{Sym}(\mathfrak{g}^\infty) \longrightarrow \mathscr{O}(\mathbb{M})$. Namely, one considers the ideal $J \subseteq \mathscr{O}(\mathbb{M})$ generated by $\mu_\infty^*(I)$, and then one sets

$$R(\mathscr{O}(\mathbb{M}), \mathfrak{g}^\infty, I) \coloneqq \mathscr{O}(\mathbb{M})^{\mathfrak{g}^\infty}\Big/J^{\mathfrak{g}^\infty},$$

where

$$\mathscr{O}(\mathbb{M})^{\mathfrak{g}^\infty} \coloneqq \{f \in \mathscr{O}(\mathbb{M}) \mid \{\mu_\infty^*(\mathfrak{g}^\infty), f\} = 0\},$$

and $J^{\mathfrak{g}^\infty} = J \cap \mathscr{O}(\mathbb{M})^{\mathfrak{g}^\infty}$. In § 2.1.6 it was shown that the reduction is Poisson. Notice that one has

$$\mathscr{O}(\mathbb{M})^{\mathfrak{g}^\infty}\Big/J^{\mathfrak{g}^\infty} \cong \Big(\mathscr{O}(\mathbb{M})/J\Big)^{\mathfrak{g}^\infty},$$

[6] This is the weakest possible notion that one finds in the literature. Some authors require the maps $\mu_i$ to be submersive. Some other require that the two subalgebras $\mu_i^*(\mathscr{O}(P_i))$ be the mutual centraliser of one another into $\mathscr{O}(M)$.

since $\mathfrak{g}^\infty$ is reductive.

The important feature of the above symplectic dual pairs is that the image of $\mu_0^*$ is contained in $\mathscr{O}(\mathbb{M})^{\mathfrak{g}^\infty}$. This provides a natural morphism of Poisson algebras

$$\varphi\colon \mathscr{O}(\mathfrak{g}^0)^m \longrightarrow R\Big(\mathscr{O}(\mathbb{M}), \mathfrak{g}^\infty, I\Big)$$

for all ideals $I \subseteq \mathrm{Sym}(\mathfrak{g}^\infty)$, by composing $\mu_0^*$ with the canonical projection to the Hamiltonian reduction $\pi\colon \mathscr{O}(\mathbb{M})^{\mathfrak{g}^\infty} \longrightarrow R(\mathscr{O}(\mathbb{M}), \mathfrak{g}^\infty, I)$.

This is the correspondence we invoke to say that $\{H_i^{\mathrm{Sch}}\}_i$ is a reduction of the classical SLIMS of a star. Indeed, we've shown that $\mu_0^*(H_i^{\mathrm{Sch}}) = H_i\colon \mathbf{B} \longrightarrow \mathscr{O}(\mathbb{M})$. Now, since actually $H_i$ takes values in the invariant part $\mathscr{O}(\mathbb{M})^{\mathfrak{g}^\infty}$, one may reduce it to a function $\pi(H_i)$ defined on any symplectic reduction $\Big(\mathbb{M}/\!/_{\mathcal{O}}G^\infty\Big) \times \mathbf{B}$ of the above trivial Poisson fibration. One then has $\varphi(H_i^{\mathrm{Sch}}) = \pi(H_i)$, by definition.

### 3.7.4 The quantum reduction

One can now use the quantum constructions which are analogue to the above classical ones. Recall that we defined quantum moments and quantum Hamiltonian reduction in § 2.1.6, and now we wish to apply that material to our context, where the group $\mathrm{GL}(W^0)$ acts on $(\mathbb{M}, \omega_a)$ with comoment $\mu_0^*\colon \mathrm{Sym}\Big((\mathfrak{g}^0)^*\Big) \longrightarrow A_0$.
One can provide a natural quantisation of it. Namely, one defines $\widehat{\mu^*}_0\colon U\Big((\mathfrak{g}^0)^*\Big) \longrightarrow A$ by showing that the auxiliary morphism

$$\widetilde{\mu^*}_0\colon \mathrm{Tens}\Big((\mathfrak{g}^0)^*\Big) \longrightarrow \mathrm{Tens}(T^*L_i), \qquad \widetilde{\mu^*}_0\colon de_{jk} \longmapsto \sum_m d(ef^{(i)})_{jm} \otimes d(f^{(i)}e)_{mk},$$

induces an associative morphism of the quantum algebras thanks to the universal property of the quotient. Just as before, one can fix $i \in I^\infty$, and admits that all results will glue as they have to, because of the universal property of the direct sum. Moreover, it turns out to be simpler to check this for the trace-dual version.

Consider thus the map

$$\alpha\colon \mathrm{Tens}(\mathfrak{g}^0) \longrightarrow \mathrm{Tens}(T^*L_i), \qquad \alpha\colon e_{jk} \longmapsto \sum_m (f^{(i)}e)_{mk} \otimes (ef^{(i)})_{jm}.$$

To get to this map, one indeed uses the dualities

$$\mathrm{Tr}\colon de_{jk} \longmapsto e_{kj}, \qquad \mathrm{Tr}\colon d(ef^{(i)})_{jk} \longmapsto (f^{(i)}e)_{kj}, \qquad \mathrm{Tr}\colon d(f^{(i)}e)_{jk} \longmapsto (ef^{(i)})_{kj},$$

provided by the trace. A last point must however be made: the linear isomorphism $\mathrm{Tr}\colon (\mathbb{M}, \omega_a) \longrightarrow (\mathbb{M}^*, \omega_a)$ is anti-symplectic, since the trace reverses the orientation chosen for the arrows of $\mathcal{G}$. To be more explicit, the isomorphism $\varphi\colon \mathbb{M}^* \longrightarrow \mathbb{M}$ provided by the symplectic pairing is canonical, whereas the isomorphism $\mathrm{Tr}\colon \mathbb{M}^* \longrightarrow \mathbb{M}$ is noncanonical (it relies on the choice of a polarisation for $(\mathbb{M}, \omega_a)$), and they differ by a sign when computed on the usual basis:

$$\varphi\colon d(ef^{(i)})_{jk} \longmapsto -(f^{(i)}e)_{kj}, \qquad \varphi\colon d(f^{(i)}e)_{jk} \longmapsto (ef^{(i)})_{kj}$$

Hence one should prove that $\alpha$ induces a morphism with the Weyl algebra for the opposite symplectic structure, and this is precisely what happens.

**Proposition 3.13.** *The map $\alpha\colon \mathrm{Tens}(\mathfrak{g}^0) \longrightarrow \mathrm{Tens}(T^*L_i)$ induces an associative morphism $\widehat{\varphi}\colon U(\mathfrak{g}^0) \longrightarrow W(T^*L_i, \omega_a^{\mathrm{op}})$, where $\omega_a^{\mathrm{op}}$ is the opposite symplectic structure.*

*Proof.* Thanks to the universal property of the quotient, it is enough to show that one has $\alpha(I_1) \subseteq J_{\omega_a}^{\mathrm{op}}$. Here $J_{\omega_a}^{\mathrm{op}} \subseteq \mathrm{Tens}(T^*L_i)$ is the bilateral ideal generated by

$$x \otimes y - y \otimes x - \omega_a^{\mathrm{op}}(x,y) = x \otimes y - y \otimes x + \omega_a(x,y),$$

for $x, y \in \mathbb{M}$, and $I_1 \subseteq U(\mathfrak{g}^0)$ is the bilateral ideal generated by

$$v \otimes w - w \otimes v - [v,w],$$

for $v, w \in \mathfrak{g}^0$. This can be shown on the usual bases $\{e_{jk}\}_{j,k} \subseteq \mathfrak{g}^0$, $\{(ef^i)_{jk}\}_{j,k} \subseteq L_i$ and $\{(f^ie)_{jk}\}_{j,k} \subseteq L_i^*$. One has:

$$\begin{aligned}
&\alpha\Big(e_{jk} \otimes e_{lm} - e_{lm} \otimes e_{jk} - [e_{jk}, e_{lm}]\Big) = \alpha\Big(e_{jk} \otimes e_{lm} - e_{lm} \otimes e_{jk} - \delta_{kl}e_{jm} + \delta_{jm}e_{lk}\Big) = \\
&= \sum_{n,o} \Big[(f^ie)_{nk} \otimes (ef^i)_{jn}, (f^ie)_{om} \otimes (ef^i)_{lo}\Big] - \delta_{kl}\delta_{no}(f^ie)_{om} \otimes (ef^i)_{jn} + \delta_{jm}\delta_{no}(f^ie)_{nk} \otimes (ef^i)_{lo} = \\
&= \sum_{n,o} (f^ie)_{nk} \otimes \Big(\Big[(ef^i)_{jn}, (f^ie)_{om}\Big] + \omega_a\Big((ef^i)_{jn}, (f^ie)_{om}\Big)\Big) \otimes (ef^i)_{lo} + \\
&+ \sum_{n,o} (f^ie)_{om} \otimes \Big(\Big[(f^ie)_{nk}, (ef^i)_{lo}\Big] + \omega_a\Big((f^ie)_{nk}, (ef^i)_{lo}\Big)\Big) \otimes (ef^i)_{jn} + \\
&+ \sum_{n,o} (f^ie)_{nk} \otimes (f^ie)_{om} \otimes \Big(\Big[(ef^i)_{jn}, (ef^i)_{lo}\Big] + \omega_a\Big((ef^i)_{jn}, (ef^i)_{lo}\Big)\Big) + \\
&+ \sum_{n,o} \Big(\Big[(f^ie)_{nk}, (f^ie)_{om}\Big] + \omega_a\Big((f^ie)_{nk}, (f^ie)_{om}\Big)\Big) \otimes (ef^i)_{lo} \otimes (ef^i)_{jn}.
\end{aligned}$$

This expansion follows from the Leibnitz rule for the commutator of $\mathrm{Tens}(T^*L_i)$, and from the canonical relations

$$\begin{gathered}
\omega_a\Big((ef^i)_{jn}, (f^ie)_{om}\Big) = \delta_{jm}\delta_{no}, \qquad \omega_a\Big((f^ie)_{nk}, (ef^i)_{lo}\Big) = -\delta_{kl}\delta_{no}, \\
\omega_a\Big((ef^i)_{jn}, (ef^i)_{lo}\Big) = 0 = \omega_a\Big((f^ie)_{nk}, (f^ie)_{om}\Big).
\end{gathered}$$

At the end one has four addends, each of which is a (bilateral) $\mathrm{Tens}(T^*L_i)$-linear combination of generators of $J_{\omega_a}^{\mathrm{op}}$. Hence the result lies in the ideal, by definition. □

This proposition shows that the definition

$$\widehat{\mu^*}_0\Big(\widehat{de}_{jk}\Big) = \sum_m \widehat{d(ef^{(i)})}_{jm} \cdot \widehat{d(f^{(i)}e)}_{mk} \in A \tag{3.8}$$

makes sense. The conceptual reason why this works is that $\mu_0^*$ is Poisson. It is straightforward to check that this maps quantises $\mu_0^*$. Moreover, it now defines a quantum $\mathfrak{g}^0$-action $\widehat{\xi}_0\colon \mathfrak{g}^0 \longrightarrow \mathrm{Der}(A)$ that quantises the classical $\xi_0\colon \mathfrak{g} \longrightarrow \mathrm{Der}(A_0)$.

**Proposition 3.14.** *One has* $\widehat{\mu_0^*}\big(\widehat{H}_i^{\mathrm{KZ}}\big) = \widehat{H}_i$.

*Proof.* This equality makes sense by using the canonical trace-duality, as above. By linearity, it will be enough to show that

$$\widehat{\mu^*}_0\Big(\operatorname{Tr}(\widehat{R}_i\widehat{R}_j)\Big) = \operatorname{Tr}(\widehat{Q}_i\widehat{P}_i\widehat{Q}_j\widehat{P}_j),$$

for $i \neq j \in I^\infty$. Indeed, one knows that the $\sum_{j\neq i}\frac{\operatorname{Tr}(\widehat{R}_i\widehat{R}_j)}{t_i-t_j} \in U(\mathfrak{g}^*)^{\otimes m}$ is the trace-dual of $\sum_{j\neq i}\frac{\Omega_{ij}}{t_i-t_j} = \widehat{H}_i^{\mathrm{KZ}}$, because KZ quantises Schlesinger. The result now follows by the straightforward expansion

$$\widehat{\mu^*}_0\Big(\operatorname{Tr}(\widehat{R}_i\widehat{R}_j)\Big) = \sum_{k,l}\widehat{\mu^*}_0\Big((\widehat{R}_i)_{kl}\Big)\cdot\widehat{\mu^*}_0\Big((\widehat{R}_j)_{lk}\Big) = \sum_{k,l}\widehat{\mu^*}_0\Big(\widehat{de^{(i)}}_{kl}\Big)\cdot\widehat{\mu^*}_0\Big(\widehat{de^{(j)}}_{lk}\Big) =$$
$$= \sum_{k,l,m,n}\widehat{d(ef^{(i)})}_{km}\cdot\widehat{d(f^{(i)}e)}_{ml}\cdot\widehat{d(ef^{(j)})}_{ln}\cdot\widehat{d(f^{(j)}e)}_{nk} = \sum_{k,l,m,n}(\widehat{Q}_i)_{km}\cdot(\widehat{P}_i)_{ml}\cdot(\widehat{Q}_j)_{ln}\cdot(\widehat{P}_j)_{nl} =$$
$$= \operatorname{Tr}(\widehat{Q}_i\widehat{P}_i\widehat{Q}_j\widehat{P}_j),$$

because by definition the quantum variables $(\widehat{R}_i)_{kl}, (\widehat{Q}_i)_{km}$ and $(\widehat{P}_i)_{ml}$ corresponds to the generators

$$\widehat{de}^{(i)}_{kl} \in U(\mathfrak{g}^*), \qquad \widehat{d(ef^{(i)})}_{kl}, \widehat{d(f^{(i)}e)}_{ml} \in A.$$

□

Hence one has lifted the classical correspondence to a quantum one. One should again perform the "quantum" change of variables $\widehat{R}_i := \widehat{Q}_i\widehat{P}_i$ in order to retrieve KZ (2.22) from the simply-laced quantum connection (3.7). We view this as a quantum reduction, in the following sense.

### 3.7.5 Howe pairs and quantum Hamiltonian reduction

We start by providing the most straightforward quantum analogue of the notion of a symplectic dual pair. Consider three associative algebras $A, B_1, B_2$.

**Definition 3.14.** [7] A pair of associative morphisms $\widehat{\mu^*}_i\colon B_i \longrightarrow A$ is a Howe dual pair if one has

$$\Big[\widehat{\mu^*}_1(B_1), \widehat{\mu^*}_2(B_2)\Big]_A = 0,$$

where $[\cdot,\cdot]_A$ is the commutator in $A$. Moreover, this is said to be a quantisation of the symplectic dual pair $\mu_i^*\colon \mathscr{O}(P_i) \longrightarrow \mathscr{O}(M)$ if $B_i$ is a quantisation of $\mathscr{O}(P_i), A$ is a quantisation of $\mathscr{O}(M)$ and the semiclassical limits $\sigma_i\colon B_i \longrightarrow \mathscr{O}(P_i)$, $\sigma\colon A \longrightarrow \mathscr{O}(M)$ are intertwined:

[7] This is again the weakest notion one finds around. One could require that $\widehat{\mu^*}_i(B_i) \subseteq A$ be the mutual centraliser of one another into $A$. This latter notion admits a representation-theoretic analogue concerning the action of a product $G_1 \times G_2$ of two (algebraic) groups on a finite-dimensional vector space, which is the actual definition of a Howe pair (see [How89]). Nevertheless, we won't be needing any of this to discuss our quantum reduction.

$$\begin{array}{ccc} B_i & \xrightarrow{\widehat{\mu^*}_i} & A \\ {\scriptstyle\sigma_i}\downarrow & \circlearrowleft & \downarrow{\scriptstyle\sigma} \\ \mathscr{O}(P_i) & \xrightarrow[\mu_i^*]{} & \mathscr{O}(M) \end{array}$$

One can show that the quantum momenta $\widehat{\mu}_0^*, \widehat{\mu}_\infty^*$ for the actions of $\mathrm{GL}(W^0)$ and $\mathrm{GL}(W^\infty)$ provide such a Howe pair.

**Proposition 3.15.** *One has*

$$\Big[\widehat{\mu}_0^*\Big(U(\mathfrak{g}^0)\Big), \widehat{\mu}_\infty^*\Big(U(\mathfrak{g}^\infty)\Big)\Big]_A = 0.$$

The proof uses the explicit formula

$$\widehat{\mu^*}_\infty(\widehat{df}_{kl}) = \sum_n \widehat{d(fe)}_{kn} \cdot \widehat{d(ef)}_{nl} \in A,$$

which is proven in the same way as (3.8).

*Proof.* One has, using the usual generators:

$$\begin{aligned} &\Big[\widehat{\mu^*}_0(\widehat{de}_{ij}), \widehat{\mu^*}_\infty(\widehat{df}_{kl})\Big]_A = \sum_{m,n} \Big[\widehat{d(ef)}_{im} \cdot \widehat{d(fe)}_{mj}, \widehat{d(fe)}_{kn} \cdot \widehat{d(ef)}_{nl}\Big]_A = \\ &= \sum_{m,n} \widehat{d(ef)}_{im} \cdot \widehat{d(fe)}_{kn} \cdot \Big[\widehat{d(fe)}_{mj}, \widehat{d(ef)}_{nl}\Big]_A + \Big[\widehat{d(ef)}_{im}, \widehat{d(fe)}_{kn}\Big]_A \cdot \widehat{d(ef)}_{nl} \cdot \widehat{d(fe)}_{mj} = \\ &= \sum_{m,n} -\delta_{ml}\delta_{jn}\widehat{d(ef)}_{im} \cdot \widehat{d(fe)}_{kn} + \delta_{in}\delta_{mk}\widehat{d(ef)}_{nl} \cdot \widehat{d(fe)}_{mj} = \\ &= -\widehat{d(ef)}_{il} \cdot \widehat{d(fe)}_{kj} + \widehat{d(ef)}_{il} \cdot \widehat{d(fe)}_{kj} = 0. \end{aligned}$$

□

.

In particular, the image of $\widehat{\mu^*}_0$ inside $A$ is contained in the invariant subalgebra

$$A^{\mathfrak{g}^\infty} := \Big\{\hat{f} \in A \Big| \Big[\widehat{\mu^*}_\infty\Big(U(\mathfrak{g}^\infty)\Big), \hat{f}\ \Big]_A\Big\} \subseteq A.$$

This can be used to provide an correspondence between the quantum algebra $U(\mathfrak{g}^0)^{\otimes m}$ and all quantum reductions $R_q\Big(A, \mathfrak{g}^\infty, I\Big)$. Recall from § 2.1.6 that this is defined using the moment $\widehat{\mu^*}_\infty \colon U(\mathfrak{g}^\infty) \longrightarrow A$ and an ideal $I \subseteq U(\mathfrak{g}^\infty)$. As in the classical case, one considers the left ideal $J := A.\widehat{\mu^*}_\infty(I) \subseteq A$ generated by $\widehat{\mu^*}_\infty(I)$. This need not be a two-sided ideal, whereas the invariant part $J^{\mathfrak{g}^\infty} \subseteq A^{\mathfrak{g}^\infty}$ must be. Thus the quotient $R_q\Big(A, \mathfrak{g}, I\Big) = A^{\mathfrak{g}^\infty}\Big/J^{\mathfrak{g}^\infty}$ is an associative algebra.

There is now a natural arrow $\widehat{\varphi}\colon U(\mathfrak{g}^0)^{\otimes m} \longrightarrow R_q(A, \mathfrak{g}^\infty, I)$, composing $\widehat{\mu^*}_0$ with the canonical projection $\pi\colon A^{\mathfrak{g}^\infty} \longrightarrow R_q(A, \mathfrak{g}^\infty, I)$. Finally, the results of the previous section show that

$$\widehat{\varphi}(\widehat{H}_i^{\mathrm{KZ}}) = \pi(\widehat{H}_i),$$

and the right-hand side is precisely the reduction of the ($G^\infty$-invariant) quantum SLIMS with respect to the $G^\infty$-action.

## 3.8 DMT and the dual star

We want to show that the DMT connection is a reduction of the SLQC connection for the Harnad-dual picture to the previous one. The main ideas of the proof are exactly as for KZ. We will prove the following statements.

**Theorem 3.6.** *The dual Schlesinger connection quantises to the DMT-Casimir connections via the standard PBW isomorphism.*

This is proven in § 3.8.3.

**Theorem 3.7.** *The SLIMS reduce to the dual Schlesinger system, in the special case of the degenerate reading of a star with no regular times.*

This is proven in § 3.8.4.

**Theorem 3.8.** *A natural correction of the SLQC reduce to the DMT system, in the same case as above. The correction produces a strongly-flat connection whose difference with the SLQC vanishes when taking the semiclassical limit. In particular, a reduction of the SLQC is a quantisation of the dual Schlesinger system.*

This is proven in § 3.8.5.

### 3.8.1 Simply-laced quantum connection of a star: dual version

One takes $k = 2, a(J) = \{\infty, 0\}$ and $T^\infty = 0$, in the general setup of § 3.1. The graph $\mathcal{G}$, and the symplectic phase-space $(\mathbb{M}, \omega_a)$ are the same as in § 3.7.1. What changes is the space of times $\mathbf{B} = \mathbb{C}^{I^0} \setminus \{\text{diags}\}$, and the space of meromorphic connections that these data code, namely:

$$\nabla = d - \left(T^0 + \frac{QP}{z}\right) dz.$$

This is a connection in the trivial vector bundle $W^0 \times \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$. One can write $T^0 = \mathrm{diag}(t_1 I_1, \ldots, t_m I_m) \in \mathbf{B}$, where $I_i$ is the idempotent for $V_i \subseteq W^0$, and $(Q, P) \in \mathbb{M}$.

The isomonodromic deformations of those connections are controlled by the simply-laced Hamiltonian system

$$\varpi = \frac{1}{2} \operatorname{Tr}\left(\widetilde{QP}QP\right) = \sum_{i \in I^\infty} H_i dt_i \in \Omega^0(\mathbb{F}_a, \pi^* T^* \mathbf{B}),$$

where $\pi\colon \mathbb{F}_a = \mathbb{M} \times \mathbf{B} \longrightarrow \mathbf{B}$ is the usual trivial symplectic fibration. The system spells out as

$$H_i(Q, P, T^0) = \sum_{i \neq j \in I^0} \frac{\mathrm{Tr}(Q_i P_j Q_j P_i)}{t_i - t_j} \in \mathrm{Sym}(\mathbb{M}^*). \tag{3.9}$$

The SLQC at hand is then

$$\widehat{\nabla} = d_{\mathbf{B}} - \widehat{\varpi} = d_{\mathbf{B}} - \sum_{i \in I^\infty} \left( \sum_{i \neq j \in I^\infty} \frac{\mathrm{Tr}(\widehat{P}_j \widehat{Q}_j \widehat{P}_i \widehat{Q}_i)}{t_i - t_j} \right) dt_i. \tag{3.10}$$

This is a connection on the trivial (quantum) vector bundle $\mathbb{E}_a = A \times \mathbf{B} \longrightarrow \mathbf{B}$, where $A \coloneqq W(\mathbb{M}^*, \omega_a)$.

The main theorem 3.2 assures that $\widehat{\nabla}$ is strongly flat. Let us set

$$\widehat{H}_i \coloneqq \sum_{i \neq j} \frac{\mathrm{Tr}(\widehat{P}_j \widehat{Q}_j \widehat{P}_i \widehat{Q}_i)}{t_i - t_j},$$

for the quantum Hamiltonians defining the SLQC. One may prove the flatness of this connection in the very same way as Prop. 3.11. Indeed, the combinatoric of the nontrivial intersections of the degenerate 4-cycles is the same as before, because it only depends on the adjacency of the quiver $\mathcal{G}$, which has not changed.

### 3.8.2 DMT connection

Here we briefly recall the construction of the Casimir connection [MT05]. Consider a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Choose a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, with associated root system $R \subseteq \mathfrak{h}^*$. Let

$$\mathfrak{h}_{\mathrm{reg}} \coloneqq \mathfrak{h} \setminus \bigcup_{\alpha \in R} \mathrm{Ker}(\sigma_\alpha)$$

be the open complement to the root hyperplanes. This means that $\sigma_\alpha\colon \mathfrak{h} \longrightarrow \mathfrak{h}$ is the reflection associated to $\alpha \in R$. Let $K\colon \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbb{C}$ be (a multiple of) the Killing form of $\mathfrak{g}$. One can now define a strongly flat connection $\widehat{\nabla}^{\mathrm{DMT}}$ on the trivial vector bundle $U(\mathfrak{g}) \times \mathfrak{h}_{\mathrm{reg}} \longrightarrow \mathfrak{h}_{\mathrm{reg}}$. Namely, for all $\alpha \in R$ one chooses a $\mathfrak{sl}_2(\mathbb{C})$-triplet of vectors $e_\alpha \in \mathfrak{g}_\alpha, f_\alpha \in \mathfrak{g}_{-\alpha}, h_\alpha = [e_\alpha, f_\alpha] \in \mathfrak{h}$, and then

$$\widehat{\nabla}^{\mathrm{DMT}} \coloneqq d_{\mathfrak{h}_{\mathrm{reg}}} - \sum_{\alpha \in R} \frac{K(\alpha, \alpha)}{2} (\widehat{e}_\alpha \cdot \widehat{f}_\alpha + \widehat{f}_\alpha \cdot \widehat{e}_\alpha) \frac{d\alpha}{\alpha},$$

where $K(\alpha, \alpha) \in \mathbb{R}_{>0}$ is the length squared of the root $\alpha$, computed with the dual of the Killing form (still abusively noted $K$). Just as for KZ, the constant in front of the 1-form

$$\widehat{\varpi}^{\mathrm{DMT}} \coloneqq \sum_{\alpha \in R} \frac{K(\alpha, \alpha)}{2} (\widehat{e}_\alpha \cdot \widehat{f}_\alpha + \widehat{f}_\alpha \cdot \widehat{e}_\alpha) d\log(\alpha) \in \Omega^1\Big(\mathbf{B}, U(\mathfrak{g})\Big)$$

is not a concern here. One could introduce any deformation parameter $\hbar \in \mathbb{C}$ in front of it. Moreover, this is the universal version of the DMT connection: one may pick any $\mathfrak{g}$-module $V$ and use it for the generic fibre of the trivial vector bundle.

We will now specialise this connection to the case of $\mathfrak{g} \coloneqq \mathfrak{gl}_l(\mathbb{C})$, noting that this Lie algebra is reductive but not semi-simple. One can just consider an invariant nondegenerate bilinear symmetric form on $\mathfrak{g}$ to perform the same construction as above, and we will use $K(X,Y) = \frac{1}{2}\operatorname{Tr}(XY)$, for $X, Y \in \mathfrak{g}$. We can choose $\mathfrak{h} \subseteq \mathfrak{g}$ to be the subalgebra of diagonal matrices. Now the root system $R = R(\mathfrak{g}, \mathfrak{h})$ reads

$$R = \{\alpha_{ij} \colon \mathfrak{h} \longrightarrow \mathbb{C}\}_{1 \leq i \neq j \leq l}, \qquad \text{where } \alpha_{ij}(\operatorname{diag}(x_1, \dots, x_l)) \coloneqq x_i - x_j.$$

Moreover, one has

$$\mathfrak{g}_{\alpha_{ij}} = \operatorname{span}_{\mathbb{C}}\{e_{ij}\}, \qquad \mathfrak{g}_{-\alpha_{ij}} = \mathfrak{g}_{\alpha_{ji}} = \operatorname{span}_{\mathbb{C}}\{e_{ji}\},$$

so that $h_{\alpha_{ij}} = [e_{ij}, e_{ji}] = e_{ii} - e_{jj} \in \mathfrak{h}$. This is indeed the dual root to $\alpha_{ij}$ with respect to Tr, as

$$\operatorname{Tr}(h_{\alpha_{ij}} e_{kl}) = \operatorname{Tr}(e_{ii}e_{kl}) - \operatorname{Tr}(e_{jj}e_{kl}) = \delta_{ik}\operatorname{Tr}(e_{il}) - \delta_{jk}\operatorname{Tr}(e_{jl}) = \delta_{ik}\delta_{il} - \delta_{jk}\delta_{jl} = \alpha_{ij}(e_{kl}).$$

The length squared of all roots is given by

$$2K(\alpha_{ij}, \alpha_{ij}) = \operatorname{Tr}(h_{\alpha_{ij}} h_{\alpha_{ij}}) = \operatorname{Tr}(e_{ii}) + \operatorname{Tr}(e_{jj}) - 2\operatorname{Tr}(e_{ii}e_{jj}) = 2.$$

Finally, notice that if one introduces the usual global coordinates $\{t_i\}_{1 \leq i \leq l}$ on $\mathfrak{h}_{\mathrm{reg}}$, then one has

$$d\log(\alpha_{ij}) = d\Big(\log(t_i - t_j)\Big) = \frac{dt_i - dt_j}{t_i - t_j}.$$

Putting all this together, one finds

$$\widehat{\nabla}^{\mathrm{DMT}} = d_{\mathfrak{h}_{\mathrm{reg}}} - \sum_i \widehat{H}_i^{\mathrm{DMT}} dt_i, \tag{3.11}$$

if one sets

$$\widehat{H}_i^{\mathrm{DMT}} \coloneqq \frac{1}{2}\sum_{j \neq i} \frac{\widehat{e}_{ij} \cdot \widehat{e}_{ji} + \widehat{e}_{ji} \cdot \widehat{e}_{ij}}{t_i - t_j} \colon \mathfrak{h}_{\mathrm{reg}} \longrightarrow U(\mathfrak{g}).$$

These are by definition the DMT Hamiltonians.

## Dual Schlesinger system

Pick a complex reductive group $G$, with Lie algebra $\mathfrak{g}$. Consider the trivial Poisson fibration $\mathfrak{g}^* \times \mathfrak{h}_{\mathrm{reg}} \longrightarrow \mathfrak{h}_{\mathrm{reg}}$, where $\mathfrak{h} \subseteq \mathfrak{g} \cong \mathfrak{g}^*$ is Cartan. One lets $(R, T^0) \in \mathfrak{g} \times \mathfrak{h}_{\mathrm{reg}}$ parametrise the following set of $\mathfrak{g}$-valued meromorphic connections on trivial $G$-bundles over $\mathbb{C}P^1$:

$$\nabla = d - \left(T^0 + \frac{R}{z}\right) dz.$$

One can show that the isomonodromy differential equations of such connections admit an Hamiltonian formulation. Namely, one defines

$$\varpi := K\left(R, \mathrm{ad}_{T^0}^{-1}[dT^0, R]\right) = K\left(R, \widetilde{R}\right),$$

where $K$ is an invariant nondegenerate bilinear symmetric form on $\mathfrak{g}$, and where $R$ is a local section of the fibration. Next, by choosing global coordinates $\{t_i\}_i$ on $\mathfrak{h}_{\mathrm{reg}}$, one writes

$$\varpi = \sum_i H_i^{\mathrm{dSch}} dt_i,$$

for functions $H_i^{\mathrm{dSch}} \colon \mathfrak{g}^* \times \mathfrak{h}_{\mathrm{reg}} \longrightarrow \mathbb{C}$. This is by definition the dual Schlesinger system. Specialising all this to $G = \mathrm{GL}_l(\mathbb{C})$ as before, one has

$$\sum_i H_i^{\mathrm{dSch}} dt_i = \frac{1}{2}\,\mathrm{Tr}\left(\widetilde{R}R\right).$$

Notice that

$$\widetilde{R}_{ij} = \begin{cases} d\log(t_i - t_j)R_{ij}, & i \neq j, \\ 0, & i = j \end{cases},$$

and so

$$\frac{1}{2}\,\mathrm{Tr}(\widetilde{R}R) = \frac{1}{2}\sum_{i,j} \widetilde{R}_{ij}R_{ji} = \frac{1}{2}\sum_{i\neq j} \frac{R_{ij}R_{ji}}{t_i - t_j}(dt_i - dt_j).$$

This provides the explicit expansion

$$H_i^{\mathrm{dSch}} = H_i^{\mathrm{dSch}}(R, T^0) = \sum_{j\neq i} \frac{R_{ij}R_{ji}}{t_i - t_j}. \tag{3.12}$$

### 3.8.3 DMT is a quantisation of dual Schlesinger

It is argued in [Boa02] that the DMT connection is a quantisation of the dual Schlesinger system. After all the above preparation, it follows from the following observation. The (classical) dual Schlesinger Hamiltonians can be written

$$H_i^{\mathrm{dSch}} = \sum_{j\neq i} \frac{e_{ij} \otimes e_{ji}}{t_i - t_j},$$

if one thinks of them as smooth sections $H_i \colon \mathfrak{h}_{\mathrm{reg}} \longrightarrow \mathrm{Sym}(\mathfrak{g}^*) \cong \mathrm{Sym}(\mathfrak{g})$. The PBW quantisation (2.8) of the numerator is by definition

$$\mathcal{Q}(e_{ij} \otimes e_{ji}) = \frac{\widehat{e}_{ij} \cdot \widehat{e}_{ji} + \widehat{e}_{ji} \cdot \widehat{e}_{ij}}{2},$$

so that one has on the whole $\mathcal{Q}(H_i^{\mathrm{dSch}}) = \widehat{H}_i^{\mathrm{DMT}}$.

### 3.8.4 The classical reduction

To get the dual Schlesinger system (3.12) from the simply-laced system (3.9), one should first restrict itself to the case where $\dim(V_i) = 1$ for all $i \in I^0$. Notice that this implies $\dim(W^0) = |I^0| =: l$. In this case $\mathbf{B} = \mathbb{C}^l \setminus \{\text{diags}\} \cong \mathfrak{h}_{\text{reg}}$, where $\mathfrak{h}$ is the standard Cartan subalgebra of $\mathfrak{g} := \mathfrak{gl}(W^0) \cong \mathfrak{gl}_l(\mathbb{C})$.

Now, as in § 3.7.2, one considers the classical (co)moment

$$\mu_0^*\colon \operatorname{Sym}(\mathfrak{g}^0) \longrightarrow \mathscr{O}(\mathbb{M})$$

for the $\mathrm{GL}(W^0)$-action. This is the dual map to

$$\mu_0\colon \mathbb{M} \longrightarrow \mathfrak{g}^0, \qquad (Q,P) \longmapsto R := QP.$$

Now one can show that $\mu_0^*(H_i^{\text{dSch}}) = H_i$ for all $i \in I^0$. Notice that in this case there are no traces to take, as the endomorphism $Q_iP_jQ_jP_i\colon V_i \longrightarrow V_i$ is a scalar for all $i \neq j$. Hence indeed

$$\mu_0^*(H_i^{\text{dSch}}) = \mu_0^*\left(\sum_{i\neq j}\frac{R_{ij}R_{ji}}{t_i-t_j}\right) = \sum_{i\neq j}\frac{Q_iP_jQ_jP_i}{t_i-t_j} = \sum_{i\neq j}\frac{\operatorname{Tr}(Q_iP_jQ_jP_i)}{t_i-t_j} = H_i,$$

by means of the change of variable $R_{ij} := Q_iP_j \in \mathbb{C}$. The conceptual justification of this computation is again the explicit formula

$$\mu_0^*(de_{ij}) = \sum_{k=1}^{\dim(W^\infty)} d(e^{(i)}f)_k \otimes d(fe^{(j)})_k,$$

where $\{de_{ij} := e_j \otimes de_i\} \subseteq \mathfrak{g}^0$ is the canonical basis (with $V_i = \operatorname{span}_{\mathbb{C}}\{e_i\}$), and with $d(e^{(i)}f)_k := e_i \otimes df_k, d(fe^{(j)})_k := f_k \otimes de_j$, fixing a basis $\{f_k\}_k$ of $W^\infty$. Indeed, with these notation one has $R_{ij} = de_{ij}(R)$, and the slightly more complicated

$$Q_iP_j = \sum_{k=1}^{\dim(W^\infty)}(Q_i)_k(P_j)_k = \sum_k d(e^{(i)}f)_k(Q)\cdot d(fe^{(j)})_k(P) = \sum_k d(e^{(i)}f)_k \otimes d(fe^{(j)})_k(QP),$$

where $(Q_i)_k\colon \mathbb{C} \longrightarrow V_i, (P_j)_k\colon V_j \longrightarrow \mathbb{C}$ are the components of $Q_i\colon W^\infty \longrightarrow V_i$ and $P_j\colon V_j \longrightarrow W^\infty$ in the subspace $\operatorname{span}_{\mathbb{C}}\{f_k\} \subseteq W^\infty$. We view this as a reduction, in the same sense in which Schlesinger (2.21) is a reduction of the SLIMS for a star (3.6) (see § 3.7.2 and § 3.7.3).

### 3.8.5 The quantum reduction

Following the same strategy as in § 3.7.4 and § 3.7.5, one would like to implement the quantum change of variable $\widehat{R}_{ij} := \widehat{Q}_i\widehat{P}_j$ to show that the formula for the DMT connection (3.11) expands to that of the simply-laced quantum connection (3.10). The precise justification for this would again be the natural quantisation $\widehat{\mu}_0^*\colon U(\mathfrak{g}^0) \longrightarrow A$ of the moment

map of the previous section. By doing this, however, one does not recover the words that make up the the Hamiltonians $\widehat{H}_i$ of the SLQC. Rather:

$$\widehat{\mu}_0^*\left(\sum_{i\neq j}\frac{\widehat{R}_{ij}\widehat{R}_{ji}}{t_i-t_j}\right)=\sum_{i\neq j}\frac{\mathrm{Tr}(\widehat{Q}_i\widehat{P}_j\widehat{Q}_j\widehat{P}_i)}{t_i-t_j},$$

where as usual one writes $\widehat{R}_{ij}=\widehat{de}_{ij}\in U\Big((\mathfrak{g}^*)^0\Big)$ and

$$\widehat{Q}_i\widehat{P}_j=\sum_k(\widehat{Q}_i)_k(\widehat{P}_j)_k=\sum_k\widehat{de^{(i)}f}_k\cdot\widehat{dfe^{(j)}}_k\in A$$

for the natural quantisations of $de_{ij}\in\mathscr{O}(\mathfrak{g}^0)$ and $d(e^{(i)}f)_k, d(fe^{(j)})_k\in\mathscr{O}(\mathbb{M})=A_0$. This means that the actual quantum connection that reduces to DMT is

$$\widehat{\nabla}'=d_{\mathfrak{h}_{\mathrm{reg}}}-\sum_{i\in I^0}\widehat{H}'_i dt_i,$$

still defined on the trivial vector bundle $\mathbb{E}_a=A\times\mathfrak{h}_{\mathrm{reg}}\longrightarrow\mathfrak{h}_{\mathrm{reg}}$, where

$$\widehat{H}'_i=\sum_{j\neq i}\frac{\mathrm{Tr}(\widehat{Q}_i\widehat{P}_j\widehat{Q}_j\widehat{P}_i)}{t_i-t_j}.$$

This connection is clearly a close relative to the simply-laced quantum connection (3.10). Indeed, it just amounts to a change of anchoring of all quantum potentials defining the Hamiltonians $\widehat{H}_i$: instead of anchoring all degenerate 4-cycles at their centre, one anchors them at their starting node.[8] The following proposition shows that the difference between $\widehat{\nabla}$ and $\widehat{\nabla}'$ is negligible as $\hbar\longrightarrow+\infty$.

**Proposition 3.16.** *The $A$-valued one-form $\widehat{\nabla}-\widehat{\nabla}'\in\Omega^1(\mathfrak{h}_{\mathrm{reg}},A)$ vanishes in the semi-classical limit.*

*Proof.* This means that the element

$$\langle\widehat{\nabla}-\widehat{\nabla}',\partial_{t_i}\rangle=H'_i-H_i=\sum_{j\neq i}\frac{\mathrm{Tr}(\widehat{Q}_i\widehat{P}_j\widehat{Q}_j\widehat{P}_i)-\mathrm{Tr}(\widehat{P}_j\widehat{Q}_j\widehat{P}_i\widehat{Q}_i)}{t_i-t_j}\in A$$

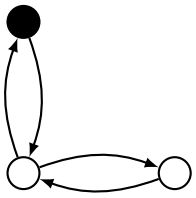
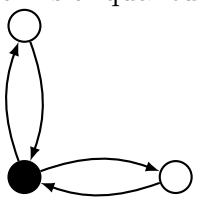
lies in the kernel of the grading map $\mathrm{gr}\colon A\longrightarrow A_0$ for all $\{t_i\}_i\in\mathfrak{h}_{\mathrm{reg}}$. In the chosen notation, this is seen to be true because the identity $\mathrm{Tr}(Q_iP_jQ_jP_j)=\mathrm{Tr}(P_jQ_jP_iQ_i)$ holds for matrices $Q_i,P_i,Q_j,P_j$ having coefficients in a commutative ring ($\mathbb{C}$, in this case), and because of the definition of quantum potentials of § 3.3: namely, the functions $\mathrm{Tr}(Q_iP_jQ_jP_j),\mathrm{Tr}(P_jQ_jP_iQ_i)\in A_0$ are the semiclassical limit of the quantum operators $\mathrm{Tr}(\widehat{Q}_i\widehat{P}_j\widehat{Q}_j\widehat{P}_i),\mathrm{Tr}(\widehat{P}_j\widehat{Q}_j\widehat{P}_i\widehat{Q}_i)\in A$.

One can moreover provide an explicit formula for the above difference, using Prop. 3.5:

$$\mathrm{Tr}(\widehat{Q}_i\widehat{P}_j\widehat{Q}_j\widehat{P}_i)-\mathrm{Tr}(\widehat{Q}_j\widehat{P}_j\widehat{Q}_i\widehat{P}_i)=\mathrm{Tr}(\widehat{P}_j\widehat{Q}_j).$$

[8] Notice that if one had defined the SLQC making this latter of anchoring, then an analogous correction would have been needed to recover KZ, which is part of the motivation for choosing the former one.

To show this, one uses the fact that $\left[(\widehat{Q}_i)_k, (\widehat{P}_i)_k\right] = 1$ in the quantum algebra $A$. This also gives an equivalent proof that the semiclassical limit of the difference must vanish: if one added the deformation parameter $\hbar$ into the picture, via the universal construction of § 2.1.4, then the operator of order two $\mathrm{Tr}(\widehat{P}_j\widehat{Q}_j) \in A$ would come with a power $\hbar^k$ of $\hbar$ having an exponent $k > 2$. □

In terms of quantum cycles, the above proof can be sketched by saying that  equals  plus  . This is a valid statement in the space $\widehat{\mathbb{C}\mathcal{G}_{\mathrm{cycl}}}$. As a final application of the material about quantum traces introduced in § 3.3, one can quickly conclude the following.

**Proposition 3.17.** *The corrected connection $\widehat{\nabla}'$ is strongly flat.*

*Proof.* The only nontrivial thing to prove is that $\left[\widehat{H}'_i, \widehat{H}'_j\right] = 0$ for $i \neq j$, as the other half of strong flatness follows trivially from that of $\widehat{\nabla}$. Now, the above picture says that we can move the anchor of all degenerate 4-cycles of $\widehat{H}'_i$ from the arrow coming out of $i$ to the centre, up to adding a 2-cycle to each of them. One thus has $\widehat{H}'_i = \widehat{H}_i + \sum_{j\neq i} \frac{\mathrm{Tr}(\widehat{C}_{ij})}{t_i - t_j}$, where $\widehat{C}_{ij}$ is a suitable quantum 2-cycle based at the centre, for all $i \neq j$. The following lemma is then enough to conclude. □

**Lemma 3.4.** *All quantum 2-cycles commute among themselves, as well as with all degenerate 4-cycles.*

*Proof.* Thanks to Prop. 3.6, these commutators are anchored cycles with the Poisson bracket of their semiclassical limit as underlying cycle. A straightforward computation then shows that these Poisson brackets vanish. □

Another meaningful comparison can be made between $\nabla^{\mathrm{DMT}}$ and the reduction of the simply-laced quantum connection. To compute the latter, notice that one has:

$$\widehat{H}_i = \sum_{j\neq i} \frac{\mathrm{Tr}(\widehat{P}_i\widehat{Q}_i\widehat{P}_j\widehat{Q}_j)}{t_i - t_j} = \sum_{j\neq i} \frac{1}{t_i - t_j}\Big(\mathrm{Tr}(\widehat{Q}_i\widehat{P}_j\widehat{Q}_j\widehat{P}_i) - \mathrm{Tr}(\widehat{Q}_j\widehat{P}_j) + 1\Big),$$

using the commutation relations of $A$. The reduction of this is

$$\pi(\widehat{H}_i) = \sum_{j\neq i} \frac{1}{t_i - t_j}\Big(\widehat{e}_{ij}\cdot\widehat{e}_{ji} - \widehat{e}_{jj} + 1\Big).$$

Notice that adding diagonal elements $\widehat{e}_{ii}$ does not tamper with flatness, because of

$$\left[e_{ij}e_{ji}, e_{kk}\right] = 0, \qquad \text{for all } k.$$

This is the reduction of the cycle-theoretic statement that all 2-cycles commute with all degenerate 4-cycles. The difference between this reduced connection and DMT is analogous to the difference between DMT and a reduction of the FMTV connection (to be explicitly written down at the end of § 3.9.3).

## 3.9 FMTV and the bipartite graph

Combining § 3.7 and § 3.8 leads to the fact that for the case of a complete bipartite graph the simply laced quantum connection reduces to the FMTV connection [Fel+00], as we shall argue momentarily. More precisely, we will show the following.

**Theorem 3.9.** *The SLIMS reduce to the JMMS system, in the special case of the degenerate reading of a complete bipartite graph.*

This is proven in § 3.9.2.

**Theorem 3.10.** *A restriction of the FMTV connection can be corrected to obtain the PBW quantisation of the JMMS system, and the difference vanishes semiclassically. Hence the FMTV system quantises the JMMS system.*

This is proven in § 3.9.3.

**Theorem 3.11.** *A natural correction of the SLQC reduces to the FMTV system, in the special case of the degenerate reading of a complete bipartite graph. The correction produces a connection whose difference with the SLQC vanishes when taking the semiclassical limit. In particular, a reduction of the SLQC is a quantisation of JMMS.*

This is proven in § 3.9.4.

### 3.9.1 Simply-laced quantum connection of a bipartite graph

The constructions of § 3.7.1 and § 3.8.1 generalise as follows. One still has $k = 2$ and $a(J) = \{+\infty, 0\}$, but $\mathcal{G}$ is now an arbitrary bipartite graph on nodes $I = I^0 \coprod I^\infty$. The base space of times is

$$\mathbf{B} = \mathbb{C}^{|I^\infty|} \setminus \{\text{diags}\} \times \mathbb{C}^{|I^0|} \setminus \{\text{diags}\}.$$

The vector phase-space is

$$\mathbb{M} = \mathrm{Hom}(W^\infty, W^0) \oplus \mathrm{Hom}(W^0, W^\infty),$$

equipped with the symplectic form $\omega_a = \mathrm{Tr}(dQ \wedge dP)$, where $Q, P$ have the same meaning as in § 3.7.1 and § 3.8.1. If $W^\infty = \bigoplus_{i\in I^\infty} V_i^\infty$ and $W^0 = \bigoplus_{i\in I^0} V_i^0$, then one will write $Q_{ij}$ for the component of $Q$ in $(V_j^\infty)^* \otimes V_i^0$, and $P_{ji}$ for the component of $P$ in $(V_j^0)^* \otimes V_i^\infty$. These data code a space of meromorphic connections of the form

$$\nabla = d - \left( T^0 + \sum_{i\in I^\infty} \frac{Q_i P_i}{z - t_i^\infty} \right) dz,$$

on the trivial vector bundle $W^0 \times \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$, with respect to the coarser decomposition $P_i \colon W^0 \longrightarrow V_i^\infty, Q_i \colon V_i^\infty \longrightarrow W^0$. The isomonodromic deformations of those connections are coded by the Hamiltonian system

$$\varpi = \frac{1}{2}\,\mathrm{Tr}\left(\widetilde{PQ}PQ\right) + \frac{1}{2}\,\mathrm{Tr}(\widetilde{QP}QP) = \sum_{i\in I^\infty} H_i^\infty dt_i^\infty + \sum_{j\in I^0} H_j^0 dt_j^0,$$

which spells out as

$$\begin{aligned} H_i^\infty(Q,P,T^\infty,T^0) &= \sum_{k\in I^\infty\setminus\{i\},j,l\in I^0} \frac{\mathrm{Tr}(P_{il}Q_{lk}P_{kj}Q_{ji})}{t_i^\infty - t_k^\infty} + \sum_{j\in I^0} t_j^0\,\mathrm{Tr}(P_{ij}Q_{ji}), \\ H_j^0(Q,P,T^\infty,T^0) &= \sum_{l\in I^0\setminus\{j\},i,k\in I^\infty} \frac{\mathrm{Tr}(Q_{ji}P_{il}Q_{lk}P_{kj})}{t_j^0 - t_l^0} + \sum_{i\in I^\infty} t_i^\infty\,\mathrm{Tr}(Q_{ji}P_{ij}). \end{aligned} \tag{3.13}$$

One now has nondegenerate 4-cycles, as well as 2-cycles. The SLQC at hand is

$$\widehat{\nabla} = d_{\mathbf{B}} - \widehat{\varpi} = d_{\mathbf{B}} - \sum_{i\in I^\infty} \widehat{H}_i^\infty dt_i^\infty - \sum_{j\in I^0} \widehat{H}_0^j dt_j^0, \tag{3.14}$$

on the trivial vector bundle $\mathbb{E}_a = A \times \mathbf{B} \longrightarrow \mathbf{B}$, where $A := W(\mathbb{M}^*, \omega_a)$. The explicit quantum Hamiltonians are

$$\begin{aligned} \widehat{H}_i^\infty = &\sum_{k\in I^\infty\setminus\{i\},j\neq l\in I^0} \frac{\mathrm{Tr}(\widehat{P}_{il}\widehat{Q}_{lk}\widehat{P}_{kj}\widehat{Q}_{ji})}{t_i^\infty - t_k^\infty} + \sum_{k\in I^\infty\setminus\{i\},j\in I^0} \frac{\mathrm{Tr}(\widehat{Q}_{jk}\widehat{P}_{kj}\widehat{Q}_{ji}\widehat{P}_{ij})}{t_i^\infty - t_k^\infty} + \\ &+ \frac{1}{2}\sum_{j\in I^0} t_j^0 \Big( \mathrm{Tr}(\widehat{P}_{ij}\widehat{Q}_{ji}) + \mathrm{Tr}(\widehat{Q}_{ji}\widehat{P}_{ij}) \Big), \\ \widehat{H}_j^0 = &\sum_{l\in I^0\setminus\{j\},i\neq k\in I^\infty} \frac{\mathrm{Tr}(\widehat{Q}_{ji}\widehat{P}_{il}\widehat{Q}_{lk}\widehat{P}_{kj})}{t_j^0 - t_l^0} + \sum_{l\in I^0\setminus\{j\},i\in I^\infty} \frac{\mathrm{Tr}(\widehat{P}_{jl}\widehat{Q}_{li}\widehat{P}_{ji}\widehat{Q}_{ji})}{t_j^0 - t_l^0} + \\ &+ \frac{1}{2}\sum_{i\in I^\infty} t_i^\infty \Big( \mathrm{Tr}(\widehat{Q}_{ji}\widehat{P}_{ij}) + \mathrm{Tr}(\widehat{P}_{ij}\widehat{Q}_{ji}) \Big). \end{aligned}$$

Here one separated nondegenerate 4-cycles (on the left) from degenerate ones, taking care to anchor the latter at their centre rather than at their starting node. The main theorem 3.2 assures that $\widehat{\nabla}$ is strongly flat.

### 3.9.2 Classical reduction: JMMS system

It is argued in [Boa12b] that (3.13) controls isomonodromic deformation equations which correspond to the lifted equations of [Jim+80] (A.5.9). Moreover, the usual change of

variable $R_i = Q_i P_i$ provides the JMMS equations themselves ([Jim+80] 4.44 or A.5.1). Just as in § 3.8.4, one must further restrict $T^0$ to have simple spectrum in order to recover the setup of JMMS. One finds then a system of time-dependent Hamiltonians $H_i^{\mathrm{JMMS},\infty}, H_j^{\mathrm{JMMS},0}$ controlling the isomonodromic deformations of meromorphic connections of the form

$$\nabla = d - \left( T^0 + \sum_{i=1}^{m} \frac{R_i}{z - t_i^\infty} \right) dz,$$

on the trivial vector bundle $W^0 \times \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$. More precisely, one lets $T^0$ and $\{t_i^\infty\}_i$ vary, and looks for residues $R_i \in \mathfrak{gl}(W^0) \cong \mathfrak{gl}_l(\mathbb{C})$ such that the monodromy of $\nabla$ stays fixed (where $\dim(W^0) = |I^0| =: l$) . This should be thought as the combination of the isomonodromy problems of § 3.7 and § 3.8, and the resulting phase-space for the JMMS Hamiltonians will be the trivial Poisson fibration $\mathfrak{g}^m \times \mathbf{B} \longrightarrow \mathbf{B}$, where $\mathfrak{g} \coloneqq \mathfrak{gl}_l(\mathbb{C})$ and $\mathbf{B} \coloneqq \mathfrak{h}_{\mathrm{reg}} \times \mathrm{Conf}_m(\mathbb{C})$. Here $\mathfrak{h} \subseteq \mathfrak{g}$ is the standard Cartan subalgebra, and $\mathrm{Conf}_m(\mathbb{C})$ is the space of configurations of $m$-tuples of points in the complex plane, so that

$$(R_i)_i \in \mathfrak{g}^m, \qquad T^0 \in \mathfrak{h}_{\mathrm{reg}}, \qquad \{t_i\}_i \in \mathrm{Conf}_m(\mathbb{C}).$$

The explicit formula for the JMMS Hamiltonians is

$$\begin{aligned} H_i^{\mathrm{JMMS},\infty} &= \sum_{1 \le k \ne i \le m} \frac{\mathrm{Tr}(R_i R_k)}{t_i^\infty - t_k^\infty} + \mathrm{Tr}(R_i T^0), \\ H_j^{\mathrm{JMMS},0} &= \sum_{1 \le k \ne j \le l} \sum_{1 \le i,n \le m} \frac{(R_i)_{jk}(R_n)_{kj}}{t_j^0 - t_k^0} + \sum_{1 \le i \le m} t_i^\infty \mathrm{Tr}(R_i e_{jj}). \end{aligned} \tag{3.15}$$

Indeed, this is precisely equation (A.5.13) of [Jim+80], rewritten according to our notation.[9] To justify why (3.13) reduces to (3.15) after changing variables, notice that one has $(R_i)_{jk} = Q_{ji} P_{ik} \in \mathbb{C}$ for all $1 \le i \le m$ and $1 \le j,k \le l$. Hence, e.g.

$$\mathrm{Tr}(R_i T^0) = \sum_{j,k} (R_i)_{jk} T^0_{jk} = \sum_{j,k} \delta_{jk} t_j^0 (R_i)_{jk} = \sum_j t_j^0 Q_{ji} P_{ij} = \sum_j t_j^0 \mathrm{Tr}(P_{ij} Q_{ji}),$$

where in the last passage one used the fact that $Q_{ji}P_{ij} \colon V_j^0 \longrightarrow V_j^0$ is a complex number. Similarly

$$\sum_{i \ne k} \mathrm{Tr}(R_i R_k) = \sum_{i \ne k, j, l} (R_i)_{jl} (R_k)_{lj} = \sum_{i \ne k, j, l} Q_{ji} P_{il} Q_{lk} P_{kj} = \sum_{i \ne k, j, l} \mathrm{Tr}(P_{il} Q_{lk} P_{kj} Q_{ji}).$$

The expansions of the terms of $H_j^{0,\mathrm{JMMS}}$ are done similarly. The justification of this formula lies as usual on the components $\mu_0^i \colon T^* \mathrm{Hom}(V_i^\infty, W^0) \longrightarrow \mathfrak{g}$ of the moment $\mu_0 \colon \mathbb{M} \longrightarrow \mathfrak{g}^m$, defined by

$$\mu_i(Q_i, P_i) \coloneqq Q_i P_i, \qquad \mu = \bigoplus_{i \in I^\infty} \mu_i \colon (Q,P) \longmapsto (Q_i P_i)_i.$$

One could pick bases $\{e_i\}_i$ of $W^0$ and $\{f_j\}_j$ of $W^\infty$ and write this down in terms of explicit generators of the algebras $\mathscr{O}(\mathfrak{g}), \mathscr{O}(\mathbb{M})$, as done in § 3.7.2 and § 3.8.4. Moreover, the same

[9] We wrote it in this half-expanded form to make it clear that this is indeed the same Hamiltonian system. One should replace $R_i, T^0, t_i^\infty, t_j^0$ with $A_i, A_\infty, c_i, a_j$, respectively.

argument of § 3.7.3 permits to rely this change of variable to the reduction of the SLIMS with respect to the action of $\mathrm{GL}(W^0)$ on $\mathbb{M}$.

As a final remark, notice that the leading term of $H_i^{\mathrm{JMMS},\infty}$ provides the Schlesinger Hamiltonian (2.21). Similarly the leading term of $H_j^{\mathrm{JMMS},0}$ provides a generalisation of the dual Schlesinger Hamiltonian (3.12). This generalisation amounts to the fact that now one has several simple poles, corresponding to several residues, instead of just one.

### 3.9.3 FMTV is a quantisation of JMMS

Here we briefly recall the construction of the dynamical connection [Fel+00]. One considers the KZ equations for a simple Lie algebra $\mathfrak{g}$, with an additional parameter $\mu \in \mathfrak{h}$ chosen in a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. The equations read

$$\frac{\partial \psi}{\partial t_i^\infty} = \left( \mu^{(i)} + \sum_{1 \leq j \neq i \leq m} \frac{\Omega_{ij}}{t_i^\infty - t_j^\infty} \right) \psi, \tag{3.16}$$

for a section of $\psi \colon \mathrm{Conf}_m(\mathbb{C}) \longrightarrow U(\mathfrak{g})^{\otimes m}$ of the universal KZ vector bundle, where the indices of $\Omega$ and $\mu$ specify on which factors of the fibre to act. This is equation 3 on page 3 of [Fel+00].[10] Eq. 4 on the same page then provides a system of differential equations which is compatible with this version of KZ: the *dynamical equations*. It is a system for $\psi$ with respect to the variable $\mu \in \mathfrak{h}$, in which one recognises a variation of the DMT connection of § 3.8.2, plus an additional term. Namely, the derivative of $\psi$ in the direction of $\mu' \in \mathfrak{h}$ is given by:

$$\frac{\partial \psi}{\partial \mu'} = \left( \sum_{1 \leq i \leq m} t_i^\infty (\mu')^{(i)} + \sum_{\alpha > 0} \frac{\langle \alpha, \mu' \rangle}{\langle \alpha, \mu \rangle} \widehat{e}_\alpha \cdot \widehat{f}_\alpha \right) \psi \tag{3.17}$$

The rightmost sum is made over positive roots $\alpha \in \mathfrak{h}^*$. The superscript of $\mu'$ still denotes the factor on which to act, and the symbols $\widehat{e}_\alpha, \widehat{f}_\alpha \in U(\mathfrak{g})$ are the standard quantisations of the vectors $e_\alpha, f_\alpha \in \mathfrak{g}$ that together with $h_\alpha$ compose a $\mathfrak{sl}_2$-triplet. The full FMTV system consists of those two sets of equations, for a function $\psi \colon \mathfrak{h} \times \mathrm{Conf}_m(\mathbb{C}) \longrightarrow U(\mathfrak{g})^{\otimes m}$. One can now interpret those linear differential equations as a flat connection $\widehat{\nabla}^{\mathrm{FMTV}}$ on the trivial vector bundle $\mathbb{E} \coloneqq U(\mathfrak{g})^{\otimes} \times \mathbf{B} \longrightarrow \mathbf{B}$, where $\mathbf{B} \coloneqq \mathfrak{h} \times \mathrm{Conf}_m(\mathbb{C})$, called the FMTV connection:

$$\widehat{\nabla}^{\mathrm{FMTV}} = d_{\mathbf{B}} - \sum_{1 \leq i \leq m} \widehat{H}_i^{\mathrm{FMTV},I} dt_i^\infty - \sum_{i \leq j \leq l} \widehat{H}_j^{\mathrm{FMTV},II} d\mu'_j, \tag{3.18}$$

where $l \coloneqq \dim(\mathfrak{h})$ is the rank of $\mathfrak{g}$, and where one picks linear coordinates $\{\mu'_j\}_j$ on $\mathfrak{h}$,with respect to a fixed basis of the Cartan subalgebra. The time-dependent quantum operators

$$\widehat{H}_i^{\mathrm{FMTV},I}, \widehat{H}_i^{\mathrm{FMTV},II} \colon \mathbf{B} \longrightarrow U(\mathfrak{g})^{\otimes m}$$

[10] One should replace $\psi, t_i^\infty, \Omega_{ij}$ with $u, z_i, \Omega^{(ij)}$. Moreover, instead of considering the universal KZ equations, one should pick highest weight $\mathfrak{g}$-modules $V_i$, and replace $U(\mathfrak{g})^{\otimes m}$ with $V = V_1 \otimes \cdots \otimes V_m$. Finally, the choice of a complex parameter $k$ is not relevant here.

are by definition the FMTV Hamiltonians. As done for KZ and DMT in § 2.3.5 and § 3.8.2, one can rewrite the system for the reductive Lie algebra $\mathfrak{g} \coloneqq \mathfrak{gl}_l(\mathbb{C})$, using (a multiple of) the canonical duality induced by the trace pairing. Now it makes sense to compare (i) the restriction of the FMTV Hamiltonians to the regular part $\mathfrak{h}_{\mathrm{reg}}$ of the Cartan subalgebra and (ii) the PBW quantisation of the JMMS system (3.15).

**Proposition 3.18.** *Take $\mathfrak{g} = \mathfrak{gl}_l(\mathbb{C})$. Then:*

1. *The FMTV Hamiltonian $\widehat{H}_i^{\mathrm{FMTV},I}$ restricts to $\mathcal{Q}\big(H_i^{\mathrm{JMMS},\infty}\big)$ on $\mathfrak{h}_{\mathrm{reg}} \times \mathrm{Conf}_m(\mathbb{C})$.*
2. *The difference*

$$\left(\widehat{H}_j^{\mathrm{FMTV},II}\Big|_{\mathfrak{h}_{\mathrm{reg}}\times\mathrm{Conf}_m(\mathbb{C})}\right) - \mathcal{Q}\big(H_j^{\mathrm{JMMS},0}\big) \colon \mathfrak{h}_{\mathrm{reg}} \times \mathrm{Conf}_m(\mathbb{C}) \longrightarrow U(\mathfrak{g})^{\otimes m}$$

*vanishes in the semiclassical limit, everywhere on the restricted JMMS base.*

*Hence the FMTV system is a quantisation of the JMMS system.*

*Proof.* [11] Let us start by showing that the quantisation of the linear addends of (3.15) yields the additional terms of (3.16) and (3.17), when restricted to the regular part $\mathfrak{h}_{\mathrm{reg}}$. Notice that the elements $\mu \in \mathfrak{h}_{\mathrm{reg}}$ now corresponds to the diagonal matrix $T^0 = \mathrm{diag}(t_1^0, \dots, t_l^0)$. Similarly, instead of differentiating along a generic direction $\mu' \in \mathfrak{h}$, it is enough to consider partial derivatives with respect to the system of fundamental coweights $\{e_{ii}\}_{1\leq i\leq l}$, which is a basis of $\mathfrak{h}$. This is a subset of the canonical basis $e_{ij} \in \mathfrak{g}$.

With these identifications, the additional term of (3.16) can be written

$$\mu^{(i)} = (T^0)^{(i)} = \sum_{1\leq j\leq l} t_j^0 \widehat{e}_{jj}^{(i)},$$

where $T^0 = \sum_j t_j^0 e_{jj}$ is the decomposition with respect to the basis of fundamental coweights. This is indeed the PBW quantisation of the linear function

$$\mathrm{Tr}(R_i T^0) = \sum_{1\leq j\leq l} t_j^0 (R_i)_{jj},$$

which appears in the JMMS Hamiltonian $H_i^{\mathrm{JMMS},\infty}$. The justification of this passes through the identification $(R_i)_{jj} = de_{jj}^{(i)}(R) \in \mathbb{C}$, where $de_{jj}^{(i)} \in (\mathfrak{g}^m)^*$ is the appropriate coordinate function. The trace duality sends this element to $e_{jj}^{(i)}$, whose standard PBW quantisation is by definition $\widehat{e}_{jj}^{(i)} \in U(\mathfrak{g})^{\otimes m}$ (see (2.8)).

[11] This proof is suggested by a slight variation of the example on page 4 of [Fel+00], where the FMTV connection is written down for the simple Lie algebra $\mathfrak{sl}_l(\mathbb{C})$, in terms of the fundamental coweights of the standard Cartan subalgebra of traceless diagonal matrices:

$$\varpi_i = \left(1 - \frac{i}{l}\right) \sum_{1\leq j\leq i} e_{jj} - \frac{i}{l} \sum_{i<j\leq l} e_{jj}, \qquad i \in \{1, \dots, l-1\}.$$

Similarly, the additional term of (3.17) becomes

$$\sum_{1\leq i\leq m} t_i^\infty (\mu')^{(i)} = \sum_{1\leq i\leq m} t_i^\infty \widehat{e}_{jj}^{(i)},$$

for $1 \leq j \leq l$. This is because, as explained above, the derivative $\partial_{\mu'}$ has been replaced by the partial derivative $\partial_{t_j^0}$ in the direction of $e_{jj} \in \mathfrak{h}$. This is the quantisation of the linear function

$$\sum_{1\leq i\leq m} t_i^\infty \operatorname{Tr}(R_i e_{jj}) = \sum_{i,k,m} t_i^\infty (R_i)_{km}(e_{jj})_{mk} = \sum_{i,k,m} t_i^\infty \delta_{kj}\delta_{mj}(R_i)_{km} = \sum_i t_i^\infty (R_i)_{jj},$$

which appears in the JMMS Hamiltonian $H_j^{\mathrm{JMMS},0}$.

Now, it has been shown in § 2.3.5 and § 3.8.3 that the standard PBW quantisation of Schlesinger (eq. (2.21)) and dual Schlesinger (Eq. (3.12)) yields KZ (Eq. (2.22)) and DMT (Eq. (3.11)), respectively. This means that the restriction to the regular part $\mathfrak{h}_{\mathrm{reg}}$ of the leading term of (3.16) should coincide with the PBW quantisation of the leading term of $H_i^{\mathrm{JMMS},\infty}$. Similarly, this suggests that the restriction to $\mathfrak{h}_{\mathrm{reg}}$ of the leading term of (3.17) should be strictly related with the PBW quantisation of the quadratic term of $H_j^{\mathrm{JMMS},0}$, as it reduces to DMT when $m = 1$. Replacing again $\mu = T^0, \mu' = \partial_{t_j^0}$ for $1 \leq j \leq l$, one has:

$$\sum_{\alpha>0} \frac{\langle \alpha, \mu'\rangle}{\langle \alpha, \mu\rangle} \widehat{e}_\alpha \cdot \widehat{f}_\alpha = \sum_{1\leq i,n\leq m} \sum_{k>l} \frac{\langle \alpha_{kl}, e_{jj}\rangle}{\langle \alpha_{kl}, T^0\rangle} \widehat{e}_{kl}^{(i)} \cdot \widehat{e}_{lk}^{(n)} =$$

$$= \sum_{i,n}\sum_{j>l} \frac{\langle \alpha_{jl}, e_{jj}\rangle}{\langle \alpha_{jl}, T^0\rangle} \widehat{e}_{jl}^{(i)} \cdot \widehat{e}_{lj}^{(n)} + \sum_{i,n}\sum_{k>j} \frac{\langle \alpha_{kj}, e_{jj}\rangle}{\langle \alpha_{kj}, T^0\rangle} \widehat{e}_{kj}^{(i)} \cdot \widehat{e}_{jk}^{(n)} =$$

$$= \sum_{i,n}\sum_{j>l} \frac{\widehat{e}_{jl}^{(i)} \cdot \widehat{e}_{lj}^{(n)}}{t_j^0 - t_l^0} - \sum_{i,n}\sum_{k>j} \frac{\widehat{e}_{kj}^{(i)} \cdot \widehat{e}_{jk}^{(n)}}{t_k^0 - t_j^0} = \sum_{i,n}\sum_{k\neq j} \frac{\widehat{e}_{jk}^{(i)} \cdot \widehat{e}_{kj}^{(n)}}{t_j^0 - t_k^0}.$$

Here one used $\langle \alpha_{kl}, e_{jj}\rangle = \delta_{kj} - \delta_{lj}$ and $\langle \alpha_{kl}, T^0\rangle = t_k^0 - t_l^0$. Also, the choice $\{\alpha_{kl}\}_{k>l}$ of positive roots has been made. This sum is not quite the PBW quantisation of the quadratic term of $H_j^{\mathrm{JMMS},0}$, which rather equals:

$$\mathcal{Q}\left(\sum_{i,n}\sum_{j\neq k} \frac{(R_i)_{jk}(R_n)_{kj}}{t_j^0 - t_k^0}\right) = \sum_{i,n}\sum_{j\neq k} \frac{1}{2(t_j^0 - t_k^0)}\left(\widehat{e}_{jk}^{(i)} \cdot \widehat{e}_{kj}^{(n)} + \widehat{e}_{kj}^{(n)} \cdot \widehat{e}_{jk}^{(i)}\right) =$$

$$= \sum_{i,n}\sum_{j\neq k} \frac{\widehat{e}_{jk}^{(i)} \cdot \widehat{e}_{kj}^{(n)}}{t_j^0 - t_k^0} + \sum_{i}\sum_{j\neq k} \frac{\widehat{e}_{kk}^{(i)} - \widehat{e}_{jj}^{(i)}}{2(t_j^0 - t_k^0)}.$$

This formula is obtained by remarking that $\left[\widehat{e}_{kj}^{(i)}, \widehat{e}_{jk}^{(n)}\right] = \delta_{in}\left(\widehat{e}_{kk}^{(i)} - \widehat{e}_{jj}^{(i)}\right)$ as elements of $U(\mathfrak{g})^{\otimes m}$. This means that the difference

$$\left(\widehat{H}_j^{\mathrm{FMTV},II}\Big|_{\mathfrak{h}_{\mathrm{reg}}\times \mathrm{Conf}_m(\mathbb{C})}\right) - \mathcal{Q}\left(H_j^{\mathrm{JMMS},0}\right) : \mathfrak{h}_{\mathrm{reg}} \times \mathrm{Conf}_m(\mathbb{C}) \longrightarrow U(\mathfrak{g})^{\otimes m}$$

is an operator of order one, when evaluated at a point of the restricted JMMS base. Hence the restricted FMTV Hamiltonian has the same semiclassical limit as $\mathcal{Q}\left(H_j^{\mathrm{JMMS},0}\right)$, which is by definition the JMMS Hamiltonian for the irregular time $t_j^0$. □

Finally, one can make sense of the statement that the DMT connection (3.11) is a variation of the FMTV connection. Indeed, by taking $m = 1$ in the Hamiltonian $\widehat{H}_j^{\mathrm{FMTV},II}$, and picking $\mathfrak{g} = \mathfrak{gl}_l(\mathbb{C})$, one finds

$$\widehat{H}_j^{\mathrm{FMTV},II} = \sum_{k\neq j} \frac{\widehat{e}_{jk}\cdot\widehat{e}_{kj}}{t_j^0 - t_k^0},$$

for $1 \leq j \leq l$. Now there is no need for superscripts, because one has only one residue. The analogous Hamiltonian of the DMT system is

$$\widehat{H}_j^{\mathrm{DMT}} = \sum_{k\neq j} \frac{\widehat{e}_{jk}\cdot\widehat{e}_{kj} + \widehat{e}_{kj}\cdot\widehat{e}_{jk}}{2(t_j^0 - t_k^0)} = \sum_{k\neq j} \frac{\widehat{e}_{jk}\cdot\widehat{e}_{kj}}{t_j^0 - t_k^0} + \sum_{k\neq j} \frac{\left[\widehat{e}_{kj}, \widehat{e}_{jk}\right]}{2(t_j^0 - t_k^0)} = \widehat{H}_j^{\mathrm{FMTV},II} + \sum_{k\neq j} \frac{\widehat{e}_{kk} - \widehat{e}_{jj}}{2(t_j^0 - t_k^0)}.$$

This difference is indeed coherent with the fact that DMT is the PBW quantisation of the dual Schlesinger system, i.e. of the JMMS Hamiltonian $H_j^{\mathrm{JMMS},0}$. Indeed, the proof of Prop. 3.18 shows that one must correct FMTV in order to get to the PBW quantisation of JMMS. Moreover, this difference is totally analogous to the difference between the simply-laced quantum connection and DMT. One thus has three different flat connections quantising the same classical system.

As a final remark, let us mention the cycle-theoretic interpretation of the difference between FMTV and the PBW quantisation of JMMS. Namely, the sum

$$\sum_{j\neq k} \widehat{e}_{jk}^{(i)}\cdot\widehat{e}_{kj}^{(n)} \in U(\mathfrak{g})^{\otimes m}$$

will expand in the Weyl algebra (via the quantum moment $\widehat{\mu}_0^*\colon U(\mathfrak{g})^{\otimes m} \longrightarrow W(\mathbb{M}, \omega_a)$ as in § 3.7.4) as the trace of a quantum 4-cycle through the nodes $i, n \in I^\infty$ and $j \neq k \in I^0$. More precisely, the expansion yields a quantum 4-cycle, which is an addend of the quantum potential for the simply-laced quantum Hamiltonian $\widehat{H}_j^0$. Now, the condition $i \neq n$ means that the cycle is nondegenerate, whereas $i = n$ makes it degenerate. It is only on degenerate 4-cycles that one must make a choice of ordering, and one basically has to either pick an anchor or to consider some symmetrisation. In this article made the first choice, because the symmetrisation seems to tamper with the flatness for more general quivers. This is thus more in line with the definition of the FMTV connection. The DMT connection, on the other way, considered a symmetric quantisation, and this is possibly one way of explaining the footnote on page 3 of [MT05] (it states that the Casimir connection is equivariant with respect to the action of the Weyl group, whereas the dynamical connection is not).

### 3.9.4 The quantum reduction

What is left is putting together the results of § 3.7.4, § 3.7.5 and § 3.8.5 to show that a correction of (3.14) reduces to FMTV when $\mathfrak{g} = \mathfrak{gl}_l(\mathbb{C})$. In the proof of Prop. 3.18 we

provided the explicit formulae:

$$\widehat{H}_i^{\mathrm{FMTV},I} = \sum_{i\neq k,j,l} \frac{\widehat{e}_{jl}^{(i)} \cdot \widehat{e}_{lj}^{(k)}}{t_i^\infty - t_k^\infty} + \sum_j t_j^0 \widehat{e}_{jj}^{(i)},$$
$$\widehat{H}_j^{\mathrm{FMTV},II} = \sum_{k\neq j,i,l} \frac{\widehat{e}_{jk}^{(i)} \cdot \widehat{e}_{kj}^{(l)}}{t_j^0 - t_k^0} + \sum_i t_i^\infty \widehat{e}_{jj}^{(i)}.$$

As argued in the aforementioned sections, the change of variable $(\widehat{R}_i)_{jk} = \widehat{Q}_{ji}\widehat{P}_{ik}$ amounts to applying the quantum (co)moment $\widehat{\mu}_0^*\colon U(\mathfrak{g})^{\otimes m} \longrightarrow A = W(\mathbb{M}, \omega_a)$. Doing this, one finds the following $A$-valued quantum Hamiltonians:

$$\widehat{H'}_i^\infty = \sum_{i\neq k,j,l} \frac{\mathrm{Tr}(\widehat{Q}_{ji}\widehat{P}_{il}\widehat{Q}_{lk}\widehat{P}_{kj})}{t_i^\infty - t_k^\infty} + \sum_j t_j^0 \,\mathrm{Tr}(\widehat{Q}_{ji}\widehat{P}_{ij}),$$
$$\widehat{H'}_j^0 = \sum_{k\neq j,i,l} \frac{\mathrm{Tr}(\widehat{Q}_{ji}\widehat{P}_{ik}\widehat{Q}_{kl}\widehat{P}_{lj})}{t_j^0 - t_k^0} + \sum_i t_i^\infty \,\mathrm{Tr}(\widehat{Q}_{ji}\widehat{P}_{ij}).$$

This produces a new connection

$$\widehat{\nabla}' := d_{\mathbf{B}} - \sum_{i\in I^\infty} \widehat{H'}_i^\infty dt_i^\infty - \sum_{j\in I^0} \widehat{H'}_j^0 dt_j^0,$$

in the trivial vector bundle $\mathbb{E}_a = A \times \mathbf{B} \longrightarrow \mathbf{B}$. This connection is again obtained by the SLQC via a change of anchors, so to write all operators as traces of quantum cycles based at nodes in the part $I^0 \subseteq I$, because such a cycle can be interpreted as a residue. Moreover, the correction is negligible in the semiclassical limit.

**Proposition 3.19.** *The $A$-valued one-form $\widehat{\nabla} - \widehat{\nabla}' \in \Omega^1(\mathbf{B}, A)$ vanishes in the semiclassical limit.*

*Proof.* The proof is analogous to that of Prop. 3.16. One can compute explicitly the differences $\langle \widehat{\nabla} - \widehat{\nabla}', \partial_{t_i^\infty} \rangle = \widehat{H'}_i^\infty - \widehat{H}_i^\infty$ and $\langle \widehat{\nabla} - \widehat{\nabla}', \partial_{t_j^0} \rangle = \widehat{H'}_j^0 - \widehat{H}_j^0$ for all $i, j$, using $\left[\widehat{Q}_{ji}, \widehat{P}_{ij}\right] = 1$. The difference of these operators of order four is an operator of order strictly less than four. Hence they have the same principal symbol, i.e. the same semiclassical limit. □

It is not relevant here, but the corrected connection might still be flat. The general proposition 3.6 can still be used to test this. Notice however that $\widehat{\nabla}'$ is less symmetric than the simply-laced quantum connection, because now on top of the classical nontrivial intersection of a nondegenerate 4-cycle with a degenerate one there are a priori different quantum intersections. Hence more minute verifications than those of § 3.6.2 are required. This phenomenon is another motivation for the definition of the simply-laced quantum connection.

## 3.10 Comparison with Nagoya–Sun.

As a last comparison, we consider the isomonodromy systems of [JMU81]. They developed a theory of monodromy preserving deformations of systems of differential equations of the form:

$$\frac{dY}{dz} = A(z)Y(z), \qquad \text{where} \qquad A(z) = \sum_{i=1}^{m}\sum_{l=0}^{r_i}\frac{A_{i,-l}}{(z-a_i)^{l+1}} - \sum_{l=1}^{r_\infty} A_{\infty,-l}z^{l-1},$$

such that the leading coefficient at each irregular pole has distinct eigenvalues. Here $z$ is a local holomorphic coordinate on $\mathbb{C}P^1$ that identifies $\mathbb{C}P^1 \cong \mathbb{C}\coprod\{\infty\}$, and $Y = Y(z), A_{i,-l}, A_{\infty,-l} \in M_n(\mathbb{C})$ are matrices of size $n > 0$ for all $i, l$. Next, $\{a_i\}_i \subseteq \mathbb{C}$ is a collection of $m$ distinct points (with $a_\infty = \infty$ implied) and $r_i, r_\infty \geq 0$ are integers for all $i$. If $r_i > 0$ then one has an irregular singularity at $a_i$ of Poincaré rank $r_i$, for $i \in \{1, \dots, m, \infty\}$. If instead $r_i = 0$ then one has a regular singularity. One further assumes $A_{i,-r_i}$ to have distinct eigenvalues (modulo integers, if $r_i = 0$).

The intersection of those systems with the SLIMS of [Boa12b] is obtained by taking $r_i = 0$ for all $1 \leq i \leq m$ and $r_\infty = 2$. Indeed, by replacing $A_{i,0}, a_i, A_{\infty,-1}, A_{\infty,-2}$ with $-Q_iP_i, t_i^\infty, B+T, A$, respectively, then the above system reduces to

$$\frac{dY}{dz} = -\left(\sum_{i=1}^{m}\frac{Q_iP_i}{z-t_i^\infty} + (B+T) + Az\right)Y(z),$$

which is equivalent to considering the meromorphic connection

$$\nabla = d - \mathcal{A} = d - \Big(Az + (B+T) + Q(z-T^\infty)^{-1}P\Big)dz$$

on a trivial vector bundle $U^\infty \times \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$, as in § 3.1. The difference is that now one considers nonresonant residues, and more importantly the diagonal matrix $A = \mathrm{diag}(a_1, \dots, a_k)$ is regular. This means taking a degenerate reading of a complete $k+1$-partite graph $\mathcal{G}$ such that only the part corresponding to $\infty$ is splayed in $|I^\infty| = m$ nodes, and to all other parts one attaches vector spaces $W^j$ of dimension one.

Now, the theory of [JMU81] allows one to consider the pole positions $t_i^\infty$ and the diagonal matrices $T, A$ as independent deformation parameters (see Eq. 4.2 of [JMU81] and the remark thereafter[12]). This falls out of the present scope, where the highest irregular type $A$ is not varied in order to enable for an explicit description of the isomonodromy Hamiltonians for the remaining deformation parameters $T, T^\infty$. One is after these explicit formulations because they provide quite straightforward quantisation: this is one of the main points of this article. The general proof that the deformation equations are symplectic can be found in [Boa01], but a general Hamiltonian theory hasn't been written down yet.

In [NS11] one finds explicit functions associated to the deformations of $T, A$ and $T^\infty$. Namely, one takes a variation of the Schlesinger Hamiltonians for the deformation of the

[12] One should replace $a_i, T^\infty_{-1}, T^\infty_{-2}$ with $t_i^\infty, T, A$, respectively.

pole positions. For the other parameters, one writes

$$\omega = \omega^{(1)} + \omega^{(2)} = \sum_{i=1}^{k} H_i^{(1)} dt_i + H_i^{(2)} da_i,$$

for suitable functions $H_i^{(1)}, H_i^{(2)}$.[13] Then Eq. 3.11 and 3.12 of [NS11] provide an explicit expansion of the one-form $\omega$.

The most interesting comparison to make is that of $H_i^{(1)}$ and $H_i$, for $1 \leq i \leq k$, since the results of § 3.7 and § 3.9 prove that the simply-laced quantum Hamiltonians can recover the KZ connection when moving the regular times $t_i^\infty$. Now, looking at the aforementioned Eq. 3.11, one sees indeed expansions of traces of cycles in $\mathcal{G}$, in terms of the entries of the matrices $B, R_i$. If one writes the first addend of $\langle \omega^{(1)}, \partial_{t_i} \rangle$, then one recognises the sum of all three cycles in $\mathcal{G}$ starting at the node $i$, with suitable weights.[14] The rest of the formula yields a weighted sum of traces of 2-cycles through the node $i$. There are no 4-cycles as it is expected in the general setup of the SLIMS. Finally, notice that eq. 3.12 has the same overall structure: each of its addend is the trace of a certain potential on $\mathcal{G}$, and now one also finds 4-cycles.

All this seems to indicate that the cycle-theoretic viewpoint of this article provides a conceptual framework that simplifies part of the exposition of [NS11], and that moreover this approach could be extended to the full Hamiltonian system at the intersection of [JMU81] and [Boa12b].

As far as quantisation is concerned, in [NS11] one fixes an order for the variables appearing in the classical system, and then systematically replaces them with operators acting on modules for a certain quantum algebra (those are the confluent Verma modules, defined in [JNS08] for $\mathfrak{sl}_2(\mathbb{C})$). The choice of an order is noncanonical, and in our setting it amounts to fix an orientation for the quiver. In this way one defines operators $\mathcal{H}_i^{(1)}, \mathcal{H}_i^{(2)}$, so that $\mathcal{H}_i^{(1)}$ is associated to the time $t_i$, and $\mathcal{H}_i^{(2)}$ to $a_i$. Ex. 3.3. on page 9 provides an expansion of these operators when there are no simple poles, in the case of rank $k = 3$. In our setting, this means passing to the generic reading of a triangle where all nodes have dimension one. Again, the most meaningful comparison to make is that of $\mathcal{H}_i^{(1)}$ and the quantum Hamiltonian $\widehat{H}_i$ 3.10, where $i \in \{1, 2, 3\}$. To write down the simply-laced quantum Hamiltonians, call $1, 2, 3$ the nodes of the triangle, and write arrows as $\alpha := (1, 2), \beta := (2, 3), \gamma := (3, 1)$. Then

$$\widehat{H}_1 = c_1 \operatorname{Tr}(\gamma\beta\underline{\alpha}) + c_2 \operatorname{Tr}(\alpha^*\beta^*\underline{\gamma^*}) + c_3 \operatorname{Tr}(\alpha^*\underline{\alpha}) + c_4 \operatorname{Tr}(\gamma\underline{\gamma^*}) + c_5,$$

where the weights $c_1, \ldots, c_4$ depend on the reading $a\colon \{1, 2, 3\} \hookrightarrow \mathbb{C}P^1$, and $c_5$ is a function used to commute the arrows of the 2-cycles. To make things more explicit, one chooses an orientation for the triangle, by saying that the arrow $(i, j)$ is positive if $i < j$. Then

[13] It is not entirely clear to us how to interpret the functions $H_i^{(2)}$ as time-dependent Hamiltonians (cf. Rem. 7.1 of [Boa01]).

[14] The matrix $B_1$ of [NS11] correspond to our $B$, the matrix $B_0$ to the sum $\sum_i Q_i P_i$ of residues, and $B_{-1}$ is the sum $\sum_i Q_i P_i t_i^\infty$. Finally, the highest irregular times $t_i^{(2)}$ now become our diagonal entries $a_i$.

one has the splitting

$$\mathbb{M} = \bigoplus_{1\leq i\neq j\leq 3} \mathrm{Hom}(W^i, W^j) \cong T^* \left( \bigoplus_{1\leq i<j\leq 3} \mathrm{Hom}(W^i, W^j) \right),$$

for which the arrows $\alpha, \beta, \gamma$ now correspond to the position variables $q_{12}, q_{23}, p_{31}$, and their opposite to the momenta variable $p_{21}, p_{32}, p_{13}$:

$$\omega_{\mathrm{can}} = \sum_{1\leq i<j\leq 3} dq_{ij} \wedge dp_{ji}.$$

One can now let $\widehat{H}_1 \colon \mathbf{B} \longrightarrow W(\mathbb{M}, \omega_a)$ act on the space of polynomials $\mathbb{C}[q]$ in the position variables, by means of $\widehat{q}_{ij} \coloneqq \mu_{q_{ij}}$ (multiplication operator) and $\widehat{p}_{ji} \coloneqq -\partial_{q_{ij}}$, so that:

$$\widehat{H}_1 = c'_1 q_{31} q_{23} q_{12} + c'_2(\partial_{q_{12}} \partial_{q_{23}} \partial_{q_{31}}) + c'_3 \partial_{q_{12}} q_{12} + c'_4 q_{13} \partial_{q_{13}} + c'_5$$

Ignoring the order-zero operator $c'_5$, this formula is an analogue of that of $\mathcal{H}_1^{(1)}$ in example 3.3 of [NS11], which seems to indicate that the two viewpoints are compatible.

# Chapter 4

# Hitchin connections

In this chapter we describe the results obtained in the direction of geometric quantisation. The primary aim was the construction of a Hitchin connection for the Kähler quantisation of moduli space of polystable $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles over a torus. The flat connection we construct is a Kähler analogue of the connection of [Wit91], defined for the geometric quantisation with respect to real polarisations, to which we will refer as the Hitchin–Witten connection (see § 4.6.4).

Consider then a real, compact orientable surface of genus $\Sigma$ of genus $g = 1$: a torus. The purpose here is to be as concrete as possible, so we can realise the surface as $\Sigma \cong \mathbb{R}^2 \big/ \mathbb{Z}^2$, where $\mathbb{Z}^2$ acts by translations on the plane. The usefulness of this flat description of the torus will become apparent as we go on. Set $K \coloneqq \mathrm{SU}(2)$, and let $G \coloneqq \mathrm{SL}(2,\mathbb{C})$ be the complexification of the compact Lie group $K$.

In § 4.1 we provide an explicit topological description of the moduli space $\mathcal{M} = \mathcal{M}_{\mathrm{fl}}(\Sigma, K)$ of polystable, flat $K$-connection on $\Sigma$.
In § 4.2 we provides a much more in-depth description of the moduli space of the complex theory $\mathfrak{M}$, which is the hyper-Kähler moduli space of polystable Hitchin pairs, introduced in § 2.3.1. We also set up all that's needed for its geometric quantisation, following the background material of § 2.2: the symplectic structure (§ 4.2.3), the prequantum data (§ 4.2.4) and the polarisations (§ 4.2.5 and 4.2.6).
Next, in § 4.3, we use all the data we introduce to achieve the Kähler quantisation of $\mathfrak{M}$. More precisely, in § 4.3.2 we construct a connection $\widehat{\nabla}$ that preserves holomorphic sections (Thm. 4.1), and in § 4.3.3 we show that the restriction of $\widehat{\nabla}$ to be subbundle of holomorphic sections is flat (Thm. 4.2).
After this first success, in § 4.4 we first consider the natural $U(1)$-action on polystable $G$-Higgs fields. After finding explicit formulae for this actions and its pull-backs, in § 4.4.4 we to set up $U(1)$-equivariant prequantum data on $\mathfrak{M}$, and in § 4.4.5 we discuss how to extend the basis of the quantum bundle to an extended Teichmüller space that takes into account the circle action.
We move on in § 4.5 by trying to construct an Hitchin connection inside the extended quantum bundle of the previous section. After a technical discussion about how to correctly extend the variation of tensors in the new context of a varying symplectic structure,

in Thm. 4.3 we achieve the construction of a connection that preserves holomorphic sections.
In the last section 4.6 instead we set up the Bargmann transform $\mathcal{B}_\tau$ in our non-flat context. The result is a transform defined on the moduli space, which identifies the quantum spaces corresponding to different polarisations (see Thm. 4.4).
The transform is introduced to transport the Hitchin–Witten connection of [Wit91] to a flat connection the Kähler-polarised side, as detailed in § 4.6.4. This is done with the clear idea to look for a new Ansatz for constructing Hitchin connection, as did in (4.15) (see also the discussion of 4.22).
Finally, in § 4.6.5 we make use of the Bargmann transform to turn the Hitchin action of § 4.4 into an operator defined on the Segal–Bargmann space: this operator is an integral transform, whose kernel generalises the reproducing kernel of coherent states (see 4.7). In the same section we compute the way in which derivatives and function multiplications commute with the new action, which is the first step to conjugate any connection defined on the bundle of holomorphic sections, like the flat connections previously constructed.

## 4.1 Compact moduli space

We start by describing more precisely the moduli space of flat connection for the group $K = \mathrm{SU}(2)$ over $\Sigma$, already introduced in § 2.3.3. The final result of this section is that the moduli space for the compact theory is topologically a torus, up to a flip.

Let $(x, y)$ the standard real coordinates on $\Sigma$. This means that we can either think of them as real coordinates defined modulo $\mathbb{Z}$ or as sincere real coordinates on a dense open set of the torus, namely on the image of a fundamental domain of the lattice $\mathbb{Z}^2 \subseteq \mathbb{R}^2$. This open set leaves behind only one parallel and one meridian on the torus, which we denote $\gamma_1$ and $\gamma_2$; their homotopy classes generate the fundamental group: $\pi_1(\Sigma) \cong \mathbb{Z}[\gamma_1] \oplus \mathbb{Z}[\gamma_2]$. The first aim is to explicitly describe the moduli spaces attached to the pairs $(\Sigma, K)$ in § 2.3.2.

To this end, let $P \coloneqq \Sigma \times K \longrightarrow \Sigma$ be the trivial smooth principal $K$-bundle on $\Sigma$. Picking trivial bundles is not a restriction, for generic topological reasons: every principal bundle over a curve that has a simply connected structure group is trivialisable. Thus we're simply assuming that an explicit global trivialisation has been given. Moreover, one may equivalently work on trivialisable, rank 2 vector bundles $\pi\colon V \longrightarrow \Sigma$, with the appropriate restriction of the transition functions to $K \subseteq GL_2(\mathbb{C})$. Let us denote $e_1, e_2\colon \Sigma \longrightarrow V$ a global trivialising frame of $V$.

As explained in § 2.3.1, we can now consider the moduli space $\mathcal{M}_{\mathrm{fl}}(\Sigma, K)$ of gauge-equivalence classes of flat, unitary connections on $V \longrightarrow \Sigma$, and the moduli space $\mathrm{Char}(\Sigma, K)$ of conjugacy classes of unitary representations of $\pi_1(\Sigma)$. The two are related by the Riemann–Hilbert correspondence, mapping a class of flat connections to their conjugacy class of monodromy representations of the fundamental group of $\Sigma$. This map is a transcendental ($\mathbb{C}$-analytic) isomorphism, as will be argued more explicitly momentarily. Moreover, $\nu$ intertwines the gauge-action on the left-hand side with the conjugation action on the right-hand side.

Now let us give a coordinate-wise description of those spaces in our particular case, starting from the de Rham viewpoint. A connection over $V \longrightarrow \Sigma$ is globally defined by a $\mathfrak{k}$-valued 1-form $A \in A^1(\Sigma, \mathfrak{k})$, setting $\nabla = d - A$, where $d$ is the ordinary de Rham differential over $A^\bullet(\Sigma, \mathbb{C})$, and $\mathfrak{k} \coloneqq \mathfrak{su}(2)$.
Using our real coordinates $x, y$, we can write an $\mathfrak{k}$-valued 1-form as $A = \alpha dx + \beta dy$, for $\alpha, \beta$ two smooth $\mathfrak{k}$-valued functions. It turns out that it is enough to consider the following space of connections, as far as gauge equivalence classes are concerned:

$$\mathcal{A}^0 \coloneqq \left\{ A = \begin{pmatrix} i\xi_1 dx + i\xi_2 dy & 0 \\ 0 & -(i\xi_1 dx + i\xi_2 dy) \end{pmatrix} \middle| \, \xi_1, \xi_2 \in \mathbb{R} \right\} \subseteq A^1(\Sigma, \mathfrak{k}).$$

This copy of $\mathbb{R}^2$ parametrises a space of translation-invariant connections. They are all flat, as it is straightforward to check that $dA = 0 = [A \wedge A]$ if $A \in \mathcal{A}^0$. Notice that they are not stable, but rather polystable, according to Def. 2.39. Indeed, it is clear that in chosen frame $\{e_1, e_2\}$ the pointwise span of $e_i$ is $\nabla$-invariant, and $V$ is the direct sum of these two (necessarily stable) subconnections of rank one.

Now, thanks to the Riemann–Hilbert correspondence of Thm. 2.8, the image of this set of flat connections in the moduli space is given by modding out those that have a trivial monodromy around both generating loops of $\pi_1(\Sigma)$. Recall from the beginning of § 2.3.5 that the monodromy data $\nu(A)\colon \pi_1(\Sigma) \longrightarrow K$ of a flat connection $A \in \mathcal{A}^0$ correspond by definition to a pair of commuting matrices $\mathrm{hol}_{\gamma_1}(A)$, $\mathrm{hol}_{\gamma_2}(A) \in K$; in this case one finds:

$$\mathrm{hol}_{\gamma_1}(A) = \begin{pmatrix} e^{i\xi_1} & 0 \\ 0 & e^{-i\xi_1} \end{pmatrix}, \qquad \mathrm{hol}_{\gamma_2}(A) = \begin{pmatrix} e^{i\xi_2} & 0 \\ 0 & e^{-i\xi_2} \end{pmatrix}. \tag{4.1}$$

One thinks here of $\mathrm{hol}_{\gamma_i}\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ as a automorphism of the fibre at $\gamma_i(0) = \gamma_i(1)$, for $i \in \{1, 2\}$. Notice that the term in $dy$ gets cancelled by the holonomy around $\gamma_1$ because this loop is defined by taking $y$ to be constant (with the homotopy class being independent of its precise value), and similarly for $\gamma_2$.

We conclude that the monodromy of $A$ is trivial if and only if $\xi_1, \xi_2 \in 2\pi\mathbb{Z}$. The quotient of $\mathcal{A}^0$ with respect to those choices of parameters gives a torus. Finally, one must not forget that the gauge group also identifies $A$ and $-A$, because:

$$hAh^{-1} + h^{-1}dh = hAh^{-1} = -A, \qquad \text{if} \qquad h \coloneqq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

One thinks of $h$ as a constant map $h\colon \Sigma \longrightarrow K$, i.e. as an element of the gauge group $\mathcal{K} \cong C^\infty(\Sigma, K)$. Now we will show that the space $\mathcal{A}^0$ surjects on the moduli space, by showing that it provides all possible monodromies.

To do this, consider the Betti viewpoint. We consider conjugacy classes of group morphisms $\mathbb{Z}^2 \cong \pi_1(\Sigma) \longrightarrow K$. Those are just given by pairs of commuting matrices $(\widetilde{A}, \widetilde{B}) \in K$ that we can assume simultaneously codiagonalised up to a conjugation in $K$. In particular, we see there are no irreducible representations of $\pi_1(\Sigma)$, up to conjugation, but rather just completely decomposable representations: this is the conceptual reason why considering polystable connections is not restrictive. Pairs of diagonal matrices in $K$ are written

$$\widetilde{A} = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_1^{-1} \end{pmatrix}, \qquad \widetilde{B} = \begin{pmatrix} \theta_2 & 0 \\ 0 & \theta_2^{-1} \end{pmatrix},$$

for suitable unitary complex numbers $\theta_1, \theta_2$. We thus find a torus $U(1) \times U(1)$ of matrices. Notice that by diagonalising $A, B$ have not considered the whole conjugation by $K$: just as above, we still need to identify matrices obtained after conjugating by $h$. This yields

$$h \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_1^{-1} \end{pmatrix} h^{-1} = \begin{pmatrix} \theta_1^{-1} & 0 \\ 0 & \theta_1 \end{pmatrix} = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_1^{-1} \end{pmatrix}^{-1},$$

and similarly for $\widetilde{B}$. Conceptually, the gauge-action $h\colon A \longmapsto -A$ on the matrix of a connection gets turned $h\colon \widetilde{A} \longmapsto \widetilde{A}^{-1}$ after exponentiating (i.e. taking monodromies).

In the end, we have the explicit description

$$\mathrm{Char}(\Sigma, K) \cong \Big(U(1) \times U(1)\Big) \Big/ (\mathbb{Z}/2\mathbb{Z}), \tag{4.2}$$

with the group of order two acting diagonally by taking inverse matrices. This is a nice finite-dimensional presentation of the moduli space as a torus, up to a flip. From this description it follows that there are 4 fixed points for the action, namely the pairs $(\widetilde{A}, \widetilde{B}) \in \{(\pm I, \pm I), (\pm I, \mp I)\} \subseteq U(1) \times U(1)$. The degenerate orbits of these points give 4 singular points on the quotient, outside which one has a two-fold covering of the torus on the moduli space.

Notice that we have also incidentally proved the above claim that the flat connections in $\mathcal{A}^0$ provide a full set of representatives for the de Rham moduli space, because these connections give all the possible monodromy representations of $\pi_1(\Sigma)$. To see this, remark that the equations

$$(\widetilde{A}, \widetilde{B}) = \left( \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_1^{-1} \end{pmatrix}, \begin{pmatrix} \theta_2 & 0 \\ 0 & \theta_2^{-1} \end{pmatrix} \right) = \Big( \mathrm{hol}_{\gamma_1}(A), \mathrm{hol}_{\gamma_2}(A) \Big)$$

have solutions $A \in \mathcal{A}^0$ for all $\theta_1, \theta_2 \in U(1)$, using (4.1): pick $\xi_1, \xi_2 \in \mathbb{R}$ such that $e^{i\xi_1} = \theta_1, e^{i\xi_2} = \theta_2 \in U(1)$.

To conclude with the gauge-equivalence description on the de Rham side, notice that the elements

$$g_{lm} \coloneqq \begin{pmatrix} e^{2\pi i(xl+ym)} & 0 \\ 0 & e^{-2\pi i(xl+ym)} \end{pmatrix} \in \mathcal{K}$$

of the gauge group generate the action on $\mathbb{R}^2 \cong \mathcal{A}^0$ by translations:

$$g_{lm} A g_{lm}^{-1} + g_{lm}^{-1} dg_{lm} = A + g_{lm}^{-1} dg_{lm} = A + \begin{pmatrix} 2\pi i(l dx + m dy) & 0 \\ 0 & -2\pi i(l dx + m dy) \end{pmatrix},$$

where we used the fact that all diagonal matrices commute. We see that this gauge action add to $A \in \mathcal{A}^0$ a matrix that corresponds to a connection with trivial holonomy. Now, just to stress, the Riemann-Hilbert map tells us that in our moduli space we must precisely consider flat connections up to those inducing a trivial monodromy representation of the torus, hence we get a finite-dimensional representation of the de Rham moduli space as

$$\mathcal{M}_{\mathrm{dR}} \cong \mathcal{A}^0 \big/ \mathcal{K}^0, \qquad \text{where} \qquad \mathcal{K}^0 \coloneqq \langle g_{10}, g_{01}, h \rangle \subseteq \mathcal{K}. \tag{4.3}$$

This is a discrete group, consisting of the standard lattice $\mathbb{Z}^2 \subseteq \mathbb{R}^2$, plus the flip $h$.

From the above descriptions of $\mathrm{Char}(\Sigma, K)$ and $\mathcal{M}_{\mathrm{fl}}(\Sigma, K)$, we see that the two moduli spaces are isomorphic as real analytic spaces, and the map from one to the other is given in coordinates by the exponential $\mathbb{R}^2 \longrightarrow U(1) \times U(1)$, which sends $(\xi_1, \xi_2)$ to $(e^{i\xi_1}, e^{i\xi_2})$. We will hereafter just write $\mathcal{M}$ to refer to this compact moduli space, when we do not wish to distinguish among its realisations.

**Remark 4.1.** This topological characterisation can be rewritten as

$$\mathcal{M} \cong \mathbb{T}_K \times \mathbb{T}_K \big/ W_K,$$

where $\mathbb{T}_K \subseteq K$ is the maximal torus of $K$, and $W_K$ its Weyl group. Indeed, if $K = \mathrm{SU}(2)$ then $\mathbb{T}_K \cong U(1)$, and $W_K$ is the group of order two. This description would still be true for any other compact, simple, simply connected Lie group $K$ (see [AM16]).

Notice how we already exploited the notion of a "constant" connection, i.e. a translation-invariant one, which only makes sense because the coordinates $x, y$ vary in a flat space. This gives a canonical identification of the fibres, and picking constant matrices makes sense.

## 4.2 Complex moduli spaces

The purpose of this section is to describe all the data required for the geometric quantisation of $\mathfrak{M}(\Sigma, G)$ in full detail, where $G = \mathrm{SL}(2, \mathbb{C})$, the complexification of $K = \mathrm{SU}(2)$ from the previous section. We will discuss the following facets:

- The topology: § 4.2.1.
- The hyper-Kähler structure: § 4.2.2.
- The symplectic structure: § 4.2.3.
- The prequantum data: § 4.2.4.
- The Kähler polarisations: § 4.2.5.
- The real polarisation: § 4.2.6.

At the end of each section we will make a remark related to the analogous piece of data on the compact moduli space $\mathcal{M}$.

Finally, in § 4.2.7 we will explain how the mapping class group of closed surfaces of genus one acts on all these data. To do this, a discussion about the parametrisation of complex structures on a closed, oriented real surface $S$ — of arbitrary genus — seems necessary, and we will delve into this immediately (the standard reference is [FM12]; see also the beginning of [Mad07]). Recall that a complex structure on the surface $S$ is an

integrable almost-complex structure $I \in C^\infty(S, TS \otimes T^*S)$ that is compatible with the orientation, i.e. such that $\{v, I(v)\}$ is an oriented $\mathbb{R}$-basis of $T_pS$ for all $p \in S$ and $v \in T_pS$. Since the integrability condition is vacuous in this low-dimensional setup, the space

$$\mathcal{C}_S \coloneqq \left\{ I \in C^\infty(S, TS \otimes T^*S) \,\middle|\, I^2 = -\operatorname{Id}, \quad \{v, I(v)\} > 0 \text{ for all } v \in TS \right\}$$

parametrises all complex structures on the oriented surface $S$: the elements of $\mathcal{C}_S$ are all the Riemann surfaces on the underlying smooth manifold $S$. The orientation of $S$ defines a restriction of the structure group of the frame bundle of $TS$ from $\mathrm{GL}(2,\mathbb{R})$ to $\mathrm{GL}_+(2,\mathbb{R})$, where $\mathrm{GL}_+(2,\mathbb{R}) = \{M \in \mathrm{GL}_2(\mathbb{R}) \mid \det(M) > 0\}$. This defines a principal $\mathrm{GL}_+(2,\mathbb{R})$-bundle over $S$. Sections of the adjoint bundle correspond to sections of $\mathrm{End}(TS)$ that preserve orientation. This space carries a natural (Whitney) topology, and $\mathcal{C}_S$ inherits the subspace topology. One can actually show that it is an infinite-dimensional complex Fréchet manifold.

Now the group $\mathrm{Diff}_+(S)$ of orientation-preserving diffeomorphisms of $S$ acts on $\mathcal{C}_S$, by push-forward of almost-complex structure, and the quotient for this action is the (coarse) Riemann moduli space of closed Riemann surfaces of genus $g$:

$$\mathcal{M}_g \coloneqq \mathcal{C}_S \big/ \mathrm{Diff}_+(S).$$

Geometrically, this quotient is a topological space with mild singularities corresponding to fixed points of the action. More precisely, it can be made into an orbifold.
Because of the GAGA correspondence, $\mathcal{M}_g$ is also the moduli space for smooth complex projective curves of genus $g$ (using the correspondence in the sense explained at the beginning of § 2.3.1).

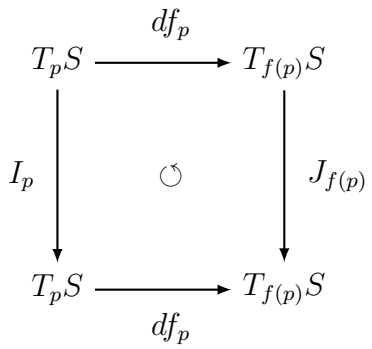
The connected component of the identity $\mathrm{Diff}_0(S) \trianglelefteq \mathrm{Diff}_+(S)$ instead acts freely on $\mathcal{C}_S$, and the quotient

$$\mathcal{T}_g \coloneqq \mathcal{C}_S \big/ \mathrm{Diff}_0(S)$$

can be made into a smooth manifold, which happens to be contractible and finite-dimensional.

**Definition 4.1.** The space $\mathcal{T}_g$ is the Teichmüller space of smooth closed surfaces of genus $g$.

If follows from the definition that Teichmüller space parametrises complex structures on $S$ up to *isotopy*. More precisely, if $I, J \in \mathcal{C}_S$ are orientation-preserving complex structures on $S$, then by definition they lie in the same $\mathrm{Diff}_0(S)$-orbit if there exists an orientation preserving diffeomorphism $f \in \mathrm{Diff}_+(S)$ which is both $(I, J)$-holomorphic and isotopic to the identity inside $\mathrm{Diff}_+(S)$. The former condition means that for all $p \in S$ one has a commutative square




The latter condition instead means that there exists a smooth map $\Gamma\colon S \times [0,1] \longrightarrow S$ such that $\Gamma(p,0) = \gamma(p)$, $\Gamma(p,1) = p$ and $\Gamma|_{S\times\{t\}} \in \mathrm{Diff}_+(S)$ for all $p \in \Sigma, t \in [0,1]$.

Another characterisation of $\mathcal{T}_g$ is via *marked* Riemann surfaces, which we recall in the form of a definition.

**Definition 4.2.** A marked Riemann surface structure over the smooth closed oriented curve $S$ is a pair $(X, f)$, where $X$ is a Riemann surface and $f\colon S \longrightarrow X$ is an orientation preserving diffeomorphism, endowing $X$ with the natural orientation defined by its holomorphic atlas. The diffeomorphism $f$ is the marking of $X$.
Two marked Riemann surfaces $(X, f), (Y, h)$ over $S$ are equivalent if there exists a biholomorphism $\varphi\colon X \longrightarrow Y$ such that $h^{-1} \circ \varphi \circ f\colon S \longrightarrow S$ is isotopic to the identity.[1]

One may then define $\mathcal{T}_g$ as the space of equivalence classes of marked Riemann surfaces of genus $g$, and show that this is consistent with the previous description.
From the former description it also follows that there is a residue action of the discrete quotient group $\Gamma_g \coloneqq \pi_0\Big(\mathrm{Diff}_+(S)\Big) = \mathrm{Diff}_+(S)\Big/\mathrm{Diff}_0(S)$ on Teichmüller space, such that $\mathcal{T}_g\Big/\Gamma_g \cong \mathcal{M}_g$.

**Definition 4.3.** The group $\Gamma_g$ is the mapping class group of smooth closed surfaces of genus $g$.

The mapping class group acts on a marked Riemann surface $(X, f)$ via $\gamma.(X, f) \coloneqq (X, f \circ g^{-1})$, where $\gamma \in \Gamma_g$ is a mapping class. The projection $\mathcal{T}_g \longrightarrow \mathcal{M}_g$ amounts to forgetting the marking, and it is a (universal) covering map.
All this discussion is done for an oriented surface. If one forgets about the orientation, then one can define an *extended* mapping class group, which contains the mapping class group as a subgroup of index 2.

Regardless of the definition used, in genus $g = 1$ the Teichmüller space can be identified with the upper-half plane

$$\mathcal{T}_1 \cong \mathbb{H} \coloneqq \{\tau_1 + i\tau_2 \in \mathbb{C} \mid \tau_2 > 0\}.$$

The way in which a point $\tau \in \mathbb{H}$ defines a complex structure is the following. If one lets again $\Sigma$ be a closed surface of genus one, together with the realisation $\Sigma \cong \mathbb{R}^2\Big/\mathbb{Z}^2$ of the

[1] It is the same to ask that the orientation preserving diffeomorphism $h \circ f^{-1}\colon X \longrightarrow Y$ be isotopic to a biholomorphism.

previous section, then one may consider the integer lattice $\Lambda_\tau = \mathrm{span}_{\mathbb{Z}}\{1, \tau\} \subseteq \mathbb{C}$, and upgrade the real quotient $\mathbb{R}^2/\mathbb{Z}^2$ to the holomorphic quotient $X = \mathbb{C}/\Lambda_\tau$. This is now a compact Riemann surface of genus one, together with the marking $g\colon \Sigma \longrightarrow X$ induced to the quotients by the map $(x, y) \longmapsto z_\tau \coloneqq x + \tau y$. Notice that translating $(x, y) \in \mathbb{R}^2$ by the vector $(k, l) \in \mathbb{Z}^2$ amounts to translating $x + \tau y \in \mathbb{C}$ by the vector $k + \tau l \in \Lambda_\tau$, and thus the transformation $g$ is well defined. One may also take $z_\tau$ as a densely defined holomorphic coordinate on $X$ (it is well defined on a fundamental domain of the lattice $\Lambda_\tau$, which descends bijectively to $X$).

The mapping class group $\Gamma_1$ is isomorphic to $\mathrm{SL}(2, \mathbb{Z})$. One can show this by considering the morphism $\Phi\colon \mathrm{SL}(2, \mathbb{Z}) \longrightarrow \Gamma_1$ that to a matrix $A \in \mathrm{SL}(2, \mathbb{Z})$ associates the diffeomorphism $f_A\colon \Sigma \longrightarrow \Sigma$ induced by the ($\mathbb{Z}$-linear) map $A\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, via the universal property of the quotient. The inverse of this morphism is obtained as follows. If $f \in \mathrm{Diff}_+(\Sigma)$ is an orientation-preserving diffeomorphism, the homology functor $H_\bullet(\cdot, \mathbb{Z})$ defines a map $f_*\colon H_1(\Sigma, \mathbb{Z}) \longrightarrow H_1(\Sigma, \mathbb{Z})$ which only depends on the mapping class of $f$. One has an identification $H_1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^2$, and thus $f_*$ is an invertible $\mathbb{Z}$-linear map of $\mathbb{Z}^2$ into itself, which has positive determinant since $f$ preserves orientation, i.e. an element of $\mathrm{SL}(2, \mathbb{Z})$ (see e.g. [FM12], Thm. 2.5). In this case the morphism $f_*$ is just the usual morphism defined by the continuous map $f$ on the *Abelian* fundamental group.
The extended mapping class group is then isomorphic to $\mathrm{GL}(2, \mathbb{Z})$.

### 4.2.1 The topology

As explained in § 2.3.1, one can attach four moduli spaces to the pair $(\Sigma, G)$. One has the de Rham moduli space $\mathcal{M}_{\mathrm{dR}}(\Sigma, G)$ of gauge-equivalence classes of flat $G$-connections on the trivial principal $G$-bundle $\pi\colon P_{\mathbb{C}} \longrightarrow \Sigma$. Choosing the trivial bundle is as restrictive as fixing a global section, because all principal $G$-bundles on $\Sigma$ are trivialisable. Working in the equivalent category of vector bundles, one considers smooth flat connections on rank 2, complex vector bundles $\pi\colon V \longrightarrow \Sigma$ having trivial determinant.
Next, there is the Betti moduli space $\mathcal{M}_{\mathrm{B}}(\Sigma, G)$ of representations of $\pi_1(\Sigma)$ in $G$, up to conjugation.

These two are the complex versions of the moduli spaces introduced in the previous section 4.1, but now one also has the moduli space $\mathcal{M}_{\mathrm{Dol}}(\Sigma, G)$ of $G$-Higgs bundles on holomorphic, principal $G$-bundles, as well as the hyper-Kähler moduli space $\mathfrak{M}$ of solutions to Hitchin's equations, which underlies the triple of de Rham, Betti and Dolbeault.

Let us start from the Betti viewpoint, considering homomorphisms $\rho\colon \pi_1(\Sigma) \longrightarrow G$ up to conjugation by $G$. As before, those are coded by a pair $\left(\widetilde{A}, \widetilde{B}\right)$ of commuting matrices of determinant one, up to conjugation. In this complexified context, there is no need for $\widetilde{A}$ and $\widetilde{B}$ to be diagonalisable. Let us first assume that both are. We will then find, up to a simultaneous diagonalisation:

$$\widetilde{A} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1^{-1} \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} \mu_2 & 0 \\ 0 & \mu_2^{-1} \end{pmatrix},$$

where $\mu_1, \mu_2 \in \mathbb{C}^*$. This space is parametrised by $\mathbb{C}^* \times \mathbb{C}^*$. Then, just as in the compact

case, we still need to consider the conjugation under the flip matrix $h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Hence, topologically, the moduli space of semisimple representations of $\pi_1(\Sigma)$ is homeomorphic to the quotient $\left(\mathbb{C}^* \times \mathbb{C}^*\right)\big/\left(\mathbb{Z}\big/2\mathbb{Z}\right)$, and this is the space of representations we want to consider, discarding the non-diagonalisable one. This means that we will restrict to polystable connections in the de Rham viewpoint, exactly as we did in § 4.1 for the compact moduli space. Similarly, we will only consider polystable Higgs bundles.

**Remark 4.2.** Considering polystable objects is reasonable, since there are no stable ones. To see this, let $\left(\widetilde{A}, \widetilde{B}\right)$ be a pair of commuting matrices. Assume that one of them is diagonalisable, say $\widetilde{A}$, which we may take to be diagonal without loss of generality. Then either $\widetilde{A} \in \{\pm \mathrm{Id}\} = Z(G)$, or $\widetilde{B}$ is also diagonal. Conversely, if none of them is diagonalisable, then we may assume that $\widetilde{A}$ is the Jordan block of size 2 for the eigenvalue $+1$ or $-1$, and one finds

$$\widetilde{A} \in \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \right\}, \qquad \widetilde{B} \in \left\{ \pm \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| \, a \in \mathbb{C} \right\}.$$

In particular, there are no irreducible $G$-representations of $\pi_1(\Sigma)$, since one always finds a common eigenvector for $\widetilde{A}, \widetilde{B}$. These quick computations also explicitly show the points that we are discarding.

The classification of polystable points up to isomorphism is the same as the classification of semistable ones up to $S$-equivalence (see Rem. 2.15). Hence we are indeed considering the usual moduli problem.

Getting back to the topology of the space of semisimple representations up to conjugation, notice that $\mathbb{C}^* \times \mathbb{C}^*$ can be identified with the cotangent bundle to $\Sigma$. Indeed, the torus is *parallelisable*, hence $T^*\Sigma \cong \Sigma \times \mathbb{R}^2 \cong U(1) \times U(1) \times \mathbb{R}^2$, and $U(1) \times \mathbb{R} \cong U(1) \times \mathbb{R}_{>0} \cong \mathbb{C}^*$, using the standard polar representation of complex numbers.

Let us now pass to the de Rham viewpoint. One can again identify flat connections on $P_{\mathbb{C}} \longrightarrow \Sigma$ with $\mathfrak{g}$-valued differential 1-forms on $\Sigma$. We will consider the following space of flat, translation-invariant, polystable connections:

$$\mathfrak{A}^0 := \left\{ A_{\mathbb{C}} = \begin{pmatrix} w_1 dx + w_2 dy & 0 \\ 0 & -(w_1 dx + w_2 dy) \end{pmatrix} \middle| \, w_1, w_2 \in \mathbb{C} \right\} \subseteq A^1(\Sigma, \mathfrak{g}).$$

The idea is again to show that this space surjects on the de Rham moduli space, via the action of a restricted gauge group, as in the previous section § 4.1. To this end, consider the holonomy of $A_{\mathbb{C}} \in \mathfrak{A}^0$ connections along $\gamma_1$ and $\gamma_2$. This holonomy is again explicitly given by exponentiating:

$$\mathrm{hol}_{\gamma_1}(A_{\mathbb{C}}) = \begin{pmatrix} e^{w_1} & 0 \\ 0 & e^{-w_1} \end{pmatrix}, \qquad \mathrm{hol}_{\gamma_2}(A_{\mathbb{C}}) = \begin{pmatrix} e^{w_2} & 0 \\ 0 & e^{-w_2} \end{pmatrix}$$

Hence we see that the connections inducing a trivial monodromy representation are those for which $e^{w_1} = 1 = e^{w_2}$. This happens for $w_1, w_2 \in 2\pi i\mathbb{Z}$. Now, the short exact sequence

$$(0) \longrightarrow 2\pi i\mathbb{Z} \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^* \longrightarrow (1),$$

where the nontrivial arrows are the natural inclusion $\iota\colon 2\pi i\mathbb{Z} \hookrightarrow \mathbb{C}$ and the exponential $\exp\colon \mathbb{C} \twoheadrightarrow \mathbb{C}^*$, implies that the space of parameters $(w_1, w_2) \in \mathbb{C}^2$ identified modulo $(2\pi i\mathbb{Z})^2$ is $\mathbb{C}^* \times \mathbb{C}^*$. Finally, just as before, we need to further identify our connections with their opposite because of the gauge group. Hence:

$$\mathcal{M}_{\mathrm{dR}}(\Sigma, \mathbb{C}) \cong (\mathbb{C}^2) \Big/ \Big((2\pi i\mathbb{Z})^2 \rtimes (\mathbb{Z}/2\mathbb{Z})\Big) \cong (\mathbb{C}^* \times \mathbb{C}^*) \Big/ (\mathbb{Z}/2\mathbb{Z}).$$

One has incidentally defined two natural complex coordinates $w_1, w_2 \in \mathbb{C}$ on the complex moduli space, defined up to translations, and a flip. They are complex because the group $G$ is, and to define them we did not appeal to the complex structure chosen for $\Sigma$.

Finally, let's describe Hitchin pairs, for which we'll need the description of $\mathcal{M}_{\mathrm{fl}}(\Sigma, K)$ from the previous section § 4.1. The claim is that all Hitchin pair are represented up to gauge equivalences by Hitchin pairs $(D, \Phi)$, with $D \in \mathcal{A}^0$ and

$$\Phi = \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi \end{pmatrix} dz, \qquad \text{where} \qquad \phi \in \mathbb{C}.$$

One has $\Phi \in H^0(\Sigma, K_\Sigma \otimes V)$, and thus this is indeed a polystable $G$-Higgs field. Once again, we've considered only constant tensors. Set also

$$\mathcal{B} \coloneqq \left\{ \Phi = \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi \end{pmatrix} dz \,\middle|\, \varphi \in \mathbb{C} \right\}.$$

To see that this gives a Hitchin pair, notice that $F_D = 0$ by definition. Thus we need to show that $[\Phi \wedge \Phi^*] = 0$ and $D^{0,1}\Phi = 0$. The former follows from the fact that diagonal matrices commute, together with $dz \wedge dz = 0$; the latter from the fact that $\Phi$ is constant.

**Proposition 4.1.** *The quotient of* $\{(D, \Phi) \mid D \in \mathcal{A}^0, \Phi \in \mathcal{B}\}$ *by the action of the gauge group gives the whole Hitchin moduli space*

*Proof.* We can do as follows. First, we map such a pair to a flat complex connection $\nabla = D + \Phi + \Phi^*$, according to the non-Abelian Hodge correspondence, and then we try to show that these connections give all possible complex monodromy representations of $\pi_1(\Sigma)$.

If we write $D = d - A$, for $A = \begin{pmatrix} i\xi_1 dx + i\xi_2 dy & 0 \\ 0 & -(i\xi_1 dx + i\xi_2 dy) \end{pmatrix}$, for $\xi_1, \xi_2 \in \mathbb{R}$, then the $\mathfrak{g}$-valued 1-form defining $\nabla$ is:

$$\begin{aligned} A_{\mathbb{C}} &= A + \Phi + \Phi^* = \\ &= \begin{pmatrix} i\xi_1 dx + i\xi_2 dy & 0 \\ 0 & -(i\xi_1 dx + i\xi_2 dy) \end{pmatrix} + \begin{pmatrix} \varphi dz + \overline{\varphi} d\overline{z} & 0 \\ 0 & -(\varphi dz + \overline{\varphi} d\overline{z}) \end{pmatrix} = \\ &= \begin{pmatrix} i\xi_1 + \varphi + \overline{\varphi} & 0 \\ 0 & -i\xi_1 - (\varphi + \overline{\varphi}) \end{pmatrix} dx + \begin{pmatrix} i\xi_2 + \tau\varphi + \overline{\tau}\overline{\varphi} & 0 \\ 0 & -i\xi_2 - (\tau\varphi + \overline{\tau}\overline{\varphi}) \end{pmatrix} dy. \end{aligned}$$

Now, the monodromy of such a connection around $\gamma_1, \gamma_2$ is given by:

$$\mathrm{hol}_{\gamma_1}(A_{\mathbb{C}}) = \begin{pmatrix} e^{i\xi_1}e^{\varphi}e^{\overline{\varphi}} & 0 \\ 0 & (e^{i\xi_1}e^{\varphi}e^{\overline{\varphi}})^{-1} \end{pmatrix}, \qquad \mathrm{hol}_{\gamma_2}(A_{\mathbb{C}}) = \begin{pmatrix} e^{i\xi_2}e^{\tau\varphi}e^{\overline{\tau\varphi}} & 0 \\ 0 & (e^{i\xi_1}e^{\tau\varphi}e^{\overline{\tau\varphi}})^{-1} \end{pmatrix}$$

Thus we need to show that the following equation has a solution in $\xi_1, \xi_2 \in \mathbb{R}, \varphi \in \mathbb{C}$ for all $\mu_1, \mu_2 \in \mathbb{C}^*$:

$$\begin{cases} e^{i\xi_1}e^{\varphi}e^{\overline{\varphi}} = \mu_1 \\ e^{i\xi_2}e^{\tau\varphi}e^{\overline{\tau\varphi}} = \mu_2 \end{cases}$$

Consider an open simply connected set of $\mathbb{C}$ that contains $\mu_1, \mu_2$. Then we have a well defined branch of the logarithm on this open set, and we can find $w_1, w_2 \in \mathbb{C}$ such that $e^{w_1} = \mu_1, e^{w_2} = \mu_2$. It is then enough to solve:

$$\begin{cases} i\xi_1 + \varphi + \overline{\varphi} = w_1 \\ i\xi_2 + \tau\varphi + \overline{\tau\varphi} = w_2. \end{cases}$$

If one writes $w_1 = u_1 + iv_1$, $w_2 = u_2 + iv_2$, then considering decompositions in real and imaginary parts provide the unique solution

$$u_1 = 2\Re(\varphi), \qquad v_1 = \xi_1, \qquad u_2 = 2\Re(\tau\varphi), \qquad v_2 = \xi_2. \tag{4.4}$$

□

**Remark 4.3.** The interest of this proof was mainly to describe natural coordinates on the moduli space. One has

- The set of four real coordinates $\xi_1, \xi_2, \varphi_1 := \Re(\varphi)$ and $\varphi_2 := \Im(\varphi)$, defined using the Hitchin pair viewpoint.
- The complex coordinates $w_1, w_2$, defined using the de Rham viewpoint.
- The real coordinates $u_1, v_1, u_2, v_2$, defined from the decomposition of $w_1, w_2$ in real and imaginary part.

The linear relation between these two sets of coordinates is coded by (4.4).

Another important remark is that the set of coordinates $\{\xi_1, \xi_2, \varphi_1, \varphi_2\}$ depends on the complex structure $\tau \in \mathbb{H}$ of $\Sigma$, because one uses Higgs fields to define them. The second set, on the contrary, does not depend on the complex structure; we shall hereafter refer to them as the $\tau$-independent coordinates, and having them will be very useful when considering derivatives with respect to $\tau$.

Once again, $\xi_1$ and $\xi_2$ are defined modulo $2\pi\mathbb{Z}$ on the quotient, and then we have to act by the group $\mathbb{Z}/2\mathbb{Z}$ generated by the gauge-flip $h$. Thus, once more:

$$\mathfrak{M} \cong \mathbb{R}^4 \Big/ \Big((2\pi\mathbb{Z})^2 \rtimes (\mathbb{Z}/2\mathbb{Z})\Big) \cong (U(1) \times U(1) \times \mathbb{R}^2) \Big/ (\mathbb{Z}/2\mathbb{Z}).$$

This is the hyper-Kähler Hitchin moduli space underlying the three nonisomorphic complex algebraic varieties $\mathcal{M}_{\mathrm{dR}}$, $\mathcal{M}_{\mathrm{B}}$ and $\mathcal{M}_{\mathrm{Dol}}$.

A last useful thing to do is to write the connection form $A_{\mathbb{C}}$ in terms of the complex coordinate $z = x + \tau dy$. One finds:

$$\begin{aligned} A_{\mathbb{C}} = &\begin{pmatrix} \varphi + \frac{i}{\overline{\tau}-\tau}(\overline{\tau}\xi_1 - \xi_2) & 0 \\ 0 & -\left(\varphi + \frac{i}{\overline{\tau}-\tau}(\overline{\tau}\xi_1 - \xi_2)\right) \end{pmatrix} dz + \\ &+ \begin{pmatrix} \overline{\varphi} + \frac{i}{\overline{\tau}-\tau}(\xi_2 - \tau\xi_1) & 0 \\ 0 & -\left(\overline{\varphi} + \frac{i}{\overline{\tau}-\tau}(\xi_2 - \tau\xi_1)\right) \end{pmatrix} d\overline{z} \end{aligned} \tag{4.5}$$

Notice that $\overline{\tau} - \tau = -2i\Im(\tau) = -2i\tau_2 \neq 0$, and thus one can divide by it.

One could again look for a restricted gauge group $\mathcal{G}^0 \subseteq \mathcal{G} \cong C^\infty(\Sigma, G)$ that provides all the necessary gauge-transformations to get the good quotient. Reasoning as in the previous section, it turns out that one can take the same group as in the compact case, i.e. :

$$\mathcal{M}_{\mathrm{dR}} \cong \mathfrak{A}^0 \big/ \mathcal{K}^0,$$

with $\mathcal{K}^0$ defined in (4.3). The point is that in the complex case too one only needs the flip $h$ and the translations $g_{01}, g_{10}$ in order to introduce all the gauge relations. Hence one also has a finite-dimensional presentation of the complex moduli space. Moreover, one sees that there is an embedding

$$\iota \colon \mathcal{M} \hookrightarrow \mathfrak{M},$$

which is well defined on $\mathcal{G}^0$-orbits. This embedding is natural, and it does not depend on the choice of a complex structure on $\Sigma$. Notice that (4.4) means that $\mathcal{M}$ is defined as the subspace of the complex moduli space where the natural coordinates $u_1, u_2$ vanish.

**Remark 4.4.** All the discussion of this section is an accordance with [FGN14], which studies Higgs bundles over elliptic curves with a classical complex reductive group as structure group.[2] In particular, Thm. 4.27 states that the moduli space of semistable $G$-Higgs bundles up to $S$-equivalence is isomorphic to $T^*\Sigma \big/ (\mathbb{Z}/2\mathbb{Z})$, where the action of the group of order two on the cotangent bundle is obtained by lifting the action on $U(1) \times U(1)$ (i.e. the action on the compact moduli space by matrix inversion). Our explicit description agrees with that result (see also [Tha01], § 2).

Moreover, we can explicitly verify the following fact.

**Proposition 4.2.** *The singular locus of* $\mathcal{M}_{\mathrm{Dol}}\big(\Sigma, \mathrm{SL}(n, \mathbb{C})\big)$ *has complex codimension* 2 *for all* $n \geq 2$.

This proposition is intended in the category of algebraic varieties. It implies that the section of any holomorphic vector bundle over the smooth locus can be extended on the whole of the moduli space. We will implicitly use this fact when considering holomorphic sections of the prequantum bundle — with respect to the choice of Kähler polarisations — without introducing further notation for the smooth locus.

[2] We can neglect the choice of a base point $p_0 \in \Sigma$, which is needed to define an elliptic curve.

To be more concrete in our case where $n = 2$, the dimension of the of the whole moduli space equals 2, and thus one expects to just have finitely many singular points. Indeed, the action of the group of order two on $\mathbb{C}^* \times \mathbb{C}^* \cong \mathbb{R}^2 \times U(1) \times U(1)$ is obtained by lifting the action on $U(1) \times U(1)$ by simultaneous inversion that we already encountered. The induced action on cotangent vectors is a change of sign, which fixes only the origin in every cotangent space. Hence the fixed points of the lifted actions come from those on the base, of which there are 4.

Finally, let us remark explicitly that since the canonical bundle $K_\Sigma$ is trivial, then a $G$-Higgs bundle is (poly)stable if and only if the underlying principal $G$-bundle is (poly)stable. Hence one abstractly expects to have a projection $\mathfrak{M} \longrightarrow \mathcal{M}$ obtained by forgetting the Higgs field in any given Hitchin pair. In our notation, this is just the restriction of the canonical projection $\pi\colon \mathfrak{A}^0 \longrightarrow \mathfrak{A}^0 \big/ \mathcal{G}^0$ to $\mathcal{A}^0 \subseteq \mathfrak{A}^0$.

### 4.2.2 The complex structures

Let us start from a general definition.

**Definition 4.4.** A hyper-Kähler manifold is a smooth Riemannian manifold $(M, g)$, together with three $\nabla^{\mathrm{LC}}$-covariantly constant, $g$-symmetric sections

$$I, J, K \in C^\infty(M, TM \otimes T^*M)$$

of the endomorphism bundle of $TM$, which satisfy the quaternionic relations $I^2 = J^2 = K^2 = IJK = -\operatorname{Id}$, where $\nabla^{\mathrm{LC}}$ denotes the Levi–Civita connection of $(M, g)$. The metric $g$ is called the hyper-Kähler metric.

The relations we quoted imply $IJ = -JI = K$, $JK = -KJ = I$ and $KI = -IK = J$. Remark that if $a, b, c \in \mathbb{R}^3$ are real numbers satisfying $a^2 + b^2 + c^2 = 1$, then the tensor

$$I_{abc} \coloneqq aI + bJ + cK$$

is still a complex structure on $M$, using the same notations as in the above definition. Its integrability is assured from that of $I, J, K$.

**Definition 4.5.** Let $(M, I, J, K, g)$ be a hyper-Kähler manifold. The set

$$\{I_{abc} \mid (a, b, c) \in \mathbb{R}^3, \quad a^2 + b^2 + c^2 = 1\} \subseteq C^\infty(M, TM \otimes T^*M)$$

is called the sphere of Kähler structures of $M$.

It is shown in [Hit87a] that $(\mathfrak{M}, I_\tau, J, K_\tau)$ admits a hyper-Kähler metric $g$. What is nice is that one can explicitly describe its sphere Kähler structures in our context, by writing down formulae for the triple $I_\tau, J, K_\tau = I_\tau J$ of complex structures.

The Dolbeault complex structure $I_\tau$ comes from the complex structure $\tau$ fixed on the surface $\Sigma$. Concretely, when we decompose a connection as $A_{\mathbb{C}} = A_1 dz + A_2 d\overline{z}$, we're using the complex structure of the curve to define coordinates $z, \overline{z}$. This in turn defines

a complex structure by declaring that the local coordinates $\varphi, \psi \coloneqq \xi_2 - \tau\xi_1$ on $\mathfrak{M}$ are holomorphic.
The associated vector fields live in the eigenbundle of the complexification of $I_\tau$ of eigenvalue $i \in \mathbb{C}$. This amounts to impose

$$d\varphi(\mathrm{Id} + iI_\tau) = d\psi(\mathrm{Id} + iI_\tau) = 0 = d\overline{\varphi}(\mathrm{Id} - iI_\tau) = d\overline{\psi}(\mathrm{Id} - iI_\tau),$$

since $\frac{1}{2}(\mathrm{Id} + iI_\tau)$ and $\frac{1}{2}(\mathrm{Id} - iI_\tau)$ are the projections of the complexified tangent bundle $T_{\mathbb{C}}\mathfrak{M}$ onto the antiholomorphic part and the holomorphic part, respectively.
The De Rham complex structure $J$ comes from the fact that $G$ itself has a complex structure. Namely, writing $A_{\mathbb{C}} = \mathrm{diag}(w_1, -w_1)dx + \mathrm{diag}(w_2, -w_2)dy$, we declare $w_1, w_2$ to be holomorphic coordinates on the moduli space. As argued just above, this amounts to imposing

$$dw_1(\mathrm{Id} + iJ) = dw_2(\mathrm{Id} + iJ) = 0 = d\overline{w_1}(\mathrm{Id} - iJ) = d\overline{w_2}(\mathrm{Id} - iJ).$$

Finally, there will be the third complex structure $K_\tau = I_\tau J$.

One may explicitly compute the following:

$$\begin{cases} I_\tau(\partial_{\xi_1}) &= \frac{\tau_1}{\tau_2}\partial_{\xi_1} + \frac{|\tau|^2}{\tau_2}\partial_{\xi_2} \\ I_\tau(\partial_{\xi_2}) &= -\frac{1}{\tau_2}\partial_{\xi_1} - \frac{\tau_1}{\tau_2}\partial_{\xi_2} \\ I_\tau(\partial_{\varphi_1}) &= \partial_{\varphi_2} \\ I_\tau(\partial_{\varphi_2}) &= -\partial_{\varphi_1} \end{cases}$$

$$\begin{cases} J(\partial_{\xi_1}) &= -\frac{1}{2}\partial_{\varphi_1} - \frac{\tau_1}{2\tau_2}\partial_{\varphi_2} \\ J(\partial_{\xi_2}) &= \frac{1}{2\tau_2}\partial_{\varphi_2} \\ J(\partial_{\varphi_1}) &= 2\partial_{\xi_1} + 2\tau_1\partial_{\xi_2} \\ J(\partial_{\varphi_2}) &= -2\tau_2\partial_{\xi_2} \end{cases}$$

$$\begin{cases} K_\tau(\partial_{\xi_1}) &= \frac{\tau_1}{2\tau_2}\partial_{\varphi_1} - \frac{1}{2}\partial_{\varphi_2} \\ K_\tau(\partial_{\xi_2} = &= -\frac{1}{2\tau_2}\partial_{\varphi_1} \\ K_\tau(\partial_{\varphi_1}) &= 2\tau_2\partial_{\xi_2} \\ K_\tau(\partial_{\varphi_2}) &= 2\partial_{\xi_1} + 2\tau_1\partial_{\xi_2} \end{cases}$$

**Remark 4.5.** One can ask what is the compatibility between the complex structure $I_\tau$ on $\mathfrak{M}$ and the complex structure on the compact moduli space $\mathcal{M}$, which also arises from an element $\tau$ in Teichmüller space (see § 2.3.3). What happens is that the complex structure $I_\tau$ identifies $\mathcal{M}_{\mathrm{Dol}}$ as the holomorphic cotangent bundle $T_{1,0}\mathcal{M}$, where $\mathcal{M}$ is given its $\tau$-dependent complex structure. Notice indeed that the first set of formulae shows in particular that $I_\tau$ restricts to a $\tau$-dependent complex structure on $\mathcal{M}$, because the vector fields $\partial_{\xi_1}, \partial_{\xi_2}$ generate the complexified tangent bundle to $\mathcal{M}$ over $\mathbb{C}$.

We now move on to define the symplectic structure of $\mathfrak{M}$.

### 4.2.3 Symplectic structure

We follow the infinite-dimensional approach by Atiyah and Bott (see 2.3.3), and then write the result explicitly in our finite-dimensional presentation. The idea is to realise $\mathfrak{M}$ as the symplectic reduction of the space $\mathcal{A}_{\mathbb{C}}$ of all $G$-flat connections, with respect to the Hamiltonian action of gauge group $\mathcal{G} \cong C^\infty(\Sigma, G)$. Recall that the moment map for the action is the curvature

$$F\colon \mathcal{A}_{\mathbb{C}} \longrightarrow A^2\big(\Sigma, \mathfrak{g}\big) \cong C^\infty\big(\Sigma, \mathfrak{g}\big)^* \cong \mathrm{Lie}(\mathcal{G})^*.$$

This means that one can use the complex analogue of the Atiyah–Bott symplectic form $\omega_{\mathbb{C}}$ on $\mathcal{A}_{\mathbb{C}}$. Since this form is gauge-invariant, it automatically induces a symplectic structure on the moduli space by Marsden–Weinstein reduction (see § 2.3.3).
Set then:

$$\omega_{\mathbb{C}}(A, B) \coloneqq -\frac{1}{2\pi}\int_\Sigma \mathrm{Tr}(A \wedge B),$$

with $A, B \in A^1(\Sigma, \mathfrak{g})$. As done in § 2.3.3, one uses a multiple of the Killing form of the simple Lie algebra $\mathfrak{g}$ to provide a pairing $A^1(\Sigma, \mathfrak{g}) \otimes A^1(\Sigma, \mathfrak{g}) \longrightarrow A^2(\Sigma, \mathbb{C})$, and then integrates the result on the surface.

We now want to compute this using our coordinates on the restricted space $\mathfrak{A}^0 \cong \mathbb{C}^2$ of flat connections that we introduced in the previous section, using the complex coordinates $w_1, w_2$. Recall that the coordinates on the Betti space are $\mu_1 = e^{w_1}$, $\mu_2 = e^{w_2}$. In these coordinates, if we use the tacitly complexified base $dx, dy$ for $A^1(\Sigma, \mathbb{C})$, we see that the basis tangent vectors are

$$\partial_{w_1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} dx, \quad \partial_{w_2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} dy.$$

Hence

$$\omega_{\mathbb{C}}(\partial_{w_1}, \partial_{w_2}) = -\frac{1}{2\pi}\int_\Sigma \mathrm{Tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) dx \wedge dy = -\frac{1}{\pi}\int_\Sigma dx \wedge dy = -\frac{1}{\pi}.$$

This means that one has

$$\omega_{\mathbb{C}} = \frac{1}{\pi}(dw_2 \wedge dw_1).$$

**Remark 4.6.** Let us verify that the symplectic form is the same when written in terms of the complex basis $dz, d\overline{z}$ for $A^1(\Sigma, \mathbb{C})$. This is expected because the symplectic structure has a topological nature: it does not depend on the complex structure chosen on the real surface.

Recall that $z = x + \tau y$, so that $dz = dx + \tau dy$ and $\overline{z} = dx + \overline{\tau} dy$. We must write again our flat connections in terms of this coordinates, i.e. solving the following linear equation for two complex diagonal matrices $A_1, A_2$ of size two:

$$A = \begin{pmatrix} w_1 & 0 \\ 0 & -w_1 \end{pmatrix} dx + \begin{pmatrix} w_2 & 0 \\ 0 & -w_2 \end{pmatrix} dy = A_1 dz + A_2 d\overline{z}.$$

One finds

$$A_1 = \frac{1}{\overline{\tau}-\tau}\begin{pmatrix} \overline{\tau}w_1 - w_2 & 0 \\ 0 & -(\overline{\tau}w_1 - w_2)\end{pmatrix} dz, \qquad A_2 = \frac{1}{\overline{\tau}-\tau}\begin{pmatrix} w_2 - \tau w_1 & 0 \\ 0 & -(w_2 - \tau w_1)\end{pmatrix} d\overline{z},$$

and thus in these coordinates one has

$$\partial_{w_1} = \frac{1}{\overline{\tau}-\tau}\begin{pmatrix} \overline{\tau}dz - \tau d\overline{z} & 0 \\ 0 & -(\overline{\tau}dz - \tau d\overline{z})\end{pmatrix}, \qquad \partial_{w_2} = \frac{1}{\overline{\tau}-\tau}\begin{pmatrix} -dz + d\overline{z} & 0 \\ 0 & -(-dz + d\overline{z})\end{pmatrix}.$$

Hence:

$$\omega_{\mathbb{C}}(\partial_{w_1}, \partial_{w_2}) = -\frac{1}{2\pi}\int_\Sigma \mathrm{Tr}\begin{pmatrix} \frac{\overline{\tau}}{(\overline{\tau}-\tau)^2} & 0 \\ 0 & \frac{\overline{\tau}}{(\overline{\tau}-\tau)^2}\end{pmatrix} dz\wedge d\overline{z} + \mathrm{Tr}\begin{pmatrix} \frac{-\tau}{(\overline{\tau}-\tau)^2} & 0 \\ 0 & \frac{-\tau}{(\overline{\tau}-\tau)^2}\end{pmatrix} dz\wedge d\overline{z} =$$

$$= -\frac{1}{\pi(\overline{\tau}-\tau)}\int_\Sigma dz\wedge d\overline{z}.$$

Now remark that $dz\wedge d\overline{z} = (\overline{\tau}-\tau)dx\wedge dy$, so that $\omega_{\mathbb{C}}(\partial_{w_1},\partial_{w_2}) = -\frac{1}{\pi}$, as it was to be shown.

We shall thereafter denote $\omega_{\mathbb{C}}$ the symplectic structure on $\mathfrak{M}$ obtained via symplectic reduction.

**Remark 4.7.** One may now ask what is the restriction $\iota^*\omega_{\mathbb{C}}$ of the symplectic structure to $\mathcal{M}$, if $\iota\colon \mathcal{M}\hookrightarrow\mathfrak{M}$ is the natural embedding of $\mathcal{M}$ into $\mathfrak{M}$. More precisely, one should consider the relation between $\omega_{\mathbb{C}}$ and the compact version $\omega\in A^2(\mathcal{M},\mathbb{R})$ of the Atiyah–Bott symplectic form presented in 2.3.3.

**Proposition 4.3.** *The symplectic form $\omega_{\mathbb{C}}$ is the complexification of $\omega$, in the sense that*

$$\iota^*\Big(\Re(\omega_{\mathbb{C}})\Big) = \omega, \qquad \textit{where} \qquad \Re(\omega_{\mathbb{C}}) \coloneqq \frac{\omega_{\mathbb{C}} + \overline{\omega_{\mathbb{C}}}}{2}.$$

This is ultimately related to the fact that $\mathfrak{g}\cong\mathfrak{k}\otimes_{\mathbb{R}}\mathbb{C}$, and that one is defining the symplectic forms via the pairing provided by the traces in both cases. Apart from this conceptual reason, one can give a proof in coordinates.

*Proof.* One has:

$$\Re(\omega_{\mathbb{C}}) = -\frac{1}{\pi}\Re(dw_1\wedge dw_2) = -\frac{1}{\pi}\Re\Big((du_1+idv_1)\wedge(du_2+idv_2)\Big) = -\frac{1}{\pi}(du_1\wedge du_2 - dv_1\wedge dv_2).$$

Now, one recalls from (4.4) that $u_1, u_2$ are the coordinates parametrising the field $\Phi$ in any Hitchin pair, and thus they vanish in the restriction to $\mathcal{M}$. On the contrary, $v_1, v_2$ are the real coordinates parametrising unitary connections, whose classes are precisely the points of $\mathcal{M}$. Hence

$$\iota^*\Re\Big(\omega_{\mathbb{C}}\Big) = \frac{1}{\pi}(dv_1\wedge dv_2) = \frac{1}{\pi}(d\xi_1\wedge d\xi_2).$$

On the other hand, the vector fields $\partial_{\xi_1}, \partial_{\xi_2}$ on the compact moduli space $\mathcal{M}$ correspond to the matrices $X \coloneqq \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} dx$ and $Y \coloneqq \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} dy$, and thus:

$$\omega(\partial_{\xi_1}, \partial_{\xi_2}) = -\frac{1}{2\pi} \int_\Sigma \mathrm{Tr}(XY) dx \wedge dy = \frac{1}{\pi},$$

which means that $\omega = \frac{1}{\pi}(d\xi_1 \wedge d\xi_2)$, as it was to be shown. □

We now move to define prequantum data on $(\mathfrak{M}, \omega_\mathbb{C})$. To do this, one first has to correct for the fact that the symplectic form is complex, by introducing a complex version of the quantum level of 2.2.5.

### 4.2.4 Prequantum data

In what follows fix a nonzero complex number $t = k + i\sigma \in \mathbb{C}^*$, with $k, \sigma \in \mathbb{R}$.[3] This complex number will play the role of what was the level of the quantisation in the compact case, and its real part will be eventually directly related to that. The idea to define prequantum data on $\mathfrak{M}$ is again to pass from our quotient description of this moduli space. All that happens in this section is independent of the complex structure of $\Sigma$, since this is still prequantisation.

Let us associate to each level $t \in \mathbb{C}^*$ a real symplectic form on $\mathfrak{A}^0$ via:

$$\omega_t \coloneqq \Re(t\omega_\mathbb{C}) = k\Re(\omega_\mathbb{C}) - \sigma\Im(\omega_\mathbb{C}).$$

We will abusively use the same symbol for the induced real symplectic structure at level $t \in \mathbb{C}^*$ on the moduli space. This is the symplectic form we're considering when looking for prequantum data.

As explained in § 2.3.3, we start by defining prequantum data $(\widetilde{L}^{(t)}, \widetilde{\nabla}^{(t)}, \widetilde{h}^{(t)})$ on the prequantisable phase-space of flat connection $(\mathfrak{A}^0, \omega_t)$, and then we make them descend on the moduli space itself via multipliers acting on $\widetilde{L}^{(t)}$.

Since $\mathfrak{A}^0$ is just a flat space, all vector bundles over it are trivialisable. We take $\widetilde{L}^{(t)}$ to be the trivial line bundle:

$$\widetilde{L}^{(t)} \coloneqq \mathbb{C} \times \mathfrak{A}^0 \longrightarrow \mathfrak{A}^0.$$

Moreover, we pick $\widetilde{h}^{(t)}$ to be the translation-invariant metric which is everywhere equal to the standard Hermitian product on $\mathbb{C}$. Then we can look for a prequantum connection of the form

$$\widetilde{\nabla}^{(t)} = d - i\alpha_t$$

where $d$ is the trivial connection and $\alpha_t \in A^1(\mathfrak{A}^0, \mathbb{R})$ Such a connection is automatically $h$-unitary. We see that the prequantum condition $F_{\widetilde{\nabla}^{(t)}} = -i\omega_t$ forces $d\alpha_t = \omega_t$, and the most natural definition is $\alpha_t \coloneqq \Re(t\alpha)$, where

$$\alpha_\nabla(B) \coloneqq -\frac{1}{4\pi} \int_\Sigma \mathrm{Tr}(A, B)$$

[3] We shall see later on that the restriction $k \in \mathbb{Z}$ is to be imposed.

for $\nabla = d - A \in \mathfrak{A}^0$, $A, B \in A^1(\Sigma, \mathfrak{g})$.

**Proposition 4.4.** *One has $d\alpha = \omega_{\mathbb{C}}$, and thus $d\alpha_t = \omega_t$.*

*Proof.* If one denotes $X \in C^\infty(\mathfrak{A}^0, T\mathfrak{A}^0)$ the tautological vector field defined by

$$X(d - A) := A,$$

then one has $2\alpha = \iota_X \omega_{\mathbb{C}}$, meaning that $2\alpha$ is the contraction of $\omega_{\mathbb{C}}$ against this $X$.

Now, one can prove that $\mathcal{L}_X \omega_{\mathbb{C}} = 2\omega_{\mathbb{C}}$, so that by the Cartan formula:

$$2\omega_{\mathbb{C}} = \mathcal{L}_X \omega_{\mathbb{C}} = d\iota_X \omega_{\mathbb{C}} + \iota_X d\omega_{\mathbb{C}} = d\iota_X \omega_{\mathbb{C}} = 2d\alpha.$$

Indeed, the flow $\varphi_t^X$ of $X$ at times $t$ is given by $\varphi_t^X(d - A) = d - (1+t)A$, for a connection $\nabla = d - A \in \mathfrak{A}^0$. Hence, omitting the evaluation of $\omega_{\mathbb{C}}$ at the point $\nabla$:

$$\left(\varphi_t^X\right)^* \omega_{\mathbb{C}}(B, C) = \omega_{\mathbb{C}}\Big(d\varphi_t^X.B, d\varphi_t^X.C\Big) = \omega_{\mathbb{C}}\Big((1+t)B, (1+t)C\Big) = (1+t)^2(\omega_{\mathbb{C}})(B, C),$$

so that

$$\frac{d}{dt}\left(\varphi_t^X\right)^* \omega_{\mathbb{C}} = 2(1+t)\omega_{\mathbb{C}}.$$

Evaluating at $t = 0$ yields $\mathcal{L}_X \omega_{\mathbb{C}} = 2\omega_{\mathbb{C}}$. □

**Remark 4.8.** The same conclusion can be obtained by computing $\alpha$ in coordinates. We consider again the tangent vectors $\partial_{w_1}, \partial_{w_2}$ to $\mathfrak{A}^0$. At a point $\nabla = d - A$, with $A$ written as usual, one finds:

$$\begin{aligned}\alpha_\nabla(\partial_{w_1}) &= -\frac{1}{4\pi}\int_\Sigma \mathrm{Tr}(A \wedge \partial_{w_1}) = -\frac{1}{4\pi}\int_\Sigma \mathrm{Tr}\left(\begin{pmatrix} w_2 & 0 \\ 0 & -w_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) dy \wedge dx = \\ &= \frac{1}{2\pi} w_2.\end{aligned}$$

Similarly $\alpha_\nabla(\partial_{w_2}) = -\frac{1}{2\pi} w_1$. Hence, in those coordinates:

$$\alpha = \frac{1}{2\pi}(w_2 dw_1 - w_1 dw_2),$$

whose differential is precisely $\omega_{\mathbb{C}} = \frac{1}{\pi}(dw_2 \wedge dw_1)$.

Now that we have defined the prequantum line bundle and its connection, we can try to define an action of the (restricted) gauge group $\mathcal{G}^0$ on $\widetilde{L}^{(t)}$ that preserves the connection. This means that our data will descend to an hermitian line bundle with connection on $\mathfrak{A}^0 \big/ \mathcal{G}^0 = \mathcal{M}_{\mathrm{dR}}$. To define this lifted action, we use *multipliers*

$$\Theta^{(t)} \colon \mathcal{G}^0 \times \mathfrak{A}^0 \longrightarrow U(1),$$

with a slight modification of those of § 2.3.3, used for the compact theory. Namely, if $g \in \mathcal{G} \cong C^\infty(\Sigma, G)$ is a gauge-transformation, then it is isotopic to the identity, i.e. the map constant to the identity of $G$. Using this, one can extend $g$ to a gauge-transformation $\widetilde{g}$ of

the pull-back bundle $\pi^*P_{\mathbb{C}} \longrightarrow X$ over to the cylinder $X \coloneqq \Sigma \times [0,1]$, where $\pi\colon X \longrightarrow \Sigma$ is the canonical projection. Namely, one lets $\widetilde{g}$ act on the restricted principal bundle $\pi^*P_{\mathbb{C}}|_{\Sigma\times\{t\}} \longrightarrow \Sigma \times \{t\} \cong \Sigma$ as $g_t\colon P \longrightarrow P$, where $\{g_t\}_{t\in[0,1]}$ is an isotopy from $g_0 = g$ to $g_1 = \mathrm{Id}$. If $\nabla = d - A \in \mathfrak{A}^0$ is a connection, then consider $\widetilde{\nabla} = \pi^*\nabla = d - \pi^*A$, which is a connection on the 3-manifold $X$. Then $\widetilde{g}.\widetilde{\nabla}$ is an extension of $g.\nabla$ to $X$, and one defines:

$$\Theta^{(t)}(g,\nabla) \coloneqq \exp\Big(2\pi i \Re\Big(tS_{\mathrm{CS}}(\widetilde{g}.\pi^*A)\Big)\Big),$$

where $S_{\mathrm{CS}}\colon A^1(X,\mathfrak{g}) \longrightarrow \mathbb{R}$ denotes the Chern–Simons action. Notice that the only difference with the exposition of § 2.3.3 is that the multiplication by the integer $k$ is now replaced by taking the real part of the multiplication by $t$.

It turns out that one can be much more explicit when considering the restricted gauge group $\mathcal{G}^0 \subseteq \mathcal{G}$, at least in genus 1, by imposing that the multipliers preserve the pre-quantum data. To this end, consider temporarily the following abstract setting (we also introduce this viewpoint since it will also be useful in § 4.6).
Let $(V,\omega)$ be a real symplectic vector space of dimension four. Thinking of $V$ as a manifold such that $TV \cong V \times V$, we set $\alpha_v(w) \coloneqq \frac{1}{2}\omega(v,w)$, which is a symplectic potential. To see it, notice that if one fixes a Darboux basis $\{x_i, y_i\}_i$ of $V$ where $\omega = \sum_i dx_i \wedge dy_i$, then one has $\alpha = \frac{1}{2}\sum_i (x_i dy_i - y_i dx_i)$ in the same set of coordinates. Now, the full lattice $\Lambda \subseteq V$ of integer translations acts on the space $V$:

$$\lambda\colon v \longmapsto v + 2\pi\lambda \qquad \text{for} \qquad \lambda \in \Lambda, v \in V.$$

This lattice is isomorphic to $\mathbb{Z}^4$ as $\mathbb{Z}$-module, and this is precisely the situation we're dealing with above, only that there we only consider a half lattice $\Lambda_0 \subseteq \Lambda$ generated by the translation along 2 directions. Nonetheless, one can define multipliers to lift the full $\Lambda$-action to the trivial complex line bundle $L \coloneqq \mathbb{C} \times V \longrightarrow V$, equipped with the connection $\nabla \coloneqq d - i\alpha$, in such a way that the connection is fixed.

**Proposition 4.5.** *The multipliers*

$$\Theta(\lambda, x) \coloneqq \varepsilon_\lambda e^{\pi i \omega(\lambda,x)}, \tag{4.6}$$

*provide a lift of the $\Lambda$-action from $V$ to $L$ that preserves $\nabla$ and the standard metric on $L$, where $\varepsilon\colon \Lambda \longrightarrow \mathbb{Z}\big/2\mathbb{Z}$ is a sign satisfying $\varepsilon_\lambda \cdot \varepsilon_\mu = \varepsilon_{\lambda+\mu} e^{\pi i\omega(\lambda,\mu)}$.*

In this proposition, one may just think of $\omega$ as an alternating bilinear form on $V$, and since $\Lambda \subseteq V$ it makes sense to evaluate it on $V \times \Lambda$. Moreover, one has $\omega\big(\Lambda \times \Lambda\big) \subseteq \mathbb{Z}$, and so the identity for the signs makes sense too. Notice that one has to add these signs into the definition in order to establish a genuine multiplier, i.e. to ensure that one has

$$\Theta(\lambda, x+\mu)\cdot\Theta(\mu,x) = \Theta(\lambda+\mu,x),$$

for $x \in V, \lambda, \mu \in \Lambda$.

*Proof.* It is clear that the metric is preserved, since we're acting fibrewise by unitary transformations of $\mathbb{C}$.

Next, the condition $\lambda^*\nabla = \nabla$ can be rewritten as follows, in terms of the connection form $\alpha$:

$$\lambda^*\Big(\Theta(\lambda,x)^{-1}d\Theta(\lambda,x) - i\Theta(\lambda,x)^{-1}\alpha_x\Theta(\lambda,x)\Big) = -i\alpha_x$$

where $\alpha_x$ means the evaluation of $\alpha$ at the point $x \in V$.
Now, the conjugation acts trivially, because $L$ has rank 1. Moreover, the pull-back $\lambda^*$ fixes the translation-invariant form $\Theta(\lambda,x)^{-1}d\Theta(\lambda,x)$, since the differential of $\lambda$ is the identity everywhere. Finally, one verifies directly that

$$\Theta(\lambda,x)^{-1}d\Theta(\lambda,x) = \pi i\omega(\lambda,dx) = i(\lambda^*\alpha_x - \alpha_x)$$

□

Considering $V = \mathfrak{A}^0$, $\omega = \omega_t$, $\alpha = \alpha_t$, and $\Lambda_0$ the $\mathbb{Z}$-module generated by the translations in the direction of a unitary connection, one has a lift of almost all the gauge-group $\mathcal{G}^0$ to the prequantum line bundle, such that the prequantum data are preserved. One can then extend this definition to the whole of $\mathcal{G}^0$ by declaring that $\Theta^{(t)}$ is trivial on the flip $h$.

**Remark 4.9.** We now see that one is forced to choose a quantum level $t = k + i\sigma$ with $k \in \mathbb{Z}$. The point is that this condition assures that $\omega_t$ takes integer values when evaluated on the half-lattice $\mathbb{Z}^2 \subseteq \mathfrak{A}^0$, since $\omega_{\mathbb{C}}$ does so.

We now apply all this to $(V,\omega) = (\mathfrak{A}^0,\omega_t)$, thereby defining $\Theta^{(t)}$ according to (4.6). We can now consider the quotient line bundle $L^{(t)} \coloneqq \tilde{L}^{(t)}/\mathcal{G}^0$, defined on the moduli space $\mathfrak{M} \cong \mathfrak{A}^0\big/\mathcal{G}^0$, and then $\nabla^{(t)}$ and $h^{(t)}$ to be the induced connection and metric downstairs on this line bundle. By construction, we have $F_{\nabla^{(t)}} = -i\omega_t$, so that we have defined prequantum data for $(\mathfrak{M},\omega_t)$ at level $t \in \mathbb{C}^*$.

**Remark 4.10.** As always, we compare this situation with that for the compact moduli space, with respect to the natural embedding $\iota\colon \mathcal{M} \hookrightarrow \mathfrak{M}$. More precisely, one ponders the relation between the prequantum line bundle : $L^{(t)} \longrightarrow \mathfrak{M}$ and the Chern–Simons one $L \longrightarrow \mathcal{M}$ over the compact moduli space, and one finds that the topological type of the restriction $L^{(t)}\big|_{\mathcal{M}} = \iota^*L^{(t)} \longrightarrow \mathcal{M}$ is that of the $k$-fold tensor power $L^{\otimes k}$ (see [AM16]).

The relation between the overall prequantum data is quite transparent when $\sigma = 0$. Indeed, one has in that case $\omega_t = \omega_k = k\Re(\omega_{\mathbb{C}})$, which restricts to $k\omega$ on $\mathcal{M}$, where $\omega \in A^2(\mathcal{M},\mathbb{R})$ is the Atiyah–Bott symplectic form on the compact moduli space (see Rem. 4.7). In the same way, the symplectic potential $\alpha$ of $\omega_{\mathbb{C}}$ restrict to the standard primitive of $\omega$, and thus $\alpha_t = \alpha_k = k\Re(\alpha)$ also descends accordingly. In this case the prequantum connection $\nabla^{(t)} = \nabla^{(k)}$ restricts to $k\nabla$ on $L^{\otimes k} \longrightarrow \mathcal{M}$, where $\nabla$ is the prequantum connection of compact Chern–Simons theory, which has the correct curvature. Finally, the Hermitian metrics also match.

If one instead chooses $\sigma \neq 0$, then the situation gets more complicated. The symplectic structure now becomes

$$\omega_t = k\Re(\omega_{\mathbb{C}}) - \sigma\Im(\omega_{\mathbb{C}}),$$

but its restriction to $\mathcal{M}$ is still equal to $k\omega$, thereby loosing a lot of information. The same happens for $\alpha_t$ and $\nabla^{(t)}$.

Now we move on from prequantisation to quantisation, following § 2.2.2 and § 2.2.3. To do this, one can introduce $\mathcal{K}^0$-invariant polarisations on $(\mathfrak{A}^0, \omega_t)$, and then make them descend to the moduli space as done in this section for the prequantum data. The invariance for the action is quite restrictive, and forces e.g. the polarisations to be translation-invariant in the direction of the compact moduli space (i.e. when varying the unitary connection in a Hitchin pair). This will however pose no issues, as we will consider polarisation which are *tout-court* translation-invariant on the whole affine space. Such polarisation on a symplectic vector space are said to be linear, because they are defined by translating a linear subspace (see Ex. 2.10), and will descend to translation-invariant polarisations on $\mathfrak{M}$.

### 4.2.5 The Kähler Polarisations

We now introduce Kähler polarisations $P_\tau$ on $\mathfrak{M}$. These polarisations depend on the (marked) complex structure $\tau$ of the surface, thereby defining a map $I^{(t)}$ from $\mathbb{H}$ to the space of Kähler complex structures of the symplectic moduli space. To describe them in more detail, consider first the following general fact.

Let $(M, g, I, J, K)$ be a hyper-Kähler manifold, and define symplectic forms as follows, for $p \in M$, $v, w \in T_pM$:

$$\omega_I(v,w) \coloneqq g(v, Iw), \qquad \omega_J(v,w) \coloneqq g(v, Jw), \qquad \omega_K(v,w) \coloneqq g(v, Kw).$$

These are the Kähler forms in the respective complex structures.

**Proposition 4.6.** *The nondegenerate complex form $\omega_{\mathbb{C}} = \omega_I + i\omega_K$ is a $J$-holomorphic symplectic structure.*

This means that $\omega_{\mathbb{C}}$ is a $J$-holomorphic nondegenerate $(2,0)$-form.

*Proof.* One has, for all $p \in M$ and $v, w \in \big(T_{\mathbb{C}}M\big)_p$ (omitting the evaluation at $p$):

$$\begin{aligned} -\omega_{\mathbb{C}}\big(v, (\mathrm{Id} + iJ)w\big) &= g\big(Iv, (\mathrm{Id} + iJ)w\big) + ig\big(Kv, (\mathrm{Id} + iJ)w\big) = \\ &= g(Iv, w) + ig(Iv, Jw) + ig(Kv, w) - g(Kv, Jw) = \\ &= g(Iv, w) - g(JKv, w) + ig(Kv, w) + ig(JIv, w) = \\ &= g(Iv, w) - g(Iv, w) + ig(Kv, w) - ig(Kv, w) = 0. \end{aligned}$$

This means that $\omega_{\mathbb{C}}$ vanishes on $T_{\mathbb{C}}M \wedge T_{(0,1)}M$, because $\mathrm{Id} + iJ \colon T_{\mathbb{C}}M \longrightarrow T_{(0,1)}M$ is the natural projection on the antiholomorphic tangent bundle to $M$. Hence indeed $\omega_{\mathbb{C}}$ is of bidegree $(2,0)$ for the complex structure $J$. □

This is exactly the situation we face. Namely, consider the complex structures $I_\tau, J, K_\tau$ of § 4.2.2, and the symplectic form $\omega_{\mathbb{C}}$ of § 4.2.3. There we provided the formula

$$\omega_{\mathbb{C}} = \frac{1}{\pi}(dw_2 \wedge dw_1),$$

using the natural, $\tau$-independent complex coordinates introduced in § 4.2.1. These coordinates are by definition the ones which are holomorphic for the complex structure $J$ arising from the group $G$, and $\omega_{\mathbb{C}}$ is the $J$-holomorphic symplectic structure that appears in the proposition, and one has $\omega_{\mathbb{C}} = \omega_{I_\tau} + i\omega_{K_\tau}$.
This accidentally provides a different proof of the fact that the real part of $\omega_{\mathbb{C}}$ restricts to the Atiyah–Bott form $\omega \in A^2(\mathcal{M}, \mathbb{R})$ on the compact moduli space. Indeed, $\Re(\omega_{\mathbb{C}}) = \omega_{I_\tau}$ is the Kähler form for the complex structure $I_\tau$, and thus $\iota^*\Re(\omega_{\mathbb{C}})$ must be the Kähler form of $\iota^* I_\tau$. However, this complex structure is precisely the one arising on $\mathcal{M}$ from $\tau$, and its associated Kähler form is $\omega$.

The above formula also shows that

$$\omega_t = k\Re(\omega_{\mathbb{C}}) - \sigma\Im(\omega_{\mathbb{C}}) = k\omega_{I_\tau} - \sigma\omega_{K_\tau}.$$

This provides a natural way to pick a complex structure $I = I^{(t)}$ such that the triple $\left(\mathfrak{M}, \omega_t, I^{(t)}\right)$ is a Kähler manifold. Namely, if we write $t' := \frac{t}{|t|} \in U(1)$ as $t' = k' + i\sigma'$, with

$$k' := \frac{k}{\sqrt{k^2+\sigma^2}}, \qquad \sigma' := \frac{\sigma}{\sqrt{k^2+\sigma^2}},$$

then one can show that the complex structure we are looking for is

$$I_\tau^{(t)} := I_{k',0,-\sigma'} = k' I_\tau - \sigma' K_\tau,$$

using the same notation as in § 4.2.2. This is because the symplectic form $\omega_t$ is of type $(1,1)$ with respect to $I^{(t)}$, and $I^{(t)}$ is the only complex structure in the Kähler sphere for which this is true. To show it, consider that the definition of the hyper-Kähler metric implies that

$$g\Big(I_\tau(\cdot), \cdot\Big) = \omega_{I_\tau}, \qquad g\Big(J(\cdot), \cdot\Big) = \omega_J, \qquad g\Big(K_\tau(\cdot), \cdot\Big) = \omega_{K_\tau}.$$

Hence by bilinearity

$$\begin{aligned} g\Big(I_{abc}, \cdot\Big) &= g\Big(aI_\tau(\cdot) + bJ(\cdot) + cK_\tau(\cdot), \cdot\Big) = ag\Big(I_\tau(\cdot), \cdot\Big) + bg\Big(K_\tau(\cdot), \cdot\Big) + cg\Big(K_\tau(\cdot)\,\cdot\Big) = \\ &= a\omega_{I_\tau} + b\omega_{K_\tau} + c\omega_{K_\tau}, \end{aligned}$$

and indeed the choice $(a,b,c) = (k', 0, -\sigma')$ is the correct one. One can also directly check the compatibility of $\omega_t$ with $I_\tau^{(t)}$.

**Lemma 4.1.** *One has* $\omega_t\Big(I_\tau^{(t)}(\cdot), I_\tau^{(t)}(\cdot)\Big) = \omega_t(\cdot,\cdot)$.

*Proof.* A straightforward expansion yields the following, omitting evaluation dots:

$$\begin{aligned} &\omega_t(I_\tau^{(t)}, I_\tau^{(t)}) = \\ &= (k')^2\Big(k'\omega_{I_\tau} - \sigma'\Big(\omega_{K_\tau}(I_\tau, I_\tau) + \omega_{I_\tau}(I_\tau, K_\tau) + \omega_{I_\tau}(K_\tau, I_\tau)\Big)\Big) + \\ &+ (\sigma')^2\Big( - \sigma'\omega_{K_\tau} + k'\Big(\omega_{I_\tau}(K_\tau, K_\tau) + \omega_{K_\tau}(K_\tau, I_\tau) + \omega_{K_\tau}(I_\tau, K_\tau)\Big)\Big). \end{aligned}$$

Hence it is enough to show that

$$\omega_{K_\tau}(I_\tau, I_\tau) + \omega_{I_\tau}(I_\tau, K_\tau) + \omega_{I_\tau}(K_\tau, I_\tau) = \omega_{K_\tau},$$

since this is equivalent to proving the identity for the expression obtained by swapping $I_\tau$ and $K_\tau$, giving

$$\omega_t(I_\tau^{(t)}, I_\tau^{(t)}) = (k')^2(k'\omega_{I_\tau} - \sigma'\omega_{K_\tau}) + (\sigma')^2(-\sigma'\omega_{K_\tau} + k'\omega_{I_\tau}) = \Big((k')^2 + (\sigma')^2\Big)\omega_t = \omega_t.$$

Now, one has

$$\omega_{I_\tau}(K_\tau, I_\tau) = g(K_\tau, \cdot) = \omega_{K_\tau}(K_\tau, K_\tau) = \omega_{K_\tau},$$

and thus we are left to prove that $\omega_{K_\tau}(I_\tau, I_\tau) + \omega_{I_\tau}(I_\tau, K_\tau) = 0$, which is seen to be true as follows:

$$\omega_{K_\tau}(I_\tau, I_\tau) = -\omega_{K_\tau}(I_\tau, K_\tau^2 I_\tau) = -\omega_{I_\tau}(I_\tau, I_\tau K_\tau I_\tau) = -\omega_{I_\tau}(I_\tau, I_\tau J) = -\omega_{I_\tau}(I_\tau, K_\tau).$$

□

**Definition 4.6.** The Kähler polarisation $P_\tau \subseteq T_{\mathbb{C}}\mathfrak{M}$, associated to $\tau \in \mathbb{H}$, on the symplectic manifold $(\mathfrak{M}, \omega_t)$, is the polarisation defined by the Kähler structure $I^{(t)}$.

We can thus define the space of the Kähler quantisation of $\mathfrak{M}$ as the space of $I_\tau^{(t)}$-holomorphic sections of $L^{(t)}$:

$$\mathcal{H}_{\mathbb{C},\tau}^{(t)} := H^0\Big(\mathfrak{M}, L^{(t)}\Big).$$

This follows the general theory presented in § 2.2.3, particularly Ex. 2.11. Right now we do not care about the exact form of the inner product of the quantum phase-space, which is induced by a natural $L^2$-norm with exponential weight. We will nonetheless deal with this later on, when discussing the Bargmann transform (see § 4.6).

To make this more explicit, some elementary linear algebra shows the following.

**Proposition 4.7.** *The matrix $M_{I^{(t)}}$ of the complex structure $I^{(t)}$ in the $\tau$-independent frame $\{\partial_{u_1}, \partial_{v_1}, \partial_{u_2}, \partial_{v_2}\}$ reads*

$$M_{I^{(t)}} = \frac{1}{\tau_2}\begin{pmatrix} -k'\tau_1 & -\sigma'\tau_1 & k' & \sigma' \\ -\sigma'\tau_1 & k'\tau_1 & \sigma' & -k' \\ -k'|\tau|^2 & -\sigma'|\tau|^2 & k'\tau_1 & \sigma'\tau_1 \\ -\sigma'|\tau|^2 & k'|\tau|^2 & \sigma'\tau_1 & -k'\tau_1 \end{pmatrix}.$$

*The eigenspace of (the complexification of) $M_{I^{(t)}}$ for the eigenvalue $-i$ is the complex span of*

$$\frac{1}{|\tau|^2}\begin{pmatrix} i\tau_2\sigma' \\ \tau_1 - i\tau_2 k' \\ 0 \\ |\tau|^2 \end{pmatrix}, \qquad \frac{1}{|\tau|^2}\begin{pmatrix} \tau_1 + i\tau_2 k' \\ i\tau_2\sigma' \\ |\tau|^2 \\ 0 \end{pmatrix}.$$

This is obtained by reading from the formulae of § 4.2.2, and conjugating with respect to the linear change of coordinates from $\{\xi_1, \xi_2, \varphi_1, \varphi_2\}$ to $\{u_1, v_1, u_2, v_2\}$. In particular one has an explicit expression for $P_\tau$ as a —constant — vector subbundle of $T_{\mathbb{C}}\mathfrak{M}$.

**Remark 4.11.** As for the prequantum data, the relation between the Kähler polarisation on $\mathfrak{M}$ and $\mathcal{M}$ (for the same fixed $\tau$) is the most natural one only for $\sigma = 0$. In that case $t = k \in \mathbb{Z}_{\geq 0}$, and thus $k' = 1$. Thus $I^{(t)} = I^{(k)} = I_\tau$, and it was shown in 4.5 that $I_\tau$ restricts to a complex structure on the compact moduli space. Moreover, this is necessarily the Kähler structure for $k\omega \in A^2(\mathcal{M}, \mathbb{R})$, i.e. the complex structure defined by the opposite of the Hodge $*$-operator (see § 2.3.3).
If $\sigma \neq 0$, then things get mixed up by $\sigma K_\tau$. There is no reason why the restriction of $I^{(t)}$ should preserve $T_{\mathbb{C}}\mathcal{M}$, and indeed it does not.

As a final remark, notice that one could make the same construction at the higher level of the flat space $(\mathfrak{A}^0, \omega_t)$, and equip it with a $\omega_t$-compatible complex structure $I^{(t)}$. This complex structure is invariant for the $\mathcal{G}^0$-action, and thus defines a complex structure on the quotient $\mathfrak{M} \cong \mathfrak{A}^0$. Hence the Kähler polarisation we constructed actually comes from a linear Kähler polarisation on a 4-dimensional (real) symplectic vector space.

We now move on to discuss the real polarisations originally considered in [Wit91].

### 4.2.6 The real polarisations

The moduli space $\mathfrak{M}$ also carries a family of real polarisations $P_{\mathbb{R},\tau}$, which still depend on the complex structure of the Riemann surface. Namely, for any Riemann surface structure on $\Sigma$ one has a Hodge splitting

$$H^1_{\mathrm{dR}}(\Sigma, \mathfrak{g}) \cong H^{1,0}_{\mathrm{Dol}}(\Sigma, \mathfrak{g}) \oplus H^{0,1}_{\mathrm{Dol}}(\Sigma, \mathfrak{g}).$$

Since $H^1_{\mathrm{dR}}(\Sigma, \mathfrak{g})$ is identified with the complexified tangent space to $\mathfrak{M}$ at every point, this decomposition defines subbundles of $T_{\mathbb{C}}\mathfrak{M}$. We now use this to define a Lagrangian subbundle which is fixed by conjugation, in the following way.

If $z = x + \tau y$ is a holomorphic coordinate on $\Sigma \cong \mathbb{C}/\Lambda_\tau$, we can write a flat $G$-connection in the form $A_{\mathbb{C}} = A_1 dz + A_2 d\bar{z}$, where $A_i \in C^\infty(\Sigma, \mathfrak{g})$ for $i = 1, 2$, as we did at the end of § 4.2.1. The idea is now to let $P_{\mathbb{R},\tau}$ be the pointwise span of the tangent vectors in the direction of the complex coordinates $A_1$ and $\overline{A_1}$, defined up to gauge (a diagonal traceless matrix of size two is the same as a complex number).

Let us now make this more precise, ultimately describing the distribution $P_{\mathbb{R},\tau}$ as the pointwise complex span of suitable linear combinations of the vector fields associated to the $\tau$-independent coordinates $w_1, w_2$ of § 4.2.1. Recall that they correspond to the decomposition of an $\mathrm{SL}(2, \mathbb{C})$-connection with respect to the real coordinates $x, y$ on $\Sigma$. In what follows we identify a diagonal matrix $\mathrm{diag}(w_j, -w_j)$ with the number $w_j$, for $j = 1, 2$. We also write $w_1 = u_1 + iv_1$, $w_2 = u_2 + iv_2$ to separate real and imaginary part.

**Proposition 4.8.** *The real polarisation $P_{\mathbb{R},\tau}$ is the complex span of the following vector fields:*

$$\begin{cases} X &= \partial_{u_1} - i\partial_{v_1} + \tau\partial_{u_2} - i\tau\partial_{v_2} \\ \overline{X} &= \partial_{u_1} + i\partial_{v_1} + \overline{\tau}\partial_{u_2} + i\overline{\tau}\partial_{v_2} \end{cases}$$

*Proof.* We have already shown that $A_1 dz + A_2 d\overline{z} = w_1 dx + w_2 dy$ implies

$$\begin{cases} A_1 &= \frac{1}{\overline{\tau}-\tau}(\overline{\tau} w_1 - w_2) \\ A_2 &= \frac{1}{\overline{\tau}-\tau}(w_2 - \tau w_1) \end{cases}$$

Now, if one expresses this in terms of $u_1, v_1, u_2, v_2$, separates real and imaginary parts, and take differentials, one finds:

$$\begin{cases} dA_1 &= \frac{1}{2\tau_2}\Big[(\tau_2 du_1 - \tau_1 dv_1 + dv_2) + i(\tau_1 du_1 + \tau_2 dv_1 - du_2)\Big] \\ dA_2 &= \frac{1}{2\tau_2}\Big[(\tau_2 du_1 + \tau_1 dv_1 - dv_2) + i(-\tau_1 du_1 + \tau_2 dv_1 + du_2)\Big] \end{cases}$$

The natural way to compute the vector fields $\partial_{A_1}, \partial_{A_2}$ associated to our coordinates is to impose the duality equations $d(A_i)\partial_{A_j} = \delta_{ij}$, for $\{i,j\} = \{1,2\}$. Since each vector field is a complex linear combination of the four vector fields above, we can expect the solution to this linear system to exist and to be unique. Some elementary linear algebra finally provides the formulae

$$\begin{cases} \partial_{A_1} &= \frac{1}{2}\partial_{u_1} - \frac{i}{2}\partial_{v_1} + \frac{\tau}{2}\partial_{u_2} + \frac{-i\tau}{2}\partial_{v_2} \\ \partial_{A_2} &= \frac{1}{2}\partial_{u_1} - \frac{i}{2}\partial_{v_1} + \frac{\overline{\tau}}{2}\partial_{u_2} + \frac{-i\overline{\tau}}{2}\partial_{v_2} \end{cases}$$

The thesis follows. □

**Corollary 4.1.** *The real polarisation $P_{\mathbb{R},\tau}$ is transverse to the compact moduli space for all $\tau \in \mathbb{H}$.*

*Proof.* The compact moduli space sits inside $\mathfrak{M}$ as the torus parametrised by the real coordinates $v_1 = \xi_1, v_2 = \xi_2$ (up to integer translations, and the flip; see § 4.2.1). Hence $T_{\mathbb{C}}\mathcal{M} \subseteq T_{\mathbb{C}}\mathfrak{M}$ is the complex span of the vector fields $\partial_{v_1}, \partial_{v_2}$. If one completes this to the set $\{\partial_{v_1}, \partial_{v_2}, X, \overline{X}\}$, with $P_{\mathbb{R},\tau} = \mathrm{span}_{\mathbb{C}}\{X, \overline{X}\}$ as in the previous proposition 4.8, one must then show that this family is $\mathbb{C}$-linearly independent.

This is equivalent to showing that the square matrix of the components of these vectors in the frame $\{\partial_{u_1}, \partial_{v_1}, \partial_{u_2}, \partial_{v_2}\}$ is nonsingular everywhere on $\mathcal{M}$. Reading again from Prop. 4.8, this matrix equals

$$M = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & -i & i \\ 0 & 0 & \tau & \overline{\tau} \\ 0 & 1 & -i\tau & i\overline{\tau} \end{pmatrix},$$

and one computes $\det(M) = \tau - \overline{\tau} = -2i\tau_2 \neq 0$, for all $\tau \in \mathbb{H}$. □

This proposition is just a way to check the computations. It was expected that the tangent space to $\mathcal{M}$ is transverse to $P_{\mathbb{R},\tau}$ at any point, since the only real form of type $(1,0)$ is the zero form.

Finally, since $P_{\mathbb{R}}$ is a real polarisation, one should be able to find an integrable Lagrangian subbundle of $T\mathfrak{M}$ that complexifies to $P_{\mathbb{R}}$. One may in this case just take the complexification of the real pointwise span of the (constant) vector fields

$$X_1 \coloneqq \begin{pmatrix} 1 \\ 0 \\ \tau_1 \\ \tau_2 \end{pmatrix}, \qquad X_2 \coloneqq \begin{pmatrix} 0 \\ -1 \\ \tau_2 \\ -\tau_1 \end{pmatrix},$$

i.e. $X_1 = \frac{1}{2}\big(X + \overline{X}\big) = \Re(X)$ and $X_2 = \frac{1}{2i}\big(X - \overline{X}\big) = \Im(X)$. One can also explicitly check that $\omega_t(X_1, X_2) = 0$ as it should, using the fact that the matrix of $\omega_t$ in the given global coordinates reads $\frac{1}{\pi}\begin{pmatrix} 0 & 0 & k & -\sigma \\ 0 & 0 & -\sigma & -k \\ -k & \sigma & 0 & 0 \\ \sigma & k & 0 & 0 \end{pmatrix}$. Moreover, one could have started from this totally real description, i.e. considering the quantum space as that of smooth sections of $L^{(t)} \longrightarrow \mathfrak{M}$ which are covariantly constant along the ($\tau$-dependent) coordinates $\Re(A_1)$ and $\Im(A_1)$ on $\mathfrak{M}$.

**Remark 4.12.** Each Lagrangian leaf of the real polarisation meets $\mathcal{M}$ in exactly one point. This provides an explicit $\tau$-dependent projection $\pi_\tau \colon \mathfrak{M} \longrightarrow \mathcal{M}$, obtained by sending a point $p \in \mathfrak{M}$ to the only point at the intersection of $\mathcal{M}$ and the leaf for the polarisation $P_{\mathbb{R},\tau}$ passing through $p$. This realises once more $\mathfrak{M}$ as a vector bundle over $\mathcal{M}$.

Finally, one can make the same remark as for the Kähler polarisation: the real polarisation we constructed comes from a linear real polarisation on a 4-dimensional (real) symplectic vector space, i.e. from the choice of a distinguished Lagrangian subspace. We now move on to describe the action of the mapping class group of closed orientable surfaces of genus one on all the data introduced so far.

### 4.2.7 Mapping class group action

Let $\Gamma = \Gamma_1$ be the mapping class group of oriented closed surfaces of genus one, as in Def. 4.3.

**Proposition 4.9.** *There is a natural action of $\Gamma$ on $\mathfrak{M}$ induced by the action on the fundamental group of the surface.*

*Proof.* Consider the Betti viewpoint: $\mathfrak{M} = \mathcal{M}_B(\Sigma, G) = \mathrm{Hom}\big(\pi_1(\Sigma), G\big)\big/G$ in one algebraic structure. If $f \in \mathrm{Diff}_+(\Sigma)$ is a diffeomorphism of $\Sigma$ (that preserves orientation), then there is an induced isomorphism $f_* \colon \pi_1(\Sigma, p) \longrightarrow \pi_1\big(\Sigma, f(p)\big)$ for all $p \in \Sigma$, defined by $f_*([\gamma]_p) = [f \circ \gamma]_{f(p)}$, where $[\cdot]_q$ denotes the homotopy class of a loop at $q \in \Sigma$. As explained in § 2.3.1, there is an identification of the fundamental group based at two different points which is well defined up to inner automorphisms, which in turn are negligible

in $\mathcal{M}_B$ since they get turned into inner automorphisms of $G$ by any representation. In this sense, the class of $f_*$ in the group of outer automorphisms of $\pi_1(\Sigma)$ is well defined, and the $G$-conjugacy orbit of the representation $\rho \circ f_*$ only depends on this class. This defines a $\mathrm{Diff}_+(\Sigma)$-action on $\mathcal{M}_B$.

This description also shows that $\mathrm{Diff}_0(\Sigma) \trianglelefteq \mathrm{Diff}_+(\Sigma)$ acts trivially on the moduli space, since a continuous map $f\colon \Sigma \longrightarrow \Sigma$ which is isotopic to the identity induces the identity on the fundamental group. Hence the quotient $\Gamma = \mathrm{Diff}_+(\Sigma)\big/\mathrm{Diff}_0(\Sigma)$ acts on the moduli space. □

**Remark 4.13.** In the proof we identified a morphism $\Gamma \longrightarrow \mathrm{Out}\big(\pi_1(\Sigma)\big)$, where we set $\mathrm{Out}\big(\pi_1(\Sigma)\big) \coloneqq \mathrm{Aut}\big(\pi_1(\Sigma)\big)\big/\mathrm{Inn}\big(\pi_1(\Sigma)\big)$ (well defined without choosing base point). One can actually show that it is injective, and that the natural extension to the extended mapping class group is an isomorphism. This is the content of Dehn–Nielsen–Baer theorem, valid for closed surfaces of genus $g \geq 1$ (see [FM12], Thm. 8.1). In genus one, the isomorphism follows from the identifications $\mathrm{Out}\big(\pi_1(\Sigma)\big) = \mathrm{Aut}\big(\pi_1(\Sigma)\big) = \mathrm{Aut}(\mathbb{Z}^2) \cong \mathrm{GL}(2,\mathbb{Z})$.

In the de Rham viewpoint, the action of $\mathrm{Diff}_+(\Sigma)$ is defined by pulling back connections on the trivial principal $G$-bundle $P_{\mathbb{C}} = G \times \Sigma$ over $\Sigma$, using that the pull-back of the trivial bundle is still the trivial bundle (e.g. since it admits a global section). Let us now describe the compatibility with the gauge action of $\mathcal{G} \cong C^\infty(\Sigma, G)$.
First, notice that $\mathrm{Diff}_+(\Sigma)$ acts on $\mathcal{G}$ by composition: $f.g \coloneqq g \circ f\colon \Sigma \longrightarrow G$, for $g \in \mathcal{G}$, $f \in \mathrm{Diff}_+(\Sigma)$. If one denotes with $g.\nabla$ the gauge-action on a $G$-connection $\nabla$, then one has

$$f^*(g.\nabla) = (g \circ f).(f^*\nabla).$$[4]

This implies that the gauge-class of $f^*(g.\nabla)$ is the same as that of $f^*\nabla$, and thus the $\mathrm{Diff}_+(\Sigma)$-action is well defined on gauge-equivalence classes of connections. Since it moreover preservers flatness, it descends to an action on the de Rham moduli space.
The above proposition 4.9 then also implies that $\mathrm{Diff}_0(\Sigma)$ acts trivially, since the monodromy representation of a pulled-back connection changes according to the natural action of $\mathrm{Diff}_+(\Sigma)$ on $\mathrm{Hom}\big(\pi_1(\Sigma), G\big)$. This claim essentially follows from

$$\big(f^*\nabla_{\dot\gamma}\big)(f^*s) = f^*\Big(\nabla_{df(\dot\gamma)}s\Big),$$

where $s$ is a local section of the trivial bundle defined on an open set containing the loop $\gamma$, and $f \in \mathrm{Diff}_+(\Sigma)$. The vector field $df(\dot\gamma) = \frac{d}{dt}(f \circ \gamma)$ is the velocity of the loop $f \circ \gamma$, and thus the parallel transport of $f^*\nabla$ along the homotopy class $[\gamma]$ is given by the parallel transport of $\nabla$ along the homotopy class $[f \circ \gamma] = f_*([\gamma])$, which means that $\rho_{f^*\nabla} = \rho_\nabla \circ f_*$, as claimed.

It is known that the mapping class group action on $\mathfrak{M}$ preserves the complex symplectic form $\omega_{\mathbb{C}}$ of § 4.2.3, and thus also $\omega_t = \frac{t\omega_{\mathbb{C}} + \overline{t\omega_{\mathbb{C}}}}{2}$: see e.g. [Gol04], which describes

[4] Both are equal to the pull-back of $\nabla$ with respect to the bundle automorphism over $f\colon \Sigma \longrightarrow \Sigma$ defined by the pair $(g, f)$. We will momentarily describe the group of automorphisms as a semidirect product of the $\mathrm{Diff}_+(\Sigma)$ and $\mathcal{G}$.

exactly the case of the group $\mathrm{SL}(2,\mathbb{C})$. In brief, one can argue as follows.
As mentioned in § 2.3.3, the tangent space to the de Rham moduli space at the gauge-equivalence class of a $G$-connection $d_A$ on $P_{\mathbb{C}} = G \times \Sigma$ is the first cohomology group $H^1(\Sigma, d_A)$ of the complex of $\mathrm{Ad}(P_{\mathbb{C}})$-valued differential forms on $\Sigma$, with the flat connection $d_A$ as differential. The nondegenerate alternating pairing

$$\omega_{\mathbb{C}} \colon H^1(\Sigma, d_A) \wedge H^1(\Sigma, d_A) \longrightarrow \mathbb{C}$$

is then defined by composing:

1. The cohomology cup product $H^1(\Sigma, d_A) \times H^1(\Sigma, d_A) \longrightarrow H^2(\Sigma, d_A)$ twisted with the $\mathrm{Ad}_G$-invariant scalar product $\mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbb{C}$ fixed at the beginning. Notice that the cup product is induced by the wedge product on cohomology classes of $\mathfrak{g}$-valued differential forms. This results in a map

$$H^1(\Sigma, d_A) \times H^1(\Sigma, d_A) \longrightarrow H^2(\Sigma, \mathbb{C}).$$

2. The isomorphism $H^2(\Sigma, \mathbb{C}) \cong \mathbb{C}$ provided by the orientation of $\Sigma$.

Now one remarks that the cup product of two de Rham cocycles is the Poincaré dual of the intersection of their dual cycles, and that these intersections are preserved under diffeomorphisms of $\Sigma$. Hence $\Gamma$ preserves the symplectic pairing.
This implies that $\Gamma$ also preserves the symplectic potential $\alpha$, since

$$(f^*\alpha)_\nabla(B) = \alpha_{f^*\nabla}(f^*B) = \omega_{\mathbb{C}}(f^*A, f^*B) = \omega_{\mathbb{C}}(A, B) = \alpha_\nabla(B),$$

for $f \in \mathrm{Diff}_+(\Sigma)$, $\nabla = d - A \in \mathcal{A}_{\mathbb{C}}$, $B \in T_\nabla \mathcal{A}_{\mathbb{C}}$. Hence $\Gamma$ also fixes the $t$-deformation $\alpha_t$. Notice also that the pull-back $f^*L^{(t)}$ of the prequantum bundle is now isomorphic to $L^{(t)}$ as a smooth line bundle, since their Chern classes coincide: these line bundles carry connections with the same curvature $-i\omega_t$.

Now that we know that $\Gamma$ acts symplectically on $(\mathfrak{M}, \omega_t)$, there is the question of lifting the action to the Chern–Simons prequantum bundle $L^{(t)}$ at level $t \in \mathbb{C}^*$. The best is to define an action on $\widetilde{L}^{(t)} = \mathcal{A}_{\mathbb{C}} \times \mathbb{C}$, and show the compatibility with the lift of the $\mathcal{G}$-action to $\widetilde{L}^{(t)}$, i.e. with the multipliers of § 4.2.4.
Consider then the trivial linearisation of the $\mathrm{Diff}_+(\Sigma)$-action defined by

$$f.(\nabla, \eta) \coloneqq (f^*\nabla, \eta) \qquad \text{for} \qquad \nabla \in \mathcal{A}_{\mathbb{C}}, \eta \in \mathbb{C}.$$

A straightforward adaptation of Lem. 3.1 of [And+17] yields the following.

**Lemma 4.2.** *The lifts of the $\mathcal{G}$-action and the $\Gamma$-action combine to provide a lift of the action of $\mathrm{Aut}(P_{\mathbb{C}}) \cong \mathcal{G} \rtimes \mathrm{Diff}_+(\Sigma)$ from $\mathcal{A}_{\mathbb{C}}$ to $\mathcal{A}_{\mathbb{C}} \times \mathbb{C}$.*

The statement is understood as follows. The automorphism group $\mathrm{Aut}(P_{\mathbb{C}})$ consists of bundle automorphisms $P_{\mathbb{C}} \longrightarrow P_{\mathbb{C}}$ covering an arbitrary orientation-preserving diffeomorphism $f \colon \Sigma \longrightarrow \Sigma$. They are thus coded by pairs $(g, f) \in \mathcal{G} \times \mathrm{Diff}_+(\Sigma)$, where

$(g,f)(p,h) \coloneqq \Big(f(p), g(p)h\Big)$ for all $p \in \Sigma$, $h \in G$. This is however not a direct product, since the composition of two such pairs is evaluated as

$$(g_2,f_2)\circ(g_1,f_1)(p,h) = \Big(f_2\circ f_1(p), g_2(f_1(p))\cdot g_1(p)h\Big),$$

i.e. $(g_2,f_2)\circ(g_1,f_1) = \Big((g_2\circ f_1)\cdot g_1, f_2\circ f_1\Big)$. This means that one has an isomorphism $\mathrm{Aut}(P_{\mathbb{C}}) \cong \mathcal{G} \rtimes_\Phi \mathrm{Diff}_+(\Sigma)$, where $\Phi\colon \mathrm{Diff}_+(\Sigma) \longrightarrow \mathrm{Aut}(\mathcal{G})$ is the action by right composition considered earlier.

*Proof.* One has to prove that

$$f^*\Big(g.(\nabla,\eta)\Big) = (g\circ f).\Big(f^*(\nabla,\eta)\Big),$$

for all $\nabla \in \mathcal{A}_{\mathbb{C}}$, $\eta \in \mathbb{C}$, $f \in \mathrm{Diff}_+(\Sigma)$ and $g \in \mathcal{G}$.

The left-hand side reads

$$f^*\Big(g.(\nabla,\eta)\Big) = f^*\Big(g.\nabla, \Theta^{(t)}(\nabla,g)\eta\Big) = \Big(f^*(g.\nabla), \Theta^{(t)}(\nabla,g)\eta\Big).$$

The right-hand instead equals:

$$(g\circ f).\Big(f^*(\nabla,\eta)\Big) = \Big((g\circ f).(f^*\nabla), \Theta^{(t)}\Big(f^*\nabla, g\circ f\Big)\eta\Big),$$

where in the second passage we used $f^*(g.\nabla) = (g\circ f).(f^*\nabla)$. Hence one can conclude by showing that $\Theta^{(t)}(f^*\nabla, g\circ f) = \Theta^{(t)}(\nabla,g)$. This follows from the invariance of integrals on the oriented 3-fold $X \coloneqq \Sigma\times[0,1]$ under the orientation preserving diffeomorphisms $F \coloneqq f\times \mathrm{Id}$, which is used in the last passage of the following suite of identities:

$$S_{\mathrm{CS}}\Big(\widetilde{(g\circ f)}.(\widetilde{f^*\nabla})\Big) = S_{\mathrm{CS}}\Big((\widetilde{g}\circ F).(F^*\widetilde{\nabla})\Big) = S_{\mathrm{CS}}\Big(F^*(\widetilde{g}.\widetilde{\nabla})\Big) = S_{\mathrm{CS}}(\widetilde{g}.\widetilde{\nabla}).$$

□

This lemma says that the lift of $\mathrm{Diff}_+(\Sigma) \cong \mathrm{Aut}(P_{\mathbb{C}})\big/\mathcal{G}$ to $\widetilde{L}^{(t)}$ is compatible with the lifted gauge action, and thus one also gets an action on the prequantum line bundle $L^{(t)} \longrightarrow \mathfrak{M}$ over the moduli space. Finally, one can conclude that $\mathrm{Diff}_0(\Sigma)$ acts trivially, as follows. If $f \in \mathrm{Diff}_0(\Sigma)$, then $f^*(\nabla,\eta) = (f^*\nabla,\eta) = (g.\nabla,\eta)$ for some element $g \in \mathcal{G}$. One can then show that $\Theta^{(t)}(\nabla,g) = 1$ for such $g$, and thus the element $f^*(\nabla,\eta)$ is equal to $(\nabla,\eta)$ in the quotient line bundle (see Prop. 3.10 of [And+17], where this is proven in the more general case of a curve with a nonnegative number of punctures).
The punchline is that there is an action of $\Gamma$ via bundle automorphisms of $L^{(t)} \longrightarrow \mathfrak{M}$ that covers the natural symplectic action of Prop. 4.9. It follows from the definition of the lift that the action preserves the Hermitian metric $h^{(t)}$ of the prequantum line bundle, and since it preserves the symplectic potential then it also fixes the prequantum connection $\nabla^{(t)}$. Hence the lift of the mapping class group action fixes all the prequantum data.

The action on the line bundle induces a $\Gamma$-action on the space of its smooth sections by pull-back: $f.s([\nabla]) = s([f^*\nabla])$ for all $s \in C^\infty\Big(\mathfrak{M}, L^{(t)}\Big)$, $[\nabla] \in \mathfrak{M}$, and $f$ an orientation-preserving diffeomorphism. The action on sections reduces to pull-back because $f$ acts

trivially on the fibres of $L^{(t)}$. Moreover, since $\Gamma$ preserves the Hermitian metric on $L^{(t)}$ and the Liouville measure on $\mathfrak{M}$, then it also preserves the $L^2$-norm of $C^\infty(\mathfrak{M}, L^{(t)})$. Finally, if $\tau \in \mathbb{H}$ is a point in Teichmüller space, then this need not be fixed by $\Gamma$. Rather, if $\gamma \in \Gamma$ is a mapping class and $s$ a $P_\tau$-polarised (resp. $P_{\mathbb{R},\tau}$-polarised) section of $L^{(t)}$, where $P_\tau$ is the Kähler polarisation of § 4.2.5 (resp. the real polarisation of § 4.2.6), then $\gamma.s$ will be a $\gamma^* P_\tau$-polarised (resp. $\gamma^* P_{\mathbb{R},\tau}$-polarised) section. This follows from the identity

$$(\gamma^* \nabla^{(t)})_X (\gamma^* s) = \gamma^* \left( \nabla^{(t)}_{d\gamma(X)} s \right).$$

Indeed, if $s$ is a $P_\tau$-polarised (resp. $P_{\mathbb{R},\tau}$-polarised) section of $L^{(t)}$, and $X$ is a section of $\gamma^* P_\tau$ (resp. $\gamma^* P_{\mathbb{R},\tau}$), then $d\gamma(X)$ will be a section of $P_\tau$ (resp. $P_{\mathbb{R},\tau}$), and the right-hand side will vanish.
In the case of Kähler polarisations, this means that $I^{(t)}_{f^*\tau} = f^* I^{(t)}_\tau$ for all $\tau \in \mathbb{H}$ and all orientation-preserving diffeomorphism $f$ of $\Sigma$, where $I^{(t)}$ is the map from $\mathbb{H}$ to the space of $\omega_t$-compatible complex structures on $(\mathfrak{M}, \omega_t)$ defined in § 4.2.5. In general, one might say that the map from Teichmüller space to polarisations on the moduli space is $\Gamma$-equivariant.

Putting all together, there is an action of the mapping class group on the quantum vector bundles of polarised sections over Teichmüller space, lifting the action of $\Gamma$ on Teichmüller space. We will resume the discussion of this section in a proposition.

**Proposition 4.10.** *The natural symplectic $\Gamma$-action of Prop. 4.9 on the prequantisable symplectic manifold $(\mathfrak{M}, \omega_t)$ lifts to the an action of the prequantum line bundle $L^{(t)}$ at level $t$ that preserves the prequantum data $(L^{(t)}, \nabla^{(t)}, h^{(t)})$ of § 4.2.4. There is a natural $\Gamma$-action on the quantum bundles $\mathcal{H}_P, \mathcal{H}_{P_\mathbb{R}}$ of $L^2$-summable polarised sections which lifts the natural $\Gamma$-action on Teichmüller space.*

In particular, if one element $\tau \in \mathbb{H}$ is fixed by $\Gamma$ then one gets a genuine infinite-dimensional representation of the mapping class group on the Hilbert spaces $\mathcal{H}_{P_\tau}$ and $\mathcal{H}_{P_{\mathbb{R}_\tau}}$. Otherwise, some identification between $\mathcal{H}_{P_\tau}$ and $\mathcal{H}_{f^*P_\tau}$ (resp. $\mathcal{H}_{P_{\mathbb{R},\tau}}$ and $\mathcal{H}_{f^*P_{\mathbb{R},\tau}}$) is needed. This is one place where the Hitchin connection comes into play, and helps defining the so-called "quantum" representations of the mapping class group in the geometric quantisation setting (see e.g. [Mas03]).

We now move on and construct such a Hitchin connection in the Kähler-polarised case, complementing the Hitchin–Witten connection [Wit91; AG14] on the quantum bundle for real polarisations.

## 4.3 The Hitchin connection before the circle action

In the previous section we attached a symplectic phase-space $(\mathfrak{M}, \omega_t)$ to a genus one surface $\Sigma$, the group $G = \mathrm{SL}(2, \mathbb{C})$ and a level $t \in \mathbb{C}^*$. Moreover, we also constructed prequantum data $(L^{(t)}, \nabla^{(t)}, h^{(t)})$ at level $t$, and a family of Kähler polarisations $P_\tau$ parametrised by the upper-half plane $\mathbb{H} \subseteq \mathbb{C}$. Everything is ready to adapt the general theory presented in § 2.2.4, and construct a Hitchin connection for the Kähler quantisation at hand.

To this end, we need some more ingredients.

### 4.3.1 Variation of tensors

Let $V \in C^\infty(\mathbb{H}, T_{\mathbb{C}}\mathbb{H})$ be a complex vector field on Teichmüller space. One can differentiate any $\tau$-dependent tensor on $\mathfrak{M}$ along $V$, and this turns out to be relevant for the construction of the Hitchin connection. We will denote $V[T]$ the variation of a tensor field $T$ on $\mathbb{H}$ along $V$. The most interesting tensors to differentiate for us are the complex structure $I^{(t)}\colon \mathbb{H} \longrightarrow C^\infty\Big(\mathfrak{M}, T\mathfrak{M} \otimes T^*\mathfrak{M}\Big)$ of 4.2.5, and the hyper-Kähler metric $g = \omega_t\Big(\cdot, I^{(t)}(\cdot)\Big)$. Let us start from understanding what kind of tensor the derivative $V\Big[I^{(t)}\Big]$ is.

The identity $\Big(I^{(t)}\Big)^2 = -\operatorname{Id}$ for almost-complex structures implies

$$V\Big[I^{(t)}\Big]I^{(t)} + I^{(t)}V\Big[I^{(t)}\Big] = 0, \tag{4.7}$$

by the Leibnitz rule. This means that the tensor $V\Big[I^{(t)}\Big] \in C^\infty\Big(\mathfrak{M}, T_{\mathbb{C}}\mathfrak{M} \otimes T^*\mathfrak{M}\Big)$ corresponds to an endomorphism of the complexified tangent bundle that exchange types. Indeed, if $p \in \mathfrak{M}$, and $v \in T_{1,0}\mathfrak{M}$, then:

$$-I^{(t)}V\Big[I^{(t)}\Big](v) = V\Big[I^{(t)}\Big]I^{(t)}(v) = iV\Big[I^{(t)}\Big](v).$$

This implies that $V\Big[I^{(t)}\Big]$ lives in the eigenbundle of $I^{(t)}$ of eigenvalue $-i$, i.e. it that it is of type $(0,1)$. The same verification shows that $V\Big[I^{(t)}\Big]\Big(T_{0,1}\mathfrak{M}\Big) \subseteq T_{1,0}\mathfrak{M}$.
This can be written as

$$V\Big[I^{(t)}\Big] \in C^\infty(\mathfrak{M}, T_{1,0} \otimes T^{0,1}) \oplus C^\infty(\mathfrak{M}, T_{0,1} \otimes T^{1,0}),$$

where one abusively omits $\mathfrak{M}$ from the notation of tangent and cotangent bundles. This decomposition defines a splitting

$$V\Big[I^{(t)}\Big] = V\Big[I^{(t)}\Big]' + V\Big[I^{(t)}\Big]'',$$

where the former addend is an endomorphism taking values in the holomorphic tangent bundle, and the latter in the antiholomorphic one.

Now we define a tensor $\widetilde{G}(V)$ on $\mathfrak{M}$ by means of the identity

$$\widetilde{G}(V).\omega_t = V\Big[I^{(t)}\Big],$$

where the dot means a contraction. By looking at types, one concludes necessarily that $\widetilde{G}(V) \in C^\infty(\mathfrak{M}, T_{\mathbb{C}} \otimes T_{\mathbb{C}})$. More precisely, if one has a real set of coordinates $\{x_i\}_{1\leq i\leq 4}$, so that $\omega_t = \sum_{i,j} \omega_{ij} dx_i \wedge dx_j$ and $\widetilde{G}(V) = \sum_{i,j} G_{ij}\partial_{x_i} \otimes \partial x_j$, then:

$$\widetilde{G}(V).\omega_t = \sum_{i,j,k,l} G_{ij}\omega_{kl}(\partial_{x_i} \otimes \partial_{x_j}) \cdot (d_k \wedge dx_l) = \sum_{i,j,l} G_{ij}\omega_{jl}\partial_{x_i} \otimes dx_l = \sum_{i,l} (G\omega)_{il}\partial_{x_i} \otimes dx_l.$$

This is indeed a tensor of type $(1,1)$, whose coefficient matrix is the product of that of $\widetilde{G}(V)$ on the left of that of $\omega_t$. Moreover, one can now use the relation $g\Big(I^{(t)}(\cdot),\cdot\Big)=\omega_t(\cdot,\cdot)$, which is equivalent to $g(\cdot,\cdot)=\omega_t\Big(\cdot,I^{(t)}(\cdot)\Big)$, to see that the variation of the hyper-Kähler metric is

$$V[g]=\omega_t.V\Big[I^{(t)}\Big]=g.\widetilde{G}(V).g,$$

because $\omega_t$ is $\tau$-independent. Hence in this context the variation of the metric and of the complex structure are related one to another via the symplectic form, and this incidentally shows that the tensor $\widetilde{G}(V)$ is symmetric.

Finally, there is a decomposition $\widetilde{G}(V)=G(V)+\overline{G}(V)$, due to the type-swapping of $V\Big[I^{(t)}\Big]$, where

$$G(V)\in C^\infty(\mathfrak{M},T_{1,0}\otimes T_{1,0}),\qquad \overline{G}(V)\in C^\infty(\mathfrak{M},T_{0,1}\otimes T_{0,1}).$$

Said differently, the tensor $\widetilde{G}(V)$ has vanishing $(1,1)$-component, and its decomposition is defined by the identity

$$G(V).\omega_t=V\Big[I^{(t)}\Big]',\qquad \overline{G}(V).\omega_t=V\Big[I^{(t)}\Big]''.$$

A crucial property for the construction of the Hitchin connection is the following.

**Definition 4.7.** The family of complex structures $\Big\{I_\tau^{(t)}\Big\}_{\tau\in\mathbb{H}}$ is rigid if $G(V)$ is a holomorphic section of $T_{1,0}\otimes T_{1,0}$ for all complex vector field $V$ on $\mathbb{H}$.

If one takes a $I^{(t)}$-holomorphic frame $\{\partial_{z_1},\partial_{z_2}\}$ for $T_{1,0}$, then $G(V)$ is identified with a function $G(V)\colon\mathfrak{M}\longrightarrow M_2(\mathbb{C})$, where $M_n(\mathbb{C})$ is the space of matrices of order $n$ with complex coefficients. Then the condition is that $\overline{\partial}G(V)=0$ for this function, which is usually a strong assumption. In our case, however, it is straightforward to see that it is verified: since $\omega_t$ and $I^{(t)}$ are translation-invariant, the same is true for $G(V)$, and a constant section is holomorphic.

### 4.3.2 Construction of the connection

Now consider the trivial prequantum bundle $C^\infty\Big(\mathfrak{M},L^{(t)}\Big)\times\mathbb{H}\longrightarrow\mathbb{H}$, as in § 2.2.4. We pick in there a connection of the form $\widehat{\nabla}=\nabla^T-u$, where $\nabla^T$ is the trivial connection, and $u\in A^1\Big(\mathbb{C},\mathcal{D}(\mathfrak{M},L^{(t)})\Big)$ a 1-form on Teichmüller space taking values in differential operators acting on smooth sections of $L^{(t)}$. Such a connection acts on sections of the trivial bundle by means of:

$$\widehat{\nabla}_V s=V[s]-u(V).s,$$

where $V$ is a complex vector space on Teichmüller, $s=s(\tau)$ is a $\tau$-dependent smooth section of $L^{(t)}$, and $u(V).s$ denotes the action of the operator on the smooth section.

Next, let $\mathcal{H}_{\mathbb{C}}^{(t)}$ be the quantum bundle fitting together the spaces $\mathcal{H}_{\mathbb{C},\tau}^{(t)}:=H_\tau^0(\mathfrak{M},L^{(t)})$ be the quantum bundle at level $t$, fitting together the spaces of the Kähler quantisation

of the moduli space with respect to the polarisation of § 4.2.5. The first objective is to find an operator $u$ such that $\mathcal{H}^{(t)}_{\mathbb{C}}$ is preserved. This means by definition that

$$\left(\nabla^{(t)}\right)^{0,1}\widehat{\nabla}s = 0, \qquad \text{if} \qquad s\colon \mathbb{H} \longrightarrow \mathcal{H}^{(t)}_{\mathbb{C}}.$$

To check this, one has the following.

**Lemma 4.3.** *A connection of the form $\widehat{\nabla} = \nabla^T - u$ preserves holomorphicity if and only if*

$$\frac{i}{2}V\left[I^{(t)}\right]\nabla^{(t)}s + \left(\nabla^{(t)}\right)^{0,1}u(V).s = 0,$$

*for all vector fields $V$ on $\mathbb{H}$ and all section $s$ taking values inside $\mathcal{H}^{(t)}_{\mathbb{C}}$.*

*Proof.* Let $V$ be a vector field of Teichmüller, and $s = s(\tau)$ a section which is everywhere holomorphic. One computes

$$\left(\nabla^{(t)}\right)^{0,1}\widehat{\nabla}^{(t)}s = \left(\nabla^{(t)}\right)^{0,1}V[s] - \left(\nabla^{(t)}\right)^{0,1}u(V).s.$$

Now, the first addend can be rewritten, differentiating the identity $\left(\nabla^{(t)}\right)^{0,1}s = 0$ along $V$. To do this recall that according to (2.12) one has

$$\left(\nabla^{(t)}\right)^{0,1} = \frac{1}{2}\left(\mathrm{Id} + iI^{(t)}\right)\nabla^{(t)}.$$

Hence:

$$0 = V\left[\left(\nabla^{(t)}\right)^{0,1}\widehat{\nabla}s\right] = \frac{1}{2}V\left[\nabla^{(t)}s\right] + \frac{i}{2}V\left[I^{(t)}\nabla^{(t)}s\right] =$$

$$= \frac{1}{2}\nabla^{(t)}V[s] + \frac{i}{2}V\left[I^{(t)}\right]\nabla^{(t)}s + \frac{i}{2}I^{(t)}\nabla^{(t)}V[s] = \left(\nabla^{(t)}\right)^{0,1}V[s] + \frac{i}{2}V\left[I^{(t)}\right]\nabla^{(t)}s.$$

If one substitutes the new identity $\left(\nabla^{(t)}\right)^{0,1}V[s] = -\frac{i}{2}V\left[I^{(t)}\right]\nabla^{(t)}s$ in the first equation, one sees that $\left(\nabla^{(t)}\right)^{0,1}\widehat{\nabla}s = 0$ is equivalent to

$$-\frac{i}{2}V\left[I^{(t)}\right]\nabla^{(t)}s - \left(\nabla^{(t)}\right)^{0,1}u(V).s = 0,$$

which is exactly the condition in the statement. □

Notice that $\nabla^{(t)}s = \left(\nabla^{(t)}\right)^{1,0}s$, since $s$ is everywhere holomorphic. Moreover, one has $V\left[I^{(t)}\right]\left(\nabla^{(t)}\right)^{1,0} = V\left[I^{(t)}\right]'\left(\nabla^{(t)}\right)^{1,0}$, by looking at types. Hence one may write the condition for preserving holomorphicity as

$$\left(\nabla^{(t)}\right)^{0,1}u(V).s = -\frac{i}{2}V\left[I^{(t)}\right]'\left(\nabla^{(t)}\right)^{1,0}s \tag{4.8}$$

We now consider the Ansatz for the operator $u$ of [And12], which is used for the Kähler quantisation of the compact theory. The prescription is to produce a differential operator of order two out of:

- The tensor $G(V)$ of the previous section § 4.3.1, encoding the holomorphic part of the variation of the complex structure.
- The prequantum connection $\nabla^{(t)}$.
- The Levi–Civita connection $\nabla^{\mathrm{LC}}$ of the Riemannian manifold $(\mathfrak{M}, g)$.
- A family of *Ricci potential* $F\colon \mathbb{H} \longrightarrow C^\infty(\mathfrak{M}, \mathbb{C})$.

In general, one defines a family of Ricci potentials for the Kähler manifold $(\mathfrak{M}, \omega_t, I^{(t)})$ as follows. The Riemannian metric $g$ defines the *Riemann curvature tensor* $R_g = F_{\nabla^{\mathrm{LC}}}$ as the curvature of the Levi–Civita connection $\nabla^{\mathrm{LC}}$ for the metric $g$. Next, one extracts the *Ricci tensor* $\mathrm{Ric} \in C^\infty(\mathfrak{M}, T_{\mathbb{C}} \otimes T_{\mathbb{C}})$ from it, via a contraction, and defines the *Ricci form* $\rho := \mathrm{Ric}\left(I^{(t)}(\cdot), \cdot\right) \in A^2(\mathfrak{M}, \mathbb{C})$. This form is a representative of the first Chern class of the complexified tangent bundle, which is by definition the Chern class of the underlying symplectic manifold. If one assumes that $\lambda\omega_t$ is also such a representative for some integer $\lambda$, then difference $\lambda\omega_t - \rho$ is exact, and one calls $F \in C^\infty(\mathfrak{M}, \mathbb{C})$ a Ricci potential if

$$\rho = \lambda\omega_t + 2i\partial\overline{\partial}F.$$

Now, since the complex and the metric structures are $\tau$-dependent, then one would like to have a family of such potentials parametrised by $\mathbb{H}$. The operator $u$ depends explicitly on $F$.

In our flat context, however, one has $R_g = \mathrm{Ric} = \rho = 0$, because there is no curvature whatsoever: the tangent bundle to $\mathfrak{M}$ is trivial, since the moduli space is topologically the cotangent bundle to a flat torus (up to an immaterial flip, see § 4.2.1). The first Chern class of the moduli space is given by the curvature of $\nabla^{\mathrm{LC}}$, which is the trivial connection; hence the Ricci potential vanishes for all $\tau \in \mathbb{H}$.
The punchline is that one might try to use the symmetric tensors $G(V)$ and the $(1,0)$-part of the prequantum connection $\nabla^{(t)}$ to construct a Hitchin connection. The fact that this works is the content of the following two theorems. This fortunate situation does not hold when one takes into account the $\mathbb{C}^*$-action on Higgs fields (see § 4.4 and 4.5).

Consider then the variation $V\left[\Delta_g\right]$ of the usual Laplace–Beltrami Laplacian coming from the Riemannian structure. One can realise this differential operator of order two more explicitly. Namely, consider the composition

$$\Delta_{G(V)}\colon \mathrm{Tr}\left(\widetilde{\nabla}^{1,0} \circ G(V) \circ \left(\nabla^{(t)}\right)^{1,0}\right)\colon C^\infty(\mathfrak{M}, L^{(t)}) \longrightarrow C^\infty(\mathfrak{M}, L^{(t)}), \tag{4.9}$$

where:

- The $(1,0)$-part of the prequantum connection acts as
$$\left(\nabla^{(t)}\right)^{1,0}\colon C^\infty(\mathfrak{M}, L^{(t)}) \longrightarrow C^\infty(\mathfrak{M}, T^{1,0} \otimes L^{(t)}).$$
- The contraction with the tensor $G(V)$ changes the types of tensor, thereby providing a map
$$G(V)\colon C^\infty(\mathfrak{M}, T^{1,0} \otimes L^{(t)}) \longrightarrow C^\infty(\mathfrak{M}, T_{1,0} \otimes L^{(t)}).$$

- One sets $\widetilde{\nabla} := \nabla^{\mathrm{LC}} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{(t)}$, which is a connection in the bundle $T_{\mathbb{C}} \otimes L^{(t)}$ over the moduli space. Hence its $(1,0)$-part provides a map

$$\left(\nabla^{\mathrm{LC}}\right)^{1,0} \otimes \mathrm{Id} + \mathrm{Id} \otimes \left(\nabla^{(t)}\right)^{1,0} : C^\infty(\mathfrak{M}, T_{1,0} \otimes L^{(t)}) \longrightarrow C^\infty(\mathfrak{M}, T^{1,0} \otimes T_{1,0} \otimes L^{(t)}).$$

- Finally, the trace means the contraction of $T^{1,0}$ against $T_{1,0}$, and one lands again in the space of smooth sections of the level $t$ prequantum bundle.

This general definition simplifies significantly, because of the aforementioned fact that the Levi–Civita connection is trivial, and that $G(V)$ is constant. Namely, if one considers a $I^{(t)}$-holomorphic frame $\{z_1, z_2\}$ of $T_{1,0}$, then one finds:

$$\Delta_{G(V)} = \sum_{i,j=1}^{2} \nabla^{(t)}_{\partial_{z_i}} G_{ij}(V) \nabla^{(t)}_{\partial_{z_j}} = \sum_{i,j} G_{ij}(V) \nabla^{(t)}_{\partial_{z_i}} \nabla^{(t)}_{\partial_{z_j}}.$$

We now set $u(V) := c\Delta_{G(V)}$, where $c \in \mathbb{C}$ is a constant, and try to fix it so to define a Hitchin connection.

**Proposition 4.11.** *One has:*

$$\left(\nabla^{(t)}\right)^{0,1} \Delta_{G(V)} s = -2iV\left[I^{(t)}\right]\left(\nabla^{(t)}\right)^{1,0} s$$

*for all* $V \in C^\infty(\mathbb{H}, T_{\mathbb{C}}\mathbb{H})$*, and for all sections* $s \colon \mathbb{H} \longrightarrow \mathcal{H}^{(t)}_{\mathbb{C}}$*.*

*Proof.* The idea is to make $\left(\nabla^{(t)}\right)^{0,1}$ commute past the Laplacian, using the commutation relations

$$\left[\left(\nabla^{(t)}\right)^{0,1}_{\partial_{\overline{z_i}}}, \left(\nabla^{(t)}\right)^{1,0}_{\partial_{z_j}}\right] = F^{1,1}_{\nabla^{(t)}}(\partial_{\overline{z_i}}, \partial_{z_j}) = -i\omega_t(\partial_{\overline{z_i}}, \partial_{z_j}),$$

for $i, j \in \{1, 2\}$, since $\left[\partial_{\overline{z_i}}, \partial_{z_j}\right] = 0$. If one omits the superscript $(t)$ and the evaluation at the vector fields associated to the complex coordinates this reads $\left[\nabla^{0,1}, \nabla^{1,0}\right] = -i\omega_t$. Hence, if one abusively writes $\nabla^{1,0}$ also for the connection $\widetilde{\nabla}$ defined above, one finds:

$$\begin{aligned}
\nabla^{0,1}\Delta_{G(V)} s &= \nabla^{0,1} \operatorname{Tr}\left(\nabla^{1,0} G(V) \nabla^{1,0}\right) s = \operatorname{Tr}\left(\nabla^{0,1}\nabla^{1,0} G(V) \nabla^{1,0}\right) s = \\
&= -i \operatorname{Tr}\left(\omega_t G(V) \nabla^{1,0}\right) s + \operatorname{Tr}\left(\nabla^{1,0} G(V) \nabla^{0,1} \nabla^{1,0}\right) s = \\
&= -i \operatorname{Tr}\left(\left(\omega_t . G(V) \nabla^{1,0}\right) + \left(\nabla^{1,0} G(V) . \omega_t\right)\right) s + \operatorname{Tr}(\nabla^{1,0} G(V) \nabla^{1,0}) \nabla^{0,1} s = \\
&= -2i\omega_t . S\Big(G(V)\Big) \nabla^{1,0} s,
\end{aligned}$$

where $S\Big(G(V)\Big)$ denotes the symmetric part of the tensor $G(V)$ (cf. § 4.5.2), and where one uses the rigidity of the complex structure to conclude that $\nabla^{0,1} G(V) = 0$. Also, in the last passage $\nabla^{0,1} s = 0$ because $s$ is taken to be holomorphic. See Rem. 4.20 for a computation in coordinates.

Since $G(V)$ is symmetric, one concludes that

$$\nabla^{0,1}\Delta_{G(V)}s = -2i\omega_t.G(V)\nabla^{1,0}s = -2iV\left[I^{(t)}\right]'\nabla^{1,0}s,$$

since $\omega_t.G(V) = V\left[I^{(t)}\right]'$ by definition (see § 4.3.1). However, this is also equal to the whole of the differential operator $V\left[I^{(t)}\right]\nabla^{1,0}s$, because the $(1,0)$-part of the prequantum connection vanishes when contracted with the antiholomorphic component $V\left[I^{(t)}\right]''$ of the variation of the complex structure. □

Putting together this proposition with (4.8), we have proved the following.

**Theorem 4.1.** *The connection*

$$\widehat{\nabla}_V := \nabla^T_V - u(V), \qquad \textit{where} \qquad u(V) := \frac{1}{4}\Delta_{G(V)}, \tag{4.10}$$

*defined in the trivial prequantum bundle* $C^\infty\left(\mathfrak{M}, L^{(t)}\right)\times\mathbb{H} \longrightarrow \mathbb{H}$, *preserves the subspaces* $\mathcal{H}^{(t)}_{\mathbb{C},\tau}$ *of* $I^{(t)}$*-holomorphic sections.*

This in particularly proves that $\mathcal{H}^{(t)}_{\mathbb{C}}$ is actually a vector sub-bundle. The program sketched in § 2.2.4 is almost completed: we still have to show that the connection is (projectively) flat.

**Remark 4.14.** A corollary of the construction is that the connection (4.10) is preserved by the group of bundle automorphisms of $\mathcal{H}^{(t)}_{\mathbb{C}}$ defined by the mapping class group $\Gamma$, according to proposition 4.10. This is the same remark that precedes Lem. 6 of [And12], transferred to our non-compact context.
Indeed, the Laplacian $\Delta_{G(V)}$ is constructed out of the $\Gamma$-invariant prequantum connection $\nabla^{(t)}$ and the symmetric tensor $G(V)$. In turn, $G(V)$ corresponds to the holomorphic part $V[I]'$ of the variation of the complex structure under the isomorphism between $T_{1,0}$ and $T^{0,1}$ induced by the $\Gamma$-invariant symplectic pairing $\omega_t\colon T_{1,0}\wedge T_{0,1} \longrightarrow \mathbb{C}$. Since the variation of the complex structure varies naturally along the $\Gamma$-action, so does $G(V)$.

**Remark 4.15.** Before discussing flatness, let us discuss the uniqueness of our construction. To this end consider any Laplacian operator built out of the $(1,0)$-part of the prequantum connection, i.e.

$$\Delta_T = \sum_{i,j}\nabla^{(t)}_{\partial_{z_i}}T_{ij}\nabla^{(t)}_{\partial_{z_j}},$$

where $T\in C^\infty(\mathfrak{M}, T_{\mathbb{C}}\otimes T_{\mathbb{C}})$ is a twice contravariant (complex) tensor on $\mathfrak{M}$. The principal symbol of $\Delta_T$ is $T$, and we may ask for conditions on $T$ such that $\nabla^T - c\Delta_T$ preserves holomorphicity, where $c\in\mathbb{C}^*$ is a constant.

According to (4.8) and the further remarks we made, one needs $c\nabla^{0,1}\Delta_T = -\frac{i}{2}V[I]'\nabla^{1,0}$. Since the right-hand side is a differential operator of order one, we need in particular that $T$ is annihilated $\nabla^{0,1}$, i.e. that the tensor $T$ is holomorphic. Using this, one finds

$$\nabla^{0,1}\Delta_T = -2i\omega_t.S(T)\nabla^{1,0},$$

and thus we see that one needs

$$\omega_t.S(T) = \frac{1}{4c}V[I]'.$$

Hence the symmetric part of the symbol is fixed by the condition that holomorphicity is preserved.

One might of course add lower order terms to the Laplacian, and consider an operator of the form

$$u'(V) = c_2\Delta_{T(V)} + c_1\nabla^{1,0}_{X(V)} + c_0 f(V),$$

where $X(V)$ is a vector field on $\mathfrak{M}$ of type $(1,0)$, and $f(V)$ is a smooth function on $\mathfrak{M}$. Looking at (4.8) one still sees the necessary conditions $\nabla^{0,1}T = 0$ and $\omega_t.S(T) = \frac{1}{4c_2}V[I]'$, and then the zero-order term $\nabla^{0,1}\Big(c_1\nabla_{X(V)} + c_2 f(V)\Big)$ must simply vanish.

In § 4.5 we will indeed have to consider an order-one correction to the Laplacian operator, in the new context where the symplectic structure and the prequantum connection vary too.

### 4.3.3 Flatness

The aim of this section is to prove the following theorem.

**Theorem 4.2.** *The connection (4.10) is flat when restricted to $\mathcal{H}^{(t)}_{\mathbb{C}}$.*

To show this, one starts by computing the curvature $F_{\widehat{\nabla}} \in A^2\Big(\mathbb{H}, \mathcal{D}(\mathfrak{M}, L^{(t)})\Big)$, finding:

$$F_{\widehat{\nabla}}(V,W) = \frac{1}{4}\Big(W\Big[\Delta_{G(V)}\Big] - V\Big[\Delta_{G(W)}\Big]\Big) + \frac{1}{16}\Big[\Delta_{G(V)}, \Delta_{G(W)}\Big],$$

for vector fields $V, W$ on Teichmüller space such that $[V,W] = 0$. This is computed by expanding the commutator $\Big[\nabla^T_V - \frac{1}{4}\Delta_{G(V)}, \nabla^T_W - \frac{1}{4}\Delta_{G(W)}\Big]$.
Notice that the complex vector fields $\partial_\tau, \partial_{\overline{\tau}}$ associated to the complex coordinate $\tau$ yield a global frame for the complexified tangent bundle $T_{\mathbb{C}}\mathbb{H}$ of $\mathbb{H}$. In particular, the curvature must be of type $(1,1)$ for the standard complex structure on the upper-half complex plane, and it is enough to show that the differential operator $F_{\widehat{\nabla}}(\partial_\tau, \partial_{\overline{\tau}}) \in \mathcal{D}(\mathfrak{M}, L^{(t)})$ vanishes on holomorphic sections, whereas a priori it acts on them via a differential operator of order at most three. In particular, the hypothesis that $V$ and $W$ commute is not restrictive.

**Lemma 4.4.** *One has $W\Big[\Delta_{G(V)}\Big]s - V\Big[\Delta_{G(W)}\Big] = 0s$ for all commuting vector fields $V, W \in C^\infty(\mathbb{H}, T_{\mathbb{C}}\mathbb{H})$, and for all holomorphic section $s$.*

*Proof.* One has

$$V\Big[\Delta_{G(W)}\Big] = V\,\mathrm{Tr}\left(\nabla^{1,0}G(W)\nabla^{1,0}\right) =$$

$$= -\frac{i}{2}\Big(V[I]G(W)\nabla^{1,0} + \nabla^{1,0}G(W)V[I]\Big) + \mathrm{Tr}\left(\nabla^{1,0}V\Big[G(W)\Big]\nabla^{1,0}\right) =$$
$$= -iG(W)V[I]'.\nabla^{1,0} + \Delta_{V\Big[G(W)\Big]}.$$

In this expansion we used

$$V\Big[\nabla^{1,0}\Big] = V\Big[\pi^{1,0}\Big]\nabla = -\frac{i}{2}V[I]\nabla,$$

where $\pi^{1,0} = \frac{1}{2}\Big(\mathrm{Id} - iI\Big)$ is the projection on the holomorphic tangent bundle. Also, the contraction $V[I].\nabla^{1,0}$ is the same as $V[I]'.\nabla^{1,0}$, since $\nabla^{1,0}$ vanishes on the antiholomorphic tangent bundle to $\mathfrak{M}$.

If one now subtracts this term from $W\Big[\Delta_{G(V)}\Big]$, one is left with the following differential operator of order 2:

$$\begin{aligned} &W\Big[\Delta_{G(V)}\Big] - V\Big[\Delta_{G(W)}\Big] = \\ &\quad = i\Big(G(W)V[I]' - G(V)W[I]'\Big)\nabla^{1,0} + \Delta_{W\Big[G(V)\Big]} - \Delta_{V\Big[G(W)\Big]}. \end{aligned}$$

To see that these two operators vanish, one can just transform using the $\tau$-independent tensor $\omega$:

$$\begin{aligned} W\Big[G(V)\Big] - V\Big[G(W)\Big] = W\Big[\omega^{-1}.V[I]'\Big] - V\Big[\omega^{-1}.W[I]'\Big] = \omega^{-1}.\Big(WV[I] - VW[I]\Big)' = \\ = \omega^{-1}.\Big([V,W][I]\Big)' = 0, \end{aligned}$$

and

$$\begin{aligned} \omega.\Big(G(W)V[I]' - G(V)W[I]'\Big) = \omega.G(W)V[I]' - \omega.G(V)W[I]' = \\ = G(W)G(V) - G(V)G(W) = 0. \end{aligned}$$

□

**Remark 4.16.** A quicker proof is given by the characterisation of $\Delta_{G(V)}$ as the the variation $V\Big[\Delta_g\Big]$ of the Laplace–Beltrami operator for the (hyper-)Kähler metric $g$, acting on the space of holomorphic sections. One can show that

$$\Delta_{G(V)}s = V\Big[\Delta_g\Big]s,$$

if $s$ is a holomorphic section. Hence

$$W\Big[\Delta_{G(V)}\Big]s - V\Big[\Delta_{G(W)}\Big]s = WV\Big[\Delta_g\Big]s - VW\Big[\Delta_g\Big]s = [W,V]\Big[\Delta_g\Big]s = 0.$$

Next comes the commutator of the Laplacians.

**Proposition 4.12.** *One has*

$$\Big[\Delta_{G(V)}, \Delta_{G(W)}\Big] = 0,$$

*for all vector fields $V, W$ on $\mathbb{H}$.*

*Proof.* One has, writing abusively $\nabla^{1,0}$ for the composite connection $\widetilde{\nabla}^{1,0}$, and omitting the superscript $t$ for the prequantum connection:

$$\left[\Delta_{G(V)}, \Delta_{G(W)}\right] = \left[\mathrm{Tr}\left(\nabla^{1,0} G(V) \nabla^{1,0}\right), \mathrm{Tr}\left(\nabla^{1,0} G(W) \nabla^{1,0}\right)\right] = \\ = \mathrm{Tr}\left(\left[\nabla^{1,0} G(V) \nabla^{1,0}, \nabla^{1,0} G(W) \nabla^{1,0}\right]\right).$$

Now, ignoring the final contraction provided by picking traces, one can just develop the inner commutator using the Leibnitz rule. This provides nine addends, which vanish because of the relations $\left[\nabla^{1,0}, \nabla^{1,0}\right] = F_\nabla^{2,0} = (-i\omega_t)^{2,0} = 0$, and $\left[\nabla^{1,0}, G(V)\right] = \left[\nabla^{1,0}, G(W)\right] = 0$, due to the fact that the tensors $G(V), G(W)$ are translation-invariant. It is also clear that $\left[G(V), G(W)\right] = 0$, since these operators are just function multiplications. □

This concludes the proof of Thm. 4.2. We have constructed a flat Hitchin connection for the Kähler quantisation of the moduli space of polystable $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles over a Riemann surface of genus one, which is invariant under the natural action of the mapping class group.

**Remark 4.17.** To construct the Hitchin connection, to prove its flatness, and to deduce $\Gamma$-invariance, we did not use any particular feature of the group $G = \mathrm{SL}(2,\mathbb{C})$, apart from it being the complexification of a simple, compact, 1-connected Lie group. The same proof would work for any group satisfying these requirements.

Now one would like to explore a feature of the moduli space $\mathfrak{M}$ which is not present in the compact case: the dilation action on Higgs fields. This is an action of the group $\mathbb{C}^*$ on $\mathfrak{M}$, and one might look for a Hitchin connection which is compatible with it. The motivation for this has been discussed in the introduction, and we now move on to describe this extension.

## 4.4 The Hitchin action

We introduce here another characteristic data of the Dolbeault moduli space, the natural $\mathbb{C}^*$-action of homothety on Higgs fields.

### 4.4.1 Definition of the action

If $\lambda \in \mathbb{C}^*$ is a nonzero complex number, then one can multiply a Higgs field $\Phi$ by $\lambda$ to get another Higgs field, because such objects are linear. This clearly breaks down for connections.

The coordinate-free formula for the action is just

$$\lambda.(A, \Phi) \coloneqq (A, \lambda\Phi),$$

for $\lambda \in \mathbb{C}^*$ acting on a Hitchin pair $(A, \Phi)$. Recall from § 4.2.1 that we had natural coordinates $\xi_1, \xi_2, \varphi_1, \varphi_2$ on the moduli space given by the decomposition

$$A + \Phi + \Phi^* =$$

$$= \begin{pmatrix} i\xi_1 + 2\varphi_1 & 0 \\ 0 & -(i\xi_1 + 2\varphi_1) \end{pmatrix} dx + \begin{pmatrix} i\xi_2 + 2(\tau_1\varphi_1 - \tau_2\varphi_2) & 0 \\ 0 & -(i\xi_2 + 2(\tau_1\varphi_1 - \tau_2\varphi_2)) \end{pmatrix} dy.$$

Now, if we write $\lambda = \lambda_1 + i\lambda_2$, then in these coordinates the action is coded by the linear transformation

$$\lambda . \begin{pmatrix} \xi_1 \\ \xi_2 \\ \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & -\lambda_2 \\ 0 & 0 & \lambda_2 & \lambda_1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \lambda\varphi_1 - \lambda\varphi_2 \\ \lambda_1\varphi_2 + \lambda_2\varphi_1 \end{pmatrix}.$$

The problem is, as always, that those coordinates depend on the choice of the complex structure $\tau \in \mathcal{T}$ chosen to define Higgs fields over $\Sigma$. Since eventually we'll have to pick derivatives with respect to the coordinate on Teichmüller space, we prefer working in the $\tau$-independent coordinates $(u_1, v_1, u_2, v_2)$ introduced in § 4.2.1.

Recall that the transformation from the atlas $(\xi_1, \xi_2, \varphi_1, \varphi_2)$ to $(u_1, v_1, u_2, v_2)$ is linear, and coded by

$$\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2\tau_1 & 2\tau_2 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 2\varphi_1 \\ \xi_1 \\ 2\tau_1\varphi_1 - 2\tau_2\varphi_2 \\ \xi_2 \end{pmatrix}.$$

One can now compute the action in the new coordinates by applying this change of coordinates, *i.e.* by conjugating the matrix of the action by the matrix of the change of basis. One finds:

$$\lambda . \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \frac{1}{\tau_2} \begin{pmatrix} (\tau_2\lambda_1 - \tau_1\lambda_2)u_1 + \lambda_2 u_2 \\ \tau_2 v_1 \\ -|\tau|^2\lambda_2 u_1 + (\tau_2\lambda_1 + \tau_1\lambda_2)u_2 \\ \tau_2 v_2 \end{pmatrix}.$$

The action fixes the coordinates $v_1$ and $v_2$. One has $v_1 = \xi_1$, $v_2 = \xi_2$, and these real coordinates parametrise the unitary connection $A$ in the Hitchin pairs $(A, \Phi)$. Since the $\mathbb{C}^*$-action is on the Higgs field only, this was expected. Notice also that the action depends on $\tau$, as it should, since Higgs field do so.

There is now a 1-dimensional family $\{\lambda \colon \mathfrak{M} \longrightarrow \mathfrak{M} \mid \lambda \in \mathbb{C}^*\}$ of diffeomorphisms of the moduli space. The next aim is to compute the pull-back of various tensor fields with respect to a diffeomorphism $\lambda$. To do this, one must compute the tangent map

$$d\lambda_p \colon T_p\mathfrak{M} \longrightarrow T_{\lambda . p}\mathfrak{M},$$

at every point $p \in \mathfrak{M}$. On the other hand, since $\lambda$ is linear in the coordinates chosen on $\mathfrak{M}$, that linear map admits the same formula as above, computed on the frame $\{\partial_{u_1}, \partial_{v_1}, \partial_{u_2}, \partial_{v_2}\}$.

### 4.4.2 Pull-back of the symplectic structure

One might hope that the dilation action preserves the symplectic form. This is far from true.

**Proposition 4.13.** *Pick $\lambda \in \mathbb{C}^*$. One has, in local, $\tau$-invariant coordinates:*

$$\begin{aligned}\lambda^*\omega_t = -\frac{1}{\pi}\Bigg[ &\left(\frac{\sigma\lambda_2}{\tau_2}\right) du_1 \wedge v_1 + \left(\frac{k(\tau_2^2\lambda_1^2 - \tau_1^2\lambda_2^2 + |\tau|^2)}{\tau_2^2}\right) du_1 \wedge du_2 + \\ &+ \left(\frac{-\sigma(\tau_2\lambda_1 - \tau_1\lambda_2)}{\tau_2}\right) du_1 \wedge dv_2 + \left(\frac{-\sigma(\tau_2\lambda_1 + \tau_1\lambda_2)}{\tau_2}\right) dv_1 \wedge du_2 - \\ &- k dv_1 \wedge dv_2 + \left(\frac{\sigma|\tau|^2\lambda_2}{\tau_2}\right) du_2 \wedge dv_2 \Bigg].\end{aligned} \tag{4.11}$$

The proof is a straightforward expansion of the formula

$$\omega_t = -\frac{1}{\pi}\Big(k(du_1 \wedge du_2 - dv_1 \wedge dv_2) - \sigma(du_1 \wedge dv_2 + dv_1 \wedge du_2)\Big).$$

The $\mathbb{C}^*$-action is thus not symplectic. This means that one cannot hope to keep the prequantum data of § 4.2.4 in order to take into account this action. Notice however that (4.11) implies that the restriction of $\lambda^*\omega_t$ on $\mathcal{M}$ equals $k\omega$, where $\omega$ is the Atiyah–Bott symplectic form on the compact moduli space. Indeed, setting $du_1 = du_2 = 0$ yields

$$\iota^*\big(\lambda^*\omega_t\big) = \frac{1}{\pi}k(dv_1 \wedge dv_2) = k\omega.$$

One can also provide a formula for the pull-back $\lambda^*\alpha_t$ of the symplectic potential $\alpha_t$. The proof would be analogous to that of Prop. 4.13. Regardless of the exact expression, one verifies that $\lambda^*\alpha$ varies linearly on the moduli space, and that it is a polynomial of degree two in $\lambda_1, \lambda_2, \tau_1, \tau_2$ and $\tau_2^{-1}$. Moreover, one can also show that $\lambda^*\alpha_t$ restricts to the canonical symplectic potential of $k\omega$ on $\mathcal{A}^0 \subseteq \mathfrak{A}^0$.

### 4.4.3 Pull-back of complex structures

We now consider the sphere of complex structures

$$I_{abc} = aI_\tau + bJ + cK_\tau,$$

with $a^2 + b^2 + c^2 = 1$ in $\mathbb{R}$, as introduced in § 4.2.2. The Kähler structure $I^{(t)}$ that defines the Kähler polarisation, defined in § 4.2.5, lives in this sphere.

Now, the pull-back of a complex structure $I_{abc}$ with respect to a diffeomorphism $\lambda$ given by the $\mathbb{C}^*$-action is defined by a conjugation:

$$(\lambda^* I_{abc})_p(v) = d\lambda^{-1}.(I_{abc})_{\lambda.p}(d\lambda.v),$$

for $p \in \mathcal{M}$ and $v \in T_p\mathfrak{M}$ (omitting the evaluation of $d\lambda$ at $p$ and $d\lambda^{-1}$ at $\lambda.p$). This definition is coherent with that of the pull-back of the associated Kähler polarisation.

The result we are after is the following.

**Proposition 4.14.** *The pull-back of the $U(1)$-action obtained by restricting the $\mathbb{C}^*$-action on the hyperkähler manifold $(\mathfrak{M}, I_\tau, J, K_\tau, g)$ moves the complex structures $I_{abc}$ along the parallels of the Kähler sphere. More precisely one has $\lambda^* I_{abc} = I_{a,b',c'}$, where $\lambda \in U(1)$, $(a,b,c) \in \mathbb{S}^2 \subseteq \mathbb{R}^3$, and*

$$b' \coloneqq \lambda_1 b + \lambda_2 c, \qquad c' \coloneqq \lambda_1 c - \lambda_2 b.$$

*Proof.* To make explicit computations, one must write the formulae for the complex structures $I_\tau, J, K_\tau$ in the $\tau$-independent complex frame $\{\partial_{u_1}, \partial_{v_1}, \partial_{u_2}, \partial_{v_2}\}$. To this end, the formulae of § 4.2.2 imply that the matrices of the complex structures in the old frame $\{\partial_{\xi_1}, \partial_{\xi_2}, \partial_{\varphi_1}, \partial_{\varphi_2}\}$ are

$$M_{I_\tau} = \frac{1}{\tau_2}\begin{pmatrix} \tau_1 & -1 & 0 & 0 \\ |\tau|^2 & -\tau_1 & 0 & 0 \\ 0 & 0 & 0 & -\tau_2 \\ 0 & 0 & \tau_2 & 0 \end{pmatrix},$$

$$M_J = \frac{1}{\tau_2}\begin{pmatrix} 0 & 0 & 2\tau_2 & 0 \\ 0 & 0 & 2\tau_1\tau_2 & -2\tau_2^2 \\ -\frac{\tau_2}{2} & 0 & 0 & 0 \\ -\frac{\tau_1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix},$$

$$M_{K_\tau} = \frac{1}{\tau_2}\begin{pmatrix} 0 & 0 & 0 & 2\tau_2 \\ 0 & 0 & 2\tau_2^2 & 2\tau_1\tau_2 \\ \tau_1 & -\frac{1}{2} & 0 & 0 \\ -\frac{\tau_2}{2} & 0 & 0 & 0 \end{pmatrix}.$$

One can check that $M_{I_\tau} M_J = M_{K_\tau}$. Moreover, the complex variable $\varphi = \varphi_1 + i\varphi_2$ is a holomorphic variable for the complex structure $I_\tau$, as it should, since $I_\tau$ is the complex structure on $\mathfrak{M}$ that corresponds to the complex algebraic Dolbeault moduli space of Higgs fields.

Now, in order to compute the matrices in the new frame above, we have to conjugate

$M_{I_\tau}$, $M_J$ and $M_{K_\tau}$ with respect to the matrix of $P$ that changes basis. One finds:

$$PM_{I_\tau}P^{-1} = \frac{1}{\tau_2}\begin{pmatrix} -\tau_1 & 0 & 1 & 0 \\ 0 & \tau_1 & 0 & -1 \\ -|\tau|^2 & 0 & \tau_1 & 0 \\ 0 & |\tau|^2 & 0 & -\tau_1 \end{pmatrix}$$

$$PM_JP^{-1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$PM_{K_\tau}P^{-1} = \frac{1}{\tau_2}\begin{pmatrix} 0 & \tau_1 & 0 & -1 \\ \tau_1 & 0 & -1 & 0 \\ 0 & |\tau|^2 & 0 & -\tau_1 \\ |\tau|^2 & 0 & -\tau_1 & 0 \end{pmatrix}$$

Notice that the complex structure $J$ is the canonical one in the $\tau$-independent frame because indeed it was defined by declaring the coordinates $u_1 + iv_1, u_2 + iv_2$ to be holomorphic (we are writing the complex structure of the group $G$ in canonical form).
This provides the following matrix for the complex structures $I_{abc}$, in the $\tau$-independent frame:

$$M_{I_{abc}} = aM_{I_\tau} + bM_J + cM_{K_\tau} = \frac{1}{\tau_2}\begin{pmatrix} -a\tau_1 & -b\tau_2 + c\tau_1 & a & -c \\ b\tau_2 + c\tau_1 & a\tau_1 & -c & -a \\ -a|\tau|^2 & c|\tau|^2 & a\tau_1 & -b\tau_2 - c\tau_1 \\ c|\tau|^2 & a|\tau|^2 & b\tau_2 - c\tau_1 & -a\tau_1 \end{pmatrix}.$$

Now we can finally compute the pull-back with respect to the linear transformation

$$\lambda = d\lambda = \frac{1}{\tau_2}\begin{pmatrix} \tau_2\lambda_1 - \tau_1\lambda_2 & 0 & \lambda_2 & 0 \\ 0 & \tau_2 & 0 & 0 \\ -|\tau|^2\lambda_2 & 0 & \tau_2\lambda_1 + \tau_1\lambda_2 & 0 \\ 0 & 0 & 0 & \tau_2 \end{pmatrix},$$

since it is already expressed in the correct set of coordinates.

The matrix obtained for $|\lambda| = 1$ is seen to be equal to $M_{I_{a,b',c'}}$, if one sets $b' \coloneqq \lambda_1 b + \lambda_2 c$, $c' \coloneqq \lambda_1 c - \lambda_2 b$. Moreover, notice that $(a, b', c') \in \mathbb{R}^3$ is still a point in the unit sphere, since $b' + ic' = \overline{\lambda}(b + ic)$, which amounts to a rotation within the sphere.

Finally, this transformation is bijective, since one immediately checks that

$$\overline{\lambda}^*\lambda^* I_{abc} = (\lambda^{-1})^*\lambda^* I_{abc} = I_{abc},$$

for $|\lambda| = 1$. □

We see in particular that the complex structures $\pm I_\tau$ are preserved.

**Remark 4.18.** In the proof we remarked that $(b', c') = \overline{\lambda}(b, c)$, if one identifies a pair $(b, c) \in \mathbb{R}^2$ with $b + ic \in \mathbb{C}$. Because of $a^2 + b^2 + c^2 = 1$, one has necessarily $b + ic \in D$, if

$D$ is the closed unit disk of $\mathbb{C}$ (the boundary $\partial D$ corresponds to the equator of $\mathbb{S}^2$). Hence the circle action can be geometrically described as follows.
If $\lambda = e^{i\theta}$, and $(a, b, c) \in \mathbb{S}^2$, then $\lambda^*(a, b, c)$ is obtained by rotating the pair $(b, c) \in D$ of $-\theta$, in the parallel at height $a \in [-1, 1]$.

This action is free on the complement of the two poles, which is homeomorphic to $\mathbb{C}^*$, using the standard stereographic projections. One can also show that the full $\mathbb{C}^*$-action is essentially the same as the restricted circle action, up to stretching some coordinates. To see this, pick $\lambda \in \mathbb{C}^*$, and consider the linear diffeomorphism $\psi$ of $\mathfrak{M}$ given in $\tau$-independent coordinates by the diagonal matrix

$$M_\psi \coloneqq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & |\lambda| & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & |\lambda| \end{pmatrix}.$$

It amounts to dilating the natural coordinates $\xi_1, \xi_2$ on $\mathfrak{M}$. This correction is clearly trivial for $|\lambda| = 1$.

One can then prove the following.

**Proposition 4.15.** *Choose $\lambda \in \mathbb{C}^*$. The pull-back $\psi^*(\lambda^* I_{abc})$ is equal to the complex structure $I_{a,b',c'}$, if one sets:*

$$b' \coloneqq \frac{\lambda_1 b + \lambda_2 c}{|\lambda|}, \quad c' \coloneqq \frac{\lambda_1 c - \lambda_2 b}{|\lambda|}.$$

*Proof.* It is enough to compute the conjugation $(M_\psi)^{-1} \lambda^* M_{I_{abc}} M_\psi$ and see that one gets the matrix $M_{I_{a,b',c'}}$ with the above definition. □

Hence, on the whole, the $\mathbb{C}^*$-action is free on the two-holed sphere

$$\{I_{abc} \mid a^2 + b^2 + c^2 = 1\} \setminus \{I_{\pm 1,0,0}\} \cong \mathbb{C}^*,$$

up to correcting with a dilation. Moreover, the full action still fixes $I_{\pm 1,0,0}$. In particular, if one takes a quantum level $t = k + i\sigma$ with vanishing imaginary part, then the Kähler polarisation on $\mathfrak{M}$ is given by the complex structure $I^{(t)} = I^{(k)} = kI_\tau$, which is fixed by the circle action. This means that if one takes $\sigma = 0$ then there is a natural $U(1)$-action on the Hilbert space $\mathcal{H}^{(k)}_{P_\tau}$ of the Kähler quantisation of the Higgs bundle moduli space, for all $\tau \in \mathbb{H}$.
The fact that this is still true in higher genus is used in [AGP16] to compute a $U(1)$-equivariant version of the Verlinde formula, as explained in the introduction to this thesis. What we would like to do instead is to allow for a generic quantum level $t \in \mathbb{C}^*$, and try to construct canonical identifications between the spaces of holomorphic sections for the complex structure $I^{(t)}_\tau$ and $\lambda^* I^{(t)}_\tau$, where $\lambda \in U(1)$. A consistent way to do that is to let the bigger space $\widetilde{\mathbb{H}} \coloneqq \mathbb{H} \times U(1)$ parametrise the enlarged family of complex structures $\lambda^* I^{(t)}_\tau$, and construct a projectively flat connection inside the bundle of holomorphic sections over $\widetilde{\mathbb{H}}$.

The first clear obstacle is Prop. 4.13: the symplectic structure of the moduli space varies along the circle. In the next section we will explain how one may deal with that issue.

### 4.4.4 Equivariant prequantum data

The aim of this section is to construct prequantum data on the moduli space which transform well under the circle action.
To this end, we choose to fix the prequantum line bundle, and to let the prequantum connection vary, together with the symplectic form. This is necessary, since its connection form is a local symplectic potential, as implied by the prequantum condition. Moreover, it is best to work on the flat space $\mathfrak{A}^0$, and define prequantum data on the moduli space via $\mathcal{G}^0$-invariant prequantum data on $\mathfrak{A}^0$, as we did in § 4.2.4.

Let then $\widetilde{L}^{(t)} := \mathbb{C} \times \mathfrak{A}^0 \longrightarrow \mathfrak{A}^0$ be the trivial line bundle on the usual space of flat connections. If $\lambda \in U(1)$, then we get a new line bundle $\lambda^*\widetilde{L}^{(t)}$ over $\mathfrak{A}^0$, which is tautologically isomorphic to $\widetilde{L}^{(t)}$ over the diffeomorphism $\lambda\colon \mathfrak{A}^0 \longrightarrow \mathfrak{A}^0$. We then prescribe a new identification of the two line bundles over the identity, thereby providing a noncanonical trivialisation of the line bundle $\widetilde{\mathbb{L}}^{(t)} \longrightarrow U(1)$ over the circle, whose fibre at $\lambda \in U(1)$ is $\lambda^*\widetilde{L}^{(t)}$. In this way, one can always pick the tautological metric $h$ on $\widetilde{L}^{(t)}$.
To conclude with the prequantum triple, we let on the contrary the prequantum connection vary, thereby getting a circle of prequantum data $\left(L^{(t)}, \widetilde{\nabla}^{(t)}_\lambda, \widetilde{h}^{(t)}\right)_{\lambda\in U(1)}$, over the circle $(\mathfrak{M}, \lambda^*\omega_t)_{\lambda\in U(1)}$ of symplectic manifolds.

We thus have to fix, for each $\lambda \in U(1)$, an isomorphism

$$\Phi_\lambda\colon \lambda^*\widetilde{L}^{(t)} \longrightarrow \widetilde{L}^{(t)}$$

that closes the following square:

$$\begin{array}{ccc} & \overset{\Phi_\lambda}{\dashrightarrow} & \\ \lambda^*\widetilde{L}^{(t)} & & \widetilde{L}^{(t)} \\ \downarrow & & \downarrow \\ (\mathfrak{A}^0, \lambda^*\omega_t) & \xrightarrow[\mathrm{Id}]{} & (\mathfrak{A}^0, \omega_t) \end{array}$$

Notice that the fibre of $\lambda^*\widetilde{L}^{(t)}$ at $A_\mathbb{C} \in \mathfrak{A}^0$ is by definition the fibre of $\widetilde{L}^{(t)}$ at $\lambda.A_\mathbb{C}$. Hence an isomorphism is the data of a linear isomorphism

$$\Phi_\lambda(A)\colon \lambda^*\widetilde{L}^{(t)}_{A_\mathbb{C}} = \widetilde{L}^{(t)}_{\lambda.A_\mathbb{C}} \longrightarrow \widetilde{L}^{(t)}_{A_\mathbb{C}}$$

for each $A \in \mathfrak{A}^0$. But every fibre is a copy of $\mathbb{C}$, hence $\Phi_\lambda$ is coded by a (smooth) map:

$$\Phi_\lambda\colon \mathfrak{A}^0 \longrightarrow \mathrm{GL}(1,\mathbb{C}) \cong \mathbb{C}^*.$$

As far as the construction of prequantum data is concerned, we may as well restrict to constant maps $\Phi_\lambda \in \mathbb{C}^*$. Finally, it is natural to ask that the association $\lambda \longmapsto \Phi_\lambda$ be multiplicative, i.e. a character $\Theta\colon \lambda \longmapsto \lambda^r$, for $r \in \mathbb{Z}$.

On the whole, having fixed a character via the integer $r \in \mathbb{Z}$, one gets a circle of connections $\left\{\widetilde{\nabla}^{(t)}_\lambda\right\}_{\lambda\in U(1)}$ in $\widetilde{L}^{(t)}$, as follows. For each $\lambda \in \mathbb{C}^*$, one first considers the pull-back $\lambda^*\widetilde{\nabla}^t$, which is a connection on $\lambda^*\widetilde{L}^{(t)}$.[5]

**Definition 4.8.** The connection $\widetilde{\nabla}^{(t)}_\lambda$ is the connection induced on $\widetilde{L}^{(t)}$ by $\lambda^*\widetilde{\nabla}^{(t)}$ via the isomorphism $\Phi_\lambda$.

This means that $\widetilde{\nabla}^{(t)}_\lambda$ is obtained by conjugating $\lambda^*\widetilde{\nabla}^{(t)}$ with respect to $\Phi_\lambda$. The curvature of this connection is the same as that of $\lambda^*\widetilde{\nabla}^t$, i.e. $\lambda^*\omega_t$. Indeed, this curvature would be given by conjugating $F_{\lambda^*\widetilde{\nabla}^t}$ with the map $\Phi_\lambda$, fibrewise, but in dimension one all conjugations are trivial. Moreover, one can provide the following explicit formula.

**Proposition 4.16.** *The $\lambda$-deformation $\widetilde{\nabla}^{(t)}_\lambda$ of the connection $\widetilde{\nabla}^{(t)} = d - i\alpha_t$ is*

$$\widetilde{\nabla}^\lambda = d - i\lambda^*\alpha_t + \Phi_\lambda^{-1}d\Phi_\lambda.$$

This boils down to the computation of a gauge transformation.

Notice that the formula simplifies to $\widetilde{\nabla}^{(t)}_\lambda = \lambda^*\nabla^{(t)}$ when $\Phi_\lambda$ is constant. This can be seen directly from the definition of $\widetilde{\nabla}^{(t)}_\lambda$, using the fact that all connections are $\mathbb{C}$-linear. In this case the isomorphism $\Phi_\lambda$ chosen is immaterial, and in what follows we'll assume to have made such a choice.

Finally, one would like to pick the quotients of these $U(1)$-equivariant prequantum data with respect to the $\mathcal{G}^0$-action, in order to define a circle of prequantum data lying over the circle $\left\{(\mathfrak{M}, \lambda^*\omega_t)\right\}_{\lambda\in U(1)}$ of prequantisable manifolds. Notice however that a priori the pull-back of the symplectic form on the moduli space may be not well defined, since one still has to show that the $U(1)$-action is compatible with the $\mathcal{G}^0$-action.

**Proposition 4.17.** *Choose $\lambda \in U(1)$, $g \in \mathcal{G}^0$ and $A_\mathbb{C}$. One has*

$$\lambda.(g.A_\mathbb{C}) = g.(\lambda.A_\mathbb{C}).$$

*Proof.* It is enough to check this on the generators $h, g_{10}, g_{01}$ of $\mathcal{G}^0$.

The result is clearly true for the flip, by $\mathbb{C}$-linearity: $\lambda.(-A_\mathbb{C}) = -\lambda.(A_\mathbb{C})$.
As for $g_{10}$ and $g_{01}$, notice that the $U(1)$-action rotates the component $\Phi$ of any given Hitchin pair $(D, \Phi)$, whereas the elements of the gauge group translate the unitary connection $D$. Hence the two actions commute. □

This result implies that the formula for circle action is the same on the moduli space, if one works on representatives. This concludes the construction of $U(1)$-equivariant prequantum data for the complex moduli space.

[5] Recall that by definition $\widetilde{\nabla}^{(t)} = d - i\alpha_t$, where $\alpha_t$ is the $t$-deformation of the canonical symplectic potential $\alpha$ of the Atiyah–Bott symplectic form on $\mathfrak{A}^0$ (see § 4.2.3). Hence $\lambda^*\widetilde{\nabla}^{(t)} = d - i\lambda^*\alpha_t$, which is compatible with the fact that $F_{\lambda^*\nabla^{(t)}} = \lambda^*F_{\nabla^{(t)}} = -i\lambda^*\omega_t$.

### 4.4.5 The extended Teichmüller and the problem of equivariance

The next step is to enlarge the complex structures considered. We set

$$\widetilde{\mathbb{H}} := U(1) \times \mathbb{H},$$

and refer to this as the *extended Teichmüller* space of $\Sigma$. Now, for $(\lambda, \tau) \in \widetilde{\mathbb{H}}$, one considers the complex structure

$$I^{(t)}_{\lambda} := \lambda^* I^{(t)},$$

where $I^{(t)}$ was defined in § 4.2.5 to be

$$I^{(t)} = I_{k',0,-\sigma'} = k' I_\tau - \sigma' K_\tau,$$

choosing from the sphere of Kähler structures defined by the hyper-Kähler triple $I_\tau, J, K_\tau = I_\tau J$. The results of § 4.4.3 (namely Prop. 4.14) yield

$$I^{(t)}_{\tau,\lambda} = I_{k',-\lambda_2\sigma',\lambda_1\sigma'} = k' I_\tau - \lambda_2 \sigma' J + \lambda_1 \sigma' K_\tau.$$

We are effectively enlarging the class of Kähler polarisation by allowing a pull-back with respect to $\lambda$. Indeed, the triple $(\mathfrak{M}, \lambda^*\omega_t, I^{(t)}_{\tau,\lambda})$ is a Kähler manifold, and the diffeomorphism

$$\lambda \colon (\mathfrak{M}, \lambda^*\omega_t, I^{(t)}_{\lambda}) \longrightarrow (\mathfrak{M}, \omega_t, I^{(t)})$$

tautologically preserves the Kähler structures.

Fixing both parameters $\lambda, \tau \in \widetilde{\mathbb{H}}$, one can define (local) holomorphic sections $s$ of $L^{(t)}$ to be smooth sections such that:

$$\left(\lambda^* \nabla^{(t)}\right)^{0,1} s = 0.$$

One can now define the usual spaces $\mathcal{H}^{(t)}_{\mathbb{C},\lambda^*\tau} \cong H^0\left(\mathfrak{M}, L^{(t)}\right)$ for the Kähler quantisation of the moduli space, and the ultimate goal would be to define a Hitchin connection $\widehat{\nabla}$ in the trivial prequantum bundle

$$C^\infty(\mathfrak{M}, L^{(t)}) \times \widetilde{\mathbb{H}} \longrightarrow \widetilde{\mathbb{H}},$$

over the extended Teichüller space, that preserves these subspaces. This bundle is still trivial, since the space of smooth sections of $L^{(t)}$ only depends on the smooth structure of $\mathfrak{M}$, and not on its symplectic structure. The choice of identifying all pull-backs $\lambda^* L^{(t)}$ was made precisely to have a trivial prequantum bundle.

**Remark 4.19.** Notice two interesting new features of this geometric quantisation. First, the quantum level $t$ is no longer fixed, since the $U(1)$-action moves the parameters $k', \sigma'$. Second, the base space is not simply connected, but rather has a fundamental group isomorphic to $\mathbb{Z}$. Hence this also opens the question to compute the monodromy of the Hitchin connection around the nontrivial loop.

In the next section we tackle this problem by looking for a different Ansatz for the Hitchin connection, admitting that the Laplacians $\Delta_G$ might be corrected by a differential operator of order one of the form $\lambda^*\nabla^{(t)}_X$, where $X \in C^\infty(\mathfrak{M}, T_{1,0})$ is a vector field of type $(1,0)$, depending on $\lambda$ and $\tau$.
The result of this approach is that indeed one can find a connection $\widehat{\nabla}$ that preserves holomorphic sections (incidentally showing that they constitute a vector sub-bundle of the prequantum one), but we have no proof that it is projectively flat.

## 4.5 The equivariant connection

Let us briefly recall the setup. One considers a genus one closed surface $\Sigma$, the group $G = \mathrm{SL}(2,\mathbb{C})$, and three parameters $t \in \mathbb{C}^*, \tau \in \mathbb{H}$ and $\lambda = e^{i\theta} \in U(1)$. To these data one attaches the following objects:

- A smooth symplectic manifold $(\mathfrak{M}, \omega)$.
- Prequantum data $(L, \nabla, h)$ for $(\mathfrak{M}, \omega)$.
- A Kähler polarisation $P \subseteq T_{\mathbb{C}}\mathfrak{M}$ on the symplectic manifold, coming from a complex structure $I$.
- The hyper-Kähler metric $g = \omega.I \in A^2(\mathfrak{M}, T^*\mathfrak{M} \otimes T^*\mathfrak{M})$.
- The trivial prequantum bundle $C^\infty(\mathfrak{M}, L) \times \widetilde{\mathbb{H}} \longrightarrow \widetilde{\mathbb{H}}$, where $\widetilde{\mathbb{H}} = \mathbb{H} \times U(1)$.
- The quantum space $\mathcal{H}_P = H^0(\mathfrak{M}, L) = \mathrm{Ker}\left(\nabla^{0,1} \colon C^\infty(\mathfrak{M}, L) \longrightarrow A^{0,1}(\mathfrak{M}, L)\right)$.

Each of these objects depends on a subset of $\{t, \tau, \lambda\}$, but we will drop their dependence from the notation in this section, to avoid cumbersome formulae. This dependence must however be taken into account when differentiating sections and tensors in the directions $\partial_\tau, \partial_{\overline{\tau}}$ and $\partial_\theta$ on the extended Teichmüller space $\widetilde{\mathbb{H}}$.

### 4.5.1 Preservation of holomorphicity

Now, just as we did in § 4.3.2, before looking at the $U(1)$-action, we consider connections $\widehat{\nabla}$ in the trivial prequantum bundle of the form

$$\widehat{\nabla} = \nabla^T - u,$$

where $\nabla^T$ is the trivial connection and $u \in A^1\left(\widetilde{\mathbb{H}}, \mathcal{D}(\mathfrak{M}, L)\right)$ a 1-form on the base space taking values in differential operators acting on smooth sections. We can now give conditions on $u$ so that $\widehat{\nabla}$ preserves the quantum spaces fibrewise, finding an equation analogous to (4.8).

**Proposition 4.18.** *The connection* $\widehat{\nabla} = \nabla^T - u$ *preserves the spaces of holomorphic sections if and only if*

$$\nabla^{0,1}u(V).s = -\pi^{0,1}V[\nabla]s - \frac{i}{2}V[I]\nabla s, \tag{4.12}$$

*for all vector fields* $V \in C^\infty(\widetilde{\mathbb{H}}, T\widetilde{\mathbb{H}})$ *and for all holomorphic sections* $s$*, where we write* $\pi^{0,1}\colon T^*_{\mathbb{C}}\mathfrak{M} \longrightarrow T^{0,1}\mathfrak{M}$ *for the projection on the antiholomorphic cotangent bundle.*

Notice the crucial fact that $V[\nabla] \neq 0$. The derivative of the prequantum connection $\nabla = d - i\alpha$ along $V$ equals the 1-form $V[\nabla] = -iV[\alpha] \in A^1(\mathfrak{M}, \mathbb{C})$. Indeed, 1-forms are the objects that linearise connections. Hence one gets the additional operator

$$\pi^{0,1}V[\nabla]\colon C^\infty(\mathfrak{M}, L) \longrightarrow A^{0,1}(\mathfrak{M}, L)$$

that one did not see in (4.8). The projection $\pi^{0,1}$ is by definition the dual of the standard projection $\pi_{0,1} \coloneqq \frac{1}{2}\Big(\mathrm{Id} + iI\Big)\colon T_{\mathbb{C}}\mathfrak{M} \longrightarrow T_{0,1}\mathfrak{M}$ on the antiholomorphic tangent bundle.

*Proof.* One imposes $\nabla^{0,1}(\widehat{\nabla}_V s) = 0$ for every vector field $V$ on $\widetilde{\mathbb{H}}$, i.e.

$$\nabla^{0,1}V[s] - \nabla^{0,1}u(V).s = 0.$$

Now, the holomorphicity of $s$ is expressed by the identity $\nabla^{0,1}s = 0$, which one can differentiate with respect to any vector field $V$ on the extended Teichmüller, in order to find a different formula for $\nabla^{0,1}V[s]$. One finds

$$0 = V[\nabla^{0,1}s] = \frac{i}{2}V[I]\nabla s + \pi^{0,1}V[\nabla]s + \nabla^{0,1}V[s],$$

because by definition $\nabla^{0,1} = \pi^{0,1}\nabla$. This expansion follows from the chain rule.
Substituting $-\frac{i}{2}V[I]\nabla s - \pi^{0,1}V[\nabla]s = \nabla^{0,1}V[s]$ in the previous equation proves the stated formula. □

As remarked in the nonequivariant case, we may replace $V[I]\nabla s$ by $V[I]'\nabla^{1,0}s$, since $s$ is holomorphic.

### 4.5.2 Variations of tensors: equivariant version

Let $V$ be a vector field on the extended Teichmüller space. The derivative of the metric $g = \omega.I$ in the direction of $V$ direction now has a further term:

$$V[g] = \omega.V[I] + V[\omega].I,$$

by the Leibnitz rule. This means that for all vector fields $X, Y$ on $\mathfrak{M}$ one has:

$$g(X,Y) = \omega\Big(X, I(Y)\Big), \qquad V[g](X,Y) = V[\omega]\Big(X, I(Y)\Big) + \omega\Big(X, V[I](Y)\Big).$$

Now there is a choice: either one keeps defining the tensor $\widetilde{G}(V)$ as before, coding only the variation of the complex structure, or we take into account the whole variation of the metric. The first choice defines $\widetilde{G}(V)$ via

$$\widetilde{G}(V).\omega = V[I], \qquad \text{which means} \qquad \iota_{\omega\otimes\omega}\widetilde{G}(V) = \omega.V[I].$$

The second choice instead defines $\widetilde{G}_{\text{tot}}(V)$ by means of

$$\iota_{\omega\otimes\omega}\widetilde{G}_{\text{tot}}(V) = V[g].$$

Notice that the tensor $\widetilde{G}_{\text{tot}}(V) \in C^\infty(\mathfrak{M}, T_{\mathbb{C}}\mathfrak{M}\otimes T_{\mathbb{C}}\mathfrak{M})$ is still symmetric. This happens since $V[g]$ is symmetric, and one applies the same isomorphism (defined by $\omega$) in both entries. There are instead no particular reasons why tensor $\widetilde{G}(V)$ should be symmetric, as it corresponds to only one part of the variation of $g$.

Another difference is also the decomposition into holomorphic and antiholomorphic types. Namely, one can write

$$\widetilde{G}(V) = G(V) + \overline{G}(V),$$

according to the decomposition $V[I] = V[I]' + V[I]''$, corresponding to $\omega.G(V) = V[I]'$, $\omega.\overline{G}(V) = V[I]''$. Notice that $G(V)$ is still translation-invariant, which implies that it is a holomorphic section of $T_{1,0}\mathfrak{M}^{\otimes 2}$. This is the property of rigidity for the extended family of complex structures $\left\{I = \lambda^* I_\tau^{(t)}\right\}_{(\tau,\lambda)\in\widetilde{\mathbb{H}}}$. On the other hand, the tensor $\widetilde{G}_{\text{tot}}(V)$ decomposes as

$$\widetilde{G}_{\text{tot}}(V) = G_{\text{tot}}(V) + G^{1,1}_{\text{tot}}(V) + \overline{G}_{\text{tot}}(V),$$

according to

$$\iota_{\omega\otimes\omega}\widetilde{G}_{\text{tot}}(V) = V[g]^{2,0} + V[g]^{1,1} + V[g]^{0,2},$$

with the addends taken in the same order.

In the next section we will try to construct a Hitchin connection by correcting the Laplacian $\Delta_{G(V)}$ associated to the nonsymmetric, holomorphic tensor $G(V)$ of type $(2,0)$.

### 4.5.3 Construction of the equivariant connection

Let $V$ be a vector field on $\widetilde{\mathbb{H}}$. Consider the second-order differential operator:

$$\Delta_{G(V)} \coloneqq \text{Tr}(\nabla^{1,0}G(V)\nabla^{1,0}),$$

defined as in (4.9). One can compute the commutator $[\nabla^{0,1}, \Delta_{G(V)}]$ on a holomorphic section $s$, and see which type of correction is required in order to satisfy 4.12.

Looking at the proof of Prop. 4.11, one sees that

$$\nabla^{0,1}\Delta_{G(V)}s = -2i\omega.S\Big(G(V)\Big)\nabla^{1,0}s,$$

for a holomorphic section $s$, where $S\Big(G(V)\Big)$ denotes the symmetrisation of the tensor $G(V)$. The symmetrisation operator, already met in the context of Weyl quantisation in

§ 2.1.5, is defined as follows. If $V$ is a vector space over $\mathbb{C}$, then $S\colon \mathrm{Tens}(V) \longrightarrow \mathrm{Tens}(V)$ is defined on homogeneous elements of degree $n$ by

$$S(v_1 \otimes \cdots \otimes v_n) \longmapsto \frac{1}{n!} \sum_{\sigma \in \Sigma_n} v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_n}.$$

The image $S\Big(\mathrm{Tens}(V)\Big)$ is identified with the symmetric algebra $\mathrm{Sym}(V)$ of $V$, which is a priori a quotient of the tensor algebra $\mathrm{Tens}(V)$. This definition extends naturally to vector bundles, and then to sections of such. If one replace $V$ by $T_{\mathbb{C}}\mathfrak{M}$ or $T^*_{\mathbb{C}}\mathfrak{M}$, then $S$ acts on sections of arbitrary tensor powers of these bundles, which are by definition tensors on $\mathfrak{M}$.

**Remark 4.20.** Let us provide a computation in $I$-holomorphic coordinates $\{z_1, z_2\}$, for the sake clarity. Write the relevant tensors as follows

$$G(V) = \sum_{i,j} G_{ij}(V) \partial_{z_i} \otimes \partial_{z_j}, \qquad \omega = \sum_{i,j} \omega_{ij} d\bar{z}_i \wedge dz_j, \qquad V[I]' = \sum_{i,j} a_{ij} d\bar{z}_i \otimes \partial_{z_j},$$

so that one has by definition

$$S\Big(G(V)\Big) = \frac{1}{2} \sum_{i,j} (G_{ij}(V) + G_{ji}(V)) \partial_{z_i} \otimes \partial_{z_j}, \qquad \text{and} \qquad a_{ij} = \sum_k \omega_{ik} G_{kj}(V).$$

Finally, in those trivialisations one can write

$$\Delta_{G(V)} = \sum_{i,j} \nabla_i G_{ij}(V) \nabla_j,$$

where $\nabla_i \coloneqq \nabla^{1,0}_{\partial z_i} = \nabla_{\partial_{z_i}}$. If one chooses $k \in \{1, 2\}$ then one finds the following, using the commutation relation $\Big[\nabla^{0,1}_{\partial_{\bar{z}_k}}, \nabla_j\Big] = -i\omega_{kj}$ provided by the prequantum condition:

$$\frac{i}{2}\Big[\nabla^{0,1}_{\partial_{\bar{z}_k}} \Delta G(V)\Big] = \frac{i}{2} \sum_{j,l} \Big[\nabla^{0,1}_{\partial_{\bar{z}_k}}, \nabla_j G_{jl}(V) \nabla_l\Big] = \frac{1}{2} \sum_{j,l} \omega_{kj} G_{jl}(V) \nabla_l + \nabla_j G_{jl}(V) \omega_{kl} =$$

$$= \frac{1}{2} \sum_{j,l} \Big(\omega_{kj} G_{jl}(V) + \omega_{kj} G_{lj}(V)\Big) \nabla_l = \frac{1}{2} \sum_{j,l} \omega_{kj} \Big(G_{jl}(V) + G_{lj}(V)\Big) \nabla_l =$$

$$= \sum_{j,l} \omega_{kj} S\Big(G(V)\Big)_{jl} \nabla_l.$$

The last term is by definition the $k$-th component of the operator $\omega.S(G(V))\nabla^{1,0}$.

The idea is now to look for a differential operator of order one that takes into account what is left to satisfy 4.12. To this end, let $\varphi \in A^{0,1}(\mathfrak{M}, \mathbb{C})$ be a $(0,1)$-form, and consider the vector field $X \in \Omega(\mathfrak{M}, T)$ defined by

$$\iota_X \omega + \varphi = \omega(X, \cdot) + \varphi = 0.$$

The vector field $X$ must then be of type $(1,0)$, since the pairing $\omega\colon T_{0,1}\mathfrak{M} \wedge T_{0,1}\mathfrak{M} \longrightarrow \mathbb{R}$ defined by the nondegenerate $(1,1)$-form $\omega$ swap types.

Let us now compute the commutator of $\nabla^{0,1}$ and the operator of covariant derivative $\nabla_X$. One finds the following, extracting a vector field from an $I$-antiholomorphic frame $\{\partial_{\overline{z_1}}, \partial_{\overline{z_2}}\}$ of $T_{0,1}\mathfrak{M}$:

$$\nabla^{0,1}_{\partial_{\bar{z}_k}} \nabla_X s = \Big[\nabla^{0,1}_{\partial\bar{z}_k}, \nabla^{1,0}_X\Big]\, s = F_\nabla(\partial_{\bar{z}_k}, X)s + \nabla_{[\partial_{\bar{z}_k}, X]} s = \Big( - i\omega(\partial_{\bar{z}_k}, X) + \nabla_{\Big[\partial_{\bar{z}_k}, X\Big]}\Big) s =$$
$$= \Big( - i\varphi(\partial_{\bar{z}_k}) + \nabla_{\Big[\partial_{\bar{z}_k}, X\Big]}\Big) s.$$

The idea is then to set $\varphi \coloneqq \pi^{0,1}V[\nabla] \in A^{0,1}(\mathfrak{M}, \mathbb{C})$, in order to recover the derivative of the prequantum connection that was necessary to satisfy Eq. (4.12). Thus one considers the $(\tau, \lambda)$-dependent vector field $X(V)$ of type $(1,0)$, satisfying $X(V).\omega = \pi^{0,1}V[\nabla]$, where as always we mean the contraction on the right.

Putting everything together one has the following, for $V \in C^\infty(\tilde{\mathbb{H}}, T\tilde{\mathbb{H}})$ and $s$ holomorphic:

$$\nabla^{0,1}\left(\frac{1}{4}\Delta_{G(V)} + i\nabla_X\right) s = -\frac{i}{2}\omega.S\Big(G(V)\Big)\nabla^{1,0}s - \pi^{0,1}V[\nabla]s + i\nabla_{\nabla^{0,1}_{\mathrm{LC}}X(V)}s,$$

where $\nabla_{\mathrm{LC}}$ denotes the (trivial) Levi–Civita connection on $T\mathfrak{M}$ for the metric $g$. This means that if one were able to show that

$$\nabla_{\nabla^{0,1}_{\mathrm{LC}}X(V)}s = -\frac{1}{2}\omega.A\Big(G(V)\Big)\nabla^{1,0}s, \tag{4.13}$$

then one would have a connection that preserves holomorphicity, using

$$-\frac{i}{2}\omega.\Big(S\Big(G(V)\Big) + A\Big(G(V)\Big)\Big) = -\frac{i}{2}\omega.G(V) = -\frac{i}{2}V[I]',$$

where $A\Big(G(V)\Big)$ is the alternating part of $G(V)$; it is defined as the symmetric part, adding the sign of each permutation involved. The normalisation factors of $S$ and $A$ are indeed chosen so that $S + A = \mathrm{Id}$.
Moreover, to show (4.13) it is enough to prove that $\nabla^{0,1}_{\mathrm{LC}}X(V) = -\frac{1}{2}\omega.A\Big(G(V)\Big)$, up to applying the $(1,0)$-part of the prequantum connection.

Now, one has

$$\nabla^{0,1}_{\mathrm{LC}}X(V) = \nabla^{0,1}_{\mathrm{LC}}(\omega.\pi^{0,1}V[\nabla]) = \omega.A\Big(\nabla^{0,1}_{\mathrm{LC}}\pi^{0,1}V[\nabla]\Big),$$

using the Leibnitz rule to differentiate the contraction $\omega.\pi^{0,1}V[\nabla]$, together with the fact that $\nabla_{\mathrm{LC}}\omega = 0$ since $\omega$ is Kähler. Notice that to be rigorous one should introduce a different symbol for the connections that $\nabla_{\mathrm{LC}}$ induces in $T^*\mathfrak{M}$, and that one has to explicitly take alternating parts, since the covariant derivative of a differential form is not automatically alternating.
Next, if $\beta$ is a differential 1-form on $\mathfrak{M}$, and $X$, $Y$ are commuting vector fields on $\mathfrak{M}$, then

$$\Big(A(\nabla_{\mathrm{LC}}\beta)\Big)(X,Y) = \frac{1}{2}\Big(\nabla_{\mathrm{LC}}\beta(X,Y) - \nabla_{\mathrm{LC}}\beta(Y,X)\Big) =$$
$$= \frac{1}{2}\Big(\nabla_{\mathrm{LC}}X[\beta(Y)] - \nabla_{\mathrm{LC}}(Y)[\beta(X)]\Big) =$$
$$= \frac{1}{2}\Big(X[\beta(Y)] - Y[\beta(X)]\Big) = \frac{1}{2}d\beta(X,Y),$$

since the (dual) Levi–Civita connection is trivial. Hence

$$\omega.A(\nabla^{0,1}_{\mathrm{LC}}\pi^{0,1}V[\nabla]) = \omega.A\Big((\pi^{0,1}\circ\nabla_{\mathrm{LC}})\pi^{0,1}V[\nabla]\Big) = \frac{1}{2}\omega.\Big((\pi^{0,1}\circ d)\pi^{0,1}V[\nabla]\Big) = \frac{1}{2}\omega.\Big(\overline{\partial}\pi^{0,1}V[\nabla]\Big).$$

On the whole, one has $\nabla^{0,1}_{\mathrm{LC}}X(V) = \frac{1}{2}\omega.(\overline{\partial}\pi^{0,1}V[\nabla])$, and thus if one were able to show that

$$\overline{\partial}\pi^{0,1}V[\nabla] = -A(\omega.V[I]') \in A^{0,2}(\mathfrak{M},\mathbb{C}),$$

then the proof would be complete, because $A(\omega.V[I]')$ corresponds to $\omega.A\Big(G(V)\Big)$ under the isomorphism defined by $\omega$ (this is clear if one computes in local normal coordinates on the Kähler manifold where $\omega = \sum_j dz_j \wedge d\overline{z_j}$).

**Lemma 4.5.** *One has* $\overline{\partial}\pi^{0,1}V[\nabla] + A\Big(\omega.V[I]'\Big) = 0.$

**Remark 4.21.** Before proving this final lemma, we remark that it expresses a necessary condition for a having a connection that preserves holomorphicity. Indeed, (4.12) implies

$$0 = \Big(\nabla^{0,1}\Big)^2 u(V).s = -\nabla^{0,1}\left(\pi^{0,1}V[\nabla] + \frac{i}{2}V[I]'\nabla^{1,0}\right)s = -\Big(\overline{\partial}\pi^{0,1}V[\nabla] + A\Big(\omega.V[I]'\Big)\Big)\,s.$$

Once again, taking alternating parts is necessary in the last passage, since $\nabla^{0,1}$ produces a $(0,2)$-form on $\mathfrak{M}$ — with values in $\mathcal{D}(\mathfrak{M},L)$ — out of the $(0,1)$-form $\nabla^{0,1}u(V)$.
To be more explicit, let $X, Y$ be commuting vector fields of type $(0,1)$ on $\mathfrak{M}$, and compute:

$$\begin{aligned}
\frac{i}{2}\Big(\nabla^{0,1}(V[I].\nabla^{1,0}s)\Big)(X,Y) &= \frac{i}{2}\nabla^{0,1}_X(\nabla^{1,0}_{V[I]'Y}s) - \frac{i}{2}\nabla^{0,1}_Y(\nabla^{1,0}_{V[I]'X}s) = \\
&= \frac{i}{2}\Big(-i\omega(X,V[I]'Y)s\Big) - \frac{i}{2}\Big(-i\omega(Y,V[I]'X)s\Big) = \\
&= \frac{1}{2}\Big(\omega(X,V[I]'Y) - \omega(Y,V[I]'X)\Big)s = \\
&= \frac{1}{2}\Big(\omega.V[I]'(X,Y) - \omega.V[I]'(Y,X)\Big)s = \\
&= A(\omega.V[I]')s,
\end{aligned}$$

from the convention taken for the alternating part.

*Proof.* Since the tensor $V[g] = V[\omega].I + \omega.V[I]$ is symmetric, one has

$$A(V[\omega].I) + A(\omega.V[I]) = 0.$$

Notice that the contraction with $I$ and $V[I]$ perturbs the skew-symmetry of $\omega$, but taking alternating parts one has genuine differential forms. There is also an analogous identity for the $(0,2)$-parts:

$$\pi^{0,2}A(V[\omega].I) + \pi^{0,2}A(\omega.V[I]) = 0.$$

Looking at bidegrees one sees that $\pi^{0,2}A(\omega.V[I]) = A(\omega.V[I]')$. Hence:

$$A(V[I]'.\omega) = -\pi^{0,2}A(V[\omega].I). \tag{4.14}$$

Now, one has

$$\overline{\partial}\pi^{0,1}V[\nabla] = \overline{\partial}(-i\pi^{0,1}V[\alpha]),$$

where $\nabla = d - i\alpha$ is — as usual — the prequantum connection written on a local patch where $L \longrightarrow \mathfrak{M}$ is trivial. Since $I$ acts on $(0,1)$-forms multiplying by $-i$, one finds

$$\overline{\partial}(-i\pi^{0,1}V[\alpha]) = \overline{\partial}(\pi^{0,1}V[\alpha].I).$$

Next, one uses the following identity of differential operators

$$\overline{\partial} \circ \pi^{0,1} = \pi^{0,2} \circ d \colon A^1(\mathcal{M}_{\mathbb{C}}, \mathbb{C}) \longrightarrow A^{0,2}(\mathcal{M}_{\mathbb{C}}, \mathbb{C}),$$

obtaining

$$\overline{\partial}(\pi^{0,1}V[\alpha].I) = \pi^{0,2}d(V[\alpha].I) = \pi^{0,2}A(dV[\alpha].I) = \pi^{0,2}A(V[d\alpha].I).$$

The last identity uses $dV[\alpha] = V[d\alpha]$, which is clearly true in any $(\tau, \lambda)$-independent local frame of the cotangent bundle, e.g. the one defined by the usual independent coordinates. On the whole

$$\overline{\partial}\pi^{0,1}V[\nabla] = \pi^{0,2}A(V[\omega].I) = -A(\omega.V[I]'),$$

using the identity (4.14). □

We have thus proven the following theorem.

**Theorem 4.3.** *Consider the connection*

$$\widehat{\nabla} = \nabla^T - u, \qquad \textit{where} \qquad u(V) \coloneqq \frac{1}{4}\Delta_{G(V)} + i\nabla^{1,0}_{X(V)},$$

*defined in the trivial bundle $C^\infty(\mathfrak{M}, L) \times \widetilde{\mathbb{H}} \longrightarrow \widetilde{\mathbb{H}}$ over the extended Teichmüller space. This connection preserves the subspaces $\mathcal{H}^{(t)}_{\mathbb{C}} = H^0(\mathfrak{M}, L)$ of holomorphic sections fibre-wise.*

This shows in particular that the spaces of holomorphic sections fit in a nontrivial vector sub-bundle $\mathcal{V} \longrightarrow \widetilde{\mathbb{H}}$. The next step to get the canonical identification of the fibres of the projectivisation $\mathbb{P}(\mathcal{V}) \longrightarrow \widetilde{\mathbb{H}}$ would be to prove that $\widehat{\nabla}$ is projectively flat, and that it has trivial monodromy around the generator of $\pi_1(\widetilde{\mathbb{H}})$.

### 4.5.4 Comments on flatness and final remarks

The curvature of the connection constructed in Thm. 4.3 is an operator of order at most two, since two holomorphic Laplacians constructed using a translation-invariant tensor always commute (see § 4.3.3). The complete expansion yields, for commuting vector fields $V, W$ on the extended Teichmüller:

$$\begin{aligned} &F_{\widehat{\nabla}}(V,W) = \\ &= \frac{i}{4}\Big(\Big[\Delta_{G(V)}, \nabla_{X(W)}\Big] - \Big[\Delta_{G(W)}, \nabla_{X(V)}\Big]\Big) - \Big[\nabla_{X(V)}, \nabla_{X(W)}\Big] + \\ &+ \frac{1}{4}\Big(W\Big[\Delta_{G(V)}\Big] - V\Big[\Delta_{G(W)}\Big]\Big) + i\Big(\nabla_{W[X(V)]} - \nabla_{V[X(W)]}\Big) + \\ &+ i\Big(\langle W[\nabla], X(V)\rangle - \langle V[\nabla], X(W)\rangle\Big). \end{aligned}$$

This is indeed obtained considering the commutators among the different operators, and the terms involving the derivatives with respect to $V$ and $W$. This expansion is straightforward, apart maybe from

$$V\Big[\nabla_{X(W)}\Big] = \langle V[\nabla], X(W)\rangle + \nabla_{V[X(W)]},$$

which follows from the fact that both $\nabla$ and $X$ vary on $\widetilde{\mathbb{H}}$. The first term means the contraction of the 1-form $V[\nabla] = -iV[\alpha]$ against the vector field $X(W)$, which defines an operator of order zero. Also, one has

$$\Big[\nabla_{X(V)}, \nabla_{X(W)}\Big] = F_\nabla\Big(X(V), X(W)\Big) - \nabla_{\Big[X(V),X(W)\Big]} = -\nabla_{\Big[X(V),X(W)\Big]},$$

since $F_\nabla = -i\omega$ has vanishing $(2,0)$-part. Finally, notice that it is no longer true that $\Delta_{G(V)} = V\Big[\Delta_g\Big]$, since the holomorphic tensor $G(V)$ now only codes the variation of the complex structure (see § 4.5.2).
There is still one general simplification that can be observed. One has:

$$V\Big[\Delta_{G(W)}\Big] = V\,\mathrm{Tr}\left(\nabla^{1,0}G(W)\nabla^{1,0}\right) =$$

$$= V\Big[\nabla^{1,0}\Big]G(W)\nabla^{1,0} + \nabla^{1,0}V\Big[G(W)\Big]\nabla^{1,0} + \nabla^{1,0}G(W)V\Big[\nabla^{1,0}\Big].$$

However, differentiating $\nabla^{1,0}G(W) = 0$ along $V$ yields $V\Big[\nabla^{1,0}\Big]G(W)+\nabla^{1,0}V\Big[G(W)\Big] = 0$, and thus

$$V\Big[\Delta_{G(W)}\Big] = \nabla^{1,0}G(W)V\Big[\nabla^{1,0}\Big].$$

**Remark 4.22.** The upshot of this section is that one should look for Hitchin connections of the form $\widehat{\nabla} = d - u$, where $u(V) = \Delta_{G(V)} + \nabla_{X(V)}$ for a suitable vector field $X(V)$, instead of just using the Laplacian operator as we did in § 4.3.2. Moreover, it could be necessary to add an operator of order zero, i.e. a function multiplication $\mu_{f(V)}$, in order to achieve flatness.
In short, one could consider the new Ansatz

$$u(V) = \Delta_{G(V)} + \nabla_{X(V)} + f(V), \tag{4.15}$$

where one still constructs the Laplacian $\Delta_{G(V)}$ using the holomorphic part of the variation of the Kähler structure (i.e. one still asks for $\omega.G(V) = V[I]'$), but now one accepts to correct it with an operator of order one. Such a differential operator can be written as the covariant derivative $\nabla_{X(V)}$ with respect to a vector field $X(V)$ of type $(1,0)$ that depends on $(\tau,\lambda)$, plus the multiplication by a function $f(V)$ which also varies on the extended Teichmüller. One can readily write some differential equations satisfied by $X(V)$ and $f(V)$, in order for $u(V)$ to satisfy (4.12), and for $\widehat{\nabla}$ to be projectively flat. This would provide the full class of equivariant Hitchin connections $\widehat{\nabla} = \nabla^T - u$ such that the principal symbol of $u(V)$ is $S\Big(G(V)\Big)$.

If one tries to attack the problem as it is stated, the differential equations obtained are difficult to solve. There is however one natural intermediate step to take. Namely,

assume Ansatz (4.15) for constructing a Hitchin connection without considering the $U(1)$-action. This means producing a connection which is different from that of Thm. 4.1, but that has the same leading differential operator $\Delta_{G(V)}$ for all vector field on the ordinary Teichmüller $\mathbb{H}$. To this end, in the next section § 4.6 we will turn to the connection of [Wit91], described in sound mathematical terms in [AG14] (and e.g. [AM16] in the explicit case of genus one), which we call the Hitchin–Witten connection.

The Hitchin–Witten connection is defined for the geometric quantisation of $\mathfrak{M}$ with respect to the real polarisations $\left\{P_{\mathbb{R},\tau}\right\}_{\tau\in\mathbb{H}}$ of § 4.2.6. In the case of genus one as we are, it is flat. Now, there exists an isomorphism between the space of real-polarised sections and that of Kähler-polarised ones, named after Bargmann: one can try to push the Hitchin–Witten connection along the Bargmann transform to define a new, necessarily flat connection that tautologically preserves holomorphicity.

## 4.6 The Bargmann transform and the Hitchin–Witten connection

In this section we set up the Bargmann transform, which relates the quantum spaces for the geometric quantisation of $\mathfrak{M}$ with respect to real and Kähler polarisations, and we use it to transfer the Hitchin–Witten connection onto the Kähler-polarised side. We will start from an abstract viewpoint, as we temporarily did in § 4.2.4, for the sake of obtaining cleaner formulae.

### 4.6.1 Setup

Let $(V,\omega)$ be a symplectic vector space of dimension four, which we consider as a manifold equipped with a constant symplectic form. Since there is no cohomology one has $[\omega]=0$, and $(V,\omega)$ is an exact symplectic manifold, hence prequantisable. Consider prequantum data $(L,\nabla,h)$ for $(V,\omega)$, where $L\coloneqq\mathbb{C}\times V\longrightarrow V$ is the trivial line bundle, $\nabla\coloneqq d-i\alpha$ the prequantum connection defined by the symplectic potential $\alpha_v(w)\coloneqq\frac{1}{2}\omega(v,w)$, for $v,w\in V$, and $h$ the translation-invariant metric such that $h(z,w)=z\overline{w}$, for $z,w\in\mathbb{C}$. Note that we define the symplectic potential as a 1-form varying linearly on $V$, using that $TV\cong V\times V$.

Finally, assume there to be two linear polarisations on $(V,\omega)$: a real polarisation $P_{\mathbb{R}}\subseteq TV$, and a Kähler polarisation $P\subseteq T_{\mathbb{C}}V$, both depending on a parameter $\tau\in\mathbb{H}$.[6]

As we know, one attaches Hilbert spaces of polarised sections to these data. First, set:

$$\widetilde{\mathcal{H}}_{\mathbb{R}}\coloneqq\left\{r\in C^\infty(V,L)\,\middle|\,\nabla_X r=0\text{ for all }X\in C^\infty(V,P_{\mathbb{R}}),\int_V h(r,r)\mu<+\infty\right\},$$

[6]This matches up perfectly with the original setup of § 4.2, by taking $V=\mathfrak{A}^0$, $\omega=\omega_t$, $\alpha=\alpha_t$, $(L,\nabla,h)=(\widetilde{L}^{(t)},\widetilde{\nabla}^{(t)},\widetilde{h}^{(t)})$, and the linear polarisations of § 4.2.6 and § 4.2.5.

and

$$\widetilde{\mathcal{H}}_{\mathbb{C}} \coloneqq \left\{ s \in C^\infty(V, L) \,\middle|\, \nabla_X s = 0 \text{ for all } X \in C^\infty(V, P_{\mathbb{C}}), \int_V h(s,s)\mu < +\infty \right\},$$

where $\mu \coloneqq \frac{\omega \wedge \omega}{2}$ is the Liouville volume form. The spaces $\widetilde{\mathcal{H}}_{\mathbb{R}}$ and $\widetilde{\mathcal{H}}_{\mathbb{C}}$ are pre-Hilbert, and one lets $\mathcal{H}_{\mathbb{R}}$ and $\mathcal{H}_{\mathbb{C}}$ be the Hilbert spaces given by their completion with respect to $L^2$-type norm (2.10) introduced in the background.

Finally, there is an action $\Lambda \colon V \longrightarrow V$ via translations

$$\lambda \colon v \longmapsto v + 2\pi\lambda,$$

where $\Lambda \cong \mathbb{Z}^4$ sits inside $V$ as an integral lattice. To get the quotient we want, we actually consider only half of that lattice, i.e. a $\mathbb{Z}$-submodule $\Lambda_0 \cong \mathbb{Z}^2$ containing only two of the generators of $\Lambda$ (this corresponds to half-lattice generated by the translations in the direction of a unitary connection, in the language of § 4.2.1).

We know from (4.6) that there exists a lift of the action on the prequantum line bundle $L$, via multipliers $\Theta \colon \mathcal{G}^0 \times V \longrightarrow U(1)$, where $\mathcal{G}^0$ is the group generated by the $\frac{1}{2}$-lattice $\Lambda_0$ and the involution $h.v = -v$, for $v \in V$. This means that

$$\lambda.s(v) = \Theta(\lambda, v-\lambda)s(v-\lambda), \qquad \text{and} \qquad h.s(v) = s(-v),$$

for all $\lambda \in \Lambda_0$. Since the lattice action preserves linear polarisations, it makes sense to consider $\mathcal{G}^0$-invariant, polarised sections of the prequantum line bundle, i.e. elements of $\mathcal{H}_{\mathbb{R}}^{\mathcal{G}^0}$ and $\mathcal{H}_{\mathbb{C}}^{\mathcal{G}^0}$. These spaces are by definition the Hilbert spaces of the quantisation of the quotient $V\big/\mathcal{G}^0$, which is the abstract avatar of the hyperKähler moduli space $\mathfrak{M}$.

**Remark 4.23.** To describe the Hilbert spaces more explicitly, choose global frames $r, s \colon V \longrightarrow L$ for the prequantum line bundle such that $r \in \mathcal{H}_{\mathbb{R}}$ and $s \in \mathcal{H}_{\mathbb{C}}$.

One can choose $r$ to be $h$-unitary, i.e. $h(r,r) = 1$ identically on $V$. With this choice, every real-polarised section can be written $t = \psi r$, for some function $\psi \in C^\infty(V, \mathbb{C})$ that is constant along the directions of $P_{\mathbb{R}}$, because of

$$0 = \nabla_X(\psi r) = X.\psi + \psi \nabla_X r = X.\psi,$$

where $X$ is tangent to $P_{\mathbb{R}}$. According to (2.10) the inner product $(\cdot, \cdot)$ of two sections of the prequantum bundle is defined by

$$\langle t \mid t' \rangle = \int_V h(t,t')\mu = \int_V \psi\overline{\psi'} \cdot h(r,r)\mu = \int_V \psi\overline{\psi'}\mu,$$

where $t = \psi r$, $t' = \psi' r$. This has no chances of converging if $t$ and $t'$ are both real-polarised and nonvanishing, because the leaves of $P_{\mathbb{R}}$ are the noncompact translations of the Lagrangian space $\mathcal{L}$ (this issue was pointed out in 2.11). To correct this problem one may replace integration on $V$ with the integration over a global transverse subspace $W \subseteq V$ to $\mathcal{L}$, and remark that $W$ is in natural correspondence with the set of leaves of the real polarisation. The advantage is that the space $W$ now carries a natural 2-dimensional Lebesgue measure $dw$, and this provides an identification of Hilbert spaces

$$\psi r \longmapsto \psi|_W : \mathcal{H}_{\mathbb{R}} \cong C^\infty(W, \mathbb{C}) \cap L^2(W, dw).$$

More precisely, $\mathcal{H}_{\mathbb{R}}$ is isomorphic as a vector space to

$$\left\{ f \in \pi_{\mathbb{R}}^*\big(C^\infty(W,\mathbb{R})\big) \,\middle|\, f|_W \in L^2(W,dx) \right\} \subseteq C^\infty(V,\mathbb{C}),$$

where $\pi_{\mathbb{R}} \colon V \longrightarrow V\big/P_{\mathbb{R}} \cong W$ is the projection of $V$ onto the space of leaf of the admissible real polarisation $P_{\mathbb{R}}$, which we identify with the global transverse. This is because a function $\psi \colon V \longrightarrow \mathbb{C}$ is constant along $P_{\mathbb{R}}$ if and only if it can be written as $\pi_{\mathbb{R}}^*\widetilde{\psi}$ for some function $\widetilde{\psi} \colon W \longrightarrow \mathbb{C}$. Next, one promotes the vector space to an Hilbert space, taking a completion with respect to the $L^2$-norm on $W$.

Similarly, one may write $t = fs$ for a generic Kähler-polarised section, using the frame $s$ of $\mathcal{H}_{\mathbb{C}}$. Note since $r$ is a frame then there exists a smooth function $\varphi \colon V \longrightarrow \mathbb{C}$ such that $s = e^{\varphi} r$, and thus the norm of $s$ is

$$h(s,s) = |e^{\varphi}|^2 h(r,r) = |e^{\varphi}|^2 = e^{2\Re(\varphi)}.$$

A section $t = fs$ is now holomorphic if and only if $\overline{\partial} f = 0$, i.e. if $f$ is an entire function on $V$ with respect to the complex structure associated to the Kähler polarisation. The inner product of $t = fs$, $t' = f's$ reads

$$\langle t \mid t' \rangle = \int_V h(t,t')\mu = \int_V f\overline{f'} \cdot h(s,s)\mu = \int_V f\overline{f'}\nu$$

where $\nu \coloneqq e^{2\Re(\varphi)}\mu$. Note that in the chosen Darboux coordinates the volume form reads $\mu = dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2$, and thus the integration with respect to $\mu$ means to integrate for the Lebesgue measure $dxdy$ on $V \cong \mathbb{R}^4$. One may finally write

$$\mathcal{H}_{\mathbb{C}} \cong \mathcal{O}\big(\mathbb{C}^2\big) \cap L^2(\mathbb{C}^2, e^{2\Re(\varphi)} dxdy).$$

### 4.6.2 The Bargmann transform: Lagrangian splitting

We will now describe a $\tau$-dependent isomorphism $\mathcal{B} \colon \mathcal{H}_{\mathbb{R}}^{\mathcal{G}^0} \longrightarrow \mathcal{H}_{\mathbb{C}}^{\mathcal{G}^0}$. To do this, it is natural to start from constructing a map $\mathcal{B} \colon \mathcal{H}_{\mathbb{R}} \longrightarrow \mathcal{H}_{\mathbb{C}}$, and to show that it commutes with the $\mathcal{G}^0$-action. This map is the Bargmann transform on flat space (cf. [GS81]), and so it make sense to start discussing the standard setup to clarify what we want to do.

With the same notions and notations of the previous section, consider a Darboux basis $\{x_j, y_j\}_j$ on $V$, so that $\omega = \sum_j dx_j \wedge dy_j$, and let $\mathcal{L}$ be the Lagrangian subspace spanned by $\{y_1, y_2\}$. One then has a symplectomorphism $(V,\omega) \cong (T^*\mathbb{R}^2, \omega_{\text{can}})$, and the symplectic potential reads $\alpha = \frac{1}{2}\sum_j (x_j dy_j - y_j dx_j)$ in the same basis. Next, define the balanced holomorphic coordinates $z_j \coloneqq \frac{x_j + iy_j}{\sqrt{2}}$, which are associated to the complex structure $I \colon V \longrightarrow V$ sending $x_j$ to $y_j$ and $y_j$ to $-x_j$.

One can show that $r(x,y) = e^{\frac{i}{2}x\cdot y}$ defines a real-polarised unitary frame, and that a holomorphic frame is provided by $s = ce^{-\frac{|z|^2}{2}}$, for every constant $c \in \mathbb{C}$.[7]

[7] Here and afterwards we denote by a dot the componentwise product of two vectors in any vector space: $x \cdot y = \sum_j x_j y_j$ and $|z|^2 = z \cdot \overline{z} = \sum_j z_j \overline{z_j} = \sum_j |z_j|^2$.

Indeed, if one denotes $\partial_{y_j}$ the partial derivative along the direction of $y_i$, then

$$\nabla_{\partial_{y_j}} r = \partial_{y_j} r - i\langle \alpha, \partial_{y_j} \rangle r = \partial_{y_j} e^{\frac{i}{2} x\cdot y} - \frac{i}{2} x_j e^{\frac{i}{2} x\cdot y} = \left(\partial_{y_j} - \frac{i}{2} x_j\right) e^{\frac{i}{2} x\cdot y} = 0.$$

Similarly, the symplectic form $\omega$ in the chosen — normalised — complex coordinates reads $\omega = i\sum_j dz_j \wedge d\overline{z_j}$, and thus $\alpha = \frac{i}{2}\sum_j z_j d\overline{z_j} - \overline{z_j} dz_j$. Then the $(0,1)$-part of the covariant derivative acts as the differential operator $\nabla^{0,1}_{\partial_{\overline{z_j}}} = \partial_{\overline{z_j}} + \frac{1}{2} z_j$, which annihilates $s$.

With these choices we may identify the spaces of polarised sections with spaces of complex valued functions, as explained in the previous section:

$$\mathcal{H}_{\mathbb{R}} \cong C^\infty(\mathbb{R}^2, \mathbb{C}) \cap L^2(\mathbb{R}^2, dx), \qquad \mathcal{H}_{\mathbb{C}} \cong \mathcal{O}(\mathbb{C}^2) \cap L^2(\mathbb{C}^2, e^{-|z|^2} dxdy),$$

where $dx$ is the Lebesgue measure on the global transverse $\{(x,0) \mid x \in \mathbb{R}^2\} \cong \mathbb{R}^2$ to the real polarisation, and $dxdy$ the standard Lebesgue measure on $\mathbb{C}^2$ which corresponds to the volume form $\mu$ on $V$.

The inner product of two holomorphic functions — noted $(\cdot,\cdot)$ — reads

$$(f,g) = |c|^2 \int_{\mathbb{C}^2} f(z)\overline{g(z)} e^{-|z|^2} dxdy,$$

We decide to take $c = \pi^{-1}$, so that the norm of the constant function $z \longmapsto 1$ equals one. This in turn corresponds to the definition of the 2-dimensional Segal–Bargmann space (see [Hal99]; the origins of the definition of the Segal–Bargmann space trace back to [Bar61; Seg63], with Segal actually working in the infinite-dimensional setting).

**Definition 4.9.** Let $d$ be a positive integer, and $q \in \mathbb{R}_{>0}$. The $q$-normalised, $d$-dimensional Segal–Bargmann space $\mathcal{F}_{\mathbb{C}}$ is the Hilbert space of holomorphic functions on $\mathbb{C}^d$ which have finite length for the norm induced by the scalar product

$$(f,g) = (q\pi)^{-d} \int_{\mathbb{C}^d} f(z)\overline{g(z)} e^{-\frac{|z|^2}{q}} dxdy.$$

The Gaussian weight assures that all polynomials will be elements of the space, with the monomials providing an orthogonal basis. Moreover, the formal adjoint of the operator $\mu_{z_j} : f \longmapsto z_j f$ is $q\partial_{z_j} : f \longmapsto q\frac{\partial f}{\partial z_j}$, as it can be shown integrating by parts:

$$\begin{aligned}(q\partial_{z_j} f, g) &= q\pi^{-2} \int_{\mathbb{C}^2} (\partial_{z_j} f)\overline{g} e^{-\frac{|z|^2}{q}} dxdy = -q\pi^{-2} \int_{\mathbb{C}^2} f \partial_{z_j}\left(\overline{g} e^{-\frac{|z|^2}{q}}\right) dxdy = \\ &= \pi^{-2} \int_{\mathbb{C}^2} f(\overline{z_j g}) e^{-\frac{|z|^2}{q}} dxdy = (f, z_j g).\end{aligned}$$

In this passages we used the fact that $z \longmapsto f(z)\overline{g}(z)e^{-|z|^2}$ goes to zero sufficiently fast for $|z| \longrightarrow +\infty$, since $(f,g) < +\infty$, and also that $\partial_{z_j}\overline{g} = 0$ since $\overline{g}$ is antiholomorphic.

It was Fock who first remarked that the equality $\left[q\partial_{z_i}, z_j\right] = \delta_{ij} q$ holds when these differential operators act on holomorphic functions defined on $\mathbb{C}^d$. Bargmann enriched

this idea by finding a suitable Hermitian product so that moreover these operators will be reciprocally adjoint.[8]

Going back to our discussion where $d = 2$ and $q = 1$ (abusively omitted from the notation), let us similarly denote by $\mathcal{F}_{\mathbb{R}}$ the space of functions $\psi\colon \mathbb{R}^4 \longrightarrow \mathbb{C}$ which correspond to real-polarised sections, i.e. functions which are constant along the $y$-direction whose restriction to $\mathbb{R}^2 \oplus (0) \subseteq V$ is square-summable for the 2-dimensional Lebesgue measure $dx$. The transform $\mathcal{B}\colon \mathcal{H}_{\mathbb{R}} \longrightarrow \mathcal{H}_{\mathbb{C}}$ corresponds to a linear map $\mathcal{I}\colon \psi \longmapsto \widehat{\psi}$ acting on functions, by means of $\mathcal{B}(\psi r) = \widehat{\psi} s$. In turn, we require the following property for $\mathcal{I}$:

$$\left(\widehat{\psi}, f\right) = \langle \psi r \mid f s\rangle \in \mathbb{C},$$

for $\psi \in \mathcal{F}_{\mathbb{R}}$ and $f \in \mathcal{F}_{\mathbb{C}}$. Since by definition $\left(\widehat{\psi}, f\right) = \langle \widehat{\psi} s \mid f s\rangle\!\rangle$, it is the same to ask that

$$\left\langle\!\left\langle \mathcal{B}(\psi r) \mid f s\right\rangle\!\right\rangle = \langle \psi r \mid f s\rangle. \tag{4.16}$$

On both sides of (4.16) one sees the restriction — to closed subspaces — of the inner product on the prequantum Hilbert space of smooth sections of $L$. On the right-hand side one considers the restriction to $\mathcal{H}_{\mathbb{R}} \times \mathcal{H}_{\mathbb{C}}$, which yields a nondegenerate pairing $\mathcal{H}_{\mathbb{R}} \otimes \mathcal{H}_{\mathbb{C}} \longrightarrow \mathbb{C}$, since the two polarisations are transverse.[9] The transversality is due to the complex span of $\{\partial_{\overline{z_1}}, \partial_{\overline{z_2}}\} = \left\{\frac{1}{\sqrt{2}}(\partial_{x_1} + i\partial_{y_1}), \frac{1}{\sqrt{2}}(\partial_{x_2} + i\partial_{y_2})\right\}$ intersecting the complex span of $\{\partial_{y_1}, \partial_{y_2}\}$ only at zero inside $V \otimes_{\mathbb{R}} \mathbb{C}$, which is identified with the complexified tangent space to $V$ at any point.
On the left-hand side of (4.16) instead one sees the restriction of $\langle \cdot \mid \cdot \rangle$ to $\mathcal{H}_{\mathbb{C}} \times \mathcal{H}_{\mathbb{C}}$, which defines the inner product on the Hilbert subspace of Kähler-polarised sections. Note that since the pairing is nondegenerate then this property determines $\mathcal{B}$ and $\mathcal{I}$ uniquely.

The computation of an explicit formula for the transform $\mathcal{I}\colon \mathcal{F}_{\mathbb{R}} \longrightarrow \mathcal{F}_{\mathbb{C}}$ is a different task. For doing it, one exploits the fact that $\mathcal{F}_{\mathbb{C}}$ is a *reproducing kernel Hilbert space*, which means that function evaluation defines continuous functionals: for all $a \in \mathbb{C}^2$ there exists a (unique) function $e_a \in \mathcal{F}_{\mathbb{C}}$ such that $f(a) = (f, e_a)$ for all $f \in \mathcal{F}_{\mathbb{C}}$.

**Definition 4.10.** The function $e_a$ is the coherent state at the point $a \in \mathbb{C}^2$.

Using (4.16) one thus finds

$$\widehat{\psi}(a) = \left(\widehat{\psi}, e_a\right) = \langle \psi r \mid e_a s\rangle,$$

and this determines the value of the function $\widehat{\psi}$ at all point. To compute the coherent states explicitly one relies on the following fact.

**Proposition 4.19.** *Polynomial functions are dense in the Segal–Bargmann space. More precisely, the monomials defined by*

$$u_{m_1,m_2}(z) = \frac{z_1^{m_1} z_2^{m_2}}{\sqrt{m_1! m_2!}},$$

[8] The idea behind this is to take $q = \hbar$ and to construct representations of the $d$-dimensional Weyl algebra $W(T^*\mathbb{R}^d, \omega_{\text{can}})$ of 2.5 on $\mathcal{F}_{\mathbb{C}}$. We will come back to that after deriving an explicit formula for the Bargmann transform.

[9] The intrinsic definition of the pairing requires a choice of half-forms which we tacitly assume. This choice influences the pairing only up to a multiplicative factor (see [GS81]).

*provide a orthonormal basis of* $\mathcal{F}_{\mathbb{C}}$*, where* $m_1, m_2 \in \mathbb{Z}_{\geq 0}$.

*Proof.* The integrals involved split into one-dimensional integrals for the two complex variables $z_1, z_2$, and thus the result follows from the analogous one on $\mathbb{C}$ (i.e. for $d = 1$).

If $m, n$ are nonnegative integers, one finds:

$$\int_{\mathbb{C}} z^m \overline{z}^n e^{-|z|^2} dxdy = \int_0^{+\infty} \int_0^{2\pi} \rho^m e^{im\theta} \rho^n e^{-in\theta} e^{-\rho^2} \rho d\rho d\theta,$$

after passing to polar coordinates. The right-hand side equals

$$\int_0^{+\infty} \rho^{m+n+1} e^{-\rho^2} \left( \int_0^{2\pi} e^{i(m-n)\theta} d\theta \right) d\rho,$$

and the integral $\int_0^{2\pi} e^{i(m-n)\theta} d\theta$ vanishes if $m - n \neq 0$; otherwise it equals $2\pi$. This establishes orthogonality.

Moreover, the orthogonal system $\{z_m\}_{m\geq 0}$ is complete. Indeed, if $f$ is any entire function defined on $\mathbb{C}$, then one can expand in power series $f = \sum_{k\geq 0} f_k z^k$, and the condition $(f, z_m) = 0$ gives $a_m = 0$. Hence the orthogonal subspace to the span of all monomials is reduced to zero.

As for orthonormality, if $n = m$ the computation above yields

$$\int_{\mathbb{C}} z^m \overline{z}^n e^{-|z|^2} dxdy = 2\pi \int_0^{+\infty} \rho^{2n+1} e^{-\rho^2} d\rho = \pi \int_0^{+\infty} r^n e^{-r} dr = \pi \Gamma(n+1) = \pi n! \ ,$$

where we substituted $r = \rho^2$ in the second passage. Hence we see that the set of monomials $\left\{ \frac{z^n}{\sqrt{n!}} \right\}_{n\geq 0}$ provides an orthonormal Hilbert basis inside $\mathcal{F}_{\mathbb{C}}$ for $d = q = 1$, and the very same proof can be used for $d = 2$, $q = 1$. □

**Corollary 4.2.** *The function* $e_a(z) \coloneqq e^{\overline{a}\cdot z}$ *is the coherent state at* $a \in \mathbb{C}^2$.

*Proof.* Consider the following expansion:

$$\sum_{m_1,m_2\geq 0} \overline{u_{m_1,m_2}(a)} u_{m_1,m_2}(z) = \sum_{m_1,m_2\geq 0} \frac{(\overline{a_1} z_1)^{m_1}}{m_1!} \frac{(\overline{a_2} z_2)^{m_2}}{m_2!} = \prod_{i=1}^{2} \sum_{m_i \geq 0} \frac{(\overline{a_i} z_i)^{m_i}}{m_i!} =$$

$$= \prod_{i=1}^{2} \exp(\overline{a_i} z_i) = \exp\left( \sum_{i=1}^{2} \overline{a_i} z_i \right) = e^{\overline{a}\cdot z} = e_a(s).$$

Infinite sums and finite products can be intertwined because all power series converge, and this means that $e_a = \sum_{m_1,m_2\geq 0} \overline{u_{m_1,m_2}(a)} u_{m_1,m_2}$.

Now introduce a 2-index $m = (m_1, m_2) \in \left(\mathbb{Z}_{\geq 0}\right)^2$, and pick any function $f \in \mathcal{F}_{\mathbb{C}}$. Its decomposition in the orthonormal basis of 4.19 is $f = \sum_m f_m u_m$, with coefficients given by $f_m = (f, u_m)$, so that one has by definition $f(a) = \sum_m f_m u_m(a)$. Hence, if $n$ is also a 2-index:

$$(f, e_a) = \sum_{m,n} (f_m u_m, \overline{u_n(a)} u_n) = \sum_{m,n} f_m u_n(a) \delta_{mn} = \sum_m f_m u_m(a) = f(a).$$

□

This proposition means that one has the integral representation formula

$$f(a) = \frac{1}{\pi^2}\int_{\mathbb{C}^2} e^{a\cdot\overline{z}} f(z) e^{-|z|^2} dxdy,$$

for all entire function $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}$ which is square-summable for the measure $e^{-|z|^2}dxdy$.

Using the explicit formula for the coherent state, one finds the following for a function $\psi \in \mathcal{F}_{\mathbb{R}}$:

$$\widehat{\psi}(a) = \Big(\widehat{\psi}, e_a\Big) = \langle \psi r \mid e_a s\rangle = \frac{1}{\pi}\int_{\mathbb{C}^2} h(\psi r, e_a s) dxdy =$$

$$= \frac{1}{\pi}\int_{\mathbb{C}^2} \psi(z)\overline{e_a(z)} h(r, e^{\varphi} r) dxdy = \frac{1}{\pi}\int_{\mathbb{C}^2} \psi(z) e^{a\cdot\overline{z}+\overline{\varphi(z)}} dxdy,$$

where we introduced the change of frame $s = e^{\varphi} r$. It follows from the definition of $r$ and $s$ that $\varphi$ is the quadratic form $\varphi(x,y) = -\frac{1}{2}\Big(|z|^2 + ix\cdot y\Big)$, and plugging-in the value of $\overline{\varphi(z)}$ finally yields the following integral transform for the smooth function $\psi$:

$$\Big(\mathcal{I}(\psi)\Big)(a) = \widehat{\psi}(a) = \frac{1}{\pi}\int_{\mathbb{C}^2} \psi(z) \exp\left\{a\cdot\overline{z} - \frac{1}{2}|z^2| + \frac{i}{2}x\cdot y\right\} dxdy. \tag{4.17}$$

One sees that the result is holomorphic, by taking the derivative with respect to the new variable $a$ under the integral sign.

**Remark 4.24.** One should spend a few words about the changing of frames. Let then $r'$ be another real-polarised unitary frame of $L$, and call $\varphi_r\colon \mathcal{H}_{\mathbb{R}} \longrightarrow \mathcal{F}_{\mathbb{R}}$ the identification $\psi r \longmapsto \psi$. The new frame $r'$ provides a second identification $\varphi_{r'}\colon \mathcal{H}_{\mathbb{R}} \longrightarrow \mathcal{F}_{\mathbb{R}}$, which differs by a phase. Namely, there exists a function $\chi\colon V \longrightarrow \mathbb{C}$ such that $r' = e^{\chi} r$, and the unitarity condition yields $\chi(V) \subseteq i\mathbb{R}$ (also $\partial_{y_j}\chi = 0$ in order that $r'$ be polarised). Now $\varphi_{r'}\colon \psi r = \psi e^{-\chi} r' \longmapsto \psi e^{-\chi} \in \mathcal{F}_{\mathbb{R}}$.
If the Kähler-polarised frame $s$ is fixed, then one has to compare the number $\widehat{\psi}(a) \in \mathbb{C}$ defined by (4.17) with the pairing $\langle \psi e^{-\chi} r' \mid e_a s\rangle$, where the coherent state at $a$ hasn't changed. The relation between $s$ and $r'$ is $s = e^{\varphi} r = e^{\varphi} e^{-\chi} r'$, and thus

$$h(\psi e^{-\chi} r', e_a s) = \psi\overline{e_a} e^{\overline{\varphi}} |e^{-\chi}|^2 h(r', r') = \psi\overline{e_a} e^{\overline{\varphi}},$$

where we used that $r'$ and $e^{\chi}$ are unitary. Hence $\widehat{\psi}(a) = \langle \psi e^{-\chi} r' \mid e_a s\rangle$, and the resulting map $\mathcal{H}_{\mathbb{R}} \longrightarrow \mathcal{H}_{\mathbb{C}}$ does not depend on the choice of a *unitary* real-polarised frame.

The choice of a holomorphic frame instead is more rigid, because one wants an identification with the Segal–Bargmann space. More precisely, one has to impose the condition that $\Big(\partial_{\overline{z_j}} + \frac{1}{2} z_j\Big)\, s = 0$ in order to have a holomorphic frame, but there is also a further constraint if one wants that the differential operators $z_j$ and $t\partial_{z_j}$ be reciprocally adjoint for $j = 1, 2$, where $t > 0$ is the parameter appearing in Def. 4.9. Namely, the weight $h(s,s) = |s|^2\colon V \longrightarrow \mathbb{R}_{>0}$ must be annihilated by the differential operator $\partial_{z_j} + \frac{1}{t}\overline{z_j}$, because this is precisely what is needed to show that $(t\partial_{z_j} f, g) = (f, z_j g)$ via an integration by parts.
Hence the choice of $s$ is fixed up to a normalisation constant, and finally we fix this constant according to Def. 4.9.

One can now move on by splitting (4.17) in an integral along the Lagrangian and the transverse. Since $\partial_{y_j}\psi = 0$ for $j = 1, 2$ then one may write

$$\widehat{\psi}(a) = \int_{\mathbb{R}^2} \psi(x) A(x, a) dx,$$

where the kernel $A\colon \mathbb{R}^2 \times \mathbb{C}^2 \longrightarrow \mathbb{C}$ is essentially the Fourier transform of a Gaussian:

$$A(x, a) \coloneqq \pi^{-1} \int_{\mathbb{R}^2} \exp\left\{a \cdot \overline{z} - \frac{1}{2}|z|^2 + ix \cdot y\right\} dy.$$

**Proposition 4.20.** *One has:*

$$A(x, a) = \frac{2}{\sqrt{\pi}} \exp\left\{-\frac{1}{2}\left(x^2 - 2\sqrt{2}a \cdot x + a^2\right)\right\}.^{10}$$

In the next section we will provide a proof in a more general context, which implies this proposition as a corollary (see Prop. 4.23). We thus recovered the usual formula for Bargmann kernel, although one has to take care of a multiplicative factor to achieve the unitarity of $\mathcal{B}$ (cf. [Hal13]).

Finally, one should recall the quantum-mechanical relevance of the transform. As mentioned before, the operators $\mu_{z_j}$ and $\partial_{z_j}$ act on $\mathcal{F}_{\mathbb{C}}$, and are reciprocally adjoint. This defines a representation of the 2-dimensional Weyl algebra on the Segal–Bargmann space (i.e. the algebra defined by the canonical commutation relations), which is the Fock representation. The Weyl algebra also acts on $\mathcal{F}_{\mathbb{R}}$, by means of the *annihilation* and *creation* operators, i.e. respectively

$$a_j \coloneqq \frac{Q_j + iP_j}{\sqrt{2}}$$
$$a_j^* \coloneqq \frac{Q_j - iP_j}{\sqrt{2}},$$

where $Q_j\mu_{x_j}$ is the *position* operator, and $P_j = -i\partial_{x_j}$ the *momentum* operator. Since $Q_j$ and $P_j$ are self-adjoint, it follows that $a_j^\dagger$ is the formal adjoint of $a_j$, as the notation suggests. These operators make $\mathcal{F}_{\mathbb{R}}$ into a module for the Weyl algebra as well, and one may show that both modules are irreducible. The Stone–Von Neumann theorem thus suggests that there should exist a unitary map $\mathcal{F}_{\mathbb{R}} \longrightarrow \mathcal{F}_{\mathbb{C}}$ that intertwines the two representations: the Schrödinger position representation corresponds to the Fock representation. The Bargmann transform (4.17) is precisely this map, up to a multiplicative constant.

**Proposition 4.21.** *One has, for $j \in \{1, 2\}$:*

$$\begin{cases} \mathcal{I} \circ a_j \circ \mathcal{I}^{-1} &= \partial_{z_j} \\ \mathcal{I} \circ a_j^\dagger \circ \mathcal{I}^{-1} &= \mu_{z_j}. \end{cases}$$

[10] Note that $x^2 = x \cdot x = |x|^2$, but $a^2 = a \cdot a \neq a \cdot \overline{a} = |a|^2$. This is important to get the commutation relations of Prop. (4.18).

Summing and subtracting the relations of Prop. (4.21), one finds:

$$\begin{cases} \mathcal{I} \circ \mu_{x_j} \circ \mathcal{I}^{-1} &= \frac{\partial_{z_j} + \mu_{z_j}}{\sqrt{2}} \\ \mathcal{I} \circ \partial_{x_j} \circ \mathcal{I}^{-1} &= \frac{\partial_{z_j} - \mu_{z_j}}{\sqrt{2}}. \end{cases} \tag{4.18}$$

These are the actual commutation relations that we will generalise to the context where the linear polarisations vary along the Teichmüller space of a genus one closed orientable surface. We will achieve this in Thm. 4.6, incidentally deriving Prop. 4.21 as a corollary.

With this aim in mind, let us consider generic linear polarisations $P_{\mathbb{R}}, P$. The space $V$ is thus equipped with a $\omega$-compatible complex structure $I$ corresponding to $P$ and a preferred Lagrangian subspace $\mathcal{L} \subseteq V$ corresponding to $P_{\mathbb{R}}$. One can turn this situation back to the previous one, up to a linear symplectomorphism. Namely, the sum $\mathcal{L} + I(\mathcal{L}) \subseteq V$ is direct, because if $g = \omega\Big(\cdot, I(\cdot)\Big)$ is the real scalar product associated to $\omega$ and $I$, then $I(\mathcal{L}) = \mathcal{L}^{\perp_g}$ is the $g$-orthogonal space to $\mathcal{L}$. Hence $V = \mathcal{L} \oplus I(\mathcal{L})$, and the (constant) tensors $\omega$ and $I$ have the canonical formulae when written in a basis $\{x_j, y_j\}_j$ of $V$ such that $y_j$ is a $g$-orthonormal basis of $\mathcal{L}$, and where $x_j \coloneqq -I(y_j)$. If one considers the basis as a set of global (linear) coordinates $dx_j, dy_j \colon V \longrightarrow \mathbb{R}$, then

$$\omega = \sum_j dx_j \wedge dy_j, \qquad I = \sum_j dx_j \otimes \partial_{y_j} - dy_j \otimes \partial_{x_j}.$$

Moreover, one has a Lagrangian splitting of $(V, \omega)$, since $\omega$ provides an identification $I(\mathcal{L}) \cong \mathcal{L}^*$ which induces an isomorphism $V \cong \mathcal{L} \oplus \mathcal{L}^* \cong T^*\mathcal{L}$. This basically amounts to the choice of global normal coordinates for the Kähler manifold $(V, \omega, I)$, and shows that $P_{\mathbb{R}}$ and $P$ are transverse.

In the next section we will pursue this idea further to get more explicit formulae, but one can already conclude at this stage that there exists a transform $\mathcal{B} \colon \mathcal{H}_{\mathbb{R}} \longrightarrow \mathcal{H}_{\mathbb{C}}$ satisfying (4.16) for all choice of $P_{\mathbb{R}}$ and $P$, because the pairing $\mathcal{H}_{\mathbb{R}} \otimes \mathcal{H}_{\mathbb{C}} \longrightarrow \mathbb{C}$ is always nondegenerate. Moreover, (4.16) alone implies that this generalised transform commutes with the gauge-action on polarised sections, i.e. with the translations $\lambda \in \Lambda_0$ and the flip $h$, acting both on $\mathcal{H}_{\mathbb{R}}$ and $\mathcal{H}_{\mathbb{C}}$.

To prove this, consider the following.

**Lemma 4.6.** *The translation operators $\lambda \colon C^\infty(V, L) \longrightarrow C^\infty(V, L)$ are $\langle \cdot \mid \cdot \rangle$-unitary for all $\lambda \in \Lambda^0$, and the flip $h$ is (formally) self-adjoint.*

*Proof.* The first statement follows from the fact that the Lebesgue measure is invariant under translations, and the cocycle $\Theta$ takes values in $U(1)$. Explicitly, if $s$ is a smooth section of $L$ and $\lambda \in \Lambda_0$, one has

$$\begin{aligned} \Big|\lambda.s\Big|^2 = \langle \lambda.s \mid \lambda.s \rangle &= \int_V h\Big(\lambda.s(v), \lambda.s(v)\Big)\mu(v) = \\ &= \int_V h\Big(\Theta(\lambda, v-\lambda)s(v-\lambda), \Theta(\lambda, v-\lambda)s(v-\lambda)\Big)\mu(v) = \\ &= \int_V \Big|\Theta(\lambda, v-\lambda)\Big|^2 h\Big(s(v), s(v)\Big)\mu(v) = \int_V h\Big(s(v), s(v)\Big)\mu(v) = \\ &= \langle s \mid s \rangle = |s|^2, \end{aligned}$$

where we used $\left|\Theta(\lambda, v-\lambda)\right| = 1$ and the change of variable $-\lambda\colon v \longmapsto v-\lambda$, satisfying $-\lambda^*(\mu) = \mu$.

The analogous proof for the flip $h.s(z) = s(-z)$ just follows from the fact that $V$ has even real dimension. □

This lemma implies that the formal adjoint to $\lambda$ is $\lambda^{-1} = -\lambda$, for all $\lambda \in \Lambda_0$. This is enough to conclude what one wants.

**Corollary 4.3.** *The Bargmann transform $\mathcal{B}\colon \mathcal{H}_{\mathbb{R}} \longrightarrow \mathcal{H}_{\mathbb{C}}$ commute with the $\mathcal{G}^0$-action for any choice of a real polarisation $P_{\mathbb{R}}$ and a Kähler polarisation $P$.*

*Proof.* Choose $\psi \in \mathcal{F}_{\mathbb{R}}$, $f \in \mathcal{F}_{\mathbb{C}}$ and $\lambda \in \Lambda_0$. One has, using (4.16) and the previous lemma twice:

$$\Big\langle \mathcal{B}(\lambda.\psi r) \mid fs \Big\rangle = \langle \lambda.(\psi r) \mid fs\rangle = \langle \psi r \mid (-\lambda).fs\rangle = \Big\langle \mathcal{B}(\psi.r) \mid (-\lambda).fs \Big\rangle = \Big\langle \lambda.\mathcal{B}(\psi r) \mid fs \Big\rangle.$$

Since $\langle \cdot \mid \cdot \rangle$ is nondegenerate when restricted to $\mathcal{H}_{\mathbb{C}}$, this proves that $\mathcal{B}(\lambda.\psi r) = \lambda.\mathcal{B}(\psi r)$. The important point is that the formal adjoint of $\lambda$ preserves polarised sections, and thus all passages makes sense.

The proof for the group element $h$ is exactly the same. □

In summary, in this section we have shown the following.

**Theorem 4.4.** *The Bargmann transform on 4-dimensional flat space induces a map $\mathcal{B}\colon \mathcal{H}_{\mathbb{R}}^{\mathcal{G}^0} \longrightarrow \mathcal{H}_{\mathbb{C}}^{\mathcal{G}^0}$ satisfying (4.16) and (4.18), for all choice of a linear real polarisation $P_{\mathbb{R}}$ and a linear Kähler polarisation $P$ on $(V, \omega)$.*

**Remark 4.25.** Let us get back to our original viewpoint, where the polarisations are attached to a point the Teichmüller space of closed surfaces of genus one.

It follows from the very definition of the Bargmann transform that $\mathcal{B}$ is compatible with the mapping class group action on the quantum bundles over Teichmüller space. Namely, recall from § 4.2.7 that the mapping class group $\Gamma \cong \mathrm{SL}(2,\mathbb{Z})$ acts symplectically on the moduli space (Prop. 4.9), that this action lifts to the Chern–Simons prequantum bundle fixing all prequantum data, and that the induced action on $L^2$-summable polarised sections of the prequantum bundle covers the standard action of $\Gamma$ on Teichmüller space (Prop. 4.9). Since this action preserves the $L^2$-inner product, it also preserves the nondegenerate pairing $\mathcal{H}_{\mathbb{R}} \otimes \mathcal{H}_{\mathbb{C}} \longrightarrow \mathbb{C}$, and thus also the transformation $\mathcal{B}$ — since this transform satisfies (4.16).

To give more details one needs to introduce quite a lot of notation. Pick $\psi \in \mathcal{F}_{\mathbb{R}}$, $f \in \mathcal{F}_{\mathbb{C}}$ and $\gamma \in \Gamma$. Let us explicitly introduce the symbol $\tau \in \mathbb{H}$ for the element of Teichmüller space used to define the polarisations. Then the sections $\gamma.r$ and $\gamma.s$ are now frames of $L$ which are horizontal for the real and Kähler polarisation — respectively — arising from $\gamma^*\tau \in \mathbb{H}$ (thinking of $\tau$ as a marked Riemann structure $\mathbb{C}\big/\Lambda_\tau \longrightarrow \mathbb{R}\big/\mathbb{Z}$ on the real torus). The section $\gamma.(\psi r)$ will be written $\gamma^*\psi(\gamma.r)$ in the new frame, and similarly $\gamma.(fs) = \gamma^* f(\gamma.s)$, where $\psi \in \mathcal{F}_{\mathbb{R}}$, $f \in \mathcal{F}_{\mathbb{C}}$.

Let us denote $\mathcal{H}_{\mathbb{R},\tau}, \mathcal{H}_{\mathbb{R},\gamma^*\tau}$ the spaces of real-polarised sections, and $\mathcal{H}_{\mathbb{C},\tau}, \mathcal{H}_{\mathbb{C},\gamma^*\tau}$ those of Kähler-polarised ones; then let $\mathcal{F}_{\mathbb{R}}$, $\mathcal{F}_{\mathbb{C}}$, $\mathcal{F}_{\mathbb{R},\gamma^*\tau}$ and $\mathcal{F}_{\mathbb{C},\gamma^*\tau}$ be the associated spaces of complex-valued functions in the frames $r, s, \gamma.\tau, \gamma.s$, respectively. Write $(\cdot,\cdot)_\tau$ and $(\cdot,\cdot)_{\gamma^*\tau}$ for the inner products of $\mathcal{H}_{\mathbb{C},\tau}$ and $\mathcal{H}_{\mathbb{C},\gamma^*\tau}$, and finally add subscripts $\mathcal{I}_\tau\colon \mathcal{F}_{\mathbb{R}} \longrightarrow \mathcal{F}_{\mathbb{C}}$, $\mathcal{I}_{\gamma^*\tau}\colon \mathcal{F}_{\mathbb{R},\gamma^*\tau} \longrightarrow \mathcal{F}_{\mathbb{C},\gamma^*\tau}$ for the integral transforms.

Then one has:

$$(\gamma^*\widehat{\psi}_\tau, \gamma^* f)_{\gamma^*\tau} = (\widehat{\psi}_\tau, f)_\tau = \langle \psi r \mid fs\rangle = \langle \gamma.(\psi r) \mid \gamma.(fs)\rangle = \langle \gamma^*\psi(\gamma.r) \mid \gamma^* f(\gamma.s)\rangle = \\ = (\widehat{\gamma^*\psi}_{\gamma^*\tau}, \gamma^* f)_{\gamma^*\tau}.$$

Using that $f$ is arbitrary, that $\gamma$ is an isomorphism of the quantum bundle for Kähler polarisations, and that $(\cdot,\cdot)_{\gamma^*\tau}$ is nondegenerate, one gets $\gamma^*\widehat{\psi}_\tau = \widehat{\gamma^*\psi}_{\gamma^*\tau}$. This means precisely that the $\Gamma$-action intertwines naturally with the Bargmann transform.
Hence, in brief, the transform of Thm. 4.4 is $\Gamma$-equivariant.

### 4.6.3 The Bargmann transform: symplectic transverse

We now need to adapt the previous abstract construction to the case which is relevant for us. The main technical issue is that we consider a global transverse to the real polarisation which is symplectic, and not Lagrangian.

Let us then fix a quantum level $t = k + i\sigma \in \mathbb{C}^*$ and a point $\tau \in \mathbb{H}$, and specialise the abstract objects of the previous section to $W = \mathcal{A}^0$ (§ 4.1), $V = \mathfrak{A}^0$ (§ 4.2.1), $\omega = \omega_t$, $\alpha = \alpha_t$ (§ 4.2.3), $(L, \nabla, h) = \left(\widetilde{L}^{(t)}, \widetilde{\nabla}^{(t)}, \widetilde{h}^{(t)}\right)$ (§ 4.2.4) and finally $P_{\mathbb{R},\tau}$ and $P_\tau$ defined as in § 4.2.6 and § 4.2.5. The Hilbert spaces $\mathcal{H}_{\mathbb{R},\tau}, \mathcal{H}_{\mathbb{C},\tau} \subseteq C^\infty\left(\mathfrak{A}^0, \widetilde{L}^{(t)}\right)$ are also defined.
The restricted gauge group $\mathcal{G}^0$ acts on these data, defining the moduli spaces

$$\mathcal{M} \cong \mathcal{A}^0\big/\mathcal{G}^0 \subseteq \mathfrak{A}^0\big/\mathcal{G}^0 \cong \mathfrak{M},$$

the prequantum data $\left(L^{(t)}, \nabla^{(t)}, h^{(t)}\right)$ and the Hilbert spaces $\mathcal{H}^{\mathcal{G}^0}_{\mathbb{R},\tau}, \mathcal{H}^{\mathcal{G}^0}_{\mathbb{C},\tau} \subseteq C^\infty\left(\mathfrak{M}, L^{(t)}\right)$, which are the spaces for the geometric quantisation of $\mathfrak{M}$.

In the previous section § 4.6.2 we constructed an isomorphism $\mathcal{B}_\tau\colon \mathcal{H}_{\mathbb{R},\tau} \longrightarrow \mathcal{H}_{\mathbb{C},\tau}$ that commutes with the $\mathcal{G}^0$-action. Its explicit formula depends on the choice of a unitary real-polarised frame $r$ and a Kähler polarised frame $s$ of $\widetilde{L}^{(t)}$. Such frames are easy to compute in a Darboux basis for $\omega_t$, but here we wish to do something different. Namely, we wish to identify $\mathcal{H}_{\mathbb{R},\tau}$ with the space of smooth functions on the symplectic global transverse $\mathcal{A}^0$ to the real polarisation $P_{\mathbb{R},\tau}$, and thus we look at the splitting $\mathfrak{A}^0 \cong \mathcal{A}^0 \oplus \mathcal{L}_\tau$, where $\mathcal{L}_\tau$ is the Lagrangian space defining $P_{\mathbb{R},\tau}$.

Let us start by getting the simplest possible expression for the symplectic form $\omega_t$ in this context.

**Proposition 4.22.** *There exists a basis $\{x_1, x_2, y_1, y_2\}$ of $\mathfrak{A}^0$ such that $\{x_1, x_2\}$ is a basis*

*of $\mathcal{A}^0$, $\{y_1, y_2\}$ is a basis of $\mathcal{L}_\tau$, and one has*

$$\omega_t = \begin{pmatrix} 0 & k & 1 & 0 \\ -k & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

*Proof.* Looking at the formulae of § 4.2.6, one sees that $\omega_t$ has matrix $\begin{pmatrix} 0 & 0 & k & -\sigma \\ 0 & 0 & -\sigma & -k \\ -k & \sigma & 0 & 0 \\ \sigma & k & 0 & 0 \end{pmatrix}$ in the usual $\tau$-independent global de Rham coordinates $u_1, v_1, u_2, v_2$, up to a scaling which cancels the factor $\pi^{-1}$ (this is the matrix of a real alternating bilinear form on $\mathfrak{A}^0$). Moreover, the Lagrangian subspace $\mathcal{L}_\tau \subseteq \mathfrak{A}^0$ that corresponds to $P_{\mathbb{R},\tau}$ is the real span of the vectors

$$X_1 \coloneqq \begin{pmatrix} 1 \\ 0 \\ \tau_1 \\ \tau_2 \end{pmatrix}, \qquad X_2 \coloneqq \begin{pmatrix} 0 \\ -1 \\ \tau_2 \\ -\tau_1 \end{pmatrix},$$

expressed in the same basis, and $\mathcal{A}^0 \subseteq \mathfrak{A}^0$ is the real span of $\{v_1, v_2\}$.

Hence the matrix of $\omega_t \in \Lambda^2\Big(\mathfrak{A}^0\Big)^*$ in the new basis $\{x_1, x_2, X_1, X_2\} \coloneqq \{v_1, -v_2, X_1, X_2\}$ becomes

$$\begin{pmatrix} 0 & k & -(\sigma\tau_1 + k\tau_2) & k\tau_1 - \sigma\tau_2 \\ -k & 0 & -\sigma & k \\ \sigma\tau_1 + k\tau_2 & \sigma & 0 & 0 \\ \sigma\tau_2 - k\tau_1 & -k & 0 & 0 \end{pmatrix}.$$

Note that the determinant of this matrix is the real number $(|t|^2\tau_2)^2 \neq 0$.

Now one can change basis within $\mathcal{L}_\tau$ to get to the matrix given in the statement. Explicitly:

$$y_1 \coloneqq -\frac{1}{|t|^2\tau_2}(kX_1 + \sigma X_2), \qquad y_2 \coloneqq \frac{1}{|t|^2\tau_2}\Big((k\tau_1 - \sigma\tau_2)X_1 + (\sigma\tau_1 + k\tau_2)X_2\Big).$$

□

This means that in the new basis one has $\omega_t = \omega_{\text{can}} + k dx_1 \wedge dx_2$, where $\omega_{\text{can}}$ is the canonical symplectic form $\omega_{\text{can}} = \sum_j dx_j \wedge dy_j$. In words, one must add to the canonical structure the real Atiyah–Bott symplectic form $\omega \coloneqq dx_1 \wedge dx_2$ on the space of unitary flat connections — at level $k > 0$ — which is precisely the expected restriction of $\omega_t$ to $\mathcal{A}^0$. The Liouville form still reads

$$\mu = \frac{\omega_t \wedge \omega_t}{2} = dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2$$

in these coordinates. Similarly, the symplectic potential defining the prequantum connection becomes $\alpha_t = \alpha_{\text{can}} + \frac{k}{2}(x_1 dx_2 - x_2 dx_1)$, where as usual $\alpha_{\text{can}} = \frac{1}{2}\sum_j x_j dy_j - y_j dx_j$.

The 1-form $\alpha \coloneqq \frac{1}{2}(x_1 dx_2 - x_2 dx_1)$ is the standard symplectic potential for $\omega$, and the prequantum connection $\widetilde{\nabla}^{(t)}$ restricts to $\nabla \coloneqq d - ik\alpha$ on $\mathcal{A}^0$. Recall that this is a connection on $\iota^* \widetilde{L}^{(t)} = \widetilde{L}^{\otimes k}$, where $\iota \colon \mathcal{A}^0 \hookrightarrow \mathfrak{A}^0$ is the $\tau$-independent de Rham embedding, and that $\widetilde{L}$ descends to the Chern–Simons prequantum line bundle $L = \widetilde{L}\big/\mathcal{G}^0 \longrightarrow \mathcal{M}$ over the compact moduli space.

From this explicit description, one sees that the covariant derivatives along the Lagrangian are the differential operators

$$\widetilde{\nabla}^{(t)}_{\partial_{y_j}} = \partial_{y_j} - i\langle \alpha_t, \partial_{y_j} \rangle = \partial_{y_j} - \frac{i}{2} x_j,$$

for $j \in \{1, 2\}$. Hence the function $r(x, y) \coloneqq e^{\frac{i}{2} x \cdot y}$ still defines a unitary real-polarised frame of $\widetilde{L}^{(t)}$ — equal to 1 on $\mathcal{A}^0$ — thereby providing the usual identification of Hilbert spaces $\mathcal{H}_{\mathbb{R},\tau} \cong C^\infty(\mathcal{A}^0, \mathbb{C}) \cap L^2(\mathcal{A}^0, dx_1 dx_2)$. Moreover, considering the gauge-group action one will have (temporarily neglecting the $L^2$-norm) $\mathcal{H}^{\mathcal{G}^0}_{\mathbb{R},\tau} \cong C^\infty(\mathcal{M}, L^{\otimes k})$, where $L \longrightarrow \mathcal{M}$ is the Chern–Simons prequantum bundle for the compact theory (cf. 2.3.3).
The computation of an $I^{(t)}$-holomorphic frame is more involute (if $\sigma \neq 0$ then $\mathcal{A}^0$ is not even preserved by the complex structure). One viable option is to look for a symplectomorphism that turns the situation back to the standard case.

Concretely, denote $g = \omega_t\Big(\cdot, I^{(t)}(\cdot)\Big)$ the Kähler scalar product associated to $\omega_t$ and construct a $g$-orthonormal basis $\{y'_1, y'_2\}$ of $\mathcal{L}$. Next, define $x'_j \coloneqq -I^{(t)}(y'_j) \in \mathfrak{A}^0$, and remark that $\{x'_1, x'_2\}$ is now a $g$-orthonormal basis of $I^{(t)}(\mathcal{L}_\tau)$, because $I^{(t)} \in O(g) \subseteq \mathrm{GL}_{\mathbb{R}}\left(\mathfrak{A}^0\right)$. As already mentioned in the previous section, one has $I^{(t)}(\mathcal{L}_\tau) \oplus \mathcal{L}_\tau \cong T^*\mathcal{L}_\tau$, and $\omega_t$ and $I^{(t)}$ tautologically have the canonical form in the new basis.
To be more precise, denote $\varphi_\tau \colon \Big(\mathcal{A}^0 \oplus \mathcal{L}_\tau, \omega_t\Big) \longrightarrow \Big(T^*\mathcal{L}_\tau, \omega_{\mathrm{can}}\Big)$ the $\tau$-dependent linear map that sends the basis $\{x_j, y_j\}_j$ to $\{x'_j, y'_j\}_j$. It satisfies $\varphi_\tau^* \omega_{\mathrm{can}} = \omega_t$ and $I^{(t)} = \varphi_\tau^{-1} \circ I \circ \varphi_\tau$ by construction, where $I \colon T^*\mathcal{L}_\tau \longrightarrow T^*\mathcal{L}_\tau$ is the standard complex structure. This symplectomorphism sends $\mathcal{L}_\tau$ to itself and $\mathcal{A}^0$ to $I^{(t)}(\mathcal{L}_\tau)$, and is encoded in by a $\tau$-dependent, invertible block-triangular matrix $M_\tau = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$, meaning that the real global coordinates $dx'_j, dy'_j \colon T^*\mathcal{L}_\tau \longrightarrow \mathbb{R}^4$ are related to $dx_j, dy_j \colon \mathcal{A}^0 \oplus \mathcal{L}_\tau \longrightarrow \mathbb{R}^4$ by

$$\begin{pmatrix} dx' \\ dy' \end{pmatrix} = M_\tau \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} Adx \\ Bdx + Cdy \end{pmatrix},$$

where $A, B, C \in M_2(\mathbb{R})$, with $\det(A)\det(C) = \det(M_\tau) \neq 0$ — all depending on $\tau$ and $t$.

The point of this is that $dz' = \frac{dx' + i dy'}{\sqrt{2}}$ defines a vector of $I$-holomorphic coordinates on $T^*\mathcal{L}_\tau$, i.e. a global $I$-holomorphic chart $dz' \colon T^*\mathcal{L}_\tau \longrightarrow \mathbb{C}^2$. Hence the composition $dz \coloneqq \varphi^* dz' = dz' \circ \varphi_\tau \colon \mathcal{A}^0 \oplus \mathcal{L}_\tau \longrightarrow \mathbb{C}^2$ will define $I^{(t)}$-holomorphic coordinates on $\mathfrak{A}^0$ which depend linearly on $dx$ and $dy$:

$$z = \frac{Ax + i\Big(Bx + Cy\Big)}{\sqrt{2}} = \frac{Zx + iCy}{\sqrt{2}}, \tag{4.19}$$

where $Z \coloneqq A + iB \in M_2(\mathbb{C})$. This generalises the standard parametrisation of linear complex structures which are compatible with the canonical symplectic structure of $\mathbb{R}^4$

(see e.g. [KW06]). In our context we cannot use the standard parametrisation, since we do not consider the canonical symplectic structure. Note however that the symplectic structure $\omega_t$ is tautologically of type $(1,1)$ when expressed in coordinates (4.19), since

$$\omega_t = \varphi_\tau^* \omega_{\text{can}} = i\varphi_\tau^*(dz' \wedge d\overline{z'}) = i\varphi_\tau^* dz' \wedge \varphi_\tau^* d\overline{z'} = id(\varphi_\tau^* z') \wedge d(\varphi_\tau^* \overline{z'}) = idz \wedge d\overline{z}.$$

This clearly imposes constraints on the matrices $A$, $B$ and $C$. In particular the following must hold.

**Lemma 4.7.** *One has $A^t = C^{-1}$.*

Hereafter a superscript "$t$" on a matrix always denotes transposition, and never to elevate to the power $t \in \mathbb{C}^*$.

*Proof.* One must have

$$i\Big(dz_1 \wedge d\overline{z_1} + dz_2 \wedge d\overline{z_2}\Big) = k\omega + \omega_{\text{can}}.$$

In particular, comparing the coefficients of $dx_i \wedge dy_j$ for $i, j \in \{1,2\}$ yields

$$\frac{-i}{2} \sum_{i,j=1}^{2} dx_i \wedge dy_j \Big(Z_{1i}C_{1j} + \overline{Z}_{1i}C_{1j} + Z_{2i}C_{2j} + \overline{Z}_{2i}C_{2j}\Big) = dx_1 \wedge dy_1 + dx_2 \wedge dy_2,$$

and the left-hand side also equals

$$-i \sum_{i,j=1}^{2} dx_i \wedge dy_j \Big(A_{1i}C_{1j} + A_{2i}C_{2j}\Big),$$

using that $Z_{ij} + \overline{Z}_{ij} = 2\Re(Z_{ij}) = 2A_{ij}$. Now notice that

$$A_{1i}C_{1j} + A_{2i}C_{2j} = A^t_{i1}C_{1j} + A^t_{i2}C_{2j} = (A^tC)_{ij}.$$

Hence the condition reads $(A^tC)_{ij} = \delta_{ij}$, i.e. $A^tC = \mathrm{Id}$, as claimed. □

Since now $\alpha_t = \frac{i}{2}\sum_j z_j d\overline{z_j} - \overline{z_j} dz_j$, we see that the function $s(z) = \pi^{-1} e^{-\frac{|z|^2}{2}}$ is a $I^{(t)}$-holomorphic frame of $\widetilde{L}^{(t)}$, and we can repeat the construction of the previous section even if the symplectic structure is not in canonical form. Set then

$$\mathcal{F}_{\mathbb{R}} := \Big\{ \psi \in C^\infty(\mathfrak{A}^0, \mathbb{C}) \,\Big|\, \partial_y \psi = 0,\, \psi|_{\mathcal{A}^0} \in L^2(\mathcal{A}^0, dx) \Big\} \cong C^\infty(\mathcal{A}^0, \mathbb{C}) \cap L^2(\mathcal{A}^0, dx).$$

and

$$\mathcal{F}_{\mathbb{C}} := \left\{ f \in \mathcal{O}(\mathfrak{A}^0) \,\middle|\, \int_{\mathfrak{A}^0} |f|^2 e^{-|z|^2} dxdy < +\infty \right\}.$$

We know from Thm. 4.4 that there exists a transform $\mathcal{I}_\tau \colon \psi \longmapsto \widehat{\psi}$ going from $\mathcal{F}_{\mathbb{R}}$ to $\mathcal{F}_{\mathbb{C}}$, and satisfying

$$\Big(\widehat{\psi}, f\Big) = \langle \psi r \mid fs \rangle \in \mathbb{C},$$

where $\langle\cdot \mid \cdot\rangle$ is the inner product of $C^\infty(\mathfrak{A}^0, \widetilde{L}^{(t)})$, and $(\cdot,\cdot)\colon \mathcal{F}_\mathbb{C} \otimes \mathcal{F}_\mathbb{C} \longrightarrow \mathbb{C}$ is obtained by restricting $\langle\cdot \mid \cdot\rangle$ to $\mathcal{H}_\mathbb{C}$. This uses once more the nondegenerate pairing $\mathcal{H}_{\mathbb{R},\tau} \otimes \mathcal{H}_{\mathbb{C},\tau} \longrightarrow \mathbb{C}$ induced by $\langle\cdot \mid \cdot\rangle$, and defines a transform $\mathcal{B}_\tau\colon \mathcal{H}_\mathbb{R} \longrightarrow \mathcal{H}_\mathbb{C}$ by plugging-in the frames $r$ and $s$:

$$\mathcal{B}_\tau(\psi r) \coloneqq \mathcal{I}_\tau(\psi)s = \widehat{\psi}s.$$

We have shown in 4.2 that the function $e_a(z) \coloneqq e^{\overline{a}\cdot z}$ is the coherent state at $a \in \mathbb{C}^2$ inside $\mathcal{F}_\mathbb{C}$, and thus reasoning as in § 4.6.2 one gets again to the integral formula (4.17), i.e.

$$\big(\mathcal{I}_\tau\psi\big)(a) = \widehat{\psi}(a) = \frac{1}{\pi}\int_{\mathfrak{A}^0} \psi(z) \exp\left\{a\cdot\overline{z} - \frac{1}{2}|z^2| + \frac{i}{2}x\cdot y\right\} dxdy,$$

with the important difference that now the coordinates $z_1, z_2$ are $\tau$-dependent linear combinations of $x_1, x_2, y_1, y_2$. To obtain a new $\tau$-dependent Bargmann kernel $A_\tau\colon \mathbb{C}^2\times\mathbb{R}^2 \longrightarrow \mathbb{C}$ one must then integrate the following quadratic function along the Lagrangian:

$$\begin{aligned}
&\exp\left\{a\cdot\overline{z} - \frac{1}{2}|z^2| + \frac{i}{2}x\cdot y\right\} = \\
&= \exp\left\{\frac{1}{\sqrt{2}}a\cdot\big(\overline{Z}x - iCy\big) - \frac{1}{4}\big(Zx + iCy\big)\cdot\big(\overline{Z}x - iCy\big) + \frac{i}{2}x\cdot y\right\} \\
&= \exp\left\{\frac{1}{\sqrt{2}}a\cdot\overline{Z}x - \frac{i}{\sqrt{2}}a\cdot Cy - \frac{1}{4}\Big(x\cdot Z^t\overline{Z}x + y\cdot C^tCy\Big) - \frac{1}{2}x\cdot B^tCy + \frac{i}{2}x\cdot y\right\},
\end{aligned}$$

where $\overline{Z}^t, B^t = \Im(Z^t)$ and $C^t$ denote the transpose of the three matrices, and the position of the dot is important when working with nonsymmetric matrices.

**Proposition 4.23.** *One has:*

$$A_\tau(a,x) = \frac{2}{\sqrt{\pi}\big|\det(C)\big|}\exp\left\{\begin{pmatrix}a\\x\end{pmatrix}\begin{pmatrix}-\frac{\mathrm{Id}}{2} & \frac{\overline{Z}}{\sqrt{2}}\\ \frac{C^{-1}+iB^t}{\sqrt{2}} & -\frac{\Xi}{2}\end{pmatrix}\begin{pmatrix}a\\x\end{pmatrix}\right\},$$

*where* $\Xi \coloneqq A^tA + 2i(A^tB + B^tA)$.

The result is the exponential of a quadratic function, as expected. Moreover, taking $Z = C = \mathrm{Id}$ (and thus $B = 0$) yields the standard Bargmann kernel of Prop. 4.20:

$$A(z,x) = \frac{2}{\sqrt{\pi}}\exp\left\{\begin{pmatrix}z\\x\end{pmatrix}\begin{pmatrix}-\frac{\mathrm{Id}}{2} & \frac{\mathrm{Id}}{\sqrt{2}}\\ \frac{\mathrm{Id}}{\sqrt{2}} & -\frac{\mathrm{Id}}{2}\end{pmatrix}\begin{pmatrix}z\\x\end{pmatrix}\right\}.$$

One might say that there exists a family of integral kernels parametrised by the Teichmüller space of a genus one closed orientable surface. Finally, remark that the real part of $\Xi$ is a symmetric positive definite matrix. Indeed one has $A^tA \geq 0$, and the definiteness follows from $\det(A) \neq 0$.

*Proof.* The term $\exp\left\{\frac{1}{\sqrt{2}}a\cdot\overline{Z}x - \frac{1}{4}x\cdot Z^t\overline{Z}x\right\}$ comes out of the integral, and one is left with computing

$$\int_{\mathcal{A}^0}\exp\left\{\frac{1}{4}y\cdot C^tCy - \frac{i}{\sqrt{2}}a\cdot Cy - \frac{1}{2}x\cdot B^tCy + \frac{i}{2}x\cdot y\right\}dy.$$

Substituting $y \longmapsto Cy$ simplifies this to

$$\Big|\det(C)\Big|^{-1} \int_{\mathcal{A}^0} \exp\left\{\frac{-y^2}{4} - \frac{i}{\sqrt{2}} a \cdot y - \frac{1}{2} x \cdot B^t y + \frac{i}{2} x \cdot C^{-1} y\right\} dy.$$

We now have two squares to complete, and then the Fourier transform of a Gaussian to compute. First:

$$-\frac{y^2}{4} - \frac{i}{\sqrt{2}} a \cdot y = -\frac{1}{4}\Big(y^2 + 2\sqrt{2} i a \cdot y\Big) = -\frac{1}{4}\Big(y + \sqrt{2} i a\Big)^2 - \frac{a^2}{2},$$

and thus the above integral becomes

$$\begin{aligned} & e^{-\frac{a^2}{2}} \int_{\mathcal{A}^0} \exp\left\{-\frac{1}{4}\Big(y + \sqrt{2} i a\Big)^2 - \frac{1}{2} x \cdot B^t y - \frac{i}{2} x \cdot C^{-1} y\right\} dy = \\ & = \exp\left\{-\frac{a^2}{2} + \frac{1}{\sqrt{2}} x \cdot \Big(C^{-1} + i B^t\Big) a\right\} \int_{\mathcal{A}^0} \exp\left\{\frac{-y^2}{4} - \frac{1}{2} x \cdot B^t y + \frac{i}{2} x \cdot C^{-1} y\right\} dy, \end{aligned}$$

where we changed variable according to $y \longmapsto y + \sqrt{2} i a$, which is a transform whose Jacobian equals the identity.

Now, similarly

$$\frac{-y^2}{4} - \frac{1}{2} x \cdot B^t y = -\frac{1}{4}\Big(y + Bx\Big)^2 + \frac{1}{4} x \cdot B^t B x,$$

and thus the integral in the last passage becomes

$$\begin{aligned} & e^{\frac{1}{4} x \cdot B^t B x} \int_{\mathcal{A}^0} \exp\left\{-\frac{1}{4}\Big(y + Bx\Big)^2 + \frac{i}{2} x \cdot C^{-1} y\right\} dy = \\ & \qquad = \exp\left\{\frac{1}{4} x \cdot B^t B x - \frac{i}{2} x \cdot C^{-1} B x\right\} \int_{\mathcal{A}^0} \exp\left\{-\frac{y^2}{4} + i \frac{(C^{-1})^t x}{2} \cdot y\right\} dy, \end{aligned}$$

by replacing the variable $y$ with the variable $y + Bx$.

The last integral to evaluate is by definition the inverse Fourier transform of the function $y \longmapsto e^{-\frac{y^2}{4}}$ evaluated at the point $\frac{1}{2}(C^{-1})^t x$, i.e.

$$\int_{\mathcal{A}^0} \exp\left\{-\frac{y^2}{4} + i \frac{(C^{-1})^t x}{2} \cdot y\right\} dy = 2\sqrt{\pi} e^{-\frac{1}{4} x \cdot C^{-1} (C^{-1})^t x}.$$

Putting all together provides the formula given in the statement. In particular, to show the formula for the matrix $\Xi$ given there observe that $C^{-1}(C^{-1})^t = A^t A$ and $C^{-1} B = A^t B$, thanks to Lem. 4.7. Then

$$\begin{aligned} \frac{1}{4}\Big( - Z^t \overline{Z} + B^t B - 2 i A^t B - A^t A\Big) &= \frac{1}{4}\Big((-A^t - i B^t)(A - i B) + B^t B - 2 i A^t B - A^t A\Big) = \\ &= \frac{1}{4}\Big((-2 A^t A + i(A^t B - B^t A) - 2 i A^t B\Big) = \\ &= -\frac{1}{2}\Big(A^t A + 2 i (A^t B - B^t A)\Big). \end{aligned}$$

□

**Theorem 4.5.** *The explicit formula for the Bargmann transform $\mathcal{B}_\tau\colon \mathcal{H}_{\mathbb{R},\tau} \longrightarrow \mathcal{H}_{\mathbb{C},\tau}$ reads as follows, in the chosen frames:*

$$e^{\frac{i}{2}x\cdot y}\psi(x,y) \longmapsto \frac{2e^{-\frac{|z|^2}{2}}}{\sqrt{\pi}\left|\det(C)\right|}\int_{\mathcal{A}^0}\psi(x)\exp\left\{-\frac{1}{2}\begin{pmatrix}z\\x\end{pmatrix}\begin{pmatrix}\mathrm{Id} & -\sqrt{2}A\\ -\sqrt{2}A^t & \Xi\end{pmatrix}\begin{pmatrix}z\\x\end{pmatrix}\right\}dx. \tag{4.20}$$

*Proof.* This result is a corollary of the previous computations. The point is that it is only the symmetric part of the matrix $\begin{pmatrix} -\frac{\mathrm{Id}}{2} & \frac{\overline{Z}}{\sqrt{2}} \\ \frac{C^{-1}+iB^t}{\sqrt{2}} & -\frac{\Xi}{2}\end{pmatrix}$ which is relevant, because we work with commutative variables. Hence one has may replace the off-diagonal blocks with the sum

$$\frac{1}{2}\left(\frac{\overline{Z}}{\sqrt{2}}+\frac{\left(C^{-1}+iB^t\right)^t}{\sqrt{2}}\right)=\frac{\overline{Z}+(C^{-1})^t+iB}{2\sqrt{2}}.$$

Using Lem. 4.7, one finds indeed

$$\overline{Z}+(C^{-1})^t+iB=A-iB+(C^{-1})^t+iB=A+(C^{-1})^t=2A.$$

In particular there is no ambiguity in the notation for the quadratic form: no dot is needed. □

Finally one can describe how the differential operators $\partial_x$ and $\mu_x$ commute past the transform $I_\tau\colon \mathcal{F}_{\mathbb{R}} \longrightarrow \mathcal{F}_{\mathbb{C}}$, generalising (4.18).

**Proposition 4.24.** *One has*

$$\begin{cases}\mathcal{I}_\tau\circ\mu_x\circ\mathcal{I}_\tau^{-1} &= \frac{1}{\sqrt{2}}A^{-1}\big(\partial_z+\mu_z\big)\\ \mathcal{I}_\tau\circ\partial_x\circ\mathcal{I}_\tau^{-1} &= \frac{1}{\sqrt{2}}\Xi A^{-1}\Big(\partial_z+\big(\mathrm{Id}-2A^t\big)\mu_z\Big)\end{cases} \tag{4.21}$$

*Proof.* The basic ingredient are the derivatives of the quadratic form in the exponent of the generalised kernel of Prop. 4.23, i.e.

$$-\frac{1}{2}\partial_x\big(z^2-2\sqrt{2}x\cdot A^tz+x\cdot\Xi x\big)=\big(\sqrt{2}A^tz-\Xi x\big),$$

and

$$-\frac{1}{2}\partial_z\big(z^2-2\sqrt{2}z\cdot Ax+x\cdot\Xi x\big)=\big(\sqrt{2}Ax-z\big).$$

Because of that, integration by parts yields

$$\begin{aligned}\big(\mathcal{I}_\tau\circ\partial_x\big)\psi(z)=\widehat{\partial_x\psi}(z)&=\int_{\mathcal{A}^0}\big(\partial_x\psi(x)\big)A_\tau(z,x)dx=-\int_{\mathcal{A}^0}\psi(x)\big(\partial_xA_\tau(z,x)\big)dx=\\ &=-\int_{\mathcal{A}^0}\psi(x)\big(\sqrt{2}A^tz-\Xi x\big)A_\tau(z,x)dx=-\sqrt{2}A^tz\widehat{\psi}+\widehat{\Xi x\psi}=\\ &=\Big(-\sqrt{2}A^t\mu_z\circ\mathcal{I}_\tau+\mathcal{I}_\tau\circ\Xi\mu_x\Big)\psi(z),\end{aligned}$$

for any $\psi \in C^\infty(\mathcal{A}^0, \mathbb{C})$. Hence

$$\mathcal{I}_\tau^{-1} \circ \mu_z \circ \mathcal{I}_\tau = \frac{1}{\sqrt{2}} (A^{-1})^t \Big( \Xi \mu_x - \partial_x \Big). \tag{4.22}$$

Next, similarly

$$\begin{aligned} \Big(\partial_z \circ \mathcal{I}_\tau\Big)\psi(z) = \partial_z \widehat{\psi}(z) = \int_{\mathcal{A}^0} \psi(x) \Big(\partial_z A_\tau(z,x)\Big) dx = \int_{\mathcal{A}^0} \psi(x) \Big(\sqrt{2} Ax - z\Big) A_\tau(z,x) dx = \\ = \sqrt{2}\widehat{Ax\psi} - z\widehat{\psi} = \Big(\sqrt{2}\mathcal{I}_\tau \circ Ax - \mu_z \circ \mathcal{I}_\tau\Big)\psi, \end{aligned}$$

and substituting the value of $\mu_z \circ \mathcal{I}_\tau$ provided by the first identity yields

$$\mathcal{I}_\tau^{-1} \circ \partial_z \circ \mathcal{I}_\tau = \frac{1}{\sqrt{2}} (A^{-1})^t \Big( \Big(2A^t A - \Xi\Big)\mu_x + \partial_x \Big). \tag{4.23}$$

Summing (4.22) and (4.23) provides:

$$\mathcal{I}_\tau^{-1} \circ (\mu_z + \partial_z) \circ \mathcal{I}_\tau = \sqrt{2} A \mu_x,$$

whence the first identity in the statement. Now one can substitute the value of $\mathcal{I}_\tau^{-1} \circ \Xi x \circ \mathcal{I}_\tau$ in the identity

$$\mathcal{I}_\tau \circ \partial_x \circ \mathcal{I}_\tau^{-1} = \mathcal{I}_\tau \circ \Xi x \circ \mathcal{I}_\tau^{-1} + \sqrt{2} A^t \mu_z,$$

finding

$$\mathcal{I}_\tau \circ \partial_x \circ \mathcal{I}_\tau^{-1} = \frac{1}{\sqrt{2}} \Xi A^{-1} \Big(z + \partial_z\Big) - \sqrt{2} A^t z = \frac{1}{\sqrt{2}} \Xi A^{-1} \Big( \Big( \mathrm{Id} - 2A^t \Big)\mu_z + \partial_z \Big),$$

which is the second identity in the statement. In all this computations one uses the $\mathbb{C}$-linearity of $\mathcal{I}_\tau$. □

Note that (4.21) specialise to (4.18) for $A = \Xi = \mathrm{Id}$, as expected. Prop. 4.24 is almost what we were after, since our transform $\mathcal{B}_\tau$ acts on sections rather then functions. To correct for this, one must plug-in the $\tau$-dependent frames $r$ and $s$ of $\widetilde{L}^{(t)}$. Let us denote $\mu_r \colon \psi \longrightarrow \psi r$ and $\mu_s \colon f \longrightarrow fs$ the frame multiplications. Clearly these maps commute with function multiplication, but the same is not true for the derivative of functions.

**Lemma 4.8.** *One has, for $j \in \{1, 2\}$:*

$$\begin{cases} \mu_r^{-1} \circ \nabla_{\partial_{x_1}} \circ \mu_r &= \partial_{x_1} + \frac{ik}{2} \mu_{x_2}, \\ \mu_r^{-1} \circ \nabla_{\partial_{x_2}} \circ \mu_r &= \partial_{x_2} - \frac{ik}{2} \mu_{x_1}, \\ \mu_s \circ \partial_{z_j} \circ \mu_s^{-1} &= \widetilde{\nabla}^{(t)}_{\partial_{z_j}} + \mu_{\overline{z_j}}. \end{cases}$$

*Proof.* The real-polarised frame is $r = e^{\frac{i}{2} x \cdot y}$, and thus e.g.

$$\begin{aligned} \Big(\widetilde{\nabla}^{(t)}_{\partial_{x_1}} \circ \mu_r - \mu_r \circ \partial_{x_1}\Big)\psi = \widetilde{\nabla}^{(t)}_{\partial_{x_1}} (\psi r) - (\partial_{x_1}\psi) r = \psi \widetilde{\nabla}^{(t)}_{\partial_{x_1}} r = \psi \Big(\partial_{x_1} r - i \langle \alpha_t, \partial_{x_1} \rangle r\Big) = \\ = i \left( y_1 + \frac{k}{2} x_2 \right) \psi r, \end{aligned}$$

where we recall that $\alpha_t = k\alpha + \alpha_{\text{can}} = \frac{k}{2}x_1 dx_2 - x_2 dx_1 + \sum_j x_j dy_j - y_j dx_j$. The restriction of this relation to $\mathcal{A}^0$ is

$$\nabla_{\partial_{x_1}} \circ \mu_r - \mu_r \circ \partial_{x_1} = \frac{ik}{2}\mu_{x_2} \circ \mu_r,$$

since $\widetilde{\nabla}^{(t)}$ restricts to $\nabla$ by definition, and the coordinates $y_j$ vanish on $\mathcal{A}^0$. This yields the first identity of the statement. Similarly one sees that $\nabla_{\partial_{x_2}}$ corresponds to $\partial_{x_2} - \frac{ik}{2}\mu_{x_1}$. Said differently, the frame $r$ restricts to the constant function 1 on $\mathcal{A}^0$ (for all $\tau$), which is not horizontal for $\nabla$, and thus one needs a correction for passing from ordinary derivatives to covariant ones.

The same happens for the Kähler-polarised frame $s = \pi^{-1}e^{-\frac{|z|^2}{2}}$. The derivative $\partial_z$ on holomorphic functions cannot correspond to $\widetilde{\nabla}^{(t)}_{\partial_z}$ under the frame multiplication $\mu_s$, since the latter differential operator does not even preserve holomorphic sections (because of $\left[\nabla^{0,1}, \nabla^{1,0}\right] \neq 0$). Rather, one has:

$$\left(\widetilde{\nabla}^{(t)}_{\partial_z} \circ \mu_s - \mu_s \circ \partial_z\right) f = \widetilde{\nabla}^{(t)}_{\partial_z}(fs) - (\partial_z f)s = f\widetilde{\nabla}^{(t)}_{\partial z} s = -\overline{z} f s,$$

which means that $\partial_z$ corresponds to the holomorphicity-preserving differential operator $\widetilde{\nabla}^{(t)}_{\partial_z} + \mu_{\overline{z}}$, since $\mu_s \circ \partial_z \circ \mu_s^{-1} = \widetilde{\nabla}^{(t)}_{\partial_z} + \mu_{\overline{z}}$. □

Hence one can upgrade prop. 4.24 to a statement about differential operators acting on smooth sections of line bundles.

**Theorem 4.6.** *One has:*

$$\begin{cases} \mathcal{B}_\tau \circ \mu_x \circ \mathcal{B}_\tau^{-1} & = \frac{1}{\sqrt{2}} A^{-1}\left(\widetilde{\nabla}^{(t)}_{\partial_z} + \mu_{\overline{z}} + \mu_z\right) \\ \mathcal{B}_\tau \circ \nabla_{\partial_{x_1}} \circ \mathcal{B}_\tau^{-1} & = \frac{1}{\sqrt{2}} \sum_j \left((\Xi A^{-1})_{1j} + \frac{ik}{2} A^{-1}_{2j}\right)\left(\widetilde{\nabla}^{(t)}_{\partial_{z_j}} + \mu_{\overline{z_j}}\right) + \\ & \quad + \frac{1}{\sqrt{2}} \sum_{j,l} \left(2(\Xi A^{-1})_{1j}(\delta_{jl} - A^t_{jl})\mu_{z_l} + \frac{ik}{2} A^{-1}_{2j}\mu_{z_j}\right), \\ \mathcal{B}_\tau \circ \nabla_{\partial_{x_2}} \circ \mathcal{B}_\tau^{-1} & = \frac{1}{\sqrt{2}} \sum_j \left((\Xi A^{-1})_{2j} - \frac{ik}{2} A^{-1}_{1j}\right)\left(\widetilde{\nabla}^{(t)}_{\partial_{z_j}} + \mu_{\overline{z_j}}\right) + \\ & \quad + \frac{1}{\sqrt{2}} \sum_{j,l} \left(2(\Xi A^{-1})_{2j}(\delta_{jl} - A^t_{jl})\mu_{z_l} - \frac{ik}{2} A^{-1}_{1j}\mu_{z_j}\right). \end{cases} \tag{4.24}$$

Note that the differential operators on the right-hand sides preserve holomorphicity, because those on the left-hand sides preserve real-polarised sections.

*Proof.* The statement is a corollary of Prop. 4.24 and the above lemma. Using the defining identity $\mathcal{B}_\tau = \mu_s \circ \mathcal{I}_\tau \circ \mu_r^{-1}$. One finds

$$\mathcal{B}_\tau \circ \mu_x \circ \mathcal{B}_\tau^{-1} = \mu_s \circ \left(\mathcal{I}_\tau \circ \mu_x \circ \mathcal{I}_\tau^{-1}\right) \circ \mu_s^{-1} = \mu_s \left(\frac{1}{\sqrt{2}} A^{-1}\left(\partial_z + \mu_z\right)\right) \mu_s^{-1} =$$
$$= \frac{1}{\sqrt{2}} A^{-1}\left(\mu_s \circ \partial_z \circ \mu_s^{-1} + \mu_z\right) = \frac{1}{\sqrt{2}} A^{-1}\left(\widetilde{\nabla}^{(t)}_{\partial_z} + \mu_{\overline{z}+z}\right)$$

where we used the commutation between function multiplications, as well as the $\mathbb{C}$-linearity of all maps involved.

Similarly, but slightly more difficultly

$$\begin{aligned}
&\mathcal{B}_\tau \circ \nabla_{\partial_{x_1}} \circ \mathcal{B}_\tau^{-1} = \mu_s \circ \mathcal{I}_\tau \circ (\mu_r^{-1} \circ \nabla_{\partial_{x_1}} \circ \mu_r) \circ \mathcal{I}_\tau^{-1} \circ \mu_s^{-1} = \\
&= \mu_s \circ \mathcal{I}_\tau \circ \left(\partial_{x_1} + \frac{ik}{2}\mu_{x_2}\right) \circ \mathcal{I}_\tau^{-1} \circ \mu_s^{-1} = \\
&= \frac{1}{\sqrt{2}} \sum_{j,l} \mu_s \circ (\Xi A^{-1})_{1j} \Big(\partial_{z_j} + (\delta_{jl} - 2A^t_{jl})\mu_{z_l}\Big) \circ \mu_s^{-1} + \frac{ik}{2\sqrt{2}} \sum_j A^{-1}_{2j} (\widetilde{\nabla}^{(t)}_{\partial_{z_j}} + \mu_{\overline{z_j}+z_j}) = \\
&= \frac{1}{2\sqrt{2}} \left( \sum_{j,l} (2\Xi A^{-1})_{1j} \Big(\widetilde{\nabla}^{(t)}_{\partial_{z_j}} + (2\delta_{jl} - 2A^t_{jl})\mu_{z_l} + \mu_{\overline{z_j}}\Big) + ik \sum_j A^{-1}_{2j} (\widetilde{\nabla}^{(t)}_{\partial_{z_j}} + \mu_{\overline{z_j}+z_j}) \right),
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{B}_\tau \circ \nabla_{\partial_{x_2}} \circ \mathcal{B}_\tau^{-1} = \mu_s \circ \mathcal{I}_\tau \circ \left(\partial_{x_2} - \frac{ik}{2}\mu_{x_1}\right) \circ \mathcal{I}_\tau^{-1} \circ \mu_s^{-1} = \\
&\quad = \frac{1}{2\sqrt{2}} \left( \sum_{j,l} (2\Xi A^{-1})_{2j} \Big(\widetilde{\nabla}^{(t)}_{\partial_{z_j}} + (2\delta_{jl} - 2A^t_{jl})\mu_{z_l} + \mu_{\overline{z_j}}\Big) - ik \sum_j A^{-1}_{1j} (\widetilde{\nabla}^{(t)}_{\partial_{z_j}} + \mu_{\overline{z_j}+z_j}) \right).
\end{aligned}$$

One can now rearrange the term, isolating the covariant derivative and the antiholomorphic multiplication on the left. This yields the latter identities of the statement. □

Finally, letting the restricted gauge group $\mathcal{G}^0$ act yields a transform

$$\mathcal{B}_\tau \colon \mathcal{H}^{\mathcal{G}^0}_{\mathbb{R},\tau} \longrightarrow \mathcal{H}^{\mathcal{G}^0}_{\mathbb{C},\tau},$$

for all $\tau \in \mathbb{H}$. Neglecting inner products, the space on the left is identified with $C^\infty(\mathcal{M}, L^{\otimes k})$, and that on the right with a subspace of $C^\infty(\mathfrak{M}, L^{(t)})$. Letting $\tau$ vary provides a map of vector bundles over Teichmüller space, and in the next section we will exploit Thm. 4.6 to conjugate the Hitchin–Witten connection through it.

### 4.6.4 The conjugation of the Hitchin–Witten connection

As explained in [Wit91; Wit11], the quantum Hilbert space for Chern–Simons theory with gauge group $\mathrm{SL}_2(\mathbb{C})$ can be identified with the prequantum Hilbert space for Chern–Simons theory with gauge group SU(2). This identification depends on the complex structure of $\Sigma$. In our setting we were able to made this quite explicit, as we now briefly recall.

In § 4.2.6 we introduced real polarisations $P_\mathbb{R}$ on $\mathfrak{M}$, and we showed in 4.1 that these polarisations are all transverse to the compact moduli space $(\mathcal{M}, \omega)$ for the gauge group SU(2), where $\omega$ is the real Atiyah–Bott symplectic form of § 2.3.3. We know that this results in an identification of all quantum spaces $\mathcal{H}_{\mathbb{R},\tau}$ with $C^\infty\big(\mathcal{M}, L^{\otimes k}\big)$ (neglecting the inner product), where $k$ is the real (positive integer) part of the quantum level $t = k + i\sigma$.[11]

[11] In this section we drop the group $\mathcal{G}^0$ from the notation. One should as always define all objects on the flat spaces $\mathcal{A}^0$ and $\mathfrak{A}^0$, where there are global symplectic potential and frames, and then argues that all data descends following the gauge-action.

Let us write $\mathcal{H}^{(t)}_{\mathbb{R}} \longrightarrow \mathbb{H}$ for the vector bundle of the geometric quantisation of $(\mathfrak{M}, \omega_t)$ —at level $t$ — with respect to the polarisations $P_{\mathbb{R},\tau}$. Then $\mathcal{H}^{(t)}_{\mathbb{R}}$ carries its own Hitchin connection, which can be pushed along the isomorphism of vector bundles

$$\mathcal{H}^{(t)}_{\mathbb{R}} \longrightarrow C^\infty\left(\mathcal{M}, L^{\otimes k}\right) \times \mathbb{H},$$

where we temporarily neglect the inner product on the right-hand side. Doing so provides the following explicit formula for a connection on the trivial prequantum bundle for the compact moduli space (see [AM16] for this formula, the original [Wit91] for other genera, and [AG14] in a more general context).

**Definition 4.11.** The Hitchin–Witten connection $\widehat{\nabla}_{\mathbb{R}}$ at level $t$, for the closed curve $\Sigma$ of genus one and the gauge group $\mathrm{SL}(2,\mathbb{C})$, is the following connection on the trivial vector bundle $C^\infty\left(\mathcal{M}, L^{\otimes k}\right) \times \mathbb{H} \longrightarrow \mathbb{H}$:

$$\widehat{\nabla}_{\mathbb{R}} = \nabla^T + \frac{1}{2t}\Delta_{G'} - \frac{1}{2\bar{t}}\Delta_{\overline{G'}}. \tag{4.25}$$

Here $\nabla^T$ is the trivial connection in the trivial bundle, and

$$G'(V) \in C^\infty(\mathcal{M}, T_{1,0}\mathcal{M} \otimes T_{1,0}\mathcal{M}), \qquad \overline{G'}(V) \in C^\infty(\mathcal{M}, T_{0,1}\mathcal{M} \otimes T_{0,1}\mathcal{M})$$

are the symmetric tensors corresponding via $\omega$ to the variation of the complex structures $I_\tau$ (arising from the Hodge star) as $\tau$ varies in $\mathbb{H}$. This means that $k\omega.G(V) = V\left[I_\tau\right]'$ and $k\omega.\overline{G'}(V) = V\left[I_\tau\right]''$. The size of their matrices in any given trivialisation is half of that of the tensors $G(V)$ and $\overline{G}(V)$ of § 4.3.1, which were instead defined for variations of the complex structure $I^{(t)} = k'I_\tau - \sigma' K_\tau$ on $\mathfrak{M}$; this means that $G'(V)$ and $\overline{G}'(V)$ are complex numbers.

If $\Sigma$ is a torus then one can be much more explicit, stating that

$$\Delta_{G'(\partial_\tau)} = \frac{i}{2\pi}\nabla_{z'}\nabla_{z'}, \qquad \Delta_{\overline{G'}(\partial_{\bar\tau})} = -\frac{i}{2\pi}\nabla_{\overline{z'}}\nabla_{\overline{z'}}, \tag{4.26}$$

and

$$\Delta_{G'(\partial_{\bar\tau})} = 0 = \Delta_{\overline{G'}(\partial_\tau)},$$

where $\nabla_{z'} \coloneqq \nabla_{\partial_{z'}}$, $\nabla_{\overline{z'}} \coloneqq \nabla_{\partial_{\overline{z'}}}$, and $z'$ is a $I_\tau$-holomorphic coordinate on $\mathcal{M}$. The vector fields $\partial_\tau$ and $\partial_{\bar\tau}$ provide a frame of $T_{\mathbb{C}}\mathbb{H}$, and this is enough to code $G'(V)$ and $\overline{G'}(V)$ for all complex vector fields $V$ on Teichmüller space.
Recall that when $\sigma \neq 0$ then $z'$ *is not* a $I^{(t)}$-holomorphic coordinate on $\mathfrak{M}$. Rather, we will keep writing $z_1$ and $z_2$ for the coordinates defined on $\mathfrak{M}$ in the previous section.

The connection $\nabla$ of (4.26) is the prequantum connection of the $k$-fold tensor power $L^{\otimes k}$ of the Chern–Simons line bundle for the compact theory (see Rem. 4.10). The same argument used for the connection of Thm. 4.1 shows that the Hitchin–Witten connection is invariant for the group of bundle automorphisms defined by the action of the mapping class group.

**Remark 4.26.** It is known that the Hitchin–Witten connection (4.25) is flat in genus one, thereby providing canonical identification between the fibres of the trivial prequantum bundle $C^\infty\big(\mathcal{M}, L^{\otimes k}\big) \times \mathbb{H} \longrightarrow \mathbb{H}$, along the lines of § 2.2.4 (see [AM16]). The point that must not be missed is that the given global trivialisation of the quantum bundle for real polarisations depends on the complex structure $\tau \in \mathbb{H}$, and thus picking the tautological identifications among the fibres of $C^\infty\big(\mathcal{M}, L^{\otimes k}\big) \times \mathbb{H} \longrightarrow \mathbb{H}$ does not get rid of the choice of a polarisation: it is only the parallel transport along $\widehat{\nabla}_{\mathbb{R}}$ that achieves that.

One may now push the Hitchin–Witten connection (4.25) onto the Kähler-polarised side, using the transform of Thm. 4.4. Concretely, this means computing the conjugate of the Hitchin–Witten connection with respect to $\mathcal{B}_\tau$, where we denote by $\mathcal{B}_\tau$ the isomorphism already defined on $C^\infty(\mathcal{M}, \mathbb{C})$, i.e. $\psi \cdot 1 \longmapsto \widehat{\psi} s$. This is not quite the same as (4.20), because here we already considered the restriction to $\mathcal{M}$. This viewpoint is better because the restriction of $r$ on $\mathcal{M}$ does not depend on $\tau$, which is useful for conjugating the trivial connection $\nabla'$, which is a connection on $C^\infty(\mathcal{M}, L^{\otimes k}) \times \mathbb{H} \longrightarrow \mathbb{H}$.

**Proposition 4.25.** *One has $\mathcal{B}_\tau \circ \nabla'_V \circ \mathcal{B}_\tau^{-1} = \nabla^T_V - D(V)$, where $\nabla^T$ is the trivial connection on $C^\infty(\mathfrak{M}, L^{(t)}) \times \mathbb{H} \longrightarrow \mathbb{H}$, and $D(V)$ is the differential operator defined by*

$$\begin{aligned} D(V) \coloneqq & -\frac{1}{2\sqrt{2}} \nabla_z \widetilde{A} \nabla_z - \\ & - \frac{1}{\sqrt{2}} \nabla_z \widetilde{A} (\mu_z + \mu_{\overline{z}}) + \sqrt{2} (\mu_{V[z]} \cdot \nabla_z + \mu_z \cdot V[A] A^{-1} \nabla_z) + \\ & + (\sqrt{2} - 1) \mu_{z V[z]} + \left( \sqrt{2} - \frac{1}{2} \right) \mu_{\overline{z} V[z]} - \frac{1}{2} \mu_{z V[\overline{z}]} + \\ & + \frac{1}{2\sqrt{2}} \mu_z \cdot \Big( 4 V[A] A^{-1} - \widetilde{A} \Big) \mu_z + \frac{1}{\sqrt{2}} \mu_z \cdot \Big( 2 V[A] A^{-1} - \widetilde{A} \Big) \mu_{\overline{z}} - \frac{1}{2\sqrt{2}} \mu_{\overline{z}} \widetilde{A} \mu_{\overline{z}} - \\ & - \mathrm{Tr}(\widetilde{A}) - \det\Big( V[C] \Big) \det(C)^{-1}, \end{aligned}$$

*where $\widetilde{A} \coloneqq (A^{-1})^t V[\Xi] A^{-1}$.*

We have written $D(V)$ in order-descending fashion: the first item is an operator of order two, the second row has order one, and then one has function multiplication in the third and fourth row (with the last one being the multiplication by a constant function). Note that the differential operators $D(V)$ themselves *do not* preserve holomorphicity.

*Proof.* The trivial connection $\nabla'$ acts on a section $\psi = \psi_\tau \cdot 1$ of $C^\infty(\mathcal{M}, L^{\otimes k}) \times \mathbb{H}$ by means of $\nabla'_V(\psi \cdot 1) = V[\psi] \cdot 1$, where $V$ is a vector field on the upper-half plane. The trivial connection $\nabla^T$ instead acts on a Kähler-polarised section $fs = f_\tau s_\tau$ by

$$\nabla^T_V \Big[ \widehat{\psi} s \Big] = V[\widehat{\psi}] s + \widehat{\psi} \nabla^T_V s = V[\widehat{\psi}] s + \widehat{\psi} V[s].$$

Hence:

$$\nabla^T_V \circ \mathcal{B}_\tau(\psi \cdot 1) - \mathcal{B}_\tau \circ \nabla'_V(\psi \cdot 1) = V[\widehat{\psi}]s + \widehat{\psi}V[s] - \widehat{V[\psi]}s = V\left[\int_{\mathcal{M}} \psi(x)A_\tau(z,x)dx\right] s(z) +$$
$$+ \left(\int_{\mathcal{M}} \psi(x)A_\tau(z,x)dx\right) V[s](z) - \left(\int_{\mathcal{M}} V[\psi](x)A_\tau(z,x)dx\right) s(z) =$$
$$= \left(\int_{\mathcal{M}} \psi(x)V\Big[A_\tau(z,x)\Big]dx\right) s(z) + \left(\int_{\mathcal{M}} \psi(x)A_\tau(z,x)dx\right) V[s](z),$$

where we used $V\left(\int_{\mathcal{M}} \psi A_\tau dx\right) = \int_{\mathcal{M}} \Big(V[\psi]A_\tau + \psi V[A_\tau]\Big)dx$, which is obtained by taking the derivative under the integral sign. Now, the derivative of the Bargmann kernel of Prop. 4.23 is

$$V\Big[A_\tau(z,x)\Big] = V\left[\frac{2}{\sqrt{\pi}|\det(C)|}\exp\Big\{-\frac{1}{2}\Big(z^2 - 2\sqrt{2}z\cdot Ax + x\cdot \Xi x\Big)\Big\}\right] =$$
$$= -\frac{2}{\sqrt{\pi}}\frac{\det\Big(V[C]\Big)}{\Big(\det(C)\Big)^2}\exp\Big\{-\frac{1}{2}\Big(z^2 - 2\sqrt{2}z\cdot Ax + x\cdot \Xi x\Big)\Big\} -$$
$$- \frac{1}{2}\Big(2zV[z] - 2\sqrt{2}V[z]\cdot Ax - 2\sqrt{2}z\cdot V\Big[A\Big]x + x\cdot V\Big[\Xi\Big]x\Big)\, A_\tau(z,x) =$$
$$= \left(-\frac{\det\Big(V[C]\Big)}{\det(C)} - \frac{1}{2}\Big(2zV[z] - 2\sqrt{2}V[z]\cdot Ax - 2\sqrt{2}z\cdot V\Big[A\Big]x + x\cdot V\Big[\Xi\Big]x\Big)\right) A_\tau(z,x),$$

where we assumed $\det(C) > 0$ for all $\tau$. Since $\tau \longmapsto \det(C_\tau)$ is a continuous nowhere vanishing function then it is always strictly positive or negative, and one may assume the former up to choosing the $g$-orthonormal basis $\{y'_1, y'_2\}$ of the Lagrangian space $\mathcal{L}_\tau$ to have the same orientation as the base $y_1, y_2$ (see § 4.6.3).

Similarly, the derivative of the holomorphic frame $s$ of § 4.6.3 is

$$V[s](z) = \pi^{-1}V\Big[e^{-\frac{|z|^2}{2}}\Big] = -\frac{1}{2\pi}\Big(zV[\overline{z}] + \overline{z}V[z]\Big)e^{-\frac{|z|^2}{2}} = -\frac{1}{2}\Big(zV[\overline{z}] + \overline{z}V[z]\Big)s(z).$$

Hence:

$$(\nabla^T_V \circ \mathcal{B}_\tau - \mathcal{B}_\tau \circ \nabla'_V)\psi =$$
$$= -\left(\frac{\det\Big(V[C]\Big)}{\det(C)} + zV[z] + \frac{1}{2}(\overline{z}V[z] + zV[\overline{z}])\right)\cdot\left(\int_{\mathcal{M}} \psi(x)A_\tau(z,x)dx\right) s(z) +$$
$$+ \left(\int_{\mathcal{M}} \left(\sqrt{2}(V[z]\cdot Ax + z\cdot V[A]x) - \frac{1}{2}xV[\Xi]x\right)\psi(x)A_\tau(z,x)dx\right) s(z) =$$
$$= \left(\mu_f \circ \mathcal{B}_\tau + \sqrt{2}\mu_{V[z]} \circ \mathcal{B}_\tau \circ A\mu_x + \sqrt{2}\mu_z \circ \mathcal{B}_\tau \circ V\Big[A\Big]\mu_x - \frac{1}{2}\mathcal{B}_\tau \circ \mu_x \cdot V\Big[\Xi\Big]\mu_x\right)\psi,$$

where $f := -\det\Big(V[C]\Big)\det(C)^{-1} - \frac{1}{2}\Big(2zV[z] + \overline{z}V[z] + zV[\overline{z}]\Big)$.

Now one can use the first identity of (4.24) to rewrite the right-hand side in the form $D(V)\circ\mathcal{B}_\tau$, where $D(V) \in \mathcal{D}(\mathfrak{M}, L^{(t)})$ is a differential operator of order two acting on smooth sections of $L^{(t)}$. This is achieved by making the Bargmann transform $\mathcal{B}_\tau$ commute to the right, passing over various differential operators of order zero.

More precisely, if one denotes $\nabla^{(t)}_{z_k}$ the covariant derivative along $\partial_{z_k}$ for the prequantum connection on $L^{(t)}$:

$$\sqrt{2}\mu_{V[z]} \circ \mathcal{B}_\tau \circ A\mu_x = \sum_{i,j} \mu_{V[z]_i} \circ \mathcal{B}_\tau \circ A_{ij}\mu_{x_j} = \sum_{i,j,k} \mu_{V[z]_i} \circ A_{ij}A^{-1}_{jk}(\nabla^{(t)}_{z_k} + \mu_{z_k} + \mu_{\overline{z_k}}) \circ \mathcal{B}_\tau =$$
$$= \sum_i \mu_{V[z]_i} \circ (\nabla^{(t)}_{z_i} + \mu_{z_i} + \mu_{\overline{z_i}}) \circ \mathcal{B}_\tau = \mu_{V[z]} \cdot (\nabla^{(t)}_z + \mu_z + \mu_{\overline{z}}) \circ \mathcal{B}_\tau,$$

and

$$\sqrt{2}\mu_z \circ \mathcal{B}_\tau \circ V\Big[A\Big]\mu_x = \sqrt{2}\sum_{i,j} \mu_{z_i} \circ \mathcal{B}_\tau \circ V\Big[A\Big]_{ij}\mu_{x_j} =$$
$$= \sum_{i,j,k} \mu_{z_i} \circ V\Big[A\Big]_{ij}A^{-1}_{jk}(\nabla^{(t)}_{z_k} + \mu_{z_k} + \mu_{\overline{z_k}}) \circ \mathcal{B}_\tau =$$
$$= \mu_z \cdot V\Big[A\Big]A^{-1}(\nabla^{(t)}_z + \mu_z + \mu_{\overline{z}}) \circ \mathcal{B}_\tau,$$

and finally the operator of order two:

$$\mathcal{B}_\tau \circ \mu_x \cdot V\Big[\Xi\Big]\mu_x = \mathcal{B}_\tau \circ \sum_{i,j} \mu_{x_i} V\Big[\Xi\Big]_{ij}\mu_{x_j} = \sum_{i,j} \mathcal{B}_\tau \circ \mu_{x_i} \circ \mathcal{B}^{-1}_\tau \circ \mathcal{B}_\tau \circ V\Big[\Xi\Big]_{ij}\mu_{x_j} =$$
$$= \frac{1}{\sqrt{2}} \sum_{i,j,k,l} A^{-1}_{ik}(\nabla^{(t)}_{z_k} + \mu_{z_k} + \mu_{\overline{z_k}}) \circ V\Big[\Xi\Big]_{ij}A^{-1}_{jl}(\nabla^{(t)}_{z_l} + \mu_{z_l} + \mu_{\overline{z_l}}) \circ \mathcal{B}_\tau =$$
$$= \frac{1}{\sqrt{2}} \sum_{i,j,k,l} (\nabla^{(t)}_{z_k} + \mu_{z_k} + \mu_{\overline{z_k}}) \circ (A^{-1})^t_{ki} V\Big[\Xi\Big]_{ij}A^{-1}_{jl}(\nabla^{(t)}_{z_l} + \mu_{z_l} + \mu_{\overline{z_l}}) \circ \mathcal{B}_\tau =$$
$$= \frac{1}{\sqrt{2}}(\nabla^{(t)}_z + \mu_z + \mu_{\overline{z}}) \cdot (A^{-1})^t V\Big[\Xi\Big]A^{-1}(\nabla^{(t)}_z + \mu_z + \mu_{\overline{z}}) \circ \mathcal{B}_\tau.$$

Hence one has $\nabla^T_V \circ \mathcal{B}_\tau - \mathcal{B}_\tau \circ \nabla'_V = D(V) + \mathcal{B}_\tau$, by defining $D(V)$ as in the statement. Applying $\mathcal{B}^{-1}_\tau$ to the right yields $\nabla^T_V - \mathcal{B}_\tau \circ \nabla'_V \circ \mathcal{B}^{-1}_\tau = D(V)$, which means precisely that the action of $\nabla'_V$ on $C^\infty(\mathcal{M}, L^{\otimes k}) \times \mathbb{H}$ gets turned by $\mathcal{B}_\tau$ into the action of $\nabla^T_V - D(V)$ on $C^\infty(\mathfrak{M}, L^{(t)}) \times \mathbb{H}$. □

We are left with computing the conjugation of the Laplace operators. For this it is enough to compute the conjugation of the covariant derivatives along the prequantum connection $\nabla$ on $L^{\otimes k}$, since

$$\frac{1}{2t}\mathcal{B}_\tau \circ \Delta_{G'(\partial_\tau)} \circ \mathcal{B}^{-1}_\tau = \frac{i}{4\pi t}\Big(\mathcal{B}_\tau \circ \nabla_{z'} \circ \mathcal{B}^{-1}_\tau\Big) \circ \Big(\mathcal{B}_\tau \circ \nabla_{z'} \circ \mathcal{B}^{-1}_\tau\Big),$$

and

$$\frac{1}{2\overline{t}}\mathcal{B}_\tau \circ \Delta_{\overline{G'}(\partial_\tau)} \circ \mathcal{B}^{-1}_\tau = \frac{i}{4\pi\overline{t}}\Big(\mathcal{B}_\tau \circ \nabla_{\overline{z'}} \circ \mathcal{B}^{-1}_\tau\Big) \circ \Big(\mathcal{B}_\tau \circ \nabla_{\overline{z'}} \circ \mathcal{B}^{-1}_\tau\Big)$$

Now we have to consider the complex structure $I_\tau$ on $\mathcal{M}$, which is compatible with the symplectic form $k\omega = kdx_1 \wedge dx_2$. Recall from § 4.2.2 that $I_\tau$ is defined by the complex coordinate $z' = \xi_2 - \tau\xi_1 = -(x_2 + \tau x_2)$, since the natural coordinates $\xi_1$ and $\xi_2$ are related

to $x_1$ and $x_2$ by the simple relations $\xi_1 = x_1$ and $\xi_2 = -x_2$ (see the proof of Prop. 4.22). One then computes the complex derivatives to be

$$\partial_{z'} = \frac{i}{2\tau_2}(\partial_{x_1} - \overline{\tau}\partial_{x_2}), \qquad \partial_{\overline{z'}} = \frac{i}{2\tau_2}(-\partial_{x_1} + \tau\partial_{x_2}).$$

Hence, by the $C^\infty$-linearity of connections on vector fields:

$$\nabla_{z'} = \frac{i}{2\tau_2}\Big(\nabla_{\partial_{x_1}} + \overline{\tau}\nabla_{\partial_{x_2}}\Big), \qquad \nabla_{\overline{z'}} = \frac{i}{2\tau_2}\Big(-\nabla_{\partial_{x_1}} + \tau\nabla_{\partial_{x_2}}\Big).$$

Now one may directly apply the second and third identity of Thm. 4.6, finding:

$$\begin{aligned}
&\mathcal{B}_\tau \circ \nabla_{\partial_{z'}} \circ \mathcal{B}_\tau^{-1} = \frac{i}{2\tau_2}\Big(\mathcal{B}_\tau \circ \nabla_{\partial_{x_1}} \circ \mathcal{B}_\tau^{-1} - \overline{\tau}\mathcal{B}_\tau \circ \nabla_{\partial_{x_2}} \circ \mathcal{B}_\tau^{-1}\Big) = \\
&= \frac{i}{2\sqrt{2}\tau_2}\cdot \\
&\cdot \left(\sum_j \left((\Xi A^{-1})_{1j} + \frac{ik}{2}A^{-1}_{2j}\right)\left(\widetilde{\nabla}^{(t)}_{\partial_{z_j}} + \mu_{\overline{z_j}}\right) + \sum_{j,l}\left(2(\Xi A^{-1})_{1j}(\delta_{jl} - A^t_{jl})\mu_{z_l} + \frac{ik}{2}A^{-1}_{2j}\mu_{z_j}\right)\right) - \\
&- \frac{i\overline{\tau}}{2\sqrt{2}\tau_2}\cdot \\
&\cdot \left(\sum_j \left((\Xi A^{-1})_{2j} - \frac{ik}{2}A^{-1}_{1j}\right)\left(\widetilde{\nabla}^{(t)}_{\partial_{z_j}} + \mu_{\overline{z_j}}\right) + \sum_{j,l}\left(2(\Xi A^{-1})_{2j}(\delta_{jl} - A^t_{jl})\mu_{z_l} - \frac{ik}{2}A^{-1}_{1j}\mu_{z_j}\right)\right),
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{B}_\tau \circ \nabla_{\partial_{\overline{z'}}} \circ \mathcal{B}_\tau^{-1} = \frac{i}{2\tau_2}\Big(\mathcal{B}_\tau \circ -\nabla_{\partial_{x_1}} \circ \mathcal{B}_\tau^{-1} + \tau\mathcal{B}_\tau \circ \nabla_{\partial_{x_2}} \circ \mathcal{B}_\tau^{-1}\Big) = \\
&= -\frac{i}{2\sqrt{2}\tau_2}\cdot \\
&\cdot \left(\sum_j \left((\Xi A^{-1})_{1j} + \frac{ik}{2}A^{-1}_{2j}\right)\left(\widetilde{\nabla}^{(t)}_{\partial_{z_j}} + \mu_{\overline{z_j}}\right) + \sum_{j,l}\left(2(\Xi A^{-1})_{1j}(\delta_{jl} - A^t_{jl})\mu_{z_l} + \frac{ik}{2}A^{-1}_{2j}\mu_{z_j}\right)\right) + \\
&+ \frac{i\tau}{2\sqrt{2}\tau_2}\cdot \\
&\cdot \left(\sum_j \left((\Xi A^{-1})_{2j} - \frac{ik}{2}A^{-1}_{1j}\right)\left(\widetilde{\nabla}^{(t)}_{\partial_{z_j}} + \mu_{\overline{z_j}}\right) + \sum_{j,l}\left(2(\Xi A^{-1})_{2j}(\delta_{jl} - A^t_{jl})\mu_{z_l} - \frac{ik}{2}A^{-1}_{1j}\mu_{z_j}\right)\right).
\end{aligned}$$

Composing these two order-one differential operators with themselves, and multiplying by the correct constants, provides a formula for the differential operator

$$E(V) \coloneqq \mathcal{B}_\tau \circ \left(-\frac{1}{2t}\Delta_{G'(V)} + \frac{1}{2\overline{t}}\Delta_{\overline{G'}(V)}\right) \circ \mathcal{B}_\tau^{-1}.$$

Recall from (4.24) that the differential operator $\widetilde{\nabla}^{(t)}_{\partial_{z_j}} + \mu_{\overline{z_j}}$ acts on a holomorphic section $fs$ by $fs \longmapsto (\partial_{z_j} f)s$. Hence $E(V)$ preserve holomorphicity by construction, and one may well add it to $D(V)$.

Finally

$$\widehat{\nabla}_{\mathbb{C}} \coloneqq \mathcal{B}_\tau \circ \widehat{\nabla}_{\mathbb{R}} \circ \mathcal{B}_\tau^{-1} = \nabla^T - D - E$$

is a mapping class group invariant flat connection inside the trivial prequantum bundle $C^\infty\left(\mathfrak{M}, L^{(t)}\right) \times \mathbb{H} \longrightarrow \mathbb{H}$ that preserves the subbundle $\mathcal{H}_{\mathbb{C}}^{(t)}$ of holomorphic sections: it is a Hitchin connection for the geometric quantisation of $(\mathfrak{M}, \omega_t)$ with respect to the natural family $P_\tau$ of Kähler polarisations on the moduli space.
It is certainly different from the connection of Thm. 4.1, since it also contains terms of order one and zero.

### 4.6.5 Circle action on holomorphic sections

As a second application of the Bargmann transform, we use it to turn Hitchin's action of § 4.4 into an action on the spaces $\mathcal{H}_{\mathbb{C}}$. In this section we denote again by $\lambda \in U(1)$ an element of the circle, acting on $\mathfrak{M}$ via $\lambda.(A, \Phi) = (A, \lambda\Phi)$, where $A$ is a unitary connection and $\Phi$ a Higgs field.

The idea is the following. For fixed $\tau \in \mathbb{H}$ and $\lambda \in U(1)$, we first consider the Bargmann transform $\mathcal{B}_\tau^{-1} \colon \mathcal{H}_{\mathbb{C},\tau} \longrightarrow \mathcal{H}_{\mathbb{R},\tau}$, then we act by $\lambda$ by pull-back on section, fixing the line bundle but moving the prequantum connection (see § 4.4.4), thereby defining an arrow $\lambda^* \colon \mathcal{H}_{\mathbb{R},\tau} \longrightarrow \mathcal{H}_{\mathbb{R},\lambda^*\tau}$; the space on the right is that of smooth sections of $L^{(t)}$ which are covariantly constant along the Lagrangian subspace $\lambda^{-1}(\mathcal{L}_\tau)$ for the pull-back connection. Indeed, if $s$ is covariantly constant along $\mathcal{L}_\tau$, and if $X \in \lambda^{-1}\mathcal{L}_\tau$ then one has

$$(\lambda^*\nabla_X^{(t)})\lambda^* s = \lambda^*\left(\nabla_{d\lambda.X}^{(t)} s\right) = 0.$$

Here again one uses the fact that we are working on a vector space, so that the linear real polarisation is the same as a Lagrangian subspace, a tangent vector field is the same as a family of endomorphisms taking values into that subspace, and $\lambda = d\lambda$ since $\lambda$ is linear.

Now one can consider $\mathcal{H}_{\mathbb{R},\lambda^*\tau}$ and $\mathcal{H}_{\mathbb{C},\tau}$ as vector subspaces of one and the same vector space, $C^\infty(\mathfrak{M}, L^{(t)})$. Then the symplectic form $\omega_t$ and the Hermitian metric $h^{(t)}$ provide an inner product $\langle \cdot \mid \cdot \rangle$ on $C^\infty(\mathfrak{M}, L^{(t)})$, which is different then the one provided by $\lambda^*\omega_t$ and $h^{(t)}$: since we change some prequantum data, the Hilbert space structure varies along the Liouville measure. This notwithstanding, the pairing between $\mathcal{H}_{\mathbb{R},\lambda^*\tau}$ and $\mathcal{H}_{\mathbb{C},\tau}$ defined by $\langle \cdot \mid \cdot \rangle$ is still nondegenerate, since all totally real subbundles of $T_{\mathbb{C}}\mathfrak{M}$ are transverse to all purely complex ones (be them Lagrangian with respect to some symplectic form or not). Hence one still has an isomorphism $\varphi = \varphi_{\tau,\lambda} \colon \mathcal{H}_{\mathbb{R},\lambda^*\tau} \longrightarrow \mathcal{H}_{\mathbb{C},\tau}$ that closes the triangle, defined by

$$\langle \varphi(\lambda^* s_1) \mid s_2 \rangle = \langle \lambda^* s_1 \mid s_2 \rangle,$$

where $s_1$ is polarised along $\mathcal{L}_\tau$, and $s_2$ is holomorphic for $I_\tau^{(t)}$.

**Definition 4.12.** The action of $\lambda \in U(1)$ on $\mathcal{H}_{\mathbb{C},\tau}$ is given by the isomorphism

$$\lambda \coloneqq \varphi_{\tau,\lambda} \circ \lambda^* \circ \mathcal{B}_\tau^{-1} \colon \mathcal{H}_{\mathbb{C},\tau} \longrightarrow \mathcal{H}_{\mathbb{C},\tau}.$$

The aim is now to compute a formula as explicit as possible for the above composition. To this end, we pick again the (local) coordinates $\{x_j, y_j\}$ on $\mathfrak{M}$ such that $\mathcal{M}$ is parametrised by the $\tau$-independent coordinates $x_j$ and $\mathcal{L}_\tau$ by the $y_j$. This splitting has the further advantage that $\lambda \in U(1)$ acts trivially on $\mathcal{M}$, and thus also on the coordinates $x_j$. On the contrary, it transforms the coordinates $y_j$ into a linear combination $y'_j$ of $x_j$ and $y_j$. Hence there exist 2-by-2 real matrices $\alpha, \beta$, depending on $\tau$ and $\lambda$, such that

$$\begin{pmatrix} dx \\ dy' \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & 0 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}, \qquad \text{i.e.} \qquad \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & 0 \\ -\beta^{-1}\alpha & \beta^{-1} \end{pmatrix} \begin{pmatrix} dx \\ dy' \end{pmatrix},$$

with $\det(\beta) \neq 0$. Now if $r = e^{\frac{i}{2}x\cdot y}$ is the usual $P_{\mathbb{R},\tau}$-polarised frame, and $\psi \in C^\infty(\mathfrak{M}, \mathbb{C})$, then by definition

$$\lambda^*(\psi r)(x,y) = \lambda^*\psi(x,y) \cdot \lambda^* r(x,y) = \psi(x,y')r(x,y') = \psi(x,y')e^{\frac{i}{2}x\cdot(\alpha x + \beta y)}.$$

Hence, introducing the coherent state $e_a(z) = e^{\overline{a}\cdot z}$ at the point $a$, where $z = \frac{Zx+iCy}{\sqrt{2}}$ as in § 4.6.3:

$$\begin{aligned} \varphi\Big(\lambda^*(\psi r)\Big)(a) = \langle \lambda^*(\psi r) \mid e_a s\rangle = \int_{\mathfrak{M}} h^{(t)}\Big(\lambda^*(\psi r), e_a s\Big) dxdy = \\ = \int_{\mathfrak{M}} \psi(x,y') e^{a\cdot\overline{z}} h^{(t)}(\lambda^* r, s) dxdy = \pi^{-1} \int_{\mathfrak{M}} \psi(x,y') e^{a\cdot\overline{z} + \frac{i}{2}x\cdot y' - \frac{|z|^2}{2}} dxdy, \end{aligned}$$

where we used $s(z) = \pi^{-1}e^{-\frac{|z|^2}{2}} = \pi^{-1}e^{-\frac{i}{2}x\cdot y' - \frac{|z|^2}{2}} r(x,y')$, as well as the $h^{(t)}$-unitarity of $r$. Now one may change variable in the above integral, going from $(x,y)$ to $(x,y')$. Correcting the measure according to $dxdy = |\det(\beta)|^{-1} dxdy'$, and considering that

$$z = \frac{Zx + iCy}{\sqrt{2}} = \frac{Zx + iC(-\beta^{-1}\alpha x + \beta^{-1}y')}{\sqrt{2}} = \frac{(Z - iC\beta^{-1}\alpha)x + iC\beta^{-1}y'}{\sqrt{2}},$$

one finds:

$$\begin{aligned} \varphi\Big(\lambda^*(\psi r)\Big)(a) = \frac{1}{\pi|\det(\beta)|} \int_{\mathfrak{M}} \psi(x,y') \cdot \\ \cdot \exp\left\{ \frac{1}{\sqrt{2}} a \cdot \Big(\overline{Z'}x - iC'y'\Big) - \frac{1}{2}\Big(Z'x + iC'y'\Big) \cdot \Big(\overline{Z'}z - iC'y'\Big) + \frac{i}{2} x\cdot y' \right\} dxdy', \end{aligned}$$

where $Z' \coloneqq Z - iC\beta^{-1}\alpha \in M_2(\mathbb{C})$ and $C' = C\beta^{-1} \in \mathrm{GL}(2,\mathbb{R})$. Notice that the real part of $Z'$ is still equal to the matrix $A \in \mathrm{GL}(2,\mathbb{R})$ of the previous two sections. Hence one can perform the integration over $y'$ to get a new $\lambda$-deformed kernel as in Prop. 4.23, and the result of the computation is obtained by a straightforward adaptation of Thm. 4.5.

**Proposition 4.26.** *The explicit formula for the map $\varphi\colon \mathcal{H}_{\mathbb{R},\lambda^*\tau} \longrightarrow \mathcal{H}_{\mathbb{C},\tau}$ reads as follows:*

$$\lambda^*\Big(e^{\frac{i}{2}x\cdot y}\psi(x,y)\Big) \longmapsto \frac{2e^{-\frac{|z|^2}{2}}}{\sqrt{\pi}\Big|\det(\beta)\det(C)\Big|} \int_{\mathcal{M}} \psi(x) \exp\left\{ -\frac{1}{2}\begin{pmatrix} z \\ x \end{pmatrix} \begin{pmatrix} \mathrm{Id} & -\sqrt{2}A \\ -\sqrt{2}A^t & \Xi' \end{pmatrix} \begin{pmatrix} z \\ x \end{pmatrix} \right\} dx,$$

*where*

$$\Xi' \coloneqq A^tA + 2i\Big((A^tB' + (B')^tA\Big).$$

*and $B' \coloneqq B - C\beta^{-1}\alpha$.*

This means that the isomorphism $\lambda$ of $\mathcal{H}_{\mathbb{C},\tau}$ acts on a Bargmann-transformed section $\widehat{\psi}s$ by

$$\lambda(\widehat{\psi}s)(z) = \left(\int_{\mathcal{M}} \psi(x) A_{\tau,\lambda}(z,x)dx\right) s(z),$$

where

$$A_{\tau,\lambda}(z,x) \coloneqq \frac{2}{\sqrt{\pi}\left|\det(\beta)\det(C)\right|} \exp\left\{-\frac{1}{2}\Big(z^2 - 2\sqrt{2}z\cdot Ax + x\Xi' x\Big)\right\}$$

is the new $\lambda$-deformed Bargmann kernel. To write the formula in terms of a generic holomorphic section $fs$ one must now invert the Bargmann transform $\mathcal{B}_\tau$. This is simple, once one has fixed the correct unitary constant to make it unitary.

**Proposition 4.27.** *Assume to have fixed $c_\tau \in \mathbb{R}$ such $c_\tau\mathcal{B}_\tau$ is $\langle\cdot \mid \cdot\rangle$-unitary for all $\tau \in \mathbb{H}$. Then one has*

$$\mathcal{B}_\tau^{-1}(fs) = \left(\frac{c_\tau}{\pi}\right)^2 \cdot \left(\int_{\mathfrak{M}} f(z)\overline{A_\tau(z,x)}e^{-|z|^2}dxdy\right) r.$$

*Proof.* If $c_\tau$ is as in the statement, one has

$$\Big\langle c_\tau\mathcal{B}_\tau(\psi r)\Big|c_\tau\mathcal{B}_\tau(\psi' r)\Big\rangle = \langle \psi r \mid \psi' r\rangle,$$

but also

$$\Big\langle c_\tau\mathcal{B}_\tau(\psi r)\Big|c_\tau\mathcal{B}_\tau(\psi' r)\Big\rangle = c_\tau^2\Big\langle \mathcal{B}_\tau^\dagger\mathcal{B}_\tau(\psi r)\Big|\psi' r\Big\rangle,$$

where $\mathcal{B}_\tau^\dagger\colon \mathcal{H}_{\mathbb{C},\tau} \longrightarrow \mathcal{H}_{\mathbb{R},\tau}$ is the Hermitian adjoint of the Bargmann transform. The nondegeneracy of the pairing $\mathcal{H}_{\mathbb{R},\tau} \otimes \mathcal{H}_{\mathbb{C},\tau} \longrightarrow \mathbb{C}$ then implies that $c_\tau^2\mathcal{B}_\tau^\dagger \circ \mathcal{B}_\tau$ is the identity on $\mathcal{H}_{\mathbb{R},\tau}$, and thus $c_\tau^2\mathcal{B}_\tau^\dagger$ is the inverse of $\mathcal{B}_\tau$.

To compute the adjoint boils down to swap two integrals. More precisely, if $\psi r$ is a real-polarised section and $fs$ a holomorphic one, then

$$\begin{aligned}\Big\langle \mathcal{B}_\tau(\psi r)\Big|fs\Big\rangle = \big(\widehat{\psi}s, fs\big) &= \pi^{-2}\int_{\mathfrak{M}}\left(\int_{\mathcal{M}}\psi(q)A_\tau(z,q)dq\right)\overline{f}(z)e^{-|z|^2}dxdy = \\ &= \pi^{-2}\int_{\mathcal{M}}\psi(q)\left(\int_{\mathfrak{M}}\overline{\overline{A_\tau(z,q)}f(z)}e^{-|z|^2}dxdy\right)dq = \langle \psi r \mid \widetilde{f}r\rangle,\end{aligned}$$

where

$$\widetilde{f}(q) \coloneqq \pi^{-2}\int_{\mathfrak{M}} f(z)\overline{A_\tau(z,q)}e^{-|z|^2}dxdy.$$

Hence $\mathcal{B}_\tau^\dagger(fs) = \widetilde{f}r$, and one recovers the formula in the statement. □

Note that we have incidentally shown that the map $f \longmapsto \widetilde{f}$ is the inverse of the integral transform $I_\tau$. Explicitly it reads:

$$\widetilde{f}(q) = \pi^{-2}\int_{\mathfrak{M}} f(z)\exp\left\{-\frac{1}{2}\overline{z}^2 + \sqrt{2}\overline{z}\cdot Aq - \frac{1}{2}q\overline{\Xi}q\right\}e^{-|z|^2}dxdy.$$

**Remark 4.27.** As a consistency check, one may to compute the value of $\mathcal{B}_\tau(\widetilde{fr})$ at a point $a \in \mathfrak{M}$. One finds, up to a multiplicative constant:

$$\mathcal{B}_\tau(\widetilde{fr})(a) = k_\tau \left( \int_{\mathfrak{M}} f(z) \left( \int_{\mathcal{M}} A_\tau(a,q)\overline{A_\tau(z,q)}dq \right) e^{-|z|^2} dxdy \right) s(a),$$

and the integral in the middle is seen to be equal to the conjugate of the coherent state $e_a = e^{\overline{a}\cdot z}$. Indeed

$$\begin{aligned}
\int_{\mathcal{M}} A_\tau(a,q)\overline{A_\tau(z,q)}dq &= e^{-\frac{1}{2}(a^2+\overline{z}^2)} \int_{\mathcal{M}} \exp\left\{ -\frac{1}{2} q(\Xi + \overline{\Xi})q + \sqrt{2}(a+\overline{z})\cdot Aq \right\} dq = \\
&= e^{-\frac{1}{2}(a^2+\overline{z}^2)} \int_{\mathcal{M}} \exp\left\{ -qA^tAq + \sqrt{2}(a+\overline{z})\cdot Aq \right\} dq = \\
&= \frac{e^{-\frac{1}{2}(a^2+\overline{z}^2)}}{|\det(A)|} \int_{\mathcal{M}} \exp\left\{ -q^2 + \sqrt{2}(a+\overline{z})q \right\} dq = \\
&= \frac{\exp\left\{ -\frac{1}{2}(a^2+\overline{z}) + \frac{1}{2}(a+\overline{z})^2 \right\}}{|\det(A)|} \int_{\mathcal{M}} e^{-q^2} dq = \frac{\sqrt{\pi}}{|\det(A)|} e^{a\cdot\overline{z}}.
\end{aligned}$$

Hence indeed $\mathcal{B}_\tau(\widetilde{fr})(a) = f(a)s(a)$, by the reproducing property of the coherent state.

This is also the description of the action of the identity $1 \in U(1)$ acting on a holomorphic section $fs$. What happens for generic $\lambda \in U(1)$ is that one still finds an integral transform, but with a different overall kernel. Hence we say that the circle action on Higgs fields gets turned into a coherent state transform on the Segal–Bargmann moduli space, and then one recovers the geometric action on sections by plugging-in the holomorphic frame.

**Theorem 4.7.** *Assume that the coefficients of the complex symmetric matrix*

$$-\frac{1}{2}(\overline{\Xi} + \Xi') = \begin{pmatrix} n_{11} & n_{12} \\ n_{12} & n_{22} \end{pmatrix}$$

*satisfy*

$$\Re\left( \frac{n_{11}n_{22} - n_{12}^2}{n_{11}} \right) > 0. \tag{4.27}$$

*Then the action of $\lambda \in U(1)$ on $\mathcal{H}_{\mathbb{C},\tau}$ is provided by the integral transform:*

$$\lambda(fs)(a) = \left( \int_{\mathfrak{M}} f(z)\widetilde{A}_{\tau,\lambda}(\overline{z},a)e^{-|z|^2} dxdy \right) s(a),$$

*where*

$$\widetilde{A}_{\tau,\lambda}(z,a) \coloneqq c_{\tau,\lambda} \exp\left\{ -\frac{1}{2} \begin{pmatrix} a \\ z \end{pmatrix} \begin{pmatrix} \mathrm{Id} + M & M \\ M & \mathrm{Id} + M \end{pmatrix} \begin{pmatrix} a \\ z \end{pmatrix} \right\},$$

*for a suitable symmetric matrix $M$ and a constant $c_{\tau,\lambda}$.*

*Proof.* One has $\lambda(fs) = \varphi \circ \lambda^* \mathcal{B}_\tau^{-1}(fs)$, by definition. Using Prop. 4.27 and 4.26 one sees that this equals

$$\varphi \circ \lambda^* \circ \mathcal{B}_\tau^{-1}(fs)(a) = \varphi \circ \lambda^*(\widetilde{fr})(a) = \varphi\Big(\lambda^*(\widetilde{fr})\Big)(a) = \left(\int_{\mathcal{M}} A_{\tau,\lambda}(a,x)\widetilde{f}(x)dx\right) s(a) =$$
$$= \left(\frac{c_\tau}{\pi}\right)^2 \cdot \left(\int_{\mathcal{M}} A_{\tau,\lambda}(a,x)\left(\int_{\mathfrak{M}} f(z)e^{-|z|^2}\overline{A_\tau(z,x)}idzd\overline{z}\right)dx\right) s(a) =$$
$$= \left(\frac{c_\tau}{\pi}\right)^2 \cdot \left(\int_{\mathfrak{M}} f(z)e^{-|z|^2}\left(\int_{\mathcal{M}} A_{\tau,\lambda}(a,x)\overline{A_\tau(z,x)}dx\right)idzd\overline{z}\right) s(a).$$

Replacing the measure $dxdy = idzd\overline{z}$ provides the formula in the statement, after calculating $\int_{\mathcal{M}} A_{\tau,\lambda}(a,x)\overline{A_\tau(z,x)}dx$. Now, up to a multiplicative constant the integrand equals

$$\exp\left\{-\frac{1}{2}\overline{z}^2 - \frac{1}{2}a^2 - \frac{1}{2}x(\overline{\Xi} + \Xi')x + \sqrt{2}(a+\overline{z})\cdot Ax\right\},$$

and after isolating $e^{-\frac{1}{2}(\overline{z}^2+a^2)}$ one is left with

$$\exp\left\{-\frac{1}{2}x(\overline{\Xi} + \Xi')x + \sqrt{2}(a+\overline{z})\cdot Ax\right\}.$$

Now there is some condition on the matrix $\overline{\Xi} + \Xi'$ to assure that this integral converges. One can split the integral along $x_1$ and $x_2$, and then notice that the integral in $dx_1$ converges because the real part of $\overline{\Xi} + \Xi'$ is the positive definite symmetric matrix $2A^tA$, and the imaginary part oscillates quickly enough. This does not however imply that the resulting integral in $dx_2$ will converge.
Rather, if one denotes $n_{ij}$ the complex coefficients of $-\frac{1}{2}(\overline{\Xi} + \Xi')$, then one finds the condition (4.27) in the statement.[12]

Let us thus assume that this condition is satisfied. Then, up to changing the previous multiplicative constant, the result of the integration is the exponential of a quadratic function of $a + \overline{z}$, which may be written

$$-\frac{1}{2}(a+\overline{z})M(a+\overline{z}) = -\frac{1}{2}\Big(aMa + 2aM\overline{z} + \overline{z}M\overline{z}\Big),$$

where one can assume that the invertible matrix $M$ is symmetric.[13] Hence one recovers the formula in the statement by picking the correct constant $c_{\tau,\lambda}$. □

**Remark 4.28.** One way of getting more control on all the constants involved would be to introduce the metaplectic correction, defining all pairings intrinsically in terms of half-forms. Doing this would not however make it possible to just set all these constants to be equal to one (see e.g. [KW06]).

[12] Note that (4.27) is not quite the same as

$$\Re(n_{11})\Re(n_{22}) - \Re(n_{12})^2 > 0,$$

which expresses the fact that $A^tA > 0$.

[13] The computation in Rem. 4.27 shows that $M = -\mathrm{Id}$ if $\lambda = 1$.

Now that we have an explicit formula for the Hitchin action on $\mathcal{H}_{\mathbb{C},\tau}$, which can be upgraded to an action on the whole of the vector bundle $\mathcal{H}_{\mathbb{C}}^{(t)} \longrightarrow \mathbb{H}$, we can study the compatibility of any connection with it. To do this, the essential problem is to conjugate the differential operators of function multiplication and covariant derivative through the action. As we did in § 4.6.3, we start by seeing how the integral transform

$$\mathcal{I}_{\tau,\lambda} \colon f \longmapsto \int_{\mathfrak{M}} f \widetilde{A}_{\tau,\lambda} dxdy,$$

commutes with function multiplication and ordinary derivative $\partial_z$. Then one may use $\lambda = \mu_s \circ \mathcal{I}_{\tau,\lambda} \circ \mu_s^{-1}$.

**Proposition 4.28.** *One has:*

$$\begin{cases} \mathcal{I}_{\tau,\lambda} \circ \partial_z \circ \mathcal{I}_{\tau,\lambda}^{-1} &= -M^{-1}\left(\frac{1}{2}(M+\mathrm{Id})\mu_a + \partial_a\right) \\ \mathcal{I}_{\tau,\lambda} \circ \mu_z \circ \mathcal{I}_{\tau,\lambda}^{-1} &= \frac{1}{2}\left(\left(\mathrm{Id} - \frac{3}{2}M + \frac{1}{2}M^{-1}\right)\mu_a + (M^{-1}+\mathrm{Id})\partial_a\right). \end{cases}$$

*Proof.* Let us compute the derivative $\partial_a$ after applying the transform. One finds:

$$\partial_a \circ \mathcal{I}_{\tau,\lambda}(f) = \partial_a\left(\int_{\mathfrak{M}} f(z)\widetilde{A}_{\tau,\lambda}(a,\overline{z})e^{-|z|^2}dxdy\right) = \int_{\mathfrak{M}} f(z)\partial_a\left(\widetilde{A}_{\tau,\lambda}(a,\overline{z})\right)e^{-|z|^2}dxdy = \\ = \int_{\mathfrak{M}} \left(-\frac{1}{2}(M+\mathrm{Id})a - M\overline{z}\right) f(z)\widetilde{A}_{\tau,\lambda}(a,\overline{z})e^{-|z|^2}dxdy.$$

This gives two addends: the former is just $-\frac{1}{2}(M+\mathrm{Id})\mu_a \circ \mathcal{I}_{\tau,\lambda}$ and for evaluating the latter one may use $-M\overline{z}e^{-|z|^2} = M\partial_z e^{-|z|^2}$, which yields:

$$\int_{\mathfrak{M}} -M\overline{z}f(z)\widetilde{A}_{\tau,\lambda}(a,\overline{z})e^{-|z|^2}dxdy = \int_{\mathfrak{M}} f(z)\widetilde{A}_{\tau,\lambda}(\overline{z},a)\left(M\partial_z e^{-|z|^2}\right)dxdy = \\ = -\int_{\mathfrak{M}} \left(M\partial_z f(z)\right)\widetilde{A}_{\tau,\lambda}(\overline{z},a)e^{-|z|^2}dxdy,$$

where we integrated by parts, also using $\partial_z \widetilde{A}(a,\overline{z}) = 0$. Hence the second addend equals $-\mathcal{I}_{\tau,\lambda} \circ M\partial_z(f)$, and one finds

$$\partial_a \circ \mathcal{I}_{\tau,\lambda} = -\frac{1}{2}(M+\mathrm{Id})\mu_a \circ \mathcal{I}_{\tau,\lambda} - \mathcal{I}_{\tau,\lambda} \circ M\partial_z,$$

whence the first identity of the statement.

Similarly we compute $\mathcal{I}_{\tau,\lambda} \circ \mu_z$ to be:

$$\mathcal{I}_{\tau,\lambda} \circ \mu_z(f) = \int_{\mathfrak{M}} zf(z)\widetilde{A}_{\tau,\lambda}(a,\overline{z})e^{-|z|^2}dxdy = -\int_{\mathfrak{M}} f(z)\widetilde{A}_{\tau,\lambda}(a,\overline{z})\left(\partial_{\overline{z}}e^{-|z|^2}\right)dxdy = \\ = \int_{\mathfrak{M}} f(z)\left(\partial_{\overline{z}}\widetilde{A}_{\tau,\lambda}(a,\overline{z})\right)e^{-|z|^2}dxdy = \\ = \int_{\mathfrak{M}} \left(-\frac{1}{2}(M+\mathrm{Id})\overline{z} - Ma\right) f(z)\widetilde{A}_{\tau,\lambda}(a,\overline{z})e^{-|z|^2}dxdy,$$

where we used $\partial_{\overline{z}} f(z) = 0$ when integrating by parts. This also gives two addends: the first one is $-M\mu_a \circ \mathcal{I}_{\tau,\lambda}$, and the second one is evaluated as before. Hence

$$\mathcal{I}_{\tau,\lambda} \circ \mu_z = -M\mu_a \circ \mathcal{I}_{\tau,\lambda} - \frac{1}{2}(M+\mathrm{Id})\mathcal{I}_{\tau,\lambda} \circ \partial_z.$$

Substituting the value of $\mathcal{I}_{\tau,\lambda} \circ \partial_z$ provided by the first identity yields the second identity.

In all integration by parts one needs to use the fact that $f$ is square-summable for the measure $e^{-|z|^2} dxdy$. □

Now one may use again Lem. 4.8 to compute analogous commutation relations after plugging-in the holomorphic frame $s$. This can be used to observe how the flat connection of Thm. 4.1 transforms under the action, and similarly for the Bargmann-conjugated Hitchin–Witten connection $\widehat{\nabla}_{\mathbb{C}}$ of § 4.6.4. The argument is analogous to the one used in § 4.6.4 to conjugate $\widehat{\nabla}_{\mathbb{R}}$ with respect to the Bargmann transform, using § 4.6.

# Appendix A

# The problem of quantisation

This chapter presents some motivational material. The aim is to introduce the general problem of quantisation, which has its roots in quantum mechanics.

After a loose terminological introduction to classical systems in § A.1, we provide their basic mathematical dictionary in § A.2.
Next, quantum system are introduced from the mathematical perspective in § A.3, together with some of the standard formalism of quantum mechanics.
Finally, we state the problem of quantisation in § A.4, relying on the definitions of the previous sections.

Another objective of this chapter is to motivate some of the terminology used throughout the whole document. Also, the notions of time-dependent Hamiltonian systems, be they classical or quantum, are assumed as standard in Chap. 3. An excellent reference for all this material is [FY09].

## A.1 Classical systems in words

The main ingredients of a classical mechanical system are

- a configuration space
- a phase-space
- states
- observables.

The *configuration space* is the spacial, geometric setting for the evolution of the system at hand. Each of its points corresponds to a different position for the overall components of the said system. For instance, the configuration space for an unrestrained point-particle in three-dimensional space is the whole of $\mathbb{R}^3$.

The *phase-space* complements the configuration space by adding kinetic information. It is a space parametrising both the positions and the velocities of the components of the mechanical system. For example, the phase-space for an unrestrained point-particle in three-dimensional space is the whole of $\mathbb{R}^3 \times \mathbb{R}^3 \cong \mathbb{R}^6$. It is a fundamental, undisputed assumption that this information is all that's needed in order to predict the evolution of the system: the totality of positions and velocities of the components of a system at some moment of time uniquely determine its motion.[1]

At each given time, the system will thus find itself in a particular mechanical condition, as described by its phase-space. The particular value of all relevant variables that make this up is the *state* of the system, which thus amounts to a point in its phase-space. This should more precisely be referred to as a *pure state*, whereas a *mixed state* is a statistical combinations of such pure states (i.e. a generic probability measure on the phase-space, instead of a Dirac $\delta$). We shall always deal with pure states, since this is not a thesis in statistical mechanics. To get back to the usual example, the point $(x, 0) \in \mathbb{R}^3 \times \mathbb{R}^3$ in the phase-space of a free point-particle correspond the state in which the particle is at rest at the position (or configuration) $x \in \mathbb{R}^3$.

An *observable* is a physical quantity that depends on the mechanical state of the system, and which can be measured to yield some real number in any fixed unit system (in the case of mixed states, one would only get an expected value). For example, the kinetic energy of a free point particle of mass $m \in \mathbb{R}_{\geq 0}$ in the pure state $(x, v) \in \mathbb{R}^6$ is the quantity $H(x, v) = \frac{m}{2}|v|^2$. One may have time-independent observables, as well as observables whose values explicitly depend on the moment in which the measure is performed.

Finally, a *classical system* is a collection of observables $H_1, \ldots, H_n$. They are meaningful, possibly interacting quantities that can be measured as time flows, and the prediction of their time evolution is the fundamental question one asks. More precisely, in non-statistical mechanics one studies the evolution of the values $H_i(p) \in \mathbb{R}$, where $p$ is a pure state. To do this, one has different dynamical pictures:

1. One may fix the state $p$, and try to predict the evolution of the values $H_i(t, p)$ by letting the time-dependence lie in this the observable entirely: this is the *Hamilton picture* of motion.

2. One may fix the observable $H_i$, and try to predict the evolution of the values $H_i\big(p(t)\big)$ by letting the time-dependence lie in the state entirely: this is the *Liouville picture* of motion.

3. One may let both the observable and the state vary: this is the *interaction picture*.

The first two pictures are equivalent, in the sense that the value of an observable $H_i$ in the state $p$ changes with time in the same way. Both pictures provide differential equations for the object that is assumed to vary, which rely on (i) the choice of a distinguished

[1] This is taken *verbatim* from [Arn89], which also adds: "One can imagine a world in which to determine the future of a system one must also know the acceleration at the initial moment, but experience shows us that our world is not like this".

observable $H$, called the *Hamiltonian* of the system (e.g. the total energy, for conservative systems), and (ii) a binary operation $\{\cdot,\cdot\}$ on the space of observables, called the Poisson bracket. With these two choices made, the Hamilton picture of motion yields the following differential equations:

$$\dot{H}_i = \{H, H_i\}, \qquad \dot{p} = 0. \tag{A.1}$$

The Liouville picture instead provides differential equations for a curve into the phase-space. If one picks local coordinates $\{x_i\}_i$ on the phase-space, so that the curve can be written $p(t) = (x_1(t), \dots, x_n(t))$ componentwise, then one has

$$\dot{H}_i = 0, \qquad \dot{x_i} = \{x_i, H\} = -\{H, x_i\}.^{2} \tag{A.2}$$

A solution to this system of nonlinear, first-order, ordinary differential equations is called a *classical trajectory*. Solving the dynamics of the system means solving an initial-value problem for these equations, collectively called the classical equations of motion.

Let us now provide a sound mathematical dictionary for all these terms.

## A.2 Mathematical dictionary for classical systems

We must first define phase-spaces, states and observables.

**Definition A.1.** A classical phase-space is a Poisson manifold $(M, \{\cdot,\cdot\})$. A (pure) classical state is a point $p \in M$. A classical observable is a smooth function $H\colon M \longrightarrow \mathbb{R}$ on the phase-space, also called a classical Hamiltonian.

Recall that a Poisson manifold $(M, \{\cdot,\cdot\})$ is a manifold $M$ together with a Poisson structure $\{\cdot,\cdot\}\colon C^\infty(M,\mathbb{R}) \wedge C^\infty(M,\mathbb{R}) \longrightarrow C^\infty(M,\mathbb{R})$, i.e. a Lie bracket which is compatible with the associative product, in the sense of the Leibnitz identity:

$$\{fg, h\} = f\{g, h\} + \{f, h\}g, \qquad \text{for} \qquad f, g, h \in C^\infty(M, \mathbb{R}).$$

**Example A.1** (Associative algebras are Poisson)**.**
Any associative algebra $A$ can be endowed with the Poisson structure provide by the commutator for its associative product:

$$\{a, b\} \coloneqq \Big[a, b\Big] = ab - ba, \qquad \text{for} \qquad a, b \in A.$$

Indeed, the Leibnitz identity is readily shown:

$$\Big[ab, c\Big] = abc - cab = abc - acb + acb - cab = a(cb - cb) + (ac - ca)b = a\Big[b, c\Big] + \Big[a, c\Big]b,$$

for all $a, b, c \in A$. This is actually the universal construction for the Poisson bracket of the algebra of quantum observables, as explained in § A.3.

[2] When expressed for a symplectic phase-space in local Darboux coordinates $(q, p)$, this yields the canonical *Hamilton equations*: $\dot{q} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial q}$.

Poisson manifolds are thus the mathematical stage of classical mechanics. In the thesis we deal with less general objects, namely *symplectic manifolds* $(M, \omega)$, i.e. smooth manifolds equipped with distinguished closed, nondegenerate 2-forms.

**Remark A.1** (Symplectic manifolds are Poisson)**.**
Let $(M, \omega)$ be a symplectic manifold, and call $\omega^\sharp \colon T^*M \longrightarrow TM$ the isomorphism of vector bundles induced by the fibrewise nondegenerate pairings $\omega_p \colon T_pM \wedge T_pM \longrightarrow \mathbb{R}$ $(p \in M)$, as follows. If $\alpha \in T_p^*M$, then there exists a unique tangent vector $v \in T_pM$ such that $\iota_v \omega_p = \omega_p(v, \cdot) = -\alpha$, and one sets $\omega_p^\sharp(\alpha) \coloneqq v$. This means that $v$ is defined by the identity $\iota_v \omega_p + \alpha = 0$.
Now, if $f, g \in C^\infty(M, \mathbb{R})$, then the differential forms $df, dg \in C^\infty(M, T^*M)$ correspond to unique vector fields $X_f \coloneqq \omega^\sharp(df), X_g \coloneqq \omega^\sharp(dg) \in C^\infty(M, TM)$. Then it makes sense to set

$$\{f, g\} \coloneqq \omega(X_f, X_g) \in C^\infty(M, \mathbb{R}),$$

and one can readily verify that the Leibnitz identity is verified.
Notice moreover that the vector field $X_f$ acts as a derivation of $C^\infty(M, \mathbb{R})$ via the Poisson bracket with respect to $f$, *on the left*:

$$X_f(g) = dg(X_f) = -\iota_{X_g}\omega(X_f) = \omega(X_f, X_g) = \{f, g\}, \qquad \text{for all } g \in C^\infty(M, \mathbb{R}).$$

This follows from the sign convention taken earlier:

$$\iota_{X_f}\omega + df = \omega(X_f, \cdot) + df = 0. \tag{A.3}$$

This convention is also used so that the map $f \longmapsto X_f \colon C^\infty(M, \mathbb{R}) \longrightarrow C^\infty(M, TM)$ is a morphism of Lie algebras:

$$X_{\{f,g\}} = \big[X_f, X_g\big], \qquad \text{for} \qquad f, g \in C^\infty(M, \mathbb{R}). \tag{A.4}$$

This remark suggests the following more general definition for the *Hamiltonian vector field* of an observable on a classical phase-space.

**Definition A.2.** Let $(M, \{\cdot, \cdot\})$ be a classical phase-space, and $f \colon M \longrightarrow \mathbb{R}$ an observable. The Hamiltonian vector field $X_f \in C^\infty(M, TM)$ of $f$ is defined as a derivation by

$$X_f(g) \coloneqq \{f, g\} \in C^\infty(M, \mathbb{R}), \qquad \text{for all } g \in C^\infty(M, \mathbb{R}).$$

One can finally define what a classical Hamiltonian system is.

**Definition A.3.** Let $(M, \{\cdot, \cdot\})$ be a classical phase-space, and $n$ a positive integer.

- A time-independent classical Hamiltonian system is a set of classical observables $\{H_i\}_i \subseteq C^\infty(M, \mathbb{R})$.
- A $n$-time-dependent Hamiltonian consist of an open set $\mathbf{B} \subseteq \mathbb{R}^n$, together with a smooth function $H \colon M \times \mathbf{B} \longrightarrow \mathbb{R}$. The natural global coordinates $t_1, \ldots, t_n$ on $\mathbf{B}$ are called the time variables, and $\mathbf{B}$ the space of times.

- A $n$-time-dependent classical Hamiltonian system is a set $H_1, \dots, H_n \colon M \times \mathbf{B} \longrightarrow \mathbb{R}$ of $n$-time-dependent Hamiltonians of cardinality $n$, all the Hamiltonians having the same space of times $\mathbf{B} \subseteq \mathbb{R}^n$.

An important notion for Hamiltonian systems is that of *integrability*, which is ultimately related to the problem of solving the dynamics of the system, as pointed out at the end of § A.1. The equations (A.2) for a classical "trajectory" $x \colon \mathbf{B} \longrightarrow M$ now read

$$\frac{\partial x_i}{\partial t_j} = \{x_i, H_j\}, \tag{A.5}$$

where $\{x_i\}_i$ are local coordinates on the phase-space, and $j \in \{1, \dots, n\}$. This is the reason why one asks for as many Hamiltonians as there are time variables.

We will not delve in the analytical interest of the notion of integrability, i.e. actually solving — integrate — the equations of motion; rather, we shall be content with the following definition.

**Definition A.4.** Let $\{H_i\}_i$ be a time-dependent classical Hamiltonian system on the phase-space $M$, with space of times $\mathbf{B}$.

- The system is said to be integrable, or flat, if the function

$$f_{ij} \coloneqq \frac{\partial H_j}{\partial t_i} - \frac{\partial H_i}{\partial t_j} - \{H_i, H_j\} \in C^\infty(M \times \mathbf{B}, \mathbb{R}) \tag{A.6}$$

  vanishes for all $i, j$. The Poisson bracket is computed for the fibrewise restrictions $H_i|_{M\times\{t\}}$, $H_j|_{M\times\{t\}} \in C^\infty(M, \mathbb{R})$, where $t \in \mathbf{B}$.

- The system is said to be strongly integrable, or strongly flat, if

$$\frac{\partial H_j}{\partial t_i} - \frac{\partial H_i}{\partial t_j} = 0 = \{H_i, H_j\},$$

  for all $i, j$.

**Remark A.2.** The definition of integrable time-independent systems is in a way simpler than that for time-dependent ones, and in a way more complicated. It is simpler, since there are no partial derivatives $\partial_{t_i} H_j$ to consider, and thus (A.6) reduces to Poisson-commutativity; it is more complicated, because in the usual notion of *Liouville integrability* one adds the condition that the Hamiltonians be functionally independent on an open dense set.

Def. A.4 has a natural geometric interpretation. Namely, consider a time-dependent Hamiltonian system $H_1, \dots, H_n \colon M \times \mathbf{B} \longrightarrow \mathbb{R}$. The product $\mathbb{F} \coloneqq M \times \mathbf{B}$, together with the canonical projection $\pi \colon \mathbb{F} \longrightarrow \mathbf{B}$ on the second factor, is a trivial Poisson fibration over the base $\mathbf{B}$. One can use now use the Hamiltonian vector fields of the fibrewise restrictions of the time-dependent Hamiltonians in order to define a *Ehresmann connection* in this

fibration. Recall in turn that a Ehresmann connection is a horizontal vector sub-bundle $\mathbb{H} \subseteq T\mathbb{F} \cong TM \times T\mathbf{B}$, i.e a vector sub-bundle of the tangent bundle to $\mathbb{F}$ satisfying

$$\mathrm{Ker}(d\pi) \oplus \mathbb{H} = T\mathbb{F},$$

where $d\pi \colon T\mathbb{F} \longrightarrow T\mathbf{B}$ is the tangent map to the fibration map. This means that $\mathbb{H}$ is a complement to the vertical subspace $\mathrm{Ker}(d\pi) = TM \subseteq T\mathbb{F}$.
Now, to define such horizontal spaces, one considers the derivatives $\partial_{t_1}, \dots, \partial_{t_n}$ associated to the time variables varying in $\mathbf{B}$. They provide a global trivialising frame for $T\mathbf{B} \subseteq T\mathbb{F}$, and this sub-bundle defines the trivial Ehresmann connection. Next, one modifies this connection by adding the following vertical components. If $t \in \mathbf{B}$, then the restrictions $H_1|_{M\times\{t\}}, \dots, H_n|_{M\times\{t\}} : M \longrightarrow \mathbb{R}$ admits Hamiltonian vector fields $X_{H_1}, \dots, X_{H_n}$; let us set

$$X_i \coloneqq \partial_{t_i} + X_{H_i} \in C^\infty(\mathbb{F}, T\mathbb{F}),$$

for $i \in \{1, \dots, n\}$, and let us define a Ehresmann connection as the pointwise span of these vector fields

$$\mathbb{H} \coloneqq \mathrm{span}_{\mathbb{R}}\{X_1, \dots, X_n\} \subseteq T\mathbb{F}.^{3}$$

Notice that a section $x \colon \mathbf{B} \longrightarrow \mathbb{F}$ of the fibration is horizontal with respect to this connection if and only if it solves the partial derivative version of the equations of motion, since

$$X_j(x_i) = \big(\partial_{t_j} + X_{H_j}\big)x_i = \frac{\partial x_i}{\partial t_j} + \{H_j, x_i\},$$

so that $X_j(x_i) = 0$ is exactly (A.5), using the alternance of the Poisson bracket. This new nonlinear connection is thus the geometric object that codes the classical dynamics of the time-dependent system.

Finally, recall that such a distribution $\mathbb{H}$ of planes is said to be integrable, or flat, if there exists a foliation of $\mathbb{F}$ consisting of integral manifolds for $\mathbb{H}$. In this case, where $\mathbb{H}$ is smooth and has constant rank, the Frobenius theorem implies that $\mathbb{H}$ is integrable if and only if it is *involutive*, i.e. if the space of smooth sections $C^\infty(\mathbb{F}, \mathbb{H}) \subseteq C^\infty(\mathbb{F}, T\mathbb{F})$ is a Lie subalgebra for the standard Lie bracket of vector fields.
One now computes, for any smooth function $f \in C^\infty(\mathbb{F}, \mathbb{R})$, and for $i, j \in \{1, \dots, n\}$:

$$\big[X_i, X_j\big](f) = \{\partial_{t_i} H_j - \partial_{t_j} H_i, f\} + \{\{H_i, H_j\}, f\}.$$

This means that the vector field $\big[X_i, X_g\big] \in C^\infty(\mathbb{F}, T\mathbb{F})$ acts via the Poisson bracket of the function

$$f_{ij} \coloneqq \frac{\partial H_j}{\partial t_i} - \frac{\partial H_i}{\partial t_j} + \{H_i, H_j\},$$

on the left. Hence $\big[X_i, X_i\big] = X_{f_{ij}}$ is the (vertical) Hamiltonian vector field of this function, and one needs it to vanish in order for $\mathbb{H}$ to be integrable. This happens if and only if $f_{ij}$ is constant on the fibre $M$ of the fibration, so that it must actually the pull-back of a function on $\mathbf{B}$ with respect to $\pi$. The first condition of A.4 amounts to the stronger

[3] This is indeed a horizontal subbubdle, since the restriction $d\pi|_{\mathbb{H}} : \mathbb{H} \longrightarrow T\mathbf{B}$ is an isomorphism of vector bundles, because of $d\pi(X_i) = \partial_{t_i}$ for $i \in \{1, \dots, n\}$.

requirement that $f_{ij} = 0$. If moreover $\{H_i, H_j\} = 0$ for all $i, j$, then one has a strongly flat connection, and this is totally equivalent to the strong integrability of the classical system.

**Remark A.3.** There exists a coordinate-independent way of phrasing the above construction. Namely, using the same notations, one can consider the following horizontal 1-form on the fibration $\mathbb{F}$:

$$\varpi \coloneqq \sum_i H_i dt_i \in C^\infty(\mathbb{F}, \pi^* T^* \mathbf{B}).$$

With this notation introduced, the dynamical equations (A.5) become $dx_i = \{x_i, \varpi\}$. If one starts from such a 1-form $\varpi$, then any vector field $X$ on the basis produces a time-depend Hamiltonian function with space of times $\mathbf{B}$, by contraction: $H_X \coloneqq \langle \varpi, X \rangle$. One can thus give the following alternative definition of a time-dependent Hamiltonian system.

**Definition A.5.** Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold, and $\mathbf{B} \subseteq \mathbb{R}^n$ an open set. A time-dependent classical Hamiltonian system is a horizontal 1-form $\varpi$ on the trivial Poisson fibration $\pi \colon M \times \mathbf{B} \longrightarrow \mathbf{B}$.

In the symplectic case, there is a natural way of thinking of $\varpi$ as a correction to the trivial Ehresmann connection. Namely, assume $(M, \omega)$ to be symplectic, so that $\mathbb{F} = M \times \mathbf{B}$ is a trivial symplectic fibration, and denote $\widehat{\omega} \in A^2(\mathbb{F}, \mathbb{R})$ the pull-back of $\omega$ with respect to the canonical projection $\mathbb{F} \longrightarrow M$ to the first factor. This is a 2-form on the fibration that restricts to $\omega$ on each fibre, and such objects define Ehresmann connections by taking the orthogonal complements of the vertical spaces $TM \subseteq T\mathbb{F}$. In particular, the form $\widehat{\omega}$ defines the trivial Ehresmann connection, and one can correct it via

$$\Omega \coloneqq \widehat{\omega} - d\varpi \in A^2(\mathbb{F}, \mathbb{R}).$$

One can indeed check that the spaces

$$\mathbb{H}_p \coloneqq T_p M^{\perp \Omega_p} = \Big\{ v \in T\mathbb{F} \mid \Omega_p(T_p M, v) = 0 \Big\} \subseteq T_p \mathbb{F}$$

satisfy $\mathbb{H}_p \oplus T_p M = T_p \mathbb{F}$, for all $p \in \mathbb{F}$. Moreover, they are the pointwise span of the vector fields $X_i = \partial_{t_i} + X_{H_i}$, where $H_i \coloneqq H_{\partial_{t_i}}$, as above.

Hence a time-dependent classical Hamiltonian system is the correction between a trivial connection on a trivial symplectic fibration, and a new, possibly interesting one. In the thesis we deal with connections arising from the isomonodromic deformations of meromorphic connections over the Riemann sphere (in Chap. 3).

**Remark A.4** (Universal classical vector bundle)**.**
Let us mention a third "universal" viewpoint on classical systems, which is suited to deformation quantisation. Namely, one can think of a time-dependent Hamiltonian function $H \colon M \times \mathbf{B} \longrightarrow \mathbb{R}$ as a section $H \colon \mathbf{B} \longrightarrow A_0$ of the trivial vector bundle $A_0 \times \mathbf{B} \longrightarrow \mathbf{B}$, where $A_0 \coloneqq C^\infty(M, \mathbb{R})$ is the Poisson algebra of classical observables. This is just the remark that the restriction $H|_t : M \longrightarrow \mathbb{R}$ is by definition a classical observable for all $t \in \mathbf{B}$. We say that this a universal construction, because one can then take any $A_0$-module to construct other vector bundles. The case discussed above correspond to the

tautological representation $\rho\colon A_0 \longrightarrow \mathrm{Der}\left(C^\infty(M,\mathbb{R})\right)$, where $H_i$ acts via its Hamiltonian vector field, i.e. its Poisson bracket.

In particular, one may also define a time-dependent classical Hamiltonian system as an $A_0$-valued 1-form on the space of times $\mathbf{B}$:

$$\varpi = \sum_i H_i dt_i \in C^\infty(\mathbf{B}, T^*\mathbf{B} \otimes A_0).$$

This shows that the formula

$$\nabla := d_{\mathbf{B}} + \varpi \tag{A.7}$$

defines a connection in the vector bundle $A_0 \times \mathbf{B} \longrightarrow \mathbf{B}$, by letting $H_i\colon \mathbf{B} \longrightarrow A_0$ act on the fibre $A_0$ via its Poisson bracket. This is just a way to implement the equation of motion (A.5). The simply-laced isomonodromy systems of § 3.1 correspond exactly to such objects. We will define the quantum analogue of those classical connections at the end of the next section § A.3.

**Remark A.5.** We can (and will) consider algebraic Poisson varieties $M$ as phase-spaces, instead of Poisson manifolds. One should then replace smooth functions with sections of the structural sheaf $\mathscr{O}_M$, and ask for it to be a sheaf of Poisson algebras. In the case where $M$ is a variety over $\mathbb{C}$, one should also allow for complex-valued observables, and for complex spaces of times $\mathbf{B} \subseteq \mathbb{C}^n$. With these small caveat, all the above material can be rephrased in the complex-algebraic category.

## A.3 Quantum Hamiltonian systems

A quantum mechanical still consists of a set of observables, whose (expectation) values depend on the state of the system. To propose a quantum analogue of the previous two sections, one must first recall the standard linear-algebraic formalism of quantum mechanics. The mathematical notions we will introduce provide a quantum analogue of the mathematical dictionary for classical mechanics presented in § A.2.

**Definition A.6.** The phase-space of a quantum mechanical system is the projective space $\mathbb{P}(\mathcal{H})$ to a separable Hilbert space $\left(\mathcal{H}, \langle\cdot \mid \cdot\rangle\right)$. A (pure) quantum state is a line $\psi \in \mathbb{P}(\mathcal{H})$, also called a wave function. A quantum observable is a self-adjoint operator $\widehat{H}\colon \mathcal{H} \longrightarrow \mathcal{H}$, also called a quantum Hamiltonian.[4]

Recall in turn that a Hilbert space is a vector space $\mathcal{H}$ over $\mathbb{C}$, together with an Hermitian product $\langle\cdot \mid \cdot\rangle\colon \mathcal{H} \otimes \overline{\mathcal{H}} \longrightarrow \mathbb{C}$, such that $\mathcal{H}$ is a complete metric space with respect to the metric induced by the inner product. Such a space is *separable* if there exists a $\langle\cdot \mid \cdot\rangle$-orthonormal countable basis (in the topological sense, when $\mathrm{Dim}_{\mathbb{C}}(\mathcal{H}) = +\infty$). Notice that a pure state is the same as a unitary vector of $\mathcal{H}$, defined up to a phase $e^{i\theta} \in U(1)$.

[4]For the sake of simplicity, we assume all operators to have the whole of $\mathcal{H}$ as domain. We however admit unbounded operators, in order to deal with the quantisation of generalised positions and momenta later on.

Exactly as in the classical context, the phase-space of the theory parametrises pure states of the system. In this case a mixed state would be a *density matrix*, i.e. an Hermitian, positive semi-definite operator of trace 1. Extremal cases are the 1-dimensional projectors, i.e. wave functions.

A quantum observable $\widehat{A} \in \mathrm{End}(\mathcal{H})$ can still be evaluated at a pure state $\psi \in \mathcal{H}$, only that this does not give a real number as result. Rather, the standard formalism instructs one to take the number

$$\langle \psi \mid \widehat{A}\psi \rangle \in \mathbb{R}$$

as the expectation value of $\widehat{A}$ in the state $\psi$: this is only notion that makes sense in the intrinsically probabilistic quantum theory. Notice that the expectation value is real because $\widehat{A}$ is self-adjoint, and it is well defined up to a change of phase on $\psi$.[5]

Just as the space of classical observables had the structure of a Poisson algebra, the same is true for quantum observables, following Ex. A.1. Namely, the commutator $\big[\cdot,\cdot\big]$ makes $\mathrm{End}(\mathcal{H})$ into a noncommutative Poisson algebra. Unfortunately, however, the commutator of two self-adjoint operators is not self-adjoint; rather, it is anti-adjoint:

$$(\widehat{A}\widehat{B} - \widehat{B}\widehat{A})^\dagger = \widehat{B}^\dagger\widehat{A}^\dagger - \widehat{A}^\dagger\widehat{B}^\dagger = \widehat{B}\widehat{A} - \widehat{A}\widehat{B} = -(\widehat{A}\widehat{B} - \widehat{B}\widehat{A}).$$

One can however correct this by multiplying it with a pure imaginary constant, usually written

$$\big\{\widehat{A},\widehat{B}\big\}_\hbar \coloneqq \frac{i}{\hbar}\big[\widehat{A},\widehat{B}\big].$$

The value of $\hbar \in \mathbb{R}$ is fixed by the experiments, not by the theory. It is not surprising that this value is that of the reduced Planck constant.

To show the analogue dynamical notions as those at the end of § A.1, one can still study the evolution of the expectation values $\langle \psi \mid \widehat{A}\psi \rangle \in \mathbb{R}$ with different pictures:

1. One may fix the state $\psi$, and try to predict the evolution of the values $\langle \psi \mid \widehat{A}(t)\psi \rangle$ by letting the time-dependence lie in this the observable entirely: this is the *Heisenberg picture* of motion.

2. One may fix the observable $\widehat{H}$, and try to predict the evolution of the values $\langle \psi(t) \mid \widehat{A}\psi(t) \rangle$ by letting the time-dependence lie in the state entirely: this is the *Schrödinger picture*.

3. One may let both the observable and the state vary: this is the *interaction picture*, or *Dirac picture*.

The first two pictures are equivalent, and rely again on (i) the choice of a distinguished quantum observable $\widehat{H}$, the Hamiltonian of the system, and (ii) the Poisson bracket $\{\cdot,\cdot\}_\hbar$ of observables. More precisely, the first picture yields the *Heisenberg equation*, for the quantum observable $\widehat{A}$:

$$\frac{d\widehat{A}}{dt} = \big\{\widehat{H},\widehat{A}\big\}_\hbar = \frac{i}{\hbar}\big[\widehat{H},\widehat{A}\big], \qquad \dot{\psi} = 0. \tag{A.8}$$

[5] More generally, if $S$ is a density matrix then the expectation value of the observable $A$ in the mixed state $S$ is the real number $\mathrm{Tr}(AS)$.

The second picture yields *Schrödinger equation*, for the wave function $\psi$:

$$\frac{d\widehat{A}}{dt} = 0, \qquad \dot{\psi} = -\frac{i}{\hbar}\widehat{H}\psi. \tag{A.9}$$

It follows that the eigenvectors of $\widehat{\psi}$ correspond to stationary states, since scalar multiplication does not affect the state of the system.

We can now provide the good quantum analogue of Def. A.5. Let us denote $A$ the noncommutative Poisson algebra of quantum observables, endowed with the bracket $\{\cdot,\cdot\}_\hbar$.

**Definition A.7.** Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space, and $\mathbf{B} \subseteq \mathbb{R}^n$ an open space. A time-dependent quantum Hamiltonian system is a $A$-valued 1-form $\widehat{\varpi}$ on $\mathbf{B}$:

$$\widehat{\varpi} \in C^\infty(\mathbf{B}, T^*\mathbf{B} \otimes A).$$

The idea is the same as in § A.2. If now $X$ is a vector field on $\mathbf{B}$, then one finds a time-dependent quantum Hamiltonian by contraction: $\widehat{H}_X := \langle\widehat{\varpi}, X\rangle \in C^\infty(\mathbf{B}, A)$, thinking of this object as a smooth section of the trivial vector bundle $A \times \mathbf{B} \longrightarrow \mathbf{B}$ over $\mathbf{B}$.
To push the geometric parallel even further, recall that, according to Def. A.5, a classical Hamiltonian system is the same as nonlinear Ehresmann connection on a trivial Poisson fibration $\pi\colon \mathbb{F} = M \times \mathbf{B} \longrightarrow \mathbf{B}$, where $M$ is the classical phase-space. In the same way, one may define a quantum system as a connection $\widehat{\nabla}$ in the trivial vector bundle $\mathbb{E} := \mathcal{H} \times \mathbf{B} \longrightarrow \mathbf{B}$, by setting

$$\widehat{\nabla} := d_{\mathbf{B}} + \widehat{\varpi},$$

where $d_{\mathbf{B}}$ is the ordinary exterior differential on the base. This may be referred to as a quantum connection, to stress that it is associated to a time-dependent quantum Hamiltonian system.

The 1-form $\widehat{\varpi}$ takes its values in the endomorphisms of the fibre, by definition, and thus this formula really defines a connection in $\mathbb{E}$. One can thus also think of a time-dependent quantum Hamiltonian system as a correction to the trivial connection on a trivial vector bundle bundle, in order to obtain new interesting ones. Just as in the classical case, a $\widehat{\nabla}$-horizontal section $\psi : \mathbf{B} \longrightarrow \mathcal{H}$ corresponds to a solution of the equations of motion, since

$$\widehat{\nabla}_{\partial_{t_i}}\psi = \partial_{t_i}\psi + \widehat{H}_i\psi,$$

so that $\widehat{\nabla}\psi = 0$ is the multi time variable case of (A.9), up to the quantum parameter $i\hbar^{-1}$.

The flatness of a quantum Hamiltonian system is equivalent to that of its associate connection, as expressed in the following definition.

**Definition A.8.** Let $\{\widehat{H}_i\}_i$ be a time-dependent quantum Hamiltonian system on the quantum phase-space $\mathbb{P}(\mathcal{H})$, with space of times $\mathbf{B}$.

- The system is integrable, or flat, if the operator

$$\widehat{f}_{ij} := \frac{\partial\widehat{H}_j}{\partial t_i} - \frac{\partial\widehat{H}_i}{\partial t_j} - \left[\widehat{H}_i, \widehat{H}_j\right] \in C^\infty(\mathbf{B}, A) \tag{A.10}$$

vanishes for all $i, j$. The commutator is computed for the fibrewise restriction $\widehat{H}_i\big|_t, \widehat{H}_j\big|_t \in A$, where $t \in \mathbf{B}$.

- The system is strongly integrable, or strongly flat, if

$$\frac{\partial \widehat{H}_j}{\partial t_i} - \frac{\partial \widehat{H}_i}{\partial t_j} = 0 = \big[\widehat{H}_i, \widehat{H}_j\big],$$

for all $i, j$.

The time dependent operator $\hat{f}_{ij}$ is simply the commutator $\big[\widehat{\nabla}_{\partial_{t_i}}, \widehat{\nabla}_{\partial_{t_j}}\big]$, i.e. one term of the curvature $F_{\widehat{\nabla}} \in A^2\Big(\mathbf{B}, \mathrm{End}(\mathcal{H})\Big)$ of $\widehat{\nabla}$.

**Remark A.6** (Projective flatness)**.**
To be totally rigorous, the quantum analogue of the trivial Poisson fibration $M \times \mathbf{B} \longrightarrow \mathbf{B}$ should be the trivial projective bundle $\mathbb{P}(\mathbb{E}) \coloneqq \mathbb{P}(\mathcal{H}) \times \mathbf{B} \longrightarrow \mathbf{B}$, since one needs the phase-spaces as fibres. All that we defined above for the vector bundle $\mathbb{E} = \mathcal{H} \times \mathbf{B} \longrightarrow \mathbf{B}$ descend to the projectivisation, because of the elementary fact that all linear operators $\widehat{H} \in \mathrm{End}(\mathcal{H})$ act naturally on the projective space. However, the induced connection on the projectivisation may be flat even if $\widehat{\nabla}$ is not.

**Definition A.9.** The quantum connection $\widehat{\nabla}$ is projectively flat if the induced connection on $\mathbb{P}(\mathbb{E})$ is flat.

This is equivalent to the fact that the curvature of $\widehat{\nabla}$ be a scalar. This definition is assumed when discussing the Hitchin connection (see § 2.2.4).

**Remark A.7** (Universal quantum vector bundle)**.**
There exists a "universal" version of the quantum vector bundle $\mathbb{E}$. Namely, one could consider the algebra $A$ of quantum observable abstractly, and use a time-dependent quantum Hamiltonian system $\widehat{H}_i \colon \mathbf{B} \longrightarrow A$ to define a connection $\widehat{\nabla}$ in the trivial vector bundle $A \times \mathbf{B} \longrightarrow \mathbf{B}$, with the same formula as before:

$$\widehat{\nabla} = d_{\mathbf{B}} + \sum_i \widehat{H}_i dt_i. \tag{A.11}$$

This is the exact quantum analogue of (A.7). The simply-laced quantum connections of § 3.4 are exactly such objects.

Now the elements $\widehat{H}_i(t) \in A$ are assumed to act on $A$ by left multiplication, for all $t \in \mathbf{B}$, thereby implementing Schrödinger equation of motion (A.9). This is a universal version of the above material, in the sense that one can now pick any $A$-module $V$ to induce a connection in the vector bundle $V \times \mathbf{B} \longrightarrow \mathbf{B}$. The tautological representation $\rho \colon A \hookrightarrow \mathrm{End}(\mathcal{H})$ is just an example.

The universal viewpoint of the above remark is more suited to deformation quantisation, which is introduced in § 2.1. Indeed, in this algebraic setting one abstractly looks at the algebra of quantum observables, and not to a concrete geometric realization of the

quantum phase-space. On the other hand, this is exactly the primary concern of geometric quantisation, discussed in § 2.2.
Both constructions aim to solve the very same fundamental problem of quantisation, and the following section, last of this motivational appendix, is devoted to introducing this problem.

## A.4 The correspondence principle and the problem of quantisation

We can now phrase the problem of the quantisation of a classical Hamiltonian system, be it time-dependent or not. The origin of this question lies in *Bohr correspondence principle*.

One can safely work under the ontological assumption that the phenomena of nature occur regardless of the theory that one uses to describe them, be it classical or quantum. It happens however that in certain regimes one theory might be better suited than another in order to provide this description. The only ground for preferring one theory over another is their respective prediction accuracy.

For instance, both Newtonian mechanics (implementing Galilean relativity) and relativistic mechanics (implementing Einstein relativity) can be used to derive a velocity-addition formula to relate the velocities of objects in different reference frames. When the velocities $v$ considered are negligible with respect to the speed of light in vacuum $c$, i.e. when the *Lorentz factor*

$$\gamma := \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

is almost equal to 1, then the two theories are in good accordance. However, as $v \longrightarrow c$, which is equivalent to $\gamma \longrightarrow +\infty$, the predictions diverge, and it is relativistic mechanics the one that passes the empirical tests. One might say that there exists a mathematical theory for all values of the parameter $\gamma \in [1, +\infty)$, that the "Newton regime" corresponds to the limit $\gamma \longrightarrow 1$, and the "relativistic regime" to the asymptotic $\gamma \longrightarrow +\infty$ (study of photons, and other massless light-like objects). It is mathematically equivalent to obtain the limit $\gamma \longrightarrow 1$ by taking $c \longrightarrow +\infty$, if one considers the speed of light in vacuum as upper bound; indeed, in Galilean relativity there is no such thing as a limit speed.[6]
The deformation approach thus indicates that there is a full family of theories underlying Newton mechanics, with relativistic mechanics being an extremal case, and one may ask the question to reconstruct the full family from the knowledge of the regime $\gamma \approx 1$.

Conceptually, the same happens for passing from classical mechanics to quantum mechanics. Bohr correspondence principle states that the quantum description underlies the classical one: it provides a finer theory whose predictions deviate from the classical one in the microscopic regime, and whose macroscopic asymptotic gives back the predictions of classical mechanics. The macroscopic asymptotic is obtained by taking the limit of one

[6] This clashes however with the interpretation of $c$ as a universal constant of nature. See also the discussion for Planck's constant $\hbar$ below.

or several quantum numbers that dictates the characteristic size of the system (e.g. the distance between two consecutive energy shells in Bohr's model of the hydrogen atom). The most important such quantum parameter is $\hbar$: the limit $\hbar \longrightarrow 0$ would correspond to the failure of the fundamental quantum hypothesis that energy is quantised, since the difference between two consecutive energy levels in the harmonic oscillator is proportional to $\hbar$. Because of this, the asymptotic $\hbar \approx 0$ is called the *semiclassical limit*, to indicate that the theory is already somewhat classical, but more quantum number might still be far from vanishing.
To resolve the question about the status of $\hbar$ — variable or constant ? — one might say the following. For every mechanical system there are usually characteristic masses, velocities, etc., from which a unit of action appropriate to the system can be derived. The semiclassical limit is applicable when Planck's constant $\hbar$ divided by this unit is much smaller than 1. Hence, mathematically, we just regard $\hbar$ as a parameter.[7]

In conclusion, one has the semiclassical regime $\hbar \longrightarrow 0$ (study of macroscopic objects), and the quantum regime $\hbar \approx 1$ (study of microscopic objects).

The problem of quantisation is the following:

*Given the semiclassical limit of a family of quantum theories, is it possible to reconstruct the whole family?*

To get back to the terminology of the previous section, one would start from a time-dependent classical Hamiltonian system $H_i \colon M \times \mathbf{B} \longrightarrow \mathbb{R}$ over the classical phase-space $M$, with space of times $\mathbf{B}$. The question is then to construct a time-dependent quantum Hamiltonian system $\widehat{H}_i \colon \mathbf{B} \longrightarrow \mathrm{End}(\mathcal{H})$ over a quantum phase-space $\mathbb{P}(\mathcal{H})$, such that the semiclassical limit of the quantum Hamiltonian $\widehat{H}_i$ is the classical $H_i$. The semiclassical limit will be a distinguished projection $\sigma \colon A \longrightarrow A_0$ sending the *noncommutative* Poisson algebra $A$ of quantum observables to the *commutative* Poisson algebra $A_0$ of classical observables, thereby implementing the aforementioned operation of taking the limit $\hbar \approx 0$. The strongest possible solution to this problem is to explicitly construct a Hilbert space $(\mathcal{H}, \langle \cdot \mid \cdot \rangle)$, together with a *quantisation map*

$$\mathcal{Q} \colon C^\infty(M, \mathbb{C}) \longrightarrow \mathrm{End}(\mathcal{H}), \qquad \mathcal{Q} \colon f \longmapsto \widehat{f},$$

satisfying the so-called *Dirac axioms*:

1. The map is $\mathbb{C}$-linear.
2. The map sends classical observables (real-valued functions) to quantum observables (selfadjoint operators): $\mathcal{Q}(f) = \mathcal{Q}(f)^\dagger$ if $f \in C^\infty(M, \mathbb{R})$. In particular, $\mathcal{Q}$ is a section of the semiclassical limit.
3. The map is normalised: $\widehat{1} = -i\hbar\, \mathrm{Id}_{\mathcal{H}}$.
4. The map respects the Lie structures: $\mathcal{Q}\big(\{f, g\}\big) = \frac{i}{\hbar}\big[\widehat{f}, \widehat{g}\big]$ for all smooth functions $f, g \in C^\infty(M, \mathbb{C})$.

[7] This is quoted almost verbatim from the introduction of [BW97].

5. The map respects irreducibility: if $\{f_i\}_i \subseteq C^\infty(M,\mathbb{C})$ is a complete set, then the quantisation $\{\widehat{f}_i\}_i \subseteq \mathrm{End}(\mathcal{H})$ acts irreducibly.[8]

Moreover, one demands that this procedure recovers the standard Dirac quantisation of the free particle in flat space $\mathbb{R}^{2n}$, which is presented in Ex. 2.12, in the context of geometric quantisation.
Finally, if the initial classical system carries a group of symmetries $G$, i.e. a group of Poisson automorphisms, then it is natural to ask that the quantum model does too. This would result in a $\langle\cdot \mid \cdot\rangle$-unitary $G$ action on $\mathcal{H}$. In the symplectic case, the group of all classical symmetries — physicists' *canonical transformations* — is by definition the group of all symplectomorphisms. On the quantum side, a quantum symmetry is an element of the projective unitary group of $(\mathcal{H}, \langle\cdot \mid \cdot\rangle)$. Since these two groups are in general not isomorphic, one sees that there is a priori no hope to have a natural correspondence of classical and quantum symmetries. When a classical symmetry is not preserved one says that it has been broken, and speaks of *quantum anomalies*.

There are then other general reasons why the program described above is destined to fail, collectively referred to as *no-go* theorems. The original results in this direction are due to Groenewold ([Gro46]) and van Howe ([Van51b; Van51a]). This is why one also speaks of Groenewold–van Howe theorems for such negative results. One may collectively express them by saying that there is no functor from the category of Poisson manifold to the category of Hilbert spaces that satisfies of all the above.

In the background chapter 2 we have introduced two sound mathematical frameworks that nonetheless aim to provide partial solutions to the problem of quantisation: deformation quantisation and geometric quantisation (see § 2.1 and § 2.2, respectively).
Loosely speaking, deformation quantisation weakens axiom 4, asking for the relevant identity to hold in the semiclassical asymptotic, thereby making sense of the following equality in the quantum algebra $A$:

$$\{f,g\} = \lim_{\hbar\longrightarrow 0} \frac{i}{\hbar}\big[\widehat{f},\widehat{g}\big], \tag{A.12}$$

where $f,g \in A_0$ (see Eq. (2.2)). Geometric quantisation instead starts by constructing the quantum phase-space $\mathbb{P}(\mathcal{H})$.

[8] A set of function $f_i\colon M \longrightarrow \mathbb{C}$ is *complete* if its centraliser equals the centre $\mathbb{C} \subseteq C^\infty(M,\mathbb{C})$ of the whole Poisson algebra, i.e. the constant functions.

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## **Titre :** Quantification d'espaces de modules et de connexions

**Mots clefs :** quantification, espaces de modules, connexions

**Résumé :** On construit de nouvelles connexions quantiques intégrables dans fibrés vectoriels au-dessus d'espaces de modules de surfaces de Riemann et de leurs généralisations sauvages, en utilisant deux approches différentes. Premièrement, on utilise la quantification par déformation pour construire de nouvelles connexions intégrables à partir d'Hamiltoniennes d'isomonodromie irrégulières, dans l'esprit de Reshetikhin de la dérivation de la connexion de Knizhnik–Zamolodchikov à partir des Hamiltoniennes de Schlesinger [Res92]. Deuxièmement, on construit une version complexe de la connexion de Hitchin [Hit90] pour la quantification géométrique de l'espace de modules de Hitchin sur une surface de genre un, par rapport au groupe $\mathrm{SL}(2,\mathbb{C})$ et à des polarisations Kählériennes, en complémentant l'approche par polarisations réelles de Witten [Wit91]. Finalement, on utilise la transformée de Bargmann pour dériver une formule pour la connexion de Hitchin–Witten [Wit91] dans le fibré vectoriel des sections holomorphes, et pour transformer l'action de Hitchin en une transformée sur l'espace de Segal–Bargmann [Seg63; Bar61], basée sur les états cohérents.

## **Title :** Quantisation of moduli spaces and connections

**Keywords :** Quantisation, moduli spaces, connections

**Abstract :** We construct new flat quantum connections on vector bundles over moduli spaces of Riemann surfaces and their wild generalisations, using two different approaches. Firstly, we use deformation quantisation to construct new flat connections from irregular isomonodromy Hamiltonians, in the spirit of Reshetikhin's derivation of the Knizhnik–Zamolodchikov connection from the Schlesinger Hamiltonians [Res92]. Secondly, we construct a complex version of the Hitchin connection [Hit90] for the geometric quantisation of the Hitchin moduli space over a surface of genus one, with respect to the group $\mathrm{SL}(2,\mathbb{C})$ and to Kähler polarisations, complementing Witten's real polarisation approach [Wit91]. Finally, we use the Bargmann transform to derive a formula for the connection of Hitchin–Witten [Wit91] on the vector bundle of holomorphic sections, and to turn Hitchin's action into a transform on the Segal–Bargmann space [Seg63; Bar61], which relies on coherent states.

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