Some contributions to the study and simulation of asymmetric diffusions

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Quelques contributions à l'étude et à la simulation des diffusions asymétriques

"Habilitation à diriger des recherches"

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Introduction (french)

Depuis l'époque de ma thèse ([19]) l'étude, la caractérisation et la simulation des diffusions *asymétriques* occupent une place centrale dans ma recherche. Par diffusion asymétrique on entend, dans un contexte unidimensionnel, un processus qui est solution d'une Équation Différentielle Stochastique (EDS) faisant intervenir le temps local du processus inconnu, et à coefficients discontinus. Ces EDS avec Temps Local (EDSTL) sont, dans le cas homogène en temps, de la forme

$$dX_t = \sigma(X_t)dW_t + b(X_t)dt + \sum_{1\le i\le I} \beta_i \, dL_t^{x_i}(X). \tag{0.0.1}$$

Ici $I \in \mathbb{N}$, les x_i's sont des points de $\mathbb{R}$, et $L_\cdot^{x_i}(X)$ désigne le temps local symétrique de X au point x_i (voir Frequently used notations). Les coefficients σ et b sont autorisés à être discontinus aux points x_i's. Ce type d'EDSTL a été étudié par J.-F. Le Gall dans l'article fondateur [43], où sont données des conditions nécessaires et suffisantes pour qu'elles satisfassent une propriété d'unicité trajectorielle (par exemple, parmi les hypothèses cruciales, on a le fait que $|\beta_i| < 1$, $1 \le i \le I$). Comme sous ces conditions on a aussi l'existence d'une solution faible pour (0.0.1), il y a en conséquence une unique solution forte à (0.0.1). Notons que, concernant l'étude théorique des telles équations, divers problèmes peuvent encore faire l'objet de recherches: par exemple on peut étudier ce qui se passe quand les x_i's sont en nombre infini et présentent un point d'accumulation (voir l'article récent [72], où ce problème est abordé à l'aide de la théorie des formes de Dirichlet, dans le cas $\sigma \equiv 1$, $b \equiv 0$).

Mais en fait notre intérêt pour les EDSTL est principalement dû à leur lien avec les opérateurs sous forme divergence à coefficients discontinus. Soient $\rho(x), a(x) \ge m > 0$ et $b(x)$ des fonctions de la variable d'espace, considérons alors l'opérateur elliptique

$$\frac{\rho}{2}\nabla\big(a\nabla\,\cdot\,\big) + \big(b - \frac{\rho a'_{x,\pm}}{2}\big)\,\nabla \tag{0.0.2}$$

(voir Frequently used notations, pour une définition précise de $a'_{x,\pm}$). Si les coefficients (en particulier a) sont suffisamment réguliers, on peut récrire (0.0.2) sous la forme non-divergence

$$\frac{1}{2}\sigma^2\,\Delta + b\,\nabla \tag{0.0.3}$$

avec $\sigma^2 = \rho a$. Alors il est bien connu ([39]) que l'opérateur (0.0.3) - et donc (0.0.2) - est le générateur infinitésimal d'un processus X qui est solution de

$$dX_t = \sigma(X_t)dW_t + b(X_t)dt. \tag{0.0.4}$$

Si maintenant les coefficients ne sont pas réguliers (en particulier a est discontinu aux points x_i, $1 \le i \le I$), on ne peut pas récrire (0.0.2) sous la forme (0.0.3). Mais (0.0.2) est encore le générateur d'un processus X et, comparé à (0.0.4), des termes de temps local vont apparaître dans la dynamique de X. Plus précisément, on aura que X est solution de (0.0.1) avec

$$\sigma^2 = \rho a \quad \text{ et } \quad \beta_i = \frac{a(x_i+) - a(x_i-)}{a(x_i+) + a(x_i-)}, \ \ \forall 1 \le i \le I. \tag{0.0.5}$$

Notons que dans (0.0.2) le signe ∇ peut désigner soit la dérivée faible, comme par exemple quand on étudie le problème dans un contexte L^2 à l'aide des formes de Dirichlet ([72]), soit la dérivée classique,

comme quand on travaille avec les semi-groupes de Feller. Notons que ces deux approches nécessitent de préciser soigneusement le domaine de tels opérateurs, pour garantir que pour toute fonction φ dans ce domaine, la dérivée faible de $a\nabla\varphi$ existe en tant que fonction.

De plus, si on suppose que les coefficients sont définis par (0.0.5), et que σ et b sont réguliers en dehors des points de singularité x_i, $1 \leq i \leq I$, on peut établir, via une formule de Feynman-Kac, le lien entre la solution X de (0.0.1) et la solution classique $u(t,x)$ d'une Équations aux Dérivées Partielles (EDP) avec conditions de transmission (appelée le problème de *Diffraction* ou le problème de *transmission* parabolique): l'EDP satisfaite par $u(t,x)$ met en jeu l'opérateur (0.0.2), et $u(t,x)$ doit satisfaire à tout instant t la condition de transmission

$$a(x_i+)u'_x(t,x_i+) = a(x_i-)u'_x(t,x_i-)$$

pour tout $1 \leq i \leq I$. En particulier, ce lien ouvre la voie à un vaste champs d'applications concernant par exemple la dispersion au franchissement d'interfaces [2], la diffusion en milieu poreux [45], la magnéto-électroencéphalographie [32] (voir aussi les références données dans [46]).

Pour les preuves concernant - dans le contexte homogène en temps évoqué plus haut - le lien entre les solutions de (0.0.1), les opérateurs sous la forme (0.0.2), et les solutions d'EDP avec conditions de transmission, on pourra consulter les articles fondateurs [62][56], les articles de synthèse [46], [71], et la série de travaux [52], [53], [20], [19], [54], où des schémas numériques sont de plus présentés et étudiés. Notons que ce type de questions semble toujours susciter un grand intérêt (voir les travaux récents [9], [17]). Notons que le lien entre EDS et EDP ouvre la voie à l'usage des méthodes de Monte Carlo dans les problèmes modélisés par ce type d'opérateur.

Durant ma thèse j'ai travaillé à produire des schémas numériques (avec erreur de discrétisation) pour la simulation des diffusions asymétriques. Ces schémas étaient basés sur des approximations par marches aléatoires (voir [20, 19] et [24]).

Après ma période de post-doc à Paris (2007-2008), où j'ai eu l'opportunité d'explorer d'autres sujets, et depuis que je suis maître de conférences à Grenoble, j'ai continué régulièrement à explorer le champ de recherche constitué par les diffusions asymétriques, en me focalisant maintenant sur d'autres questions ([27, 28, 29, 30]).

Je vais présenter dans les chapitres 1 à 3 la série de travaux [27, 28, 29, 30]. J'ai choisi de présenter les choses du plus général au plus particulier, et du plus théorique au plus numérique. Notons que de la sorte, la chronologie des publications (ou des soumissions) ne sera pas respectée. Mais la présentation sera plus satisfaisante du point de vue mathématique. Dans le chapitre 4, je résumerai plusieurs sujets de recherche sur lesquels j'ai travaillé ces dernières années, mais qui sont en dehors du champ des diffusions asymétriques.

Le chapitre 1 présente les résultats et idées du récent preprint [30]. Nous examinons une version inhomogène en temps de (0.0.1), c'est à dire

$$dX_t = \sigma(t,X_t)dW_t + b(t,X_t)dt + \sum_{i=1}^{I} \beta_i(t)dL_t^{x_i}(X) \tag{0.0.6}$$

(ici on a fait dépendre tous les coefficients du temps, y compris les $x_i(\cdot)$'s qui sont maintenant des courbes fonctions du temps; voir Frequently used notation pour une définition précise du temps local sur une courbe). On peut en premier lieu se demander si (0.0.6) a une solution. La réponse est affirmative: en adaptant la démarche de [43] au cas inhomogène en temps, on montre des résultats d'existence et d'unicité pour (0.0.6) (on utilise en particulier la récente formule d'Itô- Peskir donnée dans [61], qui est une généralisation de la formule d'Itô-Tanaka pour des fonctions dépendant du temps). Les conditions que nous mettons en évidence sur les coefficients sont très similaires à celles requises dans [43] pour le cas homogène en temps (en particulier on a encore $|\beta_i(\cdot)| < 1$).

Une fois l'existence d'une solution X à (0.0.6) établie, on cherche un lien entre X et un opérateur sous la forme (0.0.2), mais cette fois avec des coefficients $\rho(t,x)$, $a(t,x)$, $b(t,x)$ dépendant du temps (et toujours autorisés à être discontinus). On montre une formule de Feynman-Kac liant X et $u(t,x)$, la solution classique d'une EDP parabolique avec conditions de transmission. Cette EDP met en jeu (0.0.2) avec $\sigma^2 = \rho a$, et les $a(t,x_i(t)+) - a(t,x_i(t)-)$ proportionnel à $\beta_i(t)$ pour tous t,i. La condition

de transmission est maintenant inhomogène en temps, elle devient

$$\forall 1 \leq i \leq I, \quad a(t, x_i(t)+)u'_x(t, x_i(t)+) = a(t, x_i(t)-)u'_x(t, x_i(t)-), \ \ \forall t \in [0, T)$$

(T sera l'horizon temporel de l'EDP considérée). Bien sûr, pour que cette formule de Feynman-Kac soit valide, on a besoin de l'existence effective de la solution classique $u(t, x)$. Dans la littérature sur les EDP que nous avons pu trouver sur le sujet, les choses sont le plus souvent étudiées avec $\rho \equiv 1$, et des interfaces ne bougeant pas avec le temps. C'est le cas dans l'article de référence sur le sujet par Ladyzhenskaya et al. ([40]), qui traite de solutions classiques (et ne se cantonne pas aux solutions faibles). Ainsi nous avons été conduits à adapter ces résultats d'EDP à notre cas. Ces aspects EDP représentent une large part du chapitre 1 (cependant, pour un grand nombre de résultats techniques on se référera à [30]). Finalement on utilise ces résultats d'EDP pour caractériser X en tant que processus de Markov inhomogène en temps (section 1.7).

Mais l'équation générale (0.0.6) n'est pas la première EDSTL inhomogène en temps que nous avions rencontrée et traitée. Dans un de nos précédents articles [27] nous avons examiné le type le plus simple d'EDSTL inhomogène en temps:

$$dB_t^\beta = dW_t + \beta(t)dL_t^0(B^\beta).$$

On appelle B^β le Inhomogeneous Skew Brownian Motion (ISBM). Le ISBM apparait pour la première fois dans un article de S. Weinryb ([78]) où un résultat d'unicité trajectorielle est prouvé, sous la seule hypothèse que β est une fonction mesurable à valeurs dans $[-1, 1]$. Mais la question de l'existence ne paraissait pas avoir été abordée, de sorte que dans [27] nous avions travaillé à lui fournir une réponse positive, en même temps que nous travaillions à décrire certaines lois relatives à B^β (voir aussi [14]). Soulignons que l'esprit de [27] était assez différent de celui de [30]. En fait, B^β se comporte le plus souvent comme un mouvement brownien, mais il est de temps en temps perturbé par le terme de temps local. Ainsi, on peut exploiter beaucoup de lois connus concernant le mouvement brownien, pour étudier B^β. Soulignons qu'on obtient un résultat d'existence pour β non régulier, ce qui est en dehors des résultats fournis par le chapitre 1. Mais à un certain moment (début de la sous-section 2.5.2) les résultats du chapitre 1 serviront d'ingrédient de base (voir cependant remarque 2.5.3). Ainsi dans le chapitre 2 on cherche à récrire et résumer [27] à le lumière du chapitre 1. On insiste sur ce qui n'est pas contenu dans le chapitre 1, et on ne reproduit pas ce qui pourrait être redondant avec ce chapitre, ainsi que certaines preuves très techniques pour lesquelles on renvoie à [27]. Parmi les lois explicites que nous sommes capables de donner figurent la densité de probabilité de transition de B^β, et la tri-densité de B^β, son temps local time, et son dernier temps de sortie du point zéro.

Comme dit précédemment, une de mes préoccupations concernant les diffusions asymétriques est de simuler leurs trajectoires. Revenons au cas homogène en temps (c'est en partie parce que les travaux[28][29] dont il va être question maintenant ont été faits avant [30]; bien sûr on pourrait se demander maintenant ce qu'on pourrait faire pour la simulation d'EDSTL inhomogènes en temps...). Examinons un version très simple de l'équation (0.0.1),

$$dX_t^\beta = dW_t + \mathbf{b}(X_t^\beta)dt + \beta dL_t^0(X^\beta). \tag{0.0.7}$$

Ici le coefficient de dérive $\mathbf{b}$ est autorisé à avoir une discontinuité, au point zéro, le point où est pris l'unique temps local dans l'EDS. On peut se demander s'il est possible de simuler exactement les trajectoires de X^β solution de (0.0.7), dans l'esprit de Beskos et al. pour le cas C^1 et $\beta = 0$ (cf [8] [6]).

Dans le chapitre 3 on a cherché à présenter dans un même cadre les deux articles [28][29] que nous avons écrits sur le sujet ([28] traite du cas $\beta \neq 0$ et [29] traite du cas $\beta = 0$).

L'idée de Beskos et al. repose sur le fait que la loi de X solution de $dX_t = dW_t + b(X_t)dt$ est absolument continue par rapport à celle du mouvement brownien, avec une densité de type Girsanov qui peut être rendue explicite (pour ces auteurs b est de classe C^1). Grâce à des changements de probabilité supplémentaires, Beskos et al. réussissent finalement à simuler exactement les trajectoires de X en acceptant/rejetant des trajectoires de pont brownien (ils utilisent une fonction de rejet ad hoc, qui dépend de b).

Examinons (0.0.7) avec $\beta \neq 0$. Notre premier problème pour adapter la méthodologie de Beskos et al. est que la loi de X^β n'est plus absolument continue par rapport à celle du brownien (cf Théorème 2.4 dans [43]). Cependant, dans [28], nous avons réussi dans un premier temps à calculer la densité de

le loi de X^β, $\beta \neq 0$, par rapport à celle de $B^{\beta,\mu}$ solution de

$$dB_t^{\beta,\mu} = dW_t + \mu_\beta \, dt + \beta dL_t^0(B^{\beta,\mu})$$

(dans notre méthodologie μ_β depend de b et β). Dans un second temps nous calculons la densité de transition de $B^{\beta,\mu}$ (on généralise ainsi un résultat de [75]). Ainsi, on peut appliquer la méthode de Beskos et al., en utilisant une fonction de rejet similaire, et des ponts du processus $B^{\beta,\mu}$ au lieu de ponts browniens.

Dans [29] nous avons examiné le cas $\beta = 0$ dans (0.0.7) (mais avec $\mathbf{b}$ toujours discontinu). Cette fois la loi de X^0 est absolument continue par rapport à celle du brownien, mais la densité n'est pas facilement exploitable. En effet, en raison de la discontinuité de $\mathbf{b}$ au point zéro, un terme de temps local apparait dans cette densité. Nous avons choisi de contourner le problème en regardant le cas $\beta = 0$ comme limite du cas $\beta \neq 0$ pour $\beta \downarrow 0$ (ainsi cette approche est très différente de celle du très récent article [60]). Grâce à des propriétés de consistance des EDSTL (cf [43]), la solution de (0.0.7) converge en loi, quand $\beta \downarrow 0$, vers la solution de $dX_t^0 = dW_t + \mathbf{b}(X_t^0)dt$. Le miracle calculatoire que nous avons observé est que cette convergence a lieu aussi au niveau des algorithmes utilisés pour $\beta \neq 0$ (par exemple les fonctions de rejet utilisées pour $\beta \neq 0$ tendent vers des fonctions limites). Il nous reste ainsi à prouver mathématiquement que l'algorithme limite mis en évidence est bien un algorithme de simulation exacte pour $dX_t^0 = dW_t + \mathbf{b}(X_t^0)dt$.

Dans le chapitre 4, j'ai choisi trois autres sujets de recherche sur lesquels j'ai travaillé durant les dernières années, et j'en ai fait une présentation rapide et synthétique. Ils sont en dehors du cadre des diffusions asymétriques, mais représentent une expérience significative pour moi, car ils m'ont permis de découvrir d'autres champs de recherche.

Au cours de ma période de post-doc (janvier 2007-août 2008) j'ai eu l'occasion de travailler sur des méthodes de réduction de variance pour les méthodes de Monte Carlo appliquées à la finance. Plus précisément sur la méthode dite de *stratification*. Cette méthode, pour amener une bonne réduction de variance, nécessite de bien choisir les strates utilisées, ainsi que la politique d'allocation des tirages de Monte Carlo à effectuer dans ces states. Ce choix peut se faire de façon adaptative: cette problématique a fait l'objet des articles [23] et [21] (ils n'étaient pas publiés au moment de ma prise de poste MCF, c'est pourquoi je crois bon de les mentionner).

Au LJK de 2009 à 2010 j'ai eu l'occasion de travailler avec Emmanuel Gobet sur des méthodes de développement limité stochastique pour le calcul de prix d'options versant des dividendes discrets ([22]). L'idée est de faire un développement autour du modèle de Black-Scholes (BS) 1D à coefficients constants, pour lequel le prix explicite P^{BS} est connu. Le prix d'intérêt apparait alors comme la somme de P^{BS} et de termes correctifs faisant intervenir les grecques dans le modèle de BS, elles aussi connues explicitement. On obtient ainsi une méthode de calcul sans simulations Monte Carlo. Les preuves de convergence font appel au calcul de Malliavin.

De septembre 2012 à septembre 2015 j'ai co-encadré au LJK la thèse de Ester Mariucci (à 50 % avec Sana Louhichi). Cette thèse portait sur l'équivalence au sens de Le Cam d'expériences statistiques mettant en jeu des processus de diverses natures (processus de Lévy, processus de diffusion, ...). On dit que deux expériences statistiques $\mathcal{P}$ et $\mathcal{Q}$, portant sur le même paramètre f, sont équivalentes, si le risque de se tromper sur l'estimation de f dans l'expérience $\mathcal{P}$, est contrôlé par celui dans l'expérience $\mathcal{Q}$, et vice-versa (cf [50]). Un de nos premiers résultats dans cette direction a été [25] (unpublished note). De plus, pour prouver une telle équivalence, un des outils est le contrôle de la distance entre deux lois le probabilités (type distance L^1, distance de Kullback-Leibler etc...). L'article [26] établit un tel contrôle pour les lois de processus additifs avec mesure de Lévy homogène en temps (c'est à dire que ces processus ressemblent à des processus de Lévy, mais on a autorisé la dérive et le coefficient de diffusion à dépendre du temps). Les techniques de preuves font appel notamment à des changements de probabilités de type Girsanov et Esscher. Notons que Ester Mariucci a continué à utiliser ce type de calculs dans sa thèse et ses articles, notamment [49][51].

Introduction (english)

Since the time of my PhD thesis ([19]) the study, characterization and simulation of *asymmetric* diffusions, has been at the very center of my research. By asymmetric diffusion we mean, in a one-dimensional context, a process which is the solution of a Stochastic Differential Equation (SDE) that involves the local time of the process itself, and that is with discontinuous coefficients. These SDEs with Local Time (SDELTs) are, in the time-homogeneous case, of the form

$$dX_t = \sigma(X_t)dW_t + b(X_t)dt + \sum_{1\leq i\leq I} \beta_i \, dL_t^{x_i}(X). \tag{0.0.8}$$

Here $I \in \mathbb{N}$, the x_i's are points of $\mathbb{R}$, and $L_\cdot^{x_i}(X)$ stands for the symmetric local time process of X at point x_i (see Frequently used notations). The coefficients σ and b are allowed to be discontinuous at the points x_i's. This kind of SDELT has been studied by J.-F. Le Gall in the seminal paper [43], where necessary and sufficient conditions for its pathwise uniqueness property are given (for example, among the crucial assumptions, is the fact that $|\beta_i| < 1$, $1 \leq i \leq I$). As under these conditions a weak solution also exists to (0.0.8), there is consequently a unique strong solution to (0.0.8). Note that, concerning the theory of such equations, various issues can still be addressed: for example one may study what happens when the x_i's are in infinite number, and present accumulation points (see the recent paper [72], where this problem is addressed with the help of Dirichlet forms, in the case $\sigma \equiv 1$, $b \equiv 0$).

But in fact our interest in SDELTs is mainly motivated by their link with divergence form operators with discontinuous coefficients. Let $\rho(x), a(x) \geq m > 0$ and $b(x)$ be functions of the space variable, and consider the elliptic operator

$$\frac{\rho}{2}\nabla\big(a\nabla\,\cdot\,\big) + \big(b - \frac{\rho a'_{x,\pm}}{2}\big)\,\nabla \tag{0.0.9}$$

(see Frequently used notations, for a precise definition of $a'_{x,\pm}$). If the coefficients (in particular a) are smooth enough we can rewrite (0.0.9) in the non-divergence form

$$\frac{1}{2}\sigma^2\,\Delta + b\,\nabla \tag{0.0.10}$$

with $\sigma^2 = \rho a$. Then it is well known ([39]) that the operator (0.0.10) - and thus (0.0.9) - is the infinitesimal generator of a process X that solves

$$dX_t = \sigma(X_t)dW_t + b(X_t)dt. \tag{0.0.11}$$

If now the coefficients are not smooth (in particular a is discontinuous at points x_i, $1 \leq i \leq I$), we cannot rewrite (0.0.9) in (0.0.10). But (0.0.9) is still the generator of a process X and, compared to (0.0.11), local time terms will appear in the dynamic of X. More precisely, we will have that X solves (0.0.8) with

$$\sigma^2 = \rho a \quad \text{ and } \quad \beta_i = \frac{a(x_i+) - a(x_i-)}{a(x_i+) + a(x_i-)}, \;\; \forall 1 \leq i \leq I. \tag{0.0.12}$$

Note that in (0.0.9) the ∇-sign can stand either for the weak derivative, for example when one studies the problem in a L^2-context with the help of Dirichlet forms ([72]), or for the classical derivative, when one works with Feller semigroups. Note that both approaches require to carefully specify the domain of the operator, guaranteeing that for any function φ in this domain, the weak derivative of $a\nabla\varphi$ exists as a function.

Further, assuming the coefficients are defined by (0.0.12), and σ and b are smooth outside the points of singularity x_i, $1 \leq i \leq I$, one can establish, via a Feynman-Kac formula, the link between the solution X of (0.0.8) and the classical solution $u(t,x)$ of some parabolic Partial Differential Equation (PDE) with transmission conditions (the so-called *Diffraction* or *transmission parabolic problem*): the PDE satisfied by $u(t,x)$ involves the operator (0.0.9), and $u(t,x)$ has to satisfy at any time t the transmission condition

$$a(x_i+)u'_x(t,x_i+) = a(x_i-)u'_x(t,x_i-)$$

for any $1 \leq i \leq I$. In particular, this link opens an extended broadcast of applications such as dispersion across interfaces [2], diffusions in porous media [45], magneto-electroencephalography [32] (see also [46] and the references therein).

For proofs stating - in this time-homogeneous context - the link between solutions of (0.0.8), operators of the form (0.0.9), and solutions of PDE involving transmission conditions, one may refer to the seminal papers [62][56], the overviews [46], [71], and also to the series of works [52], [53], [20], [19], [54], where numerical schemes are presented and studied. Note that this kind of questions still seems to rise a lot of interest (see the recent papers [9], [17]). Note that the link between SDE and PDE aspects opens the way to the use of Monte Carlo methods in models described by such operators.

During my PhD thesis I aimed at producing numerical schemes (with discretization error) for the simulation of asymmetric diffusions. These schemes are based on random walk approximations (see the mentioned references [20, 19] and [24]).

After my period of post-doc in Paris (2007-2008), where I explored other topics, and since I am an assistant professor at Grenoble University, I have regularly continued to explore the world of asymmetric diffusions, focusing now on other questions ([27, 28, 29, 30]).

I will present from Chapter 1 to Chapter 3 the series of work [27, 28, 29, 30]. I have chosen to present things from the more general to the more particular, and from the more theoretical to the more numerical. Note that by doing so, the chronology of the publications (or submission) will not be respected, but the presentation will be more satisfying from a mathematical point of vue. In Chapter 4, I will sum up a selection of several research topics I have worked on in the past years, but that have nothing to do with asymmetric diffusions.

Chapter 1 presents the results and ideas of the recent preprint [30]. Here we have a look at a time-inhomogeneous version of (0.0.8), namely

$$dX_t = \sigma(t,X_t)dW_t + b(t,X_t)dt + \sum_{i=1}^{I} \beta_i(t)dL_t^{x_i}(X) \tag{0.0.13}$$

(here everything depends on time, even the $x_i(\cdot)$'s that are now time-curves; see Frequently used notation for a definition of the local time on a curve). One can first wonder if (0.0.13) has a solution. The answer is essentially affirmative: by adapting the methodology of [43] to the time-inhomogeneous case, we show existence and uniqueness results for (0.0.13) (we use in particular the recent Itô- Peskir formula given in [61], which is a generalization of the Itô-Tanaka formula to time-dependent functions). Our conditions on the coefficients are very similar to those required in [43] for the time-homogeneous case (in particular we have again $|\beta_i(\cdot)| < 1$).

Once the existence of a solution X to (0.0.13) is established, one seeks for a link between X and an operator of the form (0.0.9), but this time with time-dependent (and still possibly discontinuous) coefficients $\rho(t,x)$, $a(t,x)$, $b(t,x)$. We show a Feynman-Kac formula linking X and $u(t,x)$ the classical solution of a parabolic PDE with transmission conditions. This PDE involves (0.0.9) with $\sigma^2 = \rho a$, and the $a(t,x_i(t)+) - a(t,x_i(t)-)$ proportional to $\beta_i(t)$ for all t,i. The transmission condition is now time-inhomogeneous, it is

$$\forall 1 \leq i \leq I, \quad a(t,x_i(t)+)u'_x(t,x_i(t)+) = a(t,x_i(t)-)u'_x(t,x_i(t)-), \ \ \forall t \in [0,T)$$

(T will be the finite horizon of the PDE problem). Of course to ensure the validity of the Feynman-Kac formula, we need this classical solution $u(t,x)$ to exist. In the PDE litterature we could find on the subject, things are studied mostly with $\rho \equiv 1$, and with non-moving interfaces. That is the case with the reference paper on the subject by Ladyzhenskaya et al. ([40]), that deals with classical solutions (and not

only weak ones). We were thus led to adapt these PDE results to our case. These PDE aspects represent a large part of Chapter 1 (however, for a lot of technical proofs we will refer to [30]). Finally we use these PDE results in order to characterize X as a time-inhomogeneous Markov process (Section 1.7).

But the general equation (0.0.13) is not the first inhomogeneous SDELT we have dealt with. In one of our previous paper [27] we have examined to most simple type of time-inhomogeneous SDELT, namely

$$dB_t^\beta = dW_t + \beta(t)dL_t^0(B^\beta).$$

We call B^β the Inhomogeneous Skew Brownian Motion (ISBM). The ISBM appears in a seminal paper by S. Weinryb ([78]) where a pathwise uniqueness result is proved, under the sole assumption that β is a Borel function taking values in $[-1, 1]$. But the question of existence seemed not to have been addressed, so that in [27] we worked at providing a positive answer, at the same time we were working at describing some laws related to B^β (see also [14]). We stress that the spirit of [27] was rather different than the one of [30]. In fact B^β behaves mostly like a Brownian motion, except that it is sometimes perturbed by the local time term. Thus we can exploit a lot of known explicit laws about the Brownian motion, in order to study B^β. We stress that we get an existence result for a non smooth coefficient β, which is outside the results provided in Chapter 1. But at some point (beginning of Subsection 2.5.2) the results of Chapter 1 will serve as a basic ingredient (see however Remark 2.5.3). Thus in Chapter 2 we have aimed at rewriting and summing up [27], in the light of Chapter 1. We insist on what is not contained in Chapter 1, and drop what could be redundant with this chapter, along with some very technical proofs for which we refer to [27]. Among the explicit laws we are able to give are the transition function of B^β, and the trivariate density of B^β, its local time and its last exit time from point zero.

As previously said, one of my preoccupations about asymmetric diffusions is to simulate their paths. Let us come back to the time-homogeneous case (this is partly because the works [28][29] we will deal with now have been made before [30]; one could wonder now what could be done about simulation of time-inhomogeneous SDELTs...). Let us examine a very simple equation of the type (0.0.8),

$$dX_t^\beta = dW_t + \mathbf{b}(X_t^\beta)dt + \beta dL_t^0(X^\beta). \tag{0.0.14}$$

Here the drift component $\mathbf{b}$ is allowed to have one discontinuity, at point zero, the point where is taken the unique local time in the SDE. One can wonder if it is possible to simulate exactly the paths of X^β solution of (0.0.14), in the spirit of the work by Beskos et al. for the case of a drift of class C^1 and $\beta = 0$ (cf [8] [6]).

In Chapter 3 we have aimed at presenting in the same framework the two papers [28][29] we have written to address this question ([28] deals with the case $\beta \neq 0$ and [29] deals with the case $\beta = 0$).

The idea of Beskos et al. relies on the fact that the law of X solution of $dX_t = dW_t + b(X_t)dt$ is absolutely continuous w.r.t. the one of Brownian motion, with a Girsanov type density that can be made explicit (for them b is of class C^1). Thanks to additional changes of probability, Beskos et al. finally manage to simulate exactly the paths of X by accepting/rejecting Brownian bridge paths (they use an ad hoc rejection function, depending on b).

Let us examine (0.0.14) with $\beta \neq 0$. Our first problem in order to adapt the methodology of Beskos et al. is that the law of X^β is no more absolutely continuous w.r.t. the one of Brownian motion (cf Theorem 2.4 in [43]). However, in [28], we manage in a first time to compute the density of the law of X^β, $\beta \neq 0$, w.r.t. the one of $B^{\beta,\mu}$ solution of

$$dB_t^{\beta,\mu} = dW_t + \mu_\beta\, dt + \beta dL_t^0(B^{\beta,\mu})$$

(in our methodology μ_β depends on b and β). In a second time we compute the density of $B^{\beta,\mu}$ (we thus generalize a result of [75]). Thus, we can apply the method of Beskos et al., by using a similar rejection function but bridges of $B^{\beta,\mu}$ instead of Brownian bridges.

In [29] we have examined the case $\beta = 0$ in (0.0.14) (but $\mathbf{b}$ still discontinuous). This time the law of X^0 is absolutely continuous w.r.t. the one of Brownian motion, but with a density that is not much tractable. Indeed because of the discontinuity of $\mathbf{b}$ at point zero, a local time term appears in this density. We have chosen to avoid this problem by seeing the case $\beta = 0$ as the limit of the case $\beta \neq 0$ for $\beta \downarrow 0$ (thus this approach is different from the one of the very recent paper [60]). Thanks to consistency properties of SDELTs (cf [43]), the solution of (0.0.14) converges in law, as $\beta \downarrow 0$, to the solution of

$dX_t^0 = dW_t + \mathbf{b}(X_t^0)dt$. The computational miracle that we have observed is that this convergence also occurs at the level of algorithms used for $\beta \neq 0$ (for instance the rejection functions used for $\beta \neq 0$ tend to some limit functions). Then we prove mathematically that the exhibited Limit algorithm is indeed an exact simulation algorithm for $dX_t^0 = dW_t + \mathbf{b}(X_t^0)dt$.

In Chapter 4, I have chosen to present shortly three research topics I have worked on during the past years, that are outside the field of asymmetric diffusions. But they represent a significative experience for me, as they have allowed me to discover other fields of research.

During my postdoc period (2007-2008, CERMICS-ENPC then CMAP-École Polytechnique) I had the opportunity to work on variance reduction methods for Monte Carlo computations applied to Finance. More precisely on the method called *stratification*. This method, in order to yield a significant variance reduction, requires a smart choice of the strata, and of the amount of Monte Carlo drawings to be done in these strata. This choice can be done in an adaptive way: these issues have been addressed in the papers [23] and [21].

At LJK (Grenoble) from 2009 to 2010 I had the opportunity to work with Emmanuel Gobet on stochastic expansion methods for the pricing of options paying discrete dividends ([22]). The idea is to perform the expansion around the Black-Scholes (BS) 1D model with constant coefficients, for which the explicit price P^{BS} is known. The price of interest appears then as the sum of P^{BS} and of corrective terms involving greeks in the BS model, for which closed explicit formulae are also available. One thus obtains a computation method without any Monte Carlo simulation. The proofs for convergence require Malliavin Calculus.

From september 2012 to september 2015 I have co-advised the PhD of Ester Mariucci (at 50 % with Pr. Sana Louhichi). This PhD was on the equivalence in the Le Cam sense of statistical experiences involving processes of various kind (Lévy processes, diffusion processes, ...). One says that two statistical experiences $\mathcal{P}$ et $\mathcal{Q}$, with the same parameter to estimate f, are equivalent, if the risk to be wrong on the estimation of f in experience $\mathcal{P}$, is controlled by the one in experience $\mathcal{Q}$, and vice-versa (cf [50]). One of our first results in this direction has been [25] (unpublished note). Besides, in order to prove such an equivalence, one of the tools is the control of the distance between two probability measures (L^1-type distance, distance of Kullback-Leibler etc...). The paper [26] establishes such a control, for the laws of additive processes with a time-homogeneous Lévy measure (that is to say these processes are somehow similar to Lévy processes, but the drift and diffusion terms are allowed to depend on time). The proofs require Girsanov and Esscher type changes of probability measure. Note that Ester Mariucci has continued to use this type of computations in her thesis and articles, in particular [49][51].

Frequently used notations

We denote by D the space of $\mathbb{R}$-valued functions of $[0,T]$, $T<\infty$, that are right-continuous with left limits (r.c.l.l.). It will be equipped with the σ-field $\mathcal{D}$ endowed by the Skorokhod norm, so that $(D,\mathcal{D})$ is the usual Skorokhod space.

We denote by $(C,\mathcal{C})$ the space of $\mathbb{R}$-valued continuous functions of $[0,T]$, $T<\infty$, equipped with the σ-field $\mathcal{C}$ endowed by the supremum norm.

We will denote $\omega=(\omega_t)_{t\in[0,T]}$ the canonical (coordinate) process defined on $(C,\mathcal{C})$. That is to say we denote $\omega=(\omega_t)_{t\in[0,T]}$ the usual canonical process $Y=(Y_t)_{t\in[0,T]}$ defined on $(C,\mathcal{C})$ simply by $Y_\cdot(\omega)=\omega,\ \forall w\in C$ (i.e. $Y_t(\omega)=\omega_t,\ \forall\omega\in\Omega,\forall t\in[0,T]$). We claim that this will be clear from the context if ω denotes the canonical process or an element of C, so that this notation brings simplicity and clarity.

We denote $(\mathcal{C}_t)$ the canonical filtration (in fact the natural filtration of the canonical process ω, that has been modified in order to verify the usual conditions).

Instead of $[0,T]$ we will sometimes work with $[0,\infty)$, keeping the same notations. This will be clear from the context.

In Subsection 2.5.2 we will shortly consider the coordinate process $\tilde{\omega}=(\tilde{\omega}_t)_{t\geq0}$ defined in the same spirit on $\left(\mathbb{R}^{[0,\infty)},\mathcal{B}(\mathbb{R}^{[0,\infty)})\right)$ (see [39] Section 2.2).

For any semi-martingale X the process $L^0_\cdot(X)=(L^0_t(X))_{t\in[0,T]}$ is the symmetric local time at point 0 of X. And for any continuous function of bounded variation $\gamma:[0,T]\to\mathbb{R}$ we denote by $L^\gamma_\cdot(X)$ the process defined by

$$L^\gamma_t(X)=L^0_t(X-\gamma),\quad\forall t\in[0,T].$$

So that

$$\forall t\in[0,T],\quad L^\gamma_t(X)=\mathbb{P}-\lim_{\varepsilon\downarrow0}\frac{1}{2\varepsilon}\int_0^t\mathbf{1}_{|X_s-\gamma(s)|<\varepsilon}\,d\langle X\rangle_s,$$

(see [64], Exercise VI-1-25, and [61]).

For any topological spaces U,V we denote by $C(U)$ the set of continuous $\mathbb{R}$-valued functions on U, and by $C(U,V)$ the set of continuous functions from U to V.

$C_b(U)$ denotes the set of continuous bounded functions on U.

$C^p(U)$, $p\in\bar{\mathbb{N}}$, denotes the set of continuous functions on U with continuous derivatives up to order p.

$C_0(\mathbb{R})$ denotes the set on continuous functions on $\mathbb{R}$ vanishing at infinity.

In the following notations an interval $[0,T]\subset\mathbb{R}_+$ is given and kept fixed (with $0<T<\infty$).

We denote $E=[0,T]\times\mathbb{R}$ and $E^\circ=[0,T)\times\mathbb{R}$.

Let $F\subset E$ an open subset of E. We denote by $C^{p,q}(F)$ the set of continuous functions on F, with continuous derivatives up to order p in the time variable, and up to order q in the space variable (with the convention that for example $q=0$ corresponds to the continuity w.r.t. the space variable).

We denote by $C_0(E)$ the space of $\mathbb{R}$-valued continuous functions of E, vanishing at infinity, i.e. when $|x|\to\infty$, $(t,x)\in E$. We will denote this space C_0 in short when this causes no ambiguity. The spaces $C_0(\mathbb{R})$ and $C_0(E)$ are endowed with the corresponding supremum norm, for which we use the common notation $||\cdot||_\infty$ (which norm is meant will be made clear from the context).

We denote by $C_c^{\infty,\infty}(E)$ the set of $\mathbb{R}$-valued functions of E that are $C^{\infty,\infty}(E)$, and of compact support with respect to the space variable (i.e. for $f\in C_c^{\infty,\infty}(E)$, for any $t\in[0,T]$, the function $f(t,\cdot)$ is of compact support).

We denote by $C_{c,c}^{\infty,\infty}(E)$ the set of $\mathbb{R}$-valued functions of E that are in $C^{\infty,\infty}(E)$, and of compact support $K \subset (0,T)\times\mathbb{R}$.

For a function in $L^2(\mathbb{R})$ we denote by $\frac{\mathrm{d}f}{\mathrm{dx}}$ its first derivative in the distribution sense. We denote by $H^1(\mathbb{R})$ the usual Sobolev space of those functions f in $L^2(\mathbb{R})$ such that $\frac{\mathrm{d}f}{\mathrm{dx}}$ belongs to $L^2(\mathbb{R})$. We denote by $H^{-1}(\mathbb{R})$ the usual dual space of $H^1(\mathbb{R})$.

We denote $L^2(0,T;L^2(\mathbb{R}))$ the set of measurable functions $f(t,x)$ s.t.

$$\int_0^T \int_{\mathbb{R}} |f(t,x)|^2 dxdt < \infty.$$

For $f \in L^2(0,T;L^2(\mathbb{R}))$ we denote by $||f||^2$ the above quantity.

We denote by $L^2(0,T;H^1(\mathbb{R}))$ the set of mesurable functions $f(t,x)$ such that for any $t \in [0,T]$ the function $f(t,\cdot)$ is in $H^1(\mathbb{R})$ and

$$\int_0^T \int_{\mathbb{R}} |f(t,x)|^2 dxdt + \int_0^T \int_{\mathbb{R}} \big|\frac{\mathrm{d}f}{\mathrm{dx}}(t,x)\big|^2 dxdt < \infty.$$

For a function $f \in L^2(0,T;L^2(\mathbb{R}))$ we denote by $\frac{\mathrm{d}f}{\mathrm{dt}}$ its first derivative with respect to time in the distribution sense (see Remark 1.6.7 for some details).

We will denote by $H^{1,1}(E)$ the set of functions in $L^2(0,T;H^1(\mathbb{R}))$ such that $\frac{\mathrm{d}f}{\mathrm{dt}}$ belongs to $L^2(0,T;L^2(\mathbb{R}))$. It is equipped with the norm $f \mapsto \left(||f||^2 + \big|\big|\frac{\mathrm{d}f}{\mathrm{dx}}\big|\big|^2 + \big|\big|\frac{\mathrm{d}f}{\mathrm{dt}}\big|\big|^2\right)^{1/2}$.

Finally we will denote by $H_0^{1,1}(E)$ the closure in $H^{1,1}(E)$ of $C_{c,c}^{\infty,\infty}(E)$ with respect to the just above defined norm. Note that for $\varphi \in H_0^{1,1}(E)$ we have $\varphi(0,\cdot) = \varphi(T,\cdot) = 0$, and $\lim_{|x|\to\infty} \varphi(t,x) = 0$, $t \in [0,T]$.

For $0 < m < M < \infty$ we denote by $\Theta(m,M)$ the set of functions $\sigma : [0,T]\times\mathbb{R} \to [m,M]$ that are measurable. We denote by $\Xi(M)$ the set of functions $b : [0,T]\times\mathbb{R} \to [-M,M]$ that are measurable.

For any function $f : \mathbb{R} \to \mathbb{R}$ and any $x \in \mathbb{R}$ such that $f(x+) = \lim_{y\downarrow x} f(y)$ and $f(x-) = \lim_{y\uparrow x} f(y)$ both exist, we will sometimes use the following notations :

$$f_\pm(x) := \frac{f(x+)+f(x-)}{2} \quad \text{and} \quad \triangle f(x) = \frac{f(x+)-f(x-)}{2}.$$

In particular if $f : \mathbb{R} \to \mathbb{R}$ is differentiable, except on a finite number of points $x_1 < \ldots < x_I$, where $f'(x_i\pm)$, $1 \le i \le I$ exist, note that the function $f'_\pm$ is defined on the whole real line and represents the absolute part of $f'(dx)$, the derivative of f in the generalized sense; in other words,

$$f'(dx) = f'_\pm(x)dx + \sum_{i=1}^{I} 2 \triangle f(x_i)\delta_{x_i}(dx).$$

Chapter 1

Time inhomogeneous Stochastic Differential Equations involving the local time of the unknown process, and associated parabolic operators

This Chapter is based on the paper *Time inhomogeneous Stochastic Differential Equations involving the local time of the unknown process, and associated parabolic operators* ([30]), written with Miguel Martinez.

1.1 Introduction

In this chapter we investigate time-inhomogeneous versions of SDEs with local time, namely equations of the form

$$dX_t = \sigma(t, X_t)dW_t + b(t, X_t)dt + \sum_{i=1}^{I} \beta_i(t) dL_t^{x_i}(X), \quad t \in [0,T], \quad X_0 = x_0. \tag{1.1.1}$$

Here the generalization is three fold : first the coefficients σ and b are now allowed to depend on time, second the coefficients β_i are no longer constant but are also allowed to depend on time, and third the functions $x_i \ : \ t \mapsto x_i(t)$ are now time-curves, so that $(L_t^{x_i}(X))_{t\in[0,T]}$ stands for the (symmetric) local time of the unknown process $(X_t)_{t\in[0,T]}$ along the time-curve x_i (we recall that for any continuous function of bounded variation $\gamma : [0,T] \to \mathbb{R}$ we denote by $L_\cdot^\gamma(X) = L_\cdot^0(X-\gamma)$).

Then we establish the link between X solution of (1.1.1) and the solution of some parabolic PDE with (time-inhomogeneous) transmission conditions, for which we give our proper treatment (but relying deeply on [40] for the study of classical solutions). This PDE will imply operators of the form

$$\frac{\rho}{2}\nabla\big(a\nabla\cdot\big) + \big(b - \frac{\rho a'_{x,\pm}}{2}\big)\nabla \tag{1.1.2}$$

with in particular $\rho a = \sigma^2$ and $a(t, x_i(t)+) - a(t, x_i(t)-)$ proportional to $\beta_i(t)$ for any $1 \le i \le I$ and any $t \in [0,T]$. Then we exploit these results in order to characterize X as a (time-inhomogeneous) Markov process.

Organization of the chapter. In Section 1.3, we give preliminary material for the study of equation (1.1.1). This includes, results on the related martingale problem, recalls on some pathwise uniqueness results to be found in [43] (available in a time-inhomogeneous context), and a slight adaptation of the Itô-Peskir formula (in the case where the curves are C^1 functions). Since we aim at studying the generator of the solutions of equation (1.1.1), we also give introductory material to the semigroups associated to time inhomogeneous Markov processes and Feller evolution systems.

In Section 1.4 we use the result of Peskir ([61]) to prove a change of variable formula, that will be of crucial use in the rest of the chapter. Then we give conditions for the equation (1.1.1) to admit a weak or strong solution, to enjoy pathwise uniqueness. The method follows closely Le Gall [43] by means of a space transform.

Section 1.5 is devoted to the proof of the Feynman-Kac representation linking the solution of (1.1.1) and the solution of a parabolic partial differential equation with transmission conditions along the curves x_i's. It is assumed that the solution of the parabolic PDE with transmission conditions is smooth enough in order to apply the change of variable formula of Section 1.4.

Section 1.6 is naturally devoted to the study of the parabolic PDE with transmission conditions appearing in the previous section. This PDE involves (1.1.2), once it has been rewritten in a divergence-form fashion. We first study the weak interpretation of the PDE and manage to show, by adapting the arguments in [48], that a weak solution exists. As regarding classical solutions, we rely on the main result of the reference article [40], where $\rho \equiv 1$ and the sub-domains are cylindrical. Again, using the one-dimensional context of the equation, it is possible to generalize the result to the solution of the parabolic PDE with transmissions conditions, with $\rho \neq 1$ and several moving interfaces. Thus, the solution of the parabolic PDE with transmission conditions is smooth enough to assert the validity of the Feynman-Kac representation given in the previous section (see the conclusion at the end of Section 1.6).

Section 1.7 is an attempt to characterize the Markov generator of $(X_t)_{t\in[0,T]}$ solution of (1.1.1). We manage to do it fully in the case of non-moving interfaces. The case of moving interfaces seems more difficult to handle since we do no longer have the continuity of the time derivative of the associated parabolic transmission problem.

Some notations specific to the chapter are introduced in the next section.

1.2 Notations

Let $I \in \mathbb{N}^*$. For each $1 \leq i \leq I$, let $x_i : [0,T] \to \mathbb{R}$ be a continuous function of bounded variation, and assume that $x_i(t) < x_j(t)$ for all $t \in [0,T]$ and all $1 \leq i < j \leq I$.

Given such a family $(x_i)_{i=1}^I$ we will denote $D_0^x = \{(t,z) \in [0,T] \times \mathbb{R} : \ z < x_1(t)\}$, $D_I^x = \{(t,z) \in [0,T] \times \mathbb{R} : \ z > x_I(t)\}$ and, for any $1 \leq i \leq I-1$, $D_i^x = \{(t,z) \in [0,T] \times \mathbb{R} : \ x_i(t) < z < x_{i+1}(t)\}$.

We will denote

$$\Delta_{\mathbf{x}} = \{(t, x_i(t)) : 0 \leq t \leq T\}_{i=1}^I \subset E \tag{1.2.1}$$

(this will be clear from the context which family $(x_i)_{i=1}^I$ is dealt with).

We define now the $\mathbf{H}^{(x_i)}$-hypothesis for functions in $\Theta(m,M)$ in the following way:

$$(\mathbf{H}^{(x_i)}): \quad g \in \Theta(m,M) \cap C^{0,1}(E \setminus \Delta_{\mathbf{x}}), \quad \max_{1\leq i\leq I} \sup_{t\in[0,T]} \sup_{x_i(t)<x<x_{i+1}(t)} |g'_x(t,x)| < \infty$$

$$\text{and} \quad \sup_{t\in[0,T]} \sup_{x<x_1(t)} |g'_x(t,x)| < \infty, \quad \sup_{t\in[0,T]} \sup_{x>x_I(t)} |g'_x(t,x)| < \infty.$$

We define the $\mathbf{AJ}^{(x_i)}$-hypothesis (AJ for Average Jumps) in the following way

$$(\mathbf{AJ}^{(x_i)}): \ \exists 0 < C < \infty, \ \infty > C \int_0^T \sum_{x\leq z\leq y} |g^2(s,z+) - g^2(s,z-)| ds \geq \sum_{x\leq z\leq y} |g^2(t,z+) - g^2(t,z-)|,$$

for all $x, y \in \mathbb{R}$, $t \in [0,T]$.

Remark 1.2.1. *This roughly speaking, means that the size of the jumps of g are not allowed to go too far from a kind of time-averaged size jump. See Remark 1.3.5 below for a comment on why this technical hypothesis is needed.*

For the study of the PDE aspects we define the $\mathbf{H}^{(t)}$-hypothesis for functions in $\Theta(m,M)$, in $\Xi(M)$ or in $C_c(E)$, by

$$(\mathbf{H}^{(t)}): \quad g \in C^{1,0}(E \setminus \Delta_{\mathbf{x}}), \quad \max_{1\le i\le I} \sup_{t\in[0,T]} \sup_{x_i(t)<x<x_{i+1}(t)} |g'_t(t,x)| < \infty$$

$$\text{and} \quad \sup_{t\in[0,T]} \sup_{x<x_1(t)} |g'_t(t,x)| < \infty, \quad \sup_{t\in[0,T]} \sup_{x>x_I(t)} |g'_t(t,x)| < \infty.$$

Note that the same kind of notations will be used for a family $y_i : [0,T] \to \mathbb{R}$, $1 \le i \le I$, satisfying the same assumptions (for example in Corollary 1.3.8 below).

Finally, we fix notations for two sets of type $\Delta_{\mathbf{x}}$ that play a special role in the sequel. Those are

$$\Delta = \{(t,i) : 0 \le t \le T\}_{i=1}^{I} \subset E \quad \text{and} \quad \Delta_0 = \{(t,0) : 0 \le t \le T\}. \tag{1.2.2}$$

1.3 Preliminaries and known results concerning the stochastic aspects of the problem

1.3.1 Well-posedness of the martingale problem associated to discontinuous coefficients

Of crucial importance is the following result, to be found in [70].

Theorem 1.3.1 ([70], Exercise 7.3.3)**.** *Let $\bar\sigma \in \Theta(\bar m, \bar M)$ and $\bar b \in \Xi(\bar M)$ (for some $0 < \bar m < \bar M < \infty$). Then the martingale problem associated to $\bar\sigma^2$ and $\bar b$ is well-posed.*

The first important consequence of this result is that the for any $(s,y) \in E$ the SDE

$$dY_t = \bar\sigma(t,Y_t)dW_t + \bar b(t,Y_t)dt, \quad t \in [s,T], \quad Y_s = y$$

has a weak solution ([70], Theorem 4.5.1), unique in law ([70], Theorem 5.3.2). The second one is that this weak solution is (time-inhomogeneous) Markov ([70], Theorem 6.2.2; see also the forthcoming Subsection 1.3.4 for comments on time-inhomogeneous Markov processes).

Remark 1.3.2. *Note that the result of Theorem 1.3.1 is available for time-dependent coefficients, only because the dimension of the space variable is $d = 1$. For $d = 2$, up to our knowledge, such results exist but with a time-homogeneous diffusion matrix ([70], Exercise 7.3.4).*

1.3.2 Pathwise uniqueness results and strong solutions of time-inhomogeneous SDEs with discontinuous coefficients

We have the following results.

Theorem 1.3.3 (J.-F. Le Gall, [43])**.** *Let $\bar\sigma \in \Theta(\bar m, \bar M)$ and $\bar b \in \Xi(\bar M)$ for some $0 < \bar m < \bar M < \infty$. Assume further that there exists a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that*

$$|\bar\sigma(t,x) - \bar\sigma(t,y)|^2 \le |f(x) - f(y)|, \quad \forall (t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}. \tag{1.3.1}$$

Then the SDE

$$dY_t = \bar\sigma(t,Y_t)dW_t + \bar b(t,Y_t)dt, \quad t \in [0,T], \quad Y_0 = y_0 \tag{1.3.2}$$

enjoys pathwise uniqueness.

As an immediate consequence we get the following corollary.

Corollary 1.3.4. *Let $I \in \mathbb{N}^*$. For each $1 \le i \le I$, let $y_i : [0,T] \to \mathbb{R}$ be a continuous function of bounded variation, and assume that $y_i(t) < y_j(t)$ for all $t \in [0,T]$ and all $1 \le i < j \le I$.*

Let $\bar\sigma \in \Theta(\bar m, \bar M)$ and $\bar b \in \Xi(\bar M)$ for some $0 < \bar m < \bar M < \infty$.

The SDE (1.3.2) *has a weak solution.*

Assume further that $\bar\sigma$ satisfies the $\mathbf{H}^{(y_i)}$ and $\mathbf{AJ}^{(y_i)}$-hypotheses.

Then the SDE (1.3.2) *enjoys pathwise uniqueness and has in fact a unique strong solution.*

Proof. As already pointed in Subsection 1.3.1 Equation (1.3.2) has weak solutions. We aim now at using Theorem 1.3.3. Then the well known results of Yamada and Watanabe ([80]) will provide the desired conclusion.

First we notice that for all $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}$,

$$|\bar{\sigma}(t,x) - \bar{\sigma}(t,y)|^2 \leq \bar{\sigma}^2(t,x) + \bar{\sigma}^2(t,y) - 2(\bar{\sigma}^2(t,x) \wedge \bar{\sigma}^2(t,y)) = |\bar{\sigma}^2(t,y) - \bar{\sigma}^2(t,x)|.$$

Thus, to get the result by Theorem 1.3.3 it suffices to find a stricly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that

$$|\bar{\sigma}^2(t,x) - \bar{\sigma}^2(t,y)| \leq |f(x) - f(y)|, \quad \forall (t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}. \tag{1.3.3}$$

Using the $\mathbf{H}^{(y_i)}$-hypothesis, we set

$$K = \max\Big\{ \sup_{t\in[0,T]} \sup_{x<y_1(t)} |(\bar{\sigma}^2)'_x(t,x)|, \max_{1\leq i\leq I} \sup_{t\in[0,T]} \sup_{y_i(t)\leq x<y_{i+1}(t)} |(\bar{\sigma}^2)'_x(t,x)|, \sup_{t\in[0,T]} \sup_{x\geq y_I(t)} |(\bar{\sigma}^2)'_x(t,x)| \Big\} < \infty.$$

One can define a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = Kx + C \sum_{z\leq x} \int_0^T |\bar{\sigma}^2(s,z+) - \bar{\sigma}^2(s,z-)| ds,$$

where C is the constant involved in the $\mathbf{AJ}^{(y_i)}$-hypothesis (note that as $\sum_{z\leq x} |\bar{\sigma}^2(s,z+) - \bar{\sigma}^2(s,z-)|$ is finite and bounded -for any s-, Fubuni's Theorem ensures that f take finite values). Then one can use the $\mathbf{H}^{(y_i)}$ and $\mathbf{AJ}^{(y_i)}$-hypotheses to check that for $x < y$,

$$\begin{aligned} |\bar{\sigma}^2(t,x) - \bar{\sigma}^2(t,y)| &\leq K(y-x) + \textstyle\sum_{x\leq z\leq y} |\bar{\sigma}^2(t,z+) - \bar{\sigma}^2(t,z-)| \\ &\leq K(y-x) + C \textstyle\int_0^T \sum_{x\leq z\leq y} |\bar{\sigma}^2(s,z+) - \bar{\sigma}^2(s,z-)| ds \\ &= f(y) - f(x) = |f(y) - f(x)|. \end{aligned}$$

Thus f satisfies (1.3.3). □

Remark 1.3.5. *It would be tempting to set* $f(x) = Kx + \sum_{z\leq x} \sup_{s\in[0,T]} |\bar{\sigma}^2(s,z+) - \bar{\sigma}^2(s,z-)|$ *in order to try to check* (1.3.1). *But as* $\sup_{s\in[0,T]} |\bar{\sigma}^2(s,z+) - \bar{\sigma}^2(s,z-)|$ *could be non zero for non countable values of* z *the function* f *could be not well defined as a function from* $\mathbb{R}$ *to* $\mathbb{R}$. *This justifies our assumption* $\mathbf{AJ}^{(y_i)}$.

1.3.3 The Itô-Peskir formula

Our fundamental tool is the following result due to G. Peskir (see [61]).

Theorem 1.3.6 (Time inhomogeneous symmetric Itô-Tanaka formula ([61]))**.** *Let* Y *a continuous* $\mathbb{R}$*-valued semimartingale. Let* $\gamma : [0,T] \to \mathbb{R}$ *be a continuous function of bounded variation.*

Denote $C = \{(t,x) \in [0,T] \times \mathbb{R} \ : \ x < \gamma(t)\}$ *and* $D = \{(t,x) \in [0,T] \times \mathbb{R} \ : \ x > \gamma(t)\}$.

Let $r \in C(E) \cap C^{1,2}(\overline{C}) \cap C^{1,2}(\overline{D})$. *Then, for any* $0 \leq t < T$,

$$\begin{aligned} r(t,Y_t) &= r(0,Y_0) + \int_0^t \frac{1}{2}(r'_t(s,Y_s+) + r'_t(s,Y_s-))ds + \int_0^t \frac{1}{2}(r'_y(s,Y_s+) + r'_y(s,Y_s-))dY_s \\ &\quad + \frac{1}{2}\int_0^t r''_{yy}(s,Y_s)\mathbf{1}_{Y_s\neq\gamma(s)} d\langle Y\rangle_s + \frac{1}{2}\int_0^t (r'_y(s,Y_s+) - r'_y(s,Y_s-)) dL^\gamma_s(Y). \end{aligned} \tag{1.3.4}$$

Note that in the above Theorem, the assumption $r \in C^{1,2}(\overline{C}) \cap C^{1,2}(\overline{D})$ means that r restricted to C coincides with a function r_0 laying in the whole space $C^{1,2}(E)$, and r restricted to D coincides with a function r_1 laying in the whole space $C^{1,2}(E)$.

However, when dealing with PDE aspects (Sections 1.5, 1.6 and 1.7), we will need to apply the Itô-Peskir formula to functions that have less smoothness: these functions will only possess continuous partial derivatives (of order one in time and at least two in the space variable) with limits all the way up

to the boundary $\Delta_\gamma = \{(t,x) \in [0,T] \times \mathbb{R} : x = \gamma(t)\}$. The price to pay, in order to get the same formula (1.3.4), is then to require additional smoothness of the curve $\gamma(t)$: we require it to be of class C^1.

In Theorem 1.3.7 below, we give the adaptation of the Itô-Peskir formula that will be used in Sections 1.5, 1.6 and 1.7 (in fact the formula is the key the forthcoming Proposition 1.4.1, that will be used repeatedly in the sequel). Note that the assumptions on the function r in Theorem 1.3.6 imply the ones in Theorem 1.3.7. But of course, on the opposite, the fact that γ is C^1 implies the fact that it is continuous of bounded variation.

Theorem 1.3.7. *Let Y a continuous $\mathbb{R}$-valued semimartingale. Let $\gamma : [0,T] \to \mathbb{R}$ be a function of class C^1, and consider $\Delta_\gamma = \{(t,x) \in [0,T] \times \mathbb{R} : x = \gamma(t)\}$. Let $r \in C(E) \cap C^{1,2}(E^\circ \setminus \Delta_\gamma)$ such that the limits $r'_t(t,\gamma(t)\pm)$, $r'_y(t,\gamma(t)\pm)$, and $r''_{yy}(t,\gamma(t)\pm)$ exist and are continuous as functions of $t \in [0,T)$. Then, for any $0 \le t < T$, we have* (1.3.4).

Proof. See the Appendix of [30]. □

For our purpose we need a more general formula, valid for multiple curves and local times. Such an extension of the result of Theorem 1.3.7 was announced in [61] (see the Remark 2.3 therein) without proof.

Corollary 1.3.8. *Let Y a continuous $\mathbb{R}$-valued semimartingale.*

Let $I \in \mathbb{N}^*$. For each $1 \le i \le I$, let $y_i : [0,T] \to \mathbb{R}$ be a continuous function of bounded variation, and assume that $y_i(t) < y_j(t)$ for all $t \in [0,T]$ and all $1 \le i < j \le I$.

Let $r \in C(E) \cap \Big(\cap_{i=0}^I C^{1,2}(\overline{D_i^y}) \Big)$. Then, for any $0 \le t < T$,

$$\begin{aligned} r(t,Y_t) &= r(0,Y_0) + \int_0^t \frac{1}{2}(r'_t(s,Y_s+) + r'_t(s,Y_s-))ds + \int_0^t \frac{1}{2}(r'_y(s,Y_s+) + r'_y(s,Y_s-))dY_s \\ &\quad + \frac{1}{2}\int_0^t r''_{yy}(s,Y_s)\mathbf{1}_{\{Y_s \neq y_i(s),\, \forall 1 \le i \le I\}} d\langle Y \rangle_s + \frac{1}{2}\sum_{i=1}^I \int_0^t (r'_y(s,Y_s+) - r'_y(s,Y_s-))dL_s^{y_i}(Y). \end{aligned}$$

The result remains valid if the curves y_i's are of class C^1 and if $r \in C(E) \cap C^{1,2}(E^\circ \setminus \Delta_{\mathbf{y}})$ is such that for all $1 \le i \le I$, the limits $r'_t(t,y_i(t)\pm)$, $r'_y(t,y_i(t)\pm)$, and $r''_{yy}(t,y_i(t)\pm)$ exist and are continuous as functions of $t \in [0,T)$.

Proof. See the Appendix of [30]. □

1.3.4 Time-inhomogeneous Markov processes, infinitesimal generator of the associated space-time process

The presentation of Markov processes, especially when coming to the time-inhomogeneous case, varies slightly from one book to the other. Here we precise some definitions and concepts. We follow mainly [64] but we are also inspired by other references ([39], [79]; see also [13]).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, $(\mathcal{F}_t)_{t\in[0,T]}$ a filtration ($\mathcal{F}_t \subset \mathcal{F}$ for any $t \in [0,T]$) and consider $Z = (Z_t)_{t\in[0,T]}$ an adapted process defined on this probability space, taking values in a measurable space $(U,\mathcal{U})$.

We will say that Z is a $(\mathcal{F}_t)$-*Markov process* if for any $0 \le s \le t \le T$, and any $f \in C_b(U)$ we have

$$\mathbb{E}[f(Z_t) \,|\, \mathcal{F}_s] = \mathbb{E}[f(Z_t) \,|\, Z_s].$$

Denoting $\mathbb{E}^{s,x}(\cdot) = \mathbb{E}(\cdot \,|Z_s = x)$ and defining the operator $P_{s,t}$ by $P_{s,t}f(x) = \mathbb{E}^{s,x}[f(Z_t)]$, for any $f \in C_b(U)$, any $x \in U$, we clearly have $\mathbb{E}[f(Z_t) \,|\, \mathcal{F}_s] = P_{s,t}f(Z_s)$. The family $(P_{s,t})_{0\le s\le t\le T}$ is called the transition function of Z. We will say that Z is a *time-homogeneous Markov process* if $P_{s,t} = P_{0,t-s}$. In the opposite case it is called *time-inhomogeneous*.

Now to fix ideas suppose the Markov process Z is $\mathbb{R}$-valued, and denote $(P_{s,t})$ its transition function. Consider the associated E-valued *space-time process* $\tilde{Z} = ((t, Z_t))_{t\in[0,T]}$. It is an exercise ([64], Exercise III.1.10) to check that for any $\varphi \in \mathcal{C}_b(E)$ and any $0 \le s \le t \le T$,

$$\mathbb{E}[\varphi(\tilde{Z}_t) \,|\, \mathcal{F}_s] = P_{t-s}\varphi(\tilde{Z}_s)$$

with

$$\forall (s,x) \in E, \ \ \forall \varphi \in C_b(E), \ \ \forall\, 0 \leq t \leq T-s, \quad P_t\varphi(s,x) = P_{s,t+s}\varphi(t+s,x) = \mathbb{E}^{s,x}[\varphi(s+t, Z_{s+t})] \quad (1.3.5)$$

(the value of $P_t\varphi(s,x)$ for $t+s>T$ is arbitrarily set to zero; see the forthcoming Remark 1.7.3). Thus the space-time process $\tilde{Z}$ is always a time-homogeneous Markov process (Z being time-homogeneous or not), with transition function given by (1.3.5).

Note that the family $(P_{s,t})$ satisfies $P_{t,t} = \mathrm{Id}$ and, thanks to the Markov property on Z, the evolution property

$$P_{s,u} \circ P_{u,t} = P_{s,t}, \quad \forall 0 \leq s \leq u \leq t \leq T. \quad (1.3.6)$$

The family (P_t) satisfies $P_0 = \mathrm{Id}$, and thanks to the time-homogeneous Markov property on $\tilde{Z}$, the semigroup property

$$P_s \circ P_t = P_{t+s}, \quad \forall 0 \leq s \leq T, \ \ \forall\, 0 \leq t \leq T-s. \quad (1.3.7)$$

If the family $(P_{s,t})$ satisfies, in addition to (1.3.6), that for any $f \in C_0(\mathbb{R})$ we have $P_{s,t}f \in C_0(\mathbb{R})$, $||P_{s,t}f||_\infty \leq ||f||_\infty$, $P_{s,t}f \geq 0$ if $f \geq 0$, and

$$\lim_{\substack{(s,t)\to(v,w)\\ s\leq t}} ||P_{s,t}f - P_{v,w}f||_\infty = 0 \quad (1.3.8)$$

it is called a *Feller evolution system.*

If the family (P_t) satisfies, in addition to (1.3.7), that for any $\varphi \in C_0(E)$, we have $P_t\varphi \in C_0(E)$, $||P_t\varphi||_\infty \leq ||\varphi||_\infty$, $P_t\varphi \geq 0$ if $\varphi \geq 0$, and $\lim_{t\downarrow 0} ||P_t\varphi - \varphi||_\infty = 0$, then it is called a *Feller semigroup.*

We have the following result.

Theorem 1.3.9 ([13])**.** *Let Z be a Markov process with corresponding transitions $(P_{s,t})$. Let (P_t) the semigroup associated to the corresponding space-time process $\tilde{Z}$. Then the following statements are equivalent:*

i) $(P_{s,t})$ is a Feller evolution system.

ii) (P_t) is a Feller semigroup.

Proof. Note that our definition of the space-time process, which follows [64], is a bit different from the one in [13],[79], which is more canonical. But in fact, the families of operators $(P_{s,t})$ and (P_t) that we have defined above, are exactly the same than the ones in [13],[79]. Therefore is suffices to adapt the proof if [13], which is written on a infinite time interval, to the finite time interval case. □

We will say that Z is a *Feller time-inhomogeneous Markov process* if its corresponding evolution system $(P_{s,t})$ is Feller, or equivalently if the semigroup (P_t) of the corresponding space-time process $\tilde{Z}$ is Feller (note that $\tilde{Z}$ is therefore a Feller process in the sense of [64]). We will focus on this latter point of view, because we believe it provides a more synthetic setting in order to describe the operators associated to a Feller time-inhomogeneous Markov process Z. More precisely we will work at identifying the parabolic operator that is the infinitesimal generator of the space-time process $\tilde{Z}$.

At this point we recall the following definition.

Definition 1.3.10. *Let $\tilde{Z}$ a E-valued Feller process, with associated Feller semigroup (P_t). A function φ in $C_0 = C_0(E)$ is said the belong to the domain $D(\mathcal{L})$ of the infinitesimal generator of $\tilde{Z}$ if the limit*

$$\mathcal{L}\varphi = \lim_{t\downarrow 0} \frac{1}{t}(P_t\varphi - \varphi) \quad (1.3.9)$$

exists in C_0. The operator $\mathcal{L} : D(\mathcal{L}) \to C_0$ thus defined is called the infinitesimal generator of the process $\tilde{Z}$ or of the semigroup (P_t).

In order to identify such infinitesimal generators we will use the following proposition.

Proposition 1.3.11. *Let $Z = (Z_t)_{t\in[0,T]}$ a $\mathbb{R}$-valued Feller time-inhomogeneous $(\mathcal{F}_t)$-Markov process and let $\tilde{Z} = ((t, Z_t))_{t\in[0,T]}$ the E-valued corresponding space-time process. Assume $\tilde{Z}$ has generator $(\mathcal{L}, D(\mathcal{L}))$.*

If $\varphi \in C_0$, and if there exists a function $g \in C_0$ such that $M^{\varphi,g} = (M_t^{\varphi,g})_{t\in[s,T]}$ defined by

$$\forall t \in [s,T], \quad M_t^{\varphi,g} = \varphi(\tilde{Z}_t) - \varphi(\tilde{Z}_s) - \int_s^t g(\tilde{Z}_u)du$$

is a $(\mathcal{F}_t)$-martingale under $\mathbb{P}^{s,x}$ (for any $(s,x) \in E$), then $\varphi \in D(\mathcal{L})$ and $\mathcal{L}\varphi = g$.

Proof. See the proof of Proposition 2.11 in [30], where we adapt the proof of Proposition VII.1.7 in [64] to the inhomogeneous case. □

Remark 1.3.12. *In the sequel, for any $\mathbb{R}$-valued Markov process Z the family (P_t) will denote the semigroup associated with the space-time process $\tilde{Z}$. This will be clear from the context, and there will be no risk to take this semigroup for the one associated to Z, should this process be time-homogeneous Markov (as P_t will act on functions from E to $\mathbb{R}$).*

Remark 1.3.13. *For a time-inhomogeneous diffusion we can expect that $\mathcal{L}\varphi(t,\cdot) = (\partial_t + L_t)\varphi(t,\cdot)$, with L_t a second order elliptic operator in the space variable. But in our case, with discontinuous coefficients and singular terms, $D(\mathcal{L})$ will not contain $C^{1,2}(E)$ functions (cf Section 1.7).*

1.4 Getting solutions by the mean of a space transform

1.4.1 Main results

A probability measure $\mathbb{P}$ is given on $(C,\mathcal{C})$, together with a $(\mathcal{C}_t)$-brownian motion $W = (W_t)_{t\in[0,T]}$ defined on $(C,\mathcal{C},\mathbb{P})$.

We will have the following main results: the first one (Proposition 1.4.1) is a change of variable formula for time-inhomogeneous SDEs with local time (it is thus more general than the formula stated in Theorem 3.1 of [61], but our assumptions are more restrictive). Assuming a solution Y exists to the time-inhomogeneous SDE with local time (1.4.1) Proposition 1.4.1 gives the form of some transformed process $\phi(t,Y_t)$. This formula will be used extensively in the sequel. To start with, it allows to prove Theorem 1.4.5, that gives existence and uniqueness results for the solution $X = (X_t)_{t\in[0,T]}$ to Equation (1.1.1), under some conditions on the coefficients $\sigma(t,x)$, $b(t,x)$, $\beta_i(t)$, $1 \le i \le I$, and the curves $x_i(t)$. But Proposition 1.4.1 will be again used in Sections 1.6 and 1.7.

Proposition 1.4.1. *Let $I \in \mathbb{N}^*$. For each $1 \le i \le I$, let $y_i : [0,T] \to \mathbb{R}$ a function of class C^1, and assume that $y_i(t) < y_j(t)$ for all $t \in [0,T]$ and all $1 \le i < j \le I$.*

Let $Y = (Y_t)_{0\le t\le T}$ a continuous $\mathbb{R}$-valued semimartingale satisfying

$$dY_t = \bar{\sigma}(t,Y_t)dW_t + \bar{b}(t,Y_t)dt + \sum_{i=1}^{I} \bar{\beta}_i(t)dL_t^{y_i}(Y) \tag{1.4.1}$$

where $\bar{\sigma}, \bar{b} : [0,T]\times\mathbb{R} \to \mathbb{R}$ are some bounded functions, and the functions $\bar{\beta}_i : [0,T] \to (-1,1)$, $1 \le i \le I$, are of class C^1.

Let $\phi \in C(E) \cap C^{1,2}(E^\circ \setminus \Delta_{\mathbf{y}})$ such that for all $1 \le i \le I$, the limits $\phi'_t(t,y_i(t)\pm)$, $\phi'_y(t,y_i(t)\pm)$, and $\phi''_{yy}(t,y_i(t)\pm)$ exist and are continuous as functions of $t \in [0,T)$.

Set $X_t = \phi(t,Y_t)$ for any $t \in [0,T]$. Then

$$\begin{aligned} dX_t &= (\bar{\sigma}\phi'_{y,\pm})(t,Y_t)dW_t + [\phi'_{t,\pm} + \bar{b}\phi'_{y,\pm}](t,Y_t)dt + \tfrac{1}{2}(\bar{\sigma}^2\phi''_{yy})(t,Y_t)\mathbf{1}_{\{Y_t \ne y_i(t),\, \forall 1\le i\le I\}}dt \\ &\quad + \textstyle\sum_{i=1}^{I}[\triangle\, \phi'_y(t,y_i(t)) + \bar{\beta}_i(t)\phi'_{y,\pm}(t,y_i(t))]\, dL_t^{y_i}(Y). \end{aligned} \tag{1.4.2}$$

Assume further that $\phi \in C(E) \cap \left(\cap_{i=0}^{I} C^{1,2}(\overline{D_i^y}) \right)$ and

$$\phi'_y(t,y) > 0 \quad \forall (t,y) \in E \setminus \Delta_{\mathbf{y}} \tag{1.4.3}$$

and denote, for any $t \in [0,T]$, $\Phi(t,\cdot) = [\phi(t,\cdot)]^{-1}$ and

$$x_i(t) = \phi(t,y_i(t)) \tag{1.4.4}$$

for all $1 \leq i \leq I$. *Then*

$$dX_t = \sigma(t, X_t)dW_t + b(t, X_t)dt + \sum_{i=1}^{I} \beta_i(t) dL_t^{x_i}(X) \tag{1.4.5}$$

with

$$\begin{aligned} \sigma(t,x) &= (\bar{\sigma}\phi'_{y,\pm})(t, \Phi(t,x)) \\ b(t,x) &= [\phi'_{t,\pm} + \bar{b}\phi'_{y,\pm}](t,\Phi(t,x)) + \tfrac{1}{2}(\bar{\sigma}^2\phi''_{yy})(t,\Phi(t,x))\mathbf{1}_{\{x \neq x_i(t),\, \forall 1 \leq i \leq I\}} \end{aligned}$$

and

$$\beta_i(t) = \frac{\triangle\, \phi'_y(t, y_i(t)) + \bar{\beta}_i(t)\phi'_{y,\pm}(t, y_i(t))}{\phi'_{y,\pm}(t, y_i(t)) + \bar{\beta}_i(t) \triangle\, \phi'_y(t, y_i(t))} \tag{1.4.6}$$

for all $t \in [0,T]$ *and all* $x \in \mathbb{R}$.

Remark 1.4.2. *Note that the curves* x_i*'s defined by* (1.4.4) *are themselves of class* C^1*, so that the local times terms in* (1.4.5) *are still well defined.*

In order to see that, let us focus on $x_1(t) = \phi(t, y_1(t))$. *As* ϕ *is in* $C(E) \cap \left(\cap_{i=0}^{I} C^{1,2}(\overline{D_i^y}) \right)$ *one has that* ϕ *restricted to* D_0^y *coincides with a function* $\phi_0 \in C^{1,2}(E)$, *and that* ϕ *restricted to* D_1^y *coincides with a function* $\phi_1 \in C^{1,2}(E)$. *Thus, as* ϕ *is continuous, one has*

$$\phi_0(t, y_1(t)) = \phi(t, y_1(t)) = \phi_1(t, y_1(t)), \quad \forall t \in [0,T].$$

Thus in particular $x_1(t) = \phi_0(t, y_1(t))$, *and one sees by composition that* x_1 *is of class* C^1.

Remark 1.4.3. *Note that*

$$\triangle\, \phi'_y(t, y_i(t)) + \bar{\beta}_i(t)\phi'_{y,\pm}(t, y_i(t)) = \phi'_y(t, y_i(t)+)(1 + \bar{\beta}_i(t)) - \phi'_y(t, y_i(t)-)(1 - \bar{\beta}_i(t)) \tag{1.4.7}$$

and that $\phi'_{y,\pm}(t, y_i(t)) + \bar{\beta}_i(t) \triangle\, \phi'_y(t, y_i(t)) = \phi'_y(t, y_i(t)+)(1 + \bar{\beta}_i(t)) + \phi'_y(t, y_i(t)-)(1 - \bar{\beta}_i(t))$, *so that the new coefficients* $\beta_i(t)$ *in Proposition 1.4.1 may be rewritten*

$$\beta_i(t) = \frac{\phi'_y(t, y_i(t)+)(1 + \bar{\beta}_i(t)) - \phi'_y(t, y_i(t)-)(1 - \bar{\beta}_i(t))}{\phi'_y(t, y_i(t)+)(1 + \bar{\beta}_i(t)) + \phi'_y(t, y_i(t)-)(1 - \bar{\beta}_i(t))}. \tag{1.4.8}$$

Remark 1.4.4. *Note that the result of Proposition 1.4.1 is a time-inhomogeneous version of Proposition 3.1 in [20] (or equivalently Proposition 2.2.1 in [19]).*

Theorem 1.4.5. *Let* $I \in \mathbb{N}^*$. *For each* $1 \leq i \leq I$, *let* $x_i : [0,T] \to \mathbb{R}$ *be a function of class* C^1, *and assume that* $x_i(t) < x_j(t)$ *for all* $t \in [0,T]$ *and all* $1 \leq i < j \leq I$.

Let $\sigma \in \Theta(m, M)$ *and* $b \in \Xi(M)$ *for some* $0 < m < M < \infty$.

Assume that for each $1 \leq i \leq I$, *the function* $\beta_i : [0,T] \to [k, \kappa]$ *(*$-1 < k \leq \kappa < 1$*) is of class* C^1, *and that* $|\beta'_i(t)| \leq M$ *for any* $t \in [0,T]$.

Then the time inhomogeneous SDE with local time

$$dX_t = \sigma(t, X_t)dW_t + b(t, X_t)dt + \sum_{i=1}^{I} \beta_i(t) dL_t^{x_i}(X), \quad t \in [0,T], \quad X_0 = x_0$$

(i.e. Equation (1.1.1)*) has a weak solution.*

Assume further that σ *satisfies the* $\mathbf{H}^{(x_i)}$ *and* $\mathbf{AJ}^{(x_i)}$*-hypotheses.*

Then the SDE (1.1.1) *has a unique strong solution (as it enjoys pathwise uniqueness).*

Remark 1.4.6. *The conditions of Theorem 1.4.5 have to be compared to the conditions in [43]. In particular, as in [43], it is required that the* β_i*'s stay in* $(-1, 1)$.

1.4.2 Proofs

Proof of Proposition 1.4.1. Applying Corollary 1.3.8 we get

$$\begin{aligned} dX_t &= \phi'_{t,\pm}(t,Y_t)dt + \phi'_{y,\pm}(t,Y_t)dY_t + \tfrac{1}{2}\phi''_{yy}(t,Y_t)\bar{\sigma}^2(t,Y_t)\mathbf{1}_{\{Y_t\neq y_i(t),\ 1\leq i\leq I\}}dt \\ &\quad + \textstyle\sum_{i=1}^I \triangle\ \phi'_y(t,Y_t)dL_t^{y_i}(Y) \\ &= (\bar{\sigma}\phi'_{y,\pm})(t,Y_t)dW_t + [\phi'_{t,\pm} + \bar{b}\phi'_{y,\pm}](t,Y_t)dt + \tfrac{1}{2}(\bar{\sigma}^2\phi''_{yy})(t,Y_t)\mathbf{1}_{\{Y_t\neq y_i(t),\,\forall 1\leq i\leq I\}}dt \\ &\quad + \textstyle\sum_{i=1}^I [\triangle\ \phi'_y(t,y_i(t)) + \bar{\beta}_i(t)\phi'_{y,\pm}(t,y_i(t))]\, dL_t^{y_i}(Y), \end{aligned}$$

where we have used the fact that $dL_t^{y_i}(Y) = \mathbf{1}_{Y_t=y_i(t)}dL_t^{y_i}(Y)$, for any $1\leq i\leq I$.

Thus, the first part of Proposition 1.4.1 is proved. To prove the second part it suffices to use the following lemma.

Lemma 1.4.7. *In the above context and under* (1.4.3) *we have*

$$dL_t^{y_i}(Y) = \frac{dL_t^{x_i}(X)}{\phi'_{y,\pm}(t,y_i(t)) + \bar{\beta}_i(t)\ \triangle\ \phi'_y(t,y_i(t))}, \quad \forall 1\leq i\leq I.$$

Proof. Let $1\leq i\leq I$. On one side we apply the symmetric Tanaka formula ([64] Exercise VI.1.25) to the process $X - x_i$. We get

$$\begin{aligned} d|X_t - x_i(t)| &= \mathrm{sgn}(X_t - x_i(t))d(X_t - x_i(t)) + dL_t^0(X - x_i) \\ &= dL_t^{x_i}(X) - \mathrm{sgn}(Y_t - y_i(t))dx_i(t) \\ &\quad + \mathrm{sgn}(Y_t - y_i(t))\sigma(t,\phi(t,Y_t))dW_t + \mathrm{sgn}(Y_t - y_i(t))b(t,\phi(t,Y_t))dt \\ &\quad + \textstyle\sum_{j\neq i}\mathrm{sgn}(Y_t - y_i(t))[\triangle\ \phi'_y(t,y_j(t)) + \bar{\beta}_j(t)\phi'_{y,\pm}(t,y_j(t))]\, dL_t^{y_j}(Y). \end{aligned} \tag{1.4.9}$$

In the above expression we have first used the fact that $\mathrm{sgn}(X_t - x_i(t)) = \mathrm{sgn}(Y_t - y_i(t))$ for any $t\in[0,T]$ (as $\phi(t,\cdot)$ is stricly increasing). Second we have used the fact that with the symmetric sign function we have

$$\mathrm{sgn}(X_t - x_i(t)) = \mathrm{sgn}(Y_t - y_i(t)) = 0 \text{ for any } t\in[0,T] \text{ s.t. } Y_t = y_i(t).$$

Third we have used $dL_t^{y_j}(Y) = \mathbf{1}_{Y_t=y_j(t)}dL_t^{y_j}(Y)$, for any $1\leq j\leq I$.

On the other side we may apply the first part of Proposition 1.4.1 (that is Equation (1.4.2); we stress that at this stage this part is already proved) with the semimartingale Y and the function $\zeta : (t,y)\mapsto |\phi(t,y) - x_i(t)|$. We get

$$\begin{aligned} d|X_t - x_i(t)| &= d|\phi(t,Y_t) - x_i(t)| \\ &= (\bar{\sigma}\zeta'_{y,\pm})(t,Y_t)dW_t + [\zeta'_{t,\pm} + \bar{b}\zeta'_{y,\pm}](t,Y_t)dt + \tfrac{1}{2}(\bar{\sigma}^2\zeta''_{yy})(t,Y_t)\mathbf{1}_{\{Y_t\neq y_j(t),\,\forall 1\leq j\leq I\}}dt \\ &\quad + \textstyle\sum_{j=1}^I [\triangle\ \zeta'_y(t,y_j(t)) + \bar{\beta}_j(t)\zeta'_{y,\pm}(t,y_j(t))]\, dL_t^{y_j}(Y) \\ &= -\mathrm{sgn}(Y_t - y_i(t))dx_i(t) + \mathrm{sgn}(Y_t - y_i(t))\sigma(t,\phi(t,Y_t))dW_t + \mathrm{sgn}(Y_t - y_i(t))b(t,\phi(t,Y_t))dt \\ &\quad + \textstyle\sum_{j\neq i}\mathrm{sgn}(Y_t - y_i(t))[\triangle\ \phi'_y(t,y_j(t)) + \bar{\beta}_j(t)\phi'_{y,\pm}(t,y_j(t))]\, dL_t^{y_j}(Y) \\ &\quad + [\phi'_{y,\pm}(t,y_i(t)) + \bar{\beta}_i(t)\ \triangle\ \phi'_y(t,y_i(t))]dL_t^{y_i}(Y). \end{aligned} \tag{1.4.10}$$

In (1.4.10) we have used several facts (see the proof of Lemma 3.7 in [30] for details). The most crucial ones concern the local time terms.

For $j<i$, we have $y_j(t) < y_i(t)$ and thus $\phi(t,y_j(t)) < x_i(t)$ for any $t\in[0,T]$, which leads to

$$[\triangle\ \zeta'_y(t,y_j(t)) + \bar{\beta}_j(t)\zeta'_{y,\pm}(t,y_j(t))] = -[\triangle\ \phi'_y(t,y_j(t)) + \bar{\beta}_j(t)\phi'_{y,\pm}(t,y_j(t))].$$

Using $dL_t^{y_j}(Y) = \mathbf{1}_{Y_t=y_j(t)} dL_t^{y_j}(Y)$ we then get that

$$[\triangle\, \zeta'_y(t, y_j(t)) + \bar{\beta}_j(t)\zeta'_{y,\pm}(t, y_j(t))]\, dL_t^{y_j}(Y) = \mathrm{sgn}(Y_t - y_i(t))[\triangle\, \phi'_y(t, y_j(t)) + \bar{\beta}_j(t)\phi'_{y,\pm}(t, y_j(t))]\, dL_t^{y_j}(Y)$$

We have the same result for $j > i$ (plus sign replaces minus sign).

We now examine what happens for $j = i$. Because of the different sign of $\phi(t, y_i(t)\pm) - x_i(t)$ we have

$$[\triangle\, \zeta'_y(t, y_i(t)) + \bar{\beta}_j(t)\zeta'_{y,\pm}(t, y_i(t))] = [\phi'_{y,\pm}(t, y_i(t)) + \bar{\beta}_i(t) \triangle\, \phi'_y(t, y_i(t))].$$

Therefore (1.4.10).

Comparing (1.4.9) and (1.4.10) we get the desired result. □

Proof of Theorem 1.4.5. Inspired by [43], we will use the following bijection in space $r(t, \cdot)$ (for any $t \in [0, T]$), that we now define.

For any $t \in [0, T]$ we define

$$\mu(t, x) = \prod_{x_i(t) \le x} \frac{1 - \beta_i(t)}{1 + \beta_i(t)} \tag{1.4.11}$$

(with the convention that $\mu(t, x) = 1$ for any $x < x_1(t)$).

Let then

$$R(t, x) = \int_{x_1(t)}^{x} \mu(t, z)dz. \tag{1.4.12}$$

As $\mu(t, z)$ is strictly positive for any $z \in \mathbb{R}$ the function $R(t, \cdot)$ is strictly increasing. Thus we can define

$$r(t, y) = \big[R(t, \cdot)\big]^{-1}(y). \tag{1.4.13}$$

For any $1 \le i \le I$ we define

$$y_i(t) = R(t, x_i(t)) \tag{1.4.14}$$

(note that $y_1 \equiv 0$). It is easy to check that

$$r(t, y) = \int_0^y \alpha(t, z)dz + x_1(t) \tag{1.4.15}$$

with

$$\alpha(t, y) = \prod_{y_i(t) \le y} \frac{1 + \beta_i(t)}{1 - \beta_i(t)} \tag{1.4.16}$$

(with $\alpha(t, y) = 1$ for any $y < y_1(t)$). Note that the function $r(t, \cdot)$ is strictly increasing too.

Let us check that R is in $C(E) \cap \Big(\cap_{i=0}^{I} C^{1,2}(\overline{D_i^x}) \Big)$ and that r is in $C(E) \cap \Big(\cap_{i=0}^{I} C^{1,2}(\overline{D_i^y}) \Big)$. We focus on $R(t, x)$, as the computations are similar for $r(t, y)$.

Using (1.4.11)(1.4.12) it is easy to check that $R(t, x)$ coincides on D_0^x with the function $R_0(t, x) = x - x_1(t)$. On D_i^x, $1 \le i \le I$, it coincides with the function

$$R_i(t, x) = \sum_{j=1}^{i-1} \Big\{ \prod_{k \le j} \frac{1 - \beta_k(t)}{1 + \beta_k(t)} \Big\} (x_{j+1}(t) - x_j(t)) + \Big\{ \prod_{k \le i} \frac{1 - \beta_k(t)}{1 + \beta_k(t)} \Big\} (x - x_i(t)).$$

Obviously, all the functions $R_i(t, x)$, $0 \le i \le I$ are in $C^{1,2}(E)$, and thus we see that $R(t, x)$ is in $\cap_{i=0}^{I} C^{1,2}(\overline{D_i^x})$.

To see that $R(t, x)$ is in $C(E)$ it remains to prove that it is continuous at any point $(t_0, x_0) \in \Delta_{\mathbf{x}}$. For such a point we have $(t_0, x_0) = (t_0, x_i(t_0))$, for some $t_0 \in [0, T]$ and some $1 \le i \le I$. But, together with the relationship

$$R(t_0, x_i(t_0)) = R_{i-1}(t_0, x_i(t_0)) = R_i(t_0, x_i(t_0))$$

the continuity of R_{i-1} and R_i then yields the desired result. Thus, R is indeed in $C(E) \cap \Big(\cap_{i=0}^{I} C^{1,2}(\overline{D_i^x}) \Big)$. Note that this implies that the y_i's defined by (1.4.14) are of class C^1 (by the same arguments as in Remark 1.4.2).

We then set

$$\bar\sigma(t,y)=\frac{\sigma(t,r(t,y))}{r'_{y,\pm}(t,y)}\quad\text{and}\quad\bar b(t,y)=\frac{b(t,r(t,y))}{r'_{y,\pm}(t,y)}-\frac{r'_{t,\pm}(t,y)}{r'_{y,\pm}(t,y)}\tag{1.4.17}$$

It is easy to check that $\bar\sigma\in\Theta(\bar m,\bar M)$ and $\bar b\in\Xi(\bar M)$ for some $0<\bar m<\bar M<\infty$.

From now on the starting point $x_0\in\mathbb{R}$ is fixed. By Corollary 1.3.4 we have the existence of a weak solution Y to

$$dY_t=\bar\sigma(t,Y_t)dW_t+\bar b(t,Y_t)dt,\quad Y_0=R(0,x_0).\tag{1.4.18}$$

We wish now to use the second part of Proposition 1.4.1, with the function $r(t,y)$ and the process Y (and the curves y_i's). Note that by construction we have

$$(\bar\sigma r'_{y,\pm})(t,R(t,x))=\sigma(t,x),$$

$$[r'_{t,\pm}+\bar b r'_{y,\pm}](t,R(t,x))+\frac{1}{2}(\bar\sigma^2 r''_{yy})(t,R(t,x))\mathbf{1}_{\{x\neq x_i(t),\,\forall 1\le i\le I\}}=b(t,x),$$

(we have used in particular $r''_{yy}\equiv 0$ in the above expression) and

$$\frac{\triangle\, r'_y(t,y_i(t))}{r'_{y,\pm}(t,y_i(t))}=\frac{\prod_{j<i}\frac{1+\beta_j(t)}{1-\beta_j(t)}\frac12\left(\frac{1+\beta_i(t)}{1-\beta_i(t)}-1\right)}{\prod_{j<i}\frac{1+\beta_j(t)}{1-\beta_j(t)}\frac12\left(\frac{1+\beta_i(t)}{1-\beta_i(t)}+1\right)}=\frac{\frac{2\beta_i(t)}{1-\beta_i(t)}}{\frac{2}{1-\beta_i(t)}}=\beta_i(t)$$

(here we have computed (1.4.6) using the fact that there is no local time term in (1.4.18)).

So that by setting

$$X_t=r(t,Y_t),\quad\forall t\in[0,T]\tag{1.4.19}$$

we immediately see by Proposition 1.4.1 that X is a weak solution to (1.1.1).

In order to prove the last part of the theorem, we first notice that $\bar\sigma$ satisfies the $\mathbf{H}^{(y_i)}$ and $\mathbf{AJ}^{(x_i)}$-hypotheses. Thus (1.4.18) enjoys pathwise uniqueness (Corollary 1.3.4). Assume X' is a second solution to (1.1.1), then we could show that $Y'_t=R(t,X'_t)$ is a solution to (1.4.18). Thus, using the pathwise uniqueness property of (1.4.18), we would show that pathwise uniqueness holds for (1.1.1). Therefore Theorem 1.4.5 is proved.

1.5 Feynman-Kac formula: link with a parabolic transmission problem

Assume the curves x_i, $1\le i\le I$ and the coefficients β_i, $1\le i\le I$ are as in Theorem 1.4.5, b is in $\Xi(M)\cap C(E\setminus\Delta_{\mathbf{x}})$, and σ is in $\Theta(m,M)\cap C(E\setminus\Delta_{\mathbf{x}})$.

For $\lambda\ge 0$, a source term $g\in C_c(E)$ and a terminal condition $f\in C_0(\mathbb{R})\cap L^2(\mathbb{R})$, we will call a classical solution of the parabolic transmission problem $(\mathcal{P}^\lambda_{\Delta_{\mathbf{x}}}(\sigma,b,\beta))$ a function $u(t,x)$ that is of class $C(E)\cap C^{1,2}(E^\circ\setminus\Delta_{\mathbf{x}})$, is such that for all $1\le i\le I$ the limits $u'_t(t,x_i(t)\pm)$, $u'_x(t,x_i(t)\pm)$ and $u''_{xx}(t,x_i(t)\pm)$ exist and are continuous as functions of $t\in[0,T)$, and that satisfies

$$(\mathcal{P}^\lambda_{\Delta_{\mathbf{x}}}(\sigma,b,\beta))\begin{cases}\big[u'_t+\frac12\sigma^2u''_{xx}+b\,u'_x-\lambda u\big](t,x) &= g(t,x) & \forall(t,x)\in E^\circ\setminus\Delta_{\mathbf{x}}\\ (1+\beta_i(t))u'_x(t,x_i(t)+) &= (1-\beta_i(t))u'_x(t,x_i(t)-) & \forall 1\le i\le I,\,\forall t\in[0,T)\quad(\star)\\ u(T,x) &= f(x) & \forall x\in\mathbb{R}.\\ \lim_{|x|\to\infty}|u(t,x)| &= 0 & \forall t\in[0,T].\end{cases}$$

In particular we stress that the first and second line of this system of equations are satisfied in the classical sense.

The question whether a classical solution $u(t,x)$ exists to $(\mathcal{P}^\lambda_{\Delta_{\mathbf{x}}}(\sigma,b,\beta))$ will be discussed in Section 1.6, with the help of an equivalent formulation of this parabolic transmission problem, in a more divergence-like form (Subsection 1.6.1). The condition $(\star)$ will be called the *transmission condition* in the sequel.

For the moment, assuming in this section the existence of such a solution $u(t,x)$, we draw some consequences on the solution X of (1.1.1): we have a Feynman-Kac formula linking X and $u(t,x)$.

We have the following result.

Theorem 1.5.1. *Any classical solution $u(t,x)$ of $(\mathcal{P}^{\lambda}_{\Delta_{\mathbf{x}}}(\sigma,b,\beta))$ admits the stochastic representation*

$$u(t,x)=\mathbb{E}^{t,x}\Big[f(X_T)e^{-\lambda(T-t)}-\int_t^T g(s,X_s)e^{-\lambda(s-t)}ds\Big]$$

where X is the solution to (1.1.1); in particular such a classical solution $u(t,x)$ is unique.

Remark 1.5.2. *In this theorem the unicity of $u(t,x)$ comes from the uniqueness in law of the weak solution X (see Subsection 1.3.1).*

Proof. We only sketch the proof, as the details are to be found in [30] (Theorem 4.1 therein). The idea is to follow the lines of the proof of Theorem 5.7.6 in [39], and to use our Proposition 1.4.1 in the computations. Let $t\in[0,T)$. Applying Proposition 1.4.1 and Equation (1.4.7) we get for any $s\in[t,T)$,

$$\begin{aligned}u(s,X_s)e^{-\lambda(s-t)}-u(t,X_t) &= \int_t^s u'_{x,\pm}(v,X_v)e^{-\lambda(v-t)}\sigma(v,X_v)dW_v\\ &\quad+\int_t^s e^{-\lambda(v-t)}\big[u'_{t,\pm}+bu'_{x,\pm}-\lambda u\big](v,X_v)dv\\ &\quad+\tfrac12\int_s^t e^{-\lambda(v-s)}u''_{xx}(v,X_v)\sigma^2(v,X_v)\mathbf{1}_{\{X_v\neq x_i(v),1\le i\le I\}}dv\\ &\quad+\tfrac12\sum_{i=1}^I\int_t^s\big[(1+\beta_i(v))u'_x(v,X_v+)-(1-\beta_i(v))u'_x(v,X_v-)\big]dL_v^{x_i}(X)\\ &= \int_t^s u'_{x,\pm}(v,X_v)e^{-\lambda(v-t)}\sigma(v,X_v)dW_v+\int_t^s e^{-\lambda(v-t)}g(v,X_v)dv\end{aligned}\tag{1.5.1}$$

where we have used in particular the transmission condition $(\star)$ satisfied by $u(t,x)$.

Using a localizing sequence of stopping times, taking the expectation $\mathbb{E}^{t,x}(\cdot)$, and using finally convergence arguments we get the desired result. □

1.6 Parabolic transmission problem with time-dependent coefficients

1.6.1 Equivalent formulation in divergence like form and getting cylindrical subdomains by the mean of a space transform

Assume that we have curves x_i, $1\le i\le I$ satisfy the same assumptions than in Theorem 1.4.5. Let us consider coefficients $\rho,a\in\Theta(m',M')\cap C(E\setminus\Delta_{\mathbf{x}})$, and a coefficient $B\in\Xi(M')\cap C(E\setminus\Delta_{\mathbf{x}})$ (for some $0<m'<M'<\infty$).

For $\lambda\ge0$, a source term $g\in C_c(E)$ and a terminal condition $f\in C_0(\mathbb{R})\cap L^2(\mathbb{R})$, we will call a classical solution of the transmission problem in divergence form $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta_{\mathbf{x}}}(\rho,a,B))$, a function $u(t,x)$ that is of class $C(E)\cap C^{1,2}(E^\circ\setminus\Delta_{\mathbf{x}})$, is such that for all $1\le i\le I$ the limits $u'_t(t,x_i(t)\pm)$, $u'_x(t,x_i(t)\pm)$ and $u''_{xx}(t,x_i(t)\pm)$ exist and are continuous as functions of $t\in[0,T)$, and that satisfies

$$(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta_{\mathbf{x}}}(\rho,a,B))\begin{cases}\big[u'_t+\dfrac{\rho}{2}\big(au'_x\big)'_x+B\,u'_x-\lambda u\big](t,x) = g(t,x) & \forall(t,x)\in E^\circ\setminus\Delta_{\mathbf{x}}\\ a(t,x_i(t)+)u'_x(t,x_i(t)+) = a(t,x_i(t)-)u'_x(t,x_i(t)-) & \forall1\le i\le I,\ \forall t\in[0,T)\quad(\star)\\ u(T,x) = f(x) & \forall x\in\mathbb{R}.\\ \lim_{|x|\to\infty}|u(t,x)| = 0 & \forall t\in[0,T].\end{cases}$$

For any ρ,a,B with $\rho a=\sigma^2$, $a(t,x_i(t)\pm)=p_i(t)(1\pm\beta_i(t))$, $1\le i\le I$, $t\in[0,T)$ and $B=b-\rho\,a'_{x,\pm}/2$, it is clear that a classical solution to $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta_{\mathbf{x}}}(\rho,a,B))$ is a classical solution to $(\mathcal{P}^{\lambda}_{\Delta_{\mathbf{x}}}(\sigma,b,\beta))$ (here $p_i(t)$ is a non zero multiplicative factor that depends on $1\le i\le I$, $t\in[0,T)$). One may for example choose for any $(t,x)\in E$

$$a(t,x)=\prod_{x_i(t)\le x}\frac{1+\beta_i(t)}{1-\beta_i(t)},\quad \rho(t,x)=\frac{\sigma^2(t,x)}{a(t,x)},\quad B(t,x)=b(t,x)\tag{1.6.1}$$

(Note that here $p_i(t) = \dfrac{1}{1-\beta_i(t)} \prod_{j<i} \dfrac{1+\beta_j(t)}{1-\beta_j(t)}$). Note that the presence of the variable coefficient $\rho(t,x)$ is due to the fact that the coefficient $\sigma(t,x)$ has been chosen independently from the $\beta_i(t)$'s. Note also that a convenient triple (ρ, a, B) is not unique.

Conversely, it is always possible to pass from a transmission problem in the form $(\mathcal{P}^\lambda_{\mathrm{div},\Delta_{\mathbf{x}}}(\rho,a,B))$ to another one in the form $(\mathcal{P}^\lambda_{\Delta_{\mathbf{x}}}(\sigma,b,\beta))$, by setting in particular

$$\beta_i(t) = \frac{a(t,x_i(t)+) - a(t,x_i(t)-)}{a(t,x_i(t)+) + a(t,x_i(t)-)}. \tag{1.6.2}$$

In fact, in the PDE litterature, parabolic transmission problems are classically studied in the purely divergence-like form of $(\mathcal{P}^\lambda_{\mathrm{div},\Delta_{\mathbf{x}}}(\rho \equiv 1,a,B))$. Up to our knowledge fewer studies exist in the non divergence form $(\mathcal{P}^\lambda_{\Delta_{\mathbf{x}}}(\sigma,b,\beta))$. The aim of this section is to present some known results on the problem $(\mathcal{P}^\lambda_{\mathrm{div},\Delta}(\rho\equiv 1,a,B))$, and to derive new ones for the general case.

In the case $\rho \equiv 1$, the transmission problem in divergence form $(\mathcal{P}^\lambda_{\mathrm{div},\Delta_{\mathbf{x}}}(\rho,a,B))$ is well studied in the PDE litterature, concerning the existence and unicity of weak solutions (see the forthcoming Subsection 1.6.2 for a definition of weak solutions). See for instance to [41], [48], [47].

Concerning classical solutions in the presence of a discontinuous coefficient $a(t,x)$ like in our case, it seems that less references are available. In the fundamental paper [40] it is shown that, still with $\rho\equiv 1$, and in the case of cylindrical space-time subdomains (that is to say $x_i(t) = x_i$ for all $1 \le i \le I$, $0 \le t \le T$) every weak solution to $(\mathcal{P}^\lambda_{\mathrm{div},\Delta_{\mathbf{x}}}(\rho\equiv 1,a,B))$ is in fact classical. As a consequence there exists a classical solution to $(\mathcal{P}^\lambda_{\mathrm{div},\Delta_{\mathbf{x}}}(\rho\equiv 1,a,B))$.

In the case $\rho \neq 1$ and in the presence of non-cylindrical subdomains some results are announced in [40] and [41]. However they are stated without any complete proof (with the notable exception of the proof of the existence of a unique weak solution in the case of cylindrical subdomains, but with $\rho \neq 1$, pp 229-232 of [41]; see Subsection 1.6.2 for further comments).

We continue this subsection by noticing that in fact we can get rid of the difficulty of having non-cylindrical subdomains, by applying a space transform trick, available only because the space dimension is one. We choose to present things on the problem in its non-divergence form $(\mathcal{P}^\lambda_{\Delta_{\mathbf{x}}}(\sigma,b,\beta))$ again.

From now on we assume $I \ge 3$ and set

$$\forall (t,\hat{x}) \in E, \quad \psi(t,\hat{x}) = \begin{cases} x_1(t) + (x_2(t)-x_1(t))(\hat{x}-1) & \text{if} \quad \hat{x} < 1 \\ x_j(t) + (x_{j+1}(t)-x_j(t))(\hat{x}-j) & \text{if} \quad j \le \hat{x} < j+1,\ j = 1,\dots,I-2 \\ x_{I-1}(t) + (x_I(t)-x_{I-1}(t))(\hat{x}-I+1) & \text{if} \quad \hat{x} \ge I-1 \end{cases}$$

For any $t \in [0,T]$ we note $\Psi(t,\cdot) = [\psi(t,\cdot)]^{-1}(\cdot)$. Notice that

$$\Delta = \Psi(\Delta_{\mathbf{x}})$$

and that $E \setminus \Delta$ appears as the union of some open cylindrical space-time domains.

We have the following result.

Proposition 1.6.1. *A fonction $u(t,x)$ is a classical solution to $(\mathcal{P}^\lambda_{\Delta_{\mathbf{x}}})(\sigma,b,\beta)$ if and only if $\hat{u}(t,\hat{x}) := u(t,\psi(t,\hat{x}))$ is a classical solution to*

$$((\hat{\mathcal{P}}^\lambda_{\Delta})(\hat{\sigma},\hat{b},\hat{\beta})) \begin{cases} \big[\hat{u}'_t + \frac{1}{2}\hat{\sigma}^2\hat{u}''_{\hat{x}\hat{x}} + \hat{b}\,\hat{u}'_{\hat{x}} - \lambda\hat{u}\big](t,\hat{x}) &=& \hat{g}(t,\hat{x}) & \forall (t,\hat{x}) \in E\setminus\Delta \\ (1+\hat{\beta}_i(t))\hat{u}'_{\hat{x}}(t,i+) &=& (1-\hat{\beta}_i(t))\hat{u}'_{\hat{x}}(t,i-) & \forall 1 \le i \le I,\ \forall t \in [0,T) \quad (\star) \\ \hat{u}(T,\hat{x}) &=& \hat{f}(\hat{x}) & \forall \hat{x} \in \mathbb{R}. \\ \lim_{|\hat{x}|\to\infty} |\hat{u}(t,\hat{x})| &=& 0 & \forall t \in [0,T], \end{cases}$$

where

$$\hat{\sigma}(t,\hat{x}) = \sigma(t,\psi(t,\hat{x})) \times \Psi'_{x,\pm}(t,\psi(t,\hat{x})), \quad \hat{b}(t,\hat{x}) = b(t,\psi(t,\hat{x})) \times \Psi'_{x,\pm}(t,\psi(t,\hat{x})) + \Psi'_{t,\pm}(t,\psi(t,\hat{x})), \tag{1.6.3}$$

$\hat{g}(t,\hat{x}) = g(t,\psi(t,\hat{x}))$, $\hat{f}(\hat{x}) = f(\psi(T,\hat{x}))$ *and*

$$\hat{\beta}_i(t) = \frac{(1+\beta_i(t))\Psi'_x(t,x_i(t)+) - (1-\beta_i(t))\Psi'_x(t,x_i(t)-)}{(1+\beta_i(t))\Psi'_x(t,x_i(t)+) + (1-\beta_i(t))\Psi'_x(t,x_i(t)-)}. \tag{1.6.4}$$

Remark 1.6.2. *Note that*

$$\forall (t,x)\in E,\quad \Psi(t,x) = \begin{cases} (x-x_1(t))/(x_2(t)-x_1(t)) + 1 & \text{if} \quad x < x_1(t) \\ (x-x_j(t))/(x_{j+1}(t)-x_j(t)) + j & \text{if} \quad x_j(t)\le x < x_{j+1}(t),\ j=1,\dots,I-2 \\ (x-x_{I-1}(t))/(x_I(t)-x_{I-1}(t)) + I - 1 & \text{if} \quad x \ge x_{I-1}(t) \end{cases} \tag{1.6.5}$$

and that this function is of class $C(E)\cap C^{1,2}(E\setminus\Delta_{\mathbf{x}})$. *Besides, choosing* $\varepsilon < \inf_{1\le j\le I-1}\inf_{s\in[0,T]}(x_{j+1}(s)-x_j(s))$ *and using the fact that* $\varepsilon < x_{j+1}(t)-x_j(t) \le \sup_{s\in[0,T]}(x_{j+1}(s)-x_j(s))$ *we can see that there exist constants* $0<\hat{m}<\hat{M}<\infty$ *such that* $\Psi'_{x,\pm}\in\Theta(\hat{m},\hat{M})$. *In addition* $\Psi'_{t,\pm}$ *remains bounded (thanks in particular to the fact that the* $x_i:[0,T]\to\mathbb{R}$, $1\le i\le I$ *are of class* C^1*). Thus the coefficients* $\hat{\sigma}(t,\hat{x})$, $\hat{b}(t,\hat{x})$ *and* $\hat{\beta}_i(t)$, $1\le i\le I$, *still satisfy the hypotheses of Section 1.5.*

Proof of Proposition 1.6.1. We only prove the sufficient condition, the converse being proved in the same manner.

First for any $(t,x)\in E\setminus\Delta_{\mathbf{x}}$ we have

$$u'_x(t,x) = \hat{u}'_{\hat{x}}(t,\Psi(t,x))\times\Psi'_x(t,x), \tag{1.6.6}$$

and, as $\Psi''_{xx}(t,x)=0$,

$$u''_{xx}(t,x) = \hat{u}''_{\hat{x}\hat{x}}(t,\Psi(t,x))\times[\Psi'_x(t,x)]^2. \tag{1.6.7}$$

We also have

$$u'_t(t,x) = \hat{u}'_t(t,\Psi(t,x)) + \hat{u}'_{\hat{x}}(t,\Psi(t,x))\times\Psi'_t(t,x). \tag{1.6.8}$$

So that for any $(t,\hat{x})\in E\setminus\Delta$ we may use this with $(t,x)=(t,\psi(t,\hat{x}))$ in the first line of $(\mathcal{P}^\lambda_{\Delta_{\mathbf{x}}}(\sigma,b,\beta))$ and thus we get the first line of $(\hat{\mathcal{P}}^\lambda_\Delta(\hat{\sigma},\hat{b},\hat{\beta}))$, with the newly defined coefficients $\hat{\sigma}$, $\hat{b}$ and $\hat{g}$.

Concerning the transmission condition $(\hat{\star})$, we notice that we have from $(\star)$ in $(\mathcal{P}^\lambda_{\Delta_{\mathbf{x}}}(\sigma,b,\beta))$

$$\forall t\in[0,T],\quad (1+\beta_i(t))\Psi'_x(t,x_i(t)+)\hat{u}'_{\hat{x}}(t,\Psi(t,x_i(t))+) = (1-\beta_i(t))\Psi'_x(t,x_i(t)-)\hat{u}'_{\hat{x}}(t,\Psi(t,x_i(t))-)$$

for any $1\le i\le I$. As $\Psi(t,x_i(t))=i$ for any $1\le i\le I$, an easy computation shows that this is equivalent to $(\hat{\star})$, with the newly defined $\hat{\beta}_i(t)$, $1\le i\le I$.

The third and fourth lines of $(\hat{\mathcal{P}}^\lambda_\Delta(\hat{\sigma},\hat{b},\hat{\beta}))$ are straightforward. □

We can sum up the preceding discussions in the following proposition.

Proposition 1.6.3. *Assume the curves* x_i, *and the coefficients* β_i, $1\le i\le I$, *are as in Theorem 1.4.5, and that* b *is in* $\Xi(M)\cap C(E\setminus\Delta_{\mathbf{x}})$, *and* σ *is in* $\Theta(m,M)\cap C(E\setminus\Delta_{\mathbf{x}})$.

Let $\hat{\sigma}$, $\hat{b}$, $\hat{\beta}_i$, $1\le i\le I$, *defined by* (1.6.3) (1.6.4). *Let* $\hat{\rho},\hat{a},\hat{B}$ *defined by* (1.6.1), *but with* $\hat{\sigma}$, $\hat{b}$, $\hat{\beta}_i$, $1\le i\le I$ *instead of* σ, b, β_i, $1\le i\le I$.

Then $(\mathcal{P}^\lambda_{\Delta_{\mathbf{x}}}(\sigma,b,\beta))$ *has a classical solution if and only if* $(\mathcal{P}^\lambda_{\mathrm{div},\Delta}(\hat{\rho},\hat{a},\hat{B}))$ *has a classical solution* $\hat{u}(t,\hat{x})$. *This classical solution of* $(\mathcal{P}^\lambda_{\Delta_{\mathbf{x}}}(\sigma,b,\beta))$ *is given by* $u(t,x)=\hat{u}(t,\Psi(t,x))$ *with* $\Psi(t,x)$ *defined by* (1.6.5).

Without loss of generality we shall investigate the problem $(\mathcal{P}^\lambda_{\mathrm{div},\Delta}(\rho,a,B))$ (i.e. with $x_i\equiv i$, $1\le i\le I$). In Subsection 1.6.2 we deal with weak solutions. In Subsection 1.6.3 we present a way to get classical solutions in the case $\rho\neq 1$, using the results of [40] for the case $\rho\equiv 1$, and again (different) space transform tricks.

1.6.2 Weak solutions

In this subsection it is assumed $\rho, a \in \Theta(m', M')$ and $B \in \Xi(M')$ for some $0 < m' < M' < \infty$, and that the coefficient ρ satisfies the $\mathbf{H}^{(t)}$-hypothesis.

We will call a weak solution of the parabolic problem $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta}(\rho, a, B))$ a function $u(t,x)$ in the space $L^2(0,T;H^1(\mathbb{R})) \cap C([0,T];L^2(\mathbb{R}))$, with $u(T,\cdot) = f$ a.e., and satisfying for any test function $\varphi \in H_0^{1,1}(E)$ the relation

$$\int_0^T \int_{\mathbb{R}} u \frac{\mathrm{d}\varphi}{\mathrm{dt}} \rho^{-1}\, dxdt$$

$$+\frac{1}{2}\int_0^T \int_{\mathbb{R}} a \frac{\mathrm{d}u}{\mathrm{dx}} \frac{\mathrm{d}\varphi}{\mathrm{dx}}\, dxdt - \int_0^T \int_{\mathbb{R}} B \frac{\mathrm{d}u}{\mathrm{dx}} \varphi\rho^{-1} dxdt + \int_0^T \int_{\mathbb{R}} u(\lambda - \frac{\rho_t'}{\rho})\varphi\rho^{-1} dxdt = -\int_0^T \int_{\mathbb{R}} g\varphi\rho^{-1}\, dxdt. \tag{1.6.9}$$

Indeed, imagine for a while that we have a classical solution $u(t,x)$ of $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta}(\rho, a, B))$. If we formally multiply the first line of $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta}(\rho, a, B))$ by a test function φ vanishing at infinity and with $\varphi(0,\cdot) = \varphi(T,\cdot) = 0$, and integrate the resulting equation against $\rho^{-1}dxdt$ on $[0,T] \times \mathbb{R}$ we recover (1.6.9), using in particular $(\star)$ in the integration by parts formula.

We first aim at proving the following result.

Proposition 1.6.4. *The parabolic problem $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta}(\rho, a, B))$ has a unique weak solution.*

In fact this result is in essence contained in the discussion p 229-232 of [41], but we want here to give our own, new and different proof, using the tools proposed in [48]. They differ from the ones used in [41][47] but provide an elegant framework to handle the problem, and could be the starting point of the use of Generalized Dirichlet forms in these questions (see some of our concluding remarks).

In order to use the tools in [48] we denote $\mathcal{H} = L^2(0,T;L^2(\mathbb{R});\rho^{-1})$ the set of measurable functions $f(t,x)$ such that

$$\int_0^T \int_{\mathbb{R}} |f(t,x)|^2 \rho^{-1}(t,x) dxdt < \infty,$$

equipped with the scalar product

$$\forall u, v \in \mathcal{H}, \quad \langle u, v\rangle_{\mathcal{H}} = \int_0^T \int_{\mathbb{R}} u(t,x)v(t,x)\rho^{-1}(t,x) dxdt.$$

We denote $\mathcal{V} = L^2(0,T;H^1(\mathbb{R});\rho^{-1})$ the set of mesurable functions $f(t,x)$ such that for any $t \in [0,T]$ the function $f(t,\cdot)$ is in $H^1(\mathbb{R})$ and

$$\int_0^T \int_{\mathbb{R}} |f(t,x)|^2 \rho^{-1}(t,x) dxdt + \int_0^T \int_{\mathbb{R}} |\frac{\mathrm{d}f}{\mathrm{dx}}(t,x)|^2 \rho^{-1}(t,x) dxdt < \infty,$$

equipped with the scalar product

$$\forall u, v \in \mathcal{V}, \quad \langle u, v\rangle_{\mathcal{V}} = \langle u, v\rangle_{\mathcal{H}} + \langle \frac{\mathrm{d}u}{\mathrm{dx}}, \frac{\mathrm{d}v}{\mathrm{dx}}\rangle_{\mathcal{H}}.$$

We will denote by $||\cdot||_{\mathcal{H}}$ and $||\cdot||_{\mathcal{V}}$ the norms corresponding to the above defined scalar products. We denote by $\mathcal{V}'$ the dual of $\mathcal{V}$. Note that we have $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ with dense inclusions.

Remark 1.6.5. *Note that as $\rho \in \Theta(m', M')$, of course $\mathcal{H}$ (resp. $\mathcal{V}$) is, as a set, just equal to $L^2(0,T;L^2(\mathbb{R}))$ (resp. $L^2(0,T;H^1(\mathbb{R}))$). Besides, as a set, $\mathcal{V}'$ is equal to $L^2(0,T;H^{-1}(\mathbb{R}))$.*

We define a semigroup $(U_t)_{t\in[0,T]}$ of contraction on $\mathcal{V}'$ by

$$U_t f(s,\cdot) = \begin{cases} f(s+t,\cdot) & \text{if } 0 < s < T - t \\ 0 & \text{otherwise.} \end{cases}$$

We denote $(\Lambda, D(\Lambda;\mathcal{V}'))$ the infinitesimal generator of (U_t). We have the following elementary fact.

Lemma 1.6.6. *We have*

$$D(\Lambda,\mathcal{V}')=\Big\{u\,|\,u\in\mathcal{V}',\ \frac{\mathrm{d}u}{\mathrm{d}t}\in\mathcal{V}',\ u(T,\cdot)=0\Big\}$$

and $\Lambda u=\dfrac{\mathrm{d}u}{\mathrm{d}t}$ *for any* $u\in D(\Lambda,\mathcal{V}')$.

Remark 1.6.7. *In Lemma 1.6.6, the time derivative* $\dfrac{\mathrm{d}u}{\mathrm{d}t}$ *is understood in the distribution sense. For example, in the case* $u\in\mathcal{V}\cap D(\Lambda,\mathcal{V}')$, *we have* $\langle u,v\rangle_{\mathcal{V}',\mathcal{V}}=\langle u,v\rangle_{\mathcal{H}}$ *for any* $v\in\mathcal{V}$, *and for any* $\varphi\in\mathcal{C}^{\infty,\infty}_{c,c}(E)$

$$\langle\frac{\mathrm{d}u}{\mathrm{d}t},\varphi\rangle_{\mathcal{V}',\mathcal{V}}=-\int_0^T\int_{\mathbb{R}}u(\varphi\rho^{-1})'_t dxdt=-\langle u,\varphi'_t-\varphi\frac{\rho'_t}{\rho}\rangle_{\mathcal{H}}.$$

Besides, for $u\in\mathcal{V}\cap D(\Lambda,\mathcal{V}')$ *and* $\varphi\in H^{1,1}_0(E)$ *we have*

$$\langle\frac{\mathrm{d}u}{\mathrm{d}t},\varphi\rangle_{\mathcal{V}',\mathcal{V}}=-\langle u,\frac{\mathrm{d}\varphi}{\mathrm{d}t}-\varphi\frac{\rho'_t}{\rho}\rangle_{\mathcal{H}}\tag{1.6.10}$$

(using the fact that $\mathcal{C}^{\infty,\infty}_{c,c}(E)$ *is dense in* $H^{1,1}_0(E)$*). Note that* ρ'_t *exists in the classical sense, even if it is not continuous, thanks to the fact that the subdomains are cylindrical. Besides,* ρ'_t *is bounded thanks to the* $\mathbf{H}^{(t)}$*-hypothesis.*

Proof. See [48], Section 3.4.3. □

As $\rho\neq 1$ we cannot use directly Theorem 3.4.1 in [48]. We will use a natural generalization of this result, that we have proved in [30] (besides note that we deal here with backward problems with terminal condition).

Theorem 1.6.8. *Assume* $\mathcal{A}$ *is a bilinear form on* $\mathcal{V}$ *satisfying*

i) $|\mathcal{A}(u,v)|\leq C||u||_{\mathcal{V}}||v||_{\mathcal{V}}$ *for all* $u,v\in\mathcal{V}$, *where* $0<C<\infty$.

ii) $\mathcal{A}(v,v)+\lambda_0||v||^2_{\mathcal{H}}\geq\alpha_0||v||^2_{\mathcal{V}}$ *for all* $v\in\mathcal{V}$ *(for some* $\lambda_0,\alpha_0>0$*).*

Then for any $G\in\mathcal{V}'$ *and any* $f\in\mathcal{H}$ *there exists a unique* $u\in L^2(0,T;H^1(\mathbb{R}))\cap C([0;T];L^2(\mathbb{R}))$ *(in particular* u *is in* $\mathcal{V}$*) such that* $u(T,\cdot)=f$, *and with* $\dfrac{\mathrm{d}u}{\mathrm{d}t}\in L^2(0,T;H^{-1}(\mathbb{R}))$ *and*

$$\big\langle-\frac{\mathrm{d}u}{\mathrm{d}t},v\big\rangle_{\mathcal{V}',\mathcal{V}}+\mathcal{A}(u,v)=\big\langle G,v\big\rangle_{\mathcal{V}',\mathcal{V}}\quad\forall v\in\mathcal{V}.\tag{1.6.11}$$

In order to apply Theorem 1.6.8 we now define for any $u,v\in\mathcal{V}$

$$\mathcal{A}(u,v)=\frac{1}{2}\int_0^T\int_{\mathbb{R}}a(t,x)\frac{\mathrm{d}u}{\mathrm{d}x}(t,x)\frac{\mathrm{d}v}{\mathrm{d}x}(t,x)dxdt-\int_0^T\int_{\mathbb{R}}B(t,x)\frac{\mathrm{d}u}{\mathrm{d}x}(t,x)v(t,x)\rho^{-1}(t,x)dxdt+\lambda\langle u,v\rangle_{\mathcal{H}}\tag{1.6.12}$$

and for any $\lambda_0>0$

$$\mathcal{A}_{\lambda_0}(u,v)=\mathcal{A}(u,v)+\lambda_0\langle u,v\rangle_{\mathcal{H}}.\tag{1.6.13}$$

Proof. See the Appendix of [30]. □

Using the boundedness of ρ,a,b, and the strict ellipticity of ρ,a, we get the following result.

Lemma 1.6.9. *The bilinear form* $\mathcal{A}(\cdot,\cdot)$ *defined by* (1.6.12) *is continuous, i.e.*

$$\forall u,v\in\mathcal{V},\quad|\mathcal{A}(u,v)|\leq C||u||_{\mathcal{V}}||v||_{\mathcal{V}},\tag{1.6.14}$$

where $C=C(m',M',\lambda)$.

It is always possible to choose $\lambda_0>0$ *large enough such that* $\mathcal{A}_{\lambda_0}(\cdot,\cdot)$ *defined by* (1.6.12)(1.6.13) *is coercive, i.e.*

$$\forall v\in\mathcal{V},\quad\mathcal{A}_{\lambda_0}(v,v)\geq\alpha_0||v||^2_{\mathcal{V}}.\tag{1.6.15}$$

where $\alpha_0=\alpha_0(m',M')$.

Proof. See the Appendix of [30]. □

We are now in position to prove Proposition 1.6.4. Indeed, thanks to Lemma 1.6.9 we may apply Theorem 1.6.8 with $\mathcal{A}(\cdot,\cdot)$ defined by (1.6.12) and with $G \in \mathcal{V}'$ defined by $\langle G, v\rangle_{\mathcal{V}',\mathcal{V}} = -\langle g, v\rangle_{\mathcal{H}}$ for any $v \in \mathcal{V}$. For any $\varphi \in H_0^{1,1}(E) \subset \mathcal{V}$, using (1.6.10) in the computation of the term $\big\langle -\dfrac{\mathrm{d}u}{\mathrm{d}t}, \varphi\big\rangle_{\mathcal{V}',\mathcal{V}}$ appearing in (1.6.11) (φ replaces v), we get (1.6.9).

It is possible to go a bit further in the analysis of the weak solution and to prove the following lemma, that asserts that the weak solution of $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta}(\rho,a,B))$ is of class H^1 in the time variable.

Lemma 1.6.10. *The weak u solution of $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta}(\rho,a,B))$ satisfies $\dfrac{\mathrm{d}u}{\mathrm{d}t} \in L^2(0,T;L^2(\mathbb{R}))$.*

Proof. See the Appendix of [30]. □

The above result is one of the crucial steps in the study of the case $\rho \equiv 1$ in [40]. However, it seems challenging to adapt all the other steps of [40] and [41] to our case $\rho \neq 1$, see Remark 1.6.13.

1.6.3 Classical solutions by the mean of space transforms

First we summarize the results of the seminal paper [40] for the problem $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta_{\mathrm{z}}}(1,\mathbf{A},\mathbf{B}))$. In fact, for our coming purpose, we consider a slightly more general problem, that we denote by $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta_{\mathrm{z}},(l,r)}(1,\mathbf{A},\mathbf{B}))$ (with $-\infty \le l < r \le \infty$). It is defined by the following system of equations:

$$\begin{cases} \big[v'_t + \frac{1}{2}\big(\mathbf{A}v'_z\big)'_z + \mathbf{B}\,v'_z - \lambda v\big](t,z) &=& g(t,z) & \forall (t,z) \in [0,T)\times(l,r)\setminus\Delta_{\mathbf{z}} \\ \mathbf{A}(t,z_i+)v'_z(t,z_i+) &=& \mathbf{A}(t,z_i-)v'_z(t,z_i-) & \forall 1\le i\le I,\ \forall t\in[0,T)\ \ (\star) \\ v(T,z) &=& f(z) & \forall z\in(l,r). \\ v(t,l) &=& f_l(t) & \forall t\in[0,T) \\ v(t,r) &=& f_r(t) & \forall t\in[0,T). \end{cases}$$

Here we have $l < z_1 < \ldots < z_I < r$ and we have denoted $\Delta_{\mathrm{z}} = \{(t,z_i) : 0 \le t \le T\}_{i=1}^{I}$. The functions f_l, f_r giving the Dirichlet conditions are in $L^2(0,T)$. Note that the problem $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta_{\mathrm{z}}}(1,\mathbf{A},\mathbf{B}))$ corresponds simply to $l=-\infty$, $r=\infty$ and $f_l = f_r = 0$.

We should precise what we mean by a classical solution $v(t,z)$ of $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta_{\mathrm{z}},(l,r)}(1,\mathbf{A},\mathbf{B}))$. For any compact $K \subset (0,T)\times(l,r)$ this is a function of class $C(K)\cap C^{1,2}(K\setminus\Delta_{\mathrm{z}})$ such that for all $1\le i\le I$ the limits $v'_t(t,z_i\pm)$, $v'_z(t,z_i\pm)$ and $v''_{zz}(t,z_i\pm)$ exist and are continuous as functions of $t\in[0,T)$ (we assume for simplicity that K contains all the z_i's). Then $v(t,z)$ satisfies in particular the first and second line of $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta_{\mathrm{z}},(l,r)}(1,\mathbf{A},\mathbf{B}))$ in the classical sense.

Theorem 1.6.11 (O.A. Ladyzhenskaya et al., [40])**.** *For any $\mathbf{A}\in\Theta(m',M')$ satisfying the $\mathbf{H}^{(x_i)}$ and $\mathbf{H}^{(t)}$-hypotheses, any $\mathbf{B}\in\Xi(M')$ satisfying the $\mathbf{H}^{(t)}$-hypothesis, and provided that g satisfies the $\mathbf{H}^{(t)}$-hypothesis, the parabolic problem $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta_{\mathrm{z}},(l,r)}(1,\mathbf{A},\mathbf{B}))$ has a classical solution $v(t,z)$, that is Hölder continuous (see Remark 1.6.12). Besides the time derivative v'_t is itself Hölder continuous.*

Remark 1.6.12. *Here the Hölder continuity means more precisely that for any compact $K\subset(0,T)\times(l,r)$ we have*

$$\forall (t,x),(s,y)\in K,\quad |v(t,x)-v(s,y)| \le C|(t,x)-(s,y)|^{\nu} \tag{1.6.16}$$

with C,ν positive constants depending on K,m',M'.

Proof of Theorem 1.6.11. See of course [40] and Subsection 5.3 in [30], where we aim at presenting the structure and ideas of the proof of Ladyzhenskaya et al. Basically the idea is to prove that the weak solution of $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta_{\mathrm{z}},(l,r)}(1,\mathbf{A},\mathbf{B}))$, which exists (see [41] and note that our Subsection 1.6.2 presents an alternate proof), is in fact classical. □

Remark 1.6.13. *In [41][40] the authors claim that this is feasible to mimic all the steps of the proof of Theorem 1.6.11 in the case $\rho \neq 1$ (but without writing down the proofs, except for the existence of the weak solution as already mentionned). However, in our opinion, to prove directly that the weak solution $u(t,x)$ is Hölder continuous (or even continuous) presents difficulties in the case $\rho \neq 1$.*

We now aim at proving the following result.

Proposition 1.6.14. *Let $\lambda \geq 0$, a source term $g \in C_c(E)$ and a terminal condition $f \in C_0(\mathbb{R}) \cap L^2(\mathbb{R})$.*

Let $\rho, a \in \Theta(m', M')$ and $B \in \Xi(M')$ for some $0 < m' < M' < \infty$. We assume that ρ, a satisfy the $\mathbf{H}^{(i)}$ and $\mathbf{H}^{(t)}$-hypotheses, and that B and g satisfy the $\mathbf{H}^{(t)}$-hypothesis.

The problem $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta}(\rho, a, B))$ has a classical solution.

Proof of Proposition 1.6.14.

STEP1. The problem $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta}(\rho, a, B))$ has a weak solution $u(t,x)$ (see Subsection 1.6.2). We shall aim at proving that $u(t,x)$ is in fact a classical solution.

In the sequel we (arbitrarily) set $\delta = 1/4$. We denote $\sigma = \sqrt{\rho a}$.

STEP2. We treat in details what happens around the interface $\{(t,1) : 0 \leq t \leq T\}$. We set

$$\phi_1(t,x) = \int_1^x \frac{dy}{\rho(t,y)},$$

$$\mathbf{A}_1(t,z) = \frac{a}{\rho}(t, \Phi_1(t,z)) \tag{1.6.17}$$

and

$$\mathbf{B}_1(t,z) = [(\phi_1)'_{t,\pm} + B(\phi_1)'_{x,\pm}](t, \Phi_1(t,z)), \tag{1.6.18}$$

where $\Phi_1(t,\cdot) = [\phi_1(t,\cdot)]^{-1}$.

We set $z_1 = \inf_{t\in[0,T]} \phi_1(t, 2-\delta)$. We will show that $u(t,x)$ satisfies $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta}(\rho, a, B))$ in the classical sense in the subregion $\{(t,x) \in E : x \leq \Phi_1(t, z_1)\}$.

Note that for any $t \in [0,T]$ we have $\phi_1(t,1) = 0$ and $\Phi_1(t,0) = 1$, and that for any $z \leq z_1$, any $t \in [0,T]$ we have $\Phi_1(t,z) \leq \Phi_1(t,z_1) \leq 2 - \delta$. So that the sole singularity of the coefficients $\mathbf{A}_1(t,z)$ and $\mathbf{B}_1(t,z)$ in the region $\{(t,z) \in E : 0 \leq t \leq T,\, z \leq z_1\}$ is for $z = 0$.

We consider the function $v_1(t,z) = u(t, \Phi_1(t,z))$, $0 \leq t \leq T$, $z \leq z_1$. We claim that this is a weak solution to the problem $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta_0,(-\infty,z_1)}(1, \mathbf{A}_1, \mathbf{B}_1))$ defined by the system of equations

$$\begin{cases} \big[(v_1)'_t + \frac{1}{2}\big(\mathbf{A}_1 (v_1)'_z\big)'_z + \mathbf{B}_1 (v_1)'_z - \lambda v_1\big](t,z) &=& g(t, \Phi_1(t,z)) & \forall (t,z) \in [0,T) \times (-\infty, z_1) \setminus \mathbf{\Delta_0} \\ \mathbf{A}_1(t,0+)(v_1)'_z(t,0+) &=& \mathbf{A}_1(t,0-)(v_1)'_z(t,0-) & \forall t \in [0,T) \;\; (\star) \\ v_1(T,z) &=& f(\Phi_1(T,z)) & \forall z \in (-\infty, z_1) \\ \lim_{z\to-\infty} v_1(t,z) &=& 0 & \forall t \in [0,T) \\ v_1(t,z_1) &=& u(t, \Phi_1(t,z_1)) & \forall t \in [0,T) \end{cases}$$

(here, note that as $u(t,x)$ lives in particular in $L^2(0,T; L^2(\mathbb{R}))$, the function $t \mapsto u(t, \Phi_1(t,z_1))$ is in $L^2(0,T)$, as required for the Dirichlet boundary condition).

Indeed if we suppose formally that $u(t,x)$ is a classical solution to $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta}(\rho, a, B))$ in the subregion $\{(t,x) \in E : x \leq \Phi_1(t,z_1)\}$, then the reader can check by easy computations that $v_1(t,z)$ is a classical solution to $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta_0,(-\infty,z_1)}(1, \mathbf{A}_1, \mathbf{B}_1))$ - below we will detail computations for the converse. But it is possible to show that this is mathematically true at the level of weak solutions, using changes of variable inside the weak formulation (see [30] for details).

But according to the proof of Theorem 1.6.11, the weak solution $v_1(t,z)$ is in fact also a classical solution of $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta_0,(-\infty,z_1)}(1, \mathbf{A}_1, \mathbf{B}_1))$. We draw the consequences on the PDE problem solved by $u(t,x)$

in the classical sense, using again $u(t,x) = v_1(t,\phi_1(t,x))$ and the expression of the classical derivatives (for $t \in [0,T],\ x \le \Phi_1(t,z_1),\ x \neq 1$)

$$u'_x(t,x) = (v_1)'_z(t,\phi_1(t,x))(\phi_1)'_x(t,x) \tag{1.6.19}$$

$$u'_t(t,x) = (v_1)'_t(t,\phi_1(t,x)) + (v_1)'_z(t,\phi_1(t,x))(\phi_1)'_t(t,x) \tag{1.6.20}$$

$$u''_{xx}(t,x) = (v_1)''_{zz}(t,\phi_1(t,x))((\phi_1)'_x)^2(t,x) + (v_1)'_z(t,\phi_1(t,x))(\phi_1)''_{xx}(t,x). \tag{1.6.21}$$

We first identify the transmission condition at the interface $\{(t,1) : 0 \le t \le T\}$. We have, using in particular $(\phi_1)'_x(t,x) = \dfrac{1}{\rho(t,x)}$ and (1.6.19),

$$\begin{aligned} a(t,1+)u'_x(t,1+) &= \mathbf{A}_1(t,0+)\rho(t,1+)u'_x(t,1+) = \mathbf{A}_1(t,0+)(v_1)'_z(t,0+) \\ &= \mathbf{A}_1(t,0-)(v_1)'_z(t,0-) = \mathbf{A}_1(t,0-)\rho(t,1-)u'_x(t,1-) = a(t,1-)u'_x(t,1-). \end{aligned} \tag{1.6.22}$$

Second, for $t \in [0,T)$, $x \le \Phi_1(t,z_1)$, $x \neq 1$, we have

$$\begin{aligned} &\big[u'_t + \frac{\rho}{2}\big(au'_x\big)'_x + B\,u'_x - \lambda u\big](t,x) \\ &= \big[u'_t + \frac{\sigma^2}{2}u''_{xx} + \big(B + \frac{\rho a'_x}{2}\big)\,u'_x - \lambda u\big](t,x) \\ &= (v_1)'_t(t,\phi_1(t,x)) + (v_1)'_z(t,\phi_1(t,x))(\phi_1)'_t(t,x) + \big(B + \tfrac{\rho a'_x}{2}\big)(t,x)(v_1)'_z(t,\phi_1(t,x))(\phi_1)'_x(t,x) \\ &\quad + \tfrac{\sigma^2(t,x)}{2}\Big((v_1)''_{zz}(t,\phi_1(t,x))((\phi_1)'_x)^2(t,x) + v'_z(t,\phi_1(t,x))(\phi_1)''_{xx}(t,x)\Big) - \lambda v_1(t,\phi_1(t,x)) \\ &= (v_1)'_t(t,\phi_1(t,x)) + \tfrac{\sigma^2(t,x)}{2\rho^2(t,x)}(v_1)''_{zz}(t,\phi_1(t,x)) - \lambda v_1(t,\phi_1(t,x)) \\ &\quad + (v_1)'_z(t,\phi_1(t,x))\Big[(\phi_1)'_t(t,x) + \big(B + \tfrac{\rho a'_x}{2}\big)(t,x)(\phi_1)'_x(t,x) + \tfrac{\sigma^2(t,x)}{2}(\phi_1)''_{xx}(t,x)\Big] \\ &= \big[(v_1)'_t + \tfrac{1}{2}(\mathbf{A}_1(v_1)'_z)'_z - \lambda v_1\big](t,\phi_1(t,x)) \\ &\quad + \Big[\,(v_1)'_z\Big((\phi_1)'_t \circ \Phi_1 + \big(B(\phi_1)'_x + \frac{a'_x}{2}\big) \circ \Phi_1 + \frac{\sigma^2(\phi_1)''_{xx}}{2} \circ \Phi_1 - \frac{(\mathbf{A}_1)'_z}{2}\Big)\Big](t,\phi_1(t,x)) \\ &= \big[(v_1)'_t + \tfrac{1}{2}(\mathbf{A}_1(v_1)'_z)'_z + \mathbf{B}_1\,(v_1)'_z - \lambda v_1\big](t,\phi_1(t,x)) = g(t,\Phi_1(t,\phi_1(t,x)) = g(t,x). \end{aligned} \tag{1.6.23}$$

Here we have used

$$(\mathbf{A}_1)'_z(t,z) = \big(\frac{a'_x}{\rho} - \frac{a\rho'_x}{\rho^2}\big)(t,\Phi_1(t,z))\rho(t,\Phi_1(t,z)) = a'_x(t,\Phi_1(t,z)) + \sigma^2(t,\Phi_1(t,z))(\phi_1)''_{xx}(t,\Phi_1(t,z)).$$

In view of (1.6.22) and (1.6.23) we have proved that that $u(t,x)$ satisfies $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta}(\rho,a,B))$ in the classical sense in the subregion $\{(t,x) \in E :\ x \le \Phi_1(t,z_1)\}$ (we can easily that $u(t,x)$ has the required smoothness and satisfies the terminal condition).

STEP3. We repeat Step 2 around each interface $\{(t,i) : 0 \le t \le T\}$, $2 \le i \le I$. More precisely we define for any $2 \le i \le I$

$$\phi_i(t,x) = \int_i^x \frac{dy}{\rho(t,y)},\ \text{and}\ z_{i,\mathrm{d}} = \sup_{t\in[0,T]} \phi_i(t,i-1+\delta). \tag{1.6.24}$$

For $2 \le i \le I-1$ we define $z_i = \inf_{t\in[0,T]} \phi_i(t,i+1-\delta)$. By computations similar to Step 2 we will then prove that $u(t,x)$ satisfies $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta}(\rho,a,B))$ in the classical sense in each of the subregions $\{(t,x) \in E :\ \Phi_i(t,z_{i,\mathrm{d}}) \le x \le \Phi_i(t,z_i)\}$, $2 \le i \le I-1$, and in the region $\{(t,x) \in E :\ \Phi_I(t,z_{I,\mathrm{d}}) \le x\}$.

In particular, at this stage, $u(t,x)$ satisfies the transmission condition $(\star)$ in $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta}(\rho,a,B))$ in the classical sense, at each interface (for $1 \leq i \leq I$).

STEP4. The trouble is that we cannot say for the moment that the first line of $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta}(\rho,a,B))$ holds true in the whole domain $E^{\circ} \setminus \Delta$.

But using again Theorem 1.6.11, in a different manner, we can prove that $u(t,x)$ satisfies $(\mathcal{P}^{\lambda}_{\mathrm{div},\Delta}(\rho,a,B))$ in each of the subregions $\{(t,x) \in E : \, i < x < i+1\}$, $1 \leq i \leq I-1$ (see [30] for details).

Therefore Proposition 1.6.14 is proved.

We now give further properties of the solution $u(t,x)$ considered in Proposition 1.6.14.

Lemma 1.6.15. *In the above context the classical time derivative u'_t is continuous.*

Proof. That u'_t is continuous at any point $(t,x) \notin \Delta$ is clear, by definition of a classical solution. Let $(t,x) \in \Delta$, i.e. we have $(t,x) = (t,i)$ for some $1 \leq i \leq I$. Considering (1.6.20) we have

$$u'_t(t\pm, i\pm) = (v_i)'_t(t\pm, 0\pm) + (v_i)'_z(t\pm, 0\pm)(\phi_i)'_t(t\pm, 0\pm).$$

But by taking the time derivative of (1.6.24), and inverting this derivative and the integral sign, we see that we simply have $(\phi_i)'_t(t\pm, 0\pm) = 0$. And thus

$$u'_t(t\pm, i\pm) = (v_i)'_t(t\pm, 0\pm).$$

But as $(v_i)'_t$ is continuous (Theorem 1.6.11) we see that $u'_t(t\pm, i\pm) = (v_i)'_t(t,0) = u'_t(t,i)$. □

Remark 1.6.16. *Note that the result of Lemma 1.6.15 is true because the interfaces are not moving. In the case of moving interfaces u'_t will not be continuous in general, because there is no reason the second RHS term in* (1.6.8) *vanishes at the interface (contrary to what happens in* (1.6.20)*).*

Conclusion of Section 1.6. In view of Propositions 1.6.3 and 1.6.14, it is now clear that if we have the x_i's and β_i's as in Theorem 1.4.5, $\sigma \in \Theta(m,M)$, $b \in \Xi(M)$, with σ satisfiying the $\mathbf{H}^{(x_i)}$ and $\mathbf{H}^{(t)}$-hypotheses, and b and g satisfying the $\mathbf{H}^{(t)}$-hypothesis, then $(\mathcal{P}^{\lambda}_{\Delta_{\mathbf{x}}}(\sigma,b,\beta))$ has indeed a classical solution (unique thanks to Theorem 1.5.1).

1.7 Markov property, Feller semigroup and generator in the strong sense

We first have the following result.

Proposition 1.7.1. *In the context of Theorem 1.4.5, assume that σ satisfies the $\mathbf{H}^{(x_i)}$ and $\mathbf{H}^{(t)}$-hypotheses and that b satisfies the $\mathbf{H}^{(t)}$-hypothesis.*

Let $X = (X_t)_{t\in[0,T]}$ a weak solution of (1.1.1)*.*

Then X is a Feller time-inhomogeneous $(\mathcal{C}_t)$-Markov process.

Proof. Remember that for any $t \in [0,T]$, $X_t = r(t,Y_t)$ where Y is the solution of (1.3.2) with the coefficients defined by (1.4.17). As these coefficients satisfy the hypotheses of Theorem 1.3.1 we can see from Theorem 6.2.2 in [70] that Y is Markov, as already pointed in Subsection 1.3.1.

Therefore we can easily see that X is Markov and that the associated family $(P^X_{s,t})$ satisfies (1.3.6). Thus the family (P^X_t) (associated to the space time process $\tilde{X}$) satisfies (1.3.7). The only point that requires special attention is to show that (P^X_t) is a Feller semigroup. Indeed, as the coefficients $\bar{\sigma}, \bar{b}$ in (1.3.2) are not smooth, we cannot apply directly Corollary 3.1.2 in [70], to get the Feller property for the family $(P^Y_{s,t})$ associated to Y, and deduce the Feller property for $(P^X_{s,t})$.

Thus we will focus on (P^Y_t), and prove by our means that this is a Feller semigroup. We recall that

$$\forall (s,y) \in E, \;\; \forall \varphi \in C_0(E), \;\; \forall\, 0 \leq t \leq T-s, \quad P^Y_t \varphi(s,y) = P^Y_{s,t+s}\varphi(t+s,y) = \mathbb{E}^{s,y}[\varphi(s+t, Y_{s+t})]. \;\; (1.7.1)$$

Then, one may show that (P^X_t) inherits the Feller property of (P^Y_t). To that aim, one may denote now $\tilde{r}(t,y) = (t, r(t,y))$, $\tilde{R}(t,x) = (t, R(t,x))$, use the relationship

$$\forall (s,x) \in E, \;\; \forall \varphi \in C_0(E), \; \forall t \in [0, T-s], \quad P^X_t \varphi(s,x) = P^Y_t(\varphi \circ \tilde{r})(\tilde{R}(s,x)),$$

the continuity of $r(t,z)$, $R(t,x)$, and $\lim_{y\to\pm\infty} r(t,y) = \pm\infty$, $\lim_{x\to\pm\infty} R(t,x) = \pm\infty$, for any $t \in [0,T]$.

That being said, we now prove that (P_t^Y) is Feller. We denote $\Delta_{\mathbf{y}} = \tilde{R}(\Delta_{\mathbf{x}})$. Note that, thanks to the assumptions on the coefficients, and Proposition 1.6.14, we have that $(\mathcal{P}^{\lambda}_{\Delta_{\mathbf{y}}}(\bar{\sigma},\bar{b},0))$ has a classical solution for any finite time horizon, terminal condition $f \in C_0(\mathbb{R})\cap L^2(\mathbb{R})$, and $g \equiv 0$. Note that $(\mathcal{P}^{\lambda}_{\Delta_{\mathbf{y}}}(\bar{\sigma},\bar{b},0))$ is a parabolic transmission problem with discontinuous coefficients, but with no transmission condition (more precisely the transmission condition is simply of type $u'_y(t,y_i(t)+) = u'_y(t,y_i(t)-)$ for any $t \in [0,T)$).

STEP1. Pick $\varphi \in C_c^{\infty,\infty}(E)$. We will show that $P_t^Y\varphi$ is in $C_0 = C_0(E)$.

a) Let $(s,y) \in E$ fixed. We first show that $P_t^Y\varphi$ is continuous at point (s,y). Let $\delta > 0$. For any $(r,z) \in E$ (we suppose that $t+s, t+r < T$) we have

$$\begin{aligned}|P_t^Y\varphi(s,y) - P_t^Y\varphi(r,z)| \quad &\leq \quad \big|\mathbb{E}^{s,y}[\varphi(t+s,Y_{t+s})] - \mathbb{E}^{r,z}[\varphi(t+s,Y_{t+s})]\,\big| \\ &+\big|\mathbb{E}^{r,z}[\varphi(t+s,Y_{t+s})] - \mathbb{E}^{r,z}[\varphi(t+s,Y_{t+r})]\,\big| + |P^Y_{r,t+r}\varphi(t+s,z) - P^Y_{r,t+r}\varphi(t+r,z)|. \end{aligned} \tag{1.7.2}$$

Note that by virtue of Theorem 1.5.1, for any (r,z) we may regard $\mathbb{E}^{r,z}[\varphi(t+s,Y_{t+s})]$ as $u_{t+s}(r,z)$, where u_{t+s} is the classical solution of the parabolic problem $(\mathcal{P}^0_{\Delta_{\mathbf{y}}}(\bar{\sigma},\bar{b},0))$ (with time horizon $t+s \leq T$), with terminal condition $\varphi(t+s,\cdot) \in C_c^\infty(\mathbb{R}) \subset C_0(\mathbb{R})\cap L^2(\mathbb{R})$ and source term $g \equiv 0$.

As the function u_{t+s} is continuous on E we may find η_1 such that for any (r,z) with $|(s,y)-(r,z)| < \eta_1$ we have

$$\big|\mathbb{E}^{s,y}[\varphi(t+s,Y_{t+s})] - \mathbb{E}^{r,z}[\varphi(t+s,Y_{t+s})]\,\big| < \frac{\delta}{3}.$$

We now turn to the second RHS term in (1.7.2). We have,

$$\big|\mathbb{E}^{r,z}[\varphi(t+s,Y_{t+s})] - \mathbb{E}^{r,z}[\varphi(t+s,Y_{t+r})]\,\big| \leq ||\varphi'_x||_\infty \mathbb{E}^{r,z}|Y_{t+s} - Y_{t+r}|.$$

Further, we have

$$\mathbb{E}^{r,z}|Y_{t+s} - Y_{t+r}|^2 \leq 4\big(\mathbb{E}^{r,z}\big|\int_{t+s}^{t+r}\bar{\sigma}(u,Y_u)dW_u\big|^2 + \mathbb{E}^{r,z}\big|\int_{t+s}^{t+r}\bar{b}(u,Y_u)du\big|^2\big) \leq 4\bar{M}^2(|r-s| + |r-s|^2),$$

where we have used $|a+b|^2 \leq 4(|a|^2+|b|^2)$ and the fact that $\bar{\sigma},\bar{b} \in \theta(\bar{m},\bar{M})$. Thus by Jensen inequality we see that

$$\mathbb{E}^{r,z}|Y_{t+s} - Y_{t+r}| \leq C(T)|r-s|^{1/2}.$$

To sum up we may find $\eta_2 > 0$ such that for any $|(s,y)-(r,z)| < \eta_1 \wedge \eta_2$ we have

$$\big|\mathbb{E}^{r,z}[\varphi(t+s,Y_{t+s})] - \mathbb{E}^{r,z}[\varphi(t+s,Y_{t+r})]\,\big| < \frac{\delta}{3}.$$

To finish with, we turn to the third RHS term in (1.7.2). It is clear that we have

$$|P^Y_{r,t+r}\varphi(t+s,z) - P^Y_{r,t+r}\varphi(t+r,z)| \leq ||\varphi(t+s,\cdot) - \varphi(t+r,\cdot)||_\infty \leq ||\varphi'_t||_\infty\,|r-s|,$$

so that we may find $\eta_3 > 0$ such that for any $|(r,z)-(s,x)| < \eta_3$ we have

$$|P^Y_{r,t+r}\varphi(t+s,z) - P^Y_{r,t+r}\varphi(t+r,z)| < \frac{\delta}{3}.$$

Thus, setting $\eta = \eta_1 \wedge \eta_2 \wedge \eta_3$, we have

$$|P_t^Y\varphi(s,x) - P_t^Y\varphi(r,z)| < \delta$$

for any $|(r,z)-(s,x)| < \eta$. Therefore the continuity of $P_t^Y\varphi$ is established.

b) We now show that $\lim_{|y|\to\infty} P_t^Y\varphi(s,y) \to 0$ (for any $s \in [0,T]$). Again we may see $P_t^Y\varphi(s,\cdot)$ as the solution $u_{t+s}(s,\cdot)$ (at time $s \in [0,t+s]$) of $(\mathcal{P}^0_{\Delta_{\mathbf{y}}}(\bar{\sigma},\bar{b},0))$ with terminal condition $\varphi(t+s,\cdot)$ (again time horizon is $t+s$ and the source term is zero). The result then follows from the boundary condition in problem $(\mathcal{P}^0_{\Delta_{\mathbf{y}}}(\bar{\sigma},\bar{b},0))$.

STEP2. Pick $\varphi \in C_0$. We may construct a sequence (φ_n) in $C_c^{\infty,\infty}(E)$ such that $||\varphi_n - \varphi||_\infty \to 0$ as $n \to \infty$. As $||P_t^Y f||_\infty \leq ||f||_\infty$ for any $f \in C_b(E)$, we get $||P_t^Y \varphi - P_t^Y \varphi_n||_\infty \leq ||\varphi - \varphi_n||_\infty$, and we see that the sequence $(P_t^Y \varphi_n)$ in $C_b(E)$ converges uniformly to $P_t^Y \varphi$. Therefore $P_t^Y \varphi$ is in C_0, as each $P_t^Y \varphi_n$ is in C_0 by Step 1. This shows that for any $t \in [0,T]$, $P_t^Y C_0 \subset C_0$.

STEP3. Let $(s,y) \in E$ and $\varphi \in C_0$. From (1.7.1) and the continuity of Y, we easily see by dominated convergence that $P_t^Y \varphi(s,y) \to \varphi(s,y)$ as $t \downarrow 0$. Using this and the conclusion of Step 2, we deduce from Proposition III.2.4 in [64] that (P_t^Y) is a Feller semigroup. □

Therefore the corresponding space-time process $\tilde{X} = ((t, X_t))_{t\in[0,T]}$ is a E-valued Feller homogeneous $(\mathcal{C}_t)$-Markov process (cf Subsection 1.3.4). We wish to identify the infinitesimal generator of $\tilde{X}$. For technical reasons we only treat the case $\Delta_{\mathbf{x}} = \Delta$ (see Remark 1.7.4). To that aim we have to introduce further notations.

With the same assumptions on the coefficients β_i's as in Theorem 1.4.5, we define

$$\mathcal{S}^X = \Big\{ \varphi \in C(E) \cap C^{1,2}(E \setminus \Delta) : \text{ with } \varphi(T,\cdot) = 0, \quad t \mapsto \varphi'_t(t,i) \text{ is continuous on } [0,T),$$

$$\forall (t,i) \in \Delta, \ \ \varphi'_t(t,i\pm) = \varphi'_t(t,i) \text{ and } \varphi'_x(t,i\pm) \text{ and } \varphi''_{xx}(t,i\pm) \text{ exist with}$$

$$\tfrac{\sigma^2}{2}(t,i+)\varphi''_{xx}(t,i+) + b(t,x+)\varphi'_x(t,i+) = \tfrac{\sigma^2}{2}(t,i-)\varphi''_{xx}(t,i-) + b(t,i-)\varphi'_x(t,i-).$$

$$\text{Besides } \ (1+\beta_i(t))\varphi'_x(t,i+) = (1-\beta_i(t))\varphi'_x(t,i-) \ \ \forall 1 \leq i \leq I, \forall t \in [0,T) \ \ (\star)$$

$$\forall 1 \leq i \leq I, \quad \varphi'_x(t,i\pm) \text{ and } \varphi''_{xx}(t,i\pm) \text{ are continuous functions of } t \in [0,T)$$

$$\text{and } \lim_{|x|\to\infty} \big(\varphi'_t(t,x) + \tfrac{1}{2}\sigma^2(t,x)\varphi''_{xx}(t,x) + b(t,x)\varphi'_x(t,x)\big) = 0 \quad \forall t \in [0,T] \Big\}.$$

For any $\varphi \in \mathcal{S}^X$ we define $L^X\varphi$ by

$$\begin{aligned}
\forall (t,x) \in E \setminus \Delta, \quad L^X\varphi(t,x) &= \varphi'_t(t,x) + \tfrac{1}{2}\sigma^2(t,x)\varphi''_{xx}(t,x) + b(t,x)\varphi'_x(t,x) \\
\forall (t,i) \in \Delta, \qquad L^X\varphi(t,i) &= \varphi'_t(t,i) + \tfrac{\sigma^2}{2}(t,i+)\varphi''_{xx}(t,i+) + b(t,i+)\varphi'_x(t,i+) \\
&= \varphi'_t(t,i) + \tfrac{\sigma^2}{2}(t,i-)\varphi''_{xx}(t,i-) + b(t,i-)\varphi'_x(t,i-).
\end{aligned}$$

We will have the following result.

Theorem 1.7.2. *Assume $\Delta_{\mathbf{x}} = \Delta$. In the context of Proposition 1.7.1 let $X = (X_t)_{t\in[0,T]}$ the solution of* (1.1.1).
We then denote by $(\mathcal{L}^X, D(\mathcal{L}^X))$ the infinitesimal generator of the Feller space-time process $\tilde{X}$.
Then the operator $(\mathcal{L}^X, D(\mathcal{L}^X))$ is the closure of $(L^X, \mathcal{S}^X)$.

Remark 1.7.3. *Note that the condition $\varphi(T,\cdot) = 0$ in the definition of $\mathcal{S}^X$ is here because we already know that the functions φ in $D(\mathcal{L}^X)$ have to satisfy $\varphi(T,\cdot) = 0$. Indeed, as we have set $P_t^X\varphi(s,x) = 0$ for $t + s > T$, this is needed in order to have the existence of the limit in* (1.3.9) *for $s = T$. This is somehow the same issue than in the definition of the domain $D(\Lambda, \mathcal{V}')$ in Lemma 1.6.6.*

Proof of Theorem 1.7.2. Take $\varphi \in \mathcal{S}^X \subset C_0$ and notice that $L^X\varphi$ is in C_0. Then, using Proposition 1.4.1, Equation (1.4.7) and condition $(\star)$, we have for any $0 \leq s \leq t \leq T$,

$$\varphi(\tilde{X}_t) - \varphi(\tilde{X}_s) - \int_s^t L^X\varphi(\tilde{X}_u)du = \int_s^t \varphi'_{x,\pm}(u, X_u)\sigma(u, X_u)dW_u.$$

The above t-indexed process being a martingale we see by Proposition 1.3.11 that $\mathcal{S}^X \subset D(\mathcal{L}^X)$ and that $\mathcal{L}^X$ coincides with L^X on $\mathcal{S}^X$.

We shall now prove that the closure of $(L^X, \mathcal{S}^X)$ is the generator of a Feller semigroup on C_0. Indeed the result will then follow from Exercise VII.1.18 in [64] (note that in the language of [18] we have $(L^X, \mathcal{S}^X) \subset (\mathcal{L}^X, D(\mathcal{L}^X))$, and that $(\mathcal{L}^X, D(\mathcal{L}^X))$ is closed, see Proposition VII.1.3 in [64]).

The idea is to apply Theorem 1.2.12 in [18], which is an Hille-Yosida type theorem, in the Banach space C_0 (see also their Theorem 4.2.2).

STEP1. Let $g \in C_c^{1,0}(E) \subset C_0$, and $\lambda > 0$. The equation

$$\lambda u - L^X u = -g \tag{1.7.3}$$

with terminal condition $u(T, \cdot) = 0$ and with $\lim_{|x|\to\infty} u(t,x) = 0$, has a classical solution $u(t,x)$ satisfying $(\star)$, living in $C_0(E) \cap C^{1,2}(E \setminus \Delta)$, and satisfying all the other requirements for being in $\mathcal{S}^X$, thanks to the results of Subsection 1.6.3 (see in particular Proposition 1.6.14, Lemma 1.6.15 and the Conclusion). Note in particular that as $(L^X u)(t,x) = (g + \lambda u)(t,x)$ for x great enough, and as $g \in C_c(E)$ and $u \in C_0(E)$, we clearly have that $(L^X u)(t,x) \to 0$ as $x \to \infty$ (for any $t \in [0,T]$).

Remember that $C_c^{1,0}(E)$ is dense in C_0. Thus, denoting by $\mathcal{R}(\lambda I - L^X)$ the image of $\mathcal{S}^X$ by the operator $\lambda I - L^X$, we have

$$C_c^{1,0}(E) \subset \mathcal{R}(\lambda I - L^X) \subset C_0,$$

and taking closures we see that $\mathcal{R}(\lambda I - L^X)$ is dense in C_0.

STEP2. The domain $\mathcal{S}^X$ is obviously dense in C_0.

STEP3. We show now that $(L^X, \mathcal{S}^X)$ is dissipative. Let $\lambda > 0$ and pick $\varphi \in \mathcal{S}^X$.

a) Assume φ reaches a positive maximum at a point $(t_0, x_0) \in [0,T) \times \mathbb{R}$.

If $(t_0, x_0) \notin \Delta$ it is clear that $\varphi'_t(t_0,x_0) \leq 0$, $\varphi'_x(t_0,x_0) = 0$ and $\varphi''_{xx}(t_0,x_0) \leq 0$, thus $L^X\varphi(t_0,x_0) \leq 0$.

If $(t_0, x_0) \in \Delta$ (i.e. $x_0 = i$ for some $1 \leq i \leq I$) things are not so clear because of the lack of smoothness of φ on Δ. But because $(1 + \beta_{i_0}(t_0)), (1 - \beta_{i_0}(t_0)) > 0$, $\varphi'_x(t_0, x_0+)$ and $\varphi'_x(t_0, x_0-)$ share the same sign and this implies $\varphi'_x(t_0, x_0\pm) = 0$.

Let us now prove that $\varphi'_t(t_0,x_0) \leq 0$. Indeed, since $t \mapsto \varphi(t, x_0)$ is a C^1 function, we may apply the mean value theorem ensuring that for $h > 0$ there exists $\theta \in (0,1)$ such that $\frac{1}{h}(\varphi(t_0 + h, x_0) - \varphi(t_0, x_0)) = \varphi'_t(t_0,x_0) + (\varphi'_t(t_0 + \theta h, x_0) - \varphi'_t(t_0,x_0))$. Now, since φ reaches a positive maximum at a point $(t_0, x_0) \in [0,T) \times \mathbb{R}$, the left hand side of the equality is negative. Then, letting h tend to zero in the right hand side ensures that necessarily $\varphi'_t(t_0,x_0) \leq 0$.

Therefore we must have $\varphi''_{xx}(t_0, x_0\pm) \leq 0$ and consequently $L^X\varphi(t_0,x_0) \leq 0$.

Thus we have

$$||\lambda\varphi - L^X\varphi||_\infty \geq \lambda\varphi(t_0,x_0) - L^X\varphi(t_0,x_0) \geq \lambda\varphi(t_0,x_0) = \lambda||\varphi||_\infty.$$

b) Assume now φ reaches a positive maximum at a point (T, x_0), $x_0 \in \mathbb{R}$, therefore this positive maximum is in fact zero. Thus, either φ is the null function and we have automatically $\lambda||\varphi||_\infty \leq ||\lambda\varphi - L^X\varphi||_\infty$. Either this is not the case and φ reaches a strictly negative minimum on $[0,T) \times \mathbb{R}$. Thus considering $-\varphi$ and applying Subset a) we get the desired inequality.

c) If it is $-\varphi$ that reaches a positive maximum, we may repeat Substeps a)-b) to get $\lambda||\varphi||_\infty \leq ||\lambda\varphi - L^X\varphi||_\infty$.

STEP4. We apply Theorem 1.2.12 in [18] to see that the closure of $(L^X, \mathcal{S}^X)$ generates a strongly continuous, contraction semigroup (T_t) on C_0.

STEP5. It remains to see that (T_t) is positive, but this can be accomplished in the same manner than in the proof of Theorem 4.2.2 in [18] (note that (T_t) is conservative, thanks to Proposition III.2.2 in [64]). □

Remark 1.7.4. *In fact, if we do not have $\Delta_{\mathbf{x}} = \Delta$, to prove that $\varphi'_t(t_0,x_0) \leq 0$ in Step 3-b) (case $(t_0,x_0) \in \Delta$) seems more difficult. Besides, note that we would have to define the domain $\mathcal{S}^X$ in a different manner, as we would no more have the continuity of u'_t for u solving the resolvent equation* (1.7.3) *(see Remark 1.6.16).*

Chapter 2

The special case of the time-inhomogeneous Skew Brownian Motion

This Chapter is based on the paper *On the existence of a time inhomogeneous skew Brownian motion and some related laws* ([27]), written with Miguel Martinez.

Its structure differs slightly from the one of [27], as we can now rely on some results of Chapter 1. In fact we want to highlight the results of [27] that are not contained in Chapter 1 (and therefore in [30]; note however that [30] is posterior to [27]).

2.1 Introduction

Consider the equation

$$dB_t^\beta = W_t + \beta(t)dL_t^0(B^\beta), \quad t \in [0,T], \quad B_0^\beta = x_0. \tag{2.1.1}$$

If $\beta : [0,T] \to [k,\kappa]$ $(-1 < k \leq \kappa < 1)$ is of class C^1, with $|\beta'(t)| \leq M$ for any $t \in [0,T]$, we know from Chapter 1 that (2.1.1) possesses a unique strong solution B^β. We call this solution the time-Inhomogeneous Brownian Motion (ISBM).

The ISBM appears in a seminal paper by S. Weinryb ([78]), where a pathwise uniqueness property for (2.1.1) is proved. To prove this result no special smoothness of β is required: it is only assumed to be a deterministic function taking values in $[-1,1]$ (see Remark 2.2.3).

But however, concerning the existence of weak solutions to (2.1.1), and although in [78] it is said that *"partial existence results were obtained by Watanabe [76, 77]"*, we could find no satisfactory reference, by the time we were working on [27] (note that this was before working on [30]; note the recent paper [14])). Thus, in [27], apart from studying some laws related to the ISBM, we aimed at giving a positive answer to the existence of solutions to (2.1.1), under the general assumption that β is measurable and takes values in $[-1,1]$ (we are then outside the assumptions of [30] and Chapter 1).

In fact, as the process B^β behaves like a Brownian motion that is (in a time-inhomogeneous fashion) perturbed by the local time term, we can resort on known laws about the Brownian motion, in order to study B^β. Contrary to what happens in Chapter 1, where we resort to general stochastic calculus arguments that need some smoothness assumptions, and where nothing is explicit.

In this chapter we give roughly speaking three type of results: the already mentioned existence result for the solutions of (2.1.1), with a non smooth function β; the explicit transition probability density of the process B^β; the explicit trivariate density of the process B^β, its local time and its last exit time from point zero.

Organization of the chapter. In Section 2.2 we fix some notations and gather some preliminary material. In Section 2.3 we define a transition function that will reveal itself to be the one of the ISBM in the sequel. In Section 2.4 we state our main results. In Section 2.5 we give a first proof of the existence

of the ISBM: we construct in a canonical way a Markov process with transition function the one given in Section 2.3, and show that this is weak a solution of (2.1.1). The starting point of this proof will be provided by the results of Chapter 1. In Section 2.6 we prove the expression of the trivariate density of the process B^β, its local time and its last exit time from point zero. In Section 2.7 we evoke another construction of the ISBM that was given in [27], using Brownian excursions.

2.2 Notations and preliminaries

To be consistent with the study in [27], we work in a infinite time horizon context (that is to say, from now on $t \in \mathbb{R}_+$ replaces $t \in [0,T]$ in (2.1.1)).

In the following W is a standard Brownian motion defined on $(C, \mathcal{C}, \mathbb{P})$, where $\mathbb{P}$ is the Wiener measure.

Throughout the chapter, $\mathbf{e}$ denotes the exponential law of parameter 1 ; Arcsin is the standard arcsin law with density $(\pi\sqrt{y(1-y)})^{-1}$ on $[0,1]$; $\mathcal{R}(p)$ denotes the Rademacher law with p parameter *i.e.* the law of random variable Y taking values $\{-1,+1\}$ with $\mathbb{P}(Y=1) = p = 1 - \mathbb{P}(Y=-1)$.

We denote for all $t \geq 0$,

$$G_t := \sup\{s < t \ : \ |W_s| = 0\} \quad \text{and} \quad G_t^\beta := \sup\{s < t \ : \ |B_s^\beta| = 0\}.$$

It is well known that $G_1 \overset{\mathcal{L}}{\sim}$ Arcsin (see [64] Chap. III, Exercise 3.20). We will also denote for all $0 \leq u \leq 1$,

$$M_u := \left|W_{G_1+u(1-G_1)}\right| / \sqrt{1-G_1} \quad \text{and} \quad M_u^\beta := \left|B^\beta_{G_1^\beta+u(1-G_1^\beta)}\right| / \sqrt{1-G_1^\beta}.$$

The process M is called the *Brownian Meander* of length 1. It is well known that $M_1 \overset{\mathcal{L}}{\sim} \sqrt{2\mathbf{e}}$ (see [64], Chap. XII, Exercise 3.8).

As in Chapter 1, the expectation $\mathbb{E}^x$ (resp. $\mathbb{E}^{s,x}$) refers to the probability measure $\mathbb{P}^x := \mathbb{P}(\,\cdot\,|B_0^\beta = x)$ (resp. $\mathbb{P}^{s,x} := \mathbb{P}(\,\cdot\,|B_s^\beta = x)$).

Let $\beta : \mathbb{R}^+ \to [-1,1]$ a Borel function. The following fundamental facts are the key of many considerations of this paper.

Proposition 2.2.1. *(see [78] or [64] Chap. VI Exercise 2.24 p. 246) Assume* (2.1.1) *has a weak solution B^β. Then under $\mathbb{P}^0$,*

$$(|B_t^\beta|)_{t\geq 0} \overset{\mathcal{L}}{\sim} (|W_t|)_{t\geq 0}\,.$$

We recall below the main result of [78].

Theorem 2.2.2. *(see [78] or [64] Chap. VI Exercise 2.24 p. 246)*
Pathwise uniqueness holds for the weak solutions of equation (2.1.1).

Remark 2.2.3. *In the introductory article [78], it is shown that there is pathwise uniqueness for equation (2.1.1) but with a slight modification : in [78] the local time appearing in the equation is the standard right sided local time, so that the function β is supposed to take values in $(-\infty, 1/2]$. Still, all the results of [78] may be easily adapted for the case where $L^0_\cdot(B^\beta)$ stands for the symmetric local time at 0. We leave these technical aspects to the reader.*

As $L_1^0(B^\beta)$, M_1^β and G_1^β (resp. $L_1^0(W)$, M_1 and G_1) are measurable functions of the trajectories of $|B^\beta|$ (resp. $|W|$), we get immediately the following corollary.

Corollary 2.2.4. *We have*

$$(|B_1^\beta|, L_1^0(B^\beta), G_1^\beta, M_1^\beta) \overset{\mathcal{L}}{\sim} (|W_1|, L_1^0(W), G_1, M_1).$$

The following known trivariate density will play a crucial role.

Proposition 2.2.5. *i) We have*

$$(|W_1|, L_1^0(W), G_1) = (\sqrt{1-G_1}M_1, \sqrt{G_1}l^0, G_1), \tag{2.2.1}$$

where $l^0 \overset{\mathcal{L}}{\sim} \sqrt{2\mathbf{e}}$*, and with* G_1, M_1, l^0 *independent.*

ii) As a consequence, for all $t, s > 0$*, and all* $\ell, x \geq 0$*, the image measure* $\mathbb{P}^0[|W_t| \in dx, L_t^0(W) \in d\ell, G_t \in ds]$ *is given by*

$$\mathbf{1}_{s\leq t}\sqrt{\frac{2}{\pi s^3}}\ell \exp\Big(-\frac{\ell^2}{2s}\Big)\frac{x}{\sqrt{2\pi(t-s)^3}}\exp\left(-\frac{x^2}{2(t-s)}\right) ds\, d\ell\, dx. \tag{2.2.2}$$

Proof. See [64], Chap. XII, Exercise 3.8. □

Remark 2.2.6. *Note that, by integrating* (2.2.2) *with respect to* ℓ*, and using a symmetry argument we get that*

$$p(t,0,y) = \frac{|y|}{2\pi}\int_0^t \frac{1}{\sqrt{s}(t-s)^{3/2}}\exp\left(-\frac{y^2}{2(t-s)}\right) ds, \tag{2.2.3}$$

where $p(t,x,y) := \frac{1}{\sqrt{2\pi t}}\exp\big(-\frac{(y-x)^2}{2t}\big)$ *is the transition density of a Brownian motion.*

2.3 Transition probability density

All through the chapter the transition probability density of B^β will be denoted $p^\beta(s,t;x,y)$ (we show that it exists).

Let us now give the analytical expression of the function $p^\beta(s,t;x,y)$. It will be shown later (Subsection 2.5.2) that $p^\beta(s,t;x,y)$ is a transition probability function (t.f.) (in particular it satisfies the Chapman-Kolmogorov equations), and that the existing strong solution B^β of (2.1.1) is indeed an inhomogeneous Markov process with transition function $p^\beta(s,t;x,y)$.

Definition 2.3.1. *For all* $t > 0, y \in \mathbb{R}$*, we set*

$$p^\beta(0,t;0,y) := \frac{|y|}{\pi}\int_0^t \frac{1+\mathrm{sgn}(y)\beta(s)}{2}\frac{1}{\sqrt{s}(t-s)^{3/2}}\exp\left(-\frac{y^2}{2(t-s)}\right) ds. \tag{2.3.1}$$

Let us now introduce the shift operator (σ_t) acting on time dependent functions as follows:

$$\beta\circ\sigma_t(s) = \beta(t+s).$$

Assume for a moment that (2.1.1) has a solution B^β which enjoys the strong Markov property and satisfies

$$\mathbb{P}^0(B_t^\beta \in dy) = p^\beta(0,t;0,y)dy.$$

Let $x \neq 0$ be the starting point of B^β at time s. Let

$$\tau_0 := \inf(t \geq s \ :\ B_t^\beta = 0).$$

Since the local time $L_\cdot^0(B^\beta)$ does not increase until B^β reaches 0, the process B^β, heuristically speaking, behaves like a Brownian motion on time interval (s,τ_0), implying that $\mathbb{P}^{s,x}(\tau_0 \in du) = |x|\exp(-x^2/2(u-s))/\sqrt{2\pi(u-s)^3}$. Then it starts afresh from zero, behaving like an ISBM. Thus, for $t > s$,

$$\begin{aligned}
\mathbb{P}^{s,x}(B_t^\beta \in dy) &= \mathbb{P}^{s,x}(B_t^\beta \in dy\,;\, s \leq \tau_0 \leq t) + \mathbb{P}^{s,x}(B_t^\beta \in dy\,;\, \tau_0 > t) \\
&= dy\int_0^{t-s}\frac{|x|\mathrm{e}^{-x^2/2u}}{\sqrt{2\pi u^3}}p^{\beta\circ\sigma_s\circ\sigma_u}(0,t-s-u;0,y)du \\
&\quad + \frac{1}{\sqrt{2\pi(t-s)}}\left[\exp\left(-\frac{(y-x)^2}{2(t-s)}\right) - \exp\left(-\frac{(y+x)^2}{2(t-s)}\right)\right]\mathbf{1}xy > 0.
\end{aligned} \tag{2.3.2}$$

The second line is a consequence of the assumed strong Markov property, while the third line is a consequence of the reflection principle due to the fact that on the event $\{\tau_0 > t\}$ the process B^β behaves like a Brownian motion.

But using (2.3.1), a Fubini-Tonelli argument, a change of variable, and (2.2.3), we get :

$$\begin{aligned}
&\int_0^{t-s} \frac{|x|\mathrm{e}^{-x^2/u}}{\sqrt{2\pi u^3}} p^{\beta\circ\sigma_s\circ\sigma_u}(0,t-s-u;0,y)du \\
&= \int_{u=0}^{t-s}\int_{r=0}^{u} \frac{1+\mathrm{sgn}(y)\beta\circ\sigma_s(u)}{2}\sqrt{\frac{2}{\pi}}\frac{|y|}{(t-(s+u))^{3/2}} e^{-\frac{y^2}{2(t-(s+u))}} \frac{|x|}{2\pi\sqrt{r}(u-r)^{3/2}} e^{-\frac{x^2}{2(u-r)}} dr\, du \\
&= \int_0^{t-s} \frac{1+\mathrm{sgn}(y)\beta\circ\sigma_s(u)}{2}\frac{|y|}{\pi}\frac{e^{-\frac{y^2}{2(t-(s+u))}}}{\sqrt{u}(t-s-u)^{3/2}} e^{-x^2/2u} du.
\end{aligned} \tag{2.3.3}$$

This leads us to the following definition.

Definition 2.3.2. *For $t > s$, $x, y \in \mathbb{R}$, we set*

$$\begin{aligned}
p^\beta(s,t;x,y) \;:=\; &\int_0^{t-s} \frac{1+\mathrm{sgn}(y)\beta\circ\sigma_s(u)}{2}\frac{|y|}{\pi}\frac{e^{-\frac{y^2}{2(t-(s+u))}}}{\sqrt{u}(t-s-u)^{3/2}} e^{-x^2/2u} du \\
&+ \frac{1}{\sqrt{2\pi(t-s)}}\left[\exp\left(-\frac{(y-x)^2}{2(t-s)}\right) - \exp\left(-\frac{(y+x)^2}{2(t-s)}\right)\right]\mathbf{1}xy > 0.
\end{aligned} \tag{2.3.4}$$

Remark 2.3.3. *Note that in the case of Brownian motion ($\beta \equiv 0$) we have :*

$$p(t,x,y) = \int_0^t \frac{|y|}{2\pi}\frac{e^{-\frac{y^2}{2(t-u)}} e^{-\frac{x^2}{2u}}}{\sqrt{u}(t-u)^{3/2}} du + \frac{1}{\sqrt{2\pi t}}\left[\exp\left(-\frac{(y-x)^2}{2t}\right) - \exp\left(-\frac{(y+x)^2}{2t}\right)\right]\mathbf{1}xy > 0. \tag{2.3.5}$$

Thus, considering (2.3.4)*,*

$$p^\beta(s,t;x,y) = p(t-s,x,y) + \int_0^{t-s} \frac{\beta\circ\sigma_s(u)}{2}\frac{y}{\pi}\frac{e^{-\frac{y^2}{2(t-(s+u))}}}{\sqrt{u}(t-s-u)^{3/2}} e^{-x^2/2u} du. \tag{2.3.6}$$

This will be useful in forthcoming computations.

Remark 2.3.4. *When $\beta(s) \equiv \beta$ is constant, $p^\beta(s,t;x,y)$ is just the transition density of the SBM given for example in [64].*

2.4 Main results

We now state the main results of the chapter.

Proposition 2.4.1. *Let B^β a weak solution of* (2.1.1)*.*

For all $t > 0, y \in \mathbb{R}$, we have

$$\mathbb{P}^0(B_t^\beta \in dy) = p^\beta(0,t;0,y)dy,$$

where the function $p^\beta(0,t;0,y)$ is explicit in Definition 2.3.1.

Theorem 2.4.2. *Let $\beta : \mathbb{R}^+ \to [-1,1]$ a Borel function and W a standard Brownian motion. For any fixed $x \in \mathbb{R}$, there exists a unique (strong) solution to* (2.1.1)*. It is a (strong) Markov process with transition function $p^\beta(s,t;x,y)$ given by Definition 2.3.2.*

Still, a (very) little more work allows to retrieve the law of $(B_t^\beta, L_t^0(B^\beta), G_t^\beta)$ under $\mathbb{P}^0$.

Theorem 2.4.3. *For all $t, s > 0$, all $\ell \geq 0$ and all $x \in \mathbb{R}$, the image measure $\mathbb{P}^0[B_t^\beta \in dx, L_t^0(B^\beta) \in d\ell, G_t^\beta \in ds]$ is given by*

$$\mathbf{1}_{s\leq t}\frac{1+\mathrm{sgn}(x)\beta(s)}{2}\sqrt{\frac{2}{\pi s^3}}\,\ell\exp\Big(-\frac{\ell^2}{2s}\Big)\frac{|x|}{\sqrt{2\pi(t-s)^3}}\exp\left(-\frac{x^2}{2(t-s)}\right) ds\, d\ell\, dx. \tag{2.4.1}$$

2.5 A first proof of the existence of the ISBM

2.5.1 Law of the ISBM at a fixed time : proof of Proposition 2.4.1

Let $\beta : \mathbb{R}^+ \to [-1,1]$ a Borel function and assume (2.1.1) has a weak solution B^β (it will be shown in the next subsection that this is indeed the case).

In this part, we note $g_{t,x}(\lambda) := \mathbb{E}^x \exp\left(i\lambda B_t^\beta\right)$ the Fourier transform of B_t^β starting from x and $h_x(t) := \mathbb{E}^x \int_0^t \beta(s) dL_s^0(B^\beta)$. We start with two lemmas.

Lemma 2.5.1. *We have* $h_0(t) = \frac{1}{\sqrt{2\pi}} \int_0^t \frac{\beta(s)}{\sqrt{s}} ds$.

Proof. Using the symmetric Tanaka formula and Proposition 2.2.1 we get $\mathbb{E}^0(L_t^0(B^\beta)) = \mathbb{E}^0|B_t^\beta| = \mathbb{E}^0|W_t| = \sqrt{\frac{2}{\pi}}\sqrt{t}$. Consequently, we may apply Fubini's theorem and we get that, $h_0(t) = \mathbb{E}^0 \int_0^t \beta(s) dL_s^0(B^\beta) = \int_0^t \beta(s) d\left(\mathbb{E}^0 L_s^0(B^\beta)\right) = \frac{1}{\sqrt{2\pi}} \int_0^t \frac{\beta(s)}{\sqrt{s}} ds$. □

Lemma 2.5.2. *We have for all* $\lambda > 0$ *and* $t > 0$,

$$g_{t,0}(\lambda) = e^{-\lambda^2 t/2} \left(1 + \frac{i\,\lambda}{\sqrt{2\pi}} \int_0^t \frac{\beta(s)}{\sqrt{s}} e^{\lambda^2 s/2} ds\right).$$

Proof. Applying Itô's formula ensures that for any fixed $\lambda \in \mathbb{R}$ the process $(g_{t,x}(\lambda))_{t\geq 0}$ is solution of the first order differential equation :

$$g_{t,x}(\lambda) = e^{i\lambda x} - \frac{\lambda^2}{2} \int_0^t g_s(\lambda) ds + i\lambda h_x(t),$$

(see [78] or [64] Chap. VI Exercise 2.24 p. 246). Solving formally this equation, we find that for any fixed $\lambda > 0$:

$$g_{t,x}(\lambda) = e^{-\lambda^2 t/2} \left(e^{i\lambda x} + i\lambda h_x(t) e^{\lambda^2 t/2} - \frac{i\lambda^3}{2} \int_0^t h_x(s) e^{\lambda^2 s/2} ds\right). \tag{2.5.1}$$

Integrating by part, taking $x = 0$ and using Lemma 2.5.1 we get the announced result. □

Proof of Proposition 2.4.1. In the following computations we note $\widehat{\mathcal{F}}^{-1}(g)(z) := 2\pi \int_{\mathbb{R}} g(\lambda) e^{-iz\lambda} d\lambda$ the inverse Fourier transform of a function g. We will sometimes write $\widehat{\mathcal{F}}^{-1}(g(\lambda))(z)$ to make the dependence of g with respect to λ explicit.

We have for $y \in \mathbb{R}$,

$$\begin{aligned}
p^\beta(0,t;0,y) &= \widehat{\mathcal{F}}^{-1}(g_{t,0})(y) \\
&= \widehat{\mathcal{F}}^{-1}(e^{-\lambda^2 t/2})(y) + 2\pi \int_{\mathbb{R}} \frac{i\lambda}{\sqrt{2\pi}} \Big(\int_0^t \frac{\beta(s) e^{-\lambda^2(t-s)/2}}{\sqrt{s}} ds \Big) e^{-iy\lambda} d\lambda \\
&= p(t,0,y) + \frac{1}{\sqrt{2\pi}} \int_0^t \beta(s) 2\pi \Big(\int_{\mathbb{R}} i\lambda \frac{e^{(i\lambda)^2(t-s)/2}}{\sqrt{s}} e^{-iy\lambda} d\lambda \Big) ds \\
&= p(t,0,y) + \frac{1}{\sqrt{2\pi}} \int_0^t \beta(s) \widehat{\mathcal{F}}^{-1}\Big(i\lambda (t-s) \frac{e^{(i\lambda)^2(t-s)/2}}{\sqrt{s}(t-s)} \Big)(y) ds \\
&= p(t,0,y) + \frac{1}{\sqrt{2\pi}} \int_0^t \beta(s) \widehat{\mathcal{F}}^{-1}\Big(\frac{d}{d\lambda} \Big(\frac{e^{(i\lambda)^2(t-s)/2}}{\sqrt{s}(t-s)} \Big) \Big)(y) ds \\
&= p(t,0,y) + \frac{1}{\sqrt{2\pi}} \int_0^t y\beta(s) \widehat{\mathcal{F}}^{-1}\Big(\frac{e^{(i\lambda)^2(t-s)/2}}{\sqrt{s}(t-s)} \Big)(y) ds \\
&= p(t,0,y) + \frac{y}{\sqrt{2\pi}} \int_0^t \frac{\beta(s)}{\sqrt{s}(t-s)} \widehat{\mathcal{F}}^{-1}(e^{-\lambda^2(t-s)/2})(y) ds
\end{aligned}$$

so that

$$p^\beta(0,t;0,y) = p(t,0,y) + \frac{y}{\sqrt{2\pi}}\int_0^t \frac{\beta(s)}{\sqrt{s}(t-s)}\frac{1}{\sqrt{2\pi(t-s)}}\exp\big(-\frac{y^2}{2(t-s)}\big)ds$$

$$= p(t,0,y) + \frac{y}{2\pi}\int_0^t \frac{\beta(s)}{\sqrt{s}(t-s)^{3/2}}\exp\big(-\frac{y^2}{2(t-s)}\big)ds.$$

Using (2.2.3), we get the announced result. □

2.5.2 First proof of Theorem 2.4.2

Let $\beta : \mathbb{R}_+ \to [k,\kappa]$ $(-1 < k \leq \kappa < 1)$ of class C^1, with $|\beta'(t)| \leq M$ for any $t \in \mathbb{R}_+$. We know from Theorem 1.4.5 and Proposition 1.7.1 that a solution B^β exists to (2.1.1), and that this process is a Feller time-inhomogeneous Markov process (in particular it is strong Markov).

Remark 2.5.3. *In [27], to assert the existence result mentionned above, we proved the Proposition 5.1 therein. This is a proof that we have generalized in [30].*

Thus we have indeed $\mathbb{P}^0(B_t^\beta \in dy) = p^\beta(0,t;0,y)dy$ and, using the computations just after Definition 2.3.1, we can see that

$$\mathbb{P}^{s,x}(B_t^\beta \in dy) = p^\beta(s,t;x,y)dy, \tag{2.5.2}$$

that is to say $p^\beta(s,t;x,y)$ is indeed the transition function of B^β. Thus this function satisfies the Chapman-Kolmogorov equations

$$\int_{-\infty}^{\infty} p^\beta(s,t;x,y)p^\beta(t,v;y,z)dy = p^\beta(s,v;x,z), \;\; 0 < s < t < v, \;\; x,z \in \mathbb{R} \tag{2.5.3}$$

(see Chap. III, sect. 1, p. 80 of [64]). Using now convergence arguments we can get the following result.

Proposition 2.5.4. *Let $\beta : \mathbb{R}_+ \to [-1,1]$ a Borel function. The family of measures $p^\beta(s,t;x,y)dy$ of Definition 2.3.2 is a (inhomogeneous) family of transition probabilities satisfying the Chapman-Kolmogorov equation* (2.5.3)*.*

Proof. We may approximate β by a sequence of smooth functions $\beta_n(.)$ satisfying the requirements at the beginning of the subsection. As the family $p^{\beta_n}(s,t;x,y)dy$ satisfies (2.5.3) we recover the same result for $p^\beta(s,t;x,y)dy$ thanks to Lebesgue's domination theorem (used successively in (2.3.4) and (2.5.3)). □

In the following a Borel function $\beta : \mathbb{R}_+ \to [-1,1]$ and a point $x_0 \in \mathbb{R}$ are fixed.

The idea is the following: as we have by Proposition 2.5.4 a transition function $p^\beta(s,t;x,y)$ satisfying the Chapman-Kolmogorov equation, we can construct in a canonical way a Markov process with this prescribed transition function. It will remain to prove that this process is a weak solution to (2.1.1).

More precisely: we can construct a probability measure $\tilde{\mathbb{W}}_\beta^{x_0}$ on $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)}))$, such that the coordinate process $\tilde{\omega} = (\tilde{\omega}_t)_{t\geq 0}$ defined on this measurable space is a Markov process with t.f. $p^\beta(s,t;x,y)$, starting from x_0, under $\tilde{\mathbb{W}}_\beta^{x_0}$ (Theorem 1.5 Chap. III in [64]).

Then we can show that a Kolmogorov's continuity criterion holds for $(\tilde{\omega}_t)_{t\geq 0}$ under $\tilde{\mathbb{W}}_\beta^{x_0}$.

Proposition 2.5.5. *There exists a universal constant $C > 0$ (independent of the function $\beta(.)$) such that for all $\varepsilon \geq 0$ and $t \geq 0$,*

$$\mathbb{E}_{\tilde{\mathbb{W}}_\beta^{x_0}} = |\tilde{\omega}_{t+\varepsilon} - \tilde{\omega}_t|^4 \leq C\,\varepsilon^2. \tag{2.5.4}$$

Proof. Conditioning with respect to $\mathcal{B}([0,t])$ and using the Markov property and (2.5.2) we get,

$$\begin{aligned}
\mathbb{E}^x|\tilde{\omega}_{t+\varepsilon}-\tilde{\omega}_t|^4 &= \mathbb{E}^x\big[\int_{-\infty}^{\infty}(y-\tilde{\omega}_t)^4\,p^\beta(t,t+\varepsilon;\tilde{\omega}_t,y)dy\big]\\
&= \mathbb{E}^x\big[\int_{-\infty}^{\infty}(y-\tilde{\omega}_t)^4\,p(\varepsilon,\tilde{\omega}_t,y)dy\\
&\quad+\int_{-\infty}^{\infty}dy\,(y-\tilde{\omega}_t)^4\int_0^\varepsilon\frac{\beta\circ\sigma_s(u)}{2}\frac{y}{\pi}\frac{e^{-\frac{y^2}{2(\varepsilon-u)}}}{\sqrt{u}(\varepsilon-u)^{3/2}}e^{-\frac{|\tilde{\omega}_t|^2}{2u}}du\big]\\
&\le \mathbb{E}^x\big[\int_{-\infty}^{\infty}(y-\tilde{\omega}_t)^4\,p(\varepsilon,\tilde{\omega}_t,y)dy\\
&\quad+\int_{-\infty}^{\infty}dy\,(y-\tilde{\omega}_t)^4\int_0^\varepsilon\frac{|y|}{2\pi}\frac{e^{-\frac{y^2}{2(\varepsilon-u)}}}{\sqrt{u}(\varepsilon-u)^{3/2}}e^{-\frac{|\tilde{\omega}_t|^2}{2u}}du\big]\\
&\le 2\mathbb{E}^x\big[\int_{-\infty}^{\infty}(y-\tilde{\omega}_t)^4\,p(\varepsilon,\tilde{\omega}_t,y)dy\big]
\end{aligned}$$

where we have successively used (2.3.6) and (2.3.5). As for the brownian density we have $\int_{-\infty}^{\infty}(y-\tilde{\omega}_t)^4\,p(\varepsilon,\tilde{\omega}_t,y)dy\le C\varepsilon^2$, we get the desired result. □

This implies that we can construct a modification of $(\tilde{\omega}_t)_{t\ge0}$ with $\tilde{\mathbb{W}}_\beta^{x_0}$-a.s. continuous paths. Transporting the measure $\tilde{\mathbb{W}}_\beta^{x_0}$ on the set of continuous functions C (see [64] p.35 for details) we get the following:

Proposition 2.5.6. *There exists a probability measure $\mathbb{W}_\beta^{x_0}$ on $(C,\mathcal{C})$ under which the coordinate process $\omega=(\omega_t)_{t\ge0}$ is a Markov process with t.f. $p^\beta(s,t;x,y)dy$.*

Further we have:

Proposition 2.5.7. *under $\mathbb{W}_\beta^{x_0}$, the process $(|\omega|_t)_{t\ge0}$ is distributed as $(|W_t|)_{t\ge0}$ where W is a Brownian motion starting from x_0.*

Proof. Let $y,x>0$ and $0<s<t$. We compute

$$\begin{aligned}
\mathbb{W}_\beta^{x_0}(|\omega_t|\in dy\,\big|\,|\omega_s|=x) &= \mathbb{W}_\beta^{x_0}(|\omega_t|\in dy\,\big|\,\omega_s=x)\mathbb{W}_\beta^{x_0}(\omega_s>0\big|\,|\omega_s|=x)\\
&\quad+\mathbb{W}_\beta^{x_0}(|\omega_t|\in dy\,\big|\,\omega_s=-x)\mathbb{W}_\beta^{x_0}(\omega_s<0\big|\,|\omega_s|=x)\\
&= dy\Big\{\mathbb{W}_\beta^{x_0}(\omega_s>0\big|\,|\omega_s|=x)\big(p^\beta(s,t;x,y)+p^\beta(s,t;x,-y)\big)\\
&\quad+\mathbb{W}_\beta^{x_0}(\omega_s<0\big|\,|\omega_s|=x)\big(p^\beta(s,t;-x,y)+p^\beta(s,t;-x,-y)\big)\Big\}\\
&= dy\Big\{\mathbb{W}_\beta^{x_0}(\omega_s>0\big|\,|\omega_s|=x)\big(p(t-s,x,y)+p(t-s,x,-y)\big)\\
&\quad+\mathbb{W}_\beta^{x_0}(\omega_s<0\big|\,|\omega_s|=x)\big(p(t-s,-x,y)+p(t-s,-x,-y)\big)\Big\}\\
&= dy\,[p(t-s,x,y)+p(t-s,x,-y)],
\end{aligned}$$

where we have used Equation (2.3.6), and the symmetry of $(x,y)\mapsto p(t,x,y)$ in the computations. The final right hand side expression is the well-known (homogeneous) density of a reflected Brownian motion $|W|$ starting from $x>0$.

As $|\omega|$ is Markov as well as $|W|$, and since both processes have continuous sample paths, we get the desired result (see for example Theorem 1.5 Chap. III in [64]). □

We are now in position to prove the following theorem.

Theorem 2.5.8. *There exists a weak solution to* (2.1.1).

Proof. Let $s < t$. The Markov property yields:

$$\begin{aligned}\mathbb{E}_{\mathbb{W}_\beta^{x_0}}(\omega_t|\mathcal{C}_s) = \int_{-\infty}^{\infty} y\, p^\beta(s,t;\omega_s,y)dy &= \int_0^{t-s}\int_{-\infty}^{\infty} \frac{y|y|}{\sqrt{2\pi}} \frac{e^{-\frac{y^2}{2(t-s-u))}}}{(t-s-u)^{3/2}} dy \frac{e^{-|\omega_s|^2}}{\sqrt{2\pi u}} du \\ &+ \int_0^{t-s} \frac{\beta\circ\sigma_s(u)}{t-s-u}\left(\int_{-\infty}^{\infty} |y|^2 \frac{e^{-\frac{y^2}{2(t-s-u))}}}{\sqrt{2\pi(t-s-u)}} dy\right) \frac{e^{-|\omega_s|^2}}{\sqrt{2\pi u}} du \\ &+ \int_{-\infty}^{\infty} \frac{y}{\sqrt{2\pi(t-s)}} e^{-\frac{(y-\omega_s)^2}{2(t-s)}} dy \\ &= \omega_s + \int_0^{t-s} \beta\circ\sigma_s(u) \frac{\mathrm{e}^{-\frac{|\omega_s|^2}{2u}}}{\sqrt{2\pi u}} du.\end{aligned}$$

Note that ω is a Markov process and $|\omega|$ is a reflected Brownian motion. Thus, ω admits a symmetric local time, which is a continuous additive functional of ω. So that for $s < t$:

$$\mathbb{E}_{\mathbb{W}_\beta^{x_0}}\left(\int_0^t \beta(u)dL_u^0(\omega)\,|\mathcal{C}_s\right) = \int_0^s \beta(u)dL_u^0(\omega) + \mathbb{E}_{\mathbb{W}_\beta^{x_0}}\left(\int_s^t \beta(u)dL_u^0(\omega)\,|\mathcal{C}_s\right).$$

But,

$$\begin{aligned}\mathbb{E}_{\mathbb{W}_\beta^{x_0}}\left(\int_s^t \beta(u)dL_u^0(\omega)\,|\mathcal{C}_s\right) &= \mathbb{E}_{\mathbb{W}_{\beta\circ\sigma_s}^{\omega_s}}\left(\int_0^{t-s} \beta\circ\sigma_s(u)dL_u^0(\omega)\right) \\ &= \mathbb{E}^{\omega_s}\left(\int_0^{t-s} \beta\circ\sigma_s(u)dL_u^0(|W|)\,|\right) = \int_0^{t-s} \beta\circ\sigma_s(u)\frac{\mathrm{e}^{-\frac{|\omega_s|^2}{2u}}}{\sqrt{2\pi u}} du.\end{aligned}$$

(Here W is some brownian motion starting from ω_s under the expectation of interest). Combining these facts ensures that $\left\{\omega_t - \int_0^t \beta(u)dL_u^0(\omega) \;:\; t \geq 0\right\}$ is a $(\mathcal{C}_t)$ local martingale under $\mathbb{W}_\beta^{x_0}$. Since $\langle\omega\rangle_t = \langle|\omega|\rangle_t = t$, we deduce that $\left\{\omega_t - \int_0^t \beta(u)dL_u^0(\omega) \;:\; t \geq 0\right\}$ is in fact a $(\mathcal{C}_t)$ Brownian motion (under $\mathbb{W}_\beta^{x_0}$); we have thus constructed a weak solution of (2.1.1). □

The existence of a weak solution, together with the pathwise uniqueness stated in the Theorem 2.2.2, ensures the existence of a unique strong solution to (2.1.1) (see [80]). It is clear that this solution is a Markov process with t.f. $p^\beta(s,t;x,y)dy$ (as pathwise uniqueness implies uniqueness in law). Therefore Theorem 2.4.2 is proved.

Remark 2.5.9. *Note that in this approach we have used convergence arguments inside the explicit expression of the transition function given by* (2.3.4)*. One can wonder if, without any such explicit expression, it would be still possible to use regularization arguments, in order to treat the case on non smooth coefficients. That is to say if, using general consistance properties of time-inhomogeneous SDELTs, it would be possible to handle the case of non smooth coefficients. But this goes beyond the scope of the study of time-inhomogeneous SDELTs that we have led in [30]. This could be the subject of future research.*

2.6 Azema's projection of the ISBM and proof of the announced trivariate density

Let $\beta : \mathbb{R}_+ \to [-1,1]$. We consider the strong solution B^β of (2.1.1). The process B^β is adapted to $(\mathcal{C}_t)$, the filtration of the b.m. W driving (2.1.1). Recall also the definitions of

- $\mathcal{C}_{G_1^\beta}$, the σ-algebra generated by the variables $H_{G_1^\beta}$, where H ranges through all the $(\mathcal{C}_t)$ optional (and thus predictable) processes (see [64], Chap. XII p.488).
- $\mathcal{C}_{G_1^\beta+}$, the σ-algebra generated by the variables $H_{G_1^\beta}$, where H ranges through all the $(\mathcal{C}_t)$ progressively measurable processes (see [4]).

2.6.1 Azema's projection of the ISBM

Proposition 2.6.1. *We have, under* $\mathbb{P}^0$,

$$\left(\mathrm{sgn}(B_1^\beta), G_1^\beta, M_1^\beta\right) \overset{\mathcal{L}}{\sim} (Y, G_1, M_1)\,, \tag{2.6.1}$$

where $G_1 \overset{\mathcal{L}}{\sim} \mathrm{Arcsin}$, $M_1 \overset{\mathcal{L}}{\sim} \sqrt{2\mathbf{e}}$, G_1 *and* M_1 *are independent, and where* Y *denotes some r.v. independent of* M_1 *satisfying*

$$\mathcal{L}\left(Y \,|\, G_1 = s\right) \overset{\mathcal{L}}{\sim} \mathcal{R}\left(\frac{1+\beta(s)}{2}\right).$$

moreover, in fact

$$\mathbb{E}^0\left(\mathrm{sgn}(B_1^\beta) \mid \mathcal{C}_{G_1^\beta}\right) = \beta(G_1^\beta). \tag{2.6.2}$$

Remark 2.6.2. *Notice that in particular* $\mathrm{sgn}(B_1^\beta)$ *is independent of* M_1^β*; note that the* sgn *we work with here is the symmetric one, i.e. with* $\mathrm{sgn}(0) = 0$.

Proof. Let H denote an arbitrary real bounded $(\mathcal{C}_s)$ predictable process. The balayage formula implies on the one hand that

$$\begin{aligned} H_{G_t^\beta}\beta(G_t^\beta)|B_t^\beta| &= \int_0^t H_{G_u^\beta}\beta(G_u^\beta)\mathrm{sgn}(B_u^\beta)dW_u \\ &\quad + \int_0^t H_{G_u^\beta}\beta(G_u^\beta)dL_u^0(B^\beta). \end{aligned}$$

On another hand it implies that

$$\begin{aligned} H_{G_t^\beta}B_t^\beta &= \int_0^t H_{G_u^\beta}dB_u^\beta \\ &= \int_0^t H_{G_u^\beta}dW_u + \int_0^t H_{G_u^\beta}\beta(G_u^\beta)dL_u^0(B^\beta). \end{aligned}$$

Making the difference, we see that

$$H_{G_t^\beta}B_t^\beta - H_{G_t^\beta}\beta(G_t^\beta)|B_t^\beta| = \int_0^t H_{G_u^\beta}dW_u - \int_0^t H_{G_u^\beta}\beta(G_u^\beta)\mathrm{sgn}(B_u^\beta)dW_u.$$

Thus, the process

$$\left\{H_{G_t^\beta}\left(\mathrm{sgn}(B_t^\beta) - \beta(G_t^\beta)\right)|B_t^\beta| \;:\; t \geq 0\right\}$$

is a square integrable $(\mathcal{C}_t)$ martingale. In particular, we have that

$$\mathbb{E}^0\left(H_{G_t^\beta}\mathrm{sgn}(B_t^\beta)M_t^\beta\sqrt{t - G_t^\beta}\right) = \mathbb{E}^0\left(H_{G_t^\beta}\beta(G_t^\beta)\sqrt{\frac{\pi}{2}(t - G_t^\beta)}\right).$$

And since this equality is satisfied for all predictable process H,

$$\mathbb{E}^0\left(\mathrm{sgn}(B_t^\beta)M_t^\beta \mid \mathcal{C}_{G_t^\beta}\right) = \sqrt{\frac{\pi}{2}}\beta(G_t^\beta). \tag{2.6.3}$$

This proves that $\mathrm{sgn}(B_t^\beta)$ and M_t^β are conditionally uncorrelated. However, even though $\mathrm{sgn}(B_t^\beta)$ takes only values in $\{-1, 1\}$ $\mathbb{P}^0$-a.s., this equality is not enough to deduce the conditional law $\mathcal{L}\left(\mathrm{sgn}(B_t^\beta) \mid \sigma(G_t^\beta)\right)$ and we have to work a little more. In the following, we follow the lines of the article [4] p.290.

Let $(\mathcal{H}_t)$ the smallest right-continuous enlargement of $(\mathcal{C}_t)$ such that G_1^β becomes a stopping time. Then, according to Jeulin [37] p.77 and the exchange formula, we have

$$\mathcal{H}_{G_1^\beta} = \mathcal{C}_{G_1^\beta +} = \sigma\left(\mathcal{C}_{G_1^\beta}\right) \vee \bigcap_{n \geq \mathbb{N}^*} \sigma\left(W_{G_1^\beta + u} \;:\; 0 \leq u \leq \frac{1}{n}\right). \tag{2.6.4}$$

Define for $\varepsilon \in (0,1)$, $G_1^{\beta,\varepsilon} = G_1^\beta + \varepsilon(1-G_1^\beta)$; this is a family of $(\mathcal{H}_t)$ stopping times, such that : $\mathcal{H}_{G_1^{\beta,\varepsilon}} = \mathcal{C}_{G_1^{\beta,\varepsilon}}$ (see again [37]). Moreover, since $(\mathcal{H}_t)$ is right-continuous, we have :

$$\mathcal{C}_{G_1^\beta +} = \mathcal{H}_{G_1^\beta} = \bigcap_{\varepsilon\in(0,1)} \mathcal{C}_{G_1^{\beta,\varepsilon}}.$$

We now proceed to show that M_1^β is independent from $\mathcal{H}_{G_1^\beta}$. We first remark that the $(\mathcal{C}_t)$ submartingale $\mathbb{P}\left(G_1^\beta < t \mid \mathcal{C}_t\right)$ (for $t<1$) can be computed explicitly using the Theorem 2.2.1. We easily find that

$$\mathbb{P}\left(G_1^\beta < t \mid \mathcal{C}_t\right) = \Phi\left(\frac{|B_t^\beta|}{\sqrt{1-t}}\right)$$

where $\Phi(y) := \sqrt{\frac{2}{\pi}}\int_0^y \exp\left(-\frac{x^2}{2}\right)dx$. We deduce from this, using the explicit enlargement formulae that :

$$\begin{aligned}
&\left(|B^\beta_{G_1^\beta+u}| - L^0_{G_1^\beta+u}(B^\beta)\right) - \left(|B^\beta_{G_1^\beta}| - L^0_{G_1^\beta}(B^\beta)\right)\\
&= \vartheta_u + \int_0^u \frac{ds}{\sqrt{1-(G_1^\beta+s)}}\left(\frac{\Phi'}{\Phi}\right)\left(\frac{|B^\beta_{G_1^\beta+s}|}{\sqrt{1-(G_1^\beta+s)}}\right), \quad \text{for } u < 1-G_1^\beta, \qquad (2.6.5)
\end{aligned}$$

where $\{\vartheta_u \ : \ u\geq 0\}$ is a $\left(\mathcal{H}_{G_1^\beta+u},\, u\geq 0\right)$ Brownian motion, so that $\{\vartheta_u \ : \ u\geq 0\}$ is independent from $\mathcal{H}_{G_1^\beta}$.

Note that $B^\beta_{G_1^\beta}=0$ and $L^0_{G_1^\beta+u}(B^\beta) = L^0_{G_1^\beta}(B^\beta)$ for $0\leq u < 1-G_1^\beta$.

Using Brownian scaling, we deduce that

$$m_v^\beta = \gamma_v + \int_0^v \frac{dh}{\sqrt{1-h}}\left(\frac{\Phi'}{\Phi}\right)\left(\frac{m_h^\beta}{\sqrt{1-h}}\right) \quad \text{for } v<1, \qquad (2.6.6)$$

where $\gamma_v := \frac{1}{\sqrt{1-G_1^\beta}}\vartheta_{(1-G_1^\beta)v}$ is again a Brownian motion which is independent from $\mathcal{H}_{G_1^\beta}$ and $m_v^\beta := \frac{|B^\beta_{G_1^\beta+v(1-G_1^\beta)}|}{\sqrt{1-G_1^\beta}}$.

From this, we deduce that $\{m_v^\beta \ : \ v<1\}$ is the unique strong solution of a SDE driven by (γ_v). Consequently, $\{m_v^\beta \ : \ v<1\}$ is independent of $\mathcal{H}_{G_1^\beta}$ and by continuity of $(m_v^\beta)_{0\leq v\leq 1}$ so is $m_1^\beta := M_1^\beta$.

From the fact that B^β is a $(\mathcal{C}_t)$ predictable process and (2.6.4) (2.6.5), we deduce that

$$\bigcap_{n\geq \mathbb{N}^*} \sigma\left(B^\beta_{G_1^\beta+u} \ : \ 0\leq u\leq \frac{1}{n}\right) \subseteq \mathcal{C}_{G_1^\beta+}$$

and thus, since $\operatorname{sgn}(B_1^\beta) = \operatorname{sgn}(B^\beta_{G_1^\beta+1/n})$ for all $n>0$, the random variable $\operatorname{sgn}(B_1^\beta)$ is $\mathcal{C}_{G_1^\beta+}$ measurable. So that,

$$\begin{aligned}
\mathbb{E}^0\left(\operatorname{sgn}(B_1^\beta)M_1^\beta \mid \mathcal{C}_{G_1^\beta}\right) &= \mathbb{E}^0\left(\mathbb{E}^0\left(\operatorname{sgn}(B_1^\beta)M_1^\beta \mid \mathcal{C}_{G_1^\beta+}\right) \mid \mathcal{C}_{G_1^\beta}\right)\\
&= \mathbb{E}^0\left(M_1^\beta\right)\mathbb{E}^0\left(\operatorname{sgn}(B_1^\beta) \mid \mathcal{C}_{G_1^\beta}\right)\\
&= \sqrt{\frac{\pi}{2}}\mathbb{E}^0\left(\operatorname{sgn}(B_1^\beta) \mid \mathcal{C}_{G_1^\beta}\right),
\end{aligned}$$

and identifying with (2.6.3) ensures that

$$\mathbb{E}^0\left(\operatorname{sgn}(B_1^\beta) \mid \mathcal{C}_{G_1^\beta}\right) = \beta(G_1^\beta)\left(= \mathbb{E}^0\left(\operatorname{sgn}(B_1^\beta) \mid \sigma(G_1^\beta)\right)\right).$$

□

Remark 2.6.3. *The time $t = 1$ plays no role in the above reasoning so the relation $\mathbb{E}^0\left(\mathrm{sgn}(B_t^\beta) \mid \mathcal{C}_{G_t^\beta}\right) = \beta(G_t^\beta)$ holds also for any time t. This proves that, up to a modification, the dual predictable projection of the process $(\mathrm{sgn}(B_t^\beta))_{t\geq 0}$ on the filtration $(\mathcal{C}_{G_t^\beta})$ is given by the process $(\beta(G_t^\beta))_{t\geq 0}$.*

This means that the fundamental equation of the Inhomogeneous Skew Brownian motion may be re-interpreted like forcing with β a prescribed $(\mathcal{C}_{G_t^\beta})$-predictable projection for $(\mathrm{sgn}(B_t^\beta))_{t\geq 0}$ in the following equation

$$\begin{cases} B_t^\beta = W_t + \int_0^t {}^{\mathrm{p}}\left(\mathrm{sgn}(B_s^\beta)\right) dL_s^0(B^\beta) \\ {}^{\mathrm{p}}\left(\mathrm{sgn}(B_t^\beta)\right) = \beta(G_t^\beta), \end{cases} \tag{2.6.7}$$

where ${}^{\mathrm{p}}(Y_\cdot)$ is a notation for the $(\mathcal{C}_{G_t^\beta})$ predictable projection of the measurable process Y.

Remark 2.6.4. *In [27] Proposition 2.6.1 was immediately used to prove the Markov property of B^β (see Proposition 4.4 in [27]). As this fact is directly clear from the construction made in Subsection 2.5.2 we decide not to give this proof here (even if it is interesting in itself and directly available in the case of a non smooth coefficient β).*

2.6.2 Proof of Theorem 2.4.3

The result of Theorem 2.4.3 will appear as a consequence of the previous Proposition 2.6.1 and the following lemma which appears again as a consequence of Proposition 2.2.1 and known results concerning the standard Brownian motion :

Lemma 2.6.5. *Under $\mathbb{P}^0$, the process $\{|\check{B}_t^\beta| := \frac{1}{\sqrt{G_1^\beta}}|B^\beta_{t\,G_1^\beta}| \ : \ t \leq 1\}$ is the reflection (above 0) of a Brownian Bridge independent of $\mathcal{G} := \sigma\left\{G_1^\beta, B^\beta_{G_1^\beta+u} \ ; \ u \geq 0\right\}$.*

Proof. By the result of Proposition 2.2.1 and time inversion, $|B_t^\beta| := t\,|\tilde{B}^\beta_{\frac{1}{t}}|$ is a reflected Brownian motion. Note that

$$\tilde{d}_1^\beta := \inf\left(u > 1 \ : \ |\tilde{B}_u^\beta| = 0\right) = \frac{1}{G_1^\beta}.$$

So that, since $\tilde{B}^\beta_{\frac{1}{G_1^\beta}} = \tilde{B}^\beta_{\tilde{d}_1^\beta} = 0$,

$$\frac{1}{\sqrt{G_1^\beta}}|B^\beta_{t\,G_1^\beta}| = t\sqrt{G_1^\beta}|\tilde{B}^\beta_{\frac{1}{t\,G_1^\beta}}| = \frac{t}{\sqrt{\tilde{d}_1^\beta}}|\tilde{B}^\beta_{\frac{1}{G_1^\beta}+\frac{1}{G_1^\beta}\left(\frac{1}{t}-1\right)} - \tilde{B}^\beta_{\frac{1}{G_1^\beta}}| = \frac{t}{\sqrt{\tilde{d}_1^\beta}}|\tilde{B}^\beta_{\tilde{d}_1^\beta+\tilde{d}_1^\beta\left(\frac{1}{t}-1\right)} - \tilde{B}^\beta_{\tilde{d}_1^\beta}|.$$

Since $\left\{|\tilde{B}^\beta_{\tilde{d}_1^\beta+u} - \tilde{B}^\beta_{\tilde{d}_1^\beta}| \ : \ u \geq 0\right\}$ is a reflected Brownian motion independent of $\tilde{\mathcal{C}}_{\tilde{d}_1^\beta}$ and $\tilde{B}^\beta_{\tilde{d}_1^\beta} = 0$, the process $|\hat{B}_u^\beta| := \frac{1}{\sqrt{\tilde{d}_1^\beta}}|\tilde{B}^\beta_{\tilde{d}_1^\beta+\tilde{d}_1^\beta u}|$ is also a reflected Brownian motion independent of $\tilde{\mathcal{C}}_{\tilde{d}_1^\beta}$; hence, $\left(t|\hat{B}^\beta_{\frac{1}{t}-1}|\right)_{t\geq 0}$ is a reflected Brownian Bridge independent of $\tilde{\mathcal{C}}_{\tilde{d}_1^\beta}$. This implies the result. □

Corollary 2.6.6. *We have that under $\mathbb{P}^0$,*

$$L_1^0\left(B^\beta\right) = \sqrt{G_1^\beta}\ell_1^0 \tag{2.6.8}$$

where ℓ_1^0 is the symmetric local time at time 1 of a standard Brownian Bridge independent of $\mathcal{G}$.

Proof. Under $\mathbb{P}^0$, we have that $L_1^0\left(B^\beta\right) = L_1^0\left(|B^\beta|\right) = L^0_{G_1^\beta}\left(|B^\beta|\right) = L^0_{G_1^\beta}\left(\sqrt{G_1^\beta}|\check{B}^\beta_{\cdot/G_1^\beta}|\right)$.

Recall that the symmetric local time a semimartingale (Y_t) is given by

$$L_t^0(Y) = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{(-\varepsilon,\varepsilon)}(Y_s)d\langle Y\rangle_s.$$

So we find, using an obvious change of variable, that

$$L_1^0\left(B^\beta\right) = \sqrt{G_1^\beta}L_1^0\left(|\check{B}^\beta|\right) = \sqrt{G_1^\beta}\ell_1^0,$$

where the last equality comes from the result of Lemma 2.6.5. □

Combining this result and the result of Proposition 2.6.1 gives that

$$\left(G_1^\beta, L_1^0\left(B^\beta\right), B_1^\beta\right) \overset{\mathcal{L}}{\sim} \left(G_1, \sqrt{G_1}\ell_1^0, Y\sqrt{1-G_1}M_1\right)$$

where $G_1 \overset{\mathcal{L}}{\sim}$ Arcsin, $\ell_1^0 \overset{\mathcal{L}}{\sim} \sqrt{2\mathbf{e}}$, $M_1 \overset{\mathcal{L}}{\sim} \sqrt{2\mathbf{e}}$ are independent and where Y denotes a r.v. independent of M_1 and ℓ_1^0 and satisfying

$$\mathcal{L}\left(Y \,|\, G_1 = s\right) \overset{\mathcal{L}}{\sim} \mathcal{R}\left(\frac{1+\beta(s)}{2}\right).$$

This construction gives the result announced in Theorem 2.4.3.

2.7 Another construction of the ISBM: using Brownian Excursions

Here we want to evoke another construction of the ISBM we have given in [27]. Some proofs are not given as all the details are to be found in [27].

Let $\{\Pi \,:\, 0 = t_0 < t_1 < \cdots < t_i < \cdots < t_n = 1\}$ be a partition of the interval $[0,1]$.
We define a function $i : [0,1] \to [\![0 : n-1]\!]$ by

$$i(t) = \sup\{0 \le k \le n-1 :\, t_k \le t\}, \quad \forall t \in [0,1) \quad \text{and} \quad i(1) = n-1.$$

Let $\bar{\beta} : \mathbb{R}^+ \to [0,1]$ be a r.c.l.l. function with constant value in $[-1,1]$ on each interval $[t_i, t_{i+1})$. In particular $\bar{\beta}$ is a Borel function. In this section, we give a construction of a weak solution of (2.1.1) on the interval $[0,1]$, obtained by changing the sign of the excursion of a reflecting Brownian motion. We are inspired by [64], Chap. XII, Exercise 2.16 p. 487.

Let us follow the notations of [64] concerning the excursions of a Brownian motion B defined on some space $(\Omega, \mathcal{F}, \mathbb{P})$: the excursion process is denoted by $(e_s)_{s\ge 0}$, where the index s is in the local time scale. Each excursion $e_s(\omega)$ has support $[\tau_{s-}(\omega), \tau_s(\omega))$, where $\tau_s(\omega) = \sum_{u\le s} R(e_u(\omega))$, and $\tau_{s-}(\omega) = \sum_{u<s} R(e_u(\omega))$, with $R(e_s(\omega))$ the length of the excursion $e_s(\omega)$. We recall that $L_t^0(B)$ can be recovered as the inverse of τ_t.

The construction is the following : for each $0 \le i \le n-1$ let $(Y_k^i)_k$ be a sequence of independent r.v.'s, identically distributed with law $\mathcal{R}\left(\frac{1+\beta_i^n}{2}\right)$ (with $\beta_i^n := \bar{\beta}(t_i)$), and defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let B be a standard Brownian motion independent of the Y_k^i's, constructed on $(\Omega, \mathcal{F}, \mathbb{P})$. The set of its excursions $e_s(\omega)$ is countable and may be given the ordering of $\mathbb{N}$.

We define a process $X^{\bar{\beta}}$ on $[0,1]$ by putting

$$\forall t \in [0,1], \quad X_t^{\bar{\beta}}(\omega) = Y_{k_s(\omega)}^{i(\tau_{s-}(\omega))}(\omega)|e_s(t - \tau_{s-}(\omega), \omega)|,$$

if $\tau_{s-}(\omega) \le t \le \tau_s(\omega)$ and $e_s(\omega)$ is the $k_s(\omega)$-th excursion in the above ordering.

For $\tau_{s-}(\omega) \le t \le \tau_s(\omega)$ we have $\tau_{s-}(\omega) = g_t(\omega)$ where $g_t := \sup\{u < t \;:\; |B_u| = 0\}$, and $|e_s(t - \tau_{s-}(\omega), \omega)| = |B_t(\omega)|$.

Note also that for a fixed $\omega \in \Omega$, the construction does not make a use of the entire double indexed sequence $(Y_k^i(\omega))_{i\in\{0,\ldots,n-1\}, k\in\mathbb{N}}$.

Proposition 2.7.1. *The process $X^{\bar{\beta}}$ is a weak solution of equation (2.1.1) with parameter $\bar{\beta}$ and starting from zero.*

Proof. 1st step : preliminary facts Note that $X^{\bar{\beta}}$ is constructed such that

$$(|X_t^{\bar{\beta}}|)_{t\ge 0} \overset{\mathcal{L}}{\sim} (|W_t|)_{t\ge 0}.$$

Moreover, defining $G_1^{\bar{\beta}} := \sup\left(0 \le s \le 1 \;:\; X_s^{\bar{\beta}} = 0\right)$ and $M_1^{\bar{\beta}} := |X_1^{\bar{\beta}}|/\sqrt{1 - G_1^{\bar{\beta}}}$, we see from the construction of $X^{\bar{\beta}}$ that

$$\left(\mathrm{sgn}(X_1^{\bar{\beta}}), G_1^{\bar{\beta}}, M_1^{\bar{\beta}}\right) \overset{\mathcal{L}}{\sim} (Y, G_1, M_1), \tag{2.7.1}$$

where $G_1 \overset{\mathcal{L}}{\sim} \text{Arcsin}$, $M_1 \overset{\mathcal{L}}{\sim} \sqrt{2\mathbf{e}}$, G_1 and M_1 are independent, and where Y denotes some r.v. independent of M_1 satisfying

$$\mathcal{L}\left(Y \,|\, G_1 = s\right) \overset{\mathcal{L}}{\sim} \mathcal{R}\left(\frac{1+\bar{\beta}(s)}{2}\right).$$

Let $\left(\bar{\mathcal{F}}_t\right)$ the natural filtration of $X^{\bar{\beta}}$ that has been completed and augmented in order to satisfy the usual conditions. From the construction of $X^{\bar{\beta}}$, we see that

$$\mathbb{E}\left(\text{sgn}(X_t^{\bar{\beta}}) \mid \bar{\mathcal{F}}_{G_t^{\bar{\beta}}}\right) = \mathbb{E}\left(Y^{i(G_t^{\bar{\beta}})}_{k_{L_t^0(|B|)}} \mid \bar{\mathcal{F}}_{G_t^{\bar{\beta}}}\right) = \bar{\beta}(G_t^{\bar{\beta}}). \tag{2.7.2}$$

2nd step : $X^{\bar{\beta}}$ is a $\left(\bar{\mathcal{F}}_t\right)$ Markov process

Because of these preliminary facts, we may repeat the arguments of the first part in the proof of Proposition 4.4 in [27]: we see that for $s < t$ and any measurable function f :

$$\begin{aligned}
&\mathbb{E}\left(f(X_t^{\bar{\beta}})\mathbb{1}_{G_t^{\beta}>s}|\bar{\mathcal{F}}_s\right) \\
&= \int_{-\infty}^{\infty} d\xi f(\xi)\mathbb{E}^0\left[\sum_{\delta\in\{-1,1\}}\int_0^{t-s}\frac{1+\delta(\beta\circ\sigma_s)(u)}{2}\sqrt{\frac{2}{\pi}}\frac{|\xi|\mathrm{e}^{-\frac{\xi^2}{2(t-(s+u))}}}{(t-(s+u))^{3/2}}\frac{\mathrm{e}^{-\frac{|X_s^{\bar{\beta}}|^2}{2u}}}{\sqrt{2\pi u}}du\right]\mathbb{1}_{\delta\xi>0}.
\end{aligned}$$

The part $\mathbb{E}\big(f(X_t^{\bar{\beta}})\mathbb{1}_{G_t^{\beta}\leq s}|\bar{\mathcal{F}}_s\big)$ is more complicated since we cannot refer to Equation (2.1.1).

Still, for a fixed time $s > 0$ we may set

$$D_s^{\bar{\beta}} := \inf\{u \geq 0 \ : \ X_{s+u}^{\bar{\beta}} = 0\}.$$

We have :

$$\begin{aligned}
D_s^{\bar{\beta}} &= \inf\{u \geq 0 \ : \ X_{s+u}^{\bar{\beta}} = 0\} \\
&= \inf\{u \geq 0 \ : \ X_s^{\bar{\beta}} + X_{s+u}^{\bar{\beta}} - X_s^{\bar{\beta}} = 0\} \\
&= \inf\{u \geq 0 \ : \ X_s^{\bar{\beta}} + \text{sgn}(X_s^{\bar{\beta}})\left(|B_{s+u}| - |B_s|\right) = 0\} \\
&= \inf\{u \geq 0 \ : \ \left(|B_{s+u}| - |B_s|\right) = -|X_s^{\bar{\beta}}|\}.
\end{aligned}$$

But on the set $\{G_t^{\beta} \leq s\} = \{D_s^{\bar{\beta}} \geq (t-s)\}$ and for $r < D_s^{\bar{\beta}}$, the random variables B_{s+r} and B_s share the same sign. We deduce that on the set $\{G_t^{\beta} \leq s\}$,

$$D_s^{\bar{\beta}} = \begin{cases} \inf\{u \geq 0 \ : \ (B_{s+u} - B_s) = -|X_s^{\bar{\beta}}|\} := T_s^+ & \text{if } B_s \geq 0\ ; \\ \inf\{u \geq 0 \ : \ (B_{s+u} - B_s) = |X_s^{\bar{\beta}}|\} := T_s^- & \text{if } B_s < 0. \end{cases}$$

Let us introduce $\mathcal{K}_s := \bar{\mathcal{F}}_s \vee \sigma\left(B_s\right)$.

We have

$$\begin{aligned}
&\mathbb{E}\left(f(X_t^{\bar{\beta}})\mathbb{1}_{G_t^{\beta}\leq s}|\mathcal{K}_s\right) = \mathbb{E}\left(f\left(X_s^{\bar{\beta}} + \text{sgn}(X_s^{\bar{\beta}})\left(|B_{s+(t-s)}| - |B_s|\right)\right)\mathbb{1}_{D_s^{\bar{\beta}}\geq(t-s)}|\mathcal{K}_s\right) \\
&= \mathbb{1}_{B_s\geq 0}\mathbb{E}\left(f\left(X_s^{\bar{\beta}} + \text{sgn}(X_s^{\bar{\beta}})\left(B_{s+(t-s)} - B_s\right)\right)\mathbb{1}_{T_s^+\geq(t-s)}|\mathcal{K}_s\right) \\
&\quad + \mathbb{1}_{B_s<0}\mathbb{E}\left(f\left(X_s^{\bar{\beta}} - \text{sgn}(X_s^{\bar{\beta}})\left(B_{s+(t-s)} - B_s\right)\right)\mathbb{1}_{T_s^-\geq(t-s)}|\mathcal{K}_s\right).
\end{aligned}$$

Since $(B_{s+u} - B_s \ : \ u \geq 0)$ is a Brownian motion independent of B_s and of $\bar{\mathcal{F}}_s$ (and thus of $\mathcal{K}_s$), we may integrate this expression using the known laws of the Brownian motion killed when hitting 0 :

$$\begin{aligned}
&\mathbb{1}_{B_s\geq 0}\mathbb{E}\left(f\left(X_s^{\bar{\beta}} + \text{sgn}(X_s^{\bar{\beta}})\left(B_{s+(t-s)} - B_s\right)\right)\mathbb{1}_{T_s^+\geq(t-s)}|\mathcal{K}_s\right) \\
&= \mathbb{1}_{B_s\geq 0}\int_{-\infty}^{\infty} d\theta f\left(X_s^{\bar{\beta}} + \text{sgn}(X_s^{\bar{\beta}})\left(\theta - |X_s^{\bar{\beta}}|\right)\right) \\
&\quad \times \frac{1}{\sqrt{2\pi(t-s)}}\left[\exp\left(-\frac{(\theta - |X_s^{\bar{\beta}}|)^2}{2(t-s)}\right) - \exp\left(-\frac{(\theta + |X_s^{\bar{\beta}}|)^2}{2(t-s)}\right)\right]\mathbb{1}_{|X_s^{\bar{\beta}}|\,\theta>0} \\
&= \mathbb{1}_{B_s\geq 0}\int_{-\infty}^{\infty} dy f\left(y\right)\frac{1}{\sqrt{2\pi(t-s)}}\left[\exp\left(-\frac{(y - X_s^{\bar{\beta}})^2}{2(t-s)}\right) - \exp\left(-\frac{(y + X_s^{\bar{\beta}})^2}{2(t-s)}\right)\right]\mathbb{1}_{X_s^{\bar{\beta}}\,y>0}
\end{aligned}$$

where for the last line, we performed the change of variable $y = \mathrm{sgn}(X_s^{\bar\beta})\theta$. We have a similar term on the side $\{B_s < 0\}$, so that

$$\begin{aligned}
&\mathbb{E}\left(f(X_t^{\bar\beta})\mathbb{1}_{G_t^\beta \le s}|\mathcal{K}_s\right)\\
&= \int_{-\infty}^{\infty} dy f(y)\, \frac{1}{\sqrt{2\pi(t-s)}}\left[\exp\left(-\frac{(y-X_s^{\bar\beta})^2}{2(t-s)}\right) - \exp\left(-\frac{(y+X_s^{\bar\beta})^2}{2(t-s)}\right)\right]\mathbb{1}_{X_s^{\bar\beta} y>0}\\
&= \mathbb{E}\left(f(X_t^{\bar\beta})\mathbb{1}_{G_t^\beta \le s}|\bar{\mathcal{F}}_s\right).
\end{aligned}$$

Finally, adding both parts gives that $\mathbb{E}\left(f(X_t^{\bar\beta})|\bar{\mathcal{F}}_s\right) = \int_{-\infty}^{\infty} dy f(y) p^{\bar\beta}(s,t;X_s^{\bar\beta},y)$. This is enough to conclude that $X^{\bar\beta}$ is a Markov process with $p^{\bar\beta}(s,t;x,y)dy$ as its family of transition probability (satisfying (2.5.3)).

3rd step : the process $X^{\bar\beta}$ is a weak solution of equation (2.1.1)

It suffices to perform the same computations as in the proof of Theorem 2.5.8. Indeed $X^{\bar\beta}$ is Markov and, by construction, $|X^{\bar\beta}|$ is a reflected Brownian motion. □

Finally, using the above construction and convergence arguments, we were able in [27] to give a construction of the solution B^β of (2.1.1) in the case the coefficient $\beta : \mathbb{R}_+ \to [-1,1]$ is a Borel function satisfying the so-called $\mathcal{H}$-hypothesis:

"Let $\{\Pi_n \ : \ 0 = t_0^n < t_1^n < \cdots < t_i^n < \ldots t_n^n = 1, \ \ n \ge 0\}$ a sequence of partitions over $[0,1]$. Assume that

$$\sup_{0\le i\le n-1} |t_{i+1}^n - t_i^n| \xrightarrow[n\to+\infty]{} 0.$$

It is possible to construct a decreasing (resp. increasing) sequence $(\check\beta_n)$ (resp. $(\hat\beta_n)$) of r.c.l.l. step functions that are constant on each of the intervals $[t_i^n, t_{i+1}^n)$ in such a way that

$$\check\beta_n(t) \ge \beta(t) \ge \hat\beta_n(t) \quad \text{and} \quad \lim_{n\to+\infty}\check\beta_n(t) = \lim_{n\to+\infty}\hat\beta_n(t) = \beta(t), \ \ \forall t \in [0,1].$$

"

By doing so we provided another proof than the one of Subsection 2.5.2. But the assumptions were more restrictive (see Theorem 7.4 in [27]; see also [14] for further discussions).

Chapter 3

Exact simulation issues (time-homogeneous case)

This chapter is based on the papers *Exact simulation of one-dimensional stochastic differential equations involving the local time at zero of the unknown process* (2013, [28]), and *Exact simulation for solutions of one-dimensional stochastic differential equations with discontinuous drift* (2014, [29]), written with Miguel Martinez.

3.1 Introduction

In this chapter we go back to the time-homogeneous equation

$$dX_t^\beta = dW_t + \mathbf{b}(X_t^\beta)dt + \beta\, dL_t^0(X^\beta), \quad t \in [0,T], \tag{3.1.1}$$

where $|\beta| < 1$ and $\mathbf{b}$ is a bounded function, possibly discontinuous at point zero. We aim at producing an exact sampling method for (3.1.1), in the spirit of Beskos et al. ([8],[6]).

Exact simulation methods for trajectories of one-dimensional SDEs has been a subject of much interest in the last years : see for example [8], [6], [7], [63], [68]. Unlike the classical simulation methods which all involve some kind of discretization error (we mention [3] for the Euler Scheme), the exact simulation methods are constructed in such a way that they do not present any discretization error, under the strong hypothesis that the diffusion coefficient is constant and equal to one (but this is usually claimed to be without loss of generality, by the use of a Lamperti transformation).

On another hand (3.1.1) corresponds, up to a Lamperti transformation (see for instance our Example 2 in Subsection 3.8.1), to an equation of the form (1.1.1). And we have already explained the link between such equations and divergence form operators with discontinuous coefficients, and the importance of such operators (Introduction and Chapter 1).

In the one-dimensional context, various Random Walks and an Euler Scheme have been studied for the simulation of the solution of such SDEs : for Random Walks we mention [20], [19], [24], [44] ; for the Euler Scheme see [52], [53], [54] in the case where the discontinuity of the coefficient in the divergence operator appears at point zero. Of course, for such SDEs, the order of discretization error of these discretization schemes is usually greater than those obtained in a more classical context.

An important problem in adapting the methodology of Beskos et al. to Equation (3.1.1) comes from the fact the laws of the solution of such one-dimensional SDEs are no longer absolutely continuous with respect to the Wiener measure (see [43], Theorem 2.4). Thus we cannot use the Wiener measure as a reference measure in the rejection sampling procedure as in [6].

In fact, in the case $\beta \neq 0$, the law of X^β solution to (3.1.1) is absolutely continuous with respect to the law of some Skew Brownian Motion (SBM) with a constant drift component. The reason why the SBM with drift appears naturally in our computations is explained in Section 3.4.1 (see Remark 3.4.1). So, contrary to the already mentioned discretization schemes where the standard SBM is used in force, we do not longer deal with a simple SBM but with a SBM that possesses a drift component. As a consequence, in order to adapt the method of [6] in this setting, we have to be able to simulate bridges

of the SBM with drift. Therefore we will have to compute the transition function of such a SBM with drift. These are roughly speaking all the issues adressed in [28].

In [29] we have studied the case $\beta = 0$, for which we cannot use directly the method used for the case $\beta \neq 0$. In particular the convenient constant drift for the SBM with drift, used in the case $\beta \neq 0$, is no longer defined in the case $\beta = 0$ (see again Remark 3.4.1).

However, we know from [43] that X^β tends strongly to X^0 as β tends to 0 (this processes having the same variable drift $\mathbf{b}$). This leads us to examine what happens at the level of the algorithms used in the case $\beta \neq 0$, as β tends to 0. In fact, we check here by computations that there is indeed a convergence phenomenon at the level of rejection functions involved in the exact simulation algorithms given for the case $\beta \neq 0$. This convergence gives rise naturally to a nice and implementable Limit algorithm.

The main problem becomes then to prove rigorously that this Limit algorithm is indeed an exact simulation algorithm for X^0. Let us also emphasize that this new algorithm is still a rejection algorithm, and one may naturally ask for a direct interpretation of its corresponding reference measure. At the end of Subsection 3.4.2 we give an interpretation of the reference measure (corresponding to the limit rejection algorithm) in terms of a standard Brownian motion conditioned on prescribed laws for its final position and its local time at 0 at time horizon T. By our convergence arguments we avoid in fact to have to deal with this interpretation.

It is worth noting that in the very recent paper [60] - posterior to [29] - the authors, on the opposite, choose to deal in some way with conditioned paths of the couple (Brownian Motion , its local time), in order to produce exact sampling for exactly the same equation - that is $dX_t^0 = dW_t + \mathbf{b}(X_t^0)dt$ with $\mathbf{b}$ discontinuous at zero. However, their method seems computationally more complicated, and less suitable for generalizations in the presence of several discontinuities (like in the recent work [17]).

Note that the simplest case of an equation of type (3.1.1), with $\beta = 0$ and $\mathbf{b}$ discontinuous at zero, is surely the so-called 'Brownian motion with two valued drift' solution of

$$dX_t^0 = W_t + \left(\theta_0 \mathbf{1}_{X_t^0>0} + \theta_1 \mathbf{1}_{X_t^0<0}\right) dt, \quad t \in [0,T], \tag{3.1.2}$$

where $(\theta_0, \theta_1) \in \mathbb{R}^2$. For a general reference concerning these types of motions, we refer to [39] p.440-441 or [38]. These motions appear in stochastic control problems (see for example [5], [38]) and also theoretical studies concerning representations of reflected Brownian motion with drift (see [35] in the case $\theta_0 = -\theta_1$). Even though there exist explicit representation formulae for the densities of such Brownian motions with two valued drift in terms of combination of convolution integrals (see [39] p.440-441), up to our knowledge there is no exact numerical simulation algorithm for such motions available in the literature. The algorithm presented in this chapter gives an answer to this question.

Organization of the chapter. In Section 3.2 we precise the hypotheses and define the problem we will deal with. We also introduce notations used in the sequel. In Section 3.3 we state a fundamental result on abstract rejection sampling algorithms and their convergence, that will be used extensively in this chapter. In Section 3.4 we show that the law of X^β, $\beta \neq 0$ (resp. X^0), when starting from x, is absolutely continuous with respect to a reference measure $\widehat{\mathbb{Z}}_\beta^x$, $\beta \neq 0$ (resp. w.r.t. $\widehat{\mathbb{Z}}_0^x$), with the same rejection function in both cases $\beta \neq 0$ and $\beta = 0$. In the case $\beta \neq 0$, the measure $\widehat{\mathbb{Z}}_\beta^x$ is interpreted as the law of a SBM with constant drift, and such that its final position follows a prescribed law. Therefore we compute the transition function of the SBM with constant drift in Section 3.5. In Section 3.6 we then explain how to sample exactly along $\widehat{\mathbb{Z}}_\beta^x$, using in particular bridges of the SBM with constant drift. In Section 3.7 we show that $\widehat{\mathbb{Z}}_{1/n}^x \xrightarrow[n\to\infty]{w} \widehat{\mathbb{Z}}_0^x$ and explain how to sample along $\widehat{\mathbb{Z}}_0^x$, using convergence arguments. Finally, Section 3.8 is devoted to numerical experiments.

3.2 First notations and assumptions

A time horizon $0 < T < \infty$ is fixed.

We recall that C is the set of continuous mappings from $[0,T]$ to $\mathbb{R}$, and $\mathcal{C}$ the Borel σ-field on C induced by the supreme norm. We denote $(\mathcal{C}_t)_{t\in[0,T]}$ the usual canonical filtration.

We recall that for simplicity we will denote $\omega = (\omega_t)_{t\in[0,T]}$ the canonical process.

Let $\mathbb{P}^0$ the Wiener measure on $(C, \mathcal{C})$, so that $W = \omega$ (i.e. $W_t(\omega) = \omega_t$ for any $\omega \in C$, any $t \in [0, T]$) is a $(\mathcal{C}_t)$-Brownian motion starting from zero under $\mathbb{P}^0$. We will have to consider Brownian motions starting from any $x \in \mathbb{R}$ and will simply denote $\mathbb{P}^x = \mathbb{P}(\cdot \,|\, W_0 = x)$ (see Remark 3.2.1).

Throughout the whole chapter, we will make the following assumptions concerning the coefficients in (3.1.1)

– The function $\mathbf{b} : \mathbb{R} \to \mathbb{R}$ is bounded with bounded first derivative on $\mathbb{R}^{*,+}$ and $\mathbb{R}^{*,-}$ with a possible discontinuity at point $\{0\}$. We suppose that both limits $\lim_{z\to 0+} \mathbf{b}(z) = \mathbf{b}(0+)$ and $\lim_{z\to 0-} \mathbf{b}(z) = \mathbf{b}(0-)$ exist and are finite. The value $\mathbf{b}(0)$ of the function $\mathbf{b}$ at 0 is of no importance and can be fixed arbitrarily to some constant (possibly different from either $\mathbf{b}(0+)$ or $\mathbf{b}(0-)$). As usual we denote $\mathbf{b}'$ the absolute part of the derivative of $\mathbf{b}$. We set M a constant such that

$$\sup_{z\in\mathbb{R}} |\mathbf{b}(z)| \,,\ \sup_{z\in\mathbb{R}} |\mathbf{b}'(z)| \ \leq M. \tag{3.2.1}$$

– $|\beta| < 1$.

Note that the two above assumptions ensure the existence of a unique strong solution X^β to (3.1.1), associated to W defined on $(C, \mathcal{C})$ as above. We recall that X^β enjoys the strong Markov property. On these points the reader can refer for example to Chapter 1, as the time-homogeneous case is a subcase of the general case (possibly time-inhomogeneous).

We set

$$\phi(x) := \frac{\mathbf{b}^2(x) + \mathbf{b}'(x)}{2},$$
$$\tilde{\phi}(x) = \phi(x) - m \quad \text{with} \quad m = \inf_{x\in\mathbb{R}} \phi(x),$$

and

$$\theta := \frac{\mathbf{b}(0+) - \mathbf{b}(0-)}{2}. \tag{3.2.2}$$

Let $\mu \in \mathbb{R}$. We will denote $B^{\beta,\mu}$ the SBM of parameter β and drift μ. That is to say $B^{\beta,\mu}$ is the strong solution of (3.1.1) in the case $\mathbf{b} \equiv \mu$, namely :

$$dB_t^{\beta,\mu} = dW_t + \mu dt + \beta dL_t^0(B^{\beta,\mu}). \tag{3.2.3}$$

We will denote $p^{\beta,\mu}(t, x, y)$ the transition probability density of $B^{\beta,\mu}$.

Note that, with this notation, $p^{0,\mu}(t, x, y)$ is the transition probability density of the Brownian motion with constant drift $\mu \in \mathbb{R}$, namely

$$p^{0,\mu}(t, x, y) = \frac{1}{\sqrt{2\pi t}} \exp\Big\{ - \frac{(y - x - \mu t)^2}{2t} \Big\}. \tag{3.2.4}$$

Note also that $p^{\beta,0}(t, x, y)$ is the transition probability density of the SBM of parameter β (without drift), which is known (see [75], [46]). Let us now introduce the function $v^{\beta,\mu}(t, x, y)$ defined by

$$\begin{aligned} v^{\beta,\mu}(t, x, y) \quad &= \quad (1 - \exp(-\tfrac{2xy}{t}))\mathbf{1}_{xy>0} \\ &\quad + (1 + \mathrm{Sgn}(y)\beta) \exp(-\tfrac{2xy}{t}\mathbf{1}_{xy>0}) \big[1 - \beta\mu\sqrt{2\pi t} \exp\{\tfrac{(|x|+|y|+t\beta\mu)^2}{2t}\} N^c(\tfrac{\beta\mu t + |x| + |y|}{\sqrt{t}})\big], \end{aligned} \tag{3.2.5}$$

where $N^c(y) = \frac{1}{\sqrt{2\pi}} \int_y^\infty e^{-z^2/2} dz$.

One of our chief task will be to show that $p^{\beta,\mu}(t, x, y) = p^{0,\mu}(t, x, y) v^{\beta,\mu}(t, x, y)$ (see Proposition 3.5.1).

To finish with, let us define $\hat{\mathbb{P}}_\beta^x$ (resp. $\hat{\mathbb{W}}_{\beta,\mu}^x$), the probability measure induced on $(C, \mathcal{C})$ by the law of X^β (resp. $B^{\beta,\mu}$) under $\mathbb{P}^x$. The measure $\hat{\mathbb{P}}_\beta^x$ is our target distribution. More precisely we will produce exact samples of ω_T under $\hat{\mathbb{P}}_\beta^x$. Those are distributed as X_T^β, the solution of (3.1.1) starting from x (in other words as X_T^β under $\mathbb{P}^x$).

Remark 3.2.1. *Note that the definition of $\mathbb{P}^x$ is here different for example from the one of $\mathbb{P}^{s,x}$ in Chapter 1, where we had $\mathbb{P}^{s,x} = \mathbb{P}(\cdot|X_s = x)$, with X the solution of the time-inhomogeneous EDSTL (the family $(C,\mathcal{C}),(\tilde{X}_t,\mathcal{C}_t),\{\mathbb{P}^{s,x}\}$ was therefore a Markovian family) - we recall that $\tilde{X}$ denotes the space-time process associated to X. Indeed here it will be important in our computations to stress that x is the starting point the Brownian motion W in (3.1.1). But (3.1.1) has just to be understood as $X_t^\beta = W_t + \int_0^t \mathbf{b}(X_s^\beta)ds + \beta L_t^0(X^\beta)$ for any $t \in [0,T]$. So that it is clear that under $\mathbb{P}^x$ the solution X^β starts from x. But here $(C,\mathcal{C}),(X_t,\mathcal{C}_t),\{\mathbb{P}^x\}$ is not a Markovian family.*

3.3 Abstract rejection sampling algorithm and their convergence

During all this chapter we will use several times the following fundamental result.

Proposition 3.3.1. *i) Assume that we have a sequence (ξ_n) of probability measures on a measurable space $(S,\mathcal{S})$, and ξ_{dom} a probability measure on $(S,\mathcal{S})$, satisfying for any $n \in \mathbb{N}$*

$$\frac{d\xi_n}{d\xi_{dom}} = \frac{1}{\varepsilon_n} f_n,$$

with $\varepsilon_n > 0$ and $0 \leq f_n \leq 1$.

Assume that $f_n \to f$ as $n \to \infty$ point-wise on S.

Then, (ξ_n) converges towards a probability measure ξ satisfying

$$\frac{d\xi}{d\xi_{dom}} = \frac{1}{\varepsilon} f, \tag{3.3.1}$$

with $\varepsilon = \lim_{n\to\infty} \varepsilon_n$.

ii) Moreover, let $(Y_k, I_k)_{k\geq 1}$ be a sequence of i.i.d. random elements taking values in $S \times \{0,1\}$ such that $Y_1 \sim \xi_{dom}$ and $\mathbb{P}[I_1 = 1|Y_1 = y] = f(y)$ for all $y \in S$. Define $\tau := \min(k \geq 1 = I_k = 1)$. Then, $\mathbb{P}(Y_\tau \in dy) = \xi(dy)$.

Proof. For Point i) see [29], Proposition 2.1, and for Point ii) see [8], Proposition 1. □

In the sequel the considered space $(S,\mathcal{S})$ will be either $(C,\mathcal{C})$, on which we will consider several (Wiener type) probability measures (Section 3.4), or more simply $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, when we will sample the desired real random variables by rejecting simple normal random variables (see for example the sampling from the density h_β^x in Section 3.6).

3.4 Changes of probability measure on $(C,\mathcal{C})$ for the exact sampling procedure

3.4.1 Case $\beta \neq 0$

We set

$$\mu_\beta = \frac{1+\beta}{2\beta}\mathbf{b}(0+) - \frac{1-\beta}{2\beta}\mathbf{b}(0-) \tag{3.4.1}$$

and then $b_\beta(x) = \mathbf{b}(x) - \mu_\beta$ and $B_\beta(x) = \int_0^x b_\beta(z)dz$.

We have

$$\begin{aligned} dX_t^\beta &= dW_t + b_\beta(X_t^\beta)dt + \mu_\beta dt + \beta dL_t^0(X^\beta) \\ &= dW_t^{SD} + \mu_\beta dt + \beta dL_t^0(X^\beta) \end{aligned}$$

where $W_t^{SD} := W_t + \int_0^t b_\beta(X_s^\beta)ds$ is a Brownian motion (starting from x) under $\mathbb{W}^x_{\beta,\mu_\beta}$ defined by

$$\begin{aligned}\frac{d\mathbb{P}^x}{d\mathbb{W}^x_{\beta,\mu_\beta}}(\omega) &= \exp\Big\{\int_0^T b_\beta(X_t^\beta)dW_t + \tfrac{1}{2}\int_0^T b_\beta^2(X_t^\beta)dt\Big\}(\omega)\\ &= \exp\Big\{\int_0^T b_\beta(X_t^\beta)dW_t^{SD} - \tfrac{1}{2}\int_0^T b_\beta^2(X_t^\beta)dt\Big\}(\omega) \qquad (3.4.2)\\ &= \exp\Big\{\int_0^T b_\beta(X_t^\beta)(dX_t^\beta - \mu_\beta dt - \beta\, dL_t^0(X^\beta)) - \tfrac{1}{2}\int_0^T b_\beta^2(X_t^\beta)dt\Big\}(\omega)\end{aligned}$$

(this is an application of Girsanov's theorem, see Theorem 3.5.1 in [39]; note that the assumptions on $\mathbf{b}$ ensure that the Novikov condition is verified). But note that X^β under $\mathbb{W}^x_{\beta,\mu_\beta}$ is distributed as a SBM with drift B^{β,μ_β}. So that (3.4.2) may be rewritten

$$\begin{aligned}\frac{d\hat{\mathbb{P}}^x_\beta}{d\hat{\mathbb{W}}^x_{\beta,\mu_\beta}}(\omega) &= \exp\Big\{\int_0^T b_\beta(\omega_t)(d\omega_t - \mu_\beta dt - \beta\, dL_t^0(\omega)) - \tfrac{1}{2}\int_0^T b_\beta^2(\omega_t)dt\Big\}\\ &= \exp\Big\{\int_0^T b_\beta(\omega_t)dw_t^{SD} - \tfrac{1}{2}\int_0^T b_\beta^2(\omega_t)dt\Big\}\end{aligned}$$

with $dw_t^{SD} = d\omega_t - \mu_\beta dt - \beta dL_t^0(\omega)$ a Brownian motion (starting from x) under $\hat{\mathbb{W}}^x_{\beta,\mu_\beta}$.

Applying the symmetric Itô-Tanaka formula ([64], Exercise VI.1.25) we get

$$\begin{aligned}B_\beta(\omega_T) - B_\beta(\omega_0) &= \int_0^T \tfrac{b_\beta(\omega_s-)+b_\beta(\omega_s+)}{2}d\omega_s + \tfrac{1}{2}\int_{\mathbb{R}} B''_\beta(dx)L_T^x(\omega)\\ &= \int_0^T b_\beta(\omega_s)\mathbf{1}_{\omega_s\neq 0}dw_s^{SD} + \int_0^T \mu_\beta b_\beta(\omega_s)\mathbf{1}_{\omega_s\neq 0}ds + \beta\tfrac{b_\beta(0-)+b_\beta(0+)}{2}L_T^0(\omega)\\ &\quad + \int_{\mathbb{R}} \tfrac{b'_\beta(x)}{2}\mathbf{1}_{x\neq 0}L_T^x(\omega)dx + \tfrac{b_\beta(0+)-b_\beta(0-)}{2}L_T^0(\omega).\end{aligned}$$

By the occupation time formula we then get

$$\begin{aligned}B_\beta(\omega_T) - B_\beta(\omega_0) &= \int_0^T b_\beta(\omega_s)\mathbf{1}_{\omega_s\neq 0}dw_s^{SD} + \int_0^T \mu_\beta b_\beta(\omega_s)\mathbf{1}_{\omega_s\neq 0}ds + \int_0^T \tfrac{b'_\beta(\omega_s)}{2}\mathbf{1}_{\omega_s\neq 0}ds\\ &\quad + \Big(\underbrace{\beta\frac{\mathbf{b}(0-)+\mathbf{b}(0+)}{2} - \beta\mu_\beta + \frac{\mathbf{b}(0+)-\mathbf{b}(0-)}{2}}_{(*)}\Big)L_T^0(\omega)\end{aligned}$$

Note that thanks to (3.4.1) we have $(*) = 0$. Finally,

$$\int_0^T b_\beta(\omega_s)dw_s^{SD} - \frac{1}{2}\int_0^T b_\beta^2(\omega_s)ds = B_\beta(\omega_T) - B_\beta(\omega_0) - \int_0^T \phi_\beta(\omega_s)ds$$

with $\phi_\beta(z) = \frac{b_\beta^2(z)+b'_\beta(z)+2\mu_\beta b_\beta(z)}{2} = \frac{\mathbf{b}^2(x)+\mathbf{b}'(x)}{2} - \frac{\mu_\beta^2}{2} = \phi(z) - \frac{\mu_\beta^2}{2}$.

Notice that denoting $m_\beta = \inf_{x\in\mathbb{R}}\phi_\beta(x)$ we have

$$\phi_\beta(x) - m_\beta = \phi(x) - \frac{\mu_\beta^2}{2} - \inf_{x\in\mathbb{R}}(\phi(x) - \frac{\mu_\beta^2}{2}) = \tilde{\phi}(x),$$

and then

$$\frac{d\hat{\mathbb{P}}^x_\beta}{d\hat{\mathbb{W}}^x_{\beta,\mu_\beta}}(\omega) = e^{-m_\beta T}\exp\big\{B_\beta(\omega_T) - B_\beta(x) - \int_0^T \tilde{\phi}(\omega_s)ds\big\}.$$

Denote now $\hat{\mathbb{Z}}^x_\beta$ the probability measure induced on $(C,\mathcal{C})$ by the law of $B^{\beta,\mu}$ (under $\mathbb{P}^x$), conditioned to $B_T^{\beta,\mu} \sim h_\beta^x(z)dz$, where h_β^x is a probability density. Proposition 1 in [6] asserts that

$$\frac{d\hat{\mathbb{Z}}^x_\beta}{d\hat{\mathbb{W}}^x_{\beta,\mu_\beta}}(\omega) = \frac{h_\beta^x}{p^{\beta,\mu}(T,x,.)}(\omega_T).$$

Choosing then $h^x_\beta(y) = C^x_\beta \exp(B_\beta(y) - B_\beta(x))p^{\beta,\mu}(T,x,y)$ (C^x_β is the normalizing constant that makes h^x_β a density), and setting $c^x_\beta = (C^x_\beta)^{-1}e^{-m_\beta T}$ we get

$$\frac{1}{c^x_\beta}\frac{d\hat{\mathbb{P}}^x_\beta}{d\hat{\mathbb{Z}}^x_\beta}(\omega) = \exp\Big\{-\int_0^T \tilde{\phi}(\omega_t)dt\Big\} \leq 1. \tag{3.4.3}$$

Therefore the idea is to sample along $\hat{\mathbb{P}}^x$ using rejection sampling with $\hat{\mathbb{Z}}^x_\beta$ as a reference measure. One issue is to find a way to draw the Bernoulli random variables involved in Point ii) of Proposition 3.3.1. To that aim Beskos et al. proposed the following procedure in [6], that involves a Poisson point process on the plane.

Let us denote by K an upper bound for $\tilde{\phi}(x)$.

EXACT SIMULATION ALGORITHM FOR THE SOLUTION OF (3.1.1) STARTING FROM x (CASE $\beta \neq 0$)

1. SIMULATE A POISSON POINT PROCESS WITH UNIT DENSITY ON $[0,T]\times[0,K]$. THE RESULT IS A RANDOM NUMBER N OF POINTS OF COORDINATES $(t_1,z_1),\dots,(t_N,z_N)$ (SORT THESE POINTS IN ORDER TO HAVE $t_1 < \dots < t_N$).
2. SIMULATE A SKELETON $(\omega_{t_1},\dots,\omega_{t_N},\omega_T)$ WHERE $\omega \sim \widehat{\mathbb{Z}}^x_\beta$.
3. IF $\forall i \in \{1,\dots,N\}$ $\tilde{\phi}(\omega_{t_i}) \leq z_i$ ACCEPT THE SKELETON. ELSE RETURN TO STEP 1.

This algorithm produces an exact sampling of X^β_T under $\mathbb{P}^x$: it is the final instance ω_T of an accepted skeleton.

The main issue in the above algorithm is to sample a skeleton of the canonical process under $\widehat{\mathbb{Z}}^x_\beta$ (Step 2). Indeed recall that this is the law of a SBM with drift whose terminal position follows a prescribed density. This issue will be adressed in Section 3.5, where we compute $p^{\beta,\mu}(t,x,y)$, and in Section 3.6, where the algorithm to sample $(\omega_{t_1},\dots,\omega_{t_n},\omega_T)$, $\omega \sim \widehat{\mathbb{Z}}^x_\beta$, for any $n \in \mathbb{N}$, will be fully described.

Remark 3.4.1. *Note that in the case $\mathbf{b}(0\pm) = \mathbf{b}(0)$ we simply have $\mu_\beta = \mathbf{b}(0)$. Note that the definition of b_β allows to get rid of the local time term involved in the exponential martingale of Girsanov's theorem, after the application of the Itô-Tanaka formula. This makes it tractable for a numerical perspective.*

However note that in the case $\beta = 0$, the constant μ_β is no more defined. In fact in the case $\beta = 0$ and $\mathbf{b}(0+) \neq \mathbf{b}(0-)$, there is no constant μ such that proceeding as in the above computations with $b_0(x) = \mathbf{b}(x) - \mu$ we can cancel the local time term appearing in the exponential weight. Therefore the case $\beta = 0$ needs a specific approach.

3.4.2 Case $\beta = 0$

We start afresh and will define some new probability measures on $(C,\mathcal{C})$ in order to handle the case $\beta = 0$ and $\mathbf{b}(0+) \neq \mathbf{b}(0-)$.

Again by Girsanov's theorem we see that X^0 is a Brownian motion under $\mathbb{Q}^x$ defined by

$$\frac{d\mathbb{P}^x}{d\mathbb{Q}^x}(\omega) = \exp\Big\{\int_0^T \mathbf{b}(X^0_t)dX^0_t - \frac{1}{2}\int_0^T \mathbf{b}^2(X^0_t)dt\Big\}(\omega).$$

Writing again the above relation in a more canonical way we get

$$\frac{d\hat{\mathbb{P}}^x_0}{d\mathbb{P}^x}(\omega) = \exp\Big\{\int_0^T \mathbf{b}(\omega_t)d\omega_t - \frac{1}{2}\int_0^T \mathbf{b}^2(\omega_t)dt\Big\}.$$

We set $B(x) = \int_0^x \mathbf{b}(y)dy$. By Itô-Tanaka,

$$\begin{aligned} B(\omega_T) - B(\omega_0) &= \int_0^T \mathbf{b}(\omega_t)d\omega_t + \tfrac{1}{2}\int_{\mathbb{R}} B''(dx)L^x_T(\omega) \\ &= \int_0^T \mathbf{b}(\omega_t)d\omega_t + \tfrac{1}{2}\int_0^T \mathbf{b}'(\omega_t)dt + \tfrac{\mathbf{b}(0+)-\mathbf{b}(0-)}{2}L^0_T(\omega), \end{aligned}$$

therefore we get for the moment

$$\frac{d\hat{\mathbb{P}}_0^x}{d\mathbb{P}^x}(\omega) = e^{-mT} \exp\big\{B(\omega_T) - B(x) - \theta L_T^0(\omega)\big\} \exp\big\{ - \int_0^T \tilde{\phi}(\omega_t)dt\big\}.$$

Defining now

$$\frac{d\hat{\mathbb{Z}}_0^x}{d\mathbb{P}^x}(\omega) = C^x \exp\big\{B(\omega_T) - B(x) - \theta L_T^0(\omega)\big\}$$

(C^x is a normalizing constant) we get

$$\frac{1}{c_0^x}\frac{d\hat{\mathbb{P}}_0^x}{d\hat{\mathbb{Z}}_0^x}(\omega) = \exp\big\{ - \int_0^T \tilde{\phi}(\omega_t)dt\big\} \leq 1 \tag{3.4.4}$$

with $c_0^x = (C^x)^{-1}e^{-mT}$. Therefore the following exact algorithm in order to sample X_T^0 under $\mathbb{P}^x$.

EXACT SIMULATION ALGORITHM FOR THE SOLUTION OF (3.1.1) STARTING FROM x (CASE $\beta = 0$)

1. SIMULATE A POISSON POINT PROCESS WITH UNIT DENSITY ON $[0,T]\times[0,K]$. THE RESULT IS A RANDOM NUMBER N OF POINTS OF COORDINATES $(t_1,z_1),\ldots,(t_N,z_N)$ (SORT THESE POINTS IN ORDER TO HAVE $t_1 < \ldots < t_N$).
2. SIMULATE A SKELETON $(\omega_{t_1},\ldots,\omega_{t_N},\omega_T)$ WHERE $\omega \sim \widehat{\mathbb{Z}}_0^x$.
3. IF $\forall i \in \{1,\ldots,N\}$ $\tilde{\phi}(\omega_{t_i}) \leq z_i$ ACCEPT THE SKELETON. ELSE RETURN TO STEP 1.

Of course the structure of this algorithm is exactly the same than in the case $\beta \neq 0$ (!). But we just want to stress the fact that the issue here is to sample a skeleton along $\widehat{\mathbb{Z}}_0^x$. The interpretation of $\widehat{\mathbb{Z}}_0^x$ cannot be the same as the one of $\widehat{\mathbb{Z}}_\beta^x$, $\beta \neq 0$. Indeed, under $\widehat{\mathbb{Z}}_0^x$, ω is a Brownian motion conditioned on $(\omega_T, L_T^0(\omega)) \sim h(y,\ell)dyd\ell$ with

$$h(y,\ell)dyd\ell \propto \exp\left(B(y) - B(x) - \theta\ell\right) \mathbb{P}^x\left(\omega_T \in dy, L_T^0(\omega) \in d\ell\right).$$

This makes it difficult to sample exactly ω under $\widehat{\mathbb{Z}}_0^x$ (this is the approach we have avoided in [29], and that Papaspiliopoulos et al. have tackled recently in [60]. Note however that their approach seems less suitable to possible generalizations with multiple points of discontinuity - see the recent companion papers [17, 16]).

Thus, in [29], we were led to use convergence arguments to produce a limit algorithm in order to sample along $\widehat{\mathbb{Z}}_0^x$, taking advantage of our algorithm for $\widehat{\mathbb{Z}}_\beta^x$, $\beta \neq 0$, making β tend to zero, and using Proposition 3.3.1. These arguments and the corresponding Limit algorithm will be fully described in Section 3.7.

3.5 Transition function of the SBM with drift

The purpose of this section is to show the following proposition.

Proposition 3.5.1. *We have for all $t > 0$, all $x, y \in \mathbb{R}$,*

$$p^{\beta,\mu}(t,x,y) = p^{0,\mu}(t,x,y)v^{\beta,\mu}(t,x,y). \tag{3.5.1}$$

Remark 3.5.2. *It can be shown that the quantity $1-\beta\mu\sqrt{2\pi t}\exp\big\{\frac{(|x|+|y|+t\beta\mu)^2}{2t}\big\}N^c(\frac{\beta\mu t+|x|+|y|}{\sqrt{t}})$ involved in $v^{\beta,\mu}(t,x,y)$ remains strictly positive, whatever the sign of $\beta\mu$ (see Remark 3.6.3).*

In this section we will denote $B^\beta = B^{\beta,0}$ the SBM without drift of parameter β. Further we set

$$\tau_0 := \inf\Big(t > 0 \ : \ B_t^\beta = 0\Big) \quad \text{with the convention} \quad \inf(\emptyset) = +\infty.$$

We will denote by $h(x,.)$ the density of τ_0 under $\mathbb{P}^x$.

We start with two lemmas that are a consequence of the construction of B^β starting from zero that consists in changing the sign of each excursion of a reflected Brownian motion with probability $\mathbb{P}^0(B^\beta_t > 0) = (1+\beta)/2$.

Let us recall briefly this construction, that we have already generalized to the time-inhomogeneous case in Section 2.7 (the construction in itself goes back to [36]; for details see Exercise XII.2.16 in [64], and [28]). We give us $(Y_n)_{n\geq 0}$ a sequence of independent r.v.'s taking the values 1 and -1 with probabilities $(1+\beta)/2$ and $(1-\beta)/2$ and independent of some reflected Brownian Motion $|B|$ starting from zero. With the same notations as in Section 2.7 we then put

$$B^\beta_t(\omega) = Y_{k_s(\omega)}(\omega)|\mathrm{e}_s(t-\tau_{s-}(\omega),\omega)|$$

if $\tau_{s-} \leq t \leq \tau_s$.

The above process is distributed as the solution starting from zero of (3.1.1) with $\mathbf{b} \equiv 0$, therefore the common notation. Note that it is clear with the above construction that $|B^\beta|$ is distributed (under $\mathbb{P}^0$) as a reflected Brownian motion starting from zero. Of course this could also be seen from Proposition 2.2.1. A direct consequence is the following lemma (we recall that we are working with the symmetric local time).

Lemma 3.5.3. *We have for all $t > 0$,*

$$\mathbb{P}^0\big[\,|B^\beta_t| \in dy; L^0_t(B^\beta) \in d\ell\,\big] = \mathbb{P}^0\big[\,|W_t| \in dy; L^0_t(W) \in d\ell\,\big]. \tag{3.5.2}$$

Let us now state an intuitive result, which is somewhat not so easy to prove without using the above explained construction. The difficulty comes from the presence of the local time in the equalities below (for details, see the proof of Lemma 4.4 in [28]).

Lemma 3.5.4. *We have for all $t > 0$,*

$$\mathbb{P}^0\big[\,B^\beta_t \in dy; L^0_t(B^\beta) \in d\ell\,\big] = \frac{1+\beta}{2}\mathbb{P}^0\big[\,|B^\beta_t| \in dy; L^0_t(B^\beta) \in d\ell\big] + \frac{1-\beta}{2}\mathbb{P}^0\big[\,-|B^\beta_t| \in dy; L^0_t(B^\beta) \in d\ell\big].$$

Remark 3.5.5. *Note that we could have get this result by integrating* (2.4.1) *with respect to s. But this could seem counter-intuitive to resort to the time-inhomogeneous case, therefore we prefer to go back to simpler arguments.*

Using the two last lemmas we can prove the following result.

Proposition 3.5.6. *We have for all $t > 0$, $x \geq 0$,*

$$\begin{aligned}
\mathbb{P}^x\big[\,B^\beta_t \in dy; L^0_t(B^\beta) \in d\ell\,\big] \;=\;& \mathbf{1}_{y\geq 0}\mathbf{1}_{\ell>0}\tfrac{(1+\beta)(\ell+y+x)}{\sqrt{2\pi t^3}}\exp\big\{-\tfrac{(\ell+y+x)^2}{2t}\big\}dyd\ell \\
&+\mathbf{1}_{y\geq 0}\tfrac{1}{\sqrt{2\pi t}}\big(\exp\{-\tfrac{(y-x)^2}{2t}\} - \exp\{-\tfrac{(y+x)^2}{2t}\}\big)dy\delta_0(d\ell) \\
&+\mathbf{1}_{y<0}\mathbf{1}_{\ell\geq 0}\tfrac{(1-\beta)(\ell-y+x)}{\sqrt{2\pi t^3}}\exp\big\{-\tfrac{(\ell-y+x)^2}{2t}\big\}dyd\ell.
\end{aligned}$$

Proof. Step 1. Combining the results of the Lemmas 3.5.3 and 3.5.4 we have

$$\begin{aligned}
\mathbf{1}_{y\geq 0}\mathbb{P}^0\big[\,B^\beta_t \in dy; L^0_t(B^\beta) \in d\ell\,\big] &= \tfrac{1+\beta}{2}\mathbb{P}^0\big[\,|B^\beta_t| \in dy; L^0_t(B^\beta) \in d\ell\big] \\
&= \tfrac{1+\beta}{2}\mathbb{P}^0\big[\,|W_t| \in dy; L^0_t(W) \in d\ell\,\big].
\end{aligned}$$

Step 2. Let $x \geq 0$. As $\mathbf{1}_{y\geq 0}\mathbf{1}_{\ell>0}\mathbb{P}^x[B^\beta_t \in dy; L^0_t(B^\beta) \in d\ell; t < \tau_0] = 0$, we have, using the strong Markov property of B^β,

$$\begin{aligned}
\mathbf{1}_{y\geq 0}\mathbf{1}_{\ell>0}\mathbb{P}^x[\,B^\beta_t \in dy; L^0_t(B^\beta) \in d\ell\,] &= \mathbf{1}_{y\geq 0}\mathbf{1}_{\ell>0}\mathbb{P}^x[\,B^\beta_t \in dy; L^0_t(B^\beta) \in d\ell; t \geq \tau_0\,] \\
&= \mathbf{1}_{y\geq 0}\mathbf{1}_{\ell>0}\mathbb{E}^x[\mathbf{1}_{\{t\geq\tau_0\}}\mathbb{P}^x[\,B^\beta_t \in dy; L^0_t(B^\beta) \in d\ell\,|\mathcal{C}_{\tau_0}]] \\
&= \mathbf{1}_{y\geq 0}\mathbf{1}_{\ell>0}\textstyle\int_0^t \mathbb{P}^0[B^\beta_{t-s} \in dy; L^0_{t-s}(B^\beta) \in d\ell\,]h(x,s)ds.
\end{aligned}$$

But $h(x,.)$ is also the density of $T_0 = \inf(t>0\ :\ W_t = 0)$. And using the first step of the proof we have

$$\mathbf{1}_{y\geq 0}\mathbf{1}_{\ell>0}\mathbb{P}^0[B^\beta_{t-s}\in dy; L^0_{t-s}(B^\beta)\in d\ell\,] = \mathbf{1}_{\ell>0}\frac{1+\beta}{2}\mathbb{P}^0[\,|W_{t-s}|\in dy; L^0_{t-s}(W)\in d\ell\,].$$

Using again the strong Markov property (of W) we get

$$\begin{aligned}
\mathbf{1}_{y\geq 0}\mathbf{1}_{\ell>0}\mathbb{P}^x[\,B^\beta_t\in dy; L^0_t(B^\beta)\in d\ell\,] &= \mathbf{1}_{\ell>0}\mathbb{E}^x[\mathbf{1}_{\{t\geq T_0\}}\tfrac{1+\beta}{2}\mathbb{P}^x[\,|W_t|\in dy; L^0_t(W)\in d\ell\,|\mathcal{F}_{T_0}]]\\
&= \mathbf{1}_{\ell>0}\frac{1+\beta}{2}\mathbb{P}^x[\,|W_t|\in dy; L^0_t(W)\in d\ell; t\geq T_0\,]\\
&= \mathbf{1}_{\ell>0}\frac{1+\beta}{2}\mathbb{P}^x[\,|W_t|\in dy; L^0_t(W)\in d\ell\,]\\
&= \mathbf{1}_{y\geq 0}\mathbf{1}_{\ell>0}\frac{(1+\beta)(\ell+y+x)}{\sqrt{2\pi t^3}}\exp\big\{-\frac{(\ell+y+x)^2}{2t}\big\}dyd\ell
\end{aligned}$$

(see [10]).

Step 3. It is a consequence of the reflection principle for Brownian motion (and of the fact that the SBM behaves like a brownian before τ_0) that

$$\begin{aligned}
\mathbf{1}_{y\geq 0}\mathbb{P}^x[\,B^\beta_t\in dy; L^0_t(B^\beta)=0\,] &= \mathbf{1}_{y\geq 0}\mathbb{P}^x[\,B^\beta_t\in dy; t<\tau_0\,]\\
&= \mathbf{1}_{y\geq 0}\frac{1}{\sqrt{2\pi t}}\left(\exp\{-\frac{(y-x)^2}{2t}\}-\exp\{-\frac{(y+x)^2}{2t}\}\right).
\end{aligned}$$

Using Step 1 to 3 we have the result for $x\geq 0$ on $\mathbb{R}_+\times\mathbb{R}_+$. In order to retrieve the result on $\mathbb{R}^*_-\times\mathbb{R}_+$ we use Step 1 and 2 with $\frac{1+\beta}{2}$ replaced by $\frac{1-\beta}{2}$ and $\mathbf{1}_{y\geq 0}$ replaced by $\mathbf{1}_{y<0}$, and the fact that for $x\geq 0$, $\mathbf{1}_{y<0}\mathbb{P}^x[\,B^\beta_t\in dy; L^0_t(B^\beta)=0\,] = \mathbf{1}_{y<0}\mathbb{P}^x[\,B^\beta_t\in dy; t<\tau_0\,] = 0$. □

Remark 3.5.7. *Note that the result of Proposition 3.5.6 (and consequently the result stated in Proposition 3.5.1) differs slightly from results published by T. Appuhamillage et al. in [2] where there is a computational error (see also [1] for a discussion). Here our computations are detailed for the sake of completeness and clarification.*

Proof of Proposition 3.5.1. We have

$$dB^{\beta,\mu}_t = dW^\mu_t + \beta dL^0_t(B^{\beta,\mu}),$$

with $W^\mu_t = W_t+\mu t$ a Brownian motion starting from 0 under $\mathbb{Q}^0_\mu$ defined by $\dfrac{d\mathbb{Q}^0_\mu}{d\mathbb{P}^0} = \exp\{-\mu W_t - \frac{1}{2}\mu^2 t\}$.

For any bounded continuous function f and any $t\geq 0$, we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}^x}[f(B^{\beta,\mu}_t)] &= \mathbb{E}_{\mathbb{P}^0}[f(B^{\beta,\mu}_t+x)]\\
&= \mathbb{E}_{\mathbb{Q}^0_\mu}[f(B^{\beta,\mu}_t+x)\exp\{\mu W^\mu_t - \tfrac{1}{2}\mu^2 t\}]\\
&= \int\int_{\mathbb{R}^2} f(y+x)\exp\{\mu w - \tfrac{1}{2}\mu^2 t\}\mathbb{P}^0[B^\beta_t\in dy;\, W_t\in dw]\\
&= \int\int_{\mathbb{R}^2} f(y)\exp\{\mu w - \tfrac{1}{2}\mu^2 t\}\mathbb{P}^x[B^\beta_t\in dy;\, W_t - x\in dw]
\end{aligned} \tag{3.5.3}$$

Suppose $\beta>0$.

We set $\Phi_x(z,\ell) = (z, z-x-\beta\ell)$ which defines a bijection $\Phi_x : \mathbb{R}\times\mathbb{R}_+\to D_x$ where $D_x = \{(y,w)\in\mathbb{R}^2 : y-x\geq w\}$. Note that $(B^\beta_t, W_t-x) = \Phi_x(B^\beta_t, L^0_t(B^\beta))$. Besides, almost surely, $(B^\beta, L^0(B^\beta))\in\mathbb{R}\times\mathbb{R}_+$ and $(B^\beta, W-x)\in D_x$.

For $x>0$, Proposition 3.5.6 ensures that the measure $\mathbb{P}^x[\,B^\beta_t\in dy; L^0_t(B^\beta)\in d\ell\,]$ has a density with respect to $dy\,d\ell$ on $\mathbb{R}\times\mathbb{R}^{*,+}$, and gives mass to the segments of $\mathbb{R}_+\times\{0\}$ with the density

$$\mathbb{P}^x[\,B^\beta_t\in dy; L^0_t(B^\beta)=0\,] = \frac{1}{\sqrt{2\pi t}}\big(\exp\{-\frac{(y-x)^2}{2t}\}-\exp\{-\frac{(y+x)^2}{2t}\}\big)dy. \tag{3.5.4}$$

Let us denote $\Delta_x := \{(y,w) \in \mathbb{R}_+ \times \mathbb{R} : y = w + x\} = \Phi_x(\mathbb{R}_+ \times \{0\})$. The measure $\mathbb{P}^x[B_t^\beta \in dy; W_t - x \in dw]$ has a density $g^x_{B^\beta,W}(y,w)$ with respect to $dy\,dw$ on $D_x \setminus \Delta_x = \Phi_x(\mathbb{R} \times \mathbb{R}^{*,+})$. But it gives mass to the segments of the line Δ_x. Let us denote $\Phi_x^{-1}(y,w) := (\Phi_1^{-1}(y,w), \Phi_2^{-1}(y,w))$ and notice that $\Phi_1^{-1}(y,w) = y$. Let $A_1 \subset \mathbb{R}_+$ and $A = \{(y,w) \in \mathbb{R}^2 : y \in A_1, y = w + x\} \subset \Delta_x$. As $\Phi_x^{-1}(A) \subset \mathbb{R}_+ \times \{0\}$ we have

$$\begin{aligned}\mathbb{P}^x[(B_t^\beta, W_t - x) \in A] &= \mathbb{P}^x[(B_t^\beta, L_t^0(B^\beta)) \in \Phi_x^{-1}(A)] \\ &= \mathbb{P}^x[B_t^\beta \in \Phi_1^{-1}(A); L_t^0(B^\beta) = 0\,] \\ &= \mathbb{P}^x[B_t^\beta \in A_1; L_t^0(B^\beta) = 0\,].\end{aligned}$$

Using this and (3.5.4) in (3.5.3) we get

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^x}[f(B_t^{\beta,\mu})] &= \int\int_{D_x\setminus\Delta_x} f(y)\exp\{\mu w - \tfrac{1}{2}\mu^2 t\}\mathbb{P}^x[B_t^\beta \in dy;\, W_t - x \in dw] \\ &\quad + \int\int_{\Delta_x} f(y)\exp\{\mu w - \tfrac{1}{2}\mu^2 t\}\mathbb{P}^x[B_t^\beta \in dy;\, W_t - x \in dw] \\ &= \int_{\mathbb{R}} f(y)\int_{-\infty}^{y-x}\exp\{\mu w - \tfrac{1}{2}\mu^2 t\} g^x_{B^\beta,W}(y,w)dw\,dy \\ &\quad + \int_{\mathbb{R}_+} f(y)\tfrac{1}{\sqrt{2\pi t}}\exp\{\mu(y-x) - \tfrac{1}{2}\mu^2 t\}\big(\exp\{-\tfrac{(y-x)^2}{2t}\} - \exp\{-\tfrac{(y+x)^2}{2t}\}\big)dy.\end{aligned}$$

We now compute $\int_{-\infty}^{y-x}\exp\{\mu w - \frac{1}{2}\mu^2 t\} g^x_{B^\beta,W}(y,w)dw$ with a change of variable and an integration by parts. We have for $y \geq 0$,

$$\int_{-\infty}^{y-x}\exp\{\mu w - \frac{1}{2}\mu^2 t\} g^x_{B^\beta,W}(y,w)dw = \frac{e^{-\frac{1}{2}\mu^2 t}}{\beta}\int_{-\infty}^{y-x} e^{\mu w}\frac{(1+\beta)(\frac{y-w-x}{\beta} + x + y)}{\sqrt{2\pi t^3}} e^{-\frac{(\frac{y-w-x}{\beta}+x+y)^2}{2t}}dw.$$

And,

$$\int_{-\infty}^{y-x} e^{\mu w}(\tfrac{y-w-x}{\beta} + x + y)e^{-\frac{(\frac{y-w-x}{\beta}+x+y)^2}{2t}}dw = \beta e^{\mu(y-x)}\int_0^\infty e^{-\beta\mu w'}(w' + x + y)e^{-\frac{(w'+x+y)^2}{2t}}dw'$$

$$\begin{aligned}&= \beta e^{\mu(y-x)}\Big(te^{-\frac{(x+y)^2}{2t}} - \beta\mu t\int_0^\infty e^{-\beta\mu w' - \frac{(w'+x+y)^2}{2t}}dw'\Big) \\ &= \beta e^{\mu(y-x)}\Big(te^{-\frac{(x+y)^2}{2t}} - \sqrt{2\pi}\beta\mu t^{3/2}e^{\frac{\beta^2}{2}\mu^2 t}e^{\beta\mu(x+y)}N^c(\tfrac{x+y+t\beta\mu}{\sqrt{t}})\Big) \\ &= \beta\, t\, e^{\mu(y-x)}e^{-\frac{(x+y)^2}{2t}}\Big(1 - \sqrt{2\pi t}\beta\mu e^{\frac{(x+y+\beta\mu t)^2}{2t}}N^c(\tfrac{x+y+t\beta\mu}{\sqrt{t}})\Big)\end{aligned}$$

which yields the desired result. The cases $y < 0$ and $\beta < 0$ are treated in a similar way.

For the case $x < 0$, we perform the change of variable $x \to -x$, $y \to -y$, $\beta \to -\beta$ and $\mu \to -\mu$. □

3.6 Sampling a skeleton along $\widehat{\mathbb{Z}}^x_\beta$, $\beta \neq 0$

3.6.1 Bounds for the transition function of the SBM with drift

We first aim at giving bounds on the transition function of the SBM with drift, in order to get bounds for the forthcoming rejections procedures.

Let us set $\overline{\alpha} = \max(\frac{1+\beta}{2}, \frac{1-\beta}{2})$ and

$$\gamma^{\beta,\mu}(t,z) = 1 - \beta\mu\sqrt{2\pi t}\exp(\frac{(z+t\beta\mu)^2}{2t})N^c(\frac{\beta\mu t + z}{\sqrt{t}}). \tag{3.6.1}$$

We also set

$$c^{\beta,\mu}_{t,x} = \begin{cases} 2\overline{\alpha} & \text{if} \quad \beta\mu \geq 0 \\ 2\overline{\alpha}\gamma^{\beta,\mu}(t,|x|) & \text{if} \quad \beta\mu < 0. \end{cases} \tag{3.6.2}$$

We have two following lemmas.

Lemma 3.6.1. *Let $(\beta,\mu) \in (-1,1)\times\mathbb{R}$. We have*

$$v^{\beta,\mu}(t,x,y) \leq c^{\beta,\mu}_{t,x}, \quad \forall x,y \in \mathbb{R}. \tag{3.6.3}$$

Proof. *Case* $\beta\mu \geq 0$. Let $t > 0$. It suffices to notice that

$$v^{\beta,\mu}(t,x,y) \leq (1 - e^{-\frac{2xy}{t}})\mathbf{1}_{xy>0} + (1 + \mathrm{Sgn}(y)\beta)\exp\big(-\frac{2xy}{t}\mathbf{1}_{xy>0}\big)$$

and it is then easy to check that $v^{\beta,\mu}(t,x,y) \leq 2\overline{\alpha}$ for any value of $x, y \in \mathbb{R}$.

Case $\beta\mu < 0$. Let us denote $\Gamma^{\beta,\mu}(t,x,y) := 1 - \beta\mu\sqrt{2\pi t}\exp\big\{\frac{(|x|+|y|+t\beta\mu)^2}{2t}\big\}N^c(\frac{\beta\mu t+|x|+|y|}{\sqrt{t}})$. For fixed $x \in \mathbb{R}$, $y \mapsto \Gamma^{\beta,\mu}(t,x,y)$ is an even function. As we have

$$\forall z > 0, \quad z\, e^{\frac{z^2}{2}} \int_z^\infty e^{-\frac{u^2}{2}} du < 1, \tag{3.6.4}$$

the function $z \mapsto \sqrt{2\pi}\exp(\frac{z^2}{2})N^c(z)$ has negative first derivative on $\mathbb{R}^+$. Therefore $y \mapsto \Gamma^{\beta,\mu}(t,x,y)$ is decreasing on $\mathbb{R}^+$ and we have $\max_{y\in\mathbb{R}} \Gamma^{\beta,\mu}(t,x,y) = \gamma^{\beta,\mu}(t,|x|)$. Using this and $1 < \gamma^{\beta,\mu}(t,|x|)$ we have

$$v^{\beta,\mu}(t,x,y) \leq \gamma^{\beta,\mu}(t,|x|)\big[(1 - e^{-\frac{2xy}{t}})\mathbf{1}_{xy>0} + (1 + \mathrm{Sgn}(y)\beta)\exp\big(-\frac{2xy}{t}\mathbf{1}_{xy>0}\big)\big]$$

and thus we get $v^{\beta,\mu}(t,x,y) \leq 2\overline{\alpha}\gamma^{\beta,\mu}(t,|x|)$, using the same checking computations than in the case $\beta\mu \geq 0$. □

Lemma 3.6.2. *Let* $(\beta,\mu) \in (-1,1)\times\mathbb{R}$. *We have*

$$v^{\beta,\mu}(t,x,y) \leq c_{t,y}^{\beta,\mu}, \quad \forall x, y \in \mathbb{R}. \tag{3.6.5}$$

Proof. It suffices to notice that the role of x and y are symmetric in the quantity $\Gamma^{\beta,\mu}(t,x,y)$ and then to proceed similarly to the proof of Lemma 3.6.1. □

Remark 3.6.3. *Note that thanks to* (3.6.4) *we can see that* $\gamma^{\beta,\mu}(t,z) > 0$ *and thus* $v^{\beta,\mu}(t,x,y) > 0$ *for any* $t \in \mathbb{R}^{*,+}$, $x, y, z \in \mathbb{R}$, *even for large values of* μ.

3.6.2 Sampling bridges of the SBM with drift

We denote by $q^{\beta,\mu}(t,T,a,b,y)$ the density defined (for $t < T$) by

$$\mathbb{P}^{(a)}[\, B_t^{\beta,\mu} \in dy \mid B_0^{\beta,\mu} = a,\, B_T^{\beta,\mu} = b] = q^{\beta,\mu}(t,T,a,b,y)dy.$$

The function $(t,y) \mapsto q^{\beta,\mu}(t,T,a,b,y)$ is the transition density function of a bridge of a SBM with drift relating points a and b in T unit time. Remember that $B^{\beta,\mu}$ is homogeneous Markov so that

$$q^{\beta,\mu}(t,T,a,b,y) = \frac{p^{\beta,\mu}(t,a,y)p^{\beta,\mu}(T-t,y,b)}{p^{\beta,\mu}(T,a,b)}.$$

Besides $q^{0,\mu}(t,T,a,b,y) = q^{0,0}(t,T,a,b,y)$, thus by Proposition 3.5.1 we get,

$$q^{\beta,\mu}(t,T,a,b,y) = q^{0,0}(t,T,a,b,y)\frac{v^{\beta,\mu}(t,a,y)v^{\beta,\mu}(T-t,y,b)}{v^{\beta,\mu}(T,a,b)}. \tag{3.6.6}$$

Let us set

$$C_{t,T,a,b}^{\beta,\mu} = \begin{cases} 4\overline{\alpha}^2 & \text{if} \quad \beta\mu \geq 0 \\ 4\overline{\alpha}^2\gamma^{\beta,\mu}(t,|a|)\gamma^{\beta,\mu}(T-t,|b|) & \text{if} \quad \beta\mu < 0. \end{cases} \tag{3.6.7}$$

We have

$$\frac{q^{\beta,\mu}(t,T,a,b,y)}{q^{0,0}(t,T,a,b,y)} = \frac{C_{t,T,a,b}^{\beta,\mu}}{v^{\beta,\mu}(T,a,b)} f_{a,b,t}^{\mathfrak{B},\beta,\mu}(y),$$

with

$$f_{a,b,t}^{\mathfrak{B},\beta,\mu}(y) := \frac{v^{\beta,\mu}(t,a,y)v^{\beta,\mu}(T-t,y,b)}{C_{t,T,a,b}^{\beta,\mu}}, \tag{3.6.8}$$

where the superscript $\mathfrak{B}$ appears for the word "Bridge".

Considering (3.6.2), (3.6.3), (3.6.5) and (3.6.7) it is clear that

$$f_{a,b,t}^{\mathfrak{B},\beta,\mu}(y) \leq 1, \quad \forall y \in \mathbb{R}.$$

We thus propose the following rejection algorithm in order to sample along $q^{\beta,\mu}(t,T,a,b,y)dy$.

AUXILIARY ALGORITHM 1: SAMPLING ALONG $q^{\beta,\mu}(t,T,a,b,y)dy$

---

1. SAMPLE A BROWNIAN BRIDGE Y ALONG $q^{0,0}(t,T,a,b,y)$.
2. EVALUATE
$$f_{a,b,t}^{\mathfrak{B},\beta,\mu}(Y) \leq 1.$$
3. DRAW $U \sim \mathcal{U}([0,1])$. IF $U \leq f_{a,b,t}^{\mathfrak{B},\beta,\mu}(Y)$ ACCEPT THE PROPOSED VALUE Y. ELSE RETURN TO STEP 1.

Remark 3.6.4. *Note that the quantities $v^{\beta,\mu}$, $\gamma^{\beta,\mu}$, $c_{t,x}^{\beta,\mu}$, $C_{t,T,a,b}^{\beta,\mu}$, and $f_{a,b,t}^{\mathfrak{B},\beta,\mu}$ defined respectively in* (3.2.5),(3.6.1), (3.6.2) (3.6.7), *and* (3.6.8) *involved in the above algorithm depend only on μ through the product $\beta\mu$. This computational fact gives the key ensuring the construction of the limit algorithm by convergence performed at the beginning of Section 3.7.*

3.6.3 Sampling along h_β^x, $\beta \neq 0$

In order to sample a skeleton of ω under $\widehat{\mathbb{Z}}_\beta^x$, the first thing to do is to sample ω_T along $h_\beta^x(y)dy$. We now handle this task, again by rejection sampling. We have

$$\begin{aligned}
h_\beta^x(y) =& C_\beta^x \exp\big(B_\beta(y) - B_\beta(x)\big) p^{0,\mu_\beta}(T,x,y) v^{\beta,\mu_\beta}(T,x,y) \\
=& C_\beta^x \exp\big(-\mu_\beta(y-x) + \int_x^y \mathbf{b}(z)dz\big) \times \exp\big(+\mu_\beta(y-x) - \frac{\mu_\beta^2}{2}T\big) p^{0,0}(T,x,y) v^{\beta,\mu_\beta}(T,x,y) \\
=& C_\beta^x e^{-\frac{\mu_\beta^2}{2}T} \exp\big(B(y) - B(x)\big) v^{\beta,\mu_\beta}(T,x,y) p^{0,0}(T,x,y).
\end{aligned}$$

Recall that M denotes an upper bound for the function $z \mapsto |\mathbf{b}|(z)$ (see (3.2.1)). Then, using the result of Lemma 3.6.1 and performing easy computations, we easily see that for any $0 < \delta < 1$:

$$\frac{h_\beta^x(y)}{p^{0,0}(T/(1-\delta),x,y)} = C_\beta^x \frac{e^{TM^2/\delta}}{\sqrt{1-\delta}} e^{-\frac{\mu_\beta^2}{2}T} c_{T,x}^{\beta,\mu_\beta} \, f_\delta^{\mathfrak{h},\beta,\mu_\beta}(y),$$

with

$$f_{\delta,x}^{\mathfrak{h},\beta,\mu_\beta}(y) = \sqrt{1-\delta} \, \exp\left(B(y) - B(x) - \frac{TM^2}{\delta}\right) \frac{p^{0,0}(T,x,y)}{p^{0,0}(T/(1-\delta),x,y)} \frac{v^{\beta,\mu_\beta}(T,x,y)}{c_{T,x}^{\beta,\mu_\beta}}.$$

Using (3.6.3) one may easily check that $f_{\delta,x}^{\mathfrak{h},\beta,\mu_\beta}(y) \leq 1$ for any $y \in \mathbb{R}$. One might then optimize w.r.t. $\delta \in (0,1)$ in order to find $f_{\delta,x}^{\mathfrak{h},\beta,\mu_\beta}$ closest to 1.

Let us set for simplicity, $f_x^{\mathfrak{h},\beta,\mu_\beta} = f_{1/2,x}^{\mathfrak{h},\beta,\mu_\beta}$. We deduce therefore the following procedure in order to sample along $h_\beta^x(y)dy$.

AUXILIARY ALGORITHM 2: SAMPLING ALONG $h_\beta^x(y)dy$

---

1. SAMPLE $Y \sim \mathcal{N}(x, 2T)$.
2. EVALUATE
$$f_x^{\mathfrak{h},\beta,\mu_\beta}(Y) \leq 1.$$
3. DRAW $U \sim \mathcal{U}([0,1])$. IF $U \leq f_x^{\mathfrak{h},\beta,\mu_\beta}(Y)$ ACCEPT THE PROPOSED VALUE Y. ELSE RETURN TO STEP 1.

3.6.4 Sampling $(\omega_{t_1}, \dots, \omega_{t_n}, \omega_T)$ under $\widehat{\mathbb{Z}}_\beta^x$

Let $n \in \mathbb{N}$ (this represent the number N of Poisson points in $[0,T] \times [0,K]$ drawn at Step 1 of the Exact simulation algorithm of Subsection 3.4.1). Let $0 < t_1 < \ldots < t_n < T$ (this time instances are given by the reordered first coordinates of the Poisson points).

We now describe the procedure used to draw $(\omega_{t_1}, \dots, \omega_{t_n}, \omega_T)$ where $\omega \sim \widehat{\mathbb{Z}}_\beta^x$ (that is to perform Step 2 of the Exact simulation algorithm), using the results of Subsections 3.6.2 and 3.6.3.

Note that ω under $\widehat{\mathbb{Z}}_\beta^x$ is Markov, and remember that $\omega_T \sim h_\beta^x(y)dy$ by definition of $\widehat{\mathbb{Z}}_\beta^x$. Therefore the following algorithm.

Performing Step 2 of the Exact Simulation Algorithm.
Sampling $(\omega_{t_1}, \dots, \omega_{t_n}, \omega_T)$ under $\widehat{\mathbb{Z}}_\beta^x$, $\beta \neq 0$ (starting from x)$(n = N)$

---

1. Sample ω_T along $h_\beta^x(y)dy$ using the Auxiliary Algorithm 2.
2. Sample ω_{t_1} along $q^{\beta,\mu_\beta}(t_1, T, x, \omega_T, y)dy$ using the Auxiliary Algorithm 1.
3. For $i = 2, \dots, n$, sample $\omega_{t_{i+1}}$ along $q^{\beta,\mu_\beta}(t_{i+1} - t_i, T - t_i, \omega_{t_i}, \omega_T, y)dy$ using the Auxiliary Algorithm 2.

Note that, using conditioning arguments, we can see that the law of $(\omega_{t_1}, \dots, \omega_{t_n}, \omega_T)$ under $\widehat{\mathbb{Z}}_\beta^x$ is given by

$$h_\beta^x(y) \prod_{i=0}^{n-1} q^{\beta,\mu_\beta}(t_{i+1} - t_i, T - t_i, y_i, y, y_{i+1}) dy_1 \dots dy_n dy \tag{3.6.9}$$

(we have set $y_0 = x$ to simplify the notations). This consideration will be used in the forthcoming section.

3.7 Sampling a skeleton along $\widehat{\mathbb{Z}}_0^x$

We now aim at sampling $(\omega_{t_1}, \dots, \omega_{t_n}, \omega_T)$ where $\omega \sim \widehat{\mathbb{Z}}_0^x$, using convergence arguments.

We consider a sequence (β_n) converging to zero. To fix ideas we choose $\beta_n = \frac{1}{n}$. On one side we have the following proposition.

Proposition 3.7.1. *We have*

$$\widehat{\mathbb{Z}}_{1/n}^x \xrightarrow[n\to\infty]{w} \widehat{\mathbb{Z}}_0^x.$$

Lemma 3.7.2 (Le Gall [43], Theorem 3.1, 1984)**.** *For any $n \in \mathbb{N}$ consider $X^{1/n}$ the solution of*

$$dX_t^{1/n} = dW_t + \mathbf{b}(X_t^{1/n})dt + \frac{1}{n} dL_t^0(X^{1/n}), \quad X_0^{1/n} = x.$$

We have for all $0 < t < T$,

$$\mathbb{E}\big[\sup_{0 \le s \le t} |X_s^0 - X_s^{1/n}| \big] \xrightarrow[n\to\infty]{} 0.$$

Proof. We define $f_n(x) = \mathbf{1}_{x<0} + \mathbf{1}_{x \ge 0} \frac{1-1/n}{1+1/n}$ and $F_n = \int_0^x f_n(z)dz$. For $Y^n = F_n(X^n)$ we have, using Itô-Tanaka formula,

$$dY_t^n = f_n \circ F_n^{-1}(Y_t^n) dW_t + (\mathbf{b} f_n) \circ F_n^{-1}(Y_t^n) dt.$$

By dominated convergence we may show that for any $K > 0$ we have $\int_{-K}^K |f_n \circ F_n^{-1} - 1|(x)dx \to 0$ and $\int_{-K}^K |(\mathbf{b} f_n) \circ F_n^{-1} - \mathbf{b}|(x)dx \to 0$ as $n \to \infty$. Besides $0 < \varepsilon \le f_n \le M'$ and $|\mathbf{b} f_n| \le M'$ for all $n \ge 2$ (for some constants ε, M').

Thus by Theorem 1.5 in [43] we have $\mathbb{E}[\sup_{s \le t} |Y_s^n - X_s^0|] \to 0$ as $n \to \infty$. But $F_n \to \mathrm{id}$ uniformly on each compact set. Thus as at the end of Theorem 3.1 in [43]

$$\mathbb{E}[\sup_{s \le t} |X_s^n - F^{-1}(X_s^0)|] \xrightarrow[n\to\infty]{} 0.$$

But here $F^{-1} = \mathrm{id}$ and we get the desired result. □

Proof of Proposition 3.7.1. As strong convergence implies convergence in law, it is clear from Lemma 3.7.2 that

$$\widehat{\mathbb{P}}^x_{1/n} \xrightarrow[n\to\infty]{w} \widehat{\mathbb{P}}^x_0. \tag{3.7.1}$$

Let us define $\Phi : (C, \mathcal{C}) \to \mathbb{R}$ by

$$\Phi(\omega) := \exp\big\{ -\int_0^T \tilde{\phi}(\omega_t)dt\big\}, \quad \forall \omega \in C.$$

Note that $0 < \Phi(\omega) \leq 1$. Thanks to (3.4.4) and (3.4.3) we have

$$\frac{d\widehat{\mathbb{P}}^x_0}{d\widehat{\mathbb{Z}}^x_0}(\omega) = c^x_0\, \Phi(\omega) \quad \text{and} \quad \frac{d\widehat{\mathbb{P}}^x_{1/n}}{d\widehat{\mathbb{Z}}^x_{1/n}}(\omega) = c^x_{1/n}\, \Phi(\omega)$$

and thus,

$$\frac{d\widehat{\mathbb{Z}}^x_0}{d\widehat{\mathbb{P}}^x_0}(\omega) = \frac{1}{c^x_0}\,\frac{1}{\Phi(\omega)} \quad \text{and} \quad \frac{d\widehat{\mathbb{Z}}^x_{1/n}}{d\widehat{\mathbb{P}}^x_{1/n}}(\omega) = \frac{1}{c^x_{1/n}}\,\frac{1}{\Phi(\omega)}. \tag{3.7.2}$$

Under the assumptions of Section 3.2, the functional $\omega \mapsto 1/\Phi(\omega)$ is easily seen to be bounded and continuous from $(C, \mathcal{C})$ to $\mathbb{R}$ for the topology of the supreme norm. Using this and (3.7.1) we see that

$$\int_C \frac{\widehat{\mathbb{P}}^x_{1/n}(d\omega)}{\Phi(\omega)} \xrightarrow[n\to\infty]{} \int_C \frac{\widehat{\mathbb{P}}^x_0(d\omega)}{\Phi(\omega)}. \tag{3.7.3}$$

Since $\widehat{\mathbb{Z}}^x_{1/n}$ is a probability measure on $(C, \mathcal{C})$, we also have $1 = \widehat{\mathbb{Z}}^x_{1/n}(C) = \widehat{\mathbb{Z}}^x_0(C)$. In view of (3.7.2) and (3.7.3) this implies that necessarily $(1/c^x_{1/n})_n$ is a convergent sequence and that

$$\lim_n \frac{1}{c^x_{1/n}} = \frac{1}{c^x_0}. \tag{3.7.4}$$

Therefore, for any bounded and continuous funcional $\omega \mapsto F(\omega)$ from $(C, \mathcal{C})$ to $\mathbb{R}$,

$$\int_C F(\omega)\widehat{\mathbb{Z}}^x_{1/n}(d\omega) = \frac{1}{c^x_{1/n}} \int_C F(\omega)\frac{\widehat{\mathbb{P}}^x_{1/n}(d\omega)}{\Phi(\omega)} \xrightarrow[n\to\infty]{} \frac{1}{c^x_0} \int_C F(\omega)\frac{\widehat{\mathbb{P}}^x_0(d\omega)}{\Phi(\omega)} = \int_C F(\omega)\widehat{\mathbb{Z}}^x_0(d\omega)$$

and the result follows. □

On the other side we will now use the two points of Proposition 3.3.1.

Recall the definition (3.2.2) of θ. Let us denote

$$v^\theta(t,x,y) = (1 - e^{-2xy/t})\mathbf{1}_{xy>0} + e^{-2xy/t}\big[1 - \theta\sqrt{2\pi t}\exp\{\frac{(|x|+|y|+t\theta)^2}{2t}\}N^c(\frac{\theta t + |x| + |y|}{\sqrt{t}})\big],$$

$$\gamma^\theta(t,z) = 1 - \theta\sqrt{2\pi t}\exp(\frac{(z+t\theta)^2}{2t})N^c(\frac{\theta t + z}{\sqrt{t}}),$$

$$c^\theta_{t,x} = \begin{cases} 1 & \text{if} \quad \theta \geq 0 \\ \gamma^\theta(t,|x|) & \text{if} \quad \theta < 0, \end{cases} \quad \text{and} \quad C^\theta_{t,T,a,b} = \begin{cases} 1 & \text{if} \quad \theta \geq 0 \\ \gamma^\theta(t,|a|)\gamma^\theta(T-t,|b|) & \text{if} \quad \theta < 0. \end{cases}$$

Remember our definitions (3.2.5),(3.6.1),(3.6.2) and (3.6.7) and Remark 3.6.4. It is clear from (3.4.1) that $\frac{1}{n}\mu_{1/n} \to \theta$ (as n tends to $+\infty$), so that we have,

$$\begin{aligned}
&v^{\frac{1}{n},\mu_{1/n}}(t,x,y) \xrightarrow[n\to\infty]{} v^\theta(t,x,y) && \forall (t,x,y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R},\\
&\gamma^{\frac{1}{n},\mu_{1/n}}(t,z) \xrightarrow[n\to\infty]{} \gamma^\theta(t,z) && \forall (t,z) \in \mathbb{R}^+ \times \mathbb{R},\\
&c^{\frac{1}{n},\mu_{1/n}}_{t,x} \xrightarrow[n\to\infty]{} c^\theta_{t,x} && \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R},\\
&C^{\frac{1}{n},\mu_{1/n}}_{t,T,a,b} \xrightarrow[n\to\infty]{} C^\theta_{t,T,a,b} && \forall (t,T,a,b) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}.
\end{aligned}$$

Let us now examine the sequence $(f^{\mathfrak{h},\frac{1}{n},\mu_{1/n}}_{\delta,x})$ of the rejection functions used in the Auxiliary Algorithm 2. From the same reasons as above, it is clear that $(f^{\mathfrak{h},\frac{1}{n},\mu_{1/n}}_{\delta,x})$ converges towards

$$f^{\mathfrak{h},\theta}_{\delta,x}(y) = \sqrt{1-\delta}\,\exp\left(B(y) - B(x) - \frac{TM^2}{\delta}\right)\frac{p^{0,0}(T,x,y)}{p^{0,0}(T/(1-\delta),x,y)}\frac{v^\theta(T,x,y)}{c^\theta_{T,x}} \leq 1.$$

Thus, applying Point i) of Proposition 3.3.1, the sequence of laws $(h^x_{1/n}(y)dy)$ converges to some limit law $h^x_\theta(y)dy$ satisfying

$$h^x_\theta(y) \propto f^{\mathfrak{h},\theta}_{\delta,x}(y)p^{0,0}(T/(1-\delta),x,y).$$

Thus applying Point ii) of the same proposition we can see that we can sample from $h^x_\theta(y)dy$ using the following algorithm (here $f^{\mathfrak{h},\theta}_x$ simply denotes $f^{\mathfrak{h},\theta}_{1/2,x}$).

LIMIT AUXILIARY ALGORITHM 2: SAMPLING ALONG $h^x_\theta(y)dy$

---

1. SAMPLE $Y \sim \mathcal{N}(x, 2T)$.
2. EVALUATE
$$f^{\mathfrak{h},\theta}_x(Y) \leq 1.$$
3. DRAW $U \sim \mathcal{U}([0,1])$. IF $U \leq f^{\mathfrak{h},\theta}_x(Y)$ ACCEPT THE PROPOSED VALUE Y. ELSE RETURN TO STEP 1.

In the same manner, for any fixed $a, b \in \mathbb{R}$, the sequence $(f^{\mathfrak{B},\frac{1}{n},\mu_{1/n}}_{a,b,t})$ of rejection functions used in Auxiliary Algorithm 1 converges towards

$$f^{\mathfrak{B},\theta}_{a,b,t}(y) := \frac{v^\theta(t,a,y)v^\theta(T-t,y,b)}{C^\theta_{t,T,a,b}} \leq 1.$$

Consequently, the law $q^{\frac{1}{n},\mu_n}(t,T,a,b,y)dy$ converges towards a limit law $q^\theta(t,T,a,b,y)dy$, along which it can be sampled using the following algorithm.

LIMIT AUXILIARY ALGORITHM 1: SAMPLING ALONG $q^\theta(t,T,a,b,y)dy$

---

1. SAMPLE A BROWNIAN BRIDGE Y ALONG $q^{0,0}(t,T,a,b,y)$.
2. EVALUATE
$$f^{\mathfrak{B},\theta}_{a,b,t}(Y) \leq 1.$$
3. DRAW $U \sim \mathcal{U}([0,1])$. IF $U \leq f^{\mathfrak{B},\theta}_{a,b,t}(Y)$ ACCEPT THE PROPOSED VALUE Y. ELSE RETURN TO STEP 1.

Let us now suppose $n \in \mathbb{N}$ is given, together with some time values be $0 < t_1 < \ldots < t_n < T$. Consider the following global algorithm (whose name will be justified just afterwhile).

PERFORMING STEP 2 OF THE EXACT SIMULATION ALGORITHM.
SAMPLING $(\omega_{t_1}, \ldots, \omega_{t_n}, \omega_T)$ UNDER $\widehat{\mathbb{Z}}_0^x$ (STARTING FROM x)$(n = N)$

1. SAMPLE ω_T ALONG $h_\theta^x(y)dy$ USING THE LIMIT AUXILIARY ALGORITHM 2.
2. SAMPLE ω_{t_1} ALONG $q^\theta(t_1, T, x, \omega_T, y)dy$ USING THE LIMIT AUXILIARY ALGORITHM 1.
3. FOR $i = 2, \ldots, n$, SAMPLE $\omega_{t_{i+1}}$ ALONG $q^\theta(t_{i+1} - t_i, T - t_i, \omega_{t_i}, \omega_T, y)dy$ USING THE LIMIT AUXILIARY ALGORITHM 1.

It is clear that the vector $(\omega_{t_1}, \ldots, \omega_{t_n}, \omega_T)$ sampled by the above algorithm follows the distribution

$$h_\theta^x(y) \prod_{i=0}^{n-1} q^\theta(t_{i+1} - t_i, T - t_i, y_i, y, y_{i+1}) dy_1 \ldots dy_n dy$$

(again we have denoted $y_0 = x$). But this is what we get by passing to the limit (as $n \to \infty$) in (3.6.9). Thus, in view of Proposition 3.7.1, the sampled vector is distributed as $(\omega_{t_1}, \ldots, \omega_{t_n}, \omega_T)$ where $\omega \sim \widehat{\mathbb{Z}}_0^x$.

In conclusion one will use the above algorithm in order to perform Step 2 of the Exact Simulation Algorithm in the case $\beta = 0$.

3.8 Numerical experiments

3.8.1 Case $\beta \neq 0$

Example 1. We first deal with a toy example. We consider the following SDE

$$dX_t^\beta = dW_t - \frac{\pi}{2} \cos(\frac{\pi}{5} X_t^\beta) dt + \beta dL_t^0(X^\beta), \quad X_0^\beta = x, \tag{3.8.1}$$

with $\beta = 0.6$, and $x = 0.2$. Note that, here, the drift $\mathbf{b}(x) = -\frac{\pi}{2} \cos(\frac{\pi}{5} x)$ is bounded with bounded first derivative, and of class C^∞ on the whole real line. For the exact procedure the constant drift involved in Subsection 3.4.1, equals $\mu_\beta = \mathbf{b}(0) = -\frac{\pi}{2}$. We have

$$\tilde{\phi}(x) = \frac{\pi^2}{8} \cos^2(\frac{\pi}{5} x) + \frac{\pi^2}{20} \sin(\frac{\pi}{5} x) + \frac{\pi^2}{20},$$

and take $K = \frac{9\pi^2}{20}$ as un upper bound for $\tilde{\phi}$.

We plot on Figure 3.3 (top and bottom figures) the approximated density obtained with 10^6 simulations of X_T^β, sampled with our exact procedure. On the top figure we plot the approximated densities obtained with 10^6 simulations of the Euler Scheme used in [52] and [53], for decreasing time steps. We can observe the convergence of Euler type simulations to exact ones. Note that to have the Euler scheme fitting the exact procedure we have to take a fine time step (namely $\Delta t = 10^{-4}$). This is because, as shown in [52], the rate of weak convergence of the Euler scheme in this situation is of order $(\Delta t)^{1/2-\epsilon}$, for a smooth initial condition.

On the bottom figure the approximated density is compared with the approximated densities obtained with 10^6 simulations of the random walk based method studied in [24], for decreasing space steps. Again we can observe the convergence of the process with discretization error.

In Table 3.5 we report the empirical acceptance ratios for the rejection step using $\tilde{\phi}$ and the Poisson point process in the Exact Algorithm (this corresponds to the column Exact Algorithm in the table), and for the rejection sampling of bridges of the SBM with drift (this is the average acceptance ratio in this case).

In Table 3.4 we report the CPU times needed to get the 10^6 simulations, with the three different methods (and with the different discretization steps we have used). Programs were written in C-language and executed on a personal computer equipped with an Intel Core 2 duo processor, running at 2.23 Ghz.

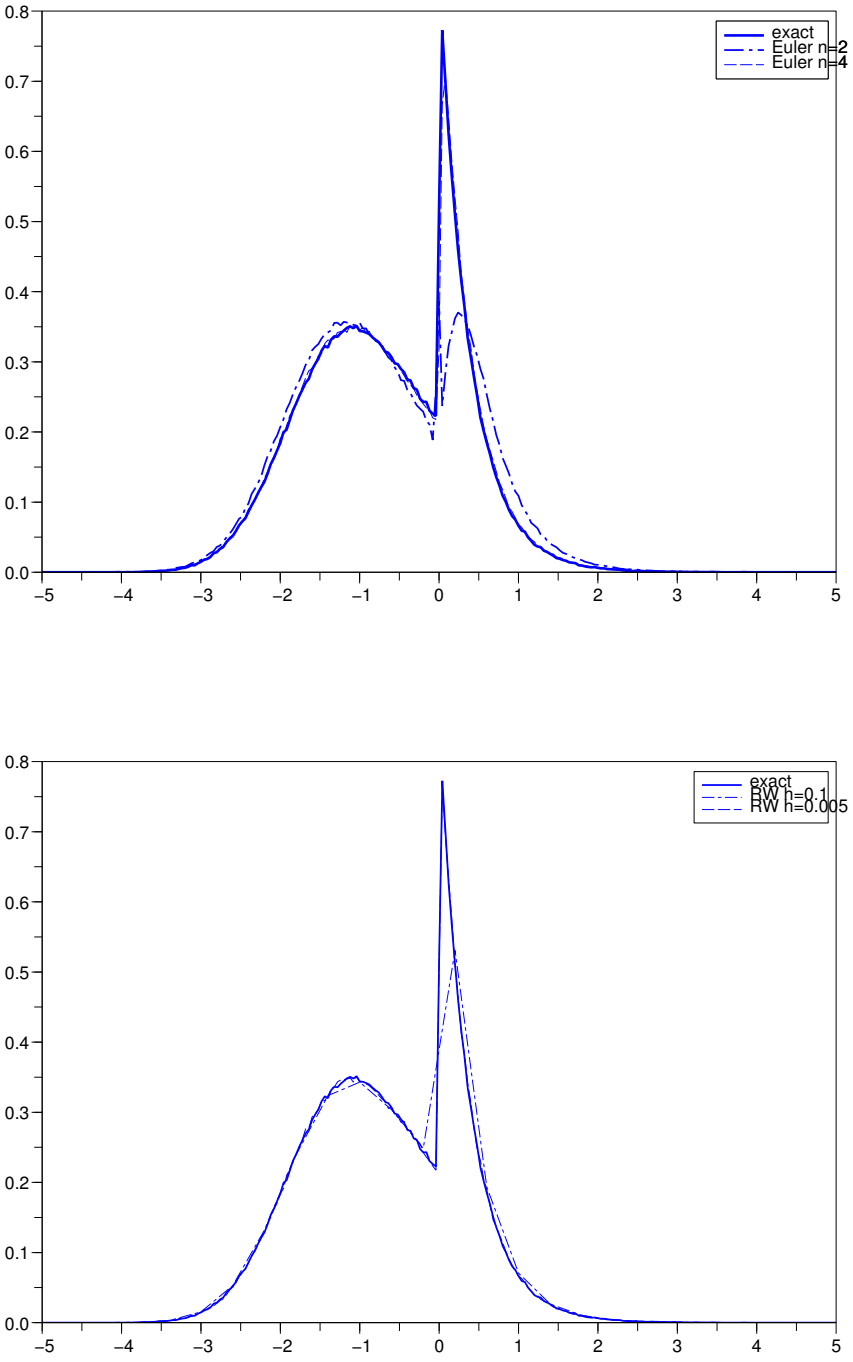

Figure 3.1: Approximated densities of the positions at time $T = 1.0$ of 10^6 paths of the solution of (3.8.1) starting from $x_0 = 0.2$: exact versus Euler with time step $\Delta t = 10^{-n}$, for $n = 2, 4$ (top) and exact versus random walk with space steps $h = \frac{1}{10}, \frac{1}{200}$ (bottom).

	Exact Algorithm	Bridges
Acceptance Ratio	0.28	0.18

Table 3.1: Acceptance ratios in Example 1.

Exact	Euler ($\Delta t = 10^{-n}$, $n = 2, 4$)	Random Walk ($h = \frac{1}{10}, \frac{1}{200}$)
239s	17s	3.52s
	1680s	1411s

Table 3.2: CPU times for 10^6 simulations of X_T.

On this example the exact simulation is competitive, compared to schemes with very fine grids.

Example 2. We want now to sample along the law of the continuous Markov process X generated by the operator defined formely in divergence form by

$$L = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dx}}\big(a\frac{\mathrm{d}}{\mathrm{dx}}\cdot\big) \tag{3.8.2}$$

with

$$a(x) = \begin{cases} \frac{x^2+x+1}{(2x+1)^2} & \text{if } x \geq 0 \\ \\ \frac{3x^2-x+2}{(6x-1)^2} & \text{if } x < 0. \end{cases}$$

Note that $a(0+) = 1 \neq 2 = a(0-)$. The coefficient $a(x)$ is of class C^1 on $\mathbb{R}^{*,-}$ and $\mathbb{R}^{*,+}$, and uniformly strictly positive and bounded, which ensures the existence of X; in addition X solves

$$dX_t = \sqrt{a(X_t)}dW_t + \frac{a'_{x,\pm}(X_t)}{2}dt + \frac{a(0+)-a(0-)}{a(0+)+a(0-)}dL_t^0(X), \tag{3.8.3}$$

(see [20] or Chapter 1). We define the Lamperti transformation $\Phi(x) = \int_0^x dz/\sqrt{a(z)}$ and set $Y_t := \Phi(X_t)$. Then

$$dY_t = dW_t + \frac{1}{2}(\sqrt{a})' \circ \Phi^{-1}(Y_t)dt + \frac{\sqrt{a(0+)}-\sqrt{a(0-)}}{\sqrt{a(0+)}+\sqrt{a(0-)}}dL_t^0(Y), \tag{3.8.4}$$

(this follows from Proposition 3.1 in [20]; see also [44] and [59]). Firstly, note that $\left|\frac{\sqrt{a(0+)}-\sqrt{a(0-)}}{\sqrt{a(0+)}+\sqrt{a(0-)}}\right| < 1$. Secondly, we have

$$(\sqrt{a})'(x) = \begin{cases} \frac{1}{2\sqrt{x^2+x+1}} - 2\frac{\sqrt{x^2+x+1}}{(2x+1)^2} & \text{if } x \geq 0 \\ \\ -\frac{1}{2\sqrt{3x^2-x+1}} + 6\frac{\sqrt{3x^2-x+2}}{(6x-1)^2} & \text{if } x < 0, \end{cases}$$

$$\Phi(x) = \begin{cases} 2\sqrt{x^2+x+1} - 2 & \text{if } x \geq 0 \\ \\ -2\sqrt{3x^2-x+1} + 2\sqrt{2} & \text{if } x < 0, \end{cases} \quad \text{and} \quad \Phi^{-1}(y) = \begin{cases} \frac{-1+\sqrt{(y+2)^2-3}}{2} & \text{if } y \geq 0 \\ \\ s\frac{1-\sqrt{1-12[2-(\sqrt{2}-y/2)^2]}}{6} & \text{if } y < 0. \end{cases}$$

As $(\sqrt{a})'(x)$ is bounded with bounded first derivative on $\mathbb{R}^{*,-}$ and $\mathbb{R}^{*,+}$, the explicitly known coefficients $\beta = \frac{\sqrt{a(0+)}-\sqrt{a(0-)}}{\sqrt{a(0+)}+\sqrt{a(0-)}}$ and $\mathbf{b}(y) = \frac{1}{2}(\sqrt{a})' \circ \Phi^{-1}(y)$ satisfy the assumptions of Section 3.2. Thus we can

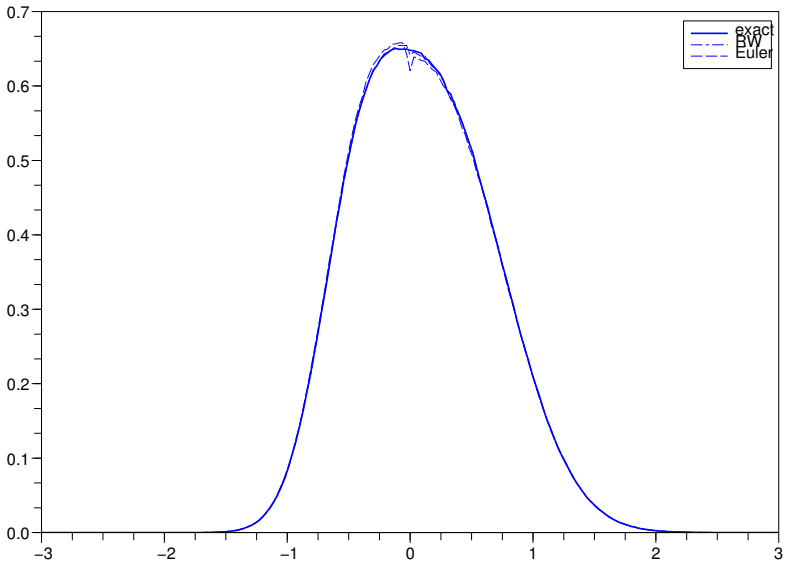

Figure 3.2: DIVERGENCE FORM OPERATOR: Approximated density of the positions at time $T = 1$ of 10^7 paths of the solution of (3.8.3) starting from $x_0 = 0.0$: exact versus random walk with space step $h = 3.10^{-3}$ and Euler scheme with $\Delta t = 10^{-4}$.

	Exact Algorithm	Bridges
Acceptance Ratio	0.017	0.5

Table 3.3: Acceptance ratios in Example 2.

perform exact sampling from (3.8.4), and, applying the exact inverse transformation Φ^{-1}, get samples from (3.8.3) with absolutely no discretization error.

Here we have,

$$\mu_\beta = \frac{1}{4}\frac{a'(0+) - a'(0-)}{\sqrt{a(0+)} - \sqrt{a(0-)}} = -\frac{26}{4(1-\sqrt{2})},$$

and

$$\tilde{\phi}(y) = \frac{((1/2)(\sqrt{a})' \circ \Phi^{-1}(y))^2 + (1/2)((\sqrt{a})''\sqrt{a}) \circ \Phi^{-1}(y)}{2},$$

with

$$(\sqrt{a})''(x) \begin{cases} -\frac{2x+1}{4(x^2+x+1)^{3/2}} - \frac{1}{(2x+1)\sqrt{x^2+x+1}} + 8\frac{\sqrt{x^2+x+1}}{(2x+1)^3} & \text{if } x \geq 0 \\ \frac{6x-1}{4(3x^2-x+2)^{3/2}} - \frac{3}{(2x+1)\sqrt{3x^2-x+2}} + 72\frac{\sqrt{3x^2-x+2}}{(6x-1)^3} & \text{if } x < 0. \end{cases}$$

We take $K = \frac{(6\sqrt{2}-1/2)^2/4+(141-1/8)/2}{2}$ as an upper bound for $\tilde{\phi}$. We plot on Figure 3.5 the approximated density computed with 10^7 simulations of X_T for $x_0 = 0.0$ and $T = 1$, obtained from the exact procedure. We plot on the same figure the approximated densities obtained with the Euler scheme and the random walk approximation mentioned in Example 1.

We report in Table 3.7 the acceptance ratios.

	Exact	Euler
CPU times	2111s	9521s

Table 3.4: CPU times for 10^6 simulations of a Brownian motion with two-valued drift with $\theta_0 = 2$ and $\theta_1 = -1$ ($x = 0.0$ and $T = 1$).

	Exact Algorithm	Bridges
Acceptance Ratio	20.4%	58,6%

Table 3.5: Acceptance ratios for the case of a Brownian motion with two-valued drift with $\theta_0 = 2$ and $\theta_1 = -1$ ($x = 0.0$ and $T = 1$).

Remark 3.8.1. *Note that the acceptance ratio for the algorithm in the first example in Table 3.5 is quite low but decreases to less than 2% in Table 3.7 in the context of the second example. These figures are closely related to the measurement of the "distance" between the measure of the initial process from the reference measure and so these limitations of the algorithm arise even in the "classical" setting of the reference article [6] (for example with a rapidly varying drift). Nevertheless, in terms of CPU time, the performance of the algorithm seems quite competitive, in comparison with those of discretization schemes.*

Remark 3.8.2. *Note that, at least graphically and contrary to what we can see on Figure 3.3, the transition density plotted on Figure 3.5 seems to be continuous at* 0 *: this matches the well-known theoretical result, which asserts that the transition density of diffusion semigroups corresponding to elliptic divergence form operator of the form* (3.8.2) *is always continuous. We refer to Stroock [69] for a proof based on the self-adjoint properties of these semi-groups and Nash's inequality.*

3.8.2 Case $\beta = 0$

Example 1: Exact simulation of a Brownian motion with two-valued (or alternate) drift.

In this paragraph, we choose to exhibit numerical results obtained with the exact limit algorithm for the simplest non-trivial cases

$$dX_t^0 = dW_t \pm \mathrm{sgn}(X_t^0)dt, \quad X_0^0 = 0,$$

corresponding to either $\theta_0 = -\theta_1 = \pm 1$ in (3.1.2) ($\mathbf{b}(y) = \pm \mathrm{Sgn}(y)$ in (3.1.1)). Indeed, in this symmetric case a benchmark is provided by the explicit and computable density of X_T^0 given in [39] p. 440-441.

We draw the renormalized histogram of 10^6 samples of X_T^0 and compare it to the explicit density of X_T^0 (Figure 3.3 for the outgoing case $\theta_0 = 1$ and Figure 3.4 for the incoming case $\theta_0 = -1$).

In the non-symmetric case we can still use our limit algorithm but the density of X_T^0 becomes less explicit (see formula (6.5.12) in [39]). Thus we will use as a benchmark the renormalized histogram of 10^6 samples an Euler Scheme with time step $\Delta = T.10^{-5}$. We chose $\theta_0 = 2$, $\theta_1 = -1$, $T = 1$ and $x = 0.0$. We plot the corresponding renormalized histograms on Figure 3.5.

In Table 3.4 we report the CPU times needed to get the 10^6 samples, with the exact limit algorithm and the Euler scheme. Programs were written in C-language and executed on a personal computer equipped with an Intel Core 2 duo processor, running at 2.23 Ghz. We report in Table 3.5 the acceptance ratios.

On this example the acceptance ratios are good and the exact method is nearly four times faster than the Euler scheme with time step $\Delta = T.10^{-5}$.

Example 2: Exact simulation of an SDE with a discontinuous drift coefficient.

We consider now the SDE (3.1.1) with $\beta = 0$ and with

$$\mathbf{b}(x) = \begin{cases} -\frac{\pi}{2}\cos\left(\frac{\pi}{5}x\right) & \text{if} \quad x \geq 0 \\ \\ \frac{3\pi}{2} - \frac{\pi}{2}\cos\left(\frac{\pi}{5}x\right) & \text{if} \quad x < 0. \end{cases} \tag{3.8.5}$$

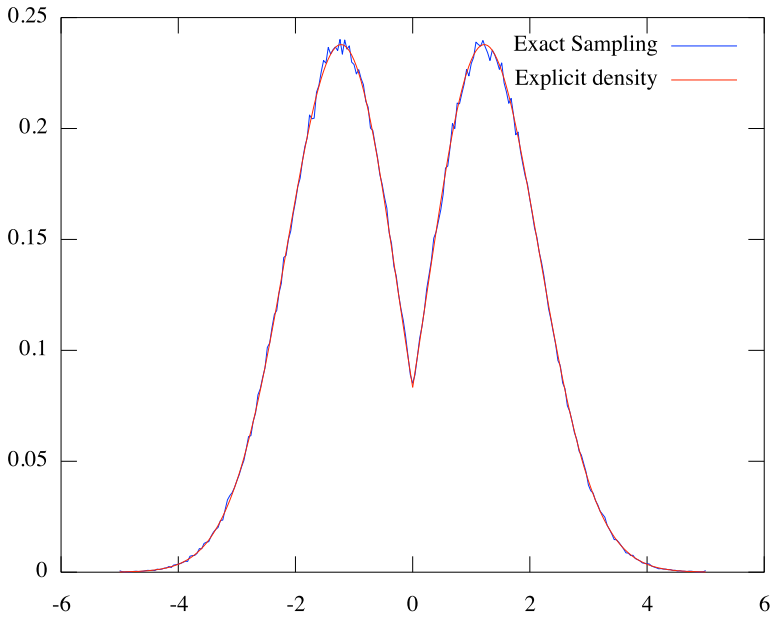

Figure 3.3: Brownian motion with two-valued drift, case $\theta_0 = -\theta_1 = 1$ ($T = 1$).

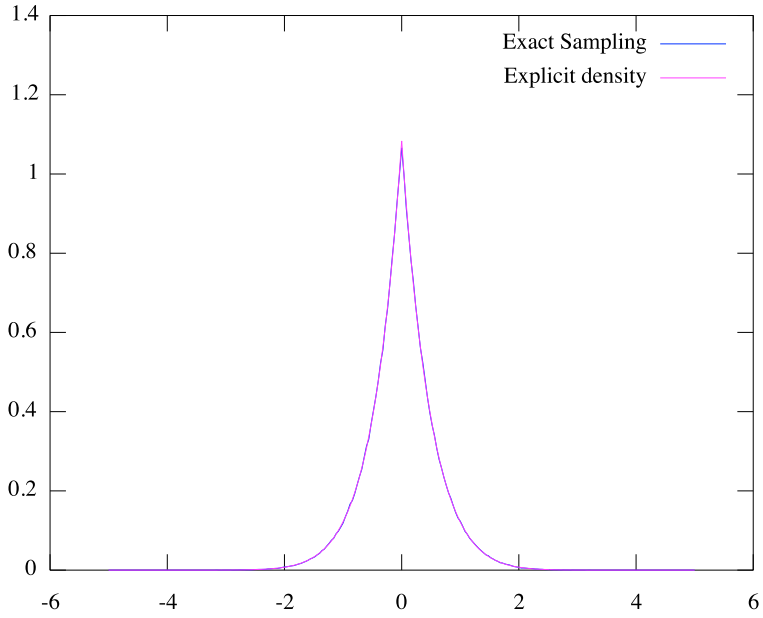

Figure 3.4: Brownian motion with two-valued drift, case $\theta_0 = -\theta_1 = -1$ ($T = 1$).

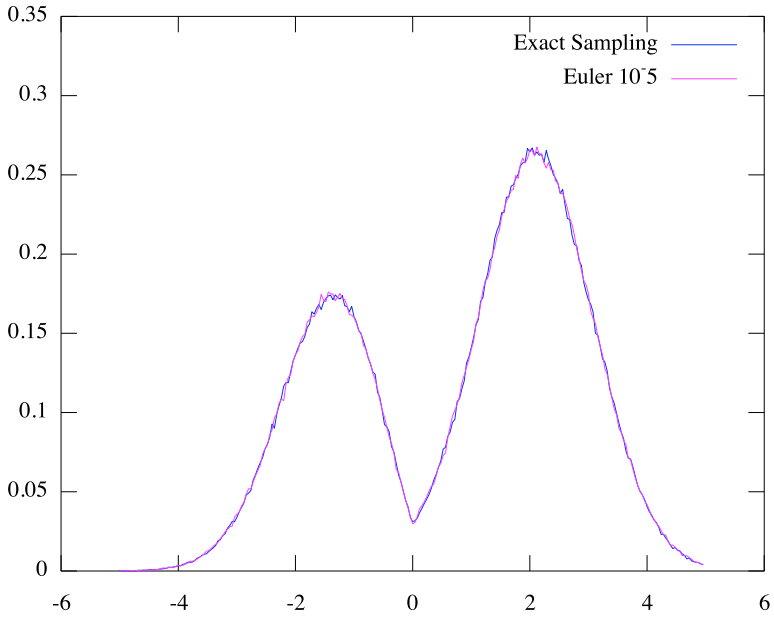

Figure 3.5: Limit algorithm v.s. Euler Scheme for Brownian motion with two-valued drift with $\theta_0 = 2$ and $\theta_1 = -1$ ($x = 0.0$ and $T = 1$).

	Exact	Euler ($\Delta t = 10^{-n}$, $n = 2, 5$)
CPU times	11813s	20s
		12952s

Table 3.6: CPU times for 10^6 simulations of X_T for the case where $\mathbf{b}$ is given by (3.8.5) ($x = 0.0$ and $T = 1$).

Let $0 < T < \infty$. We wish to sample along X_T^0.

We have $\theta = -3\pi/4$ and

$$\tilde{\phi}(x) = \frac{\mathbf{b}^2(x) + \mathbf{b}'(x)}{2} + \frac{\pi^2}{20}.$$

We take $K = 2\pi^2 + \frac{\pi^2}{10}$ as an upper bound for $\tilde{\phi}$. This allows to use the limit Algorithm.

Figure 3.5 shows a comparison between a renormalized histogram of 10^6 samples of X_T^0 obtained with the exact limit algorithm, and a renormalized histogram of 10^6 samples of an Euler Scheme with time step Δ. We chose $x = 0.0$, $T = 1$ and time-steps $\Delta = T.10^{-2}$ and $\Delta = T.10^{-5}$. In Table 3.6 we report the CPU times needed to get the 10^6 samples, with the exact limit algorithm and the Euler scheme (and, for the later one, with the different time steps we have used). We report in Table 3.7 the acceptance ratios.

On this example the exact simulation is competitive, compared to schemes with very fine grids.

	Exact Algorithm	Bridges
Acceptance Ratio	3.6%	50,7%

Table 3.7: Acceptance ratios for the case where $\mathbf{b}$ is given by (3.8.5) ($x = 0.0$ and $T = 1$).

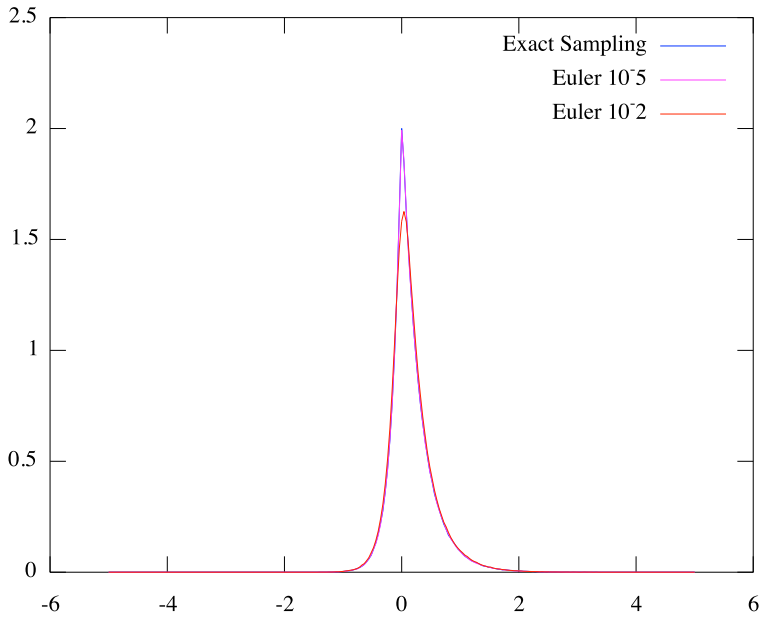

Figure 3.6: Limit algorithm v.s. Euler Scheme for the case where $\mathbf{b}$ is given by (3.8.5) ($x = 0.0$ and $T = 1$).

Chapter 4

Other research topics: a selection

This Chapter presents shortly some other research topics I have been involved in during the past years. It is divided in three parts: adaptive stratification ([23][21]), stochastic expansion for options paying discrete dividends ([22]) and equivalence of statistical experiences involving stochastic processes ([25][26]).

4.1 Adaptive stratification

Stratified sampling belongs to the panel of variance reduction techniques, used for example widely in quantitative finance ([33]). In order to fix ideas let X a random variable taking values in A, and $f : A \to \mathbb{R}$ s.t. $\mathbb{E}^2 f(X) < \infty$. We wish to estimate $c = \mathbb{E}f(X)$ by Monte Carlo computations.

We partition A into strata A_i's. Thus, instead of approximating c by the standard Monte Carlo estimator

$$\frac{1}{N}\sum_{j=1}^{N} f(X^j), \tag{4.1.1}$$

(here the X^j's are i.i.d. drawings of X) one writes $c = \sum_{i=1}^{I} \mathbb{P}(X \in A_i)\mathbb{E}[f(X)|X \in A_i]$. Then one computes the stratified estimator

$$\sum_{i=1}^{I} \frac{p_i}{N_i} \sum_{j=1}^{N_i} f(X_i^j) = \frac{1}{N}\sum_{i=1}^{I} \frac{p_i}{q_i} \sum_{j=1}^{N_i} f(X_i^j) \tag{4.1.2}$$

where $\sum_i N_i = N$, one has denoted $p_i = \mathbb{P}(X \in A_i)$, and the X_i's are distributed as $X|\{X \in A_i\}$. The drawings X_i^j, $1 \le j \le N^i$, $1 \le i \le I$, are independent. The variance of (4.1.2) is given by $\frac{1}{N}\sum_{i=1}^{I} \frac{p_i^2\sigma_i^2}{q_i}$, where $\sigma_i^2 = \mathbb{V}(X|X \in A_i)$.

Assume the strata are fixed. The choice of the proportions q_i's in (4.1.2) is what we call the *allocation* (or *allocation policy*) issue. By a convexity argument one may see that $q_i = p_i$ (*proportional* allocation) reduces that variance, compared to the one of (4.1.1) . But the optimal allocation is

$$q_i^* = \frac{p_i\sigma_i}{\sum_{j=1}^{I} p_j\sigma_j} \tag{4.1.3}$$

(see [33]). This means that we have better to invest drawings in strata with high probability and where the approximation of $\mathbb{E}f(X_i)$ by $\frac{1}{N_i}\sum_{j=1}^{N_i} f(X_i^j)$ is the less precise. The reached *optimal variance* of (4.1.2) is then

$$\frac{\sigma_*^2}{N} = \frac{1}{N}\Big(\sum_{i=1}^{I} p_i\sigma_i\Big)^2 \le \frac{\mathbb{V}(f(X))}{N}. \tag{4.1.4}$$

But in practice the σ_i^2 are not known, and one is simply led to use proportional allocation ([34]). But one may construct a stratified estimator with *adaptive* allocation policy, which uses the already done

drawings of the X_i's, in order to estimate the σ_i's. The amount of the next drawings to do in each stratum are then corrected, and so on (therefore the term adaptive).

This was the subject of the paper [23], written with Benjamin Jourdain. In this paper we prove a Central Limit Theorem (CLT) for the adaptive stratified estimator, which involves the optimal variance σ_*^2 (Theorem 3.1 in [23]): this means that the estimator is asymptotically optimal. The proof uses the Law of Large Numbers and the CLT for martingales (these tools come from the world of stochastic algorithms and optimization).

Further, in [21], we addressed the more difficult problem of finding the best way to stratify the state space A (with Benjamin Jourdain, Gersende Fort and Éric Moulines).

In order to precise the ideas of this paper we have to precise some notations.

We have $X \sim \mathcal{N}_d(0, I_d)$ (in the applications of interest X is a gaussian vector). We are given $\big(\mathcal{A}_i\big)_{i=1}^I$ a partition of $\mathbb{R}$ and $\mu \in \mathbb{R}^d$ with $||\mu|| = 1$.

The strata are then defined by $A_i^{\mathcal{A},\mu} = \{x \in \mathbb{R}^d : \mu' x \in \mathcal{A}_i\}$.

Note that it is easy to calculate $p_i = \mathbb{P}(X \in A_i^{\mathcal{A},\mu})$ as $\mu' X \sim \mathcal{N}(0,1)$.

Besides, on knows how to sample along $X|X \in A_i^{A,\mu} \sim X|\mu' X \in \mathcal{A}_i$ as $X|\mu' X = z \sim \mathcal{N}_d(uz, (I_d - uu'))$.

We are then in the framework of the reference paper [34].

One may construct a partition $(\mathcal{A}_i)$ of $\mathbb{R}$, using the definition

$$\mathcal{A}_i = \{x \in \mathbb{R} : G^{-1}(\frac{i-1}{I}) < x \leq G^{-1}(\frac{i}{I})\}$$

with $G(x) = \int_{-\infty}^x g(y)dy$ and g a probability density on $\mathbb{R}$.

Besides the allocations q_i are determined by $q_i = \int_{A_i} \chi(y)dy$ where χ is another probability density.

We have the first following result (we recall that the variance of the stratified estimator of $c = \mathbb{E}f(X)$ is proportional to $\sum_i (p_i^2 \sigma_i^2)/q_i$).

Theorem 4.1.1. *Under mild assumptions on f one has*

$$\lim_{I \to \infty} \Big(\sum_{i=1}^I \frac{p_i^2 \sigma_i^2}{q_i} \Big)(\mu, g, \chi) = \int_{\mathbb{R}} [\phi^2 (\zeta_\mu - \psi_\mu^2) \chi^{-1}](y) dy, \tag{4.1.5}$$

where ϕ is the density of $\mathcal{N}(0,1)$ and

$$\zeta_\mu(x) = \mathbb{E}[f(X)|\mu' X = x], \quad et \quad \psi_\mu(x) = \mathbb{E}[f^2(X)|\mu' X = x].$$

This result indicates that, for a large number of strata, the variance of the stratified estimator depends on the allocation policy (through χ) and of the choice of the direction μ. But it is not sensitive to the choice of the partition $(\mathcal{A}_i)$ of $\mathbb{R}$ (as g disappears in the RHS of (4.1.5)).

One thus sets

$$V(\mu) := \big(\sum_{i=1}^I p_i \sigma_i \big)(\mu),$$

it is the optimal standard deviation of the stratified estimator. One will seek for $\text{argmin}_\mu V(\mu)$, using a stochastic gradient type method. In the same time we will use again the used drawings in order to estimate c, by an adaptive stratified estimator (adapting in the same time the allocations and the direction μ).

An arbitrary partition $\big((a_{i-1}, a_i]\big)_{i=1}^I$ of $\mathbb{R}$ is chosen, with large I.

One notes $\nu_i(h, \mu) := \mathbb{E}[h(X)\mathbf{1}_{\{\mu' X \in (a_{i-1}, a_i]\}}]$. One has

$$\sigma_i^2(\mu) = \frac{\nu_i(f^2, \mu)}{p_i(\mu)} - \Big(\frac{\nu_i(f, \mu)}{p_i(\mu)}\Big)^2, \quad V(\mu) = \sum_{i=1}^I \sqrt{\nu_i(1,\mu)\nu_i(f^2,\mu) - \nu_i^2(f,\mu)}, \quad \text{and}$$

$$\nabla_\mu V(\mu) = \sum_{i=1}^I \frac{\nabla_\mu \nu_i(1,\mu)\, \nu_i(f^2,\mu) + p_i(\mu) \nabla_\mu \nu_i(f^2,\mu) - 2\nu_i(f,\mu)\nabla_\mu \nu_i(f,\mu)}{2 p_i(\mu) \sigma_i(\mu)}$$

Thanks to the following result we may compute the gradient by Monte Carlo.

Proposition 4.1.2. *Under mild assumptions on $h : \mathbb{R}^d \to \mathbb{R}$ one has*

$$\nabla_\mu \nu_i(h,\mu) = \phi(a_{i-1})\mathbb{E}[Xh(X)|\mu' X = a_{i-1}] - \phi(a_i)\mathbb{E}[Xh(X)|\mu' X = a_i].$$

Using these considerations we produce a global algorithm that shows good behavior compared to the initial approach in [34] (see tables at the end of [21]).

4.2 Stochastic expansion for the pricing of call options with discrete dividends

This part is based on the paper [22] written with Emmanuel Gobet.

A maturity $T < \infty$ is fixed. We consider European Call options on an underlying asset (stock) that pays discrete dividends. More precisely, at the dividend payment dates $0 < t_1 < \ldots < t_n \leq T$, the stock, whose price process is denoted $S^{(y,\delta)}$, pays

$$\delta_i + y_i S^{(y,\delta)}_{t_i-}.$$

So that

$$S^{(y,\delta)}_{t_i} = S^{(y,\delta)}_{t_i-} - [\delta_i + y_i S^{(y,\delta)}_{t_i-}] = S^{(y,\delta)}_{t_i-}(1-y_i) - \delta_i. \tag{4.2.1}$$

Between two dates t_i's the process $S^{(y,\delta)}$ follows a Black-Scholes dynamic

$$dS^{(y,\delta)}_t = \sigma_t S^{(y,\delta)}_t dW_t + (r_t - q_t)S^{(y,\delta)}_t dt. \tag{4.2.2}$$

Here W is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}$ (the coefficients σ_t, r_t, q_t are deterministic). We denote S_0 the initial value of $S^{(y,\delta)}$.

We introduce the notations $D_t = \exp(-\int_0^t (r_s - q_s)ds)$ and $M_t = \exp(\int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t \sigma_s^2 ds)$.

Note that $(S_0 M_t/D_t)_{t\geq 0}$ solves the SDE given by (4.2.2), with initial condition S_0. We stress that $(S_0 M_t/D_t)_{t\geq 0}$ follows this Black-Scholes dynamic also at times t_i's, so that its paths are continuous, contrary to those of $S^{(y,\delta)}$.

We will denote $\pi_{0,n} = \prod_{i=1}^n (1-y_i)$ and $\hat{\delta}_i = \frac{D_{t_i}}{D_T}\delta_i \prod_{j=i+1}^n (1-y_j)$.

Using (4.2.1) and the dynamic (4.2.2) it is possible to show by induction the following lemma.

Lemma 4.2.1. *We have* $S^{(y,\delta)}_T = \pi_{0,n} S_0 \dfrac{M_T}{D_T} - \sum_{i=1}^n \hat{\delta}_i \dfrac{M_T}{M_{t_i}}$.

The idea is then the following (we present things at order 2 in order to simplify).

We take h_N smooth, and we want to compute $\mathbb{E}(e^{-\int_0^T r_s ds} h_N(S^{(y,\delta)}_T - K))$.

Afterwards we will make h_N tend to $h(x) = (x)_+$ in order to get the price of the Call option. One has

$$S^{(y,\delta)}_T = \pi_{0,n} S_0 \frac{M_T}{D_T} - \sum_{i=1}^n \hat{\delta}_i (1 + \frac{M_T}{M_{t_i}} - 1) = \pi_{0,n} S_0 \frac{M_T}{D_T} - \sum_{i=1}^n \hat{\delta}_i + \sum_{i=1}^n \hat{\delta}_i (\frac{M_T}{M_{t_i}} - 1).$$

$$\begin{aligned}\text{Thus,}\quad \mathbb{E}\big[e^{-\int_0^T r_s ds} h_N(S^{(y,\delta)}_T - K)\big] &= \mathbb{E}\big[e^{-\int_0^T r_s ds} h_N(\pi_{0,n} S_0 \frac{M_T}{D_T} - K^{(y,\delta)})\big] \\ &\quad - \sum_{i=1}^n \hat{\delta}_i \mathbb{E}\left[e^{-\int_0^T r_s ds} h'_N(\pi_{0,n} S_0 \frac{M_T}{D_T} - K^{(y,\delta)})(\frac{M_T}{M_{t_i}} - 1)\right] \\ &\quad + \mathrm{Error}_2(h_N)\end{aligned}$$

with $K^{(y,\delta)} = K + \sum_{i=1}^n \hat{\delta}_i$. Using

$$\mathbb{E}[e^{-\int_0^T r_s ds} h_N^{(m)}(\alpha S_0 \frac{M_T}{D_T} - k)] = (-1)^m \partial_k^m \mathbb{E}[e^{-\int_0^T r_s ds} h_N(\alpha S_0 \frac{M_T}{D_T} - k)],$$

one has for example

$$-\mathbb{E}[e^{-\int_0^T r_s ds} h'_N(\pi_{0,n} S_0 \frac{M_T}{D_T} - K^{(y,\delta)})] = \partial_k \mathbb{E}[e^{-\int_0^T r_s ds} h_N(\pi_{0,n} S_0 \frac{M_T}{D_T} - k)]|_{k=K^{(y,\delta)}}.$$

For terms of type $\mathbb{E}[e^{-\int_0^T r_s ds} h'_N(\pi_{0,n} S_0 \frac{M_T}{D_T} - K^{(y,\delta)}) \frac{M_T}{M_{t_i}}]$, one may see $\frac{M_T}{M_{t_i}}$ as a Girsanov type density. One thus obtains

$$\mathbb{E}[e^{-\int_0^T r_s ds} h'_N(\pi_{0,n} S_0 \frac{M_T}{D_T} - K^{(y,\delta)}) \frac{M_T}{M_{t_i}}] = \mathbb{E}[e^{-\int_0^T r_s ds} h'_N(\pi_{0,n} e^{\int_{t_i}^T \sigma_s^2 ds} S_0 \frac{M_T}{D_T} - K^{(y,\delta)})],$$

and finally

$$\mathbb{E}[e^{-\int_0^T r_s ds} h'_N(\pi_{0,n} S_0 \frac{M_T}{D_T} - K^{(y,\delta)}) \frac{M_T}{M_{t_i}}] = -\partial_k \mathbb{E}\big[e^{-\int_0^T r_s ds} h_N(\pi_{0,n} e^{\int_{t_i}^T \sigma_s^2} S_0 \frac{M_T}{D_T} - k)\big]\big|_{k=K^{(y,\delta)}}.$$

We get the following result.

Theorem 4.2.2. *For smooth h_N one has*

$$\begin{aligned}
&\mathbb{E}\big[e^{-\int_0^T r_s ds} h_N(S_T^{(y,\delta)} - K)\big] = \mathbb{E}\big[e^{-\int_0^T r_s ds} h_N(\pi_{0,n} S_0 \frac{M_T}{D_T} - K^{(y,\delta)})\big] \\
&+ \sum_{i=1}^n \hat{\delta}_i \Big(\partial_k \mathbb{E}\big[e^{-\int_0^T r_s ds} h_N(\pi_{0,n} e^{\int_{t_i}^T \sigma_s^2} S_0 \frac{M_T}{D_T} - k)\big]\big|_{k=K^{(y,\delta)}} - \partial_k \mathbb{E}\big[e^{-\int_0^T r_s ds} h_N(\pi_{0,n} S_0 \frac{M_T}{D_T} - k)\big]\big|_{k=K^{(y,\delta)}} \Big) \\
&\quad + \mathrm{Error}_2(h_N) \qquad (4.2.3)
\end{aligned}$$

In order to get an approximated formula for the Call option price one needs to control $\mathrm{Error}_2(h_N)$ uniformly in N. One has the following result.

Proposition 4.2.3. *For smooth h_N with $|h'_N|_\infty = 1$ one has (with $\underline{\sigma} \leq \sigma_t \leq \overline{\sigma}$),*

$$|\mathrm{Error}_2(h_N)| \leq c(1 + S_0^p) \sup_i \Big(\delta_i \overline{\sigma} \sqrt{T - t_i} \Big)^2.$$

In the above proposition the parameter p is involved in our smoothness assumptions on h_N. The idea of the proof of the proposition is the following. From our second order Taylor expansion comes the fact that

$$\mathrm{Error}_2(h_N) = \mathbb{E}\left(e^{-\int_0^T r_s ds} \Big(\sum_{i=1}^n \hat{\delta}_i (\frac{M_T}{M_{t_i}} - 1) \Big)^2 \int_0^1 \frac{(1-\lambda)^2}{2} h''_N(F_T^\lambda - K) d\lambda \right)$$

where $\forall 0 \leq \lambda \leq 1$ one has defined

$$F_T^\lambda := S_0 \frac{M_T}{D_T} - \sum_{i=1}^n \hat{\delta}_i - \lambda \sum_{i=1}^n \hat{\delta}_i (\frac{M_T}{M_{t_i}} - 1).$$

The issue is thus now to control quantities of type

$$\mathbb{E}\big[(\frac{M_T}{M_{t_i}} - 1)(\frac{M_T}{M_{t_j}} - 1) h''_N(F_T^\lambda - K) \big].$$

As we wish to take $h_N(x) \to h(x) = (x)_+$ we cannot use estimates on h''_N (they will explode as N tends to infinity).

But one may use Malliavin calculus to go back to an expression involving h'_N (the interest is that this function satisfies $|h'_N|_\infty = 1$). One has

$$\mathbb{E}\big[(\frac{M_T}{M_{t_i}} - 1)(\frac{M_T}{M_{t_j}} - 1) h''_N(F_T^\lambda - K) \big] = \mathbb{E}[h'_N(F_T^\lambda - K) H_{ij}^{1,\lambda}],$$

where $H_{ij}^{1,\lambda}$ is the Malliavin weight of order 1 (we have used the Malliavin integration by parts formula; see [58]). These weights do not depend on h_N, neither on N. The most technical part of [22] (10 pages) consists then in controlling the Hölder norm of weights $H_{ij}^{1,\lambda}$. The computations imply the norms $\mathbb{D}^{k,p}$, in particular the one of $\mathcal{D}_\cdot(F_T^\lambda)$ (we use the common notations in [58]).

Then it remains to make h_N tend to h, so that we can replace $h_N(x)$ by $h(x) = (x)_+$ in Theorem 4.2.2.

We denote by $\mathrm{Call}^{BS}(x,k)$ the price of a Call option with spot x and strike k in the Black-Scholes model with coefficients σ_t, r_t, q_t. We have clearly for example

$$\mathbb{E}\big[e^{-\int_0^T r_s ds} h(\pi_{0,n} S_0 \frac{M_T}{D_T} - K^{(y,\delta)})\big] = \mathrm{Call}^{BS}(S_0, K^{(y,\delta)})$$

(this comes from the remark on the dynamic of $(S_0 \frac{M_t}{D_t})_{t \geq 0}$). One finally gets the following result.

Theorem 4.2.4. *One has*

$$\begin{aligned}\mathbb{E}(e^{-\int_0^T r_s ds}(S_T^{(y,\delta)}-K)_+) &= \mathrm{Call}^{BS}\left(\pi_{0,n}S_0, K^{(y,\delta)}\right)\\ &+\sum_{i=1}^n \hat{\delta}_i\left(\partial_k \mathrm{Call}^{BS}(\pi_{0,n}S_0 e^{\int_{t_i}^T \sigma_s^2 ds}, K^{(y,\delta)}) - \partial_k \mathrm{Call}^{BS}(\pi_{0,n}S_0, K^{(y,\delta)})\right)\\ &+\mathrm{Error}_2(\mathrm{Call}),\end{aligned}$$

with $|\mathrm{Error}_2(\mathrm{Call})| \leq c\sup_i\left(\frac{\delta_i}{S_0}\sqrt{1-\frac{t_i}{T}}\right)^2 S_0\overline{\sigma}\sqrt{T}$.

That is to say, in order to get the (approximated) price of the option of interest (the one on the asset with discrete dividends), one has to compute a linear combination of price and greeks in the Black-Scholes model (with no discrete dividends). But we have closed formulae for such quantities. Thus, in terms of computation time, this method is particularly competitive (the method is nearly instantaneous). In terms of precision the numerical tests are encouraging if we go up to order 3 or 4. This is what we have done in [22]: the ideas are the same than at order 2, formulae and computations are much longer, but the computation time remains very short (see Tables in [22]; we compare our method to available references on the subject, in particular [11],[73] and [74]).

4.3 Equivalence of statistical experiences involving stochastic processes

One considers two statistical experiences $\mathcal{P} = (A, \mathcal{A}, (P_f,\ f \in F))$ and $\mathcal{Q} = (B, \mathcal{B}, (Q_f,\ f \in F))$, on the same parameter of interest $f \in F$, but for which the observations can be of very different nature (for example A can be a functional space, and B a vectorial space with finite dimension).

Lucien Le Cam ([42]) has developed the notion of equivalence between two statistical experiences: $\mathcal{P}$ and $\mathcal{Q}$ are equivalent if the risk of being wrong about the estimation of f in experience $\mathcal{P}$ is controlled by the one in experience $\mathcal{Q}$, and vice versa (see [50] for a summary about equivalence in the sense of Le Cam). One has then $\Delta(\mathcal{P}, \mathcal{Q}) = 0$, where $\Delta(\cdot,\cdot)$ denotes the (pseudo)-distance of Le Cam. These tools allow for example to "transport" known statistical mathematical results on problem $\mathcal{P}$ towards $\mathcal{Q}$. For example in [15], this type of (asymptotical) equivalence is proved, between a white noise model with unknown non parametric drift function $f(t)$ to estimate, and a gaussian regression model with the same parameter of interest $f(t)$. Thus, one can use known results on the white noise model towards the regression one.

In this spirit we have proved in [25] asymptotical equivalence results between jumps Lévy processes and their discrete counterpart. A horizon $T < \infty$ is fixed and we recall that we denote $D = D([0,T];\mathbb{R})$. For ν in the set of Lévy measure $\mathcal{M}'_{\tilde{\nu}}$ (see [25] for a precise definition; in particular $\tilde{\nu}$ is some dominating measure and for ν in $\mathcal{M}'_{\tilde{\nu}}$ the quantity $\eta_\nu = \int_{|y|\leq 1}|y|\nu(dy)$ is finite) we denote $P^{(\eta_\nu,0,\nu)}$ the law induced on $(D,\mathcal{D})$ by a pure jump Lévy process X with characteristic function

$$\forall u \in \mathbb{R},\ \forall 0 < t \leq T, \quad \mathbb{E}(e^{iuX_t}) = \exp\left(t\int_{\mathbb{R}}(e^{iuy}-1)\nu(dy)\right).$$

Then the first statistical experience we define is

$$\mathcal{P}'^{(\eta_\nu,0,\nu)} = \left(D, \mathcal{D}, \left\{P^{(\eta_\nu,0,\nu)}, \nu \in \mathcal{M}'_{\tilde{\nu}}\right\}\right).$$

The second one (the discrete counterpart) is

$$\mathcal{Q}'^R_m = (\mathbb{N}^{2m^2}, \mathcal{P}(\mathbb{N}^{2m^2}), \{Q^{m,R}_\nu, \nu \in \mathcal{M}'_{\tilde{\nu}}\}),$$

where $Q^\nu_{m,R}$ is the law on $(\mathbb{N}^{2m^2}, \mathcal{P}(\mathbb{N}^{2m^2}))$ of the random vector

$$(R_{-\infty}, \ldots, R_{j,k}, \ldots, R_\infty), \quad j = 1,\ldots,m,\ k = -m,\ldots,m-1$$

where $R_{-\infty} \sim \mathcal{P}oi\big(T\nu((-\infty,m])\big)$, $R_\infty \sim \mathcal{P}oi\big(T\nu((m,\infty))\big)$ and $R_{j,k} \sim \mathcal{P}oi\big(T\nu((k+\frac{j-1}{m}, k+\frac{j}{m}])\big)$ for all $j = 1,\ldots,m,\ k = -m,\ldots,m-1$, these r.v. being independent. We have then:

Theorem 4.3.1. $\Delta(\mathcal{P}'^{(\eta_\nu,0,\nu)}, \mathcal{Q}'^R_m) \xrightarrow{m\to\infty} 0$.

Which means that $\mathcal{P}'^{(\eta_\nu,0,\nu)}$ and $\mathcal{Q}'^R_m$ are *asymptotically* equivalent. One possible interpretation is the following:

Corollary 4.3.2. *So far as the estimation of ν is concerned, observing a Lévy process X of characteristic $(\eta_\nu, 0, \nu)$ asymptotically gives the same amount of information as the* a priori *coarser process $\Big\{\sum_{t\le T} \mathbf{1}_A(\Delta X_t)\Big\}_{A\in\mathcal{A}^m}$ where $\mathcal{A}^m$ is the set defined by $\big\{(-\infty,m], (m,\infty),\ (k+\frac{j-1}{m}, k+\frac{j}{m}],\ k=-m,\dots,m,\ j=1,\dots,m,\ (j,k)\neq(1,0),(m,-1)\big\}$.*

We shortly give an idea of the proof of the theorem (in [25] things are a bit more intricate as we show in the same time two very similar results, varying the assumptions on the considered family of measures). One introduces an intermediate problem $\mathcal{P}_m = (D, \mathcal{D}, \{P^{(\eta_{\bar\nu_m},0,\bar\nu_m)}, \nu\in\mathcal{M}'_{\bar\nu}\})$, where $\bar\nu_m$ is a piecewise constant approximation of ν. Thanks to an Esscher type formula one can compute the density of $P^{(\eta_{\bar\nu_m},0,\bar\nu_m)}$ w.r.t. $P^{(\eta_\nu,0,\nu)}$, and thus control the L^1-distance between $P^{(\eta_{\bar\nu_m},0,\bar\nu_m)}$ and $P^{(\eta_\nu,0,\nu)}$: this distance tends to zero, which yields $\Delta(\mathcal{P}'^{(\eta_\nu,0,\nu)}, \mathcal{P}_m)\to 0$ (thanks to one of the properties of the Le Cam distance). Then, in order to conclude by triangular inequality, it remains to show that at fixed m one has $\Delta(\mathcal{P}_m, \mathcal{Q}'^R_m) = 0$. In order to do that, one shows, using again Esscher type arguments, that the density of $P^{(\eta_{\bar\nu_m},0,\bar\nu_m)}$ w.r.t. $P^{(\eta_{\bar\nu},0,\bar\nu)}$ exhibits a sufficient statistics that has law $Q^\nu_{m,R}$ under $P^{(\eta_{\bar\nu_m},0,\bar\nu_m)}$ (this kind of factorization argument is very common in the literature about Le Cam equivalence).

Ultimately we decided to leave unpublished the note [25]. Indeed we were maybe a bit green on the subject by the time of this work, and several referees complained that the result was pointless from an application perspective (...).

However this work has not been totally lost. First: Ester Mariucci has succeeded in recycling in her further publications some of the computations and ideas in [25] (see in particular [49][51], and [26] that I comment below).

Second: by reorganizing and extending the computation of the L^1-distance in [25], we manage to show a little result that has an interest *per se* about the L^1-distance between the laws of *additive* processes with time-homogeneous Lévy measure. This has been the subject of the paper [26]. By an additive process with time-homogeneous Lévy measure, with characteristic triplet $(f(\cdot), \sigma^2(\cdot), \nu)$, we mean a process X with independent increments, r.c.l.l. paths, and characteristic function

$$\mathbb{E}\big[e^{iuX_t}\big] = \exp\Big(iu\int_0^t f(r)dr - \frac{u^2}{2}\int_0^t \sigma^2(r)dr - t\int_{\mathbb{R}}(1-e^{iuy}+iuy\mathbf{1}_{|y|\le 1})\nu(dy)\Big)$$

(note that here the Lévy measure ν is leaved time-homogeneous, therefore the terminology). We denote $P_T^{(f,\sigma^2,\nu)}$ the law induced on $(D,\mathcal{D})$ by such a process (the subscript stresses the time horizon dependency). Our result is the following. We denote ϕ the density function of the standard normal law.

Theorem 4.3.3. *Let ν_1 and ν_2 two Lévy measures such that ν_1 is absolutely continuous with respect to ν_2 and satisfying: $L^1(\nu_1,\nu_2)<\infty$ Then, the following upper bounds hold, for any $0<T<\infty$.*

If $\sigma^2>0$ then

$$L^1\Big(P_T^{(f_1,\sigma^2,\nu_1)}, P_T^{(f_2,\sigma^2,\nu_2)}\Big) \le 2\sinh\Big(TL_1(\nu_1,\nu_2)\Big) + 2\Big[1-2\phi\Big(-\frac{\xi}{2}\Big)\Big].$$

If $\sigma^2=0$ and $f_1-f_2\equiv\gamma^{\nu_1}-\gamma^{\nu_2}$, then

$$L^1\Big(P_T^{(f_1,0,\nu_1)}, P_T^{(f_2,0,\nu_2)}\Big) \le 2\sinh\Big(TL^1(\nu_1,\nu_2)\Big).$$

This result has to be compared with the one of Mémin and Shiryayev in [57]. Our assumptions (see [26] for the very precise ones) are a bit more restrictive, but when applicable our bound is sharper. And our proof is much simpler as, instead of relying on the general theory of (r.c.l.l.) semimartingales, it only uses Girsanov and Esscher type changes of probability, in order to explicit the density of $P_T^{(f_1,\sigma^2,\nu_1)}$ w.r.t. $P_T^{(f_2,\sigma^2,\nu_2)}$, which allows to compute and bound the L^1-distance of interest (we have used the tools to be found in [66, 67]).

Concluding remarks and perspectives

Concerning asymmetric diffusions, and now that their existence and their link with PDE is established in a one-dimensional and time-inhomogeneous context, one could ask what could be done about simulation in this context. What comes immediately in mind is to use an adaptation of the method in [55]: indeed we have identified in the proof of Theorem 1.4.5 the time-inhomogeneous transformation that removes the local time in (1.1.1), and we know that the Euler scheme for a SDE with time-inhomogeneous discontinuous coefficients converges ([81]). Another arising question is if we are able to simulate exactly the ISBM, and maybe to take advantage of this to simulate exactly some simple time-inhomogeneous SDELT. On a more theoretical level one could wonder if the theory generalized Dirichlet forms coud help to study time-inhomogeneous SDELT with coefficients having even less smoothness (the publication [65] presents some results in this direction; see some of our comments in [30]).

But in fact the most relevant challenge now is to handle the problem of simulation of asymmetric diffusions in a multi-dimensional context. Indeed, for the time being, most of the works on asymmetric diffusions (simulation with or without discretization error, theoretical existence issues, etc...) are done in dimension one. This is partly because in order to manipulate SDELTs, they use tools that are particular to dimension one (such as the Itô-Tanaka formula). But divergence form operators of type $\frac{\rho}{2}\nabla(\mathbf{a}\nabla)$ still make sense in dimension $d > 1$, and still generate a Markov process (here $\mathbf{a}$ is some $\mathbb{R}^{d\times d}$-valued discontinuous coefficient). But the stochastic dynamic of such a process is then difficult to describe. Among the very few papers exploring this direction we can cite [12], where $\mathbf{a}(x) = a(x)I$, with $a(x)$ a discontinuous coefficient with scalar values (I is the identity matrix). The case of a non-diagonal matrix valued coefficient remains largely to explore. A possibility would be to use an Euler scheme, and to correct its behavior when it crosses the interfaces where $\mathbf{a}$ is discontinuous. To that aim, we could look at the analysis of the Euler scheme performed in [55] for $d = 1$, and read on this error analysis what is the right correction to apply on the scheme to ensure its convergence. Miguel Martinez and I are planning to investigate these questions.

After discussions with researchers from the PDE and SDE worlds, the idea of a companion project has emerged in my mind: this would be to re-visit the fundamental paper [40] on the transmission PDE problem. Indeed, as already mentioned, things are done in this paper with $\rho \equiv 1$. Bur for us, taking $\rho \neq 1$ makes sense. We managed to treat the case $d = 1$, using change of variable tricks (Chapter 1). But to treat the case $d > 1$ requires to do again the analysis performed in [40]. This is not clear that we will recover in this general multi-dimensional setting all the results obtained in [40] for the case $\rho \equiv 1$. Besides, the proofs of [40] deserve to be rewritten in more modern terms. I have noticed this paper is often cited, but not totally understood. Of course this is a bit far from the world of Probabilities. But I am in contact on this point with Faouzi Triki (LJK), a colleague of Gen Nakamura (cf [31]).

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