Differential games with incomplete information : signals and revelation

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Résumé

Cette thèse concerne les jeux différentiels à somme nulle et à deux joueurs avec information incomplète. La structure de l'information est liée à un signal que reçoivent les joueurs. Cette information est dite symétrique quand la connaissance du signal est la même pour les deux joueurs (il s'agit un signal public), et asymétrique quand les signaux reçus par les joueurs peuvent être différents (il s'agit de signaux privés). Ces signaux sont révélés au cours du jeu. Dans plusieurs situations de tels jeux, il est montré dans cette thèse, l'existence d'une valeur du jeu et sa caractérisation comme unique solution d'une équation aux dérivées partielles.

Le principal apport de notre travail concerne la prise en compte de signaux et de leur révélation pour des jeux différentiels à information incomplète étudiés par Cardaliaguet en 2007 dont les travaux généralisaient un modèle proposé par Aumann-Maschler en 1968 et, dans le cas à information des deux côtés, par Mertens-Zamir en 1972.

Précisons la forme générale de structure d'information des jeu de ce mémoire.

1. Avant que le jeu commence, chaque joueur reçoit un signal aléatoire qui dépend de données, inconnues de lui, que sont la dynamique, les données initiales et les coûts.

2. Pendant le jeu, tous les joueurs observent, avec mémoire parfaite, un signal public qui est fonction de l'état du système et des actions jouées.

Un premier type particulier de structure d'information concerne le cas asymétrique où les deux joueurs n'observent que des actions jouées pendant le jeu. Nous prouvons que ce jeu admet une valeur pour les stratégies mixtes non anticipatives avec délai.

Un second type de structure d'information concerne le cas symétrique où le signal est réduit à la connaissance par les joueurs de l'état du système au moment où celui-ci atteint une cible donnée (les données initiales inconnues sont alors révelées). Pour ce type du jeu, nous avons introduit des stratégies non anticipatives qui dépendent du signal et nous avons obtenu l'existence d'une valeur.

Les coûts des jeux étudiés sont en général de forme coût intégral (en horizon fini ou infini) et coût terminal.

Comme les fonctions valeurs sont en général irrégulières (seulement continues), un des points clefs de notre approche est de prouver des résultats d'unicité et des principes de comparaison pour des solutions de viscosité lipschitziennes de nouveaux types d'équation d'Hamilton-Jacobi-Isaacs associées aux jeux étudiés.

Summary

In this thesis we investigate two-person zero-sum differential games with incomplete information. The information structure is related to a signal communicated to the players during the game. In such games, the information is symmetric if both players receive the same signal (namely it is a public signal). Otherwise, if the players could receive different signals (i.e. they receive private signals), the information is asymmetric. We prove in this thesis the existence of value and the characterization of the value function by a partial differential equation for various types of such games.

Our main contribution is to take into account the notion of signals and their revelation for differential games with incomplete information studied by Cardaliaguet in 2007 which generalized the game models proposed by Aumann-Maschler in 1968 (for the case with incomplete information on one side) and by Mertens-Zamir in 1972 (for the case with incomplete information on both sides).

The general form of information structure of the games studied here is as follows:

1. Before the game begins, each player receives a signal that depends on the unknown data which are the dynamical system, the initial data and the cost functions.

2. During the game, all players observe with perfect memory a public signal in function of the state of the system and the played actions.

A particular situation of such information structure is the case of asymmetric information where both players only observe the played actions during the game. We prove that the value exists for such games with random non-anticipative strategies with delay.

Another type of such information structure is the symmetric case in which the players receive as their signal the current state of the dynamical system at the moment when the state of the dynamic hits a fixed target set (the unknown initial data are then revealed to both players). For this type of games, we introduce the notion of signal-depending non-anticipative strategies with delay and we prove the existence of value with such strategies.

The pay-offs of the games studied here consist of in general a running cost (of finite or infinite horizon) and a terminal cost.

As the value functions are in general irregular (at most continuous), a crucial step of our approach is to prove the uniqueness results and the comparison principles for viscosity solutions of new types of Hamilton-Jacobi-Isaacs equation associated to the games studied in this thesis.

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Introduction

0.1 General Presentation of Differential Games

Differential games consist of a class of continuous time dynamic games. In this manuscript, we are only interested by two-person zero-sum differential games which two players named Player 1 and Player 2. In this section, we present basic concepts and results for differential games of finite horizon. We will comment the infinite horizon case later on.

0.1.1 Game model with complete information

Let us consider some arbitrary finite-dimensional vector space identified as $\mathbb{R}^n$ and a following dynamical system:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), v(t)), & \text{for any } t \in [t_0, T]; \\ x(t_0) = x_0, \end{cases} \tag{P_T}$$

where times $t_0 < T$ are fixed. The function $f : \mathbb{R}^n \times \mathbb{U} \times \mathbb{V} \to \mathbb{R}$ is assumed regular enough (bounded, continuous and Lipschitz continuous on x) and $\mathbb{U}$, $\mathbb{V}$ are some compact metric spaces. The initial state of system (P_T) at t_0 is set to be x_0.

We assume that the above dynamic and the triplet (t_0, T, x_0) are commonly known by both players. The players control the evolution of the system by choosing the pair of controls (u, v): Player 1 chooses the control $u : [t_0, T] \to \mathbb{U}$ while Player 2 chooses the control $v : [t_0, T] \to \mathbb{V}$. To assure the existence of a solution of system (P_T), we impose that admissible controls for the players are Lebesgue measurable controls, namely u belongs to $\mathcal{U}(t_0) := L^1([t_0, T]; \mathbb{U})$ and v belongs to $\mathcal{V}(t_0) := L^1([t_0, T]; \mathbb{V})$. We denote by $t \mapsto X_t^{t_0,x_0,u,v}$ the unique trajectory of the solution of system (P_T) associated to the pair of controls $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$.

We suppose that both players observe all played actions with perfect memory during the game. In other words, for any moment $t \in [t_0, T]$, both players have knowledge of the function $s \mapsto (u, v)|_{[t_0,t]}(s)$. For each pair of admissible controls (u, v), the pay-off of the players is described by the following cost function:

$$\mathcal{J}(t_0, x_0, u, v) := \int_{t_0}^{T} \ell(X_t^{t_0,x_0,u,v}, u(t), v(t))dt + g(X_T^{t_0,x_0,u,v}).$$

The cost function $\mathcal{J}$ consists of a running cost $\ell : \mathbb{R}^n \times \mathbb{U} \times \mathbb{V} \to \mathbb{R}$ and of a terminal cost $g : \mathbb{R}^n \to \mathbb{R}$ at the moment $t = T$. For our two-person zero-sum game, the goal of Player 1 is to minimize the cost $\mathcal{J}(t_0, x_0, u, v)$ while Player 2 tries to maximize it.

Observe that in the game described above, both players have access to all information needed for the optimization of the outcome so we say that it is a differential game with complete information.

For game theory, there are two important questions for such game model:

- To determine the guaranteed result for the players while each player plays optimally.
- To determine the strategy of each player achieving the guaranteed outcome.

In this thesis, we are particularly interested by the first question. Let us denote by α a strategy for Player 1, namely a process that chooses an action to play for each instant -i.e. a measurable control $v \in \mathcal{V}(t_0)$ - in function of the action $u \in \mathbb{U}$ of his opponent. We observe that α can be viewed as a map from $\mathcal{V}(t_0)$ into $\mathcal{U}(t_0)$. Similarly, we call $\beta : \mathcal{U}(t_0) \to \mathcal{V}(t_0)$ a strategy for Player 2. Let us assume that to each pair of strategies (α, β), there exists a unique pair of admissible controls $(u_{\alpha,\beta}, v_{\alpha,\beta})$ compatible with (α, β) (namely, satisfying $\alpha(v_{\alpha,\beta}) = u_{\alpha,\beta}$ and $\beta(u_{\alpha,\beta}) = v_{\alpha,\beta}$). Consequently, one can furthermore assign to each pair of strategies (α, β) a unique pay-off, denoted by $\mathcal{J}(t_0, x_0, \alpha, \beta) := \mathcal{J}(t_0, x_0, u_{\alpha,\beta}, v_{\alpha,\beta})$. We introduce below the upper- and lower-value functions.

Definition 0.1.1 (Value functions)**.** *For any $(t_0, x_0) \in [0, T) \times \mathbb{R}^n$, the upper-value of the differential game is defined by:*

$$V^+(t_0, x_0) := \inf_{\alpha} \sup_{\beta} \mathcal{J}(t_0, x_0, \alpha, \beta),$$

and its lower-value is defined by:

$$V^-(t_0, x_0) := \sup_{\beta} \inf_{\alpha} \mathcal{J}(t_0, x_0, \alpha, \beta).$$

By the above definition, the upper-value $V^+(t_0, x_0)$ is the best result that Player 1 can guarantee during the game and the lower-value $V^-(t_0, x_0)$ is the best result that Player 2 can guarantee. When $V^+ = V^-$, the game is said to have a value. This is an important property for two-person zero-sum differential games since it modelizes the fact that both players choose their actions simultaneously. Thus, if the game has a value, $V(t_0, x_0)$ is a reasonable choice for expected outcome of the game while both players play rationally.

0.1.2 Strategies

Upper- and lower-values for a differential game may depend on the chosen type of strategy for each player. A first example of strategies for Player 1 in the above game model is that Player 1 will always play a fixed control $u \in \mathcal{U}(t_0)$ no matter what is played by his adversary. Thus elements of $\mathcal{U}(t_0)$ can be considered as "constant" strategies of Player 1. Similarly, Player 2 can always choose to play a fixed control v while ignoring the opponent's actions.

There are various definitions of strategies for differential games in the literature. Here we list several main types of them. We begin with the notion of so-called feedback strategy, which is widely used in the theory of optimal control.

Definition 0.1.2 (Feedback strategy)**.** *A feedback strategy for Player 1 is an application* $\alpha : [t_0, T]\times\mathbb{R}^n \to \mathbb{U}$ *and a feedback strategy for Player 2 is an application* $\beta : [t_0, T]\times\mathbb{R}^n \to \mathbb{V}$.

With this type of strategy, for each moment $t \in [t_0, T]$ the players choose their actions according to their observation of $(t, x(t))$ and thus they are able to profit from their knowledge about the current states of the trajectory. But there are also two major problems. The first one is that such strategies do not depend on the history of the trajectory and thus they neglect part of the information of the players that may be useful. The second one lies in the fact that one can not assure the existence of a solution for system (P_T) while a pair of feedback strategies (α, β) are employed by the players, namely a solution of the following dynamic system may not exist in the used sense [62]:

$$\begin{cases} \dot{x}(t) = f(x(t), \alpha(t, x(t)), \beta(t, x(t))), & \text{for any } t \in [t_0, T]; \\ x(t_0) = x_0. \end{cases}$$

The second issue can be overcome by introducing the notion of positional strategies that we briefly describe now. Let α be a feedback strategy and let Δ: $t_0 = t_1 < t_2 < ... < t_m = T$ be a finite partition of the interval $[t_0, T]$ with $\tau_\Delta > 0$ being the diameter of Δ, namely $\tau_\Delta = \max_{2\leq k\leq m}(t_k - t_{k-1})$. We denote by $\mathcal{X}_\Delta(t_0, x_0, \alpha)$ the set of all trajectories $t \mapsto x(t)$ satisfying:

- There exists a stage control $u_\Delta : [t_0, T] \to \mathbb{U}$ such that for any $1 \leq k \leq m-1$, one has $u_\Delta|_{[t_k,t_{k+1}]} = \alpha(t_k, x(t_k))$.
- The differential inclusion: $x'(t) \in \{f(x(t), u_\Delta(t), v) \mid v \in \mathbb{V}\}$.
- $x(t_0) = x_0$.

Let $\mathcal{D}(t_0)$ be the set of all finite partition of the interval $[t_0, T]$. By the Ascoli's theorem, the set $\cup_{\Delta\in\mathcal{D}(t_0)}\mathcal{X}_\Delta(t_0, x_0, \alpha)$ is relatively compact. We denote by $\mathcal{X}(t_0, x_0, \alpha)$ the set of all trajectories $t \mapsto x(t)$ such that there exist $(\Delta_k)_{k\in\mathbb{N}} \subset \mathcal{D}(t_0)$ and $(x_k(\cdot))_{k\in\mathbb{N}}$ with:

- $\lim_{k\to\infty} \tau_{\Delta_k} = 0$;
- for any $k \in \mathbb{N}$, $x_k(\cdot) \in \mathcal{X}_{\Delta_k}(t_0, x_0, \alpha)$ and $(x_k(\cdot))_{k\in\mathbb{N}}$ converge uniformly to $x(\cdot)$.

It is clear that for any feedback strategy α, the set $\mathcal{X}(t_0, x_0, \alpha)$ is non-empty.

Definition 0.1.3. *[Positional Strategy, Krasovskii and Subbotin [47]] A positional strategy for Player 1 is an application* $\alpha : [t_0, T] \times \mathbb{R}^n \to \mathbb{U}$. *To each positional strategy* α *the set* $\mathcal{X}(t_0, x_0, \alpha)$ *is associated.*

We suppose here that $\ell(\cdot) = 0$ and therefore the cost function becomes $\mathcal{J}(t_0, x_0) = g(X_T^{t_0,x_0,u,v})$. The upper-value function under positional strategies is defined by:

$$V^+(t_0, x_0) := \min_{\alpha} \sup_{x(\cdot)\in\mathcal{X}(t_0,x_0,\alpha)} g(x(T)).$$

The corresponding lower-value function can be defined symmetrically.

Another type of strategies is the following:

Definition 0.1.4 (Non-anticipative strategy, Varaiya [69], Roxin [62], Elliott and Kalton [32]). *A non-anticipative strategy for Player 1 is an application: $\alpha : \mathcal{V}(t_0) \to \mathcal{U}(t_0)$ such that for any $t \in [t_0, T]$ and v_1, $v_2 \in \mathcal{V}(t_0)$, if $v_1|_{[t_0,t]} = v_2|_{t_0,t}$, then $\alpha(v_1)|_{[t_0,t]} = \alpha(v_2)|_{[t_0,t]}$.*

A non-anticipative strategy for Player 2 can be defined symmetrically. By playing non-anticipative strategies, the players chooses their action according to the history of observations. Thus the class of non-anticipative strategies is richer than that of feedback strategies. But still, it is possible that one can not associate to all pair of non-anticipative strategies (α, β) a pair of admissible controls (u, v) such that $\alpha(v) = u$ and $\beta(u) = v$. To overcome this, we restrict ourselves to a smaller class of strategies, i.e. the following:

Definition 0.1.5. *[Non-anticipative strategy with delay (NAD Strategy), cf. Cardaliaguet and Quincampoix [21]] An NAD strategy for Player 1 is an application $\alpha : \mathcal{V}(t_0) \to \mathcal{U}(t_0)$ such that there exists $\tau > 0$, for any $t \in [t_0, T)$ and v_1, $v_2 \in \mathcal{V}(t_0)$, if $v_1|_{[t_0,t]} = v_2|_{[t_0,t]}$ a.e., then $\alpha(v_1)|_{[t_0,t+\tau]} = \alpha(v_2)|_{[t_0,t+\tau]}$ a.e..*

NAD strategies for Player 2 are defined similarly. Let us denote the set of NAD strategies of Player 1 by $\mathcal{A}_d(t_0)$ and that of Player 2 by $\mathcal{B}_d(t_0)$. With NAD strategies, the game can be written into a normal form.

Lemma 0.1.6 (cf. [21]). *For any pair of NAD strategies (α, β), there exists a unique pair of admissible controls $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ such that $\alpha(v) = u$ and $\beta(u) = v$.*

For $(\alpha, \beta) \in \mathcal{A}(t_0) \times \mathcal{B}(t_0)$, let (u, v) defined as in the above lemma. We denote by $\mathcal{J}(t_0, x_0, \alpha, \beta) := \mathcal{J}(t_0, x_0, u, v)$. This allows us to treat both players symmetrically while defining and investigating the two value functions.

$$\begin{aligned} V^+(t_0, x_0) &= \inf_{\alpha \in \mathcal{A}_d(t_0)} \sup_{\beta \in \mathcal{B}_d(t_0)} \mathcal{J}(x_0, t_0, \alpha, \beta); \\ V^-(t_0, x_0) &= \sup_{\beta \in \mathcal{B}_d(t_0)} \inf_{\alpha \in \mathcal{A}_d(t_0)} \mathcal{J}(x_0, t_0, \alpha, \beta). \end{aligned}$$

In addition, since all NAD strategies are non-anticipative strategies, they preserve the history of observation. We will see in the next section that one can also extend naturally the notion of NAD strategies to define random strategies or mixed strategies.

0.1.3 Methods and Related Works

The theory of differential games is studied since the 40s' by Isaacs [40] and Pontryagin [57, 58, 59]. It has a wide range of applications, in particular, for military issues and in investment theory.

Isaacs has investigated differential games with players employing feedback strategies. Suppose that the following Isaacs condition holds:

$$\forall (x,\xi) \in (\mathbb{R}^n)^2,\ \inf_{u\in\mathbb{U}}\sup_{v\in\mathbb{V}}\langle \xi, f(x,u,v)\rangle + \ell(x,u,v) = \sup_{v\in\mathbb{V}}\inf_{u\in\mathbb{U}}\langle \xi, f(x,u,v)\rangle + \ell(x,u,v). \quad (0.1.1)$$

He proved that, under (0.1.1), if the game has a value and its value function V (defined by $V := V^+ = V^-$) is regular enough (in particular, continuously differentiable), then V is the solution of the following PDE (called the Hamilton-Jacobi-Isaacs equation):

$$\partial_t V(t,x) + \inf_{u\in\mathbb{U}}\sup_{v\in\mathbb{V}}\left[\langle D_x V(t,x), f(x,u,v)\rangle + \ell(x,u,v)\right] = 0. \quad (0.1.2)$$

Isaacs also provided a verification theorem, i.e. any C^1 solution of the above PDE is the value of the game.

Differential games are classified into two groups according to Isaacs. The first type is called the qualitative games. In such games, we are only interested in determining whether one of the players can guarantee the victory (by verifying certain conditions). The second one is called quantitative games in which players aim to optimise some pay-off functions. In the latter case, we are able to define value functions. Let us take a pursuit-evasion game as an example with the assumption that Player 1 is the chaser. If we are only interested in knowing whether Player 1 can catch Player 2, then it is a qualitative game. In contrary, if we want to further determine the minimal duration needed for Player 1 to guarantee the capture of Player 2, then it belongs to the quantitative games.

In this manuscript, we only consider quantitative games. For further references on qualitative games, readers may refer to [47, 15] or the survey paper [23].

There are mainly two methods for establishing the existence of value for the game presented in this section. The first one lies on introducing a discretization of the game by choosing controls according to some points of a partition of the time interval $[t_0, T]$. This approach were employed in, for instance, Fleming [34, 35], Friedman [37], Krasovskii & Subbotin [47] and Petrosjan [55]. As we have seen previously in the definition of positional strategies (cf. Definition 0.1.3), in this case, the value functions are defined as the optimal pay-off that a player can guarantee while the diameter of the time partition tends to zero.

Another approach for studying differential games were proposed by Elliott & Kalton [32], Roxin [61], and Varaiya & Lin [70]. It considers the cases with non-anticipative strategies. The principal interest of this method is its relation with the notion of viscosity solutions of PDEs introduced by Crandall and Lions (cf. [29, 30]). It was established by Evans & Souganidis [33] in the 80s'. With this approach, we are able to characterize the value function of a differential game as the unique viscosity solution of a Hamilton-Jacobi equation.

It has been proved that under regularity assumptions on the game, the value functions introduced in Section 0.1.2 (with different types of strategy) are the same. More precisely, they all satisfy the same Hamilton-Jacobi equation.

We present briefly the approach for studying differential games with NAD strategies. It relies on proving a dynamic programming principle respectively for the upper- and lower-value functions, namely for any $t \in (t_0, T)$:

$$V^+(t_0, x_0) = \inf_{\alpha \in \mathcal{A}_d(t_0)} \sup_{\beta \in \mathcal{B}_d(t_0)} \Big(V^+(t, X_t^{t_0,x_0,\alpha,\beta}) + \int_{t_0}^{t} \ell(X_s^{t_0,x_0,\alpha,\beta}, \alpha, \beta) ds \Big)$$

and

$$V^-(t_0, x_0) = \sup_{\beta \in \mathcal{B}_d(t_0)} \inf_{\alpha \in \mathcal{A}_d(t_0)} \Big(V^-(t, X_t^{t_0,x_0,\alpha,\beta}) + \int_{t_0}^{t} \ell(X_s^{t_0,x_0,\alpha,\beta}, \alpha, \beta) ds \Big).$$

Here we denote by $t \mapsto X_t^{t_0,x_0,\alpha,\beta}$ the trajectory of the solution of system (P_T) while the pair of NAD strategies (α, β) is played by the players. An interpretation of the above equations is that to obtain the optimal outcome guaranteed, both players will have to play optimally during $[t_0, t]$ such that the state $x(t)$ optimizes the optimal outcome guaranteed of the game at $(t, x(t))$, namely $V^+(t, x(t))$ for Player 1 and $V^-(t, x(t))$ for Player 2.

Heuristically, by dividing both sides of each equation above by $t - t_0$ and then passing $t \to t_0^+$, one obtain that V^+ is a viscosity sub-solution of:

$$\partial_t W(t, x) + H^+(W(t, x), D_x W(t, x)) = 0,$$

and V^- is a super-viscosity solution of:

$$\partial_t W(t, x) + H^-(W(t, x), D_x W(t, x)) = 0.$$

In these PDEs, the hamiltonians are defined by:

$$H^+(x, \xi) := \inf_{\mathbb{U}} \sup_{\mathbb{V}} \big[\langle f(x, u, v), \xi \rangle + \ell(x, u, v) \big];$$
$$H^-(x, \xi) := \sup_{\mathbb{V}} \inf_{\mathbb{U}} \big[\langle f(x, u, v), \xi \rangle + \ell(x, u, v) \big].$$

The next step is to impose that $H^+ = H^-$ (Isaacs' condition) which implies that V^+ and V^- are respectively viscosity sub- and super-solution of a same PDE:

$$\begin{cases} \partial_t W(t, x) + H(W(t, x), D_x W(t, x)) = 0, \\ W(T, x) = g(x), \text{ for all } x \in \mathbb{R}^n. \end{cases} \tag{0.1.3}$$

where $H := H^+ = H^-$. We notice that, under the Isaacs condition, this is the PDE introduced by Isaacs (0.1.2). Furthermore, one can prove a comparison principle for this PDE: a viscosity sub-solution W_1 of (0.1.3) is always inferior to a viscosity super-solution W_2. Thus $V^+ = V^-$ and $V := V^+$ is the unique viscosity solution of the equation (0.1.3).

Since the existence of a value of the game has been established, now we turn to the determination of optimal strategies for the players if they exist. If $V \in C^1([t_0, T] \times \mathbb{R}^n, \mathbb{R})$, then the optimal feedbacks can be obtained by using the PDE (0.1.3). Otherwise, one can find ϵ-optimal strategies as for example, in [47].

0.2 Differential games with incomplete information

In the previous subsection the presented differential game has complete information. More precisely, both players know as common knowledge the initial data, the dynamic, the cost function, and the history of all played actions. But for economic models, participants of games (negotiations, business competitions etc.) often do not know the cost functions of their adversary or even that of themselves, and not all actions are monitored and memorized. It is under such background that games of incomplete information are introduced and studied.

The theory of two-person zero-sum differential games of incomplete information on both sides was studied by Cardaliaguet [16] (cf. also [10, 5, 6, 11, 12, 17, 19, 21, 25, 24, 26, 27, 43, 54, 55, 56, 67, 66]). It generalizes the notion of two-person zero-sum repeated games of Aumann and Maschler [2] to the context of differential games of Isaacs [40]. This section is devoted to a brief introduction of the theory of differential games with incomplete information and its recent developments.

0.2.1 Game model

For any $m \in \mathbb{N}_*$, we denote by $\Delta(m)$ the $(m-1)$-dimensional simplex, i.e. $\Delta(m) := \{(y_k)_{1\leq k\leq m} \in [0,1]^m \mid \sum_{1\leq k\leq m} y_k = 1\}$. Let us recall the game model proposed by Cardaliaguet [16]. Such a game is defined by:

- The dynamic system (P_T) with the initial state $x_0 \in \mathbb{R}^n$ at the moment $t = t_0 < T$.
- Two finite index sets $I := \{1, 2, ..., I\}$ and $J := \{1, 2, ..., J\}$. (Here the notation I (resp. J) denotes both the index set and its cardinal.)
- A probability measure p on I and a probability measure q on J. We assume that p (resp. q) is an element of $\Delta(I)$ (resp. $\Delta(J)$).
- To each pair $(i, j) \in I \times J$, we associate a terminal cost $g^{ij} : \mathbb{R}^n \mapsto \mathbb{R}$ which is assumed to be bounded and Lipschitz continuous.

We recall that the map $t \mapsto X_t^{t_0,x_0,u,v}$ denotes the trajectory of the solution of the dynamic (P_T). The game is played as follows. Before the game begins, a pair of index (i, j) is chosen randomly according to the probability measure $\pi := p \otimes q \in \Delta(I \times J)$. Its first coordinate i is communicated to Player 1 but not to Player 2. Conversely, the second coordinate j is communicated to Player 2 but not to Player 1. As in the differential game presented in Section 0.1.1, Player 1 chooses the control $u \in \mathcal{U}(t_0)$ in order to minimize the terminal cost $g^{ij}(X_T^{t_0,x_0,u,v})$ while Player 2 aims to maximize $g^{ij}(X_T^{t_0,x_0,u,v})$ by choosing the control $v \in \mathcal{V}(t_0)$. In addition, we suppose that both players observe all played actions with perfect memory.

In this game, both players have their own private a-priori information about the chosen cost function. Therefore the players will try to optimize the pay-off by choosing their actions according to their private information and the observation of the played actions. Hence their choices of actions will reveal in a certain degree their private knowledge to the opponent. To avoid that too much information is transmitted to their adversary, both players need to adopt random (or "mixed") strategies instead of pure strategies (cf. NAD strategies in Definition 0.1.5). Let us consider the following set of probability spaces:

$$\mathbb{S} := \{([0,1]^m, \mathcal{B}([0,1]^m), \mathcal{L}([0,1]^m) \mid m \in \mathbb{N}_*)\}$$

with $\mathcal{B}([0,1]^m)$ and $\mathcal{L}([0,1]^m)$ being respectively the Borel σ-algebra on $[0,1]^m$ and the corresponding Lebesgue measure on $[0,1]^m$.

Definition 0.2.1 (Random non-anticipative strategy with delay (RNAD Strategy) - Finite horizon, cf. [16])**.** *A RNAD strategy for Player 1 is a pair $((\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha), \alpha)$ with $(\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha) \in \mathbb{S}$ and the application $\alpha : \Omega_\alpha \times \mathcal{V}(t_0) \to \mathcal{U}(t_0)$ such that:*

- *α is Borel measurable with Ω_α equipped with $\mathcal{F}_\alpha$ and $\mathbb{U}(t_0)$ equipped with the L^1-distance.*
- *There exists $\tau_\alpha > 0$ such that for any $\omega \in \Omega_\alpha$, $t \in [t_0, T)$ and v_1, $v_2 \in \mathcal{V}(t_0)$, $v_1|_{[t_0,t]} = v_2|_{[t_0,t]}$ a.e. implies that:*

$$\alpha(\omega, v_1)|_{[t_0,t+\tau_\alpha]} = \alpha(\omega, v_2)|_{[t_0,t+\tau_\alpha]}, \ \text{a.e..}$$

A RNAD strategy for Player 2 can be defined similarly.

Throughout the manuscript, the set of RNAD strategies of Player 1 defined above is denoted by $\mathcal{A}_r(t_0)$ and that of Player 2 is denoted by $\mathcal{B}_r(t_0)$. As the pure NAD strategies presented in the previous section, the game can be written into a normal form with RNAD strategies in view of the following:

Lemma 0.2.2 ([16])**.** *For any $(\alpha, \beta) \in \mathcal{A}_r(t_0) \times \mathcal{B}_r(t_0)$ and any $\omega := (\omega_\alpha, \omega_\beta) \in \Omega_\alpha \times \Omega_\beta$, there exists a unique pair of controls $(u_\omega, v_\omega) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ such that:*

$$\alpha(\omega_\alpha, v_\omega) = u_\omega \text{ and } \beta(\omega_\beta, u_\omega) = v_\omega.$$

In addition, the map $\omega \mapsto (u_\omega, v_\omega)$ is measurable with $\mathcal{U}(t_0) \times \mathcal{V}(t_0)$ equipped with the Borel σ-algebra induced by the L^1-distance.

With the notations in the above lemma, we denote for any $(\alpha, \beta) \in \mathcal{A}_r(t_0) \times \mathcal{B}_r(t_0)$ the trajectory $(t, \omega) \mapsto X_t^{t_0,x_0,\alpha,\beta}(\omega) := X_t^{t_0,x_0,u_\omega,v_\omega}$. Furthermore, we define for any $(i,j) \in I \times J$:

$$\mathbb{E}_{\alpha,\beta}[g^{ij}(X_T^{t_0,x_0,\alpha,\beta})] := \int_{\Omega_\alpha \times \Omega_\beta} g^{ij}(X_T^{t_0,x_0,\alpha,\beta}(\omega)) d\mathbb{P}_\alpha \otimes d\mathbb{P}_\beta(\omega).$$

We notice that in order to profit from their private information, Player 1 should choose a vector of RNAD strategies $A := (\alpha_i)_{i\in I} \in \mathcal{A}_r(t_0)^I$ (an RNAD strategy for each $i \in I$) and Player 2 should adopt a vector $B := (\beta_j)_{j\in J} \in \mathcal{B}_r(t_0)^J$. In this case, for a pair of strategies (A, B), the associated expectation of outcome is defined by:

$$\mathcal{J}(t_0, x_0, p, q, A, B) := \sum_{(i,j)\in I\times J} p_i q_j \mathbb{E}_{\alpha_i,\beta_j}[g^{ij}(X_T^{t_0,x_0,\alpha_i,\beta_j})].$$

We suppose the following Isaacs condition: for any $(x, \xi) \in (\mathbb{R}^n)^2$,

$$H^+(x,\xi) := \inf_{\mathbb{U}} \sup_{\mathbb{V}} \langle f(x,u,v),\xi\rangle = \sup_{\mathbb{V}} \inf_{\mathbb{U}} \langle f(x,u,v),\xi\rangle =: H^-(x,\xi). \tag{0.2.1}$$

The main result in [16] is the following:

Theorem 0.2.3 (Existence of value)**.** *Under Isaacs' condition* (0.2.1)*, the game has a value, Namely:*

$$\begin{aligned} V(t_0,x_0,p,q) &:= \inf_{A\in\mathcal{A}_r(t_0)^I} \sup_{B\in\mathcal{B}_r(t_0)^J} \mathcal{J}(t_0,x_0,p,q,A,B) \\ &= \sup_{B\in\mathcal{B}_r(t_0)^J} \inf_{A\in\mathcal{A}_r(t_0)^I} \mathcal{J}(t_0,x_0,p,q,A,B). \end{aligned}$$

In addition, V is bounded, Lipschitz continuous, convex on p, and concave on q.

The major difficulty in obtaining the above result comes from the asymmetry of the information structure. Cardaliaguet has proved in [16] that, instead of a dynamic programming principle for the value function itself as in Section 0.1.3, one can obtain respectively a super-dynamic programming principle for the concave conjugate of V on q and a sub-dynamic programming principle for the convex conjugate of V on p. Such method was introduced by De Meyer [31] for dealing with similar information structure in the theory of repeated games.

Now let us turn to the characterization of the value function V. There are two types of characterization which are equivalent. Let us denote, for any $(x_0, \xi) \in (\mathbb{R}^n)^2$:

$$(x_0,\xi) \mapsto H^*(x_0,\xi) := -H(x_0,-\xi).$$

We remark that V is convex on p and concave on q. Its Fenchel convex and concave conjugates are defined as follows:

$$\begin{aligned} V^*(t_0,x_0,\hat{p},q) &:= \sup_{p\in\Delta(I)} \big\{\langle \hat{p},p\rangle - V(t_0,x_0,p,q)\big\}; \\ V^{\#}(t_0,x_0,p,\hat{q}) &:= \inf_{q\in\Delta(J)} \big\{\langle \hat{q},q\rangle - V(t_0,x_0,p,q)\big\}. \end{aligned}$$

The first characterization is proposed in [16], it relies on the notion of dual viscosity solutions of a Hamilton-Jacobi equation:

Proposition 0.2.4 ([16]). *For any $(\hat{p}, q) \in \mathbb{R}^I \times \Delta(J)$ fixed, the function $(t_0, x_0) \mapsto V^*(t_0, x_0, \hat{p}, q)$ is a viscosity sub-solution of the dual Hamilton-Jacobi equation:*

$$\partial_t W + H^*(x, D_x W) = 0. \tag{0.2.2}$$

For any $(p, \hat{q}) \in \Delta(I) \times \mathbb{R}^J$ fixed, the function $(t_0, x_0) \mapsto V^{\#}(t_0, x_0, p, \hat{q})$ is a viscosity super-solution of equation (0.2.2).

In addition, V satisfies the boundary condition:

$$V(T, x, p, q) = \sum_{(i,j) \in I \times J} p_i q_j g^{ij}(x), \text{for any } x \in \mathbb{R}^n. \tag{0.2.3}$$

V is said to be the unique dual viscosity solution of equation (0.1.3) *with the boundary condition* (0.2.3).

The second characterization is proposed in [17]. Instead of the first-order PDE (0.2.2), Cardaliaguet demonstrated that V can be characterized as unique viscosity solution of a double-obstacle second-order Hamilton-Jacobi-Isaacs equation:

$$\begin{cases} \min\left\{\lambda_{\min}(D^2_{pp} V, p); \max\{\lambda_{\max}(D^2_{qq} W, q); \partial_t W + H(x, D_x W)\}\right\} = 0; \\ \max\left\{\lambda_{\max}(D^2_{qq} V, q); \min\{\lambda_{\min}(D^2_{pp} W, p); \partial_t W + H(x, D_x W)\}\right\} = 0; \\ W(T, x, p, q) = \sum_{(i,j) \in I \times J} p_i q_j g^{ij}(x). \end{cases} \tag{0.2.4}$$

Proposition 0.2.5 ([17]). *The value function V is the unique viscosity solution of* (0.2.4) *in the following sense:*

- *For any $\varphi \in \mathcal{C}^2([0,T] \times \mathbb{R}^n \times \Delta(J), \mathbb{R})$ such that the map $(t, x, q) \mapsto V(t, x, p, q) - \varphi(t, x, q)$ has a local maximum at some point (t_0, x_0, q_0) for some $p \in \Delta(I)$, one has:*

$$\max\{\lambda_{\max}(D^2_{qq}\varphi(t_0, x_0, q_0), q_0); \partial_t \varphi(t_0, x_0, q_0) + H(x_0, D_x \varphi(t_0, x_0, q_0))\} \geq 0.$$

- *Symmetrically, for any $\varphi \in \mathcal{C}^2([0,T] \times \mathbb{R}^n \times \Delta(I), \mathbb{R})$ such that the map $(t, x, p) \mapsto V(t, x, p, q) - \varphi(t, x, p)$ has a local maximum at some point (t_0, x_0, p_0) for some $q \in \Delta(J)$, one has:*

$$\min\{\lambda_{\min}(D^2_{pp}\varphi(t_0, x_0, p_0), p_0); \partial_t \varphi(t_0, x_0, p_0) + H(x_0, D_x \varphi(t_0, x_0, p_0))\} \leq 0.$$

- *For any $x \in \mathbb{R}^n$, $V(T, x, p, q) = \sum_{(i,j) \in I \times J} p_i q_j g^{ij}(x)$.*

0.2.2 Recent developments:

Before we state the results of this PhD thesis, let us present in this subsection some recent developments in the literature of differential games of incomplete information.

The previous subsection was only devoted to the case when the private information $i \in I$ and $j \in J$ for the players are chosen independently from two finite index sets. It was proved in [54] that the game has a value even the pair is chosen according to some probability measure $\pi \in \Delta(I \times J)$ (thus not necessarily independently) under the same regularity assumptions and the Isaacs condition as the game model in [16]. This result was generalized in [74].

For the case where $x_0 \in \mathbb{R}^n$ is chosen randomly among a compact subset of $\mathbb{R}^n$ according to some probability measure ν on $\mathbb{R}^n$ (not necessarily with finite support) and the choice is only communicted to Player 1 but not to Player 2, the existence of value under Isaacs' condition was first proved in [19] and a characterization of the value function was later proposed in [42].

The proofs of the results cited above all rely on finding the proper Isaacs condition. The existence of value for differential games with incomplete information without Isaacs' condition was investigated in [12] and [43]. Results about stochastic differential games of incomplete information can be found in [25, 24, 26].

0.3 Results of the PhD Thesis

In this section we present the main results of this PhD thesis. Each subsection provides a brief introduction to the results of the article included in the associated chapter. All chapters following this introduction are self-contained and they consist of articles that are published, submitted or that will be submitted.

0.3.1 Differential games with asymmetric information - a general case (Chapter 1 and [74])

In this subsection, we present our results of the existence of value for a differential game with asymmetric information that generalizes the game models proposed in in [16] and [54]. The main novelty of our game model is that the private signals communicated to the players are generated randomly according to the unknown initial data. Let us explain describe this differential game in details.

Let $t_0 \in [0, T]$, $n \in \mathbb{N}_*$, $I \in \mathbb{N}_*$, and $X_0 = (x_0^i)_{i \in I} \in (\mathbb{R}^n)^I$. (We recall that I also denotes the set $I := \{1, 2, ..., I\}$.) We consider a family of I two-person zero-sum differential games $\big(\mathcal{G}_i(t_0, x_0^i)\big)_{i \in I}$. Each differential game $\mathcal{G}_i(t_0, x_0^i)$ is defined by:

- a dynamical system:

$$\begin{cases} x'(t) = f^i\big(x(t), u(t), v(t)\big),\ t \in [t_0, T], \\ x(t_0) = x_0^i, \end{cases} \tag{P_i}$$

- a payoff function:

$$\mathcal{J}^i(t_0, x_0^i, u, v) := \int_{t_0}^T \ell^i\big(X_s^{t_0,x_0^i,u,v}, u(s), v(s)\big)ds + g^i(X_T^{t_0,x_0^i,u,v}).$$

As in Section 0.2.1, $\mathbb{U}$, $\mathbb{V}$ are some compact metric spaces. The functions $(f^i : \mathbb{R}^n \times \mathbb{U} \times \mathbb{V} \to \mathbb{R}^n)_{i\in I}$ are supposed regular enough (bounded, continuous and Lipschitz continuous on x uniformly on (u, v)) such that for any $i \in I$, the dynamical system (P_i) admits a unique solution, the trajectory of which is denoted by $t \mapsto X_t^{t_0,x_0^i,u,v}$. The set of admissible controls for Player 1 is $\mathcal{U}(t_0)$ and that of Player 2 is $\mathcal{V}(t_0)$. The functions $\big(\ell^i \in L^1(\mathbb{R}^n \times \mathbb{U} \times \mathbb{V})\big)_{i\in I}$ and $(g^i : \mathbb{R}^n \to \mathbb{R})_{i\in I}$ are respectively the running costs and the terminal costs. In this subsection, we assume that for any $i \in I$:

- ℓ^i is bounded, continuous, and in addition Lipschitz continuous on x uniformly for all $(u, v) \in \mathbb{U} \times \mathbb{V}$.

- g^i is bounded and Lipschitz continuous.

Let $K = \{1, 2, \ ... \ , K\}$ and $L = \{1, 2, \ ... \ , L\}$ be two finite signal sets with K, $L \in \mathbb{N}_*$. We consider $h_1 : I \to K$ and $h_2 : I \to L$ two signal generators. For any $p \in \Delta(I)$, we define a differential game $\tilde{\mathcal{G}}(t_0, X_0, p)$ of incomplete information on both sides by:

1. Before the game begins, an element $i \in I$ is chosen randomly according to the probability $p \in \Delta(I)$. The signal $k = h_1(i)$ is communicated to Player 1 but not to Player 2, and the signal $l = h_2(i)$ is communicated to Player II but not to Player I.

2. The game $\mathcal{G}_i(t_0, x_0^i)$ is played. More precisely, Player 1 chooses the control u and aims to minimize $\mathcal{J}^i(t_0, x_0^i, u, v)$. By choosing the control v, Player 2 aims to maximize $\mathcal{J}^i(t_0, x_0^i, u, v)$.

In addition, we suppose that the probability p and the family $\big(\mathcal{G}_i(t_0, x_0^i)\big)_{i\in I}$ are commonly known by the players, and both players observe all played actions with perfect memory in all sub-games $\mathcal{G}_i(t_0, x_0^i)$.

Since both players receive a private signal before the game begins, they should choose for each possible signal a RNAD strategy. Thus a strategy of Player 1 for game $\tilde{\mathcal{G}}(t_0, X_0, p)$ is a vector in $\mathcal{A}_r(t_0)^K$ and that of Player 2 is a vector in $\mathcal{B}_r(t_0)^L$.

Now let us show that game $\tilde{\mathcal{G}}(t_0, X_0, p)$ consists of a game model with a more general information structure than that of the games in [16, 54]. On one hand, to obtain the differential game defined in [54], we only need to set, in game $\tilde{\mathcal{G}}(t_0, X_0, p)$, $I = K \times L$ and for any $i = (k, l) \in K \times L$:

$$h_1(i) = k \text{ and } h_2(i) = l.$$

On the other hand, let us define, in game $\tilde{\mathcal{G}}(t_0, X_0, p)$, the following function $p \mapsto \pi_p \in \Delta(K \times L)$ with π_p defined as follows:

$$\pi_p(k, l) := \sum_{i \in (h_1, h_2)^{-1}(k,l)} p_i, \text{ for any } (k, l) \in K \times L.$$

It is clear that $\pi_p(k, l)$ describes the probability of a pair of private information (k, l) being communicated to the players before the game begins. But the map $p \mapsto \pi_p$ is in general not an injection. Consequently, in game $\tilde{\mathcal{G}}(t_0, X_0, p)$, the triplet (p, h_1, h_2) can not be replaced by (π_p, h_1, h_2). In other words, the signal structure of game $\tilde{\mathcal{G}}(t_0, X_0, p)$ is "strictly" richer than that in [16, 54] and results from these papers can not be applied directly on our game model.

To adopt the method of De Meyer [31] for treating the information structure of game $\tilde{\mathcal{G}}(t_0, X_0, p)$, we will need to consider a larger class of differential games. For any $\pi \in \Delta(I \times K \times L)$, we define a two-person zero-sum differential game of incomplete information $\mathcal{G}(t_0, X_0, \pi)$ as follows:

1. Before the game begins, an element $(i, k, l) \in I \times K \times L$ is chosen randomly according to the probability π. The second component k is communicated to Player 1 but not to Player 2, and the third component l is communicated to Player II but not to Player I. Both players do not know the exact choice of i.

2. The game $\mathcal{G}_i(t_0, x_0^i)$ is played.

It is clear that by choosing $\pi \in \Delta(I \times K \times L)$ such that $\pi(i, k, l) = p_i$ for all (i, k, l) satisfying $(h_1, h_2)(i) = (k, l)$, the game $\mathcal{G}(t_0, X_0, \pi)$ is equivalent to game $\tilde{\mathcal{G}}(t_0, X_0, p)$.

Similarly as in Section 0.2.1, we associate any $(t_0, X_0, \pi) \in [0, T] \times (\mathbb{R}^n)^I \times \Delta(I \times K \times L)$ and any pair of strategies $(A, B) \in \mathcal{A}_r(t_0)^K \times \mathcal{B}_r(t_0)^L$ a pay-off defined by:

$$\mathcal{J}(t_0, X_0, \pi, A, B) := \sum_{i,k,l} \pi(i, k, l) \mathbb{E}_{\alpha_k, \beta_l} \Big[\int_{t_0}^{T} \ell^i \big(X_s^{t_0, x_0^i, \alpha_k, \beta_l}, \alpha_k, \beta_l \big) ds + g^i(X_T^{t_0, x_0^i, \alpha_k, \beta_l}) \Big]$$

Let us introduce the following Isaacs condition for game $\mathcal{G}(t_0, X_0, \pi)$: for any $(X = (x^i)_{i\in I}, \xi = (\xi^i)_{i\in I}) \in \mathbb{R}^{nI} \times \mathbb{R}^{nI}$ and $\pi \in \Delta(I \times K \times L)$:

$$\begin{aligned} H(X,\xi,\pi) &:= \inf_{u\in\mathbb{U}} \sup_{v\in\mathbb{V}} \sum_{i,k,l} \Big\{ \langle f^i(x^i,u,v), \xi^i\rangle + \pi(i,k,l)\ell^i(x^i,u,v) \Big\} \\ &= \sup_{v\in\mathbb{V}} \inf_{u\in\mathbb{U}} \sum_{i,k,l} \Big\{ \langle f^i(x^i,u,v), \xi^i\rangle + \pi(i,k,l)\ell^i(x^i,u,v) \Big\}. \end{aligned} \tag{0.3.1}$$

Our main result about game $\mathcal{G}(t_0, X_0, \pi)$ is the following:

Theorem 0.3.1. *Under Isaacs' condition* (0.3.1)*, game* $\mathcal{G}(t_0, X_0, \pi)$ *has a value and so does game* $\tilde{\mathcal{G}}(t_0, X_0, p)$*. More precisely:*

$$V(t_0,X_0,\pi) := \inf_{A\in\mathcal{A}_r(t_0)^I} \sup_{B\in\mathcal{B}_r(t_0)^J} \mathcal{J}(t_0,X_0,\pi,A,B) = \sup_{B\in\mathcal{B}_r(t_0)^J} \inf_{A\in\mathcal{A}_r(t_0)^I} \mathcal{J}(t_0,X_0,\pi,A,B).$$

The scheme of proof for the above theorem is as follows:

- Prove that the upper- and lower-value functions are viscosity solutions of the same Hamilton-Jacobi-Isaacs equation under Isaacs' condition (0.3.1).
- Prove that this Hamilton-Jacobi-Isaacs equation has a unique viscosity solution.
- Thus the upper- and lower-value functions must coincide.

The associated Hamilton-Jacobi-Isaacs equation is a second-order double-obstacle PDE taking a similar form as (0.2.4). It will be defined properly in Chapter 1 of this manuscript.

For proving that the upper- and lower-value functions verify the same Hamilton-Jacobi-Isaacs equation, we first establish a sub-dynamic programming principle. The Hamilton-Jacobi-Isaacs equation is obtained by passing to the limit on both sides of the sub-dynamic programming principle. Due the complexity of the information structure, this requires to introduce proper notions of convexity and concavity of V on π [31]. However, in the particular case while $I = K \times L$ and $\pi((k,l),k',l') = 0$ for all $(k,l) \neq (k',l')$, our equation reduces to PDE (0.2.4).

Once we have determined a suitable Hamilton-Jacobi-Isaacs equation and the notion of its viscosity solutions, we state and prove a comparison principle for this equation. It is worth pointing out that the proof of this comparison principle is much more technical than those of the games without signals (cf. [16, 54]). The comparison principle enables us to deduce the uniqueness property of the viscosity solution of the Hamilton-Jacobi-Isaacs equation and consequently we obtain the existence of a value for game $\mathcal{G}(t_0, X_0, \pi)$.

0.3.2 Differential games with incomplete information and signal revelation - the symmetric case (Chapter 2 and [73])

Now we investigate the case while both players observe a more general signal function during the game, taking the form for example:

$$(t, x(t), u(t), v(t)) \mapsto h(t, x(t), u(t), v(t)),$$

with $h : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{V} \to S$ being some measurable application with S a signal space. But such extension also brings a few major difficulties:

1. Firstly, we will need to introduce new notions of strategy with which players can choose their actions in accordance with the observed signals.

2. Secondly, it will be difficult to obtain the continuity of the upper- and lower-value functions of games with strategies that depend on signals. (cf. [75]).

3. Thirdly, with state-depending signals, the game is often non-markov. In such cases, it will be difficult to determine the dynamic programming principles and therefore the associated PDEs.

In view of these above obstacles, we restrict ourselves to the study of differential games with a particular type of signal functions: signal revelation. Let us describe our game model in detail.

We consider the differential game with incomplete information $\mathcal{G}(X_0, r_0, p)$ with dynamic (P'):

$$\begin{cases} \dot{x}(t) = f\big(x(t), u(t), v(t)\big), & t \geq 0; \\ \dot{r}(t) = g\big(r(t), u(t), v(t)\big), & t \geq 0; \\ x(0) = x_0;\ r(0) = r_0. \end{cases} \tag{P'}$$

We assume that both $f : \mathbb{R}^n \times \mathbb{U} \times \mathbb{V} \to \mathbb{R}^n$ and $g : \mathbb{R} \times \mathbb{U} \times \mathbb{V} \to \mathbb{R}^+$ are regular enough such that (P') has a unique solution. To any pair of admissible controls $(u, v) \in \mathcal{U} \times \mathcal{V}$, the following pay-off is associated:

$$\mathcal{J}(x_0, r_0, u, v) := \int_0^\infty e^{-\lambda t} \ell\big((X, R)_t^{x_0, r_0, u, v}, u(t), v(t)\big) dt, \tag{0.3.2}$$

where the map $t \mapsto (X, R)_t^{x_0, r_0, u, v} = (X_t^{x_0, u, v}, R_t^{r_0, u, v})$ denotes the unique solution of dynamic (P') and $\lambda > 0$ is fixed. Let $p \in \Delta(I)$ be a probability measure on I. The game is played in the following way:

(A) Before the game begins, the first initial state $x_0 \in \mathbb{R}^n$ is chosen randomly among I points x_0^1, x_0^2, ..., x_0^I, according to a probability measure $p \in \Delta(I)$. The set $X_0 = \{x_0^1, x_0^2, ..., x_0^I\}$, the other initial state $r_0 \in \mathbb{R}$, and p are common knowledge of both players. However, x_0 is not communicated to both players.

(B) Player 1 chooses the control u from $\mathcal{U}$ in the dynamic and aims to minimize $\mathcal{J}(x_0, r_0, u, v)$. In contrary, Player 2 tries to maximize $\mathcal{J}(x_0, r_0, u, v)$ by choosing the control $v \in \mathcal{V}$. We assume that during the game both players observe all played actions with perfect memory (i.e. for any $t > 0$, the set $\{(s, u(s), v(s)) \mid 0 \leq s < t\}$ is considered as common knowledge of both players).

(C) During the game, if $t \mapsto R_t^{r_0,u,v}$ reaches a fixed target $M_0 > r_0$, the current $X_t^{x_0,u,v}$ is announced publicly to both players at the moment $t = \mathcal{T}(r_0, u, v) := \inf\{t > 0 \mid R_t^{r_0,u,v} = M_0\}$.

Example 0.3.2. *An interesting and simple example of system (P') is the case where g is a constant equal to 1. In this case, if $r_0 = 0$, both players know that they have to play during a time M_0 before having the unknown current states revealed.*

We are interested in the existence of value of the above game and the characterization of its value function as the unique solution of a PDE (in the sense of viscosity solutions).

In contrast to classic pursuit-evasion games (cf. [4, 22, 32, 40, 47, 63, 69, 77]), players in the game $\mathcal{G}(X_0, r_0, p)$ have incomplete information about the initial states and it is with different types of pay-offs. We point out that this game is different from the state constrained differential games (cf. [7, 22]) since players in the game $\mathcal{G}(X_0, r_0, p)$ do not necessarily need to choose their controls such that the dynamic never hits the target set.

We notice that neither player receive private information before the game begins and thus in game $\mathcal{G}(X_0, r_0, p)$, the information structure is symmetric.

Our main interest is to prove that, under Isaacs' condition, the game has a value which is the unique bounded continuous viscosity solution of the following Hamilton-Jacobi-Isaacs equation with boundary condition:

$$\begin{cases} -\lambda V(X,r) + H_p(X, r, D_{X,r}V(X,r)) = 0, & (X,r) \in (\mathbb{R}^n)^I \times (-\infty, M_0); \\ V(X, M_0) = \sum_{i \in I} p_i \tilde{V}(X^i, M_0), & X \in (\mathbb{R}^n)^I. \end{cases} \tag{0.3.3}$$

Let we assume the following Isaacs condition:

$$\forall p \in \Delta(I),\ \forall (X, r, q) \in (\mathbb{R}^n)^I \times \mathbb{R} \times \left((\mathbb{R}^n)^I \times \mathbb{R}\right),\ H_p^+(X, r, q) = H_p^-(X, r, q), \tag{0.3.4}$$

with the hamiltonians defined by:

$$H_p^+(X,r,q) := \inf_{u\in U}\sup_{v\in V}\Big\langle q, \big((f(X^i,u,v))_{i\in I}, g(r,u,v)\big)\Big\rangle + \sum_{i\in I} p_i \ell(X^i,r,u,v);$$
$$H_p^-(X,r,q) := \sup_{v\in V}\inf_{u\in U}\Big\langle q, \big((f(X^i,u,v))_{i\in I}, g(r,u,v)\big)\Big\rangle + \sum_{i\in I} p_i \ell(X^i,r,u,v).$$

Our result is the following:

Theorem 0.3.3. *Under Isaacs' condition* (0.3.4)*, the game has a value. Moreover for any $p \in \Delta(I)$, the value function $(X_0, r_0) \mapsto V(X_0, r_0, p)$ is the unique bounded continuous viscosity solution of the Hamilton-Jacobi-Isaacs equation* (0.3.3)*.*

In the above equation, the function $y \to \tilde{V}(y, M_0)$ is the value function of a classic differential game of infinite horizon (cf. [3] for a detail description). It can also be obtained as a particular case of the differential game of incomplete information in Section 0.3.2 (by setting $I = J = 1$ and $\ell^{1,1} = \ell$ in the value function of the game described in Section 0.3.4).

Such a boundary condition can be interpreted as follows: as $r_0 \to M_0^-$, the value of the game converges to the value of a complete information game (i.e. the choice of x_0 is revealed as soon as the game begins).

As mentioned at the beginning of Section 0.3, we are facing two major difficulties: the first consists in determining proper notions of non-anticipative strategies with delays for modelling behaviours of the players at the "hitting time"; the second one concerns the regularity of the value functions.

0.3.3 Differential games with incomplete information and signal revelation - the general symmetric case (Chapter 3 and [75])

Instead of (P'), we consider again the dynamic system (P) in Section 0.3.2. To each triplet of (x_0, u, v), we associate a following running cost:

$$\mathcal{J}(x_0,u,v) := \int_0^{+\infty} e^{-\lambda t}\ell\big(X_t^{x_0,u,v}, u(t), v(t)\big)dt, \tag{0.3.5}$$

where ℓ is a real value function that is bounded, continuous and Lipschitz continuous on x uniformly with respect of (u, v). For $p \in \Delta(I)$, let us consider $\mathcal{G}(X_0, p)$ a two-person zero-sum differential game with incomplete information and signal revelation played as follows:

(A) Before the game begins, an index $i \in I$ is chosen randomly according to $p = \big(p(i)\big)_{i\in I} \in \Delta(I)$. The chosen index i is not communicated to any player. The set X_0 and the probability measure p are assumed to be common knowledge of both players.

(B) The game begins with $x_0 = x_0^i$ in (P). Player 1 chooses the control u in order to minimize running cost $\mathcal{J}(x_0^i, u, v)$ and Player 2 aims to maximize $\mathcal{J}(x_0^i, u, v)$ by choosing the control v. During the game both players observe all played actions with perfect memory.

(C) During the game, both players also observe with perfect memory a signal:

$$s_{x_0^i,u,v}(t) = \begin{cases} (0,0) \in \mathbb{R} \times \mathbb{R}^n, & \text{if } t < \mathcal{T}(x_0^i, u, v); \\ (1, X^{x_0^i,u,v}_{\mathcal{T}(x_0^i,u,v)}), & \text{if } t \geq \mathcal{T}(x_0^i, u, v). \end{cases}$$

with $\mathcal{T}(x_0^i, u, v) := \inf\{t \geq 0 \mid X_t^{x_0^i,u,v} \in K\}$.

Now we explain the signal structure. During the game, the signal observed by the players has two components: the first one takes binary values which is an indicator for whether the target K has been hit: it is 0 before hitting on K and it turns to 1 after the hitting occurs; the second one indicates the current state of the system at the moment $\mathcal{T}(x_0^i, u, v)$ and it remains constant vector 0 if the system has not reached K. Before the dynamic (P) hits the target K, both players receive no further information and observe a constant pair $(0, 0)$. As soon as $X_t^{x_0^i,u,v}$ reaches K (at the moment $\mathcal{T}(x_0^i, u, v)$), both players receive the pair $(1, X^{x_0^i,u,v}_{\mathcal{T}(x_0^i,u,v)})$ where the integer 1 indicates the "target hit". This avoids the possible confusion while $X^{x_0^i,u,v}_{\mathcal{T}(x_0^i,u,v)} = 0$.

We notice that here the public signal remains the same constant $(0, 0)$ until the target set is hit and then it reveals all information about the current state. This consists of the same type of signal function adopted in [64] and [53] for repeated games.

When the target set is $K = \mathbb{R}^n$ (the initial state is revealed immediately as the game begins) or $K = \emptyset$ (the initial state is never revealed), our game model reduces to a classic differential game with complete information. For non-trivial subset K of $\mathbb{R}^n$, game $\mathcal{G}(X_0, p)$ can be interpreted as a differential game with incomplete information and revelation since the step (C) in the game procedure above is in fact equivalent to:

(C') When $X_t^{x_0^i,u,v}$ hits the target K, the current state (i.e $X_{\mathcal{T}}^{x_0^i,u,v}$) is announced publicly to both players.

This furthermore indicates that the game model described in the previous subsection is as a particular case of game $\mathcal{G}(X_0, p)$. During $\mathcal{G}(X_0, p)$, for profiting from their information, both players should choose their actions in function of the triplet $(u, v, s_{x_0^i,u,v})|_{[0,t]}$ at any moment $t \geq 0$. Let $\mathcal{S}$ denote the set of all possible signals during the game:

$$\mathcal{S} := \{t \mapsto s_{x,u,v}(t),\ (x, u, v) \in \mathbb{R}^n \times \mathcal{U} \times \mathcal{V}\}.$$

We will define in detail a proper metric on $\mathcal{S}$. Here we assume that $\mathcal{S}$ is a metric space equipped with the associated Borel σ-algebra. Inspired by the notion of NAD strategies, we introduce the following:

Definition 0.3.4 (Signal-depending non-anticipative strategy (SNAD-Strategy) of Player 1)**.** *An SNAD strategy of Player 1 is a Borel-measurable map: $\alpha : \mathcal{S} \times \mathcal{V} \to \mathcal{U}$ such that: there exists $\tau_\alpha > 0$, for any v_1, $v_2 \in \mathcal{V}$, s_1, $s_2 \in \mathcal{S}$ and $t \geq 0$, if $v_1|_{[0,t]} = v_2|_{[0,t]}$ a.e. and $s_1|_{[0,t]} = s_2|_{[0,t]}$, then $\alpha(s_1, v_1)|_{[0,t+\tau_\alpha]} = \alpha(s_2, v_2)|_{[0,t+\tau_\alpha]}$ a.e.. The set of such maps is denoted by $\mathcal{A}_s$.*

SNAD strategies of Player 2 can be defined similarly and we denote by $\mathcal{B}_s$ the set of all such strategies of Player 2.

Remark 0.3.5. *It is clear that any NAD strategy can also be viewed as a signal-depending one, thus we have $\mathcal{A}_d \subset \mathcal{A}_s$ and $\mathcal{B}_d \subset \mathcal{B}_s$.*

As NAD strategies, with SNAD strategies, we are able to put game $\mathcal{G}(X_0, p)$ into a normal form: for any pair of SNAD strategies $(\alpha, \beta) \in \mathcal{A}_s \times \mathcal{B}_s$ and $x_0 \in \mathbb{R}^n$, there exists a unique pair of controls $(u_{x_0,\alpha,\beta}, v_{x_0,\alpha,\beta})$ such that:

$$\alpha(s_{x_0,u_{x_0,\alpha,\beta},v_{x_0,\alpha,\beta}}, v_{x_0,\alpha,\beta}) = u_{x_0,\alpha,\beta} \text{ and } \beta(s_{x_0,u_{x_0,\alpha,\beta},v_{x_0,\alpha,\beta}}, u_{x_0,\alpha,\beta}) = v_{x_0,\alpha,\beta}.$$

Thus for all (x_0, α, β), there exists a unique solution of system (P) that is compatible with the triplet and we associated to any $(X_0 = (x_0^i)_{i\in I}, \alpha, \beta)$ a following cost:

$$\mathcal{J}(X_0, p, \alpha, \beta) := \sum_{i\in I} p(i) J\big(x_0^i, u_{x_0^i,\alpha,\beta}, v_{x_0^i,\alpha,\beta}\big).$$

Furthermore we define the upper- and lower-value functions of the game by:

$$V^+(X_0, p) = \inf_{\alpha\in\mathcal{A}_s} \sup_{\beta\in\mathcal{B}_s} \mathcal{J}(X_0, p, \alpha, \beta);$$
$$V^-(X_0, p) = \sup_{\beta\in\mathcal{B}_s} \inf_{\alpha\in\mathcal{A}_s} \mathcal{J}(X_0, p, \alpha, \beta).$$

Before we state our main result about the existence of the game, let us first describe the Isaacs condition:

$$\text{for any } X_0 = (x_0^i)_{i\in I} \in (\mathbb{R}^n)^I,\ \xi \in (\mathbb{R}^n)^I \text{ and } p \in \Delta(I), H_p^+(X_0, \xi) = H_p^-(X_0, \xi), \qquad \text{(IC)}$$

where:

$$H_p^+(X_0, \xi) := \inf_{u\in U} \sup_{v\in V} \big\{\langle (f(x_0^i, u, v))_{i\in I}, \xi\rangle + \sum_{i\in I} p(i)\ell(x_0^i, u, v)\big\};$$
$$H_p^-(X_0, \xi) := \sup_{v\in V} \inf_{u\in U} \big\{\langle (f(x_0^i, u, v))_{i\in I}, \xi\rangle + \sum_{i\in I} p(i)\ell(x_0^i, u, v)\big\}.$$

Theorem 0.3.6. *Under Isaacs' condition* (IC)*, if the function $x \mapsto \min\{\mathcal{T}(x, u, v), T\}$ is locally Lipschitz continuous for any $T \geq 0$, game $\mathcal{G}(X_0, p)$ has a value. Furthermore, restricted to $\big(\mathbb{R}^n\backslash K\big)^I$, the value function $X_0 \mapsto V(X_0, p)$ is the unique bounded continuous viscosity solution of the Hamilton-Jacobi-Isaacs equation* (0.3.6)*:*

$$-\lambda V(X) + H_p(X, D_X V(X)) = 0. \qquad (0.3.6)$$

There are three main novelties of this new game model. Firstly, we do not require $\mathcal{T}(x_0^i, \alpha, \beta)$ to be bounded (for example, in [73]), i.e. the players are able to stay outside of the target set given proper dynamic system. Secondly, we have introduced the notion of SNAD strategies with delay which allows the players to choose their actions according to the observation of played actions and the signal. Such strategies are more complicate than classic NAD strategies and we will see that one can extend it naturally to define the notion of mixed strategies which plays an important role in the next subsection. Thirdly, the equality between the upper-value and the lower-value does not automatically hold while X_0 is on the boundary of the open set $(\mathbb{R}^n \backslash K)^I$ (in comparison with Bolza problems in [16, 54, 74] or the pursuit-evasion games listed in Section 0.3.3), this means that the upper- and lower-value functions do not automatically verify the boundary condition in the comparison principle for the Hamilton-Jacobi-Isaacs equation (0.3.6), which brings technical obstacles in the proof of Theorem 0.3.6.

To establish the main result, we prove that, for any $p \in \Delta(I)$ upper-value function $V^+(\cdot, p)$ is a continuous viscosity sub solution of the PDE (0.3.6) on the open set $(\mathbb{R}^n \backslash K)^I$ while the lower-value function $V^-(\cdot, p)$ is a continuous viscosity super-solution on the same open set following the classic scheme in [33]. The major difficulty here is to obtain the regularity of V^+ and V^-. This is due to the fact that $X_0 \mapsto \mathcal{J}(X_0, p, \alpha, \beta)$ is not necessarily continuous for all (α, β). To overcome such obstacle, we prove a two-step programming principle for obtaining an alternative formula of V^+ and V^-:

$$\begin{aligned} V^+(X_0, p) = \inf_{\alpha \in \mathcal{A}_d} \sup_{\beta \in \mathcal{B}_d} \sum_{i \in I} p(i) \Big\{ \int_0^{\mathcal{T}(x_0^i, \alpha, \beta)} e^{-\lambda t} \ell\big(X_t^{x_0^i, \alpha, \beta}, \alpha, \beta\big) dt + \\ e^{-\lambda \mathcal{T}(x_0^i, \alpha, \beta)} \tilde{V}^+(X^{x_0^i, \alpha, \beta}_{\mathcal{T}(x_0^i, \alpha, \beta)}) \Big\}; \\ V^-(X_0, p) = \sup_{\beta \in \mathcal{B}_d} \inf_{\alpha \in \mathcal{A}_d} \sum_{i \in I} p(i) \Big\{ \int_0^{\mathcal{T}(x_0^i, \alpha, \beta)} e^{-\lambda t} \ell\big(X_t^{x_0^i, \alpha, \beta}, \alpha, \beta\big) dt + \\ e^{-\lambda \mathcal{T}(x_0^i, \alpha, \beta)} \tilde{V}^-(X^{x_0^i, \alpha, \beta}_{\mathcal{T}(x_0^i, \alpha, \beta)}) \Big\}. \end{aligned}$$

In the above equations, $\mathcal{T}(x_0^i, \alpha, \beta)$ denotes the "hitting time" of the dynamical system on the target set K while i is chosen before the game begins and the pair of strategies (α, β) is played by the players. It is not difficult to prove that the right-hand sides of both equations above are continuous on X_0 given the regularity assumption on $x_0^i \mapsto \mathcal{T}(x_0^i, \alpha, \beta)$.

Next, we prove a comparison principle about the PDE (0.3.6) which indicates that $V^+ \leq V^-$ if $V^+ = V^-$ on $\big(\partial(\mathbb{R}^n \backslash K)^I\big) \times \Delta(I)$. In the end, as an application of the above two-step programming principles, we are able prove that $V^+ = V^-$ on $\big(\partial(\mathbb{R}^n \backslash K)^I\big) \times \Delta(I)$ by recurrence on I, which is the required boundary condition of the comparison principle.

0.3.4 Games with asymmetric information - the infinite horizon case (Chapter 4 and [76])

Since in this thesis we will be working on differential games of infinite horizon, we provide here the results for the existence of a value for infinite horizon differential games with a information structure of the type of Section 0.2.1. Existence of a value for two-person zero-sum differential games of complete information is a classic result, for a proof with non-anticipative strategies, readers may refer to [3]. Further results about other types of infinite horizon differential games can be found in, for instance, [3] and [14].

Let us consider a two-person zero-sum differential game with incomplete information $\mathcal{G}(x_0, p, q)$ defined by:

- The dynamical system (P):

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), v(t)), & \text{for any } t \geq 0; \\ x(0) = x_0. \end{cases} \tag{P}$$

- A pair of finite index sets I, J defined as in Section 0.2.1.
- A pair of probability measures $p \in \Delta(I)$ and $q \in \Delta(J)$.
- A family of running costs: $(\ell^{ij} : \mathbb{R}^n \times \mathbb{U} \times \mathbb{V} \to \mathbb{R})_{(i,j)\in I\times J}$.
- A fixed constant $\lambda > 0$.

Here we suppose that the application $f : \mathbb{R}^n \times \mathbb{U} \times \mathbb{V} \to \mathbb{R}^n$ is bounded, continuous and in particular Lipschitz continuous on x uniformly on (u, v). Under such condition, the dynamic (P) has a unique solution for any pair of Lebesgue measurable controls (u, v) and we denote it by $t \mapsto X_t^{x_0,u,v}$. Let us denote by $\mathcal{U}$ the set of Lebesgue measurable control $u : \mathbb{R}_+ \to \mathbb{U}$ and we denote by $\mathcal{V}$ the set of Lebesgue measurable control $v : \mathbb{R}_+ \to \mathbb{V}$.

The game is played as follows. Before the game begins, a pair of index $(i, j) \in I \times J$ is chosen randomly according to the probability measure $\pi := p \otimes q$. i is communicated to Player 1 but not to Player 2. j is communicated to Player 2 but not to Player 1. During the game, Player 1 chooses a control $u \in \mathcal{U}$ to minimize a following pay-off:

$$\mathcal{J}^{ij}(x_0, u, v) := \int_0^{+\infty} e^{-\lambda t} \ell^{ij}(X_t^{x_0,u,v}, u(t), v(t))dt.$$

Player 2 chooses a control $v \in \mathcal{V}$ to maximize $\mathcal{J}^{ij}(x_0, u, v)$. We assume as in the previous subsections that both players observe all played actions with perfect memory.

We equip the space $\mathcal{U}$ with the following metric and the associated Borel σ-algebra.

$$d_{\mathcal{U}}(u_1, u_2) := \int_0^{+\infty} e^{-t} d_{\mathbb{U}}(u_1(t), u_2(t)) dt.$$

Similarly, $\mathcal{V}$ is considered a metric space with

$$d_{\mathcal{V}}(v_1, v_2) := \int_0^{+\infty} e^{-t} d_{\mathbb{V}}(v_1(t), v_2(t)) dt.$$

The definition of RNAD strategies for the players extends naturally to the case of infinite horizon:

Definition 0.3.7 (Random non-anticipative strategy with delay (RNAD Strategy) - Infinite horizon)**.** *A RNAD strategy for Player 1 is a pair* $((\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha), \alpha)$ *with* $(\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha) \in \mathbb{S}$ *and the application* $\alpha : \Omega_\alpha \times \mathcal{V} \to \mathcal{U}$ *such that:*

- α *is Borel measurable with* Ω_α *equipped with* $\mathcal{F}_\alpha$ *and* $\mathbb{U}$ *equipped with the Borel* σ*-algebra induced by the distance* $d_{\mathcal{U}}$*.*
- *There exists* $\tau_\alpha > 0$ *such that for any* $\omega \in \Omega_\alpha$*,* $t \geq 0$ *and* v_1*,* $v_2 \in \mathcal{V}$*,* $v_1|_{[0,t]} = v_2|_{[0,t]}$ *a.e. implies that:*

$$\alpha(\omega, v_1)|_{[0,t+\tau_\alpha]} = \alpha(\omega, v_2)|_{[0,t+\tau_\alpha]}, \ \text{a.e.}.$$

A RNAD strategy for Player 2 can be defined similarly. We denote by $\mathcal{A}_r$ *the set of RNAD strategies (for the case of infinite horizon) of Player 1 and that of Player 2 is denoted by* $\mathcal{B}_r$*.*

As RNAD strategies for the case with finite horizon in Section 0.2.1, we have that for any $(\alpha, \beta) \in \mathcal{A}_r \times \mathcal{B}_r$ and any $\omega := (\omega_\alpha, \omega_\beta) \in \Omega_\alpha \times \Omega_\beta$, there exists a unique pair of controls $(u_\omega, v_\omega) \in \mathcal{U} \times \mathcal{V}$ such that:

$$\alpha(\omega_\alpha, v_\omega) = u_\omega \text{ and } \beta(\omega_\beta, u_\omega) = v_\omega.$$

In addition, the map $\omega \mapsto (u_\omega, v_\omega)$ is measurable.

Since both players should choose an RNAD strategy for each possible private signal, a strategy of Player 1 for the game is thus a vector $A \in \mathcal{A}_r^I$ and Player 2 chooses a strategy among $\mathcal{B}_r^J$. We can therefore define similarly the expectation of outcome while a pair of strategies $(A, B) = ((\alpha_i)_{i \in I}, (\beta_j)_{j \in J})$ is played by the players:

$$\mathcal{J}(x_0, p, q, A, B) := \sum_{(i,j) \in I \times J} p_i q_j \mathbb{E}_{\alpha_i, \beta_j} \Big[\int_0^{+\infty} e^{-\lambda t} \ell^{ij}(X_t^{x_0, \alpha_i, \beta_j}, \alpha_i, \beta_j) dt \Big].$$

The Isaacs condition associated to this model takes the form below: for any $(x,\xi,p,q) \in \mathbb{R}^n \times \mathbb{R}^n \times \Delta(I) \times \Delta(J)$:

$$\begin{aligned} H(x,\xi,p,q) &:= \inf_{u\in\mathbb{U}} \sup_{v\in\mathbb{V}} \big\{ \langle f(x,u,v),\xi\rangle + \sum_{(i,j)\in I\times J} p_i q_j \ell^{ij}(x,u,v) \big\} \\ &= \sup_{v\in\mathbb{V}} \inf_{u\in\mathbb{U}} \big\{ \langle f(x,u,v),\xi\rangle + \sum_{(i,j)\in I\times J} p_i q_j \ell^{ij}(x,u,v) \big\}. \end{aligned} \tag{0.3.7}$$

Our main result of this section is the following:

Theorem 0.3.8 (Existence of value)**.** *Under Isaacs' condition* (0.3.7)*, the game has a value:*

$$V(x_0,p,q) := \inf_{A\in\mathcal{A}_r^I} \sup_{B\in\mathcal{B}_r^J} \mathcal{J}(x_0,p,q,A,B) = \sup_{B\in\mathcal{B}_r^J} \inf_{A\in\mathcal{A}_r^I} \mathcal{J}(x_0,p,q,A,B).$$

In addition, V is bounded, continuous, convex on p, and concave on q and it is the unique viscosity solution of the following second-order double-obstacle Hamilton-Jacobi-Isaacs equation:

$$\begin{cases} \min\Big\{\lambda_{\min}(D^2_{pp}V,p); \max\{\lambda_{\max}(D^2_{qq}W,q); -\lambda W + H(x,D_xW)\}\Big\} = 0; \\ \max\Big\{\lambda_{\max}(D^2_{qq}V,q); \min\{\lambda_{\min}(D^2_{pp}W,p); -\lambda W + H(x,D_xW)\}\Big\} = 0. \end{cases} \tag{0.3.8}$$

We will prove in detail the above theorem in Chapter 4. The main difficulty in its proof is the comparison principle for viscosity solutions of (0.3.8). We will see that as soon as the hamiltonian H is regular enough, the PDE (0.3.8) has at most one viscosity solution.

0.4 Perspectives:

For differential games with incomplete information on both sides introduced in Section 0.3.1-0.3.4, there are several interesting directions for future investigations.

1. The results about the existence of a value for the differential games studied in this PhD thesis all rely on a certain Isaacs condition. We have mentioned in Section 0.2.2 that for differential games with an information structure of the type in [16], existence of a value without Isaacs' condition has been proved in [12, 43]. For differential games with incomplete information and signal revelation, the existence and characterization of the value function for the cases without Isaacs' condition remain an open problem.

2. For the differential game with asymmetric information presented in Section 0.3.1, we have assumed that the initial states are chosen among a finite set (taking the form of a vector $X_0 \in \mathbb{R}^{nI}$). A direction for future research is to investigate the case where the initial states are chosen in accordance to some probability measure μ on $\mathbb{R}^n$ which is not necessarily of finite support.

3. In Chapter 2 and Chapter 3, in order to obtain the existence of value, we have required that the function of "hitting time" $(x, u, v) \mapsto \mathcal{T}(x, u, v)$ (defined in Section 0.3.3) is locally Lipschitz continuous in its domain. This consists of a rather strong condition on the dynamical system. Thus a direction of generalization is to verify the existence of value under weakened conditions or even for the case with more general type of signal functions.

4. Differential games presented in Section 0.3.2-0.3.3 are with information structure similar to repeated games with incomplete information whose information structures are symmetric. In the theory of repeated games, the existence of value for games of the symmetric case was studied in [36, 44, 45, 52, 53]. An open problem here is to determine the relation between such repeated games of incomplete information to our game models in Section 0.3.2 and 0.3.3.

5. Another problem for differential games with signal revelation is to study the case with asymmetric infomration and to determine their relation with repeated game of incomplete information (cf. [51, 64, 72]).

6. In this thesis, we are focused on studying the existence of value of differential games with incomplete information. For the games in Section 0.3.1-0.3.4, we have not answered the question of determining a pair of optimal strategies for the players (or calculating ϵ-optimal strategies).

7. A particular challenging but widely open question concerns the study of non-zero-sum differential games with incomplete information (cf. [8, 9, 20] for the complete information case).

Contents

Chapter 1

Existence of Value for Differential Games with Asymmetric Information and Signals

Abstract:
This article is devoted to the existence of the value for a two-players differential game with incomplete information and a signal structure on the data (initial positions, dynamics, and payoffs). Before the game starts, the data are chosen randomly in a finite set, the players are not informed on the exact data but they receive a signal. The main result of the paper says that the value of the game does exist for suitably defined nonanticipative random strategies. To obtain such a result, we introduce and study a new type of second-order double-obstacle Hamilton-Jacobi-Isaacs equation. We also prove that the value is the unique viscosity solution of this partial differential equation.

Key words: Differential Games; Incomplete Information; Signal; Hamilton-Jacobi-Isaacs Equation.

1.1 Introduction

The theory of differential games with incomplete information was first introduced in [16], which is a generalization of the two-person zero-sum repeated games with incomplete information of Aumann and Maschler [2]. For a glimpse of the results in the domain of differential games with incomplete information, readers may refer to the following articles [12, 17, 19, 21, 24, 25, 26, 43, 54]. Our work consists in investigating a precise structure of information with signals.

The notion of signals is commonly studied in the theory of repeated games with incomplete information, related results for repeated games can be found in, for example, [38, 49, 65]. Here we investigate the concept of signals for differential games.

Let us now explain our game model in details. Let $t_0 \in [0,1]$, $n \in \mathbb{N}_*$, $I \in \mathbb{N}_*$, and $X_0 = (x_0^i)_{i\in I} \in (\mathbb{R}^n)^I$. (In this paper, we abuse slightly the notation by denoting a finite set and its cardinal by the same symbol. For example, in this case, we set $I = \{1, 2, 3, \ ... \ I\}$) We consider a family of I two-person zero-sum differential games $\big(\mathcal{G}_i(t_0, x_0^i)\big)_{i\in I}$. Each differential game $\mathcal{G}_i(t_0, x_0^i)$ is defined by:

- a dynamical system:

$$\begin{cases} x'(t) = f^i\big(x(t), u(t), v(t)\big), \ u(t) \in U, \ v(t) \in V, \ t \in [t_0, 1], \\ x(t_0) = x_0^i, \end{cases} \tag{P_i}$$

- a payoff function:

$$\mathcal{J}^i(t_0, x_0^i, u, v) := \int_{t_0}^1 \gamma^i\big(X_s^{t_0,x_0^i,u,v}, u(s), v(s)\big)ds + g^i(X_1^{t_0,x_0^i,u,v}).$$

Here, U, V are some compact metric spaces. The functions $(f^i : \mathbb{R}^n \times U \times V \to \mathbb{R}^n)_{i\in I}$ are supposed to satisfy standard regularity assumptions ensuring that for any $i \in I$, the dynamical system (P_i) admits a unique solution, denoted by $t \mapsto X_t^{t_0,x_0^i,u,v}$. The controls $u : [t_0, 1] \to U$ and $v : [t_0, 1] \to V$ are both Lebesgue measurable. The functions $\big(\gamma^i \in L^1(\mathbb{R}^n \times U \times V)\big)_{i\in I}$ and $(g^i : \mathbb{R}^n \to \mathbb{R})_{i\in I}$ are respectively the running costs and the terminal costs.

Let us consider two finite signal sets $K = \{1, 2, \ ... \ , K\}$ and $L = \{1, 2, \ ... \ , L\}$, with K, $L \in \mathbb{N}_*$. Let $h_1 : I \to K$ and $h_2 : I \to L$ be two signal generators. Let $\Delta(I)$ denote the set of all probability measures on I, or equivalently, the I-simplex. For any $p \in \Delta(I)$, we define a differential game $\tilde{\mathcal{G}}(t_0, X_0, p)$ with incomplete information and signals by:

1. Before the game begins, an element $i \in I$ is chosen randomly according to the probability $p \in \Delta(I)$. The signal $k = h_1(i)$ is communicated to Player I but not to Player II, and the signal $l = h_2(i)$ is communicated to Player II but not to Player I.

2. The game $\mathcal{G}_i(t_0, x_0^i)$ is played. More precisely, Player I chooses the control $t \mapsto u(t)$ and aims to minimize $\mathcal{J}^i(t_0, x_0^i, u, v)$. By choosing the control $t \mapsto v(t)$, Player II aims to maximize $\mathcal{J}^i(t_0, x_0^i, u, v)$.

In addition, we suppose that the probability p and the family $\big(\mathcal{G}_i(t_0, x_0^i)\big)_{i\in I}$ are commonly known by the players, and both players observe the actions played by his opponent, i.e., we suppose perfect monitoring in $\mathcal{G}_i(t_0, x_0^i)$, for any $i \in I$.

During the game, each player will try to guess the chosen game $\mathcal{G}_i(t_0, x_0^i)$ by observing their own signal and the actions of his opponent. In the meantime, they will try to hide their own information from his opponent by playing random strategies.

In this paper we investigate a more general structure of signals which includes in a particular case the notion of signals described above. We will describe later on this general structure, which is adapted from the theory of repeated games (cf. [49]).

Differential games with incomplete information without signals investigated in [16, 54] can be considered as particular cases of the game $\tilde{\mathcal{G}}(t_0, X_0, p)$.

The goal of this paper is to prove the existence of a value for differential games with incomplete information and signals. For the case without signals, it has been proved in [16, 54] that the value of the game exists and this value can be characterized as the unique viscosity solution of a Hamilton-Jacobi-Isaacs equation. But these results can not apply to the case with signals. There are two main difficulties: The first one is to find a suitable notion of nonanticipative random strategies that characterizes the fact that both players choose their strategies according to their own signals. The second one lays on the definition of the value functions over an information space which is much bigger than those of [16, 54].

To obtain the wished result, in this paper we will follow the classic scheme of [33]:

- Prove that the upper- and lower-value functions are viscosity solutions of the same Hamilton-Jacobi-Isaacs equation under the Isaacs condition introduced in Section 2.
- Prove that this Hamilton-Jacobi-Isaacs equation has a unique viscosity solution.
- Thus the upper- and lower-value functions must coincide.

For proving that the upper- and lower-value functions verify the same Hamilton-Jacobi-Isaacs equation, we first establish a sub-dynamic programming principle. This method is proposed in [31] for repeated games and it is generalized to the theory of differential games with incomplete information in [16]. The main difficulty is that, to state such sub-dynamic programming principle, we will have to prove that the value functions possess certain convexity property over the information space which is more complicated than that in [16]. The Hamilton-Jacobi-Isaacs equation is obtained by passing to the limit on both sides of the sub-dynamic programming principle. Due to the differences between the information structures, this equation takes a form which is more complicated and it is slightly different

from the ones in [16, 54]. However, in the particular case without signals, our equation reduces to the PDE proposed in [17].

Once a suitable notion of viscosity solution is introduced for the Hamilton-Jacobi-Isaacs equation, we state and prove a comparison principle for this equation. It is worth pointing out that the proof of this comparison principle is much more technical than those of the games without signals (cf. [16, 54]). The comparison principle enables us to deduce the uniqueness property of the viscosity solution of the Hamilton-Jacobi-Isaacs equation and consequently we obtain the existence of a value for the game, which is our main result.

We finish this introduction by presenting the organization of this paper. Section 2 is devoted to detailed hypothesis of the game model and formal definitions of random strategies and value functions. Regularity properties and convexity properties of the value functions are studied in Section 3. In Section 4, we prove that the Fenchel conjugate of V^- satisfies a sub-dynamic programming principle. In Section 5, as a consequence of the sub-dynamic programming principle, we deduce that the upper- and lower-value functions are respectively sub-solution and super-solution of the associated Hamilton-Jacobi-Isaacs equation. The existence of a value for the game $\tilde{\mathcal{G}}(t_0, X_0, p)$ is deduced from a new comparison principle for viscosity solutions of the Hamilton-Jacobi-Isaacs equation.

1.2 Game Model, Strategies and Value Functions

Before we present the differential games with a general signal structure, let us first introduce some useful notions and the assumptions imposed on the dynamics $(P_i)_{i\in I}$ and the cost functions.

For any integer $m > 0$, we denote by $\Delta(m) = \big\{(p_1, p_2, \ ... \ p_m) \mid \sum_{i=1}^m p_i = 1, \ p_i \geq 0, \ \forall i \in \{1, 2, \ ... \ m\}\big\}$ the set of probability measures on $\{$ 1,2, ... m $\}$.

Let $I = \{1, 2, \ ... \ , I\}$, $K = \{1, 2, \ ... \ , K\}$ and $L = \{1, 2, \ ... \ , L\}$ be finite signal sets and $n \in \mathbb{N}_*$. We assume the following hypothesis throughout this paper:

- U, V are two compact metric spaces endowed respectively with the Borel σ-algebra.
- $(f^i)_{i\in I}$ is a family of maps such that for any $i \in I$, $f^i : \mathbb{R}^n \times U \times V \to \mathbb{R}^n$ is bounded, continuous, and Lipschitz continuous uniformly with respect to x.
- $(\gamma^i)_{i\in I}$ is a family of functions such that for any $i \in I$, $\gamma^i : \mathbb{R}^n \times U \times V \to \mathbb{R}$ is bounded, continuous, and Lipschitz continuous with respect to the first variable.
- $(g^i)_{i\in I}$ is a family of functions such that for any $i \in I$, $g^i : \mathbb{R}^n \to \mathbb{R}$ is bounded and Lipschitz continuous.
- $X_0 = (x_0^i)_{i\in I}$ is an arbitrary vector in $(\mathbb{R}^n)^I$.

We suppose the following Isaacs condition: For any $(x^i)_{i\in I}$, $(\xi^i)_{i\in I} \in (\mathbb{R}^n)^I$ and $p \in \Delta(I)$,

$$\begin{aligned}&\inf_{u\in U}\sup_{v\in V}\sum_{i\in I}\big(\langle f^i(x^i,u,v),\xi^i\rangle + p(i)\gamma^i(x^i,u,v)\big)\\ =&\sup_{v\in V}\inf_{u\in U}\sum_{i\in I}\big(\langle f^i(x^i,u,v),\xi^i\rangle + p(i)\gamma^i(x^i,u,v)\big).\end{aligned} \tag{1.2.1}$$

Let $[0,1]$ be endowed with the Lebesgue measure. For $t_0 \in [0,1]$, the set of all admissible controls for Player I and that for Player II are defined respectively as follows:

$$\begin{aligned}\mathcal{U}(t_0) &= \{u : [t_0,1] \to U \mid u \textit{ is Lebesgue measurable}\}\\ \mathcal{V}(t_0) &= \{v : [t_0,1] \to V \mid v \textit{ is Lebesgue measurable}\}\end{aligned}$$

Thanks to the above suppositions, for any $i \in I$, given a pair of controls $(u,v) \in \mathcal{U}(t_0)\times\mathcal{V}(t_0)$, there exists a unique solution $t \mapsto X_t^{t_0,x_0^i,u,v}$ of the dynamical system (P_i). Let $\big(\mathcal{G}_i(t_0,x_0^i)\big)_{i\in I}$ be defined as in the introduction. Thus for any $i \in I$ the payoff associated to $\mathcal{G}_i$ is:

$$\forall (u,v) \in \mathcal{U}(t_0)\times\mathcal{V}(t_0),\ \mathcal{J}^i(t_0,x_0^i,u,v) = \textstyle\int_{t_0}^1 \gamma^i\big(X_s^{t_0,x_0^i,u,v},u(s),v(s)\big)ds + g^i(X_1^{t_0,x_0^i,u,v}).$$

Now we generalize the information structure of the game $\tilde{\mathcal{G}}(t_0,X_0,p)$ by adopting a more general signal structure.

A two-person zero-sum differential game with incomplete information and signals $\mathcal{G}(t_0,X_0,\pi)$ is characterized by the family $\big(\mathcal{G}_i(t_0,x_0^i)\big)_{i\in I}$, the signal sets K and L, and a probability measure $\pi \in \Delta(I\times K\times L)$. More precisely, the game $\mathcal{G}(t_0,X_0,\pi)$ involves two steps:

1. Before the game starts, an element $(i,k,l) \in I\times K\times L$ is chosen randomly according to the probability $\pi \in \Delta(I\times K\times L)$. The signal k is communicated to Player I but not to Player II, and the signal l is communicated to Player II but not to Player I.

2. The game $\mathcal{G}_i(t_0,x_0^i)$ is played. Namely the Player I chooses the control u and aims to minimize the payoff $\mathcal{J}^i(t_0,x_0^i,u,v)$, while the Player II chooses the control v and aims to maximize $\mathcal{J}^i(t_0,x_0^i,u,v)$.

The probability π and the family $\big(\mathcal{G}_i(t_0,x_0^i)\big)_{i\in I}$ are commonly known by the players, and for each differential game $\mathcal{G}_i(t_0,x_0^i)$, we suppose perfect monitoring of the actions played for both players.

The game $\tilde{\mathcal{G}}(t_0,X_0,p)$ presented in the introduction is a particular case of the above model, and thus the game $\mathcal{G}(t_0,X_0,\pi)$ is a generalization of the differential games with incomplete information and without signals in [16, 54]. This can be proved by verifying the equivalence between the game $\tilde{\mathcal{G}}(t_0,X_0,p)$ and the game $\mathcal{G}(t_0,X_0,\pi_0)$ with $\pi_0 \in \Delta(I\times K\times L)$ satisfying:

$$\pi_0(i,k,l) = p(i), \text{ if } (k,l) = h(i);\ \pi_0(i,k,l) = 0, \text{ else.}$$

For future convenience, we define the Hamiltonian $H_{\mathcal{G}}$ associated to the game $\mathcal{G}(t_0, X_0, \pi)$: for any $X = (x^i)_{i\in I} \in (\mathbb{R}^n)^I$, $(\xi^i)_{i\in I} \in (\mathbb{R}^n)^I$,

$$\begin{aligned} H_{\mathcal{G}}(X, (\xi_i)_{i\in I}, \pi) :=& \inf_{u\in U}\sup_{v\in V}\sum_{i\in I}\langle f^i(x^i, u, v), \xi^i\rangle + \sum_{(i,k,l)\in I\times K\times L}\pi(i,k,l)\gamma^i(x^i, u, v) \\ =& \sup_{v\in V}\inf_{u\in U}\sum_{i\in I}\langle f^i(x^i, u, v), \xi^i\rangle + \sum_{(i,k,l)\in I\times K\times L}\pi(i,k,l)\gamma^i(x^i, u, v). \end{aligned} \tag{1.2.2}$$

The last equality is a consequence of the Isaacs condition (1.2.1).

Now we introduce the notions of pure and random strategies for the two players, which are based on the notion of nonanticipative strategies with delay (NAD strategies). (Readers may refer to, for instance [16, 32, 62]). Let the sets $\mathcal{U}(t_0)$ and $\mathcal{V}(t_0)$ be respectively equipped with the L^1-topologies associated.

Definition 1.2.1. *A pure strategy for Player I in the game $\mathcal{G}(t_0, X_0, \pi)$ is a Borel measurable map $\alpha : \mathcal{V}(t_0) \to \mathcal{U}(t_0)$ such that there exist $J \in \mathbb{N}_*$ and $t_0 = \tau_0 \le \tau_1 \le \ \ldots \ \le \tau_{J-1} \le \tau_J = 1$, for any $(v, v') \in \big(\mathcal{V}(t_0)\big)^2$, for all $1 \le j \le J$, if $v|_{[t_0,\tau_{j-1}]} = v'|_{[t_0,\tau_{j-1}]}$ a.e., then $\alpha(v)|_{[t_0,\tau_j]} = \alpha(v')|_{[t_0,\tau_j]}$ a.e..*

The definition of a pure strategy $\beta : \mathcal{U}(t_0) \to \mathcal{V}(t_0)$ for Player II in the game $\mathcal{G}(t_0, X_0, \pi)$ is similar (cf. [21]). We denote the set of all pure strategies for Player I and that for Player II in the game $\mathcal{G}(t_0, X_0, \pi)$ respectively by $\mathcal{A}(t_0)$ and by $\mathcal{B}(t_0)$.

Now we define the notion of random strategies for the players in the game $\mathcal{G}(t_0, X_0, \pi)$. For simplicity, we consider the set of probability spaces: $\mathcal{S} = \big\{\big([0,1]^m, \mathcal{B}([0,1]^m), \mathcal{L}([0,1]^m)\big), m \in \mathbb{N}_*\big\}$, where $\mathcal{B}([0,1]^m)$ is the Borel σ-algebra and $\mathcal{L}([0,1]^m)$ denotes the Lebesgue measure on $\mathcal{B}([0,1]^m)$. Then $\mathcal{S}$ is clearly stable under finite products of its elements.

Definition 1.2.2. *A random strategy for Player I in the game $\mathcal{G}(t_0, X_0, \pi)$ is an element of the set $\big(\mathcal{A}_r(t_0)\big)^K$. $\mathcal{A}_r(t_0)$ is defined as the set of all pairs $\big((\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha), \alpha\big)$ with $(\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha) \in \mathcal{S}$ and with the Borel measurable map $\alpha : \Omega_\alpha \times \mathcal{V}(t_0) \to \mathcal{U}(t_0)$ (with Ω_α endowed with the σ-algebra $\mathcal{F}_\alpha$) satisfying: $\exists J \in \mathbb{N}_*$ and $t_0 = \tau_0 \le \tau_1 \le \ \ldots \ \le \tau_{J-1} \le \tau_J = 1$, such that*

$$\begin{aligned} &\forall \omega \in \Omega_\alpha,\ \forall 1 \le j \le J,\ \forall v_1 \in \mathcal{V}(t_0),\ \forall v_2 \in \mathcal{V}(t_0),\ v_1|_{[t_0,\tau_{j-1}]} = v_2|_{[t_0,\tau_{j-1}]}\ a.e. \\ &\qquad \Longrightarrow \alpha(\omega, v_1)|_{[t_0,\tau_j]} = \alpha(\omega, v_2)|_{[t_0,\tau_j]}\ a.e\ on\ [t_0, \tau_j]. \end{aligned}$$

In the above definition, we have equivalently, for any $\omega \in \Omega_\alpha$, the map $v \mapsto \alpha(\omega, v)$ is a pure strategy for Player I in the game $\mathcal{G}(t_0, X_0, \pi)$. Respectively:

Definition 1.2.3. *A random strategy for Player II in the game $\mathcal{G}(t_0, X_0, \pi)$ is an element of the set $\big(\mathcal{B}_r(t_0)\big)^L$. $\mathcal{B}_r(t_0)$ is the set of all pairs $\big((\Omega_\beta, \mathcal{F}_\beta, \mathbb{P}_\beta), \beta\big)$ with $(\Omega_\beta, \mathcal{F}_\beta, \mathbb{P}_\beta) \in \mathcal{S}$ and with the Borel measurable map $\beta : \Omega_\beta \times \mathcal{U}(t_0) \to \mathcal{V}(t_0)$ satisfying: $\exists J \in \mathbb{N}_*$ and $t_0 = \tau_0 \le \tau_1 \le \ \ldots \ \le \tau_{J-1} \le \tau_J = 1$, such that*

$$\forall \omega \in \Omega_\beta,\ \forall 1 \le j \le J,\ \forall u_1 \in \mathcal{U}(t_0),\ \forall u_2 \in \mathcal{U}(t_0),\ u_1|_{[t_0,\tau_{j-1}]} = u_2|_{[t_0,\tau_{j-1}]}\ a.e.$$
$$\Longrightarrow \beta(\omega,u_1)|_{[t_0,\tau_j]} = \beta(\omega,u_2)|_{[t_0,\tau_j]}\ a.e\ on\ [t_0,\tau_j].$$

If there is no confusion, we often denote shortly the pair $\big((\Omega_\alpha,\mathcal{F}_\alpha,\mathbb{P}_\alpha),\alpha\big) \in \mathcal{A}_r(t_0)$ (resp. $\big((\Omega_\beta,\mathcal{F}_\beta,\mathbb{P}_\beta),\beta\big) \in \mathcal{B}_r(t_0)$) by the symbol of the map, i.e. by α (resp. by β). Clearly, $\mathcal{A}(t_0) \subset \mathcal{A}_r(t_0)$ and $\mathcal{B}(t_0) \subset \mathcal{B}_r(t_0)$.

For a glimpse of random strategies for differential games with incomplete information without signals, readers may refer to, for instance, [21]. The above notions of random strategies for the game $\mathcal{G}(t_0,X_0,\pi)$ have two important aspects: first, the players will choose their actions by observing their own signals and the actions of their opponents; second, in order to hide their own signals from each other, the players will play random strategies. It is worth pointing out that, for the particular case without signals, the notions of random strategies of our game reduce to those in [16, 54].

The following two lemmas give the properties of pure and random strategies of the game $\mathcal{G}(t_0,X_0,\pi)$ which are crucial for putting the game into a normal form:

Lemma 1.2.4. *(cf. [21]) For any pair of pure strategies $(\alpha,\beta) \in \mathcal{A}(t_0)\times\mathcal{B}(t_0)$, there exists a unique pair of controls $(u,v) \in \mathcal{U}(t_0)\times\mathcal{V}(t_0)$ such that*

$$\alpha(v) = u,\ and\ \beta(u) = v.$$

Lemma 1.2.5. *(cf. [16]) For any $\big((\Omega_\alpha,\mathcal{F}_\alpha,\mathbb{P}_\alpha),\alpha\big) \in \mathcal{A}_r(t_0)$, $\big((\Omega_\beta,\mathcal{F}_\beta,\mathbb{P}_\beta),\beta\big) \in \mathcal{B}_r(t_0)$, and $\omega = (\omega_\alpha,\omega_\beta) \in \Omega_\alpha\times\Omega_\beta$, there exits a unique pair $(u_\omega,v_\omega) \in \mathcal{U}(t_0)\times\mathcal{V}(t_0)$ such that:*

$$\alpha(\omega_\alpha,v_\omega) = u_\omega\ and\ \beta(\omega_\beta,u_\omega) = v_\omega.$$

Moreover, the map $\omega \mapsto (u_\omega,v_\omega)$ is Borel measurable from $\Omega_\alpha\times\Omega_\beta$ endowed with the σ-algebra $\mathcal{F}_\alpha\otimes\mathcal{F}_\beta$ into $\mathcal{U}(t_0)\times\mathcal{V}(t_0)$ endowed with the σ-algebra induced by the $\mathcal{L}^1$-topology.

For a detailed proof of the above two lemmas, we refer the readers to [16].

We fix $(\alpha,\beta) \in \mathcal{A}_r(t_0)\times\mathcal{B}_r(t_0)$. For any $\omega \in \Omega_\alpha\times\Omega_\beta$, let $(u_\omega,v_\omega) \in \mathcal{U}(t_0)\times\mathcal{V}(t_0)$ be defined as in Lemma 1.2.5. We denote by $\theta(\alpha,\beta)$ the map $\omega \mapsto \theta(u_\omega,v_\omega)$, for any $\theta : \mathcal{U}(t_0)\times\mathcal{V}(t_0) \to \mathbb{R}^m$. For any $i \in I$, the expectation of payoff of the game $\mathcal{G}_i(t_0,x_0^i)$ associated to (α,β) is defined by:

$$\mathbb{E}_{\alpha,\beta}\mathcal{J}^i(t_0,x_0^i,\alpha,\beta) := \int_{\Omega_\alpha\times\Omega_\beta} \mathcal{J}^i(t_0,x_0^i,u_\omega,v_\omega)d\mathbb{P}_\alpha(\omega)\otimes\mathbb{P}_\beta(\omega).$$

When $\alpha \in \mathcal{A}(t_0)$ and $\beta \in \mathcal{B}_r(t_0)$, we write shortly $\mathbb{E}_\beta\mathcal{J}^i(t_0,x_0^i,\alpha,\beta)$, and we write $\mathbb{E}_\alpha\mathcal{J}^i(t_0,x_0^i,\alpha,\beta)$ for short when $\alpha \in \mathcal{A}_r(t_0)$ and $\beta \in \mathcal{B}(t_0)$.

Let us introduce a normal form of the game $\mathcal{G}(t_0,X_0,\pi)$ by associating to each pair of strategies $\big((\alpha_k)_{k\in K},(\beta_l)_{l\in L}\big) \in \big(\mathcal{A}_r(t_0)\big)^K \times \big(\mathcal{B}_r(t_0)\big)^L$ the corresponding payment:

$$\mathcal{J}\big(t_0,X_0,\pi,(\alpha_k)_{k\in K},(\beta_l)_{l\in L}\big) := \sum_{(i,k,l)\in I\times K\times L} \pi(i,k,l)\mathbb{E}_{\alpha_k,\beta_l}\mathcal{J}^i(t_0,x_0^i,\alpha_k,\beta_l).$$

The upper-value function V^+ and lower-value function V^- of the game $\mathcal{G}(t_0, X_0, \pi)$ are consequently defined as follows:

$$V^+(t_0, X_0, \pi) := \inf_{(\alpha_k)_{k\in K}\in(\mathcal{A}_r(t_0))^K} \sup_{(\beta_l)_{l\in L}\in(\mathcal{B}_r(t_0))^L} \mathcal{J}\big(t_0, X_0, \pi, (\alpha_k)_{k\in K}, (\beta_l)_{l\in L}\big),$$
$$V^-(t_0, X_0, \pi) := \sup_{(\beta_l)_{l\in L}\in(\mathcal{B}_r(t_0))^L} \inf_{(\alpha_k)_{k\in K}\in(\mathcal{A}_r(t_0))^K} \mathcal{J}\big(t_0, X_0, \pi, (\alpha_k)_{k\in K}, (\beta_l)_{l\in L}\big).$$

If $V^+ = V^-$, we say that the game admits a value and we denote its value function by $V = V^+ = V^-$. One can observe that $V^+ \geq V^-$. To obtain the existence of a value for $\mathcal{G}(t_0, X_0, \pi)$, we only need to prove $V^+ \leq V^-$.

Remark 1.2.6. *The function $-V^+$ can be viewed as the lower-value function of another differential game with information incomplete and signals. Namely:*

$$\begin{aligned} -V^+(t_0, X_0, \pi) &= -\inf_{(\alpha_k)_{k\in K}\in(\mathcal{A}_r(t_0))^K} \sup_{(\beta_l)_{l\in L}\in(\mathcal{B}_r(t_0))^L} \mathcal{J}\big(t_0, X_0, \pi, (\alpha_k)_{k\in K}, (\beta_l)_{l\in L}\big) \\ &= \sup_{(\alpha_k)_{k\in K}\in(\mathcal{A}_r(t_0))^K} \inf_{(\beta_l)_{l\in L}\in(\mathcal{B}_r(t_0))^L} \sum_{(i,k,l)\in I\times K\times L} \pi(i,k,l)\mathbb{E}_{\alpha_k,\beta_l}\big(-\mathcal{J}^i(t_0, x_0^i, \alpha_k, \beta_l)\big). \end{aligned}$$

For any $i \in I$, let $-\mathcal{G}_i(t_0, x_0^i)$ be the differential game with dynamical system (P_i), but with the roles of the two players exchanged and with payoff $-\mathcal{J}^i$.

For $-\mathcal{G}(t_0, X_0, \pi)$, the differential game with information incomplete and signals associated to the family of differential games $\big(-\mathcal{G}_i(t_0, x_0^i)\big)_{i\in I}$, we have clearly that its lower-value function is $-V^+$. Symmetrically, the function $-V^-$ is its upper-value function.

1.3 Regularities of the Values

In this section, we establish the regularity and convexity properties of the two value functions. The following easy result will be used several times.

Lemma 1.3.1. *Let A and B be some sets and let f, $g : A \times B \to \mathbb{R}$ be two maps. If there exists a constant $C \geq 0$ such that*

$$\sup_{a\in A,\ b\in B} |f(a,b) - g(a,b)| \leq C.$$

Then as soon as $\inf_{a\in A}\sup_{b\in B} f(a,b)$ and $\inf_{a\in A}\sup_{b\in B} g(a,b)$ are both finite, we have:

$$|\inf_{a\in A}\sup_{b\in B} f(a,b) - \inf_{a\in A}\sup_{b\in B} g(a,b)| \leq C.$$

Proposition 1.3.2. *The functions V^+ and V^- are both bounded and Lipschitz continuous.*

Proof. In view of Remark 1.2.6, $-V^-$ can be viewed as the upper-value function of the game $-\mathcal{G}(t_0, X_0, \pi)$, we only need to prove the proposition for V^+.

Observe that the boundedness of V^+ is a direct consequence of the boundedness of the functions $(\mathcal{J}^i)_{i\in I}$.

Let us prove the Lipschitz continuity of V^+.

Due to standard estimations on trajectories of differential equations, for any $(u, v) \in \mathcal{U}(t_0)\times\mathcal{V}(t_0)$, for any $i \in I$, the map $(x_0^i, s) \mapsto X_s^{t_0,x_0^i,u,v}$ is Lipschitz continuous on $\mathbb{R}^n\times[t_0, 1]$.

By the regularity imposed on g^i and γ^i, one has $x_0^i \mapsto \mathcal{J}^i(t_0, x_0^i, u, v)$ is Lipschitz continuous, for any $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$. Furthermore, for any $\big((\alpha_k)_{k\in K}, (\beta_l)_{l\in L}\big) \in \big(\mathcal{A}_r(t_0)\big)^K \times \big(\mathcal{B}_r(t_0)\big)^L$, we have the Lipschitz continuity of $X_0 \mapsto \mathcal{J}\big(t_0, X_0, \pi, (\alpha_k)_{k\in K}, (\beta_l)_{l\in L}\big)$, which implies the Lipschitz continuity of $X_0 \mapsto V^+(t_0, X_0, \pi)$ in view of Lemma 1.3.1.

On the other hand, since for any $\big((\alpha_k)_{k\in K}, (\beta_l)_{l\in L}\big) \in \big(\mathcal{A}_r(t_0)\big)^K \times \big(\mathcal{B}_r(t_0)\big)^L$, $\pi \mapsto \mathcal{J}\big(t_0, X_0, \pi, (\alpha_k)_{k\in K}, (\beta_l)_{l\in L}\big)$ is linear and bounded on $\Delta(I \times K \times L)$, thus Lipschitz continuous. The Lipschitz continuity of $\pi \mapsto V^+(t_0, X_0, \pi)$ follows from the Lemma 1.3.1.

It remains to prove the Lipschitz continuity of V^+ with respect to t_0.

We claim that there exists $C \in \mathbb{R}_*^+$ independent of X_0 and π such that, for any $0 \le t_0 < t_1 \le 1$:

$$-C(t_1 - t_0) \le V^+(t_0, X_0, \pi) - V^+(t_1, X_0, \pi), \tag{1.3.1}$$

Let $\big((\Omega_{\hat\alpha_k}, \mathcal{F}_{\hat\alpha_k}, \mathbb{P}_{\hat\alpha_k}, \hat\alpha_k)\big)_{k\in K} \in \big(\mathcal{A}_r(t_0)\big)^K$ and $\bar v \in V$ be fixed. We construct a new random strategy $\big((\Omega_{\alpha_k}, \mathcal{F}_{\alpha_k}, \mathbb{P}_{\alpha_k}, \alpha_k)\big)_{k\in K} \in \big(\mathcal{A}_r(t_1)\big)^K$ by defining:

- For any $k \in K$, $(\Omega_{\alpha_k}, \mathcal{F}_{\alpha_k}, \mathbb{P}_{\alpha_k}) = (\Omega_{\hat\alpha_k}, \mathcal{F}_{\hat\alpha_k}, \mathbb{P}_{\hat\alpha_k})$.
- For any $k \in K$, $\omega \in \Omega_{\alpha_k}$ and $v(\cdot) \in \mathcal{V}(t_1)$, $\alpha_k\big(\omega, v(\cdot)\big) = \hat\alpha_k\big(\omega, \tilde v(\cdot)\big)|_{[t_1,1]}$, where:

$$\tilde v(t) = \begin{cases} \bar v, & \text{for } t \in [t_0, t_1), \\ v(t), & \text{for } t \in [t_1, 1]. \end{cases}$$

One can check that $\big((\Omega_{\alpha_k}, \mathcal{F}_{\alpha_k}, \mathbb{P}_{\alpha_k}, \alpha_k)\big)_{k\in K}$ is an admissible random strategy.

Fix $\epsilon > 0$. Choose a strategy for Player II $(\beta_l)_{l\in L} \in \big(\mathcal{B}_r(t_1)\big)^L$ such that:

$$\begin{aligned}\mathcal{J}\big(t_1, X_0, \pi, (\alpha_k)_{k\in K}, (\beta_l)_{l\in L}\big) &\ge \sup_{(\tilde\beta_l)_{l\in L}\in(\mathcal{B}_r(t_1))^L} \mathcal{J}\big(t_1, X_0, \pi, (\alpha_k)_{k\in K}, (\tilde\beta_l)_{l\in L}\big) - \epsilon \\ &\ge V^+(t_1, X_0, \pi) - \epsilon\end{aligned}$$

We construct a random strategy $\big((\Omega_{\hat\beta_l}, \mathcal{F}_{\hat\beta_l}, \mathbb{P}_{\hat\beta_l}, \hat\beta_l)\big)_{l\in L} \in \big(\mathcal{B}_r(t_0)\big)^L$ for Player II by defining:

- For any $l \in L$, $(\Omega_{\hat\beta_l}, \mathcal{F}_{\hat\beta_l}, \mathbb{P}_{\hat\beta_l}) = (\Omega_{\beta_l}, \mathcal{F}_{\beta_l}, \mathbb{P}_{\beta_l})$.
- For any $l \in L$, $\omega \in \Omega_{\hat\beta_l}$ and $u(\cdot) \in \mathcal{U}(t_0)$,

$$\hat{\beta}_l\big(\omega, u(.)\big)(t) = \begin{cases} \bar{v}, & \text{for } t \in [t_0, t_1), \\ \beta_l\big(\omega, u|_{[t_1,1]}(.)\big)(t), & \text{for } t \in [t_1, 1]. \end{cases}$$

Then $(\hat{\beta}_l)_{l\in L}$ is an admissible random strategy. By the boundedness of $(\gamma_i)_{i\in I}$ and the Lipschitz continuity of $x_0^i \mapsto J^i(t_0, x_0^i, u, v)$, for any $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$. There exists $C > 0$ depending only on $(f^i)_{i\in I}$, $(\gamma^i)_{i\in I}$ and $(g^i)_{i\in I}$ such that:

$$\begin{aligned}
&\mathcal{J}\big(t_0, X_0, \pi, (\hat{\alpha}_k)_{k\in K}, (\hat{\beta}_l)_{l\in L}\big) = \sum_{(i,k,l)\in I\times K\times L} \pi(i,k,l)\mathbb{E}_{\hat{\alpha}_k,\hat{\beta}_l}\mathcal{J}^i(t_0, x_0^i, \hat{\alpha}_k, \hat{\beta}_l) \\
&= \sum_{(i,k,l)\in I\times K\times L} \pi(i,k,l)\mathbb{E}_{\hat{\alpha}_k,\hat{\beta}_l}\Big(\int_{t_0}^{t_1} \gamma(X_s^{t_0,x_0^i,\hat{\alpha}_k,\hat{\beta}_l}, \hat{\alpha}_k, \hat{\beta}_l)ds + \mathcal{J}^i(t_1, X_{t_1}^{t_0,x_0^i,\hat{\alpha}_k,\hat{\beta}_l}, \alpha_k, \beta_l)\Big) \\
&\geq \sum_{(i,k,l)\in I\times K\times L} \pi(i,k,l)\mathbb{E}_{\alpha_k,\beta_l}\big(- C(t_1 - t_0) + \mathcal{J}^i(t_1, x_0^i, \alpha_k, \beta_l)\big) \\
&\geq - C(t_1 - t_0) + \mathcal{J}\big(t_1, X_0, \pi, (\alpha_k)_{k\in K}, (\beta_l)_{l\in L}\big).
\end{aligned} \tag{1.3.2}$$

Hence, by (1.3.2) and the choice of $(\beta_l)_{l\in L}$:

$$\sup_{(\bar{\beta}_l)_{l\in L}\in(\mathcal{B}_r(t_0))^L} \mathcal{J}\big(t_0, X_0, \pi, (\hat{\alpha}_k)_{k\in K}, (\bar{\beta}_l)_{l\in L}\big) \geq -C(t_1 - t_0) + V^+(t_1, X_0, \pi) - \epsilon.$$

Taking the infimum over $(\hat{\alpha}_k)_{k\in K}$, we obtain:

$$V^+(t_0, X_0, \pi) \geq -C(t_1 - t_0) + V^+(t_1, X_0, \pi) - \epsilon.$$

Since ϵ is arbitrary, we obtain the wished claim (1.3.1). A similar argument gives that:

$$C(t_1 - t_0) \geq V^+(t_0, X_0, \pi) - V^+(t_1, X_0, \pi)$$

The proof is complete.

□

Now we investigate the convexity property of $\pi \mapsto V^{+/-}(t_0, X_0, \pi)$. This requires a suitable notion of convexity which relies on two decompositions of $\pi \in \Delta(I \times K \times L)$.

Let us define the following operators:

$$\begin{aligned}
\otimes_K : \Delta(K) \times \big(\Delta(I \times L)\big)^K &\to \Delta(I \times K \times L) \\
\big(\pi^K, (\Pi_k^K)_{k\in K}\big) &\mapsto \big(\pi^K(k)\Pi_k^K(i,l)\big)_{(i,k,l)\in I\times K\times L} \\
\otimes_L : \Delta(L) \times \big(\Delta(I \times K)\big)^L &\to \Delta(I \times K \times L) \\
\big(\pi^L, (\Pi_l^L)_{l\in L}\big) &\mapsto \big(\pi^L(l)\Pi_l^L(i,k)\big)_{(i,k,l)\in I\times K\times L}
\end{aligned}$$

We can therefore obtain two decompositions of π into $\Delta(K)\times\big(\Delta(I\times L)\big)^K$ and $\Delta(L)\times\big(\Delta(I\times K)\big)^L$ by taking, for any $(i,k,l)\in I\times K\times L$:

$$\begin{cases}\pi^K(k)=\displaystyle\sum_{(i,l)\in I\times L}\pi(i,k,l), & \Pi^K_k(i,l)=\pi\big((i,k,l)|k\big),\\ \pi^L(l)=\displaystyle\sum_{(i,k)\in I\times K}\pi(i,k,l), & \Pi^L_l(i,k)=\pi\big((i,k,l)|l\big).\end{cases}$$

By denoting shortly $\Pi^K=(\Pi^K_k)_{k\in K}$ and $\Pi^L=(\Pi^L_l)_{l\in L}$, we have $\pi=\pi^K\otimes_K\Pi^K$ and $\pi=\pi^L\otimes_L\Pi^L$.[1] In fact, the vectors Π^K and Π^L characterize respectively the conditional information of Player I and that of Player II while a pair of signals is given.

Now we can define the K-convexity and L-concavity for a function $\phi:\Delta(I\times K\times L)\to\mathbb{R}$. These notions, which date back to the 70s' (see [39, 50]), are proved to be powerful tools in the analysis of differential games with incomplete and correlated information in [54].

For a function $\phi:\Delta(I\times K\times L)\to\mathbb{R}$, we define the following functions:

$$\begin{aligned}\phi_K:\Delta(K)\times\big(\Delta(I\times L)\big)^K&\to\mathbb{R}\\ (\pi^K,\Pi^K)&\mapsto\phi(\pi^K\otimes_K\Pi^K),\\ \phi_L:\Delta(L)\times\big(\Delta(I\times K)\big)^L&\to\mathbb{R}\\ (\pi^L,\Pi^L)&\mapsto\phi(\pi^L\otimes_L\Pi^L),\end{aligned}$$

Definition 1.3.3. *$\phi:\Delta(I\times K\times L)\to\mathbb{R}$ is said to be K-convex if for any $\Pi^K\in\big(\Delta(I\times L)\big)^K$, $\phi_K(\pi^K,\Pi^K)$ is convex with respect to π^K. Similarly, ϕ is said to be L-concave if for any $\Pi^L\in\big(\Delta(I\times K)\big)^L$, $\phi_L(\pi^L,\Pi^L)$ is concave with respect to π^L.*

The following proposition gives the convexity property of the value functions.

Proposition 1.3.4. *The functions V^+ and V^- are both K-convex and L-concave on π.*

Proof. Let $(t_0,X_0)\in[0,1]\times(\mathbb{R}^n)^I$ be fixed. By Remark 1.2.6, we only need to prove the proposition for V^+.

First of all, let us prove that $\pi\mapsto V^+(t_0,X_0,\pi)$ is L-concave. By definition, we have:

$$\begin{aligned}&V^+(t_0,X_0,\pi^L\otimes_L\Pi^L)\\ =&\inf_{(\alpha_k)_{k\in K}\in(\mathcal{A}_r(t_0))^K}\sup_{(\beta_l)_{l\in L}\in(\mathcal{B}_r(t_0))^L}\sum_{l\in L}\pi^L(l)\sum_{(i,k)\in I\times K}\Pi^L_l(i,k)\mathbb{E}_{\alpha_k,\beta_l}\mathcal{J}^i(t_0,x^i_0,\alpha_k,\beta_l)\\ =&\inf_{(\alpha_k)_{k\in K}\in(\mathcal{A}_r(t_0))^K}\sum_{l\in L}\pi^L(l)\sup_{\beta\in\mathcal{B}_r(t_0)}\sum_{(i,k)\in I\times K}\Pi^L_l(i,k)\mathbb{E}_{\alpha_k,\beta}\mathcal{J}^i(t_0,x^i_0,\alpha_k,\beta).\end{aligned}$$

[1] Thus the operators $\otimes_K$ and $\otimes_L$ are in fact bijections.

$V_L^+(t_0, X_0, \pi^L, \Pi^L)$ being the infimum of a family of linear functions of π^L, this gives the L-concavity of $\pi \mapsto V^+(t_0, X_0, \pi)$.

Now we turn to the proof of the K-convexity of $\pi \mapsto V^+(t_0, X_0, \pi)$. Let $\Pi^K \in \big(\Delta(I\times L)\big)^K$ be fixed. Without loss of generality, let us consider $\pi_1^K \in \Delta(K)$, $\pi_2^K \in \Delta(K)$ such that $\pi_1^K(k) + \pi_2^K(k) > 0$ for any $k \in K$. Indeed, if this is not true, we only need to eliminate from the set K the elements $k \in K$ satisfying $\pi_1^K(k)+\pi_2^K(k) = 0$. For $\lambda \in (0,1)$, we denote:

$$\pi_\lambda^K = \lambda\pi_1^K + (1-\lambda)\pi_2^K.$$

Fix $\epsilon > 0$, let $(\alpha_k^1)_{k\in K}$ be the ϵ-optimal strategy of Player I for $V_K^+(t_0, X_0, \pi_1^K, \Pi^K)$ and respectively let $(\alpha_k^2)_{k\in K}$ be the ϵ-optimal strategy of Player I for $V_K^+(t_0, X_0, \pi_2^K, \Pi^K)$. We construct a strategy of Player I $(\hat{\alpha}_k)_{k\in K}$ by setting:

$$(\Omega_{\hat{\alpha}_k}, \mathcal{F}_{\hat{\alpha}_k}, \mathbb{P}_{\hat{\alpha}_k}) = \big([0,1]\times\Omega_{\alpha_k^1}\times\Omega_{\alpha_k^2}, \mathcal{B}([0,1])\otimes\mathcal{F}_{\alpha_k^1}\otimes\mathcal{F}_{\alpha_k^2}, \mathcal{L}([0,1])\otimes\mathbb{P}_{\alpha_k^1}\otimes\mathbb{P}_{\alpha_k^2}\big);$$

$$\forall(\omega_1,\omega_2,\omega_3,v)\in\Omega_{\hat{\alpha}_k}\times\mathcal{V}(t_0),\ \hat{\alpha}_k(\omega_1,\omega_2,\omega_3,v) = \begin{cases} \alpha_k^1(\omega_2,v), & \text{if } \omega_1\in[0,\dfrac{\lambda\pi_1^K(k)}{\pi_\lambda^K(k)}), \\ \alpha_k^2(\omega_3,v) & \text{if } \omega_1\in[\dfrac{\lambda\pi_1^K(k)}{\pi_\lambda^K(k)},1]. \end{cases}$$

One can check that $(\hat{\alpha}_k)_{k\in K}$ is an admissible random strategy for Player I. Therefore:

$$\begin{aligned}
&\sup_{(\beta_l)_{l\in L}\in(\mathcal{B}_r(t_0))^L} \mathcal{J}\big(t_0, X_0, \pi_\lambda^K\otimes_K\Pi^K, (\hat{\alpha}_k)_{k\in K}, (\beta_l)_{l\in L}\big) \\
=&\sup_{(\beta_l)_{l\in L}\in(\mathcal{B}_r(t_0))^L} \sum_{(i,k,l)\in I\times K\times L} \pi_\lambda^K(k)\Pi_k^K(i,l)\big(\frac{\lambda\pi_1^K(k)}{\pi_\lambda^K(k)}\mathbb{E}_{\alpha_k^1,\beta_l}\mathcal{J}^i(t_0,x_0^i,\alpha_k^1,\beta_l)+ \\
&\frac{(1-\lambda)\pi_2^K(k)}{\pi_\lambda^K(k)}\mathbb{E}_{\alpha_k^2,\beta_l}\mathcal{J}^i(t_0,x_0^i,\alpha_k^2,\beta_l)\big) \\
\le&\lambda V_K^+(t_0,X_0,\pi_1^K,\Pi^K)+(1-\lambda)V_K^+(t_0,X_0,\pi_2^K,\Pi^K)+2\epsilon.
\end{aligned}$$

The last inequality above is a result of the choice of $(\alpha_k^1)_{k\in K}$ and $(\alpha_k^2)_{k\in K}$. Thus,

$$V_K^+(t_0,X_0,\pi_\lambda^K,\Pi^K) \le \lambda V_K^+(t_0,X_0,\pi_1^K,\Pi^K)+(1-\lambda)V_K^+(t_0,X_0,\pi_2^K,\Pi^K)+2\epsilon,$$

Since ϵ is arbitrarily small, V_K^+ is convex with respect to π^K. The proof is complete. □

1.4 Sub-dynamic Programming Principle

In this section, we prove that the Fenchel conjugate of the map $\pi^K \mapsto V_K^-(t_0, X_0, \pi^K, \Pi^K)$ satisfies a sub-dynamic programming principle. The technic of taking the convex conjugates of the value functions was introduced in [31] for repeated games.

We denote $\bar{\mathbb{R}}$ the set: $\mathbb{R}\bigcup\{+\infty,-\infty\}$. For $m\in\mathbb{N}_*$, the convex conjugate of $\varphi:\mathbb{R}^m\to\bar{\mathbb{R}}$ is defined as:

$$\varphi^*(\xi) := \sup_{q\in\mathbb{R}^m} \langle \xi, q\rangle - \varphi(q),\ \forall \xi \in \mathbb{R}^m.$$

By applying an extension of the map $\pi^K \mapsto V_K^-(t_0, X_0, \pi^K, \Pi^K)$ to $\mathbb{R}^K\backslash\Delta(K)$ by $+\infty$, i.e.,

$$V_K^-(t_0, X_0, q, \Pi^K) = +\infty, \text{ for any } q \in \mathbb{R}^K\backslash\Delta(K),$$

we define its convex conjugate by:

$$V_K^{-*}(t_0, X_0, \xi, \Pi^K) := \sup_{\pi^K\in\Delta(K)} \langle \xi, \pi^K\rangle - V_K^-(t_0, X_0, \pi^K, \Pi^K),\ \forall \xi \in \mathbb{R}^K.$$

Since the map $\pi^K \mapsto V_K^-(t_0, X_0, \pi^K, \Pi^K)$ is bounded, Lipschitz continuous and convex, then its extended version above is proper, lower semicontinuous (l.s.c) and convex.

For $\Pi^K \in \big(\Delta(I\times L)\big)^K$, $\alpha \in \mathcal{A}(t_0)$ and $(\beta_l)_{l\in L} \in \big(\mathcal{B}_r(t_0)\big)^L$, we define, for $\bar{k} \in K$:

$$\mathcal{J}_{\bar{k}}\big(t_0, X_0, \Pi^K, \alpha, (\beta_l)_{l\in L}\big) := \sum_{(i,l)\in I\times L} \Pi_{\bar{k}}^K(i,l)\mathbb{E}_{\beta_l}\mathcal{J}^i(t_0, x_0^i, \alpha, \beta_l).$$

The main result of this section is the following:

Proposition 1.4.1. *(Sub-dynamic Programming) For any* $\big(t_0, (x_0^i)_{i\in I}, \zeta, \Pi^K\big) \in [0,1)\times(\mathbb{R}^n)^I \times \mathbb{R}^K \times \Delta(I\times L)^K$ *and* $h \in [0,1]$ *such that* $t_0 + h \in [0,1]$*, we have:*

$$V_K^{-*}\big(t_0, (x_0^i)_{i\in I}, \zeta, \Pi^K\big) \le \inf_{\beta\in\mathcal{B}(t_0)} \sup_{\alpha\in\mathcal{A}(t_0)} V_K^{-*}\big(t_0+h, (X_{t_0+h}^{t_0,x_0^i,\alpha,\beta})_{i\in I}, \zeta(t_0+h,\alpha,\beta), \Pi^K\big),$$

where $\zeta(t_0+h,\alpha,\beta) = \zeta - \Big(\sum_{(i,l)\in I\times L} \Pi_k^K(i,l)\int_{t_0}^{t_0+h}\gamma^i\big(X_s^{t_0,x_0^i,\alpha,\beta},\alpha,\beta\big)ds\Big)_{k\in K}$.

To prove Proposition 1.4.1, we first introduce a reformulation of V_K^{-*} (cf. [65]).

Lemma 1.4.2. *For any* $\zeta \in \mathbb{R}^K$, $t_0 \in [0,1]$, $X_0 = (x_0^i)_{i\in I} \in (\mathbb{R})^I$, *and* $\Pi^K \in \big(\Delta(I\times L)\big)^K$*:*

$$V_K^{-*}(t_0, X_0, \zeta, \Pi^K) = \inf_{(\beta_l)_{l\in L}\in(\mathcal{B}_r(t_0))^L} \sup_{\alpha\in\mathcal{A}(t_0)} \max_{k\in K}\Big(\zeta_k - \mathcal{J}_k\big(t_0, X_0, \Pi^K, \alpha, (\beta_l)_{l\in L}\big)\Big).$$

The proof of the previous lemma is standard. We refer the interested readers to [16, 65]. Now, we are ready to prove Proposition 1.4.1.

Proof (of Proposition 1.4.1). Let (t_0, X_0, ζ, Π^K) and h be as in the statement of the proposition. The proof is divided into two steps. In Step 1, we prove the inequality for a particular case where $(\gamma^i)_{i\in I}$ are reduced to 0. In Step 2, by applying the result of Step 1, we complete the proof of the proposition.

Step 1:
Let us suppose that for any $i \in I$, $\gamma^i = 0$. The inequality in the proposition becomes:

$$V_K^{-*}(t_0, X_0, \zeta, \Pi^K) \leq \inf_{\beta \in \mathcal{B}(t_0)} \sup_{\alpha \in \mathcal{A}(t_0)} V_K^{-*}\big(t_0 + h, (X_{t_0+h}^{t_0,x_0^i,\alpha,\beta})_{i\in I}, \zeta, \Pi^K\big).$$

Since $\gamma^i = 0$, for any $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$, we have $\mathcal{J}^i(t_0, x_0^i, u, v) = g^i(X_1^{t_0,x_0^i,u,v})$.

Firstly, one can easily obtain that for any $(\beta_l)_{l\in L} \in \big(\mathcal{B}_r(t_0)\big)^L$ fixed, the map

$$X_0 \to \sup_{\alpha\in\mathcal{A}(t_0)} \max_{k\in K} \Big(\zeta_k - \mathcal{J}_k\big(t_0, X_0, \Pi^K, \alpha, (\beta_l)_{l\in L}\big)\Big)$$

is Lipschitz continuous. Using Lemma 1.4.2, we deduce that $V_K^{-*}(t_0, X_0, \zeta, \Pi^K)$ is Lipschitz continuous with respect to X_0. To finish the proof of Step 1, we need to construct a family of random strategies of Player II, more precisely:

Fix $\epsilon > 0$. Let M be an upper bound of $\big(\|f^i(\mathbb{R}^n)\|\big)_{i\in I}$. For any $Y \in B(X_0; M)$, let $(\hat{\beta}_l^Y)_{l\in L}$ be an ϵ-optimal strategy for $V_K^{-*}(t_0 + h, Y, \zeta, \Pi^K)$. By the Lipschitz continuity of V_K^{-*}, there exists $\delta_Y > 0$ such that for any $Y' \in B(Y; \delta_Y)$, $(\hat{\beta}_l^Y)_{l\in L}$ remains a 2ϵ-optimal strategy for $V_K^{-*}(t_0 + h, Y', \zeta, \Pi^K)$.

The set $A = \{(X_{t+h}^{t_0,x_0^i,u,v})_{i\in I} |\ u \in \mathcal{U}(t_0),\ v \in \mathcal{V}(t_0)\}$ is contained in the compact $B(X_0; M)$, which implies that there exists a finite set $\{Y_1, Y_2, \ ..., \ Y_N\} \subset B(X_0; M)$ with $N \in \mathbb{N}_*$ such that $\cup_{m=1,2,\ ...,\ N} B(Y_m; \frac{\delta_{Y_m}}{2}) \supset B\big((x_0^i)_{i\in I}; M\big) \supset A$. By defining $E_0 = \emptyset$, let us construct a Borel partition $(E_m)_{m=1,\ ...,\ N}$ of $B(X_0; M)$ as follows:

$$\forall m \in \{1, 2, \ ... \ N\},\ E_m = \big(B(X_0; M) \cap B(Y_m; \tfrac{\delta_{Y_m}}{2})\big) \setminus (E_1 \cup E_2 \cup ... \cup E_{m-1}).$$

We define now a random strategy $(\beta_l^\epsilon)_{l\in L} \in \big(\mathcal{B}_r(t_0)\big)^L$ of Player II :

For any $l \in L$ and $m \in \{1, 2, \ ... \ N\}$, let $(\Omega_m^l, \mathcal{F}_m^l, \mathbb{P}^l{}_m) = (\Omega_{\hat{\beta}_l^{Y_m}}, \mathcal{F}_{\hat{\beta}_l^{Y_m}}, \mathbb{P}_{\hat{\beta}_l^{Y_m}})$, and let $t_0 + h = \tau_0' < \tau_1' < \ ... \ < \tau_J' = 1$ be a common partition for all $\beta_l^{Y_m}$, $1 \leq m \leq N$, $l \in L$ (This is possible since the number of strategies are finite). For any $l \in L$, we denote $(\Omega_{\beta_l^\epsilon}, \mathcal{F}_{\beta_l^\epsilon}, \mathbb{P}_{\beta_l^\epsilon})$ the probability space $(\Pi_{m\leq N}\Omega_m^l, \otimes_{m\leq N}\mathcal{F}_m^l, \otimes_{m\leq N}\mathbb{P}^l{}_m) \in \mathcal{S}$.

Let $\beta_0 \in \mathcal{B}(t_0)$ be an ϵ-optimal strategy for $\inf_{\beta\in\mathcal{B}(t_0)} \sup_{\alpha\in\mathcal{A}(t_0)} V_K^{-*}\big(t_0+h, (X_{t_0+h}^{t_0,x_0^i,\alpha,\beta})_{i\in I}, \zeta, \Pi^K\big)$. For any $\omega^l = (\omega_m^l)_{m\leq N} \in \Omega_{\beta_l^\epsilon}$, $u \in \mathcal{U}(t_0)$ and $s \in [t_0, 1]$, we define the following map:

$$\beta_l^\epsilon(\omega^l, u)(s) = \begin{cases} \beta_0(u)(s), & if\ s \in [t_0, t_0 + h], \\ \hat{\beta}_l^{Y_m}(\omega_m^l, u|_{[t_0+h,1]})(s), & \text{if } s \in (t_0 + h, 1],\ and\ (X_{t_0+h}^{t_0,x_0^i,\beta_0,u})_{i\in I} \in E_m, \end{cases}$$

where $\omega = (\omega_l)_{l\in L}$. One can check that $(\beta_l^\epsilon)_{l\in L} \in \big(\mathcal{B}_r(t_0)\big)^L$.

Then for any $\alpha \in \mathcal{A}(t_0)$ and $(i, l) \in I \times L$,

$$\mathbb{E}_{\hat{\beta}_l^\epsilon} g^i(X_1^{t_0,x_0^i,\alpha,\beta_l^\epsilon}) = \sum_{m\leq N} \mathbb{E}_{\beta_l^{Y_m}} g^i(X_1^{t_0+h, X_{t_0+h}^{t_0,x_0^i,\alpha,\beta_0}, \alpha, \beta_l^{Y_m}}) \mathbf{1}_{F_m},$$

where $F_m = \{(X_{t_0+h}^{t_0,x_0^i,\alpha,\beta_0})_{i\in I} \in E_m\}$, for any $1 \le m \le N$. Thus,

$$\begin{aligned}
&\max_{k\in K}\Big(\zeta_k - \mathcal{J}_k\big(t_0, X_0, \Pi^K, \alpha, (\beta_l^\epsilon)_{l\in L}\big)\Big)\\
=&\max_{k\in K}\big(\zeta_k - \sum_{(i,l)\in I\times L}\Pi_k^K(i,l)\sum_{m\le N}E_{\beta_l^{Y_m}}g^i(X_1^{t_0+h,X_{t_0+h}^{t_0,x_0^i,\alpha,\beta_0},\alpha,\beta_l^{Y_m}})\mathbf{1}_{F_m}\big)\\
\le&\sum_{m\le N}\sup_{\alpha'\in\mathcal{A}(t_0+h)}\max_{k\in K}\Big(\zeta_k - \mathcal{J}_k\big(t_0,(X_{t_0+h}^{t_0,x_0^i,\alpha,\beta_0})_{i\in I},\Pi^K,\alpha',(\beta_l^{Y_m})_{l\in L}\big)\Big)\mathbf{1}_{F_m}\\
\le&\sum_{m\le N}\Big(V_K^{-*}\big(t_0+h,(X_{t_0+h}^{t_0,x_0^i,\alpha,\beta_0})_{i\in I},\zeta,\Pi^K\big)+2\epsilon\Big)\mathbf{1}_{F_m}\\
=&V_K^{-*}\big(t_0+h,(X_{t_0+h}^{t_0,x_0^i,\alpha,\beta_0})_{i\in I},\zeta,\Pi^K\big)+2\epsilon,
\end{aligned}$$

By Lemma 1.4.2, we deduce:

$$\begin{aligned}
&V_K^{-*}(t_0,X_0,\zeta,\Pi^K)\\
=&\inf_{(\beta_l)_{l\in L}\in(\mathcal{B}_r(t_0))^L}\sup_{\alpha\in\mathcal{A}(t_0)}\max_{k\in K}\Big(\zeta_k-\mathcal{J}_k\big(t_0,X_0,\Pi^K,\alpha,(\beta_l)_{l\in L}\big)\Big)\\
\le&\sup_{\alpha\in\mathcal{A}(t_0)}V_K^{-*}\big(t_0+h,(X_{t_0+h}^{t_0,x_0^i,\alpha,\beta_0})_{i\in I},\zeta,\Pi^K\big)+2\epsilon\\
\le&\inf_{\beta\in\mathcal{B}(t_0)}\sup_{\alpha\in\mathcal{A}(t_0)}V_K^{-*}\big(t_0+h,(X_{t_0+h}^{t_0,x_0^i,\alpha,\beta})_{i\in I},\zeta,\Pi^K\big)+3\epsilon,
\end{aligned}$$

where the last inequality holds by the choice of β_0. Since ϵ is arbitrary, we have finished the proof of Step 1.

Step 2:

Now we consider the case where the running costs $(\gamma^i)_{i\in I}$ do not necessarily reduce to 0.

For some $(\rho_0^i)_{i\in I} \in \mathbb{R}^I$ fixed, let us define $\bar{X}_0 = (\bar{x}_0^i)_{i\in I} = \big((x_0^i,\rho_0^i)\big)_{i\in I}$, $(\bar{f}^i)_{i\in I} = \big((f^i,\gamma^i)\big)_{i\in I}$, and $\bar{g}^i(x,\rho) = g^i(x)+\rho$, for any $(x,\rho)\in\mathbb{R}^n\times\mathbb{R}$ and any $i\in I$. We introduce I new dynamic systems $(P_i')_{i\in I}$ by setting:

$$\left\{\begin{array}{l} x'(t) = f^i\big(x(t),u(t),v(t)\big),\ t\in[t_0,1],\\ \rho'(t) = \gamma^i\big(x(t),u(t),v(t)\big),\ t\in[t_0,1],\\ \big(x(t_0),\rho(t_0)\big) = (x_0^i,\ \rho_0^i),\end{array}\right. \tag{P_i'}$$

We denote $\big(\bar{\mathcal{G}}_i(t_0,\bar{x}_0^i)\big)_{i\in I}$ the family of differential games with the dynamical system (P_i') and the payment function: $\bar{\mathcal{J}}^i(t_0,\bar{x}_0^i,u,v) := \bar{g}^i(X_1^{t_0,\bar{x}_0^i,u,v})$, where the map $s\mapsto X_s^{t_0,\bar{x}_0^i,u,v}$ denotes the unique solution of the dynamical system (P_i').

Let $\bar{\mathcal{G}}(t_0,\bar{X}_0,\pi)$ denote the differential game with incomplete information and signals associated to the family of differential games $\big(\bar{\mathcal{G}}_i(t_0,\bar{x}_0^i)\big)_{i\in I}$ and the probability $\pi\in\Delta(I\times$

$K \times L$). Then the upper-value function of $\bar{\mathcal{G}}(t_0, \bar{X}_0, \pi)$, denoted by $\bar{V}^-(t_0, \bar{X}_0, \pi)$, satisfies:

$$\bar{V}^-(t_0, \bar{X}_0, \pi) - \sum_{(i,k,l)\in I\times K\times L} \pi(i,k,l)\rho_0^i = V^-(t_0, X_0, \pi). \tag{1.4.1}$$

We notice that the right-hand side of (1.4.1) is independent of $(\rho_0^i)_{i\in I}$.

Then, we obtain, for any $(\zeta, \Pi^K) \in \mathbb{R}^K \times \big(\Delta(I\times L)\big)^K$:

$$V_K^{-*}(t_0, X_0, \zeta, \Pi^K) = \bar{V}_K^{-*}\Big(t_0, \bar{X}_0, \zeta - \big(\sum_{(i,l)\in I\times L} \Pi_k^K(i,l)\rho_0^i\big)_{k\in K}, \Pi^K\Big). \tag{1.4.2}$$

From Step 1 and (1.4.2) we have:

$$\begin{aligned}
&\bar{V}_K^{-*}\Big(t_0, \bar{X}_0, \zeta - \big(\sum_{(i,l)\in I\times L} \Pi_k^K(i,l)\rho_0^i\big)_{k\in K}, \Pi^K\Big)\\
&\le \inf_{\beta\in\mathcal{B}(t_0)} \sup_{\alpha\in\mathcal{A}(t_0)} \bar{V}_K^{-*}\Big(t_0+h, (\bar{X}_{t_0+h}^{t_0,\bar{x}_0^i,\alpha,\beta})_{i\in I}, \zeta - \big(\sum_{(i,l)\in I\times L} \Pi_k^K(i,l)\rho_0^i\big)_{k\in K}, \Pi^K\Big) \qquad (1.4.3)\\
&= \inf_{\beta\in\mathcal{B}(t_0)} \sup_{\alpha\in\mathcal{A}(t_0)} V_K^{-*}\big(t_0+h, (X_{t_0+h}^{t_0,x_0^i,\alpha,\beta})_{i\in I}, \zeta(t_0+h,\alpha,\beta), \Pi^k\big).
\end{aligned}$$

The last equality holds by choosing $\rho_0^i = \int_{t_0}^{t_0+h} \gamma^i\big(X_s^{t_0,x_0^i,\alpha,\beta}, \alpha, \beta\big)ds$, $\forall i \in I$. By combining (1.4.2) and (1.4.3), the proof is complete. □

1.5 Viscosity Solutions and Existence of Value

In this section, we introduce a new Hamilton-Jacobi-Isaacs equation which is associated to the sub-dynamic programming principle in Section 4, and we prove that the upper- and lower-value functions are respectively viscosity sub-solution and super-solution of this equation. The existence of a value for the game $\mathcal{G}(t_0, X_0, \pi)$ comes from a comparison principle.

Let $\mathcal{S}(K)$ denote the set of all $K \times K$ symmetric real matrices, and for any $(A, \pi^K) \in \mathcal{S}(K) \times \Delta(K)$, let us set as in [17]:

$$\begin{aligned}
\lambda_{min}(A, \pi^K) &:= \min\{\langle Az, z\rangle \mid z \in T_{\Delta(K)}(\pi^K),\ \|z\| = 1\};\\
\lambda_{max}(A, \pi^K) &:= \max\{\langle Az, z\rangle \mid z \in T_{\Delta(K)}(\pi^K),\ \|z\| = 1\},
\end{aligned}$$

where $T_{\Delta(K)}(\pi^K)$ is the tangent cone associated to $\Delta(K)$ at π^K, i.e.,

$$T_{\Delta(K)}(\pi^K) = \{z \in \mathbb{R}^K \mid z_k < 0 \Longrightarrow \pi^K(k) > 0,\ \forall k \in K \text{ and } \textstyle\sum_{k\in K} z_k = 0\}.$$

Similarly, we set, for $(B, \pi^L) \in \mathcal{S}(L) \times \Delta(L)$:

$$\lambda_{min}(B,\pi^L) := \min\{\langle Bz,z\rangle \mid z\in T_{\Delta(L)}(\pi^L),\ \|z\|=1\};$$
$$\lambda_{max}(B,\pi^L) := \max\{\langle Bz,z\rangle \mid z\in T_{\Delta(L)}(\pi^L),\ \|z\|=1\},$$

Let $H : [0,1]\times(\mathbb{R}^n)^I\times\Delta(I\times K\times L)\to\mathbb{R}$ be a continuous function. We consider the following double-obstacle second-order Hamilton-Jacobi-Isaacs equation in $[0,1]\times(\mathbb{R}^n)^I\times\Delta(I\times K\times L)$:

$$\begin{aligned}&\min\big\{\lambda_{min}(D^2_{\pi^K\pi^K}W_K,\pi^K);\ \max\{\lambda_{max}(D^2_{\pi^L\pi^L}W_L,\pi^L);\ \partial_t W+H(t,X,D_XW,\pi)\}\big\}=0;\\ &\max\big\{\lambda_{max}(D^2_{\pi^L\pi^L}W_L,\pi^L);\ \min\{\lambda_{min}(D^2_{\pi^K\pi^K}W_K,\pi^K);\ \partial_t W+H(t,X,D_XW,\pi)\}\big\}=0,\end{aligned} \tag{1.5.1}$$

where, for any $(k,l)\in K\times L$,

$$\pi^K(k)=\textstyle\sum_{(i,l')\in I\times L}\pi(i,k,l'),\ \pi^L(l)=\sum_{(i,k')\in I\times K}\pi(i,k',l).$$

The equation (1.5.1) is a generalization of the PDE studied in [17]. We now define the viscosity sub-solution and super-solution for the Hamilton-Jacobi-Isaacs equation (1.5.1).

Definition 1.5.1. *A function $W : [0,1]\times(\mathbb{R}^n)^I\times\Delta(I\times K\times L)\to\mathbb{R}$ is a viscosity sub-solution of the equation* (1.5.1) *if it is Lipschitz continuous, and K-convex and L-concave in $\Delta(I\times K\times L)$, and if for any test function $\varphi\in\mathcal{C}^2\big((0,1)\times(\mathbb{R}^n)^I\times\mathbb{R}^L\big)$ such that the map*

$$(t,X,\pi^L)\mapsto W(t,X,\pi^L\otimes_L\bar\Pi^L)-\varphi(t,X,\pi^L)$$

has a local maximum at some point $(\bar t,\bar X,\bar\pi^L)\in(0,1)\times(\mathbb{R}^n)^I\times\Delta(L)$ for some $\bar\Pi^L\in\big(\Delta(I\times K)\big)^L$, one has:

$$\max\{\lambda_{max}(D^2_{\pi^L\pi^L}\varphi,\bar\pi^L);\ \partial_t\varphi+H(\bar t,\bar X,D_X\varphi,\bar\pi^L\otimes_L\bar\Pi^L)\}\geq 0\ \text{ at }(\bar t,\bar X,\bar\pi^L).$$

Symmetrically, W is a viscosity super-solution of the equation (1.5.1) *if it is Lipschitz continuous, and K-convex and L-concave in $\Delta(I\times K\times L)$, and if for any test function $\varphi\in\mathcal{C}^2\big((0,1)\times(\mathbb{R}^n)^I\times\mathbb{R}^K\big)$ such that the map*

$$(t,X,\pi^K)\mapsto W(t,X,\pi^K\otimes_K\bar\Pi^K)-\varphi(t,X,\pi^K)$$

has a local minimum at some point $(\bar t,\bar X,\bar\pi^K)\in(0,1)\times(\mathbb{R}^n)^I\times\Delta(K)$ for some $\bar\Pi^K\in\big(\Delta(I\times L)^K\big)$, one has:

$$\min\{\lambda_{min}(D^2_{\pi^K\pi^K}\varphi,\bar\pi^K);\ \partial_t\varphi+H(\bar t,\bar X,D_X\varphi,\bar\pi^K\otimes_K\bar\Pi^K)\}\leq 0\ \text{ at }(\bar t,\bar X,\bar\pi^K).$$

Finally, if W is both viscosity sub-solution and super-solution of the equation (1.5.1)*, we say that it is a viscosity solution of the equation.*

In the above definition, we only need to consider strict local extrema of the functions $W_K(\centerdot,\Pi^K)-\varphi(\centerdot)$ and $W_L(\centerdot,\Pi^L)-\varphi(\centerdot)$, by the following standard lemma (cf. [3]).

Lemma 1.5.2. *A function $W : [0,1] \times (\mathbb{R}^n)^I \times \Delta(I \times K \times L) \to \mathbb{R}$ is a viscosity sub-solution of the equation* (1.5.1) *if and only if it is Lipschitz continuous, and K-convex and L-concave in $\Delta(I \times K \times L)$, and for any test function $\varphi \in \mathcal{C}^2\big((0,1) \times (\mathbb{R}^n)^I \times \mathbb{R}^L\big)$ such that the map*

$$(t, X, \pi^L) \mapsto W(t, X, \pi^L \otimes_L \bar{\Pi}^L) - \varphi(t, X, \pi^L)$$

has a strict local maximum at some point $(\bar{t}, \bar{X}, \bar{\pi}^L) \in (0,1) \times (\mathbb{R}^n)^I \times \Delta(L)$ for some $\bar{\Pi}^L \in \big(\Delta(I \times K)\big)^L$, one has:

$$\max\{\lambda_{max}(D^2_{\pi^L \pi^L}\varphi, \bar{\pi}^L);\ \partial_t\varphi + H(\bar{t}, \bar{X}, D_X\varphi, \bar{\pi}^L \otimes_L \bar{\Pi}^L)\} \geq 0 \text{ at } (\bar{t}, \bar{X}, \bar{\pi}^L).$$

Similarly, W is a viscosity super-solution of the equation (1.5.1) *if and only if it is Lipschitz continuous, and K-convex and L-concave in $\Delta(I \times K \times L)$, and for any test function $\varphi \in \mathcal{C}^2\big((0,1) \times (\mathbb{R}^n)^I \times \mathbb{R}^K\big)$ such that the map*

$$(t, X, \pi^K) \mapsto W(t, X, \pi^K \otimes_K \bar{\Pi}^K) - \varphi(t, X, \pi^K)$$

has a strict local minimum at some point $(\bar{t}, \bar{X}, \bar{\pi}^K) \in (0,1) \times (\mathbb{R}^n)^I \times \Delta(K)$ for some $\bar{\Pi}^K \in \big(\Delta(I \times L)^K\big)$, one has:

$$\min\{\lambda_{min}(D^2_{\pi^K \pi^K}\varphi, \bar{\pi}^K);\ \partial_t\varphi + H(\bar{t}, \bar{X}, D_X\varphi, \bar{\pi}^K \otimes_K \bar{\Pi}^K)\} \leq 0 \text{ at } (\bar{t}, \bar{X}, \bar{\pi}^K).$$

Before we prove that V^+ and V^- are respectively viscosity sub-solution and viscosity super-solution of the equation (1.5.1), we recall the definition of convex sub-differential. The convex sub-differential of φ at $q \in \mathbb{R}^m$ is defined by:

$$\partial^-\varphi(q) = \{q^* \in \mathbb{R}^m \mid \varphi(q) + \langle q^*, q' - q\rangle \leq \varphi(q'),\ \textit{for any } q' \in \mathbb{R}^m\}.$$

Let us first of all prove that:

Proposition 1.5.3. *The lower-value function V^- is a viscosity super-solution of Hamilton-Jacobi equation* (1.5.1) *with $H(t, .) = H_{\mathcal{G}}(.)$ for any $t \in [0,1]$, where $H_{\mathcal{G}}$ is defined as in* (1.2.2).

Before proving Proposition 1.5.3, we state the following technical lemma:

Lemma 1.5.4. *(cf. [17]) Let $W(t, x, \pi^K)$ be a continuous function on $[0,1] \times \mathbb{R}^m \times \Delta(K)$ and convex with respect to π^K. Let $\varphi \in \mathcal{C}^2\big((0,1) \times \mathbb{R}^m \times \Delta(K)\big)$ be a test function such that $W - \varphi$ has a local minimum on $(0,1) \times \mathbb{R}^m \times \Delta(K)$ at some point $(\bar{t}, \bar{x}, \bar{\pi}^K)$. If*

$$\lambda_{min}\langle D^2_{\pi^K \pi^K}\varphi(\bar{t}, \bar{x}, \bar{\pi}^K), \bar{\pi}^K\rangle > 0,$$

then there exist some δ, $\eta > 0$ such that

$$W(t, x, \pi^K) \geq \varphi(t, x, \bar{\pi}^K) + \langle D_{\pi^K}\varphi(t, x, \bar{\pi}^K), \pi^K - \bar{\pi}^K\rangle + \tfrac{\eta}{2}\|\pi^K - \bar{\pi}^K\|^2,$$

for any $(t, x) \in B(\bar{t}, \bar{x}; \delta)$, and $\pi^K \in \Delta(K)$.

Proof of Proposition 1.5.3. Let $\bar{\Pi}^K \in \big(\Delta(I \times L)\big)^K$ be fixed. Suppose that for some $\varphi \in \mathcal{C}^2\big([0,1] \times (\mathbb{R}^n)^I \times \mathbb{R}^K\big)$, the map

$$(t, X, \pi^K) \mapsto V_K^-(t, X, \pi^K, \bar{\Pi}^K) - \varphi(t, X, \pi^K)$$

has a local minimum on $(0,1) \times (\mathbb{R}^n)^I \times \Delta(K)$ at some point $(\bar{t}, \bar{X}, \bar{\pi}^K)$. Without loss of generality, we suppose in addition that $V_K^-(\bar{t}, \bar{X}, \bar{\pi}^K, \bar{\Pi}^K) - \varphi(\bar{t}, \bar{X}, \bar{\pi}^K) = 0$ and that

$$\lambda_{min}\langle D^2_{\pi^K\pi^K}\varphi(\bar{t}, \bar{X}, \bar{\pi}^K), \bar{\pi}^K\rangle > 0.$$

Then we only need to prove that:

$$\partial_t\varphi(\bar{t}, \bar{X}, \bar{\pi}^K) + H\big(\bar{t}, \bar{X}, D_X\varphi(\bar{t}, \bar{X}, \bar{\pi}^K), \bar{\pi}^K \otimes_K \bar{\Pi}^K\big) \leq 0.$$

By Lemma 1.5.4, there exist some $\delta,\ \eta > 0$ such that, for any $(t, X) \in B(\bar{t}, \bar{X}; \delta)$ and $\pi^K \in \Delta(K)$:

$$V_K^-(t, X, \pi^K, \bar{\Pi}^K) \geq \varphi(t, X, \bar{\pi}^K) + \langle D_{\pi^K}\varphi(t, X, \bar{\pi}^K), \pi^K - \bar{\pi}^K\rangle + \frac{\eta}{2}\|\pi^K - \bar{\pi}^K\|^2. \quad (1.5.2)$$

Let $\hat{\zeta} = D_{\pi^K}\varphi(\bar{t}, \bar{X}, \bar{\pi}^K)$. By (1.5.2), for any $(t, X, \pi^K, \hat{\zeta}') \in B(\bar{t}, \bar{X}; \delta) \times \Delta(K) \times \mathbb{R}^K$, we have:

$$\begin{aligned}\langle \pi^K, \hat{\zeta}'\rangle - V_K^-(t, X, \pi^K, \bar{\Pi}^K) \leq &- \varphi(t, X, \bar{\pi}^K) - \langle D_{\pi^K}\varphi(t, X, \bar{\pi}^K) - \hat{\zeta}', \pi^K - \bar{\pi}^K\rangle + \langle \hat{\zeta}', \bar{\pi}^K\rangle - \\ &\frac{\eta}{2}\|\pi^K - \bar{\pi}^K\|^2.\end{aligned}$$

By taking the supremum with respect to $\pi^K \in \Delta(K)$ in the above inequality, we obtain:

$$V_K^{-*}(t, X, \hat{\zeta}', \bar{\Pi}^K) \leq -\varphi(t, X, \bar{\pi}^K) + \frac{1}{2\eta}\|D_{\pi^K}\varphi(t, X, \bar{\pi}^K) - \hat{\zeta}'\|^2 + \langle \hat{\zeta}', \bar{\pi}^K\rangle. \quad (1.5.3)$$

In addition, (1.5.2) implies $\hat{\zeta} \in \partial^- V_K^-(\bar{t}, \bar{X}, \bar{\pi}^K, \bar{\Pi}^K)$. Then the Fenchel-duality yields:

$$\langle \hat{\zeta}, \bar{\pi}^K\rangle - V_K^{-*}(\bar{t}, \bar{X}, \hat{\zeta}, \bar{\Pi}^K) = V_K^-(\bar{t}, \bar{X}, \bar{\pi}^K, \bar{\Pi}^K). \quad (1.5.4)$$

Let $0 < h < \delta$ be small enough. We apply the sub-dynamic programming (Proposition 1.4.1). By (1.5.3), we have, for $\bar{X} = (\bar{x}^i)_{i\in I}$,

$$\begin{aligned}&V_K^{-*}(\bar{t}, \bar{X}, \hat{\zeta}, \bar{\Pi}^K)\\ \leq &\inf_{\beta\in\mathcal{B}(\bar{t})}\sup_{\alpha\in\mathcal{A}(\bar{t})} V_K^{-*}\big(\bar{t}+h, (X^{\bar{t},\bar{x}^i,\alpha,\beta}_{\bar{t}+h})_{i\in I}, \hat{\zeta}(\bar{t}+h, \alpha, \beta), \bar{\Pi}^K\big)\\ \leq &\inf_{\beta\in\mathcal{B}(\bar{t})}\sup_{\alpha\in\mathcal{A}(\bar{t})} -\varphi\big(\bar{t}+h, (X^{\bar{t},\bar{x}^i,\alpha,\beta}_{\bar{t}+h})_{i\in I}, \bar{\pi}^K\big) + \frac{1}{2\eta}\|D_{\pi^K}\varphi\big(\bar{t}+h, (X^{\bar{t},\bar{x}^i,\alpha,\beta}_{\bar{t}+h})_{i\in I}, \bar{\pi}^K\big) - \\ &\hat{\zeta}(\bar{t}+h, \alpha, \beta)\|^2 + \langle \hat{\zeta}(\bar{t}+h, \alpha, \beta), \bar{\pi}^K\rangle.\end{aligned} \quad (1.5.5)$$

By combining (1.5.4) and (1.5.5), we obtain:

$$\begin{aligned}0 \geq \sup_{\beta\in\mathcal{B}(\bar t)}\inf_{\alpha\in\mathcal{A}(\bar t)}&\varphi\big(\bar t+h,(X^{\bar t,\bar x^i,\alpha,\beta}_{\bar t+h})_{i\in I},\bar\pi^K\big)-\varphi(\bar t,\bar X,\bar\pi^K)-\\&\frac{1}{2\eta}\|D_{\pi^K}\varphi\big(\bar t+h,(X^{\bar t,\bar x^i,\alpha,\beta}_{\bar t+h})_{i\in I},\bar\pi^K\big)-\hat\zeta(\bar t+h,\alpha,\beta)\|^2+\\&\sum_{k\in K}\bar\pi^K(k)\Big(\sum_{(i,l)\in I\times L}\bar\Pi^K_k(i,l)\int_{\bar t}^{\bar t+h}\gamma^i\big(X^{\bar t,\bar x^i,\alpha,\beta}_s,\alpha,\beta\big)ds\Big),\end{aligned}\tag{1.5.6}$$

We fix $\epsilon>0$. By (1.5.6), for any $v\in V$(time-independent control), there exists $u^{v,h}_\epsilon\in\mathcal{U}(\bar t)$ such that ($u^{v,h}_\epsilon$ and v can be viewed respectively as elements of $\mathcal{A}(\bar t)$ and $\mathcal{B}(\bar t)$):

$$\begin{aligned}\epsilon h\geq&\varphi\big(\bar t+h,(X^{\bar t,\bar x^i,u^{v,h}_\epsilon,v}_{\bar t+h})_{i\in I},\bar\pi^K\big)-\varphi\big(\bar t,\bar X,\bar\pi^K\big)-\\&\frac{1}{2\eta}\|D_{\pi^K}\varphi\big(\bar t+h,(X^{\bar t,\bar x^i,u^{v,h}_\epsilon,v}_{\bar t+h})_{i\in I},\bar\pi^K\big)-\hat\zeta(\bar t+h,u^{v,h}_\epsilon,v)\|^2+\\&\sum_{k\in K}\bar\pi^K(k)\Big(\sum_{(i,l)\in I\times L}\bar\Pi^K_k(i,l)\int_{\bar t}^{\bar t+h}\gamma^i\big(X^{\bar t,\bar x^i,u^{v,h}_\epsilon,v}_s,u^{v,h}_\epsilon(s),v\big)ds\Big).\end{aligned}\tag{1.5.7}$$

We have:

$$\begin{aligned}&\varphi\big(\bar t+h,(X^{\bar t,\bar x^i,u^{v,h}_\epsilon,v}_{\bar t+h})_{i\in I},\bar\pi^K\big)-\varphi(\bar t,\bar X,\bar\pi^K)=\int_{\bar t}^{\bar t+h}\partial_t\varphi\big(s,(X^{\bar t,\bar x^i,u^{v,h}_\epsilon,v}_s)_{i\in I},\bar\pi^K\big)+\\&\sum_{i\in I}D_{x^i}\varphi\big(s,(X^{\bar t,\bar x^i,u^{v,h}_\epsilon,v}_s)_{i\in I},\bar\pi^K\big)f^i\big(X^{\bar t,\bar x^i,u^{v,h}_\epsilon,v}_s,u^{v,h}_\epsilon(s),v\big)ds.\end{aligned}\tag{1.5.8}$$

The following estimations are direct consequences of the hypothesis about the regularity of φ, of $(f^i)_{i\in I}$ and of $(\gamma^i)_{i\in I}$:

$$\begin{aligned}o(h)\geq&\|D_{\pi^K}\varphi\big(\bar t+h,(X^{\bar t,\bar x^i,u^{v,h}_\epsilon,v}_{\bar t+h})_{i\in I},\bar\pi^K\big)-\hat\zeta(\bar t+h,u^{v,h}_\epsilon,v)\|^2.\\o(h)\geq&\int_{\bar t}^{\bar t+h}\partial_t\varphi\big(s,(X^{\bar t,\bar x^i,u^{v,h}_\epsilon,v}_s)_{i\in I},\bar\pi^K\big)ds-h\partial_t\varphi(\bar t,\bar X,\bar\pi^K)\\o(h)\geq&\int_{\bar t}^{\bar t+h}\sum_{i\in I}D_{x^i}\varphi\big(s,(X^{\bar t,\bar x^i,u^{v,h}_\epsilon,v}_s)_{i\in I},\bar\pi^K\big)f^i\big(X^{\bar t,\bar x^i,u^{v,h}_\epsilon,v}_s,u^{v,h}_\epsilon(s),v\big)ds-\\&\int_{\bar t}^{\bar t+h}\sum_{i\in I}D_{x^i}\varphi(\bar t,\bar X,\bar\pi^K)f^i(\bar x^i,u^{v,h}_\epsilon(s),v)ds\\o(h)\geq&\sum_{k\in K}\bar\pi^K(k)\Big(\sum_{(i,l)\in I\times L}\bar\Pi^K_k(i,l)\int_{\bar t}^{\bar t+h}\gamma^i\big(X^{\bar t,\bar x^i,u^{v,h}_\epsilon,v}_s,u^{v,h}_\epsilon(s),v\big)ds\Big)-\\&\sum_{k\in K}\bar\pi^K(k)\Big(\sum_{(i,l)\in I\times L}\bar\Pi^K_k(i,l)\int_{\bar t}^{\bar t+h}\gamma^i\big(\bar x^i,u^{v,h}_\epsilon(s),v\big)ds\Big)\end{aligned}\tag{1.5.9}$$

Combining (1.5.7), (1.5.8), and (1.5.9) gives:

$$\begin{aligned}\epsilon h + o(h) \geq & h\partial_t\varphi(\bar{t},\bar{X},\bar{\pi}^K) + \int_{\bar{t}}^{\bar{t}+h}\sum_{i\in I} D_{x^i}\varphi(\bar{t},\bar{X},\bar{\pi}^K) f^i\big(\bar{x}^i, u_\epsilon^{v,h}(s), v\big)ds + \\ & \sum_{k\in K}\bar{\pi}^K(k)\Big(\sum_{(i,l)\in I\times L}\bar{\Pi}_k^K(i,l)\int_{\bar{t}}^{\bar{t}+h}\gamma^i\big(\bar{x}^i, u_\epsilon^{v,h}(s), v\big)ds\Big) \\ \geq & h\partial_t\varphi(\bar{t},\bar{X},\bar{\pi}^K) + h\inf_{u\in U}\Big(\sum_{i\in I} D_{x^i}\varphi(\bar{t},\bar{X},\bar{\pi}^K) f^i(\bar{x}^i,u,v) \\ & + \sum_{k\in K}\bar{\pi}^K(k)\big(\sum_{(i,l)\in I\times L}\bar{\Pi}_k^K(i,l)\gamma^i(\bar{x}^i,u,v)\big)\Big)\end{aligned} \tag{1.5.10}$$

Dividing both sides of (1.5.10) by h and letting $h \to 0$, we obtain:

$$\epsilon \geq \partial_t\varphi(\bar{t},\bar{X},\bar{\pi}^K) + \inf_{u\in U}\Big(\sum_{i\in I} D_{x^i}\varphi(\bar{t},\bar{X},\bar{\pi}^K) f^i(\bar{x}^i,u,v) + \sum_{k\in K}\bar{\pi}^K(k)\big(\sum_{(i,l)\in I\times L}\bar{\Pi}_k^K(i,l)\gamma^i(\bar{x}^i,u,v)\big)\Big)$$

Since ϵ is arbitrary, taking the supremum of v over V in both sides gives:

$$0 \geq \partial_t\varphi(\bar{t},\bar{X},\bar{\pi}^K) + H\big(\bar{t},\bar{X}, D_X\varphi(\bar{t},\bar{X},\bar{\pi}^K), \bar{\pi}^K \otimes_K \bar{\Pi}^K\big),$$

which is our wished result. The proof is complete. □

Symmetrically, in view of Remark 1.2.6, by interchanging Player I and Player II, we obtain:

Proposition 1.5.5. *The upper-value function V^+ is a viscosity sub-solution of Hamilton-Jacobi-Isaacs equation* (1.5.1) *with $H(t,.) = H_{\mathcal{G}}(.)$ for any $t \in [0,1]$.*

Now we state the main result of this paper:

Theorem 1.5.6. *The game $\mathcal{G}(t_0, X_0, \pi)$ has a value which is the unique viscosity solution of the Hamilton-Jacobi-Isaacs equation* (1.5.1) *with $H(t,.) = H_{\mathcal{G}}(.)$ for any $t \in [0,1]$ and with the boundary condition: for all $(X,\pi) \in (\mathbb{R}^n)^I \times \Delta(I\times K\times L)$,*

$$W(1,X,\pi) = \textstyle\sum_{(i,k,l)\in I\times K\times L}\pi(i,k,l)g(x^i).$$

We deduce this theorem from the following:

Proposition 1.5.7. *(Comparison Principle) Let W_1, $W_2 : [0,1]\times(\mathbb{R}^n)^I\times\Delta(I\times K\times L) \to \mathbb{R}$ be respectively a viscosity sub-solution and a viscosity super-solution of equation* (1.5.1). *Suppose that there exists a constant $C > 0$ such that for any $\pi \in \Delta(I\times K\times L)$, any (t_1,X_1), $(t_2,X_2) \in [0,1]\times(\mathbb{R}^n)^I$ and any ζ_1, $\zeta_2 \in (\mathbb{R}^n)^I$,*

$$|H(t_1,X_1,\zeta_1,\pi) - H(t_2,X_2,\zeta_1,\pi)| \leq C\|(t_1,X_1)-(t_2,X_2)\|(1+\|\zeta_1\|); \tag{1.5.11}$$

$$|H(t_1,X_1,\zeta_1,\pi) - H(t_1,X_1,\zeta_2,\pi)| \leq C\|\zeta_1-\zeta_2\|. \tag{1.5.12}$$

If for any $(X,\pi) \in (\mathbb{R}^n)^I \times \Delta(I\times K\times L)$,

$$W_1(1, X, \pi) \leq W_2(1, X, \pi)$$

Then $W_1 \leq W_2$ *in* $[0,1] \times (\mathbb{R}^n)^I \times \Delta(I \times K \times L)$.

Proof of Proposition 1.5.7. The following technical lemma allows us to restrict the state variable in a bounded area.

Lemma 1.5.8. *If* W *is a viscosity sub-solution (resp. super-solution) of* (1.5.1) *with Hamiltonian satisfying* (1.5.12) *on* $[0,1] \times (\mathbb{R}^n)^I \times \Delta(I \times K \times L)$*, then for any* $(t_0, X_0, \pi_0) \in [0,1) \times (\mathbb{R}^n)^I \times \Delta(I \times K \times L)$*,* W *is still a viscosity sub-solution (resp. super-solution) of* (1.5.1) *on* $C_{t_0,X_0} \times \Delta(I \times K \times L)$*, where:*

$$C_{t_0,X_0} = \{(t, X) \in [t_0, 1] \times (\mathbb{R}^n)^I \mid \|X - X_0\| \leq C(t - t_0)\}.$$

Proof of Lemma 1.5.8. We only prove the lemma for the super-solutions, since the other part can be proved by a similar argument.

In view of Lemma 1.5.2, let us suppose that for a test function $\phi \in \mathcal{C}^2\big([0,1]\times(\mathbb{R}^n)^I\times\mathbb{R}^K\big)$, and for some $\Pi^K \in \big(\Delta(I \times L)\big)^K$, the map $(t, X, \pi^K) \mapsto W(t, X, \pi^K \otimes_K \Pi^K) - \phi(t, X, \pi^K)$ has a strict local minimum on $C_{t_0,X_0} \times \Delta(K)$ at some point $(\bar{t}, \bar{X}, \bar{\pi}^K)$, where $\bar{t} < 1$.

If $\lambda_{min} D^2_{\pi^K \pi^K} \phi(\bar{t}, \bar{X}, \bar{\pi}^K) \leq 0$, then by definition, the proof is complete. Let us suppose that $\lambda_{min} D^2_{\pi^K \pi^K} \phi(\bar{t}, \bar{X}, \bar{\pi}^K) > 0$.

For $\sigma > 0$, let us consider a minimizer $(s_\sigma, Y_\sigma, \pi^{K,\sigma}) \notin \partial(C_{t_0,X_0}) \times \Delta(K)$ of the map:

$$(s, Y, \pi^K) \mapsto \Phi(s, Y, \pi^K) := W(s, Y, \pi^K \otimes_K \Pi^K) - \phi(s, Y, \pi^K) - \tfrac{\sigma}{2} \ln\big(C^2(s - t_0)^2 - \|Y - X_0\|^2\big)$$

The existence of $(s_\sigma, Y_\sigma, \pi^{K,\sigma})$ for any $\sigma > 0$ is due to the fact that $C_{t_0,X_0} \times \Delta(K)$ is compact and that:

$$\Phi(s, Y, \pi^K) \to +\infty, \text{ as } (s, Y) \to \partial(C_{t_0,X_0}).$$

One can verify easily that $(s_\sigma, Y_\sigma, \pi^{K,\sigma}) \to (\bar{t}, \bar{X}, \bar{\pi}^K)$, as $\sigma \to 0^+$, which leads to, in particular, that $s_\sigma < 1$ for σ small enough[2]. Since W is a super-solution of (1.5.1) and that $\lambda_{min} D^2_{\pi^K \pi^K} \phi(\bar{t}, \bar{X}, \bar{\pi}^K) > 0$, we have, for σ sufficiently small:

$$\begin{aligned} 0 \geq & \partial_t \phi(s_\sigma, Y_\sigma, \pi^{K,\sigma}) + \frac{\sigma C^2 (s_\sigma - t_0)}{C^2(s_\sigma - t_0)^2 - \|Y_\sigma - X_0\|^2} + \\ & H\Big(s_\sigma, Y_\sigma, D_X \phi(s_\sigma, Y_\sigma, \pi^{K,\sigma}) - \frac{\sigma(Y_\sigma - X_0)}{C^2(s_\sigma - t_0)^2 - \|Y_\sigma - X_0\|^2}, \pi^{K,\sigma} \otimes_K \Pi^K\Big). \end{aligned} \tag{1.5.13}$$

We notice that (1.5.13) is well-defined since that $C^2(s_\sigma - t_0)^2 - \|Y_\sigma - X_0\|^2 > 0$. By (1.5.12), we obtain:

$$\partial_t \phi(s_\sigma, Y_\sigma, \pi^{K,\sigma}) + \frac{\sigma C\big(C(s_\sigma - t_0) - \|Y_\sigma - X_0\|\big)}{C^2(s_\sigma - t_0)^2 - \|Y_\sigma - x_0\|^2} + H\big(s_\sigma, Y_\sigma, D_X \phi(s_\sigma, Y_\sigma, \pi^{K,\sigma}), \pi^{K,\sigma} \otimes_K \Pi^K\big) \leq 0,$$

[2]Because of the compacity of the set $C_{t_0,X_0} \times \Delta(I \times K \times L)$, we can in fact suppose that $W(\bar{t}, \bar{X}, \bar{\pi}^K \otimes_K \Pi^K) - \phi(\bar{t}, \bar{X}, \bar{\pi}^K)$ is a strict global minimum. (cf. [3])

which implies, since $C(\bar{t}-t_0)-\|\bar{X}-X_0\|>0$, by letting $\sigma\to 0^+$:

$$\partial_t\phi(\bar{t},\bar{X},\bar{\pi}^K)+H\big(\bar{t},\bar{X},D_x\phi(\bar{t},\bar{X},\bar{\pi}^K),\bar{\pi}^K\otimes_K\Pi^K\big)\leq 0.$$

The proof is complete. □

Let us return to the proof of Proposition 1.5.7. By contradiction, let us suppose that

$$\sup_{(t,X,\pi)\in[0,1]\times(\mathbb{R}^n)^I\times\Delta(I\times K\times L)}(W_1-W_2)(t,X,\pi)>0.$$

Then there exists $(t_0,X_0)\in[0,1)\times(\mathbb{R}^n)^I$, such that for σ, $\delta>0$ sufficiently small:

$$M:=\sup_{(t,X,\pi)\in C_{t_0,X_0}\times\Delta(I\times K\times L)}W_1(t,X,\pi)-W_2(t,X,\pi)-\sigma(1-t)+\delta(\|\pi^K\|^2+\|\pi^L\|^2)>0$$

where C_{t_0,X_0} is defined as in Lemma 1.5.8 and π^K, π^L are respectively the marginal probability of π on K and L.

We apply the double-variable technique: for $\epsilon>0$, let us define:

$$\phi_\epsilon\big((t,X),(s,Y),\pi\big)=$$
$$W_1(t,X,\pi)-W_2(s,Y,\pi)-\tfrac{1}{2\epsilon}\|(s,Y)-(t,X)\|^2-\sigma(1-s)+\delta(\|\pi^K\|^2+\|\pi^L\|^2).$$

By setting:

$$M_\epsilon:=\sup_{((t,X),(s,Y),\pi)\in(C_{t_0,X_0})^2\times\Delta(I\times K\times L)}\phi_\epsilon\big((t,X),(s,Y),\pi\big),$$

we have that $M_\epsilon\geq M$ for any $\epsilon>0$ and there exist $((t_\epsilon,X_\epsilon),(s_\epsilon,Y_\epsilon),\pi_\epsilon)\in(C_{(t_0,X_0)})^2\times\Delta(I\times K\times L)$ where the maximum in the definition of M_ϵ is achieved.

We will use now the following technical result, of which the proof is postposed after the proof of Proposition 1.5.7:

Lemma 1.5.9. *1.* $\lim_{\epsilon\to0^+}M_\epsilon=M$*;*

2. $\lim_{\epsilon\to0^+}\frac{1}{\epsilon}\|(t_\epsilon,X_\epsilon)-(s_\epsilon,Y_\epsilon)\|^2=0$*;*

3. for $\delta>0$, $\sigma>0$ *and* ϵ *sufficiently small,* $t_\epsilon<1$ *and* $s_\epsilon<1$.

Let us finish the proof of Proposition 1.5.7:
For $\epsilon>0$, we decompose $\pi_\epsilon=\pi_\epsilon^K\otimes_K\Pi_\epsilon^K=\pi_\epsilon^L\otimes_L\Pi_\epsilon^L$ as in Section 3. For any $\bar{\pi}\in\Delta(I\times K\times L)$ such that $\bar{\pi}^K\otimes_K\bar{\Pi}^K=\bar{\pi}^L\otimes_L\bar{\Pi}^L=\bar{\pi}$, let $A^K(\bar{\Pi}^K)$ be the $L\times K$-real matrix such that for $(l,k)\in L\times K$, $A^K(\bar{\Pi}^K)_{l,k}=\sum_{i\in I}\bar{\Pi}^K_k(i,l)$, then $\bar{\pi}^L=A^K(\bar{\Pi}^K)\bar{\pi}^K$. Similarly, let $A^L(\bar{\Pi}^L)$ be the $K\times L$-real matrix such that for $(k,l)\in K\times L$, $A^L(\bar{\Pi}^L)_{k,l}=\sum_{i\in I}\bar{\Pi}^L_l(i,k)$, then $\bar{\pi}^K=A^L(\bar{\Pi}^L)\bar{\pi}^L$.

Since the map $(t,X,\pi^L)\mapsto \phi_\epsilon\big((t,X),(s_\epsilon,Y_\epsilon),\pi^L\otimes_L\Pi^L_\epsilon\big)$ reaches its maximum at the point $(t_\epsilon,X_\epsilon,\pi^L_\epsilon)\in C_{t_0,X_0}\times\Delta(L)$, for any $(t,X,\pi^L)\in C_{t_0,X_0}\times\Delta(L)$ and any $\hat{\zeta}_\epsilon\in\partial^+W_{2L}(s_\epsilon,Y_\epsilon,\pi^L_\epsilon,\Pi^L_\epsilon)$, we have[3]:

$$\begin{aligned}
&W_{1L}(t,X,\pi^L,\Pi^L_\epsilon)\\
\leq&W_{1L}(t_\epsilon,X_\epsilon,\pi^L_\epsilon,\Pi^L_\epsilon)+W_{2L}(s_\epsilon,Y_\epsilon,\pi^L,\Pi^L_\epsilon)-W_{2L}(s_\epsilon,Y_\epsilon,\pi^L_\epsilon,\Pi^L_\epsilon)+\frac{1}{2\epsilon}\big(\|(s_\epsilon,Y_\epsilon)-(t,X)\|^2-\\
&\|(s_\epsilon,Y_\epsilon)-(t_\epsilon,X_\epsilon)\|^2\big)-\delta\big(\|A^L(\Pi^L_\epsilon)\pi^L\|^2+\|\pi^L\|^2-\|A^L(\Pi^L_\epsilon)\pi^L_\epsilon\|^2-\|\pi^L_\epsilon\|^2\big)\\
\leq&W_{1L}(t_\epsilon,X_\epsilon,\pi^L_\epsilon,\Pi^L_\epsilon)+\langle\hat{\zeta}_\epsilon,\pi^L-\pi^L_\epsilon\rangle+\frac{1}{2\epsilon}\big(\|(s_\epsilon,Y_\epsilon)-(t,X)\|^2-\|(s_\epsilon,Y_\epsilon)-(t_\epsilon,X_\epsilon)\|^2\big)-\\
&\delta\big(\|A^L(\Pi^L_\epsilon)\pi^L\|^2+\|\pi^L\|^2-\|A^L(\Pi^L_\epsilon)\pi^L_\epsilon\|^2-\|\pi^L_\epsilon\|^2\big).
\end{aligned}$$

We denote $\varphi(t,X,\pi^L)$ the right-hand side of the last inequality. Then $\varphi\in\mathcal{C}^2\big([0,1]\times(\mathbb{R}^n)^I\times\mathbb{R}^L\big)$ satisfies clearly:

$$\varphi(t_\epsilon,X_\epsilon,\pi^L_\epsilon)=W_1(t_\epsilon,X_\epsilon,\pi_\epsilon).$$

In addition, $(t,X,\pi^L)\mapsto(W_{1L}-\varphi)(t,X,\pi^L)$ has its local maximum at $(t_\epsilon,X_\epsilon,\pi^L_\epsilon)$. Let σ and δ be small enough such that $t_\epsilon<1$. By hypothesis, W_1 is a viscosity sub-solution and $\lambda_{max}\big(D^2_{\pi^L\pi^L}\varphi(t_\epsilon,X_\epsilon,\pi^L_\epsilon),\pi^L_\epsilon\big)=\lambda_{max}\Big(-2\delta\big((A^L(\Pi^L_\epsilon))^TA^L(\Pi^L_\epsilon)+I_L\big),\pi^L_\epsilon\Big)<0$, then Lemma 1.5.8 yields:

$$\frac{t_\epsilon-s_\epsilon}{\epsilon}+H(t_\epsilon,X_\epsilon,\frac{X_\epsilon-Y_\epsilon}{\epsilon},\pi_\epsilon)\geq 0. \tag{1.5.14}$$

Symmetrically, by observing that $(s,Y,\pi^K)\mapsto\phi_\epsilon\big((t_\epsilon,X_\epsilon),(s,Y),\pi^K\otimes_K\Pi^K_\epsilon\big)$ has a maximum at $\big((s_\epsilon,Y_\epsilon),\pi^K_\epsilon\big)$, we obtain:

$$\frac{t_\epsilon-s_\epsilon}{\epsilon}+\sigma+H(s_\epsilon,Y_\epsilon,\frac{X_\epsilon-Y_\epsilon}{\epsilon},\pi_\epsilon)\leq 0. \tag{1.5.15}$$

By combining (1.5.14) and (1.5.15), one has:

$$-\sigma+H(t_\epsilon,X_\epsilon,\frac{X_\epsilon-Y_\epsilon}{\epsilon},\pi_\epsilon)-H(s_\epsilon,Y_\epsilon,\frac{X_\epsilon-Y_\epsilon}{\epsilon},\pi_\epsilon)\geq 0. \tag{1.5.16}$$

By (1.5.11), the inequality (1.5.16) gives:

$$-\sigma+C\|(t_\epsilon,X_\epsilon)-(s_\epsilon,Y_\epsilon)\|(1+\|\frac{X_\epsilon-Y_\epsilon}{\epsilon}\|)\geq 0.$$

By Lemma 1.5.9, let $\epsilon\to 0^+$ in the above inequality, we have:

$$\sigma\leq 0,$$

which is a contradiction, since $\sigma>0$. The proof is complete. □

[3]Since $\pi^L\to W_{2L}(s_\epsilon,Y_\epsilon,\pi^L,\Pi^L_\epsilon)$ is Lipschitz continuous and L-concave, $\partial^+W_{2L}(s_\epsilon,Y_\epsilon,\pi^L_\epsilon,\Pi^L_\epsilon)\neq\emptyset$, cf. [60].

Proof of Lemma 1.5.9. It is clear that for any $\epsilon > 0$, $M_\epsilon \geq M$. Let R be an upper bound for $W_1 - W_2$ on $C_{t_0,X_0} \times \Delta(I \times K \times L)$. Then we have:

$$\begin{aligned}
0 \leq & M \leq M_\epsilon \\
= & W_1(t_\epsilon, X_\epsilon, \pi_\epsilon) - W_2(s_\epsilon, Y_\epsilon, \pi_\epsilon) - \frac{1}{2\epsilon}\|(s_\epsilon, Y_\epsilon) - (t_\epsilon, X_\epsilon)\|^2 - \sigma(1 - s_\epsilon) + \delta(\|\pi^K_\epsilon\|^2 + \|\pi^L_\epsilon\|^2) \\
\leq & R + 2\delta - \frac{1}{2\epsilon}\|(s_\epsilon, Y_\epsilon) - (t_\epsilon, X_\epsilon)\|^2.
\end{aligned}$$

This implies that $\frac{1}{2\epsilon}\|(s_\epsilon, Y_\epsilon) - (t_\epsilon, X_\epsilon)\|^2$ is bounded and that $(s_\epsilon, Y_\epsilon) - (t_\epsilon, X_\epsilon) \to 0$ as $\epsilon \to 0^+$. Let (t, X, π) be a cluster point of the sequences $(s_\epsilon, Y_\epsilon, \pi_\epsilon)$, as $\epsilon \to 0^+$. Then there exists $(\epsilon_n)_{n\in\mathbb{N}}$ such that $\epsilon_n > 0$, for any $n \in \mathbb{N}$ satisfying:

$$\epsilon_n \to 0,\ (t_{\epsilon_n}, X_{\epsilon_n}) \to (t, X),\ \pi_{\epsilon_n} \to \pi,\ as\ n \to \infty.$$

We obtain:

$$\begin{aligned}
M \leq \lim_{n\to\infty} M_{\epsilon_n} \leq & \liminf_{n\to\infty} W_1(t_{\epsilon_n}, X_{\epsilon_n}, \pi_{\epsilon_n}) - W_2(s_{\epsilon_n}, Y_{\epsilon_n}, \pi_{\epsilon_n}) - \sigma(1 - s_{\epsilon_n}) + \delta(\|\pi^K_{\epsilon_n}\|^2 + \|\pi^L_{\epsilon_n}\|^2) \\
\leq & \limsup_{n\to\infty} W_1(t_{\epsilon_n}, X_{\epsilon_n}, \pi_{\epsilon_n}) - W_2(s_{\epsilon_n}, Y_{\epsilon_n}, \pi_{\epsilon_n}) - \sigma(1 - s_{\epsilon_n}) + \delta(\|\pi^K_{\epsilon_n}\|^2 + \|\pi^L_{\epsilon_n}\|^2) \\
\leq & W_1(t, X, \pi) - W_2(t, X, \pi) - \sigma(1 - t) + \delta(\|\pi^K\|^2 + \|\pi^L\|^2) \leq M.
\end{aligned}$$

Thus $M_{\epsilon_n} \to M$ as $n \to \infty$. Since that $C_{t_0,X_0} \times \Delta(I \times K \times L)$ is compact, we have $\lim_{\epsilon\to 0^+} M_\epsilon = M$. In addition, the above inequalities give:

$$\begin{aligned}
& \lim_{n\to\infty} W_1(t_{\epsilon_n}, X_{\epsilon_n}, \pi_{\epsilon_n}) - W_2(s_{\epsilon_n}, Y_{\epsilon_n}, \pi_{\epsilon_n}) - \sigma(1 - s_{\epsilon_n}) + \delta(\|\pi^K_{\epsilon_n}\|^2 + \|\pi^L_{\epsilon_n}\|^2) \\
= & W_1(t, X, \pi) - W_2(t, X, \pi) - \sigma(1 - t) + \delta(\|\pi^K\|^2 + \|\pi^L\|^2) = M.
\end{aligned} \tag{1.5.17}$$

This leads to

$$\begin{aligned}
& \lim_{n\to\infty} \frac{1}{2\epsilon_n}\|(s_{\epsilon_n}, Y_{\epsilon_n}) - (t_{\epsilon_n}, X_{\epsilon_n})\|^2 \\
= & \lim_{n\to\infty} W_1(t_{\epsilon_n}, X_{\epsilon_n}, \pi_{\epsilon_n}) - W_2(s_{\epsilon_n}, Y_{\epsilon_n}, \pi_{\epsilon_n}) - \sigma(1 - s_{\epsilon_n}) + \delta(\|\pi^K_{\epsilon_n}\|^2 + \|\pi^L_{\epsilon_n}\|^2) - M_{\epsilon_n} \\
= & \lim_{n\to\infty} W_1(t_{\epsilon_n}, X_{\epsilon_n}, \pi_{\epsilon_n}) - W_2(s_{\epsilon_n}, Y_{\epsilon_n}, \pi_{\epsilon_n}) - \sigma(1 - s_{\epsilon_n}) + \delta(\|\pi^K_{\epsilon_n}\|^2 + \|\pi^L_{\epsilon_n}\|^2) - \lim_{n\to\infty} M_{\epsilon_n} \\
= & M - M = 0.
\end{aligned}$$

Thus, we have $\lim_{\epsilon\to 0^+} \frac{1}{2\epsilon}\|(s_\epsilon, X_\epsilon) - (t_\epsilon, X_\epsilon)\|^2 = 0$.

Finally, let us prove that $t < 1$. Indeed, if this is not true, we can choose $(\epsilon_n)_{n\in\mathbb{N}}$ such that $t = 1$ in the above argument, then by (1.5.17):

$$M \leq W_1(1, X, \pi) - W_2(1, X, \pi) + \delta(\|\pi^K\|^2 + \|\pi^L\|^2) \leq \delta(\|\pi^K\|^2 + \|\pi^L\|^2) \leq 2\delta.$$

But for σ and δ sufficiently small:

$$M \geq \frac{1}{2} \sup_{(t,X,\pi)\in C_{t_0,X_0}\times\Delta(I\times K\times L)} (W_1 - W_2)(t, X, \pi) =: m > 0.$$

Then we obtain $m \leq 2\delta$ and thus we have $m \leq 0$ since $\delta > 0$ is arbitrary, which is a contradiction.

Then, for any ϵ small enough, $t_\epsilon < 1$ and $s_\epsilon < 1$. The proof is complete. □

Finally, let us prove Theorem 1.5.6.

Proof of Theorem 1.5.6. We have first of all, the Hamiltonian $H_{\mathcal{G}}$ satisfies (1.5.11) and (1.5.12). We know that V^- is a super-solution (Proposition 1.5.3) and V^+ is a sub-solution (Proposition 1.5.5) of the Hamilton-Jacobi-Isaacs equation (1.5.1) with $H(t, .) = H_{\mathcal{G}}(.)$, $\forall t \in [0, 1]$. Since clearly $V^+(1, X_0, \pi) = V^-(1, X_0, \pi) = \sum_{(i,k,l)\in I\times K\times L} \pi(i, k, l)g(x_0^i)$ which satisfies the boundary condition in the theorem, the comparison principle yields $V^- \geq V^+$. Because we already knew that $V^- \leq V^+$, the proof is complete. □

1.6 Conclusion

We have proved in this paper that the differential game with incomplete information and a signal structure has a value. Moreover this value function is the unique viscosity solution of a Hamilton-Jacobi-Isaacs equation with a boundary condition. For proving this result, we have obtained a new comparison result for a class of Hamilton-Jacobi-Isaacs equations.

The structure of signals that we have studied in the paper concerns mainly signals on payoffs and initial positions *before the game has started*. The case where the players receive signals *during the game* is of great interest. This is still widely open for differential games while it is partially studied for repeated games.

Chapter 2

Existence of Value for Differential Games with Incomplete Information and with Signal Revelation

Abstract:
This article is devoted to the existence of value for a zero-sum differential game with incomplete information on initial state with partial revealing of information and with a running cost. Before the game begins, an initial state is chosen among several ones and both players do not know this initial state until the dynamic system reaches a fixed target during the game. The main result is the existence of value for the game defined with non-anticipative strategies with delay. Moreover, the value is the unique bounded continuous viscosity solution of a suitable Hamilton-Jacobi-Isaacs equation.

Key words: Differential Games; Incomplete Information; Hamilton-Jacobi Equations; Revealing.
AMS Classification: 49N30; 49N70; 91A05; 91A10; 91A23.

Introduction

We consider the differential game with incomplete information $\mathcal{G}(X_0, r_0, p)$ whose dynamic system is given by:

$$\begin{cases} \dot{x}(t) = f\big(x(t), u(t), v(t)\big), & t \geq 0; \\ \dot{r}(t) = g\big(r(t), u(t), v(t)\big), & t \geq 0; \\ x(0) = x_0;\ r(0) = r_0. \end{cases} \tag{P}$$

With any pair of controls (u, v), the following pay-off is associated:

$$J(x_0, r_0, u, v) := \int_0^\infty e^{-\lambda t} \ell\big((X, R)_t^{x_0, r_0, u, v}, u(t), v(t)\big) dt, \tag{2.0.1}$$

where $\lambda > 0$ is a constant and the map $t \mapsto (X, R)_t^{x_0, r_0, u, v} = (X_t^{x_0, u, v}, R_t^{r_0, u, v})$ denotes the unique solution of dynamic (P). We denote, for all $I \in \mathbb{N}_*$, the set $\{(p_1, p_2, ..., p_I) \in {\mathbb{R}_+}^I \mid \sum_{i=0}^I p_i = 1\}$ by $\Delta(I)$. The game is played in the following way:

(A) Before the game begins, the first initial state $x_0 \in \mathbb{R}^n$ is chosen randomly among I points x_0^1, x_0^2, ..., x_0^I, according to a probability measure $p \in \Delta(I)$. The set $X_0 = \{x_0^1, x_0^2, ..., x_0^I\}$, the other initial state $r_0 \in \mathbb{R}$, and p are common knowledge of Player 1 and Player 2. However, x_0 is not communicated to any players.

(B) Player 1 chooses the measurable control $t \mapsto u(t)$ in the dynamic and wants to minimize $J(x_0, r_0, u, v)$. In contrary, Player 2 aims at the maximization of $J(x_0, r_0, u, v)$ by choosing the control $t \mapsto v(t)$. We assume that during the game both players observe all played actions with perfect memory (i.e. for any $t > 0$, the set $\{(s, u(s), v(s)) \mid 0 \leq s < t\}$ is considered as common knowledge of both players).

(C) During the game, if $t \mapsto R_t^{r_0, u, v}$ reaches a fixed target $M_0 > r_0$, the current state $X_t^{x_0, u, v}$ is announced publicly to both players at the moment $t = T(r_0, u, v) := \inf\{t > 0 \mid R_t^{r_0, u, v} = M_0\}$.

An interesting and simple example of such games is while g in system (P) is a constant equal to 1. In this case, if $r_0 = 1$, both players know that they have to play during a time M_0 before having full revealing.

Our main motivation is to study games in which players can modify the timing of the revealing of unknown information by choosing their actions. In particular, let us consider a differential game with dynamic:

$$\begin{cases} \dot{x}(t) = u(t), & t \geq 0; \\ \dot{y}(t) = v(t), & t \geq 0; \\ (x, y)(0) = (x_0, y_0). \end{cases}$$

The Player 1 chooses the control u and he wants to remain close with Player 2 during the game by minimizing the cost:

$$\int_0^{+\infty} e^{-\lambda t}\|x(t) - y(t)\|dt.$$

Player 2 chooses the control v and aims to maximize the above cost. We suppose that the position x_0 is common knowledge of the players while y_0 is chosen randomly before the pursuit begins and it is not revealed to any players. We assume that they are both connected to a public localization system (for example, a GPS system) which will publicly provide all the positional information to both players after it collects an enough quantity of data $M_0 > 0$. Here the quantity of collected data at the moment $t > 0$ is $r(t) := \int_0^t c + \|v(s)\|ds$ with $c > 0$ a fixed constant. At any moment $t \geq 0$ the localization system can work only if $r(t) \geq M_0$. Namely, in this differential game, the current state of Player 2 will be revealed to both players before the moment $t = \frac{M_0}{c}$ but Player 2 can control the timing of the revealing by choosing the control v.

This paper concerns the existence of value of the above game and the characterization of its value function as the unique solution of a PDE (in the sense of viscosity solutions).

The signal structure in our game model is deeply related with that of repeated games with a symmetric information structure (cf. [36, 45, 52, 53]). In contrast to classic pursuit-evasion games (cf. [4, 22, 32, 40, 47, 63, 69, 77]), players in game $\mathcal{G}(X_0, r_0, p)$ have incomplete information about the initial states and the cost function here is not about the optimization of the hitting time. We point out that this game is different from the state constrained differential games (cf. [7, 22]) since players in game $\mathcal{G}(X_0, r_0, p)$ do not necessarily need to choose their controls such that the dynamic never hits the target set.

Differential games with incomplete information was introduced in [16] as a generalization of the repeated games with incomplete information of Aumann and Maschler in [2] and the differential games of Isaacs in [40]. Several articles have been dedicated to further investigation of the game model in [16] (such results can be found in [12, 17, 19, 21, 43]) and the existence of a value of related game models (for example, see [24, 25, 54, 74]). For such differential games, information about the chosen initial state is communicated to the players before (and only before) the game begins in the form of private signals. Compared with these games, in the problem investigated in our paper, players do not receive private information and they are not informed about the initial state until the dynamic hits the target (i.e. after the game begins). As a consequence, notions of strategies in these above papers can not be directly applied to our game model.

As a main result of this manuscript, we prove that, under Isaacs' condition, the game has a value which is the unique bounded continuous viscosity solution of the following Hamilton-Jacobi-Isaacs equation with boundary condition:

$$\begin{cases} -\lambda V(X,r) + H_p(X,r,D_{X,r}V(X,r)) = 0, & (X,r) \in (\mathbb{R}^n)^I \times (-\infty, M_0); \\ V(X,M_0) = \sum_{i\in I} p_i \tilde{V}(X^i, M_0), & X \in (\mathbb{R}^n)^I. \end{cases} \tag{2.0.2}$$

In the above equation, the function $y \to \tilde{V}(y, M_0)$ is the value function of a classic differential game that we will describe later. Such a boundary condition can be interpreted as follows: as $r_0 \to M_0^-$, the value of the game converges to the value of a complete information game (i.e. the choice of x_0 is revealed as soon as the game begins).

We are facing two major difficulties: the first consists in determining proper notions of non-anticipative strategies with delays for modelling behaviours of the players at the "hitting time"; the second one concerns the regularity of the value functions.

In this paper, to prove the main result, we follow the classic scheme in [33]. We first prove that the upper-value function and the lower-value function of our game model are respectively bounded sub- and super-viscosity solutions of the same PDE (2.0.2) under the Isaacs condition (see Section 1). Secondly, we prove that the PDE has a unique bounded continuous viscosity solution by proving a comparison principle; thus, the upper- and lower-values of our game must coincide and consequently the game has a value.

Our approach can also be applied to the study of other differential games with a similar information structure. We will provide an example of its applicability by proving the existence of value for a differential game with dynamic:

$$\begin{cases} \dot{x}(t) = f\big(x(t), u(t), v(t)\big), & t \geq 0; \\ x(0) = x_0, \end{cases}$$

where the initial state is chosen randomly among 2 points ($I = 2$) according to step (A). The game is played and $i \in \{1, 2\}$ is revealed to both players when the state $X_t^{x_0,u,v}$ reaches a target set $C \subset \mathbb{R}^n$ given in advance. We will show that the information structure of such games is close to game $\mathcal{G}(X_0, r_0, p)$.

After the preliminaries section, the article is organized as follows. In Section 2, the upper- and lower-value functions for the game are defined and we furthermore study their properties to obtain two corresponding dynamic programming principles. Section 3 is dedicated to the introduction of viscosity solutions for PDE and the proof of a comparison principle. Existence and characterization of the value function are established in Section 4. Finally, in Section 5, to demonstrate the applicability of our method, we prove the existence of value for another type of differential games with incomplete information and revealing when the state variable hits a given target set.

2.1 Statement of Preliminaries

In the finite dimensional space $\mathbb{R}^n$, we denote by $\|x\|$ the euclidean norm of $x \in \mathbb{R}^n$, and $\langle x, y\rangle$ the associated scalar product of any points x, $y \in \mathbb{R}^n$. An open ball with center $x \in \mathbb{R}^n$ and radius a is denoted by $B(x;a)$, and its closure by $\bar{B}(x;a)$. We present in this section some preliminaries and hypothesis and we also introduce appropriate strategies for game $\mathcal{G}(X_0, r_0, p)$ and the associated normal form. Let us consider:

(i) U, V are two compact metric spaces, both endowed with the Borel σ-algebra;

(ii) $f : \mathbb{R}^n \times U \times V \to \mathbb{R}^n$ is a bounded, continuous function which is Lipschitz continuous with respect to the first variable with a Lipschitz constant $L_f > 0$;

(iii) $g : \mathbb{R} \times U \times V \to (0, +\infty)$ is a bounded, continuous function which is Lipschitz continuous with respect to the first variable with a Lipschitz constant $L_g > 0$;

(iv) $\ell : \mathbb{R}^n \times \mathbb{R} \times U \times V \to \mathbb{R}$ is bounded, continuous, and Lipschitz continuous with respect to the first two variables with a Lipschitz constant $L_\ell > 0$;

(v) $\mathcal{U}$ denotes the set of measurable controls $u : \mathbb{R}^+ \to U$ while $\mathcal{V}$ stands for the set of measurable controls $v : \mathbb{R}^+ \to V$;

(vi) $\lambda > 0$ and $M_0 \in \mathbb{R}$ are fixed constants.

Elements of $\mathcal{U}$ (resp. $\mathcal{V}$) are called admissible controls of Player 1 (resp. of Player 2). In addition, we suppose that both $\mathcal{U}$ and $\mathcal{V}$ are equipped with the corresponding L^1-topology. Throughout this paper, both players in game $\mathcal{G}(X_0, r_0, p)$ choose their controls respectively in $\mathcal{U}$ and $\mathcal{V}$. By the regularity hypothesis on f and g, for any $(x_0, r_0, u, v) \in \mathbb{R}^n \times (-\infty, M_0) \times \mathcal{U} \times \mathcal{V}$, the system (P) has a unique solution denoted by $t \mapsto (X_t^{x_0,u,v}, R_t^{r_0,u,v})$. Moreover, when (u, v) is fixed, it is well-known that $(t, x_0, r_0) \mapsto (X, R)_t^{x_0,r_0,u,v}$ is locally Lipschitz continuous. We recall that the hitting time of the system (P) on the target M_0 is defined as:

$$T(r_0, u, v) = \inf\{t > 0 \mid R_t^{r_0,u,v} = M_0\}$$

As a result of hypothesis (i) - (vi), we deduce the following:

Lemma 2.1.1. *For any pair of admissible controls $(u, v) \in \mathcal{U} \times \mathcal{V}$, the function $r_0 \mapsto T(r_0, u, v)$ is continuous. In addition, for any $S < M_0$, $T(\cdot, u, v)$ is Lipschitz continuous on $[S, M_0)$ with a Lipschitz constant $C_S > 0$ independent from u and v, i.e., for any $S < r_1 < r_2 < M_0$ and any $(u, v) \in \mathcal{U} \times \mathcal{V}$:*

$$|T(r_1, u, v) - T(r_2, u, v)| \leq C_S |r_1 - r_2|. \tag{2.1.1}$$

Proof. Let $S < M_0$ be fixed. Consider $\epsilon_S > 0$ such that $g(r,\cdot,\cdot) \geq \epsilon_S$ for all $r \in [S, M_0)$. Since for any $(t, r_0, u, v) \in \mathbb{R}^+ \times [S, M_0) \times \mathcal{U} \times \mathcal{V}$, $R_t^{r_0,u,v} \geq r_0 + t\epsilon_S \geq S + t\epsilon_S$, we have $T(r_0, u, v) \leq (M_0 - S)/\epsilon_S$ and thus $r_0 \mapsto T(r_0, u, v)$ is a real value function defined on $(-\infty, M_0)$. Let us fix $S \leq r_2 < r_1 \leq M_0$. By Grönwall's inequality, one has:

$$\forall t > 0,\ |R_t^{r_1,u,v} - R_t^{r_2,u,v}| \leq e^{L_g t}|r_1 - r_2|,$$

Thus for any $t \in (0, (M_0 - S)/\epsilon_S]$, we have:

$$|R_t^{r_1,u,v} - R_t^{r_2,u,v}| \leq e^{\frac{L_g(M_0-S)}{\epsilon_S}}|r_1 - r_2|. \tag{2.1.2}$$

In particular, since $T(r_1, u, v) \leq (M_0 - S)/\epsilon_S$, we take $t = T(r_1, u, v)$ the above inequality and obtain:

$$|R_{T(r_1,u,v)}^{r_2,u,v} - M_0| \leq e^{\frac{L_g(M_0-S)}{\epsilon_S}}|r_1 - r_2|. \tag{2.1.3}$$

Since $r_0 \mapsto R_{T(r_1,u,v)}^{r_0,u,v}$ is increasing, one has $R_T^{r_2,u,v}(r_1, u, v) \leq M_0$. With $g(r,\cdot,\cdot)$ bounded from below by $\epsilon_S > 0$ on $[S, M_0) \times U \times V$, inequality (2.1.3) implies that:

$$|T(r_2, u, v) - T(r_1, u, v)| \leq \frac{1}{\epsilon_S} e^{\frac{L_g(M_0-S)}{\epsilon_s}}|r_1 - r_2|.$$

By taking $C_S = \frac{1}{\epsilon_S} e^{\frac{L_g(M_0-S)}{\epsilon_S}}$, we have obtained (2.1.1). The proof is complete. □

Remark 2.1.2. *By Grönwall's inequality, one also has, for x_1, $x_2 \in \mathbb{R}^n$ and fixed $(u, v) \in \mathcal{U} \times \mathcal{V}$:*

$$\forall t > 0,\ \|X_t^{x_1,u,v} - X_t^{x_2,u,v}\| \leq e^{L_f t}\|x_1 - x_2\|,$$

We can thus obtain the following estimation as that in (2.1.2)*. Let $\epsilon_S > 0$ be a lower bound of g on $[S, M_0) \times U \times V$. For any $t \in [0, (M_0 - S)/\epsilon_S]$,*

$$\|X_t^{x_1,u,v} - X_t^{x_2,u,v}\| \leq e^{\frac{L_f(M_0-S)}{\epsilon_S}}\|x_1 - x_2\|. \tag{2.1.4}$$

The above inequality is useful in the proof of Proposition 2.2.8 in the next section.

Throughout the paper, we suppose the following Isaacs condition:

$$\forall p \in \Delta(I),\ \forall (X, r, \xi) \in (\mathbb{R}^n)^I \times \mathbb{R} \times \big((\mathbb{R}^n)^I \times \mathbb{R}\big),\ H_p^+(X, r, \xi) = H_p^-(X, r, \xi), \tag{2.1.5}$$

where the hamiltonians are given by:

$$H_p^+(X, r, \xi) := \inf_{u\in U}\sup_{v\in V}\Big\langle \xi, \big((f(X^i, u, v))_{i\in I}, g(r, u, v)\big)\Big\rangle + \sum_{i\in I} p_i \ell(X^i, r, u, v);$$

$$H_p^-(X, r, \xi) := \sup_{v\in V}\inf_{u\in U}\Big\langle \xi, \big((f(X^i, u, v))_{i\in I}, g(r, u, v)\big)\Big\rangle + \sum_{i\in I} p_i \ell(X^i, r, u, v).$$

We denote:

$$H_p(X, r, \xi) := H_p^+(X, r, \xi) = H_p^-(X, r, \xi). \tag{2.1.6}$$

Now we turn to the notion of strategies. Let us recall the definition of:

Definition 2.1.3 (Non-anticipative Strategies with Delay (in short, NAD Strategies) of Player 1, [21]). *An NAD strategy of Player 1 is a Borel-measurable map: $\alpha : \mathcal{V} \to \mathcal{U}$ such that: there exists $\tau_\alpha > 0$, for any v_1, $v_2 \in \mathcal{V}$, $t \geq 0$, if $v_1|_{[0,t]} = v_2|_{[0,t]}$ a.e., then $\alpha(v_1)|_{[0,t+\tau_\alpha]} = \alpha(v_2)|_{[0,t+\tau_\alpha]}$ a.e.. The set of such maps is denoted by $\mathcal{A}_d$.*

We define an NAD strategy of Player 2 similarly and let $\mathcal{B}_d$ denote the set of NAD strategies of Player 2. There are two main advantages for using NAD strategies. First, by playing NAD strategies, players can choose their actions in according to their history of observation. Second, with NAD strategies we are able to define the upper- and lower-value functions of the game in a symmetric fashion, which follows from the below result.

Lemma 2.1.4 (cf. [21]). *For any $(\alpha, \beta) \in \mathcal{A}_d \times \mathcal{B}_d$, there exists a unique pair of admissible controls $(u_{\alpha,\beta}, v_{\alpha,\beta})$ such that:*

$$\alpha(v_{\alpha,\beta}) = u_{\alpha,\beta} \text{ and } \beta(u_{\alpha,\beta}) = v_{\alpha,\beta}.$$

But Definition 2.1.3 is not adapted to the game in this paper: Player 1 should be able to choose his actions according to the information revealed at the hitting time. Accordingly, we introduce the following:

Definition 2.1.5 (Pure Strategy of Player 1). *A pure strategy of Player 1 is an element of $\mathcal{A}_d \times (\mathcal{A}_d)^{\mathbb{R}^n}$.*

The above definition is interpreted as follows: after the game begins, Player 2 plays an admissible control v and Player 1 chooses his actions with NAD strategy α while observing the actions of his opponent until the hitting time $T(r_0, \alpha(v), v)$. At the moment $t = T\big(r_0, \alpha(v), v\big)$, Player 1 chooses a new strategy $\alpha_{x(T)} \in \mathcal{A}_d$ according to $x(T) := X^{x_0,\alpha(v),v}_{T(r_0,\alpha(v),v)}$ and plays afterwards $\alpha_{x(T)}\big(v(\cdot + T(r_0, \alpha(v), v))\big)$.

Similarly, a pure strategy of Player 2 is defined as an element of $\mathcal{B}_d \times (\mathcal{B}_d)^{\mathbb{R}^n}$.

Lemma 2.1.4 enables us to write the game into a normal form . Let $A = \big(\alpha, (\alpha_x)_{x\in\mathbb{R}^n}\big)$ be a strategy of Player 1 and $B = \big(\beta, (\beta_x)_{x\in\mathbb{R}^n}\big)$ a strategy of Player 2. For any $x \in \mathbb{R}^n$, let $(u_{\alpha,\beta}, v_{\alpha,\beta})$ and $(u_{\alpha_x,\beta_x}, v_{\alpha_x,\beta_x})$ be defined as in Lemma 2.1.4. We denote by $t \mapsto X^{x_0,\alpha,\beta}_t$ the unique solution of dynamic (P) with $(u, v) = (u_{\alpha,\beta}, v_{\alpha,\beta})$. With $Z = X^{x_0,\alpha,\beta}_{T(r_0,u_{\alpha,\beta},v_{\alpha,\beta})}$, we denote by $t \mapsto (X, R)^{x_0,r_0,A,B}_t = (X^{x_0,A,B}_t, R^{r_0,A,B}_t)$ the unique solution of system (P) with (u, v) defined by:

$$u(t) = \begin{cases} u_{\alpha,\beta}(t), & t \in [0, T(r_0, u_{\alpha,\beta}, v_{\alpha,\beta})); \\ u_{\alpha_Z,\beta_Z}(t - T(r_0, u_{\alpha,\beta}, v_{\alpha,\beta})), & t \in [T(r_0, u_{\alpha,\beta}, v_{\alpha,\beta}), +\infty); \end{cases}$$

$$v(t) = \begin{cases} v_{\alpha,\beta}(t), & t \in [0, T(r_0, u_{\alpha,\beta}, v_{\alpha,\beta})); \\ v_{\alpha_Z,\beta_Z}(t - T(r_0, u_{\alpha,\beta}, v_{\alpha,\beta})), & t \in [T(r_0, u_{\alpha,\beta}, v_{\alpha,\beta}), +\infty). \end{cases}$$

Let $(x_0, r_0, t) \mapsto \big((X, R)_t^{x_0, r_0, A, B}, A, B\big)$ denotes the map $(x_0, r_0, t) \mapsto \big((X, R)_t^{x_0, r_0, u, v}, u(t), v(t)\big)$ with (u, v) defined as above. We associate to this pair of strategies (A, B) in game $\mathcal{G}(X_0, r_0, p)$ the pay-off below:

$$\mathcal{J}(X_0, r_0, p, A, B) := \sum_{i \in I} p_i \int_0^\infty e^{-\lambda t} \ell\big((X, R)_t^{x_0^i, r_0, A, B}, A, B\big) dt.$$

With the above cost function, we are able to write game $\mathcal{G}(X_0, r_0, p)$ into a normal form.

Remark 2.1.6. *In this paper, we do not need to use the notions of mix strategies (or random strategies) in [16]. This is because although the game is with incomplete information (lack of information about the initial state on both sides), neither Player 1 nor Player 2 has private information during the game and thus there is no need to employ mix or random strategies (which serve to protect private information from the opponent). The case of partial revealing when the state hits the target (thus with private signals for the players) and the case with asymmetric information (both players receive private information before the game begins) remain open problems.*

2.2 Value Functions

In this section, we study the upper- and lower-value functions of game $\mathcal{G}(X_0, r_0, p)$ which are defined respectively by:

$$V^+(X_0, r_0, p) = \inf_A \sup_B \mathcal{J}(X_0, r_0, p, A, B),$$
$$V^-(X_0, r_0, p) = \sup_B \inf_A \mathcal{J}(X_0, r_0, p, A, B).$$

Obviously, we have:

$$\forall (X_0, r_0, p) \in (\mathbb{R}^n)^I \times (-\infty, M_0) \times \Delta(I),\ V^+(X_0, r_0, p) \geq V^-(X_0, r_0, p). \tag{2.2.1}$$

When $V^+ = V^-$, we say that game $\mathcal{G}(X_0, r_0, p)$ has a value. If this is the case, we define the value function of the game by $V(X_0, r_0, p) := V^+(X_0, r_0, p) = V^-(X_0, r_0, p)$.

Remark 2.2.1. *Since $-V^-(X_0, r_0, p) = \inf_B \sup_A -\mathcal{J}(X_0, r_0, p, A, B)$, we define a new game, denoted by $-\mathcal{G}(X_0, r_0, p)$, by replacing ℓ by $-\ell$ in game $\mathcal{G}(X_0, r_0, p)$ and by interchanging the roles of Player 1 and Player 2 (Thus Player 1 wants to maximize $-\mathcal{J}(X_0, r_0, p, A, B)$ and Player 2 wants to minimize $-\mathcal{J}(X_0, r_0, p, A, B)$). We can easily prove that $-V^-$ is the upper-value function of game $-\mathcal{G}(X_0, r_0, p)$.*

In this section, by writing $(u, (u_x)_{x \in \mathbb{R}^n})$ (resp. $(v, (v_x)_{x \in \mathbb{R}^n})$), we refer to a strategy of Player 1 $A = (\alpha, (\alpha_x)_{x \in \mathbb{R}^n}) \in \mathcal{A}_d \times (\mathcal{A}_d)^{\mathbb{R}^n}$ such that $\forall (x, v) \in \mathbb{R}^n \times \mathcal{V}$, $\alpha(v) = u$ and

$\alpha_x(v) = u_x$ (resp. a strategy of Player 2 $B = (\beta, (\beta_x)_{x\in\mathbb{R}^n}) \in \mathcal{B}_d \times (\mathcal{B}_d)^{\mathbb{R}^n}$ such that $\forall (x,u) \in \mathbb{R}^n \times \mathcal{U}$, $\beta(u) = v$ and $\beta_x(u) = v_x$). Let us point out that:

$$V^+(X_0, r_0, p) = \inf_A \sup_{(v,(v_x)_{x\in\mathbb{R}^n})} \mathcal{J}\big(X_0, r_0, p, A, (v, (v_x)_{x\in\mathbb{R}^n})\big), \tag{2.2.2}$$

$$V^-(X_0, r_0, p) = \sup_B \inf_{(u,(u_x)_{x\in\mathbb{R}^n})} \mathcal{J}\big(X_0, r_0, p, (u, (u_x)_{x\in\mathbb{R}^n}), B\big). \tag{2.2.3}$$

Proof of equalities (2.2.2) *and* (2.2.3)*:* Thanks to Remark 2.2.1, we only need to prove (2.2.2). Let us fix $A = \big(\alpha, (\alpha_x)_{x\in\mathbb{R}^n}\big) \in \mathcal{A}_d \times (\mathcal{A}_d)^{\mathbb{R}^n}$. Since:

$$\sup_{(v,(v_x)_{x\in\mathbb{R}^n})} \mathcal{J}\big(X_0, r_0, p, A, (v, (v_x)_{x\in\mathbb{R}^n})\big) \le \sup_B \mathcal{J}(X_0, r_0, p, A, B),$$

we have:

$$\inf_A \sup_{(v,(v_x)_{x\in\mathbb{R}^n})} \mathcal{J}\big(X_0, r_0, p, A, (v, (v_x)_{x\in\mathbb{R}^n})\big) \le V^+(X_0, r_0, p). \tag{2.2.4}$$

For any $B = (\beta, (\beta_x)_{x\in\mathbb{R}^n}) \in \mathcal{B}_d \times (\mathcal{B}_d)^{\mathbb{R}^n}$, by Lemma 2.1.4, there exists $(u_{\alpha,\beta}, (u_{\alpha_x,\beta_x})_{x\in\mathbb{R}^n})$ and $(v_{\alpha,\beta}, (v_{\alpha_x,\beta_x})_{x\in\mathbb{R}^n})$ such that for any $x \in \mathbb{R}^n$:

$$\begin{cases} \alpha(v_{\alpha,\beta}) = u_{\alpha,\beta}, \\ \beta(u_{\alpha,\beta}) = v_{\alpha,\beta}, \end{cases} \text{ and } \begin{cases} \alpha_x(v_{\alpha_x,\beta_x}) = u_{\alpha_x,\beta_x}, \\ \beta_x(u_{\alpha_x,\beta_x}) = v_{\alpha_x,\beta_x}. \end{cases}$$

This implies:

$$\mathcal{J}(X_0, r_0, p, A, B) = \mathcal{J}(X_0, r_0, p, A, (v_{\alpha,\beta}, (v_{\alpha_x,\beta_x})_{x\in\mathbb{R}^n})),$$

and since B is arbitrary:

$$\sup_{(v,(v_x)_{x\in\mathbb{R}^n})} \mathcal{J}(X_0, r_0, p, A, (v, (v_x)_{x\in\mathbb{R}^n})) \ge \sup_B \mathcal{J}(X_0, r_0, p, A, B).$$

By taking the infimum of $A \in \mathcal{A}_d \times (\mathcal{A}_d)^{\mathbb{R}^n}$ on both sides in the above inequality, we obtain:

$$\inf_A \sup_{(v,(v_x)_{x\in\mathbb{R}^n})} \mathcal{J}(X_0, r_0, p, A, (v, (v_x)_{x\in\mathbb{R}^n})) \ge V^+(X_0, r_0, p). \tag{2.2.5}$$

Combining (2.2.4) and (2.2.5), we obtain (2.2.2). The proof is complete. □

In this section, we aim to establish the following:

Proposition 2.2.2 (Dynamic Programming Principles)**.** *For any* $(X_0, r_0, p) \in (\mathbb{R}^n)^I \times (-\infty, M_0) \times \Delta(I)$ *and* $0 < h < (M_0 - r_0)/\|g\|_\infty$,

$$\begin{aligned} V^+(X_0, r_0, p) = \inf_{\alpha\in\mathcal{A}_d} \sup_{v\in\mathcal{V}} \sum_{i\in I} p_i \int_0^h e^{-\lambda t} \ell\big((X,R)_t^{x_0^i, r_0, \alpha(v), v}, \alpha(v)(t), v(t)\big) dt \\ + e^{-\lambda h} V^+\big((X_h^{x_0^i,\alpha(v),v})_{i\in I}, R_h^{r_0,\alpha(v),v}, p\big), \end{aligned} \tag{2.2.6}$$

$$\begin{aligned} V^-(X_0, r_0, p) = \sup_{\beta\in\mathcal{B}_d} \inf_{u\in\mathcal{U}} \sum_{i\in I} p_i \int_0^h e^{-\lambda t} \ell\big((X,R)_t^{x_0^i, r_0, u, \beta(u)}, u(t), \beta(u)(t)\big) dt \\ + e^{-\lambda h} V^-\big((X_h^{x_0^i,u,\beta(u)})_{i\in I}, R_h^{r_0,u,\beta(u)}, p\big). \end{aligned} \tag{2.2.7}$$

2.2.1 Two-step Programming Principles

Let us begin by the following:

Lemma 2.2.3 (Two-step Programming Principles)**.** *For any* $(X_0, r_0, p) \in (\mathbb{R}^n)^I \times (-\infty, M_0) \times \Delta(I)$*, we have:*

$$\begin{aligned} V^+(X_0, r_0, p) = \inf_{\alpha \in \mathcal{A}_d} \sup_{v \in \mathcal{V}} \sum_{i \in I} p_i \Big\{ \int_0^{T(r_0, \alpha(v), v)} e^{-\lambda t} \ell\big((X, R)_t^{x_0^i, r_0, \alpha(v), v}, \alpha(v)(t), v(t)\big) dt \\ + e^{-\lambda T(r_0, \alpha(v), v)} \tilde{V}(X^{x_0^i, \alpha(v), v}_{T(r_0, \alpha(v), v)}, M_0) \Big\}, \end{aligned} \tag{2.2.8}$$

$$\begin{aligned} V^-(X_0, r_0, p) = \sup_{\beta \in \mathcal{B}_d} \inf_{u \in \mathcal{U}} \sum_{i \in I} p_i \Big\{ \int_0^{T(r_0, u, \beta(u))} e^{-\lambda t} \ell\big((X, R)_t^{x_0^i, r_0, u, \beta(u)}, u(t), \beta(u)(t)\big) dt \\ + e^{-\lambda T(r_0, u, \beta(u))} \tilde{V}(X^{x_0^i, u, \beta(u)}_{T(r_0, u, \beta(u))}, M_0) \Big\}. \end{aligned} \tag{2.2.9}$$

In the above equalities, the function $(y, M_0) \mapsto \tilde{V}(y, M_0)$ *is defined by:*

$$\begin{aligned} \tilde{V}(y, M_0) &:= \inf_{\alpha \in \mathcal{A}_d} \sup_{v \in \mathcal{V}} \int_0^{+\infty} e^{-\lambda t} \ell\big((X, R)_t^{y, M_0, \alpha(v), v}, \alpha(v)(t), v(t)\big) dt \\ &= \sup_{\beta \in \mathcal{B}_d} \inf_{u \in \mathcal{U}} \int_0^{+\infty} e^{-\lambda t} \ell\big((X, R)_t^{y, M_0, u, \beta(u)}, u(t), \beta(u)(t)\big) dt. \end{aligned} \tag{2.2.10}$$

Remark 2.2.4. $\tilde{V}(y, M_0)$ *is the value function of game* $\tilde{\mathcal{G}}(y, M_0)$ *with complete information with dynamic:*

$$\begin{cases} \dot{x}(t) = f\big(x(t), u(t), v(t)\big), & t \geq 0; \\ \dot{r}(t) = g\big(r(t), u(t), v(t)\big), & t \geq 0; \\ x(0) = y;\ r(0) = M_0; \end{cases}$$

and pay-off:

$$\tilde{J}(y, u, v) := J(y, M_0, u, v) = \int_0^{\infty} e^{-\lambda t} \ell\big((X, R)_t^{y, M_0, u, v}, u(t), v(t)\big) dt.$$

The last equality of (2.2.10) *(i.e existence of a value of* $\tilde{\mathcal{G}}(y, M_0)$ *with NAD strategies under Isaacs' condition* (2.1.5) *is a classic result of differential games, cf. [3, p. 431-470]). Equality* (2.2.10) *will play a crucial role in the proof of our main result (i.e. the existence of value).*

Proof. In view of Remark 2.2.1, we only need to prove (2.2.8). For any (α, v), let $x_T^i(\alpha, v)$ denote $X^{x_0^i, \alpha(v), v}_{T(r_0, \alpha(v), v)}$. By definition of upper-value function of game $\mathcal{G}(X_0, r_0, p)$, we have from

one hand:

$$
\begin{aligned}
&V^+(X_0, r_0, p)\\
&= \inf_{\alpha} \inf_{(\alpha_x)_{x\in\mathbb{R}^n}} \sup_{v} \Big\{ \sum_{i\in I} p_i \int_0^{T(r_0,\alpha(v),v)} e^{-\lambda t} \ell\big((X,R)_t^{x_0^i,r_0,\alpha(v),v}, \alpha(v)(t), v(t)\big) dt\\
&+ e^{-\lambda T(r_0,\alpha(v),v)} \sup_{(v_x)_{x\in\mathbb{R}^n}} \sum_{i\in I} p_i J\big(x_T^i(\alpha,v), M_0, \alpha_{x_T^i(\alpha,v)}, v_{x_T^i(\alpha,v)}\big) \Big\}\\
&\geq \inf_{\alpha} \sup_{v} \Big\{ \sum_{i\in I} p_i \int_0^{T(r_0,\alpha(v),v)} e^{-\lambda t} \ell\big((X,R)_t^{x_0^i,r_0,\alpha(v),v}, \alpha(v)(t), v(t)\big) dt\\
&+ e^{-\lambda T(r_0,\alpha(v),v)} \inf_{(\alpha_x)_{x\in\mathbb{R}^n}} \sup_{(v_x)_{x\in\mathbb{R}^n}} \sum_{i\in I} p_i J\big(x_T^i(\alpha,v), M_0, \alpha_{x_T^i(\alpha,v)}, v_{x_T^i(\alpha,v)}\big) \Big\}\\
&= \inf_{\alpha} \sup_{v} \Big\{ \sum_{i\in I} p_i \int_0^{T(r_0,\alpha(v),v)} e^{-\lambda t} \ell\big((X,R)_t^{x_0^i,r_0,\alpha(v),v}, \alpha(v)(t), v(t)\big) dt\\
&+ e^{-\lambda T(r_0,\alpha(v),v)} \sum_{i\in I} p_i \inf_{\alpha'} \sup_{v'} J\big(x_T^i(\alpha,v), M_0, \alpha'(v'), v'\big) \Big\}
\end{aligned}
$$

which, by the definition of $\tilde{V}$,

$$
\begin{aligned}
&= \inf_{\alpha} \sup_{v} \sum_{i\in I} p_i \Big\{ \int_0^{T(r_0,\alpha(v),v)} e^{-\lambda t} \ell(X_t^{x_0^i,\alpha(v),v}, R_t^{r_0,\alpha(v),v}, \alpha(v)(t), v(t)) dt\\
&+ e^{-\lambda T(r_0,\alpha(v),v)} \tilde{V}(X_{T(r_0,\alpha(v),v)}^{x_0^i,\alpha(v),v}, M_0) \Big\}.
\end{aligned}
$$

So we have proved one inequality in (2.2.8). From the other hand, let us prove that of the reverse direction. For any $x \in \mathbb{R}^n$ and $\epsilon > 0$, let $\alpha_x^\epsilon \in \mathcal{A}_d$ be a ϵ-optimal strategy for $\tilde{V}(x, M_0)$ such that:

$$
\tilde{V}(x, M_0) \leq \sup_{v} J(x, M_0, \alpha_x^\epsilon, v) dt \leq \tilde{V}(x, M_0) + \epsilon.
$$

Thus one has, for any $v \in \mathcal{V}$ and $\alpha \in \mathcal{A}_d$:

$$\begin{aligned}
&\inf_{(\alpha_x)_{x\in\mathbb{R}^n}} \sup_v \Big\{ \sum_{i\in I} p_i \int_0^{T(r_0,\alpha(v),v)} e^{-\lambda t} \ell\big((X,R)_t^{x_0^i,r_0,\alpha(v),v}, \alpha(v)(t), v(t)\big) dt \\
&+ e^{-\lambda T(r_0,\alpha(v),v)} \sup_{(v_x)_{x\in\mathbb{R}^n}} \sum_{i\in I} p_i J(x_T^i(\alpha,v), M_0, \alpha_{x_T^i(\alpha,v)}, v_{x_T^i(\alpha,v)}) \Big\} \\
&\leq \sup_v \Big\{ \sum_{i\in I} p_i \int_0^{T(r_0,\alpha(v),v)} e^{-\lambda t} \ell\big((X,R)_t^{x_0^i,r_0,\alpha(v),v}, \alpha(v)(t), v(t)\big) dt \\
&+ e^{-\lambda T(r_0,\alpha(v),v)} \sup_{(v_x)_{x\in\mathbb{R}^n}} \sum_{i\in I} p_i J(x_T^i(\alpha,v), M_0, \alpha^\epsilon_{x_T^i(\alpha,v)}, v_{x_T^i(\alpha,v)}) \Big\} \\
&\leq \sup_v \Big\{ \sum_{i\in I} p_i \int_0^{T(r_0,\alpha(v),v)} e^{-\lambda t} \ell\big((X,R)_t^{x_0^i,r_0,\alpha(v),v}, \alpha(v)(t), v(t)\big) dt \\
&+ e^{-\lambda T(r_0,\alpha(v),v)} \Big(\sum_{i\in I} p_i \tilde{V}(X^{x_0^i,\alpha(v),v}_{T(r_0,\alpha(v),v)}, M_0) + \epsilon\Big) \Big\}.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, by passing $\epsilon \to 0^+$ and then taking the infimum of $\alpha \in \mathcal{A}_d$ on both sides of the above inequality, we have:

$$\begin{aligned}
V^+(X_0, r_0, p) \leq &\inf_\alpha \sup_v \sum_{i\in I} p_i \int_0^{T(r_0,\alpha(v),v)} e^{-\lambda t} \ell\big((X,R)_t^{x_0^i,r_0,\alpha(v),v}, \alpha(v)(t), v(t)\big) dt \\
&+ e^{-\lambda T(r_0,\alpha(v),v)} \tilde{V}(X^{x_0^i,\alpha(v),v}_{T(r_0,\alpha(v),v)}, M_0),
\end{aligned}$$

which is the missing inequality of (2.2.8). The proof is complete. □

2.2.2 Regularity of the Values

In this subsection we investigate the regularity properties of the upper- and lower-value functions of game $\mathcal{G}(X_0, r_0, p)$. We will need the following easy lemma:

Lemma 2.2.5 ([18]). *Let Γ, Λ be arbitrary sets and $h_1 : \Gamma \times \Lambda \to \mathbb{R}$, $h_2 : \Gamma \times \Lambda \to \mathbb{R}$ some functions on $\Gamma \times \Lambda$ such that*

$$|\inf_{c\in\Gamma} \sup_{l\in\Lambda} h_1(c,l)| + |\inf_{c\in\Gamma} \sup_{l\in\Lambda} h_2(c,l)| < \infty.$$

If there exists some $k > 0$ such that: $\sup_{c\in\Gamma, l\in\Lambda} |h_1(c,l) - h_2(c,l)| \leq k$, then:

$$|\inf_{c\in\Gamma} \sup_{l\in\Lambda} h_1(c,l) - \inf_{c\in\Gamma} \sup_{l\in\Lambda} h_2(c,l)| \leq k.$$

Before we investigate the regularity of V^+ and V^-, we recall the regularity property of $\tilde{V}$.

Lemma 2.2.6 ([3]). *The function $y \to \tilde{V}(y, M_0)$ is Hölder continuous on $\mathbb{R}^n$. More precisely, there exist $C > 0$ and $0 < \gamma \leq 1$ such that for any $(y_1, y_2) \in (\mathbb{R}^n)^2$, one has:*

$$|\tilde{V}(y_1, M_0) - \tilde{V}(y_2, M_0)| \leq C\|y_1 - y_2\|^\gamma.$$

Without the above lemma, the regularity of V^+ and V^- with respect to (X_0, r_0) will be much more difficult to prove. Let us consider the following:

Example 2.2.7. *We consider the case where $n = I = 2$. We suppose that in game $\mathcal{G}(X_0, r_0, p)$: $g \equiv 1$, $f = u + v$ with $U = V = \bar{B}(0; 1) \subset \mathbb{R}^2$ and $\ell = \|u\|$. Let us consider a strategy of Player 1 $(u, (u_x)_{x\in\mathbb{R}^2})$ with $u = u_0 \equiv 0$ and $u_x \equiv (0, 1)$ for any $x \neq 0$. It is clear that for the strategy of Player 2 $v \equiv 0$:*

$$\mathcal{J}\big(X_0, 0, p, (u, (u_x)_{x\in\mathbb{R}^2}), v\big) = \sum_{i=1,2} p_i \mathbf{1}_{\{0\}}(x_0^i)\frac{e^{-\lambda}}{\lambda}.$$

Therefore $X_0 \mapsto \mathcal{J}\big(X_0, 0, p, (u, (u_x)_{x\in\mathbb{R}^2}), v\big)$ is discontinuous at $X_0 = (0, 0)$.

The above example shows that we can not obtain the regularity of V^+ with respect to X_0 by simply applying Lemma 2.2.5 on $(A, B) \mapsto \mathcal{J}(X_0, r_0, p, A, B)$. We conclude this subsection by giving the following estimations:

Proposition 2.2.8. *Both $V^+(X_0, r_0, p)$ and $V^-(X_0, r_0, p)$ are bounded and continuous. Moreover, for any $S < M_0$, there exists $\tilde{C}_S > 0$ and $\gamma \in (0, 1]$ such that: $\forall S < r_1 \leq r_2 < M_0$, $\forall X_1,\ X_2 \in (\mathbb{R}^n)^I$ and $\forall p \in \Delta(I)$,*

$$\begin{aligned}|V^+(X_1, r_1, p) - V^+(X_2, r_2, p)| \leq& \tilde{C}_S(|r_1 - r_2| + |r_1 - r_2|^\gamma + \|X_1 - X_2\| + \|X_1 - X_2\|^\gamma);\\ |V^-(X_1, r_1, p) - V^-(X_2, r_2, p)| \leq& \tilde{C}_S(|r_1 - r_2| + |r_1 - r_2|^\gamma + \|X_1 - X_2\| + \|X_1 - X_2\|^\gamma).\end{aligned}$$

Proof. The boundedness of both V^+ and V^- is direct result of the boundedness of ℓ. Their continuity on p can be deduced from Lemma 2.2.5 (cf. [16]). To finish the proof of the proposition, we only need to prove the first of the two inequalities above, since the proof of the second is similar.

Let $\gamma > 0$ be defined as in Lemma 2.2.6. To obtain the result, we prove respectively that: there exists $\tilde{C}_S > 0$ such that $\forall S < r_1 \leq r_2 < M_0$, $\forall X_1,\ X_2 \in (\mathbb{R}^n)^I$ and $\forall p \in \Delta(I)$,

$$|V^+(X_1, r_1, p) - V^+(X_1, r_2, p)| \leq \tilde{C}_S(|r_1 - r_2| + |r_1 - r_2|^\gamma); \tag{2.2.11}$$

$$|V^+(X_1, r_1, p) - V^+(X_2, r_1, p)| \leq \tilde{C}_S(\|X_1 - X_2\| + \|X_1 - X_2\|^\gamma). \tag{2.2.12}$$

Step 1: proof of inequality (2.2.11). By Lemma 2.2.6, there exists $C > 0$ such that

$$|\tilde{V}(x, M_0) - \tilde{V}(y, M_0)| \leq C\|x - y\|^\gamma. \tag{2.2.13}$$

Let $(u,v)\in\mathcal{U}\times\mathcal{V}$ be a fixed pair of admissible open-loop controls. For $X_1=(x_1^i)_{i\in I}$, on one hand, we observe that: $\forall i\in I$,

$$\begin{aligned}
&\Big|\int_0^{T(r_2,u,v)} e^{-\lambda t}\ell\big((X,R)_t^{x_1^i,r_2,u,v},u,v\big)dt-\int_0^{T(r_1,u,v)} e^{-\lambda t}\ell\big((X,R)_t^{x_1^i,r_1,u,v},u,v\big)dt\Big|\\
\leq&\int_0^{T(r_2,u,v)} e^{-\lambda t}\Big|\ell\big((X,R)_t^{x_1^i,r_2,u,v},u,v\big)-\ell\big((X,R)_t^{x_1^i,r_1,u,v},u,v\big)\Big|dt\\
+&\Big|\int_{T(r_1,u,v)}^{T(r_2,u,v)} e^{-\lambda t}\ell\big((X,R)_t^{x_1^i,r_1,u,v},u,v\big)dt\Big|
\end{aligned}$$

in view of Lemma (2.1.1) and inequality (2.1.2):

$$\begin{aligned}
\leq&\frac{M_0-S}{\epsilon_S}L_\ell e^{\frac{Lg(M_0-S)}{\epsilon_0}}|r_1-r_2|+C_S\|\ell\|_\infty|r_1-r_2|\\
=&\Big(\frac{M_0-S}{\epsilon_S}L_\ell e^{\frac{Lg(M_0-S)}{\epsilon_S}}+C_S\|\ell\|_\infty\Big)|r_1-r_2|.
\end{aligned}$$

On the other hand, by combining (2.2.13), Lemma 2.1.1 and the regularity of the map f, we have:

$$\begin{aligned}
|\tilde V(X^{x_1^i,u,v}_{T(r_2,u,v)},M_0)-\tilde V(X^{x_1^i,u,v}_{T(r_1,u,v)},M_0)|\leq&C(2\|f\|_\infty|T(r_1,u,v)-T(r_2,u,v)|)^\gamma\\
\leq&C(2\|f\|_\infty C_S)^\gamma|r_1-r_2)|^\gamma.
\end{aligned}$$

Thus the above two inequalities together yield:

$$\begin{aligned}
&\sup_{(\alpha,v)}\Big|\sum_{i\in I}p_i\Big\{\int_0^{T(r_1,\alpha(v),v)} e^{-\lambda t}\ell\big((X,R)_t^{x_1^i,r_1,\alpha(v),v},\alpha(v)(t),v(t)\big)dt+e^{-\lambda T(r_1,\alpha(v),v)}\times\\
&\tilde V(X^{x_1^i,\alpha(v),v}_{T(r_1,\alpha(v),v)},M_0)-\int_0^{T(r_2,\alpha(v),v)} e^{-\lambda t}\ell\big((X,R)_t^{x_1^i,r_2,\alpha(v),v},\alpha(v)(t),v(t)\big)dt-\\
&e^{-\lambda T(r_2,\alpha(v),v)}\tilde V(X^{x_1^i,\alpha(v),v}_{T(r_2,\alpha(v),v)},M_0)\Big\}\Big|\leq\tilde C'_S(|r_1-r_2|+|r_1-r_2|^\gamma).
\end{aligned}\tag{2.2.14}$$

In the last inequality, $\tilde C'_S\geq\max\Big\{\Big(\frac{M_0-S}{\epsilon_S}L_\ell e^{\frac{Lg(M_0-S)}{\epsilon_S}}+C_S\|\ell\|_\infty\Big),C(2\|f\|_\infty C_S)^\gamma\Big\}>0$.

With Lemma 2.2.3, Lemma 2.2.5 and (2.2.14), we have proved inequality (2.2.11).

Step 2: proof of inequality (2.2.12)**.** Let us turn to (2.2.12). For $X_2=(x_2^i)_{i\in I}$, one has:

$$\begin{aligned}
&\Big|\int_0^{T(r_1,u,v)} e^{-\lambda t}\ell\big((X,R)_t^{x_1^i,r_1,u,v},u,v\big)dt-\int_0^{T(r_1,u,v)} e^{-\lambda t}\ell\big((X,R)_t^{x_2^i,r_1,u,v},u,v\big)dt\Big|\\
\leq&\int_0^{T(r_1,u,v)} e^{-\lambda t}\Big|\ell\big((X,R)_t^{x_1^i,r_1,u,v},u,v\big)-\ell\big((X,R)_t^{x_2^i,r_1,u,v},u,v\big)\Big|dt\\
\leq&\frac{L_\ell(M_0-S)}{\epsilon_S}e^{\frac{L_f(M_0-S)}{\epsilon_S}}\|x_1^i-x_2^i\|.
\end{aligned}\tag{2.2.15}$$

The last inequality is from the regularity of ℓ and (2.1.4). In addition, we have:

$$\begin{aligned}|\tilde{V}(X^{x_1^i,u,v}_{T(r_1,u,v)},M_0)-\tilde{V}(X^{x_2^i,u,v}_{T(r_1,u,v)},M_0)|\leq& C(e^{\frac{L_f(M_0-S)}{\epsilon_S}}\|x_1^i-x_2^i\|)^\gamma\\ =&C(e^{\frac{L_f(M_0-S)}{\epsilon_S}})^\gamma\|x_1^i-x_2^i\|^\gamma.\end{aligned}\tag{2.2.16}$$

Let us define

$$\tilde{C}_S:=\max\Big\{\tilde{C}'_S,\tfrac{L_\ell(M_0-S)}{\epsilon_S}e^{\frac{L_f(M_0-S)}{\epsilon_S}},C(e^{\frac{L_f(M_0-S)}{\epsilon_S}})^\gamma\Big\}.$$

By combining (2.2.15) and (2.2.16), we obtain:

$$\begin{aligned}&\sup_{(\alpha,v)}\Big|\sum_{i\in I}p_i\int_0^{T(r_1,\alpha(v),v)}e^{-\lambda t}\ell\big((X,R)_t^{x_1^i,r_1,\alpha(v),v},\alpha(v)(t),v(t)\big)dt+e^{-\lambda T(r_1,\alpha(v),v)}\times\\ &\tilde{V}(X^{x_1^i,\alpha(v),v}_{T(r_1,\alpha(v),v)},M_0)-\int_0^{T(r_1,\alpha(v),v)}e^{-\lambda t}\ell\big((X,R)_t^{x_2^i,r_1,\alpha(v),v},\alpha(v)(t),v(t)\big)dt-\\ &e^{-\lambda T(r_1,\alpha(v),v)}\tilde{V}(X^{x_2^i,\alpha(v),v}_{T(r_1,\alpha(v),v)},M_0)\Big|\leq\tilde{C}_S(\|X_1-X_2\|+\|X_1-X_2\|^\gamma).\end{aligned}\tag{2.2.17}$$

By Lemma 2.2.3 and Lemma 2.2.5, we have proved (2.2.12). The proof of Proposition 2.2.8 is complete. □

Corollary 2.2.9. *For any $(X_0,p)\in(\mathbb{R}^n)^I\times\Delta(I)$, one has:*

$$\lim_{r_0\to M_0^-}V^+(X_0,r_0,p)=\lim_{r_0\to M_0^-}V^-(X_0,r_0,p)=\sum_{i\in I}p_i\tilde{V}(x_0^i,M_0).$$

Moreover, these above limits are uniform with respect to (X_0,p).

Proof. We only prove the part of V^+, since that of V^- is symmetric. For any $0<\epsilon<1$ and $M_0-\epsilon<r_\epsilon<M_0$, by taking $S=M_0-1$, $X_1=X_0$, $r_1=r_\epsilon$ and $r_2=M_0$ in (2.2.14), one has:

$$\begin{aligned}&\sup_{(\alpha,v)}\Big|\sum_{i\in I}p_i\Big\{\int_0^{T(r_\epsilon,\alpha(v),v)}e^{-\lambda t}\ell\big((X,R)_t^{x_1^i,r_\epsilon,\alpha(v),v},\alpha(v)(t),v(t)\big)dt+e^{-\lambda T(r_\epsilon,\alpha(v),v)}\times\\ &\tilde{V}(X^{x_1^i,\alpha(v),v}_{T(r_\epsilon,\alpha(v),v)},M_0)\Big\}-\sum_{i\in I}p_i\tilde{V}(x_0^i,M_0)\Big|\leq\tilde{C}'_S(\epsilon+\epsilon^\gamma).\end{aligned}$$

In view of Lemma 2.2.3 and Lemma 2.2.5, this yields:

$$|V^+(X_0,r_\epsilon,p)-\sum_{i\in I}p_i\tilde{V}(x_0^i,M_0)|\leq\tilde{C}'_S(\epsilon+\epsilon^\gamma).$$

The proof is complete. □

Let us denote, for all $(X_0, r_0, p, \alpha, v) \in (\mathbb{R}^n)^I \times (-\infty, M_0) \times \Delta(I) \times \mathcal{A}_d \times \mathcal{V}$:

$$\bar{\mathcal{J}}(X_0, r_0, p, \alpha, v) := \sum_{i\in I} p_i \Big\{ \int_0^{T(r_0,\alpha(v),v)} e^{-\lambda t} \ell\big((X,R)_t^{x_0^i, r_0, \alpha(v), v}, \alpha(v)(t), v(t)\big) dt + e^{-\lambda T(r_0,\alpha(v),v)} \tilde{V}(X_{T(r_0,\alpha(v),v)}^{x_0^i,\alpha(v),v}, M_0) \Big\}.$$

Corollary 2.2.10. *For any $\epsilon > 0$ and $(X_0, r_0) \in (\mathbb{R}^n)^I \times (-\infty, M_0)$, there exists $\delta^{\epsilon}_{X_0,r_0} > 0$ such that, if $\alpha \in \mathcal{A}_d$ is a ϵ-optimal strategy for $V^+(X_0, r_0, p)$ in the sense of (2.2.8), then for any $(Y_0, s_0) \in B((X_0, r_0); \delta^{\epsilon}_{X_0,r_0})$, α is still a 2ϵ-optimal strategy for $V^+(Y_0, s_0, p)$ in the sense of (2.2.8).*

Proof. Since α is a ϵ-optimal strategy of $V^+(X_0, r_0, p)$ in the sense of (2.2.8), one has:

$$\sup_v \bar{\mathcal{J}}(X_0, r_0, p, \alpha, v) \le V^+(X_0, r_0, p) + \epsilon. \tag{2.2.18}$$

Let us take $S = r_0 - 2$ in Proposition 2.2.8. Then for any $(Y_0, s_0) \in B((X_0, r_0); M_0 - r_0)$ such that $\|(Y_0, s_0) - (X_0, r_0)\| \le \min\big(2, \frac{\epsilon}{8\tilde{C}_S}, (\frac{\epsilon}{8\tilde{C}_S})^{\frac{1}{\gamma}}\big)$, we have:

$$|V^+(X_0, r_0, p) - V^+(Y_0, s_0, p)| \le 4\tilde{C}_S \frac{\epsilon}{8\tilde{C}_S} = \frac{\epsilon}{2}; \tag{2.2.19}$$

$$|\sup_v \bar{\mathcal{J}}(X_0, r_0, p, \alpha, v) - \sup_v \bar{\mathcal{J}}(Y_0, s_0, p, \alpha, v)| \le 4\tilde{C}_S \frac{\epsilon}{8\tilde{C}_S} = \frac{\epsilon}{2}. \tag{2.2.20}$$

The last inequality is a result of (2.2.14) and (2.2.17). Combining (2.2.18) with (2.2.19) and (2.2.20), we obtain:

$$\sup_v \bar{\mathcal{J}}(Y_0, s_0, p, \alpha, v) \le \sup_v \bar{\mathcal{J}}(X_0, r_0, p, \alpha, v) + \frac{\epsilon}{2} \le V^+(X_0, r_0, p) + \frac{3\epsilon}{2} \le V^+(Y_0, s_0, p) + 2\epsilon.$$

Thus α is indeed a 2ϵ-optimal strategy for $V^+(Y_0, s_0, p)$ in the sense of (2.2.8) for any $(Y_0, s_0) \in B\big((X_0, r_0); \min(2, M_0 - r_0, \frac{\epsilon}{8\tilde{C}_S}, (\frac{\epsilon}{8\tilde{C}_S})^{\frac{1}{\gamma}})\big)$. Set $\delta^{\epsilon}_{X_0,r_0} = \min(2, M_0 - r_0, \frac{\epsilon}{8\tilde{C}_S}, (\frac{\epsilon}{8\tilde{C}_S})^{\frac{1}{\gamma}})$, and the proof is complete. □

2.2.3 Proof of the Dynamic Programming Principles

Now we are ready to prove Proposition 2.2.2.

Proof. In view of Remark 2.2.1, we only prove the proposition for V^+. Let:

$$W(X_0, r_0, p) =: \inf_\alpha \sup_v \sum_{i\in I} p_i \int_0^h e^{-\lambda t} \ell\big((X,R)_t^{x_0^i, r_0, \alpha(v), v}, \alpha(v)(t), v(t)\big) dt + e^{-\lambda h} V^+\big((X_h^{x_0^i,\alpha(v),v})_{i\in I}, R_h^{r_0,\alpha(v),v}, p\big).$$

Step 1: $V^+ \geq W$**.**

For $\epsilon > 0$, let $\alpha \in \mathcal{A}_d$ be a ϵ-optimal strategy of Player 1 such that:

$$\sup_v \bar{\mathcal{J}}(X_0, r_0, p, \alpha, v) \leq V^+(X_0, r_0, p) + \epsilon.$$

For $\bar{v} \in \mathcal{V}$ fixed, let us define $\bar{\alpha} \in \mathcal{A}_d$ such that, for any $v \in \mathcal{V}$:

$$\bar{\alpha}(v)(t) = \alpha(\tilde{v})(t+h),$$

where:

$$\tilde{v}(s) = \begin{cases} \bar{v}(s), & s \in [0,h]; \\ v(s), & s > h. \end{cases}$$

Let us denote by $\mathcal{V}(\bar{v}, h)$ the set $\{v \in \mathcal{V} \mid v|_{[0,h]} = \bar{v}|_{[0,h]}\}$. Hence:

$$\sup_{\tilde{v} \in \mathcal{V}(\bar{v},h)} \bar{\mathcal{J}}(X_0, r_0, p, \alpha, \tilde{v}) = \sup_{\tilde{v} \in \mathcal{V}(\bar{v},h)} \sum_{i \in I} p_i \int_0^h e^{-\lambda t} \ell\big((X,R)_t^{x_0^i, r_0, \alpha(\bar{v}), \bar{v}}, \alpha(\bar{v}), \bar{v}\big) dt +$$

$$e^{-\lambda h} \bar{\mathcal{J}}\big((X_h^{x_0^i, \alpha(\bar{v}), \bar{v}})_{i \in I}, R_h^{r_0, \alpha(\bar{v}), \bar{v}}, p, \bar{\alpha}, \tilde{v}(\cdot + h)\big) = \sum_{i \in I} p_i \int_0^h e^{-\lambda t} \ell\big((X,R)_t^{x_0^i, r_0, \alpha(\bar{v}), \bar{v}}, \alpha(\bar{v}), \bar{v}\big) dt +$$

$$e^{-\lambda h} \sup_v \bar{\mathcal{J}}\big((X_h^{x_0^i, \alpha(\bar{v}), \bar{v}})_{i \in I}, R_h^{r_0, \alpha(\bar{v}), \bar{v}}, \bar{\alpha}, v\big) \geq \sum_{i \in I} p_i \int_0^h e^{-\lambda t} \ell\big((X,R)_t^{x_0^i, r_0, \alpha(\bar{v}), \bar{v}}, \alpha(\bar{v}), \bar{v}\big) dt +$$

$$e^{-\lambda h} V^+\big((X_h^{x_0^i, \alpha(\bar{v}), \bar{v}})_{i \in I}, R_h^{r_0, \alpha(\bar{v}), \bar{v}}, p\big)$$

Since in the above inequality, $\bar{v}$ is arbitrary, we deduce

$$\begin{aligned} V^+(X_0, r_0, p) + \epsilon \geq & \sup_v \bar{\mathcal{J}}(X_0, r_0, p, \alpha, v) \geq \sup_{\bar{v}} \sup_{\tilde{v} \in \mathcal{V}(\bar{v},h)} \bar{\mathcal{J}}(X_0, r_0, p, \alpha, \tilde{v}) \\ \geq & \sup_{\bar{v}} \sum_{i \in I} p_i \int_0^h e^{-\lambda t} \ell\big((X,R)_t^{x_0^i, r_0, \alpha(\bar{v}), \bar{v}}, \alpha(\bar{v}), \bar{v}\big) dt + e^{-\lambda h} V^+\big((X_h^{x_0^i, \alpha(\bar{v}), \bar{v}})_{i \in I}, R_h^{r_0, \alpha(\bar{v}), \bar{v}}, p\big) \\ \geq & W(X_0, r_0, p). \end{aligned}$$

Thus, since $\epsilon > 0$ is arbitrary, we have proved $V^+ \geq W$.

Step 2: $V^+ \leq W$**.**

For fixed (X_0, r_0, p), let $\bar{\alpha} \in \mathcal{A}_d$ be a ϵ-optimal strategy for $W(X_0, r_0, p)$:

$$\begin{aligned} W(X_0, r_0, p) + \epsilon \geq & \sup_v \sum_{i \in I} p_i \int_0^h e^{-\lambda t} \ell\big((X,R)_t^{x_0^i, r_0, \bar{\alpha}(v), v}, \bar{\alpha}(v)(t), v(t)\big) dt + \\ & e^{-\lambda h} V^+\big((X_h^{x_0^i, \bar{\alpha}(v), v})_{i \in I}, R_h^{r_0, \bar{\alpha}(v), v}, p\big). \end{aligned}$$

To all $(Y,s)\in\bar{B}(X_0;h\|f\|_\infty)\times[r_0,r_0+h\|g\|_\infty]$, we associate $\alpha^{Y,s}\in\mathcal{A}_d$, a ϵ-optimal strategy for $V^+(Y,s,p)$ in the sense of (2.2.8). By Corollary 2.2.10, there exists $\delta_\epsilon^{Y,s}>0$ such that $\alpha^{Y,s}$ remains a 2ϵ-optimal strategy for $V^+(Y',s',p)$ for any $(Y',s')\in B((Y,s);\delta_\epsilon^{Y,s})$. Thus, we can construct a finite partition $(E_j)_{j=1}^K$ of $\bar{B}(X_0;h\|f\|_\infty)\times[r_0,r_0+h\|g\|_\infty]$, such that there exists, for any $1\le j\le K$, $(Y_j,s_j)\in E_j$ with $\alpha^j=\alpha^{Y_j,s_j}$ being a 2ϵ-optimal strategy for $V^+(Y,s,p)$ if $(Y,s)\in E_j$.

Let us construct a strategy $\alpha\in\mathcal{A}_d$ for Player 1 by defining:

$$\forall v\in\mathcal{V},\alpha(v)(t)=\begin{cases}\bar{\alpha}(v)(t), & t\in[0,h];\\ \alpha^j\big(v(\cdot+h)\big)(t-h), & \text{if } t>h \text{ and } \big((X_h^{x_0^i,\bar{\alpha}(v),v})_{i\in I},R_h^{r_0,\bar{\alpha}(v),v}\big)\in E_j.\end{cases}$$

One can check easily that α is indeed a pure strategy for Player 1. Thus we have:

$$\begin{aligned}V^+(X_0,r_0,p)&\le\sup_v\mathcal{J}(X_0,r_0,p,\alpha,v)=\sup_{v\in V}\Big\{\sum_{i\in I}p_i\int_0^h e^{-\lambda t}\ell\big((X,R)_t^{x_0^i,r_0,\bar{\alpha}(v),v},\bar{\alpha}(v),v\big)dt+\\&\sum_{1\le j\le K}e^{-\lambda h}\bar{\mathcal{J}}\big((X_h^{x_0^i,\bar{\alpha}(v),v})_{i\in I},R_h^{r_0,\bar{\alpha}(v),v},p,\alpha^j,v(\cdot+h)\big)\mathbf{1}_{E_j}\big((X_h^{x_0^i,\bar{\alpha}(v),v})_{i\in I},R_h^{r_0,\bar{\alpha}(v),v}\big)\Big\}\\&\le\sup_{v\in V}\Big\{\sum_{i\in I}p_i\int_0^h e^{-\lambda t}\ell\big((X,R)_t^{x_0^i,r_0,\bar{\alpha}(v),v},\bar{\alpha}(v),v\big)dt+\\&\sum_{1\le j\le K}e^{-\lambda h}\sup_{v'\in\mathcal{V}}\bar{\mathcal{J}}\big((X_h^{x_0^i,\bar{\alpha}(v),v})_{i\in I},R_h^{r_0,\bar{\alpha}(v),v},p,\alpha^j,v'\big)\mathbf{1}_{E_j}\big((X_h^{x_0^i,\bar{\alpha}(v),v})_{i\in I},R_h^{r_0,\bar{\alpha}(v),v}\big)\Big\}\\&\le\sup_{v\in V}\Big\{\sum_{i\in I}p_i\int_0^h e^{-\lambda t}\ell\big((X,R)_t^{x_0^i,r_0,\bar{\alpha}(v),v},\bar{\alpha}(v),v\big)dt+e^{-\lambda h}V^+\big((X_h^{x_0^i,\bar{\alpha}(v),v})_{i\in I},R_h^{r_0,\bar{\alpha}(v),v},p\big)+2\epsilon\Big\}.\end{aligned}$$

Hence $V^+\le W+3\epsilon$. Since $\epsilon>0$ is arbitrary, we have obtained $V^+\le W$. The proof is complete. □

Passing to the limit as $h\to0^+$ in the dynamic programming principle, we will prove in Section 4 that the value is a solution of a PDE. In the next section, we investigate properties of such PDE.

2.3 Hamilton-Jacobi-Isaacs Equations

In this section, we introduce and study the following Hamilton-Jacobi-Isaacs equation with boundary condition:

$$\begin{cases}H\big((x,r),W(x,r),D_{(x,r)}W(x,r)\big)=0, & (x,r)\in\mathbb{R}^m\times(-\infty,M);\\ W(x,M)=\hat{V}(x).\end{cases}\tag{2.3.1}$$

In (2.3.1), both $H:\mathbb{R}^{2m+3}\to\mathbb{R}$ and $\hat{V}:\mathbb{R}^m\to\mathbb{R}$ are continuous functions and $m>0$ some fixed natural number.

Definition 2.3.1 (Viscosity Solutions). *A function $W : \mathbb{R}^m \times (-\infty, M] \to \mathbb{R}$ is:*

- *a viscosity super-solution of Hamilton-Jacobi-Isaacs equation* (2.3.1) *if and only if: it is lower semi-continuous (l.s.c.), it verifies $\forall x \in \mathbb{R}^m,\ W(x, M) = \hat{V}(x)$, and for any function $\varphi \in C^1(\mathbb{R}^m \times (-\infty, M))$ such that $W - \varphi$ has a local minimum at some point $(\bar{x}, \bar{r}) \in \mathbb{R}^m \times (-\infty, M)$, one has:*

$$H\big((\bar{x}, \bar{r}), W(\bar{x}, \bar{r}), D_{(x,r)}\varphi(\bar{x}, \bar{r})\big) \leq 0.$$

- *a viscosity sub-solution of Hamilton-Jacobi-Isaacs equation* (2.3.1) *if and only if: it is upper semi-continuous (u.s.c.), it verifies $\forall x \in \mathbb{R}^m,\ W(x, M) = \hat{V}(x)$, and for any function $\varphi \in C^1(\mathbb{R}^m \times (-\infty, M))$ such that $W - \varphi$ has a local maximum at some point $(\bar{x}, \bar{r}) \in \mathbb{R}^m \times (-\infty, M)$, one has:*

$$H\big((\bar{x}, \bar{r}), W(\bar{x}, \bar{r}), D_{(x,r)}\varphi(\bar{x}, \bar{r})\big) \geq 0.$$

- *a viscosity solution of Hamilton-Jacobi-Isaacs equation* (2.3.1) *if it is both viscosity super-solution and viscosity sub-solution of equation* (2.3.1).

The following lemma is an immediate consequence of the above definition:

Lemma 2.3.2. *Let V be a viscosity super-solution of equation* (2.3.1). *Then $-V$ is a viscosity sub-solution of:*

$$\begin{cases} \bar{H}\big((x, r), W(x, r), D_{(x,r)}W(x, r)\big) = 0, \quad (x, r) \in \mathbb{R}^m \times (-\infty, M); \\ W(x, M) = -\hat{V}(x). \end{cases} \tag{2.3.2}$$

where the hamiltonian $\bar{H}$ is defined by:

$$\forall (x, r, s, q) \in \mathbb{R}^m \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{m+1},\ \bar{H}\big((x, r), s, q\big) = -H\big((x, r), -s, -q\big).$$

We assume that the hamiltonian H satisfies the following conditions: there exist constants $\eta > 0$ and $\bar{C} > 0$ such that for any r, r_1, r_2, s, $s_1 \geq s_2 \in \mathbb{R}$, x, x_1, $x_2 \in \mathbb{R}^m$ and q, q_1, $q_2 \in \mathbb{R}^{m+1}$:

$$H\big((x, r), s_1, q\big) - H\big((x, r), s_2, q\big) \leq -\eta(s_1 - s_2); \tag{2.3.3}$$

$$|H\big((x_1, r_1), s, q\big) - H((x_2, r_2), s, q)| \leq \bar{C}(1 + \|q\|)\|(x_1, r_1) - (x_2, r_2)\|; \tag{2.3.4}$$

$$|H\big((x, r), s, q_1\big) - H\big((x, r), s, q_2\big)| \leq \bar{C}\|q_1 - q_2\|. \tag{2.3.5}$$

Let us prove the main result of this section:

Theorem 2.3.3 (Comparison Principle). *Let W_1 and W_2 be respectively bounded viscosity sub-solution and bounded viscosity super-solutions of equation* (2.3.1) *with H verifying conditions* (2.3.3), (2.3.4) *and* (2.3.5), *one has in $\mathbb{R}^m \times (-\infty, M]$, $W_1 \leq W_2$.*

Proof. Let us argue by contradiction. We suppose that:

$$N := \sup_{(x,r)\in\mathbb{R}^m\times(-\infty,M)} (W_1 - W_2)(x,r) > 0. \tag{2.3.6}$$

Then, for $a > 0$ small enough,

$$N_a := \sup_{(x,r)\in\mathbb{R}^m\times(-\infty,M)} (W_1 - W_2)(x,r) - a\|(x,r)\|^2 > 0, \tag{2.3.7}$$

and in addition, $N = \lim_{a\to 0^+} N_a$.

For $\epsilon > 0$, let us define:

$$W_{a,\epsilon}(x,r,y,s) := W_1(x,r) - W_2(y,s) - \frac{\|(x,r)-(y,s)\|^2}{\epsilon} - \frac{a}{2}(\|(x,r)\|^2 + \|(y,s)\|^2).$$

We have immediately by (2.3.7):

$$N_{a,\epsilon} := \sup_{(x,r,y,s)\in(\mathbb{R}^m\times(-\infty,M))^2} W_{a,\epsilon}(x,r,y,s) \geq N_a > 0.$$

Since $W_{a,\epsilon}$ is upper semi-continuous and coercive, we have that $W_{a,\epsilon}(x,r,y,s)$ achieves its maximum at some point $(x_{a,\epsilon}, r_{a,\epsilon}, y_{a,\epsilon}, s_{a,\epsilon})$. We state a technical lemma, the proof of which is postponed after the proof of this theorem.

Lemma 2.3.4. *(a)* $\lim_{\epsilon\to 0^+} N_{a,\epsilon} = N_a$*;*

(b) For any $a > 0$ *small enough,* $\lim_{\epsilon\to 0^+} \frac{1}{\epsilon}(\|(x_{a,\epsilon}, r_{a,\epsilon}) - (y_{a,\epsilon}, s_{a,\epsilon})\|^2) = 0$*;*

(c) There exists a constant $\bar{C}' > 0$ *such that* $a(\|(x_{a,\epsilon}, r_{a,\epsilon})\| + \|(y_{a,\epsilon}, s_{a,\epsilon})\|) \leq \bar{C}'\sqrt{a}$*;*

(d) For $\epsilon > 0$ *sufficiently small,* $r_{a,\epsilon} < M$ *and* $s_{a,\epsilon} < M$*.*

Let us return to the proof of Theorem 2.3.3. Since $(x_{a,\epsilon}, r_{a,\epsilon})$ is a maximizer of $(x,r) \mapsto W_{a,\epsilon}(x,r,y_{a,\epsilon},s_{a,\epsilon})$, we have, for any (x,r):

$$\begin{aligned} W_1(x,r) \leq & W_1(x_{a,\epsilon}, r_{a,\epsilon}) + \frac{1}{\epsilon}\big(\|(x,r)-(y_{a,\epsilon},s_{a,\epsilon})\|^2 - \|(x_{a,\epsilon},r_{a,\epsilon})-(y_{a,\epsilon},s_{a,\epsilon})\|^2\big) + \\ & \frac{a}{2}\big(\|(x,r)\|^2 - \|(x_{a,\epsilon},r_{a,\epsilon})\|^2\big). \end{aligned}$$

Let us denote by $\varphi(x,r)$ the right-hand side of the above inequality, and we have that $(x,r) \mapsto \varphi(x,r)$ is an element of $C^1(\mathbb{R}^m \times (-\infty, M))$ and that $W_1 - \varphi$ has a global maximum at $(x,r) = (x_{a,\epsilon}, r_{a,\epsilon})$. Since W_1 is a viscosity sub-solution of equation (2.3.1), by definition, we have:

$$H\big((x_{a,\epsilon}, r_{a,\epsilon}), W_1(x_{a,\epsilon}, r_{a,\epsilon}), D_{(x,r)}\varphi(x_{a,\epsilon}, r_{a,\epsilon})\big) \geq 0.$$

Thus:

$$H\Big((x_{a,\epsilon}, r_{a,\epsilon}), W_1(x_{a,\epsilon}, r_{a,\epsilon}), \frac{2}{\epsilon}\big((x_{a,\epsilon}, r_{a,\epsilon}) - (y_{a,\epsilon}, s_{a,\epsilon})\big) + a(x_{a,\epsilon}, r_{a,\epsilon})\Big) \geq 0. \tag{2.3.8}$$

In the same way, since $(y_{a,\epsilon}, s_{a,\epsilon})$ is a maximizer of $(y, s) \mapsto W_{a,\epsilon}(x_{a,\epsilon}, r_{a,\epsilon}, y, s)$, we have, for any (y, s):

$$\begin{aligned} W_2(y, s) \geq & W_2(y_{a,\epsilon}, s_{a,\epsilon}) + \frac{1}{\epsilon}\big(\|(x_{a,\epsilon}, r_{a,\epsilon}) - (y_{a,\epsilon}, s_{a,\epsilon})\|^2 - \|(x_{a,\epsilon}, r_{a,\epsilon}) - (y, s)\|^2\big) + \\ & \frac{a}{2}\big(\|(y_{a,\epsilon}, s_{a,\epsilon})\|^2 - \|(y, s)\|^2\big). \end{aligned}$$

Let us denote by $\phi(y, s)$ the right-hand side of the above inequality. $(y, s) \mapsto \phi(y, s)$ is an element of $C^1(\mathbb{R}^m \times (-\infty, M))$ and $W_2 - \phi$ has a global minimum at $(y, s) = (y_{a,\epsilon}, s_{a,\epsilon})$. Because W_2 is a viscosity super-solution of equation (2.3.1), we have:

$$H((y_{a,\epsilon}, s_{a,\epsilon}), W_2(y_{a,\epsilon}, s_{a,\epsilon}), D_{(y,s)}\phi(y_{a,\epsilon}, s_{a,\epsilon})) \leq 0.$$

Thus:

$$H\Big((y_{a,\epsilon}, s_{a,\epsilon}), W_2(y_{a,\epsilon}, s_{a,\epsilon}), \frac{2}{\epsilon}\big((x_{a,\epsilon}, r_{a,\epsilon}) - (y_{a,\epsilon}, s_{a,\epsilon})\big) - a(y_{a,\epsilon}, s_{a,\epsilon})\Big) \leq 0 \tag{2.3.9}$$

Combining (2.3.8) with (2.3.9), we have:

$$\begin{aligned} 0 \leq & H\Big((x_{a,\epsilon}, r_{a,\epsilon}), W_1(x_{a,\epsilon}, r_{a,\epsilon}), \frac{2}{\epsilon}\big((x_{a,\epsilon}, r_{a,\epsilon}) - (y_{a,\epsilon}, s_{a,\epsilon})\big) + a(x_{a,\epsilon}, r_{a,\epsilon})\Big) - \\ & H\Big((y_{a,\epsilon}, s_{a,\epsilon}), W_2(y_{a,\epsilon}, s_{a,\epsilon}), \frac{2}{\epsilon}\big((x_{a,\epsilon}, r_{a,\epsilon}) - (y_{a,\epsilon}, s_{a,\epsilon})\big) - a(y_{a,\epsilon}, s_{a,\epsilon})\Big). \end{aligned}$$

Applying the estimations (2.3.5), (2.3.3) and (2.3.4) to the right-hand side of the above inequality, we deduce that

$$\begin{aligned} 0 \leq & H\Big((x_{a,\epsilon}, r_{a,\epsilon}), W_1(x_{a,\epsilon}, r_{a,\epsilon}), \frac{2}{\epsilon}\big((x_{a,\epsilon}, r_{a,\epsilon}) - (y_{a,\epsilon}, s_{a,\epsilon})\big)\Big) - \\ & H\Big((y_{a,\epsilon}, s_{a,\epsilon}), W_2(y_{a,\epsilon}, s_{a,\epsilon}), \frac{2}{\epsilon}\big((x_{a,\epsilon}, r_{a,\epsilon}) - (y_{a,\epsilon}, s_{a,\epsilon})\big)\Big) + \bar{C}a(\|(x_{a,\epsilon}, r_{a,\epsilon})\| + \|(y_{a,\epsilon}, s_{a,\epsilon})\|) \\ \leq & H\Big((x_{a,\epsilon}, r_{a,\epsilon}), W_2(y_{a,\epsilon}, s_{a,\epsilon}), \frac{2}{\epsilon}\big((x_{a,\epsilon}, r_{a,\epsilon}) - (y_{a,\epsilon}, s_{a,\epsilon})\big)\Big) - \\ & H\Big((y_{a,\epsilon}, s_{a,\epsilon}), W_2(y_{a,\epsilon}, s_{a,\epsilon}), \frac{2}{\epsilon}\big((x_{a,\epsilon}, r_{a,\epsilon}) - (y_{a,\epsilon}, s_{a,\epsilon})\big)\Big) + \bar{C}a(\|(x_{a,\epsilon}, r_{a,\epsilon})\| + \|(y_{a,\epsilon}, s_{a,\epsilon})\|) \\ & - \eta\big(W_1(x_{a,\epsilon}, r_{a,\epsilon}) - W_2(y_{a,\epsilon}, s_{a,\epsilon})\big) \\ \leq & \bar{C}\big(1 + \frac{2}{\epsilon}\|(x_{a,\epsilon}, r_{a,\epsilon}) - (y_{a,\epsilon}, s_{a,\epsilon})\|\big)\|(x_{a,\epsilon}, r_{a,\epsilon}) - (y_{a,\epsilon}, s_{a,\epsilon})\| + \\ & \bar{C}a(\|(x_{a,\epsilon}, r_{a,\epsilon})\| + \|(y_{a,\epsilon}, s_{a,\epsilon})\|) - \eta\big(W_1(x_{a,\epsilon}, r_{a,\epsilon}) - W_2(y_{a,\epsilon}, s_{a,\epsilon})\big). \end{aligned}$$

Finally, by the very definition of $N_{a,\epsilon}$:

$$\begin{aligned}0 \leq & \bar{C}\big(1+\frac{2}{\epsilon}\|(x_{a,\epsilon},r_{a,\epsilon})-(y_{a,\epsilon},s_{a,\epsilon})\|\big)\|(x_{a,\epsilon},r_{a,\epsilon})-(y_{a,\epsilon},s_{a,\epsilon})\|+\\ & \bar{C}a(\|(x_{a,\epsilon},r_{a,\epsilon})\|+\|(y_{a,\epsilon},s_{a,\epsilon})\|)-\eta N_{a,\epsilon}\end{aligned}$$

Letting $\epsilon \to 0^+$ in the above inequality, we obtain, in view of Lemma 2.3.4:

$$0 \leq \bar{C}\bar{C}'\sqrt{a}-\eta N_a.$$

Letting $a \to 0^+$, this yields: $\eta N \leq 0$, which is a contradiction with (2.3.6). Thus $W_1 \leq W_2$. The proof is complete. □

Proof of Lemma 2.3.4. Since we have, for $K > \max\{\|W_1\|_\infty, \|W_2\|_\infty\}$:

$$\begin{aligned}&0 < N_a \leq N_{a,\epsilon} = W_1(x_{a,\epsilon},r_{a,\epsilon}) - W_2(y_{a,\epsilon},s_{a,\epsilon}) - \frac{\|(x_{a,\epsilon},r_{a,\epsilon})-(y_{a,\epsilon},s_{a,\epsilon})\|^2}{\epsilon} - \\ &\frac{a}{2}\big(\|(x_{a,\epsilon},r_{a,\epsilon})\|^2+\|(y_{a,\epsilon},s_{a,\epsilon})\|^2\big) \leq 2K - \frac{\|(x_{a,\epsilon},r_{a,\epsilon})-(y_{a,\epsilon},s_{a,\epsilon})\|^2}{\epsilon} - \\ &\frac{a}{2}\big(\|(x_{a,\epsilon},r_{a,\epsilon})\|^2+\|(y_{a,\epsilon},s_{a,\epsilon})\|^2\big).\end{aligned}$$

This proves that (c) holds and that $\frac{\|(x_{a,\epsilon},r_{a,\epsilon})-(y_{a,\epsilon},s_{a,\epsilon})\|^2}{\epsilon}$ is bounded provided that a is small enough. Thus $\|(x_{a,\epsilon},r_{a,\epsilon})-(y_{a,\epsilon},s_{a,\epsilon})\|$ tends to 0 as $\epsilon \to 0^+$. Let us fix $a > 0$ small enough. By the fact that $(x_{a,\epsilon},r_{a,\epsilon},y_{a,\epsilon},s_{a,\epsilon})$ is bounded, let (x_a,r_a) be a cluster point of $\big((x_{a,\epsilon},r_{a,\epsilon})\big)_{\epsilon>0}$ and $\big((y_{a,\epsilon},s_{a,\epsilon})\big)_{\epsilon>0}$ as $\epsilon \to 0^+$. Let $(\epsilon_n)_{n\in\mathbb{N}}$ be a sequence such that $\epsilon_n \to 0^+$ and $\lim_{n\to\infty}(x_{a,\epsilon_n},r_{a,\epsilon_n}) = \lim_{n\to\infty}(y_{a,\epsilon_n},s_{a,\epsilon_n}) = (x_a,r_a)$. We have:

$$\begin{aligned}&N_a \leq \lim_{n\to\infty} N_{a,\epsilon_n}\\ \leq &\liminf_{n\to\infty}\Big[W_1(x_{a,\epsilon_n},r_{a,\epsilon_n}) - W_2(y_{a,\epsilon_n},s_{a,\epsilon_n}) - \frac{a}{2}(\|(x_{a,\epsilon_n},r_{a,\epsilon_n})\|^2+\|(y_{a,\epsilon_n},s_{a,\epsilon_n})\|^2)\Big]\\ \leq &\limsup_{n\to\infty}\Big[W_1(x_{a,\epsilon_n},r_{a,\epsilon_n}) - W_2(y_{a,\epsilon_n},s_{a,\epsilon_n}) - \frac{a}{2}(\|(x_{a,\epsilon_n},r_{a,\epsilon_n})\|^2+\|(y_{a,\epsilon_n},s_{a,\epsilon_n})\|^2)\Big]\\ \leq &W_1(x_a,r_a) - W_2(x_a,r_a) - a\|(x_a,r_a)\|^2 \leq N_a.\end{aligned}$$

The last two inequalities are due to the fact that $W_1 - W_2$ is upper semi-continuous and $W_1(x_a,M) = W_2(x_a,M)$ (i.e. $r_a < M$, since if this is not true, we would have $N_a < 0$, a contradiction). Thus, we have $\lim_{n\to\infty} N_{a,\epsilon_n} = N_a$. In addition, the above inequalities yield:

$$\begin{aligned}&\lim_{n\to\infty}\Big[W_1(x_{a,\epsilon_n},r_{a,\epsilon_n}) - W_2(y_{a,\epsilon_n},s_{a,\epsilon_n}) - \frac{a}{2}(\|(x_{a,\epsilon_n},r_{a,\epsilon_n})\|^2+\|(y_{a,\epsilon_n},s_{a,\epsilon_n})\|^2)\Big]\\ =&W_1(x_a,r_a) - W_2(x_a,r_a) - a\|(x_a,r_a)\|^2 = N_a.\end{aligned}$$

This implies that:

$$\lim_{n\to\infty} \frac{\|(x_{a,\epsilon_n}, r_{a,\epsilon_n}) - (y_{a,\epsilon_n}, s_{a,\epsilon_n})\|^2}{\epsilon_n} = \lim_{n\to\infty} \big[W_1(x_{a,\epsilon_n}, r_{a,\epsilon_n}) - W_2(y_{a,\epsilon_n}, s_{a,\epsilon_n}) - \frac{a}{2}(\|(x_{a,\epsilon_n}, r_{a,\epsilon_n})\|^2 + \|(y_{a,\epsilon_n}, s_{a,\epsilon_n})\|^2) - N_{a,\epsilon_n}\big] = 0.$$

Since this is true for any $\epsilon_n \to 0$ such that $(x_{a,\epsilon_n}, r_{a,\epsilon_n}, y_{a,\epsilon_n}, s_{a,\epsilon_n})$ is converging, by a compactness argument, we have finally (a), (b) and (d) hold and thus the proof of the lemma is complete. □

A direct consequence of the above comparison principle is the following:

Corollary 2.3.5. *The Hamilton-Jacobi-Isaacs equation* (2.3.1) *has at most one bounded continuous viscosity solution.*

2.4 Existence and Characterization of Value

In this section, we present the main result of this paper: the existence of value for game $\mathcal{G}(X_0, r_0, p)$ and its characterization. By Corollary 2.2.9, we can extend V^+ and V^- by continuity to $(\mathbb{R}^n)^I \times (-\infty, M_0] \times \Delta(I)$ by defining, for any $X_0 \in (\mathbb{R}^n)^I$ and $p \in \Delta(I)$:

$$V^+(X_0, M_0, p) := \sum_{i\in I} p_i \tilde{V}(x_0^i, M_0);\ V^-(X_0, M_0, p) := \sum_{i\in I} p_i \tilde{V}(x_0^i, M_0).$$

Throughout this section, we do not distinguish these value functions from their extended versions above and thus both V^+ and V^- are bounded continuous functions on $(\mathbb{R}^n)^I \times (-\infty, M_0] \times \Delta(I)$. Let us state the main result:

Theorem 2.4.1. *Under Isaacs' condition* (2.1.5)*, game $\mathcal{G}(X_0, r_0, p)$ has a value. Moreover for any $p \in \Delta(I)$, the value function $(X_0, r_0) \mapsto V(X_0, r_0, p)$ is the unique bounded continuous viscosity solution of the Hamilton-Jacobi-Isaacs equation* (2.0.2)*.*

Proof. Since the hamiltonian H_p satisfies the assumptions (2.3.3), (2.3.4) and (2.3.5), the Corollary 2.3.5 implies the uniqueness property of bounded viscosity solution for the equation (2.0.2). To end the proof we only need to obtain the existence property for the PDE (2.0.2) which is a consequence of the Proposition 2.4.2 stated below. □

Proposition 2.4.2. *For any $p \in \Delta(I)$, the value functions $V^+(\cdot, p)$ and $V^-(\cdot, p)$ are respectively viscosity sub-solution and viscosity super-solution of equation* (2.0.2)*.*

Proof. By Remark 2.2.1 and Lemma 2.3.2, we only need to prove the proposition for V^+.

Let $\varphi \in C^1\big((\mathbb{R}^n)^I \times (-\infty, M_0)\big)$ be a test function such that $V^+(\cdot, p) - \varphi$ admits a local maximum at some point $(X_0, r_0) \in (\mathbb{R}^n)^I \times (-\infty, M_0)$. Since V^+ is continuous, to conclude the proof, we only need to prove that:

$$-\lambda V^+(X_0, r_0, p) + H_p(X_0, r_0, D_{X,r}\varphi(X_0, r_0)) \geq 0. \tag{2.4.1}$$

Since $(V^+(\cdot, p) - \varphi)(X_0, r_0)$ is a local maximum, there exists some $\delta > 0$ small enough such that for any $(X, r) \in B\big((X_0, r_0); \delta\big)$, $V^+(X, r, p) \leq V^+(X_0, r_0, p) - \varphi(X_0, r_0) + \varphi(X, r)$. Let $h_0 = \min(\frac{\delta}{\sqrt{\|f\|_\infty^2 + \|g\|_\infty^2 + 1}}, \frac{M_0 - r_0}{\|g\|_\infty})$. Thus, by the principle of dynamic programming for V^+ (Proposition 2.2.2), for any $h \in (0, h_0)$:

$$\begin{aligned} V^+(X_0, r_0, p) = \inf_\alpha \sup_v \sum_{i \in I} p_i \int_0^h e^{-\lambda t} \ell\big((X, R)_t^{x_0^i, r_0, \alpha(v), v}, \alpha(v)(t), v(t)\big) dt \\ + e^{-\lambda h} V^+\big((X_h^{x_0^i, \alpha(v), v})_{i \in I}, R_h^{r_0, \alpha(v), v}, p\big). \end{aligned}$$

But we have $\big((X_h^{x_0^i, \alpha(v), v})_{i \in I}, R_h^{r_0, \alpha(v), v}\big) \in B\big((X_0, r_0); \delta\big)$ for all (α, v) and all $h \in (0, h_0)$, thus:

$$\begin{aligned} (1 - e^{-\lambda h}) V^+(X_0, r_0, p) \leq \inf_\alpha \sup_v \sum_{i \in I} p_i \int_0^h e^{-\lambda t} \ell\big((X, R)_t^{x_0^i, r_0, \alpha(v), v}, \alpha(v)(t), v(t)\big) dt \\ + e^{-\lambda h} \Big(\varphi\big((X_h^{x_0^i, \alpha(v), v})_{i \in I}, R_h^{r_0, \alpha(v), v}\big) - \varphi(X_0, r_0)\Big). \end{aligned} \tag{2.4.2}$$

Let us make some estimates on (2.4.2). We have first of all:

$$(1 - e^{-\lambda h}) V^+(X_0, r_0, p) = h\lambda V^+(X_0, r_0, p) + o(h) \tag{2.4.3}$$

Secondly, let $K > 0$ be a upper bound of $\sqrt{I\|f\|_\infty^2 + \|g\|_\infty^2}\|D_{(X,r)}\varphi\|$ on $B\big((X_0, r_0); \delta\big)$. For any $(u, v) \in \mathcal{U} \times \mathcal{V}$, we have:

$$\Big|(e^{-\lambda h} - 1)\Big(\varphi\big((X_h^{x_0^i, u, v})_{i \in I}, R_h^{r_0, u, v}\big) - \varphi(X_0, r_0)\Big)\Big| \leq h(e^{-\lambda h} - 1)K + o(h) = o(h). \tag{2.4.4}$$

By (2.4.2), for any fixed constant control $u_0 \in U$, there exists $v^{h,\epsilon} \in \mathcal{V}$ such that:

$$\begin{aligned} (1 - e^{-\lambda h}) V^+(X_0, r_0, p) - \epsilon h \leq \sum_{i \in I} p_i \int_0^h e^{-\lambda t} \ell\big((X, R)_t^{x_0^i, r_0, u_0, v^{h,\epsilon}}, u_0, v^{h,\epsilon}(t)\big) dt \\ + e^{-\lambda h} \Big(\varphi\big((X_h^{x_0^i, u_0, v^{h,\epsilon}})_{i \in I}, R_h^{r_0, u_0, v^{h,\epsilon}}\big) - \varphi(X_0, r_0)\Big). \end{aligned} \tag{2.4.5}$$

For convenience, let us denote by $F(X_0, r_0, u_0, v_0)$ $(v_0 \in V)$ the map:

$$(X_0, r_0, u_0, v_0) \mapsto \Big(\big(f(x_0^i, u_0, v_0)\big)_{i \in I}, g(r_0, u_0, v_0)\Big).$$

Since $D_{(X,r)}\varphi$ is uniformly continuous on $B((X_0,r_0);\delta)$ and F, ℓ are Lipschitz continuous with respect to the state variables, this implies:

$$\begin{aligned}
&\varphi((X_h^{x_0^i,u_0,v^{h,\epsilon}})_{i\in I}, R_h^{r_0,u_0,v^{h,\epsilon}}) - \varphi(X_0,r_0)\\
=&\int_0^h \big\langle D_{(X,r)}\varphi\big((X_t^{x_0^i,u_0,v^{h,\epsilon}})_{i\in I}, R_t^{r_0,u_0,v^{h,\epsilon}}\big), F\big((X_t^{x_0^i,u_0,v^{h,\epsilon}})_{i\in I}, R_t^{r_0,u_0,v^{h,\epsilon}}, u_0, v^{h,\epsilon}(t)\big)\big\rangle dt\\
=&\int_0^h \big\langle D_{(X,r)}\varphi(X_0,r_0), F\big(X_0,r_0,u_0,v^{h,\epsilon}(t)\big)\big\rangle dt + o(h),
\end{aligned} \tag{2.4.6}$$

and that for any $i\in I$,

$$\begin{aligned}
&\int_0^h e^{-\lambda t}\ell\big((X,R)_t^{x_0^i,r_0,u_0,v^{h,\epsilon}}, u_0, v^{h,\epsilon}(t)\big)dt \le \int_0^h \ell\big((X,R)_t^{x_0^i,r_0,u_0,v^{h,\epsilon}}, u_0, v^{h,\epsilon}(t)\big)dt\\
=&\int_0^h \ell\big(x_0^i,r_0,u_0,v^{h,\epsilon}(t)\big)dt + o(h).
\end{aligned} \tag{2.4.7}$$

Combining (2.4.6) with (2.4.7), we have:

$$\begin{aligned}
&\sum_{i\in I} p_i \int_0^h e^{-\lambda t}\ell\big((X,R)_t^{x_0^i,r_0,u_0,v^{h,\epsilon}}, u_0, v^{h,\epsilon}(t)\big)dt + \varphi\big((X_h^{x_0^i,u_0,v^{h,\epsilon}})_{i\in I}, R_h^{r_0,u_0,v^{h,\epsilon}}\big) - \varphi(X_0,r_0)\\
\le&\int_0^h \Big\langle D_{(X,r)}\varphi(X_0,r_0), F\big(X_0,r_0,u_0,v^{h,\epsilon}(t)\big)\Big\rangle + \sum_{i\in I} p_i\ell\big(x_0^i,r_0,u_0,v^{h,\epsilon}(t)\big)dt + o(h)\\
\le&h\max_{v\in V}\Big\{\big\langle D_{(X,r)}\varphi(X_0,r_0), F(X_0,r_0,u_0,v)\big\rangle + \sum_{i\in I} p_i\ell(x_0^i,r_0,u_0,v)\Big\} + o(h).
\end{aligned}$$

Applying the above estimation with (2.4.3) and (2.4.4) to the right-hand side of (2.4.5), we obtain:

$$\begin{aligned}
-\epsilon h + o(h) \le h\Big(&-\lambda V^+(X_0,r_0,p) + \max_{v\in V}\Big\{\big\langle D_{(X,r)}\varphi(X_0,r_0), F(X_0,r_0,u_0,v)\big\rangle +\\
&\sum_{i\in I} p_i\ell(x_0^i,r_0,u_0,v)\Big\}\Big)
\end{aligned}$$

Dividing both sides of the above inequality by h and letting h tend to 0 gives:

$$-\epsilon \le -\lambda V^+(X_0,r_0,p) + \max_{v\in V}\Big\{\big\langle D_{(X,r)}\varphi(X_0,r_0), F(X_0,r_0,u_0,v)\big\rangle + \sum_{i\in I} p_i\ell(x_0^i,r_0,u_0,v)\Big\}.$$

Since $\epsilon>0$ is arbitrary, by taking the infimum on u_0 in the right-hand side of the above inequality, we have:

$$0 \le -\lambda V^+(X_0,r_0,p) + H_p^+\big((X_0,r_0), D_{(X,r)}\varphi(X_0,r_0)\big).$$

According to Isaacs' condition (2.1.5), the above inequality reduces to (2.4.1), which is the wished result. The proof is complete. □

2.5 Applicability of the Approach

In this section, we demonstrate the applicability of our approach by studying a differential game with incomplete information (denoted by $\hat{\mathcal{G}}(X_0, q)$). In game $\hat{\mathcal{G}}(X_0, q)$, the initial state of the dynamic is chosen randomly among only two points x_0^1 and x_0^2 and the information will be revealed to both players when and only when the dynamic hits the closed unit ball $\bar{B}(0;1) \subset \mathbb{R}^n$.

The information structure of game $\hat{\mathcal{G}}(X_0, q)$ is of a different form to that of game $\mathcal{G}(X_0, r_0, p)$. However, we will show in this section the similarity between the information structures of these two games and we will demonstrate that our approach developed in previous sections can be applied to game $\hat{\mathcal{G}}(X_0, q)$ to obtain the existence of a value of the game. Let us consider a following dynamic:

$$\begin{cases} \dot{x}(t) = \hat{f}\big(x(t), u(t), v(t)\big), \quad t \geq 0; \\ x(0) = x_0 \in \mathbb{R}^N. \end{cases} \tag{2.5.1}$$

Let $t \mapsto \hat{X}_t^{\bar{x}_0,u,v}$ denote the trajectory of the unique solution for system (2.5.1). We associate to this differential game a pay-off:

$$\hat{J}(x_0, u, v) := \int_0^\infty e^{-\lambda t} \hat{\ell}\big(\hat{X}_t^{x_0,u,v}, u(t), v(t)\big) dt. \tag{2.5.2}$$

The game is played as follows:

1. Before the game begins, an initial state x_0 is picked randomly from the set $X_0 := \{x_0^1, x_0^2\}$ according to the probability measure $(q, 1-q) \in \Delta(2)$ with $q \in (0,1)$. In addition, we suppose that for any $i \in \{1,2\}$, $x_0^i \in \mathcal{O} \subset \mathbb{R}^N$ with $\mathcal{O} := \mathbb{R}^N \backslash \bar{B}(0;1)$. Player 1 and Player 2 both know as common knowledge $(q, 1-q)$ and X_0, but they are not informed about the chosen x_0.

2. Player 1 chooses the control u to minimize $\hat{J}(x_0, u, v)$ while Player 2 choosing the control v to maximize $\hat{J}(x_0, u, v)$. As usual, both players observe all played actions with perfect memory.

3. If $\hat{T}(x_0, u, v) := \inf\{t > 0 \mid \hat{X}_t^{x_0,u,v} \in \mathcal{O}^c\} < \infty$, the current state $\hat{X}_{\hat{T}(x_0,u,v)}^{x_0,u,v}$ is publicly announced to all players at the hitting time $t = \hat{T}(x_0, u, v)$.

To guarantee the existence of value of the above game, we suppose in this section that:

(i') $\bar{f} : \mathbb{R}^N \times U \times V \to \mathbb{R}$ is a bounded, continuous function which is Lipschitz continuous with respect to the first variable; in addition, for any $(x, u, v) \in \mathbb{R}^N \backslash \{0\} \times U \times V$, $\langle f(\bar{x}, u, v), x \rangle < 0$;

(ii') $\bar{\ell} : \mathbb{R}^N \times U \times V \to \mathbb{R}$ is bounded, continuous and Lipschitz continuous with respect to the first variable;

(iii') Isaac's condition: for any $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$, $\xi = (\xi_1, \xi_2) \in \mathbb{R}^N \times \mathbb{R}^N$, $q \in (0, 1)$:

$$\begin{aligned} &\inf_{u\in U}\sup_{v\in V}\left\langle \xi, \left(\hat{f}(x,u,v), \hat{f}(y,u,v)\right)\right\rangle + q\hat{\ell}(x,u,v) + (1-q)\hat{\ell}(y,u,v) \\ =&\sup_{v\in V}\inf_{u\in U}\left\langle \xi, \left(\hat{f}(x,u,v), \hat{f}(y,u,v)\right)\right\rangle + q\hat{\ell}(x,u,v) + (1-q)\hat{\ell}(y,u,v). \end{aligned} \tag{2.5.3}$$

For future convenience, we define:

$$\hat{H}_q(X = (x, y), \xi) := \inf_{u\in U}\sup_{v\in V}\left\langle \xi, \left(\hat{f}(x,u,v), \hat{f}(y,u,v)\right)\right\rangle + q\hat{\ell}(x,u,v) + (1-q)\hat{\ell}(y,u,v).$$

Now let us demonstrate the relation between game $\hat{\mathcal{G}}$ and game $\mathcal{G}$ discussed in the previous sections. Let us introduce, for any $(X_0, u, v) \in \mathbb{R}^{2N} \times \mathcal{U} \times \mathcal{V}$, the following function:

$$t \mapsto \hat{R}_t^{X_0,u,v} := -\min_{i\in\{1,2\}} \|X_t^{x_0^i,u,v}\|.$$

By the assumption (i') on $\hat{f}$, $\hat{R}_\cdot^{X_0,u,v}$ is strictly increasing within $[0, \hat{\mathcal{T}}(X_0, u, v)]$ where $\hat{\mathcal{T}}(X_0, u, v) = \inf_{i\in\{1,2\}} \hat{T}(x_0^i, u, v)$. Furthermore, it is clear that during game $\hat{G}(X_0, q)$, both players will be informed about the exact choice of x_0 at $\hat{\mathcal{T}}(X_0, u, v)$ no matter which $i \in \{1, 2\}$ is chosen before the game begins. Suppose that $1 = \arg\min_{i\in\{1,2\}} \hat{T}(x_0^i, u, v)$, then either $X^{x_0^1,u,v}_{\hat{T}(x_0^1,u,v)}$ is revealed at $\hat{T}(x_0^1, u, v)$ or nothing (or $X^{x_0^2,u,v}_{\hat{T}(x_0^2,u,v)}$) is revealed at $\hat{T}(x_0^1, u, v)$ and both players know that $i = 2$. Hence, step 3 of game $\hat{\mathcal{G}}(X_0, q)$ is equivalent to the following:

3' During the game, if $t \mapsto \hat{R}_t^{X_0,u,v}$ hits a fixed target $\hat{M}_0 = -1$, the current state $\hat{X}^{x_0,u,v}_{\hat{\mathcal{T}}(X_0,u,v)}$ is publicly announced to all players at the moment $t = \hat{\mathcal{T}}(X_0, u, v)$.

Although the function $\hat{R}_\cdot^{X_0,u,v}$ is not differentiable with respect to t (in comparison to $R_\cdot^{r_0,u,v}$ for game $\mathcal{G}(X_0, r_0, p)$), one can check that:

Lemma 2.5.1. *For any $T > 0$, the function $X_0 \mapsto \hat{R}_t^{X_0,u,v}$ is locally Lipschitz continuous uniformly for any $(t, u, v) \in [0, T] \times \mathcal{U} \times \mathcal{V}$. In addition, given $(X_0, u, v) \in \mathbb{R}^{2n} \times \mathcal{U} \times \mathcal{V}$ and $0 \le t < \hat{\mathcal{T}}(X_0, u, v)$, for any $0 < \Delta t \le \hat{\mathcal{T}}(X_0, u, v) - t$ and $0 < \delta < 1$:*

$$\hat{R}_{t+\Delta t}^{X_0,u,v} - \hat{R}_t^{X_0,u,v} > -2 \max_{(\bar{B}(0;\|X_0\|)\setminus B(0;\delta))\times U\times V} \langle x, \hat{f}(x, u', v')\rangle \Delta t > 0.$$

Therefore, as in game $\mathcal{G}$, a strategy for Player 1 in game $\hat{\mathcal{G}}(X_0, q)$ is defined as an element $A = (\alpha, (\alpha_x)_{x\in\mathbb{R}^N})$ of $\mathcal{A}_d \times (\mathcal{A}_d)^{\mathbb{R}^N}$ and that for Player 2 is an element $B = (\beta, (\beta_x)_{x\in\mathbb{R}^N})$ of $\mathcal{B}_d \times (\mathcal{B}_d)^{\mathbb{R}^N}$. For any $X_0 \in \mathbb{R}^{2N}$ and $i \in \{1, 2\}$, we associate to each triplet (x_0^i, A, B) the trajectory:

$$t \mapsto \hat{X}_t^{x_0^i, A, B} := \hat{X}_t^{x_0^i, u_{A,B}^i, v_{A,B}^i}$$

where with $Z_i := X^{x_0^i, u_{\alpha,\beta}, v_{\alpha,\beta}}_{\hat{\mathcal{T}}(X_0, u_{\alpha,\beta}, v_{\alpha,\beta})}$,

$$u_{A,B}^i(t) = \begin{cases} u_{\alpha,\beta}(t), & t \in [0, \hat{\mathcal{T}}(X_0, u_{\alpha,\beta}, v_{\alpha,\beta})); \\ u_{\alpha_{Z_i},\beta_{Z_i}}(t - \hat{\mathcal{T}}(X_0, u_{\alpha,\beta}, v_{\alpha,\beta})), & t \in [\hat{\mathcal{T}}(X_0, u_{\alpha,\beta}, v_{\alpha,\beta}), +\infty); \end{cases}$$

$$v_{A,B}^i(t) = \begin{cases} v_{\alpha,\beta}(t), & t \in [0, \hat{\mathcal{T}}(X_0, u_{\alpha,\beta}, v_{\alpha,\beta})); \\ v_{\alpha_{Z_i},\beta_{Z_i}}(t - \hat{\mathcal{T}}(X_0, u_{\alpha,\beta}, v_{\alpha,\beta})), & t \in [\hat{\mathcal{T}}(X_0, u_{\alpha,\beta}, v_{\alpha,\beta}), +\infty). \end{cases}$$

In the above formulas, the controls $(u_{\alpha,\beta}, v_{\alpha,\beta})$ and $(u_{\alpha_{Z_i},\beta_{Z_i}}, v_{\alpha_{Z_i},\beta_{Z_i}})$ are defined as in Lemma 2.1.4. We can furthermore write the game into a normal form by associated to each pair of strategies (A, B) a following pay-off:

$$\hat{\mathcal{J}}(X_0, q, A, B) := q\int_0^\infty e^{-\lambda t}\hat{\ell}\big(\hat{X}_t^{x_0^1, A, B}, A, B\big)dt + (1-q)\int_0^\infty e^{-\lambda t}\hat{\ell}\big(\hat{X}_t^{x_0^2, A, B}, A, B\big)dt.$$

We thus define the upper- and lower-value functions of game $\hat{\mathcal{G}}(x_0, q)$ as follows:

$$V^+(X_0, q) = \inf_A \sup_B \hat{\mathcal{J}}(X_0, q, A, B),\ V^-(X_0, q) = \sup_B \inf_A \hat{\mathcal{J}}(X_0, q, A, B).$$

Let us present briefly the procedure for proving that game $\hat{\mathcal{G}}(X_0, q)$ has a value. Thanks to Lemma 2.5.1, we can prove a following lemma in a similar way as in the proof of Lemma 2.2.3.

Lemma 2.5.2 (Two-step Programming Principles for game $\hat{\mathcal{G}}(X_0, q)$)**.** *For any $(X_0, q) \in \mathcal{O}^2 \times [0, 1]$, we have:*

$$V^+(X_0, q) = \inf_{\alpha\in\mathcal{A}_d}\sup_{v\in\mathcal{V}} q\Big\{\int_0^{\hat{\mathcal{T}}(X_0,\alpha,v)} e^{-\lambda t}\hat{\ell}\big(\hat{X}_t^{x_0^1,\alpha,v}, \alpha, v\big)dt + e^{-\lambda\hat{\mathcal{T}}(X_0,\alpha,v)}\hat{V}(\hat{X}^{x_0^1,\alpha,v}_{\hat{\mathcal{T}}(X_0,\alpha,v)}, M_0)\Big\} +$$
$$(1-q)\Big\{\int_0^{\hat{\mathcal{T}}(X_0,\alpha,v)} e^{-\lambda t}\hat{\ell}\big(\hat{X}_t^{x_0^2,\alpha,v}, \alpha, v\big)dt + e^{-\lambda\hat{\mathcal{T}}(X_0,\alpha,v)}\hat{V}(\hat{X}^{x_0^2,\alpha,v}_{\hat{\mathcal{T}}(X_0,\alpha,v)}, M_0)\Big\},$$

$$V^-(X_0, q) = \sup_{\beta\in\mathcal{B}_d}\inf_{u\in\mathcal{U}} q\Big\{\int_0^{\hat{\mathcal{T}}(X_0,u,\beta)} e^{-\lambda t}\hat{\ell}\big(\hat{X}_t^{x_0^1,u,\beta}, u, \beta\big)dt + e^{-\lambda\hat{\mathcal{T}}(X_0,u,\beta)}\hat{V}(\hat{X}^{x_0^1,u,\beta}_{\hat{\mathcal{T}}(X_0,u,\beta)}, M_0)\Big\} +$$
$$(1-q)\Big\{\int_0^{\hat{\mathcal{T}}(X_0,u,\beta)} e^{-\lambda t}\hat{\ell}\big(\hat{X}_t^{x_0^2,u,\beta}, u, \beta\big)dt + e^{-\lambda\hat{\mathcal{T}}(X_0,u,\beta)}\hat{V}(\hat{X}^{x_0^2,u,\beta}_{\hat{\mathcal{T}}(X_0,u,\beta)}, M_0)\Big\}.$$

In the above equalities, the function $(y) \mapsto \hat{V}(y)$ *is defined by:*

$$\begin{aligned}\hat{V}(y) &:= \inf_{\alpha\in\mathcal{A}_d}\sup_{v\in\mathcal{V}}\int_0^{+\infty} e^{-\lambda t}\hat{\ell}\big(\hat{X}_t^{y,\alpha(v),v},\alpha(v)(t),v(t)\big)dt\\ &= \sup_{\beta\in\mathcal{B}_d}\inf_{u\in\mathcal{U}}\int_0^{+\infty} e^{-\lambda t}\hat{\ell}\big((\hat{X}_t^{y,u,\beta(u)},u(t),\beta(u)(t)\big)dt.\end{aligned}$$

As in Section 2.3, we can deduce from the above lemma the dynamic programming principles for the value functions of game $\hat{\mathcal{G}}(X_0,q)$.

Proposition 2.5.3 (Dynamic Programming Principles for $\hat{\mathcal{G}}(X_0,q)$)**.** *For any* $(X_0,q)\in\mathcal{O}^2\times[0,1]$ *(thus,* $\hat{R}_0^{X_0,u,v}<-1$ *for any* $(u,v)\in\mathcal{U}\times\mathcal{V}$*) and* $h>0$ *small enough,*

$$\begin{aligned}V^+(X_0,q) = \inf_{\alpha\in\mathcal{A}_d}\sup_{v}\Big\{&q\int_0^h e^{-\lambda t}\hat{\ell}\big(\hat{X}_t^{x_0^1,\alpha,v},\alpha,v\big)dt+(1-q)\int_0^h e^{-\lambda t}\hat{\ell}\big(\hat{X}_t^{x_0^2,\alpha,v},\alpha,v\big)dt+\\ &e^{-\lambda h}V^+\big((\hat{X}_h^{x_0^i,\alpha,v})_{i\in\{1,2\}},q\big)\Big\},\end{aligned}\tag{2.5.4}$$

$$\begin{aligned}V^-(X_0,q) = \sup_{\beta\in\mathcal{B}_d}\inf_{u}\Big\{&q\int_0^h e^{-\lambda t}\hat{\ell}\big(\hat{X}_t^{x_0^1,u,\beta},u,\beta\big)dt+(1-q)\int_0^h e^{-\lambda t}\hat{\ell}\big(\hat{X}_t^{x_0^2,u,\beta},u,\beta\big)dt+\\ &e^{-\lambda h}V^+\big((\hat{X}_h^{x_0^i,u,\beta})_{i\in\{1,2\}},q\big)\Big\},\end{aligned}\tag{2.5.5}$$

By repeating the process in Section 3 and Section 4, as a consequence of the above proposition and Lemma 2.5.1, we can obtain the existence of a value for game $\hat{\mathcal{G}}(X_0,q)$.

Theorem 2.5.4 (Existence of a Value for Game $\hat{\mathcal{G}}(X_0,q)$)**.** *Under Isaacs' condition* (2.5.3)*, game* $\hat{\mathcal{G}}(X_0,q)$ *has a value and for any* $q\in[0,1]$ *fixed, the value function* $X_0\mapsto V(X_0,q):=V^+(X_0,q)=V^-(X_0,q)$ *is the unique bounded continuous viscosity solution of the Hamilton-Jacobi-Isaacs equation:*

$$\begin{cases}-\lambda W(X)+\hat{H}_q(X,DW(X))=0, & X\in\mathcal{O}^2;\\ W(X=(x_1,x_2))=q\hat{V}(x_1)+(1-q)\hat{V}(x_2), & \min\limits_{i\in\{1,2\}}\|x_i\|=-1.\end{cases}\tag{2.5.6}$$

2.6 Conclusion

We have proved in this paper the existence of the value for game $\mathcal{G}(X_0,r_0,p)$ under an Isaacs' condition and we have furthermore characterized its value function as the unique bounded viscosity solution of a Hamilton-Jacobi-Isaacs equation. We have also shown that our approach is powerful enough to obtain the existence of a value for several games with incomplete information and revealing.

Chapter 3

Existence of Value for Differential Games with Incomplete Information and with Signal Revelation - A General Case

Abstract:
In this paper, we investigate the existence of value for a two-person zero-sum differential game with incomplete information and with signal revelation. Before the game begins, the initial state of the dynamic is chosen randomly among a finite number of points in $\mathbb{R}^n$ while both players do not know the chosen initial state. During the game, if the system reaches a fixed closed target set K, the current state of the system at the hitting time is revealed to both players. We prove in this manuscript that this game has a value and its value function is characterized as unique bounded continuous viscosity solution of a suitable Hamilton-Jacobi-Isaacs equation.

Key words: Differential Games; Incomplete Information; Hamilton-Jacobi-Isaacs Equations; Signals.
AMS Classification: 49N30; 49N70; 91A05; 91A10; 91A23.

Introduction

We consider the following dynamic system in $\mathbb{R}^n$:

$$\begin{cases} \dot{x}(t) = f\big(x(t), u(t), v(t)\big), & t \geq 0; \\ x(0) = x_0. \end{cases} \tag{P}$$

In the above system, (u, v) is a pair of controls. We suppose that the map f is regular enough such that for any pair of admissible (measurable) controls (u, v) and any initial constants x_0, the dynamic (P) has a unique solution denoted by $t \mapsto X_t^{x_0,u,v}$. To each triplet of (x_0, u, v), we associate a following running cost:

$$J(x_0, u, v) := \int_0^{+\infty} e^{-\lambda t} \ell\big(X_t^{x_0,u,v}, u(t), v(t)\big) dt, \tag{3.0.1}$$

where ℓ is a real value function that is regular enough. Let I be positive integer and let $X_0 = \{x_0^i\}_{i \in I}$ be a subset of $\mathbb{R}^n$ indexed. Here we apply a slight abuse of notations by denoting the sets $\{1, 2, ..., I\}$ by I. We fix $\Delta(I) := \{p \in (\mathbb{R}^+)^I \mid \sum_{i=1}^I p(i) = 1\}$ the set of probability measures on I, and $K \subseteq \mathbb{R}^n$ a closed target set. For $p \in \Delta(I)$, let us consider $\mathcal{G}(X_0, p)$ a two-person zero-sum differential game with incomplete information and signals played as follows:

(A) Before the game begins, an index $i \in I$ is chosen randomly according to $p = \big(p(i)\big)_{i \in I} \in \Delta(I)$. The chosen index i is not communicated to any player. The set X_0 and the probability measure p are supposed to be common knowledge of both players.

(B) The game begins with $x_0 = x_0^i$ in (P). Player 1 chooses the control u in order to minimize running cost $J(x_0^i, u, v)$ and Player 2 aims to maximize $J(x_0^i, u, v)$ by choosing the control v. Both players observe during the game all played actions with perfect memory.

(C) During the game, both players also observe with perfect memory a signal:

$$s_{x_0^i,u,v}(t) = \begin{cases} (0, 0) \in \mathbb{R} \times \mathbb{R}^n, & \text{if } t < \mathcal{T}; \\ (1, X_{\mathcal{T}}^{x_0^i,u,v}), & \text{if } t \geq \mathcal{T}. \end{cases}$$

with $\mathcal{T} := \inf\{t \geq 0 \mid X_t^{x_0^i,u,v} \in K\}$.

Here by saying "with perfect memory", we mean that at any moment $t \geq 0$, the player knows the restriction of the application $t \mapsto \big(u(t), v(t)\big)$ on the interval $[0, t]$.

Now we explain the signal structure. During the game, the signal observed by the players is composed of two components: the first one takes binary values which is an indicator for whether the target K has been hit: it is 0 before K is hit and it turns to 1 after the hitting

occurs; the second one indicates the current state of the system at the moment $\mathcal{T}$ and it is the constant vector 0 if the system has not reached K. Before the dynamic (P) hits the target K, both players receive no further information and observe a constant pair $(0,0)$. As soon as $X_t^{x_0^i,u,v}$ reaches K (at the moment $\mathcal{T}$), both players receive the pair $(1, X_{\mathcal{T}}^{x_0^i,u,v})$ where the integer 1 serves as a sign of "target hit". This avoids the possible confusion while $X_{\mathcal{T}}^{x_0^i,u,v} = 0$.

When the target set is $K = \mathbb{R}^n$ (the initial state is revealed immediately as the game begins) or $K = \emptyset$ (the initial state is never revealed), our game model reduces to a classic differential game with complete information. For non-trivial subset K of $\mathbb{R}^n$, game $\mathcal{G}(X_0, p)$ can be interpreted as a differential game with incomplete information and revealing since the step (C) above is in fact equivalent to:

(C') When $X_t^{x_0^i,u,v}$ hits the target K, the current state (i.e $X_{\mathcal{T}}^{x_0^i,u,v}$) is announced publicly to both players.

This furthermore indicates that the game models studied in [73] can be viewed as a particular cases of game $\mathcal{G}(X_0, p)$. During $\mathcal{G}(X_0, p)$, for profiting from their information, both players should choose their actions according to the triplet $(u, v, s_{x_0^i,u,v})|_{[0,t]}$ at any moment $t \geq 0$.

In this paper, we define the upper- and lower-value functions of the game under suitable notion of strategies and we investigate the existence of a value (i.e. the upper- and lower-value functions coincide) and the characterization of its value function.

Differential games with incomplete information was studied in [16] which generalized the repeated games with incomplete information of Aumann and Maschler in [2] and the theory of differential games of Isaacs in [40]. In recent years, further investigation of the game model in [16] brought forward several results for these game models (such results can be found in [12, 17, 19, 21, 43]). In particular, the existence of value of related game models are investigated in [24, 25, 54, 74]). In comparison with these above articles, our game model concerns the case where players receive signals generated by the current states of the system during the game. Thus suitable notion of strategies for the players is necessary for the study of the game model.

Compared with pursuit-evasion games (for example, in [4, 22, 32, 40, 47, 55, 63, 69, 77]), players in our game model do not necessarily aim for optimizing $\mathcal{T}$ itself. It is not necessary that they will always try to avoid or reach the target and this requires appropriate hypothesis on the boundary of the target set.

In this manuscript, we prove that the game has a value and that this value is the unique continuous viscosity solution of a Hamilton-Jacobi-Isaacs equation on the open set $(\mathbb{R}^n \backslash K)^I$. To obtain the existence of value for our game model, we follow the classic scheme in [33]: We first prove that the upper- and lower-value functions of game $\mathcal{G}(X_0, p)$ are respectively viscosity sub- and super-solutions of the same Hamilton-Jacobi-Isaacs equation restricted to $(\mathbb{R}^n \backslash K)^I$. Then we prove that this equation has a unique bounded viscosity solution with a

boundary condition. Thus, the upper- and lower-value functions of the game coincide and there follows the existence of value.

There are three main novelties of this new game model. Firstly, we do not require the hitting time $\mathcal{T}$ always finite (for example, in [73]), i.e. the players are able to stay outside of the target set given proper dynamic system. Secondly, we introduce in this paper the notion of non-anticipative signal-depending strategies with delay which allows the players to choose their actions according to the observation of played actions and the signal. Such strategies are more complicate than classic non-anticipative strategies with delay and we shall show that under such strategies, the normal form of the game could lack for continuity which poses one of the main difficulties in this paper. Thirdly, the equality between the upper-value and the lower-value does not automatically hold while X_0 is on the boundary of the open set $(\mathbb{R}^n \backslash K)^I$ (in comparison with Bolza problems in [16, 54, 74] or the pursuit-evasion games listed above), this means that the upper- and lower-value functions do not automatically verify the boundary condition in the comparison principle for the Hamilton-Jacobi-Isaacs equation, which brings technical obstacles in the proof of the main results in this manuscript.

This article is organized as follows. After the preliminary section, we investigate the notion of non-anticipative signal-depending strategies with delay in Section 2 and we define the normal form of the game under such strategies. In Section 3, we study the continuity of the upper- and lower-value functions of the game and we prove furthermore a dynamic programming principle for its upper-value function. Finally, the last section is devoted to the proof of the main theorem.

3.1 Preliminaries

In this paper, we denote by $\|x\|$ the euclidean norm of $x \in \mathbb{R}^n$, and $\langle x, y \rangle$ the associated scalar product of any points x, $y \in \mathbb{R}^n$. An open ball with center $x \in \mathbb{R}^n$ and radius a is denoted by $B(x; a)$, and its closure by $\bar{B}(x; a)$. Also we identify probability measures on the finite set I with elements of $\Delta(I)$. The support of $p \in \Delta(I)$ is defined by $supp(p) := \{i \in I \mid p(i) > 0\}$. Let us suppose the following hypothesis:

(i) U, V are two compact metric spaces, both endowed with the Borel σ-algebra;

(ii) $f : \mathbb{R}^n \times U \times V \to \mathbb{R}^n$ is a bounded, continuous function which is Lipschitz continuous on the first variable uniformly in (u, v) with a Lipschitz constant $L_f > 0$;

(iii) $\ell : \mathbb{R}^n \times U \times V \to \mathbb{R}$ is bounded, continuous, and Lipschitz continuous on the first variable with Lipschitz constant $L_\ell > 0$;

(iv) $\mathcal{U}$ denotes the set of measurable maps $u : \mathbb{R}^+ \to U$ while $\mathcal{V}$ stands for the set of measurable maps $v : \mathbb{R}^+ \to V$;

(v) the constant $\lambda > 0$ is fixed and K is a closed subset of $\mathbb{R}^n$.

(vi) the following Isaacs condition holds true:

$$\text{for any } X_0 = (x_0^i)_{i\in I} \in (\mathbb{R}^n)^I,\ \xi \in (\mathbb{R}^n)^I \text{ and } p \in \Delta(I), H_p^+(X_0,\xi) = H_p^-(X_0,\xi), \tag{3.1.1}$$

where:

$$H_p^+(X_0,\xi) := \inf_{u\in U}\sup_{v\in V}\big\{\langle (f(x_0^i,u,v))_{i\in I},\xi\rangle + \sum_{i\in I} p(i)\ell(x_0^i,u,v)\big\};$$
$$H_p^-(X_0,\xi) := \inf_{u\in U}\sup_{v\in V}\big\{\langle (f(x_0^i,u,v))_{i\in I},\xi\rangle + \sum_{i\in I} p(i)\ell(x_0^i,u,v)\big\}.$$

Elements of $\mathcal{U}$ (resp. $\mathcal{V}$) are called admissible open-loop controls of Player 1 (resp. of Player 2). In addition, we suppose that both $\mathcal{U}$ and $\mathcal{V}$ are equipped with the corresponding L^∞-topology.

By Hypothesis (ii), for any $(x_0,u,v) \in \mathbb{R}^n \times \mathcal{U} \times \mathcal{V}$, the system (P) has a unique solution defined on $[0,+\infty)$. As in the introduction, to each solution of (P), $t \mapsto X_t^{x_0,u,v}$, we associate the running cost $J(x_0,u,v)$ defined by (3.0.1).

Let us recall a useful property of the dynamic (P). For any $(u,v) \in \mathcal{U}\times\mathcal{V}$ and $(t,x_0,x_0') \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n$, we have:

$$\|X_t^{x_0,u,v} - X_t^{x_0',u,v}\| \le e^{L_f t}\|x_0 - x_0'\|. \tag{3.1.2}$$

To any $(x_0,u,v) \in \mathbb{R}^n \times \mathcal{U} \times \mathcal{V}$, we define the corresponding "hitting time" of system (P) at the closed target set K by $\mathcal{T}(x_0,u,v) := \inf\{t \ge 0 \mid X_t^{x_0,u,v} \in K\}$. Here we allow $\mathcal{T}$ to take its values in $\mathbb{R}\cup\{+\infty\}$. We suppose in addition the following hypothesis:

(vii) For any $0 < T < +\infty$, $\mathcal{T}_T(x_0,u,v) := \min\{\mathcal{T}(x_0,u,v),T\}$ is locally Lipschitz continuous on x_0 uniformly with respect to (u,v). More precisely, for any $x_0 \in \mathbb{R}^n$ and $0 < T < +\infty$, there exists an open neighbourhood B_{x_0} of x_0 and a constant $L_T(B_{x_0}) > 0$ such that for any $(u,v) \in U \times V$ and x, $y \in B_{x_0}$:

$$|\mathcal{T}_T(x,u,v) - \mathcal{T}_T(y,u,v)| \le L_T(B_{x_0})\|x-y\|.$$

When $K = \emptyset$ or $K = \mathbb{R}^n$, the above hypothesis holds. For the case K being non-trivial subset of $\mathbb{R}^n$, we have the following:

Corollary 3.1.1 (cf. [71]). *Let $K = \{x \in \mathbb{R}^n \mid \Phi(x) \le 0\}$ with $\Phi \in \mathcal{C}^2(\mathbb{R}^n;\mathbb{R})$ such that $\nabla\Phi(x) \ne 0$ if $\Phi(x) = 0$. If for any $x \in \partial K$ and any $(u,v) \in U\times V$, $\langle f(x,u,v),\nabla\Phi(x)\rangle < 0$, then for any $T > 0$, the function $x_0 \mapsto \mathcal{T}_T(x_0,u,v)$ is locally Lipschitz continuous uniformly with respect to (u,v).*

Remark 3.1.2. *This lemma shows that Hypothesis (vii) can be fulfilled by setting the boundary of K being "absorbent". We remark that the condition proposed in the above lemma is stronger than those for differential games with state constraints (cf. [22]). This is because in this game model there is not any state constraint.*

Now we introduce the signal space. Let $\mathcal{S}^1$ denote the set of all functions $s^1_T : \mathbb{R}^+ \to \{0,1\}$ with $T \in [0,+\infty]$ such that:

$$s^1_T(t) = \mathbf{1}_{[T,+\infty)}(t).$$

Let $\mathcal{S}^2$ denote the set all functions: $s^2_{x,T} : \mathbb{R}^+ \to \mathbb{R}^n$ with $x \in \mathbb{R}^n$ and $T \in [0,+\infty]$ such that:

$$s^2_{x,T}(t) = x\mathbf{1}_{[T,+\infty)}(t).$$

We define the signal space $\mathcal{S} := \{(s^1_T, s^2_{x,T}) \in \mathcal{S}^1 \times \mathcal{S}^2 \mid T \in [0,+\infty] \text{ and } x \in \mathbb{R}^n\}$. Clearly for any $i \in I$ and $(u,v) \in \mathcal{U} \times \mathcal{V}$, $s_{x^i_0,u,v} \in \mathcal{S}$.

Let us define respectively for $\mathcal{S}^1$ and $\mathcal{S}^2$ a distance by setting :

$$d_{\mathcal{S}^1}(s^1_{T_1}, s^1_{T_2}) = |T_1 - T_2|$$

$$\text{and } d_{\mathcal{S}^2}(s^2_{x_1,T_1}, s^2_{x_2,T_2})\big) = \begin{cases} \|x_1 - x_2\|, & T_1 \neq +\infty \text{ and } T_2 \neq +\infty; \\ 0, & \text{if } T_1 = T_2 = +\infty; \\ +\infty, & \text{if } T_1 = +\infty \text{ and } T_2 \neq +\infty; \\ +\infty, & \text{if } T_1 \neq +\infty \text{ and } T_2 = +\infty.. \end{cases}$$

In this manuscript, we equipped $\mathcal{S}$ with the topology induced by the product topology on $\mathcal{S}^1 \times \mathcal{S}^2$ as product space of the topological metric spaces $(\mathcal{S}^1, d_{\mathcal{S}^1})$ and $(\mathcal{S}^2, d_{\mathcal{S}^2})$. An interpretation of the above distances is that two signal functions are close if and only if the revealing time and the revealed information are both close.

Remark 3.1.3. *As defined above, we allow here the distance between two points to be infinite. In this case, a subset S of $\mathcal{S}^1$ is open if and only if for any $s^1 \in S^1$, there exists $\delta > 0$ such that the open ball $B(s^1;\delta) \subset S$. The sigleton $\{0\} \subset \mathcal{S}^1$ is an open neighbourhood of the constant function equals to 0 and thus both open and closed. For more properties of such metric topological spaces, readers may refer to [13].*

3.2 Strategies and Normal Form

Before we introduce the notion of strategies for game $\mathcal{G}(X_0,p)$, we first recall the following:

Definition 3.2.1 (Non-anticipative Strategies with Delay (in short, NAD Strategies) of Player 1, [21])**.** *An NAD strategy of Player 1 is a Borel-measurable map: $\alpha : \mathcal{V} \to \mathcal{U}$ such that: there exists $\tau_\alpha > 0$, for any v_1, $v_2 \in \mathcal{V}$, $t \geq 0$, if $v_1|_{[0,t]} = v_2|_{[0,t]}$ a.e., then $\alpha(v_1)|_{[0,t+\tau_\alpha]} = \alpha(v_2)|_{[0,t+\tau_\alpha]}$ a.e.. The set of such maps is denoted by $\mathcal{A}_d$.*

We define an NAD strategy of Player 2 similarly and let $\mathcal{B}_d$ denote the set of NAD strategies of Player 2. Below is a useful property of NAD strategies:

Lemma 3.2.2 (cf. [21]). *For any $(\alpha, \beta) \in \mathcal{A}_d \times \mathcal{B}_d$, there exists a unique pair of admissible controls $(u_{\alpha,\beta}, v_{\alpha,\beta})$ such that:*

$$\alpha(v_{\alpha,\beta}) = u_{\alpha,\beta} \text{ and } \beta(u_{\alpha,\beta}) = v_{\alpha,\beta}.$$

Since NAD strategies do not depend on the observation of signals, to characterize the signal structure of game $\mathcal{G}(X_0, p)$, we need a signal-depending version:

Definition 3.2.3 (Signal-depending NAD Strategies of Player 1). *A signal-depending NAD strategy of Player 1 is a Borel-measurable map: $\alpha : \mathcal{S} \times \mathcal{V} \to \mathcal{U}$ such that: there exists $\tau_\alpha > 0$, for any v_1, $v_2 \in \mathcal{V}$, s_1, $s_2 \in \mathcal{S}$ and $t \geq 0$, if $v_1|_{[0,t]} = v_2|_{[0,t]}$ a.e. and $s_1|_{[0,t]} = s_2|_{[0,t]}$, then $\alpha(s_1, v_1)|_{[0,t+\tau_\alpha]} = \alpha(s_2, v_2)|_{[0,t+\tau_\alpha]}$ a.e.. The set of such maps is denoted by $\mathcal{A}_s$.*

Signal-depending NAD strategies of Player 2 can be defined similarly and we denote by $\mathcal{B}_s$ the set of all such strategies of Player 2.

Remark 3.2.4. *It is clear that any NAD strategy can also be viewed as a signal-depending one, thus we have $\mathcal{A}_d \subset \mathcal{A}_s$ and $\mathcal{B}_d \subset \mathcal{B}_s$.*

We recall that, for any $(x_0, u, v) \in \mathbb{R}^n \times \mathcal{U} \times \mathcal{V}$, we denote by $s_{x_0,u,v}$ the signal function observed by the players during game $\mathcal{G}(X_0, p)$ while x_0 is chosen as initial state of dynamic (P) and the pair of controls (u, v) is played. Observe that $s_{x_0,u,v} = (s^1_{\mathcal{T}(x_0,u,v)}, s^2_{X^{x_0,u,v}_{\mathcal{T}(x_0,u,v)}, \mathcal{T}(x_0,u,v)}) \in \mathcal{S}$. In this paper, we denote by $\mathcal{U}(0,t)$ for some $t > 0$ (resp. $\mathcal{V}(0,t)$) the set of all measurable controls $u : [0,t] \mapsto U$ (resp. $v : [0,t] \mapsto V$). We have the following:

Lemma 3.2.5. *For any $(x_0, \alpha, \beta) \in \mathbb{R}^n \times \mathcal{A}_s \times \mathcal{B}_s$, there exists a unique pair of controls $(u_{x_0,\alpha,\beta}, v_{x_0,\alpha,\beta})$ such that:*

$$\alpha(s_{x_0,u_{x_0,\alpha,\beta},v_{x_0,\alpha,\beta}}, v_{x_0,\alpha,\beta}) = u_{x_0,\alpha,\beta} \text{ and } \beta(s_{x_0,u_{x_0,\alpha,\beta},v_{x_0,\alpha,\beta}}, u_{x_0,\alpha,\beta}) = v_{x_0,\alpha,\beta}.$$

Proof. Let us fix (x_0, α, β) with $\tau > 0$ a common delay of α and β. Without loss of generality, we suppose that $x_0 \notin K$ (for if $x_0 \in K$, $s_{x_0,u,v}$ is constant) and we prove the lemma by recurrence.

Since $s_{x_0,u,v}(0) = 0$ for any (u, v) and the pair of strategies (α, β) is non-anticipative with delay τ, there exists $(\bar{u}_0, \bar{v}_0) \in \mathcal{U}(0,\tau) \times \mathcal{V}(0,\tau)$ such that for any $(u, v) \in \mathcal{U} \times \mathcal{V}$:

$$\alpha(s_{x_0,u,v}, v)|_{[0,\tau]} = \bar{u}_0 \text{ and } \beta(s_{x_0,u,v}, u)|_{[0,\tau]} = \bar{v}_0.$$

Let us assume that for $k \in \mathbb{N}_*$, there exists $(\bar{u}_k, \bar{v}_k) \in \mathcal{U}(0, k\tau) \times \mathcal{V}(0, k\tau)$ such that: for any $(u, v) \in \mathcal{U} \times \mathcal{V}$, if $(u, v)|_{[0,(k-1)\tau]} = (\bar{u}_k, \bar{v}_k)|_{[0,(k-1)\tau]}$ a.e., one has:

$$\alpha(s_{x_0,u,v}, v)|_{[0,k\tau]} = \bar{u}_k \text{ and } \beta(s_{x_0,u,v}, u)|_{[0,k\tau]} = \bar{v}_k.$$

We prove that there exists $(\bar{u}_{k+1}, \bar{v}_{k+1}) \in \mathcal{U}\big(0,(k+1)\tau\big) \times \mathcal{V}\big(0,(k+1)\tau\big)$ such that if $(u,v) \in \mathcal{U}\times\mathcal{V}$ satisfies $(u,v)|_{[0,k\tau]} = (\bar{u}_k, \bar{v}_k)$, then

$$\alpha(s_{x_0,u,v}, v)|_{[0,(k+1)\tau]} = \bar{u}_{k+1} \text{ and } \beta(s_{x_0,u,v}, u)|_{[0,(k+1)\tau]} = \bar{v}_{k+1}.$$

Let $\hat{u} \in U$ and $\hat{v} \in V$ be fixed and we denote respectively by $u_k \in \mathcal{U}$ and $v_k \in \mathcal{V}$ the following controls:

$$u_k(t) := \begin{cases} \bar{u}_k(t), & \text{if } t \le k\tau; \\ \hat{u}, & \text{else;} \end{cases} \text{ and } v_k(t) := \begin{cases} \bar{v}_k(t), & \text{if } t \le k\tau; \\ \hat{v}, & \text{else.} \end{cases}$$

If $(u,v)|_{[0,k\tau]} = (\bar{u}_k, \bar{v}_k)$, then we have:

$$s_{x_0,u,v}|_{[0,k\tau]} = s_{x_0,u_k,v_k}|_{[0,k\tau]}.$$

This implies that:

$$\begin{aligned} \alpha(s_{x_0,u,v}, v)|_{[0,(k+1)\tau]} &= \alpha(s_{x_0,u_k,v_k}, v_k)|_{[0,(k+1)\tau]}; \\ \beta(s_{x_0,u,v}, u)|_{[0,(k+1)\tau]} &= \beta(s_{x_0,u_k,v_k}, u_k)|_{[0,(k+1)\tau]}, \end{aligned}$$

and therefore, by defining $\bar{u}_{k+1} := \alpha(s_{x_0,u_k,v_k}, v_k)|_{[0,(k+1)\tau]}$ and $\bar{v}_{k+1} := \beta(s_{x_0,u_k,v_k}, u_k)|_{[0,(k+1)\tau]}$ we have proved our claim.

By the hypothesis of recurrence, for any $m \in \mathbb{N}_*$, there exists a unique pair of controls $(\bar{u}_m, \bar{v}_m) \in \mathcal{U}([0,m\tau]) \times \mathcal{V}([0,m\tau])$ such that if $(u,v) \in \mathcal{U}\times\mathcal{V}$ satisfies $(u,v)|_{[0,m\tau]} = (\bar{u}_m, \bar{v}_m)$, one has:

$$\alpha(s_{x_0,u,v}, v)|_{[0,m\tau]} = \bar{u}_m \text{ and } \beta(s_{x_0,u,v}, u)|_{[0,m\tau]} = \bar{v}_m.$$

Let us define:

$$(u_{x_0,\alpha,\beta}, v_{x_0,\alpha,\beta})(t) := (\bar{u}_m, \bar{v}_m)(t), \text{ if } t \in [0, m\tau).$$

By the uniqueness property of $(\bar{u}_m, \bar{v}_m)$, the pair of controls $(u_{x_0,\alpha,\beta}, v_{x_0,\alpha,\beta})$ is well defined. By definition of $(u_{x_0,\alpha,\beta}, v_{x_0,\alpha,\beta})$, one can easily verify that:

$$\alpha(s_{x_0,u_{x,\alpha,\beta},v_{x,\alpha,\beta}}, v_{x_0,\alpha,\beta}) = u_{x_0,\alpha,\beta} \text{ and } \beta(s_{x_0,u_{x,\alpha,\beta},v_{x,\alpha,\beta}}, u_{x_0,\alpha,\beta}) = v_{x_0\alpha,\beta}.$$

The proof is complete. □

In this manuscript, for any $(x_0,\alpha,\beta) \in \mathbb{R}^n \times \mathcal{A}_s \times \mathcal{B}_s$, we denote by $(\alpha(x_0), \beta(x_0))$ the application $t \mapsto (u_{x_0,\alpha,\beta}, v_{x_0,\alpha,\beta})(t)$. For the function $t \mapsto X_t^{x_0,\alpha(x_0),\beta(x_0)}$, we write $t \mapsto X_t^{x_0,\alpha,\beta}$ for short.

With Lemma 3.2.5, we are able to put the game in a normal form. For any fixed $(X_0,p) \in (\mathbb{R}^n)^I \times \Delta(I)$, we associate to a pair of strategies $(\alpha,\beta) \in \mathcal{A}_s \times \mathcal{B}_s$ the following pay-off:

$$\mathcal{J}(X_0,p,\alpha,\beta) := \sum_{i\in I} p(i) J\big(x_0^i, \alpha(x_0^i), \beta(x_0^i)\big).$$

As mentioned in the introduction, the above function is not necessarily semi-continuous. In fact, the continuity of map $x_0 \mapsto J(x_0, \alpha(x_0), \beta(x_0))$ depends on the pair of strategies (α, β) as shown in the following:

Counterexample 3.2.6. *Let us consider the case where $n = 2$, $K = U = V = \bar{B}(0;1) \subset \mathbb{R}^2$ and:*

$$f(x,u,v) = -x + \frac{1}{3}(u+v);$$
$$\ell(x,u,v) = \langle u, (1,1)\rangle.$$

Obviously f and ℓ verifies Hypothesis (ii), (iii) and (vii). Let us consider a strategy of Player 1 $\alpha \in \mathcal{A}_s$ with small delay $\tau > 0$ defined as follows: for any $x_0 = (x_{0,1}, x_{0,2}) \in \mathbb{R}^2$, $(s^1_T, s^2_{x_0,T}) \in \mathcal{S}$ and $v \in \mathcal{V}$,

$$\alpha\big((s^1_T, s^2_{x,T}), v\big)(t) = \begin{cases} (0,0), & \text{if } T > 1 \text{ or } t < 1+\tau; \\ (0,1), & \text{if } T \le 1,\ t \ge 1+\tau \text{ and } x_{0,2} > 0; \\ (0,-1), & \text{if } T \le 1,\ t \ge 1+\tau \text{ and } x_{0,2} < 0; \\ (0,0), & \text{if } T \le 1,\ t \ge 1+\tau \text{ and } x_{0,2} = 0 \end{cases}$$

Let us consider two sequences: $\big\{e^2(\cos\frac{1}{n}, \sin\frac{1}{n})\big\}_{n\in\mathbb{N}_*}$ and $\big\{e^2(\cos -\frac{1}{n}, \sin -\frac{1}{n})\big\}_{n\in\mathbb{N}_*}$. It is clear that both sequences has as limit $(e^2, 0)$. For any $n \in \mathbb{N}_*$, let $x_n = e^2(\cos\frac{1}{n}, \sin\frac{1}{n})$ and $y_n = e^2(\cos -\frac{1}{n}, -\sin\frac{1}{n})$. For $v = (0,0)$ the constant control, we have:

$$J(x_n, \alpha, v) = \int_{1+\tau}^{+\infty} e^{-\lambda t}(-1)dt = -\frac{1}{\lambda}e^{-\lambda(1+\tau)};$$
$$J(y_n, \alpha, v) = \int_{1+\tau}^{+\infty} e^{-\lambda t}dt = \frac{1}{\lambda}e^{-\lambda(1+\tau)};$$
$$J\big((e^2,0), \alpha, v\big) = 0.$$

Thus $J(\cdot, \alpha, v)$ is neither upper semi-continuous nor lower semi-continuous at $(e^2, 0)$.

Before we move on to the next section, we introduce a new signal function which is an important technical tool in Section 3. Let us fix $T > 0$. Suppose that during game $\mathcal{G}(X_0, p)$, the current state of (P) at $t = T$ will be revealed publicly if the target set K has not been hit before T. This means that instead of $s_{x_0,u,v}$, both players observe, for $(x_0, T, u, v) \in \mathbb{R}^n \times (0, +\infty) \times \mathcal{U} \times \mathcal{V}$, a function $s_{x_0,T,u,v} \in \mathcal{S}$ defined as below:

$$s_{x_0,T,u,v}(t) = \begin{cases} (0,0), & \text{if } t < \mathcal{T}_T(x_0,u,v); \\ (1, X^{x_0,u,v}_{\mathcal{T}_T(x_0,u,v)}), & \text{else.} \end{cases}$$

As direct consequence of Lemma 3.2.5, we have the following:

Corollary 3.2.7. *For any $(x_0, T, \alpha, \beta) \in \mathbb{R}^n \times (0, +\infty) \times \mathcal{A}_s \times \mathcal{B}_s$, there exists a unique pair of controls $(u_{x_0,T,\alpha,\beta}, v_{x_0,T,\alpha,\beta})$ such that:*

$$\alpha(s_{x_0,T,u_{x_0,T,\alpha,\beta},v_{x_0,T,\alpha,\beta}}, v_{x_0,T,\alpha,\beta}) = u_{x_0,T,\alpha,\beta} \text{ and } \beta(s_{x_0,T,u_{x_0,T,\alpha,\beta},v_{x_0,T,\alpha,\beta}}, u_{x_0,T,\alpha,\beta}) = v_{x_0,T,\alpha,\beta}.$$

In addition, we have $u_{x_0,\alpha,\beta}|_{[0,T]} = u_{x_0,T,\alpha,\beta}|_{[0,T]}$ and $v_{x_0,\alpha,\beta}|_{[0,T]} = v_{x_0,T,\alpha,\beta}|_{[0,T]}$ a.e. on $[0,T]$ with $(u_{x_0,\alpha,\beta}, v_{x_0,\alpha,\beta})$ defined as in Lemma 3.2.5.

We denote by $(\alpha^T(x_0), \beta^T(x_0))$ the application $t \mapsto (u_{x_0,T,\alpha,\beta}, v_{x_0,T,\alpha,\beta})(t)$. Similarly, $t \mapsto X_t^{x_0,\alpha^T(x_0),\beta^T(x_0)}$ is denoted by $t \mapsto X_t^{x_0,\alpha^T,\beta^T}$ for short. The above corollary allows us to define the following function:

$$\mathcal{J}_T(X_0, p, \alpha, \beta) := \sum_{i \in I} p(i) J\big(x_0^i, \alpha^T(x_0^i), \beta^T(x_0^i)\big).$$

The family $(\mathcal{J}_T)_{T>0}$ provides an approximation to the cost $\mathcal{J}$. In fact, we have:

Remark 3.2.8. *It is clear that the functions $\mathcal{J}_T$ converge uniformly to $\mathcal{J}$ as T tends to $+\infty$. More precisely, for any $(X_0, p, \alpha, \beta) \in (\mathbb{R}^n)^I \times \Delta(I) \times \mathcal{A}_s \times \mathcal{B}_s$, one has:*

$$|\mathcal{J}_T(X_0, p, \alpha, \beta) - \mathcal{J}(X_0, p, \alpha, \beta)| \leq 2\|\ell\|_\infty \int_T^{+\infty} e^{-\lambda t} dt = \frac{2\|\ell\|_\infty e^{-\lambda T}}{\lambda}.$$

3.3 Value Functions

Since we have put game $\mathcal{G}(X_0, p)$ into normal form in the previous section, we can define the upper-value function and the lower-value function of the game respectively by:

$$V^+(X_0, p) = \inf_{\alpha \in \mathcal{A}_s} \sup_{\beta \in \mathcal{B}_s} \mathcal{J}(X_0, p, \alpha, \beta);$$
$$V^-(X_0, p) = \sup_{\beta \in \mathcal{B}_s} \inf_{\alpha \in \mathcal{A}_s} \mathcal{J}(X_0, p, \alpha, \beta).$$

It is clear that $V^+ \geq V^-$. We say that game $\mathcal{G}(X_0, p)$ has a value if in addition $V^+ = V^-$.

Let the functions $\tilde{V}^+ : \mathbb{R}^n \to \mathbb{R}$ and $\tilde{V}^- : \mathbb{R}^n \to \mathbb{R}$ be defined as follows:

$$\tilde{V}^+(x_0) := \inf_{\alpha \in \mathcal{A}_d} \sup_{\beta \in \mathcal{B}_d} J(x_0, \alpha, \beta);$$
$$\tilde{V}^-(x_0) := \sup_{\beta \in \mathcal{B}_d} \inf_{\alpha \in \mathcal{A}_d} J(x_0, \alpha, \beta).$$

These above functions are respectively the upper- and lower-value functions of a class of infinite horizon differential games with complete information. We recall that both $\tilde{V}^+$ and $\tilde{V}^-$ are locally Hölder continuous and under Isaacs' condition (3.1.1), $\tilde{V}^+ = \tilde{V}^-$ cf. [3, p. 431-470]. The following lemma gives a relation between $V^\pm$ and $\tilde{V}^\pm$.

Lemma 3.3.1. *Let $X_0 \in (\mathbb{R}^n)^I$ and $I' \subset I$ be the set of all $j \in I$ such that $x_0^j \notin K$. Then:*

$$V^+(X_0,p) = \inf_{\alpha\in\mathcal{A}_s}\sup_{\beta\in\mathcal{B}_s}\Big\{\sum_{j\in I'} p(j)J\big(x_0^j,\alpha(x_0^j),\beta(x_0^j)\big)\Big\} + \sum_{i\in I\setminus I'} p(i)\tilde{V}^+(x_0^i); \tag{3.3.1}$$

$$V^-(X_0,p) = \sup_{\beta\in\mathcal{B}_s}\inf_{\alpha\in\mathcal{A}_s}\Big\{\sum_{j\in I'} p(j)J\big(x_0^j,\alpha(x_0^j),\beta(x_0^j)\big)\Big\} + \sum_{i\in I\setminus I'} p(i)\tilde{V}^-(x_0^i). \tag{3.3.2}$$

From this lemma and the fact that $\tilde{V}^+ = \tilde{V}^-$, we deduce that

$$V^+(X_0,p) = V^-(X_0,p), \text{ for any } (X_0,p) \in K^I \times \Delta(I). \tag{3.3.3}$$

Proof of Lemma 3.3.1. We only need to prove (3.3.1) since the other equality can be established symmetrically. Without loss of generality, let us suppose that $x_0^1 \in K$ and that $x_0^1 \neq x_0^i$ for any $i \in I\setminus\{1\}$ (If this is not true, we only need to reindex them).

Let us fix $\epsilon > 0$ small enough. For any $x \in \mathbb{R}^n$, let $\alpha_x \in \mathcal{A}_d$ be a $\frac{\epsilon}{2}$-optimal strategy for $\tilde{V}(x)$. Thus there exists $\delta_x > 0$ such that $\alpha_x \in \mathcal{A}_d$ is still a ϵ-optimal strategy for $\tilde{V}(y)$ with $y \in B(x;\delta_x)$. For any $\alpha \in \mathcal{A}_s$, let us construct a new strategy $\alpha_{\epsilon,n} \in \mathcal{A}_s$ for $n \in \mathbb{N}$ big enough by setting: for any $(s_T^1, s_{x,T}^2) \in \mathcal{S}$ and $v \in \mathcal{V}$:

$$\alpha_{\epsilon,n}\big((s_T^1,s_{x,T}^2),v\big)(t) = \begin{cases} \alpha\big((s_T^1,s_{x,T}^2),v\big)(t), & \text{if } T=0 \text{ and } t < \dfrac{1}{n};\\ \alpha_{x_0^1}\big(v(\cdot+\dfrac{1}{n})\big)(t-\dfrac{1}{n}), & \text{if } T=0 \text{ and } t \geq \dfrac{1}{n} \text{ and } x = x_0^1;\\ \alpha\big((s_T^1,s_{x,T}^2),v\big)(t), & \text{else.}\end{cases}$$

One can check that, for n big enough, this strategy belongs to $\mathcal{A}_s$ and in addition, $X_{\frac{1}{n}}^{x_0^1,u,v} \in B(x_0^1;\delta_{x_0^1})$ with $X_{\frac{1}{n}}^{x_0^i,u,v} \notin B(x_0^1;\delta_{x_0^1})$ for any $i \neq 1$ and $(u,v) \in \mathcal{U}\times\mathcal{V}$. Similarly, we can construct for any $\beta \in \mathcal{B}_s$ and $\beta_0 \in \mathcal{B}_d$ a new strategy $\beta_{n,\beta_0} \in \mathcal{B}_s$ such that, for any $(s_T^1, s_{x,T}^2) \in \mathcal{S}^0$ and $u \in \mathcal{U}$:

$$\beta_{n,\beta_0}\big((s_T^1,s_{x,T}^2),u\big)(t) = \begin{cases} \beta\big((s_T^1,s_{x,T}^2),u\big)(t), & \text{if } T=0 \text{ and } t < \dfrac{1}{n};\\ \beta_0\big(u(\cdot+\dfrac{1}{n})\big)(t-\dfrac{1}{n}), & \text{if } T=0 \text{ and } t \geq \dfrac{1}{n} \text{ and } x = x_0^1;\\ \beta\big((s_T^1,s_{x,T}^2),u\big)(t), & \text{else.}\end{cases}$$

Thus we have, if $\alpha = \alpha_\epsilon$ with α_ϵ being a ϵ-optimal strategy for $V^+(X_0, p)$, namely:

$$
\begin{aligned}
V^+(X_0,p) &\geq \sup_{\beta\in\mathcal{B}_s} \mathcal{J}(X_0,p,\alpha_\epsilon,\beta) - \epsilon = \sup_{\beta\in\mathcal{B}_s} \sum_{i\in I} p(i) J\big(x_0^i, \alpha_\epsilon(x_0^i), \beta(x_0^i)\big) - \epsilon \\
&\geq \sup_{\beta\in\mathcal{B}_s} \sup_{\beta_0\in\mathcal{B}_d} \Big\{ \sum_{i>1} p(i) J\big(x_0^i, \alpha_\epsilon(x_0^i), \beta_{n,\beta_0}(x_0^i)\big) + p(1) J\big(x_0^1, \alpha_\epsilon(x_0^1), \beta_{n,\beta_0}\big((s_0^1, s^2_{0,x_0^1}), \cdot\big)\Big\} - \epsilon \\
&\geq \sup_{\beta\in\mathcal{B}_s} \Big\{ \sum_{i>1} p(i) J\big(x_0^i, \alpha_\epsilon(x_0^i), \beta(x_0^i)\big) + p(1) \sup_{\beta_0\in\mathcal{B}_d} \big(\int_0^{\frac{1}{n}} e^{-\lambda t} \ell(X^{x_0^1,\alpha_\epsilon(x_0^1),\beta(x_0^1)}, \alpha_\epsilon(x_0^1), \beta(x_0^1)) dt + \\
&\quad e^{-\frac{\lambda}{n}} J(X^{x_0^1,\alpha_\epsilon(x_0^1),\beta(x_0^1)}_{\frac{1}{n}}, \alpha_{x_0^1}, \beta_0)\big)\Big\} - \epsilon \\
&\geq \sup_{\beta\in\mathcal{B}_s} \Big\{ \sum_{i>1} p(i) J\big(x_0^i, \alpha_\epsilon(x_0^i), \beta(x_0^i)\big) + p(1)\big(-\frac{1}{n}\|\ell\|_\infty + \\
&\quad e^{-\frac{\lambda}{n}} \inf_{\alpha_0\in\mathcal{A}_d} \sup_{\beta_0\in\mathcal{B}_d} J(X^{x_0^1,\alpha_\epsilon(x_0^1),\beta(x_0^1)}_{\frac{1}{n}}, \alpha_0, \beta_0)\big)\Big\} - 2\epsilon \\
&\geq \inf_{\alpha\in\mathcal{A}_s} \sup_{\beta\in\mathcal{B}_s} \Big\{ \sum_{i>1} p(i) J\big(x_0^i, \alpha(x_0^i), \beta(x_0^i)\big) + p(1) e^{-\frac{\lambda}{n}} \tilde{V}^+(X^{x_0^1,\alpha,\beta}_{\frac{1}{n}})\Big\} - 2\epsilon - \frac{\|\ell\|_\infty}{n}
\end{aligned}
$$

By passing $n \to +\infty$ (this is possible by the continuity of $\tilde{V}^+$ and that of $t \mapsto X_t^{x_0^1,\alpha,\beta}$), we obtain:

$$
V^+(X_0,p) \geq \inf_{\alpha\in\mathcal{A}_s} \sup_{\beta\in\mathcal{B}_s} \Big\{ \sum_{i>1} p(i) J\big(x_0^i, \alpha(x_0^i), \beta(x_0^i)\big) + p(1)\tilde{V}^+(x_0^1)\Big\} - 2\epsilon,
$$

and since $\epsilon > 0$ is arbitrary, we have:

$$
V^+(X_0,p) \geq \inf_{\alpha\in\mathcal{A}_s} \sup_{\beta\in\mathcal{B}_s} \Big\{ \sum_{i>1} p(i) J\big(x_0^i, \alpha(x_0^i), \beta(x_0^i)\big)\Big\} + p(1)\tilde{V}^+(x_0^1).
$$

Conversely, for any $\alpha \in \mathcal{A}_s$, we have:

$$
\begin{aligned}
V^+(X_0,p) &\leq \sup_{\beta\in\mathcal{B}_s} \mathcal{J}(X_0,p,\alpha_n^\epsilon,\beta) = \sup_{\beta\in\mathcal{B}_s} \sum_{i\in I} p(i) J\big(x_0^i, \alpha_n^\epsilon(x_0^i), \beta(x_0^i)\big) \\
&\leq \sup_{\beta\in\mathcal{B}_s} \Big\{ \sum_{i>1} p(i) J\big(x_0^i, \alpha(x_0^i), \beta(x_0^i)\big) + p(1) J\big(x_0^1, \alpha_n^\epsilon(x_0^1), \beta\big((s_0^1, s^2_{0,x_0^1}), \cdot\big)\Big\} \\
&\leq \sup_{\beta\in\mathcal{B}_s} \Big\{ \sum_{i>1} p(i) J\big(x_0^i, \alpha(x_0^i), \beta(x_0^i)\big) + p(1)\big(\frac{1}{n}\|\ell\|_\infty + \\
&\quad e^{-\frac{\lambda}{n}} \inf_{\alpha_0\in\mathcal{A}_d} \sup_{\beta_0\in\mathcal{B}_d} J(X^{x_0^1,\alpha,\beta}_{\frac{1}{n}}, \alpha_0, \beta_0)\big)\Big\} + 2\epsilon \\
&\leq \sup_{\beta\in\mathcal{B}_s} \Big\{ \sum_{i>1} p(i) J\big(x_0^i, \alpha(x_0^i), \beta(x_0^i)\big) + p(1) e^{-\frac{\lambda}{n}} \tilde{V}^+(X^{x_0^1,\alpha,\beta}_{\frac{1}{n}})\Big\} + 2\epsilon + \frac{\|\ell\|_\infty}{n}
\end{aligned}
$$

Passing to the infimum of α of both sides and passing $n \to +\infty$, $\epsilon \to 0^+$, we obtain:

$$V^+(X_0,p) \leq \inf_{\alpha\in\mathcal{A}_s}\sup_{\beta\in\mathcal{B}_s}\Big\{\sum_{i>1}p(i)J\big(x_0^i,\alpha(x_0^i),\beta(x_0^i)\big)\Big\}+p(1)\tilde{V}^+(x_0^1).$$

Thus:

$$V^+(X_0,p) = \inf_{\alpha\in\mathcal{A}_s}\sup_{\beta\in\mathcal{B}_s}\Big\{\sum_{i>1}p(i)J\big(x_0^i,\alpha(x_0^i),\beta(x_0^i)\big)\Big\}+p(1)\tilde{V}^+(x_0^1).$$

By iterating the above procedure, we obtain the wished result. The proof is complete. □

Now we state several corollaries of Lemma 3.3.1 that are useful in later sections.

Corollary 3.3.2. *For any $(X_0,p)\in(\mathbb{R}^n)^I\times\Delta(I)$, let I' be defined as in Lemma 3.3.1. If $supp(p)\cap I'\neq\emptyset$, there exists $p'\in\Delta(I)$ with $supp(p')\subset I'\cap supp(p)$ such that:*

$$V^+(X_0,p) = (1-\sum_{i\in I\setminus I'}p(i))V^+(X_0,p')+\sum_{i\in I\setminus I'}p(i)\tilde{V}^+(x_0^i); \tag{3.3.4}$$

$$V^-(X_0,p) = (1-\sum_{i\in I\setminus I'}p(i))V^-(X_0,p')+\sum_{i\in I\setminus I'}p(i)\tilde{V}^-(x_0^i). \tag{3.3.5}$$

Proof. Let $p'\in\Delta(I)$ with $p'(i)=\frac{p(i)}{\sum_{j\in I'}p(j)}$ for all $i\in I'$. Then (3.3.4) and (3.3.5) follow as direct consequences of (3.3.1) and (3.3.7). The proof is complete. □

Correspondingly to the upper- and lower-value functions, let us define, for any $T>0$, the following two functions:

$$V_T^+(X_0,p) := \inf_{\alpha\in\mathcal{A}_s}\sup_{\beta\in\mathcal{B}_s}\mathcal{J}_T(X_0,p,\alpha,\beta);$$
$$V_T^-(X_0,p) := \sup_{\beta\in\mathcal{B}_s}\inf_{\alpha\in\mathcal{A}_s}\mathcal{J}_T(X_0,p,\alpha,\beta).$$

By Remark 3.2.8 and Corollary 3.3.1, we have:

Corollary 3.3.3. *The functions $V_T^+(X_0,p)$ converge uniformly to $V^+(X_0,p)$ as T tends to $+\infty$. The functions $V_T^-(X_0,p)$ converge uniformly to $V^-(X_0,p)$ as T tends to $+\infty$. Furthermore, for any $T>0$ and $(X_0,p)\in(\mathbb{R}^n)^I\times\Delta(I)$:*

$$V_T^+(X_0,p) = \inf_{\alpha\in\mathcal{A}_s}\sup_{\beta\in\mathcal{B}_s}\Big\{\sum_{j\in I'}p(j)J\big(x_0^j,\alpha^T(x_0^j),\beta^T(x_0^j)\big)\Big\}+\sum_{i\in I\setminus I'}p(i)\tilde{V}^+(x_0^i); \tag{3.3.6}$$

$$V_T^-(X_0,p) = \sup_{\beta\in\mathcal{B}_s}\inf_{\alpha\in\mathcal{A}_s}\Big\{\sum_{j\in I'}p(j)J\big(x_0^j,\alpha^T(x_0^j),\beta^T(x_0^j)\big)\Big\}+\sum_{i\in I\setminus I'}p(i)\tilde{V}^-(x_0^i). \tag{3.3.7}$$

We have seen in Example 3.2.6 that the cost $X_0 \mapsto \mathcal{J}(X_0, p, \alpha, \beta)$ could be neither upper semi-continuous nor lower semi-continuous given certain pair of strategies (α, β). This is because with a pair of signal-depending NAD strategies $(\alpha, \beta) \in \mathcal{A}_s \times \mathcal{B}_s$, the application $x_0 \mapsto \big(\alpha(x_0), \beta(x_0)\big)$ is not necessarily continuous and thus one can not deduct the continuity of the value functions from that of the cost function. But to obtain the dynamic programming principle for these value functions, we need to prove that they are both continuous on X_0. To achieve this, we will need the following:

3.3.1 Two-step Programming Principle

We set $e^{-(+\infty)} = 0$ and for any $a > 0$, $a * (+\infty) = +\infty$. While $T = +\infty$, we set $\mathcal{T}_T = \mathcal{T}$.

Lemma 3.3.4 (Two-step Programming Principle)**.** *For any $(X_0, p) \in (\mathbb{R}^n)^I \times \Delta(I)$ and $T \in [0, +\infty]$ one has:*

$$
\begin{aligned}
V_T^+(X_0, p) = \inf_{\alpha \in \mathcal{A}_d} \sup_{\beta \in \mathcal{B}_d} \sum_{i \in I} p(i) \Big\{ \int_0^{\mathcal{T}_T(x_0^i, \alpha, \beta)} & e^{-\lambda t} \ell\big(X_t^{x_0^i, \alpha, \beta}, \alpha, \beta\big) dt + \\
& e^{-\lambda \mathcal{T}_T(x_0^i, \alpha, \beta)} \tilde{V}^+\big(X^{x_0^i, \alpha, \beta}_{\mathcal{T}_T(x_0^i, \alpha, \beta)}\big) \Big\}; \\
V_T^-(X_0, p) = \sup_{\beta \in \mathcal{B}_d} \inf_{\alpha \in \mathcal{A}_d} \sum_{i \in I} p(i) \Big\{ \int_0^{\mathcal{T}_T(x_0^i, \alpha, \beta)} & e^{-\lambda t} \ell\big(X_t^{x_0^i, \alpha, \beta}, \alpha, \beta\big) dt + \\
& e^{-\lambda \mathcal{T}_T(x_0^i, \alpha, \beta)} \tilde{V}^-\big(X^{x_0^i, \alpha, \beta}_{\mathcal{T}_T(x_0^i, \alpha, \beta)}\big) \Big\}.
\end{aligned}
$$

Proof. By (3.3.6) and (3.3.7), we only need to prove the lemma for $X_0 \in (\mathbb{R}^n \backslash K)^I$ and $T > 0$. We prove here the first equality and the second one can be obtained similarly. Let us denote the right-hand side of the first equality by $W_T^+(X_0, p)$

Step 1: $V_T^+ \leq W_T^+$ for $T \in (0, +\infty)$. Let us fix $T \in (0, +\infty)$ and let $\epsilon > 0$ be arbitrary. We choose for any $x \in \mathbb{R}^n$, $\alpha_x \in \mathcal{A}_d$ a ϵ-optimal strategy for $\tilde{V}^+(x)$. By the continuity of $\tilde{V}^+$, there exists, for any $x \in \mathbb{R}^n$, $\delta_x > 0$, such that $\alpha_x \in \mathcal{A}_d$ is still a 2ϵ-optimal strategy of $\tilde{V}^+$ at any $y \in \bar{B}(x; \delta_x)$. We can therefore construct a finite Borel partition $\{E_j\}_{1 \leq j \leq m}$ of $\cup_{i \in I} \bar{B}(x_0^i; T\|f\|_\infty) = \cup_{j=1}^m E_j$ with $m \in \mathbb{N}_*$ such that for any $1 \leq j \leq m$, there exists $x_j \in \mathbb{R}^n$ with $E_j \subset B(x_j; \frac{\delta_{x_j}}{2})$. We denote simply by α^j the strategy α_{x_j} for $1 \leq j \leq m$ and we define $E_0 = (\cup_{j=1}^m E_j)^C$. For $\bar{\alpha} \in \mathcal{A}_d$, let us construct a strategy $\alpha \in \mathcal{A}_s$. Without loss of generality, we can choose $\tau \leq \min_{1 \leq j \leq m}(\frac{\delta_{x_j}}{2\|f\|_\infty})$ a small enough common delay of $(\alpha_j)_{1 \leq j \leq m}$

and $\bar{\alpha}$. We set, for any $(s^1_R, s^2_{x,R}) \in \mathcal{S}$ and $v \in \mathcal{V}$:

$$\alpha\big((s^1_R, s^2_{x,R}), v\big)(t) = \begin{cases} \bar{\alpha}(v)(t), & \text{if } x \in E_0 \text{ or } R \in \{0, +\infty\}; \\ \bar{\alpha}(v)(t), & \text{if } R \in [\frac{k}{2}\tau, \frac{k+1}{2}\tau),\ t < \frac{2+k}{2}\tau; \\ \alpha_{x_j}\big(v(\cdot + \frac{2+k}{2}\tau)\big)(t - \frac{2+k}{2}\tau), & \text{if } R \in [\frac{k}{2}\tau, \frac{k+1}{2}\tau),\ t \geq \frac{2+k}{2}\tau \\ & \text{and } x \in E_j. \end{cases}$$

One can check that $\alpha \in \mathcal{A}_s$. Let $k_R \in \mathbb{N}$ be such that $R \in [\frac{k_R}{2}\tau, \frac{k_R+1}{2}\tau)$ in the above definition of α. By definition of V^+_T, we have:

$$\begin{aligned} V^+_T(X_0, p) &\leq \sup_{\beta \in \mathcal{B}_s} \mathcal{J}_T(X_0, p, \alpha, \beta) \\ &= \sup_{\beta \in \mathcal{B}_s} \sum_{i \in I} p(i) \int_0^{+\infty} e^{-\lambda t} \ell\big(X_t^{x^i_0, \alpha^T, \beta^T}, \alpha^T(x^i_0), \beta^T(x^i_0)\big) dt \end{aligned}$$

Let us denote, for any $\beta \in \mathcal{B}_s$, the application $u \mapsto \beta\big((0,0), u\big)$ by $u \mapsto \beta(0, u)$. One can check that $\beta(0, \cdot) \in \mathcal{B}_d$. Let $T^i(\hat{\beta})$ denote the hitting time $\mathcal{T}_T(x^i_0, \bar{\alpha}, \hat{\beta})$ for any $\hat{\beta} \in \mathcal{B}_d$ and we have:

$$\begin{aligned} V^+_T(X_0, p) \leq \sup_{\beta \in \mathcal{B}_s} \sum_{i \in I} p(i) \Big\{ & \int_0^{\frac{k_{T^i(\beta(0,\cdot))}+2}{2}\tau} e^{-\lambda t} \ell\big(X_t^{x^i_0, \alpha^T, \beta^T}, \alpha^T(x^i_0), \beta^T(x^i_0)\big) dt + \\ & \int_{\frac{k_{T^i(\beta(0,\cdot))}+2}{2}\tau}^{+\infty} e^{-\lambda t} \ell\big(X_t^{x^i_0, \alpha^T, \beta^T}, \alpha^T(x^i_0), \beta^T(x^i_0)\big) dt \Big\} \\ \leq \sup_{\beta \in \mathcal{B}_s} \sum_{i \in I} p(i) \Big\{ & \int_0^{T^i(\beta(0,\cdot))} e^{-\lambda t} \ell\big(X_t^{x^i_0, \bar{\alpha}, \beta(0,\cdot)}, \bar{\alpha}, \beta(0,\cdot)\big) dt + \tau \|\ell\|_\infty + \\ & \int_{\frac{k_{T^i(\beta(0,\cdot))}+2}{2}\tau}^{+\infty} e^{-\lambda t} \ell\big(X_t^{x^i_0, \alpha^T, \beta^T}, \alpha^T(x^i_0), \beta^T(x^i_0)\big) dt \Big\} \end{aligned}$$

The last inequality above holds since $\big(\alpha^T(x), \beta^T(x)\big)|_{[0, \mathcal{T}_T(x, \alpha, \beta)]} = \big(\bar{\alpha}, \beta(0,\cdot)\big)|_{[0, \mathcal{T}_T(x, \alpha, \beta)]}$ for any $\beta \in \mathcal{B}_s$.

For any $\beta \in \mathcal{B}_s$, let us define, for any $i \in I$ and $u \in \mathcal{U}$:

$$\beta^i(u)(t) = \beta(s_{x^i_0, \alpha, \beta}, \tilde{u})(t + \frac{k_{T^i(\beta(0,\cdot)} + 2}{2}\tau),$$

$$\text{where } \tilde{u}(t) = \begin{cases} u_{\bar{\alpha}, \beta(0,\cdot)}, & \text{if } t < \frac{2 + k_{T^i(\beta(0,\cdot)}}{2}\tau; \\ u(t - \frac{2 + k_{T^i(\beta(0,\cdot)}}{2}\tau), & \text{if } t \geq \frac{2 + k_{T^i(\beta(0,\cdot)}}{2}\tau. \end{cases}$$

One can check that for any $i \in I$, $\beta^i \in \mathcal{B}_d$. Thus, by denoting $X_i^{\bar{\alpha},\beta} := X^{x_0^i,\bar{\alpha},\beta(0,\cdot)}_{\frac{k_{T^i(\beta(0,\cdot)}+2}{2}\tau}$, we have:

$$\begin{aligned}
V_T^+(X_0,p) \leq & \tau\|\ell\|_\infty + \sup_{\beta\in\mathcal{B}_s}\sum_{i\in I} p(i)\Big\{\int_0^{T^i(\beta(0,\cdot)} e^{-\lambda t}\ell\big(X_t^{x_0^i,\bar{\alpha},\beta(0,\cdot)},\bar{\alpha},\beta(0,\cdot)\big)dt + \\
& \sum_{1\leq j\leq m} e^{-\frac{k_{T^i(\beta(0,\cdot)}+2}{2}\lambda\tau}\mathbf{1}_{E_j}(X_{T_i}^{x_0^i,\bar{\alpha},\beta(0,\cdot)})\int_0^{+\infty} e^{-\lambda t}\ell\big(X_t^{X_i^{\bar{\alpha},\beta},\alpha_{x_j},\beta^i},\alpha_{x_j},\beta^i\big)dt\Big\} \\
\leq & \sup_{\beta\in\mathcal{B}_s}\sum_{i\in I} p(i)\Big\{\int_0^{T^i(\beta(0,\cdot)} e^{-\lambda t}\ell\big(X_t^{x_0^i,\bar{\alpha},\beta(0,\cdot)},\bar{\alpha},\beta(0,\cdot)\big)dt + \\
& e^{-\frac{k_{T^i(\beta(0,\cdot)}+2}{2}\lambda\tau}\tilde{V}^+(X_i^{\bar{\alpha},\beta(0,\cdot)})\Big\} + \tau\|\ell\|_\infty + \epsilon
\end{aligned} \tag{3.3.8}$$

Since $|\frac{(k_{T^i(\beta(0,\cdot)}+2)\tau}{2} - T^i(\beta(0,\cdot)| \leq \tau$. We obtain:

$$\lim_{\tau\to 0^+}\frac{(k_{T^i(\beta(0,\cdot)}+2)\tau}{2} = T^i(\beta(0,\cdot).$$

Since the above convergence is uniform, passing τ to the limit as $\tau \to 0^+$ in inequality (3.3.8):

$$\begin{aligned}
& V_T^+(X_0,p) \\
\leq & \sup_{\beta\in\mathcal{B}_s}\sum_{i\in I} p(i)\Big\{\int_0^{\mathcal{T}_T(x_0^i,\bar{\alpha},\beta(0,\cdot))} e^{-\lambda t}\ell\big(X_t^{x_0^i,\bar{\alpha},\beta(0,\cdot)},\bar{\alpha},\beta(0,\cdot)\big)dt + \\
& e^{-\lambda\mathcal{T}_T(x_0^i,\bar{\alpha},\beta(0,\cdot))}\tilde{V}^+(X^{x_0^i,\bar{\alpha},\beta(0,\cdot)}_{\mathcal{T}_T(x_0^i,\bar{\alpha},\beta(0,\cdot))})\Big\} + \epsilon \\
\leq & \sup_{\beta\in\mathcal{B}_d}\sum_{i\in I} p(i)\Big\{\int_0^{\mathcal{T}_T(x_0^i,\bar{\alpha},\beta)} e^{-\lambda t}\ell\big(X_t^{x_0^i,\bar{\alpha},\beta},\bar{\alpha},\beta\big)dt + e^{-\lambda\mathcal{T}_T(x_0^i,\bar{\alpha},\beta)}\tilde{V}^+(X^{x_0^i,\bar{\alpha},\beta}_{\mathcal{T}_T(x_0^i,\bar{\alpha},\beta)})\Big\} + \epsilon.
\end{aligned}$$

Since $\bar{\alpha} \in \mathcal{A}_d$ and $\epsilon > 0$ are arbitrary, by taking the infimum on $\bar{\alpha} \in \mathcal{A}_s$ on the right-hand side of the above inequality and passing $\epsilon \to 0^+$, we obtain:

$$V_T^+(X_0,p) \leq W_T^+(X_0,p).$$

Let us prove the reverse direction of the above inequality.
Step 2: $V_T^+ \leq W_T^+$ for $T \in (0,+\infty)$ Let us choose, for $\epsilon > 0$, $\alpha_\epsilon \in \mathcal{A}_s$, a ϵ-optimal strategy for $V_T^+(X_0,p)$. One has then:

$$V_T^+(X_0,p) \geq \sup_{\beta\in\mathcal{B}_s}\mathcal{J}_T(X_0,p,\alpha_\epsilon,\beta) - \epsilon.$$

Let us fix $\bar{\beta} \in \mathcal{B}_d$. For any $B = (\beta_i)_{1\le i\le I} \in \mathcal{B}_d^I$ a family of NAD strategies, we can construct a strategy $\beta_B^\tau \in \mathcal{B}_s$ by setting, with a small enough $\tau > 0$: for any $(s_R^1, s_{x,R}^2) \in \mathcal{S}$ and $u \in \mathcal{U}$,

$$\beta_B^\tau\big((s_R^1, s_{x,R}^2), u\big)(t) = \begin{cases} \bar{\beta}(u)(t), & \text{if } R \in \{0, +\infty\}; \\ \bar{\beta}(u)(t), & \text{if } R \in [\frac{k}{2}\tau, \frac{k+1}{2}\tau),\ t < \frac{2+k}{2}\tau; \\ \bar{\beta}(u)(t), & \text{if } R \in [\frac{k}{2}\tau, \frac{k+1}{2}\tau),\ t \ge \frac{2+k}{2}\tau \\ & \text{and } x \ne X^{x_0^i,\alpha_\epsilon,\bar{\beta}}_{\mathcal{T}_T(x_0^i,\alpha_\epsilon,\bar{\beta})},\ \forall i \in I; \\ \beta_i\big(u(\cdot + \frac{2+k}{2}\tau)\big)(t - \frac{2+k}{2}\tau), & \text{if } R \in [\frac{k}{2}\tau, \frac{k+1}{2}\tau),\ t \ge \frac{2+k}{2}\tau \\ & \text{and } x = X^{x_0^i,\alpha_\epsilon,\bar{\beta}}_{\mathcal{T}_T(x_0^i,\alpha_\epsilon,\bar{\beta})}. \end{cases}$$

One can check that $\beta_B^\tau \in \mathcal{B}_s$ and thus we have:

$$V_T^+(X_0, p) \ge \sup_{B\in\mathcal{B}_d^I} \mathcal{J}_T(X_0, p, \alpha_\epsilon, \beta_B^\tau) - \epsilon.$$

We denote $\mathcal{T}_T(x_0^i, \alpha_\epsilon, \bar{\beta})$ by T_i for short. Let us define, for any $i \in I$ and $v \in \mathcal{V}$:

$$\alpha^i(v)(t) = \alpha(s_{x_0^i,\alpha_\epsilon,\bar{\beta}}, \tilde{v})(t + \frac{k_{T^i}+2}{2}\tau),$$
$$\text{where } \tilde{v}(t) = \begin{cases} v_{\alpha_\epsilon,\bar{\beta}}, & \text{if } t < \frac{2+k_{T^i}}{2}\tau; \\ v(t - \frac{2+k_{T^i}}{2}\tau), & \text{if } t \ge \frac{2+k_{T^i}}{2}\tau. \end{cases}$$

We have then, with $X_i^{\alpha_\epsilon,\bar{\beta}} := X^{x_0^i,\alpha_\epsilon,\bar{\beta}}_{\frac{2+k_{T^i}}{2}\tau}$:

$$\begin{aligned} V_T^+(X_0, p) &\ge \sup_{B\in\mathcal{B}_d^I} \mathcal{J}(X_0, p, \alpha_\epsilon, \beta_B^\tau) - \epsilon \\ &= \sup_{B\in\mathcal{B}_d^I} \sum_{i\in I} p(i)\Big\{ \int_0^{\frac{2+k_{T^i}}{2}\tau} e^{-\lambda t}\ell(X_t^{x_0^i,\alpha_\epsilon,\beta_B^\tau}, \alpha_\epsilon, \beta_B^\tau)dt + \int_{\frac{2+k_{T^i}}{2}\tau}^{+\infty} e^{-\lambda t}\ell(X_t^{x_0^i,\alpha_\epsilon,\beta_B^\tau}, \alpha_\epsilon, \beta_B^\tau)dt\Big\} - \epsilon \\ &\ge \sup_{B\in\mathcal{B}_d^I} \sum_{i\in I} p(i)\Big\{ \int_0^{T^i} e^{-\lambda t}\ell(X_t^{x_0^i,\alpha_\epsilon,\bar{\beta}}, \alpha_\epsilon, \bar{\beta})dt + \int_{\frac{2+k_{T^i}}{2}\tau}^{+\infty} e^{-\lambda t}\ell(X_t^{x_0^i,\alpha_\epsilon,\beta_B^\tau}, \alpha_\epsilon, \beta_B^\tau)dt\Big\} - \epsilon - \tau\|\ell\|_\infty \end{aligned}$$

by a change of variable:

$$
\begin{aligned}
&= \sup_{B\in\mathcal{B}_d^I}\sum_{i\in I} p(i)\Big\{\int_0^{T^i} e^{-\lambda t}\ell(X_t^{x_0^i,\alpha_\epsilon,\bar\beta},\alpha_\epsilon,\bar\beta)dt + e^{-\lambda\frac{2+k_{T^i}}{2}\tau}\int_0^{+\infty} e^{-\lambda t}\ell(X_t^{X_i^{\alpha_\epsilon,\bar\beta},\alpha_\epsilon,\beta_B^\tau},\alpha^i,\beta_i)dt\Big\}-\\
&\quad \epsilon-\tau\|\ell\|_\infty\\
&= \sum_{i\in I} p(i)\Big\{\int_0^{T^i} e^{-\lambda t}\ell(X_t^{x_0^i,\alpha_\epsilon,\bar\beta},\alpha_\epsilon,\bar\beta)dt + e^{-\lambda\frac{2+k_{T^i}}{2}\tau}\sup_{\beta\in\mathcal{B}_d}\int_0^{+\infty} e^{-\lambda t}\ell(X_t^{X_i^{\alpha_\epsilon,\bar\beta},\alpha_\epsilon,\beta_B^\tau},\alpha^i,\beta)dt\Big\}-\\
&\quad \epsilon-\tau\|\ell\|_\infty\\
&\geq \sum_{i\in I} p(i)\Big\{\int_0^{T^i} e^{-\lambda t}\ell(X_t^{x_0^i,\alpha_\epsilon,\bar\beta},\alpha_\epsilon,\bar\beta)dt + e^{-\lambda\frac{2+k_{T^i}}{2}\tau}\tilde V^+(X^{x_0^i,\alpha_\epsilon,\bar\beta}_{\frac{2+k_{T^i}}{2}\tau})\Big\}-\epsilon-\tau\|\ell\|_\infty
\end{aligned}
$$

Letting $\tau\to 0^+$ in the right-hand side of the above inequality yields:

$$
V_T^+(X_0,p)\geq\sum_{i\in I} p(i)\Big\{\int_0^{T^i} e^{-\lambda t}\ell(X_t^{x_0^i,\alpha_\epsilon,\bar\beta},\alpha_\epsilon,\bar\beta)dt + e^{-\lambda T^i}\tilde V^+(X_{T^i}^{x_0^i,\alpha_\epsilon,\bar\beta})\Big\}-\epsilon
$$

Since the NAD strategy $\bar\beta$ is chosen arbitrarily, by taking the supremum on $\bar\beta\in\mathcal{B}_d$ on the right-hand side of the above inequality, we have:

$$
\begin{aligned}
V_T^+(X_0,p)\geq\sup_{\beta\in\mathcal{B}_d}\sum_{i\in I} p(i)\Big\{&\int_0^{\mathcal{T}_T(x_0^i,\alpha_\epsilon,\beta)} e^{-\lambda t}\ell(X_t^{x_0^i,\alpha_\epsilon,\beta},\alpha_\epsilon,\beta)dt+\\
&e^{-\lambda\mathcal{T}_T(x_0^i,\alpha_\epsilon,\beta)}\tilde V^+(X^{x_0^i,\alpha_\epsilon,\beta}_{\mathcal{T}_T(x_0^i,\alpha_\epsilon,\beta)})\Big\}-\epsilon.
\end{aligned}
$$

We notice that for any $(s_T^1,s_{x,T}^2\in\mathcal{S})$, $\alpha(s,\cdot)|_{[0,T]}=\alpha\big((0,0),\cdot\big)|_{[0,T]}$ and that $\alpha\big((0,0),\cdot\big)\in\mathcal{A}_d$. Since $\epsilon>0$ is also arbitrary, we take the infimum on $\alpha\in\mathcal{A}_d$ on the right-hand side of the inequality and then letting $\epsilon\to 0^+$ to finally obtain:

$$
V_T^+(X_0,p)\geq W_T^+(X_0,p)
$$

Since we have established both inequalities, we have proved the wished equality $V_T^+=W_T^+$ for any $T\in[0,+\infty)$.

Step 3: the case $T=+\infty$. By Corollary 3.3.3, we have:

$$
V^+=\lim_{T\to+\infty} V_T^+.
$$

But we have proved that $V_T^+=W_T^+$ for any $T<+\infty$. Thus the functions W_T^+ converge uniformly to V^+ as $T\to+\infty$. One can check that $\lim_{T\to+\infty}W_T^+=W_{+\infty}^+$ and here follows the wished result. □

3.3.2 Regularity of the Value Functions

With Lemma 3.3.4, we are able to prove the regularity of V^+ and V^-:

Proposition 3.3.5. *For any $T \in [0, +\infty]$, both V_T^+ and V_T^- are bounded and continuous. In addition, they are both Lipschitz continuous on p.*

Proof. The boundedness and the Lipschitz continuity on p are clear. By Corollary 3.3.3, we only need to prove the continuity of V_T^+ and V_T^- for $T < +\infty$. We claim that for any $T \in (0, +\infty)$, V_T^+ and V_T^- verify: for any $D \geq 1$ there exists constants $C_D > 0$ and $L_D > 0$ such that, for any X_1, $X_2 \in (\mathbb{R}^n)^I$ with $\|X_1\| \leq D$ and $\|X_2\| \leq D$, one has, for any $p \in \Delta(I)$:

$$|V_T^+(X_1, p) - V_T^+(X_2, p)| \leq C_D(\|X_1 - X_2\| + \|X_1 - X_2\|^{\frac{\lambda}{L_D}}); \tag{3.3.9}$$

$$|V_T^-(X_1, p) - V_T^-(X_2, p)| \leq C_D(\|X_1 - X_2\| + \|X_1 - X_2\|^{\frac{\lambda}{L_D}}) \tag{3.3.10}$$

We only prove (3.3.9) since the other part can be proved similarly. Let us fix X_1 and X_2 such that $\|X_1\| \leq D$ and $\|X_2\| \leq D$. Since $T > 0$ is a upper bound of $\{\mathcal{T}_T(x, u, v) \mid x \in \mathbb{R}^n \text{ and } (u, v) \in \mathcal{U} \times \mathcal{V}\}$. As a result of (3.1.2), we have:

$$\text{for any } i \in I,\ (u, v) \in \mathcal{U} \times \mathcal{V},\ \text{and } t \leq T,\ \|X_t^{x_1^i, u, v} - X_t^{x_2^i, u, v}\| \leq e^{L_f T}\|X_1 - X_2\|,$$

and by hypothesis (vii), there exists $\bar{C}_D > 0$ such that:

$$\text{for any } i \in I,\ \text{and } (u, v) \in \mathcal{U} \times \mathcal{V},\ |\mathcal{T}_T(x_1^i, u, v) - \mathcal{T}_T(x_2^i, u, v)| \leq \bar{C}_D\|X_1 - X_2\|.$$

Since $\tilde{V}^+$ is bounded and locally Hölder continuous [3], without loss of generality, let us suppose that there exists $L_D > 0$ such that:

$$\text{for any } x_1,\ x_2 \in \bar{B}(0; D),\ |\tilde{V}^+(x_1) - \tilde{V}^+(x_2)| \leq \bar{C}_D\|x_1 - x_2\|^{\frac{\lambda}{L_D}}.$$

Thus, for any $i \in I$ and $(u, v) \in \mathcal{U} \times \mathcal{V}$, we have, with $M > 0$ being an upper bound of $\tilde{V}^+$:

$$\begin{aligned}
&|e^{-\lambda \mathcal{T}_T(x_1^i, u, v)}\tilde{V}^+(X^{x_1^i, u, v}_{\mathcal{T}_T(x_1^i, u, v)}) - e^{-\lambda \mathcal{T}_T(x_2^i, u, v)}\tilde{V}^+(X^{x_2^i, u, v}_{\mathcal{T}_T(x_2^i, u, v)})| \\
\leq& M|e^{-\lambda \mathcal{T}_T(x_1^i, u, v)} - e^{-\lambda \mathcal{T}_T(x_2^i, u, v)}| + |\tilde{V}^+(X^{x_1^i, u, v}_{\mathcal{T}_T(x_1^i, u, v)}) - \tilde{V}^+(X^{x_2^i, u, v}_{\mathcal{T}_T(x_2^i, u, v)})| \\
\leq& \lambda M\bar{C}_D\|X_1 - X_2\| + \bar{C}_D(e^{L_f T}\|X_1 - X_2\|)^{\frac{\lambda}{L_D}} + \bar{C}_D(2\|f\|_\infty \bar{C}_D\|X_1 - X_2\|)^{\frac{\lambda}{L_D}} \\
=& \lambda M\bar{C}_D\|X_1 - X_2\| + \bar{C}_D\big(e^{\frac{\lambda L_f T}{L_D}} + (2\|f\|_\infty \bar{C}_D)^{\frac{\lambda}{L_D}}\big)\|X_1 - X_2\|^{\frac{\lambda}{L_D}}.
\end{aligned}$$

Similarly, we have, for any $i \in I$ and $(u, v) \in \mathcal{U} \times \mathcal{V}$:

$$\begin{aligned}
&|\int_0^{\mathcal{T}_T(x_1^i, u, v)} e^{-\lambda t}\ell(X_t^{x_1^i, u, v}, u, v)dt - \int_0^{\mathcal{T}_T(x_2^i, u, v)} e^{-\lambda t}\ell(X_t^{x_2^i, u, v}, u, v)dt| \\
\leq& \bar{C}_D\|\ell\|_\infty\|X_1 - X_2\| + \frac{L_\ell e^{L_f T}}{\lambda}\|X_1 - X_2\| = \big(\bar{C}_D\|\ell\|_\infty + \frac{L_\ell e^{L_f T}}{\lambda}\big)\|X_1 - X_2\|.
\end{aligned}$$

Combining these two estimations, for:

$$C_D = 2\max\{\bar{C}_D\lambda M, \bar{C}_D e^{\frac{\lambda L_f T}{L_D}} + \bar{C}_D(2\|f\|_\infty \bar{C}_D)^{\frac{\lambda}{L_D}}, \bar{C}_D\|\ell\|_\infty + \frac{L_\ell e^{L_f T}}{\lambda})\},$$

we have, for all $(u,v) \in \mathcal{U}\times\mathcal{V}$:

$$\begin{aligned}
&\big|\sum_{i\in I} p(i)\big(\int_0^{\mathcal{T}_T(x_1^i,u,v)} e^{-\lambda t}\ell(X_t^{x_1^i,u,v},u,v)dt + e^{-\lambda\mathcal{T}_T(x_1^i,u,v)}\tilde{V}^+(X^{x_1^i,u,v}_{\mathcal{T}_T(x_1^i,u,v)})\big) - \\
&\sum_{i\in I} p(i)\big(\int_0^{\mathcal{T}_T(x_2^i,u,v)} e^{-\lambda t}\ell(X_t^{x_2^i,u,v},u,v)dt + e^{-\lambda\mathcal{T}_T(x_2^i,u,v)}\tilde{V}^+(X^{x_2^i,u,v}_{\mathcal{T}_T(x_2^i,u,v)})\big| \\
\leq& C_D(\|X_1 - X_2\| + \|X_1 - X_2\|^{\frac{\lambda}{L_D}})\big).
\end{aligned}$$

By the two-step programming principle (Lemma 3.3.4), this implies (3.3.9). The proof is complete. □

3.3.3 Dynamic Programming Principle

In this section, we prove a dynamic programming principle for $V^+(\cdot,p)$.

Proposition 3.3.6. *For $h>0$ small enough and $X_0 \in \big(\mathbb{R}^n\backslash K\big)^I$, one has:*

$$V^+(X_0,p) = \inf_{\alpha\in\mathcal{A}_d}\sup_{v\in\mathcal{V}}\sum_{i\in I} p(i)\Big\{\int_0^h e^{-\lambda t}\ell(X_t^{x_0^i,\alpha,v},\alpha,v)dt + e^{-\lambda h}V^+\big((X_h^{x_0^i,\alpha,v})_{i\in I},p\big)\Big\} \tag{3.3.11}$$

Proof. By (3.1.2), we can choose $h>0$ small such that the vector $(X_h^{x_0^i,\alpha,v})_{i\in I}$ is still inside $\big(\mathbb{R}^n\backslash K\big)^I$ for any (α,v). Let us denote the right-hand side of (3.3.11) by $W^+(X_0,p)$. We have, by the two-step programming principle (Lemma 3.3.4):

$$\begin{aligned}
V^+(X_0,p) = \inf_{\alpha\in\mathcal{A}_d}\sup_{v\in\mathcal{V}}\sum_{i\in I} p(i)\Big\{&\int_0^{\mathcal{T}(x_0^i,\alpha,v)} e^{-\lambda t}\ell\big(X_t^{x_0^i,\alpha,v},\alpha,v\big)dt + \\
&e^{-\lambda\mathcal{T}(x_0^i,\alpha,v)}\tilde{V}^+(X^{x_0^i,\alpha,v}_{\mathcal{T}(x_0^i,\alpha,v)})\Big\}
\end{aligned}$$

Step 1: $V^+ \leq W^+$. For any $X=(x^i)_{i\in I}\in(\mathbb{R}^n)^I$, let α_X be a ϵ-optimal strategy for $V^+(X,p)$ in the sense of the above equality, namely:

$$\begin{aligned}
V^+(X,p)+\epsilon \geq \sup_{v\in\mathcal{V}}\sum_{i\in I} p(i)\Big\{&\int_0^{\mathcal{T}(x^i,\alpha_X,v)} e^{-\lambda t}\ell\big(X_t^{x^i,\alpha_X,v},\alpha_X,v\big)dt + \\
&e^{-\lambda\mathcal{T}(x^i,\alpha_X,v)}\tilde{V}^+(X^{x^i,\alpha_X,v}_{\mathcal{T}(x^i,\alpha_X,v)})\Big\}.
\end{aligned}$$

Since V^+ is continuous on X_0, there exists, $\delta_X > 0$ such that for any $Y \in B(X;\delta_X)$, α_X is still a 2ϵ-optimal strategy for $V^+(Y,p)$. We can therefore construct a Borel partition of $\bar{B}(X_0;1) \subset (\mathbb{R}^n)^I$:

$$\bar{B}(X_0;1) = \cup_{j=1}^m E_j,$$

such that for any $1 \le j \le m$, there exists $X_j \in \bar{B}(X_0;1)$ with $E_j \subset B(X_j; \frac{\delta_{X_j}}{2})$. For any $\alpha \in \mathcal{A}_d$, let τ be a common delay of α and $(\alpha_{X_j})_{1\le j\le m}$ (without loss of generality, we set $\tau < h$) and we construct a new strategy $\bar{\alpha}$ by setting, for any $v \in \mathcal{V}$:

$$\bar{\alpha}(v)(t) = \begin{cases} \alpha(v)(t), & \text{if } t < h; \\ \alpha_{X_j}\big(v(\cdot + h)\big)(t-h), & \text{if } t \ge h \text{ and } (X_{h-\tau}^{x_0^i,\alpha,v})_{i\in I} \in E_j. \end{cases}$$

We remark that we can always choose τ small enough to assure that if $(X_{h-\tau}^{x_0^i,\alpha,v})_{i\in I} \in E_j$, then $(X_h^{x_0^i,\alpha,v})_{i\in I} \in B(X_j;\delta_{X_j})$. One can check that $\bar{\alpha} \in \mathcal{A}_d$ and we have:

$$\begin{aligned} V^+(X_0,p) &\le \sup_{v\in\mathcal{V}} \sum_{i\in I} p(i)\Big\{ \int_0^{\mathcal{T}(x_0^i,\bar{\alpha},v)} e^{-\lambda t}\ell\big(X_t^{x_0^i,\bar{\alpha},v},\bar{\alpha},v\big)dt + e^{-\lambda\mathcal{T}(x_0^i,\bar{\alpha},v)}\tilde{V}^+(X_{\mathcal{T}(x_0^i,\bar{\alpha},v)}^{x_0^i,\bar{\alpha},v})\Big\} \\ &= \sup_{v\in\mathcal{V}} \sum_{i\in I} p(i)\Big\{ \int_0^{h} e^{-\lambda t}\ell\big(X_t^{x_0^i,\bar{\alpha},v},\bar{\alpha},v\big)dt + \int_h^{\mathcal{T}(x_0^i,\bar{\alpha},v)} e^{-\lambda t}\ell\big(X_t^{x_0^i,\bar{\alpha},v},\bar{\alpha},v\big)dt + \\ &\quad e^{-\lambda\mathcal{T}(x_0^i,\bar{\alpha},v)}\tilde{V}^+(X_{\mathcal{T}(x_0^i,\bar{\alpha},v)}^{x_0^i,\bar{\alpha},v})\Big\} \end{aligned}$$

by denoting $x_{v,h}^i = X_h^{x_0^i,\alpha,v}$:

$$\begin{aligned} &= \sup_{v\in\mathcal{V}} \sum_{i\in I} p(i)\Big\{ e^{-\lambda h}\sum_{j=1}^m \Big[\int_0^{\mathcal{T}(x_{v,h}^i,\alpha_{X_j},v|_{[h,+\infty)})} e^{-\lambda t}\ell\big(X_t^{x_{v,h}^i,\alpha_{X_j},v|_{[h,+\infty)}},\alpha_{X_j},v|_{[h,+\infty)}\big)dt + \\ &\quad e^{-\lambda\mathcal{T}(x_{v,h}^i,\alpha_{X_j},v|_{[h,+\infty)})}\tilde{V}^+(X_{\mathcal{T}(x_{v,h}^i,\alpha_{X_j},v|_{[h,+\infty)})}^{x_{v,h}^i,\alpha_{X_j},v|_{[h,+\infty)}})\Big]\mathbf{1}_{E_j}\big((X_{h-\tau}^{x_0^i,\alpha,v})_{i\in I}\big) + \int_0^h e^{-\lambda t}\ell\big(X_t^{x_0^i,\alpha,v},\alpha,v\big)dt\Big\} \end{aligned}$$

But any $v \in \mathcal{V}$ can be viewed as being formed by "gluing" two controls at $t = h$ in the following way:

$$v(t) = \begin{cases} v(t), & \text{if } t < h; \\ v(\cdot + h)(t-h), & \text{if } t \ge h, \end{cases}$$

thus, we have:

$$V^+(X_0,p) \le \sup_{v_0\in\mathcal{V}} \sum_{i\in I} p(i)\Big\{e^{-\lambda h}\sum_{j=1}^{m}\sup_{v\in\mathcal{V}}\big[\int_0^{\mathcal{T}(x^i_{v_0,h},\alpha_{X_j},v)} e^{-\lambda t}\ell\big(X_t^{x^i_{v_0,h},\alpha_{X_j},v},\alpha_{X_j},v\big)dt+$$
$$e^{-\lambda\mathcal{T}(x^i_{v_0,h},\alpha_{X_j},v)}\tilde{V}^+(X^{x^i_{v_0,h},\alpha_{X_j},v}_{\mathcal{T}(x^i_{v_0,h})},\alpha_{X_j},v)\big]\mathbf{1}_{E_j}\big((X^{x^i_0,\alpha,v_0}_{h-\tau})_{i\in I}\big)+\int_0^h e^{-\lambda t}\ell\big(X_t^{x^i_0,\alpha,v_0},\alpha,v_0\big)dt\Big\}$$
$$\le \sup_{v_0\in\mathcal{V}}\sum_{i\in I} p(i)\Big\{e^{-\lambda h}V^+\big((X_h^{x^i_0,\alpha,v_0})_{i\in I},p\big)+\int_0^h e^{-\lambda t}\ell\big(X_t^{x^i_0,\alpha,v_0},\alpha,v_0\big)dt\Big\}+\epsilon$$

But $\alpha\in\mathcal{A}_d$ is chosen arbitrarily, thus taking the infimum on $\alpha\in\mathcal{A}_d$ on the right-hand side of the last inequality above yields:

$$V^+(X_0,p)\le \inf_{\alpha\in\mathcal{A}_d}\sup_{v\in\mathcal{V}}\sum_{i\in I}p(i)\Big\{e^{-\lambda h}V^+\big((X_h^{x^i_0,\alpha,v})_{i\in I},p\big)+\int_0^h e^{-\lambda t}\ell\big(X_t^{x^i_0,\alpha,v},\alpha,v\big)dt\Big\}+\epsilon.$$

Since $\epsilon>0$ is arbitrary, the above inequality implies $V^+(X_0,p)\le W^+(X_0,p)$. Let us now turn to the other direction.

Step 2: $V^+\ge W^+$**.** We keep $h>0$ small enough as in Step 1. Let $\alpha_\epsilon\in\mathcal{A}_d$ be a ϵ-optimal strategy for $V^+(X_0,p)$, then:

$$V^+(X_0,p)\ge\sup_{v\in\mathcal{V}}\sum_{i\in I}p(i)\Big\{\int_0^{\mathcal{T}(x^i_0,\alpha_\epsilon,v)}e^{-\lambda t}\ell\big(X_t^{x^i_0,\alpha_\epsilon,v},\alpha_\epsilon,v\big)dt+$$
$$e^{-\lambda\mathcal{T}(x^i_0,\alpha_\epsilon,v)}\tilde{V}^+(X^{x^i_0,\alpha_\epsilon,v}_{\mathcal{T}(x^i_0,\alpha_\epsilon,v)})\Big\}-\epsilon$$

Let us fix $v_0\in\mathcal{V}$ and for any $v\in\mathcal{V}$, we define a new control:

$$\tilde{v}(t)=\begin{cases}v_0(t), & \text{if } t<h;\\ v(t-h), & \text{if } t\ge h.\end{cases}$$

We have then:

$$V^+(X_0,p)\ge\sup_{v\in\mathcal{V}}\sum_{i\in I}p(i)\Big\{\int_0^{\mathcal{T}(x^i_0,\alpha_\epsilon,\tilde{v})}e^{-\lambda t}\ell\big(X_t^{x^i_0,\alpha_\epsilon,\tilde{v}},\alpha_\epsilon,\tilde{v}\big)dt+$$
$$e^{-\lambda\mathcal{T}(x^i_0,\alpha_\epsilon,\tilde{v})}\tilde{V}^+(X^{x^i_0,\alpha_\epsilon,\tilde{v}}_{\mathcal{T}(x^i_0,\alpha_\epsilon,\tilde{v})})\Big\}-\epsilon$$
$$=\sup_{v\in\mathcal{V}}\sum_{i\in I}p(i)\Big\{\big[\int_h^{\mathcal{T}(x^i_0,\alpha_\epsilon,\tilde{v})}e^{-\lambda t}\ell\big(X_t^{x^i_0,\alpha_\epsilon,\tilde{v}},\alpha_\epsilon,\tilde{v}\big)dt$$
$$e^{-\lambda\mathcal{T}(x^i_0,\alpha_\epsilon,\tilde{v})}\tilde{V}^+(X^{x^i_0,\alpha_\epsilon,\tilde{v}}_{\mathcal{T}(x^i_0,\alpha_\epsilon,\tilde{v})})\big]+\int_0^h e^{-\lambda t}\ell\big(X_t^{x^i_0,\alpha_\epsilon,v_0},\alpha_\epsilon,v_0\big)dt\Big\}-\epsilon$$

Let us define a new strategy $\bar{\alpha}_\epsilon \in \mathcal{A}_d$ from α_ϵ: for any $v \in \mathcal{V}$, we set:

$$\bar{\alpha}_\epsilon(v) = \alpha_\epsilon(\tilde{v})(\cdot + h).$$

One can check that $\bar{\alpha}_\epsilon \in \mathcal{A}_d$. Thus, we have, from the above inequality by denoting $x_h^i = X_h^{x_0^i,\alpha_\epsilon,v_0}$:

$$\begin{aligned} V^+(X_0,p) \geq & \sup_{v\in\mathcal{V}} \sum_{i\in I} p(i)\Big\{ e^{-\lambda h}\big[\int_0^{\mathcal{T}(x_h^i,\bar{\alpha}_\epsilon,v)} e^{-\lambda t}\ell\big(X_t^{x_h^i,\bar{\alpha}_\epsilon,v},\bar{\alpha}_\epsilon,v\big)dt \\ & e^{-\lambda\mathcal{T}(x_h^i,\bar{\alpha}_\epsilon,v)}\tilde{V}^+(X^{x_h^i,\bar{\alpha}_\epsilon,v}_{\mathcal{T}(x_h^i,\bar{\alpha}_\epsilon,v)})\big] + \int_0^h e^{-\lambda t}\ell\big(X_t^{x_0^i,\alpha_\epsilon,v_0},\alpha_\epsilon,v_0\big)dt\Big\} - \epsilon \\ \geq & \sum_{i\in I} p(i)\Big\{ \int_0^h e^{-\lambda t}\ell\big(X_t^{x_0^i,\alpha_\epsilon,v_0},\alpha_\epsilon,v_0\big)dt + e^{-\lambda h}V^+\big((X_h^{x_0^i,\alpha_\epsilon,v_0})_{i\in I},p\big)\Big\} - \epsilon. \end{aligned}$$

But since the v_0 in the above inequalities is arbitrary, we obtain:

$$\begin{aligned} V^+(X_0,p) \geq & \sup_{v\in\mathcal{V}} \sum_{i\in I} p(i)\Big\{ \int_0^h e^{-\lambda t}\ell\big(X_t^{x_0^i,\alpha_\epsilon,v},\alpha_\epsilon,v\big)dt + e^{-\lambda h}V^+\big((X_h^{x_0^i,\alpha_\epsilon,v})_{i\in I},p\big)\Big\} - \epsilon \\ \geq & W^+(X_0,p) - \epsilon. \end{aligned}$$

By passing $\epsilon \to 0^+$, the above inequality implies $V^+ \geq W$. Combining these two steps, we have established $V^+ = W^+$ on $(\mathbb{R}^n\backslash K)^I$, which is the wished result. The proof is complete. □

We will show in the next section that, by dividing both sides of the equality (3.3.11) by h and by passing $h \to 0^+$, the upper-value function V^+ is bounded viscosity sub-solution of a Hamilton-Jacobi-Isaacs equation on $\big(\mathbb{R}^n\backslash K\big)^I$.

3.4 Hamilton-Jacobi-Isaacs Equations and Existence of a Value

In this section, we introduce a Hamilton-Jacobi-Isaacs equation and we are going to prove that, on $\big(\mathbb{R}^n\backslash K\big)^I$, $V^+(\cdot,p)$ is a viscosity sub-solution of this equation (thus $V^-(\cdot,p)$ a super-solution). We will also prove a comparison principle, which allows us, under Isaacs' condition (3.1.1), to obtain the existence of value for game $\mathcal{G}(X_0,p)$.

With Isaacs' condition, we define $H_p := H_p^+ = H_p^-$ for any $p \in \Delta(I)$ and we consider the following Hamilton-Jacobi-Isaacs equation:

$$-\lambda V(X) + H_p(X, D_X V(X)) = 0. \tag{3.4.1}$$

Let us first recall the notion of viscosity solutions of the above PDE:

Definition 3.4.1 (Viscosity Solutions)**.** *A function $V : (\mathbb{R}^n)^I \to \mathbb{R}$ is:*

- *a viscosity super-solution of Hamilton-Jacobi-Isaacs equation* (3.4.1) *on $(\mathbb{R}^n\backslash K)^I$ if and only if: it is lower semi-continuous (l.s.c.), and for any function $\varphi \in C^1((\mathbb{R}^n)^I;\mathbb{R})$ such that $V-\varphi$ has a local minimum at some point $X \in (\mathbb{R}^n\backslash K)^I$, one has:*

$$-\lambda V(X) + H_p(X, D_X\varphi(X)) \leq 0.$$

- *a viscosity sub-solution of Hamilton-Jacobi-Isaacs equation* (3.4.1) *if and only if: it is upper semi-continuous (u.s.c.), and for any function $\varphi \in C^1((\mathbb{R}^n)^I;\mathbb{R})$ such that $V-\varphi$ has a local maximum at some point $X \in (\mathbb{R}^n\backslash K)^I$, one has:*

$$-\lambda V(X) + H_p(X, D_X\varphi(X)) \geq 0.$$

- *a viscosity solution of Hamilton-Jacobi-Isaacs equation* (3.4.1) *if it is both viscosity super-solution and viscosity sub-solution of equation* (3.4.1)*.*

The following lemma is an immediate consequence of the above definition:

Lemma 3.4.2. *Let V be a viscosity super-solution of equation* (3.4.1)*. Then $-V$ is a viscosity sub-solution of:*

$$-\lambda V(X) + \hat{H}_p(X, D_X V(X)) = 0 \tag{3.4.2}$$

where the hamiltonian $\hat{H}_p$ is defined by:

$$\forall (X,q) \in (\mathbb{R}^n)^I \times (\mathbb{R}^n)^I,\ \hat{H}_p(X,q) = -H_p(X,-q).$$

The main result of this note is the following:

Theorem 3.4.3. *Under Isaacs' condition* (3.1.1)*, game $\mathcal{G}(X_0,p)$ has a value. Furthermore, restricted to $\big(\mathbb{R}^n\backslash K\big)^I$, the value function $X_0 \mapsto V(X_0,p)$ is the unique bounded continuous viscosity solution of the Hamilton-Jacobi-Isaacs equation* (3.4.1)*.*

Let us first prove the following:

Proposition 3.4.4. *$V^+(\cdot,p)$ is a viscosity sub-solution of* (3.4.1) *on $\big(\mathbb{R}^n\backslash K\big)^I$.*

Proof. The continuity of V^+ has been proved in Proposition 3.3.5. Let $\varphi \in \mathcal{C}^1\big((\mathbb{R}^n)^I;\mathbb{R}\big)$ be a test function such that $V^+(\cdot,p)-\varphi$ has a local maximum at $X_0 \in \big(\mathbb{R}^n\backslash K\big)^I$. By the dynamic programming principle (Proposition 3.3.6), we have for $h>0$ small enough:

$$\begin{aligned} V^+(X_0,p) &= \inf_{\alpha\in\mathcal{A}_d}\sup_{v\in\mathcal{V}}\sum_{i\in I}p(i)\Big\{\int_0^h e^{-\lambda t}\ell(X_t^{x_0^i,\alpha,v},\alpha,v)dt + e^{-\lambda h}V^+\big((X_h^{x_0^i,\alpha,v})_{i\in I},p\big)\Big\} \\ &\leq \inf_{\alpha\in\mathcal{A}_d}\sup_{v\in\mathcal{V}}\sum_{i\in I}p(i)\Big\{\int_0^h e^{-\lambda t}\ell(X_t^{x_0^i,\alpha,v},\alpha,v)dt + e^{-\lambda h}\big[\varphi\big((X_h^{x_0^i,\alpha,v})_{i\in I}\big)+ \\ &\qquad V^+(X_0,p)-\varphi(X_0)\big]\Big\}. \end{aligned}$$

The above inequality yields:

$$(1-e^{-\lambda h})V^+(X_0,p) \leq \inf_{\alpha\in\mathcal{A}_d}\sup_{v\in\mathcal{V}}\sum_{i\in I}p(i)\Big\{\int_0^h e^{-\lambda t}\ell(X_t^{x_0^i,\alpha,v},\alpha,v)dt+ e^{-\lambda h}\big[\varphi\big((X_h^{x_0^i,\alpha,v})_{i\in I}\big)-\varphi(X_0)\big]\Big\}.$$

Let us fix $u_0\in U$ a constant control and $v\in\mathcal{V}$ a ϵh-optimal control against u_0 for the right-hand side of the above inequality. This implies:

$$(1-e^{-\lambda h})V^+(X_0,p)-\epsilon h \leq \sum_{i\in I}p(i)\Big\{\int_0^h e^{-\lambda t}\ell(X_t^{x_0^i,u_0,v},u_0,v)dt+ e^{-\lambda h}\big[\varphi\big((X_h^{x_0^i,u_0,v})_{i\in I}\big)-\varphi(X_0)\big]\Big\}.$$

Let us do some estimations of the terms in the above inequality:

$$\begin{aligned}(1-e^{-\lambda h})V^+(X_0,p) &= \lambda h V^+(X_0,p)+o(h);\\ \int_0^h e^{-\lambda t}\ell(X_t^{x_0^i,\alpha,v},\alpha,v)dt &\leq \int_0^h \ell\big(x_0^i,u_0,v(t)\big)dt+o(h);\\ e^{-\lambda h}\big[\varphi\big((X_h^{x_0^i,\alpha,v})_{i\in I}\big)-\varphi(X_0)\big] &= \int_0^h\langle\big(f(x_0^i,u_0,v(t))\big)_{i\in I},D_X\varphi(X_0)\rangle dt+o(h).\end{aligned}$$

By applying these estimations, we obtain:

$$\begin{aligned}&-\epsilon h+o(h)\\ \leq& -\lambda h V^+(X_0,p)+\sup_{v_0\in V}\int_0^h dt\big[\sum_{i\in I}\ell(x_0^i,u_0,v_0)+\langle\big(f(x_0^i,u_0,v_0)\big)_{i\in I},D_X\varphi(X_0)\rangle\big]\\ =& -\lambda h V^+(X_0,p)+h\sup_{v_0\in V}\big[\sum_{i\in I}\ell(x_0^i,u_0,v_0)+\langle\big(f(x_0^i,u_0,v_0)\big)_{i\in I},D_X\varphi(X_0)\rangle\big].\end{aligned}$$

But since u_0 is arbitrary, we take the infimum on $u_0\in U$ on both sides of the above inequality, which yields:

$$-\epsilon h+o(h)\leq -\lambda h V^+(X_0,p)+hH_p\big(X_0,D_X\varphi(X_0)\big).$$

By dividing both sides by h and letting $h\to 0^+$, we obtain:

$$0\leq -\lambda V^+(X_0,p)+H_p\big(X_0,D_X\varphi(X_0)\big),$$

and thus $V^+(\cdot,p)$ is indeed a viscosity sub-solution of EDP (3.4.1) on $\big(\mathbb{R}^n\backslash K\big)^I$. □

Corollary 3.4.5. *$V^-(\cdot,p)$ is a viscosity super-solution of* (3.4.1) *on* $\big(\mathbb{R}^n\backslash K\big)^I$.

Proof. Let us define another differential game $-\mathcal{G}(X_0,p)$ of the same game procedure as $\mathcal{G}(X_0,p)$ by associating to each solution of dynamic (P) $t\mapsto X_t^{x_0,u,v}$ the pay-off $-J(x_0,u,v)$ and exchanging the roles of the players. One can check that $-V^-(X_0,p)$ is the upper-value function of this game and this furthermore yields that $-V^-(X_0,p)$ is the viscosity sub-solution of a PDE: $-\lambda V(X)-H_p(X,-D_XV(X))=0$, which implies the wished result. □

Let us prove the following property of viscosity solutions of (3.4.1) on $(\mathbb{R}^n\backslash K)^I$.

Proposition 3.4.6 (Comparison Principle)**.** *Let W_1 and W_2 be respectively bounded continuous viscosity sub-solution and bounded continuous viscosity super-solution of* (3.4.1) *such that $W_1=W_2$ on $\partial\big(\mathbb{R}^n\backslash K\big)^I$. Then one has $W_1\leq W_2$ on $\big(\mathbb{R}^n\backslash K\big)^I$.*

Proof. Let us prove the proposition by contradiction. By denoting $O=\big(\mathbb{R}^n\backslash K\big)^I$, we suppose that:

$$N:=\sup_{X\in O}W_1(X)-W_2(X)>0.$$

Then we have, for $a>0$ small enough:

$$N_a:=\sup_{X\in O}W_1(X)-W_2(X)-a\|X\|^2>0,$$

and $\lim_{a\to 0}N_a=N$. We employ the double-variable technique by defining, for any $\epsilon>0$

$$W_{a,\epsilon}(X,Y)=W_1(X)-W_2(Y)-\frac{\|X-Y\|^2}{\epsilon}-\frac{a}{2}(\|X\|^2+\|Y\|^2),$$

and

$$N_{a,\epsilon}:=\sup_{X,Y\in O}W_{a,\epsilon}(X,Y).$$

We have clearly $N_{a,\epsilon}\geq N_a>0$ and that $W_{a,\epsilon}$ is upper semi-continuous and coercive. This implies that there exists $D>0$ such that for any (a,ϵ), there exists $(X_{a,\epsilon},Y_{a,\epsilon})\in\bar{O}\cap\bar{B}(0;D)$ such that:

$$N_{a,\epsilon}=W_{a,\epsilon}(X_{a,\epsilon},Y_{a,\epsilon}).$$

Before we continue our proof, we state the following technical lemma, the proof of which is postponed after the proof of this proposition.

Lemma 3.4.7. *(a) There exists $C>0$ such that for any X, Y, ξ, and $\eta\in(\mathbb{R}^n)^I$:*

$$|H_p(X,\xi)-H_p(X,\eta)|\leq C\|\xi-\eta\|;$$
$$|H_p(X,\xi)-H_p(Y,\xi)|\leq C(1+\|\xi\|)\|X-Y\|.$$

(b) $\lim_{\epsilon\to 0^+} N_{a,\epsilon} = N_a$.

(c) For any $a > 0$ *small enough,* $\lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\|X_{a,\epsilon} - Y_{a,\epsilon}\|^2 = 0$.

(d) There exists $C' > 0$ *such that* $a(\|X_{a,\epsilon}\| + \|Y_{a,\epsilon}\|) \le C'\sqrt{a}$.

(e) For $a > 0$ *and* $\epsilon > 0$ *small enough,* $X_{a,\epsilon} \in O$ *and* $Y_{a,\epsilon} \in O$.

Since $X_{a,\epsilon}$ is a maximizer of $X \mapsto W_{a,\epsilon}(X, Y_{a,\epsilon})$, we obtain, for any $X \in O$:

$$W_1(X) \le W_1(X_{a,\epsilon}) + \frac{\|X - Y_{a,\epsilon}\|^2 - \|X_{a,\epsilon} - Y_{a,\epsilon}\|^2}{\epsilon} + \frac{a}{2}(\|X\|^2 - \|X_{a,\epsilon}\|^2).$$

We denote by $X \mapsto \varphi(X)$ the right-hand side of the above inequality and thus we have $W_1 - \varphi$ has a global maximum at $X_{a,\epsilon}$. Since W_1 is a sub-solution of (3.4.1), we have:

$$-\lambda W_1(X_{a,\epsilon}) + H_p\big(X_{a,\epsilon}, \frac{2}{\epsilon}(X_{a,\epsilon} - Y_{a,\epsilon}) + aX_{a,\epsilon}\big) \ge 0. \tag{3.4.3}$$

Similarly, we have:

$$W_2(Y) \ge W_2(Y_{a,\epsilon}) + \frac{\|X_{a,\epsilon} - Y_{a,\epsilon}\|^2 - \|X_{a,\epsilon} - Y\|^2}{\epsilon} + \frac{a}{2}(\|Y_{a,\epsilon}\|^2 - \|Y\|^2).$$

By denoting the right-hand side by $\phi(Y)$, we obtain $W_2 - \phi$ has a global minimum at $Y_{a,\epsilon}$. Since W_2 is a viscosity super-solution of (3.4.1), the above inequality implies:

$$-\lambda W_2(X_{a,\epsilon}) + H_p\big(Y_{a,\epsilon}, \frac{2}{\epsilon}(X_{a,\epsilon} - Y_{a,\epsilon}) - aY_{a,\epsilon}\big) \le 0. \tag{3.4.4}$$

Combining (3.4.3) with (3.4.4), we have:

$$\begin{aligned}
0 \le & -\lambda\big(W_1(X_{a,\epsilon}) - W_2(Y_{a,\epsilon})\big) + H_p\big(X_{a,\epsilon}, \frac{2}{\epsilon}(X_{a,\epsilon} - Y_{a,\epsilon}) + aX_{a,\epsilon}\big) - \\
& H_p\big(Y_{a,\epsilon}, \frac{2}{\epsilon}(X_{a,\epsilon} - Y_{a,\epsilon}) - aY_{a,\epsilon}\big) \\
\le & -\lambda N_{a,\epsilon} + aC(\|X_{a,\epsilon}\| + \|Y_{a,\epsilon}\|) + C(1 + \frac{2}{\epsilon}\|X_{a,\epsilon} - Y_{a,\epsilon}\|)\|X_{a,\epsilon} - Y_{a,\epsilon}\| \\
\le & -\lambda N_{a,\epsilon} + CC'\sqrt{a} + C(1 + \frac{2}{\epsilon}\|X_{a,\epsilon} - Y_{a,\epsilon}\|)\|X_{a,\epsilon} - Y_{a,\epsilon}\|
\end{aligned}$$

Passing $\epsilon \to 0^+$ and then passing $a \to 0^+$ in the above inequality yields:

$$0 \le -\lambda N.$$

But this contradicts $\lambda N > 0$, thus $W_1 \le W_2$ on O and the proof is complete. □

Proof of Lemma 3.4.7. (a) is a direct result of the definition of H_p and the regularity of f and ℓ.

Since we have, for $K > \max\{\|W_1\|_\infty, \|W_2\|_\infty\}$:

$$\begin{aligned}0 < N_a \leq N_{a,\epsilon} = W_1(X_{a,\epsilon}) - W_2(Y_{a,\epsilon}) - \frac{\|X_{a,\epsilon} - Y_{a,\epsilon}\|^2}{\epsilon} - \frac{a}{2}\big(\|X_{a,\epsilon}\|^2 + \|Y_{a,\epsilon}\|^2\big)\\ \leq 2K - \frac{\|X_{a,\epsilon} - Y_{a,\epsilon}\|^2}{\epsilon} - \frac{a}{2}\big(\|X_{a,\epsilon}\|^2 + \|Y_{a,\epsilon}\|^2\big).\end{aligned}$$

This proves that (d) holds and that $\frac{\|X_{a,\epsilon}-Y_{a,\epsilon}\|^2}{\epsilon}$ is bounded independently from a. Thus $\|X_{a,\epsilon} - Y_{a,\epsilon}\|$ tends to 0 as $\epsilon \to 0^+$. By the fact that $(X_{a,\epsilon}, Y_{a,\epsilon})$ is bounded, let X_a be a cluster point of $(X_{a,\epsilon})_{\epsilon>0}$ and $(Y_{a,\epsilon})_{\epsilon>0}$ as $\epsilon \to 0^+$. Let $(\epsilon_n)_{n\in\mathbb{N}}$ be a sequence such that $\epsilon_n \to 0^+$ and $\lim_{n\to\infty} X_{a,\epsilon_n} = \lim_{n\to\infty} Y_{a,\epsilon_n} = X_a$. We have:

$$\begin{aligned}N_a \leq \liminf_{n\to\infty} N_{a,\epsilon_n}\\ \leq \liminf_{n\to\infty} \Big[W_1(X_{a,\epsilon_n}) - W_2(Y_{a,\epsilon_n}) - \frac{a}{2}(\|X_{a,\epsilon_n}\|^2 + \|Y_{a,\epsilon_n}\|^2)\Big]\\ \leq \limsup_{n\to\infty} \Big[W_1(X_{a,\epsilon_n}) - W_2(Y_{a,\epsilon_n}) - \frac{a}{2}(\|X_{a,\epsilon_n}\|^2 + \|Y_{a,\epsilon_n}\|^2)\Big]\\ \leq W_1(X_a) - W_2(X_a) - a\|X_a\|^2 \leq N_a.\end{aligned}$$

The last two inequalities are due to the fact that $W_1 - W_2$ is upper semi-continuous. This yields in addition $X_a \in O$, since if this is not true, we would have $W_1(X_a) = W_2(X_a)$ and $N_a < 0$, a contradiction. Thus, we have $\lim_{n\to\infty} N_{a,\epsilon_n} = N_a$. In addition, the above inequalities yield:

$$\begin{aligned}\lim_{n\to\infty} \Big[W_1(X_{a,\epsilon_n}) - W_2(Y_{a,\epsilon_n}) - \frac{a}{2}(\|X_{a,\epsilon_n}\|^2 + \|Y_{a,\epsilon_n}\|^2)\Big]\\ = W_1(X_a) - W_2(X_a) - a\|X_a\|^2 = N_a.\end{aligned}$$

This implies that:

$$\begin{aligned}\lim_{n\to\infty} \frac{\|X_{a,\epsilon_n} - Y_{a,\epsilon_n}\|^2}{\epsilon_n} = \lim_{n\to\infty} W_1(X_{a,\epsilon_n}) - W_2(Y_{a,\epsilon_n}) -\\ \frac{a}{2}(\|X_{a,\epsilon_n}\|^2 + \|Y_{a,\epsilon_n}\|^2) - N_{a,\epsilon_n} = 0.\end{aligned}$$

Since this is true for any $\epsilon_n \to 0$ such that $(X_{a,\epsilon_n}, Y_{a,\epsilon_n})$ is converging, by a compactness argument, we have finally (b), (c) and (e) hold and thus the proof of the lemma is complete. □

Now we are ready to prove our main theorem:

Proof of Theorem 3.4.3. Let us prove by recurrence on I that $V^+(\cdot, p) = V^-(\cdot, p)$ on $\mathbb{R}^{nI}$ for any $p \in \Delta(I)$. By (3.3.3), we only need to prove that $V^+(\cdot, p) = V^-(\cdot, p)$ on $\mathbb{R}^{nI}\backslash K^I$.

For the case $I = 1$, one has that, for any $(X_0, p) \in \mathbb{R}^n \times \Delta(1)$:

$$V^+(X_0, p) = \inf_{\alpha \in \mathcal{A}_s} \sup_{v \in \mathcal{V}} J(X_0, \alpha(X_0), v).$$

For any $X_0 \in K$, we have that, by Lemma 3.3.1, $V^+(X_0, 1) = \tilde{V}^+(X_0)$. The comparison principle (Proposition 3.4.6) yields that $V^+(X_0, 1) = \tilde{V}^+(X_0)$ for any $X_0 \in \mathbb{R}^n \backslash K$. Hence we have $V^+(\cdot, 1) = \tilde{V}^+(\cdot)$ on $\mathbb{R}^n$.

Similarly, one can easily check that $V^-(X_0, 1) = \tilde{V}^-(X_0)$. Since $\tilde{V}^+ = \tilde{V}^-$, we have that $V^+(\cdot, p) = V^-(\cdot, p)$ on $\mathbb{R}^{nI}$ for any $p \in \Delta(I)$ with $I = 1$.

Now let us assume that for some $k \in \mathbb{N}_*$, the game has a value with any $I \leq k$. We prove that, under such assumption, the game has a value for $I = k + 1$.

Let us denote $O := (\mathbb{R}^n \backslash K)^I$ and let us prove that $V^+(\cdot, p) = V^-(\cdot, p)$ on O^c. Let $X_0 \in O^c$ and $I' = \{i \in I \mid x_0^i \notin K\} \subset I$. It is clear that $I' \neq \emptyset$. By Corollary 3.3.2, there exists $p' \in \Delta(I)$ with $supp(p') \subset I' \cap supp(p)$ such that:

$$\begin{aligned} V^+(X_0, p) &= (1 - \sum_{i \in I \backslash I'} p(i)) V^+(X_0, p') + \sum_{i \in I \backslash I'} p(i) \tilde{V}^+(x_0^i); \\ V^-(X_0, p) &= (1 - \sum_{i \in I \backslash I'} p(i)) V^-(X_0, p') + \sum_{i \in I \backslash I'} p(i) \tilde{V}^-(x_0^i). \end{aligned} \tag{3.4.5}$$

Let us prove now that $V^+(X_0, p') = V^-(X_0, p')$. Without loss of generality, we suppose that $I' = \{1, 2, \ ..., \ |I'|\}$ (if this is not true, one can always rearrange the indices of the set I). Then by definition, we have:

$$\begin{aligned} V^+(X_0, p') &= \inf_{\alpha \in \mathcal{A}_s} \sup_{\beta \in \mathcal{B}_s} \sum_{i \in I'} p'(i) J\big(x_0^i, \alpha(x_0^i), \beta(x_0^i)\big); \\ V^-(X_0, p') &= \sup_{\beta \in \mathcal{B}_s} \inf_{\alpha \in \mathcal{A}_s} \sum_{i \in I'} p'(i) J\big(x_0^i, \alpha(x_0^i), \beta(x_0^i)\big). \end{aligned}$$

Let us denote $\bar{X}_0 := (x_0^i)_{i \in I'} \in \mathbb{R}^{nI'}$ and $\bar{p} := (p'(i))_{i \in I'} \in \Delta(I')$. We write:

$$\begin{aligned} W^+(\bar{X}_0, \bar{p}) &= \inf_{\alpha \in \mathcal{A}_s} \sup_{\beta \in \mathcal{B}_s} \sum_{i \in I'} \bar{p}(i) J\big(x_0^i, \alpha(x_0^i), \beta(x_0^i)\big); \\ W^-(\bar{X}_0, \bar{p}) &= \sup_{\beta \in \mathcal{B}_s} \inf_{\alpha \in \mathcal{A}_s} \sum_{i \in I'} \bar{p}(i) J\big(x_0^i, \alpha(x_0^i), \beta(x_0^i)\big). \end{aligned}$$

Then we have $W^+(\bar{X}_0, \bar{p}) = V^+(X_0, p')$ and $W^-(\bar{X}_0, \bar{p}) = V^-(X_0, p')$. In addition, one can easily check that $W^+(\bar{X}_0, \bar{p})$ and $W^-(\bar{X}_0, \bar{p})$ are respectively the upper- and lower-value of game $\mathcal{G}(\bar{X}_0, \bar{p})$. Since $I' \leq k$, by the recurrence assumption, one has:

$$V^+(X_0, p') = W^+(\bar{X}_0, \bar{p}) = W^-(\bar{X}_0, \bar{p}) = V^-(X_0, p')$$

Since $X_0 \in O^c$ is arbitrary, for any $p \in \Delta(I)$, $V^+(\cdot, p) = V^-(\cdot, p)$ on O^c thanks to (3.4.5).

It remains to prove that $V^+(\cdot, p) = V^-(\cdot, p)$ on O. Because $V^+(\cdot, p) = V^-(\cdot, p)$ on O^c, it is in particular also valid on ∂O. By Proposition 3.4.4 and Corollary 3.4.5, $V^+(\cdot, p)$ and $V^-(\cdot, p)$ are respectively viscosity sub- and super-solution of equation (3.4.1) on $(\mathbb{R}^n \backslash K)^I = O$. Thus the comparison principle yields that $V^+(\cdot, p) = V^-(\cdot, p)$ on O. The proof is complete. □

Chapter 4

Existence of Value for Infinite Horizon Differential Games with Asymmetric Information

Abstract:
This article concerns the existence of value for a zero-sum differential game with asymmetric information on initial state with an infinite horizon running cost. Before the game begins, an initial state is chosen from a finite set and each players receives a private signal generated by the chosen initial state. The main result is the existence of value for the game with random non-anticipative strategies with delay and its characterization as the unique bounded continuous viscosity solution of a suitable Hamilton-Jacobi-Isaacs equation.

Key words: Differential Games; Incomplete Information; Infinite Horizon; Hamilton-Jacobi Equations.
AMS Classification: 49N30; 49N70; 91A05; 91A10; 91A23.

4.1 Introduction

Since the 1950s, many papers have been devoted to the study of differential games with incomplete information (cf. [28, 41, 46, 47, 48, 55, 68]). In recent years, a particular type of differential games with asymmetric information was investigated in [16]. Such games can be considered as an extension of repeated games of incomplete information studied by Aumann and Maschler [2] to the framework of differential games introduce in [57, 58]. Several further results on differential games with asymmetric information can be found in [6, 12, 17, 19, 24, 26, 43, 54].

Infinite horizon differential games with exponentially discounted costs consist of an important class of differential games (cf. [3, 14]). Compared with the finite horizon case, the subject infinite horizon differential games with incomplete information is less explored in the literature (cf. [1]). In this paper we investigate the existence of value for a differential game with asymmetric information. We consider the following dynamic:

$$\begin{cases} \dot{x}(t) = f\big(x(t), u(t), v(t)\big), \quad t \geq 0; \\ x(0) = x_0. \end{cases} \tag{P}$$

Here $u : \mathbb{R}_+ \to U$ and $v : \mathbb{R}_+ \to V$ are measurable controls with U and V being some compact metric action spaces. We assume that the application $f : \mathbb{R}^n \times U \times V \to \mathbb{R}^\varkappa$ is regular enough such that the dynamic has a unique solution associated to the triplet (x_0, u, v), denoted by $t \mapsto X_t^{x_0,u,v}$. For $\lambda > 0$ a fixed constant, we associate to each (x_0, u, v) a following running cost:

$$J(x_0, u, v) := \int_0^{+\infty} e^{-\lambda t} \ell\big(X_t^{x_0,u,v}, u(t), v(t)\big) dt.$$

Let I, J be positive integers and $X_0 = \{x_0^{i,j}\}_{(i,j)\in I\times J}$ be a subset of $\mathbb{R}^n$ doubly indexed. (Here we adopt a slight abuse of notations by denoting the sets $\{1, 2, ..., I\}$ and $\{1, 2, ..., J\}$ respectively by I and J.) We fix $p \in \Delta(I) = \{p \in \mathbb{R}_+^I \mid \sum_{i\in I} p_i = 1\}$, $q \in \Delta(J) = \{q \in \mathbb{R}_+^J \mid \sum_{j\in J} q_j = 1\}$, probability measures respectively on I and on J. Let us consider a differential game $\mathcal{G}(X_0, p, q)$ defined as follows:

(A) Before the game begins, a pair of index $(i, j) \in I \times J$ is chosen randomly according to $\pi = p \otimes q$. The chosen index i is communicated to Player 1 but not to Player 2; the chosen index j is communicated to Player 2 but not to Player 1. The set X_0 and the probability measures p and q are supposed to be a common knowledge of both players.

(B) The game begins with $x_0 = x_0^{i,j}$ in (P). Player 1 chooses the control $u \in \mathcal{U}$ in order to minimize the running cost $J(x_0^{i,j}, u, v)$ while Player 2 aims to maximize $J(x_0^{i,j}, u, v)$ by choosing the control $v \in \mathcal{V}$. Both players observe all played actions with perfect memory.

In the above game, both players only have partial information about the initial state throughout the game. In order to optimize their costs, the players will choose their actions according to the pair of private information (i, j) and their observation of the played actions. Since both players could learn about the private information of their adversary by observing the played actions, they will play mixed strategies to protect their own private information.

The main purpose of this manuscript is to prove that game $\mathcal{G}(X_0, p, q)$ has a value and to provide a characterization for its value function as unique viscosity solution of a suitable partial differential equation.

Existence of a value for game $\mathcal{G}(X_0, p, q)$ has already been proved in [1] and the author gave a characterization of the value function as dual viscosity solution of a first-order Hamilton-Jacobi-Isaacs equation. However, in this paper we prove that its value function can also be characterized as unique bounded continuous viscosity solution of a new second-order double-obstacle Hamilton-Jacobi-Isaacs equation.

In this paper, to prove the existence of value for the infinite horizon case, we adapt the approach employed in [16] to our game model. We first treat the information structure with the method proposed in [31] to obtain respectively a super-dynamic programming principle and a sub-dynamic programming principle for the value functions. Then we follow the scheme below (cf. [33]):

1. We prove that under Isaacs' condition (introduced in Section 5) the upper- and lower-value functions are respectively viscosity sub- and super-solutions of a Hamilton-Jacobi-Isaacs equation.

2. We prove that this Hamilton-Jacobi-Isaacs equation has a unique viscosity solution.

3. Conclusion, the upper- and lower-value functions of game $\mathcal{G}(X_0, p, q)$ coincide.

There are two main difficulties in this approach. The first is to introduce suitable notion random non-anticipative strategies with delay and thus to define the corresponding normal form of the game. The second is to determine the Hamilton-Jacobi-Isaacs equation associated to the value function of our game model and prove a comparison principle for this partial differential equation.

We conclude this introduction with a presentation of the organization of this paper. After the preliminaries section, we introduce in Section 3 the notion of strategies for game $\mathcal{G}(X_0, p, q)$ and we define the value functions of the game with such strategies. Regularity and convexity of the value functions are also proved in Section 3. In Section 4, we prove a sub-dynamic programming principle for the lower-value function of game $\mathcal{G}(X_0, p, q)$ and we prove furthermore that each value function satisfies a Hamilton-Jacobi-Isaacs equation in the viscosity sense. The uniqueness of viscosity solution for the Hamilton-Jacobi-Isaacs equations is established in the last section and we conclude that under the Isaacs condition,

the game has a value which is the unique bounded continuous viscosity solution of the partial differential equation defined in Section 4.

4.2 Preliminaries

In the finite dimensional space $\mathbb{R}^n$, we denote by $\|x\|$ the euclidean norm of $x \in \mathbb{R}^n$, and $\langle x, y\rangle$ the associated scalar product of any points x, $y \in \mathbb{R}^n$. An open ball with center $x \in \mathbb{R}^n$ and radius a is denoted by $B(x;a)$, and its closure by $\bar{B}(x;a)$. Let us consider:

(i) U, V be two compact metric spaces, both endowed with the Borel σ-algebra;

(ii) $f : \mathbb{R}^n \times U \times V \to \mathbb{R}^n$ be a bounded, uniformly continuous function which is Lipschitz continuous on the first variable with a Lipschitz constant $L_f > 0$;

(iii) $\ell : \mathbb{R}^n \times \mathbb{R} \times U \times V \to \mathbb{R}$ be bounded, uniformly continuous, and Lipschitz continuous on the first two variables with Lipschitz constant $L_\ell > 0$;

(iv) $\mathcal{U}$ denotes the set of measurable maps $u : \mathbb{R}^+ \to U$ while $\mathcal{V}$ stands for the set of measurable maps $v : \mathbb{R}^+ \to V$;

(v) the constants $\lambda > 0$ and $M_0 \in \mathbb{R}$ be fixed.

Elements of $\mathcal{U}$ (resp. $\mathcal{V}$) are called admissible open-loop controls of Player 1 (resp. of Player 2). In addition, we suppose that both $\mathcal{U}$ and $\mathcal{V}$ are equipped with the corresponding L^∞-topology, namely, for any $(u_1, u_2) \in \mathcal{U}^2$ and $(v_1, v_2)\mathcal{V}^2$, we define:

$$d(u_1, u_2) := \|u_1 - u_2\|_\infty, \text{ and } d(v_1, v_2) := \|v_1 - v_2\|_\infty.$$

Lemma 4.2.1. *For all fixed $(t, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$, the map $(u, v) \mapsto X_t^{x_0,u,v}$ is uniformly continuous with $\mathcal{U} \times \mathcal{V}$ endowed with the L^∞-metrics.*

Proof. Let us fix (t, x_0) and $(u_0, v_0) \in \mathcal{U} \times \mathcal{V}$. For all $\epsilon > 0$, since f is both uniformly continuous, there exists $\delta > 0$ such that for u, $u' \in U$ and v, $v' \in V$, if $d(u, u') \le \delta$ and $d(v, v') \le \delta$, one has $\|f(x, u, v) - f(x, u', v')\| \le \frac{\epsilon}{2te^{L_f t}}$ for all $x \in \mathbb{R}^n$. We prove that, for any $(u_1, v_1) \in B\big((u_0, v_0); \frac{\epsilon\delta}{e^{L_f t}}\big)$, one has:

$$\|X_t^{x_0,u_0,v_0} - X_t^{x_0,u_1,v_1}\| \le \epsilon.$$

By the regularity of f with respect to the state variable, we deduce that:

$$\begin{aligned}
&\int_0^t \|f(X_s^{x_0,u_0,v_0}, u_0(s), v_0(s)) - f(X_s^{x_0,u_1,v_1}, u_1(s), v_1(s))\| ds \\
\le &\int_0^t L_f \|X_s^{x_0,u_0,v_0} - X_s^{x_0,u_1,v_1}\| ds + \\
&\int_0^t \|f\big(X_s^{x_0,u_1,v_1}, u_0(s), v_0(s)\big) - f\big(X_s^{x_0,u_1,v_1}, u_1(s), v_1(s)\big)\| ds
\end{aligned}$$

By setting:

$$\rho(t) := \|X_t^{x_0,u_0,v_0} - X_t^{x_0,u_1,v_1}\|,$$
$$b(t) := \|f\big(X_t^{x_0,u_1,v_1}, u_0(t), v_0(t)\big) - f\big(X_t^{x_0,u_1,v_1}, u_1(t), v_1(t)\big)\|,$$

we obtain: $\rho(t) \leq \int_0^t \big(L_f\rho(s) + b(s)\big)ds + \rho(0)$. Grönwall's inequality yields:

$$\rho(t) \leq \int_0^t e^{L_f(t-s)}b(s)ds. \tag{4.2.1}$$

Since we have:

$$\begin{aligned}\int_0^t b(s)ds &= \int_0^t \|f\big(X_s^{x_0,u_1,v_1}, u_0(s), v_0(s)\big) - f\big(X_s^{x_0,u_1,v_1}, u_1(s), v_1(s)\big)\|ds \\ &\leq \int_0^t \frac{\epsilon}{te^{L_f t}}ds \leq \frac{\epsilon}{e^{L_f t}},\end{aligned}$$

we deduce from (4.2.1) that:

$$\|X_t^{x_0,u_0,v_0} - X_t^{x_0,u_1,v_1}\| \leq e^{L_f t}\int_0^t b(s)ds \leq \epsilon.$$

Thus $(u,v) \mapsto X_t^{x_0,u,v}$ is uniformly continuous. The proof is complete. □

By the above lemma and the regularity assumptions on the functions f and ℓ, we obtain the following:

Corollary 4.2.2. *For all $x_0 \in \mathbb{R}^n$ fixed, $(u,v) \mapsto J(x_0,u,v)$ is uniformly continuous with $\mathcal{U} \times \mathcal{V}$ endowed with the L^∞-metrics.*

4.3 Strategies and Values

The notion of non-anticipative strategies was introduced into the literature of differential games in [32, 62, 69]. The main advantage of using non-anticipative strategies with delay is twofold: it not only models the fact that players should choose their controls according to the observed played actions but also allows us to define the upper- and lower-value functions is a symmetric fashion.

In this section, we introduce the notion of random non-anticipative strategies with delay for game $\mathcal{G}(X_0,p,q)$. Such strategies were introduced in [21] for differential with finite horizon. Here we adapt the definition of random non-anticipative strategies with delay introduced in [1] to the case where $\mathcal{U}$ and $\mathcal{V}$ are equipped with the L^∞-metric. Let us first give the definition of pure non-anticipative strategies with delay.

Definition 4.3.1 (Non-anticipative Strategies with Delay (in short, NAD Strategies) of Player 1, [1, 21]). *An NAD strategy of Player 1 is a Borel-measurable map: $\alpha : \mathcal{V} \to \mathcal{U}$ such that: there exists $\tau_\alpha > 0$, for any v_1, $v_2 \in \mathcal{V}$, $t \geq 0$, if $v_1|_{[0,t]} = v_2|_{[0,t]}$ a.e., then $\alpha(v_1)|_{[0,t+\tau_\alpha]} = \alpha(v_2)|_{[0,t+\tau_\alpha]}$ a.e.. The set of such maps is denoted by $\mathcal{A}_d$.*

We define an NAD strategy of Player 2 similarly and let $\mathcal{B}_d$ denote the set of NAD strategies of Player 2. The main interest of employing such strategies is the following:

Lemma 4.3.2 (cf. [1, 21]). *For any $(\alpha, \beta) \in \mathcal{A}_d \times \mathcal{B}_d$, there exists a unique pair of admissible controls $(u_{\alpha,\beta}, v_{\alpha,\beta})$ such that:*

$$\alpha(v_{\alpha,\beta}) = u_{\alpha,\beta} \text{ and } \beta(u_{\alpha,\beta}) = v_{\alpha,\beta}.$$

Since both players chooses their actions according to their received private signals, both players need to play random strategies in order to protect their private information. For simplicity, we denote $\mathcal{S} = \left\{ \left([0,1]^n, \mathcal{B}([0,1]^n), \mathcal{L}([0,1]^n)\right), n \in \mathbb{N}_* \right\}$ a set of probability spaces, where $\mathcal{B}([0,1]^n)$ is the Borel σ-algebra and $\mathcal{L}([0,1]^n)$ denotes the Lebesgue measure on $\mathcal{B}([0,1]^n)$. Then $\mathcal{S}$ is stable under finite products of its elements. Let us introduce the following:

Definition 4.3.3 (Random Non-anticipative Strategies with Delay (radom NAD strategies) of Player 1, cf. [16, 1]). *A random NAD strategy for Player 1 is a pair $\big((\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha), \alpha\big)$ with $(\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha) \in \mathcal{S}$ and the map $\alpha : \Omega_\alpha \times \mathcal{V} \to \mathcal{U}$ is measurable with Ω_α endowed with the σ-algebra $\mathcal{F}_\alpha$ and satisfying: there exists $\tau_\alpha > 0$ such that for all $\omega \in \Omega_\alpha$ and $t > 0$, if one has $v_1|_{[0,t]} = v_2|_{[0,t]}$ a.e. for v_1, $v_2 \in \mathcal{V}$, then:*

$$\alpha(\omega, v_1)|_{[0,t+\tau_\alpha]} = \alpha(\omega, v_2)|_{[0,t+\tau_\alpha]}, \text{ a.e..}$$

One can define similarly random NAD strategies of Player 2. We denote the set of random NAD strategies of Player 1 by $\mathcal{A}_r$ and that of Player 2 by $\mathcal{B}_r$. For simplicity of notations, we often write α (resp. β) instead of $\big((\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha), \alpha\big)$ (resp. $\big((\Omega_\beta, \mathcal{F}_\beta, \mathbb{P}_\beta), \beta\big)$) for short.

Since any admissible control can be viewed as an NAD strategy. We have $\mathcal{U} \in \mathcal{A}_d \subset \mathcal{A}_r$ and $\mathcal{V} \in \mathcal{B}_d \subset \mathcal{B}_r$.

Similarly to Lemma 4.3.2, we have the following

Lemma 4.3.4. *[cf. [16]] For all pair of random NAD strategies $(\alpha, \beta) \in \mathcal{A}_r \times \mathcal{B}_r$, for any $\omega = (\omega_\alpha, \omega_\beta) \in \Omega_\alpha \times \Omega_\beta$, there exists a unique pair of controls $(u_{\alpha,\beta,\omega}, v_{\alpha,\beta,\omega}) \in \mathcal{U} \times \mathcal{V}$ such that:*

$$\alpha(\omega_\alpha, v_{\alpha,\beta,\omega}) = u_{\alpha,\beta,\omega}, \text{ and } \beta(\omega_\beta, u_{\alpha,\beta,\omega}) = v_{\alpha,\beta,\omega}.$$

In addition, for all $T > 0$, the map $\omega \mapsto (u_{\alpha,\beta,\omega}, v_{\alpha,\beta,\omega})|_{[0,T]}$ is measurable.

By the above lemma, we denote, for $\alpha \in \mathcal{A}_r$ and $\beta \in \mathcal{B}_r$, by (α, β, ω) (or simply (α, β)) the map $\omega \mapsto (u_{\alpha,\beta,\omega}, v_{\alpha,\beta,\omega})$.

Definition 4.3.5 (Strategies for the Game $\mathcal{G}(X_0, p, q)$)**.** *A strategy of Player 1 is an element of $\mathcal{A}_r^I$ and that of Player 2 is an element of $\mathcal{B}_r^J$.*

Observe that, as in [16], both players in the game chooses their strategies according to the signals communicated and they play random strategies to protect their private information from their opponents.

In order to write the game into normal form, let us define for each pair of strategies $(\alpha_i)_{i\in I} \in \mathcal{A}_r^I$, $(\beta_j)_{j\in J} \in \mathcal{B}_r^J$ an associated pay-off, namely a following function:

$$\big((\alpha_i)_{i\in I}, (\beta_j)_{j\in J}\big) \mapsto \mathcal{J}\big(X_0, p, q, (\alpha_i)_{i\in I}, (\beta_j)_{j\in J}\big)$$

.

By Lemma 4.3.4, for any $\big((\alpha_i)_{i\in I}, (\beta_j)_{j\in J}\big)$ and $(i,j) \in I \times J$, one has that the map $J(x_0^{i,j}, \alpha_i, \beta_j)$ is well-defined, but the measurability of the function $(\omega_{\alpha_i}, \omega_{\beta_j}) \to J(x_0^{ij}, \alpha_i, \beta_j)$ is not a trivial issue.

For proving that $(\omega_{\alpha_i}, \omega_{\beta_j}) \to J(x_0^{ij}, \alpha_i, \beta_j)$ is measurable, let us define a new cost function with a fixed horizon T:

$$J_T(x_0, u, v) := \int_0^T e^{-\lambda t} \ell\big(X_t^{x_0,u,v}, u(t), v(t)\big) dt.$$

Since $J_T(x_0^{i,j}, \alpha_i, \beta_j)$ is bounded and measurable in $\Omega_{\alpha_i} \times \Omega_{\beta_j}$, the Lebesgue integral

$$\mathbb{E}_{\alpha_i,\beta_j}\big(J_T(x_0^{i,j}, \alpha_i, \beta_j)\big) := \int_{\Omega_{\alpha_i}\times\Omega_{\beta_j}} J_T(x_0^{i,j}, \alpha_i, \beta_j) d\mathbb{P}_{\alpha_i} \otimes \mathbb{P}_{\beta_j}$$

is thus well-defined. In addition, we have:

$$J(x_0, u, v) = \lim_{T\to+\infty} J_T(x_0, u, v)$$

for all fixed (x_0, u, v) and that the above convergence is point-wise and uniform with respect to the controls (u, v) (because $|J_T(x_0, u, v) - J(x_0, u, v)| \leq \frac{e^{-\lambda T}\|\ell\|_\infty}{\lambda}$). Thus for any $(i,j) \in I \times J$, $J(x_0^{i,j}, \alpha_i, \beta_j) = \lim_{N\to+\infty} J_N(x_0^{i,j}, \alpha_i, \beta_j)$ must be measurable on $\Omega_{\alpha_i} \times \Omega_{\beta_j}$.

In view of this, we are able to write the game $\mathcal{G}(X_0, p, q)$ into a normal form by associating to all pair of random strategies $\big((\alpha_i)_{i\in I}, (\beta_j)_{j\in J}\big)$ the following pay-off:

$$\mathcal{J}\big(X_0, p, q, (\alpha_i)_{i\in I}, (\beta_j)_{j\in J}\big) := \sum_{i\in I, j\in J} p_i q_j \mathbb{E}_{\alpha_i,\beta_j}\big(J(x_0^{i,j}, \alpha_i, \beta_j)\big).$$

Furthermore, since there exists $C,\ L > 0$, for all $T > 0$, $(u, v) \in \mathcal{U} \times \mathcal{V}$ and $(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n$:

$$|J(x_1, u, v) - J(x_2, u, v)| \leq C\|x_1 - x_2\|^{\frac{\lambda}{L}},$$

we have, by definition:

$$|\mathcal{J}\big(X_0, p, q, (\alpha_i)_{i\in I}, (\beta_j)_{j\in J}\big) - \mathcal{J}\big(X_1, p, q, (\alpha_i)_{i\in I}, (\beta_j)_{j\in J}\big)| \leq C\|X_0 - X_1\|^{\frac{\lambda}{L}}, \qquad (4.3.1)$$

for any $X_0,\ X_1 \in (\mathbb{R}^n)^{I\times J}$ and any $(p, q, (\alpha_i)_{i\in I}, (\beta_j)_{j\in J})$ (i.e., the function $\mathcal{J}$ is Hölder continuous on X_0). By (4.3.1), we have the following

Lemma 4.3.6. *$\mathcal{J}\big(X_0, p, q, (\alpha_i)_{i\in I}, (\beta_j)_{j\in J}\big)$ is bounded and continuous with respect to (X_0, p, q). More precisely, $\mathcal{J}$ is Hölder continuous with respect to X_0 and Lipschitz continuous with respect to (p, q).*

We can now define the following upper- and lower-value functions of the game $\mathcal{G}(X_0, p, q)$.

$$V^+(X_0, p, q) := \inf_{(\alpha_i)_{i\in I}} \sup_{(\beta_j)_{j\in J}} \mathcal{J}\big(X_0, p, q, (\alpha_i)_{i\in I}, (\beta_j)_{j\in J}\big);$$
$$V^-(X_0, p, q) := \sup_{(\beta_j)_{j\in J}} \inf_{(\alpha_i)_{i\in I}} \mathcal{J}\big(X_0, p, q, (\alpha_i)_{i\in I}, (\beta_j)_{j\in J}\big).$$

Observe that clearly $V^- \leq V^+$. When $V^- = V^+$, the game is said to have a value and we define its value function $V := V^+$.

Remark 4.3.7. *As usual, $-V^+(X_0, p, q)$ can be viewed as the lower-value function of a differential game denoted by $-\mathcal{G}(X_0, p, q)$ which has the same dynamic but which running cost is $-J(x_0, u, v)$ where with the roles of the two players are interchanged.*

We can obtain, as in [16], the regularity of both V^+ and V^-.

Lemma 4.3.8. *V^+ and V^- are both bounded and continuous with respect to (X_0, p, q), Hölder continuous with respect to the X variable and Lipschitz continuous with respect to (p, q).*

Proof. The regularity of both V^+ and V^- can be easily obtained from Lemma 4.3.6. □

In order to associate the value functions to a Hamilton-Jacobi-Isaacs equation in the viscosity sense, we will need to obtain a sub-dynamic programming principle for the convex conjugate of the lower-value function on p. Let us state the following convexity result for V^+ and V^-.

Lemma 4.3.9. *V^+ and V^- are both convex with respect to p and concave with respect to q.*

Proof. We only need to prove the lemma for V^+. Since $\mathcal{J}\big(X_0, p, q, (\alpha_i)_{i\in I}, (\beta_j)_{j\in J}\big)$ is linear with respect to (p, q), we have firstly:

$$\begin{aligned} V^+(X_0,p,q) &= \inf_{(\alpha_i)_{i\in I}} \sup_{(\beta_j)_{j\in J}} \mathcal{J}\big(X_0,p,q,(\alpha_i)_{i\in I},(\beta_j)_{j\in J}\big) \\ &= \inf_{(\alpha_i)_{i\in I}} \sum_{j\in J} q_j \sup_{\beta\in\mathcal{B}_r} \sum_{i\in I} p_i \mathbb{E}_{\alpha_i,\beta}\big(J(x_0^{i,j},\alpha_i,\beta)\big), \end{aligned}$$

and thus V^+ is concave on q (because $q \mapsto \sum_{j\in J} q_j \sup_{\beta\in\mathcal{B}_r} \sum_{i\in I} p_i \mathbb{E}_{\alpha_i,\beta_j}\big(J(x_0^{i,j},\alpha_i,\beta_j)\big)$ is linear, thus concave, on q).

Secondly, let us prove that $p \mapsto V^+(X_0,p,q)$ is convex. We fix (X_0, q). Let $\lambda > 0$ and $p^\lambda = \lambda p^1 + (1-\lambda)p^2$ with p^1, $p^2 \in \Delta I$. Without loss of generality, we suppose that $p_i^\lambda > 0$ for any $i \in I$. For any $\epsilon > 0$, let $(\bar{\alpha}_i^\epsilon)_{i\in I} \in \mathcal{A}_r^I$ and $(\tilde{\alpha}_i^\epsilon)_{i\in I} \in \mathcal{A}_r^I$ be respectively ϵ-optimal strategies for $V^+(X_0,p^1,q)$ and $V^+(X_0,p^2,q)$. Let us construct a new strategy $(\alpha_i^\epsilon)_{i\in I} \in \mathcal{A}_r^I$ by defining:

$$\Omega_{\alpha_i^\epsilon} = [0,1]\times\Omega_{\bar{\alpha}_i^\epsilon}\times\Omega_{\tilde{\alpha}_i^\epsilon}; \mathcal{F}_{\alpha_i^\epsilon} = \mathcal{B}([0,1])\otimes\mathcal{F}_{\bar{\alpha}_i^\epsilon}\otimes\mathcal{F}_{\tilde{\alpha}_i^\epsilon}; \mathbb{P}_{\alpha_i^\epsilon} = \mathcal{L}([0,1])\otimes\mathbb{P}_{\bar{\alpha}_i^\epsilon}\otimes\mathbb{P}_{\tilde{\alpha}_i^\epsilon},$$

and

$$\text{for any } v\in\mathcal{V},\ \alpha_i^\epsilon(\omega_1,\omega_2,\omega_3,v) = \begin{cases} \bar{\alpha}_i^\epsilon(\omega_2,v), \text{ if } \omega_1 \in [0, \dfrac{\lambda p_i^1}{p_i^\lambda}); \\ \tilde{\alpha}_i^\epsilon(\omega_3,v), \text{ if } \omega_1 \in [\dfrac{\lambda p_i^1}{p_i^\lambda}, 1]. \end{cases}$$

One can check that $(\alpha_i^\epsilon)_{i\in I} \in \mathcal{A}_r^I$. Thus, we have:

$$\begin{aligned} &\sum_{j\in J} q_j \sup_{\beta\in\mathcal{B}_r} \sum_{i\in I} p_i^\lambda \mathbb{E}_{\alpha_i^\epsilon,\beta}\big(J(x_0^{i,j},\alpha_i^\epsilon,\beta)\big) \\ =&\sum_{j\in J} q_j \sup_{\beta\in\mathcal{B}_r} \sum_{i\in I} p_i^\lambda \Big\{ \frac{\lambda p_i^1}{p_i^\lambda} \mathbb{E}_{\bar{\alpha}_i^\epsilon,\beta}\big(J(x_0^{i,j},\bar{\alpha}_i^\epsilon,\beta)\big) + \frac{(1-\lambda)p_i^2}{p_i^\lambda} \mathbb{E}_{\tilde{\alpha}_i^\epsilon,\beta}\big(J(x_0^{i,j},\tilde{\alpha}_i^\epsilon,\beta)\big)\Big\} \\ =&\sum_{j\in J} q_j \sup_{\beta\in\mathcal{B}_r} \sum_{i\in I} \Big\{ \lambda p_i^1 \mathbb{E}_{\bar{\alpha}_i^\epsilon,\beta}\big(J(x_0^{i,j},\bar{\alpha}_i^\epsilon,\beta)\big) + (1-\lambda)p_i^2 \mathbb{E}_{\tilde{\alpha}_i^\epsilon,\beta}\big(J(x_0^{i,j},\tilde{\alpha}_i^\epsilon,\beta)\big)\Big\} \\ \le&\lambda \sum_{j\in J} q_j \sup_{\beta\in\mathcal{B}_r} \sum_{i\in I} p_i^1 \mathbb{E}_{\bar{\alpha}_i^\epsilon,\beta}\big(J(x_0^{i,j},\bar{\alpha}_i^\epsilon,\beta)\big) + (1-\lambda) \sum_{j\in J} q_j \sup_{\beta\in\mathcal{B}_r} \sum_{i\in I} p_i^2 \mathbb{E}_{\tilde{\alpha}_i^\epsilon,\beta}\big(J(x_0^{i,j},\tilde{\alpha}_i^\epsilon,\beta)\big) \\ \le&\lambda V^+(X_0,p^1,q) + (1-\lambda)V^+(X_0,p^2,q) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, taking the infimum of both sides in the above inequality of $\mathcal{A}_r^I$, we obtain:

$$V^+(X_0,p^\lambda,q) \le \lambda V^+(X_0,p^1,q) + (1-\lambda)V^+(X_0,p^2,q).$$

Thus V^+ is convex on p. The proof is complete. □

Let us extend $p \mapsto V^-(X_0, p, q)$ to $\mathbb{R}^J$ by $+\infty$ outside of Δ, i.e. we define:

$$V^-(X_0, \zeta, q) := \begin{cases} V^-(X_0, \zeta, q), & \text{if } \zeta \in \Delta(I); \\ +\infty, & \textbf{else.} \end{cases}$$

Similarly, we extend $q \mapsto V^+(X_0, p, q)$ to $\mathbb{R}^I$ by $-\infty$ outside of $\Delta(J)$. We have, by the above two lemmas, that after such extensions, V^- is still convex and lower semi-continuous with respect to p and V^+ is still concave and upper semi-continuous with respect to q. From now on, we do not distinguish the value functions from their extended versions.

We recall the definition of the Fenchel convex and concave conjugate of V^+ and V^+ respectively.

$$V^{-,*}(X_0, \xi, q) := \sup_{p \in \Delta(I)} \langle p, \xi \rangle - V^-(X_0, p, q), \text{ for any } \xi \in \mathbb{R}^I;$$
$$V^{+,\#}(X_0, p, \hat{q}) := \inf_{q \in \Delta(J)} \langle q, \hat{q} \rangle - V^+(X_0, p, q), \text{ for any } \hat{q} \in \mathbb{R}^J.$$

The following technical result provides a reformation of $V^{-,*}$ (cf. [65]).

Lemma 4.3.10. *We have, for any* $(X_0, \xi, q) \in (\mathbb{R}^n)^{I \times J} \times \mathbb{R}^I \times \Delta(J)$*:*

$$V^{-,*}(X_0, \xi, q) = \inf_{(\beta_j)_{j \in J}} \sup_{\alpha \in \mathcal{A}_r} \max_{i \in I} \Big\{ \xi_i - \textstyle\sum_{j \in J} q_j \mathbb{E}_{\alpha, \beta_j} \big(J(x_0^{i,j}, \alpha, \beta_j) \big) \Big\}$$

Proof. Let $z(\xi)$ denote the right hand side of the inequality. By definition, z is Lipschitz continuous (cf. Lemma 4.3.8). Let us prove that z is convex. As in the proof of Lemma 4.3.9, we use a "splitting procedure". For $\xi^\lambda = \lambda \xi^1 + (1-\lambda)\xi^2$ with ξ^1, $\xi^2 \in \mathbb{R}^I$ and $\epsilon > 0$, let $(\beta_j^1)_{j \in J}$ and $(\beta_j^2)_{j \in J}$ be respectively ϵ-optimal strategies for $z(\xi^1)$ and $z(\xi^2)$. We construct a new strategy $(\beta^\lambda) j)_{j \in J} \in \mathcal{B}_r^J$ by defining:

$$\Omega_{\beta_j^\lambda} = [0,1] \times \Omega_{\beta_j^1} \times \Omega_{\beta_j^2}; \mathcal{F}_{\beta_j^\lambda} = \mathcal{B}([0,1]) \otimes \mathcal{F}_{\beta_j^1} \otimes \mathcal{F}_{\beta_j^2}; \mathbb{P}_{\beta_j^\lambda} = \mathcal{L}([0,1]) \otimes \mathbb{P}_{\beta_j^1} \otimes \mathbb{P}_{\beta_j^2},$$

and

$$\text{for any } u \in \mathcal{U},\ \beta_j^\lambda(\omega_1, \omega_2, \omega_3, u) = \begin{cases} \beta_j^1(\omega_2, u), \text{ if } \omega_1 \in [0, \lambda); \\ \beta_j^2(\omega_3, u), \text{ if } \omega_1 \in [\lambda, 1]. \end{cases}$$

One can check that $(\beta_j^\lambda)_{j \in J} \in \mathcal{B}_r^J$. Thus, we have:

$$\begin{aligned} &\sup_{\alpha \in \mathcal{A}_r} \max_{i \in I} \Big\{ \xi_i^\lambda - \sum_{j \in J} q_j \mathbb{E}_{\alpha, \beta_j^\lambda} \big(J(x_0^{i,j}, \alpha, \beta_j^\lambda) \big) \Big\} \\ =& \sup_{\alpha \in \mathcal{A}_r} \max_{i \in I} \Big\{ \lambda \Big(\xi_i^1 - \sum_{j \in J} q_j \mathbb{E}_{\alpha, \beta_j^1} \big(J(x_0^{i,j}, \alpha, \beta_j^1) \big) \Big) + (1-\lambda) \Big(\xi_i^2 - \sum_{j \in J} q_j \mathbb{E}_{\alpha, \beta_j^2} \big(J(x_0^{i,j}, \alpha, \beta_j^2) \big) \Big) \Big\} \\ \leq& \lambda z(\xi^1) + (1-\lambda) z(\xi^2) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we obtain the convexity of $z(\xi)$. By definition, we have furthermore:

$$\begin{aligned}
z^*(p) &= \sup_{\xi\in\mathbb{R}^I} \langle p,\xi\rangle - z(\xi)\\
&= \sup_{\xi\in\mathbb{R}^I}\Big\{\langle p,\xi\rangle - \inf_{(\beta_j)_{j\in J}}\sup_{\alpha\in\mathcal{A}_r}\max_{i\in I}\Big\{\xi_i - \sum_{j\in J} q_j\mathbb{E}_{\alpha,\beta_j}\big(J(x_0^{i,j},\alpha,\beta_j)\big)\Big\}\Big\}\\
&= \sup_{\xi\in\mathbb{R}^I}\sup_{(\beta_j)_{j\in J}}\Big\{\langle p,\xi\rangle - \max_{i\in I}\sup_{\alpha\in\mathcal{A}_r}\Big\{\xi_i - \sum_{j\in J} q_j\mathbb{E}_{\alpha,\beta_j}\big(J(x_0^{i,j},\alpha,\beta_j)\big)\Big\}\Big\}\\
&= \sup_{(\beta_j)_{j\in J}}\sup_{\xi\in\mathbb{R}^I}\Big\{\langle p,\xi\rangle - \max_{i\in I}\sup_{\alpha\in\mathcal{A}_r}\Big\{\xi_i - \sum_{j\in J} q_j\mathbb{E}_{\alpha,\beta_j}\big(J(x_0^{i,j},\alpha,\beta_j)\big)\Big\}\Big\}\\
&= \sup_{(\beta_j)_{j\in J}}\sum_{i\in I} p_i \inf_{\alpha\in\mathcal{A}_r}\sum_{j\in J} q_j\mathbb{E}_{\alpha,\beta_j}\big(J(x_0^{i,j},\alpha,\beta_j)\big)\\
&= \sup_{(\beta_j)_{j\in J}}\inf_{(\alpha_i)_{i\in I}}\sum_{i\in I} p_i\sum_{j\in J} q_j\mathbb{E}_{\alpha,\beta_j}\big(J(x_0^{i,j},\alpha,\beta_j)\big) = V^-(X_0,p,q).
\end{aligned}$$

Thus $z^*(p) = V^-(X_0,p,q)$. By the Fenchel-Moreau theorem, $V^{-,*}(X_0,\xi,q) = z^{**}(\xi) = z(\xi)$ and our wished result follows. The proof is complete. □

4.4 Dynamic Programming and Hamilton-Jacobi-Isaacs Equation

In this section, we prove that V^+ and V^- are respectively viscosity sub- and super-solution of a Hamilton-Jacobi-Isaacs equation. To achieve this goal, let us first prove a sub-dynamic programming principle for the convex conjugate of V^-.

Proposition 4.4.1 (Sub-dynamic Programming Principle for $V^{-,*}$)**.** *For any $h > 0$ small enough, we have:*

$$V^{-,*}(X_0,\xi,q) \le e^{-\lambda h}\inf_{\beta\in\mathcal{B}_d}\sup_{\alpha\in\mathcal{A}_d} V^{-,*}\big((X_h^{x_0^{i,j},\alpha,\beta})_{i,j},\xi(h,\alpha,\beta),q\big), \tag{4.4.1}$$

where $\xi(h,\alpha,\beta) = e^{\lambda h}\big(\xi_i - \sum_{j\in J} q_j\int_0^h e^{-\lambda t}\ell(X_t^{x_0^{i,j},\alpha,\beta},\alpha,\beta)dt\big)_{i\in I}$.

Proof. Let us point out that:

$$V^{-,*}(X_0,\xi,q) = \inf_{(\beta_j)_{j\in J}}\sup_{\alpha\in\mathcal{A}_d}\max_{i\in I}\Big\{\xi_i - \sum_{j\in J} q_j\mathbb{E}_{\alpha,\beta_j}\big(J(x_0^{i,j},\alpha,\beta_j)\big)\Big\}.$$

As in Lemma 4.3.8, one can prove that $V^{-,*}(X_0,\xi,q)$ is Hölder continuous on X_0 and Lipschitz continuous on (ξ,q). For any $(X,\bar p)\in(\mathbb{R}^n)^{I\times J}\times\mathbb{R}^I$, let $(\beta_j^{X,\bar p})_{j\in J}\in\mathcal{B}_r^J$ be a ϵ-optimal strategy for $V^{-,*}(X,\bar p,q)$. By the regularity of $V^{-,*}$, there exists $\delta_{X,\bar p} > 0$ such that

for any $(Y,\bar{p}') \in B\big((X,\bar{p});\delta_{X,\bar{p}}\big)$, $(\beta_j^{X,\bar{p}})_{j\in J}$ is still a 2ϵ-optimal strategy for $V^{-,*}(Y,\bar{p}',q)$. Let $\cup_{1\le k\le m}E_k = B\big((X_0,\xi);\|f\|_\infty+1\big)$ be a finite Borel partition such that for any $1\le k\le m$, there exists (Y_k,ξ^k) with $E_k\subset B\big((Y_k,\xi^k);\frac{\delta_{Y_k,\xi^k}}{2}\big)$. For future convenience, for any $1\le k\le m$, we denote by $(\beta_j^k)_{j\in J}$ the strategy $(\beta_j^{Y_k,\xi^k})_{j\in J}$ and let $\tau>0$ be a common delay of $(\beta_j^k)_{j\in J}$, for any $1\le k\le m$. Without loss of generality, we suppose that for any $1\le k\le m$, $\tau\le\frac{\delta_{Y_k,\xi^k}}{2}$. Let $h>0$ be small enough. For any $\beta\in\mathcal{B}_d$, let us construct a new strategy $(\beta_j)_{j\in J}\in\mathcal{B}_r^J$ by setting:

$$\Omega_{\beta_j}=\Pi_{k=1}^m\Omega_{\beta_j^k};\mathcal{F}_{\beta_j}=\otimes_{k=1}^m\mathcal{F}_{\beta_j^k};\mathbb{P}_{\beta_j}=\otimes_{k=1}^m\mathbb{P}_{\beta_j^k};$$

and for any $\omega=(\omega_k)_{1\le k\le m}\in\Omega_{\beta_j}$, $u\in\mathcal{U}$, and $t\ge 0$:

$$\beta_j(\omega,u)(t)=\begin{cases}\beta(u)(t), & \text{if } t\le h;\\ \beta_j^k(\omega_k,u(\cdot+h))(t-h), & \text{if } t>h \text{ and } \big((X_{h-\tau}^{x_0^{i,j},u,\beta(u)})_{i,j},\xi(h,u,\beta)\big)\in E_k.\end{cases}$$

One can easily check that $(\beta_j)_{j\in J}\in\mathcal{B}_r^J$. For any $\alpha\in\mathcal{A}_d$, we define the strategy $\bar{\alpha}\in\mathcal{A}_d$ as follows. For any $v\in\mathcal{V}$, $\bar{\alpha}(v)=\alpha(\tilde{v})(\cdot+h)$ with

$$\tilde{v}(t)=\begin{cases}v_{\alpha,\beta}(t), & \text{if } t\in[0,h];\\ v(t-h), & \text{if } t>h.\end{cases}$$

Then we have:

$$\begin{aligned}
&\sup_{\alpha\in\mathcal{A}_d}\max_{i\in I}\Big\{\xi_i-\sum_{j\in J}q_j\mathbb{E}_{\alpha,\beta_j}\big(J(x_0^{i,j},\alpha,\beta_j)\big)\Big\}\\
=&\sup_{\alpha\in\mathcal{A}_d}\max_{i\in I}\Big\{\xi_i-\sum_{j\in J}q_j\mathbb{E}_{\beta_j}\big(\int_0^{+\infty}e^{-\lambda t}\ell\big(X_t^{x_0^{i,j},\alpha,\beta_j},\alpha,\beta_j\big)dt\big)\Big\}\\
=&\sup_{\alpha\in\mathcal{A}_d}\max_{i\in I}\Big\{\big(\xi_i-\sum_{j\in J}q_j\int_0^h e^{-\lambda t}\ell\big(X_t^{x_0^{i,j},\alpha,\beta},\alpha,\beta\big)dt\big)-\\
&\sum_{j\in J}q_j\mathbb{E}_{\beta_j}\big(\int_h^{+\infty}e^{-\lambda t}\ell\big(X_t^{x_0^{i,j},\alpha,\beta_j},\alpha,\beta_j\big)dt\big)\Big\}\\
=&e^{-\lambda h}\sup_{\alpha\in\mathcal{A}_d}\max_{i\in I}\sum_{1\le k\le m}\Big\{\xi(\alpha,\beta,h)_i-\sum_{j\in J}q_j\mathbb{E}_{\beta_j^k}\big(\int_0^{+\infty}e^{-\lambda t}\ell\big(X_t^{X_h^{x_0^{i,j},\alpha,\beta},\bar{\alpha},\beta_j^k},\bar{\alpha},\beta_j^k\big)dt\big)\Big\}\mathbf{1}_{E_k}\\
\le&e^{-\lambda h}\sup_{\alpha\in\mathcal{A}_d}V^{-,*}\big((X_h^{x_0^{i,j},\alpha,\beta})_{i,j},\xi(h,\alpha,\beta),q\big)+2\epsilon.
\end{aligned}$$

Since $\epsilon>0$ is arbitrary, letting $\epsilon\to 0^+$ and taking the infimum of $\beta\in\mathcal{B}_d$ on both sides of the above inequality, we obtain (4.4.1) and the proof is thus complete. □

Let $\mathcal{S}(I)$ and $\mathcal{S}(J)$ be respectively the sets of all $I \times I$ symmetric real matrix and that of all $J \times J$ symmetric real matrix. Let us set, for any $(A, p) \in \mathcal{S}(I) \times \Delta(I)$:

$$\lambda_{min}(A,p) := min\{\langle Az, z\rangle \mid z \in T_{\Delta(I)}(p), \|z\| = 1\};$$
$$\lambda_{max}(A,p) := max\{\langle Az, z\rangle \mid z \in T_{\Delta(I)}(p), \|z\| = 1\},$$

where:

$$T_{\Delta(I)}(p) = \{z \in \mathbb{R}^I \mid \sum_{i\in I} z_i = 0 \text{ and } \forall i \in I,\ z_i < 0 \Rightarrow p_i > 0\}.$$

Similarly, we set, for any $(B, q) \in \mathcal{S}(J) \times \Delta(J)$:

$$\lambda_{min}(B,q) := min\{\langle Bz, z\rangle \mid z \in T_{\Delta(J)}(q), \|z\| = 1\};$$
$$\lambda_{max}(B,q) := max\{\langle Bz, z\rangle \mid z \in T_{\Delta(J)}(q), \|z\| = 1\}.$$

In this section, we will need the following technical result, of which a detailed proof can be found in, for instance [74].

Lemma 4.4.2. *Let $W(x,p)$ be a continuous function on $\mathbb{R}^m \times \Delta(I)$ which is convex with respect to p. Let $\varphi \in \mathcal{C}^2(\mathbb{R}^m \times \Delta(I))$ be a test function such that $W - \varphi$ has a local minimum on $\mathbb{R}^m \times \Delta(I)$ at some point (x_0, p_0). If:*

$$\lambda_{min}(D^2_{pp}\varphi(x_0,p_0), p_0) > 0, \tag{4.4.2}$$

then there exists some $\delta,\ \eta > 0$ such that:

$$W(x,p) \geq \varphi(x,p_0) + W(x_0,p_0) - \varphi(x_0,p_0) + \langle D_p\varphi(x,p_0), p - p_0\rangle + \frac{\eta}{2}\|p - p_0\|^2,$$

for any $x \in B(x_0;\delta)$ and $p \in \Delta(I)$.

Let us define the Hamiltonians: for any $(W, X_0, p, q, \zeta) \in \mathbb{R} \times (\mathbb{R}^n)^{I\times J} \times \Delta(I) \times \Delta(J) \times (\mathbb{R}^n)^{I\times J}$, we set:

$$H^-(W,X_0,p,q,\zeta) := -\lambda W + \sup_{v\in V}\inf_{u\in U}\Big\{\langle\zeta, \big(f(x_0^{i,j},u,v)\big)_{i,j\in I\times J}\rangle + \sum_{i,j} p_i q_j \ell\big(x_0^{i,j},u,v\big)\Big\};$$
$$H^+(W,X_0,p,q,\zeta) := -\lambda W + \inf_{u\in U}\sup_{v\in V}\Big\{\langle\zeta, \big(f(x_0^{i,j},u,v)\big)_{i,j\in I\times J}\rangle + \sum_{i,j} p_i q_j \ell\big(x_0^{i,j},u,v\big)\Big\}.$$

By dividing both sides of the sub-dynamic programming principle of $V^{-,*}$ by h and passing $h \to 0^+$, we obtain the following:

Proposition 4.4.3. *For any $q_0 \in \Delta(J)$ fixed, $V^-(\cdot, q_0)$ satisfies:*

$$\min\Big\{\lambda_{min}\big(D^2_{pp}V^-(X,p,q_0),p\big), H^-\big(V^-(X,p,q_0),X,p,q_0,D_XV^-(X,p,q_0)\big)\Big\} \leq 0$$

in the viscosity sense with state constraint in $(\mathbb{R}^n)^{I\times J}\times\Delta(I)$. More precisely, for any test function $\varphi \in \mathcal{C}^2\big((\mathbb{R}^n)^{I\times J}\times\Delta(I)\big)$ such that $V^-(X,p,q_0)-\varphi(X,p)$ has a local minimum at $(X_0,p_0)\in(\mathbb{R}^n)^{I\times J}\times\Delta(I)$, one has:

$$\min\Big\{\lambda_{min}\big(D^2_{pp}\varphi(X_0,p_0),p_0\big), H^-\big(V^-(X_0,p_0,q_0),X_0,p_0,q_0,D_X\varphi(X_0,p_0)\big)\Big\} \leq 0.$$

Proof. Let $\varphi \in \mathcal{C}^2\big((\mathbb{R}^n)^{I\times J}\times\Delta(I)\big)$ be a test function such that $V^-(X,p,q)-\varphi(X,p)$ has a local minimum at (X_0,p_0). Without loss of generality, let us suppose that:

$$\lambda_{min}(D^2_{pp}\varphi(X_0,p_0),p_0) > 0.$$

Then by Lemma 4.4.2, we have: there exists $\delta,\ \eta>0$ such that:

$$\begin{aligned} V^-(X,p,q) \geq& \varphi(X,p_0)+V^-(X_0,p_0,q)-\varphi(X_0,p_0)+\langle D_p\varphi(X,p_0),p-p_0\rangle+\\ &\frac{\eta}{2}\|p-p_0\|^2, \end{aligned} \tag{4.4.3}$$

for any $X\in B(X_0;\delta)$ and $p\in\Delta(I)$. The above inequality yields: for any $\xi\in\mathbb{R}^I$,

$$\begin{aligned} \langle\xi,p\rangle - V^-(X,p,q) \leq& -\varphi(X,p_0)-V^-(X_0,p_0,q)+\varphi(X_0,p_0)-\langle D_p\varphi(X,p_0)-\xi,p-p_0\rangle\\ &-\frac{\eta}{2}\|p-p_0\|^2+\langle\xi,p_0\rangle. \end{aligned}$$

By taking the supremum of $p\in\Delta(I)$ on both sides of the above inequality, we obtain:

$$\begin{aligned} V^{-,*}(X,\xi,q) \leq& \sup_{p\in\Delta(I)}\Big\{-\langle D_p\varphi(X,p_0)-\xi,p-p_0\rangle-\frac{\eta}{2}\|p-p_0\|^2\Big\}-\varphi(X,p_0)-\\ &V^-(X_0,p_0,q)+\varphi(X_0,p_0)+\langle\xi,p_0\rangle\\ \leq& -\varphi(X,p_0)-V^-(X_0,p_0,q)+\varphi(X_0,p_0)+\langle\xi,p_0\rangle+\frac{1}{2\eta}\|D_p\varphi(X,p_0)-\xi\|^2. \end{aligned}$$

By the sub-dynamic principle for $V^{-,*}$, we have, for $h>0$ small enough:

$$V^{-,*}(X_0,\xi,q) \leq e^{-\lambda h}\inf_{\beta\in\mathcal{B}_d}\sup_{\alpha\in\mathcal{A}_d}V^{-,*}\big((X_h^{x_0^{i,j},\alpha,\beta})_{i,j},\xi(h,\alpha,\beta),q\big),$$

with $\xi(h,\alpha,\beta)=e^{\lambda h}\big(\xi_i-\sum_{j\in J}q_j\int_0^h e^{-\lambda t}\ell(X_t^{x_0^{i,j},\alpha,\beta},\alpha,\beta)dt\big)_{i\in I}$. Combining these above two inequality, we have that for $h>0$ small enough (such that $(X_h^{x_0^{i,j},\alpha,\beta})_{i,j}\in B(X_0,\delta)$):

$$\begin{aligned} V^{-,*}(X_0,\xi,q) \leq& e^{-\lambda h}\inf_{\beta\in\mathcal{B}_d}\sup_{\alpha\in\mathcal{A}_d}V^{-,*}\big((X_h^{x_0^{i,j},\alpha,\beta})_{i,j},\xi(h,\alpha,\beta),q\big)\\ \leq& e^{-\lambda h}\inf_{\beta\in\mathcal{B}_d}\sup_{\alpha\in\mathcal{A}_d}\Big\{-\varphi\big((X_h^{x_0^{i,j},\alpha,\beta})_{i,j},p_0\big)-V^-(X_0,p_0,q)+\varphi(X_0,p_0)+\\ &\langle\xi(h,\alpha,\beta),p_0\rangle+\frac{1}{2\eta}\|D_p\varphi\big((X_h^{x_0^{i,j},\alpha,\beta})_{i,j},p_0\big)-\xi(h,\alpha,\beta)\|^2\Big\}. \end{aligned} \tag{4.4.4}$$

In addition, by taking $X = X_0$ in (4.4.3) and $\bar{\xi} = D_p\varphi(X_0, p_0)$, we have $\bar{\xi} \in \partial_p^- V^-(X_0, p_0, q)$, which implies:

$$V^{-,*}(X_0, \bar{\xi}, q) = \langle \bar{\xi}, p_0 \rangle - V^-(X_0, p_0, q).$$

Combining the above equality with (4.4.4), we obtain:

$$\begin{aligned}
&(1 - e^{-\lambda h})V^-(X_0, p_0, q) \\
\geq &\sup_{\beta \in \mathcal{B}_d} \inf_{\alpha \in \mathcal{A}_d} \Big\{ e^{-\lambda h}\big(\varphi\big((X_h^{x_0^{i,j},\alpha,\beta})_{i,j}, p_0\big) - \varphi(X_0, p_0)\big) + \langle p_0, \bar{\xi} - e^{-\lambda h}\bar{\xi}(h, \alpha, \beta)\rangle - \\
&\frac{e^{-\lambda h}}{2\eta}\|D_p\varphi\big((X_h^{x_0^{i,j},\alpha,\beta})_{i,j}, p_0\big) - \bar{\xi}(h, \alpha, \beta)\|^2 \Big\}.
\end{aligned} \tag{4.4.5}$$

For $\epsilon > 0$, let $\bar{v} \in V$ be a constant control and $\alpha^{\epsilon h} \in \mathcal{A}_d$ be an ϵh-optimal strategy for the right-hand side of the above inequality against $\bar{v}$. (4.4.5) yields:

$$\begin{aligned}
\epsilon h \geq &e^{-\lambda h}\big(\varphi\big((X_h^{x_0^{i,j},\alpha^{\epsilon h},\bar{v}})_{i,j}, p_0\big) - \varphi(X_0, p_0)\big) + \langle p_0, \bar{\xi} - e^{-\lambda h}\bar{\xi}(h, \alpha^{\epsilon h}, \bar{v})\rangle - \\
&\frac{2e^{-\lambda h}}{\eta}\|D_p\varphi\big((X_h^{x_0^{i,j},\alpha^{\epsilon h},\bar{v}})_{i,j}, p_0\big) - \bar{\xi}(h, \alpha^{\epsilon h}, \bar{v})\|^2 + (e^{-\lambda h} - 1)V^-(X_0, p_0, q).
\end{aligned} \tag{4.4.6}$$

Let us do some estimations of the terms in the right-hand side of (4.4.6). First of all, we have:

$$(e^{-\lambda h} - 1)V^-(X_0, p_0, q) = -\lambda h V^-(X_0, p_0, q) + o(h). \tag{4.4.7}$$

Secondly,

$$\begin{aligned}
&\langle p_0, \bar{\xi} - e^{-\lambda h}\bar{\xi}(h, \alpha^{\epsilon h}, \bar{v})\rangle = \sum_{i,j} p_{0,i}q_j \int_0^h e^{-\lambda t}\ell\big(X_t^{x_0^{i,j},\alpha^{\epsilon h}(\bar{v}),\bar{v}}, \alpha^{\epsilon h}(\bar{v})(t), \bar{v}\big)dt \\
=&o(h) + \sum_{i,j} p_{0,i}q_j \int_0^h e^{-\lambda t}\ell\big(x_0^{i,j}, \alpha^{\epsilon h}(\bar{v})(t), \bar{v}\big)dt \\
=&o(h) + \sum_{i,j} p_{0,i}q_j \int_0^h \ell\big(x_0^{i,j}, \alpha^{\epsilon h}(\bar{v})(t), \bar{v}\big)dt.
\end{aligned} \tag{4.4.8}$$

Thirdly,

$$\begin{aligned}
&\|D_p\varphi\big((X_h^{x_0^{i,j},\alpha^{\epsilon h},\bar{v}})_{i,j}, p_0\big) - \bar{\xi}(h, \alpha^{\epsilon h}, \bar{v})\|^2 = \|D_p\varphi\big((X_h^{x_0^{i,j},\alpha^{\epsilon h},\bar{v}})_{i,j}, p_0\big) - \\
&e^{\lambda h}\big(D_p\varphi(X_0, p_0)_i - \sum_{j \in J} q_j \int_0^h e^{-\lambda t}\ell(X_t^{x_0^{i,j},\alpha^{\epsilon h},\bar{v}}, \alpha^{\epsilon h}(\bar{v})(t), \bar{v})dt\big)_{i \in I}\|^2 \\
\leq &\|D_p\varphi\big((X_h^{x_0^{i,j},\alpha^{\epsilon h},\bar{v}})_{i,j}, p_0\big) - e^{\lambda h}D_p\varphi(X_0, p_0)\|^2 + o(h) \\
\leq &\|D_p\varphi\big((X_h^{x_0^{i,j},\alpha^{\epsilon h},\bar{v}})_{i,j}, p_0\big) - D_p\varphi(X_0, p_0)\|^2 + o(h) = o(h).
\end{aligned} \tag{4.4.9}$$

Finally, we have:

$$\begin{aligned}
&\varphi\big((X_h^{x_0^{i,j},\alpha^{\epsilon h},\bar v})_{i,j},p_0\big)-\varphi(X_0,p_0)\\
=&\int_0^h\langle D_X\varphi\big((X_t^{x_0^{i,j},\alpha^{\epsilon h},\bar v})_{i,j},p_0\big),\big(f(X_t^{x_0^{i,j},\alpha^{\epsilon h},\bar v},\alpha^{\epsilon h}(\bar v)(t),\bar v)\big)_{i,j\in I\times J}\rangle dt\\
=&o(h)+\int_0^h\langle D_X\varphi(X_0,p_0),\big(f(X_t^{x_0^{i,j},\alpha^{\epsilon h},\bar v},\alpha^{\epsilon h}(\bar v)(t),\bar v)\big)_{i,j\in I\times J}\rangle dt\\
=&o(h)+\int_0^h\langle D_X\varphi(X_0,p_0),\big(f(x_0^{i,j},\alpha^{\epsilon h}(\bar v)(t),\bar v)\big)_{i,j\in I\times J}\rangle dt
\end{aligned}\tag{4.4.10}$$

Combining (4.4.6) with the estimations (4.4.7)-(4.4.10), we obtain:

$$\begin{aligned}
&\epsilon h+o(h)\geq-\lambda hV^-(X_0,p_0,q)+e^{-\lambda h}\int_0^h\Big\{\langle D_X\varphi(X_0,p_0),\big(f(x_0^{i,j},\alpha^{\epsilon h}(\bar v)(t),\bar v)\big)_{i,j\in I\times J}\rangle+\\
&\sum_{i,j}p_{0,i}q_j\ell\big(x_0^{i,j},\alpha^{\epsilon h}(\bar v)(t),\bar v\big)\Big\}dt\geq he^{-\lambda h}\inf_{u\in U}\Big\{\langle D_X\varphi(X_0,p_0),\big(f(x_0^{i,j},u,\bar v)\big)_{i,j\in I\times J}\rangle+\\
&\sum_{i,j}p_{0,i}q_j\ell\big(x_0^{i,j},u,\bar v\big)\Big\}-\lambda hV^-(X_0,p_0,q)
\end{aligned}$$

Since $\bar v$ is arbitrary, this implies:

$$\begin{aligned}
&\epsilon h+o(h)\geq-\lambda hV^-(X_0,p_0,q)+he^{-\lambda h}\sup_{v\in V}\inf_{u\in U}\Big\{\langle D_X\varphi(X_0,p_0),\big(f(x_0^{i,j},u,v)\big)_{i,j\in I\times J}\rangle+\\
&\sum_{i,j}p_{0,i}q_j\ell\big(x_0^{i,j},u,v\big)\Big\}
\end{aligned}$$

Dividing both sides of the above inequality by h and passing $h\to0^+$ yields:

$$\epsilon\geq-\lambda V^-(X_0,p_0,q)+\sup_{v\in V}\inf_{u\in U}\Big\{\langle D_X\varphi(X_0,p_0),\big(f(x_0^{i,j},u,v)\big)_{i,j\in I\times J}\rangle+\sum_{i,j}p_{0,i}q_j\ell\big(x_0^{i,j},u,v\big)\Big\}.$$

But $\epsilon>0$ is arbitrary, thus we have obtained the wished result. The proof is complete. □

Before we prove a similar result for V^+, let us first state a technical lemma.

Lemma 4.4.4. *For any $q_0\in\Delta(J)$ fixed, if a function $(X,p)\mapsto W(X,p,q_0)$ satisfies:*

$$\min\Big\{\lambda_{min}\big(D^2_{pp}W(X,p,q_0),p\big),H^-\big(W(X,p,q_0),X,p,q_0,D_XW^-(X,p,q_0)\big)\Big\}\leq0$$

in the viscosity sense with state constraint in $(\mathbb{R}^n)^{I\times J}\times\Delta(I)$. Then for any $q_0\in\Delta(J)$ fixed, $(X,p)\mapsto-W(X,p,q_0)$ satisfies:

$$\max\Big\{\lambda_{max}\big(D^2_{pp}(-W)(X,p,q_0),p\big),H^-_*\big(-W(X,p,q_0),X,p,q_0,D_X(-W)^-(X,p,q_0)\big)\Big\}\geq0$$

in the viscosity sense with state constraint in $(\mathbb{R}^n)^{I\times J}\times\Delta(I)$ and with $H^-_*(W,X,p,q,\zeta):=-H^-(-W,X,p,q,-\zeta)$.

Proof. Let $\varphi \in \mathcal{C}^2\big((\mathbb{R}^n)^{I\times J}\times\Delta(I)\big)$ be a test function such that $-W(X,p,q_0)-\varphi(X,p)$ has a local maximum at $(X_0,p_0)\in(\mathbb{R}^n)^{I\times J}\times\Delta(I)$. Then $W(X,p,q_0)-\big(-\varphi(X,p)\big)$ has a local minimum at (X_0,p_0). Thus one has:

$$\min\Big\{\lambda_{min}\big(D^2_{pp}(-\varphi)(X_0,p_0),p_0\big),H^-\big(W(X_0,p_0,q_0),X_0,p_0,q_0,D_X(-\varphi)(X_0,p_0)\big)\Big\}\leq 0.$$

The above inequality yields:

$$\min\Big\{-\lambda_{max}\big(D^2_{pp}(\varphi)(X_0,p_0),p_0\big),H^-\big(W(X_0,p_0,q_0),X_0,p_0,q_0,-D_X\varphi(X_0,p_0)\big)\Big\}\leq 0,$$

and thus:

$$\max\Big\{\lambda_{max}\big(D^2_{pp}(\varphi)(X_0,p_0),p_0\big),-H^-\big(W(X_0,p_0,q_0),X_0,p_0,q_0,-D_X\varphi(X_0,p_0)\big)\Big\}\geq 0.$$

This is exactly our wished inequality. The proof is complete. □

The above lemma, combined with Proposition 4.4.3, allows us to associate the upper-value function of the game to another Hamilton-Jacobi-Isaacs equation as its viscosity sub-solution.

Corollary 4.4.5. *For any $p_0\in\Delta(I)$ fixed, $V^+(\cdot,p_0)$ satisfies:*

$$\max\Big\{\lambda_{max}\big(D^2_{qq}V^+(X,p_0,q),q\big),H^+\big(V^+(X,p_0,q),X,p_0,q,D_XV^+(X,p_0,q)\big)\Big\}\geq 0$$

in the viscosity sense with state constraint in $(\mathbb{R}^n)^{I\times J}\times\Delta(J)$. More precisely, for any test function $\varphi\in\mathcal{C}^2\big((\mathbb{R}^n)^{I\times J}\times\Delta(J)\big)$ such that $V^+(X,p_0,q)-\varphi(X,q)$ has a local maximum at $(X_0,q_0)\in(\mathbb{R}^n)^{I\times J}\times\Delta(J)$, one has:

$$\max\Big\{\lambda_{max}\big(D^2_{qq}\varphi(X_0,q_0),q_0\big),H^+\big(V^+(X_0,p_0,q_0),X_0,p_0,q_0,D_X\varphi(X_0,q_0)\big)\Big\}\geq 0.$$

Proof. By Remark 4.3.7 and Proposition 4.4.3, we have, for any $p_0\in\Delta(I)$ fixed, $-V^+(\cdot,p_0)$ satisfies:

$$\min\Big\{\lambda_{min}\big(D^2_{qq}(-V)^+(X,p_0,q),q\big),-H^+\big(V^+(X,p_0,q),X,p_0,q,-D_XV^+(X,p_0,q)\big)\Big\}\leq 0$$

in the viscosity sense with state constraint in $(\mathbb{R}^n)^{I\times J}\times\Delta(J)$. By Lemma 4.4.4, this implies: for any $p_0\in\Delta(I)$ fixed, $(X,q)\mapsto -W(X,p_0,q)$ satisfies:

$$\max\Big\{\lambda_{max}\big(D^2_{qq}V^+(X,p_0,q),p\big),H^+\big(V^+(X,p_0,q),X,p_0,q,D_XV^+(X,p_0,q)\big)\Big\}\geq 0$$

in the viscosity sense with state constraint in $(\mathbb{R}^n)^{I\times J}\times\Delta(J)$. □

4.5 Existence of a Value

Before we state the main result of this chapter, let us first introduce a second order double-obstacle Hamilton-Jacobi equation and the corresponding notion of viscosity solutions. Let us consider a Hamiltonian $\bar{H} \in \mathcal{C}^0(\mathbb{R} \times \mathbb{R}^m \times \Delta(I) \times \Delta(J) \times \mathbb{R}^m)$. We define the following second order double-obstacle Hamilton-Jacobi-Isaacs equation:

$$\begin{cases} \min\Big\{\lambda_{min}\big(D^2_{pp}W, p\big), \max\Big\{\lambda_{max}\big(D^2_{qq}W, q\big), \bar{H}(W, x, p, q, D_xW)\Big\}\Big\} = 0; \\ \max\Big\{\lambda_{max}\big(D^2_{qq}W, q\big), \min\Big\{\lambda_{min}\big(D^2_{pp}W, p\big), \bar{H}(W, x, p, q, D_xW)\Big\}\Big\} = 0. \end{cases} \tag{4.5.1}$$

The notion of viscosity solutions of partial differential equations was first introduced by Crandall and Lions [30, 29]. Let us recall the definition of viscosity super- and sub-solutions for the above Hamilton-Jacobi-Isaacs equation.

Definition 4.5.1. *A function $W : \mathbb{R}^m \times \Delta(I) \times \Delta(J) \to \mathbb{R}$ is:*

- *a viscosity super-solution of* (4.5.1) *if it is lower semi-continuous and convex on $\Delta(I)$, and if for any fixed $\bar{q} \in \Delta(J)$, $W(\cdot, \bar{q})$ satisfies, in the viscosity sense:*

$$\min\Big\{\lambda_{min}\big(D^2_{pp}W(x, p, \bar{q}), p\big), \bar{H}(W(x, p, \bar{q}), x, p, \bar{q}, D_xW(x, p, \bar{q}))\Big\} \leq 0;$$

- *a viscosity sub-solution of* (4.5.1) *if it is upper semi-continuous and concave on $\Delta(J)$, and if for any fixed $\bar{p} \in \Delta(I)$, $W(\cdot, \bar{p})$ satisfies, in the viscosity sense:*

$$\max\Big\{\lambda_{max}\big(D^2_{qq}W(x, \bar{p}, q), q\big), \bar{H}(W(x, \bar{p}, q), x, \bar{p}, q, D_xW(x, \bar{p}, q))\Big\} \geq 0;$$

- *a viscosity solution of* (4.5.1) *if it is both a viscosity super-solution and a viscosity sub-solution of* (4.5.1).

Let us state a comparison principle for viscosity solutions of the Hamilton-Jacobi-Isaacs equation (4.5.1).

Proposition 4.5.2 (Comparison Principle)**.** *Let W_1 and W_2 be respectively a bounded continuous viscosity sub-solution and a bounded continuous viscosity super-solution of* (4.5.1). *Then we have $W_1 \leq W_2$ if they are both Lipschitz continuous with respect to (p, q) and if there exists $C > 0$ and $\gamma > 0$ such that the following conditions are satisfied:*

(C1) for any $(x, p, q, \zeta) \in \mathbb{R}^m \times \Delta(I) \times \Delta(J) \times \mathbb{R}^m$ and $s_1 \geq s_2$:

$$H(s_1, x, p, q, \zeta) - H(s_2, x, p, q, \zeta) \leq -\gamma(s_1 - s_2);$$

(C2) for any $(s,p,q,\zeta) \in \mathbb{R}\times\Delta(I)\times\Delta(J)\times\mathbb{R}^m$ *and* $x_1,\, x_2 \in \mathbb{R}^m$*:*

$$|H(s,x_1,p,q,\zeta) - H(s,x_2,p,q,\zeta)| \leq C(1+\|\zeta\|)\|x_1 - x_2\|;$$

(C3) for any $(s,x,p,q) \in \mathbb{R}\times\mathbb{R}^m\times\Delta(I)\times\Delta(J)$ *and* $\zeta_1,\, \zeta_2 \in \mathbb{R}^m$*:*

$$|H(s,x,p,q,\zeta_1) - H(s,x,p,q,\zeta_2)| \leq C\|\zeta_1 - \zeta_2\|.$$

Proof. Let us prove the proposition by contradiction. We suppose that:

$$N := \sup_{(x,p,q)} W_1(x,p,q) - W_2(x,p,q) > 0.$$

Then for $a > 0$ small enough, we have:

$$N_a := \sup_{(x,p,q)} W_1(x,p,q) - W_2(x,p,q) - a\|x\|^2 + a\|p\|^2 + a\|q\|^2 > 0,$$

and consequently $\lim_{a\to 0^+} N_a = N$. We employ the double-variable technique by defining, for any $\epsilon > 0$:

$$\begin{aligned} W_{a,\epsilon}(x,y,p,q) :=& W_1(x,p,q) - W_2(y,p,q) - \frac{\|x-y\|^2}{\epsilon} - \frac{a}{2}(\|x\|^2 + \|y\|^2) + \\ & a(\|p\|^2 + \|q\|^2). \end{aligned}$$

Since $N_a = \sup_{(x,p,q)} W_{a,\epsilon}(x,x,p,q)$, by defining: $N_{a,\epsilon} := \sup_{(x,y,p,q)} W_{a,\epsilon}(x,y,p,q)$, we have:

$$N_{a,\epsilon} \geq N_a > 0.$$

Because W_1 is upper semi-continuous and W_2 is lower semi-continuous by the definition of viscosity sub-solution and viscosity super-solution of (4.5.1), we have that $W_{a,\epsilon}$ is upper semi-continuous and coercive. Thus $N_{a,\epsilon} = \max_{(x,y,p,q)} W_{a,\epsilon}(x,y,p,q)$. Let us set, for any $a > 0$ and $\epsilon > 0$:

$$(x_{a,\epsilon}, y_{a,\epsilon}, p_{a,\epsilon}, q_{a,\epsilon}) := \arg\max_{(x,y,p,q)} W_{a,\epsilon}(x,y,p,q).$$

Since $(x_{a,\epsilon}, q_{a,\epsilon})$ is a global maximiser of $(x,q) \mapsto W_{a,\epsilon}(x, y_{a,\epsilon}, p_{a,\epsilon}, q)$, we obtain:

$$\begin{aligned} & W_1(x,p_{a,\epsilon},q) - W_2(y_{a,\epsilon},p_{a,\epsilon},q) - \frac{\|x-y_{a,\epsilon}\|^2}{\epsilon} - \frac{a}{2}(\|x\|^2 + \|y_{a,\epsilon}\|^2) + a(\|p_{a,\epsilon}\|^2 + \|q\|^2) \\ \leq & W_1(x_{a,\epsilon},p_{a,\epsilon},q_{a,\epsilon}) - W_2(y_{a,\epsilon},p_{a,\epsilon},q_{a,\epsilon}) - \frac{\|x_{a,\epsilon}-y_{a,\epsilon}\|^2}{\epsilon} - \frac{a}{2}(\|x_{a,\epsilon}\|^2 + \|y_{a,\epsilon}\|^2) + \\ & a(\|p_{a,\epsilon}\|^2 + \|q_{a,\epsilon}\|^2), \end{aligned}$$

which yields:

$$W_1(x,p_{a,\epsilon},q) \leq W_1(x_{a,\epsilon},p_{a,\epsilon},q_{a,\epsilon}) - W_2(y_{a,\epsilon},p_{a,\epsilon},q_{a,\epsilon}) + W_2(y_{a,\epsilon},p_{a,\epsilon},q)+ \frac{\|x-y_{a,\epsilon}\|^2 - \|x_{a,\epsilon}-y_{a,\epsilon}\|^2}{\epsilon} + \frac{a}{2}(\|x\|^2 - \|x_{a,\epsilon}\|^2) + a(\|q_{a,\epsilon}\|^2 - \|q\|^2).$$

Since W_2 is Lipschitz continuous and concave with respect to q, $\partial_q^+ W_2(y_{a,\epsilon},p_{a,\epsilon},q_{a,\epsilon}) \neq \emptyset$. Let $\hat{\xi}_{a,\epsilon} \in \partial_q^+ W_2(y_{a,\epsilon},p_{a,\epsilon},q_{a,\epsilon})$ and we obtain:

$$\begin{aligned} W_1(x,p_{a,\epsilon},q) \leq & W_1(x_{a,\epsilon},p_{a,\epsilon},q_{a,\epsilon}) + \langle \hat{\xi}_{a,\epsilon}, q-q_{a,\epsilon}\rangle + \frac{\|x-y_{a,\epsilon}\|^2 - \|x_{a,\epsilon}-y_{a,\epsilon}\|^2}{\epsilon} + \\ & \frac{a}{2}(\|x\|^2 - \|x_{a,\epsilon}\|^2) + a(\|q_{a,\epsilon}\|^2 - \|q\|^2). \end{aligned}$$

By denoting by $\phi(x,q)$ the right-hand side of the above inequality, we have $\phi(x,q) \in \mathcal{C}^2(\mathbb{R}^m \times \Delta(J))$ and $W_1 - \phi$ has a global maximum at $(x_{a,\epsilon},q_{a,\epsilon})$. But W_1 is a viscosity sub-solution of (4.5.1) and $D^2_{qq}\phi(x_{a,\epsilon},q_{a,\epsilon}) = -2aId_J$ is negative-definite, we have:

$$\bar{H}\big(W_1(x_{a,\epsilon},p_{a,\epsilon},q_{a,\epsilon}), x_{a,\epsilon}, p_{a,\epsilon}, q_{a,\epsilon}, \frac{2}{\epsilon}(x_{a,\epsilon}-y_{a,\epsilon}) + ax_{a,\epsilon}\big) \geq 0. \tag{4.5.2}$$

Similarly, since $(y_{a,\epsilon},p_{a,\epsilon})$ is a global maximiser of $(y,p) \mapsto W_{a,\epsilon}(x_{a,\epsilon},y,p,q_{a,\epsilon})$, we obtain:

$$\begin{aligned} & W_1(x_{a,\epsilon},p,q_{a,\epsilon}) - W_2(y,p,q_{a,\epsilon}) - \frac{\|x_{a,\epsilon}-y\|^2}{\epsilon} - \frac{a}{2}(\|x_{a,\epsilon}\|^2 + \|y\|^2) + a(\|p\|^2 + \|q_{a,\epsilon}\|^2) \\ \leq & W_1(x_{a,\epsilon},p_{a,\epsilon},q_{a,\epsilon}) - W_2(y_{a,\epsilon},p_{a,\epsilon},q_{a,\epsilon}) - \frac{\|x_{a,\epsilon}-y_{a,\epsilon}\|^2}{\epsilon} - \frac{a}{2}(\|x_{a,\epsilon}\|^2 + \|y_{a,\epsilon}\|^2) + \\ & a(\|p_{a,\epsilon}\|^2 + \|q_{a,\epsilon}\|^2), \end{aligned}$$

which yields:

$$W_2(y,p,q_{a,\epsilon}) \geq W_1(x_{a,\epsilon},p,q_{a,\epsilon}) - W_1(x_{a,\epsilon},p_{a,\epsilon},q_{a,\epsilon}) + W_2(y_{a,\epsilon},p_{a,\epsilon},q_{a,\epsilon})+ \frac{\|x_{a,\epsilon}-y_{a,\epsilon}\|^2 - \|x_{a,\epsilon}-y\|^2}{\epsilon} + \frac{a}{2}(\|y_{a,\epsilon}\|^2 - \|y\|^2) + a(\|p\|^2 - \|p_{a,\epsilon}\|^2).$$

Since W_1 is Lipschitz continuous and convex with respect to p, $\partial_p^- W_1(x_{a,\epsilon},p_{a,\epsilon},q_{a,\epsilon}) \neq \emptyset$. Let $\hat{\zeta}_{a,\epsilon} \in \partial_p^- W_1(x_{a,\epsilon},p_{a,\epsilon},q_{a,\epsilon})$ and we obtain:

$$\begin{aligned} W_2(y,p,q_{a,\epsilon}) \geq & W_2(y_{a,\epsilon},p_{a,\epsilon},q_{a,\epsilon}) + \langle \hat{\zeta}_{a,\epsilon}, p-p_{a,\epsilon}\rangle + \frac{\|x_{a,\epsilon}-y_{a,\epsilon}\|^2 - \|x_{a,\epsilon}-y\|^2}{\epsilon} + \\ & \frac{a}{2}(\|y_{a,\epsilon}\|^2 - \|y\|^2) + a(\|p\|^2 - \|p_{a,\epsilon}\|^2)). \end{aligned}$$

By denoting by $\varphi(y,p)$ the right-hand side of the above inequality, we have $\varphi(y,p) \in \mathcal{C}^2(\mathbb{R}^m \times \Delta(I))$ and $W_2 - \varphi$ has a global minimum at $(y_{a,\epsilon}, p_{a,\epsilon})$. But W_2 is a viscosity super-solution of (4.5.1) and $D^2_{pp}\varphi(y_{a,\epsilon}, p_{a,\epsilon}) = 2aId_I$ is positive-definite, we have:

$$\bar{H}\big(W_2(y_{a,\epsilon}, p_{a,\epsilon}, q_{a,\epsilon}), y_{a,\epsilon}, p_{a,\epsilon}, q_{a,\epsilon}, \frac{2}{\epsilon}(x_{a,\epsilon} - y_{a,\epsilon}) - ay_{a,\epsilon}\big) \leq 0.$$

Combining the above inequality with (4.5.2), one has:

$$\begin{aligned} 0 \leq & \bar{H}\big(W_1(x_{a,\epsilon}, p_{a,\epsilon}, q_{a,\epsilon}), x_{a,\epsilon}, p_{a,\epsilon}, q_{a,\epsilon}, \frac{2}{\epsilon}(x_{a,\epsilon} - y_{a,\epsilon}) + ax_{a,\epsilon}\big) - \\ & \bar{H}\big(W_2(y_{a,\epsilon}, p_{a,\epsilon}, q_{a,\epsilon}), y_{a,\epsilon}, p_{a,\epsilon}, q_{a,\epsilon}, \frac{2}{\epsilon}(x_{a,\epsilon} - y_{a,\epsilon}) - ay_{a,\epsilon}\big). \end{aligned}$$

Applying condition (C3), this implies:

$$\begin{aligned} 0 \leq & \bar{H}\big(W_1(x_{a,\epsilon}, p_{a,\epsilon}, q_{a,\epsilon}), x_{a,\epsilon}, p_{a,\epsilon}, q_{a,\epsilon}, \frac{2}{\epsilon}(x_{a,\epsilon} - y_{a,\epsilon})\big) - \\ & \bar{H}\big(W_2(y_{a,\epsilon}, p_{a,\epsilon}, q_{a,\epsilon}), y_{a,\epsilon}, p_{a,\epsilon}, q_{a,\epsilon}, \frac{2}{\epsilon}(x_{a,\epsilon} - y_{a,\epsilon})\big) + Ca(\|x_{a,\epsilon}\| + \|y_{a,\epsilon}\|). \end{aligned}$$

By condition (C1), the above inequality yields:

$$\begin{aligned} 0 \leq & \bar{H}\big(W_2(y_{a,\epsilon}, p_{a,\epsilon}, q_{a,\epsilon}), x_{a,\epsilon}, p_{a,\epsilon}, q_{a,\epsilon}, \frac{2}{\epsilon}(x_{a,\epsilon} - y_{a,\epsilon})\big) - \\ & \bar{H}\big(W_2(y_{a,\epsilon}, p_{a,\epsilon}, q_{a,\epsilon}), y_{a,\epsilon}, p_{a,\epsilon}, q_{a,\epsilon}, \frac{2}{\epsilon}(x_{a,\epsilon} - y_{a,\epsilon})\big) + Ca(\|x_{a,\epsilon}\| + \|y_{a,\epsilon}\|) - \\ & \gamma(W_1(x_{a,\epsilon}, p_{a,\epsilon}, q_{a,\epsilon}) - W_2(y_{a,\epsilon}, p_{a,\epsilon}, q_{a,\epsilon})). \end{aligned}$$

Finally, we apply condition (C2) in the above inequality and obtain:

$$0 \leq C(1 + \frac{2}{\epsilon}\|x_{a,\epsilon} - y_{a,\epsilon}\|)\|x_{a,\epsilon} - y_{a,\epsilon}\| + Ca(\|x_{a,\epsilon}\| + \|y_{a,\epsilon}\|) - \gamma(N_{a,\epsilon} - 2a). \tag{4.5.3}$$

To finish the proof, we claim the following technical lemma whose proof is postponed.

Lemma 4.5.3. *(a)* $\lim_{\epsilon \to 0^+} N_{a,\epsilon} = N_a$*;*

(b) For any $a > 0$ *small enough,* $\lim_{\epsilon \to 0^+} \frac{1}{\epsilon}(\|x_{a,\epsilon} - y_{a,\epsilon}\|^2) = 0$*;*

(c) There exists a constant $C' > 0$ *such that* $a(\|x_{a,\epsilon}\| + \|y_{a,\epsilon}\|) \leq C'\sqrt{a}$*.*

By the above lemma, passing to the limit $\epsilon \to 0^+$ in (4.5.3) yields:

$$0 \leq CC'\sqrt{a} - \gamma N_a + 2\gamma a.$$

Letting $a \to 0^+$ in the above inequality, we have:

$$0 \leq -\gamma N,$$

which is a contradiction. The proof is complete. □

Let us prove the technical lemma:

Proof of Lemma 4.5.3. Let $K > 0$ be a common upper bound of $|W_1|$ and $|W_2|$. We have, by definition:

$$\begin{aligned}0 <& N_a \leq N_{a,\epsilon} = W_1(x_{a,\epsilon}, p_{a,\epsilon}, q_{a,\epsilon}) - W_2(y_{a,\epsilon}, p_{a,\epsilon}, q_{a,\epsilon}) - \frac{\|x_{a,\epsilon} - y_{a,\epsilon}\|^2}{\epsilon} - \\ & \frac{a}{2}(\|x_{a,\epsilon}\|^2 + \|y_{a,\epsilon}\|^2) + a(\|p_{a,\epsilon}\|^2 + \|q_{a,\epsilon}\|^2) \leq 2K + 2a - \frac{\|x_{a,\epsilon} - y_{a,\epsilon}\|^2}{\epsilon} - \\ & \frac{a}{2}(\|x_{a,\epsilon}\|^2 + \|y_{a,\epsilon}\|^2).\end{aligned}$$

This proves that (c) holds for $a > 0$ small enough and (for any $a \leq 1$) $\frac{\|x_{a,\epsilon} - y_{a,\epsilon}\|^2}{\epsilon} \leq 2K + 2$. In addition, we have that $\|x_{a,\epsilon} - y_{a,\epsilon}\|$ tends to 0, as $\epsilon \to 0^+$. Since $(x_{a,\epsilon}, p_{a,\epsilon}, q_{a,\epsilon})_{\epsilon>0}$ is bounded for $a > 0$ fixed and small enough, let (x_a, p_a, q_a) be one of its cluster point as $\epsilon \to 0^+$ and let $(\epsilon_n)_{n\in\mathbb{N}_*}$ be a sequence of positive numbers such that $\lim_{n\to+\infty} \epsilon_n = 0$ and $\lim_{n\to+\infty}(x_{a,\epsilon_n}, p_{a,\epsilon_n}, q_{a,\epsilon_n}) = \lim_{n\to+\infty}(y_{a,\epsilon_n}, p_{a,\epsilon_n}, q_{a,\epsilon_n}) = (x_a, p_a, q_a)$. Then we have:

$$\begin{aligned}& N_a \leq \liminf_{n\to+\infty} N_{a,\epsilon_n} \\ \leq & \liminf_{n\to+\infty} \Big\{ W_1(x_{a,\epsilon_n}, p_{a,\epsilon_n}, q_{a,\epsilon_n}) - W_2(y_{a,\epsilon_n}, p_{a,\epsilon_n}, q_{a,\epsilon_n}) - \frac{a}{2}(\|x_{a,\epsilon_n}\|^2 + \|y_{a,\epsilon_n}\|^2) + \\ & a(\|p_{a,\epsilon_n}\|^2 + \|q_{a,\epsilon_n}\|^2) \Big\} \\ \leq & \limsup_{n\to+\infty} \Big\{ W_1(x_{a,\epsilon_n}, p_{a,\epsilon_n}, q_{a,\epsilon_n}) - W_2(y_{a,\epsilon_n}, p_{a,\epsilon_n}, q_{a,\epsilon_n}) - \frac{a}{2}(\|x_{a,\epsilon_n}\|^2 + \|y_{a,\epsilon_n}\|^2) + \\ & a(\|p_{a,\epsilon_n}\|^2 + \|q_{a,\epsilon_n}\|^2) \Big\} \\ \leq & W_1(x_a, p_a, q_a) - W_2(x_a, p_a, q_a) - a\|x_a\|^2 + a(\|p_a\| + \|q_a\|^2) \leq N_a.\end{aligned}$$

These above inequalities yields:

$$\begin{aligned}N_a = \lim_{n\to+\infty} & \Big\{ W_1(x_{a,\epsilon_n}, p_{a,\epsilon_n}, q_{a,\epsilon_n}) - W_2(y_{a,\epsilon_n}, p_{a,\epsilon_n}, q_{a,\epsilon_n}) - \frac{a}{2}(\|x_{a,\epsilon_n}\|^2 + \|y_{a,\epsilon_n}\|^2) + \\ & a(\|p_{a,\epsilon_n}\|^2 + \|q_{a,\epsilon_n}\|^2) \Big\},\end{aligned}$$

and thus $N_a \leq \liminf_{n\to+\infty} N_{a,\epsilon_n} \leq \limsup_{n\to+\infty} N_{a,\epsilon_n} \leq N_a$, which implies

$$\lim_{n\to+\infty} N_{a,\epsilon_n} = N_a.$$

But we have in addition:

$$\begin{aligned}0 =& N_a - N_a \\ =& \lim_{n\to+\infty}\Big\{W_1(x_{a,\epsilon_n}, p_{a,\epsilon_n}, q_{a,\epsilon_n}) - W_2(y_{a,\epsilon_n}, p_{a,\epsilon_n}, q_{a,\epsilon_n}) - \frac{a}{2}(\|x_{a,\epsilon_n}\|^2 + \|y_{a,\epsilon_n}\|^2) + \\ & a(\|p_{a,\epsilon_n}\|^2 + \|q_{a,\epsilon_n}\|^2)\Big\} - \lim_{n\to+\infty} N_{a,\epsilon_n} \\ =& \lim_{n\to+\infty}(\frac{\|x_{a,\epsilon_n} - y_{a,\epsilon_n}\|^2}{\epsilon}).\end{aligned}$$

Because the above calculation is true for any converging $(x_{a,\epsilon_n}, y_{a,\epsilon_n}, p_{a,\epsilon_n}, q_{a,\epsilon_n})_{n\in\mathbb{N}_*}$ with $\lim_{n\to+\infty}\epsilon_n = 0$, by a compactness argument, we have (a) and (b) hold. The proof is complete. □

We suppose the following Isaacs condition:

$$H^- = H^+. \tag{IC}$$

Let us set $H := H^+$ and let us consider the Hamilton-Jacobi-Isaacs equation below:

$$\begin{cases}\min\Big\{\lambda_{min}\big(D^2_{pp}W, p\big), \max\Big\{\lambda_{max}\big(D^2_{qq}W, q\big), H(W, X, p, q, D_xW)\Big\}\Big\} = 0; \\ \max\Big\{\lambda_{max}\big(D^2_{qq}W, q\big), \min\Big\{\lambda_{min}\big(D^2_{pp}W, p\big), H(W, X, p, q, D_xW)\Big\}\Big\} = 0.\end{cases} \tag{4.5.4}$$

The main result of this chapter is the following:

Theorem 4.5.4. *Under Isaacs' condition* (IC)*, the game $\mathcal{G}(X_0, p, q)$ has a value. Moreover, its value function is the unique bounded and Lipschitz continuous viscosity solution of HJ equation* (4.5.4)*.*

Proof. Since we already know that $V^- \le V^+$, we only need to prove the converse inequality. Under Isaacs' condition (IC), with Proposition 4.4.3 and Corollary 4.4.5, we have that V^- and V^+ are respectively bounded, continuous viscosity super-solution and bounded, continuous viscosity sub-solution of equation (4.5.4). In addition, one can verify that the Hamiltonian H satisfies conditions (C1)-(C3) in Proposition 4.5.2, thus the comparison principle yields that $V^+ \le V^-$ and the game has a value. The proof is complete. □

4.6 Conclusion and Perspectives

In this paper, we have proved the existence of value under the Isaacs condition for an infinite horizon differential game with asymmetric information $\mathcal{G}(X_0, p, q)$ and we have given a characterization of its value function as the unique bounded Lipschitz continuous viscosity solution of a new second-order Hamilton-Jacobi-Isaacs equation (4.5.4).

There are a few open problems related to this game model. The first one is to prove the existence of value for the game without Isaacs' condition. Although finite horizon differential games with asymmetric information have already been studied in [12], the infinite horizon case remain a challenging problem. The second one is to determine optimal strategies for the players in game $\mathcal{G}(X_0, p, q)$. The third one is to investigate infinite horizon differential games with incomplete information in which players observe a signal function during the game (for example, as in [6], both players observe at each moment part of the coordinates of the current state of the dynamical system).

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Titre : Jeux Différentiels avec Information Incomplète: Signaux et Révélation

Mots clés : Jeux Différentiels; Information Incomplète; Equations d'Hamilton-Jacobi; Révélation; Signaux.

Résumé : Cette thèse concerne les jeux différentiels à somme nulle et à deux joueurs avec information incomplète. La structure de l'information est liée à un signal que reçoivent les joueurs. Cette information est dite symétrique quand la connaissance du signal est la même pour les deux joueurs (le signal est public), et asymétrique quand les signaux reçus par les joueurs peuvent être différents (le signal est privé). Ces signaux sont révélés au cours du jeu. Dans plusieurs situations de tels jeux, il est montré dans cette thèse, l'existence d'une valeur du jeu et sa caractérisation comme unique solution d'une équation aux dérivées partielles.
Un type de structure d'information concerne le cas symétrique où le signal est réduit à la connaissance par les joueurs de l'état du système au moment où celui-ci atteint une cible donnée (les données initiales inconnues sont alors révélées). Pour ce type du jeu, nous avons introduit des stratégies non anticipatives qui dépendent du signal et nous avons obtenu l'existence d'une valeur.
Comme les fonctions valeurs sont en général irrégulières (seulement continues), un des points clefs de notre approche est de prouver des résultats d'unicité et des principes de comparaison pour des solutions de viscosité lipschitziennes de nouveaux types d'équation d'Hamilton-Jacobi-Isaacs associées aux jeux étudiés.

Title : Differential Games with Incomplete Information: Signals and Revelation

Keywords : Differential Games; Incomplete Information; Hamilton-Jacobi Equations; Revealing; Signals.

Abstract : In this thesis we investigate two-person zero-sum differential games with incomplete information. The information structure is related to a signal communicated to the players during the game. In such games, the information is symmetric if both players receive the same signal (namely it is a public signal). Otherwise, if the players could receive different signals (i.e. they receive private signals), the information is asymmetric. We prove in this thesis the existence of value and the characterization of the value function by a partial differential equation for various types of such games.
A particular type of such information structure is the symmetric case in which the players receive as their signal the current state of the dynamical system at the moment when the state of the dynamic hits a fixed target set (the unknown initial data are then revealed to both players). For this type of games, we introduce the notion of signal-depending non-anticipative strategies with delay and we prove the existence of value with such strategies.
As the value functions are in general irregular (at most continuous), a crucial step of our approach is to prove the uniqueness results and the comparison principles for viscosity solutions of new types of Hamilton-Jacobi-Isaacs equation associated to the games studied in this thesis.