Non-perturbative renormalisation group approach to some out of equilibrium systems : diffusive epidemic process and fully developped turbulence

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Contents

Chapter 1

General introduction

This manuscript presents the study of two physical systems belonging to the field of out of equilibrium statistical physics: the diffusive epidemic process, and homogeneous isotropic fully developed turbulence. The former is a simplified model for the diffusion of an epidemic in a population. More specifically, we focus on the continuous phase transition it undergoes when the population density is varied. The second system is a fluid in a turbulent stationary state, as described by the Navier-Stokes equation subjected to a random forcing. Both systems, in addition to share the property of being intrinsically out of equilibrium, are examples of critical phenomena. In this work, the study of each system is conducted using the tools coming from the framework known as the non-perturbative (or functional) renormalization group. Before delving into the particular physics of each system, let us present in this general introduction the more general context of universality and critical phenomena in statistical physics, with an emphasis on the case of systems out of equilibrium, as well as the field theoretical methods to address them.

Statistical physics is the study of systems containing a large number of degrees of freedom. Its aim is to derive a description of the global macroscopic phenomena emerging from the properties of the microscopic elementary constituents of the system and of their interactions. Due to the incomplete knowledge of the system, one is faced with the existence of fluctuations, which have to be taken into account. One often aims at building a minimal microscopic model, in the sense that it should reproduce all the known macroscopic features of the statistical system under study in the simplest way and with the least possible amount of ingredients needed. This approach is made possible because the macroscopic observables are built up by the contributions of a large number of microscopic degrees of freedom. Thus they should not be too sensitive to the microscopic details. The resolution of the model can in turn lead to new predictions and guide experiments. In this back-and-forth process, one hopes to find unifying pictures or mechanisms which shed light on universal phenomena in physics.

A prominent example of such universality is given by critical phenomena, such as continuous phase transitions. Indeed, in such case, the degrees of freedom become correlated over all the range of scales spanned by the system. As a consequence, the long distance behavior of the system loses memory of the microscopic scales and does not depend on the details of the model under study. This is reflected for example in the universality of exponents related to singular quantities at a continuous phase transition. Unfortunately, such singularities plague the approaches usually applied

to derive the macroscopic behavior from a microscopic model. Critical phenomena in statistical physics were identified to be closely related to the problem of renormalization in quantum field theory. Thus, it was tempting to apply the methods developed in this framework (Dyson 1949; Stueckelberg and Petermann 1953; Bogolyubov and Shirkov 1959), to study the critical properties of such systems. In the case of equilibrium physics, this bridge was made by (Wilson and Kogut 1974; Fisher 1974), building on earlier work by (Kadanoff 1966). They interpreted the early renormalization schemes developed for quantum field theory in a new framework, the Renormalisation Group (RG). The general idea of this method is to constuct the effective theory by integrating the degrees of freedom present in the system on an infinitesimal shell of wave-numbers, beginning from the ultraviolet (UV) cutoff of the system (the inverse of the smallest size in the system), up to the infrared (IR). If the system is at a critical point, the integration of all degrees of freedom from the UV to the IR generates singularities. To do the integration infinitesimaly allows one to understand how these singularities appear. This operation can be formulated as a differential equation giving the evolution of the system under a change of the RG scale. An exact equation for the RG flow was given by Polchinski (Polchinski 1984). In the following decade, this exact RG flow was reformulated in terms of the effective action by Wetterich, Morris and Ellwanger (Wetterich 1993; Morris 1994; Ellwanger 1994). This approach is now called the Non-Perturbative (also named functional) Renormalisation Group (NPRG).

Now, let us emphasize somes specificities of out of equilibrium systems in statistical physics. The most successful framework to take into account fluctuations is the theory of systems at equilibrium with a thermal bath, whose dynamics satisfies the detailed balance. For a system at equilibrium, the logarithm of the probability of a given configuration is assumed to be proportional to the energy associated to the configuration (Gibbs 1902). But a large part of phenomena in statistical physics do not fit in the framework of equilibrium. For example, the correlation and response functions of those systems are found not to satisfy the fluctuation-dissipation theorem. Because the dynamics of out of equilibrium systems are not constrained to satisfy detailed balance, they describe a richer physics than the one accessible to systems at or dissipating to equilibrium. For example, these systems can exhibit continuous phase transitions between fluctuating and non-fluctuating steady states (Hinrichsen 2000), which is impossible at equilibrium. Moreover, numerous out of equilibrium systems are found to be generically critical. For systems at equilibrium, critical behaviors are generally associated to continuous phase transitions. In these cases the critical behavior emerges from a fine-tuning of some parameters of the theory. However, for many out of equilibrium systems, this critical behavior emerges without any fine-tuning of the parameters. This phenomenon, sometimes termed self-organized criticality (Bak, Tang, and Wiesenfeld 1987), has maybe its famous examples interface growth, modeled by the celebrated Kardar-Parisi-Zhang equation (Kardar, Parisi, and Zhang 1986). These peculiarities of classical out of equilibrium systems make it an exciting playground to study the physics of critical phenomena using the tools of the RG (Cardy and Täuber 1998; Täuber 2014) and the NPRG (Canet, Chaté, and Delamotte 2004; Canet et al. 2010).

Unfortunately, contrary to equilibrium systems, for out of equilibrium systems there does not exist a general framework to account for fluctuations. Thus one has to average directly over the noise chosen in the model. There are two traditional ways

to generate such stochastic dynamics for classical systems. The first one is to write a spatio-temporal partial differential equation whose solutions are the space-time trajectories of the system. To account for the stochasticity, this differential equation couples the observables to a noise, which is generally assumed to be Gaussian. This facilitating hypothesis is justified by viewing the noise as emerging from the sum of many unknown small independent effects. This setup, which can be seen as a generalization of the Langevin equation, is named stochastic partial differential equation (SPDE) in the following. The second way to build a dynamics is to assume that the process can be described by a time-continuous Markov chain and in this case, the prescription of the dynamics is done by giving the master equation of the process. However, both of these formulations are not directly amenable to the treatment by renormalization methods. This gap led to many developments in the '60, notably for applications to turbulence (Kraichnan 1961; Wyld 1961), which were synthetized and popularized by (Martin, Siggia, and Rose 1973). It was realized that the dynamics of a statistical system given by a SPDE could be written in formal closeness with quantum field theory at the price of introducing an extra field for each degree of freedom of the theory. This breakthrough, named the response field formalism, enabled the use of the tools of quantum field theory to tackle critical phenomena in out-of-equilibrium statistical systems. Later it was shown by (Janssen 1976) and (De Dominicis 1976) that the field theory for the observables and the response fields could be formulated as a partition function, summing over configurations in space-time weighted by the exponential of an action. This formulation opens the way to powerful approximations relying on saddle-point methods and to a systematic way to account for particular symmetries of the model. The mapping from a SPDE to a partition function for a field theory is known collectively as the Martin-Siggia-Rose-Janssen-de Dominicis (MSRJD) formalism. Although it is not the focus of this work, let us note that in the case of quantum systems there also exist a formalism to write a partition function when the system is not at equilibrium. It is known as the Schwinger-Keldysh formalism (Schwinger 1960; Schwinger 1961; Keldysh 1964) and its semi-classical limit gives back the action of MSRJD (Kamenev 2011). In the same decade, starting directly from the evolution of the probability distribution of the observables, Doi devised another method to map statistical systems into field theory akin to the "second-quantization" in quantum systems (Doi 1976). This method leads to a different field theory from the one obtained by the MSRJD formalism and the link between the two is still subject to some discussions. The method was presented and refined in the case of reaction-diffusion processes (also named birth-death processes) on a lattice in (Peliti 1985) and since then bears the name of Doi-Peliti formalism. In the present days, out of equilibrium statistical physics is often presented as subdivided in two categories. On the one hand, processes which are defined in the continuum through their SPDE and casted to a field theory using the MSRJD formalism. On the other hand, jump processes with countable state space such as reaction-diffusion processes on a lattice, defined by their master equation and whose continuum limit is taken at the level of the Lagrangian, after using Doi-Peliti formalism. This separation is represented in this work, as the forced Navier-Stokes equation is a SPDE, and the diffusive epidemic process is a reaction-diffusion process on a lattice.

In Chap. 2 we will present the physics of both systems and the open problems which motivated our study. In Chap. 3, we will make a short presentation of the

framework of the NPRG, with an emphasis on its application to out of equilibrium field theories. Finally, in Chap. 4 and Chap. 5, we will present respectively our take on the characterization of the phase transition of the diffusive epidemic process, and on the breaking of scale invariance in homogeneous isotropic fully developed turbulence.

Version française

Ce manuscrit présente l'étude de deux systèmes physiques appartenant au domaine de la physique statistique hors équilibre : le processus épidémique diffusif et la turbulence isotrope homogène et pleinement développée. Le premier est un modèle simplifié pour la diffusion d'une épidémie dans une population. Plus spécifiquement, nous nous concentrons sur la transition de phase continue qu'il subit lorsque la densité de population est variée. Le second système est un fluide à l'état stationnaire turbulent, tel que décrit par l'équation de Navier-Stokes soumise à un forçage aléatoire. Les deux systèmes, en plus de partager la propriété d'être intrinsèquement hors équilibre, sont des exemples de phénomènes critiques. Dans ce travail, l'étude de chaque système est réalisée à l'aide des outils issus du formalisme dit de la renormalisation non perturbative (ou fonctionnelle).

Dans le Chap. 2 nous présenterons la physique des deux systèmes et les problèmes ouverts qui ont motivé notre étude. Dans le Chap. 3, nous ferons une brève présentation du formalisme du NPRG, en mettant l'accent sur son application aux théories des champs hors équilibre. Enfin, dans le Chap. 4 et Chap. 5, nous présenterons notre contribution respectivement sur la caractérisation de la transition de phase du processus épidémique diffusif, et sur la rupture de l'invariance d'échelle en turbulence isotrope homogène et pleinement développée.

Chapter 2

Universal behaviors in the diffusive epidemic process and in fully developed turbulence

In this chapter, the phenomenology and challenges of the two systems studied as part of the thesis work are presented. Firstly, in Sec. 2.1 we give a short account on the physics of the diffusive epidemic process and in particular of the phase transition between a fluctuating state and an absorbing state that this system undergoes. We take the time to present the existing literature on the subject and to uncover some remaining issues in the description of this sytem. Secondly, in Sec. 2.2 after giving the general phenomenology and challenges of fully developed turbulence in fluids, we focus on the two subjects studied here: the time-dependence of correlation functions in both two- and three-dimensional turbulence, and the existence of intermittency in two-dimensional turbulence.

Version française

Ce chapitre présente la phénoménologie et les défis des deux systèmes étudiés dans le cadre des travaux de thèse. Premièrement, dans la Sec. 2.1, nous donnons un bref compte- rendu de la physique du processus épidémique diffusif et en particulier de la transition de phase entre un état fluctuant et un état absorbant que subit ce système. Nous prenons le temps de présenter la littérature existante sur le sujet et de mettre en lumière certaines questions en suspens. Deuxièmement, dans la Sec. 2.2, après avoir présenté la phénoménologie générale et les défis de la turbulence pleinement développée, nous nous concentrons sur les deux sujets étudiés ici : la dépendance temporelle des fonctions de corrélation dans la turbulence bidimensionnelle et tridimensionnelle, et l'existence de l'intermittence en turbulence bidimensionnelle.

2.1 The absorbing phase transition in the diffusive epidemic process

The first part of the manuscript is devoted to studying the absorbing phase transition occuring in the diffusive epidemic process. Absorbing phase transitions are phase transitions to a state from which the system cannot escape, a phenomenon exclusive

to out of equilibrium physics. The diffusive epidemic process (DEP), proposed in (Wijland, Oerding, and Hilhorst 1998), is the simplest model available to describe the propagation of an epidemy in a population without immunization. The absorbing phase in this case is the state without any individuals infected such that the epidemy has disappeared. As announced in the introduction, DEP belongs to the class of reaction-diffusion processes. Let us first give a brief general introduction to these models.

Reaction-diffusion processes are defined in this work as discrete stochastic processes continuous in time, given by a set of particles hopping on a lattice and undergoing reactions randomly, such as annihilation of two particles on the same site. These processes are very simple to formulate in the form of a master equation but they can describe an extremely rich diversity of physics. The particles can be of different types and the reaction rates can be chosen such as to account for additional effects such as exclusion. The hoppings and reactions are most often chosen local, such that the particles perform random walks and the reactions happen between particles on the same site. Because each reaction can be chosen individually to model a particular process, reaction-diffusion processes are a rich playground to formulate and test minimal models. To fix the idea, let us already give the set of reactions and diffusions defining DEP:

$$\begin{aligned}
&\textbf{Infection} && A + B \xrightarrow{k} B + B \\
&\textbf{Recovery} && B \xrightarrow{1/\tau} A \\
&\textbf{Diffusion of A} && A + \emptyset \xrightarrow{D_A} \emptyset + A \\
&\textbf{Diffusion of B} && B + \emptyset \xrightarrow{D_B} \emptyset + B
\end{aligned} \tag{2.1}$$

When a A and a B individuals are on the same site, B can infect A at the rate k. B individuals recover at rate $1/\tau$. Finally, A and B diffuse at rate D_A and D_B respectively.

Going back to general reaction-diffusion processes, the first level approximation of their dynamics, the mean field approximation, gives partial differential equations which are a subject of study in themselves (Kolmogorov et al. 1991; Turing 1952). For homogeneous systems and with reaction rates chosen to model elementary processes, the mean-field approximation is simply the law of mass action for chemical reactions.

In order to theoretically study such systems further than their mean field description, one generically has to solve the infinite hierarchy of the coupled temporal evolutions obeyed by the moments of the observables of the system. One way to tackle this difficulty is to turn to numerical studies. These systems are conceptually simple to simulate numerically and can be implemented with cost-effective methods. In general, these methods are based on the Monte-Carlo algorithm to explore the phase space of the system through jumps between states (Marro and Dickman 1999).

A subset of the reaction-diffusion processes are said to be integrable. Formally, these are systems which possess enough conserved quantities such that one can decouple all the degrees of freedom of the system. For these systems, one can hope to find closed analytical expressions for any averaged observables. Integrable stochastic

processes are closely linked to quantum integrable models. In particular, stochastic processes in one spatial dimension and with exclusion can often be mapped to a quantum spin chain problem. The methods developed in this field are thus closely related to their quantum mechanical counterparts (Babelon, Bernard, and Talon 2003).

However, the largest part of physically relevant reaction-diffusion processes do not satisfy the conditions of integrability. As a consequence, one has to devise approximations to go further. Many approaches to get an approximate picture from the exact hierarchy of equations have been devised over the years. Among these and without exhaustivity, let us cite two approaches which have been successful in describing critical phenomena in reaction-diffusion systems. The first kind of approaches consist in modifying the reaction rates such as to inhibit the propagation of the correlations. In this family one finds for example the cluster mean-field method (Gutowitz, Victor, and Knight 1987). Another type of approach aims at devising mesoscopic Langevin equations for the coarse-grained observables. These Langevin equations are often justified on phenomenological grounds (Janssen 1981; Wiese 2016), but they can sometimes be derived rigorously (Kampen 2007; Gardiner et al. 1976; Kurtz 1978). Finally, reaction-diffusion processes can be mapped to a field theory in order to use the tools of RG and NPRG. As announced in the introduction, this is our choice in this work.

After this general survey on the different approaches to tackle reaction-diffusion processes, let us turn to their physics. However, before expounding on DEP, we present in the next section the directed percolation process, which is a simpler and well-studied one-species model.

2.1.1 Directed percolation

Let us give a brief summary of the directed percolation process. Indeed, the directed percolation process (DP) is the most paradigmatic model for transitions to an absorbing state and DEP can be seen as a extension of it. Furthermore, it can serve as a pedagogical introduction to the framework of reaction-diffusion processes for unfamiliar readers.

A simple way to define DP is as the continuous-time evolution of a population of particles, noted X, which are distributed on the sites of a lattice (most generally a d-dimensional hypercubic one). Each particle can hop to neighbouring sites with diffusion rate D. Moreover each particle can replicate itself with rate σ and disintegrate with rate μ, and two particles can merge with rate 2λ upon encountering. These rules are symbolically summarized in the following table:

Branching	$X \xrightarrow{\sigma} X + X$
Disintegration	$X \xrightarrow{\mu} \emptyset$
Coagulation	$2\,X \xrightarrow{2\lambda} X$
Diffusion	$X + \emptyset \xrightarrow{D} \emptyset + X$

(2.2)

The names branching, coagulation, and directed percolation come from an equivalent formulation of this model in term of a percolation problem with a preferred direction.

We refer to (Hinrichsen 2000) for an extensive review of the subject of DP. This set of rules can lead the system to a state with zero particles but do not allow to escape from it. The zero particle state is the simplest example of what is called an absorbing state. The natural question at this point is whether and for which initial conditions and values of the parameters the system will fall into this absorbing state.

Let us give a first rule of thumb answer. We assume that the diffusion rate is much faster than the branching, disintegration and coagulation rates, such that the system can be considered as homogeneous and can be described by only one observable: the total number of particles, N. For N initially large, the branching and disintegration rates should be proportional to N and the coagulation rate should be proportional to N^2, as it is proportional to the number of pairs. In this approximation, the variation of N is given by

$$\partial_t N = (\sigma - \mu)N - \lambda N^2 \,. \tag{2.3}$$

A quick stability analysis of this equation tells us that for $\sigma - \mu < 0$, the only stationary state is $N = 0$, which turns out to be stable, while for $\sigma - \mu > 0$ the only stable stationary state is $N = (\sigma - \mu)/\lambda \equiv N^*$. The system thus undergoes a phase transition from a fluctuating state ($N = N^*$) to an absorbing state ($N = 0$). Furthermore, it is readily calculated that for $\sigma - \mu \neq 0$, (2.3) gives an exponential approach to the stationary state, with a typical time $\tau = (\sigma - \mu)^{-1}$ while it acquires an algebraic behavior in t^{-1} at $\sigma - \mu = 0$. This critical slowing down of the dynamics is typical of second-order phase transitions. Of course, this first approximation has the obvious drawback that it describes a transition to a state with zero particles by making the assumption of a large number of particles in presence and by neglecting the pair correlations.

These findings prompt to give a more precise description of the model. The rules (2.2) can be interpreted in terms of a Markov process, whose master equation is given in Appendix A. In that appendix, we explain how the time evolution of the averaged observables can be derived using the generating function. In particular, the equation for the mean occupation number at the site k reads

$$\partial_t \langle N_k(t) \rangle = \Big\{ (\sigma - \mu)\langle N_k(t) \rangle - \lambda \langle N_k(t)^2 \rangle + D\big[\sum_{j/<k,j>} \langle N_j(t) \rangle - 2d \, \langle N_k(t) \rangle \big] \Big\} \tag{2.4}$$

where $N^{\underline{\ell}} = N!/(N-\ell)!$ is the falling factorial and $< i, j >$ for two neighbouring sites i and j.s Let us interpret this result. We found an evolution for $\langle N_k(t) \rangle$ which couples $\langle N_k(t) \rangle$ to a higher order moment $\langle N_k(t)^2 \rangle$. This coupling appears because the coagulation rule involves pairs of particles, thus the pair correlation information is necessary. If we were to derive the evolution equation for $\langle N_k(t)^2 \rangle$, it would involve $\langle N_k(t)^3 \rangle$ and so on. Thus we would be left with an infinite hierarchy of equations coupling the moments of the process.

The most straightforward approximation is at this point to neglect the pair correlations; this is the mean-field approximation. It is often justified when the mean number of particle is large with respect to its fluctuations. Thus one writes $\langle N_k(t) \rangle \equiv N \rho_k(t)$, where N is a large number of the order of the number of particles in the systems and $\rho_k(t)$ has a finite limit when $N \to \infty$. The mean-field approximation consists here in writing

$$\langle N_k(t)^2 \rangle \sim \langle N_k(t) \rangle^2 = N^2 \rho_k(t)^2 \,. \tag{2.5}$$

Noting $\lambda' = \lambda N$, which is of order one, one obtains a closed expression for the densities at each site of the lattice $\{\rho_k(t)\}$:

$$\partial_t \rho_k(t) = (\sigma - \mu)\rho_k(t) - \lambda' \rho_k(t)^2 + D\big[\sum_{j/<k,j>} \rho_j(t) - 2d\,\rho_k(t)\big]\,. \tag{2.6}$$

For homogeneous fields, we recover the same behavior as was derived in first approximation. However (2.4) allows us to make a more refined approximation. If we want our derivation to be valid at the transition, the number of particle per site cannot be large. This shortcoming could be patched up by not looking directly at the number of particles per site but at a coarse-grained density averaged over neighbouring sites. In doing so, one obtains the equation for the density field $\rho(\mathbf{x})$ depending on space-time: $\mathbf{x} \equiv (t, \vec{x})$. Under the same assumptions as for the mean-field approximation, one can hope for an end result looking like

$$\partial_t \rho(\mathbf{x}) = \tilde{\kappa}\rho(\mathbf{x}) - \tilde{\lambda}\rho(\mathbf{x})^2 + \tilde{D}\,\partial^2 \rho(\mathbf{x})\,, \tag{2.7}$$

with the tilde denoting effective rate due to the coarse-graining. This type of approach is fruitful in certain systems such as one-dimensional random walk with exclusion and leads to strong mathematical results (Bertini et al. 2015). Note that we recover formally the same equation directly from (2.6) if we let the space between sites go to zero and if we assume that the interpolating density is sufficiently smooth.

If one is only interested in the universal quantities associated to the absorbing phase transition of DP, this kind of approach is in fact valid in dimensions $d \geq 4$, the (upper) critical dimension of DP. The mean-field equation (2.7) allows one to refine the previous picture of the transition in $d \geq 4$. Because of the diffusive scaling between space and time, if we note ξ the typical length of the correlations of spatial fluctuations (the correlation length), we have $\xi \sim \sqrt{\tilde{D}\tau}$, where in this expression τ is the correlation time. As τ diverges as $\tau \sim \tilde{\kappa}^{-1}$ at the transition, where $\tilde{\kappa} \to 0$, we deduce that ξ diverges as $\xi \sim \tilde{\kappa}^{-1/2}$ in this approximation.

Let us list the algebraic behaviors characterizing the continuous absorbing phase transition. The transition is driven by the control parameter $\tilde{\kappa}$, its order parameter is the spatially averaged density ρ. Using standard notations, we have

$$\begin{aligned} &\textbf{Order parameter} && \rho \sim \tilde{\kappa}^{\beta}\,, \\ &\textbf{Correlation length} && \xi \sim \tilde{\kappa}^{-\nu}\,, \\ &\textbf{Correlation time} && \tau \sim \xi^z \sim \tilde{\kappa}^{-z\nu}\,. \end{aligned} \tag{2.8}$$

In the mean field approximation, we have

$$\beta = 1\,, \quad \nu = 1/2\,, \quad \text{and } z = 2\,. \tag{2.9}$$

Furthermore, it can be shown using saddle-point methods in the field theory of DP that above $d = 4$, the fluctuations of the field at the phase transition (which cannot be captured by our mean-field approach) are given by the standard Gaussian field theory with heat kernel (see Chap. 3). This fact is reflected in the scaling of the

two-point correlation C and response (or Green) functions G of the theory, which are defined respectively as

$$C(\mathbf{r}) = \langle \rho(\mathbf{x}+\mathbf{r})\rho(\mathbf{x})\rangle\,,$$
$$G(\mathbf{r}) = \frac{\delta}{\delta h(\mathbf{x})}\langle \rho(\mathbf{x}+\mathbf{r})\rangle\,. \tag{2.10}$$

The response function measures the variation in the mean density at a space-time point $\mathbf{x}+\mathbf{r}$ due to a Dirac delta perturbation h of the system at a point $\mathbf{x}$. At a second order phase transition, these functions are scale invariant. This defines two critical exponents, named the anomalous dimensions and noted here η and $\bar{\eta}$, such that in the general case one has the property

$$C(b^z t, b\vec{r}) = b^{-d-\eta}\, C(\mathbf{r})\,,$$
$$G(b^z t, b\vec{r}) = b^{-d-\frac{\eta+\bar{\eta}}{2}}\, G(\mathbf{r})\,, \quad b>0\,. \tag{2.11}$$

For $d \geq 4$, when the fluctuations are described by the Gaussian field theory with heat kernel, both anomalous dimensions are found to be zero: $\eta = \bar{\eta} = 0$.

For $d < 4$, although the continuous phase transition still exists, the critical exponents differ from their mean-field values and depend on the spatial dimension. We can try to cure this problem by modifying the partial differential mean-field equation (2.7). Although we have in essence derived an equation for the mean density, we can try to incorporate fluctuations by hand by adding a noise η to the above equation. Assuming that the fluctuations are produced by a large amount of uncorrelated events, this noise is chosen to be a centered Gaussian process. To preserve the feature of DP that $\rho = 0$ is an absorbing state, the variance of the noise has to vanish if the density is zero. The simplest choice thus reads

$$\partial_t \rho = \kappa\rho - \lambda_1\rho^2 + D\,\partial^2\rho + \eta$$
$$\langle \eta(\mathbf{x})\eta(\mathbf{x}')\rangle = 2\lambda_2\rho(\mathbf{x})\delta(t-t')\delta^d(\vec{x}-\vec{x}') \tag{2.12}$$

with some effective parameters $\{\kappa, D, \lambda_1, \lambda_2\}$. Although it is customary in the physics literature, for the more mathematically inclined reader the noise with a variance proportional to the density seems ill-defined. In the case of a Gaussian noise this problem can be cured by a rescaling, see Chap. 3. There exist more rigorous tools to extract expressions for the correlation of the noise around the deterministic solution from the master equation (Gardiner et al. 1976).

It turns out that as long as one is concerned with only the universal properties of the transition to the absorbing state, the process defined by (2.12) is equivalent to the initial DP reaction-diffusion process. The exponents obtained from simulations of (2.12) (Dickman 1994) are the same as those obtained from simulations of the initial master equation for DP (Jensen 1999). In other words, they belong to the same universality class. This fact was explained using RG power-counting arguments by (Janssen 1981) and may seem not surprising from this background. However, it begs the question as to whether we can generally describe the critical points of reaction-diffusion processes by SPDE and if there is a systematic way to do so. This question finds an answer within perturbative RG. However, we will see below that outside of the validity of perturbation theory, subtleties may arise.

The DP universality class appears in fact to represent an attractor for a large class of theoretical models. This has led to a conjecture (Janssen 1981; Grassberger 1982)

stating that any model exhibiting a phase transition to one absorbing state with a single scalar order parameter, with no other conserved quantities or symmetries and described by local dynamics belongs to the DP universality class. Paradoxically, the definite evidences of the DP exponents in experimental realizations are not numerous (Hinrichsen 2000). This fact led to investigate models with phase transitions to absorbing state which do not fall in the DP universality class. The reaction-diffusion processes with one type of particles have been substantially studied and universality classes different from DP have been identified (Hinrichsen 2000; Ódor 2004; Elgart and Kamenev 2006). For example, it is the case for processes which conserve the parity of the number of particles (Cardy and Täuber 1998; Canet et al. 2005). The universality classes associated to absorbing phase transitions with more than one type of particles are far less explored. DEP, which is one of the simplest two-species process with a transition to absorbing states fit in this context.

2.1.2 Diffusive epidemic process

The DEP is a toy model of propagation of an epidemy without immunization. It was first proposed by (Wijland, Oerding, and Hilhorst 1998) in 1997 and we will stick to their presentation. It involves particles of two types moving on a lattice, the healthy individuals and the sick ones, that we shall denote A and B respectively. Each state of the system is then described by the occupation number of A and B at each site. We will not take into account any exclusion effects.

Let us recall the set of reactions of DEP:

$$\begin{aligned}
&\textbf{Infection} && A + B \xrightarrow{k} B + B \\
&\textbf{Recovery} && B \xrightarrow{1/\tau} A \\
&\textbf{Diffusion of A} && A + \emptyset \xrightarrow{D_A} \emptyset + A \\
&\textbf{Diffusion of B} && B + \emptyset \xrightarrow{D_B} \emptyset + B
\end{aligned} \tag{2.13}$$

One notices that as DP, DEP possesses an absorbing state. Indeed, the species B can become extinct leaving only freely diffusing A particles. However, regarding the conjecture of Janssen and Grassberger, DEP differs from DP in a crucial feature: the total number of particles is conserved by the reactions. The master equation of DEP and the time evolution of the corresponding generating functional are given in Appendix A as well. The mean-field equations for the densities $\rho_A(\vec{x}, t)$ and $\rho_B(\vec{x}, t)$ are deduced following the same methods as in the previous section. They read

$$\begin{aligned}
\partial_t \rho_A &= -k\,\rho_A \rho_B + \tau^{-1} \rho_B + D_A\,\partial^2 \rho_A \\
\partial_t \rho_B &= k\,\rho_A \rho_B - \tau^{-1} \rho_B + D_B\,\partial^2 \rho_B
\end{aligned} \tag{2.14}$$

By summing both lines, it is seen that these equations conserve the total number of particle as well. Noting ρ_0 the initial spatially averaged total density, $c = \rho_A + \rho_B - \rho_0$ and renaming $\rho_B = \rho$, (2.14) is equivalent to

$$\begin{aligned}
\partial_t \rho &= k(c + \sigma)\rho - k\rho^2 + D_B\,\partial^2 \rho\,, \\
\partial_t c &= D_A\,\partial^2 (c - \mu\,\rho)\,,
\end{aligned} \tag{2.15}$$

where we have defined $\sigma \equiv \rho_0 - (k\tau)^{-1}$ and the following crucial parameter

$$\mu \equiv \frac{D_A - D_B}{D_A}\,. \tag{2.16}$$

One can interpret (2.15) as a field ρ evolving according to the dynamics of DP but whose control parameter is modulated by a conserved field. Thus, looking at homogeneous solutions, it is readily seen that the mean field exponents of DEP are the same as those of DP: $\beta = 1$, $\nu = 1/2$ and $z = 2$. Furthermore, it can be shown that the two processes share the same upper critical dimension $d_c = 4$.

It is tempting to upgrade (2.15) to a set of two coupled Langevin equations to account for the fluctuations, as was done in Sec. 2.1.1. The noise acting on the first line must account for the existence of an absorbing state at $\rho = 0$ as in DP and the noise acting on the second line must be conservative, so as not to break the conservation of the number of particles. Thus the simplest coupled Langevin equations that one can propose are

$$\begin{aligned} \partial_t \rho &= k(c+\sigma)\rho - k\rho^2 + D_B\,\partial^2\rho + \eta^\rho\,, \\ \partial_t c &= D_A\,\partial^2(c - \mu\,\rho) + \eta^c\,, \end{aligned} \tag{2.17}$$

with the following covariance for the noises

$$\begin{aligned} \langle \eta^\rho(\mathbf{x})\eta^\rho(\mathbf{x}')\rangle &= 2k'\rho(\mathbf{x})\delta(t-t')\delta^d(\vec{x}-\vec{x}')\,, \\ \langle \eta^c(\mathbf{x})\eta^c(\mathbf{x}')\rangle &= 2\mu' D_A(-\partial^2)\delta(t-t')\delta^d(\vec{x}-\vec{x}')\,, \\ \langle \eta^\rho(\mathbf{x})\eta^c(\mathbf{x}')\rangle &= 0\,. \end{aligned} \tag{2.18}$$

In fact, this model was proposed in (Janssen 2001) under the name of the directed percolation with a conserved quantity (DP-C) and its equivalence, in the sense of being in the same universality class, with DEP was justified using the same type of arguments used to prove the equivalence between the DP reaction-diffusion process (2.2) and the DP Langevin equation (2.12). We will expound these arguments in the case of DEP in Chap. 4 after having introduced the field theories of those processes. For now, it is enough to have in mind that at all orders of perturbative RG, DEP and DP-C belong to the same universality class.

At this point, let us make a short semantic note. In the sandpile community, the name C-DP (for *conserved directed percolation*) is often used to refer to the specific case $D_A = 0$ of DP-C, see (Janssen and Stenull 2016). This case is of high interest because it is known to belong to the Manna universality class describing a certain type of avalanche phenomena. Furthermore, it was shown that C-DP and the quenched Edwards-Wilkinson model for an interface moving in a quenched disorder belonged to the same universality class (Le Doussal and Wiese 2015; Janssen and Stenull 2016). It would be interesting to try to recover this result in our framework. However, to stick with the historical choice in the study of DEP and DP-C, we will absorb D_A in a redefinition of time. Because we only probe the stationary state of the system, the limit $D_A = 0$ is inaccessible in our framework.

Let us concentrate now on $D_A \neq 0$. The particular case of two diffusion constants taken equal ($\mu = 0$) had been introduced in (Kree, Schaub, and Schmittmann 1989) before the work of van Wijland et al. to model the effect of pollution on a population and had been studied using perturbative RG. The authors found that a

continuous absorbing phase transition indeed occurs in spatial dimensions $d_c < 4$. The corresponding critical point was new. In particular, it was different from the ubiquitous DP universality class. The critical exponents of this KSS universality class were found to be in most part fixed exactly by the symmetries. To all order in perturbation, they found

$$\nu = 2/d\,, \quad z = 2\,, \quad \bar{\eta} = \eta\,, \quad \text{and} \quad \beta = \nu(d+\eta)/2\,. \tag{2.19}$$

These relations leave only one exponent to be determined, η. Its value was calculated in a first-order expansion in $\epsilon = 4 - d$, which yielded

$$\eta = -\epsilon/8\,. \tag{2.20}$$

Van Wijland et al. confirmed the previous results for $\mu = 0$ and studied the case $\mu \neq 0$. Although their starting point was the master equation of DEP, in their work they truncated the resulting action to a form equivalent to DP-C. For $\mu \neq 0$, they found a new fixed point of the RG flow equation, signaling yet a new universality class (distinct from DP and KSS). For $\mu < 0$, ($D_A < D_B$), they predicted the existence of a continuous phase transition to an absorbing state described by this new (WOH) universality class. For $\mu > 0$ ($D_A > D_B$), however, the fixed point is not reachable from physical initial conditions within pertubative RG. Indeed, in order to reach the fixed point for $\mu > 0$, one of the microscopic reaction rate would have to be negative. This fact was interpreted as the existence of a discontinuous phase transition for $\mu > 0$ induced by fluctuations. This conjecture was backed up by semi-phenomenological arguments and seemingly confirmed by numerical simulations of DEP in $d = 2$ (Oerding et al. 2000).

Almost every exponent of the WOH universality class is fixed by the following relations

$$\nu = 2/d\,, \quad z = 2\,, \quad \eta = 0\,, \quad \text{and} \quad \beta = \nu(d+\eta)/2 = 1\,. \tag{2.21}$$

to all orders in ϵ. Although the relation $\nu = 2/d$ was not made explicit in (Wijland, Oerding, and Hilhorst 1998), it was shown in (Janssen 2001) that the DP-C model implied this relation for $\mu \neq 0$ as well. The only independent exponent which is not fixed in this case is $\bar{\eta}$. It was found to be to first-order in $\epsilon = 4 - d$ given by

$$\begin{aligned} &\bar{\eta} = -\frac{\lambda^*}{3\lambda^* + 1}\epsilon\,, \text{ with } \quad \lambda^* = \frac{1}{(2+\sqrt{3})^{\frac{1}{3}} + (2-\sqrt{3})^{\frac{1}{3}} - 2} \\ &\textit{i.e.} \quad \bar{\eta} \simeq -0.313\,\epsilon \end{aligned} \tag{2.22}$$

However, some of the RG predictions seem to be invalidated by further numerical simulations of DEP, reported in Tab. 2.1. The three issues concern

i) the nature of the phase transition in the case $\mu > 0$,

ii) the value of ν in the cases $\mu < 0$ and $\mu = 0$,

iii) the value of β for $\mu \neq 0$.

Regarding i), all simulations performed after (Oerding et al. 2000), both in $d = 1$ (Fulco, Messias, and Lyra 2001a; Maia and Dickman 2007) and $d = 2$ (Dickman

d	μ	β/ν	β	ν	z
1	0	0.197(2)	0.435(14)	2.21(5)	–
		0.226(20)	0.452(40)	–	–
		0.192(4)	0.384(46)	2.0(2)	2.02(4)
		–	–	2.037	1.980
		0.3125		*2*	*2*
	>0	0.113(8)	0.20(2)	1.77(3)	1.6(2)
		0.165(22)	0.330(44)	–	–
		–	–	2.0	1.992
		first order			
	<0	0.404(10)	0.929(144)	2.3(3)	2.01(4)
		0.336(15)	0.672(30)	–	–
		–	–	2.0	1.992
		1/2	*1*	*2*	*2*
2	0	0.856(4)	0.797(8)	0.932(5)	–
		0.875		*1*	*2*
	>0	0.88(5)	0.93(9)	1.06(4)	1.89(8)
		first order			

Table 2.1: Critical exponents of DEP from Monte Carlo simulations and field theoretical analyses. The values in gray are deduced assuming $\nu = 2/d$ and the values in italic are theoretical predictions. Reproduced from (Tarpin et al. 2017)

and Maia 2008), strongly indicate that the phase transition is continuous also in the case $\mu > 0$, checking in particular the absence of hysteresis (Dickman and Maia 2008). Regarding ii), early simulations in $d = 1$ for $\mu = 0$ (Freitas et al. 2000) found $\nu = 2.21(5)$. This result was criticized by Janssen (Janssen 2001) using the exact result $\nu = 2/d = 2$ from the DP-C model. The authors of (Freitas et al. 2001) replied by observing that the full DEP action includes terms which violate the symmetries which fix $\nu = 2/d$. Although these terms are irrelevant in a perturbative RG analysis near the upper critical dimension $d_c = 4$, they could become relevant away from it and in particular at $d = 1$. If the transition appearing in simulations of DEP is not driven by the DP-C fixed point, but instead by another one having less symmetries, the argument of (Janssen 2001) does not hold and the value of ν is not fixed. It could depart from $2/d$ and possibly be compatible with values from simulations. Subsequent simulations reported in (Fulco, Messias, and Lyra 2001b) partially reconcile both results suggesting that the discrepancy could be imputed to corrections to scaling. However the debate exposed in (Janssen 2001; Freitas et al. 2001) is still unresolved. Numerical simulations in $d = 2$ for equal diffusion constants $\mu = 0$ convincingly ruled out the DP exponents, but could not settle on whether $\nu = 1$ (with possible logarithmic correction) in accordance with the pertubative results, or $\nu < 1$ (Bertrand et al. 2007). Finally, regarding iii), Tab. 2.1 shows that for $\mu < 0$ in $d = 1$, either $\nu \neq 2$ or $\beta \neq 1$, and likewise for $\mu > 0$. Yet, if the transition is controlled by the DP-C fixed point, the symmetries constraints imply $\nu = 2/d = 2$ and $\beta = 1$.

In Chap. 4, we expound on our take on these issues, which was published in (Tarpin et al. 2017) We applied the tools of NPRG, in the form of the modified

local potential approximation to both DP-C and DEP to try to clarify the relation between both systems. The NPRG calculations are guided by a careful analysis of symmetries of both models. Now, let us turn to the second system studied in the thesis, that is fully developed turbulence.

2.2 Breaking of scale invariance in fully developed turbulence

The second part of the manuscript is devoted to the study of incompressible fully developed turbulence within the NPRG formalism. In this section, we take some time to present the physics of turbulence. In Sec. 2.2.1 we give a short historical introduction to the Navier-Stokes equation and to the development of the study of turbulence, up to its scale-invariant description by Kolmogorov in 1941. In Sec. 2.2.2, we present the phenomenon of intermittency in three-dimensional (3-D) turbulence and we give a brief summary of the theoretical efforts aimed at understanding it. Finally, in section Sec. 2.2.3 and Sec. 2.2.4 we focus on the two particular subjects of the thesis: the time dependence of the correlation functions in turbulence and the intermittency in two-dimensional (2-D) turbulence respectively.

2.2.1 The Navier-Stokes equation and scale-invariance in turbulence

Let us give a selected historical account on the steps which led to the discovery of power-law behavior and universality in turbulence. This account is mainly an abridged version of (Davidson et al. 2011). The equation which models the motion of the fluid is Navier-Stokes (NS) equation, supplemented with the continuity equation. Due to the finite viscosity of the fluid, the NS equation is dissipative. As a consequence, in order to probe the stationary state one has to inject energy into the fluid. This is done by adding a forcing term to the NS equation, making it a prime example of driven-dissipative system. We will limit ourselves to the study of flows with constant density, such that the continuity equation is equivalent to the condition that the velocity flow is divergenceless. The Navier-Stokes equation and the incompressibility condition read

$$\begin{aligned} \partial_t \vec{v} + (\vec{v}\cdot\vec{\partial})\vec{v} &= -\frac{1}{\rho}\vec{\partial} p + \nu \partial^2 \vec{v} + \vec{f}\,, \\ \vec{\partial}\cdot\vec{v} &= 0\,. \end{aligned} \tag{2.23}$$

In this equation, $\vec{v}$ is the velocity field, p is the pressure field (which is fixed by the incompressibility condition) and $\vec{f}$ is a forcing per unit of mass acting in the bulk (all the potential forces such as gravity are already included in the pressure term). ρ is the density and ν is the kinematic viscosity of the fluid. This equation dates back to the early 19th century. Indeed it was derived in 1822 by (Navier 1823), building upon the work of (Euler 1757). Euler derived the equation of conservation of momentum in the continuum for perfect fluids and Navier extended it to the viscous case by assuming the linearity of the stress tensor. The derivation was revisited by Stokes, in a more readable form (Stokes 1845). However, the progress

on the understanding of this equation and its relation to turbulence has been slow. Stokes limited himself to the study of flows whose velocity is the gradient of a scalar potential or in the limiting case of laminar flow, where dissipative effects dominate. However, let us note that he is the first to derive an energy balance for fluid motion in (Stokes 1850), that we write below in the case of incompressible turbulence. We note

$$E(t) = \frac{1}{2}\int_\Omega v^2\,, \tag{2.24}$$

the energy per unit of density, where the integration is done over the spatial domain Ω of the fluid. If we stipulate that the velocity is zero at the boundary of Ω, the energy balance equation simply reads

$$\partial_t E = W - \bar{\varepsilon}\,. \tag{2.25}$$

In this expression, $W = \int_\Omega f \cdot v$ is the power furnished per unit of density and the total dissipation $\bar{\varepsilon}$ is given by

$$\bar{\varepsilon} = \int_\Omega \varepsilon = \int_\Omega \frac{\nu}{2}\,\mathrm{tr}\, e^2\,, \tag{2.26}$$

where e is twice the Cauchy infinitesimal strain tensor:

$$e_{\alpha\beta} = \partial_\alpha v_\beta + \partial_\beta v_\alpha\,. \tag{2.27}$$

The convective term $\vec{v}\cdot\vec{\partial}\vec{v}$ does not bring in nor dissipate energy out of the sytem, thus its name of inertial term. One can already make the elementary observation that in order to have a stationary state of the fluid, the dissipation must be equal to the injected power.

It is only in 1883 that Reynolds investigated the transition to turbulent flows, where Stoke's solutions would not apply (Reynolds 1883). A fluid initially at rest which is stirred or accelerated will develop turbulent features such as the random apparition of swirls above a certain threshold of velocity. An adimensionned number characterizes the strength of the turbulence, named since the Reynolds number. It is constructed from the kinematical viscosity of the fluid, ν, the typical large length scale of the system, L and the typical velocity V of the flow at scale L. The Reynolds number is the ratio of the strength of the convective force felt by a fluid element over the strength of the dissipative force. It reads

$$\mathrm{Re} = \frac{VL}{\nu}\,. \tag{2.28}$$

Thus, in a flow with a low Reynolds number ($\mathrm{Re} \lesssim 2000$) the dissipative effects dominate and the turbulent swirls are damped down, the flow is said to be laminar. For example honey, with its viscosity of $\nu_{honey} = 10\,m^2.s^{-1}$ is always laminar in everyday life situations. A laminar flow is entirely determined by its boundary conditions. On the contrary, when the Reynolds number is high, the convective effects dominate and the velocity field becomes turbulent. The transition to turbulence was studied in (Reynolds 1895) by the same author, using the now called Reynolds-averaged NS equations, which marks the beginning of the statistical approach to turbulence. We will concentrate our study on fully developed turbulence, that is a statistically stationary flow at very high Reynolds number.

Following (Lesieur 2008), we can define a turbulent flow as having the following three characteristics: it is unpredictable, it possesses good mixing properties, and it involves a large range of scales. Example of natural realizations of fully developed turbulence are numerous. The first example which may come to mind is the flow downstream of a high throughput structure such as a dam ($\mathrm{Re} \sim 10^4$). However, let us not forget that the prominent example is simply atmospheric air. Indeed the very low viscosity of air ($\nu_{air} \sim 10^{-5}\, m^2.s^{-1}$) implies that a turbulent flow is produced even for low velocities. In the atmospheric boundary layer, the Reynolds can be as high as 10^7 (although at these scales the density cannot be considered constant anymore). Due to the ubiquity of turbulent flows, a better understanding of the mechanisms of turbulence is a high stake subject for many fields of science such as geophysics and astrophysics as well as for the industry (aeronautic, wind and hydrolic energy generation,...).

However, with the exception of the isolated work of Richardson in (Richardson 1922; Richardson 1926) which introduced the idea of energy cascade and self-similarity, it took some forty years to see progresses in the statistical theory of turbulence. Under the impulsion of Prandtl and Taylor, the two-point correlations in turbulence and the energy spectrum were investigated experimentally in wind tunnels using hot-air anemometry (Prandtl and Reichardt 1934; Simmons and Salter 1934; L. et al. 1937). These experimental data were interpreted by Taylor using the hypothesis that the fluctuations of the velocity in the turbulent region of the fluid could be well approximated as isotropic, even if the mechanism for producing the turbulence was not (Taylor 1935a; Taylor 1935b; Taylor 1935c; Taylor 1935d; Taylor 1936). Following this insight, we will concentrate as well on homogeneous and isotropic turbulence.

At the end of the '30, hints for the scale-invariance of turbulence were piling up. When the Reynolds number is high enough, the scale at which the energy is injected, named the integral scale and noted L, and the viscous scale are well separated. The range between these two scales is named the inertial range, because "inertial" convective effects dominate. In the inertial range, the power spectrum was found to be self-similar and universal (Simmons, Salter, and Taylor 1938; Taylor 1938b; Prandtl 1938). The idea that vortex stretching, through the non-linear convective term of the NS equation, was the main transfer mechanism in the inertial range received support from the theoretical side, after Kármán and Howarth derived the exact equation giving the evolution of the two-point correlation (Kármán and Howarth 1938). Due to the non-linearity present in the Navier-Stokes equation, the Kármán-Howarth equation brings into play the three-point correlation function, thus it is not a closed equation (see also (Kármán 1937; Taylor and E. 1937; Taylor 1937; Taylor 1938a) for an earlier discussion on the importance of the convective term in turbulence).

This series of results paved the way for the decisive step made by (Kolmogorov 1941b; Kolmogorov 1941a) and (Obukhov 1941), and independently and later by (Onsager 1949; Prandtl and Wieghardt 1945; Heisenberg 1948; Weizsäcker 1948). For the first time, the hypothesis of universality of turbulence in the inertial range was clearly formulated. In (Kolmogorov 1941b), the author stated the bold hypothesis that the equal-time statistics of fully developed homogeneous isotropic 3-D turbulence for scales much smaller than L was independent of how energy was supplied to the flow and depended only on the total energy dissipation $\bar{\varepsilon}$, and on the viscosity.

By dimensional analysis, this gives a scale at which the viscous effects dominate. This scale, named the Kolmogorov scale and noted η has the following expression

$$\eta \equiv \left(\frac{\nu^3}{\bar{\varepsilon}}\right)^{\frac{1}{4}} . \tag{2.29}$$

Futhermore, the second assumption that for scales much larger than η, the statistics do not depend on the viscosity fixes the scaling of every n-point velocity correlations in the inertial range, that is for scales ℓ such that $\eta \ll \ell \ll L$. We define the longitudinal velocity increment, $\delta v_{\parallel}(\ell)$, as

$$\delta v_{\parallel}(\ell) = \frac{\vec{\ell}}{\ell} \cdot \left[\vec{v}(t, \vec{x} + \vec{\ell}) - \vec{v}(t, \vec{x})\right] . \tag{2.30}$$

The structure function of order n, noted $S_n(\ell)$, is the n-th moment of the longitudinal velocity increment. The universality hypothesis implies that in the inertial range,

$$S_n(\ell) = \langle \left[\delta v_{\parallel}(\ell)\right]^n \rangle \sim C_n (\bar{\varepsilon}\ell)^{\frac{n}{3}} . \tag{2.31}$$

where the C_n are universal constants. In particular, it implies that the isotropic Fourier spectrum of the energy, noted $E(p)$ has the following behavior

$$E(p) \sim p^{-5/3} \tag{2.32}$$

in the inertial range.

A strong support to these hypotheses was given by the exact result for S_3 derived in (Kolmogorov 1941a) from the Kármán-Howarth equation. Looking back at the balance equation of energy (2.25), the condition of stationarity implies that the total dissipation $\bar{\varepsilon}$ is equal to the total injected power and has a finite limit when the viscosity goes to zero. This fact is known as the dissipative anomaly. Kolmogorov found that at infinite Reynolds number,

$$S_3(\ell) = \langle \delta v_{\parallel}(\ell)^3 \rangle \underset{\ell \to 0}{\sim} -\frac{4}{5} \bar{\varepsilon} \ell . \tag{2.33}$$

This result is the first exact law derived from the NS equation in the regime of fully developed turbulence. It is remarkable because the numerical prefactor is universal, thus its name of "four-fifths law". This law can be understood as stemming from the existence of an energy cascade in 3-D turbulence: at a wave-number k such that $L^{-1} \ll k \ll \eta^{-1}$, the energy injection and dissipation occcuring at scale k can be neglected. As a consequence, in a stationary flow, the energy flux through the momentum scale k must be independent of k and equal to the total dissipation $\bar{\varepsilon}$. A modern and pedagogical derivation of the four-fifth law is given in (Frisch 1995). Adding the hypothesis of scale invariance to the four-fifths law immediately gives back the Kolmogorov scaling (2.31) for all structure functions. The Kolmogorov hypothesis and scaling are referred to as K41 in the following. Although the hypotheses leading to K41 was questioned early on (see (Frisch 1995) for a historical account), K41 theory was considered as the standard phenomenological tool to describe turbulence and it remains useful in the present days in applied turbulence as a valid first approximation.

2.2.2 The phenomenon of intermittency in turbulence

In spite of the success of the K41 theory, more precise measurements by (Batchelor and Townsend 1949), showed that the equal-time statistics cannot be strictly scale invariant. Indeed, the authors investigated the behavior of the flatness of the velocity n-th derivative, defined as

$$\alpha_n = \frac{\langle (\partial_x^n v)^4 \rangle}{\langle (\partial_x^n v)^2 \rangle^2}. \tag{2.34}$$

While α_0 seems more or less independent of the Reynolds number, for higher values of n, α_n grows with the Reynolds number. Moreover, the dependence on the Reynolds number grows with n. As higher values of n probe variations of the velocity on smaller spatial scales, it signals that extreme turbulent events are more susceptible to happen at small scales. This phenomenon is termed intermittency. Although such findings (see also (Kuo and Corrsin 1971)) do not directly invalidate K41 because they concern the dissipative range (Kraichnan 1967b; Frisch and Morf 1981), it hinted at a possible deviation from K41 in the inertial range. The existence of intermittency in the inertial range was proven beyond doubt in (Anselmet et al. 1984), whose result is reproduced in Fig. 2.1. The authors measured the scaling exponents ζ_n of the structure functions $S_n(\ell)$ as function of n. Intermittency translates into the multiscaling of the structure functions:

$$S_n(\ell) \sim \ell^{\zeta_n}, \tag{2.35}$$

for $\eta \ll \ell \ll L$, where the ζ_n do not depend linearly on n. The exponent $\zeta_3 = 1$ is fixed by the four-fifth law, but $\zeta_{n\neq 3} \neq n/3$.

The clearest way to picture this phenomenon is to plot the distribution of probability of the velocity increment $\delta v_\parallel(\ell)$ for different scales ℓ. Today, these distributions are accessible experimentally. We reproduce in Fig. 2.2 a figure from (Chevillard et al. 2012), using data from (Kahalerras et al. 1998). The Gaussian distribution at large spatial scales (lowest curve) reflects the Gaussian nature of the forcing. When going to smaller and smaller scales, the distribution moves away from a Gaussian one as extreme events become more and more probable.

Many phenomenological models of turbulence which account for this phenomenon have been put forward. As noted by Landau and Onsager (Eyink and Sreenivasan 2006; Frisch 1995), and by (Kraichnan 1974), the K41 phenomenology relied on using the mean field approximation for the local dissipation:

$$\langle \varepsilon^n \rangle \approx \langle \varepsilon \rangle^n \propto \bar{\varepsilon}^n. \tag{2.36}$$

Multiscaling can be modeled by taking into account the fluctutations of ε. In this setting, the inertial-range intermittency is thus contained entirely in the intermittent fluctuations of ε. If we note ε_ℓ the local dissipation averaged over a scale ℓ, the intermittent behavior of ε means that the fluctuations of ε_ℓ will depend on ℓ. This is the content of the refined similarity hypothesis:

$$S_n(\ell) \sim \langle \varepsilon_\ell^{\frac{n}{3}} \rangle \ell^{\frac{n}{3}}, \tag{2.37}$$

proposed by Obukhov and Kolmogorov. In their phenomenology, the fluctuations of $\log(\varepsilon_\ell)$ are postulated to follow a Gaussian distribution with variance proportional to $\ln(L/\ell)$. This hypothesis readily gives the behavior (2.35), with the ζ_n following

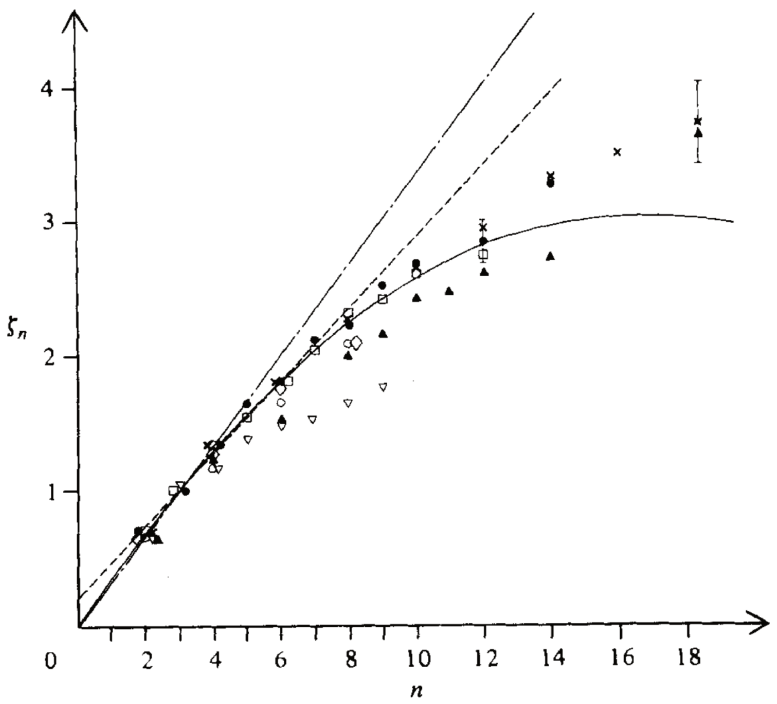

Figure 2.1: ζ_n as a function of n, from (Anselmet et al. 1984, p. 24). The points are experimental measurements. The •, □ and × are from the authors. The chain-dotted line is the prediction from K41, the dashed one from the β-model and the full one from the log-normal model (see below for an explanation).

a parabola as a function of n (Obukhov 1962; Kolmogorov 1962). The intermittency of ε can also be modeled by a multiplicative stochastic process for ε_ℓ. This approach was pioneered in (Novikov and Stewart 1964). Such phenomenological model can be reformulated as taking litteraly the idea of Richardon of a cascade of dissipative structures (Richardson 1922), but where the dissipative structures at scale ℓ have a fractal dimension $d_F < 3$ (Mandelbrot 1974). A constant fractal dimension at all scale give the β-model (Frisch, Sulem, and Nelkin 1978), equivalent to the process defined in (Novikov and Stewart 1964). However, this model gives corrections to the scaling exponents ζ_n which are linear in n. To obtain a non-linear dependency, Parisi and Frish postulated the existence of a set of such fractal structures (Parisi and Frisch 1985). They showed that in this framework the ζ_n are the Legendre transform of the singularity spectrum $d(h)$ giving the fractal dimension of the structure as a function of the associated scaling exponent. In turn, the singularity spectrum can be related to the fluctuations of ε_ℓ in the corresponding random cascade process (Benzi et al. 1984). Predictions for the fractal dimension can be made using phenomenological guess of the exact dissipative structure at play in the cascade process (Corrsin 1962; Tennekes 1968). For example, it is the case of the She-Lévêque model (She and Lévèque 1994), which assumes that the most dissipative structures are vortex filaments. The She-Lévêque model has been understood as the Poissonian limit of a random cascade process with infinite number of steps, it was generalized as a particular case of log-infinitely divisible process (Dubrulle 1994; She and Waymire 1995; Bacry and Muzy 2003). Finally, the multifractal formalism was given its modern probabilistic formulation in (Mandelbrot 1991).

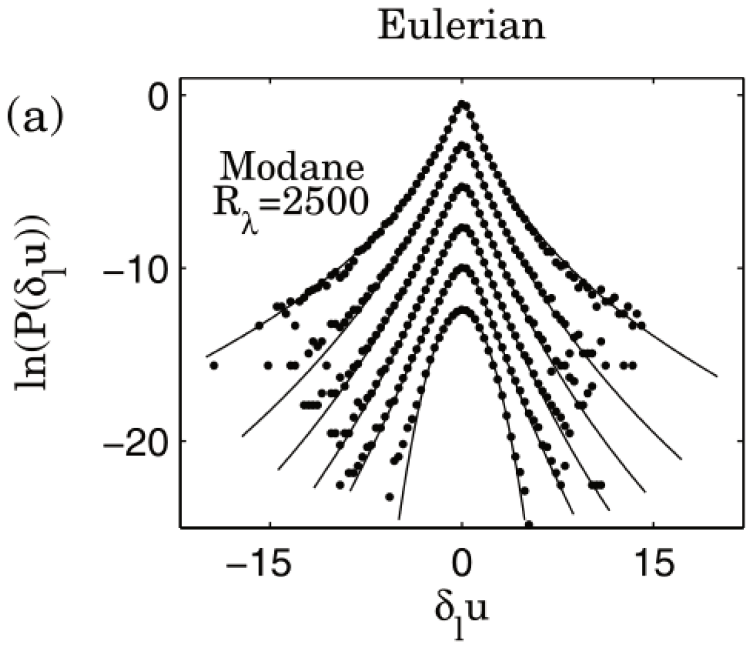

Figure 2.2: Logarithm of the distribution of the $\delta v_{\|}(\ell)$ for different value of ℓ, taken from (Chevillard et al. 2012). Each distribution has its variance normalized to unity and has been shifted vertically for readability purpose, with a higher curve corresponding to a smaller ℓ. The lowest curve corresponds to $\ell = 1.13L$ and the highest one to $\ell = L/610$.

The multifractal formalism is sufficiently powerful to describe intermittency in turbulence (Halsey et al. 1986; Meneveau and Sreenivasan 1991; Muzy, Bacry, and Arneodo 1993). Interestingly, it gives predictions for finite-Reynolds effects although it was developed to describe inertial range intermittency (Paladin and Vulpiani 1987; Nelkin 1990; Frisch and Vergassola 1991; Meneveau 1996). It can also be extended to directly describe the intermittency in the velocity increments distribution, shown in Fig. 2.2, or in the velocity gradient (Castaing, Gagne, and Hopfinger 1990; Benzi et al. 1991; Chevillard et al. 2006). However, there is no hope presently to derive the singularity spectrum of turbulence from the Navier-Stokes equation. Furthermore, these models say nothing of the peculiar features of the tensorial structure of turbulent flow, such as the preferred alignement of the vorticity with the second eigenvector of the strain tensor (Ashurst et al. 1987). More recently, a more mathematically-minded phenomenological approach to intermittency has been put forward: the construction of synthetic velocity field having the correct properties (Benzi et al. 1993; Bacry and Muzy 2003; Robert and Vargas 2008; Chevillard, Robert, and Vargas 2010). These approaches may lead to a complete description of turbulence relying on a single parameter, characterizing the strength of the intermittency. However, at this point the mathematical complexity of these phenomenological models seems to rivalize with the Navier-Stokes equation. Finally, let us stress that all these approaches have in common to focus on describing the equal-time statistics of the flow, leaving aside its non-zero time-delay properties.

To obtain intermittent behavior from the Navier-Stokes equation is the holy grail of the theoretical research in turbulence post-K41. The literature is profuse and we will focus only on field-theoretical approaches, and in particular the RG ones, as

our own work descend from it. We will not discuss here the various approaches consisting in solving approximations of the Navier-Stokes equation. In this category are the shell models (Desnianskii and Novikov 1974; Ohkitani and Yamada 1989), which are approximated dyadic versions of the Navier-Stokes equation and which show intermittent behavior. We will neither dwell on the closure approach to turbulence, which consists in closing the hierachy of equations for the correlation functions of turbulence by approximating higher order moments in terms of lower ones. This method has been explored in (Millionschikov 1941; Heisenberg 1948; Chandrasekhar 1955), but the first valid approximation of such kind was proposed by (Kraichnan 1959; Kraichnan 1965). A review of closure models and the issue of their realizability is given in (Bowman, Krommes, and Ottaviani 1993). Closure models were succesfully applied to plasma physics (Krommes 2002) but to our knowledge, none of them is able to predict intermittency corrections for turbulence.

Before describing the field-theoretical attempts, let us make a comment on the forcing $\vec{f}$ appearing in the Navier-Stokes equation (2.23). To model the non-deterministic nature of turbulent flow, without having to introduce randomness from the initial conditions, or from the way motion is conveyed at the boundary of the fluid (which would break homogeneity and isotropy), a convenient way is to use a random forcing acting on the bulk of the fluid. By asking for a statistically stationnary, homogeneous and isotropic forcing, one enforces the (statistical) stationarity, homogeneity and isotropy of the velocity field. Furthermore, because it is known experimentally that the turbulent state of the flow does not depend on the details of the forcing as long as it is concentrated on the large scales, $\vec{f}$ can be modeled as a memoryless centered Gaussian process for convenience,

$$\langle f_\alpha(t,\vec{x}) f_\beta(t',\vec{x}')\rangle = \delta(t-t') D_{\alpha\beta}(|\vec{x}-\vec{x}'|)\,. \tag{2.38}$$

Furthermore, $\vec{f}$ can be chosen solenoidal ($\partial_\alpha D_{\alpha\beta} = 0$) without loss of generality. These properties of the forcing are common to all the field-theoretic treatment of fully developed turbulence that we will present. Let us point out that a recent work undertook to analyzse the case of a forcing with small time correlations (Antonov et al. 2018). These correlations break the Galilean invariance originally present in the NS equation. However, the authors found that this symmetry is restored at large scales, such that the presence of small time correlations of the forcing do not alter the properties of the turbulent state. Hence, we restrict in the following to a forcing delta-correlated in time.

Although it has been stressed that fully developed turbulence is not a scale-invariant theory, it is an example of critical phenomena, in the sense that its physics is nonetheless controlled by a scale-invariant theory. This justifies to attack the problem with the tools of the RG. The first application of the tools of renormalization to randomly stirred fluids dynamics was done in (Forster, Nelson, and Stephen 1977). Their analysis did not describe the fully developed turbulent cascade but only the case of equipartition of energy, where the spectrum $E(k)$ goes like $E(k) \sim k^2$ in 3-D. However, it was the first time that invariance of the Navier-Stokes equation under Galilean transformations

$$\forall\, V \in \mathbb{R}^3,\ \vec{v}(t,\vec{x}) \to \vec{v}(t,\vec{x}-\vec{V}t) + \vec{V}\,, \tag{2.39}$$

was exploited to deduce non-renormalization theorems. The case of fully developed turbulence was pioneered by (De Dominicis and Martin 1979). They chose a power-

law for the spectrum of the forcing $\tilde{D}_{\alpha\beta}(q)$, defined as the Fourier transform of the forcing correlation $D_{\alpha\beta}(|\vec{x}-\vec{x}'|)$:

$$\tilde{D}_{\alpha\beta}(q) \sim q^{4-d-2\epsilon}\,, \text{ for } q \gg L^{-1} \tag{2.40}$$

where L^{-1} is an IR cutoff and ϵ is a real parameter not to be confused with the energy dissipation ε. This choice of spectrum for the forcing is necessary in order to carry on with pertubative RG. Indeed, contrary to standard critical phenomena in equilibrium statistical physics, the Navier-Stokes equation does not possess an upper critical dimension d_c above which the theory is described by Gaussian fluctuations. As a consequence, one cannot treat the problem using calculations perturbative in the distance to d_c. For the choice of the forcing above, ϵ plays the same role as $d_c - d$ in standard critical phenomena. The picture of the energy cascade applies when all the energy input is concentrated at large scale. This happens for $\epsilon > 2$. The authors of (De Dominicis and Martin 1979) found a fixed point of the RG flow. They obtained that the critical exponents obtained at first order in ϵ were in fact exact at all order. Specifically looking for scale-invariant solutions for the energy spectrum gave

$$E(p) \sim p^{1-4\epsilon/3}\,. \tag{2.41}$$

They used the transformation

$$\vec{v}(t,\vec{x}) \to \vec{v}(t,\vec{x}-\vec{R}(t)) + \dot{\vec{R}}(t)\,, \tag{2.42}$$

which is a time-gauged version of Galilean invariance (2.39), to argue how this result could hold up to $\epsilon = 2$, where one recovers the same scaling as in K41: $E(k) \sim k^{-5/3}$. This result may seem underwhelming in regard of the heavy RG machinery involved, as the critical exponents of the Navier-Stokes field theory are completely fixed by the Galilean symmetry and the requirement of stationarity. Therefore, to ask for scale invariance cannot give something else than the K41 scaling (Kraichnan 1982). The statement becomes thus that K41 is an attractive fixed point of RG trajectories for the Navier-Stokes equation. However, this result brings two comments. First, at this point it is not so clear why the critical exponents should "freeze" at their value for $\epsilon = 2$ for other IR-concentrated forcing spectrum with $\epsilon > 2$ (Fournier and Frisch 1983). Secondly, following de Dominicis and Martin, we note that as we need to reach $\epsilon > 2$ to make contact with physical energy cascade, non-perturbative effects may occur. This possibility opens the road to intermittency corrections to the K41 scaling. This research program was undertaken by Adzhemyan et al. (Adzhemyan, Vasil'ev, and Pis'mak 1983; Adzhemyan, Vasil'ev, and Gnatich 1988; Adzhemyan, Antonov, and Vasil'ev 1989), as well as in (Antonov 1991; Antonov, Borisenok, and Girina 1996) using perturbative RG and operator product expansion (Collins 1984). Let us state their findings. First, in the physical region $\epsilon > 2$, the scale-invariant correlation functions, and in particular the spectrum, are finite when $\nu \to 0$ at fixed dissipation energy ε. This shows that intermittency corrections do not come from UV singularities. Secondly, they showed how in this framework the exponents freeze at their $\epsilon = 2$ value for $\epsilon > 2$. Thirdly, they proved that for *equal-time quantities*, at $\epsilon < 2$, scale-invariant correlations functions are finite when the IR cutoff L^{-1} goes to zero. However, they were not able to extend this result for $\epsilon > 2$. In this range, the situation can be summarized as follows on the structure functions:

$$S_n(\ell) = C_n(\bar{\varepsilon}\ell)^{\frac{n}{3}}(\ell/L)^{\eta_n}\,, \text{ for } \ell \ll L\,. \tag{2.43}$$

Crucially, the authors used the symmetry(2.42) to discard a potential L^{-1} singularity for $\epsilon > 3/2$. More technically, at $\epsilon = 3/2$, a family of operators become *dangerous*, that is their scaling dimensions in the IR cutoff L^{-1} become negative. The time-gauged Galilean invariance is used to show that these operators do not participate in the operator product expansion of equal-time quantities, and thus do not play a role in their scaling. The intermittency corrections, given by η_n were out of reach of the method. See also (L'vov and Lebedev 1993; Antonov 1994) for a shorter derivation of the same result.

This conclusion was rederived in (L'vov and Procaccia 1995) with a diagrammatic method, without the need of renormalization and working directly with an IR-concentrated forcing. To bypass the IR divergences appearing at $\epsilon = 3/2$, they worked with a transformed velocity field invariant under (2.42), the *quasi-Lagrangian* variables (Belinicher and L'vov 1987). They were able to obtain expressions linking the different η_n together but no predictions for their values was obtained. Finally let us signal an isolated work conducted in (Giles 2001), which combines self-consistent determination of the effective viscosity and noise, elimination of the sweeping effect by random Galilean invariance, the operator product expansion and truncation of the renormalization flow to obtain analytical predictions for the intermittency exponents η_n, which quantitatively agree at low order ($n \leq 6$) with experiments. The attempts to tackle fully developed turbulence with the NPRG are described in Sec. 2.2.3.

At this point, let us make a quick digression on another model exhibiting intermittency, but where field-theoretical methods were successfully applied. Instead of looking directly at the velocity field, Kraichnan proposed a model of a passive scalar submitted to diffusion and advected by a memoryless (white-noise in time) Gaussian velocity field (Kraichnan 1968). The velocity field is chosen incompressible and is given a roughness exponent ξ. This model reads for a generic space dimension d

$$\begin{aligned} &\partial_t \theta + \vec{v} \cdot \vec{\partial} \theta = \kappa \, \partial^2 \theta \,, \\ &\langle v_\alpha(t, \vec{x}) v_\beta(t', \vec{x}') \rangle = D_0 \, \delta(t - t') \int_{\vec{q}} P_{\alpha\beta}(q) \frac{e^{i\vec{q}\cdot(\vec{x}-\vec{x}')}}{(q^2 + m^2)^{\frac{d}{2}+\frac{\xi}{2}}} \,, \end{aligned} \tag{2.44}$$

where

$$P_{\alpha\beta}(q) = (\delta_{\alpha\beta} - q_\alpha q_\beta / q^2) \tag{2.45}$$

is the transverse projector ensuring incompressibility. With the choice $d = 3$ and $\xi = 4/3$, one recovers the K41 velocity field (the m^2 is an IR cutoff, necessary to prevent the integral from diverging at $q = 0$ for $\xi > 0$). In this setting, Kraichnan was able to make an educated guess for the anomalous scaling of the θ structure functions (Kraichnan 1994), which was put on a more rigorous footing a year later using the so-called *zero-mode* method (Gawędzki and Kupiainen 1995). A review of this approach is given in (Falkovich, Gawędzki, and Vergassola 2001). This led to develop a new theoretical framework to understand the origin of the intermittency in the Navier-Stokes equation. However, for the original stochastic Navier-Stokes (SNS) problem, the method did not suceed to produce predictions for the exponents. Another more recent promising approach is the application of the instanton method, pioneered for turbulence in (Falkovich et al. 1996). This approach allowed to obtain the intermittency exponents in the Kraichnan model, see (Dombre 2010) for a recent development, linking instanton and zero-mode method. However, the situation is the same as for the latter as no predictions could be made in the case of the Navier-Stokes equation. It is worth noting that the perturbative RG approach described

above was also able to give results for intermittency corrections in the Kraichnan model (Adzhemyan, Antonov, and Vasil'ev 1998; Antonov 2006). These works were successfully reformulated in the NPRG framework in (Pagani 2015), although only the perturbative regime was considered.

Let us wrap up this section by a comment. Kolmogorov original hypothesis concerned only equal-time statistics. Indeed, at unequal times, other effects play a role which may severely spoil scale invariance. Accordingly, all the phenomelogical models of intermitency and fields theoretical studies that were reviewed here have in common to concentrate on equal-time statistics. Symptomatically, the only available rigorous predictions concerning fully developed turbulence concern equal-time statistics as well. Thus, it seems worthwhile to stress that the breaking of scale-invariance exhibited by unequal-time statistics in turbulence is far from trivial within the theory of critical phenomena. As a consequence, these effects were the subject of some controversy (Yakhot, Orszag, and She 1989; Nelkin and Tabor 1990; Chen and Kraichnan 1989) (see next section). Furthermore, new predictions in this direction may help to broaden the contact of theory with experiments and simulations (Canet et al. 2017; Debue et al. 2018). Unequal-time properties of fully developed turbulence are the subject of the next section.

2.2.3 Time dependence of correlation functions in turbulence

The typical decorrelation time of the velocity, and also the behavior of the frequency energy spectrum, have early been debated. Indeed, the extension of Kolmogorov original local similarity hypothesis to unequal-time statistics, as proposed in (Monin and Yaglom 1973), leads to a typical time at wave-number p which goes like

$$\tau_{\mathrm{K}} \sim \bar{\varepsilon}^{-1/3} p^{-2/3} \,. \tag{2.46}$$

This scaling is in contradiction with considerations stemming from other semi-phenomenological arguments taking into account the sweeping of small eddies by larger ones (Heisenberg 1948; Kraichnan 1959; Tennekes 1975). The corresponding typical time is in this case the following

$$\tau_{\mathrm{eddy}} \sim (\bar{\varepsilon} L)^{-1/3} p^{-1} \,. \tag{2.47}$$

In the above expression, $(\bar{\varepsilon} L)^{1/3}$ is of the order of the rms velocity of the fluid. This second option breaks K41 self-similarity hypothesis, as it involves explicitly the integral scale L. From the analysis of a simplified model of advection, Kraichnan deduced that the sweeping effect should yield for the two-point correlation function a Gaussian behavior in the variable pt where p is the wave-number and t the time delay. Such time-dependence would have a direct consequence for the energy spectrum. Whereas the local similarity hypothesis extended to the time domain predicts an energy spectrum behaving as ω^{-2} for Eulerian velocities, the sweeping effect leads instead to the power law $\omega^{-5/3}$, see also (Nelkin and Tabor 1990) on the subject.

The sweeping effect was identified early on by Kraichnan as an obstacle to prove K41 scaling or corrections to it (Kraichnan 1964). To overcome this difficulty, he was led to develop a Lagrangian version of his closure model (Kraichnan 1965). The same difficulty appears in the field-theoretical approach and is at the origin of

the quasi-Lagrangian variables of Belinicher and L'vov (Belinicher and L'vov 1987). However, the first RG studies seem to overlook this problem and obtain the K41 spectrum although they work with Eulerian velocities and do not seem to address sweeping (De Dominicis and Martin 1979; Fournier and Frisch 1983; Yakhot and Orszag 1986). In fact, the authors of (Yakhot, Orszag, and She 1989) claimed that scale invariance would hold also in the time domain, questioning the result of Kraichnan. (Chen and Kraichnan 1989) argued that the RG analysis of (Yakhot, Orszag, and She 1989), which used the power-law forcing of (2.40), could only hold at small ϵ and was not valid for $\epsilon > 2$. In the light of the result obtained by the RG method in the previous section, this concern seems justified.

This controversy was finally settled: the Gaussian behavior predicted from sweeping has now been observed in many numerical simulations (Orszag and Patterson 1972; Sanada and Shanmugasundaram 1992; He, Wang, and Lele 2004; Favier, Godeferd, and Cambon 2010; Canet et al. 2017) and also in experiments (Poulain et al. 2006). Let us also point out that the Gaussian in pt for the two-point function was confirmed in a RG study of a modified NS equation with an effective viscosity verifying K41(Antonov 1994). In (Kraichnan 1964), the analysis of sweeping was extended to a particular three-point correlation, but the general expression of the sweeping effect for n-point correlation functions was not known. Furthermore, approximations had to be made in order to derive such expressions from the NS equation. In conclusion, while today the sweeping effect is well accepted in fully developed turbulence, we are still missing a satisfactory explanation for it from first principles and until recently its consequences for general correlation functions were not known.

At this point, let us present the results obtained in fully developed turbulence using the method of the NPRG. This approach was pioneered by (Tomassini 1997). Because the NPRG method does not rely on a perturbative expansion, the choice of a power-law (2.40) for the forcing is not necessary anymore. The spectrum of the forcing can thus have a more physical form, regular and concentrated around the IR cutoff L^{-1}. A common choice is

$$\tilde{D}_{\alpha\beta}(q) \sim q^{2m} e^{-(qL)^{2n}} , \text{ with } m, n \in \mathbb{N}^* , \tag{2.48}$$

but the result should be independent of the precise form of the forcing. In order to integrate numerically the NPRG flow, (Tomassini 1997) proposes that the renormalization of the Navier-Stokes action is described by an effective viscosity and an effective noise, which both acquire non-trivial wave-number dependency. This ansatz was used to integrate numerically the RG flow equation and to reach a fixed point of the flow. At the fixed point, the author obtained the functional form of the effective viscosity and noise. Although, as explained above, the critical dimensions of a scale-invariant theory of fully developed turbulence cannot differ from their K41 value, a very peculiar and meaningful feature of the renormalized viscosity obtained in (Tomassini 1997) was observed but not commented upon. In the inertial range, the effective viscosity is found to have the following power-law behavior as a function of the wave-number

$$\nu_{\text{eff}}(p) \sim p^{-1} . \tag{2.49}$$

This result is in stark contrast with the scaling obtained by assuming scale invariance, or equivalently the K41 scaling, which is $\nu_{\text{eff}} \sim p^{-4/3}$. As expected, this anomalous behavior disappears in equal-time quantities and in particular the $-5/3$

scaling for the energy spectrum is recovered, in accordance with K41[1]. Nonetheless, it is the first time that a RG treatment of turbulence indicates breaking of scale invariance. In hindsight, this behavior corresponds to the sweeping effect present in turbulence. The power-law forcing (2.40) was revisited within the NPRG formalism in (Mejía-Monasterio and Muratore-Ginanneschi 2012), with the same anzatz as above. The main point of the paper was to investigate the freezing of exponents above $\epsilon = 2$, which is a necessary feature of pertubative RG, but was questioned within the nonperturbative formalism. However, it appears that its most interesting result was the behavior of $\nu_{\text{eff}}(p)$ obtained by integrating the RG flow:

$$\nu_{\text{eff}}(p) \sim \begin{cases} p^{-\frac{2\epsilon}{3}}, & \text{if } 0 < \epsilon < 3/2\,, \\ p^{-1}, & \text{if } \epsilon > 3/2\,. \end{cases}$$

This behavior was interpreted correctly by the authors as a sweeping effect. Furthermore, they show that from a RG perspective, it arises from the following equation verified by ν_{eff} at the RG fixed point:

$$\big(p\partial_p + 2\epsilon/3\big)\nu_{\text{eff}} \sim \frac{1}{p^2\nu_{\text{eff}}}\,, \text{ for } p \gg L^{-1}. \tag{2.50}$$

Looking for power-law solutions $\nu_{\text{eff}} \sim p^{-\alpha}$, one sees that as long as $\alpha < 1$, the right-hand side (r.h.s.) can be neglected in the large p behavior and one recovers standard scaling: $\alpha = 2\epsilon/3$. This scaling is not consistent anymore if $\epsilon > 3/2$ and in this case the left-hand side (l.h.s.) and the r.h.s. need to have the same scaling, leading to $\alpha = 1$. In standard critical phenomena, the r.h.s. of the RG fixed point equations is always sub-leading at large p, enforcing standard scaling. However in turbulence (at least for unequal time quantities), the appearance of a non-negligible r.h.s. can lead to corrections to the scaling exponents and more generally to breaking of scale invariance. This is a genuine non-perturbative effects, which explains why it was missed by perturbative RG approaches using the same kind of anzatz (Yakhot, Orszag, and She 1989). The value $\epsilon = 3/2$ at which this crossover happens corresponds to the value of ϵ at which certain dangerous operators have to be controlled by the time-gauged Galilean symmetry in the perturbative RG related in previous section, this fact leads to conjecture that both phenomena are two manifestation of the same swepping effect.

The above results were followed by the work of Canet et al. (Canet, Delamotte, and Wschebor 2015; Canet, Delamotte, and Wschebor 2016; Canet et al. 2017). The main technical innovation is that the incompressibility condition is not enforced by projecting the Navier-Stokes equation on transverse fields using the projector defined in (2.45), but by enforcing it dynamically with the help of a new field playing the role of a Lagrange multiplier for the incompressibility condition. This has the advantage that the interaction vertex is local, at the price of introducing two new scalar fields in the action. This step allowed the authors to discover a new time-gauged symmetry of the Navier-Stokes field theory. Related Schwinger-Dyson equations were used in (L'vov and Lebedev 1993) but it was the first time that the corresponding change of variable was identified and used as a symmetry of the Navier-Stokes field theory. This new symmetry allowed the authors to generalize exact relations which had

[1] Note that in order for the K41 spectrum to be recovered, the effective noise has also to receive an anomalous correction due to sweeping.

been derived previously in (Falkovich, Fouxon, and Oz 2010). In the second paper, the same ansatz as in (Tomassini 1997; Canet et al. 2011; Mejía-Monasterio and Muratore-Ginanneschi 2012) is used to integrate numerically the RG flow and find the K41 fixed point. More importantly, in the same paper it is shown that the exact RG flow equation of the two-point functions can be closed in the limit of wave-numbers large compared to the IR cutoff. This result uses the newfound symmetry and the time-gauged Galilean one and, contrarily to previous analysis, makes no reference to any particular ansatz. The RG flow equation obtained this way is exact in the limit of large wave-numbers. Contrary to what is usually termed as closures in turbulence, this result does not depend on any uncontrolled heuristic and is exactly controlled by an expansion in large wave-number. In the last paper, the resolution of these equations at the RG fixed point gave the leading behavior of the correlation and response functions. It was found that at small time delay and large wave-number, both functions were Gaussian in the variable pt, with p the wave-number amplitude and t the time delay. This result put on a firm ground the early phenomenological predictions based on the sweeping effect and was found to agree remarkably well with DNS (Canet et al. 2017)

In the first part of Chap. 5, we give an account on the work published in (Tarpin, Canet, and Wschebor 2018). We extended the above result by deriving the leading RG flow equation for any n-point generalized (velocity and response) correlation function, for any time delays as long as they are controlled by the fixed point equation. The corrections to this leading behavior are controlled by the inverse of the minimum wave-number appearing in the correlation functions, measured in term of the IR cutoff. We solved the corresponding fixed point equation and obtained an analytical expression for n-point correlation functions which is exact in the limit of large wave-numbers (and for non-exceptional wave-vector configurations), in both regimes of small and large time delays. Such rigorous theoretical results are scarce in the context of turbulence. The expressions obtained for generalized n-point correlation functions are identified as the manifestation of the sweeping effect at small time delays, but takes a different form at large time delays, suggesting a more general mechanism of the NS field theory at play. The calculated effect disappears when looking at equal-time quantities. This seems to be in accordance the previous RG analyses. However, we cannot exclude that sub-leading terms which are not captured by our result at leading order play a role at equal times and lead to intermittency corrections. This idea is more easily explored in 2-D, for reasons that will become clear in the follwing. Thus, let us turn to 2-D turbulence.

2.2.4 The question of intermittency in the direct cascade of 2D turbulence

Before giving the motivation of our studies of 2-D turbulence, let us present quickly its peculiarities. Up to now, we have not mentioned historical developments in 2-D turbulence. However, they ran in parallel to the 3-D ones. The fundamental fact of 2-D turbulence was noted by Taylor in 1917 (Taylor 1960): the vorticity $\omega = \mathrm{curl}\,\vec{v}$ is conserved in a perfect fluid. As a consequence, in 2-D turbulence the total energy dissipation, which can be rewritten

$$\bar{\varepsilon} \propto \nu \int_\Omega \frac{1}{2}\omega^2 \,, \tag{2.51}$$

cannot stay finite in the limit $\nu \to 0$, contrary to what happens in 3-D turbulence. There is no energy dissipative anomaly in 2-D. The integrand in the expression above is called the enstrophy. This fact prevented the study of 2-D turbulence as it was thought that it could not develop a stationary energy cascade. This argument was put on a more rigorous footing in (Lee 1951), where it was shown that there cannot be direct energy cascade (a cascade with energy transfer from low to high wave-number), in 2-D turbulence. A hint of the peculiar behavior at large scale was given in (Onsager 1949), which considered a gas of point-vortices in a plane. This apparently unrelated system can be shown to approximate the 2-D Euler equation when the number of point-vortices goes to infinity (Eyink and Sreenivasan 2006). His surprising finding is that this equilibrium system can be in a negative temperature state, where the vortices aggregate. This feature reminds of the appearance of large, coherent structure known to appear in 2-D flow (such as stable streams and vortices in Earth atmosphere, or the spectacular vortex in Jupiter's one).

In the breakthrough paper (Kraichnan 1967a), the author argued that in 2-D, there was in fact two separate cascades. For scales much smaller than the integral scale, that is for $k \gg L^{-1}$, there is an *enstrophy direct* cascade towards the small scale, where it is dissipated by viscosity, while the energy transfer is zero. The corresponding energy spectrum goes like $E(k) \sim k^{-3}$ (actually, it was argued by the same author that this spectrum was corrected by a logarithmic factor in the form $E(k) \sim k^{-3}(\ln kL)^{-1/3}$ (Kraichnan 1971)). For scales between the integral scale and the system size L_0, that is for $L_0^{-1} \ll k \ll L^{-1}$, the enstrophy transfer is zero and there is an *inverse energy* cascade, towards the low wave-numbers, giving a energy spectrum going like $E(k) \sim k^{-5/3}$. This prediction spawned a flurry of activity on the subject, to begin with the exploration of Batchelor and his student Bray (Batchelor 1969), in fact undertook earlier than Kraichnan's paper, and the one of Leith (Leith 1968). As in 3-D turbulence, an exact law similar to the four-fifth law in 3-D can be derived from the equation of motion, although it took more time to be recognized (Bernard 1999).

Due to the inverse energy cascade, in the absence of a term to dissipate energy at large scales, the flow cannot reach a steady state. Such a term can be derived by considering a physical situation, where the 2-D flow is embedded in a 3-D space. This has the effect to add a friction term acting in the bulk, named the Ekman drag. The presence of such term gives a well understood intermittent correction to the above Kraichnan scaling (Nam et al. 2000; Bernard, D. 2000; Boffetta et al. 2002). In what follows, we will not consider such term and concentrate instead on the case where the energy sink is limited to happen at an IR friction scale $L_0^{-1} \leq L^{-1}$. Many results seem to point to the fact that in such configuration, there is no intermittency in the direct cascade. First, there exist mathematical results which bound the magnitude of the hypothetical intermittency exponents (Eyink 1996). Second, a work using diagrammatic methods borrowed from 3-D, such as the quasi-Lagrangian variables, obtained that the two-point functions of the monomes of the vorticity did not receive intermittency corrections apart for the dimensional logarithm factor mentioned above (Falkovich and Lebedev 1994a; Falkovich and Lebedev 1994b). Third, experiments (Paret, Jullien, and Tabeling 1999) and simulations (Babiano, Dubrulle, and Frick 1995), both find weak or zero intermittency corrections in the direct cascade (see also the more recent work of (Bruneau and Kellay 2005)).

Study of 2-D turbulence using perturbative RG method are scarce. To the

difficulties inherent to the choice of a power-law forcing, the dimension 2 adds new difficulties which plagued the first attempts. In 2-D, new IR divergences appear, such that the RG approaches have to be adapted. A review of these attempts is made in (Mazzino, Muratore-Ginanneschi, and Musacchio 2009). However, in the 2-D case, we discovered new symmetries specific to 2-D turbulence which gave us hope to continue the line of work started in (Tarpin, Canet, and Wschebor 2018). As presented in Sec. 2.2.3, in the previous works, we obtained the leading term of the exact RG flow equation at large wave-numbers with respect to the IR cutoff. This term was found to vanish for equal-times quantities. In Chap. 3, we will show that in order to investigate possible intermittency corrections, one thus has to obtain the sub-leading order of the RG flow. In 3-D an exact calculation seems possible for the moment. However in 2-D, the newfound symmetries may allow to obtain exact expressions. If the next-to-leading order terms of the flow equation can be given a closed form and do not vanish at equal-time, they give a way to calculate intermittency corrections. If it can be shown that the next-to-leading order terms vanish at equal time, this would constitute a proof of the absence of intermittency correction in the direct cascade of 2-D turbulence.

This work is presented in the second part of Chap. 5. Because our method probes the regime of wave-numbers large compared to the IR cutoff, we concentrate on the direct cascade and we set the integral scale equal to the IR friction scale. First, we use the particularity of 2-D to express the NS action in terms of a scalar field, the stream function. We use this formalism to express the known and new symmetries of the 2-D NS action. The symmetries are then used to simplify the RG flow.

In conclusion, we presented in this chapter two examples of out of equilibrium physical systems exhibiting critical phenomena. The first system is the absorbing phase transition in DEP. We have shown that the current theoretical descriptions available are not entirely satisfactory. In Chap. 4 we present our take on this subject using the NPRG. The second system is homogeneous isotropic fully developed turbulence. It should be clear at this point that a theoretical description of turbulence is missing, and in particular of the intermittency effects. Our study, presented in Chap. 5 is restrained to the two subjects outlined above: time-dependence of correlation functions, and intermittency in the direct cascade in 2-D. Before delving into each of these subjects, let us first give in the next chapter a short introduction to the framework of NPRG in the context of out of equilibrium field theories.

Version française

En conclusion, nous avons présenté dans ce chapitre deux exemples de systèmes physiques hors de l'équilibre présentant des phénomènes critiques. Le premier système est la transition de phase absorbante dans le DEP. Nous avons montré que les descriptions théoriques actuellement disponibles ne sont pas entièrement satisfaisantes. Dans le Chap. 4, nous présentons notre point de vue sur ce sujet à l'aide du NPRG. Le second système est la turbulence isotrope homogène et pleinement développée. Il devrait être clair à ce stade qu'une compréhension théorique complète de la turbulence reste à bâtir, en particulier concernant les effets de l'intermittence. Notre étude, présentée au Chap. 5, se limite aux deux sujets décrits ci-dessus : la dépendance temporelle des fonctions de corrélation et l'intermittence dans la cascade directe en 2D. Avant de se plonger dans chacun de ces sujets, le chapitre suivant est

consacré à une brève introduction au NPRG dans le contexte des théories de champs hors de l'équilibre.

Chapter 3

Introduction to non-perturbative renormalization group for out of equilibrium field theories

In this chapter, we present the framework of the NPRG applied to out of equilbrium field theories. In order to introduce out of equilibrium field theories, we first present in Sec. 3.2 the mapping from a SPDE with Gaussian noise to an action functional known as the MSRJD formalism. This allows us to discuss the specificities of out of equilibrium field theories, in particular the properties of causality of such field theories. Then in Sec. 3.3 we give a short presentation of the saddle point method in statistical field theories as well as its shortfalls. This prepares and motivates the introduction of the NPRG in Sec. 3.4. Finally in Sec. 3.5 we spend some time on the treatment of causality in this setting. Prior to this, let us introduce some notations used throughout the manuscript.

Version française

Dans ce chapitre, nous présentons le formalisme du NPRG appliqué aux théories des champs hors équilibre. Afin d'introduire les théories des champs hors d'équilibre, nous présentons d'abord dans la Sec. 3.2 la mise en correspondance d'une SPDE avec bruit gaussien à une action fonctionnelle, connue sous le nom de formalisme MSRJD. Cela nous permet de discuter des spécificités des théories de champ hors d'équilibre, en particulier les propriétés de causalité de ces théories de champ. Ensuite, à la Sec. 3.3, nous donnons une brève présentation de la méthode du point-col dans les théories statistiques des champs, ainsi que de ses lacunes. Ceci prépare et motive l'introduction du NPRG dans la Sec. 3.4. Enfin, à la Sec. 3.5, nous nous attardons sur le traitement de la causalité dans ce contexte. Avant cela, introduisons quelques notations utilisées tout au long du manuscrit.

3.1 Notations

In this work, the vectors are noted with an arrow as $\vec{x}$ and their components with greek indices, x_α or x^α indifferently. The Einstein notation for the summations is used. Bold symbols indicate both a space coordinate and the corresponding time

coordinate, with the conventions: $\mathbf{x} = (t, \vec{x})$, $\mathbf{y} = (u, \vec{y})$, $\mathbf{z} = (v, \vec{z})$ for real space and $\mathbf{p} = (\omega, \vec{p})$, $\mathbf{q} = (\varpi, \vec{q})$, $\mathbf{k} = (\nu, \vec{k})$ in Fourier space. The integrals are abbreviated according to the following notation:

$$\begin{aligned} \int_{\mathbf{x}} &\equiv \int \mathrm{d}^d x \, \mathrm{d}t \quad \text{in real space,} \\ \int_{\mathbf{q}} &\equiv \int \frac{\mathrm{d}^d q \, \mathrm{d}\omega}{(2\pi)^{d+1}} \quad \text{in Fourier space.} \end{aligned} \tag{3.1}$$

The Fourier transform is noted with a tilde and the following sign convention is used

$$\begin{aligned} f(\mathbf{x}) &= \int_{\mathbf{q}} \tilde{f}(\mathbf{q}) e^{i\mathbf{p}\cdot\mathbf{x}} \\ \tilde{f}(\mathbf{p}) &= \int_{\mathbf{x}} f(\mathbf{x}) e^{-i\mathbf{p}\cdot\mathbf{x}} \,, \end{aligned} \tag{3.2}$$

with the shorthand notation

$$\mathbf{p}\cdot\mathbf{x} = \vec{p}\cdot\vec{x} - \omega\, t \tag{3.3}$$

and with the obvious generalisation for many-point functions. Due to invariance under space-time translations, functions of many momenta such as the Fourier transforms of correlation functions are often proportional to the delta-fonction of conservation. The Fourier transform with the delta extracted is noted with a bar as follows

$$\tilde{F}(\mathbf{p}_1, \ldots, \mathbf{p}_n) = (2\pi)^{d+1} \delta\left(\sum \omega_i\right) \delta^d\left(\sum \vec{p}_i\right) \bar{F}(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) \,. \tag{3.4}$$

3.2 Response field formalism for Langevin equation

To present the MSRJD formalism we will concentrate on the simple example of the Langevin equation for DP, while the general setting, with more than one field and accounting for explicit constraints along the time evolution, is deferred to Appendix B.1. The Langevin equation for DP reads

$$\partial_t \rho = \kappa\rho - \lambda_1 \rho^2 + D\,\partial^2 \rho + \eta \,, \tag{3.5}$$

where η is a centered Gaussian random field of covariance

$$\langle \eta(\mathbf{x}) \eta(\mathbf{x}') \rangle = 2\lambda_2 \rho(\mathbf{x}) \delta(t - t') \delta^d(\vec{x} - \vec{x}') \,. \tag{3.6}$$

This definition can seem somewhat problematic as the covariance of η depends on ρ, which is a process defined through η. In fact, the right way to read this Langevin equation would be to discretize it, using a forward discretization scheme. In this particular case of a Gaussian noise η, this problem can be solved in the following way. We can rescale the noise such that the Langevin equation reads

$$\partial_t \rho = \kappa\rho - \lambda_1 \rho^2 + D\,\partial^2 \rho + \sqrt{2\lambda_2 \rho}\tilde{\eta} \,, \tag{3.7}$$

with the centered Gaussian noise $\tilde{\eta}$ having now the following variance:

$$\langle \tilde{\eta}(\mathbf{x})\tilde{\eta}(\mathbf{x}')\rangle = \delta(t-t')\delta^d(\vec{x}-\vec{x}')\,, \tag{3.8}$$

or equivalently, the probability $\mathbb{P}[\tilde{\eta}]$ of a given trajectory for $\tilde{\eta}$ is given by

$$\mathbb{P}[\tilde{\eta}] \propto e^{-\frac{1}{2}\int_{\mathbf{x}} \tilde{\eta}(\mathbf{x})^2}\,. \tag{3.9}$$

The Itô calculus is the mathematical apparatus which has been developed for this formulation, when a forward discretization scheme is chosen (Gardiner 2009). In the following, we will work with the continuum SPDE (3.7) using formal functional manipulations, having in mind that what we really mean is a discretized version of the process. The first step of the MSRJD formalism is to add a linear spacetime-dependent perturbation, $\bar{\jmath}$, to the Langevin equation:

$$\partial_t \rho = \kappa\rho - \lambda_1\rho^2 + D\,\partial^2\rho + \sqrt{2\lambda_2\rho}\tilde{\eta} + \bar{\jmath}\,. \tag{3.10}$$

and we note $\langle \mathcal{F}[\rho]\rangle_{\bar{\jmath}}$, with $\mathcal{F}$ a functional of ρ, the mean value of the (functional of the) solution of the Langevin equation (3.10) above over realizations of the noise $\tilde{\eta}$. This will allow us to get informations on the response of the system to infinitesimal perturbations. For example,

$$G(\mathbf{x},\mathbf{x}') = \left.\frac{\delta\langle\rho(\mathbf{x})\rangle_{\bar{\jmath}}}{\delta\bar{\jmath}(\mathbf{x}')}\right|_{\bar{\jmath}=0}\,, \tag{3.11}$$

the functional derivative of $\langle\rho\rangle$ with respect to $\bar{\jmath}$, is the Green function, the response of the system at the point $\mathbf{x}$ to an infinitesimal perturbation concentrated at the point $\mathbf{x}'$.

Our goal is now to find an expression for the generating functional of the process ρ:

$$\mathcal{Z}[j,\bar{\jmath}] = \langle e^{\int_{\mathbf{x}} j(\mathbf{x})\rho(\mathbf{x})}\rangle_{\bar{\jmath}}\,. \tag{3.12}$$

We see that j and $\bar{\jmath}$ play similar roles. For example, the Green function can be rewritten as

$$G(\mathbf{x},\mathbf{x}') = \left.\frac{\delta^2\mathcal{Z}[j,\bar{\jmath}]}{\delta j(\mathbf{x})\delta\bar{\jmath}(\mathbf{x}')}\right|_{j=\bar{\jmath}=0} \tag{3.13}$$

The first step is to write $\mathcal{Z}$ as a functional integral

$$\mathcal{Z}[j,\bar{\jmath}] = \int \mathcal{D}[\rho] e^{\int_{\mathbf{x}} j\rho} \langle\delta[\rho-\rho_{\tilde{\eta}}]\rangle_{\bar{\jmath}}\,. \tag{3.14}$$

The integration measure $\mathcal{D}[\cdot]$ and the functional Dirac delta $\delta[\cdot]$ are to be understood as the continuum limit of their discretized versions in space and time. $\rho_{\tilde{\eta}}$ is the weak solution of the Langevin equation (3.10) for a given $\tilde{\eta}$. The second step is to replace the constraint $\langle\delta[\rho-\rho_{\tilde{\eta}}]\rangle_{\bar{\jmath}}$ by the explicit equation of motion of ρ in the presence of the perturbation $\bar{\jmath}$, which can be written as $\mathcal{F}(\cdot)=0$, with

$$\mathcal{F}(\mathbf{x}) = \partial_t\rho - \kappa\rho + \lambda_1\rho^2 - D\,\partial^2\rho - \bar{\jmath} - \sqrt{2\lambda_2\rho}\tilde{\eta}\,. \tag{3.15}$$

Assuming existence and unicity of weak solutions of (3.10) (at least almost everywhere for the measure $\mathcal{D}[\rho]$ for a given initial condition), one obtains

$$\mathcal{Z}[j,\bar{\jmath}] = \int \mathcal{D}[\rho] e^{\int_{\mathbf{x}} j\rho} \langle\delta[\mathcal{F}]\rangle \times \mathcal{J}\,, \tag{3.16}$$

with $\mathcal{J}$ the Jacobian of the transformation, $\mathcal{J} = |\det\left(\frac{\delta \mathcal{F}(\mathbf{x})}{\delta \rho(\mathbf{x}')}\right)|$, which depends on the choice of discretization of (3.10). In this work, we always assume that it has to be understood in the Itô convention, which amounts to have a forward discretization scheme of the differential equation. As a consequence, $\mathcal{J}$ does not depend on the fields (Täuber 2014). Accordingly, it can be ignored it the calculation of correlation and response functions and it is absorbed in the functional measure.

The last step is to use the Fourier representation of the functional Dirac deltas.

$$\begin{aligned}
\mathcal{Z}[j,\bar{j}] &= \int \mathcal{D}[\rho,\bar{\rho}] e^{\int_{\mathbf{x}} j\rho} \langle e^{-i\int_{\mathbf{x}} \bar{\rho}\mathcal{F}} \rangle \\
&= \int \mathcal{D}[\rho,\bar{\rho}] e^{\int_{\mathbf{x}} j\rho + i\bar{j}\bar{\rho}} e^{-i\int_{\mathbf{x}} \bar{\rho}(\partial_t \rho - \kappa\rho + \lambda_1\rho^2 - D\,\partial^2\rho)} \langle e^{i\int_{\mathbf{x}} \bar{\rho}\sqrt{2\lambda_2\rho}\tilde{\eta}} \rangle \\
&= \int \mathcal{D}[\rho,\bar{\rho}] e^{\int_{\mathbf{x}} j\rho + i\bar{j}\bar{\rho}} e^{-\mathcal{S}[\rho,\bar{\rho}]} \\
\text{with } \mathcal{S}[\rho,\bar{\rho}] &= \int_{\mathbf{x}} \left\{ i\bar{\rho}(\partial_t\rho - \kappa\rho + \lambda_1\rho^2 - D\,\partial^2\rho) + \lambda_2\rho\bar{\rho}^2 \right\},
\end{aligned} \tag{3.17}$$

using the property (3.9) to compute the average value in the second line. Finally, one usually absorbs the complex i in a redefinition of the response fields $i\bar{\rho} \equiv \tilde{\rho}$ yielding

$$\mathcal{S} = \int_{\mathbf{x}} \left\{ \tilde{\rho}(\partial_t\rho - \kappa\rho + \lambda_1\rho^2 - D\,\partial^2\rho) - \lambda_2\rho\tilde{\rho}^2 \right\}. \tag{3.18}$$

This concludes the MSRJD mapping. By rescaling $\rho \equiv \sqrt{\frac{\lambda_2}{\lambda_1}}\phi$ and $\tilde{\rho} = \sqrt{\frac{\lambda_1}{\lambda_2}}\bar{\phi}$, rescaling the time $t \to D^{-1}t$ and noting $\sigma = \kappa D^{-1}$, $\lambda = \sqrt{\lambda_1\lambda_2}$, one obtains without loss of generality

$$\begin{aligned}
\mathcal{Z}[j,\bar{j}] &= \int \mathcal{D}[\phi,\bar{\phi}] e^{\int_{\mathbf{x}} j\phi + \bar{j}\bar{\phi}} e^{-\mathcal{S}[\phi,\bar{\phi}]} \\
\mathcal{S}[\phi,\bar{\phi}] &= \int_{\mathbf{x}} \left\{ \bar{\phi}(\partial_t - \partial^2 - \sigma)\phi + \lambda\phi\bar{\phi}(\phi - \bar{\phi}) \right\}.
\end{aligned} \tag{3.19}$$

This action is known in high energy physics as the Reggeon field theory, its mapping with DP was understood in (Cardy and Sugar 1980). One sees that it depends on two scalar fields, although we have started with a Langevin equation for the density only. This is a general feature of the action obtained through this formalism: they contain twice the number of degrees of freedom present in the deterministic limit of the process. These new fields, associated to each equation of motion or constraints are called the response fields and we call the field associated to the original degrees of freedom the observable fields. From (3.19), we can rewrite the Green function as

$$G(\mathbf{x},\mathbf{x}') = \left.\frac{\delta^2 \mathcal{Z}[j,\bar{j}]}{\delta j(\mathbf{x})\delta\bar{j}(\mathbf{x}')}\right|_{j=\bar{j}=0} = \langle \phi(\mathbf{x})\bar{\phi}(\mathbf{x}') \rangle . \tag{3.20}$$

In the following, we will call the average of an arbitrary product of observable and response fields a generalized correlation function. The matrix of two-point generalized correlation function is often called the propagator, somewhat abusively as it contains also the two-point correlation of the observable field.

Now, let us discuss general properties of out of equilibrium field theories on the above example. By construction, the action vanishes when the response fields are set

to zero. Furthermore, due to the choice of discretization of the Langevin equation, the response fields are evaluated at a later time in the action than the observable field. These two properties are in fact more general than the case of Gaussian noise. Both of them are necessary to have a causal process, in the sense that

$$\frac{\delta^m}{\delta\bar{\jmath}(\mathbf{x}_1)\dots\bar{\jmath}(\mathbf{x}_m)}\langle\phi(\mathbf{x}'_1)\dots\phi(\mathbf{x}'_n)\rangle_{\bar{\jmath}} = 0 \quad \text{if } \exists\, k,\, \forall\, , l \in ,\ t_k > t'_l\,. \tag{3.21}$$

In other words, the process is causal if any multi-times response function vanishes when its largest time is the one of a response field. The treatment of this causality property within the framework of NPRG will be discussed further in Sec. 3.5. In the following, we always assume that these two properties hold, whatever the way we obtained the action. It will be the case for all actions obtained through a response field formalism generalizing MSRJD. Whether and how to interpret the action obtained using the Doi-Peliti formalism, to be presented in Chap. 4, in this setting may be more delicate, at least non-perturbatively.

3.3 Statistical physics and mean-field theories

The Chap. 2 of this memoir has been devoted to the presentation of two examples of systems belonging to the field of statistical physics. However these examples were peculiar in the sense that the emphasis was given on correlations that develop at all scales in these systems, either because we placed ourselves at a particular point in the parameter space or because it was a generic feature of the theory. It is worthwhile at this point to take a step back and review first the statistical physics of non-interacting or weakly interacting degrees of freedom. This will allow us to motivate and introduce the general formalism of the NPRG.

We will restrict ourselves to stochastic systems which can be described by a (non-normalized) functional partition function of the following form:

$$\mathcal{Z}[\mathrm{j}] = \int \mathcal{D}[\phi] e^{-\mathcal{S}[\phi]+\mathrm{j}\cdot\phi}\,. \tag{3.22}$$

For field theories obtained through a response field formalism, ϕ contains both the observable and response fields. The above formal notation has to be understood in the following sense. The symbol with brackets represent a functional dependency in its argument. The symbol $\mathcal{D}$ is a measure in a certain space of functions. The dot is a formal scalar product in this space of functions. For example, for a n-component field theory living in a d-dimensional box of linear size L and from the time 0 to τ, it writes

$$\mathrm{j}\cdot\phi = \sum_{i=1}^{n}\int_{[0,\tau]\times[0,L]^d} \mathrm{d}t\,\mathrm{d}^d x\,\mathrm{j}_i(t,\vec{x})\phi_i(t,\vec{x}) = \int_{\mathbf{x}} \mathrm{j}_i(\mathbf{x})\phi_i(\mathbf{x})\,. \tag{3.23}$$

Let us introduce two more shorthand notations. For any linear operator $\mathcal{A}$ and field ϕ, we write $\mathcal{A}\cdot\phi$ the field defined as

$$\big[\mathcal{A}\cdot\phi\big]_i(\mathbf{x}) = \int_{\mathbf{y}} \mathcal{A}_{ij}(\mathbf{x},\mathbf{y})\phi_j(\mathbf{y})\,, \tag{3.24}$$

and if $\mathcal{A}$ is symmetric,

$$\phi \cdot \mathcal{A} \cdot \phi' = \phi \cdot \big[\mathcal{A} \cdot \phi'\big] = \big[\mathcal{A} \cdot \phi\big] \cdot \phi' \,. \tag{3.25}$$

To give a rigorous meaning to these notations is often a hard task that we do not aim nor need to undertake. Indeed, in statistical physics, these generating functionals are often defined with a natural cutoff for the small length scales, or equivalently for high wave-numbers. For theories defined on a lattice for example, this cutoff corresponds to the inverse lattice spacing (in the language of field theory, an UV cutoff, generically noted Λ). When the cutoff is not explicitly present, it has to be understood as the limit of validity of the model used to write the theory. The operation to take the UV cutoff to infinity is called the continuous limit. While the continuous limit often makes the task of computing quantities easier, it is in fact not a necessary step. As a consequence, we will manipulate formally the functional expressions in the continuum, having in mind that they contain a UV cutoff which regularizes the theory. Now, let us begin by an introductive section on the calculation of statistical field theories when the tools of the renormalization group are not needed. This will also be the occasion to set up some notations.

3.3.1 Free theories and saddle point methods

A particular case where the integration over the fields can be done exactly is the case of non-interacting, or free, theories. The theory is said to be free if $\mathcal{S}$ is at most quadratic in the fields. A linear term can be absorbed in a shift of j, so we write in all generality

$$\mathcal{S}[\phi] = \frac{1}{2}\,\phi \cdot \mathcal{S}^{(2)} \cdot \phi = \frac{1}{2}\int_{\mathbf{x},\mathbf{y}} \phi_i(\mathbf{x}) \mathcal{S}^{(2)}_{ij}(\mathbf{x},\mathbf{y}) \phi_j(\mathbf{y}) \tag{3.26}$$

with $\mathcal{S}^{(2)}$ the symmetric linear operator defined as the Hessian of $\mathcal{S}$:

$$\mathcal{S}^{(2)}_{ij}(\mathbf{x},\mathbf{y}) = \left.\frac{\delta^2 \mathcal{S}[\phi]}{\delta\phi_i(\mathbf{x})\delta\phi_j(\mathbf{y})}\right|_{\phi=0} . \tag{3.27}$$

If $\mathcal{S}^{(2)}$ is definite positive, we note $G_0 = \big[\mathcal{S}^{(2)}\big]^{-1}$ and the partition function is readily calculated to be

$$\mathcal{Z}[\mathrm{j}] = \mathcal{N}\, e^{\frac{1}{2}\,\mathrm{j}\cdot G_0 \cdot \mathrm{j}} \,, \tag{3.28}$$

where $\mathcal{N}$ is independent of j. From the expression of $\mathcal{Z}[\mathrm{j}]$, one obtains the full set of generalized correlation functions defining the theory:

$$\begin{aligned} \langle \phi_{i_1}(\mathbf{x}_1)\phi_{i_2}(\mathbf{x}_2)\dots\phi_{i_k}(\mathbf{x}_k)\rangle_{\mathrm{j}} &= \frac{\int \mathcal{D}[\phi] e^{-\mathcal{S}[\phi]+\mathrm{j}\cdot\phi} \phi_{i_1}(\mathbf{x}_1)\phi_{i_2}(\mathbf{x}_2)\dots\phi_{i_k}(\mathbf{x}_k)}{\int \mathcal{D}[\phi] e^{-\mathcal{S}[\phi]+\mathrm{j}\cdot\phi}} \\ &= \frac{1}{\mathcal{Z}[\mathrm{j}]} \frac{\delta^n \mathcal{Z}[\mathrm{j}]}{\delta \mathrm{j}_{i_1}(\mathbf{x}_1)\delta \mathrm{j}_{i_2}(\mathbf{x}_2)\dots\delta \mathrm{j}_{i_k}(\mathbf{x}_k)} \,. \end{aligned} \tag{3.29}$$

Note that we have defined the correlation functions without setting the sources j to zero, thus these are still functionals of j. This is why we have to divide by $\mathcal{Z}[\mathrm{j}]$ to get a normalized expression. This dependency is made explicit in the j subscript of the mean value. The physical content of the theory is written more compactly using

the complex logarithm of the partition function, that we shall note $\mathcal{W}$ following common usage in the field. It is the analogous of the Helmoltz free energy for out of equilibrium systems. While functional derivatives of $\mathcal{Z}$ give the correlation functions, functional derivatives of $\mathcal{W}$ give connected correlation functions - the functional generalisation of cumulants (Le Bellac 1998). Let us introduce more notations. The mean value of the fields is denoted as

$$\Phi_i(\mathbf{x}) = \langle \phi_i(\mathbf{x}) \rangle_{\mathrm{j}} = \frac{1}{\mathcal{Z}[\mathrm{j}]} \frac{\delta \mathcal{Z}[\mathrm{j}]}{\delta \mathrm{j}_i(\mathbf{x})} = \frac{\delta \mathcal{W}}{\delta \mathrm{j}_i(\mathbf{x})}\,, \tag{3.30}$$

and an arbitrary n-point generalized connected correlation functions is defined as

$$G^{(n)}_{i_1 \dots i_n}[\{\mathbf{x}_\ell\}_{1\le\ell\le n}; \mathrm{j}] = \frac{\delta^n \mathcal{W}}{\delta \mathrm{j}_{i_1}(\mathbf{x}_1) \dots \delta \mathrm{j}_{i_n}(\mathbf{x}_n)}\,. \tag{3.31}$$

Note that in this definition as well, $G^{(n)}_{i_1 \dots i_n}$ is still a functional of the sources, which is materialized by the square brackets and the explicit j dependency. We indicate that a correlation function is evaluated at constant fields j_0 using the notation

$$G^{(n)}_{i_1 \dots i_n}(\{\mathbf{x}_\ell\}_{1\le\ell\le n}, \mathrm{j}_0) \equiv G^{(n)}_{i_1 \dots i_n}[\{\mathbf{x}_\ell\}; \mathrm{j} = \mathrm{j}_0]\,. \tag{3.32}$$

In the case of $\mathrm{j}_0 = 0$, we write simply $G^{(n)}_{i_1 \dots i_n}(\{\mathbf{x}_\ell\}_{1\le\ell\le n})$. The Fourier transforms of these functions follow the conventions given in the introduction:

$$\tilde{G}^{(n)}(\{\mathbf{p}_\ell\}_{1\le\ell\le n}, \mathrm{j}_0) = \int_{\{\mathbf{x}_\ell\}} G^{(n)}(\{\mathbf{x}_\ell\}_{1\le\ell\le n}, \mathrm{j}_0) e^{-i\mathbf{p}_\ell \cdot \mathbf{x}_\ell}\,. \tag{3.33}$$

and for systems invariant under space-time translations, extracting the delta function of conservation of wave-vector and frequency gives

$$\tilde{G}^{(n)}(\{\mathbf{p}_\ell\}_{1\le\ell\le n}, \mathrm{j}_0) = (2\pi)^{d+1} \delta^d(\sum_{k=1}^n \vec{p}_k) \delta(\sum_{k=1}^n \omega_k) \bar{G}^{(n)}(\{\mathbf{p}_\ell\}_{1\le\ell\le n-1}, \mathrm{j}_0)\,. \tag{3.34}$$

Going back to the Gaussian theory (3.28), we obtain simply

$$\begin{aligned} \Phi_i(\mathbf{x}) &= \big[G_0 \cdot \mathrm{j}\big]_i(\mathbf{x}) \\ G^{(2)}_{ij}[\mathbf{x}, \mathbf{y}; \mathrm{j}] &= \big[G_0\big]_{ij}(\mathbf{x}, \mathbf{y}) \\ G^{(n)}_{i_1 \dots i_n}[\{\mathbf{x}_\ell\}_{1\le\ell\le n}; \mathrm{j}] &= 0\,, \text{ for } n > 2\,. \end{aligned} \tag{3.35}$$

In the context of out of equilibrium field theories, we will often use the name of Gaussian theory and Gaussian fields for the particular case of

$$\mathcal{S}_{\text{Gaussian}} = \int_{\mathbf{x}} \Big\{ \bar{\phi}(\partial_t - \partial^2 + \sigma)\phi - \Delta \bar{\phi}^2 \Big\} \tag{3.36}$$

with σ and Δ two positive numbers. This MSRJD action corresponds to the following Langevin equation:

$$\begin{aligned} &\partial_t \phi = (\partial^2 - \sigma)\phi + \eta \\ &\langle \eta(\mathbf{x}) \eta(\mathbf{x}') \rangle = 2\Delta \delta(\mathbf{x} - \mathbf{x}')\,, \end{aligned} \tag{3.37}$$

which describes for example the propagation of heat in a material subjected to heat loss and noise. Following the previous discussion, this field is generally only defined up to an UV cutoff.

However, most theories of interest are not free. As a consequence one has to find a way to calculate approximately the partition function. Let us assume that the non-quadratic part of $\mathcal{S}$ is proportional to a small coupling g. In this case, one can try to approximate the partition function using a functional saddle-point. Let us assume $\mathcal{S}$ is smooth and expand the argument of the exponential in powers of the fields around an arbitrary value ϕ^*:

$$\begin{aligned} -\mathcal{S}[\phi] + \mathrm{j}\cdot\phi = -\mathcal{S}[\phi^*] + \left(\mathrm{j} - \frac{\delta\mathcal{S}}{\delta\phi^*}[\phi^*]\right)\cdot(\phi-\phi^*) + \mathrm{j}\cdot\phi^* \\ - \frac{1}{2}(\phi-\phi^*)\cdot\mathcal{S}^{(2)}[\phi^*]\cdot(\phi-\phi^*) + g\, S_{\mathrm{int}}[\phi^*, \phi-\phi^*]\,, \end{aligned} \tag{3.38}$$

where $S_{\mathrm{int}}[\phi^*, \phi-\phi^*]$ is at least cubic in $\phi-\phi^*$. Let us choose $\phi^* = \Phi_0[\mathrm{j}]$ the solution of the classical equation of motion

$$\frac{\delta\mathcal{S}}{\delta\phi^*}[\phi^*] = \mathrm{j}\,, \tag{3.39}$$

such that the term linear in ϕ in (3.38) vanishes. It can be shown that Φ_0 diverges when g goes to zero at j fixed. For example, in the case of a polynomial interaction, $\Phi_0 \sim g^{\frac{1}{2-n}}$, where n is the degree of the interaction. As a consequence, the first two terms of (3.38) diverge with g, the quadratic term is of order g^0 and the interaction part has positive powers of g. The partition reads, neglecting S_{int} for the moment,

$$\mathcal{Z}[\mathrm{j}] \simeq e^{-\mathcal{S}[\Phi_0]+\mathrm{j}\cdot\Phi_0}\int\mathcal{D}[\phi]e^{-\frac{1}{2}(\phi-\Phi_0)\cdot\mathcal{S}^{(2)}[\Phi_0]\cdot(\phi-\Phi_0)}\,. \tag{3.40}$$

Assuming that the Gaussian integration can be done, it gives

$$\int\mathcal{D}[\phi]e^{-\frac{1}{2}(\phi-\Phi_0)\cdot\mathcal{S}^{(2)}[\Phi_0]\cdot(\phi-\Phi_0)} = \left(\det\frac{\mathcal{S}^{(2)}[\Phi_0]}{2\pi}\right)^{-\frac{1}{2}} = \mathcal{N}\,e^{-\frac{1}{2}\mathrm{tr}\ln\mathcal{S}^{(2)}[\Phi_0]}\,. \tag{3.41}$$

The argument of the exponential goes as g^0. Furthermore, if one calculates perturbatively the higher order terms which have been neglected, they appear in the exponential with a positive power of g. In the limit $g \ll 1$, one is thus left with

$$\mathcal{W}[\mathrm{j}] = \ln\mathcal{Z}[\mathrm{j}] \approx -\mathcal{S}[\Phi_0] + \mathrm{j}\cdot\Phi_0\,, \tag{3.42}$$

where Φ_0 depends on j through (3.39). At this point, let us make a note about out of equilibrium field theory. In equilibrium field theory, the action $\mathcal{S}$ in (3.22) is real. In this case, the saddle point calculation is done exactly as described above. However, actions of out of equilibrium field theories are complex and cannot generally be deformed to the pure real case. We have thus hidden the choice of a path of constant phase in our symbolic notations, as well as the task of proving the positivity of the Hessian. At least in the case of actions obtained from MSRJD formalism, this can be checked rigorously. However, it is a subtle matter that we will not undertake here.

Let us calculate the mean value of the fields from the definition (3.30) and the expression (3.42)

$$\begin{aligned}\Phi_i(\mathbf{x}) &= \frac{\delta}{\delta j_i(\mathbf{x})}\Big(-\mathcal{S}[\Phi_0] + \mathrm{j}\cdot\Phi_0\Big)\\ &= -\left[\frac{\delta\mathcal{S}}{\delta\Phi_0}[\Phi_0]\cdot\frac{\delta\Phi_0}{\delta\mathrm{j}}\right]_i(\mathbf{x}) + \Phi_{0,i}(\mathbf{x}) + \left[\mathrm{j}\cdot\frac{\delta(\Phi_0)}{\delta\mathrm{j}}\right]_i(\mathbf{x})\\ &= \left[\Big(\mathrm{j}-\frac{\delta\mathcal{S}}{\delta\Phi_0}[\Phi_0]\Big)\cdot\frac{\delta\Phi_0}{\delta\mathrm{j}}\right]_i(\mathbf{x}) + \Phi_{0,i}(\mathbf{x})\\ &= \Phi_{0,i}(\mathbf{x})\end{aligned} \tag{3.43}$$

Furthermore, in this approximation,

$$G^{(2)}_{ij}[\mathbf{x},\mathbf{y};\mathrm{j}] = \frac{\delta\Phi_j(\mathbf{y})}{\delta\mathrm{j}_i(\mathbf{x})} = \frac{\delta\Phi_{0,j}(\mathbf{y})}{\delta\mathrm{j}_i(\mathbf{x})} = \left[\frac{\delta\mathrm{j}}{\delta\Phi_0}\right]^{-1}_{ij}[\mathbf{x},\mathbf{y};\mathrm{j}] = \big[\mathcal{S}^{(2)}\big]^{-1}_{ij}[\mathbf{x},\mathbf{y};\Phi]\,. \tag{3.44}$$

In other words, the propagator of the theory is G_0, the inverse of the Hessian of the action. In the last equality, the dependency on j appears implicitely through the dependency on $\Phi = \Phi_0[\mathrm{j}]$. Taking a derivative of the equation above with respect to $\mathrm{j}_k(\mathbf{z})$ and using the formula

$$\frac{\delta}{\delta\phi_k(\mathbf{z})}\big[\mathcal{A}[\phi]\big]^{-1}_{ij}(\mathbf{x},\mathbf{y}) = -\left[\mathcal{A}^{-1}\cdot\frac{\delta\mathcal{A}[\phi]}{\delta\phi_k(\mathbf{z})}\cdot\mathcal{A}^{-1}\right]_{ij}(\mathbf{x},\mathbf{y})\,, \tag{3.45}$$

for the derivative of the inverse of an arbitrary linear operator $\mathcal{A}$ depending on a field, one obtains

$$\begin{aligned}G^{(3)}_{ij}[\mathbf{x},\mathbf{y};\mathrm{j}] &= \frac{\delta}{\delta\mathrm{j}_k(\mathbf{z})}G^{(2)}_{ij}[\mathbf{x},\mathbf{y};\mathrm{j}] = \frac{\delta\Phi}{\delta\mathrm{j}_k(\mathbf{z})}\cdot\frac{\delta}{\delta\Phi}\big[\mathcal{S}^{(2)}\big]^{-1}_{ij}[\mathbf{x},\mathbf{y};\Phi]\\ &= -\int_{\mathbf{x}',\mathbf{y}',\mathbf{z}'}\big[G_0\big]_{ia}[\mathbf{x},\mathbf{x}';\mathrm{j}]\big[G_0\big]_{jb}[\mathbf{y},\mathbf{y}';\mathrm{j}]\big[G_0\big]_{kc}[\mathbf{z},\mathbf{z}';\mathrm{j}]\mathcal{S}^{(3)}_{abc}[\mathbf{x}',\mathbf{y}',\mathbf{z}';\Phi]\,.\end{aligned} \tag{3.46}$$

In the saddle-point approximation, all the correlation functions can be constructed from the derivatives of $\mathcal{S}$. In fact, the result is more precise: an arbitrary generalized connected correlation function $G^{(n)}$ is given by the sum over all tree diagrams whose edges are the G_0 and whose vertices are the $\mathcal{S}^{(k)}$, $3\le k\le n$. Each of this diagram appears with a signed symmetry factor which will be irrelevant in the following. This property is easily demonstrated iteratively, starting from $G^{(n-1)}$ and taking one j derivative:

$$\begin{aligned}G^{(n)}_{i_1\dots i_n}[\{\mathbf{x}_\ell\}_{1\le\ell\le n};\mathrm{j}] &= \frac{\delta}{\delta\mathrm{j}_{i_n}(\mathbf{x}_n)}G^{(n-1)}_{i_1\dots i_{n-1}}[\{\mathbf{x}_\ell\}_{1\le n-1};\mathrm{j}]\\ &= \frac{\delta\Phi}{\delta\mathrm{j}_{i_n}(\mathbf{x}_n)}\cdot\frac{\delta}{\delta\Phi}\mathcal{T}^{(n-1)}_{i_1\dots i_{n-1}}[\{\mathbf{x}_\ell\}_{1\le n-1};\Phi]\\ &= \Big[G_0\cdot\frac{\delta}{\delta\Phi}\mathcal{T}^{(n-1)}_{i_1\dots i_{n-1}}[\{\mathbf{x}_\ell\}_{1\le n-1};\Phi]\Big]_{i_n}(\mathbf{x}_n)\,,\end{aligned} \tag{3.47}$$

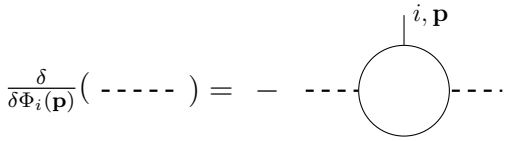

Figure 3.1: Field derivative of the propagator

where $\mathcal{T}^{(n-1)}[\Phi]$ is the tree decomposition of $G^{(n-1)}$. Either the derivative hits on one of the $\mathcal{S}^{(k)}$ or on one of the G_0 composing $\mathcal{T}^{(n-1)}[\Phi]$. In the first case, it gives $\mathcal{S}^{(k+1)}$, without changing the tree structure. In the second case, the equation (3.45) for the derivative of the inverse applies and a $\mathcal{S}^{(3)}$ vertex is inserted with a minus sign. This operation is represented diagrammaticaly in Fig. 3.1. Then the propagator coming from $\frac{\delta\Phi}{\delta \mathrm{j}_{i_n}(\mathbf{x}_n)}$ is attached to where the field derivative has hit to give back a sum of trees $\mathcal{T}^{(n)}[\Phi]$ as announced.

3.3.2 The effective action

The saddle point calculation is known as the Landau approximation (Landau 1937). Within this approximation, the microscopic action generates the equations of motion for the averaged degrees of freedom. In fact, these properties can be formalized and generalized out of the mean field approximation using the Legendre-Fenchel transform of $\mathcal{W}$, which is traditionally noted Γ and named the effective action. It is defined as

$$\Gamma[\Phi] = \sup_{\mathrm{j}} \left\{ \mathrm{j}\cdot\Phi - \mathcal{W}[\mathrm{j}] \right\} = -\inf_{\mathrm{j}} \left\{ \mathcal{W}[\mathrm{j}] - \mathrm{j}\cdot\Phi \right\}. \tag{3.48}$$

In general, it may be difficult to define the infinum if $\mathcal{W}$ is not real as it is in equilibrium field theories. This difficulty can be circumvented in the saddle-point approximation.

Let us rewrite the properties of the previous section in terms of the effective action. Within the saddle-point approximation, we can write

$$\mathcal{W}[\mathrm{j}] - \mathrm{j}\cdot\Phi = -\mathcal{S}[\Phi] + \frac{1}{2}\left(\frac{\delta\mathcal{S}}{\delta\Phi} - \mathrm{j}\right)\cdot\left[\mathcal{S}^{(2)}\right]^{-1}[\Phi]\cdot\left(\frac{\delta\mathcal{S}}{\delta\Phi} - \mathrm{j}\right). \tag{3.49}$$

The r.h.s. reaches its minimum for j such that Φ satisfies the classical equation of motion and one obtains simply that

$$\Gamma[\Phi] = \mathcal{S}[\Phi]\,. \tag{3.50}$$

Thus one sees that Γ contains all the information necessary to reconstruct the correlation functions of the theory. This fact is more general than the saddle-point approximation. Assuming $\mathcal{W}$ is sufficiently differentiable and have a single non-degenerate saddle-point, Γ can be defined by

$$\Gamma[\Phi] + \mathcal{W}[\mathrm{j}] = \mathrm{j}\cdot\Phi\,, \tag{3.51}$$

where j is determined implicitely by

$$\frac{\delta\mathcal{W}}{\delta \mathrm{j}_i(\mathbf{x})} = \Phi_i(\mathbf{x})\,. \tag{3.52}$$

In this case, j can be interpreted as the linear bias necessary to add to the action such that Φ is the mean value of the field. Taking a Φ derivative of (3.51) and using the above definition of j, one obtains that

$$\frac{\delta\Gamma}{\delta\Phi} = \mathrm{j}\,. \tag{3.53}$$

Thus, Γ plays the role of the classical action for the mean value Φ in the sense that it generates its equation of motion from a minimization principle. Furthermore

$$G^{(2)}_{ij}[\mathbf{x},\mathbf{y};\mathrm{j}] = \frac{\delta\Phi_j(\mathbf{y})}{\delta \mathrm{j}_i(\mathbf{x})} = \frac{\delta\Phi_j(\mathbf{y})}{\delta \mathrm{j}_i(\mathbf{x})} = \left[\frac{\delta \mathrm{j}}{\delta\Phi}\right]^{-1}_{ij}(\mathbf{x},\mathbf{y}) = \left[\Gamma^{(2)}\right]^{-1}_{ij}[\mathbf{x},\mathbf{y};\Phi]\,, \tag{3.54}$$

where $\Gamma^{(2)}$ is the Hessian of Γ (evaluated at the field $\Phi[\mathrm{j}]$).

In the same fashion as in the mean field approximation, all the connected correlation functions can be constructed from the derivatives of Γ, called the vertex functions. The vertex functions are often presented as the one particle-irreducible (1-PI) correlation functions for diagrammatic reasons, see (Amit 1984). Let us take this occasion to introduce useful notations for the following. The n-point vertex (1-PI) functions are defined with the same conventions as the connected correlation functions:

$$\Gamma^{(n)}_{i_1\dots i_n}[\mathbf{x}_1,\dots,\mathbf{x}_n;\Phi] = \frac{\delta^n\Gamma}{\delta\Phi_{i_1}(\mathbf{x}_1)\dots\delta\Phi_{i_n}(\mathbf{x}_n)}\,, \tag{3.55}$$

Accordingly, we define $\Gamma^{(n)}_{i_1\dots i_n}(\mathbf{x}_1,\dots,\mathbf{x}_n,\Phi_0)$ and $\Gamma^{(n)}_{i_1\dots i_n}(\mathbf{x}_1,\dots,\mathbf{x}_n)$ as the previous vertex functions evaluated at constant (resp. zero) field. Finally we define as well the Fourier transforms before and after having extracted the delta function of conservation of wave-vector and frequency, respectively $\tilde{\Gamma}^{(n)}(\{\mathbf{p}_\ell\}_{1\le\ell\le n},\Phi_0)$ and $\bar{\Gamma}^{(n)}(\{\mathbf{p}_\ell\}_{1\le\ell\le n-1},\Phi_0)$. Using (3.54) and the rule for the derivative of the inverse (3.45), the same demonstration as in the mean-field approximation goes through to show the following property: any n-point connected correlation function $G^{(n)}$ can be constructed as a sum of tree diagrams whose vertices are the 1-PI functions $\Gamma^{(k)}$, $3\le k\le n$ and whose edges are the propagators $G^{(2)} = [\Gamma^{(2)}]^{-1}$. Thus Γ generalizes $\mathcal{S}$ when the saddle-point approximation is not valid. Of course, to obtain the exact expression of Γ is as difficult as the original problem of calculating the correlation functions. Thus one has to devise approximations.

3.3.3 Corrections to the mean field approximation

Finally, let us understand how the mean field approximation can fail. To do this, let us calculate the first correction to the saddle-point.

$$\Gamma[\Phi] = \mathcal{S}[\Phi] + \frac{1}{2}\mathrm{tr}\ln\mathcal{S}^{(2)}[\Phi] + O(g^\alpha) \tag{3.56}$$

where we remember that $\mathcal{S}[\Phi]$ diverges when $g\to 0$ at fixed j and $\alpha>0$. The first order correction comes from the normalization of the Gaussian integration in (3.41). Let us for example use this expression to calculate the two-point vertex function of the theory at zero field. In order to do this we take two functional derivatives of

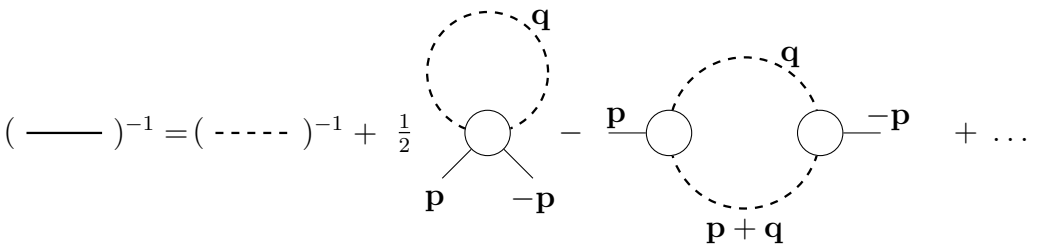

Figure 3.2: First order correction to $\Gamma^{(2)}$

the equation above. For a interaction polynomial of degree n, writing explicitly the trace and using the rule (3.45) for the derivative of the inverse, the result reads

$$\Gamma^{(2)}_{ij}(\mathbf{x},\mathbf{y}) = \mathcal{S}^{(2)}_{ij}(\mathbf{x},\mathbf{y}) + \frac{1}{2}\int_{\mathbf{z}_1,\mathbf{z}_2} \big[G_0\big]_{kl}(\mathbf{z}_1,\mathbf{z}_2)\mathcal{S}^{(4)}_{klij}(\mathbf{z}_1,\mathbf{z}_2,\mathbf{x},\mathbf{y}) - \int_{\mathbf{z}_1\dots\mathbf{z}_4} \mathcal{S}^{(3)}_{ikl}(\mathbf{x},\mathbf{z}_1,\mathbf{z}_2)\big[G_0\big]_{km}(\mathbf{z}_1,\mathbf{z}_3)\big[G_0\big]_{ln}(\mathbf{z}_1,\mathbf{z}_4)\mathcal{S}^{(3)}_{mnj}(\mathbf{z}_1,\mathbf{z}_2,\mathbf{y}) + o(g^{\frac{2}{n-2}})\,. \tag{3.57}$$

The mean field corrections are more simply expressed in Fourier space, using invariance under space-time translations:

$$\bar\Gamma^{(2)}_{ij}(\mathbf{p}) = \bar{\mathcal{S}}^{(2)}_{ij}(\mathbf{p}) + \frac{1}{2}\int_{\mathbf{q}} \big[\bar G_0\big]_{lk}(\mathbf{q})\bar{\mathcal{S}}^{(4)}_{klij}(\mathbf{q},-\mathbf{q},\mathbf{p}) - \int_{\mathbf{q}} \bar{\mathcal{S}}^{(3)}_{ikl}(\mathbf{p},\mathbf{q})\big[\bar G_0\big]_{km}(-\mathbf{q})\big[\bar G_0\big]_{ln}(\mathbf{p}+\mathbf{q})\bar{\mathcal{S}}^{(3)}_{mnj}(-\mathbf{q},\mathbf{p}+\mathbf{q}) + \dots\,. \tag{3.58}$$

The correction can also be represented in a diagrammatic form. Using a full line for the two-point connected correlation function $G^{(2)} = [\Gamma^{(2)}]^{-1}$, a dashed line for its mean field counterpart $G_0 = [\mathcal{S}^{(2)}]^{-1}$ and circles for the vertices $\mathcal{S}^{(n)}$, the above equation is represented in Fig. 3.2. At this point, let us take the example of the following generic action corresponding to an out of equilibrium field theory for one scalar defined by a Langevin equation

$$\mathcal{S} = \int_{\mathbf{x}} \Big\{ \bar\phi(\partial_t - \partial^2 + \sigma)\phi - \Delta\,(\partial_x^a\bar\phi)^2 + \bar\phi\, U_1(\phi) - \frac{1}{2}\bar\phi^2\, U_2(\phi) \Big\}\,, \tag{3.59}$$

with U^1 and U^2 are polynomials beginning at quadratic and linear order respectively and σ, $\Delta \geq 0$, such that the classical homogeneous stationary solution is $\phi = \bar\phi = 0$. The parameter a can take the value zeo or one. $a = 0$ corresponds to a non-conservative noise, such as the one appearing in the Langevin equation (3.37), while $a = 1$ corresponds to a conservative noise. An example of such noise was introduced for the DP-C model in Sec. 2.1. The case $U_2 = 0$ is analysed in details in (Täuber 2014). From the above equation, we obtain that

$$\begin{aligned} \big[\bar G_0\big]_{\phi\phi}(\mathbf{p}) &= \frac{2\Delta\, p^{2a}}{\omega^2 + (p^2+\sigma)^2}\,, \\ \big[\bar G_0\big]_{\phi\bar\phi}(\mathbf{p}) &= \frac{1}{-i\omega + p^2 + \sigma}\,, \\ \big[\bar G_0\big]_{\bar\phi\bar\phi}(\mathbf{p}) &= 0\,. \end{aligned} \tag{3.60}$$

In the following, we use the shorthand notations

$$\begin{aligned} u_{i,n} &= \partial_\phi^n U_i(\phi)\big|_{\phi=0}\,, \\ \big[\bar{G}_0\big]_{\phi\phi}(\mathbf{q}) &= C_\mathbf{q}\,, \\ \big[\bar{G}_0\big]_{\phi\bar{\phi}}(\mathbf{q}) &= R_\mathbf{q}\,. \end{aligned} \tag{3.61}$$

We will concentrate on the inverse propagator $\bar{\Gamma}^{(2)}_{\bar{\phi}\phi}(\mathbf{p})$ at $\mathbf{p}=0$. This should give the first corrections to σ. First, we look at the corrections proportional to $\mathcal{S}^{(4)}$. They read

$$\bar{\Gamma}^{(2)}_{\bar{\phi}\phi}(\mathbf{0})\big|_4 = \frac{1}{2}\int_\mathbf{q}\Big\{C_\mathbf{q}u_{1,3} + R_\mathbf{q}u_{2,2} + c.c.\Big\} \tag{3.62}$$

where $c.c.$ stands for the complex conjugate of the second term in the curly braces. It will be shown in Sec. 3.5 that due to the causality property of the action, the last two terms are both zero after integrating over the frequency. Furthermore,

$$\int_\varpi C_\mathbf{q} = \frac{\Delta\, q^{2a}}{q^2+\sigma} \equiv \Delta\, q^{2a} c_q\,, \tag{3.63}$$

where c_q is the correlator of the equilibrium Gaussian field theory. For $q \gg \sqrt{\sigma}$, *i.e.* short distances, the correlations decay with a power-law behavior. For $q \ll \sqrt{\sigma}$, *i.e.* long distances, the power law is cut off by $\sqrt{\sigma}$. In the language of Euclidean field theory $\sqrt{\sigma}$ is the mass of the excitations of the field. Now looking at the corrections proportional to $\mathcal{S}^{(3)}$, they read

$$\begin{aligned} \bar{\Gamma}^{(2)}_{\bar{\phi}\phi}(\mathbf{0})\big|_3 &= -\int_\mathbf{q}\Big\{u_{1,2}u_{2,1}R_{-\mathbf{q}}R_\mathbf{q} + u_{1,2}u_{2,1}R_\mathbf{q}R_\mathbf{q} + c.c. + \\ &\quad + u_{1,2}^2 R_{-\mathbf{q}}C_\mathbf{q} + c.c.\Big\} \\ &= -\frac{1}{2}\int_{\vec{q}}\Big\{u_{1,2}u_{2,1}\,c_q + \Delta\, u_{1,2}^2\, q^{2a} c_q^2\Big\} \end{aligned} \tag{3.64}$$

where again, the integration over frequencies is done using the causality property of the action. Inserting everything back into the expansion, we obtain

$$\bar{\Gamma}^{(2)}_{\bar{\phi}\phi}(\mathbf{0}) = \sigma + \frac{1}{2}\int_{\vec{q}}\Big\{\big(\Delta\, q^{2a}u_{1,3} - u_{1,2}u_{2,1}\big)c_q - \Delta\, u_{1,2}^2\, q^{2a}c_q^2\Big\} + \dots \tag{3.65}$$

Now integrating over the wave-vector, we use invariance under spatial rotations of the integrand to rewrite each term of the correction in the following form:

$$\int_{\vec{q}} q^{2k} c_q^\ell = 2v_d \int_{L^{-2}}^{\Lambda^2} \mathrm{d}y\, \frac{y^{\frac{d}{2}+k-1}}{(y+\sigma)^\ell}\,, \tag{3.66}$$

with $v_d = 1/(2^{d+1}\pi^{\frac{d}{2}}\Gamma(\frac{d}{2}))$. The first term of (3.65) corresponds to the above formula with $k=a$, $\ell=1$, the second to $k=0$, $\ell=1$ and the third to $k=a$, $\ell=2$. The behavior of this integral depends drastically on the dimension. For $d+2k>2\ell$, it is dominated by the UV scale and goes like $\Lambda^{d+2k-2\ell}$. This is not so surprising, as the original couplings of the theory are defined at the scale Λ, and if we want to take the

limit $\Lambda \to \infty$, we should scale the couplings accordingly. Let us examine the case $d+2k < 2\ell$, where the integrand is UV convergent, such that we can "forget" about the dependency on the UV scale and let $\Lambda \to \infty$. The integral can be calculated and have the following behavior when the linear size of the system, L, goes to infinity

$$\int_0^\infty \mathrm{d}y \, \frac{y^{\frac{d}{2}+k-1}}{(y+\sigma)^\ell} \propto \sigma^{\frac{d}{2}+k-\ell} \,. \tag{3.67}$$

However, we see that for the massless case, $\sigma = 0$, the integral behaves as $L^{-d-2k+2\ell}$. In this case, the correction term becomes dominant in the limit $L \to \infty$. For $a = 0$, this problem appears for d crossing 4. For $a = 1$, it will appear only below $d = 2$.

This divergence signals a breakdown of the mean-field approximation. In high enough dimensions, the contribution of the fluctuations of the fields which are neglected in the saddle point calculation can be taken into account by expanding the interaction terms and calculating pertubatively the corrections. However, below a certain critical dimension d_c which depends on the theory, if the theory is massless the corrections to mean field diverge faster than the leading term in the large volume "thermodynamic" limit even if the coupling of the interaction is weak. At this point we direct the reader to a more in depth presentation of the subject in the context of out of equilibrium field theories in (Täuber 2014).

Let us note also that in the case of strong coupling $g \gg 1$, the corrections can be strong enough to change qualitatively the result of the mean field calculation, even above the critical dimension.

3.4 Introduction to the non-perturbative renormalization group

We have seen in the previous section that the mean field approximation is not valid below a certain spatial dimension if we want to study a theory whose action has a Hessian that possesses zero-modes in the IR, or in other terms when the mass of the theory is zero. This is in particular the case of critical theories, that we will define more precisely later. When a theory is at such critical point of its parameter space, the integrals appearing in the corrections to the mean-field may diverge in the "thermodynamic" limit $L \to \infty$ faster than the leading term, which signals a breakdown of the approximation. As a consequence, the saddle-point calculations are not directly possible. The framework of the renormalization group was developed to bypass this problem.

3.4.1 The Wilson renormalization group

In Sec. 3.3.3 it was shown that irrespectively of the value of the microscopic coupling of the interaction, below a certain dimension, the corrections to the mean field diverge in the thermodynamics limit if the action is at a critical point. This fact can be understood in terms of the fluctuations of the fields. At a critical point, the correlations of the fluctuations are not damped by a mass and span the whole system. The interaction terms probe the correlations of the fluctuating fields at the same space-time point. If the fluctuations are critical, the corrections coming from the interaction will receive contributions from all Fourier modes, from the UV cutoff

Λ to the IR one L^{-1}. When the limit $L \to \infty$ is taken, the correction may thus diverge in the IR in low enough dimensions.

The idea of Wilson is thus not to do the integration on all modes directly but to do it progressively, starting with the UV ones. By doing this, one constructs an effective theory for the remaining IR modes. This idea can be traced back to the block spin renormalization *à la* Kadanoff. This method was originally devised for Ising models or its generalizations. The idea is to obtain an effective theory for "mesoscopic" spins, which are averaged over the microscopic initial degrees of freedom. For example, from an infinite d-dimensional lattice with lattice spacing a, one forms hypercubic blocks of 2^d adjacent spins and assigns a spin value to each block, representing the values of the spins composing the block. The mesoscopic spins thus defined live on a lattice of spacing $a' = 2a$ and interact through an effective Hamilonian that has to be determined. Then, rescaling the distances and the fields by the new lattice spacing a', one obtains a new theory for effective degrees of freedom whose number of modes is divided by two. By studying the effect of the renormalization group transformation on the Hamiltonian or the action defining the theory, one can understand how divergences appear in physical quantities at a critical point. More precisely, the different critical exponents characterizing the critical point under study are encoded in the behavior of the theory under the transformation at or near the critical point, which is a fixed point of the transformation.

The method proposed by Wilson follows the same principles, although directly in Fourier space. In the modern reformulation, one writes the field as a sum of the slow and the fast degrees of freedom:

$$\phi = \phi_< + \phi_> \,, \tag{3.68}$$

where $\phi_>$ contains only the part of the Fourier modes of the original field which are in an UV shell from Λ to Λb, with b smaller but close to 1, and $\phi_<$ contains the modes from Λb to the IR cutoff. Let us assume that j couples only to the slow degrees of freedom in the partition functional. Then it can be rewritten as follows

$$\mathcal{Z}[\mathrm{j}] = \int \mathcal{D}[\phi_<] e^{\mathrm{j}\cdot\phi_<} \int \mathcal{D}[\phi_>] e^{-\mathcal{S}[\phi_< + \phi_>]} = \int \mathcal{D}[\phi_<] e^{\mathrm{j}\cdot\phi_<} e^{-\mathcal{S}_b^{\mathrm{eff}}[\phi_<]} \,. \tag{3.69}$$

where the second equality defines $\mathcal{S}_{\mathrm{eff}}$. The heart of the difficulties lies in calculating $\mathcal{S}_{\mathrm{eff}.}$. However, the situation is different from the calculation of perturbations in the previous section. Indeed the fluctuating field we are integrating on, $\phi_>$, possesses Fourier modes only in the small shell $[\Lambda b, \Lambda]$. As a consequence, the saddle point method of the previous section can be applied as long as the coupling is small enough and the corrections from the interactions can be calculated without risk of divergences, either pertubatively as in Sec. 3.3.3 or with other mean. The next step is to change to the rescaled field variables, which read in Fourier space as

$$\tilde{\phi}_<(\vec{q}) = b^\alpha \tilde{\phi}'(\vec{q}/b) \,. \tag{3.70}$$

The exponent α is linked to the dimension of the field and is generally necessary for the transformation to have a fixed point. Rescaling the wave-numbers to get back Λ as UV cutoff, one obtains finally

$$\mathcal{Z}[\mathrm{j}] = \int \mathcal{D}[\phi'] e^{\mathrm{j}'\cdot\phi'} e^{-\mathcal{S}'[\phi']} \,, \tag{3.71}$$

where

$$\begin{aligned} \mathcal{S}'[\phi'] &= \mathcal{S}_{\text{eff}}[\phi_<] \\ \tilde{\mathrm{j}}'(\vec{q}) &= b^{d+\alpha}\tilde{\mathrm{j}}(b\vec{q})\,. \end{aligned} \tag{3.72}$$

If the initial action is invariant under the consecutive integration on the fast degrees of freedom and the rescaling, $\mathcal{S}[\phi] = \mathcal{S}'[\phi']$, then

$$\mathcal{Z}[\mathrm{j}] = \mathcal{Z}[\mathrm{j}']\,, \tag{3.73}$$

from which one deduces that the correlation functions of the system are scale invariant. This method presents the advantage that it can be done infinitesimaly. Setting $b = e^{-s}$ and letting s go to zero, one obtains the infinitesimal RG transformation.

3.4.2 The regulator and the Wetterich equation

It was realized that one can write down an exact differential equation for the effective action $\mathcal{S}^{\text{eff}}_{\Lambda b}$ under a change of parameter b (Wegner and Houghton 1973). One starts from the mean field effective action $\mathcal{S}^{\text{eff}}_{\Lambda}$ at $b = 1$ and ends at the effective action $\mathcal{S}^{\text{eff}}_{0}$ when all modes have been integrated. Later it was understood that the sharp cutoff induces spurious effects and an exact evolution equation for a smoother cutoff between fast and slow modes was proposed in (Polchinski 1984). A modern formulation of this idea is to add a non-local quadratic term, called the regulator and noted here $\Delta\mathcal{S}_\kappa$ to the action (or Hamiltonian in an equilibrium context). This term depends on a new scale that is introduced by this procedure: the renormalization scale κ. The role of the regulator is to act as a mass for the degrees of freedom of the field with wave-number below κ. In the same time, the regulator should have no effect on the degrees of freedom with wave-numbers above κ, with the additional requirement that the transition between these two regimes is sufficiently smooth.

To fix the idea, let us write the regulator as

$$\begin{aligned} \Delta\mathcal{S}_\kappa &= \frac{1}{2}\int_{\mathbf{x},\mathbf{y}} \left[\mathcal{R}_\kappa\right]_{ij}(\mathbf{x},\mathbf{y})\phi(\mathbf{x})\phi(\mathbf{y}) = \frac{1}{2}\int_{\mathbf{q},\mathbf{q}'} \left[\tilde{\mathcal{R}}_\kappa\right]_{ij}(\mathbf{q},\mathbf{q}')\phi(-\mathbf{q})\phi(-\mathbf{q}') \\ &= \frac{1}{2}\,\phi\cdot\mathcal{R}_\kappa\cdot\phi\,. \end{aligned} \tag{3.74}$$

In order not to break artificially the invariance under spatio-temporal translations and spatial rotations which could be present in the microscopic action or are expected to be realized in the IR, we will always choose a regulator of the form

$$\left[\tilde{\mathcal{R}}_\kappa\right]_{ij}(\mathbf{q},\mathbf{q}') = (2\pi)^{d+1}\delta(\varpi+\varpi')\delta^d(\vec{q}+\vec{q}')\left[\bar{\mathcal{R}}_\kappa\right]_{ij}(\varpi,q^2)\,. \tag{3.75}$$

Looking at the requirements stated above, $\bar{\mathcal{R}}_\kappa$ as a function of q should have the behavior sketched in Fig. 3.4.2 (and the same type of behavior as a function of ϖ). By varying smoothly κ from the UV cutoff of the theory, where the regulator ensures that the saddle-point is exact (see below), to its IR cutoff, one integrates smoothly fluctuations of the fields.

In the presence of the regulator $\Delta\mathcal{S}_\kappa$, the generating functional defined in (3.22) becomes scale dependent, and is denoted $\mathcal{Z}_\kappa$. Accordingly, we define a scale dependent "free energy" $\mathcal{W}_\kappa = \ln\mathcal{Z}_\kappa$. The average value of the fields in the presence of the

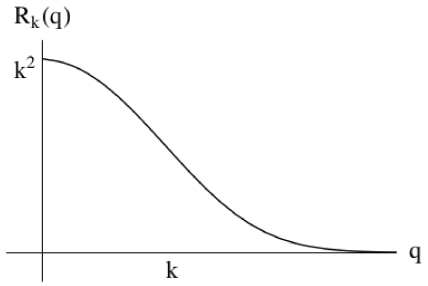

Figure 3.3: Typical behavior of the regulator (from Delamotte 2007)

sources, Φ, and all the derivatives of $\mathcal{W}_\kappa$ become scale dependent. When the renormalization scale κ varies, $\mathcal{W}_\kappa$ evolves according to the following exact flow equation (this equation is reminiscent of the original Polchinski equation (Polchinski 1984), which was formulated in terms of $\mathcal{S}_\kappa^{\text{eff}}$):

$$\begin{aligned}\partial_\kappa \mathcal{W}_\kappa &= -\frac{1}{2}\int_{\mathbf{x},\mathbf{y}} \left[\partial_\kappa \mathcal{R}_\kappa\right]_{ij}(\mathbf{x}-\mathbf{y})\left\{\frac{\delta^2 \mathcal{W}_\kappa}{\delta \mathrm{j}_i(\mathbf{x})\delta \mathrm{j}_j(\mathbf{y})} + \frac{\delta \mathcal{W}_\kappa}{\delta \mathrm{j}_i(\mathbf{x})}\frac{\delta \mathcal{W}_\kappa}{\delta \mathrm{j}_j(\mathbf{y})}\right\} \\ &= -\frac{1}{2}\,\mathrm{tr}\Big[\partial_\kappa \mathcal{R}_\kappa \cdot (G^{(2)} + \Phi\otimes\Phi)\Big]\,. \end{aligned} \tag{3.76}$$

In the last line, tr is the trace and $\otimes$ is the tensor product: $\left[\Phi\otimes\Phi\right]_{ij}(\mathbf{x},\mathbf{y}) = \Phi_i(\mathbf{x})\Phi_j(\mathbf{y})$. As $\mathcal{W}_\kappa$ is the generating functional of the generalized connected correlation functions, $G_\kappa^{(n)}$, the flow of an arbitrary $G_\kappa^{(n)}$ is obtained by taking n derivatives of the flow equation with respect to j_{i_k}, $1 \le k \le n$.

The operation of varying κ is the counterpart of the operation of integrating a momentum shell in the Wilson RG. This integrationg was followed by the additional operation of rescaling the distances and the fields. Only then, a fixed point of the transformation was reached in the case of a critical action. Within the NPRG, the situation is equivalent. Because we have introduced the IR cutoff κ, it is only once properly measured in units of κ that the NPRG flow of $\mathcal{W}_\kappa$ and its derivatives have the possibility to reach a fixed point, if one starts from a critical action. We expound on this in Sec. 3.4.3. From the characteristic of this fixed point, universal properties of the critical field theory can be obtained.

It was realized by (Wetterich 1993; Morris 1994; Ellwanger 1994) that the exact renormalization group equation could be more easily formulated and interpreted in terms of the effective action. The effective average action (EAA), Γ_κ is defined as the modified Legendre transform of the scale-dependent $\mathcal{W}_\kappa$, where the regulator term has been substracted:

$$\Gamma_\kappa[\Phi] + \mathcal{W}_\kappa[\mathrm{j}] = \mathrm{j}\cdot\Phi - \Delta\mathcal{S}_\kappa[\Phi]\,, \tag{3.77}$$

and where j is defined implicitely by the relation

$$\frac{\delta \mathcal{W}_\kappa}{\delta \mathrm{j}_i(\mathbf{x})} = \Phi_i(\mathbf{x})\,. \tag{3.78}$$

Let us note that we have defined Γ_κ as the Legendre transform of $\mathcal{W}_\kappa$ and not as the more general Legendre-Fenchel transform as in Sec. 3.3.2. This is justified by the

presence of the regulator, which ensures that $\mathcal{W}_\kappa$ is differentiable and has a single non-degenerate saddle-point. The formula of the previous section for the Legendre transform applies with Γ replaced by $\Gamma_\kappa + \Delta\mathcal{S}_\kappa[\Phi]$. Notably,

$$\mathrm{j}_i(\mathbf{x}) = \frac{\delta\Gamma_\kappa}{\delta\Phi_i(\mathbf{x})} + \big[\mathcal{R}_\kappa \cdot \Phi\big]_i(\mathbf{x})\,. \tag{3.79}$$

and

$$G^{(2)}_{ij}[\mathbf{x},\mathbf{y};\mathrm{j}] = \big[\Gamma^{(2)}_\kappa + \mathcal{R}_\kappa\big]^{-1}_{ij}[\mathbf{x},\mathbf{y};\Phi]\,. \tag{3.80}$$

More generally, the (scale dependent) generalized connected correlation functions can be constructed from the (scale dependent) vertex functions of $\Gamma_\kappa + \Delta\mathcal{S}_\kappa[\Phi]$.

The flow of Γ_κ under variation of the RG scale κ is given by the Wetterich equation (Wetterich 1993)

$$\begin{aligned}\partial_\kappa\Gamma_\kappa &= \frac{1}{2}\int_{\mathbf{x},\mathbf{y}} \big[\partial_\kappa\mathcal{R}_\kappa\big]_{ij}(\mathbf{x}-\mathbf{y})\Big[\Gamma^{(2)}_\kappa + \mathcal{R}_\kappa\Big]^{-1}_{ji}(\mathbf{y},\mathbf{x}) \\ &= \frac{1}{2}\,\mathrm{tr}\Big[\partial_\kappa\mathcal{R}\cdot(\Gamma^{(2)}_\kappa + \mathcal{R}_\kappa)^{-1}\Big]\,.\end{aligned} \tag{3.81}$$

The reader can turn to (Delamotte 2007) for a pedagogical introduction and derivation of this equation. As well as Eq. (3.76), the RG flow equation (3.81) is exact. Its initial condition corresponds to the 'microscopic' model, which is the action $\mathcal{S}$. Indeed, the flow is initiated at the UV cutoff Λ which is the inverse lattice spacing or the scale at which the continuous description in terms of a SPDE starts to be valid. At this scale, Γ_Λ identifies with the microscopic action $\Gamma_\Lambda = \mathcal{S}$, since no fluctuation is yet incorporated.

To show this, let us take the exponential of the EAA using the definition (3.77)

$$e^{-\Gamma_\kappa[\Phi]} = \int \mathcal{D}[\phi] e^{-\mathcal{S}[\phi]-\Delta\mathcal{S}_\kappa[\phi]+\mathrm{j}\cdot\phi} e^{-\mathrm{j}\cdot\Phi} e^{\Delta\mathcal{S}_\kappa[\Phi]}\,. \tag{3.82}$$

Inserting the explicit definition of j, (3.79) and using the change of variable $\phi = \Phi+\chi$, we arrive to an alternative expression for Γ_κ:

$$e^{-\Gamma_\kappa[\Phi]} = \int \mathcal{D}[\chi] e^{-\mathcal{S}[\Phi+\chi]-\Delta\mathcal{S}_\kappa[\chi]+\frac{\delta\Gamma_\kappa}{\delta\Phi}\cdot\chi}\,. \tag{3.83}$$

Let us apply the saddle point method for the regulated action. For this, we expand the argument of the exponential in the integrand to second order:

$$\begin{aligned}-\mathcal{S}[\Phi+\chi]-\Delta\mathcal{S}_\kappa[\chi]+\frac{\delta\Gamma_\kappa}{\delta\Phi}\cdot\chi &= -\mathcal{S}[\Phi] + \Big(\frac{\delta\Gamma_\kappa}{\delta\Phi} - \frac{\delta\mathcal{S}}{\delta\Phi}\Big)\cdot\chi \\ &\quad + \frac{1}{2}\,\chi\cdot\Big[\mathcal{S}^{(2)} + \mathcal{R}_\kappa\Big]\cdot\chi + o(\|\chi\|^2)\,.\end{aligned} \tag{3.84}$$

For $\kappa \sim \Lambda$, the regulator gives a large mass to all the modes present in the theory, thus even if the microscopic action is critical, the saddle-point method is well-controlled and gives

$$\Gamma_\Lambda[\Phi] = \mathcal{S}[\Phi]\,. \tag{3.85}$$

From the starting point $\kappa = \Lambda$, the renormalization scale is lowered until it reaches 0 or a natural IR scale L^{-1} of the model. At this point the regulator is

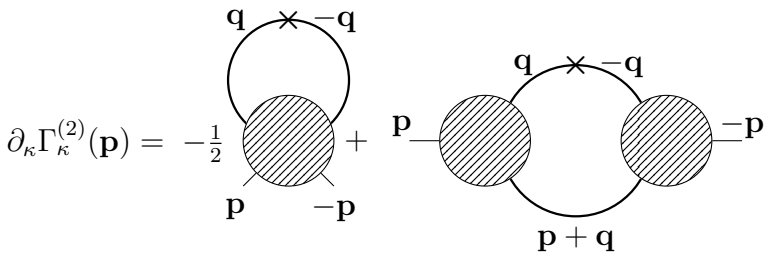

Figure 3.4: Diagrammatic representation of the flow of $\Gamma_\kappa^{(2)}$

either zero for all wave-numbers or reaches its original physical value, such that the scale dependent EAA becomes equal to the original EAA: $\Gamma_0 = \Gamma$. At the end of the flow, one recovers the actual properties of the model, when all fluctuations up to the physical IR cutoff have been integrated over. The interest of Eq. (3.81) is that it provides an exact smooth interpolation between these two scales. Due to the presence of the derivative of the regulator in the integral, the contribution at the scale κ is dominated by values of $q \lesssim \kappa$. This property and the smoothness of the EAA for $\kappa \neq 0$ allows to devise approximation schemes out of the usual perturbative calculations. When the initial theory is at a critical point, by looking at the properties of the flow one can thus hope to catch how the fluctuations at all scales build up progressively the singularities of Γ which lead to phenomena such as scale invariance.

As well as for the flow equation of $\mathcal{W}_\kappa$, one can deduce from the flow equation of Γ_κ (3.81) the flow equation for a generic n-point vertex function by taking the n corresponding functional derivatives. This yields equations which are exact, but which involve $(n+1)$ and $(n+2)$ vertex functions, such that if no approximations are made, one has to solve an infinite hierarchy of flow equations. For example, for the two-point function, writing the flow equation in Fourier space and using the conservation of wave-vector and frequency, it reads

$$\begin{aligned}\partial_s \bar{\Gamma}^{(2)}_{mn}(\mathbf{p}) = \int_{\mathbf{q}} \big[\partial_s \bar{\mathcal{R}}\big]_{ij}(\mathbf{q}) \bar{G}^{(2)}_{jk}(\mathbf{q}) \Big[-\frac{1}{2} \bar{\Gamma}^{(4)}_{klmn}(\mathbf{q}, -\mathbf{q}, \mathbf{p}) \\ + \bar{\Gamma}^{(3)}_{kms}(\mathbf{q}, \mathbf{p}) \bar{G}^{(2)}_{st}(\mathbf{q}+\mathbf{p}) \bar{\Gamma}^{(3)}_{tnl}(\mathbf{q}+\mathbf{p}, -\mathbf{p}) \Big] \bar{G}^{(2)}_{li}(\mathbf{q}) \,. \end{aligned} \tag{3.86}$$

The r.h.s. is represented diagrammatically in Figure 3.4: the dashed circles are the vertex functions, the thick lines are propagators and the cross is the derivative of the regulator. Notice the similarity with Fig. 3.2. The terms in the r.h.s. depend on a wave-vector and frequency of integration, $\mathbf{q}$, called internal, besides to depend on the external wave-number and frequency $\mathbf{p}$ at which the vertex function is evaluated. In order to go further in the study of the infinite hierarchy of flow equations, we are guided by the properties of the regulator which ensure that the integral in the r.h.s. is dominated by values of $q \lesssim \kappa$.

In most applications, the property $q \lesssim \kappa$ is used to approximately close the hierarchy by simply truncating higher-order vertices, or proposing an ansatz for Γ_κ (Berges, Tetradis, and Wetterich 2002). In the work on DEP, presented in Chap. 4, we indeed use such an ansatz for the EAA, named the modified local potential approximation (LPA'), which consists in approximating the EAA by its behavior at small wave-numbers. In the work on SNS (in Chap. 5), we concentrate on the

behavior at large external wave-numbers. We will see that in this limit, it is not necessary to make an ansatz for the EAA due to some properties that are particular to the SNS action.

3.4.3 Fixed point solutions of the flow and scale invariance

Having established the general formalism of NPRG, we can explain the link between scale invariance and fixed-point solutions of the RG flow equation. For this, we follow the presentation of (Delamotte, Tissier, and Wschebor 2016). Let us first examine the case of a critical theory. The generalized correlation functions are expected to exhibit scale invariance, which means that they are expected to be invariant under the following change of variables:

$$\phi_i(t,\vec{x}) \to b^{d_i}\phi_i(b^z t, b\vec{x})\,, \tag{3.87}$$

with d_i, z the scaling exponents of the corresponding critical theory. Let us perform this change of variable inside $\mathcal{Z}_\kappa[\mathrm{j}]$, and take the infinitesimal limit $b = e^\epsilon = 1+\epsilon+o(\epsilon)$.

$$\begin{aligned} &\phi_i(\mathbf{x}) \to \phi_i(\mathbf{x}) + \delta_\epsilon\phi_i(\mathbf{x}) + o(\epsilon)\,,\\ \text{with}\quad &\delta_\epsilon\phi_i(\mathbf{x}) = \epsilon\,(d_i + \vec{x}\cdot\partial_{\vec{x}} + z\,t\partial_t)\phi_i(\mathbf{x})\\ &\qquad\equiv \epsilon\,\big[\mathrm{D}\cdot\phi\big]_i(\mathbf{x})\,. \end{aligned} \tag{3.88}$$

The jacobian of the change of variables is non-zero but it is constant in the fields so it does not play a role in the correlation functions and it is omitted in the following. Writing that the partition function is invariant under a change of dummy variables, we obtain

$$\langle\delta_\epsilon\mathcal{S}\rangle_\mathrm{j} + \langle\delta_\epsilon\Delta\mathcal{S}_\kappa\rangle_\mathrm{j} = \mathrm{j}\cdot\delta_\epsilon\Phi\,, \tag{3.89}$$

where $\delta_\epsilon\mathcal{X}$ is the first order variation of the functional $\mathcal{X}[\phi]$ under the change of variable (3.88). This derivation will be written in more details in the general case of a change of variables linear in the fields in Sec. 3.6. The regulator term $\Delta\mathcal{S}_\kappa$ depends explicitly on the scale κ, thus its variation is non-zero. However, the κ-dependency of $\Delta\mathcal{S}_\kappa$ is chosen in a standard way such that

$$\delta_\epsilon\Delta\mathcal{S}_\kappa = -\epsilon\,\partial_s\Delta\mathcal{S}_\kappa\,, \tag{3.90}$$

where s is the "RG-time" $s\equiv\ln(\kappa/\Lambda)$. It amounts to choosing a power-law dependence in κ for the regulator. Explicitly,

$$\begin{aligned} \big[\mathcal{R}_\kappa\big]_{ij}(\mathbf{x},\mathbf{y}) &= R_\kappa^{ij}\,\hat{r}_{ij}\big(\hat{\mathbf{x}},\hat{\mathbf{y}}\big)\,,\\ \text{with } \hat{\mathbf{x}} &= (\kappa^z t, \kappa\vec{x})\\ \text{and } R_\kappa^{ij} &\sim \kappa^{2d+2z-d_i-d_j}\,. \end{aligned} \tag{3.91}$$

This means that, for this form of regulator, the variation due to dilatation of space-time and fields is equal to (minus) the variation due to a dilatation of the renormalization scale. Taking a derivative with respect to s of the partition function at fixed sources, the variation of the regulator can thus be rewritten as

$$\langle\delta_\epsilon\Delta\mathcal{S}_\kappa\rangle_\mathrm{j} = \epsilon\,\partial_s\mathcal{W}_\kappa[\mathrm{j}]|_\mathrm{j}\,. \tag{3.92}$$

Replacing in (3.89), we obtain first the Ward identity of dilatations for $\mathcal{W}_\kappa$:

$$\epsilon\,\partial_s\mathcal{W}_\kappa + \langle\delta_\epsilon\mathcal{S}\rangle_{\mathrm{j}} = \epsilon\,\mathrm{j}\cdot\Big[\mathrm{D}\cdot\frac{\delta}{\delta\mathrm{j}}\mathcal{W}_\kappa\Big]\,. \tag{3.93}$$

Furthermore, using the modified Legendre transform (3.77) in (3.92), we have also that

$$\langle\delta_\epsilon\Delta\mathcal{S}_\kappa\rangle_{\mathrm{j}} = -\epsilon\,\partial_s\Big(\Gamma_\kappa[\Phi]+\Delta\mathcal{S}_\kappa[\Phi]\Big)\Big|_{\Phi}\,, \tag{3.94}$$

which leads to the Ward identity of dilatations for Γ_κ:

$$\epsilon\,\frac{\delta\Gamma_\kappa}{\delta\Phi}\cdot\big[\mathrm{D}\cdot\Phi\big] = \langle\delta_\epsilon\mathcal{S}\rangle_{\mathrm{j}} - \epsilon\,\partial_s\Gamma_\kappa[\Phi]\,. \tag{3.95}$$

This equation states that Γ_κ is scale invariant up to the two terms of the r.h.s.: the scale invariance-breaking terms of the microscopic action, and the regulator. Even if the theory is invariant under dilatations (*i.e.* if $\delta_\epsilon\mathcal{S}=0$), the presence of the renormalization scale κ breaks scale invariance. Thus, in the presence of the regulator, one has to choose κ as the unit of scale for space-time and the fields, and to introduce dimensionless quantities in order to recover scale invariance. Defining

$$\Gamma_\kappa[\Phi] = \hat\Gamma_\kappa[\hat\Phi]\,,\ \text{with}\ \hat\Phi_i(\hat{\mathbf{x}}) = \kappa^{-d_i}\Phi_i(\hat x/\kappa,\hat t/\kappa^z)\,, \tag{3.96}$$

and replacing in (3.95), one obtains that

$$\epsilon\,\partial_s\hat\Gamma_\kappa[\hat\Phi] = \langle\delta_\epsilon\mathcal{S}\rangle_{\mathrm{j}}\,. \tag{3.97}$$

We say in this work that a theory is critical if the microscopic action is scale invariant, or if the r.h.s. of the above equation goes to zero for $\kappa\to 0$. In short, (3.97) states that critical phenomena, in the sense above, is equivalent to having a fixed points of the dimensionless RG flow in the IR. This is the motivation to look at fixed point of $\hat\Gamma_\kappa$. From (3.81), we finally write down the RG flow equation for $\hat\Gamma_\kappa$:

$$\partial_s\hat\Gamma_\kappa[\hat\Phi] = \frac{\delta\hat\Gamma_\kappa}{\delta\hat\Phi}\cdot\big[\mathrm{D}\cdot\hat\Phi\big] + \frac{1}{2}\,\mathrm{tr}\Big[\mathrm{D}^R\hat r\cdot(\hat\Gamma^{(2)}_\kappa+\hat r)^{-1}\Big]\,, \tag{3.98}$$

with

$$\mathrm{D}^R\hat r_{ij}(\mathbf{x},\mathbf{y}) = \big(2d+2z-d_i-d_j+\hat{\vec x}\cdot\partial_{\hat{\vec x}}+z\,\hat t\partial_{\hat t}+\hat{\vec y}\cdot\partial_{\hat{\vec y}}+z\,\hat u\partial_{\hat u}\big)\hat r_{ij}(\mathbf{x},\mathbf{y})\,. \tag{3.99}$$

For a given set of scaling dimensions d_i and z, using (3.97), the flow equation can be reinterpreted as a condition on the IR EAA of the critical theory. Indeed, for critical phenomena according to our definition, the r.h.s. of (3.98) is equal to zero in the IR. Of course, in the particular case of a scale-invariant Γ_κ, the first term of the r.h.s. vanishes by definition.

In standard critical phenomena, the second term of the r.h.s. can be shown to go to zero compared to the first one for external wave-numbers $p_i\gg\kappa$. In this case, the RG flow equation decouples the fast variables of the system from the slow ones. This decoupling property is satisfied by the RG flows corresponding to equilibrium phase transitions for example (Berges, Tetradis, and Wetterich 2002), and also to many critical phenomena out-of-equilibrium (Canet, Chaté, and Delamotte 2004; Canet 2005; Canet et al. 2010). If the RG flow has this decoupling property, only the first term of the r.h.s. remains when κ goes to zero and one deduces that the EAA of the theory is scale invariant, in the precise sense of invariance under (3.88). However, as it was hinted at in Sec. 2.2, it may happen that the RG flow does not have the decoupling property. In this case, even if the NPRG flow equation (3.98) reaches a fixed point, it does not necessarily entails the scale-invariance of $\hat\Gamma_\kappa$.

3.4.4 Running scaling dimensions and dimensionless quantities

In most situations, the scaling dimensions of the critical theory are not exactly known. Moreover, for a given scale κ, the EAA Γ_κ may not be scale-invariant. In this case, there are no well defined scaling dimensions of the fields and of time. One idea at this point is to define scale dependent scaling dimensions which take a constant value if the RG flow reaches a fixed point.

For a scale invariant theory with given scaling dimensions d_i and z, one has for the two-point functions

$$\Gamma^{(2)}_{ij}[\mathbf{x},\mathbf{y};\Phi] = \kappa^{-d_i-d_j}\hat{\Gamma}^{(2)}_{ij}[\hat{\mathbf{x}},\hat{\mathbf{y}};\hat{\Phi}]\,. \tag{3.100}$$

One would thus define the scale-dependent, or running dimensions of the fields as

$$d_i(\kappa) + d_j(\kappa) \equiv -\partial_s \ln \Gamma^{(2)}_{ij}[\mathbf{x},\mathbf{y};\Phi]\,. \tag{3.101}$$

However, if Γ_κ is not scale invariant, this definition is ambiguous as in fact the running dimensions may depend on the values of the field and the spacetime variables at which the two-point function is evaluated. Thus one has to choose a particular value of the field and a particular configuration in space-time. This step is analogous to normalization conditions in perturbative RG. In most situations, we choose $\Phi = \Phi_0$ homogeneous and stationary (generally, $\Phi_0 = 0$). Furthermore, because we are interested in the IR behavior of the theory, we define the running dimensions, at zero frequency and wave-number in Fourier space. Taking the Fourier transform of (3.100) and assuming invariance under space-time translations, one has

$$\bar{\Gamma}^{(2)}_{ij}(\mathbf{p},\Phi_0) = \kappa^{z+d-d_i-d_j}\hat{\bar{\Gamma}}^{(2)}_{ij}(\hat{\mathbf{p}},\Phi_0)\,. \tag{3.102}$$

As a consequence, the running field dimensions are defined as

$$\begin{aligned} d_i(\kappa) + d_j(\kappa) - d &\equiv -\partial_s \ln Z^{ij}_\kappa \\ \text{with } Z^{ij}_\kappa &= \partial_{i\omega}\bar{\Gamma}^{(2)}_{ij}(\mathbf{p},\Phi_0)|_{\mathbf{p}=0}\,. \end{aligned} \tag{3.103}$$

In the same spirit, the running scaling dimension of time is defined as

$$\begin{aligned} z(\kappa) - 2 &\equiv \partial_s \ln D^{ij}_\kappa \\ \text{with } D^{ij}_\kappa &= \frac{1}{Z^{ij}_\kappa}\partial_{p^2}\bar{\Gamma}^{(2)}_{ij}(\mathbf{p},\Phi_0)|_{\mathbf{p}=0}\,. \end{aligned} \tag{3.104}$$

Again, this definition may depend on i, j and Φ_0 but the limit $\kappa \to 0$ should be the same for all fields with non-trivial dynamics. The κ dependence of d_i and d_j is implicit in the following. Now, we can define a "running dimensionless EAA" as

$$\Gamma_\kappa[\Phi] = \hat{\Gamma}_\kappa[\hat{\Phi}]\,,\ \text{with } \hat{\Phi}_i(\hat{\mathbf{x}}) = Z^i_\kappa \Phi_i\Big(\frac{\hat{x}}{\kappa}, \frac{\hat{t}}{\kappa^2 D^i_\kappa}\Big)\,, \tag{3.105}$$

where the Z^i_κ are constructed from the Z^{ij}_κ such that $Z^i_\kappa \sim \kappa^{-d_i}$ and the D^i_κ are constructed in the same fashion from the D^{ij}_κ. The RG flow equation of $\hat{\Gamma}_\kappa$ reads

$$\partial_s \hat{\Gamma}_\kappa[\hat{\Phi}] = \frac{\delta\hat{\Gamma}_\kappa}{\delta\hat{\Phi}}\cdot\big[\mathrm{D}_\kappa\cdot\hat{\Phi}\big] + \frac{1}{2}\,\mathrm{tr}\Big[\mathrm{D}^R_\kappa\hat{r}\cdot(\hat{\Gamma}^{(2)}_\kappa + \hat{r})^{-1}\Big]\,, \tag{3.106}$$

which is the same as in the previous section except that the dilatation operators are now running with the renormalization scale. This evolution equation, as well as the prescription for the scaling dimensions (3.103) and (3.104), is our starting point for the study of DEP in the next chapter.

3.5 Causality and Itô prescription in NPRG

Let us point out a general property of out of equilibrium field theories. This will be done in the context of the forward discretization scheme for SPDE, such as Itô prescription for Langevin equations, in continuum field theories. To do this, let us use the example of the action of Sec. 3.3.3

$$\mathcal{S}[\phi,\bar{\phi}] = \int_{\mathbf{x}} \left\{ \bar{\phi}(\partial_t - \partial^2 + \sigma)\phi - \Delta\,(\partial_x^a \bar{\phi})^2 + \bar{\phi}\,U^1(\phi) - \frac{1}{2}\bar{\phi}^2\,U^2(\phi) \right\}. \tag{3.107}$$

This action was derived from an Itô Langevin equation with the MSRJD response field formalism, and as such it satisfies the two properties associated with causality. The first property is that $\mathcal{S}[\phi,0]=0$. The second property, which is hidden in the time-continuous limit, is that the response field is always evaluated at a later time than the observable field in the discretized version of the action. Let us expound on the consequences. First, let us calculate the mean field propagator of the theory:

$$\left[G_0^{(2)}\right]_{\phi\bar{\phi}}(\mathbf{x},\mathbf{y}) = \langle \phi(\mathbf{x})\bar{\phi}(\mathbf{y})\rangle_c = \left[\mathcal{S}^{(2)}\right]^{-1}_{\phi\bar{\phi}}(\mathbf{x},\mathbf{y}), \tag{3.108}$$

where $\mathcal{S}^{(2)}$ is the Hessian of the action above evaluated at the classical stationary solution for the sources set to zero. From the action, we obtain in Fourier space (after having extracted the delta functions of conservation of frequency and wave-number)

$$\bar{\mathcal{S}}^{(2)}_{\bar{\phi}\phi}(\mathbf{p}) = -i\omega + p^2 + \sigma\,. \tag{3.109}$$

As a consequence,

$$\left[G_0^{(2)}\right]_{\phi\bar{\phi}}(\mathbf{x},\mathbf{y}) = \int_{\mathbf{p}} \left[\bar{\mathcal{S}}^{(2)}_{\bar{\phi}\phi}(\mathbf{p})\right]^{-1} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} = \int_{\mathbf{p}} \frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{-i\omega + p^2 + \sigma}\,. \tag{3.110}$$

Let us recall that we note $\mathbf{x} = (t,\vec{x})$ and $\mathbf{y} = (u,\vec{y})$. For $t \neq u$, the frequency integration is done using Jordan's lemma and the residue theorem and gives

$$\left[G_0^{(2)}\right]_{\phi\bar{\phi}}(\mathbf{x},\mathbf{y}) = \Theta(t-u) \int_{\vec{p}} e^{i\vec{p}\cdot(\vec{x}-\vec{y}) - (p^2+\sigma)(t-u)}\,. \tag{3.111}$$

The Heaviside Θ function which appears in the above expression is the consequence of the causality of the initial stochastic process, as the propagator, or Green function of the theory vanishes if the response of the system is measured before the perturbation has happened. However, for the continuous description there is an ambiguity at $t=u$. Formally, the propagator is proportional to the ill-defined value $\Theta(0)$. This ambiguity is in fact a general feature of out of equilibrium field theories and it is cured by going back to the time-discretized version of the action. In a perturbative setting, it can be shown that the Itô prescription amounts to take $\Theta(0)=0$. An explanation of this prescription is given in Appendix B.2.

Let us see how the causality is treated in the NPRG setting, with the notations of a general out of equilibrium field theory as defined in Sec. 3.4. For a given theory $\mathcal{Z}[\mathrm{j}]$, let us say that j satisfies the causality property if any generalized correlation function evaluated at j vanishes when its largest time is the one of a response field. In particular, for such a configuration of j, the mean value of the response fields are zero. Using the decomposition of $G^{(n)}$ as a sum of trees whose vertices are the $\Gamma^{(k)}$,

it can be shown that the causality property of j translates for the vertex functions as the following: if the largest time is not associated to a response field, the vertex function vanishes.

To be more concret, let us look at the case of the Gaussian scalar field. In this case, the j which satisfy the causality property on the correlation functions are $\mathrm{j} = \{0, \bar{\jmath}\}$, for any $\bar{\jmath}$. Accordingly, any configuration of the fields $\Phi = \{\phi, 0\}$ satisfies the causality property on the vertex functions. For interacting theories, we ask that the causality property for vertex is verified along the RG flow in any configuration of the fields for which the response fields are set to zero. It can be checked that it is verified at the initial condition of the flow. It has been shown in a NPRG setting in (Canet, Chaté, and Delamotte 2011; Benitez and Wschebor 2013) that this property is conserved along the RG flow on the condition that the regulator itself satisfies it. Explicitly, we must have

$$\begin{aligned}
&\left[\bar{\mathcal{R}}_\kappa\right]_{ij}(\mathbf{x},\mathbf{y}) = 0\,, \text{ if both } i \text{ and } j \text{ are observable fields}\\
&\left[\bar{\mathcal{R}}_\kappa\right]_{ij}(\mathbf{x},\mathbf{y}) = 0\,, \text{ if } i \text{ is an observable field, } j \text{ a response fields and } t \geq u\,.
\end{aligned} \tag{3.112}$$

At this point, let us elaborate on our implementation of the NPRG. Although it would be desirable to have a regulator depending on the frequencies as well as on the wave-numbers, in practice this turns out to be a difficult task to set up. Notably, the introduction of a frequency regulator may breaks symmetries related to time in certain models. This is the case for example for SNS. In the present work we choose to have a regulator independent (or minimally dependent, see later) of the frequency. As a consequence, the frequency integral of the flow equation is not constrained a priori. Some progresses have been made recently to include a frequency-dependent regulator by (Duclut and Delamotte 2017).

Going back to the causality condition (3.112), the last constraint means that the off-diagonal elements of the regulator cannot be exactly instantaneous, as they would be if the regulator was frequency independent. The response field has to be evaluated at a later time than the observable fields instead of them being evaluated in the regulator at the same time. For example, for the case of a one-component scalar field theory and its response field, the causality condition (3.112) reads

$$\begin{aligned}
\Delta\mathcal{S}_\kappa[\Phi,\bar{\Phi}] &= \int_{\mathbf{x},\mathbf{y}} \mathcal{R}_\kappa(\mathbf{x},\mathbf{y})\Phi(\mathbf{x})\bar{\Phi}(\mathbf{y})\\
&= \int_{\mathbf{x},\mathbf{y}} \mathcal{R}_\kappa(|\vec{x}-\vec{y}|)\delta(t+\epsilon-u)\Phi(\mathbf{x})\bar{\Phi}(\mathbf{y})\\
&= \int_{\vec{x},\vec{y},t} \mathcal{R}_\kappa(|\vec{x}-\vec{y}|)\Phi(\mathbf{x})\bar{\Phi}(\vec{y},t+\epsilon)\\
&= \int_{\mathbf{q}} \bar{\mathcal{R}}_\kappa(q)e^{i\varpi\epsilon}\Phi(\mathbf{q})\bar{\Phi}(-\mathbf{q})
\end{aligned} \tag{3.113}$$

with $\epsilon > 0$. Note that within the derivative expansion of Chap. 4, the prescription above is equivalently taken into account by simply shifting the response field in the propagator at an infinitesimal later time:

$$\left[G_0^{(2)}\right]_{\phi\bar{\phi}}(\mathbf{x},\mathbf{y})\big|_{\mathrm{Ito}} = \langle\phi(\mathbf{x})\bar{\phi}(\mathbf{y})\rangle_c\big|_{\mathrm{Ito}} = \langle\phi(\mathbf{x})\bar{\phi}(u+\epsilon,\vec{y})\rangle_c\,, \text{ with } \epsilon > 0\,. \tag{3.114}$$

However, one has to keep in mind that to replace the propagators by their shifted version amounts to the Itô prescription only when they are integrated against vertices of the theory. Going to Fourier space, the prescription rewrites as

$$\left[\bar{G}_0^{(2)}\right]_{\phi\bar{\phi}}(\mathbf{p})\Big|_{\text{Itô}} = e^{i\omega\epsilon}\left[\bar{G}_0^{(2)}\right]_{\phi\bar{\phi}}(\mathbf{p})\,. \tag{3.115}$$

After the integration in frequency, ϵ can safely be set to 0.

3.6 Ward identities and dualities in NPRG

Both works on DEP and on SNS rely heavily on the tools of Ward identities, which relate symmetries of the microscopic action $\mathcal{S}$ to symmetries of Γ_κ. Let us expound here on this formalism.

Within the NPRG framework, in the presence of the infrared regulator $\Delta\mathcal{S}_\kappa$, Ward identities can be derived by considering a change of variables $\phi \to \phi'$ in $\mathcal{Z}_\kappa$ which is at most linear in the fields and leaves the functional measure invariant. Denoting $\delta\mathcal{X}[\phi] = \mathcal{X}[\phi'] - \mathcal{X}[\phi]$, we have

$$\delta\phi_i(\mathbf{x}) = [A\cdot\phi]_i(\mathbf{x}) + B_i(\mathbf{x})\,, \tag{3.116}$$

where A is a linear operator acting on ϕ. We have

$$\begin{aligned}
\mathcal{Z}_\kappa[\mathrm{j}] &= \int \mathcal{D}[\phi'] e^{-\mathcal{S}[\phi'] - \Delta\mathcal{S}_\kappa[\phi'] + \mathrm{j}\cdot\phi'} \\
&= \int \mathcal{D}[\phi] e^{-\mathcal{S}[\phi+\delta\phi] - \Delta\mathcal{S}_\kappa[\phi+\delta\phi] + \mathrm{j}\cdot\phi + \mathrm{j}\cdot\delta\phi} \\
&= \int \mathcal{D}[\phi] e^{-\mathcal{S}[\phi] - \Delta\mathcal{S}_\kappa[\phi] + \mathrm{j}\cdot\phi} e^{-\delta(\mathcal{S}+\Delta\mathcal{S}_\kappa)[\phi] + \mathrm{j}\cdot\delta\phi} \\
&= \mathcal{Z}_\kappa[\mathrm{j}] \langle e^{-\delta(\mathcal{S}+\Delta\mathcal{S}_\kappa)[\phi] + \mathrm{j}\cdot\delta\phi}\rangle_\mathrm{j}\,,
\end{aligned} \tag{3.117}$$

where $\langle\cdot\rangle_\mathrm{j}$ is the mean value in the presence of the sources. Thus,

$$\langle e^{-\delta(\mathcal{S}+\Delta\mathcal{S}_\kappa)[\phi] + \mathrm{j}\cdot\delta\phi}\rangle_\mathrm{j} = 1\,. \tag{3.118}$$

In the particular case of $\delta\mathcal{S}[\phi] = \delta\Delta\mathcal{S}_\kappa[\phi] = 0$ and if the operator $\mathrm{S} : \phi \to \phi'$ admits a dual operator for the scalar product, such that $\mathrm{j}\cdot\mathrm{S}\phi = (\mathrm{S}^*\mathrm{j})\cdot\phi$, the identity above reads

$$\mathcal{Z}_\kappa[\mathrm{j}] = \mathcal{Z}_\kappa[\mathrm{S}^*\mathrm{j}]\,. \tag{3.119}$$

Using the definition of the EAA (3.77) as the modified Legendre transform of $\mathcal{W}_\kappa = \ln\mathcal{Z}_\kappa$ and the invariance of the regulator, one obtains readily that

$$\Gamma_\kappa[\mathrm{S}\Phi] = \Gamma_\kappa[\Phi] \tag{3.120}$$

i.e. the symmetries of $\mathcal{S}$ are symmetries of Γ_κ.

However, if the change of variables is not an exact symmetry of the action and of the regulator, not much more can be said at this point in general. Hence let us specify that the change of variables is controlled by an infinitesimal parameter ϵ: $\delta\phi = \delta_\epsilon\phi = O(\epsilon)$. As a consequence of the linearity, $\langle\delta_\epsilon\phi\rangle_\mathrm{j} = \delta_\epsilon\Phi$. At linear order in ϵ, one obtains from Eq. (3.118)

$$\langle\delta_\epsilon\mathcal{S}\rangle_\mathrm{j} + \langle\delta_\epsilon\Delta\mathcal{S}_\kappa\rangle_\mathrm{j} = \mathrm{j}\cdot\delta_\epsilon\Phi\,. \tag{3.121}$$

where $\delta_\epsilon \mathcal{X}$ is the part of $\delta\mathcal{X}$ linear in ϵ. Using the definition (3.74) of the regulator term, its variation can be written as

$$\delta_\epsilon \Delta\mathcal{S}_\kappa = \delta_\epsilon\phi \cdot \mathcal{R}_\kappa \cdot \phi \,, \tag{3.122}$$

Inserting this expression in (3.121), along with the definition of j (3.79) and using the definition of the change of variables (3.116), it follows that

$$\delta_\epsilon \Gamma_\kappa[\Phi] = \langle \delta_\epsilon \mathcal{S} \rangle_{\mathrm{j}} + \epsilon \,\mathrm{tr}\Big[\mathcal{R}_\kappa \cdot A \cdot G^{(2)}[\Phi]\Big] \,. \tag{3.123}$$

Thus, the variation of the EAA Γ_κ is equal to the mean of the variation of $\mathcal{S}$ plus the term coming from a possible symmetry breaking regulator. Interestingly, due to the definition of Γ_κ, the variation of the regulator under the shift part of the change of variables (3.116) does not appear in the final result.

For changes of variables which are exact symmetries of the action and of the regulator, Eq. (3.123) simply translates into $\delta_\epsilon \Gamma_\kappa[\Phi] = 0$, which is the infinitesimal version of (3.120). When the variation of the action is non-zero but linear in the fields, we call the change of variables an extended symmetry, following (Teodorovich 1989). In this case, the mean and the variation commute and the identity reads

$$\delta_\epsilon \Gamma_\kappa[\Phi] = \delta_\epsilon \mathcal{S}[\Phi] \,. \tag{3.124}$$

This provides non-renormalization theorems which fix a sector of Γ_κ to its initial value and shows the usefulness of extended symmetries.

Let us present a special case of (3.121) of particular interest, which is not strictly speaking a Ward identity but a duality. In certain situations, the variation of the action $\delta_\epsilon \mathcal{S}$ can be re-rexpressed as a variation under a change of one of its couplings. Noting explicitly the dependence in the microscopic coupling, say σ, as $\mathcal{S}[\phi;\sigma]$ we have in this case

$$\delta_\epsilon \mathcal{S}[\phi;\sigma] = \epsilon\, \partial_\sigma \mathcal{S}[\phi;\sigma] \,. \tag{3.125}$$

As a consequence,

$$\langle \delta_\epsilon \mathcal{S} \rangle_{\mathrm{j}} = -\partial_\sigma \mathcal{W}[\mathrm{j};\sigma]\big|_{\mathrm{j}} \,, \tag{3.126}$$

which is easily verified by taking a σ derivative of the partition function at fixed sources j. Using the relation

$$\partial_\sigma|_\Phi = \partial_\sigma|_{\mathrm{j}} + \partial_\sigma \mathrm{j} \cdot \frac{\delta}{\delta \mathrm{j}} \tag{3.127}$$

and the definition of the EAA, we obtain that

$$\partial_\sigma \mathcal{W}[\mathrm{j};\sigma]\big|_{\mathrm{j}} = -\partial_\sigma \Gamma[\Phi;\sigma]\big|_{\Phi} \,. \tag{3.128}$$

Finally, the duality identity reads in this case

$$\delta_\epsilon \Gamma_\kappa[\Phi;\sigma] = \epsilon\, \partial_\sigma \Gamma[\Phi;\sigma] + \epsilon \,\mathrm{tr}\Big[\mathcal{R}_\kappa \cdot A \cdot G^{(2)}[\Phi]\Big] \,. \tag{3.129}$$

In conclusion, in this chapter we have motivated and introduced the framework of the NPRG in the context of out of equilibrium field theories. We have tried to show how the EAA is the natural object to look at in most situations. This has

led to discuss the relation between fixed point of the NPRG equation and scale-invariance, as well as our general strategy to find such fixed point. Finally, we have discussed two topics which are necessary for our studies: the issue of causality in out of equilibrium field theories and its treatment by the NPRG, and the derivation of Ward identities and duality identities within the NPRG framework. Now, let us apply these tools to the problem of the absorbing phase transition in DEP.

3.6.0.1 Version française

En conclusion, dans ce chapitre, nous avons motivé et présenté le formalisme du NPRG dans le contexte des théories de champ hors équilibre. Nous avons essayé de montrer comment l'EAA est l'objet naturel à considérer dans la plupart des situations. Cela a conduit à discuter de la relation entre le point fixe de l'équation du NPRG et l'invariance d'échelle, ainsi que de notre stratégie générale pour trouver ce point fixe. Enfin, nous avons abordé deux sujets qui sont nécessaires à nos études : la question de la causalité dans les théories de champ hors équilibre et son traitement par le NPRG, ainsi que la dérivation des identités de Ward et des identités de dualité dans le cadre du NPRG. Maintenant, appliquons ces outils au problème de la transition de phase absorbante dans DEP.

Chapter 4

Study of the absorbing phase transition in DEP

In Sec. 2.1 we motivated a study of the absorbing phase transition occuring in the DEP model with the tools of the NPRG. Notably we insisted on the uncertainty as to whether DEP and DP-C, its naive coarse-grained counterpart, were belonging to the same universality class. This chapter is an account on the preliminar results for the study of DEP which were published in (Tarpin et al. 2017). While the field theory of DP-C has already received considerable interest, it is the first time that the DEP field theory is studied directly. Indeed, within the perturbative approach of all the precedent works, both models are in the same universality class. Although we were not able to give a definitive answer, we hope that at least our study renewed interest in this problem and paved the way for future works on the subject.

In Sec. 4.1, we present the different field theories appearing in the study of DEP and DP-C. We study the symmetries of these actions in Sec. 4.2 and use them to obtain exact results for DEP and DP-C in the framework of NPRG. This is the occasion to discuss how both theories may differ in their phase transition. Using the constraints coming from the symmetries, we construct an ansatz for both theories in Sec. 4.3. The results obtained with the NPRG for the DP-C ansatz are gathered in Sec. 4.4. We finish in Sec. 4.5 by studying the RG flow of the DEP ansatz. Unfortunately, this ansatz turns out not to be sufficiently flexible to clarify completely the situation. We end this section by showing why it is the case.

Version française

Dans Sec. 2.1 nous avons motivé une étude de la transition de phase absorbante de DEP avec les outils du NPRG. Notamment, nous avons insisté sur l'incertitude quant à savoir si DEP et DP-C, sa formulation mésoscopique naïve, appartenaient à la même classe d'universalité. Ce chapitre est un compte-rendu des résultats préliminaires de l'étude de DEP qui ont été publiés dans (Tarpin et al. 2017). Alors que la théorie des champs de DP-C a déjà reçu un intérêt considérable, c'est la première fois que la théorie des champs de DEP est étudiée directement. En effet, dans l'approche perturbative contenues dans les travaux précédents, les deux modèles sont dans la même classe d'universalité. Bien que nous n'ayons pas été en mesure de donner une réponse définitive, nous espérons qu'au moins notre étude a suscité un regain d'intérêt pour ce problème et ouvert la voie à de futurs travaux sur le sujet.

Dans la Sec. 4.1, nous présentons les différentes théories de champ apparaissant dans l'étude de DEP et DP-C. Nous étudions les symétries de ces actions dans la Sec. 4.2 et les utilisons pour obtenir des résultats exacts pour DEP et DP-C dans le cadre du NPRG. C'est l'occasion de discuter de la façon dont les deux théories peuvent différer dans leur transition de phase. En utilisant les contraintes provenant des symétries, nous construisons un ansatz pour les deux théories dans la Sec. 4.3. Les résultats obtenus avec le NPRG pour l'ansatz pour DP-C sont rassemblés dans la Sec. 4.4. Nous terminons dans la Sec. 4.5 en étudiant le flot RG de l'ansatz pour DEP. Malheureusement, cet ansatz ne s'avère pas suffisamment flexible pour clarifier complètement la situation. Nous terminons cette section en montrant pourquoi c'est le cas.

4.1 The field theories of DEP and DP-C

4.1.1 Response field action for DP-C

First, let us use the general MSRJD formalism, presented in Chap. 3 and Appendix B.1, to write down the action of DP-C from its Langevin equation, that we recall here:

$$\begin{aligned} \partial_t \rho &= k(c+\sigma)\rho - k\rho^2 + D_B\,\partial^2\rho + \eta^\rho\,, \\ \partial_t c &= D_A\,\partial^2(c-\mu\,\rho) + \eta^c\,, \end{aligned} \tag{4.1}$$

with

$$\begin{aligned} \langle \eta^\rho(\mathbf{x})\eta^\rho(\mathbf{x}')\rangle &= 2\,k'\rho(\mathbf{x})\delta(t-t')\delta^d(\vec{x}-\vec{x}')\,, \\ \langle \eta^c(\mathbf{x})\eta^c(\mathbf{x}')\rangle &= 2\,\mu' D_A(-\partial^2)\delta(t-t')\delta^d(\vec{x}-\vec{x}')\,, \end{aligned} \tag{4.2}$$

and η^ρ and η^c independent of each other. Using the formula of Appendix B.1, the partition function of DP-C reads

$$\mathcal{Z}[j,h,\bar{\jmath},\bar{h}] = \int \mathcal{D}[\rho,c,\bar{\rho},\bar{c}]e^{\int_{\mathbf{x}}\{j\rho+hc+\bar{\jmath}\bar{\rho}+\bar{h}\bar{c}\}}e^{-\mathcal{S}_{\text{DP-C}}} \tag{4.3}$$

with the action

$$\begin{aligned} \mathcal{S}_{\text{DP-C}} = \int_{\mathbf{x}}\Big\{ &\bar{\rho}\big[\partial_t\rho - D_B\,\partial^2\rho - k(c+\sigma)\rho + k\rho^2\big] - k'\rho\bar{\rho}^2 \\ &+ \bar{c}\big[\partial_t c - D_A\,\partial^2(c-\mu\,\rho)\big] - \mu' D_A(\partial_x\bar{c})^2\Big\}\,. \end{aligned} \tag{4.4}$$

It has to be noted that the time-boundary terms of the action corresponding to initial conditions have been dropped. This is justified because we are interested only in studying stationary states of the process.

Because we are interested only in $D_A \neq 0$ in the following, we rescale the time $t \to D_A^{-1}t$. This historical choice in the study of DEP is not without consequences. Indeed, as we are limited to the study of stationnary regime, it forbids to study the case of $D_A = 0$. Following the notations of (Wijland, Oerding, and Hilhorst 1998),

we write $\lambda \equiv D_B/D_A$. Rescaling the fields as in DP (see Sec. 2.1), we arrive to the action proposed by (Kree, Schaub, and Schmittmann 1989; Janssen 2001):

$$\mathcal{S}_{\text{DP-C}} = \int_{\mathbf{x}} \left\{ \bar{s}(\partial_t - \lambda\,\partial^2 - \sigma)s + g\, s\bar{s}(s - \bar{s} - f\, c) + \bar{c}(\partial_t - \partial^2)c + \mu\,\bar{c}\,\partial^2\, s - (\partial_x \bar{c})^2 \right\} \tag{4.5}$$

with

$$\frac{k}{D_A}\sigma \to \sigma\,, \quad g = \frac{\sqrt{kk'}}{D_A}\,, \quad f = \sqrt{\mu' \frac{k}{k'}}\,. \tag{4.6}$$

4.1.2 Coherent field action for DEP

Now, let us concentrate on DEP, which is defined by the following reactions

$$\begin{aligned} &\textbf{Infection} && A + B \xrightarrow{k} B + B \\ &\textbf{Recovery} && B \xrightarrow{1/\tau} A \\ &\textbf{Diffusion of A} && A + \emptyset \xrightarrow{D_A} \emptyset + A \\ &\textbf{Diffusion of B} && B + \emptyset \xrightarrow{D_B} \emptyset + B\,. \end{aligned} \tag{4.7}$$

The field theory for the process can be casted in the form of a coherent field action using the Doi-Peliti construction. A pedagogical presentation of this formalism is presented in Appendix C.1. We now apply this procedure to One obtains (the index i indicates the site on the lattice $\mathbb{L} \in \mathbb{Z}^d$ and the initial condition has been chosen to be Poissonian on each site)

$$\begin{aligned} \mathcal{S}_{\text{DEP}}[\{a, \bar{a}, b, \bar{b}\}] = \int_0^T \mathrm{d}t \sum_{i \in \mathbb{L}} \Big\{ \bar{a}_i(t)\partial_t a_i(t) + \bar{b}_i(t)\partial_t b_i(t) - \mathcal{H}_i[\{\bar{a}, \bar{b}, a, b\}](t) \Big\} \\ + \sum_{i \in \mathbb{L}} \left[\bar{a}_i(0)(a_i(0) - \mu^A_{i,0}) + \bar{b}_i(0)(b_i(0) - \mu^B_{i,0}) \right] + o(\tau T)\,. \end{aligned} \tag{4.8}$$

The density of the Doi-shifted time-evolution operator, $\mathcal{H}$, is given by

$$\begin{aligned} \mathcal{H}_i[\{\bar{a}, \bar{b}, a, b\}] = \sum_{j/<i,j>} \left[D_A(\bar{a}_i - \bar{a}_j)a_j + D_B(\bar{b}_i - \bar{b}_j)b_j \right] \\ + k\,(\bar{b}_i - \bar{a}_i)(\bar{b}_i + 1)a_i b_i + \tau^{-1}(\bar{a}_i - \bar{b}_i)b_i\,. \end{aligned} \tag{4.9}$$

The original coherent fields a (resp. b) and a^* (resp. b^*) of the Doi-Peliti construction are complex conjugate of each other. However in Appendix C.1 we have performed the so-called Doi shift

$$a^* = \bar{a} + 1\,, \quad b^* = \bar{b} + 1\,, \tag{4.10}$$

in the partition functional. This change of variable has the advantage that the action now vanishes for $\bar{a} = \bar{b} = 0$. This property, coupled to the fact that the $\bar{a}$ (resp. $\bar{b}$) field is always evaluated at a later time that the a (resp. b) one in the action allows one to interpret the shifted action as a response field action, leaving aside the fact that a and $\bar{a}$ (resp. b and $\bar{b}$) are not independent. This seemingly ad-hoc

change of variables can be understood by using an alternative formalism to construct the above action. This formalism was proposed by Gardiner and uses the "positive Poisson representation" to write a stochastic process for a complex field, and results in the coherent field action above written directly in terms of the Doi-shifted fields. This mapping is briefly presented in Appendix C.2.

We now take the formal continuous limit in space of (4.8). Specifying to a hypercubic lattice of lattice spacing h, using interpolating fields as for the time continuous limit in Appendix C.1 and assuming that they are sufficiently smooth, both diffusion terms rewrite as

$$\sum_{j/<i,j>} D_\Phi(\bar\phi_i - \bar\phi_j)\phi_j = h^2 D_\Phi \phi\, \partial^2 \bar\phi + o(h^2 D_X) \tag{4.11}$$

with $\phi = \{a, b\}$. Setting the appropriate scaling of the fields and of the coupling constants with h and taking the limit $h \to 0$, we finally arrive to the DEP continuous coherent field action

$$\mathcal{S}_{\text{DEP}}[a,\bar a,b,\bar b] = \int_{\mathbf{x},t} \bar a(\partial_t - D_A\,\partial^2)a + \bar b(\partial_t - D_B\,\partial^2)b - k\,ab(\bar b - \bar a)(\bar b + 1) - \frac{1}{\tau} b(\bar a - \bar b)\,, \tag{4.12}$$

where we have dropped the initial conditions because we are interested only in the stationary state. The above action was derived in the first work on DEP (Wijland, Oerding, and Hilhorst 1998). We now follow its authors by rescaling the time $t \to D_A^{-1} t$ and making the following change of variables:

$$\begin{aligned} \varphi = \rho_0^{-\frac{1}{2}}(a + b - \rho_0), \quad \psi = \rho_0^{-\frac{1}{2}} b, \\ \bar\varphi = \rho_0^{\frac{1}{2}} \bar a, \quad \bar\psi = \rho_0^{\frac{1}{2}}\,(\bar b - \bar a)\,, \end{aligned} \tag{4.13}$$

where ρ_0 is the initial total density. One arrives at the following action:

$$\begin{aligned} \mathcal{S}^{\text{WOH}}_{\text{DEP}} = \int_{\mathbf{x}} \Big\{ & \bar\varphi(\partial_t - \partial^2)\varphi + \bar\psi(\partial_t - \lambda\,\partial^2 - \sigma)\psi + \mu\,\bar\varphi\,\partial^2\psi \\ & + \psi\bar\psi\Big[g(\psi - \bar\psi - \varphi - \bar\varphi) + v(\psi\bar\psi + \psi\bar\varphi - \bar\psi\varphi - \varphi\bar\varphi)\Big]\Big\}\,, \end{aligned} \tag{4.14}$$

with parameters

$$g = \frac{k\sqrt{\rho_0}}{D_A}, \quad v = \frac{k}{D_A}, \quad \sigma = k(\rho_0 - (k\tau)^{-1})/D_A\,. \tag{4.15}$$

This action will be our starting point in our NPRG study of DEP.

In (Wijland, Oerding, and Hilhorst 1998), the quartic terms of the above action are dropped. This simplification is justified by the authors, using arguments from perturbative RG. If one makes the change of variables $\varphi \to \varphi - \bar\varphi$ in the action (4.14) with the quartic terms truncated:

$$\mathcal{S}^{\text{WOH}}_{\text{DP-C}} = \int_{\mathbf{x}} \Big\{\bar\varphi(\partial_t - \partial^2)\varphi + \bar\psi(\partial_t - \lambda\,\partial^2 - \sigma)\psi + \mu\,\bar\varphi\,\partial^2\psi + g\,\psi\bar\psi(\psi - \bar\psi - \varphi - \bar\varphi)\Big\}\,, \tag{4.16}$$

one recognizes the DP-C action (4.5) with $f = 1$. Although it was obtained from the Doi-Peliti formalism, at this point it is only when interpreted as a response field

action that it can make sense. Indeed, we will see that as a coherent field action, it does not conserve the total number of particles. However, at the level of formal manipulations, (4.5) and (4.16) are completely equivalent. Thus it should be clear now that the system studied in (Wijland, Oerding, and Hilhorst 1998) is DP-C. In order to stick with this paper, when studying DP-C the action (4.16) will be used, although the physical content is clearer in (4.5).

4.1.3 Response field action for DEP

The Doi-Peliti coherent field action has the disadvantage that its fields do not represent the physical observables of the system. In fact, the example of the calculation of the mean value of the field with the Doi-Peliti formalism in Appendix C.1 is the only case where the coherent field can be identified with the observable. To overcome this, a change to so-called Grassberger variables was used in this context in (Janssen and Stenull 2016) to obtain a response field action from the coherent field one. Going back to the original coherent field action (4.12), the Grassberger variables read as follows

$$\begin{aligned} \mu_\phi &= \bar{\phi}\phi \\ \theta_\phi &= \frac{1}{i}\ln\bar{\phi} \end{aligned} \tag{4.17}$$

where $\phi = \{a, b\}$. It was shown in (Andreanov et al. 2006) that this mapping can be justified by writing down directly a response field action for the counting processes *à la* MSRJD. A work on the subject has been undertaken, notably to show how the Doi-Peliti coherent field action can be recovered from this formalism (Guioth, Lecomte, and Tarpin in prep.). Using the change of variables above, the response field action of DEP reads (the i's have been hidden in a redefinition of the fields $i\theta \to \theta$)

$$\mathcal{Z}[h_a, h_b, s_a, s_b] = \int \mathcal{D}[\mu_a, \mu_b, \theta_a, \theta_b] e^{\int_{\mathbf{x}} \{h_a\mu_a + h_b\mu_b + s_a\theta_b + s_b\theta_b\}} e^{-\mathcal{S}^{\mathrm{G}}_{\mathrm{DEP}}} \tag{4.18}$$

with the following action in the space-time continuum limit

$$\begin{aligned} \mathcal{S}^{\mathrm{G}}_{\mathrm{DEP}} = \int_{\mathbf{x}} \Big\{ &\theta_a(\partial_t - D_A\,\partial^2)\mu_a - D_A\,\mu_a(\vec{\partial}\,\theta_a)^2 + \theta_b(\partial_t - D_B\,\partial^2)\mu_b - D_B\,\mu_b(\vec{\partial}\,\theta_b)^2 \\ &- k\,\mu_a\mu_b\big(e^{\theta_b - \theta_a} - 1\big) - \frac{1}{\tau}\mu_b\big(e^{\theta_a - \theta_b} - 1\big)\Big\}, \end{aligned} \tag{4.19}$$

and where new sources $\{h_\phi, \theta_\phi\}$ couple directly to the new fields. This action was used in (Janssen and Stenull 2016) as a starting point to show perturbatively that DP-C and DEP belonged to the same universality class. This argument will be presented in Sec. 4.2.3, along with the study of the symmetries of the action (4.19).

4.2 Symmetries, Ward identities and exact results for DEP and DP-C

4.2.1 Symmetries of the DP-C action

Let us first study the symmetries of the DP-C action given in (4.16), following (Wijland, Oerding, and Hilhorst 1998; Janssen and Stenull 2016). We have seen in

Sec. 3.6 that in fact, we are not only interested in exact symmetries but also in changes of variables for which the variation of the action is not zero but linear in the fields. As explained in the previous section, these changes of variables can be exploited as well and lead to powerful non-renormalization theorems. We will use the following notation for the mean values of the different fields in presence:

$$\Phi = \langle\varphi\rangle_{\mathrm{j}}\,,\ \bar{\Phi} = \langle\bar{\varphi}\rangle_{\mathrm{j}}\,,\ \Psi = \langle\psi\rangle_{\mathrm{j}}\,,\ \bar{\Psi} = \langle\bar{\psi}\rangle_{\mathrm{j}}\,, \tag{4.20}$$

with the subscript j indicating that the mean value is taken in the presence of the various sources. Furthermore, we write $\Phi = \{\Phi, \bar{\Phi}, \Psi, \bar{\Psi}\}$ for the whole field content of the theory.

To begin with, the fact that $\varphi + \bar{\varphi} = c$ is a conserved field translates in the following extended symmetry of the DP-C action.

$$\varphi(\mathbf{x}) \to \varphi(\mathbf{x}) - \epsilon(t)\,, \quad \bar{\varphi}(\mathbf{x}) \to \bar{\varphi}(\mathbf{x}) + \epsilon(t)\,. \tag{4.21}$$

Indeed, the variation of this action under this change of variables reduces to

$$\delta(\mathcal{S}^{\mathrm{WOH}}_{\text{DP-C}}) = \int_{\mathbf{x}} \epsilon(t)\partial_t\big[\varphi(\mathbf{x}) + \bar{\varphi}(\mathbf{x})\big]\,. \tag{4.22}$$

Recalling that $\varphi + \bar{\varphi}$ is the total number of particles in the DP-C interpretation, this extended invariance under the time-gauged shift means that the variation of the total number of particles is only due to the external sources (Janssen 2001). In fact, noticing that the combination $\varphi - \bar{\varphi}$ appears only in quadratic terms, we can promote the time-gauged shift above to a shift gauged in space-time:

$$\varphi(\mathbf{x}) \to \varphi(\mathbf{x}) - \epsilon(\mathbf{x})\,, \quad \bar{\varphi}(\mathbf{x}) \to \bar{\varphi}(\mathbf{x}) + \epsilon(\mathbf{x})\,. \tag{4.23}$$

The variation of the action reads

$$\delta(\mathcal{S}^{\mathrm{WOH}}_{\text{DP-C}}) = \int_{\mathbf{x}} \epsilon\Big\{(\partial_t - \partial^2)\varphi + (\partial_t + \partial^2)\bar{\varphi} + \mu\,\partial^2\psi\Big\}\,, \tag{4.24}$$

which following Sec. 3.6, gives the Ward identity

$$\Big\{(\partial_t - \partial^2)\Phi(\mathbf{x}) + (\partial_t + \partial^2)\bar{\Phi}(\mathbf{x}) + \mu\,\partial^2\Psi(\mathbf{x})\Big\} = \frac{\delta\Gamma_\kappa}{\delta\bar{\Phi}(\mathbf{x})} - \frac{\delta\Gamma_\kappa}{\delta\Phi(\mathbf{x})}\,. \tag{4.25}$$

This identity can be integrated into

$$\Gamma_\kappa[\Phi] = \int_{\mathbf{x}} \Big\{\bar{\Phi}(\partial_t - \partial^2)\Phi + \mu\,\bar{\Phi}\,\partial^2\Psi\Big\} + \Gamma'_\kappa[\Phi + \bar{\Phi}, \Psi, \bar{\Psi}]\,, \tag{4.26}$$

where $\Gamma'_\kappa[\Phi + \bar{\Phi}, \Psi, \bar{\Psi}]$ is an arbitrary functional. The identity (4.25) is extremely powerful as it fixes many of the exponents of DP-C. Indeed, it implies

$$\begin{aligned}
\bar{\Gamma}^{(2)}_{\varphi\bar{\varphi}}[\mathbf{p};\Phi] &= i\omega + p^2\,,\\
\bar{\Gamma}^{(2)}_{\bar{\varphi}\bar{\varphi}}[\mathbf{p};\Phi] &= 0\,,\\
\bar{\Gamma}^{(2)}_{\psi\bar{\varphi}}[\mathbf{p};\Phi] &= -\mu\,p^2\,,\\
\bar{\Gamma}^{(n+1)}_{\varphi i_1\ldots i_n}[\{\mathbf{p}_\ell\};\Phi] &= \bar{\Gamma}^{(n+1)}_{\bar{\varphi} i_1\ldots i_n}[\{\mathbf{p}_\ell\};\Phi]\,,\ \text{for } n \geq 2\,.
\end{aligned} \tag{4.27}$$

Using the definition of the running scaling dimensions defined in Sec. 3.4.4, the above identities respectively give

$$\begin{aligned} d_\varphi + d_{\bar\varphi} = d \ \text{ and } z = 2\,, \\ d_\varphi = d_{\bar\varphi} \\ \text{if } \mu \neq 0 \quad d_{\bar\varphi} + d_\psi = d\,, \end{aligned} \tag{4.28}$$

that is

$$\begin{aligned} d_\varphi = d_{\bar\varphi} = \frac{d}{2}\,, \\ z = 2\,, \\ \text{if } \mu \neq 0 \quad d_\psi = \frac{d}{2}\,. \end{aligned} \tag{4.29}$$

For the particular case $\mu = 0$, the action possesses a time-reversal symmetry. This symmetry is an extension of the time-reversal symmetry present in the DP action including the fields φ, $\bar\varphi$. For a given space-time field ϕ, we note $\mathrm{T}\phi : \mathbf{x} \to \phi(-t, \vec{x})$. The symmetry reads:

$$\begin{aligned} \psi \to -\mathrm{T}\bar\psi, \quad \bar\psi \to -\mathrm{T}\psi\,, \\ \varphi \to \mathrm{T}\bar\varphi, \quad \bar\varphi \to \mathrm{T}\varphi\,. \end{aligned} \tag{4.30}$$

This symmetry would be broken by the initial conditions. However, as we are only interested in the stationary state and have dropped the initial conditions from the action, it has no implication here. For $\mu = 0$, the third line of (4.29) does not hold. However, in this case the time-reversal (4.67) symmetry translates into

$$\text{if } \mu = 0\,, \quad \Gamma_\kappa[\Phi, \bar\Phi, \Psi, \bar\Psi] = \Gamma_\kappa[\mathrm{T}\bar\Phi, \mathrm{T}\Phi, -\mathrm{T}\bar\Psi, -\mathrm{T}\Psi]\,. \tag{4.31}$$

In particular, it implies

$$\text{if } \mu = 0\,, \quad d_\varphi = d_{\bar\varphi}\,, \quad d_\psi = d_{\bar\psi}\,. \tag{4.32}$$

With (4.29) and (4.32) in mind, let us define the anomalous dimensions

$$d_\psi = \frac{d+\eta}{2} \ \text{ and } d_{\bar\psi} = \frac{d+\bar\eta}{2}\,, \tag{4.33}$$

such that $\eta = \bar\eta = 0$ at the fixed point, if the transition is described by a Gaussian theory. Using this definition, (4.29) and (4.32) imply that

$$\begin{aligned} \text{if } \mu = 0 \quad \eta = \bar\eta\,, \\ \text{else} \quad \eta = 0\,. \end{aligned} \tag{4.34}$$

Let us now give a short proof of the hyperscaling relation $\beta = \nu(d+\eta)/2$. This relation is well known and is quite generic for phase transitions in reaction-diffusion systems, but its demonstration is of pedagogical interest for later purpose. The exponent β is defined by

$$\langle n_B \rangle \sim (\sigma - \sigma_c)^\beta\,. \tag{4.35}$$

It can be shown (see Appendix C.1) that within the Doi-Peliti formalism

$$\langle n_B \rangle = \langle \bar{\psi}\psi \rangle = \langle \psi \rangle = \Psi \,. \tag{4.36}$$

Furthermore, if we suppose that there is a value σ_c such that the action is scale invariant in the infrared, we have shown in Chap. 3 that, at the critical point $\sigma = \sigma_c$, the fields scale with the renormalization scale κ as $\Psi \sim \kappa^{d_\psi}$. If we detune the action from its critical point, in other words if $\sigma - \sigma_c$ is small but different from zero, the system will have a finite correlation length behaving as

$$\xi \sim (\sigma - \sigma_c)^{-\nu} \,. \tag{4.37}$$

For $\kappa \gtrsim \xi^{-1}$, the RG does not feel yet the effect of the detuning off criticality and one has $\Psi \sim \kappa^{d_\psi}$. However, for RG scales $\kappa \lesssim \xi^{-1}$, the correlations are negligible and the RG flow freezes. Thus we are left in the infrared with

$$\langle n_B \rangle \sim \Psi \sim \left(\xi^{-1}\right)^{d_\psi} \sim (\sigma - \sigma_c)^{\nu d_\psi} \,, \tag{4.38}$$

in other words, $\beta = \nu d_\psi = \nu(d+\eta)/2$. We readily see that for $\mu \neq 0$, it simplifies to $\beta = \nu d/2$.

Finally, the action (4.16) possesses another property, which is not a symmetry in the traditional sense, but rather a duality between a field and a coupling. Indeed, the action is invariant under the following simultaneous shift of the field φ and of the mass of the bare action σ.

$$\varphi \to \varphi + \epsilon, \quad \sigma \to \sigma - g\epsilon \,. \tag{4.39}$$

In essence, this duality accounts for the fact that the mass of the field ψ comes from the non-zero total number of particles. Note that we could have replaced φ by $\bar{\varphi}$ in the change of variables. Because of the symmetry of Γ'_κ, this does not change the result. Using the formula of Sec. 3.6, it leads to the following duality identity

$$g\,\partial_\sigma \Gamma_\kappa = \int_{\mathbf{x}} \frac{\delta \Gamma_\kappa}{\delta \Phi(\mathbf{x})} \,. \tag{4.40}$$

If we assume that the system undergoes a continuous phase transition for $\sigma = \sigma_c$, that is if there is only one repulsive perturbation from the critical point which does not break the symmetries, the identity above gives us exactly this relevant eigenperturbation. We will thus be able to parametrize Γ_κ such as to place ourselves within the critical surface, where the RG flow is purely attractive. Furthermore, it fixes exactly the value of the exponent ν. Let us give here a handwaving argument.

For the action detuned from criticality by $\sigma - \sigma_c$, using the same arguments as in the above paragraph, for $\kappa \lesssim \xi^{-1}$ the r.h.s. of (4.40) scales as

$$\int_{\mathbf{x}} \frac{\delta \Gamma_\kappa}{\delta \Phi(\mathbf{x})} \sim \kappa^{-d_\varphi} \sim (\sigma - \sigma_c)^{-\nu d_\varphi} \,, \tag{4.41}$$

if we assume that the l.h.s. goes like

$$g\,\partial_\sigma \Gamma_\kappa \sim (\sigma - \sigma_c)^{-1} \,, \tag{4.42}$$

by equalizing both scalings, we find directly that

$$\nu d_\varphi = 1 \,. \tag{4.43}$$

This relation, in conjunction with the exact relations (4.29) from the shift symmetry fixes the exponent ν to

$$\nu = \frac{2}{d}\,. \tag{4.44}$$

As an immediate consequence, we have that $\beta = 1 + \eta/d$, which simplifies to $\beta = 1$ for $\mu \neq 0$. A more rigourous demonstration is given in Appendix D under technical assumptions on the minimum of Φ.

This finishes to list the exact relations for the critical exponents of DP-C. These exact relations were already known in the context of perturbative RG. However, we have seen in Sec. 2.1.2 that they do not describe satisfactorily the existing simulations of DEP.

4.2.2 Symmetries of the response field action of DEP and equivalence with DP-C

Now, let us come back to our main interest, the action of DEP. First, let us study the action obtained using the response field formalism (or the Grassberger change of variables). The response field action reads

$$\begin{aligned}\mathcal{S}^{\mathrm{G}}_{\mathrm{DEP}} = \int_{\mathbf{x}} \Big\{ &\theta_a(\partial_t - D_A\,\partial^2)\mu_a - D_A\,\mu_a(\vec{\partial}\,\theta_a)^2 + \theta_b(\partial_t - D_B\,\partial^2)\mu_b - D_B\,\mu_b(\vec{\partial}\,\theta_b)^2 \\ &- k\,\mu_a\mu_b\big(e^{\theta_b-\theta_a} - 1\big) - \frac{1}{\tau}\mu_b\big(e^{\theta_a-\theta_b} - 1\big)\Big\}\,.\end{aligned} \tag{4.45}$$

As DP-C, DEP conserves the total number of particles. This translates into the following extended symmetry:

$$\theta_a \to \theta_a + \epsilon(t)\,, \quad \theta_b \to \theta_b + \epsilon(t)\,. \tag{4.46}$$

The corresponding variation of the action is

$$\delta(\mathcal{S}^{\mathrm{G}}_{\mathrm{DEP}}) = \int_{\mathbf{x}} \epsilon(t)\big[\partial_t\mu_a(\mathbf{x}) + \partial_t\mu_b(\mathbf{x})\big] = \int_t \epsilon(t)\partial_t \int_{\vec{x}} \big[\mu_a(\mathbf{x}) + \mu_b(\mathbf{x})\big]\,. \tag{4.47}$$

However, and contrary to DP-C, the space-time gauged shift

$$\theta_a \to \theta_a + \epsilon(\mathbf{x})\,, \quad \theta_b \to \theta_b + \epsilon(\mathbf{x})\,. \tag{4.48}$$

is not an extended symmetry of the action. Indeed, we have seen that the continuum action of reaction-diffusion processes in the response field formalism contains conservative noise terms, which are cubic in the fields.

Now, let us give an account of the argument that DEP is in the DP-C universality class using (4.45). First, let us make the following change of variables:

$$\begin{aligned}\mu_c = \mu_a + \mu_b - \rho_0\,, \quad \theta_c = \theta_a\,,\\ \mu_s = \mu_b\,, \quad \theta_s = \theta_b - \theta_a\,.\end{aligned} \tag{4.49}$$

This change of variable is the exact translation of the steps presented in Sec. 2.1.2 to obtain the mean-field equations of DP-C in terms of the fields c and s. The resulting

action reads

$$\begin{aligned}\mathcal{S}_{\mathrm{DEP}}^{\mathrm{G}} = \int_{\mathbf{x}} \Big\{ & \theta_c(\partial_t - D_A\,\partial^2)\mu_c - D_A(\mu_c + \rho_0 - \mu_s)(\vec{\partial}\,\theta_c)^2 \\ & + \theta_s(\partial_t - D_B\,\partial^2)\mu_s - D_B\,\mu_s(\vec{\partial}\,\theta_c + \vec{\partial}\,\theta_s)^2 + (D_A - D_B)\,\theta_c\,\partial^2\mu_s \\ & - k\,(\mu_c + \rho_0 - \mu_s)\mu_s\big(e^{\theta_s} - 1\big) - \frac{1}{\tau}\mu_s\big(e^{\theta_s} - 1\big)\Big\}\,. \end{aligned} \tag{4.50}$$

Under renormalization, the mean value of the fields and the time dependency may acquire a non-trivial scaling with κ, thus we write

$$\mu_c(\mathbf{x}) = \kappa^{d_c}\hat{\mu}_c(\kappa\vec{x}, \kappa^z t) = \kappa^{d_c}\hat{\mu}_c(\hat{\mathbf{x}})\,, \tag{4.51}$$

and accordingly for the other fields. If the transition is well described by the mean-field approximation, the renormalized propagator keeps its bare form, thus the fluctuations of the fields are described by the quadratic part of the action. It reads, making explicit the κ dependence,

$$\begin{aligned}\mathcal{S}_{\mathrm{DEP}}^{\mathrm{Gaussian}} = \int_{\hat{\mathbf{x}}} \kappa^{-d-z}\Big\{ & \kappa^{d_c+d_{\tilde{c}}}\hat{\theta}_c(\kappa^z\partial_{\hat{t}} - D_A\,\kappa^2\hat{\partial}^2)\hat{\mu}_c + \kappa^{d_s+d_{\tilde{s}}}\hat{\theta}_s(\kappa^z\partial_{\hat{t}} - D_B\,\kappa^2\hat{\partial}^2)\hat{\mu}_s \\ & + \kappa^{2+d_{\tilde{c}}+d_s}(D_A - D_B)\,\hat{\theta}_c\,\hat{\partial}^2\hat{\mu}_s - \kappa^{d_s+d_{\tilde{s}}}k\,\hat{\theta}_s\hat{\mu}_s\big[\,\rho_0 - (k\tau)^{-1}\big]\,. \end{aligned} \tag{4.52}$$

Asking for the kinetic terms to be scale invariant and using the symmetry between the fields and their corresponding response fields, one readily obtains

$$z = 2\,, \quad d_c = d_s = d_{\tilde{c}} = d_{\tilde{s}} = d/2\,. \tag{4.53}$$

Now, let us expand the third line of the full action of DEP (4.50) in powers of θ_s:

$$\begin{aligned} & - k\,(\mu_c + \rho_0 - \mu_s)\mu_s\big(e^{\theta_s} - 1\big) - \frac{1}{\tau}\mu_s\big(e^{-\theta_s} - 1\big) \\ & = -\theta_s\big[k\,(\mu_c + \rho_0 - \mu_s)\mu_s - \frac{1}{\tau}\mu_s\big] - \frac{1}{2}\theta_s^2\big[k\,(\mu_c + \rho_0 - \mu_s)\mu_s + \frac{1}{\tau}\mu_s\big] + o(\theta_s^2\mu_s) \\ & = -k\,\theta_s\mu_s\big[\,\rho_0 - (k\tau)^{-1}\big] + k\,\theta_s\mu_s\big[\mu_s - \hat{\mu}_c - \frac{\rho_0 + (k\tau)^{-1}}{2}\theta_s\big] - \frac{k}{2}\theta_s^2\mu_s(\mu_c - \mu_s) + o(\theta_s^2\mu_s)\,. \end{aligned} \tag{4.54}$$

Inserting the Gaussian scaling (4.53) in the expanded action leads to

$$\begin{aligned}\mathcal{S}_{\mathrm{DEP}}^{\text{mean-field}} = \int_{\hat{\mathbf{x}}} \Big\{ & \hat{\theta}_c(\partial_{\hat{t}} - D_A\,\hat{\partial}^2)\hat{\mu}_c - D_A(\kappa^{d/2}\hat{\mu}_c + \rho_0 - \kappa^{d/2}\hat{\mu}_s)(\hat{\vec{\partial}}\,\hat{\theta}_c)^2 \\ & + \hat{\theta}_s(\partial_{\hat{t}} - D_B\,\hat{\partial}^2)\hat{\mu}_s - D_B\,\kappa^{d/2}\hat{\mu}_s(\hat{\vec{\partial}}\,\hat{\theta}_c + \hat{\vec{\partial}}\,\hat{\theta}_s)^2 + (D_A - D_B)\,\hat{\theta}_c\,\hat{\partial}^2\hat{\mu}_s \\ & - \kappa^{-2}k\,\hat{\theta}_s\hat{\mu}_s\big[\,\rho_0 - (k\tau)^{-1}\big] + \kappa^{d/2-2}k\,\hat{\theta}_s\hat{\mu}_s\big[\hat{\mu}_s - \mu_c - \frac{\rho_0 + (k\tau)^{-1}}{2}\hat{\theta}_s\big] \\ & + O(\kappa^{d-2})\Big\}\,. \end{aligned} \tag{4.55}$$

Looking at the IR behavior of the field theory means to take the limit $\kappa \to 0$. One sees that the quadratic "mass term" in $\theta_s\mu_s$ blows up with an exponent -2 (giving for the critical exponent ν the value $\nu = 1/2$ in the mean-field approximation). Furthermore the derivative terms cubic in the fields, which were coming from the

diffusion noise, have a positive scaling with κ in all dimensions and thus vanish when $\kappa \to 0$. However, the behavior of the reaction terms depend on the dimension. For $d > 4$, all the terms have a positive scaling with κ, while for $d \leq 4$, the cubic terms begin to have negative scalings. Thus, the upper critical dimension of the model is $d_c = 4$. When studying the theory for d lower but near d_c, one can thus drop the cubic derivative terms and the quartic terms, which are irrelevant in the IR limit. As a consequence, to study the DEP phase transition as a perturbation of the Gaussian field theory, one can start from the following truncated action, which describes the same transition, at least near $d = 4$:

$$\mathcal{S}_{\text{DEP}}^{\text{trunc.}} = \int_{\mathbf{x}} \Big\{ \theta_c(\partial_t - D_A\,\partial^2)\mu_c - D_A\,\rho_0(\vec{\partial}\,\theta_c)^2 + \theta_s(\partial_t - D_B\,\partial^2)\mu_s + (D_A - D_B)\,\theta_c\,\partial^2\mu_s \\ - k\,\theta_s\mu_s\big[\,\rho_0 - (k\tau)^{-1}\big] + k\,\theta_s\mu_s\big[\mu_s - \mu_c - \frac{\rho_0 + (k\tau)^{-1}}{2}\theta_s\big]\Big\}\,. \tag{4.56}$$

Up to a rescaling of time and of the fields, we have recovered the DP-C field theory. This argument is valid as long as DEP can be described as a perturbation of the Gaussian field theory, that is within the perturbative RG framework. This is certainly true for d close enough to $d_c = 4$. However, the existing simulations of DEP are done in $d = 1$ and 2. This prompts the question as to whether the argument still holds in these dimensions and motivates a non-perturbative study of this problem.

In fact, although the action (4.45) is the starting point to derive the coarse-grained action of DP-C, it is not clear at all what are the precursors of the duality and of the time-reversal symmetry present in DP-C. Let us nonetheless signal a symmetry of (4.45) which exists for $\mu = 0$. For an infinitesimal parameter $\vec{v}$,

$$\theta_a \to -\frac{x_\alpha v_\alpha}{2} + t\,v_\alpha\partial_\alpha\theta_a\,, \quad \mu_a \to \mu_a + t\,v_\alpha\partial_\alpha\mu_a \\ \theta_b \to -\frac{x_\alpha v_\alpha}{2} + t\,v_\alpha\partial_\alpha\theta_b\,, \quad \mu_b \to \mu_b + t\,v_\alpha\partial_\alpha\mu_b\,. \tag{4.57}$$

This symmetry is reminiscent of the well-known tilt (or Galilean) symmetry present in the Kardar-Parisi-Zhang (KPZ) equation (Kardar, Parisi, and Zhang 1986). Note that the gauged-in-time version of (4.57) also yields a variation of the action linear in the fields, and thus leads to new Ward identities, as derived in (Canet et al. 2011) for KPZ. This symmetry may prove very useful for future studies of the DEP transition starting from the response field action (4.45).

4.2.3 Symmetries of the coherent field action of DEP

Finally, let us go back to the complete coherent-field action of DEP obtained in Sec. 4.1.2 and study its symmetries. The action reads

$$\mathcal{S}_{\text{DEP}}^{\text{WOH}} = \int_{\mathbf{x}} \Big\{ \bar\varphi(\partial_t - \partial^2)\varphi + \bar\psi(\partial_t - \lambda\,\partial^2 - \sigma)\psi + \mu\,\bar\varphi\,\partial^2\psi \\ + \psi\bar\psi\Big[g(\psi - \bar\psi - \varphi - \bar\varphi) + v(\psi\bar\psi + \psi\bar\varphi - \bar\psi\varphi - \varphi\bar\varphi)\Big]\Big\}\,. \tag{4.58}$$

First, we can look for the symmetry encoding the conservation of the number of particles. We remember that at the level of the Doi-Peliti operator formalism of

Appendix C.1, this translated into the property that in each monome of the time-evolution generator, there was an equal number of creation and annihilation operators, so we expect the action to be invariant under the following rescaling of the fields:

$$\begin{aligned} \bar{\varphi} &\to \Lambda\bar{\varphi} \quad \varphi \to \Lambda^{-1}\varphi \\ \bar{\psi} &\to \Lambda\bar{\psi}, \quad \psi \to \Lambda^{-1}\psi\,. \end{aligned} \tag{4.59}$$

However, this symmetry is broken by the initial conditions. This breaking of the rescaling symmetry survives in the stationary state because of the conservation of the number of particles. Following (Wijland, Oerding, and Hilhorst 1998), in Sec. 4.1.2 we explicitly took it into account by shifting the initial fields using the initial density ρ_0. As a consequence, the symmetry of conservation of the number of particles is a bit more complicated, but it exists and reads, for a given Λ:

$$\begin{aligned} \bar{\varphi} &\to \Lambda\bar{\varphi} + (\Lambda-1)\sqrt{\rho_0} \quad \varphi \to \Lambda^{-1}\varphi + (\Lambda^{-1}-1)\sqrt{\rho_0} \\ \bar{\psi} &\to \Lambda\bar{\psi}, \quad \psi \to \Lambda^{-1}\psi\,. \end{aligned} \tag{4.60}$$

This change of variables is an exact symmetry of the action. Moreover, if we time-gauge its infinitesimal version:

$$\begin{aligned} \bar{\varphi} &\to \bar{\varphi} + \epsilon(t)\big(\bar{\varphi} + \sqrt{\rho_0}\big) \quad \varphi \to \varphi - \epsilon(t)\big(\bar{\varphi} + \sqrt{\rho_0}\big) \\ \bar{\psi} &\to \bar{\psi} + \epsilon(t)\bar{\psi}, \quad \psi \to \psi - \epsilon(t)\psi\,, \end{aligned} \tag{4.61}$$

we obtain the following variation of the action:

$$\delta(\mathcal{S}_{\mathrm{DEP}}) = \int_{\mathbf{x}} \epsilon\, \partial_t \Big[\bar{\varphi}\varphi + \bar{\psi}\psi\Big]\,. \tag{4.62}$$

We see that the number of particles is conserved as well. However in the Doi-Peliti formalism, the number of particles is not an observable linear in the fields, thus in this formalism the conservation of the number of particles do not translate directly into a non-renormalization theorem as before. Furthermore, we see now that a truncation to cubic order, as the one done in (Wijland, Oerding, and Hilhorst 1998), breaks the symmetry by rescaling, thus breaks the conservation of the number of particles, if interpreted as a coherent field action.

Although we cannot gauge (4.60) in time without losing the linearity, we can still write the Ward identity for the ungauged symmetry. It gives

$$\int_{\mathbf{x}} \left\{ \big(\sqrt{\rho_0} + \Phi\big)\frac{\delta\Gamma_\kappa}{\delta\Phi} - \big(\sqrt{\rho_0} + \bar{\Phi}\big)\frac{\delta\Gamma_\kappa}{\delta\bar{\Phi}} + \Psi\frac{\delta\Gamma_\kappa}{\delta\Psi} - \bar{\Psi}\frac{\delta\Gamma_\kappa}{\delta\bar{\Psi}} \right\} = 0\,. \tag{4.63}$$

Let us write explicitly the scaling in κ:

$$\int_{\hat{\mathbf{x}}} \left\{ \big(\kappa^{-d_\varphi}\sqrt{\rho_0} + \hat{\Phi}\big)\frac{\delta\Gamma_\kappa}{\delta\hat{\Phi}} - \big(\kappa^{-d_{\bar{\varphi}}}\sqrt{\rho_0} + \hat{\bar{\Phi}}\big)\frac{\delta\Gamma_\kappa}{\delta\hat{\bar{\Phi}}} + \hat{\Psi}\frac{\delta\Gamma_\kappa}{\delta\hat{\Psi}} - \hat{\bar{\Psi}}\frac{\delta\Gamma_\kappa}{\delta\hat{\bar{\Psi}}} \right\} = 0\,. \tag{4.64}$$

Because of the explicit dependence in κ, this identity cannot describe a scale invariant EAA. Physically, it stems from the fact that we placed ourselves at a finite total density of particles ρ_0, following (Wijland, Oerding, and Hilhorst 1998). The scale invariance can only be exact at the degenerate point $\rho_0 = 0$. However, looking at

the scaling of each terms we see that in the IR limit $\kappa \to 0$, the $\sqrt{\rho_0}$-terms dominate and one is left with

$$\int_{\hat{\mathbf{x}}} \left\{ \frac{\delta \Gamma_\kappa}{\delta \hat{\Phi}} - \frac{\delta \Gamma_\kappa}{\delta \hat{\bar{\Phi}}} \right\} = 0 \,. \tag{4.65}$$

In other words, one obtains in the IR an action with the symmetry

$$\varphi \to \varphi - \epsilon \,, \bar{\varphi} \to \bar{\varphi} + \epsilon \,. \tag{4.66}$$

This symmetry is the shift symmetry (4.23) of DP-C for a constant ϵ, thus it seems to support the idea that in the IR the action of DEP acquires the symmetry of DP-C. However, contrary to what happens when starting from the DP-C action, we do not obtain the full space-time gauged version (4.23), as it comes from the rescaling symmetry (4.60). As a consequence, although the non-gauged shift symmetry is recovered in the IR, we do not have for DEP non-renormalization theorems which would fix the anomalous dimensions of the φ, $\bar{\varphi}$ fields and the scaling of the time to their Gaussian values. Near the critical dimension $d_c = 4$, the perturbative results are well controlled and we expect DEP to be in the same universality class as DP-C because the terms of the DEP action which break the shift symmetry are irrelevant, using the same reasoning as in Sec. 4.2.2. However for $d = 1$ or 2 (the dimensions at which the system has been simulated), DEP and DP-C could belong to two different universality classes. A sufficiently simple way to adress this uncertainty is to simulate directly the Langevin equations corresponding to DP-C. To our knowledge, this has never been done and we have recently undertaken a project in this direction. For the time being, it seems worthwhile to study the field theory of DEP on its own. Let us continue with its Ward identities.

Contrary to the response field action of DEP (4.45), the coherent field action (4.58) possesses the time-reversal symmetry for $\mu = 0$ and a duality reminiscent of DP-C. The time-reversal symmetry reads

$$\begin{aligned} \psi \to -\mathrm{T}\bar{\psi}, \quad \bar{\psi} \to -\mathrm{T}\psi \,, \\ \varphi \to \mathrm{T}\bar{\varphi}, \quad \bar{\varphi} \to \mathrm{T}\varphi \,. \end{aligned} \tag{4.67}$$

This time-reversal symmetry is the same symmetry as for DP-C in the previous section and has the same consequence

$$\text{if } \mu = 0 \,, \quad \Gamma_\kappa[\Phi, \bar{\Phi}, \Psi, \bar{\Psi}] = \Gamma_\kappa[\mathrm{T}\bar{\Phi}, \mathrm{T}\Phi, -\mathrm{T}\bar{\Psi}, -\mathrm{T}\Psi] \,. \tag{4.68}$$

The duality of DEP is the invariance of the action under the following shift of the fields and of the parameters

$$\begin{aligned} \varphi \to \varphi + \epsilon \,, \quad \bar{\varphi} \to \bar{\varphi} + \epsilon \,, \\ \sigma \to \sigma - 2\epsilon g \,, \quad g \to g - \epsilon v \,. \end{aligned} \tag{4.69}$$

Using the expressions of the parameters σ and g, one can generate the shift of the parameters σ and g by shifting ρ_0 as $\rho_0 \to \rho_0 - 2\epsilon$. Thus we see again that the duality comes from the non-zero initial value of the total density, which is conserved by the dynamics. The duality (4.69) leads to the following identity:

$$\int_{\mathbf{x}} \left\{ \frac{\delta \Gamma_\kappa}{\delta \Phi} + \frac{\delta \Gamma_\kappa}{\delta \bar{\Phi}} \right\} = 2\sqrt{\rho_0}\, \partial_{\rho_0} \Gamma_\kappa . \tag{4.70}$$

The shift symmetry present in the IR gives us that $d_\varphi = d_{\bar\varphi}$. Assuming that in DEP, the control parameter of the continuous phase transition is the total density, the same reasoning as Appendix D can be used to show that

$$\nu d_\varphi = 1\,, \tag{4.71}$$

which is the same relation as in DP-C. However, as d_φ may be different from $d/2$, ν can depart from its DP-C value. This gives a possible scenario to reconcile simulations of DEP with the theory.

4.3 Modified Local potential approximation for DEP and DP-C

Given that a DP-C fixed point would be characterized by a larger symmetry group than the DEP one, it is unclear if one should expect the infrared DEP fixed point to be the same as the DP-C one for dimensions far from the upper critical dimension d_c. On top of this, as reviewed in Sec. 2.1, some DP-C predictions for critical exponents (such as $\beta = 1$ for $\mu \neq 0$) seem to be inconsistent with the results of lattice simulations. Independently of this last point, it is unsatisfactory from a NPRG point of view to rely on the perturbative argument given in Sec. 4.2.2 to explain how the DP-C symmetries can emerge along the flow from the microscopic action of DEP lacking these symmetries. This motivated a study of both field theories using the tools of the NPRG. We choose to work with the simplest ansatz developed in NPRG which turns out to be sufficient in most cases to get an accurate picture of the phase diagram as well as a reasonable estimate of the critical exponents of the continuous phase transitions. This ansatz named the modified local potential approximation relies on the fact that the phase transition is controlled by the low wave-number and low frequency content of the EAA. In this section, we motivate this approximation and write down the corresponding ansätze for DP-C and DEP. This allows us to derive some analytical results within this approximation. Finally, we expound on our choice of regulator and on a further approximation which allowed us to integrate numerically the flow equation.

4.3.1 The zeroth order of the derivative expansion

Let us rewrite the NPRG flow equation for Γ_κ (3.81) in Fourier space, making explicit the invariance under space rotations and translations, and the frequency independence of the regulator:

$$\partial_s \Gamma_\kappa[\Phi] = \frac{1}{2}\int_{\mathbf{q}} \big[\partial_s \bar{\mathcal{R}}_\kappa\big]_{ij}(q)\, \tilde{G}^{(2)}_{ji}[\mathbf{q},-\mathbf{q};\Phi]\,, \tag{4.72}$$

where $\bar{G}^{(2)}_{ji}$ is obtained by inverting the Fourier transform of the Hessian of $\Gamma_\kappa + \Delta\mathcal{S}_\kappa$. We have seen in Chap. 3 that the wave-number integration of the r.h.s. is limited to values $q \lesssim \kappa$ due to the presence of the derivative of the regulator. Because of this property we can expand the integrand of Γ_κ in powers of the wave-numbers and of the frequencies. The expansion reads at the first non-trivial order

$$\Gamma_\kappa[\Phi] = \int_{\mathbf{x}} \Big\{ U_\kappa(\Phi) + \sum_{ij} Z^{ij}_\kappa(\Phi)\, \Phi_i \big[\partial_t - D^{ij}_\kappa(\Phi)\partial^2\big]\Phi_j + o(\partial^2) \Big\}\,. \tag{4.73}$$

Some remarks are in order here. The first one is that we have assumed that the renormalized propagators of the theory $\bar{G}^{(2)}_{ij}(\mathbf{p})$ at wave-number $p \lesssim \kappa$ have their dominant contribution for frequencies of the order $\omega \lesssim \kappa^2$. Without this assumption, we cannot justify to keep only the first order in time. Indeed, the regulator chosen in this work acts only on the wave-numbers and leaves the frequency sector uncontrolled. Thus, a priori, the high frequency sector could couple in the RG flow to the low frequency one. We expect the RG flow to reach a decoupling fixed point, such that for $p \gg \kappa$ the propagators takes a scale invariant form

$$\bar{G}^{(2)}_{ij}(\mathbf{p}) \sim p^{-2-d+d_i+d_j} g\big(\omega/p^z\big)\,, \tag{4.74}$$

with the dominant contribution of the scaling function g given by values of order unity. In this case, the integral is dominated by the values $\omega \lesssim \kappa^z \sim \kappa^2$ and the assumption is justified.

The second remark is that the symmetries can greatly help to constrain the ansatz. For example, the causality ensures that the vertex function without response field legs are zero. Furthermore, instead of having a generic parametrization in terms of the fields $\{\Phi_i\}$, one can generally use the invariants of the theory. For example for DP-C at $\mu = 0$, the functions $U_\kappa\,, Z_\kappa$ and D_κ would only depend on $\Phi + \bar{\Phi}$, $\Psi\bar{\Psi}$ and $\Psi - \bar{\Psi}$.

The above ansatz is called $O(\partial^2)$ and gives generally very accurate results. However, it still requires heavy calculations, as the $U_\kappa\,, Z_\kappa$ and D_κ are unknown functions of many variables. A further approximation is to assume that the kinetic part of the theory keeps in fact its bare form. This approximation is called the local potential approximation (LPA) and sometimes denoted $O(\partial^0)$, as only the potential U_κ is renormalized. In the case of DP-C it reads

$$\Gamma_\kappa[\Phi] = \int_{\mathbf{x}} \Big\{ \bar{\Phi}(\partial_t - \partial^2)\Phi + \mu\,\bar{\Phi}\,\partial^2\Psi + \bar{\Psi}(\partial_t - \lambda\partial^2 - \sigma)\Psi + U_\kappa(\Phi + \bar{\Phi}, \Psi, \bar{\Psi}) \Big\}\,. \tag{4.75}$$

This ansatz allows in most cases to get a first picture of the phase diagram. However it fixes the scaling dimension of time and of the fields to be the Gaussian ones. Indeed, the definition of the running scaling dimensions of Sec. 3.4.4 implies that in the LPA ansatz (4.75), for example for the $\bar{\Psi}$-Ψ two-point function,

$$\begin{aligned} Z_\kappa^{\bar{\psi}\psi} &= \partial_{i\omega} \bar{\Gamma}^{(2)}_{\bar{\psi}\psi}(\mathbf{p}, \Phi_0)|_{p=0} = 1\,, \\ D_\kappa^{\bar{\psi}\psi} &= \partial_{p^2} \bar{\Gamma}^{(2)}_{\bar{\psi}\psi}(\mathbf{p}, \Phi_0)|_{p=0} = 1\,, \end{aligned} \tag{4.76}$$

where $\Phi_0 = \{\Phi_0, \bar{\Phi}_0, \Psi_0, \bar{\Psi}_0\}$ is an unspecified configuration of stationary and uniform fields. As a consequence, the LPA ansatz gives directly $\eta + \bar{\eta} = 0$ and $z = 2$.

An efficient way to upgrade the LPA approximation in order to account for field renormalizations is to take Z_κ and D_κ to be independent of the fields (but still depending on the renormalization scale). This ansatz, introduced in (Tetradis and Wetterich 1994), is named modified LPA (LPA') or leading order. Note that due to the particular symmetry of DP-C fixing the $\bar{\Phi}$-Φ and $\bar{\Phi}$-Ψ propagators to their bare value, the only propagator which has the freedom to be renormalized is $\bar{\Psi}$-Ψ.

Wrapping up, we finally give the ansatz chosen in this work for DP-C. It reads

$$\Gamma^{\text{DP-C}}_\kappa[\Phi] = \int_{\mathbf{x}} \Big\{ \bar{\Phi}(\partial_t - \partial^2)\Phi + \mu\,\bar{\Phi}\,\partial^2\Psi + Z_\kappa\bar{\Psi}(\partial_t - \lambda_\kappa\partial^2)\Psi + U_\kappa(\Phi + \bar{\Phi}, \Psi, \bar{\Psi}) \Big\} \tag{4.77}$$

for $\mu \neq 0$ and

$$\Gamma_\kappa^{\text{DP-C}}[\Phi] = \int_{\mathbf{x}} \Big\{ \bar{\Phi}(\partial_t - \partial^2)\Phi + Z_\kappa \bar{\Psi}(\partial_t - \lambda_\kappa \partial^2)\Psi + U_\kappa(\Phi + \bar{\Phi}, \Psi\bar{\Psi}, \Psi - \bar{\Psi}) \Big\} \quad (4.78)$$

for $\mu = 0$. Let us make a remark on the ansatz. Because the scaling between space and time is already fixed by the $\bar{\Phi}$-Φ propagator, that is

$$D_\kappa^{\bar{\varphi}\varphi} = \frac{\partial_{p^2} \bar{\Gamma}^{(2)}_{\bar{\varphi}\varphi}(0, \vec{p}, \Phi_0)|_{\vec{p}=0}}{\partial_{i\omega} \bar{\Gamma}^{(2)}_{\bar{\varphi}\varphi}(\omega, \vec{0}, \Phi_0)|_{\omega=0}} = 1\,, \quad (4.79)$$

we must have as well that

$$\lim_{\kappa \to 0} \frac{\mathrm{d}}{\mathrm{d}\ln\kappa} \ln D_\kappa^{\bar{\psi}\psi} = \lim_{\kappa \to 0} \frac{\mathrm{d}}{\mathrm{d}\ln\kappa} \ln \lambda_\kappa = 0\,. \quad (4.80)$$

Thus λ_κ can contain at most a sub-leading logarithmic dependency in κ.

Now, let us give the LPA' ansatz for DEP. Given that the model is less constrained by its symmetries, one must allow a priori a more general renormalization of the propagator of the theory. It reads

$$\Gamma_\kappa^{\text{DEP}}[\Phi] = \int_{\mathbf{x}} \Big\{ Z_\kappa^\varphi \bar{\Phi}(\partial_t - D_\kappa \partial^2)\Phi + \mu_\kappa D_\kappa \bar{\Phi}\, \partial^2 \Psi + Z_\kappa^\psi \bar{\Psi}(\partial_t - \lambda_\kappa D_\kappa \partial^2)\Psi + U_\kappa(\Phi) \Big\}\,, \quad (4.81)$$

for $\mu \neq 0$, with μ_κ and λ_κ having scaling dimension zero, and

$$\Gamma_\kappa^{\text{DEP}}[\Phi] = \int_{\mathbf{x}} \Big\{ Z_\kappa^\varphi \bar{\Phi}(\partial_t - D_\kappa \partial^2)\Phi + Z_\kappa^\psi \bar{\Psi}(\partial_t - \lambda_\kappa D_\kappa \partial^2)\Psi + U_\kappa(\Phi) \Big\} \quad (4.82)$$

for $\mu = 0$. Furthermore U_κ must satisfy the additional constraints coming from symmetries (4.60) and (4.67):

$$\begin{aligned} &\Big[\big(\sqrt{\rho_0} + \varphi\big)\partial_\varphi - \big(\sqrt{\rho_0} + \bar{\varphi}\big)\partial_{\bar{\varphi}} + \psi\partial_\psi - \bar{\psi}\partial_{\bar{\psi}} \Big] U_\kappa(\varphi, \bar{\varphi}, \psi, \bar{\psi}) = 0\,, \\ &\text{and for } \mu = 0\,, \quad U_\kappa(\bar{\varphi}, \varphi, -\bar{\psi}, -\psi) = U_\kappa(\varphi, \bar{\varphi}, \psi, \bar{\psi})\,. \end{aligned} \quad (4.83)$$

Both for DEP and DP-C, the RG flow of the theory is entirely described at the level of the LPA' ansatz, by the flow of the potential

$$\partial_s U_\kappa(\Phi_0) = \frac{1}{2} \int_{\mathbf{q}} \big[\partial_\kappa \bar{\mathcal{R}}_\kappa\big]_{ij}(q)\, \bar{G}^{(2)}_{ji}(\mathbf{q}, \Phi_0)\,, \quad (4.84)$$

and that of the two-point function at constant fields, which reads

$$\begin{aligned} \partial_s \bar{\Gamma}^{(2)}_{mn}(\mathbf{p}, \Phi_0) = \int_{\mathbf{q}} \big[\partial_s \bar{\mathcal{R}}\big]_{ij}(q) \bar{G}^{(2)}_{jk}(\mathbf{q}, \Phi_0) \Big[&- \frac{1}{2} U^{(4)}_{klmn}(\Phi_0) \\ &+ U^{(3)}_{kms}(\Phi_0) \bar{G}^{(2)}_{st}(\mathbf{q} + \mathbf{p}, \Phi_0) U^{(3)}_{tnl}(\Phi_0) \Big] \bar{G}^{(2)}_{li}(\mathbf{q}, \Phi_0)\,. \end{aligned} \quad (4.85)$$

More precisely, we extract information only from the low frequency and low wave-number behavior of the flow of the two-point functions. At the level of the LPA' ansatz, all the information is contained solely in the flow of the couplings Z_κ^φ, Z_κ^ψ, D_κ, λ_κ a,d μ_κ, the wave-number and frequency dependence are the bare ones. Now, let us discuss in detail our choice of the stationary homogeneous configuration Φ_0 at which these couplings are defined.

4.3.2 Choice of Φ_0 as a minimum configuration

Let us come back to the explicit definition of Z_κ^φ, Z_κ^ψ, D_κ, λ_κ and μ_κ. They are given respectively by

$$\begin{aligned} Z_\kappa^\varphi &\equiv \partial_{i\omega}\bar{\Gamma}^{(2)}_{\bar{\varphi}\varphi}(\mathbf{p},\Phi_0)|_{\mathbf{p}=0}\,, \quad D_\kappa \equiv (Z_\kappa^\varphi)^{-1}\partial_{p^2}\bar{\Gamma}^{(2)}_{\bar{\varphi}\varphi}(\mathbf{p},\Phi_0)|_{\mathbf{p}=0}\,, \\ Z_\kappa^\psi &\equiv \partial_{i\omega}\bar{\Gamma}^{(2)}_{\bar{\psi}\psi}(\mathbf{p},\Phi_0)|_{\mathbf{p}=0}\,, \quad \lambda_\kappa D_\kappa \equiv (Z_\kappa^\psi)^{-1}\partial_{p^2}\bar{\Gamma}^{(2)}_{\bar{\psi}\psi}(\mathbf{p},\Phi_0)|_{\mathbf{p}=0}\,, \\ &\mu_\kappa D_\kappa \equiv \partial_{p^2}\bar{\Gamma}^{(2)}_{\bar{\varphi}\psi}(\mathbf{p},\Phi_0)|_{\mathbf{p}=0}\,. \end{aligned} \tag{4.86}$$

Within the LPA' ansatz, we have to project explicitly these couplings to a constant value because they acquire generically a field dependence with the RG flow. Thus we have to explicitly choose a constant field configuration at which we define them. Because DP-C is a response field theory and DEP a coherent field one, the interpretation of the fields is not the same and the choices of constant field configuration differ. Let us first discuss the case of DP-C. The mean value of the response fields are zero in the stationary state, so we choose $\bar{\Psi}_0 = \bar{\Phi}_0 = 0$. Furthermore, we choose $\Psi_0 = 0$ for simplicity. Finally, guided by the duality symmetry (4.39) for DP-C which relates the mass of the $\bar{\Psi}$-Ψ propagator to the mean value of the field Φ, we set $\Phi_0 = \chi_\kappa^{\text{DP-C}}$. In the DP-C context, we thus note

$$\text{Min} : \{\Phi, \bar{\Phi}, \Psi, \bar{\Psi}\} \to \{\chi_\kappa^{\text{DP-C}}, 0, 0, 0\}\,. \tag{4.87}$$

Now, let us do the same for DEP. We have seen in Appendix C.1 that within the Doi-Peliti formalism, the original density n_i field is given in terms of its associated coherent fields by the product $\bar{\phi}_i\phi_i$. For the Ψ and $\bar{\Psi}$ fields, we choose for simplicity $\Psi = \bar{\Psi} = 0$ as in DP-C. However, once again guided by the duality symmetry (4.69) for DEP, we set $\Phi = \bar{\Phi} = \chi_\kappa^{\text{DEP}}$, such that $\bar{\Phi}\Phi = (\chi_\kappa^{\text{DEP}})^2 \neq 0$ without breaking the symmetry between Φ and $\bar{\Phi}$. Wrapping up, the constant field configuration in the context of DEP reads

$$\text{Min} : \{\Phi, \bar{\Phi}, \Psi, \bar{\Psi}\} \to \{\chi_\kappa^{\text{DEP}}, \chi_\kappa^{\text{DEP}}, 0, 0\}\,. \tag{4.88}$$

In both cases, the parametrization allows one to absorb the mass of the $\bar{\Psi}$-Ψ propagator in the following way. We assume χ_κ can be implicitely defined such that

$$\bar{\Gamma}^{(2)}_{\bar{\psi}\psi}(\mathbf{p},\Phi_0)|_{\mathbf{p}=0,\text{Min}} = \partial^2_{\psi\bar{\psi}}U_\kappa|_{\text{Min}} = 0\,. \tag{4.89}$$

This choice of parametrization and the assumption it contains allow us to use a central property of both DP-C and DEP, that is the duality (4.39) and (4.69) respectively. The general duality identity derived in Appendix D can be rewritten in the particular case of our ansatz. For example, let us concentrate on DP-C. We expand partially the potential as

$$U_\kappa^{\text{DP-C}}(\varphi+\bar{\varphi},\psi,\bar{\psi}) = \sum_n u_n^{\text{DP-C}}(\psi,\bar{\psi})(\varphi+\bar{\varphi}-\chi_\kappa^{\text{DP-C}})^n\,, \tag{4.90}$$

where the κ dependence of the $u_n^{\text{DP-C}}$ is implicit. The duality relation (D.6) then reads

$$\forall n \in \mathbb{N}, \quad \partial_\sigma u_n^{\text{DP-C}} = (n+1)u_{n+1}^{\text{DP-C}}\Big(\frac{1}{g} + \partial_\sigma \chi_\kappa^{\text{DP-C}}\Big)\,. \tag{4.91}$$

In particular, by taking derivatives of this expression with respect to ψ and $\bar{\psi}$, and evaluating it at $\psi, \bar{\psi} = 0$, we obtain

$$\forall n \in \mathbb{N}, \quad \partial_\sigma \partial^2_{\psi\bar{\psi}} u_n^{\text{DP-C}}(0,0) = (n+1)\partial^2_{\psi\bar{\psi}} u_{n+1}^{\text{DP-C}}(0,0)\big(\frac{1}{g} + \partial_\sigma \chi_\kappa^{\text{DP-C}}\big)\,. \tag{4.92}$$

Because

$$\partial^2_{\psi\bar{\psi}} u_0^{\text{DP-C}}(0,0) = \partial^2_{\psi\bar{\psi}} U_\kappa^{\text{DP-C}}|_{\text{Min}} = 0\,, \tag{4.93}$$

there exists a n such that

$$\partial^2_{\psi\bar{\psi}} u_n^{\text{DP-C}}(0,0) = 0 \text{ and } \partial^2_{\psi\bar{\psi}} u_{n+1}^{\text{DP-C}}(0,0) \neq 0\,. \tag{4.94}$$

We deduce from there that

$$\frac{1}{g} + \partial_\sigma \chi_\kappa^{\text{DP-C}} = 0\,. \tag{4.95}$$

and

$$\forall n \in \mathbb{N}, \quad \partial_\sigma u_n^{\text{DP-C}} = 0\,. \tag{4.96}$$

Remembering that σ is the control parameter of the transition, these equations mean that $\chi_\kappa^{\text{DP-C}}$ is the only relevant coupling. In other words, the RG flow can be projected on the critical surface by considering the flow of all couplings except $\chi_\kappa^{\text{DP-C}}$. If there is a critical point, the flow will then be fully attractive except for the coupling χ_κ. This is a huge simplification. Indeed in most cases when one has to integrate numerically the RG flow of the couplings, the relevant direction is not identified or cannot be parametrized so easily. In these cases, one has to fine-tune a control parameter in order to approach the critical fixed point, for example by using a dichotomy algorithm. The same can be done for DEP, using this time the following partial expansion the partial expansion

$$U_\kappa^{\text{DEP}}(\varphi, \bar{\varphi}, \psi, \bar{\psi}) = \sum_{m,n} u_{mn}^{\text{DEP}}(\psi, \bar{\psi})(\bar{\varphi} - \chi_\kappa^{\text{DEP}})^m (\varphi - \chi_\kappa^{\text{DEP}})^n\,. \tag{4.97}$$

The same steps as for the DP-C case show that

$$\frac{1}{2\sqrt{\rho_0}} + \partial_{\rho_0} \chi_\kappa^{\text{DEP}} = 0 \tag{4.98}$$

and

$$\forall m, n \in \mathbb{N}, \quad \partial_{\rho_0} u_{mn}^{\text{DEP}} = 0\,, \tag{4.99}$$

with the same consequences. However, note that the equivalence between the inverse recovery rate τ and the initial total density ρ_0 as microscopic control parameters of the critial phase transition, which is true perturbatively, may not be valid anymore.

The flow equation of the minimum χ_κ is deduced by taking a derivative of the above expression with respect to the renormalization scale. For DP-C, it reads

$$0 = \Big[\partial_\kappa \partial^2_{\psi\bar{\psi}} U_\kappa^{\text{DP-C}}\Big]_{\text{Min}} + \partial^3_{\psi\bar{\psi}\varphi} U_\kappa^{\text{DP-C}}\Big|_{\text{Min}} \partial_\kappa \chi^{\text{DP-C}}\,. \tag{4.100}$$

If $u_{111}(\kappa) \equiv \partial^3_{\psi\bar{\psi}\varphi} U_\kappa^{\text{DP-C}}\big|_{\text{Min}} = 0$, the Min configuration cannot be defined. This prevents this coupling from changing sign along the RG flow. It means that from physically admissible initial conditions (which correspond to $u_{111}(\Lambda) < 0$), the RG flow cannot reach a fixed point candidate to describe the continuous phase transition

if $u^*_{111} > 0$ at this fixed point. The perturbative fixed point found in (Wijland, Oerding, and Hilhorst 1998) has $u^*_{111}/\mu > 0$, thus it cannot be reached for $\mu > 0$. This prompted its authors to conjecture the existence of a first order phase transition in this case. However, this conclusion does not hold if $u^*_{111}/\mu > 0$ is only a one-loop or a perturbative feature or if there exist other fixed points with $u^*_{111} < 0$ independently of the sign of μ.

A remark is of order here. As long as we are truncating the potential U_κ, it is always possible to find χ_κ such that the above equation is verified (leaving aside the problem of complex roots and non-unicity of the solutions). However, the full potential $\partial^2_{\psi\bar{\psi}} U_\kappa|_{\mathrm{Min}}$ as a function of χ_κ may not vanish for finite χ_κ. While we feel this difficulty would be worth investigating, we will not delve further into the subject here. Indeed, in the following, we always truncate the potential before any actual integration of the flow.

In order to go further and explicitly calculate the RG flow equations (4.84) and (4.85), one has to specify the form of the regulator. This is the topic of next section.

4.3.3 The Litim Θ regulator

When we have introduced the general setting of the NPRG in Chap. 3 , we have insisted on the fact that the RG flow equations for Γ_κ and its vertex functions were exact. In particular, the result of the integration of these equations do not depend on the exact structure of the regulator matrix and on the form of its wave-number dependence as long as it satisfies the required properties. However, as soon as we make approximations of the flow equations, this property is lost. Thus the choice and optimization of the regulator becomes an important part of NPRG studies (Canet et al. 2004). We have already stated that we will limit ourselves to regulator of the form

$$\Delta\mathcal{S}_\kappa[\Phi] = \frac{1}{2}\int_{\mathbf{q}} \left[\bar{\mathcal{R}}_\kappa\right]_{ij}(q)\Phi_i(\mathbf{q})\Phi_j(-\mathbf{q})\,, \tag{4.101}$$

where $i,j \in \{\varphi,\bar{\varphi},\psi,\bar{\psi}\}$. As the bare action does not have a quadratic coupling in $\bar{\Psi}$-$\bar{\Psi}$ and in $\bar{\Phi}$-$\bar{\Phi}$, we have only to add a regulator to the $\bar{\Psi}$-Ψ and $\bar{\Phi}$-Φ sectors. Because the regulators depend only on the modulus of the wave-number, $\left[\bar{\mathcal{R}}_\kappa\right]_{\bar{\varphi}\varphi}(q) = \left[\bar{\mathcal{R}}_\kappa\right]_{\varphi\bar{\varphi}}(q)$, and accordingly for the $\bar{\Psi}$-Ψ one, (4.101) simply reads

$$\Delta\mathcal{S}_\kappa[\Phi] = \int_{\mathbf{q}} \left\{ \left[\bar{\mathcal{R}}_\kappa\right]_{\bar{\varphi}\varphi}(q)\bar{\Phi}(\mathbf{q})\Phi(-\mathbf{q}) + \left[\bar{\mathcal{R}}_\kappa\right]_{\bar{\psi}\psi}(q)\bar{\Psi}(\mathbf{q})\Psi(-\mathbf{q}) \right\}. \tag{4.102}$$

Now, following the discussion of Sec. 3.4.3, we have to choose a κ dependence of the regulators in accordance with the definition of the running scaling dimensions. For DP-C, it reads

$$\left[\bar{\mathcal{R}}_\kappa\right]_{\bar{\varphi}\varphi}(q) = q^2 r_\varphi\left(q^2/\kappa^2\right), \quad \left[\bar{\mathcal{R}}_\kappa\right]_{\bar{\psi}\psi}(q) = \lambda_\kappa Z_\kappa q^2 r_\psi\left(q^2/\kappa^2\right), \tag{4.103}$$

while for DEP it takes the following form

$$\left[\bar{\mathcal{R}}_\kappa\right]_{\bar{\varphi}\varphi}(q) = D_\kappa Z^\varphi_\kappa q^2 r_\varphi\left(q^2/\kappa^2\right), \quad \left[\bar{\mathcal{R}}_\kappa\right]_{\bar{\psi}\psi}(q) = \lambda_\kappa D_\kappa Z^\psi_\kappa q^2 r_\psi\left(q^2/\kappa^2\right). \tag{4.104}$$

The freedom of choice lies thus in the functional form of r_φ and r_ψ. In this work, we choose to use exclusively the Litim Θ regulator (Litim 2001):

$$r_\varphi(y) = r_\psi(y) = r(y) = \left(\frac{1}{y} - 1\right)\Theta(1-y)\,, \tag{4.105}$$

where Θ is the Heaviside function. This regulator has the huge advantage that the wave-number integration can be done analytically. On the two-point functions, say the $\bar{\Psi}$-Ψ one for example, the regulator has the following effect:

$$\left[\bar{\Gamma}^{(2)}+\bar{\mathcal{R}}_\kappa\right]_{\psi\bar{\psi}}(\varpi,q)=Z_\kappa\left[i\varpi+\lambda_\kappa q^2(1+r(q^2/\kappa^2))\right]=\begin{cases}\bar{\Gamma}^{(2)}_{\bar{\psi}\psi}(\varpi,q)\,, & \text{if } q<\kappa\\ \bar{\Gamma}^{(2)}_{\bar{\psi}\psi}(\varpi,\kappa)\,, & \text{if } q>\kappa\end{cases}.$$

At higher order of the derivative expansion, the non-analyticity coming from the Heaviside function can prevent from using this regulator. However, this is not the case here. We have not tested the independency of our results on the choice of the regulator in this work. This effect would have to be studied eventually. The frequency integrals in the flow equations (4.84) and (4.85) are carried out in accordance with causality as explained in Sec. 3.5. As a consequence of the causality prescription and of the structure of the regulator for DEP and DP-C, the integration on ϖ in the flow amounts to a sum on the residues of $G^{(2)}_{\varphi\bar{\varphi}}$ and $G^{(2)}_{\psi\bar{\psi}}$ in the upper-half plane. We note Res_+ the operation of summing the residues in the upper-half plane and multiplying by i. For example, the flow of the potential reads with this notation

$$\begin{aligned}\partial_s U_\kappa(\Phi_0)&=\frac{1}{2}\int_{\mathbf{q}}\left[\partial_\kappa\bar{\mathcal{R}}_\kappa\right]_{ij}(\mathbf{q})\,\bar{G}^{(2)}_{ji}(\mathbf{q},\Phi_0)\\&=\int_{\mathbf{q}}\left\{\left[\partial_\kappa\bar{\mathcal{R}}_\kappa\right]_{\varphi\bar{\varphi}}(\mathbf{q})\bar{G}^{(2)}_{\varphi\bar{\varphi}}(\mathbf{q},\Phi_0)+\left[\partial_\kappa\bar{\mathcal{R}}_\kappa\right]_{\psi\bar{\psi}}(\mathbf{q})\bar{G}^{(2)}_{\psi\bar{\psi}}(\mathbf{q},\Phi_0)\right\}\\&=\int_{\mathbf{q}}\left\{\left[\partial_\kappa\bar{\mathcal{R}}_\kappa\right]_{\varphi\bar{\varphi}}(q)e^{i\varpi\epsilon}\bar{G}^{(2)}_{\varphi\bar{\varphi}}(\mathbf{q},\Phi_0)+\left[\partial_\kappa\bar{\mathcal{R}}_\kappa\right]_{\psi\bar{\psi}}(q)e^{i\varpi\epsilon}\bar{G}^{(2)}_{\psi\bar{\psi}}(\mathbf{q},\Phi_0)\right\}\\&=\int_{\vec{q}}\mathrm{Res}_+\left\{\left[\partial_\kappa\bar{\mathcal{R}}_\kappa\right]_{\varphi\bar{\varphi}}(q)\bar{G}^{(2)}_{\varphi\bar{\varphi}}(\mathbf{q},\Phi_0)+\left[\partial_\kappa\bar{\mathcal{R}}_\kappa\right]_{\psi\bar{\psi}}(q)\bar{G}^{(2)}_{\psi\bar{\psi}}(\mathbf{q},\Phi_0)\right\}.\end{aligned}\tag{4.106}$$

4.3.4 Truncation of the potential

With the explicit expression for the regulator, all is set to integrate numerically the flow equations, at least in principle. One would have to parametrize the potential in terms of the symmetry invariants and integrate numerically the system constituted by the ordinary differential equations (ODE) for the scaling factors (4.86) and the non-linear partial differential flow equation of the potential, which is integro-differential in the field variables. This task is slightly simplified by the fact that both for DEP and DP-C, the prefactor $\bar{\Psi}\Psi$ which appears in the bare potential is preserved by the renormalization flow in this setting:

$$U_\kappa(\varphi,\bar{\varphi},\psi,\bar{\psi})=\bar{\psi}\psi V_\kappa(\varphi,\bar{\varphi},\psi,\bar{\psi})\,.\tag{4.107}$$

This property of the LPA' ansatz at the minimum configuration for analytic potentials is shown in Appendix E.1.

However, it is extremely costly to take into account the full field dependence of the potential. This has been done when there is only one field invariant to parametrize the potential (Berges, Tetradis, and Wetterich 2002; Canet et al. 2003). Here there are three field invariants. As a consequence, we choose as a first approach to expand the potential in powers of the field and to truncate it. This idea is suggested by perturbative RG, as higher monomes of the fields are less relevant as long as their dimensions is positive. One expects to see an numerical convergence

of the values obtained for the critical exponents with the order of truncation. This convergence means that the critical exponents are only sensitive to the first coefficients of the potential expansion, which validates the method. This approach was used successfuly to study the DP phase transition with the same tools in (Canet et al. 2004). The potential truncated at order N reads for DP-C

$$U_\kappa^{\text{DP-C}}(\varphi+\bar\varphi,\psi,\bar\psi)=\sum_{\substack{a,b,c\\ b,c\neq 0\\ 3\leq a+b+c\leq N}} u_{abc}^{\text{DP-C}}(\kappa)(\varphi+\bar\varphi-\chi_\kappa^{\text{DP-C}})^a\bar\psi^b\psi^c\,, \tag{4.108}$$

and for DEP

$$U_\kappa^{\text{DEP}}(\varphi,\bar\varphi,\psi,\bar\psi)=\sum_{\substack{s,p,q,r\\ q,r\neq 0\\ 3\leq s+p+q+r\leq N}} u_{spqr}^{\text{DEP}}(\kappa)(\varphi-\chi_\kappa^{\text{DEP}})^s(\bar\varphi-\chi_\kappa^{\text{DEP}})^p\psi^q\psi^r\,, \tag{4.109}$$

with the following constraint for the coefficients (omitting the κ dependence and the superscript for simplicity):

$$(s-p+q-r)u_{spqr}+(\sqrt{\rho_0}+\chi_\kappa)\big[(s+1)u_{s+1pqr}-(p+1)u_{sp+1qr}\big]=0\,. \tag{4.110}$$

The truncation has the effect to turn the partial differential equation for the potential into a set of ODE for each couplings, thus offering a huge simplification in the numerical integration of the flow. For example, the cubic order of DP-C reads

$$U_\kappa^{\text{DP-C}}(\varphi+\bar\varphi,\psi,\bar\psi)=\psi\bar\psi\Big[u_{021}\,\psi+u_{012}\,\bar\psi+u_{111}(\varphi+\bar\varphi-\chi_\kappa)\Big]\,, \tag{4.111}$$

with $u_{012}=-u_{021}$ for $\mu=0$. One recognizes in this ansatz the DP-C action whose bare couplings have been replaced by running couplings depending on the renormalization scale κ. The flow equations for the scaling factors (4.86) and the potential couplings from the flow equations (4.85) and (4.84) were obtained using the computer algebra system Mathematica. The extraction of the upper half plane residues corresponding to the frequency integral is done using partial fraction decomposition. This is possible because the poles appearing in the integrand of the flow equations are well identified. The wave-number integral is done analytically as well thanks to the choice of the Θ regulator. Finally, we express the flow equation for the dimensionless couplings. As expounded on in Sec. 3.4.4, one looks for fixed points of the dimensionless Γ_κ, expressed in terms of the dimensionless fields $\hat\Phi_i(\hat t,\hat{\vec x})=\kappa^{-d_{\phi_i}}\Phi_i(\kappa^{-z}\hat t,\kappa\hat{\vec x})$. One thus defines the dimensionless couplings

$$u_{abc}=\kappa^{d_{abc}}\hat u_{abc} \tag{4.112}$$

such that if the $\hat u_{abc}$ reach a fixed point, the dimensionless potential

$$\hat U_s(\hat\Phi_0)=\kappa^{-d-z}U_\kappa(\kappa^{d_\phi}\hat\Phi_0) \tag{4.113}$$

reaches a fixed point. These steps are written in detail in Appendix E.2 for the flow of $\hat\chi_\kappa^{\text{DP-C}}$. The system of flow equations for the potential couplings is closed by adding the flow of λ_κ as well as the determination of the running anomalous dimension $\eta\,(=\bar\eta)$ or $\bar\eta$ for $\mu=0$ and $\mu\neq 0$ respectively. Both of them are determined from

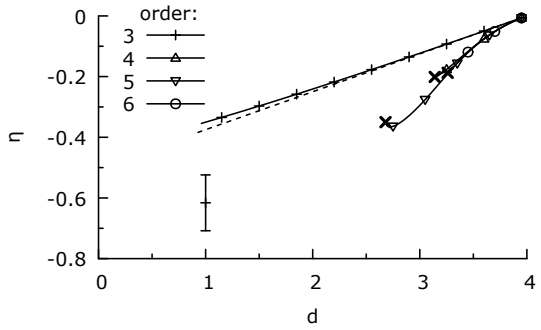

Figure 4.1: Values of η as a function of the dimension d for $\mu = 0$ calculated from the LPA' truncations compared to the one-loop result (dashed line) and to the simulation (Maia and Dickman 2007) (isolated point at $d = 1$). Thick saltires mark the disappearance of the fixed point.

the flow equation of the 2-point function (4.85) from their definitions in (4.86) and following the same steps as for the couplings.

Before going to the results of the numerical integration, let us already signal the limitation of the truncation. It turns out that the complexity of the flow equations for the couplings rises substantially with the number of couplings to consider. Furthermore, due to the high number of fields, the number of couplings grows rapidly with the order. For $\mu \neq 0$, at the truncation orders $N = 3$, 4, 5 and 6 we have to consider respectively 3, 8, 19 and 34 couplings. Above $N = 6$, the number of couplings grows faster than N^2. At this point, to obtain and treat the lengthy expressions for the flow equations of the couplings becomes too demanding on computer RAM and we could not go further than order 6. First, let us discuss the result of the integration in the DP-C case.

4.4 Results of the numerical integration of the DP-C flow

After having obtained the closed system of flow equations for λ_κ and the potential couplings, supplemented with the running of the anomalous dimension defined through Z_κ^ψ, one has to integrate them numerically. We verify that $\hat{\chi}$ does not appear in the flow of the other couplings, which implies that it is an unstable direction of the flow with eigenvalue $d/2$ and that, once its flow is excluded, the flow of the other couplings is fully attractive (if a critical fixed point exists). As a consequence, the integration of the flow can be done without the need to perform any dichotomy fine-tuning of the control parameter. This was done in C using the tools of the GNU scientific library (GSL). We use the built-in ODE solver *gsl_odeiv2* with a Runge-Kutta-Fehlberg (RKF45) numerical scheme. We integrate the flow from physically admissible initial conditions. If the flow reaches a fixed point, we record the critical exponent characterizing it, $\eta\,(= \bar{\eta})$ for $\mu = 0$ and $\bar{\eta}$ for $\mu \neq 0$. We do this operation for d ranging from $d = 3.95$, where we expect the perturbative results to hold, to $d = 1$, which is relevant to make contact with the simulations. The results are displayed in Fig. 4.1 and Fig. 4.2 respectively.

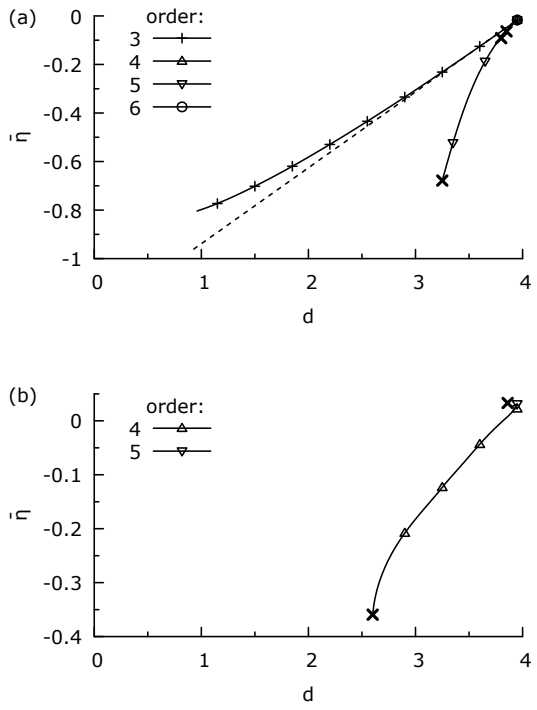

Figure 4.2: Values of $\bar{\eta}$ as a function of the dimension for (a) $\mu = -1$ and (b) $\mu = 0.5$ calculated from the LPA' truncations compared to the one-loop result (dashed line). Thick saltires mark the disappearance of the fixed point.

Let us first discuss the case $\mu = 0$. At third order, written explicitly in (4.111), (which corresponds to a renormalization of the coupling constants of the vertices of the DP-C bare action), we find a critical fixed point for any dimension below $d = 4$. It corresponds to the fixed point precedently described perturbatively in (Kree, Schaub, and Schmittmann 1989; Wijland, Oerding, and Hilhorst 1998). At fourth order, we find a fixed point only for $d \gtrsim 3$. For d below 3, the flow diverges without reaching any fixed point. At fifth order, the domain of convergence to the fixed point extends up to a value $d \lesssim 3$. There is a striking difference between the value of η at the third order and at higher orders. For $d = 3$, we obtain the value $\eta = -0.3$. This value already differs from the perturbative prediction by 50% so it would be interesting to compare it to numerical simulations in $d = 3$. If extrapolated to $d = 1$, the value of η seems to be in accordance with the simulations in $d = 1$. However, at sixth order the situation worsen and the fixed point can be reached only near $d = 4$. As we were not able to push the truncation to seventh and higher orders, it is difficult to assess at this point whether the domain of convergence to this fixed point ends up reaching $d = 1$ or stablizes at a value $3 \lesssim d < 4$. At this point, as discussed below, it is not clear whether the fact that the flow does not reach the fixed point in low dimension is an artifact due to the finite-order truncation of the LPA', or if it is related to the intrinsic nature of the DP-C fixed point and its relation to DEP.

In the $\mu < 0$ case, this problem seems even sharper. At third order, we recover the perturbative result. At higher orders, we also find the same fixed point near $d = 4$. The domain of convergence to this fixed point is maximum at fifth order and diminishes at sixth order. However, the domain of convergence in this case is smaller

than for $\mu = 0$ and does not reach $d = 3$. Let us note that for $\mu \neq 0$, as $\eta = 0$, the only non-trivial exponent for the DP-C class is $\bar{\eta}$. This exponent is related to θ', the critical initial slip exponent (Wijland, Oerding, and Hilhorst 1998). Unfortunately no numerical determination of θ' exists in the literature for $\mu \neq 0$.

For $\mu > 0$, at third order, the same fixed point as for $\mu < 0$ is present but the flow cannot reach it from physically admissible initial conditions since this would require the coupling $\hat{u}_{111}$ to change sign along the flow, leading to a divergence of $\hat{\chi}$, as explained in Sec. 4.3.2. At this order, there exists no other fixed point, such that we recover the same scenario as the perturbative study. At fourth order in the field expansion, we find a new fixed point in the physical region of parameters. In contrast with the perturbative calculations but in agreement with the numerical simulations, this finding supports the existence of a second order phase transition for $\mu > 0$. Interestingly, at fifth order the fixed point exists in $d = 3$, giving the prediction $\bar{\eta} = -0.2$ which could be compared to numerical determinations of $\bar{\eta}$, if these were to exist. However, we are not able to convincingly assess the convergence of this result with the order of the truncation in the field expansion, since the domain of convergence to this fixed point also varies significantly as the order is increased.

In both cases, $\mu < 0$ and $\mu > 0$, the divergence of the DP-C flow below a certain dimension gives rise to the same question as in the $\mu = 0$ case. We have checked that it was caused by the fixed point acquiring a new unstable direction below a certain dimension. Either it is an artifact to be imputed to the finite-order truncation, or it signals that the continuous transition observed in the simulations of DEP in $d = 1$ and 2 cannot be described by the DP-C class in low dimensions. As the equivalence between DEP and DP-C universality class is true in dimensions close enough to $d_c = 4$, in the second scenario there would thus exist a lower critical dimension of DP-C, $d_c' < 4$, below which the RG flow of the DEP action would lead to a new fixed point different from the DP-C one. Recalling the discussion of Sec. 4.2.3, this truly DEP fixed point would share the shift symmetry (4.66) with DP-C but crucially d_φ and z would be free to differ from their respective Gaussian values $d_\varphi = d/2$ and $z = 2$. This prompts to turn to the study of the RG flow of the DEP action.

4.5 Integration of the DEP flow and shortfalls of the LPA'

Let us first use the truncated DEP ansatz to reexamine the argument of the passage from the rescaling symmetry (4.60) to the shift symmetry (4.66) presented in Sec. 4.2.3. At quartic order, one obtains the following ansatz for the potential:

$$U_\kappa^{\mathrm{DEP}}(\varphi, \bar{\varphi}, \psi, \bar{\psi}) = \psi\bar{\psi}\Big\{u_{1111}\big[(\varphi + \sqrt{\rho_0})(\bar{\varphi} + \sqrt{\rho_0}) - (\chi_\kappa + \sqrt{\rho_0})^2\big] + u_{1012}\bar{\psi}(\varphi + \sqrt{\rho_0}) + u_{0121}\psi(\bar{\varphi} + \sqrt{\rho_0}) + u_{0022}\bar{\psi}\psi\Big\}. \quad (4.114)$$

Due to the presence of $\sqrt{\rho_0}$, this potential cannot be made scale independent. Let us write down the above expression in terms of the dimensionless variables, remem-

bering that $d_\varphi = d_{\bar\varphi}$ is already fixed by (4.65):

$$\hat U_\kappa^{\mathrm{DEP}}(\hat\varphi,\hat{\bar\varphi},\hat\psi,\hat{\bar\psi}) = \kappa^{-d-z+d_\psi+d_{\bar\psi}}\hat\psi\hat{\bar\psi}\Big\{ \\ \kappa^{d_{1111}}\hat u_{1111}\big[(\kappa^{d_\varphi}\hat\varphi+\sqrt{\rho_0})(\kappa^{d_\varphi}\hat{\bar\varphi}+\sqrt{\rho_0})-(\kappa^{d_\varphi}\hat\chi_\kappa+\sqrt{\rho_0})^2\big] \\ +\kappa^{d_{1012}+d_{\bar\psi}}\hat u_{1012}\hat{\bar\psi}(\kappa^{d_\varphi}\hat\varphi+\sqrt{\rho_0})+\kappa^{d_{0121}+d_\psi}\hat u_{0121}\hat\psi(\kappa^{d_\varphi}\hat{\bar\varphi}+\sqrt{\rho_0})+\kappa^{d_{0022}+d_\psi+d_{\bar\psi}}\hat u_{0022}\hat{\bar\psi}\hat\psi\Big\}. \tag{4.115}$$

The scaling dimension of u_{0022} is fixed straightforwardly to be

$$d_{0022} = d+z-2d_\psi-2d_{\bar\psi}\,. \tag{4.116}$$

Because d_φ is expected to be positive, for the terms proportional to the $\hat u_{1012}$, $\hat u_{0121}$ and $\hat u_{1111}$ couplings to have a non-trivial limit when $\kappa\to 0$, one has to choose respectively

$$\begin{aligned} d_{1012} &= d+z-d_\psi-2d_{\bar\psi}\,,\\ d_{0121} &= d+z-2d_\psi-d_{\bar\psi}\,,\\ d_{1111} &= d+z-d_\psi-d_{\bar\psi}-d_\varphi\,. \end{aligned} \tag{4.117}$$

With these choices, in the limit $\kappa\to 0$, one is left with

$$\hat U_\kappa^{\mathrm{DEP}}(\hat\varphi,\hat{\bar\varphi},\hat\psi,\hat{\bar\psi}) \underset{\kappa\to 0}{\sim} \hat\psi\hat{\bar\psi}\Big\{\hat u_{1111}\sqrt{\rho_0}(\hat\varphi+\hat{\bar\varphi}-2\hat\chi_\kappa) \\ +\hat u_{1012}\sqrt{\rho_0}\hat{\bar\psi}+\hat u_{0121}\sqrt{\rho_0}\hat\psi+\hat u_{0022}\hat{\bar\psi}\hat\psi\Big\}, \tag{4.118}$$

which has the DP-C shift symmetry (4.66). Let us check that this scenario is realized by actually integrating the flow.

The system of flow equations for the quartic DEP ansatz (4.114) is composed of the flow of the potential couplings, the flow of the kinetic couplings λ_κ and μ_κ and the running of the anomalous dimensions defined through Z_κ^ψ, Z_κ^φ and D_κ. These flow equations are integrated following the same method as above.

For all values of μ and for d in the domain of existence of the stable DP-C fixed point explored above, the couplings of DEP flow to this fixed point. To illustrate this, we have plotted in Fig. 4.3 the logarithm of the norm of the variation of the DEP potential under the rescaling symmetry (4.60) and the shift one (4.66) along the RG flow. The norm of variation under the shift symmetry saturates to a small value, indicating that the shift symmetry is recovered in the IR. This confirms the scenario above. However it turns out that Z_κ^φ, D_κ and μ_κ do not have a RG flow and keep their bare values $Z_\kappa^\varphi = D_\kappa = 1$ and $\mu_\kappa = \mu$. While in DP-C, this property was ensured by the symmetries of the model, for DEP this seems to be an accidental feature coming from the approximation. In fact, we show in Appendix E.1.3 that it is the case for the LPA' ansatz, even without truncating the potential at the condition that the latter is analytic.

As a consequence, within this ansatz, one cannot hope to reach a truly DEP fixed point for which $d_\varphi \neq d/2$ and which would explain some discrepancies between the theoretical predictions and numerical simulations. Furthermore, for values of the dimension out of the domain of convergence to the DP-C fixed point, as the DEP flow cannot escape the DP-C symmetries, it encounters the same problem

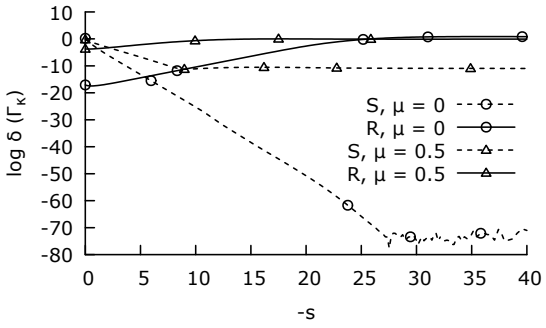

Figure 4.3: Logarithm of the Euclidean norm of the variation of $\hat{U}_\kappa$ under the rescaling (R) symmetry (4.60) (full line) and shift (S) one (4.66) (dashed line) as a function of minus the RG time, for $\mu = 0$ (circle) and $\mu = 0.5$ (triangle) and for $d = 3.2$.

as the DP-C ansatz and diverges. At this point, it is unclear how this problem could be cured. Looking at our choice of ansatz, maybe the most questionable step, after the expansion in frequency in the derivative expansion and the polynomial expansion of the potential, was the definition of the minimum configuration. Indeed, as mentioned in Sec. 4.3.2, this definition implicitely assumes that one can always follow continuously a zero of the function $\partial^2_{\psi\psi} U_\kappa|_{\rm Min}$ of χ_κ along the flow and that this root stays finite. However, if one examines in more detail Appendix E.1, it appears that it is in fact sufficient to have $\partial^2_{\psi\psi} U_\kappa|_{\rm Min} > 0$ for the proof of the accidental non-renormalization to go through. To have the property $\partial^2_{\psi\psi} U_\kappa|_{\rm Min} > 0$ for some χ_κ is necessary in order to be able to define a stationary state. In conclusion, within the LPA' it seems difficult to evade the conclusion of Appendix E.1.

The non-renormalization of the kinetic part lies crucially on the fact that the three- and four-point vertex functions do not have any frequency dependence. One possible lead would thus be to devise an ansatz for DEP which takes into account the frequency and wave-number dependence of higher order vertex functions. Maybe in conjunction with this approach, it would be fruitful to take as a starting point the DEP action in terms of the observable and response fields (4.45). Interestingly this action possesses already at its bare level functional dependence in the fields and wave-number dependence in a cubic term. On the one hand, these features complicate the NPRG treatment as they prevent from using simple approximations such as the LPA'. On the other hand, they could be the source of new physics which is missed by the simpler approximation available in the coherent field action. This study may be facilitated by the newfound symmetry of the response field action (4.57).

In summary, in this chapter we presented our take on the study of the absorbing phase transition of DEP and its coarse-grained counterpart, DP-C. An exhaustive analysis of the symmetries of both models allowed us to clarify the discussion as whether both models belong to the same universality class. On the one hand for the DP-C model, we have used a LPA' anstaz to obtain some new results, notably the indication of a continuous phase transition for $\mu > 0$, which is missed by perturbative RG at one loop. We have shown that the existence of a DP-C universality class in any dimension is not trivially reachable. It is still an open question if this difficulty would be alleviated by using a better suited NPRG ansatz or if it has

a physical origin. On the other hand for the DEP model, using the LPA' ansatz we have confirmed our scenario that the (global) shift symmetry emerges from the rescaling symmetry of DEP. However, we have also analytically shown how accidental non-renormalizations inherent to the LPA' ansatz chosen here prevented to infirm or confirm the perturbative scenario of equivalence between DEP and DP-C as universality classes.

The prospects in the study of this model include to understand better the role that the choice of a minimum configuration χ plays in the non-renormalization theorems and in the observed instabilities of the RG flow below a certain dimension This study should help to devise a more powerful ansatz for both models, such as LPA' with a functional dependence of the potential in well-chosen invariants. It may be worthwhile to also use as starting point the response field action (4.19), whose physical content is more straightforward. The ansatz in this case would exploit the uncovered tilt symmetry (4.57) which exists in this action at $\mu = 0$.

Version française

En résumé, dans ce chapitre, nous avons présenté notre point de vue sur l'étude de la transition de phase absorbante de DEP et de son homologue mésoscopique, DP-C. Une analyse exhaustive des symétries des deux modèles nous a permis de clarifier la discussion pour déterminer si les deux modèles appartiennent à la même classe d'universalité. D'une part pour le modèle DP-C, nous avons utilisé un anstaz nommé LPA' pour obtenir de nouveaux résultats, notamment l'indication d'une transition de phase continue pour $\mu > 0$, qui n'apparaît pas dans le RG perturbatif à une boucle. Nous avons montré que l'existence d'une classe d'universalité DP-C dans n'importe quelle dimension n'est pas trivialement accessible. La question reste ouverte de savoir si cette difficulté serait atténuée par l'utilisation d'un ansatz NPRG mieux adapté ou si elle a une origine physique. D'autre part, pour le modèle DEP, en utilisant le même type d'ansatz, nous avons confirmé notre scénario selon lequel la symétrie (globale) de DPC émerge de la symétrie de DEP. Cependant, nous avons aussi montré analytiquement comment les non-renormalisations accidentelles inhérentes à l'ansatz LPA' choisi ici empêchént d'infirmer ou de confirmer le scénario perturbatif de l'équivalence entre DEP et DP-C au niveau de la classes d'universalité.

Les perspectives futures dans l'étude de ce modèle incluent de mieux comprendre le rôle que joue le choix d'une configuration minimale χ dans les théorèmes de non-renormalisation et dans les instabilités observées du flot RG en-dessous d'une certaine dimension Cette étude devrait permettre de concevoir un ansatz plus flexible pour les deux modèles, tel que par exemple la LPA' avec une dépendance fonctionnelle du potentiel dans les invariants bien choisis. Il peut être utile d'utiliser également comme point de départ l'action du champ de réponse (4.19), dont le contenu physique est plus directement lisible. L'ansatz dans ce cas exploiterait la symétrie de "tilt" (4.57) qui a été découverte dans cette action à $\mu = 0$.

Chapter 5

Breaking of scale invariance in correlation functions of turbulence

In this chapter we turn to the study of the stochastic Navier-Stokes field theory. As in the previous chapter, we work in the framework of the NPRG. However, the tools used are completely different. We use a large wave-number expansion of the exact RG flow equation in order to investigate the time-dependence of (generalized) correlation functions in 2- and 3-D turbulence, and the possibility of intermittency in the direct cascade of 2-D turbulence.

In Sec. 5.1, the Lagrangian of the SNS field theory is given. We also specify the 2-D case, where the SNS field theory is in fact a scalar field theory. In Sec. 5.2, the exact and extended symmetries of SNS are listed in both formulations and we derive the corresponding Ward identities in Sec. 5.3. After having given a general introduction to the large wave-number expansion in Sec. 5.4, in Sec. 5.5, the large wave-number expansion and the obtained Ward identities are used to derive the leading time-dependence of the correlation functions. In Sec. 5.6, the leading order at equal times in the 2-D case is investigated and the consequences for intermittency corrections are discussed.

Version française

Dans ce chapitre, nous passons à l'étude de la théorie des champs de l'équation de Navier-Stokes stochastique. Comme dans le chapitre précédent, nous travaillons dans le cadre du formalisme du NPRG. Cependant, les outils utilisés sont complètement différents. Nous utilisons un développement à grand nombre d'ondes de l'équation de flot de RG exacte afin d'étudier la dépendance temporelle des fonctions de corrélation (généralisées) dans la turbulence en 2- et 3-D, et la possibilité d'intermittence dans la cascade directe en turbulence 2-D.

Dans la Sec. 5.1, le Lagrangien de la théorie des champs de SNS est donné. Nous décrivons en particuler le cas 2-D, où la théorie des champs SNS est en fait une théorie des champs scalaires. Dans la Sec. 5.2, les symétries exactes et étendues de SNS sont listées dans les deux formulations et nous dérivons les identités de Ward correspondantes dans la Sec. 5.3. Après avoir donné une introduction générale au développement à grand nombre d'ondes dans la Sec. 5.4, dans la Sec. 5.5, le développement à grand nombre d'ondes et les identités de Ward obtenues sont utilisées pour dériver la dépendance temporelle dominante des fonctions de corrélation.

Dans la Sec. 5.6, l'ordre dominant à temps égaux dans le cas 2-D est exploré et les conséquences pour la possibilité de corrections d'intermittences sont examinées.

5.1 The field theory of the stochastic Navier-Stokes equation

5.1.1 SNS action in the velocity formulation

First, let us start by recalling the setup of the SNS field theory given in Sec. 2.2. Our starting point is the Navier-Stokes equation, which reads component-by-component

$$\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha = \nu \partial^2 v_\alpha - \frac{1}{\rho}\partial_\alpha p + f_\alpha \,, \tag{5.1}$$

where $\vec{f}$ symbolizes additional external forces and with the additional constraint of incompressibility:

$$\partial_\alpha v_\alpha = 0 \,. \tag{5.2}$$

In order to study a turbulent stationary state, one has to add a forcing term $\vec{f}$ acting at large scales to simulate the addition of energy coming from the stirring. In this context, large scales means that the integral scale L at which energy is injected must be very large compared to the Kolmogorov scale η, at which viscous and convective effects are in competition. This condition implies that there exists a whole range of wave-number between L^{-1} and η^{-1} where the convection term dominates the dynamics. Because of the universality of turbulence in the inertial range, one can average over different realizations of the forcing. These stochastic forcings must share the following characteristics: their power spectrum must be concentrated on L^{-1}, such as not to spoil the universal properties of the turbulent flow at intermediary scales between η and L, and they must as well not impose a non-zero mean velocity to the flow. The weight is chosen to be Gaussian for convenience, with zero mean and covariance

$$\langle f_\alpha(\mathbf{x}) f_\beta(\mathbf{x}')\rangle \equiv D_{\alpha\beta}(\mathbf{x}-\mathbf{x}') = 2\,\delta_{\alpha\beta}\,\delta(t-t')N_{L^{-1}}(|\vec{x}-\vec{x}'|) \,. \tag{5.3}$$

The function $N_{L^{-1}}$ is chosen such that its Fourier transform is smooth, peaked at L^{-1}, is zero at zero wave-number and decays exponentially fast at large wave-number. Note that we do not need to impose the forcing to be solenoidal, as we explicitly enforce the incompressibility along each realization of the flow. As a consequence, the forcing can be chosen diagonal in component space without loss of generality. This setup is sufficient to study the Navier-Stokes equation in 3-D. However the situation is different in 2-D. In 2-D, the vorticity, defined as

$$\omega = \partial_1 v_2 - \partial_2 v_1 \,, \tag{5.4}$$

is conserved in a perfect fluid. As stressed in Sec. 2.2.4, this implies that the energy cascade is inverted, meaning that energy is transferred by the flow to larger and larger spatial scales. Thus, in order to reach a steady state, it is necessary to add a friction term to the Navier-Stokes equation in order to act as a sink for the energy. An example of such a term, called the Ekman friction, or Ekman drag, can be

derived by considering the 3-D space in which the 2-D flow is embedded. This gives a friction term acting in the 2-D bulk which reads simply

$$\vec{f}_{\text{Ekman}} = -\alpha\,\vec{v}\,. \tag{5.5}$$

As explained in Sec. 2.2.4, such term has a strong effect on the physics of the energy cascade. The presence of the Ekman friction gives intermittent corrections which are well understood (Boffetta et al. 2002): in this case, the velocity has Gaussian fluctuations and the vorticity plays the role of a passive scalar advected by the velocity field. The calculation of the intermittency corrections of the vorticity are related in this case to the calculation of the intermittency corrections in the Kraichnan model. In this work, we do not include an Ekman friction. Instead, we consider a non-local damping term acting at the scale L_0. It reads

$$\vec{f}_{\text{friction}}(\mathbf{x}) = -\int_{\vec{x}'} R_{L_0^{-1}}(|\vec{x}-\vec{x}'|)\vec{v}(t,\vec{x}')\,, \tag{5.6}$$

with the function $R_{L_0^{-1}}$ chosen such that its Fourier transform is smooth, decays exponentially fast for wave-number larger than L_0^{-1} and takes a finite value for wave-number going to zero at L_0 fixed. In other words, this term plays the role of a viscosity which would act only at the boundary of the flow to damp Fourier modes below L_0^{-1}. The stationary state of the modified Navier-Stokes equation with the stochastic forcing and the non-local friction defines our field theory of turbulence.

At this point, it is more convenient for the application of field theoretical tools to cast it into the form of a functional integral. This step is achieved using the MSRJD formalism. The general formula of the mapping is given in Appendix B.1. In the literature, the incompressibility constraint is enforced explicitly by replacing the pressure in (5.1) by its expression in term of the velocity field, given by the solution of the following Poisson equation:

$$\partial^2 p = -\rho\,\partial_\alpha v_\beta \partial_\beta v_\alpha\,. \tag{5.7}$$

However, following (Canet, Delamotte, and Wschebor 2016), we choose here another road and instead ensure the incompressibility condition (5.2) along the space-time trajectory of the velocity field. This is done by introducing a new field, that we name the response pressure and that we note $\bar{p}$, which acts as a Lagrange multiplier for (5.2), in analogy with the response velocity $\vec{\bar{v}}$ of the MSRJD formalism which is the Lagrange multiplier of the equation of motion (5.1). In this setting, the SNS partition function reads

$$\mathcal{Z}^{\text{v}}[\vec{J},\vec{\bar{J}},K,\bar{K}] = \int \mathcal{D}[\vec{v},\,p,\,\vec{\bar{v}},\,\bar{p}]\; e^{-\mathcal{S}^{\text{v}}[\vec{v},\vec{\bar{v}},p,\bar{p}]-\Delta\mathcal{S}^{\text{v}}[\vec{v},\vec{\bar{v}}]} e^{\int_{\mathbf{x}}\{\vec{J}\cdot\vec{v}+\vec{\bar{J}}\cdot\vec{\bar{v}}+Kp+\bar{K}\bar{p}\}}\,, \tag{5.8}$$

with respectively

$$\begin{aligned}
\mathcal{S}^{\text{v}}[\vec{v},\vec{\bar{v}},p,\bar{p}] &= \int_{\mathbf{x}} \Big\{ \bar{v}_\alpha(\mathbf{x})\big[\partial_t v_\alpha(\mathbf{x}) - \nu\nabla^2 v_\alpha(\mathbf{x}) + v_\beta(\mathbf{x})\partial_\beta v_\alpha(\mathbf{x}) + \frac{1}{\rho}\partial_\alpha p(\mathbf{x})\big] \\
&\qquad + \bar{p}(\mathbf{x})\,\partial_\alpha v_\alpha(\mathbf{x})\Big\} \\
\Delta\mathcal{S}^{\text{v}}[\vec{v},\vec{\bar{v}}] &= \int_{t,\vec{x},\vec{x}'} \Big\{ \bar{v}_\alpha(t,\vec{x}) R_{L_0^{-1}}(|\vec{x}-\vec{x}'|) v_\alpha(t,\vec{x}') \\
&\qquad - \bar{v}_\alpha(t,\vec{x}) N_{L^{-1}}(|\vec{x}-\vec{x}'|)\bar{v}_\alpha(t,\vec{x}')\Big\}\,.
\end{aligned} \tag{5.9}$$

We have chosen to set the quadratic, non-local part of the action apart for latter purpose. Let us make two comments for the more mathematicaly-oriented reader. First, note that the standard derivation of the MSRJD action for SNS implicitly assumes existence and unicity of weak solutions of the Navier-Stokes equations. This is a delicate point from a mathematical point of view, see for example the recent work of (Buckmaster and Vicol 2017). However, the assumption underlying the MSRJD derivation is a little weaker than strict uniqueness, since for a typical set of initial conditions, there may exist a set of velocity configurations spoiling unicity, as long as they are of zero measure with respect to the realizations of the forcing. Second, we have not made any mention of the spatial domain on which live the fields. In fact, in the following we always assume that the boundary terms which appear when doing integration by part can be discarded. This does not seem to be too strong of an assumption due to the presence of the damping term R.

5.1.2 Interpretation of $\Delta\mathcal{S}$ as a regulator

As explained in Chap. 3, a preliminary of the NPRG treatment is the choice of a well-suited regulator. Let us first examine the properties that it must satisfy. In the velocity formulation (5.9), the regulator is a priori a 8×8 matrix with entries $\vec{v}, \vec{\bar{v}}, p, \bar{p}$. In order to respect causality, the velocity-velocity sector must be set to 0, as seen in Sec. 3.5. Furthermore, because of a the particular symmetry of the action in the p and $\bar{p}$ sector, the pressure dependence of the EAA is equal to its value in the bare action $\mathcal{S}$ (as will be shown in the next section Sec. 5.3). As a consequence, there is no need to add regulator terms depending on p or $\bar{p}$. Moreover, because of the explicit incompressibility constraint along the trajectory, only the transverse part of the regulator will contribute. Thus it can be chosen diagonal in component space without loss of generality. Finally, in order not to break the symmetries of the action, it is easier to use a regulator diagonal in time and invariant under space translations and rotations. Gathering these observations, the form of the regulator must be

$$\Delta\mathcal{S}_\kappa^{\mathrm{v}} = \int_{t,\vec{x},\vec{x}'} \left\{ \bar{v}_\alpha(t,\vec{x}) \mathcal{R}_\kappa^{\bar{v}v}(|\vec{x}-\vec{x}'|) v_\alpha(t,\vec{x}') - \frac{1}{2} \bar{v}_\alpha(t,\vec{x}) \mathcal{R}_\kappa^{\bar{v}\bar{v}}(|\vec{x}-\vec{x}'|) \bar{v}_\alpha(t,\vec{x}') \right\}. \quad (5.10)$$

The sign before $\mathcal{R}_\kappa^{\bar{v}\bar{v}}$ is chosen according to the fact that the response field is imaginary.

In the Navier-Stokes action, Eq. (5.9), it turns out that terms which could play the role of the regulator are already present for physical reasons. Indeed, the functions $R_{L_0^{-1}}$ and $N_{L^{-1}}$ satisfy all the requirements to play the role of regulators of the theory. Their Fourier transforms are smooth functions, vanish exponentially fast for wave-numbers large compared to L_0^{-1} (resp. L^{-1}) and regularize the fluctuating field for small wave-numbers (Berges, Tetradis, and Wetterich 2002; Canet, Delamotte, and Wschebor 2016). Note here that although by construction $N_{L^{-1}}$ does not have an effect on the field in the IR, the presence of $R_{L_0^{-1}}$ is sufficient to do so in the SNS field theory. In other word, the SNS field theory is already well regulated.

We focus in this work in the direct cascade, that is in the behavior at wave-numbers much larger than both the integral scale L^{-1} and the friction scale L_0^{-1}. Thus, in the following, we identify each scale in both terms with the renormalization scale κ: $L^{-1} = L_0^{-1} = \kappa$. In this setting, the term $\Delta\mathcal{S}_\kappa$ of the SNS fits exactly as the

regulator (5.10) in the NPRG formalism. With the scales identified to κ, it reads

$$\Delta\mathcal{S}_\kappa^{\mathrm{v}}[\vec{v},\vec{\bar{v}}] = \int_{t,\vec{x},\vec{x}'} \Big\{ \bar{v}_\alpha(t,\vec{x}) R_\kappa(|\vec{x}-\vec{x}'|) v_\alpha(t,\vec{x}') - \bar{v}_\alpha(t,\vec{x}) N_\kappa(|\vec{x}-\vec{x}'|) \bar{v}_\alpha(t,\vec{x}') \Big\} . \tag{5.11}$$

Note that if we were to study the inverse cascade, which corresponds to wave-numbers between L_0^{-1} and L^{-1}, we would have to fix L and to reintegrate the non-local term N in the SNS action, while $L_0^{-1} = \kappa$ would be running.

Up to this point, the number of spatial dimensions needed not to be specified. Indeed, the action above is our starting point to derive results valid in any spatial dimensions d, and in particular for $d = 2$ and $d = 3$, which are the relevant ones for the study of physically realized turbulence. However, in $d = 2$, a simplification occurs. This is the subject of the next section.

5.1.3 Stream function formulation in 2-D

In 2-D, a solenoidal vector field can be written as the two-dimensional curl of a pseudo-scalar and not the curl of a vector as it would be the case in 3-D. Indeed, in 2-D, the Helmoltz decomposition for a generic vector field $\vec{v}$ reads

$$v_\alpha = \partial_\alpha \phi + \epsilon_{\alpha\beta}\partial_\beta \psi . \tag{5.12}$$

In this equation, ϕ is a scalar field and ψ a pseudo-scalar field. A pseudo-scalar is a one component field which behaves as scalar under rotations but change sign under mirror symmetries. Finally, the $\epsilon_{\alpha\beta}$'s are the components of the antisymmetric tensor with two indices and with $\epsilon_{12} = 1$. It verifies the following identity that we will use throughout this work:

$$\epsilon_{\alpha\gamma}\epsilon_{\beta\gamma} = \delta_{\alpha\beta} . \tag{5.13}$$

The incompressibility of $\vec{v}$ with adapted boundary conditions enforces that $\phi = 0$. In the problem of 2-D turbulence, ψ is usually called the stream function. It is related to the vorticity field through a Laplacian:

$$\omega = \epsilon_{\alpha\beta}\partial_\alpha v_\beta = \epsilon_{\alpha\beta}\epsilon_{\beta\gamma}\partial_\alpha\partial_\gamma\psi = -\partial^2\psi . \tag{5.14}$$

From Eq. (5.9), setting the sources K and $\bar{K}$ to zero and integrating on p and $\bar{p}$ shows that both $\vec{v}$ and $\vec{\bar{v}}$ are solenoidal. As a consequence, both fields can be expressed through their respective stream function:

$$\begin{aligned} v_\alpha &= \epsilon_{\alpha\beta}\partial_\beta\psi \\ \bar{v}_\alpha &= \epsilon_{\alpha\beta}\partial_\beta\bar{\psi} . \end{aligned} \tag{5.15}$$

Inserting the above expressions in the SNS action (5.9) yields

$$\begin{aligned} \mathcal{S}^{\mathrm{s}}[\psi,\bar{\psi}] &= \int_{\mathbf{x}} \partial_\alpha\bar{\psi}(\mathbf{x}) \Big[\partial_t\partial_\alpha\psi(\mathbf{x}) - \nu\nabla^2\partial_\alpha\psi(\mathbf{x}) + \epsilon_{\beta\gamma}\,\partial_\gamma\psi(\mathbf{x})\,\partial_\beta\partial_\alpha\psi(\mathbf{x}) \Big] \\ \Delta\mathcal{S}^{\mathrm{s}}[\psi,\bar{\psi}] &= \int_{t,\vec{x},\vec{x}'} \Big\{ \partial_\alpha\bar{\psi}(t,\vec{x}) R_{L_0^{-1}}(|\vec{x}-\vec{x}'|) \partial'_\alpha\psi(t,\vec{x}') \\ &\qquad - \partial_\alpha\bar{\psi}(t,\vec{x}) N_{L^{-1}}(|\vec{x}-\vec{x}'|) \partial'_\alpha\bar{\psi}(t,\vec{x}') \Big\} . \end{aligned} \tag{5.16}$$

Finally, we introduce the new partition function $\mathcal{Z}^{\mathrm{s}}[J,\bar{J}]$ as

$$\mathcal{Z}^{\mathrm{v}}[\vec{J},\vec{\bar{J}},0,0] = \mathcal{Z}^{\mathrm{s}}[-\epsilon_{\alpha\beta}\partial_\beta J_\alpha, -\epsilon_{\alpha\beta}\partial_\beta \bar{J}_\alpha]\,, \tag{5.17}$$

such that derivatives of $\mathcal{Z}^{\mathrm{s}}$ with respect to J and $\bar{J}$ give back moments of ψ and $\bar{\psi}$ respectively. It is worth pointing out that this action is often obtained by taking the curl of the SNS equation before casting it into a functional form (Olla 1991; Honkonen 1998; Mayo 2005). Here, this operation appears naturally as the consequence of the incompressibility constraint for $\vec{v}$. This shows that in 2-D, the velocity field action (5.9) and the stream function one (5.16) are equivalent. Now that the actions of SNS have been written down, we can look for their symmetries. However, before doing so, we have to spend some time on the nature of the non-local part of the action $\Delta\mathcal{S}$ within the NPRG framework.

5.2 Symmetries and extended symmetries of SNS

As in the study of the previous chapter on the DEP field theory, Chap. 4, we are not only interested in exact symmetries but also in extended symmetries, which give a variation linear in the fields. First, let us list them in the case of the velocity field action (5.9).

5.2.1 Extended symmetries in the velocity formulation

The exact symmetries of the SNS action (5.9) are familiar: space-time translations, space rotations and Galilean invariance. Its extended symmetries were studied in the context of the NPRG in (Canet, Delamotte, and Wschebor 2015; Canet, Delamotte, and Wschebor 2016). First, let us notice that both terms containing p and $\bar{p}$ are quadratic. Explicitly, one finds

$$\frac{\delta\mathcal{S}^{\mathrm{v}}[\vec{v},s\vec{v},p,\bar{p}]}{\delta p(\mathbf{x})} = -\frac{1}{\rho}\partial_\alpha\bar{v}_\alpha(\mathbf{x})\,, \quad \frac{\delta\mathcal{S}^{\mathrm{v}}[\vec{v},\vec{\bar{v}},p,\bar{p}]}{\delta\bar{p}(\mathbf{x})} = \partial_\alpha v_\alpha\,. \tag{5.18}$$

Second, we have mentioned in Sec. 2.2 that the SNS action also possesses a time-gauged Galilean symmetry. This time-gauged Galilean symmetry is well-known in the context of field theoretical studies of turbulence (De Dominicis and Martin 1979; Adzhemyan, Antonov, and Kim 1994; Adzhemyan, Antonov, and Vasil'ev 1999; Antonov, Borisenok, and Girina 1996; Berera and Hochberg 2007). It corresponds to the following field transformation

$$\begin{aligned} \delta v_\alpha(\mathbf{x}) &= -\dot{\epsilon}_\alpha(t) + \eta_\beta(t)\partial_\beta v_\alpha(\mathbf{x})\,, \quad \delta p(\mathbf{x}) = \eta_\beta(t)\partial_\beta p(\mathbf{x})\,, \\ \delta\bar{v}_\alpha(\mathbf{x}) &= \eta_\beta(t)\partial_\beta\bar{v}_\alpha(\mathbf{x})\,, \quad \delta\bar{p}(\mathbf{x}) = \eta_\beta(t)\partial_\beta\bar{p}(\mathbf{x})\,, \end{aligned} \tag{5.19}$$

where $\vec{\eta}(t)$ is an infinitesimal function of time, and $\dot{\eta}_\alpha = \partial_t\eta_\alpha$. The special case of a constant $\vec{\eta}$ corresponds to a translation in space, and the linear case $\vec{\eta}(t) = \vec{\eta}\,t$ to the usual (non-gauged) Galilean symmetry. Let us calculate the variation of the SNS action (5.9) under (5.19). The action is invariant under time-independent translations, thus the only non-zero variation must come from the time-derivative

in the action hitting on $\vec{\eta}$ and from the additional shift in the velocity $-\dot{\eta}_\alpha(t)$. The terms of the action (5.9) undergoing a variation are

$$\begin{aligned}\partial_t v_\alpha(\mathbf{x}) &\to -\ddot{\eta}_\alpha(t) + \dot{\eta}_\beta(t)\partial_\beta v_\alpha(\mathbf{x}) \\ v_\beta(\mathbf{x})\partial_\beta v_\alpha(\mathbf{x}) &\to -\dot{\eta}_\beta(t)\partial_\beta v_\alpha(\mathbf{x}) \\ \Delta\mathcal{S}^{\mathrm{v}}[\vec{v},\vec{\bar{v}}] &\to -\int_{t,\vec{x},\vec{x}'} \bar{v}_\alpha(t,\vec{x}) R_{L_0^{-1}}(|\vec{x}-\vec{x}'|)\eta_\alpha(t)\,. \end{aligned} \tag{5.20}$$

Plugging them back into the variation of the action, one obtains

$$\begin{aligned}&\mathcal{S}^{\mathrm{v}} + \Delta\mathcal{S}^{\mathrm{v}} \to \mathcal{S}^{\mathrm{v}} + \delta_\eta(\mathcal{S}^{\mathrm{v}} + \Delta\mathcal{S}^{\mathrm{v}}) + o(\|\bar{\eta}\|) \\ &\delta_\eta\big(\mathcal{S}^{\mathrm{v}} + \Delta\mathcal{S}^{\mathrm{v}}\big) = -\int_{\mathbf{x}} \bar{v}_\alpha(\mathbf{x})\ddot{\eta}_\alpha(t) - \int_{t,\vec{x},\vec{x}'} \bar{v}_\alpha(t,\vec{x}') R_{L_0^{-1}}(|\vec{x}-\vec{x}'|)\dot{\eta}_\alpha(t)\,. \end{aligned} \tag{5.21}$$

The variation of the action under (5.19) is thus non-zero, but linear in the fields.

Third, another extended symmetry of the SNS action was identified in (Canet, Delamotte, and Wschebor 2015). It is obtained by a simultaneous shift of both response fields. Let us perform the following space-time gauged change of variable in the action:

$$\delta\bar{v}_\alpha(\mathbf{x}) = \bar{\eta}_\alpha(\mathbf{x})\,, \quad \delta\bar{p}(\mathbf{x}) = v_\beta(\mathbf{x})\bar{\eta}_\beta(\mathbf{x})\,. \tag{5.22}$$

The corresponding variation of the action reads

$$\begin{aligned}\delta_\eta\big(\mathcal{S}^{\mathrm{v}} + \Delta\mathcal{S}^{\mathrm{v}}\big) = &\int_{\mathbf{x}} \bar{\eta}_\alpha(\mathbf{x})\Big\{\partial_t v_\alpha(\mathbf{x}) - \nu\nabla^2 v_\alpha(\mathbf{x}) + \partial_\beta\big[v_\alpha(\mathbf{x})v_\beta(\mathbf{x})\big] + \frac{1}{\rho}\partial_\alpha p(\mathbf{x})\Big\} \\ &+ \int_{t,\vec{x},\vec{x}'} \bar{\eta}_\alpha(t,\vec{x})\Big\{R_{L_0^{-1}}(|\vec{x}-\vec{x}'|)v_\alpha(t,\vec{x}') - 2\,N_{L^{-1}}(|\vec{x}-\vec{x}'|)\bar{v}_\alpha(t,\vec{x}')\Big\}\,. \end{aligned} \tag{5.23}$$

The non-linear (thus problematic) term of this variation, which comes from the interaction, may vanish for particular space dependence of $\vec{\eta}$. The most simple choice is to take

$$\eta_\alpha(\mathbf{x}) = \eta_\alpha(t) \tag{5.24}$$

independent of space. In this case, all the terms in curly braces vanish by integration by parts except the time-derivative and one is left with a variation of the action linear in the fields. We found that one can also choose

$$\eta_\alpha(\mathbf{x}) = \epsilon_{\alpha\beta\gamma} x_\beta \eta_\gamma(t)\,, \tag{5.25}$$

where the $\epsilon_{\alpha\beta\gamma}$ are the components of the Levi-Civita fully antisymmetric tensor with three indices and with $\epsilon_{123} = 1$. Once again, the last three terms in curly braces vanish after integration by parts, and this time using also the antisymmetry of $\epsilon_{\alpha\beta\gamma}$. We have not identified higher order space dependence which gives a variation of the action linear in the fields.

5.2.2 Extended symmetries of the stream function action

Now, let us do the same for the stream function action. First, due to the definition (5.15) of the stream function and its response field as space primitives, these fields are defined up to a constant of space. As the functional integration does not fix this

gauge invariance, the partition function is invariant under the following change of variable

$$\begin{aligned}\psi(\mathbf{x}) &\to \psi(\mathbf{x}) + \eta(t)\,,\\ \bar{\psi}(\mathbf{x}) &\to \bar{\psi}(\mathbf{x}) + \bar{\eta}(t)\,,\end{aligned} \tag{5.26}$$

for two functions η and $\bar{\eta}$.

Second, let us recapitulate all the extended symmetries of SNS listed in the previous section, that can be reformulated for the stream function action of (5.16). To begin with, the time-gauged Galilean symmetry reads in the stream function variables as follows

$$\begin{aligned}\psi &\to \psi + \epsilon_{\alpha\beta}x_\alpha\dot{\eta}_\beta(t) + \eta_\alpha(t)\partial_\alpha\psi\\ \bar{\psi} &\to \bar{\psi} + \eta_\alpha(t)\partial_\alpha\bar{\psi}\,.\end{aligned} \tag{5.27}$$

Let us calculate the variation of the action (5.16) As the action is invariant under translations of space independent of time, the only variation of the action comes from the time derivative hitting on $\vec{\eta}$ and from the shift in ψ. Moreover, as the shift in ψ is linear in space, the only term where it does not vanish is the interaction term:

$$\delta_\eta(\mathcal{S}^{\rm s} + \Delta\mathcal{S}^{\rm s}) = \int_{\mathbf{x}} \partial_\alpha\bar{\psi}\Big[\epsilon_{\alpha\beta}\ddot{\eta}_\beta(t) + \dot{\eta}_\beta(t)\partial_\beta\partial_\alpha\psi + \epsilon_{\alpha\beta}\epsilon_{\beta\gamma}\dot{\eta}_\gamma(t)\partial_\beta\partial_\alpha\psi\Big] = 0\,, \tag{5.28}$$

where the last two terms cancel each other using $\epsilon_{\alpha\beta}\epsilon_{\beta\gamma} = -\delta_{\alpha\gamma}$. Interestingly this change of variables makes $\delta(\mathcal{S}^{\rm s} + \Delta\mathcal{S}^{\rm s})$ vanish, so it corresponds to an exact symmetry in this formulation while it was an extended symmetry in the velocity formulation. We conjecture that this apparent paradox would disappear if we were to fix the gauge degree of freedom remaining in the stream function action. However we do not have checked this explicitly.

Now, let us investigate the effect of a space-time gauged shift of the response field on the action

$$\begin{aligned}\delta_\eta(\mathcal{S}^{\rm s} + \Delta\mathcal{S}^{\rm s}) = -\int_{\mathbf{x}} \bar{\eta}(\mathbf{x})\Bigg\{&\partial^2(\partial_t - \nu\partial^2)\psi + \partial_\alpha\partial_\beta(\epsilon_{\beta\gamma}\partial_\gamma\psi\partial_\alpha\psi)\\ &+ \int_{\vec{x}'}\Big[R_{L_0^{-1}}(|\vec{x}-\vec{x}'|)\partial'^2\psi(t,\vec{x}') - 2N_{L^{-1}}(|\vec{x}-\vec{x}'|)\partial'^2\bar{\psi}(t,\vec{x}')\Big]\Bigg\}\,.\end{aligned} \tag{5.29}$$

The choice $\bar{\eta}(\mathbf{x}) = \bar{\eta}(t)$ is simply the gauge-invariance of ψ and $\bar{\psi}$ discussed above and in that case $\delta_\eta(\mathcal{S}^{\rm s} + \Delta\mathcal{S}^{\rm s}) = 0$. The choice $\bar{\eta}(\mathbf{x}) = x_\alpha\bar{\eta}_\alpha(t)$ corresponds to the pure time-gauged shift in the velocity formulation (5.24) and the corresponding variation of the action is zero. It is thus also an exact symmetry in this formulation.

The choice $\bar{\eta}(\mathbf{x}) = \frac{x^2}{2}\bar{\eta}(t)$ gives

$$\begin{aligned}
&\delta_\eta(\mathcal{S}^{\mathrm{s}} + \Delta\mathcal{S}^{\mathrm{s}}) \\
&\quad = -\int_{\mathbf{x}} 2\bar{\eta}(\mathbf{x})\Bigg\{\partial_t\psi + \epsilon_{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi \\
&\quad + \int_{\vec{x}'}\Big[R_{L_0^{-1}}(|\vec{x}-\vec{x}'|)\psi(t,\vec{x}') - 2N_{L^{-1}}(|\vec{x}-\vec{x}'|)\bar{\psi}(t,\vec{x}')\Big]\Bigg\} \\
&\quad = -\int_{\mathbf{x}} 2\bar{\eta}(\mathbf{x})\Bigg\{\partial_t\psi + \int_{\vec{x}'}\Big[R_{L_0^{-1}}(|\vec{x}-\vec{x}'|)\psi(t,\vec{x}') - 2N_{L^{-1}}(|\vec{x}-\vec{x}'|)\bar{\psi}(t,\vec{x}')\Big]\Bigg\},
\end{aligned} \tag{5.30}$$

where the variation coming from the interaction cancels by antisymmetry of $\epsilon_{\alpha\beta}$. This choice corresponds to the shift linear in space in the velocity formulation (5.25) and the corresponding variation of the stream action is linear in the fields as well.

Third, the SNS action enjoys a supplementary extended symmetry in 2-D, which is more straightforward in the stream function formulation. This new symmetry can be understood as a time-gauged rotation in the same way as extended Galilean symmetry is a time-gauged translation in space. The corresponding change of variable reads

$$\begin{aligned}
\psi &\to \psi - \frac{x^2}{2}\dot{\eta}(t) + \eta(t)\epsilon_{\alpha\beta}x_\beta\partial_\alpha\psi \\
\bar{\psi} &\to \bar{\psi} + \eta(t)\epsilon_{\alpha\beta}x_\beta\partial_\alpha\bar{\psi}\,.
\end{aligned} \tag{5.31}$$

To check that it is a symmetry, we use the same line of reasoning as for the extended Galilean invariance above. As the action is invariant under rotations independent of time, the only variation comes from the time derivative hitting on η and from the shift of ψ:

$$\begin{aligned}
\delta_\eta(\mathcal{S}^{\mathrm{s}} + \Delta\mathcal{S}^{\mathrm{s}}) &= \int_{\mathbf{x}}\Bigg\{\partial_\alpha\bar{\psi}\Big[-\ddot{\eta}(t)x_\alpha + \dot{\eta}(t)\epsilon_{\beta\alpha}\partial_\beta\psi + \dot{\eta}(t)\epsilon_{\beta\gamma}x_\gamma\partial_\beta\partial_\alpha\psi \\
&\quad - \dot{\eta}(t)\epsilon_{\beta\gamma}x_\gamma\partial_\beta\partial_\alpha\psi - \dot{\eta}(t)\epsilon_{\alpha\beta}\partial_\beta\psi\Big] - \int_{\vec{x}'}\partial_\alpha\bar{\psi}(t,\vec{x})R_{L_0^{-1}}(|\vec{x}-\vec{x}'|)x'_\alpha\dot{\eta}(t)\Bigg\} \\
&= \int_{\mathbf{x}}\bar{\psi}\Big[2\ddot{\eta}(t) + 2\dot{\eta}(t)\epsilon_{\alpha\beta}\partial_\alpha\partial_\beta\psi + 2\dot{\eta}(t)\bar{\psi}(t,\vec{x})\int_{\vec{x}'}R_{L_0^{-1}}(|\vec{x}-\vec{x}'|)\Big] \\
&= 2\int_{\mathbf{x}}\eta(t)\Big[\partial_t^2\bar{\psi} - \partial_t\bar{\psi}\int_{\vec{x}'}R_{L_0^{-1}}(|\vec{x}-\vec{x}'|)\Big]\,.
\end{aligned} \tag{5.32}$$

Thus the variation is linear in the fields. This symmetry can be written in the velocity formulation as well, at the cost of having to deal with a non-local shift of the pressure. However this additional extended symmetry is specific to 2-D. To our knowledge, this symmetry has never been explicitly identified before in the literature.

To recapitulate, the following infinitesimal change of variables are extended sym-

metries of the SNS action in 2-D, in the stream function formulation:

$$\begin{array}{lll} a) & \delta\psi = \eta(t)\,, & \delta\bar{\psi} = \bar{\eta}(t) \\ b) & \delta\psi = 0\,, & \delta\bar{\psi} = x_\alpha\bar{\eta}_\alpha(t) \\ c) & \delta\psi = 0\,, & \delta\bar{\psi} = \frac{x^2}{2}\bar{\eta}(t) \\ d) & \delta\psi = \epsilon_{\alpha\beta}x_\alpha\dot{\eta}_\beta(t) + \eta_\alpha(t)\partial_\alpha\psi\,, & \delta\bar{\psi} = \eta_\alpha(t)\partial_\alpha\bar{\psi} \\ e) & \delta\psi = -\dot{\eta}(t)\frac{x^2}{2} + \eta(t)\epsilon_{\alpha\beta}x_\beta\partial_\alpha\psi\,, & \delta\bar{\psi} = \eta(t)\epsilon_{\alpha\beta}x_\beta\partial_\alpha\bar{\psi} \end{array} \quad . \tag{5.33}$$

Following the same methodology as in Chap. 4, the extended symmetries of the velocity action and the stream function one are translated into Ward identities in the next section.

5.3 Ward identities for the field theory of SNS

In this section, we use the symmetries deduced in the previous sections to derive constraints on the EAA in the form of Ward identities. In essence, they state that for linear changes of variables, the variation of the EAA is equal to the mean of the variation of the action. The general formula was established in Sec. 3.6:

$$\delta_\eta\Gamma_\kappa[\Phi] = \langle\delta_\eta\mathcal{S}\rangle_{\rm j}\,, \tag{5.34}$$

where $\delta_\eta X$ is the linear variation of the functional X under an infinitesimal change of variables of parameter η. This formula applies both for $\mathcal{S}^{\rm v}$ and $\mathcal{S}^{\rm s}$, with their respective set of averaged fields Φ, and corresponding currents j. For the above formula to hold in the presence of a regulator, one of the following conditions has to be satisfied: either the change of variable is a shift, or it leaves the regulator invariant. The only changes of variables considered above which are not pure shifts are the time-gauged Galilean and rotation symmetries. The regulator is invariant under translations and rotations by design. Furthermore, it has been chosen delta-correlated in time. As a consequence, it is also invariant under the time-gauged versions of both symmetries. In conclusion, the above formula holds in all cases.

Before writing down these identities, let us define our notations: the averaged fields are noted respectively

$$\Phi = \{\vec{u}, \vec{\bar{u}}, \pi, \bar{\pi}\}\,, \tag{5.35}$$

in the velocity formulation, and

$$\Phi = \{\Psi, \bar{\Psi}\}\,, \tag{5.36}$$

in the stream function one. Furthermore, we choose not to differentiate the EAA corresponding to the respective formulations, as the context is enough to lift any ambiguities.

5.3.1 Ward identities in the velocities formulation

Let us begin with the Ward identities related to the pressure and response pressure shift. Using the formulas (5.18), they read respectively

$$\begin{aligned} \int_{\mathbf{x}} \eta(\mathbf{x})\frac{\delta\Gamma_\kappa}{\delta\pi(\mathbf{x})} &= -\int_{\mathbf{x}} \eta(\mathbf{x})\frac{1}{\rho}\partial_\alpha\bar{u}_\alpha(\mathbf{x}) \\ \int_{\mathbf{x}} \eta(\mathbf{x})\frac{\delta\Gamma_\kappa}{\delta\bar{\pi}(\mathbf{x})} &= \int_{\mathbf{x}} \eta(\mathbf{x})\partial_\alpha u_\alpha\,. \end{aligned} \tag{5.37}$$

Taking a derivative with respect to η, we obtain

$$\frac{\delta\Gamma_\kappa}{\delta\pi(\mathbf{x})} = -\frac{1}{\rho}\partial_\alpha \bar{u}_\alpha(\mathbf{x})\,, \qquad \text{and } \frac{\delta\Gamma_\kappa}{\delta\bar{\pi}(\mathbf{x})} = \partial_\alpha u_\alpha\,, \tag{5.38}$$

or equivalently in an integrated form:

$$\Gamma_\kappa[\vec{u},\vec{\bar{u}},\pi,\bar{\pi}] = \int_{\mathbf{x}} \left\{ \frac{1}{\rho}\bar{u}_\alpha(\mathbf{x})\partial_\alpha\pi(\mathbf{x}) + \bar{\pi}(\mathbf{x})\partial_\alpha u_\alpha \right\} + \Gamma_\kappa[\vec{u},\vec{\bar{u}},0,0]\,. \tag{5.39}$$

Thus, these symmetries simply state that the pressure/response pressure sector is not renormalized. Thus in the following, we will consider the reduced EAA depending only on $\vec{u}$ and $\vec{\bar{u}}$.

First, the Ward identity corresponding to the time-gauged Galilean symmetry (5.19) reads

$$\int_{\vec{x}} \left\{ \left(\delta_{\alpha\beta}\partial_t + \partial_\beta u_\alpha\right)\frac{\delta\Gamma_\kappa}{\delta u_\alpha} + \partial_\beta \bar{u}_\alpha \frac{\delta\Gamma_\kappa}{\delta \bar{u}_\alpha} \right\} = -\int_{\vec{x}} \partial_t^2 \bar{u}_\beta\,.$$

As the small parameter of the change of variable is only a function of time, the corresponding identity is only local in time. By taking functional derivatives of this identity with respect to velocity and response velocity fields, and setting the fields to zero, one can derive exact identities for the Fourier transforms of 1-PI generalized correlation functions. These identites are derived in Appendix F.1.1 and reads for $(m,n) \neq (0,1)$

$$\tilde{\Gamma}^{(m+1,n)}_{\alpha\alpha_1\dots\alpha_{m+n}}(\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\leq\ell\leq m+n})\Big|_{\vec{q}=0} = \tilde{\mathcal{D}}_\alpha(\varpi)\tilde{\Gamma}^{(m,n)}_{\alpha_1\dots\alpha_{m+n}}(\{\mathbf{p}_\ell\}_{1\leq\ell\leq m+n})\,. \tag{5.40}$$

with $\tilde{\mathcal{D}}_\alpha(\varpi)$ an operator acting on functions of n frequencies and wave-vectors as the following:

$$\tilde{\mathcal{D}}_\alpha(\varpi)F(\{\mathbf{p}_\ell\}_{1\leq\ell\leq n}) \;\; = -\sum_{k=1}^{n} p_k^\alpha \frac{F(\{\mathbf{p}_\ell\}_{1\leq\ell\leq k-1},\,\omega_k+\varpi,\vec{p}_k,\{\mathbf{p}_\ell\}_{k+1\leq\ell\leq n})}{\varpi} \tag{5.41}$$

The action of $\tilde{\mathcal{D}}_\alpha(\omega)$ is hence to successively shift the frequencies of the function on which it acts and to multiply by the corresponding wave-number. The identity (5.40) exactly relates an arbitrary $(m+1,n)$-point vertex function with one vanishing wave-vector carried by a velocity field u_α to a lower-order (m,n)-point vertex function. To check that the above expression has a well-defined limit when $\varpi \to 0$, let us use the constraint coming from invariance under-space time translations:

$$\tilde{\Gamma}^{(m,n)}_{\alpha_1\dots\alpha_{m+n}}(\{\mathbf{p}_\ell\}_{1\leq\ell\leq m+n}) = (2\pi)^{d+1}\delta\left(\sum\omega_i\right)\delta^d\left(\sum\vec{p}_i\right)\bar{\Gamma}^{(m,n)}_{\alpha_1\dots\alpha_{m+n}}(\{\mathbf{p}_\ell\}_{1\leq\ell\leq m+n-1})\,, \tag{5.42}$$

to write the corresponding identity in term of the $\bar{\Gamma}^{(n)}_\kappa$:

$$\bar{\Gamma}^{(m+1,n)}_{\alpha\alpha_1\dots\alpha_{m+n}}(\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\leq\ell\leq m+n-1})\Big|_{\vec{q}=0} = \mathcal{D}_\alpha(\varpi)\bar{\Gamma}^{(m,n)}_{\alpha_1\dots\alpha_{m+n}}(\{\mathbf{p}_\ell\}_{1\leq\ell\leq m+n-1})\,, \tag{5.43}$$

where now

$$\begin{aligned}&\mathcal{D}_\alpha(\varpi)F(\{\mathbf{p}_\ell\}_{1\leq\ell\leq n})\\ &= -\sum_{k=1}^{n} p_k^\alpha \frac{F(\{\mathbf{p}_\ell\}_{1\leq\ell\leq k-1},\,\omega_k+\varpi,\vec{p}_k,\{\mathbf{p}_\ell\}_{k+1\leq\ell\leq n}) - F(\{\mathbf{p}_\ell\}_{1\leq\ell\leq n})}{\varpi}\,.\end{aligned} \tag{5.44}$$

In this case, $\mathcal{D}(\varpi)$ appears explicitly as a finite difference operator and

$$\mathcal{D}_\alpha F(\{\mathbf{p}_\ell\}_{1\le\ell\le n}) \equiv \lim_{\varpi\to 0} \mathcal{D}_\alpha(\varpi) F(\{\mathbf{p}_\ell\}_{1\le\ell\le n}) \equiv \Big(-\sum_{k=1}^{n} p_k^\alpha \partial_{\omega_k} \Big) F(\{\mathbf{p}_\ell\}_{1\le\ell\le n}) \,. \tag{5.45}$$

Now, let us turn to the Ward identities corresponding to the shifts of the response fields (5.22), respectively independent of space (5.24) or linear in space (5.25). The first one gives

$$\int_{\vec{x}} \Big\{ \frac{\delta\Gamma_\kappa}{\delta\bar{u}_\beta} + u_\beta \partial_\gamma u_\gamma \Big\} = \int_{\vec{x}} \partial_t u_\beta \,, \tag{5.46}$$

where we have used (5.38) to re-express the pressure term (Canet, Delamotte, and Wschebor 2015). Taking functional derivatives with respect to velocity and response velocity fields and evaluating at zero fields, one can deduce again exact identities for vertex functions (Canet, Delamotte, and Wschebor 2016). They give the expression of any $\tilde{\Gamma}^{(m,n)}$ with one vanishing wave-vector carried by a response velocity, which reads

$$\tilde{\Gamma}^{(m,n+1)}_{\alpha_1\dots\alpha_m\alpha\alpha_{m+1}\dots\alpha_n}(\{\mathbf{p}_\ell\}_{1\le\ell\le m}, \varpi, \vec{q}, \{\mathbf{p}_\ell\}_{1\le\ell\le n})\Big|_{\vec{q}=0} = 0 \tag{5.47}$$

for all (m,n) except for the two of lowest order, which keep their bare form. Using (5.42), they read

$$\begin{aligned} \bar{\Gamma}^{(1,1)}_{\alpha\beta}(\omega,\vec{0}) &= i\omega\delta_{\alpha\beta}, \\ \bar{\Gamma}^{(2,1)}_{\alpha\beta\gamma}(\omega_1,\vec{p}_1,\omega_2,-\vec{p}_1) &= ip_1^\alpha\delta_{\beta\gamma} - ip_1^\beta\delta_{\alpha\gamma} \,. \end{aligned} \tag{5.48}$$

For completeness, we write the Ward identities corresponding to the shift linear in space, although it will not be used in the following. The Ward identity reads

$$\int_{\vec{x}} \epsilon_{\alpha\beta\gamma} x_\beta \Big\{ \frac{\delta\Gamma_\kappa}{\delta\bar{u}_\alpha} + u_\alpha \partial_\sigma u_\sigma \Big\} = \int_{\vec{x}} \epsilon_{\alpha\beta\gamma} x_\beta \partial_t u_\alpha \,, \tag{5.49}$$

The corresponding identity for the vertex functions reads this time

$$\Big(\delta_{\alpha\gamma}\frac{\partial}{\partial q^\beta} - \delta_{\beta\gamma}\frac{\partial}{\partial q^\alpha}\Big) \tilde{\Gamma}^{(m,n+1)}_{\alpha_1\dots\alpha_m\gamma\alpha_{m+1}\dots\alpha_n}(\{\mathbf{p}_\ell\}_{1\le\ell\le m}, \varpi, \vec{q}, \{\mathbf{p}_\ell\}_{1\le\ell\le n})\Big|_{\vec{q}=0} = 0 \,, \tag{5.50}$$

for all (m,n) except for the vertex function corresponding to the bare vertex, which reads

$$\Big(\delta_{\alpha\gamma}\frac{\partial}{\partial q^\beta} - \delta_{\beta\gamma}\frac{\partial}{\partial q^\alpha}\Big) \bar{\Gamma}^{(2,1)}_{\mu\nu\gamma}(\omega_1,\vec{p}_1+\vec{q},\omega_2,-\vec{p}_1)\Big|_{\vec{q}=0} = -i(\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}) \,. \tag{5.51}$$

Now, let us turn to the specific case of 2-D, in the stream function formulation.

5.3.2 Ward identities for the SNS field theory in 2-D

The list of Ward identities can be read from the list of symmetries (5.33):

$$
\begin{aligned}
&a) \quad \int_{\vec{x}} \frac{\delta\Gamma_\kappa}{\delta\Psi(\mathbf{x})} = 0 \text{ and } \int_{\vec{x}} \frac{\delta\Gamma_\kappa}{\delta\bar{\Psi}(\mathbf{x})} = 0 \\
&b) \quad \int_{\vec{x}} x_\alpha \frac{\delta\Gamma_\kappa}{\delta\bar{\Psi}(\mathbf{x})} = 0 \\
&c) \quad \int_{\vec{x}} \frac{x^2}{2} \frac{\delta\Gamma_\kappa}{\delta\bar{\Psi}(\mathbf{x})} = -2\int_{\vec{x}} \partial_t\Psi \\
&d) \quad \int_{\vec{x}} \Big\{ \big(-\epsilon_{\beta\alpha} x_\beta \partial_t + \partial_\alpha\Psi \big) \frac{\delta\Gamma_\kappa}{\delta\Psi(\mathbf{x})} + \partial_\alpha\bar{\Psi} \frac{\delta\Gamma_\kappa}{\delta\bar{\Psi}(\mathbf{x})} \Big\} = 0 \\
&e) \quad \int_{\vec{x}} \Big\{ \big(\frac{x^2}{2}\partial_t + \epsilon_{\alpha\beta} x_\beta \partial_\alpha\Psi \big) \frac{\delta\Gamma_\kappa}{\delta\Psi(\mathbf{x})} + \epsilon_{\alpha\beta} x_\beta \partial_\alpha\bar{\Psi} \frac{\delta\Gamma_\kappa}{\delta\bar{\Psi}(\mathbf{x})} \Big\} = 2\int_{\vec{x}} \partial_t^2\bar{\Psi} \,.
\end{aligned}
\tag{5.52}
$$

Now, each of these functional identities can be transformed into a hierarchy of identities for the vertex functions, which are more easily written in Fourier space:

$$
\begin{aligned}
&a) \quad \tilde{\Gamma}_\kappa^{(m,n)}(\ldots,\varpi,\vec{q},\ldots)\Big|_{\vec{q}=0} = 0 \\
&b) \quad \frac{\partial}{\partial q^i}\tilde{\Gamma}_\kappa^{(m,n+1)}(\{\mathbf{p}_\ell\}_{1\le\ell\le m},\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\le\ell\le n})\Big|_{\vec{q}=0} = 0 \\
&c) \quad \frac{\partial^2}{\partial q^2}\tilde{\Gamma}_\kappa^{(m,n+1)}(\{\mathbf{p}_\ell\}_{1\le\ell\le m},\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\le\ell\le n})\Big|_{\vec{q}=0} = 0 \\
&\qquad \text{except } \frac{\partial^2}{\partial q^2}\tilde{\Gamma}_\kappa^{(1,1)}(\varpi',\vec{q}',\varpi,\vec{q})\Big|_{\vec{q}=0} = 4i\varpi\delta^d(\vec{q}')\delta(\varpi+\varpi') \\
&d) \quad \frac{\partial}{\partial q^i}\tilde{\Gamma}_\kappa^{(m+1,n)}(\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\le\ell\le m+n})\Big|_{\vec{q}=0} = i\epsilon_{\alpha\beta}\tilde{\mathcal{D}}_\beta(\varpi)\tilde{\Gamma}_\kappa^{(m,n)}(\{\mathbf{p}_\ell\}_{1\le\ell\le m+n}) \\
&e) \quad \frac{\partial^2}{\partial q^2}\tilde{\Gamma}_\kappa^{(m+1,n)}(\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\le\ell\le m+n})\Big|_{\vec{q}=0} = \tilde{\mathcal{R}}(\varpi)\tilde{\Gamma}_\kappa^{(m,n)}(\{\mathbf{p}_\ell\}_{1\le\ell\le m+n}) \\
&\qquad \text{except } \frac{\partial^2}{\partial q^2}\tilde{\Gamma}_\kappa^{(1,1)}(\varpi,\vec{q},\varpi',\vec{q}')\Big|_{\vec{q}=0} = -4i\varpi\delta^d(\vec{q}')\delta(\varpi+\varpi')
\end{aligned}
\tag{5.53}
$$

With the following definition for the new operator $\tilde{\mathcal{R}}(\varpi)$:

$$
\tilde{\mathcal{R}}(\varpi)F(\{\mathbf{p}_\ell\}_{1\le\ell\le n}) \equiv \left(\frac{2i\epsilon_{\alpha\beta}}{\varpi} \sum_{k=1}^{n} p_k^\alpha \frac{\partial}{\partial p_k^\beta} \right. \frac{F(\{\mathbf{p}_\ell\}_{1\le\ell\le k-1},\, \omega_k+\varpi, \vec{p}_k, \{\mathbf{p}_\ell\}_{k+1\le\ell\le n}) - F(\{\mathbf{p}_\ell\}_{1\le\ell\le n})}{\varpi} \,. \tag{5.54}
$$

The derivation of the identities d) and e) are a bit lengthy and are reported in Appendix F.1.2. The same identities can be written in terms of the $\bar{\Gamma}_\kappa^{(m,n)}$ by extracting the delta functions of conservation of momenta and frequencies to get fully smooth

functions. The identities read as follows

$$
\begin{aligned}
&a) \quad \bar{\Gamma}_\kappa^{(m,n)}(\dots,\varpi,\vec{q},\dots)\Big|_{\vec{q}=0} = 0 \\
&b) \quad \frac{\partial}{\partial q^i}\bar{\Gamma}_\kappa^{(m,n+1)}(\{\mathbf{p}_\ell\}_{1\le\ell\le m},\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\le\ell\le n-1})\Big|_{\vec{q}=0} = 0 \\
&c) \quad \frac{\partial^2}{\partial q^2}\tilde{\Gamma}_\kappa^{(m,n+1)}(\{\mathbf{p}_\ell\}_{1\le\ell\le m},\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\le\ell\le n-1})\Big|_{\vec{q}=0} = 0 \\
&\qquad \text{except } \frac{\partial^2}{\partial q^2}\bar{\Gamma}_\kappa^{(1,1)}(\varpi,\vec{q})\Big|_{\vec{q}=0} = -4i\varpi \\
&d) \quad \frac{\partial}{\partial q^i}\bar{\Gamma}_\kappa^{(m+1,n)}(\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\le\ell\le m+n-1})\Big|_{\vec{q}=0} = i\epsilon_{\alpha\beta}\mathcal{D}_\beta(\varpi)\bar{\Gamma}_\kappa^{(m,n)}(\{\mathbf{p}_\ell\}) \\
&e) \quad \frac{\partial^2}{\partial q^2}\bar{\Gamma}_\kappa^{(m+1,n)}(\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\le\ell\le m+n-1})\Big|_{\vec{q}=0} = \mathcal{R}(\varpi)\bar{\Gamma}_\kappa^{(m,n)}(\{\mathbf{p}_\ell\})
\end{aligned}
\tag{5.55}
$$

where the operators $\mathcal{R}(\varpi)$ is the finite-difference version of $\tilde{\mathcal{R}}(\varpi)$:

$$
\mathcal{R}(\varpi)F(\{\mathbf{p}_\ell\}_{1\le\ell\le n}) \equiv 2i\epsilon_{\alpha\beta}\sum_{k=1}^{n} p_k^\alpha \frac{\partial}{\partial p_k^\beta}\frac{F(\{\mathbf{p}_\ell\}_{1\le\ell\le k-1},\,\omega_k+\varpi,\vec{p}_k,\{\mathbf{p}_\ell\}_{k+1\le\ell\le n}) - F(\{\mathbf{p}_\ell\}_{1\le\ell\le n})}{\varpi}
\tag{5.56}
$$

The passage from the $\tilde{\Gamma}_\kappa^{(m,n)}$ to the $\bar{\Gamma}_\kappa^{(m,n)}$ is not entirely trivial and is expounded on in Appendix F.1.2.

5.4 Expansion at large wave-number of the RG flow equation

After having examined in details the constraints coming from the symmetries or extended symmetries of SNS, we can finally turn to the RG flow equation of the theory. More specifically, to the infinite hierarchy of flow equations for the functional moments of the generating functionals. As was expounded on in Sec. 3.4, this can be done either using the flow of the EAA Γ_κ and its functional derivatives, the vertex functions $\Gamma_\kappa^{(n)}$, or the flow of $\mathcal{W}_\kappa$ and its functional derivatives, the connected correlation functions $G_\kappa^{(n)}$. To present the method chosen here, it is easier to use the flow equation of the $\Gamma_\kappa^{(n)}$. For example, let us recall the flow of $\Gamma_\kappa^{(2)}$:

$$
\begin{aligned}
\partial_s\bar{\Gamma}_{mn}^{(2)}(\mathbf{p}) = \int_{\mathbf{q}} \big[\partial_s\bar{\mathcal{R}}\big]_{ij}(\mathbf{q})\bar{G}_{jk}^{(2)}(\mathbf{q})\Big[&-\frac{1}{2}\bar{\Gamma}_{klmn}^{(4)}(\mathbf{q},-\mathbf{q},\mathbf{p}) \\
&+\bar{\Gamma}_{kms}^{(3)}(\mathbf{q},\mathbf{p})\bar{G}_{st}^{(2)}(\mathbf{q}+\mathbf{p})\bar{\Gamma}_{tnl}^{(3)}(\mathbf{q}+\mathbf{p},-\mathbf{p})\Big]\bar{G}_{li}^{(2)}(\mathbf{q})\,,
\end{aligned}
\tag{5.57}
$$

and its diagrammatic representation in Figure 3.4: the dashed circles are the vertex functions, the thick lines are propagators and the cross is the derivative of the regulator. The central property of the regulator is that it limits the wave-number q to be of order κ or lower. This fact associated with the regularity of the vertex functions can be used to derive controlled approximations of the flow equation.

Here, we follow the stategy pioneered by (Blaizot, Méndez-Galain, and Wschebor 2006; Benitez et al. 2012) and that was already used in (Canet, Delamotte, and

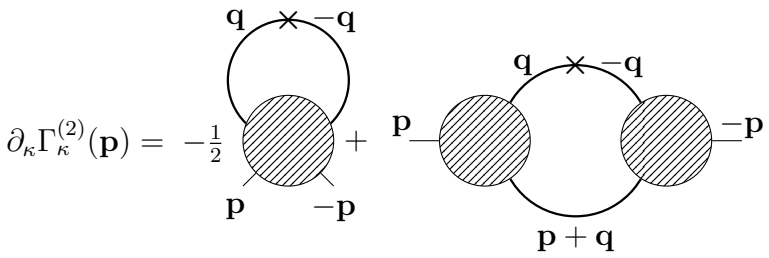

Figure 5.1: Diagrammatic representation of the flow of $\Gamma_\kappa^{(2)}$

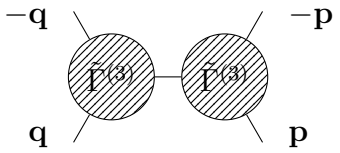

Figure 5.2: Diagram contributing to $\tilde{G}^{(4)}(\mathbf{q}, -\mathbf{q}, \mathbf{p}, -\mathbf{p})$

Wschebor 2016) in the context of turbulence. The idea is to use the two following properties of the regulator: on the one hand, its insertion in the integration loop on the r.h.s. of (5.57) limits the wave-number $\vec{q}$ to be of order κ or lower. As a consequence, if the system is probed at scales much higher in wave-number than the renormalization scale, that is if $p \gg \kappa$, there is a clear separation of scales in the flow equations: $q/p \ll 1$. On the other hand, the regulator ensures as well that the vertex functions are sufficiently smooth, which allows one to approximate them by their Taylor series in $\vec{q}$. The vertex functions are expected to depend on the internal wave-number only through scale invariant ratios of the type q/p. As a consequence, this expansion is controlled by the external wave-number and an expansion in q around 0 is equivalent to looking at some form of expansion in p around its asymptotic behavior when going to infinity.

As in (Canet, Delamotte, and Wschebor 2016), the results we obtain in the following are more conveniently expressed in terms of the flow of connected correlation functions, thus our starting point is the flow equation of $\mathcal{W}_\kappa$:

$$\partial_\kappa \mathcal{W}_\kappa = -\frac{1}{2} \operatorname{tr}\Big[\partial_\kappa \mathcal{R}_\kappa \cdot (G^{(2)} + \Phi \otimes \Phi)\Big]. \tag{5.58}$$

The above justification to expand the r.h.s. of the flow equation (5.57) applies as well to the flow of correlation functions derived from (5.58). However it must be stressed that it is only in the 1-PI vertex functions $\Gamma^{(k)}$ composing the $G^{(n)}$ that the expansion in $\vec{q}$ is controlled by the external ones. For example, the diagram of Figure 5.2 is part of the correlation functions $\tilde{G}^{(4)}(\mathbf{q}, -\mathbf{q}, \mathbf{p}, -\mathbf{p})$. One sees that the left part of this diagram contains a wave-vector $\vec{q}$ which is not controlled by $\vec{p}$. Thus one has first to decompose the $G^{(n)}$ into $\Gamma^{(k)}$, in order to do the wave-number expansion only where it is controlled.

Before using the Blaizot-Mendez-Wschebor (BMW) approximation, let us obtain the flow of the correlation functions in suitable form for the calculations. Applying n functional derivatives $\mathrm{j}_{i_k}(\mathbf{x}_k)$, with $1 \leq k \leq n$ and $i_k \in \{1, 2\}$ the field index, and taking the Fourier transform with respect to $\mathbf{x}_1, \ldots \mathbf{x}_n$, one obtains the RG flow equation for the n-point correlation function $\tilde{G}^{(n)}_{i_1 \ldots i_n}(\{\mathbf{p}_\ell\})$. In Appendix F.2 it is

shown that in the regime of large wave-numbers, that is when all the $\mathbf{p}_\ell$, $1 \leq \ell \leq n$ are large compared to κ, as well as all their partial sums, the flow equation reduces to

$$\partial_\kappa \tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\}_{1\leq\ell\leq n}) = \frac{1}{2}\int_{\mathbf{q}_1,\mathbf{q}_2} \tilde{\partial}_\kappa \tilde{G}^{(2)}_{ij}(-\mathbf{q}_1,-\mathbf{q}_2)\left[\frac{\delta^2}{\delta\Phi_i(\mathbf{q}_1)\delta\Phi_j(\mathbf{q}_2)}\tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\};\mathrm{j}]\right]_{\Phi=0} . \tag{5.59}$$

where the dependency of $\bar{G}^{(n)}$ in the fields appears through the implicit dependency $\mathrm{j} = \mathrm{j}[\Phi]$ and with the differential operator $\tilde{\partial}_\kappa$ defined as:

$$\tilde{\partial}_\kappa \equiv \partial_\kappa R_\kappa \frac{\delta}{\delta R_\kappa} + \partial_\kappa N_\kappa \frac{\delta}{\delta N_\kappa} . \tag{5.60}$$

This result is exact up to terms going to zero faster than any power of the p_ℓ and holds for both formulations, either the velocity one with

$$\Phi = \{\vec{u}, \vec{\tilde{u}}\} , \tag{5.61}$$

or the stream function one

$$\Phi = \{\Psi, \bar{\Psi}\} . \tag{5.62}$$

In the latter case, the indices i_ℓ carry only the information on the nature of the field (observable or response) while in the former, they also carry the vector component. The expression (5.59) is in the right form to use the BMW approximation. Indeed, it ensures that in the decomposition of the expression in square brackets in terms of its 1-PI vertices, the wave-numbers q_1 and q_2 are always controlled by an external wave-number and thus can be set to zero to get the leading order term in a large wave-number expansion.

5.5 Leading order at unequal time in 2- and 3-D

Let us first examine the leading order of the large wave-number expansion at unequal times and in any dimension. This entire section is taken directly from (Tarpin, Canet, and Wschebor 2018). As explained in the previous section, it amounts to set q_1 and q_2 to zero in the term in square bracket of (5.82). The derivation is made in Appendix F.3 and makes use of the Ward identities (5.40) and (5.47). The leading contribution of the flow equation for large wave-numbers reads

$$\partial_\kappa \bar{G}^{(n)}_{\alpha_1\dots\alpha_n}(\mathbf{p}_1,\dots,\mathbf{p}_{n-1}) = \frac{d-1}{2d}\int_{\mathbf{q}} \tilde{\partial}_\kappa \bar{C}(\varpi,\vec{q})\mathcal{D}_\mu(\varpi)\mathcal{D}_\mu(-\varpi)\bar{G}^{(n)}_{\alpha_1\dots\alpha_n}(\mathbf{p}_1,\dots,\mathbf{p}_{n-1}) , \tag{5.63}$$

where $\bar{C}$ is the transverse part of the velocity-velocity correlation function $\bar{G}^{(2)}_{v_\alpha v_\beta}$: $\bar{G}^{(2,0)}_{\mu\nu}(\mathbf{p}) = P^\perp_{\mu\nu}(\vec{p})\bar{C}(\mathbf{p})$ where the transverse projector is defined by

$$P^\perp_{\mu\nu}(\vec{p}) = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} . \tag{5.64}$$

The flow equation for any generalized correlation function $\bar{G}^{(n)}$ is hence closed, in the sense that it does not depend any longer on higher-order correlation functions. This closure is exact in the limit of large wave-numbers, when the modulus of all

wave-vectors and of all their partial sums are large compared to κ, which excludes exceptional configurations where a partial sum vanishes. We emphasize that this closure involves no arbitrary truncation or selection of certain diagrams rather than others. Its rationale is to retain only the leading order contribution at large wave-number in the flow equation, and this contribution is calculated exactly. Note that the $G_\kappa^{(n)}$ in the r.h.s. of Eq. (5.63) does not depend on the internal wave-vector $\vec{q}$, and is only integrated over the internal frequency ϖ. This result generalizes a previous result obtained in (Canet, Delamotte, and Wschebor 2016) for 2−point functions to generic n−point connected functions. In the next sections, we study some aspects of the solution of this flow equation at the fixed point, and in particular the general form of the space and time dependence of the correlation functions.

Let us already emphasize a very unusual feature of the flow equations obtained in this paper. The large wave-number part of the flow equation (determined exactly using the BMW framework) is not negligible compared to the rest of the flow. This behavior is referred to as non-decoupling (Collins 1984; Canet, Delamotte, and Wschebor 2016; Canet et al. 2017). It implies that the RG flow of the UV modes depends on the IR ones. This is in sharp contrast with what occurs in standard critical phenomena, where the large wave-number part of the flow equation decouples from the small wave-number one. This decoupling precisely entails standard scale invariance, as explained in Sec. 3.4.3. On the contrary, the non-decoupling induces a violation of standard scale invariance. Indeed, the main result of (Tarpin, Canet, and Wschebor 2018) is to derive the exact equation which replaces scale invariance for unequal-time quantities in turbulence.

It is important to stress that, similarly to standard critical phenomena where there are finite-size corrections to scale invariance, there are also finite-size corrections to this equation in turbulence, and hence to the leading behaviour at large wave-numbers of the correlations functions. In particular, these corrections, not captured by the present analysis, may determine the intermittency effects *at equal time*. Indeed, at equal times, the r.h.s. of (5.63) vanishes, thus the leading behavior is not captured.

5.5.1 Solution for the 2-point functions in 3-D

The leading contribution of the flow equation for the 2-point functions in the limit of large wave-numbers was already derived in (Canet, Delamotte, and Wschebor 2016). In the notation of the present paper, Eq. (5.63), it reads

$$\partial_\kappa \bar{G}^{(2)}_{\alpha_1\alpha_2}(\mathbf{p}) = \frac{d-1}{2d} \int_\varpi \mathcal{D}_\mu(\varpi)\mathcal{D}_\mu(-\varpi)\bar{G}^{(2)}_{\alpha_1\alpha_2}(\mathbf{p}) \int_{\vec{q}} \tilde{\partial}_\kappa \bar{C}(\varpi,\vec{q})\,. \tag{5.65}$$

This flow equation encompasses both the flow of the correlation function and of the response function. Let us denote $\bar{G}$ the transverse part of the Green function, that is $\bar{G}^{(1,1)}_{\mu\nu}(\mathbf{p}) = P^\perp_{\mu\nu}(\vec{p})\bar{G}(\mathbf{p})$. Let us focus on the flow equation for the transverse velocity-velocity correlation function. Using the explicit expression (5.40) for $\mathcal{D}_\mu$, one obtains:

$$\kappa\partial_\kappa \bar{C}(\omega,\vec{p}) = -\frac{2}{3}p^2 \int_\varpi \frac{\bar{C}(\omega+\varpi,\vec{p}) - \bar{C}(\omega,\vec{p})}{\varpi^2} J_\kappa(\varpi)\,, \tag{5.66}$$

with $J_\kappa(\varpi) \equiv -\int_{\vec{q}} \tilde{\partial}_\kappa \bar{C}(\varpi, \vec{q})$ given by

$$J_\kappa(\varpi) = -2 \int_{\vec{q}} \left\{ \kappa \partial_\kappa N_\kappa(\vec{q}) \, |\bar{G}(\varpi, \vec{q})|^2 - \kappa \partial_\kappa R_\kappa(\vec{q}) \, \bar{C}(\varpi, \vec{q}) \Re\big[\bar{G}(\varpi, \vec{q})\big] \right\}, \quad (5.67)$$

(which coincides with the equations given in (Canet et al. 2017)). An important feature of Eq. (5.66), already emphasized, is the non-decoupling, which means that $\kappa \partial_\kappa \bar{C}/\bar{C}$ does not vanish in the limit of large wave-numbers $|\vec{p}| \gg \kappa$. As a consequence, the behavior of the correlation functions at $t \neq 0$ shows non-standard scale invariance. On the other hand, at equal times, that is once integrated over the external frequency ω, the leading non-decoupling behavior cancels out (the r.h.s. of Eq. (5.66) vanishes when integrated over ω). For equal-times quantities, any possible non-decoupling must come from sub-leading terms at large wave-numbers. This implies in particular that intermittency corrections to exponents of equal-time quantities, such as the second order structure function, are absent in the leading order behavior at large $|\vec{p}|$, *i.e.* not included in these flow equations. This is made explicit in the solution below.

Let us derive the solutions of the flow equation (5.66) at the fixed point. It is convenient to first perform the inverse Fourier transform on ω, which yields

$$\kappa \partial_\kappa C(t, \vec{p}) = -\frac{2}{3} p^2 C(t, \vec{p}) \int_\varpi \frac{\cos(\varpi t) - 1}{\varpi^2} \, J_\kappa(\varpi) \, . \quad (5.68)$$

The regimes of small t and large t are studied separately in the following. The solution of Eq. (5.68) at the fixed point and in the limit of small time delays (or equivalently large frequencies) has been obtained in (Canet et al. 2017). For small t, $(\cos(\varpi t) - 1)/\varpi^2 \sim -t^2/2$ in the integrand. The general solution of the resulting fixed-point equation, which we index with the subscript 'S' for 'short time', follows:

$$\log\left[\frac{C_S(t, \vec{p})}{\varepsilon^{2/3} L^{11/3}}\right] = -\alpha_S (\varepsilon L)^{2/3} t^2 p^2 - \frac{11}{3} \log(pL) + F_S(\varepsilon^{1/3} p^{2/3} t) + \mathcal{O}(pL) \, , \quad (5.69)$$

where ε is the mean energy injection rate, L the integral scale, α_S a non-universal constant depending on the forcing profile (see Appendix F.4), and F_S is a regular function, universal up to a pre-factor. Note that we have included explicitly in (5.69) sub-leading terms stemming from the resolution of the fixed-point equation at leading order, although they are of the same order as the error term $\mathcal{O}(pL)$. The reason is that they correspond to Kolmogorov scaling solution, and facilitate the discussion of the result. Indeed, at equal-time, one recovers from (5.69) the Kolmogorov prediction $C_S(t = 0, \vec{p}) = H_S(0) \varepsilon^{2/3} p^{-11/3}$. However, these terms are only approximate in this calculation. If the sub-leading terms in the flow equation (5.68), neglected here, are non-decoupling, then they will induce corrections (of order at most pL) to this scaling, that is, intermittency corrections to the exponent of the structure function.

On the other hand, at finite time delays $t \neq 0$, the leading term in (5.69) explicitly breaks scale invariance. Indeed, it does not depend on the scaling variable tp^z, where z is the dynamical exponent $z = 2/3$ for NS in $d = 3$, and thus involves a scale L. This leading term conveys an effective exponent $z = 1$, which indicates significant corrections to standard scale invariance. Its physical origin is the sweeping effect, which is the random advection of small-scale velocities by large-scale eddies

(Tennekes 1975; Yakhot, Orszag, and She 1989; Chen and Kraichnan 1989; Nelkin and Tabor 1990; Gotoh et al. 1993; Chevillard et al. 2005). The typical time-scale appearing in the exponential (5.69) is the sweeping time $\tau_S \sim (\varepsilon L)^{-1/3} p^{-1} = 1/(u_{\rm rms} p)$. One of its manifestation is that the spectrum measured in frequency has the same exponent -5/3 as when measured in wave-numbers (we consider flows with zero mean velocity, this is not related to Taylor's hypothesis of frozen turbulence).

The behavior (5.69) has been observed in many numerical simulations of the NS equation (Orszag and Patterson 1972; Sanada and Shanmugasundaram 1992; He, Wang, and Lele 2004; Favier, Godeferd, and Cambon 2010; Canet et al. 2017) as well as in experiments (Poulain et al. 2006). Indeed, in Fig. 1 of (Orszag and Patterson 1972), and Fig. 5 of (He, Wang, and Lele 2004), a reasonable collapse is obtained for the quantity $R(t,p) = C(t,p)/C(0,p)$ as a function of $p \int_t u_{rms}$ for different times and different values of wave-numbers respectively, from simulations of decaying turbulence. In Fig. 6 and 7 of (Sanada and Shanmugasundaram 1992), the collapse for the same quantity as a function of time is qualitatively better when normalizing by the sweeping time τ_S rather than the eddy turn-over time $\tau_e \sim 1/(\varepsilon^{1/3} p^{2/3})$. In Fig. 7, the typical time scale of $R(t,p)$ is shown to scale linearly in p for large wave-numbers. The same analysis is carried out in Fig. 4 of (Favier, Godeferd, and Cambon 2010). The time dependence of the two-point function is explicitly tested in (Canet et al. 2017), where a Gaussian form of $R(t,p)$ in the variable $u_{rms}\, p\, t$ is very accurately found in numerical simulations. The Gaussian behavior and the linear dependency in p of the decorrelation time of $R(t,p)$ is also found in acoustic scattering measurements, see Fig. 5 and 6 in (Poulain et al. 2006). As mentioned in the introduction, such a Gaussian dependence in tp for large p and small t was predicted early on by Kraichnan within the DIA approximation (Kraichnan 1959), and later confirmed by RG approaches under some assumptions on the effective viscosity (Antonov 1994).

Let us briefly mention another feature of the fixed point solution of Eq. (5.66). It was shown in (Canet et al. 2017) that, under some additional assumptions, taking the appropriate $t \to 0$ limit, this solution predicts for the kinetic energy spectrum, a crossover from the $p^{-5/3}$ decay in the inertial range, to a stretched exponential decay in the dissipative range, on the scale $p^{2/3}$

$$E(\vec{p}) \propto p^{-5/3} \exp\left[-\mu p^{2/3}\right] , \tag{5.70}$$

with μ a non-universal constant. This prediction was precisely confirmed in direct numerical simulation of NS equation (Canet et al. 2017), and also observed in experiments on turbulent swirling flows (Debue et al. 2018). It was pointed out that this prediction does not show any bottleneck effect (Lohse and Müller-Groeling 1995), thus it may be interesting to have a better understanding of the crossover between the two regimes.

The flow equation (5.68) is valid for a large wave-number p, but for an arbitrary time delay t. Now, we study the opposite limit of asymptotically large t, which was not considered previously. As shown in Appendix F.4, the flow equation (5.68) simplifies in the limit $t \gg \kappa^{2/3}$ to

$$\kappa \partial_\kappa C(t,\vec{p}) = \frac{J_\kappa(0)}{3} |t|\, p^2\, C(t,\vec{p}). \tag{5.71}$$

As for the flow equation at small t, this equation can be solved at the fixed point (see Appendix F.4). The solution, indexed by the subscript 'L' for 'long' time, reads

$$\log\left[\frac{C_L(t,\vec{p})}{\varepsilon^{2/3}L^{11/3}}\right] = -\alpha_L\varepsilon^{1/3}L^{4/3}|t|\,p^2 - \frac{11}{3}\log(pL) + F_L(\varepsilon^{1/3}k^{2/3}t) + \mathcal{O}(pL)\,, \quad (5.72)$$

with α_L a non-universal constant, and sub-leading terms corresponding to Kolmogorov solution again included explicitly. To the best of our knowledge, this regime was not predicted before. The corresponding time-scale in the exponential is $\tau_L = (u_{\rm rms}Lp^2)^{-1}$.

Interestingly, a similar crossover from the Gaussian in tp at short time to a behavior $\exp(-|t|/\tau_{\rm exp})$ at long times was observed in (Poulain et al. 2006). However, in this paper, $\tau_{\rm exp} \propto (u_{\rm rms})^{-1}L$, that is the p^2 in τ_L is replaced by L^{-2}. This indicates that the crossover seen in the experiments is dominated by the small wave-numbers, so it is likely to differ from ours. One can compare the related time-scales of these two crossovers. The crossover between the two (short and long time) regimes occurs typically when the exponents in the two exponentials are equal. For our work, matching the exponents in (5.69) and (5.72) yields $\tau_{\rm cross} \propto L$. In the experimental paper, the crossover time is given by $\tau_{\rm cross} \propto L^2/p = L/(pL^{-1})$. Hence, at large $p \gg L^{-1}$, this crossover time is shorter than the second crossover, and may dominate over it.

5.5.2 Form of the solution for generic correlation functions in 3-D

In this section, we work out the general form of the fixed point solution of the flow equation (5.63) for any generalized $n-$point correlation functions. From equation (5.63), the first step is to perform the inverse Fourier transform in frequency in order to get the flow equation for the hybrid wave-vector and time correlation functions. One obtains for the leading contribution of the flow equation at large wave-numbers

$$\partial_\kappa G^{(n)}_{\alpha_1\dots\alpha_n}(\{t_i,\vec{p}_i\}) = \frac{1}{3}\,G^{(n)}_{\alpha_1\dots\alpha_n}(\{t_i,\vec{p}_i\})\sum_{k,\ell}\vec{p}_k\cdot\vec{p}_\ell\int_\varpi J_\kappa(\varpi)\,\frac{e^{i\varpi(t_k-t_\ell)} - e^{i\varpi t_k} - e^{-i\varpi t_\ell} + 1}{\varpi^2}\,. \quad (5.73)$$

The solution of the corresponding fixed-point equation is derived in the following in both limits $t \to 0$ and $t \to \infty$.

For $t_i \ll \kappa^{-2/3}$, the flow equation (5.73) simplifies to (see Appendix F.4)

$$\Big(\partial_\kappa - \frac{I_\kappa}{3}\,|\vec{p}_k t_k|^2\Big)G^{(n)}_{\alpha_1\dots\alpha_n}(t_1,\vec{p}_1,\cdots,t_{n-1},\vec{p}_{n-1}) = 0\,, \quad (5.74)$$

with $I_\kappa = \int_\varpi J_\kappa(\varpi)$, and J_κ defined in Eq. (5.67), and with Einstein summation convention. The corresponding fixed point equation can be solved exactly, leaving as unknown a scaling function of particular dimensionless variables. Let us present

this solution, which is derived in details in Appendix F.4, and which reads

$$\begin{aligned}
&\log\left[\varepsilon^{\frac{\bar{m}-m}{3}} L^{-d_G} G^{(n)}_{S\,\alpha_1\dots\alpha_n}(t_1,\vec{p}_1,\cdots,t_{n-1},\vec{p}_{n-1})\right]\\
&= -\alpha_S \varepsilon^{2/3} L^{2/3}\ |t_k\vec{p}_k|^2 - d_G \log(\rho_1 L)\\
&+ F^{(n)}_{S\,\alpha_1\dots\alpha_n}\left(\varepsilon^{1/3}\rho_1^{2/3} t_1, \frac{\vec{\rho}_1}{\rho_1},\cdots,\varepsilon^{1/3}\rho_1^{2/3} t_{n-1}, \frac{\vec{\rho}_{n-1}}{\rho_1}\right)\\
&+ \mathcal{O}(p_{\max} L)\,.
\end{aligned} \tag{5.75}$$

In this expression, d_G is the scaling dimension of $G^{(n)}$ (given in Appendix F.4), m (resp. $\bar{m}$) is the number of velocity (resp. response velocity) fields in this generalized correlation function, with $m+\bar{m}=n$, and α_S is the same non-universal constant as in Eq. (5.69). $F^{(n)}_S$ is a regular function of its arguments, which cannot be determined by the fixed point equation at large wave-number alone, but requires the integration of the full flow equation. As for the two-point functions, sub-leading terms which correspond to Kolmogorov scaling solution are included in this expression, although they are only approximate and could receive corrections from the neglected $\mathcal{O}(p_{\max}L)$ terms. The variables $\vec{\rho}_k$ are defined by $\vec{p}_i = \mathcal{R}_{ij}\vec{\rho}_j$ where $\mathcal{R}_{ij}$ is a rotation matrix which has to be explicitly constructed for each correlation function such that

$$\vec{\rho}_1 = \frac{t_k\vec{p}_k}{\sqrt{t_\ell t_\ell}}\,. \tag{5.76}$$

Finally, $p_{\max}$ in (5.75) is the maximum modulus of the $\vec{p}_i$ and their partial sums. The above expression provides the leading time and wave-vector dependence of any correlation functions, which is exact in the regime of small time differences and large wave-numbers. The combination of time and space appearing in the exponential part of the expression (5.75) is $|\vec{p}_k t_k|^2$. This combination breaks scale invariance and is the generalization to generic n-point correlation functions of the Gaussian dependence in the variable $(p\,t)$ for the 2−point correlation function, which is related to the sweeping effect. This breaking yields the dependence on the integral scale L of this leading term. Furthermore, in the range of validity of this solution, one has $p^{2/3}t \ll p\,t\,L^{1/3}$. Hence, because of the regularity of $F^{(n)}_S$, the leading time contribution is due in this regime to the variable $\vec{p}_k t_k$, except for exceptional configurations where $\vec{p}_k t_k \simeq 0$. The leading time-dependence of the correlation function hence takes the form of a Gaussian

$$G^{(n)}_S \sim e^{-\alpha_S(\varepsilon L)^{2/3}|\vec{p}_k t_k|^2}\,. \tag{5.77}$$

If wave-vectors are measured in units of η^{-1} as is usually the case, the resulting typical time scale is the sweeping time $\tau_{\rm s} = \eta/u_{\rm rms} = \eta/(\varepsilon L)^{1/3}$, which differs from the Kolmogorov time $\tau_K = (\nu/\varepsilon)^{1/2}$.

On the other hand, at equal times, one is left with

$$G^{(n)}_{S\,\alpha_1\dots\alpha_n}(\{0,\vec{p}_i\}) = \varepsilon^{\frac{m-\bar{m}}{3}}\rho_1^{-d_G}\ \exp F^{(n)}_{S\,\alpha_1\dots\alpha_n}\left(\left\{0,\frac{\vec{\rho}_i}{\rho_1}\right\}\right) \tag{5.78}$$

which corresponds to Kolmogorov-like solution : a power-law behavior with a dimensional exponent times a scaling function. However, our calculation shows that this is not exact *a priori*, and is thus compatible with the existence of intermittency corrections. Indeed, these terms can receive corrections from the neglected

$\mathcal{O}(p_{\max}L)$ terms in the flow equation. This terms could in particular modify the exponent of the power-law, that is yield intermittency correction to the structure functions. These corrections should be given by the next order term in the flow equation, provided it does not to vanish at equal time. This direction is left for future work.

We now specialize to the three-velocity correlation. In the regime of small time differences and large wave-numbers, we obtain

$$G^{(3)}_{S\ \alpha\beta\gamma}(t_1,\vec{p}_1,t_2,\vec{p}_2) \sim G^{(3)}_{S\ \alpha\beta\gamma}(0,\vec{p}_1,0,\vec{p}_2)\exp\Big(-\alpha_S(\varepsilon L)^{2/3}\,|\vec{p}_1 t_1+\vec{p}_2 t_2|^2\Big)\,. \tag{5.79}$$

This prediction can be tested in direct numerical simulations of the NS equation or in experiments. For example, one can construct a scalar function from the 3-velocity correlation, such as $p_1^\alpha G^{(3,0)}_{\beta\beta\alpha}(\vec{p}_1,t,\vec{p}_2,t)$, and measure its dependence in the time difference t in the stationary state. Normalizing the constructed function by its value at $t=0$, one should obtain a Gaussian dependence in the variable $|\vec{p}_1+\vec{p}_2|\,t$.

We consider again the flow equation Eq. (5.73). In the limit of large times, *i.e.* all times $t_k \gg \kappa^{2/3}$ as well as all differences $(t_k - t_\ell) \gg \kappa^{2/3}$, this equation simplifies to (see Appendix F.4)

$$\partial_\kappa G^{(n)}_{\alpha_1\dots\alpha_n}(\{t_i,\vec{p}_i\}) = \frac{J_\kappa(0)}{6}\sum_{k,\ell}\vec{p}_k\cdot\vec{p}_\ell\Big(|t_k|+|t_\ell|-|t_k-t_\ell|\Big)\bar{G}^{(n)}_{\alpha_1\dots\alpha_n}(\{t_i,\vec{p}_i\})\,. \tag{5.80}$$

We focus on the special case where all the time delays are equal $t_k \equiv t$ for $k = 1,\dots,n-1$. In this case, the analytical solution of the corresponding fixed-point equation can be straightforwardly derived (see Appendix F.4). One obtains, keeping again the sub-leading Kolmogorov scaling terms,

$$\begin{aligned}
&\log\Big[\varepsilon^{\frac{\bar{m}-m}{3}}L^{-d_G}G^{(n)}_{\alpha_1\dots\alpha_n}(t,\vec{p}_1,\cdots,\vec{p}_{n-1})\Big]\\
&= -\alpha_L\varepsilon^{1/3}L^{4/3}\,|t|\,\Big|\sum\vec{p}_k\Big|^2 - d_G\log(\varrho_1 L)\\
&+ F_L{}^{(n)}_{\alpha_1\dots\alpha_n}\left(\varrho_1^{2/3}\varepsilon^{1/3}t,\frac{\vec{\varrho}_1}{\varrho_1},\cdots,\frac{\vec{\varrho}_{n-1}}{\varrho_1}\right)\\
&+ \mathcal{O}(p_{\max}L)\,,
\end{aligned} \tag{5.81}$$

where the variables $\vec{\varrho}_k$ are obtained by a transformation of the wave-vectors satisfying $\vec{\varrho}_1 = \sum\vec{p}_k$, which can be explicitly constructed for each n. The example of $G^{(3)}$ is explicitly given in Appendix F.4. The crossover, evidenced for the two-point function, also emerges for generic n-point functions. The quadratic dependence in t in the exponential at small time delays is changed at large time delays to a linear one.

In conclusion, we have obtained the leading behavior of the correlation functions of turbulence at unequal times. It generalizes to any time difference and for any generalized correlation functions a known peculiarity of turbulence, the sweeping effect. This effect disappears at equal times. As a consequence the leading order is not sufficient to explain a possible deviation to Kolmogorov exponents for the structure functions. In the following, we will investigate the leading order terms

at equal time in the large wave-number expansion, in the case of 2-D turbulence. These terms would be responsible in our scenario for intermittency of the structure functions in turbulence. The investigation of the next-to-leading terms has more potential for success in 2-D due to the use of the new uncovered symmetry (5.31) of the SNS field theory which is not realized in 3-D.

5.6 Large wave-number expansion in the stream function formulation

5.6.1 Leading order of the flow equation at unequal times

The results related the in previous section in the velocity formulation hold in both 2-D and 3-D. However, to make use of the supplementary symmetry existing only in 2-D, it appears simpler to work in the stream function formulation. As a consequence, our first task is to rederive the leading order result in this formulation.

Let us first recall the flow equation for the correlation functions, now to be understood in the stream function formulation

$$\partial_\kappa \tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\}_{1\le\ell\le n}) = \frac{1}{2}\int_{\mathbf{q}_1,\mathbf{q}_2} \tilde{\partial}_\kappa \tilde{G}^{(2)}_{ij}(-\mathbf{q}_1,-\mathbf{q}_2)\left[\frac{\delta^2}{\delta\Phi_i(\mathbf{q}_1)\delta\Phi_j(\mathbf{q}_2)}\tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\};\mathrm{j}]\right]_{\Phi=0}. \tag{5.82}$$

The leading order term in wave-number would normally be obtained by setting q_1, q_2 to zero in the terms in bracket as in the velocity formulation. In fact, due the gauge degree of freedom of the stream function action manifested by Eq. (5.33) line a), there is no information at this order. Indeed, the symmetry implies that the vertex functions are zero if one of their wave-number is set to zero, Eq. (5.53) line a). Thus,

$$\left[\frac{\delta^2}{\delta\varphi_i(\mathbf{q}_1)\delta\varphi_j(\mathbf{q}_2)}\tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\}_{1\le\ell\le n};\mathrm{j}]\right]_{\varphi=0}\Bigg|_{\vec{q}_1=\vec{q}_2=0} = 0 \tag{5.83}$$

As a consequence, one has to go further in the expansion in q_1, q_2 to get the leading order term in the high wave-number expansion. The odd terms of this expansion vanishes after integration due to the parity of $\tilde{\partial}_\kappa \tilde{G}^{(2)}_{ij}(-\mathbf{q}_1,-\mathbf{q}_2)$. We deduce that the leading order term of the flow is

$$\begin{aligned}\partial_\kappa \tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\}_{1\le\ell\le n})\Big|_{leading} &= \frac{1}{2}\int_{\mathbf{q}_1,\mathbf{q}_2} \tilde{\partial}_\kappa \tilde{G}^{(2)}_{ij}(-\mathbf{q}_1,-\mathbf{q}_2)\\ &\times \frac{q_a^\alpha q_b^\beta}{2}\frac{\partial^2}{\partial q_a^\alpha \partial q_b^\beta}\left[\frac{\delta^2}{\delta\varphi_i(\mathbf{q}_1)\delta\varphi_j(\mathbf{q}_2)}\tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\};\mathrm{j}]\right]_{\varphi=0}\Bigg|_{\vec{q}_1=\vec{q}_2=0},\end{aligned} \tag{5.84}$$

where a, b take value in $\{1,2\}$. It is checked in Appendix F.5 that the leading order term at unequal times, already known in the velocity formulation, is recovered in the stream function formulation:

$$\partial_\kappa \tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\}_{1\le\ell\le n}) = \frac{1}{2}\int_{\mathbf{q}_1,\mathbf{q}_2} \tilde{\partial}_\kappa \tilde{G}^{(2)}_{v_\mu v_\nu}(-\mathbf{q}_1,-\mathbf{q}_2)\tilde{\mathcal{D}}_\mu(\varpi_1)\tilde{\mathcal{D}}_\nu(\varpi_2)\tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\}). \tag{5.85}$$

These flow equations can be solved at the fixed point, as was done in the 3-D case in the previous section. However, in 2-D the scaling exponents are corrected by logarithmic factors. This has the effect to produce additional dependencies in the logarithm of the wave-numbers in the fixed point solutions. These solutions, assuming no intermittency corrections, are given in (Tarpin et al. submitted).

5.6.2 Next-to-leading order of the flow equation

For equal times (or equivalently in Fourier space, after integrating all external frequencies) the calculated leading term vanishes. Thus one has to go further in the high wave-number expansion in order to calculate the leading term at equal times. This term is partially controlled in 2-D due to the supplementary symmetry, namely the time-gauged rotation. Concentrating on the equal-time correlation function, one has to go to the next-to-leading order (NLO) in the $\vec{q}$ derivatives and the leading order term of the flow reads

$$\partial_\kappa \int_{\{\omega_\ell\}} \tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\}_{1\le\ell\le n})\Big|_{NLO} = \frac{1}{2}\int_{\mathbf{q}_1,\mathbf{q}_2} \tilde{\partial}_\kappa \tilde{G}^{(2)}_{ij}(-\mathbf{q}_1,-\mathbf{q}_2) \times \int_{\{\omega_\ell\}} \frac{q_a^\mu q_b^\nu q_c^\rho q_d^\sigma}{4!} \frac{\partial^4}{\partial q_a^\mu \partial q_b^\nu \partial q_c^\rho \partial q_d^\sigma} \left[\frac{\delta^2}{\delta\varphi_i(\mathbf{q}_1)\delta\varphi_j(\mathbf{q}_2)} \tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\};\mathrm{j}]\right]_{\varphi=0}\Bigg|_{\vec{q}_1=\vec{q}_2=0}, \tag{5.86}$$

where as before a, b, c, d take value in $\{1,2\}$. The main result of this work is to show that only a certain combinations of $\vec{q}$ derivatives survives after integration over the external frequencies. The first step is to show that among the different combination of $\vec{q}_1$ and $\vec{q}_2$ derivatives only the ones with two $\vec{q}_1$ and two $\vec{q}_2$ survives the integration over the frequencies. The terms with only $\vec{q}_1$ or only $\vec{q}_2$ disappear due to (5.53), line a), stating that a vertex function with a wave-number evaluated at zero is zero. In Appendix F.6, it is shown that the terms with only one $\vec{q}_1$ or only one $\vec{q}_2$ disappear as well. This is essentially due to the fact that this derivative translates to an overall $\tilde{\mathcal{D}}$ operator as in the previous section, which vanishes after frequency integration. Thus at this point one is left with

$$\partial_\kappa \int_{\{\omega_\ell\}} \tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\}_{1\le\ell\le n})\Big|_{NLO} = \frac{1}{2}\int_{\mathbf{q}_1,\mathbf{q}_2} \tilde{\partial}_\kappa \tilde{G}^{(2)}_{ij}(-\mathbf{q}_1,-\mathbf{q}_2) \times \int_{\{\omega_\ell\}} \frac{q_1^\mu q_1^\nu q_2^\rho q_2^\sigma}{4} \frac{\partial^4}{\partial q_1^\mu \partial q_1^\nu \partial q_2^\rho \partial q_2^\sigma} \left[\frac{\delta^2}{\delta\varphi_i(\mathbf{q}_1)\delta\varphi_j(\mathbf{q}_2)} \tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\};\mathrm{j}]\right]_{\varphi=0}\Bigg|_{\vec{q}_1=\vec{q}_2=0}. \tag{5.87}$$

Using space translations and rotations invariance of $\tilde{\partial}_\kappa \tilde{G}^{(2)}$, the leading term can be rewritten as

$$\partial_\kappa \int_{\{\omega_\ell\}} \tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\}_{1\le\ell\le n})\Big|_{leading} = \frac{1}{2}\int_{\varpi_1,\varpi_2} \tilde{K}_{ij}(\varpi_1,\varpi_2) \times \int_{\{\omega_\ell\}} \left(\frac{\partial^4}{\partial q_1^\mu \partial q_1^\mu \partial q_2^\nu \partial q_2^\nu} + 2\frac{\partial^4}{\partial q_1^\mu \partial q_2^\mu \partial q_1^\nu \partial q_2^\nu}\right) \left[\frac{\delta^2}{\delta\varphi_i(\mathbf{q}_1)\delta\varphi_j(\mathbf{q}_2)} \tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\};\mathrm{j}]\right]_{\varphi=0}\Bigg|_{\vec{q}_1=\vec{q}_2=0}. \tag{5.88}$$

with

$$\tilde{K}_{ij}(\varpi_1,\varpi_2) \equiv \frac{1}{32}\int_{\vec{q}} \tilde{\partial}_\kappa \tilde{G}^{(2)}_{ij}(-\varpi_1,-\varpi_2,q^2)(q^2)^2\,. \tag{5.89}$$

The last two parts of Appendix F.6 are devoted to show that

$$\begin{aligned}\frac{\partial^4}{\partial q_1^\mu \partial q_1^\mu \partial q_2^\nu \partial q_2^\nu}\Big[\frac{\delta^2}{\delta\varphi_i(\mathbf{q}_1)\delta\varphi_j(\mathbf{q}_2)}\tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\}_{1\le\ell\le n};\mathrm{j}]\Big]_{\varphi=0}\Big|_{\vec{q}_1=\vec{q}_2=0}\\ = \delta_{i\psi}\delta_{j\psi}\tilde{\mathcal{R}}(\varpi_1)\tilde{\mathcal{R}}(\varpi_2)\tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\})\,.\end{aligned} \tag{5.90}$$

This term vanishes after integration over frequencies by conservation of angular momentum.

$$\begin{aligned}&\int_{\{\omega_\ell\}}\tilde{\mathcal{R}}(\varpi_1)\frac{2i\epsilon_{\alpha\beta}}{\varpi_2}\sum_{k=1}^{n}p_k^a\frac{\partial}{\partial p_k^b}\tilde{G}^{(n)}_{i_1\dots i_n}(\dots,\omega_k+\varpi_2,\vec{p}_k,\dots)\\ &=\int_{\{\omega_\ell\}}\tilde{\mathcal{R}}(\varpi_1)\frac{2i\epsilon_{\alpha\beta}}{\varpi_2}\sum_{k=1}^{n-1}p_k^a\frac{\partial}{\partial p_k^b}\Big[\tilde{G}^{(n)}_{i_1\dots i_n}(\dots,\omega_k+\varpi_2,\vec{p}_k,\dots)-\tilde{G}^{(n)}_{i_1\dots i_n}(\dots,\omega_n+\varpi_2,\vec{p}_n)\Big]\\ &=0\,.\end{aligned} \tag{5.91}$$

and finally one is left with

$$\begin{aligned}&\partial_\kappa\int_{\{\omega_\ell\}}\tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\}_{1\le\ell\le n})\Big|_{NLO}\\ &\quad=\int_{\varpi_1,\varpi_2}\tilde{K}_{ij}(\varpi_1,\varpi_2)\int_{\{\omega_\ell\}}\frac{\partial^4}{\partial q_1^\mu\partial q_2^\mu\partial q_1^\nu\partial q_2^\nu}\Big[\frac{\delta^2}{\delta\varphi_i(\mathbf{q}_1)\delta\varphi_j(\mathbf{q}_2)}\tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\};\mathrm{j}]\Big]_{\varphi=0}\Big|_{\vec{q}_1=\vec{q}_2=0}\,.\end{aligned} \tag{5.92}$$

The message of this derivation is that among all the terms contributing to the large wave-number leading order RG flow equation, all the terms which are controlled by the symmetries turn out to vanish after integration over external frequencies. This leaves for possible candidates for intermittency at equal times only terms which cannot be treated analytically as of now. We are now investigating the possibility that the expression above can be controlled for particular objects, such as the relevant structure functions appearing in 2-D turbulence.

In conclusion of this chapter, let us recapitulate our findings. Certainly the central result is the derivation of the large wave-number limit of the exact RG flow equation for correlation functions in turbulence. This allowed us to make new predictions on the time-dependence of such functions. The obtained behavior has been identified to generalize the well-known sweeping effect and is a clear example of breaking of scale-invariance in turbulence. At equal times, the leading order vanishes and the result is not controlled anymore. We have shown that in 2-D we may be able to control in the same way the NLO. This would give strong predictions concerning the characterization of intermittency in the direct cascade of 2-D turbulence. More precisely, if the NLO is shown to vanishes at equal times, this implies that there is no intermittency correction. The calculations did not lead to a conclusive answer at the time of the writing of thesis. However, they may give predictions for a more restrained set of observables and guide future investigations.

Version française

En conclusion de ce chapitre, récapitulons nos résultats. Le plus important est certainement la dérivation de la limite à grand nombre d'ondes de l'équation de flot de RG exacte pour les fonctions de corrélation en turbulence. Cela nous a permis de faire de nouvelles prédictions sur la dépendance temporelle de ces fonctions. Le comportement obtenu a été identifié comme généralisant le "sweeping effect" bien connu et est un exemple clair de brisure de l'invariance d'échelle en turbulence. A temps égaux, l'ordre dominant à grand nombre d'ondes devient nul et le résultat n'est plus contrôlé. Nous avons montré qu'en 2-D nous pourrions être capables de contrôler de la même manière le NLO. Cela donnerait de fortes prédictions concernant la caractérisation de l'intermittence dans la cascade directe de la turbulence de 2-D. Plus précisément, s'il est démontré que le NLO disparaît à des temps égaux, cela signifie qu'il n'y a pas de correction intermittente. Les calculs n'ont pas abouti à une réponse concluante au moment de la rédaction de la thèse. Toutefois, ils peuvent donner des prédictions pour certaines observables et guider les recherches futures.

Chapter 6

General conclusion

6.1 Summary

This manuscript describes the application of the tools of the NPRG framework to two out of equilibrium systems. On the one hand, we used the modified local potential approximation to investigate the absorbing phase transition occuring in the diffusive epidemic process and its coarse-grained counterpart, the directed percolation with a conserved quantity. On the other hand, we investigated the large wave-number expansion of the exact RG flow equation of the correlation functions in fully developed turbulence.

First, the study of DEP and DP-C was the occasion to discuss in details the respective symmetries of both models and to challenge the standard perturbative lore concerning them. Namely, it was predicted that both models belonged to the same universality class and that for certain values of the parameter μ of the models, the transition was of first order. We recovered the previous perturbative results at the lowest order of our approximation. At higher orders, we obtained hints that a continuous phase transition exists for all values of μ, in accordance with simulations. We were also able to explore the relation between DEP and DP-C within the NPRG. However, the chosen ansatz was demonstrated not to be powerful enough to give a definitive answer to the questions we raised.

Second, the large wave-number expansion of the exact RG flow equation in homogeneous isotropic fully developed turbulence led to a closed equation satisfied by any correlation functions in this system. This closed equation has a peculiar property named non-decoupling, which induces the breaking of scale-invariance in the temporal dependence of the correlation functions in turbulence. This is the highlight of the manuscript. These equations were solved both in the regime of large and small time-delays to obtain the temporal behavior of the Fourier modes of any correlation functions. For small time delays, the temporal decay of the Fourier modes of the correlation functions is Gaussian, with a characteristic time going like the inverse of the wave-number. This behavior is identified with the sweeping effect: the random advection of small eddies by larger ones. At large time differences, the Fourier modes of the correlations have an exponential decay. A similar behavior was already observed in experiments but it is the first time it is predicted from the Navier-Stokes equation. Finally, the large wave-number expansion was pushed to next-to-leading order in 2-D. The next-to-leading order term should give information on the equal-time behavior of the correlation functions, and thus on intermittency,

in the direct cascade in 2-D turbulence. Newly discovered symmetries allowed to control partially this term. However, uncontrolled terms remain, preventing to give a definitive conclusion at this point.

6.2 Prospects

Let us expound on possible leads to continue the work undertaken in this thesis. Concerning the study of the reaction-diffusion processes DEP and DP-C, the most straightforward approach would be to devise an ansatz more suited to capture the possible differences between the two models. In a first step, one should investigate the benefit of keeping a total or partial functional dependence of the potential in the field invariants of the theory, instead of doing a systematic truncation in the fields. This question is linked to the role of the choice of a minimum configuration in the ansatz and maybe would provide some answers to the instabilities observed in the flow in low dimensions. However, another approach would be to start from the response field action of DEP. The nature of the fields, the interaction terms and the symmetries are completely different in this case and they would require a completely different ansatz. A work has already been undertaken to understand the relation between both actions and to investigate the response field formalism for Poisson process in general (Guioth, Lecomte, and Tarpin in prep.). In particular, for the parameter μ set to zero, a time-gauged tilt symmetry is present in the response field action, which is reminiscent of the time-gauged tilt symmetry in the KPZ action (Canet et al. 2011). This property may guide the choice of an ansatz for this action. This NPRG study should be made in parallel of simulations of the Langevin equation of the DP-C model and simulations of the DEP model in 3-D, which are sorely lacking for the moment.

Now, let us turn to the possible ways to go further in the study of fully developed turbulence using NPRG. First, the exact closure for the large wave-number leading order RG flow equation could be complemented with the previous results obtained by integrating numerically the NPRG flow (Canet, Delamotte, and Wschebor 2016). This should allow us for example to give a prediction for the crossover between the Gaussian and the exponential decorrelation in time. It may be interesting also to make a bridge between the time-dependence obtained here for the correlation functions and the assumptions which are made in phenomenological closure models such as the EDQNM. More ambitiously, a new ansatz for the numerical integration of the flow could be proposed, taking into account our exact result, in order to investigate features of the dissipative range in turbulence with the NPRG. Indeed, it seems that some of these features, notably the stretch-exponential decay of the energy spectrum in the mid-dissipative range, are already contained in the fixed point equation (Canet et al. 2017). This fact calls for further investigations, for example to determine whether our approach predict a bottleneck effect (Lohse and Müller-Groeling 1995) in the spectrum. Second, we hope that the partial closure of the RG flow equation obtained in the direct cascade of 2-D turbulence could be exploited further. Although a closure of the remaining term in the general case seems out of reach, the obtained expression may simplify in particular cases relevant to the experiments. Third, let us point out that the framework of the large wave-number expansion for turbulence explored in this work can be generalized to other convective models. In particular we think it could lead to new predictions in the case of the

Kraichnan model of advected passive scalars. It may be fruitful to make the bridge with another approach of the Kraichnan model in the framework of the NPRG, using operator product expansion (Pagani 2015). A work has been undertaken in this direction. Fourth and finally, there exists a toy-model of intermittency in turbulence, it is the Burgers equation describing 1-D fully compressible flows. In this case, the intermittency is well-understood and stems from the shock waves which develop at finite times (Bec and Khanin 2007). To be able to rederive the intermittency corrections in this model would constitute a nice proof of concept of our approach.

Version française

The manuscrit décrit l'application des outils du NPRG à deux systèmes hors équilibre. Nous avons décrit premièrement l'utlisation d'un ansatz nommé l'approximation du potentiel local modifié pour étudier la transition de phase absorbante présente dans le processus de diffusion épidémique. Ceci nous a fait questionner ce qui était communément admis à propos de ce système. Notamment, nous trouvons une transition de phase continue pour n'importe quelle valeur du paramètre μ, et ce en accord avec les simulations. Ce travail reste néanmoins exploratoire et ne permet pas de donner de réponses définitives aux questions soulevées. Nous avons dans un second décrit le développement à grand nombre d'ondes de l'équation de flot de RG exacte pour les fonctions des corrélation généralisées dans la turbulence pleinement développée. Ce travail a permis en particulier d'obtenir une équation fermée donnant le comportement temporel dominant à grand nombre d'ondes, identifié comme le "sweeping effect". En 2-D, ce développement a pu être poussé jusqu'à l'ordre sous-dominant grâce à de nouvelles symétries découvertes durant cette étude. Malheureusement, certains termes du développement non contrôlés par ces symétries empêchent de conclure pour le moment.

Dans le future, ces deux travaux peuvent être continués dans plusieurs directions. Concernant le processus de diffusion épidémique, l'ansatz devrait être amélioré en prenant en compte une dépendance fonctionelle du potentiel dans les champs, ou du moins dans certains invariants bien choisis. L'introduction d'un régulateur en fréquence pourrait aussi grandement améliorer la convergence des résultats. Enfin, changer de point de départ et utiliser la formulation de l'action en terme de champs de réponses pourrait s'avérer fructueux. Concernant la turbulence, notre résultat pourrait être utilisé en complément d'intégration numérique du flot de RG. Ceci pourrait mener à de nouvelles prédictions concernant la transition entre les deux comportements temporels asymptotiques qui ont été découverts. Il serait très intéressant de faire le lien entre notre approche et d'autres tel que l'EDQNM dans le traitement des corrélations temporelles. Enfin, le même formalisme pourrait être testé dans des systèmes analogues mieux contrôlés tels que le modèle de Kraichnan ou la turbulence de Burgers.

Appendix A

Master equation, generating function and mean-field equations for reaction-diffusion processes

In this appendix, we expound on the formalism of the master equation and the tool of the generating function on the example of DP and give the master equation of DEP. A state of DP is given by the occupation number on each site of the lattice. The reaction defining DP, namely

$$\begin{array}{ll} \textbf{Branching} & X \xrightarrow{\sigma} X + X \\ \textbf{Disintegration} & X \xrightarrow{\mu} \emptyset \\ \textbf{Coagulation} & 2\,X \xrightarrow{2\lambda} X \\ \textbf{Diffusion} & X + \emptyset \xrightarrow{D} \emptyset + X \end{array} \tag{A.1}$$

can be translated in term of the master equation for the random process. The master equation is the time evolution of the probability to find the system in a particular state. We note $\mathbb{L}$ the subset of $\mathbb{Z}^d$ which indices the sites of the lattice and $\{N_\ell\}_{\ell\in\mathbb{L}}$, with $\forall \ell \in \mathbb{L}$, $N_\ell : t \to N_\ell(t) \in \mathbb{N}$, the random process given by the occupation number at each site of the lattice along the time. The probability that $\{N_\ell\}_{\ell\in\mathbb{L}}$ takes a given value $\{n_\ell\}_{\ell\in\mathbb{L}}$ at the time t is abbreviated with the following notation:

$$\mathbb{P}\big[\{N_\ell(t)\}_{\ell\in\mathbb{L}} = \{n_\ell\}_{\ell\in\mathbb{L}}\big] = \mathbb{P}_t\big[\{n_\ell\}\big]\,. \tag{A.2}$$

In the following, we will note $\{n_\ell\}_{i\pm} = \{n_\ell \pm \delta_{i,\ell}\}_{\ell\in\mathbb{L}}$, with the convention that $\mathbb{P}_t\big[\{n_\ell\}\big]$ is zero if one of the occupation number is strictly negative. Furthermore, we note $< i, j >$ nearest neighbor sites of the lattice. With these notations, the

master equation reads

$$\begin{aligned}\partial_t \mathbb{P}_t\big[\{n_\ell\}\big] = \sum_{i\in\mathbb{L}} \Big\{ & \sigma\Big[(n_i-1)\mathbb{P}_t\big[\{n_\ell\}_{i-}\big] - n_i\mathbb{P}_t\big[\{n_\ell\}\big]\Big] \\ & + \mu\Big[(n_i+1)\mathbb{P}_t\big[\{n_\ell\}_{i+}\big] - n_i\mathbb{P}_t\big[\{n_\ell\}\big]\Big] \\ & + \lambda\Big[(n_i+1)n_i\mathbb{P}_t\big[\{n_\ell\}_{i+}\big] - n_i(n_i-1)\mathbb{P}_t\big[\{n_\ell\}_{i}\big]\Big] \\ & + D\sum_{j/<i,j>}\Big[(n_j+1)\mathbb{P}_t\big[\{n_\ell\}_{j+,i-}\big] - n_j\mathbb{P}_t\big[\{n_\ell\}\big]\Big]\Big\}. \end{aligned} \tag{A.3}$$

Each line represents a different elementary process. For each line, the first and second term represent respectively the inflow and outflow of probability for the state $\{n_\ell\}_{\ell\in\mathbb{L}}$. The factors depending on the occupation number simply come from the contribution of each particle present at each site to the total reaction. This equation conserves the total probability. It means that the r.h.s. is zero if summed over the states:

$$\sum_{\{n_\ell\}} \partial_t \mathbb{P}_t\big[\{n_\ell\}\big] = 0\,, \tag{A.4}$$

in agreement with the normalisation of probability $\sum_{\{n_\ell\}} \mathbb{P}_t\big[\{n_\ell\}\big] = 1$. The master equation (A.3) is the starting point to derive the evolution equations for all the averaged observables of DP. This could be done directly. For example, multiplying the equation by n_k and summing over the state space gives the evolution of the average occupation number at the site k, noted $\langle N_k(t)\rangle$. However, it is worthwhile at this point to make an intermediate step by introducing the generating function of the process:

$$G_t(\{z_\ell\}_{\ell\in\mathbb{L}}) \equiv \langle \prod_{\ell\in\mathbb{L}} z_\ell^{N_\ell(t)} \rangle = \sum_{\{n_\ell\}} \mathbb{P}_t\big[\{n_\ell\}\big] \prod_{\ell\in\mathbb{L}} z_\ell^{n_\ell(t)}\,. \tag{A.5}$$

G_t is a multivariate function of $\mathbb{C}^{\mathbb{L}}$ to $\mathbb{C}$ with the property that $G_t(\{1\}_{\ell\in\mathbb{L}}) = 1$. To obtain averaged observables, one has to take the appropriate derivatives of G_t with respect to the z_ℓ's and to evaluate the resulting expression at $\{z_\ell\} = \{1\}$. The general rule for equal-time correlation functions is written more simply in terms of the falling factorial $n^{\underline{k}} = n!/(n-k)!$:

$$\langle \prod_{\alpha=1}^{m} N_{i_\alpha}(t)^{\underline{k_\alpha}} \rangle = \prod_{\alpha=1}^{m} \partial_{z_{i_\alpha}}^{k_\alpha} G_t(\{z_\ell\})\Big|_{\{z_\ell\}=\{1\}} \tag{A.6}$$

One can derive the time evolution of G_t. From (A.3), it reads

$$
\begin{aligned}
\partial_t G_t(\{z_\ell\}) &= \sum_{\{n_\ell\}} \sum_{i\in\mathbb{L}} \Big\{ \sigma\Big[(n_i-1)\mathbb{P}_t\big[\{n_\ell\}_{i-}\big] - n_i\mathbb{P}_t\big[\{n_\ell\}\big]\Big] \\
&+ \mu\Big[(n_i+1)\mathbb{P}_t\big[\{n_\ell\}_{i+}\big] - n_i\mathbb{P}_t\big[\{n_\ell\}\big]\Big] \\
&+ \lambda\Big[(n_i+1)n_i\mathbb{P}_t\big[\{n_\ell\}_{i+}\big] - n_i(n_i-1)\mathbb{P}_t\big[\{n_\ell\}\big]\Big] \\
&+ D\sum_{j/<i,j>}\Big[(n_j+1)\mathbb{P}_t\big[\{n_\ell\}_{j+,i-}\big] - n_j\mathbb{P}_t\big[\{n_\ell\}\big]\Big]\Big\}\prod_{\ell\in\mathbb{L}} z_\ell^{n_\ell} \\
&= \sum_{\{n_\ell\}}\sum_{i\in\mathbb{L}}\Big\{\sigma\Big[z_i^2\partial_{z_i}\mathbb{P}_t\big[\{n_\ell\}_{i-}\big]z_i^{n_i-1} - z_i\partial_{z_i}\mathbb{P}_t\big[\{n_\ell\}\big]z_i^{n_i}\Big]\prod_{\ell\neq i} z_\ell^{n_\ell} \\
&+ \mu\Big[\partial_{z_i}\mathbb{P}_t\big[\{n_\ell\}_{i+}\big]z_i^{n_i+1} - z_i\partial_{z_i}\mathbb{P}_t\big[\{n_\ell\}\big]z_i^{n_i}\Big]\prod_{\ell\neq i} z_\ell^{n_\ell} \\
&+ \lambda\Big[z_i\partial_{z_i}^2\mathbb{P}_t\big[\{n_\ell\}_{i+}\big]z_i^{n_i+1} - z_i^2\partial_{z_i}^2\mathbb{P}_t\big[\{n_\ell\}_{i}\big]z_i^{n_i}\Big]\prod_{\ell\neq i} z_\ell^{n_\ell} \\
&+ D\sum_{j/<i,j>}\Big[z_i\partial_{z_j}\mathbb{P}_t\big[\{n_\ell\}_{j+,i-}\big]z_j^{n_j+1}z_i^{n_i-1} - z_j\partial_{z_j}\mathbb{P}_t\big[\{n_\ell\}\big]z_j^{n_j}z_i^{n_i}\Big]\prod_{\ell\neq i,j} z_\ell^{n_\ell}\Big\} \\
&= \sum_{i\in\mathbb{L}}\Big\{\sigma(z_i^2-z_i)\partial_{z_i} + \mu(1-z_i)\partial_{z_i} \\
&+ \lambda(z_i-z_i^2)\partial_{z_i}^2 + D\sum_{j/<i,j>}(z_i-z_j)\partial_{z_j}\Big\}G_t(\{z_\ell\}) \\
&\equiv H_{\mathrm{DP}}\big(\{z_\ell\},\{\partial_{z_\ell}\}\big)G_t(\{z_\ell\})\,. \qquad \text{(A.7)}
\end{aligned}
$$

Because the original master equation conserves the probability, H_{DP}, the generator of the time-evolution of G_t, is zero when evaluated at $\{z_\ell\}=\{1\}$. This property helps us to calculate the evolution equation for mean values. Indeed, it implies that the terms where no z-derivatives hit on H_{DP} do not give any contribution. For example,

$$
\begin{aligned}
\partial_t\langle N_k(t)\rangle &= \sum_{i\in\mathbb{L}}\Big\{\delta_{ik}\big[\sigma(2z_i-1)\partial_{z_i} - \mu\partial_{z_i} + \lambda(1-2z_i)\partial_{z_i}^2\big] \\
&+ D\sum_{j/<i,j>}(\delta_{ik}-\delta_{jk})\partial_{z_j}\Big\}G_t(\{z_\ell\})\Big|_{\{z_\ell\}=\{1\}} \\
&= \Big\{\sigma(2z_k-1)\partial_{z_k} - \mu\partial_{z_k} + \lambda(1-2z_k)\partial_{z_k}^2 \\
&+ D\big[\sum_{j/<k,j>}\partial_{z_j} - 2d\,\partial_{z_k}\big]\Big\}G_t(\{z_\ell\})\Big|_{\{z_\ell\}=\{1\}} \\
&= \Big\{(\sigma-\mu)\langle N_k(t)\rangle - \lambda\langle N_k(t)^2\rangle + D\big[\sum_{j/<k,j>}\langle N_j(t)\rangle - 2d\,\langle N_k(t)\rangle\big]\Big\}\,.
\end{aligned}
\qquad \text{(A.8)}
$$

Now, let us make the same steps for DEP. We first write down the master equation of the process. We will need two integers per site: m_ℓ and n_ℓ for the

occupation numbers of A and B respectively. The master equation of DEP reads

$$\begin{aligned}\partial_t \mathbb{P}_t\big[\{m_\ell\},\{n_\ell\}\big] = \sum_{i\in\mathbb{L}} \Big\{ & \\ & k\Big[(m_i+1)(n_i-1)\mathbb{P}_t\big[\{m_\ell\}_{i+},\{n_\ell\}_{i-}\big] - m_i n_i \mathbb{P}_t\big[\{m_\ell\},\{n_\ell\}\big]\Big] \\ & + \tau^{-1}\Big[(n_i+1)\mathbb{P}_t\big[\{m_\ell\}_{i-},\{n_\ell\}_{i+}\big] - n_i \mathbb{P}_t\big[\{m_\ell\},\{n_\ell\}\big]\Big] \\ & + \sum_{j/<i,j>} \Big\{ D_A\Big[(m_j+1)\mathbb{P}_t\big[\{m_\ell\}_{j+,i-},\{n_\ell\}\big] - m_j \mathbb{P}_t\big[\{m_\ell\},\{n_\ell\}\big]\Big] \\ & + D_B\Big[(n_j+1)\mathbb{P}_t\big[\{m_\ell\},\{n_\ell\}_{j+,i-}\big] - n_j \mathbb{P}_t\big[\{m_\ell\},\{n_\ell\}_{j+,i-}\big]\Big]\Big\}\Big\}. \end{aligned} \tag{A.9}$$

The generating function depends on two complex numbers a_ℓ and b_ℓ for each site, corresponding to m_ℓ and n_ℓ respectively. Its equation of evolution reads

$$\begin{aligned}\partial_t G_t(\{a_\ell\},\{b_\ell\}) &= \sum_{i\in\mathbb{L}} \Big\{ k(b_i^2 - a_i b_i)\partial_{a_i}\partial_{b_i} + \tau^{-1}(a_i - b_i)\partial_{b_i} \\ &\quad + \sum_{j/<i,j>} \big[D_A(a_i - a_j)\partial_{a_j} + D_B(b_i - b_j)\partial_{b_j}\big]\Big\} G_t(\{a_\ell\},\{b_\ell\}) \\ &\equiv H_{\mathrm{DEP}}\, G_t(\{a_\ell\},\{b_\ell\})\,. \end{aligned} \tag{A.10}$$

Interestingly, the conservation of the total number of particles is expressed at this step by the fact that in each term of H_{DEP}, the number of derivatives with respect to a_ℓ or b_ℓ is equal to the number of multiplication by a_ℓ or b_ℓ. The time evolution for the mean occupation number $\langle M_k\rangle$ and $\langle N_k\rangle$ are obtained by differentiating with respect to a_k and b_k respectively. From them, the mean-field equation for the mean densities $\rho_A(\vec{x},t)$ and $\rho_B(\vec{x},t)$ are obtained using the same argument as for DP.

Appendix B

Out of equilibrium field theories and NPRG

B.1 The Martin-Siggia-Rose, Janssen, de Dominicis formalism

Let us describe the MSRJD formalism for the general situation of a set of n generic stochastic fields $\{\Phi_i(\mathbf{x})\}_{1\leq i\leq n}$ defined through a set of m stochastic partial differential equations with n constraints and l noises

$$\begin{aligned} &\partial_t \Phi_i = F_i(\Phi) + G_{ij}(\Phi)\xi_j\,, && 1\leq i\leq m \\ &\Phi_i(0,\vec{x}) = \phi_{i,0}(\vec{x})\,, && 1\leq i\leq m \\ &H_i(\Phi) = 0\,, && 1\leq i\leq n \end{aligned} \tag{B.1}$$

where the F_i, G_{ij} and H_i are functions of the Φ_i's and their spatial derivatives and the $\{\xi_j\}_{1\leq j\leq l}$ are centered stationary Gaussian fields of correlator D_{ij}:

$$\langle \xi_i(\mathbf{x})\xi_j(\mathbf{x}')\rangle = D_{ij}(\mathbf{x},\mathbf{x}')\,. \tag{B.2}$$

In order to obtain the generating functional of the Φ_i, as well as their dynamical responses, one introduces linear source terms $\bar{j}_i, \bar{k}_i$ to the right-hand sides of equations (B.1):

$$\begin{aligned} &\partial_t \Phi_i = F_i(\Phi) + G_{ij}(\Phi)\xi_j + \bar{j}_i\,, && 1\leq i\leq m \\ &H_i(\Phi) = \bar{k}_i\,, && 1\leq i\leq n\,. \end{aligned} \tag{B.3}$$

The corresponding generalized generating functional reads

$$\mathcal{Z}[j,\bar{j},\bar{k}] = \langle e^{\int_{\mathbf{x}} j_i\Phi_i}\rangle_{\bar{j},\bar{k}}\,, \tag{B.4}$$

where $\langle\cdot\rangle_{\bar{j},\bar{k}}$ denotes a mean on the stochastic equation in the presence of the sources $\bar{j}_i, \bar{k}_i$. Following the same steps as in the main text, we have

$$\mathcal{Z}[j,\bar{j},\bar{k}] = \int \mathcal{D}[\phi] e^{\int_{\mathbf{x}} j_i\phi_i}\delta[H(\phi)-\bar{k}]\langle\delta[\phi-\Phi_\xi]\rangle_{\bar{j}}\,, \tag{B.5}$$

where Φ_ξ is the weak solution of (B.1) for a given ξ. Replacing the constraint $\langle\delta[\phi-\Phi_\xi]\rangle_{\bar{j}}$ by the explicit equation of motion of Φ in the presence of the sources $\bar{j}_i$, noted $\mathcal{F}(\cdot)=0$, with

$$\mathcal{F}_i(\mathbf{x}) = \partial_t\Phi_i - F_i(\Phi) - G_{ij}(\Phi)\xi_j - \bar{j}_i\,, \tag{B.6}$$

and by the initial conditions, noted collectively as $\phi|_{t=0} = \phi_0$, one then obtains

$$\mathcal{Z}[j,\bar{j},\bar{k}] = \int \mathcal{D}[\phi] e^{\int_{\mathbf{x}} j_i \phi_i} \delta[H(\phi) - \bar{k}] \delta[\phi|_{t=0} - \phi_0] \langle \delta[\mathcal{F}] \rangle \times \mathcal{J} \,. \tag{B.7}$$

$\mathcal{J}$ is the Jacobian of the transformation, $\mathcal{J} = |\det\left(\frac{\delta \mathcal{F}_i(\mathbf{x})}{\delta \phi_j(\mathbf{x}')}\right)|$, which depends on the choice of discretization of (B.1). In a forward discretization, $\mathcal{J}$ does not depend on the fields (Täuber 2014), and can be absorbed in the functional measure.

Finally, using the Fourier representation of the functional Dirac deltas:

$$\begin{aligned}
\mathcal{Z}[j,\bar{j},\bar{k}] &= \int \mathcal{D}[\phi,\bar{\phi},\bar{h}] e^{\int_{\mathbf{x}} j_i \phi_i} e^{-i\int_{\mathbf{x}} \{\bar{h}_i(H_i - \bar{k}_i) + \delta(t)\bar{\phi}_{i,0}(\phi_i - \phi_{i,0})\}} \langle e^{-i\int_{\mathbf{x}} \bar{\phi}_i \mathcal{F}_i} \rangle \\
&= \int \mathcal{D}[\phi,\bar{\phi},\bar{h}] e^{\int_{\mathbf{x}} \{j_i\phi_i + i\bar{j}_i\bar{\phi}_i + i\bar{k}_i\bar{h}_i\}} e^{-i\int_{\mathbf{x}} \{\bar{h}_i H_i + \delta(t)\bar{\phi}_{i,0}(\phi_i - \phi_{i,0})\}} e^{-i\int_{\mathbf{x}} \bar{\phi}_i(\partial_t \phi_i - F_i)} \langle e^{i\int_{\mathbf{x}} \bar{\phi}_i G_{ij} \xi_j} \rangle \\
&= \int \mathcal{D}[\phi,\bar{\phi},\bar{h}] e^{\int_{\mathbf{x}} \{j_i\phi_i + \bar{j}_i\bar{\phi}_i + \bar{k}_i\bar{h}_i\}} e^{-\mathcal{S}} \\
\text{with } \mathcal{S} &= \int_{\mathbf{x}} \Big\{ \bar{\phi}_i(\mathbf{x}) \big[\partial_t \phi_i(\mathbf{x}) - F_i(\mathbf{x})\big] + \bar{h}_i H_i + \delta(t)\bar{\phi}_{i,0}(\phi_i - \phi_{i,0}) \Big\} \\
&\quad - \frac{1}{2} \int_{\mathbf{x},\mathbf{x}'} (\bar{\phi}_i G_{ij})(\mathbf{x}) D_{jk}(\mathbf{x} - \mathbf{x}') (G_{k\ell}\bar{\phi}_\ell)(\mathbf{x}') \,,
\end{aligned} \tag{B.8}$$

using the property (B.2) to compute the average value in the second line. In the last line, the response field have been redefined to absorb the i.

B.2 The prescription $\Theta(0) = 0$ in perturbation theory

It is often stated, notably in perturbative RG, that the Itô prescription amounts to take the prescription $\Theta(0) = 0$ when necessary in calculating loop integrals. This prescription comes from the fact that in the vertices of the microscopic action, the response fields are always evaluated at a later time than the observable fields. We go back to the general response field action of Sec. 3.3.3, corresponding to a Langevin equation for a scalar field. Its discretized version reads

$$\begin{aligned}
\mathcal{S}[\phi,\bar{\phi}] = \sum_{t=0}^{T-1} \int_{\vec{x}} \Big\{ \bar{\phi}_{t_+}(\phi_{t_+} - \phi_t) + \mathrm{d}t \big[\bar{\phi}_{t_+}(-\partial^2 + \sigma)\phi_t - \Delta\, (\partial_x^a \bar{\phi}_{t_+})^2 \\
+ \bar{\phi}_{t_+}\, U^1(\phi_t) - \frac{1}{2}\bar{\phi}_{t_+}^2\, U^2(\phi_t) \big] \Big\} \,,
\end{aligned} \tag{B.9}$$

with the notation $t_- = t - 1$ and with σ, $\Delta > 0$. Let us look back at the integrals appearing in the perturbative corrections to the mean field calculated in Sec. 3.3.3, for example the second term of

$$\bar{\Gamma}^{(2)}_{\bar{\phi}\phi}(\mathbf{p})\big|_4 = \frac{1}{2} \int_{\mathbf{q}} \Big\{ C_{\mathbf{q}} u_{1,3} + R_{\mathbf{q}} u_{2,2} + c.c. \Big\} \,. \tag{B.10}$$

On the one hand, we can try to calculate directly the integral in the continuous limit. Using the mean field propagator, which reads

$$\big[G_0^{(2)}\big]_{\phi\bar{\phi}}(\mathbf{x},\mathbf{y}) = \Theta(t-u) \int_{\vec{p}} e^{i\vec{p}\cdot(\vec{x}-\vec{y}) - (p^2+\sigma)(t-u)} \tag{B.11}$$

we obtain

$$\bar{\Gamma}^{(2)}_{\bar{\phi}\phi}(\mathbf{p})\big|_{2,2} \propto \int_{\mathbf{q}} R_{\mathbf{q}} \propto \big[G_0^{(2)}\big]_{\phi\bar{\phi}}(\mathbf{0}) \propto \Theta(0)\,. \tag{B.12}$$

On the other hand, going back to the integration in space and time and using the discretization, we have

$$\bar{\Gamma}^{(2)}_{\bar{\phi}\phi}(\mathbf{x},\mathbf{y})\big|_{2,2} = \frac{1}{2}\int_{\mathbf{z}_1,\mathbf{z}_2} \big[G_0\big]_{\phi\bar{\phi}}(\mathbf{z}_1,\mathbf{z}_2)\,.\mathcal{S}^{(4)}_{\phi\bar{\phi}\phi\bar{\phi}}(\mathbf{z}_1,\mathbf{z}_2,\mathbf{x},\mathbf{y})\,. \tag{B.13}$$

Due to the choice of forward discretization, the interaction term is non-zero only if the times are well ordered. Recalling that $\mathbf{x} = (t,\vec{x})$, $\mathbf{y} = (u,\vec{y})$ and $\mathbf{z} = (v,\vec{z})$, the interaction term is non-zero only if

$$v_1 < v_2 \text{ and } v_1 < u \text{ and } t < v_2 \text{ and } t < u\,. \tag{B.14}$$

However, the propagator is non-zero only if $v_1 \geq v_2$ due to the Θ-function. Thus the integral is zero. This result can be encoded in the prescription "$\Theta(0) = 0$". The same result can be derived directly in the frequency integral, although the calculations are more cumbersome. To do so, we need the expression of the discretized propagator. The space-time Fourier transform of the Hessian of the discretized action reads

$$\bar{\mathcal{S}}^{(2)}_{\bar{\phi}\phi}(\mathbf{p}) = \frac{1 + e^{i\omega\,\mathrm{d}t}\big[-1 + (p^2+\sigma)\,\mathrm{d}t\big]}{\mathrm{d}t} = e^{i\omega\,\mathrm{d}t}\left[\frac{e^{-i\omega\,\mathrm{d}t}-1}{\mathrm{d}t} + p^2 + \sigma\right]. \tag{B.15}$$

Thus the discretized propagator reads

$$\big[G_0^{(2)}\big]_{\phi\bar{\phi}}(\mathbf{p}) = \frac{e^{-i\omega\,\mathrm{d}t}}{\frac{e^{-i\omega\,\mathrm{d}t}-1}{\mathrm{d}t} + p^2 + \sigma}\,. \tag{B.16}$$

Plugging this expression in the second term of (B.10) and replacing the vertex by its discretized version:

$$u_{2,2} \to u_{2,2}\, e^{i\omega\,\mathrm{d}t + i\varpi\,\mathrm{d}t}\,, \tag{B.17}$$

the frequency integral to be calculated becomes

$$\int_{\varpi} u_{2,2} R_{\mathbf{q}} \to u_{2,2}\, e^{i\omega\,\mathrm{d}t} \int_{|\varpi|<\pi/\mathrm{d}t} \frac{1}{\frac{e^{-i\varpi\,\mathrm{d}t}-1}{\mathrm{d}t} + p^2 + \sigma}\,. \tag{B.18}$$

In the limit $\mathrm{d}t \to 0$, the frequency contour has to be closed in the upper-half complex plane for Jordan's lemma to apply and the integral vanishes as the (discretized) propagator has only poles in the lower-half complex plane. However, it is worthwhile to note that the prescription $\Theta(0) = 0$ cannot be applied outside of the loop integrals. Indeed, the equal-time propagator is

$$\big[G_0^{(2)}\big]_{\phi\bar{\phi}}(0,\vec{x}-\vec{y}) = \int_{\vec{p}} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \lim_{\mathrm{d}t\to 0} \int_{|\omega|<\pi/\mathrm{d}t} \frac{e^{-i\omega\,\mathrm{d}t}}{\frac{e^{-i\omega\,\mathrm{d}t}-1}{\mathrm{d}t} + p^2 + \sigma}\,.$$

We check that Jordan's lemma applies by closing the contour in the lower-half of the complex plane. The integrand has only one pole, at

$$\omega^* = i\ \mathrm{d}t^{-1} \ln\big[1 - \mathrm{d}t(p^2+\sigma)\big]\,, \tag{B.19}$$

which is in the contour. Using the residue theorem, the ω integral is calculated to be unity and one is left with

$$\big[G_0^{(2)}\big]_{\phi\bar{\phi}}(0,\vec{x}-\vec{y}) = \int_{\vec{p}} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = \delta^d(\vec{x}-\vec{y}) \neq 0\,. \tag{B.20}$$

Appendix C

Mappings to field theories

C.1 The Doi-Peliti construction

In this appendix, we expound on the Doi-Peliti construction of a coherent field action, which is the canonical formalism for constructing an action from the master equation of reaction-diffusion processes. For the sake of simplicity, we will use the example of the one-site reaction $2A \xrightarrow{\lambda} A$. Following the notations of Appendix A, one readily writes the corresponding master equation giving the evolution of the probabilities associated to the process:

$$\partial_t \mathbb{P}_t\big[n\big] = \lambda\Big\{(n+1)n\,\mathbb{P}_t\big[n+1\big] - n(n-1)\mathbb{P}_t\big[n\big]\Big\}\,. \tag{C.1}$$

Accordingly, the evolution of the generating functional G, defined in (A.5) is given by

$$\partial_t G(t,z) = \lambda(z-z^2)\partial_z^2 G(t,z) \equiv H_\lambda(z,\partial_z)G(t,z)\,. \tag{C.2}$$

The conservation of probability is enforced by the property $H_\lambda(1,\partial_z) = 0$. This equation can be read in the framework of the Bargmann Hilbert space $\mathscr{H}$ of analytic functions of the complex plane with finite norms for the following scalar product

$$(f_1, f_2) = \int_{\mathbb{C}} \frac{\mathrm{d}^2 z}{\pi} e^{-z^* z} f_1^*(z) f_2(z) = \langle f_1 | f_2 \rangle\ . \tag{C.3}$$

In the second equality, the Dirac notation is used, with the linear functionals on $\mathscr{H}$ given by their representations through the scalar product. A complete orthogonal family for this scalar product are the monomes $\{|n\rangle : z \mapsto z^n\}_{n\geq 0}$,

$$\langle m|n\rangle = n!\delta_{m,n}\,. \tag{C.4}$$

We note $\hat{c}$ and $\hat{a}$ the following operators acting on $f \in \mathscr{H}$, which are adjoint of one another for the scalar product:

$$\hat{c} : f \mapsto zf \tag{C.5}$$

$$\hat{a} : f \mapsto \partial_z f \tag{C.6}$$

These operators act on the monomes as $\hat{c}\,|n\rangle = |n+1\rangle$ and $\hat{a}\,|n\rangle = n\,|n-1\rangle$. As a consequence, it is readily shown that $\hat{c}$ and $\hat{a}$ verify the commutation relation of ladder operators:

$$[\hat{c}, \hat{a}] = 1\,. \tag{C.7}$$

The operator H_λ giving the evolution of $|G(t)\rangle : z \mapsto G(t,z)$ can be reinterpreted replacing the z by $\hat{c}$ and the ∂_z by $\hat{a}$ to give the operator $H_\lambda(\hat{c},\hat{a})$. Due to the way it was obtained, it is already normal-ordered, that is all the annihilation operators $\hat{a}$ act before the creation operators $\hat{c}$. Now, let us construct the coherent states corresponding to these operators, that is vector of $\mathscr{H}$ which are eigenstates of $\hat{a}$: for $\phi \in \mathbb{C}$,

$$\hat{a}\,|\phi\rangle = \phi\,|\phi\rangle\ . \tag{C.8}$$

Functionaly, it amounts to look for $f \in \mathscr{H}$ such that $\partial_z f = \phi f$, which is solved by $|\phi\rangle : z \mapsto \phi_0 e^{\phi z}$. Now, we make the choice of $\phi_0 = 1$ for commodity. This choice leads to a dual family of linear functionals which have a very simple expression:

$$\langle\phi| : f \mapsto \langle\phi|f\rangle = \int_{\mathbb{C}} \frac{\mathrm{d}^2 z}{\pi} e^{-z^* z} e^{\phi^* z^*} f(z) = f(\phi^*) \tag{C.9}$$

with the property

$$\forall f \in \mathscr{H}, \quad \langle\phi|\,\hat{c}\,|f\rangle = \phi^*\,\langle\phi|f\rangle\ . \tag{C.10}$$

However, note that this choice prevents us from interpreting the coherent states as generating functionals of a probability distribution as they are not equal to 1 at $z = 1$. A particularly useful coherent linear functional is $\langle 1|$. Indeed, the normalisation of the generating functional can be rewritten as, $\langle 1|G(t)\rangle = G(t,1) = 1$ and the property of conservation of probability of H_λ amounts to have $\langle 1|\,H_\lambda(\hat{c},\hat{a}) = \langle 1|\,H_\lambda(1,\hat{a}) = 0$. There is a non-zero overlap between the coherent states and their duals:

$$\langle\phi_1|\phi_2\rangle = e^{\phi_1^* \phi_2}\,. \tag{C.11}$$

It is straightforwardly shown that this overcomplete family gives a resolution of the identity on $\mathscr{H}$: for $f \in \mathscr{H}$

$$|f\rangle = \int_{\mathbb{C}} \frac{\mathrm{d}^2\phi}{\pi} e^{-\phi^*\phi}\,|\phi\rangle\,\langle\phi|f\rangle\ . \tag{C.12}$$

Now that the framework is fully set, we can write $|G(t)\rangle$ as a path integral. First, let us perturb the infinitesimal evolution operator by adding a coupling to sources:

$$H_\lambda(\hat{c},\hat{a}) \to H'_\lambda(t,\hat{c},\hat{a}) = H_\lambda(\hat{c},\hat{a}) + j^*(t)\hat{c} + j(t)\hat{a}\,. \tag{C.13}$$

H'_λ does not conserve probability and $\mathcal{Z}[j,\bar{j}] \equiv \langle 1|G(t)\rangle \neq 1$ but these sources allow one to calculate averages on the initial process. For example,

$$\frac{\delta\mathcal{Z}[j,j^*]}{\delta j(t')}\Big|_{j,j^*=0} = \langle 1|\,e^{(t-t')H_\lambda(\hat{c},\hat{a})}\,\hat{a}\,e^{t'H_\lambda(\hat{c},\hat{a})}\,|G(0)\rangle = \langle 1|\,\hat{c}\hat{a}\,|G(t')\rangle\ , \tag{C.14}$$

where we have used in the second equality that $\langle 1|\,\hat{c} = \langle 1|$ to cancel the evolution from t' to t and to insert a $\hat{c}$. Using the definition of $G(t)$, we obtain

$$\frac{\delta\mathcal{Z}[j,j^*]}{\delta j(t')}\Big|_{j,j^*=0} = \langle 1|\,\hat{c}\hat{a}\sum_{n\leq 0}\mathbb{P}_{t'}\big[n\big]\,|n\rangle = \sum_{n\leq 0} n\,\mathbb{P}_{t'}\big[n\big] = \langle N(t')\rangle\,. \tag{C.15}$$

Thus, one functional derivative with respect to $j(t')$ gives the mean occupation number at the time t'. We do not underake a more complete discussion of the relation

between observables and the introduced sources here. The modified generating vector $|G(t)\rangle$ can be expressed as

$$|G(t)\rangle = \mathcal{T}\Big\{ e^{\int_t H'_\lambda(t,\hat{c},\hat{a})} \Big\} |G(0)\rangle = \lim_{T\to\infty} \mathcal{T}\Big\{ \prod_{t=1}^{T} \big(1 + H'_\lambda(t,\hat{c},\hat{a})\tau\big) \Big\} |G(0)\rangle \tag{C.16}$$

where the $\mathcal{T}$ stands for "time-ordered product", that is the terms of the product are ordered from right to left, from the most anterior to the most posterior. Now, using this expression in the partition function and inserting the resolution of the identity (C.12) at each time slice, one gets

$$\begin{aligned}
&\mathcal{Z}[j,j^*] \\
&= \int_{\mathbb{C}} \frac{\mathrm{d}^2\phi_T}{\pi} e^{-\phi_T^*\phi_T} \langle 1|\phi_T\rangle \\
&\times \langle \phi_T| \mathcal{T}\Big\{ \prod_{t=1}^{T} \big(1 + H'_\lambda(t,\hat{c},\hat{a})\tau\big) \int_{\mathbb{C}} \frac{\mathrm{d}^2\phi_{t-1}}{\pi} e^{-\phi_{t-1}^*\phi_{t-1}} |\phi_{t-1}\rangle \langle\phi_{t-1}| \Big\} |G(0)\rangle \\
&= \int \mathcal{D}[\phi,\phi^*] \langle 1|\phi_T\rangle \Big\{ \prod_{t=1}^{T} e^{-\phi_t^*\phi_t} \langle\phi_t| \big(1 + H'_\lambda(t,\hat{c},\hat{a})\tau + o(\tau)\big) |\phi_{t-1}\rangle \Big\} e^{-\phi_0^*\phi_0} \langle\phi_0|G(0)\rangle
\end{aligned} \tag{C.17}$$

where

$$\int \mathcal{D}[\phi,\phi^*] = \int_{\mathbb{C}^T} \prod_{t=0}^{T} \frac{\mathrm{d}^2\phi_t}{\pi} \,. \tag{C.18}$$

Now, using the properties of the coherent states, we have

$$\begin{aligned}
\langle\phi_t| \big(1 + H'_\lambda(t,\hat{c},\hat{a})\tau + o(\tau)\big) |\phi_{t-1}\rangle &= e^{\phi_t^*\phi_{t-1}} \big(1 + H'_\lambda(t,\phi_t^*,\phi_{t-1})\tau + o(\tau)\big) \\
&= e^{\phi_t^*\phi_{t-1}} e^{H'_\lambda(t,\phi_t^*,\phi_{t-1})\tau + o(\tau)} \,.
\end{aligned} \tag{C.19}$$

Inserting this expression in the partition function, we readily find

$$\mathcal{Z}[j,j^*] = \int \mathcal{D}[\phi,\phi^*] e^{i\, j^*\cdot\phi^* + i\, j\cdot\phi_-} e^{-\mathcal{S}[\phi,\phi^*]} \tag{C.20}$$

with

$$\begin{aligned}
\mathcal{S}[\phi,\phi^*] &= -\phi_T + \sum_{t=1}^{T} \Big\{ \phi_t^*(\phi_t - \phi_{t-1}) - H(\phi_t^*,\phi_{t-1})\tau \Big\} + \phi_0^*\phi_0 - \ln\langle(\phi_0^*)^{N_0}\rangle + o(\tau T) \\
j^*\cdot\phi^* &= \sum_{t=1}^{T} j_t\phi_t\tau \,, \\
j\cdot\phi_- &= \sum_{t=1}^{T} j_t\phi_{t-1}\tau \,,
\end{aligned} \tag{C.21}$$

and N_0 denotes the (possibly random) initial occupation number. If N_0 is a Poisson variable of parameter μ_0, the initial condition simplifies into

$$\phi_0^*\phi_0 - \ln\langle(\phi_0^*)^{N_0}\rangle = \phi_0^*\phi_0 - \mu_0(\phi_0^* - 1) = \phi_0^*(\phi_0 - \mu_0) + \mu_0 \,. \tag{C.22}$$

The construction of the coherent field action generalizes straightforwardly to m types of particle $X_{1\leq i\leq m}$ (which can represent sites) and r reactions. If the reactions are written symbolically as

$$\forall\, k \leq r\,, \quad \sum_{i=1}^{m} q_k^i X_i \xrightarrow{q_k!\lambda_k} \sum_{i=1}^{m} p_k^i X_i\,, \tag{C.23}$$

the action then reads

$$\begin{aligned}\mathcal{S}[\{\phi,\phi^*\}] = &-\sum_{i=1}^{m}\phi_T^i + \sum_{t=1}^{T}\Big\{\phi_{i,t}^*(\phi_{i,t}-\phi_{i,t-1}) - H(\{\phi_t^*,\phi_{t-1}\})\tau\Big\}\\ &+\phi_{i,0}^*\phi_{i,0} - \ln\langle\prod_{i=1}^{m}(\phi_{i,0}^*)^{N_0^i}\rangle + o(\tau T)\,,\end{aligned} \tag{C.24}$$

with the following infinitesimal generator of the time evolution

$$H(\{\phi_t^*,\phi_{t-1}\}) = \sum_{k=1}^{r}\lambda_k\left[\prod_{i=1}^{m}(\phi_{i,t}^*)^{p_k^i} - \prod_{i=1}^{m}(\phi_{i,t}^*)^{q_k^i}\right]\prod_{i=1}^{m}(\phi_{i,t-1})^{q_k^i}\,. \tag{C.25}$$

This action possesses a similarity with the action obtained through the response field formalism, namely the ϕ fields are always evaluated before the ϕ^* (it is also the case between ϕ and its source, to our knowledge this particularity has never been mentioned). This fact suggests to interpret the ϕ fields as observable fields and the ϕ^* ones as response fields. However, the above action vanishes for $\phi^* = 1$ due to conservation of probability, but generally it does not vanish for $\phi^* = 0$ as it is the case for response field actions. To cure this problem, one can shift the ϕ^* in the following way:

$$\phi^* = \bar{\phi} + 1\,, \tag{C.26}$$

such that the shifted action vanishes for $\bar{\phi} = 0$. Specifying to independent Poisson distributed initial occupation numbers, with parameters $\{\mu_{i,0}\}_{1\leq i\leq m}$, the shifted action reads

$$\mathcal{S}[\{\phi,\bar{\phi}\}] = \sum_{t=1}^{T}\Big\{\bar{\phi}_{i,t}(\phi_{i,t}-\phi_{i,t-1}) - H(\{\bar{\phi}_t+1,\phi_{t-1}\})\tau\Big\} + \bar{\phi}_{i,0}(\phi_{i,0}-\mu_{i,0}) + o(\tau T)\,. \tag{C.27}$$

Let us point out that contrary to the response field formalism, ϕ and ϕ^* are complex conjugate. As a consequence, ϕ and $\bar{\phi}$ are not independent of each other. While it is possible to show perturbatively that one can deform the fields to be real and imaginary respectively, there is no general non-pertubative result in this direction. Indeed, a careless application of this deformation famously leads to negative variances for the Gaussian noise of the corresponding stochastic process in certain cases (Howard and Täuber 1997; Muñoz 1998). Let us note that (Benitez et al. 2016) were able to give a deformation applicable for one species bimolecular processes, but where ϕ^* plays the role of the observable field and ϕ that of the response field of a dual process of the original one. Nonetheless, at this point we choose to leave this problem aside as it is done traditionally and interpret the coherent field action in the same settings as a response field action with real fields.

A further step to get a workable action is to take the continuous time limit, that is to take $T = t_f/\tau \to \infty$, keeping t_f fixed, or equivalently to take $\tau \to 0$. We introduce interpolating fields of ϕ and $\bar{\phi}$:

$$\begin{aligned} &\phi_{i,t} \equiv \tilde{\phi}_i(\tilde{t})\,, \quad \bar{\phi}_{i,t} \equiv \tilde{\bar{\phi}}_i(\tilde{t})\,, \\ &\text{with } \tilde{t} \equiv t_f \frac{t}{T}\,. \end{aligned} \tag{C.28}$$

Assuming that we are interested in differentiable paths, the finite difference rewrites as

$$\phi_{i,t} - \phi_{i,t-1} = \partial_{\tilde{t}} \tilde{\phi}_i(\tilde{t})\tau + o(\tau)\,. \tag{C.29}$$

The difference of evaluation time between $\bar{\phi}$ and ϕ in the evolution operator and between j and ϕ vanishes in the limit $\tau \to 0$:

$$\sum_{t=1}^{T} H(\{\bar{\phi}_t + 1, \phi_{t-1}\})\tau = \sum_{t=1}^{T} H(\{\tilde{\bar{\phi}}(\tilde{t}) + 1, \tilde{\phi}(\tilde{t})\})\tau + o(\tau T)\,, \quad j \cdot \phi_- = j \cdot \tilde{\phi} + o(\tau T)\,. \tag{C.30}$$

Finally, the time continuous action reads, dropping the tilde symbol

$$\mathcal{S}[\{\phi, \phi^*\}] = \int_0^{t_f} \mathrm{d}t \Big\{ \bar{\phi}_i(t)\partial_t \phi_i(t) - H[\{\bar{\phi} + 1, \phi\}](t) \Big\} + \bar{\phi}_i(0)\big[\phi_i(0) - \mu_{i,0}\big]\,. \tag{C.31}$$

C.2 Gardiner's Poisson Representation

We give here a short account on an alternative derivation of coherent field which can shed light on the Doi shift. This derivation was developed in (Gardiner and Chaturvedi 1977) (see Gardiner 2009 for a more in depth presentation). To simplify the notation we will work in the zero-dimensional case. The first step is to define the (positive) Poisson representation of a probability distribution for the number of particles for the N species:

$$\mathbb{P}_t[\{n_\ell\}] = \int_{\mathbb{C}^{\mathbb{N}}} \prod_i \frac{\mathrm{d}^2\alpha_i}{2\pi} \frac{e^{-\alpha_i}\alpha_i^{n_i}}{n_i!} f(t, \{\alpha_\ell\})\,. \tag{C.32}$$

That is, we expand the probability distribution on a sum of "complex Poisson distribution" of parameter $\{\alpha\}$. Alternatively, we can directly use the Poisson representation on the generating function:

$$G(\{z\}, t) = \sum_{\{n\}} \mathbb{P}_t[\{n_\ell\}] \prod_i z_i^{n_i} = \int_{\mathbb{C}^{\mathbb{N}}} \prod_i \frac{\mathrm{d}^2\alpha_i}{2\pi} e^{\alpha_i(z_i - 1)} f(t, \{\alpha_\ell\})\,. \tag{C.33}$$

Using the equation for the time evolution of G, integration by parts and unicity of the Poisson representation, one obtains an evolution equation for f. For example for the pure fusion rule $2A \xrightarrow{2\lambda} A$, the time evolution of G reads

$$\partial_t G(t, z) = \lambda(z - z^2)\partial_z^2 G(t, z) \equiv H_\lambda(z, \partial_z) G(t, z)\,, \tag{C.34}$$

and the corresponding time evolution of f is

$$\partial_t f(t, \alpha) = \lambda\big[(1 - \partial_\alpha) - (1 - \partial_\alpha)^2\big]\alpha^2 f(t, \alpha) = H_\lambda(1 - \partial_\alpha, \alpha) f(t, \alpha)\,. \tag{C.35}$$

If the reaction-diffusion process involves at most bimolecular reactions, the master equation translates into a Fokker-Planck equation for f. In the case of the pure fusion, it reads

$$\partial_t f(t,\alpha) = \lambda\big[\partial_\alpha - \partial_\alpha^2\big]\alpha^2 f(t,\alpha) = \partial_\alpha[A\,f](t,\alpha) + \frac{1}{2}\partial_\alpha^2[B\,f](t,\alpha) \tag{C.36}$$

with in this example

$$A = \lambda\alpha^2\,, \quad B = -2\lambda\alpha^2\,. \tag{C.37}$$

If α is constrained to be real, (C.36) is not a Fokker-Planck equation as it is impossible to find a root of the non-positive B. However, we can write a Langevin equation associated to (C.36) for a complex random variable. It reads

$$\begin{aligned} \partial_t\alpha &= -\lambda\alpha^2 + i\sqrt{2\lambda}\alpha\xi \\ \langle\xi(t)\xi(t')\rangle &= \delta(t-t') \end{aligned} \tag{C.38}$$

where ξ is a real noise. Using the MSRJD formalism (see Appendix B.1) on this Langevin equation yields the following action

$$\mathcal{S}[\alpha,\bar\alpha] = \int_t \big\{\bar\alpha(\partial_t + \lambda\alpha^2) + \lambda\alpha^2\bar\alpha^2\big\}\,, \tag{C.39}$$

which is exactly the Doi shifted coherent field action for the same process. In the case of processes involving more than two particles, the construction of higher order noise have been explored in (Gardiner 2009). This sketch of derivation leaves a lot of things unclear, notably the vanishing of boundary terms in the complex integral and the unicity of the Poisson representation (Droz and McKane 1994; Drummond 2004). However, we will not dwell further on the subject and simply comment that in the case when this derivation can be put on a more stable ground, it gives an interpretation of the coherent field action in terms of a dual continuous stochastic process. If α stays in the positive real line, it corresponds to a process whose instantaneous probability can always be decomposed in a basis of (real) Poisson distribution, while a complex α corresponds to a narrower distribution as noticed in (Wiese 2016). As an application, this gives a definition of multi-time expectation values of the original process, which is more difficult to define in the standard Doi-Peliti formalism (Chaturvedi and Gardiner 1978).

Appendix D

Consequence of the duality identity

In this appendix, we expound on the consequences of the duality identity (4.40) of DP-C, which reads

$$g\,\partial_\sigma \Gamma_\kappa = \int_{\mathbf{x}} \frac{\delta \Gamma_\kappa}{\delta \Phi(\mathbf{x})}\,. \tag{D.1}$$

Let us partially expand Γ_κ in powers of Φ around a stationary homogeneous value χ_κ. Noting

$$[\Phi - \chi_\kappa]^{\otimes n}(\{\mathbf{x}_\ell\}_{1\le\ell\le n}) = \prod_k \left[\Phi(\mathbf{x}_k) - \chi_\kappa\right], \tag{D.2}$$

the expansion reads

$$\Gamma_\kappa[\Phi] = \sum_n \frac{1}{n!} \int_{\{\mathbf{x}_\ell\}} \Gamma_\kappa^{(n)}[\{\mathbf{x}_\ell\}_{1\le\ell\le n};\Phi]\big|_{\Phi\to\chi_\kappa} [\Phi-\chi_\kappa]^{\otimes n}(\{\mathbf{x}_\ell\}_{1\le\ell\le n})\,. \tag{D.3}$$

Plugging the expansion in the identity (D.1) gives on the one hand for the l.h.s.

$$\begin{aligned} g\,\partial_\sigma\Gamma_\kappa = \sum_n \frac{1}{n!} \int_{\{\mathbf{x}_\ell\}} \Big\{ & g\,\partial_\sigma\Big[\Gamma_\kappa^{(n)}[\{\mathbf{x}_\ell\}_{1\le\ell\le n};\Phi]\big|_{\Phi\to\chi_\kappa}\Big][\Phi-\chi_\kappa]^{\otimes n}(\{\mathbf{x}_\ell\}_{1\le\ell\le n}) \\ & + n\,\Gamma_\kappa^{(n)}[\{\mathbf{x}_\ell\}_{1\le\ell\le n};\Phi]\big|_{\Phi\to\chi_\kappa}[\Phi-\chi_\kappa]^{\otimes n-1}(\{\mathbf{x}_\ell\}_{1\le\ell\le n-1})(-g\,\partial_\sigma\chi_\kappa)\Big\}\,. \end{aligned} \tag{D.4}$$

and the other hand for the r.h.s.

$$\int_{\mathbf{x}} \frac{\delta\Gamma_\kappa}{\delta\Phi(\mathbf{x})} = \sum_n \frac{1}{n!}\int_{\{\mathbf{x}_\ell\}} n\,\Gamma_\kappa^{(n)}[\{\mathbf{x}_\ell\}_{1\le\ell\le n};\Phi]\big|_{\Phi\to\chi_\kappa}[\Phi-\chi_\kappa]^{\otimes n-1}(\{\mathbf{x}_\ell\}_{1\le\ell\le n-1})\,, \tag{D.5}$$

using the symmetry of the vertex functions. Equalizing the two sides and using unicity of the expansion, one obtains that

$$\begin{aligned} &\forall\, n,\ \{\chi_\kappa,\bar\Phi,\Psi,\bar\Psi\},\ \{\mathbf{x}_\ell\}_{1\le\ell\le n}, \\ &g\,\partial_\sigma\Big[\Gamma_\kappa^{(n)}[\{\mathbf{x}_\ell\}_{1\le\ell\le n};\Phi]\big|_{\Phi\to\chi_\kappa}\Big] = (n+1)(1+g\,\partial_\sigma\chi_\kappa)\int_{\mathbf{x}_{n+1}} \Gamma_\kappa^{(n+1)}[\{\mathbf{x}_\ell\}_{1\le\ell\le n+1};\Phi]\big|_{\Phi\to\chi_\kappa}\,. \end{aligned} \tag{D.6}$$

Now, if we assume the following:

$$\begin{aligned}\exists\, n,\ \{\chi_\kappa, \bar{\Phi}, \Psi, \bar{\Psi}\},\ \{\mathbf{x}_\ell\}_{1\leq\ell\leq n},& \\ \partial_\sigma\Big[\Gamma_\kappa^{(n)}[\{\mathbf{x}_\ell\}_{1\leq\ell\leq n};\Phi]\big|_{\Phi\to\chi_\kappa}\Big] &= 0 \\ \text{and}\quad \int_{\mathbf{x}_{n+1}} \Gamma_\kappa^{(n+1)}[\{\mathbf{x}_\ell\}_{1\leq\ell\leq n+1};\Phi]\big|_{\Phi\to\chi_\kappa} &\neq 0\,, \end{aligned} \tag{D.7}$$

then from (D.6) one gets

$$1 + g\,\partial_\sigma\chi_\kappa = 0\,, \tag{D.8}$$

and finally

$$\forall\, n,\ \{\chi_\kappa, \bar{\Phi}, \Psi, \bar{\Psi}\},\ \{\mathbf{x}_\ell\}_{1\leq\ell\leq n},\ \ \partial_\sigma\Big[\Gamma_\kappa^{(n)}[\{\mathbf{x}_\ell\}_{1\leq\ell\leq n};\Phi]\big|_{\Phi\to\chi_\kappa}\Big] = 0\,. \tag{D.9}$$

On the one hand, from (D.8) and the scaling $\chi_\kappa \sim \kappa^{d_\varphi} \sim (\sigma - \sigma_c)^{\nu d_\varphi}$, one readily obtains

$$\nu d_\varphi = 1\,. \tag{D.10}$$

On the other hand, (D.9) states exactly that the only relevant eigenperturbation is χ_κ. The same derivation can be followed for the duality identity of DEP. In Sec. 4.3, we will see how the assumption (D.7) translates within the chosen ansatz for Γ_κ.

Appendix E

LPA' for DEP and DP-C

E.1 Proof of the accidental non-renormalization within LPA'

In this appendix, we prove that in the framework of the LPA', for both DEP and DP-C, the $\bar{\psi}\psi$ prefactor appearing in the bare potential is preserved by the renormalization and that only the $\bar{\psi}$-ψ propagator can be renormalized. We call these results accidental non-renormalizations because they are not directly linked with identified symmetries of the model, in particular in the case of DEP. Thus they may be a spurious effect of the approximation. Let us first show how these non-renormalizations occur perturbatively.

E.1.1 Pertubative proof

The propagators and vertices of the DP-C and DEP action are listed in Fig. E.1.1 and Fig. E.1.1 respectively. The ψ and $\bar{\psi}$ legs and propagators are noted with a full line, the φ and $\bar{\varphi}$ legs and propagators are noted with a dashed one. The response field legs are noted with an arrow and are said to be exiting legs while the observable field legs do not have arrows and are said to be entering legs.

Any perturbative contribution to the renormalization of the theory is given by a graph whose vertices are chosen in the vertices of Fig. E.1.1 and whose possible edges are given by the propagators of Fig. E.1.1. Explicitly, a full line exiting leg can be linked to a full line or a dashed line entering leg and a dashed line exiting leg can be linked to a dashed line entering leg. The external entering and exiting legs of the graph give the vertex function to which it contributes. Because each edge has a direction - from a response field to an observable field - these graphs are naturally directed. Furthermore, because of the property of causality of the

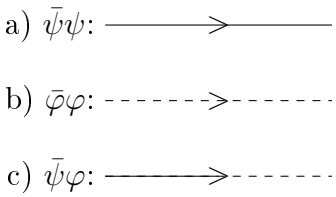

Figure E.1: List of the propagators of the DP-C and DEP theories

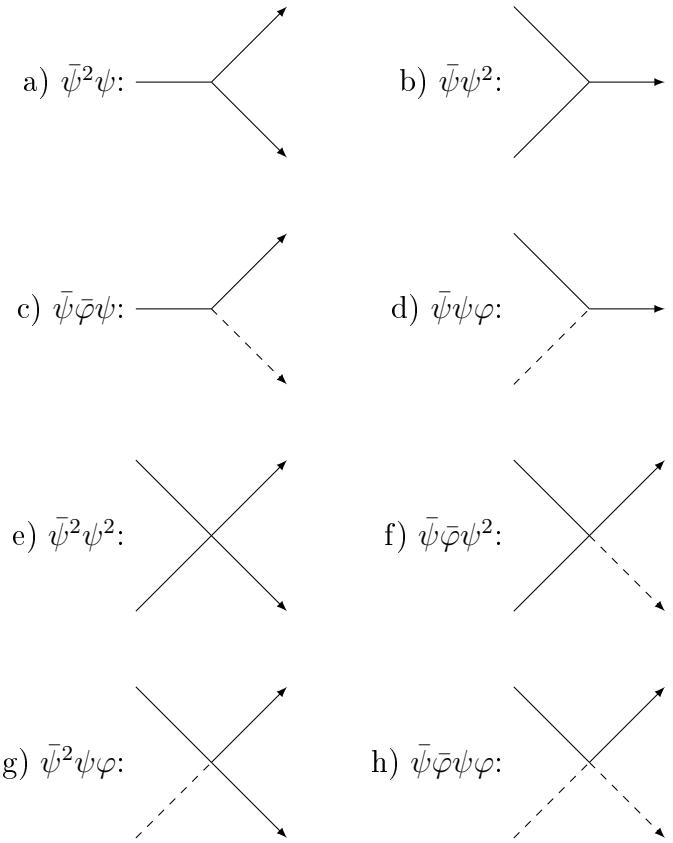

Figure E.2: List of the vertices of the DP-C and DEP theories. The quartic vertices e) to h) are present only in DEP.

theory, as discussed in Sec. 3.5, only directed acyclic graphs contribute. It means that one cannot start from a given vertex, follow the arrows and come back to the same vertex. Finally, one notices that due to the $\bar{\psi}\psi$ prefactor of the potential of the initial action, all vertices have at least one full line exiting leg, corresponding to a $\bar{\psi}$ field, and one full line entering leg, corresponding to a ψ one.

These properties are enough to show that the $\bar{\psi}\psi$ prefactor of the potential is preserved by the renormalization and that only the $\bar{\psi}$-ψ propagator can be renormalized. Indeed, the vertices of a directed acyclic graphs are partially ordered, with the order relation defined as $u \leq v$ for two vertices u and v if one can go from u to v following the arrows. Every finite graph will have at least one minimal element, that is a vertex which cannot be reached from any other vertices of the graph. This vertex has the property that all its entering legs are external. Because every bare vertex has at least one full line entering leg, every diagram contributes to a vertex function with at least one ψ field. In the same manner, every finite graph will have at least one maximal element, that is a vertex from which no vertices of the graph can be reached. As a consequence, all the exiting legs of such a vertex are external and because at least one of them is a full line, it contributes to a vertex function with at least one $\bar{\psi}$ field. In conclusion, every finite graph contribute to the renormalization of a vertex function with at least one $\bar{\psi}$ and one ψ field, which is the property we wanted to prove.

E.1.2 Non-renormalization of the $\bar{\Psi}\Psi$ factor of the LPA' potential

Now, let us see how this translates in the framework of the LPA' approximation for the NPRG flow. First, let us show that the potential U_κ is proportional to $\bar{\Psi}$:

$$U_\kappa(\varphi,\bar{\varphi},\psi,\psi) = \bar{\psi} A_\kappa(\varphi,\bar{\varphi},\psi,\bar{\psi})\,. \tag{E.1}$$

Let us remember that we have shown in Sec. 3.5 for a generic out of equilibrium field theory that the action vanishes when the response fields are set to zero. Using the analyticity of Γ_κ, this implies in particular that U_κ is proportional either to $\bar{\Psi}$ or to $\bar{\Phi}$. Now we use the respective symmetries of DP-C and DEP. For DP-C, the symmetry (4.23) implies that the potential is a function of Φ and $\bar{\Phi}$ only through their sum:

$$U_\kappa^{\text{DP-C}}(\varphi,\bar{\varphi},\psi,\psi) = U_\kappa(\varphi+\bar{\varphi},\psi,\bar{\psi})\,. \tag{E.2}$$

As a consequence, if U_κ is not proportional to $\bar{\Psi}$, it will not vanish when the response fields are set to zero. Now, for the case of DEP, the rescaling symmetry for the potential reads:

$$\Big[\big(\sqrt{\rho_0}+\varphi\big)\partial_\varphi - \big(\sqrt{\rho_0}+\bar{\varphi}\big)\partial_{\bar{\varphi}} + \psi\partial_\psi - \bar{\psi}\partial_{\bar{\psi}}\Big] U_\kappa^{\text{DEP}}(\varphi,\bar{\varphi},\psi,\bar{\psi}) = 0\,. \tag{E.3}$$

It is realized for an analytic potential as

$$U_\kappa^{\text{DEP}}(\varphi,\bar{\varphi},\psi,\bar{\psi}) = \sum_{spqr} u_{spqr}^{\text{DEP}} (\sqrt{\rho_0}+\varphi)^s (\sqrt{\rho_0}+\bar{\varphi})^p \psi^q \bar{\psi}^r$$
$$\text{with } s-p+q-r=0\,. \tag{E.4}$$

If $u_{spq0}^{\text{DEP}} \neq 0$ for $s-p+q=0$, the above expression will not vanish when the response fields are set to zero Thus we have proven that for both DP-C and DEP, the potential is proportional to $\bar{\Psi}$.

Now, let us prove that within the LPA', both for DP-C and DEP, the potential is also proportional to Ψ. Assuming that at the scale κ, the potential can be written as

$$U_\kappa(\phi) = \psi B_\kappa(\phi) \tag{E.5}$$

where $\phi = \{\varphi,\bar{\varphi},\psi,\bar{\psi}\}$ is an arbitrary configuration of constant fields, we want to show that this property is preserved by the renormalization flow:

$$\partial_s U_\kappa(\phi)\big|_{\psi=0} = 0\,. \tag{E.6}$$

The kinetic part of the LPA' ansatz for DP-C is a special case of the LPA' ansatz for DEP, thus we work directly with the latter, only specifying the symmetry of the potential when it is necesssary. The flow equation for the potential reads

$$\partial_s U_\kappa(\phi) = \int_{\vec{q}} \text{Res}_+ \Big\{ \big[\partial_\kappa \bar{\mathcal{R}}_\kappa\big]_{\varphi\bar{\varphi}}(q) \bar{G}^{(2)}_{\varphi\bar{\varphi}}(\mathbf{q},\phi) + \big[\partial_\kappa \bar{\mathcal{R}}_\kappa\big]_{\psi\bar{\psi}}(q) \bar{G}^{(2)}_{\psi\bar{\psi}}(\mathbf{q},\phi) \Big\}\,. \tag{E.7}$$

First, let us write down the general expression of the regulated two-point vertex function in the LPA' approximation:

$$\bar{\Gamma}^{(2)}_\kappa(\mathbf{p},\phi) + \bar{\mathcal{R}}_\kappa(p) = \begin{pmatrix} 0 & \gamma_\varphi & 0 & 0 \\ \gamma_\varphi^* & 0 & -\mu_\kappa D_\kappa p^2 & 0 \\ 0 & -\mu_\kappa D_\kappa p^2 & 0 & \gamma_\psi \\ 0 & 0 & \gamma_\psi^* & 0 \end{pmatrix} + \left(\frac{\partial^2 U_\kappa(\phi)}{\partial\phi_i \partial\phi_j} \right)_{i,j\in\{\varphi,\bar{\varphi},\psi,\bar{\psi}\}}$$

with the lines and columns ordered as: $\varphi, \bar{\varphi}, \psi, \bar{\psi}$ and where

$$\begin{aligned}\gamma_\varphi &= Z_\kappa^\varphi \left[i\omega + D_\kappa p^2 (1+r) \right] \\ \gamma_\psi &= Z_\kappa^\psi \left[i\omega + \lambda_\kappa D_\kappa p^2 (1+r) \right] .\end{aligned} \tag{E.8}$$

Importantly, both of these expressions vanish for a value of ω in the upper half complex plane. In principle, we could have chosen a more general kinetic structure in the DEP case (for example adding $\bar{\Psi}$-$\bar{\Psi}$ or $\bar{\Psi}$-Φ terms) but we will show in the next section that the structure of the $\bar{\Gamma}_\kappa^{(2)}$ matrix (E.8) is in fact protected by the renormalization flow as well. For the mass of the propagators, using the hypothesis (E.5), we obtain the following matrix

$$\left(\frac{\partial^2 U_\kappa(\phi)}{\partial \phi_i \partial \phi_j} \right)_{i,j} \Bigg|_{\psi=0} = \begin{pmatrix} 0 & 0 & u_{\varphi,0} & 0 \\ 0 & 0 & u_{\bar{\varphi},0} & 0 \\ u_{\varphi,0} & u_{\bar{\varphi},0} & 2u_{\psi,0} & u_{\bar{\psi},0} \\ 0 & 0 & u_{\bar{\psi},0} & 0 \end{pmatrix} , \tag{E.9}$$

where we have used the shorthand notation $u_{\phi_i,0} = \partial_{\phi_i} B_\kappa(\phi)|_{\psi=0}$. The inversion of the two-point vertex function is easy as the last line and the last column only contain one non-zero entry, the propagators thus read simply

$$\bar{G}^{(2)}_{\varphi\bar{\varphi}}(\mathbf{q},\phi)|_{\psi=0} = \left(\gamma_\varphi^*\right)^{-1} \text{ and } \bar{G}^{(2)}_{\psi\bar{\psi}}(\mathbf{q},\phi)|_{\psi=0} = \left(\gamma_\psi^* + u_{\bar{\psi},0}\right)^{-1} . \tag{E.10}$$

The pole of the first propagator is in the lower half-plane, such that the frequency integration gives zero. We are thus left with

$$\partial_s U_\kappa(\phi)\big|_{\psi=0} = \int_{\vec{q}} \left[\partial_\kappa \bar{\mathcal{R}}_\kappa\right]_{\psi\bar{\psi}}(q) \, \mathrm{Res}_+ \left\{ \left(\gamma_\psi^* + u_{\bar{\psi},0}\right)^{-1} \right\} . \tag{E.11}$$

At this point, we use the analyticity of Γ_κ and the minimum configuration Min defined as

$$\partial^2_{\psi\bar{\psi}} U_\kappa|_{\mathrm{Min}} = 0 \tag{E.12}$$

to write $U_\kappa(\phi)\big|_{\psi=0}$ as an analytic series. In the DP-C case, it reads

$$U_\kappa^{\text{DP-C}}(\phi)\big|_{\psi=0} = \sum_{km} u_{km}^{\text{DP-C}} (\varphi + \bar{\varphi} - \chi_\kappa^{\text{DP-C}})^k \bar{\psi}^m \tag{E.13}$$

with

$$\forall\, k,m,\ u_{km}^{\text{DP-C}} = \frac{1}{k!m!} \partial^{k+m}_{\varphi^k \bar{\psi}^m} U_\kappa^{\text{DP-C}}(\phi)\big|_{\text{Min}} . \tag{E.14}$$

In the case of DEP, we have

$$U_\kappa^{\text{DEP}}(\phi)\big|_{\psi=0} = \sum_{klm} u_{klm}^{\text{DEP}} (\bar{\varphi} - \chi_\kappa^{\text{DEP}})^k (\varphi - \chi_\kappa^{\text{DEP}})^l \bar{\psi}^m \tag{E.15}$$

such that (E.3) is verified and with

$$\forall\, k,l,m,\ u_{klm}^{\text{DEP}} = \frac{1}{k!l!m!} \partial^{k+l+m}_{\varphi^k \bar{\varphi}^l \bar{\psi}^m} U_\kappa^{\text{DEP}}(\phi)\big|_{\text{Min}} . \tag{E.16}$$

Because of analyticity, to show that $\partial_s U_\kappa(\phi)\big|_{\psi=0} = 0$ is equivalent to show that $\partial_s u_{km}^{\text{DP-C}} = 0$ for all k, m (resp. $\partial_s u_{klm}^{\text{DEP}} = 0$ for all k, l, m). Applying a κ-derivative to the definition of the couplings (E.14) for DP-C or (E.15) for DEP and using the definition of the minimum configuration (E.12), we have respectively

$$\forall\, k, m,\ \partial_\kappa u_{km}^{\text{DP-C}} = \left\{ \frac{1}{k!m!} \partial_s \left[\partial^{k+m}_{\varphi^k \bar{\psi}^m} U_\kappa^{\text{DP-C}}(\phi)\big|_{\psi=0} \right] \right\}_{\text{Min}} + (k+1) u_{k+1m}^{\text{DP-C}} \partial_s \chi_\kappa^{\text{DP-C}} \tag{E.17}$$

and

$$\begin{aligned} \forall\, k, l, m,\ \partial_\kappa u_{klm}^{\text{DEP}} = &\left\{ \frac{1}{k!l!m!} \partial_\kappa \left[\partial^{k+l+m}_{\varphi^k \bar{\varphi}^l \bar{\psi}^m} U_\kappa^{\text{DEP}}(\phi)\big|_{\psi=0} \right] \right\}_{\text{Min}} \\ &+ \left[(k+1) u_{k+1lm}^{\text{DEP}} + (l+1) u_{kl+1m}^{\text{DEP}} \right] \partial_\kappa \chi_\kappa^{\text{DEP}} \,. \end{aligned} \tag{E.18}$$

In both cases, the second term of the r.h.s. is zero because we assumed

$$U_\kappa(\phi)\big|_{\psi=0} = 0 \,. \tag{E.19}$$

Now, let us examinate the first term of each equation. Taking the adequate derivatives of the flow equation (E.11), we obtain

$$\forall k, l, m,\ \partial_\kappa \left[\partial^{k+l+m}_{\varphi^k \bar{\varphi}^l \bar{\psi}^m} U_\kappa(\phi)\big|_{\psi=0} \right] = \int_{\vec{q}} \left[\partial_\kappa \bar{\mathcal{R}}_\kappa \right]_{\psi\bar{\psi}}(q) \text{Res}_+ \left\{ \sum_{n \geq 2} \frac{V_n(\phi)}{(\gamma^*_\psi + u_{\bar{\psi},0})^n} \right\}, \tag{E.20}$$

where the V_n are functions of the field configuration whose exact form is not necessary to specify. Evaluating at Min and using (E.12), the denominators appearing in the integrand are simply integer powers of $(\gamma^*_\psi)^{-1}$, whose pole is in the lower half plane. As a consequence, the first term of the r.h.s. of (E.17) and (E.18) vanishes as well. This finishes to prove that in the LPA' approximation, both for DEP and DP-C, the prefactor $\bar{\psi}\psi$ is preserved by the renormalization:

$$U_\kappa(\phi) = \bar{\psi}\psi V_\kappa(\phi) \,. \tag{E.21}$$

E.1.3 Accidental non-renormalization of the LPA' kinetic part

Let us show that within the LPA' approximation, only the $\bar{\Psi}$-Ψ propagator can be renormalized. For DP-C, this fact is a direct consequence of the symmetries of the action. However, for DEP there is no identified symmetry which would enforce this result. We assume that at the scale κ, the structure of the derivative terms of $\bar{\Gamma}^{(2)}_\kappa(\mathbf{p})$ is given by the matrix (E.8) of the previous section. We want to show that both for DP-C and DEP,

$$\partial_s \bar{\Gamma}^{(2)}_{ij}(\mathbf{p}, \phi)\big|_{\text{Min}} = 0 \,. \tag{E.22}$$

if $ij \neq \bar{\psi}\psi$. Using the structure of the regulator given in Sec. 4.3.3, the flow of a general two-point function evaluated at an arbitrary field configuration is given by

$$\begin{aligned}\partial_s \bar{\Gamma}^{(2)}_{mn}(\mathbf{p},\phi) = 2\int_{\vec{q}} \mathrm{Res}_+ \Big\{ & \\ \big[\partial_s \bar{\mathcal{R}}\big]_{\varphi\bar{\varphi}}(q)\bar{G}^{(2)}_{\varphi k}(\mathbf{q},\phi)\Big[-\frac{1}{2}U^{(4)}_{klmn}(\phi) + U^{(3)}_{kms}(\phi)\bar{G}^{(2)}_{st}(\mathbf{q}+\mathbf{p},\phi)U^{(3)}_{tnl}(\phi)\Big]\bar{G}^{(2)}_{l\bar{\varphi}}(\mathbf{q},\phi) & \\ + \big[\partial_s \bar{\mathcal{R}}\big]_{\psi\bar{\psi}}(q)\bar{G}^{(2)}_{\psi k}(\mathbf{q},\phi)\Big[-\frac{1}{2}U^{(4)}_{klmn}(\phi) + U^{(3)}_{kms}(\phi)\bar{G}^{(2)}_{st}(\mathbf{q}+\mathbf{p},\phi)U^{(3)}_{tnl}(\phi)\Big]\bar{G}^{(2)}_{l\bar{\psi}}(\mathbf{q},\phi)\Big\}\,. & \end{aligned} \tag{E.23}$$

Now, let us evaluate the above equation in the configuration Min. First, we invert the kinetic part of (E.8) to obtain

$$\bar{G}^{(2)}_{\kappa}(\mathbf{q},\phi)\big|_{\mathrm{Min}} \equiv G^{\mathrm{Min}}(\mathbf{q}) = \begin{pmatrix} 0 & \frac{1}{\gamma^*_\varphi} & 0 & \frac{\mu_\kappa D_\kappa p^2}{\gamma^*_\varphi \gamma^*_\psi} \\ \frac{1}{\gamma_\varphi} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma^*_\psi} \\ \frac{\mu_\kappa D_\kappa p^2}{\gamma^*_\varphi \gamma^*_\psi} & 0 & \frac{1}{\gamma^*_\psi} & 0 \end{pmatrix},$$

which is simply the bare propagator with scaling factors. All the poles of the upper triangular elements of the matrix above are situated in the lower half plane. As a first consequence, the contributions of the flow proportional to $U^{(4)}$ vanish and we are left with

$$\begin{aligned}\partial_s \bar{\Gamma}^{(2)}_{mn}(\mathbf{p},\phi)\big|_{\mathrm{Min}} = 2\int_{\vec{q}} \mathrm{Res}_+ \Big\{ & \\ & + \big[\partial_s \bar{\mathcal{R}}\big]_{\varphi\bar{\varphi}}(q) G^{\mathrm{Min}}_{\varphi k}(\mathbf{q}) U^{(3)}_{kms}(\phi)\big|_{\mathrm{Min}} G^{\mathrm{Min}}_{st}(\mathbf{q}+\mathbf{p}) U^{(3)}_{tnl}(\phi)\big|_{\mathrm{Min}} G^{\mathrm{Min}}_{l\bar{\varphi}}(\mathbf{q}) \\ & + \big[\partial_s \bar{\mathcal{R}}\big]_{\psi\bar{\psi}}(q) G^{\mathrm{Min}}_{\psi k}(\mathbf{q}) U^{(3)}_{kms}(\phi)\big|_{\mathrm{Min}} G^{\mathrm{Min}}_{st}(\mathbf{q}+\mathbf{p}) U^{(3)}_{tnl}(\phi)\big|_{\mathrm{Min}} G^{\mathrm{Min}}_{l\bar{\psi}}(\mathbf{q})\Big\}\,. \end{aligned} \tag{E.24}$$

Secondly, due to the $\bar{\psi}\psi$ factor in the potential, most contributions proportional to $U^{(3)}$ vanish as well. We know already that due to causality, at least one of m, n must be a response field, say n. Let us assume $m \neq \psi$. In both lines, as $k \neq \psi$ due to the structure of the propagator, we must have $s = \psi$ and then, $t = \bar{\psi}$. As a consequence, in both lines all the propagators belong to the upper triangular part of (E.24) and have their poles in the lower half plane. Thus both lines vanish. Now, let us assume $m = \psi$ and $n = \bar{\varphi}$. In the first line, $l = \varphi$, such that

$$U^{(3)}_{tnl}(\phi)\big|_{\mathrm{Min}} = U^{(3)}_{t\bar{\varphi}\varphi}(\phi)\big|_{\mathrm{Min}} = 0\,. \tag{E.25}$$

In the second line, $l = \psi$ and thus $t = \bar{\psi}$. Again, all the propagators have their respective poles in the lower half plane and the flow is zero. In conclusion, we proved that only $\bar{\Gamma}^{(2)}_{\psi\bar{\psi}}$ has a renormalization flow within the LPA'.

E.2 Derivation of the NPRG flow of the couplings

In this appendix, we explicitly write down the steps necesssary to obtain the flow of the couplings. We do this on the example of the flow of χ_κ in the DP-C case. The starting point is the flow of the potential

$$\partial_s U_\kappa(\Phi_0) = \int_{\vec{q}} \mathrm{Res}_+ \Big\{ \big[\partial_\kappa \bar{\mathcal{R}}_\kappa\big]_{\varphi\bar{\varphi}}(q) \bar{G}^{(2)}_{\varphi\bar{\varphi}}(\mathbf{q}, \Phi_0) + \big[\partial_\kappa \bar{\mathcal{R}}_\kappa\big]_{\psi\bar{\psi}}(q) \bar{G}^{(2)}_{\psi\bar{\psi}}(\mathbf{q}, \Phi_0) \Big\} . \quad \text{(E.26)}$$

To obtain the propagators, we invert the matrix $\bar{\Gamma}^{(2)}_\kappa + \mathcal{R}$ written in (E.8). Specifying to the DP-C ansatz, we set

$$\begin{aligned} Z^\varphi_\kappa &= D_\kappa = 1 \,, \\ \mu_\kappa &= \mu \,, \end{aligned} \quad \text{(E.27)}$$

and we abbreviate $Z^\psi_\kappa = Z_\kappa$. Finally, we write explicitly the scaling factors of the regulators as in (4.104):

$$\big[\bar{\mathcal{R}}_\kappa\big]_{\bar{\varphi}\varphi}(q) = q^2 r\big(q^2/\kappa^2\big) \,, \quad \big[\bar{\mathcal{R}}_\kappa\big]_{\bar{\psi}\psi}(q) = \lambda_\kappa Z^\psi_\kappa q^2 r\big(q^2/\kappa^2\big) \,, \quad \text{(E.28)}$$

where r is the dimensionless Θ regulator defined in (4.105). The flow equation reads

$$\begin{aligned} \partial_\kappa U^{\text{DP-C}}_\kappa(\Phi_0) = \int_{\vec{q}} \mathrm{Res}_+ \Bigg\{ & \frac{q^2 \partial_\kappa r}{\Delta} \Big[\big(i\varpi + q^2(1+r) + U^{(2,0,0)}_\kappa\big) P_\Psi \\ & + \mu q^2 \Big(U^{(1,0,1)}_\kappa \big(- Z_\kappa i\varpi + h\big) - U^{(0,0,2)}_\kappa U^{(1,1,0)}_\kappa \Big) + Q \Big] \\ & + \frac{q^2 \partial_\kappa (\lambda_\kappa Z_\kappa r)}{\Delta} \Big[\big(Z_\kappa i\varpi + h\big) P_\Phi + \mu q^2 U^{(1,0,1)}_\kappa \big(- i\varpi \\ & + q^2(1+r)\big) - 2q^2(1+r) U^{(1,0,1)}_\kappa U^{(1,1,0)}_\kappa \Big] \Bigg\} , \end{aligned} \quad \text{(E.29)}$$

with

$$U^{(k,l,m)}_\kappa = \frac{\partial^{k+l+m}}{\partial \Phi_0^k \partial \Psi_0^l \partial \bar{\Psi}_0^m} U^{\text{DP-C}}_\kappa(\Phi_0) \,, \quad \text{(E.30)}$$

and with the following shorthand notations:

$$\begin{aligned} h &= \lambda_\kappa Z_\kappa q^2(1+r) + U^{(1,1,0)}_\kappa \,, \\ Q &= U^{(0,2,0)}_\kappa \big(U^{(1,0,1)}_\kappa\big)^2 + U^{(0,0,2)}_\kappa \big(U^{(1,1,0)}_\kappa\big)^2 - 2h U^{(1,1,0)}_\kappa U^{(1,0,1)}_\kappa \,, \\ P_\Phi &= \varpi^2 + q^2(1+r)\big(q^2(1+r) + 2U^{(2,0,0)}_\kappa\big) \,, \\ P_\Psi &= \big(Z_\kappa \varpi\big)^2 + h^2 - U^{(0,0,2)}_\kappa U^{(0,2,0)}_\kappa \,, \\ \Delta &= P_\Psi P_\Phi - 2Z_\kappa U^{(1,0,1)}_\kappa \mu q^2 \varpi^2 + 2q^2 Q(1+r) \\ &\quad + 2\mu q^4 (1+r)\big(h U^{(1,0,1)}_\kappa - U^{(0,0,2)}_\kappa U^{(1,1,0)}_\kappa\big) + \mu^2 q^4 \big((U^{(1,0,1)}_\kappa)^2 - U^{(0,0,2)}_\kappa U^{(2,0,0)}_\kappa\big) . \end{aligned} \quad \text{(E.31)}$$

By differentiating U_κ and evaluating the result in the configuration Min, one is left with only two possible poles: $iq^2(1+r)$ and $i\lambda_\kappa q^2(1+r)$. More precisely, the ϖ dependence of the integrand is as a rational function of

$$\gamma = i\varpi + q^2(1+r) \,, \quad \gamma_\lambda = i\varpi + \lambda_\kappa q^2(1+r) \,, \quad \text{(E.32)}$$

and their respective complex conjugate. The non-zero contributions to the integral arise from one or more $\gamma_{(\lambda)}$ factors in the denominator. Using the notation

$$u_{klm} \equiv \frac{1}{k!l!m!} U_\kappa^{(k,l,m)}\Big|_{\text{Min}}\,, \tag{E.33}$$

one obtains for example for u_{011}

$$\begin{aligned}
\big[\partial_\kappa u_{011}\big]_{\text{Min}} &= \int_{\mathbf{q}} \text{Res}_+ \Bigg\{ q^2 \partial_\kappa r \Big[\frac{q^2 u_{111}^2 \big(\mu - 2\lambda(r+1)Z\big) - 2Z\gamma_\lambda^* \big(u_{211} Z \gamma_\lambda^* + \mu q^2 u_{112}\big)}{Z^2 \big(\gamma_\lambda^*\big)^2 \big(\gamma^*\big)^2} \\
&+ \frac{2u_{111}\big(\mu q^2 u_{012}\gamma_\lambda^* + 2\lambda^2 q^4 (r+1)^2 u_{111} Z\big)}{\gamma_\lambda Z^2 \big(\gamma_\lambda^*\big)^2 \big(\gamma^*\big)^2} \Big] \\
&+ q^2 \partial_\kappa (\lambda_\kappa Z r) \Big[\frac{2q^2 u_{111}\big(2\mu u_{012}\gamma_\lambda^* + \big(\lambda^2 - 1\big) q^2 (r+1)^2 u_{111} Z\big)}{\gamma(\lambda-1) Z^3 \big(\gamma_\lambda^*\big)^3 \gamma^*} \\
&+ \frac{4u_{012}\big((\lambda-1)u_{021}\gamma^* - \mu q^2 u_{111}\big)}{(\lambda-1)\gamma_\lambda Z^3 \big(\gamma_\lambda^*\big)^2 \gamma^*} - 2\frac{Z\gamma_\lambda^* \big(u_{121}\gamma^* + \mu q^2 u_{112}\big) + q^2 u_{111}^2 \big((r+1)Z - \mu\big)}{Z^3 \big(\gamma_\lambda^*\big)^3 \gamma^*} \Big] \Bigg\} \\
&= \int_{\mathbf{q}} \Bigg\{ q^2 \partial_\kappa r \frac{u_{111}\big(\lambda(r+1)u_{111} Z + \mu u_{012}\big)}{\lambda(\lambda+1)^2 q^4 (r+1)^3 Z^2} \\
&+ q^2 \partial_\kappa (\lambda Z r) \frac{\lambda^2 (r+1) u_{111}^2 Z + u_{012}\big((\lambda+1)^2 (r+1) u_{021} + (2\lambda+1)\mu u_{111}\big)}{\lambda^2(\lambda+1)^2 q^4 (r+1)^3 Z^3} \Bigg\}\,,
\end{aligned} \tag{E.34}$$

where the dependence in κ is implicit for notational simplicity. At this point, the procedure is the same as for NPRG applied to equilibrium systems. With our choice of regulator, the momentum integral can be performed analytically and, following the previous example, one is left with

$$\begin{aligned}
\big[\partial_s u_{011}\big]_{\text{Min}} = 8\kappa^{d-2} v_d \Bigg\{ &\frac{\mu u_{012} u_{111} \big[\lambda\big(8 + 2d - \bar\eta + 2\partial_s\lambda - \eta\big) + 2\lambda^2\big(4 + d - \bar\eta - \eta\big) + \partial_s\lambda\big]}{(d+2)(d+4)\lambda^2(\lambda+1)^2 Z^2} + \\
&\frac{u_{012} u_{021}\big[\lambda\big(2 + d - \bar\eta - \eta\big) + \partial_s\lambda\big]}{d(d+2)\lambda^2 Z^2} + \frac{u_{111}^2\big[\lambda\big(2 + d - \bar\eta - \eta\big) + d + \partial_s\lambda + 2\big]}{d(d+2)(\lambda+1)^2 Z} \Bigg\}\,,
\end{aligned} \tag{E.35}$$

where $s = \ln(\kappa/\Lambda)$, d is the spatial dimension and $v_d = 1/(2^{d+1}\pi^{\frac{d}{2}}\Gamma(\frac{d}{2}))$. From the above equation, one deduces the flow equation of the minimum using (4.100):

$$\partial_s \chi^{\text{DP-C}} = -\frac{1}{u_{111}} \big[\partial_s u_{011}\big]_{\text{Min}}\,. \tag{E.36}$$

The last step is to introduce dimensionless variables, as explained in Sec. 3.4.3:

$$\begin{aligned}
u_{ijl} &= \begin{cases} Z^{\frac{j+l}{2}} \kappa^{2+d-(i+j+l)\frac{d}{2}} \hat u_{ijl}, & \text{if } \mu = 0 \\ Z^l \kappa^{2+d-(i+j+l)\frac{d}{2}} \hat u_{ijl}, & \text{otherwise} \end{cases} \\
\chi^{\text{DP-C}} &= \kappa^{\frac{d}{2}} \hat\chi^{\text{DP-C}}\,.
\end{aligned} \tag{E.37}$$

For $\mu = 0$, one obtains

$$\partial_s \hat{\chi} = -\frac{d}{2}\hat{\chi} - 8v_d \Bigg\{ \frac{\hat{u}_{111}(2+d+\lambda(2+d-2\eta)+\partial_s\lambda)}{d(d+2)(\lambda+1)^2} - \frac{\hat{u}_{021}^2(\lambda(2+d-2\eta)+\partial_s\lambda)}{d(d+2)\lambda^2\hat{u}_{111}} \Bigg\}. \tag{E.38}$$

In the case $\mu \neq 0$, one obtains

$$\partial_s \hat{\chi} = -\frac{d}{2}\hat{\chi} - 8v_d \Bigg\{ \frac{\hat{u}_{111}\big(2+d+\lambda\big(2+d-\bar{\eta}\big)+\partial_s\lambda\big)}{d(d+2)(\lambda+1)^2} + \frac{\hat{u}_{021}\hat{u}_{012}\big(\lambda\big(2+d-\bar{\eta}\big)+\partial_s\lambda\big)}{d(d+2)\lambda^2\hat{u}_{111}} + \frac{\mu\hat{u}_{012}\big(2(d+4)\lambda(\lambda+1)+(2\lambda+1)(\partial_s\lambda-\lambda\bar{\eta})\big)}{(d+2)(d+4)\lambda^2(\lambda+1)^2} \Bigg\}. \tag{E.39}$$

To get the full set of coupled ordinary differential equation for the coefficients $\hat{u}_{ijl}$ up to the truncation order, the above procedure is systematically implemented using Mathematica.

Appendix F

Large wave-number expansion of the RG flow equation of SNS

F.1 Ward identities for the vertex functions

F.1.1 Ward identities in velocity formulation

We derive the Ward identities for the vertex function associated with the extended Galilean symmetry for an arbitrary vertex function $\Gamma^{(m,n)}$. We consider the functional Ward identity (5.40) derived in Sec. 5.3.1

$$\int_{\vec{x}} \Big\{ \big(\delta_{\alpha\beta}\partial_t + \partial_\beta u_\alpha\big) \frac{\delta\Gamma_\kappa}{\delta u_\alpha} + \partial_\beta \bar{u}_\alpha \frac{\delta\Gamma_\kappa}{\delta \bar{u}_\alpha} \Big\} = -\int_{\vec{x}} \partial_t^2 \bar{u}_\beta \,,$$

where the pressure terms are omitted since they give no contribution in the following derivation. Taking m functional derivatives of this identity with respect to velocity fields $u_{\alpha_i}(\mathbf{x}_i)$ – $i = 1, \ldots, m$ – and n with respect to response velocity fields $\bar{u}_{\alpha_j}(\mathbf{x}_j)$ – $j = m, \ldots, m+n$ –, and setting the fields to zero yields

$$\begin{aligned}
&\int_{\vec{x}} \partial_t \Gamma^{(m+1,n)}_{\alpha\alpha_1\ldots\alpha_{m+n}}(\mathbf{x}, \{\mathbf{x}_\ell\}_{1\le\ell\le m+n}) \\
&= \int_{\vec{x}} \sum_{k=1}^{m+n} \delta(t-t_k)\delta^d(\vec{x}-\vec{x}_k) \partial_\alpha \Gamma^{(m,n)}_{\alpha_1\ldots\alpha_{m+n}}(\{\mathbf{x}_\ell\}_{1\le\ell\le k-1}, \mathbf{x}, \{\mathbf{x}_\ell\}_{k+1\le\ell\le m+n}) \,. \qquad \text{(F.1)}
\end{aligned}$$

This identity can be expressed in Fourier space. It yields in terms of the Fourier transforms $\tilde{\Gamma}^{(k,\ell)}$

$$\begin{aligned}
&\int_{\vec{x}} \int_{\mathbf{q},\mathbf{p}_i} \Bigg\{ -i\varpi \tilde{\Gamma}^{(m+1,n)}_{\alpha\alpha_1\ldots\alpha_{m+n}}(\mathbf{q}, \{\mathbf{p}_\ell\}_{1\le\ell\le m+n}) e^{i(\vec{q}\cdot\vec{x}-\varpi t)+i\sum_{i=1}^{m+n}(\vec{p}_i\cdot\vec{x}_i-\omega_i t_i)} \\
&- ip^\alpha \sum_{k=1}^{m+n} e^{i\vec{p}_k(\vec{x}-\vec{x}_k)-i\omega_k(t-t_k)} \\
&\times \tilde{\Gamma}^{(m,n)}_{\alpha_1\ldots\alpha_{m+n}}(\{\mathbf{p}_\ell\}_{1\le\ell\le k-1}, \mathbf{q}, \{\mathbf{p}_\ell\}_{k+1\le\ell\le m+n}) e^{i(\vec{q}\cdot\vec{x}-\varpi t)+i\sum_{\substack{i=1\\ i\neq k}}^{m+n}(\vec{p}_i\cdot\vec{x}_i-\omega_i t_i)} \Bigg\} = 0 \,. \qquad \text{(F.2)}
\end{aligned}$$

Performing the integration over $\vec{x}$ and $\vec{p}$, and shifting the frequency ω_k by ω, one obtains

$$\begin{aligned}
&\tilde{\Gamma}^{(m+1,n)}_{\alpha\alpha_1\dots\alpha_{n+m}}(\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\le\ell\le m+n})\big|_{\vec{q}=0}\\
&=-\sum_{k=1}^{m+n}\frac{p_k^\alpha}{\varpi}\tilde{\Gamma}^{(m,n)}_{\alpha_1\dots\alpha_{n+m}}(\{\mathbf{p}_\ell\}_{1\le\ell\le k-1},\,\omega_k+\varpi,\vec{p}_k,\{\mathbf{p}_\ell\}_{k+1\le\ell\le m+n})\\
&=\tilde{\mathcal{D}}_\alpha(\varpi)\tilde{\Gamma}^{(m,n)}_{\alpha_1\dots\alpha_{n+m}}(\{\mathbf{p}_\ell\}_{1\le\ell\le m+n})\,.
\end{aligned}\tag{F.3}$$

F.1.2 Ward identities in the stream function formulation

F.1.2.1 Time-gauged Galilean identity

The derivation of the Ward identities is a bit less involved in the case of the stream formulation because the vertex functions are scalar or pseudo-scalar and not tensors. Let us first consider the time-gauged Galilean identity Eq. (5.52), line d)

$$\int_{\vec{x}}\Big\{\big(-\epsilon_{\gamma\beta}x_\gamma\partial_t+\partial_\beta\Psi\big)\frac{\delta\Gamma_\kappa}{\delta\Psi(\mathbf{x})}+\partial_\beta\bar{\Psi}\frac{\delta\Gamma_\kappa}{\delta\bar{\Psi}(\mathbf{x})}\Big\}=0\,.\tag{F.4}$$

Multiplying by $\epsilon_{\alpha\beta}$, taking m functional derivatives with respect to the stream function $\Psi(\mathbf{x}_i)$ – $i=1,\dots,m$ – and n with respect to response response stream $\bar{\Psi}(\mathbf{x}_j)$ – $j=m,\dots,m+n$ –, and setting the fields to zero yields

$$\begin{aligned}
&\int_{\vec{x}}\Big\{-x_\alpha\partial_t\Gamma^{(m+1,n)}_\kappa(\mathbf{x},\{\mathbf{x}_\ell\}_{1\le\ell\le m+n})\\
&-\epsilon_{\alpha\beta}\sum_{k=1}^{m+n}\delta(t-t_k)\delta^d(\vec{x}-\vec{x}_k)\partial_\beta\Gamma^{(m,n)}_\kappa(\{\mathbf{x}_\ell\}_{1\le\ell\le k-1},\,\mathbf{x},\{\mathbf{x}_\ell\}_{k+1\le\ell\le m+n})\Big\}=0\,.
\end{aligned}\tag{F.5}$$

Going to the Fourier transforms of the vertex functions and of the delta functions gives

$$\begin{aligned}
&\int_{\vec{x}}\int_{\mathbf{q},\mathbf{p}_i}\Big\{i\varpi x_\alpha e^{i(\vec{q}\cdot\vec{x}-\varpi t)+i\sum_{i=1}^{m+n}(\vec{p}_i\cdot\vec{x}_i-\omega_i t_i)}\tilde{\Gamma}^{(m+1,n)}_\kappa(\mathbf{q},\{\mathbf{p}_\ell\}_{1\le\ell\le m+n})\\
&\quad-i\epsilon_{\alpha\beta}q_\beta\sum_{k=1}^{m+n}e^{i[\vec{p}_k(\vec{x}-\vec{x}_k)-\omega_k(t-t_k)+\vec{q}\cdot\vec{x}-\varpi t+\sum_{\substack{i=1\\i\neq k}}^{m+n}(\vec{p}_i\cdot\vec{x}_i-\omega_i t_i)]}\\
&\quad\times\tilde{\Gamma}^{(m,n)}_\kappa(\{\mathbf{p}_\ell\}_{1\le\ell\le k-1},\,\mathbf{q},\{\mathbf{p}_\ell\}_{k+1\le\ell\le m+n})\Big\}=0\,.
\end{aligned}\tag{F.6}$$

In the first line, using

$$\int_{\vec{q}}x_\alpha e^{i\vec{q}\cdot\vec{x}}f(\vec{q})=\int_{\vec{q}}\frac{1}{i}\frac{\partial}{\partial q^\alpha}e^{i\vec{q}\cdot\vec{x}}f(\vec{q})=\int_{\vec{q}}e^{i\vec{q}\cdot\vec{x}}i\frac{\partial}{\partial q^\alpha}f(\vec{q}),\tag{F.7}$$

then integrating on $\vec{x},\vec{q}$, shifting the frequencies ω_k by ϖ in the second line and identifying the Fourier transforms, one finally gets

$$\begin{aligned}
&\frac{\partial}{\partial q^\alpha}\tilde{\Gamma}^{(m+1,n)}_\kappa(\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\le\ell\le m+n})\Big|_{\vec{q}=0}\\
&\quad=-i\epsilon_{\alpha\beta}\sum_{k=1}^{m+n}\frac{p_k^\beta}{\varpi}\tilde{\Gamma}^{(m,n)}_\kappa(\{\mathbf{p}_\ell\}_{1\le\ell\le k-1},\,\omega_k+\varpi,\vec{p}_k,\{\mathbf{p}_\ell\}_{k+1\le\ell\le m+n})\\
&\quad=i\epsilon_{\alpha\beta}\tilde{\mathcal{D}}_\beta(\varpi)\tilde{\Gamma}^{(m,n)}_\kappa(\{\mathbf{p}_\ell\}_{1\le\ell\le m+n})\,.
\end{aligned}\tag{F.8}$$

To show that the limit $\varpi\to 0$ of the above equation is finite, one has to make use of the invariance under pure translation, which states that

$$\tilde{\Gamma}^{(m,n)}_\kappa(\{\mathbf{p}_\ell\}_{1\le\ell\le m+n})=\delta^d\Big(\sum_{k=1}^{m+n}\vec{p}_k\Big)\delta\Big(\sum_{k=1}^{m+n}\omega_k\Big)\bar{\Gamma}^{(m,n)}_\kappa(\{\mathbf{p}_\ell\}_{1\le\ell\le m+n-1})\,,\tag{F.9}$$

The l.h.s. of Eq. (F.8) thus reads

$$\begin{aligned}
&\frac{\partial}{\partial q^\alpha}\tilde{\Gamma}^{(m+1,n)}_\kappa(\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\le\ell\le m+n})\Big|_{\vec{q}=0}\\
&\quad=\frac{\partial}{\partial q^\alpha}\Big[\delta^d\big(\vec{q}+\sum_{k=1}^{m+n}\vec{p}_k\big)\delta\big(\varpi+\sum_{k=1}^{m+n}\omega_k\big)\bar{\Gamma}^{(m+1,n)}_\kappa(\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\le\ell\le m+n-1})\Big]\Big|_{\vec{q}=0}\\
&\quad=\frac{\partial}{\partial q^\alpha}\delta^d\big(\vec{q}+\sum_{k=1}^{m+n}\vec{p}_k\big)\Big|_{\vec{q}=0}\delta\big(\varpi+\sum_{k=1}^{m+n}\omega_k\big)\bar{\Gamma}^{(m+1,n)}_\kappa(\varpi,\vec{0},\{\mathbf{p}_\ell\}_{1\le\ell\le m+n-1})\\
&\qquad+\delta^d\Big(\sum_{k=1}^{m+n}\vec{p}_k\Big)\delta\big(\varpi+\sum_{k=1}^{m+n}\omega_k\big)\frac{\partial}{\partial q^\alpha}\bar{\Gamma}^{(m+1,n)}_\kappa(\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\le\ell\le m+n-1})\Big|_{\vec{q}=0}\\
&\quad=\delta^d\Big(\sum_{k=1}^{m+n}\vec{p}_k\Big)\delta\big(\varpi+\sum_{k=1}^{m+n}\omega_k\big)\frac{\partial}{\partial q^\alpha}\bar{\Gamma}^{(m+1,n)}_\kappa(\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\le\ell\le m+n-1})\Big|_{\vec{q}=0}\,.
\end{aligned}\tag{F.10}$$

In the last line, we have used Eq. (5.55), line *a*), *i.e.* that

$$\bar{\Gamma}^{(m+1,n)}_\kappa(\varpi,\vec{0},\{\mathbf{p}_\ell\}_{1\le\ell\le m+n-1})=0\,.\tag{F.11}$$

As for the r.h.s. of Eq. (F.8), we have

$$\begin{aligned}
&-i\epsilon_{\alpha\beta}\sum_{k=1}^{m+n}\frac{p_k^\beta}{\varpi}\tilde{\Gamma}^{(m,n)}_\kappa(\{\mathbf{p}_\ell\}_{1\le\ell\le k-1},\,\omega_k+\varpi,\vec{p}_k,\{\mathbf{p}_\ell\}_{k+1\le\ell\le m+n})\\
&=-i\epsilon_{\alpha\beta}\delta^d\Big(\sum_{k=1}^{m+n}\vec{p}_k\Big)\delta\big(\varpi+\sum_{k=1}^{m+n}\omega_k\big)\\
&\times\Big[\sum_{k=1}^{m+n-1}\frac{p_k^\beta}{\varpi}\bar{\Gamma}^{(m,n)}_\kappa(\{\mathbf{p}_\ell\}_{1\le\ell\le k-1},\,\omega_k+\varpi,\vec{p}_k,\{\mathbf{p}_\ell\}_{k+1\le\ell\le m+n-1})\\
&\qquad-\frac{\sum_{k=1}^{m+n-1}p_k^\beta}{\varpi}\bar{\Gamma}^{(m,n)}_\kappa(\{\mathbf{p}_\ell\}_{1\le\ell\le m+n-1})\Big]\\
&=\delta^d\Big(\sum_{k=1}^{m+n}\vec{p}_k\Big)\delta\big(\varpi+\sum_{k=1}^{m+n}\omega_k\big)i\epsilon_{\alpha\beta}\mathcal{D}_\beta(\varpi)\bar{\Gamma}^{(m,n)}_\kappa(\{\mathbf{p}_\ell\}_{1\le\ell\le m+n-1})\,.
\end{aligned}\tag{F.12}$$

Identifying in both sides the regular part, one obtains finally

$$\frac{\partial}{\partial q^\alpha}\bar{\Gamma}_\kappa^{(m+1,n)}(\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\le\ell\le m+n-1})\Big|_{\vec{q}=0} = i\epsilon_{\alpha\beta}\mathcal{D}_\beta(\varpi)\bar{\Gamma}_\kappa^{(m,n)}(\{\mathbf{p}_\ell\}_{1\le\ell\le m+n-1})\,. \quad \text{(F.13)}$$

F.1.2.2 Time-gauged rotation identity

Now, let us consider the time-gauged rotation Eq. (5.52), line e)

$$\int_{\vec{x}}\Big\{(\frac{x^2}{2}\partial_t+\epsilon_{\alpha\beta}x_\beta\partial_\alpha\Psi)\frac{\delta\Gamma_\kappa}{\delta\Psi(\mathbf{x})}+\epsilon_{\alpha\beta}x_\beta\partial_\alpha\bar{\Psi}\frac{\delta\Gamma_\kappa}{\delta\bar{\Psi}(\mathbf{x})}\Big\} = 2\int_{\vec{x}}\partial_t^2\bar{\Psi}\,. \quad \text{(F.14)}$$

First, let us study the case of one $\bar{\Psi}(\mathbf{x}')$ derivative. After setting the fields to zero, one gets

$$\int_{\vec{x}}\Big\{\frac{x^2}{2}\partial_t\Gamma_\kappa^{(1,1)}(\mathbf{x},\mathbf{x}')-\delta^d(\vec{x}-\vec{x}')\delta(t-t')\epsilon_{\alpha\beta}x_\beta\partial_\alpha\Gamma_\kappa^{(0,1)}(\mathbf{x})\Big\} = 2\int_{\vec{x}}\delta^d(\vec{x}-\vec{x}')\partial_t^2\delta(t-t')\,. \quad \text{(F.15)}$$

The invariance under translation gives $\partial_\alpha\Gamma_\kappa^{(0,1)}(\mathbf{x})=0$. Going to Fourier space yields

$$\int_{\vec{x}}\int_{\mathbf{p},\mathbf{p}'} i\omega\frac{x^2}{2}e^{i[\vec{p}\cdot\vec{x}+\vec{p}'\cdot\vec{x}'-\omega t-\omega' t']}\tilde{\Gamma}_\kappa^{(1,1)}(\mathbf{p},\mathbf{p}') = -2\int_{\vec{x}}\int_{\mathbf{p}}\omega^2 e^{i[\vec{p}\cdot(\vec{x}-\vec{x}')-\omega(t-t')]} \quad \text{(F.16)}$$

and by unicity of the Fourier transform, one obtains

$$\frac{\partial^2}{\partial p^2}\tilde{\Gamma}_\kappa^{(1,1)}(\mathbf{p},\mathbf{p}')\Big|_{\vec{p}=0} = -4i\omega\delta^d(\vec{p}')\delta(\omega+\omega')\,. \quad \text{(F.17)}$$

This result can be interpreted as the non-renormalization of the kinetic term in the bare action. Now, going to higher derivatives as in the preceding section,

$$\begin{aligned}&\int_{\vec{x}}\Big\{\frac{x^2}{2}\partial_t\Gamma_\kappa^{(m+1,n)}(\mathbf{x},\{\mathbf{x}_\ell\}_{1\le\ell\le m+n})\\&-\sum_{k=1}^{m+n}\delta(t-t_k)\delta^d(\vec{x}-\vec{x}_k)\epsilon_{\alpha\beta}x_\beta\partial_\alpha\Gamma_\kappa^{(m,n)}(\{\mathbf{x}_\ell\}_{1\le\ell\le k-1},\,\mathbf{x},\{\mathbf{x}_\ell\}_{k+1\le\ell\le m+n})\Big\} = 0\,.\end{aligned} \quad \text{(F.18)}$$

Going to the Fourier transforms of the vertex functions and of the delta functions gives

$$\begin{aligned}\int_{\vec{x}}\int_{\mathbf{q},\mathbf{p}_i}\Big\{&-i\varpi\frac{x^2}{2}e^{i(\vec{q}\cdot\vec{x}-\varpi t)+i\sum_{i=1}^{m+n}(\vec{p}_i\cdot\vec{x}_i-\omega_i t_i)}\tilde{\Gamma}_\kappa^{(m+1,n)}(\mathbf{q},\{\mathbf{p}_\ell\}_{1\le\ell\le m+n})\\&-i\epsilon_{\alpha\beta}x_\beta q^\alpha\sum_{k=1}^{m+n}e^{i[\vec{p}_k(\vec{x}-\vec{x}_k)-\omega_k(t-t_k)+\vec{q}\cdot\vec{x}-\varpi t+\sum_{\substack{i=1\\i\neq k}}^{m+n}(\vec{p}_i\cdot\vec{x}_i-\omega_i t_i)]}\\&\times\tilde{\Gamma}_\kappa^{(m,n)}(\{\mathbf{p}_\ell\}_{1\le\ell\le k-1},\,\mathbf{q},\{\mathbf{p}_\ell\}_{k+1\le\ell\le m+n})\Big\} = 0\,.\end{aligned} \quad \text{(F.19)}$$

Following the same steps as in the previous section, the identity for the Fourier transform reads

$$\begin{aligned}
&\frac{\partial^2}{\partial q^2}\tilde{\Gamma}^{(m+1,n)}_{\kappa}(\mathbf{q},\{\mathbf{p}_\ell\}_{1\leq\ell\leq m+n})\Big|_{\vec{q}=0}\\
&\quad=\frac{2i\epsilon_{\alpha\beta}}{\varpi}\sum_{k=1}^{m+n}p_k^\alpha\frac{\partial}{\partial p_k^\beta}\tilde{\Gamma}^{(m,n)}_{\kappa}(\{\mathbf{p}_\ell\}_{1\leq\ell\leq k-1},\,\omega_k+\varpi,\mathbf{p}_k,\{\mathbf{p}_\ell\}_{k+1\leq\ell\leq m+n})\\
&\quad\equiv\tilde{\mathcal{R}}(\varpi)\tilde{\Gamma}^{(m,n)}_{\kappa}(\{\mathbf{p}_\ell\}_{1\leq\ell\leq m+n})\,.
\end{aligned}\tag{F.20}$$

As for the time-gauged Galilean identity, it is possible to express this identity as finite differences extracting the delta of conservation. For the l.h.s., it reads

$$\begin{aligned}
&\frac{\partial}{\partial q^2}\tilde{\Gamma}^{(m+1,n)}_{\kappa}(\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\leq\ell\leq m+n})\Big|_{\vec{q}=0}\\
&=\frac{\partial^2}{\partial q^2}\Big[\delta^d\big(\vec{q}+\sum_{k=1}^{m+n}\vec{p}_k\big)\delta\big(\varpi+\sum_{k=1}^{m+n}\omega_k\big)\bar{\Gamma}^{(m+1,n)}_{\kappa}(\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\leq\ell\leq m+n-1})\Big]\Big|_{\vec{q}=0}\\
&=\delta^d\big(\sum_{k=1}^{m+n}\vec{p}_k\big)\delta\big(\varpi+\sum_{k=1}^{m+n}\omega_k\big)\frac{\partial^2}{\partial q^2}\bar{\Gamma}^{(m+1,n)}_{\kappa}(\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\leq\ell\leq m+n-1})\Big|_{\vec{q}=0}\\
&\quad+2\frac{\partial}{\partial q^\alpha}\delta^d\big(\vec{q}+\sum_{k=1}^{m+n}\vec{p}_k\big)\Big|_{\vec{q}=0}\delta\big(\varpi+\sum_{k=1}^{m+n}\omega_k\big)\frac{\partial}{\partial q^\alpha}\bar{\Gamma}^{(m+1,n)}_{\kappa}(\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\leq\ell\leq m+n-1})\Big|_{\vec{q}=0}\\
&\quad+\frac{\partial^2}{\partial q^2}\delta^d\big(\vec{q}+\sum_{k=1}^{m+n}\vec{p}_k\big)\Big|_{\vec{q}=0}\delta\big(\varpi+\sum_{k=1}^{m+n}\omega_k\big)\bar{\Gamma}^{(m+1,n)}_{\kappa}(\varpi,\vec{0},\{\mathbf{p}_\ell\}_{1\leq\ell\leq m+n-1})\\
&=\delta^d\big(\sum_{k=1}^{m+n}\vec{p}_k\big)\delta\big(\varpi+\sum_{k=1}^{m+n}\omega_k\big)\frac{\partial^2}{\partial q^2}\bar{\Gamma}^{(m+1,n)}_{\kappa}(\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\leq\ell\leq m+n-1})\Big|_{\vec{q}=0}\\
&\quad+2\frac{\partial}{\partial q^\alpha}\delta^d\big(\vec{q}+\sum_{k=1}^{m+n}\vec{p}_k\big)\Big|_{\vec{q}=0}\delta\big(\varpi+\sum_{k=1}^{m+n}\omega_k\big)i\epsilon_{\alpha\beta}\mathcal{D}_\beta(\varpi)\bar{\Gamma}^{(m,n)}_{\kappa}(\{\mathbf{p}_\ell\}_{1\leq\ell\leq m+n-1})\,,
\end{aligned}\tag{F.21}$$

where in the last equality *a*) and *d*) of Eq. (5.55) have been used. As for the r.h.s., the substitution reads

$$\begin{aligned}
&\frac{2i\epsilon_{\alpha\beta}}{\varpi}\sum_{k=1}^{m+n}p_k^\alpha\frac{\partial}{\partial p_k^\beta}\tilde{\Gamma}^{(m,n)}_{\kappa}(\{\mathbf{p}_\ell\}_{1\leq\ell\leq k-1},\,\omega_k+\varpi,\mathbf{p}_k,\{\mathbf{p}_\ell\}_{k+1\leq\ell\leq m+n})\\
&=\delta\big(\varpi+\sum_{k=1}^{m+n}\omega_k\big)\frac{2i\epsilon_{\alpha\beta}}{\varpi}\Bigg\{\\
&\sum_{k=1}^{m+n-1}p_k^\alpha\Bigg[\frac{\partial}{\partial p_k^\beta}\delta^d\big(\sum_{k=1}^{m+n}\vec{p}_k\big)\bar{\Gamma}^{(m,n)}_{\kappa}(\{\mathbf{p}_\ell\}_{1\leq\ell\leq k-1},\,\omega_k+\varpi,\vec{p}_k,\{\mathbf{p}_\ell\}_{k+1\leq\ell\leq m+n-1})\\
&+\delta^d\big(\sum_{k=1}^{m+n}\vec{p}_k\big)\frac{\partial}{\partial p_k^\beta}\bar{\Gamma}^{(m,n)}_{\kappa}(\{\mathbf{p}_\ell\}_{1\leq\ell\leq k-1},\,\omega_k+\varpi,\vec{p}_k,\{\mathbf{p}_\ell\}_{k+1\leq\ell\leq m+n-1})\Bigg]\\
&+p_{m+n}^\alpha\frac{\partial}{\partial p_{m+n}^\beta}\delta^d\big(\sum_{k=1}^{m+n}\vec{p}_k\big)\bar{\Gamma}^{(m,n)}_{\kappa}(\{\mathbf{p}_\ell\}_{1\leq\ell\leq m+n})\Bigg\}\,.
\end{aligned}\tag{F.22}$$

Using the delta of conservation on p^α_{m+n} in the last line and rewriting

$$\frac{\partial}{\partial p_k^\beta}\delta^d\big(\sum_{k=1}^{m+n}\vec{p}_k\big) = \frac{\partial}{\partial q^\beta}\delta^d\big(\vec{q}+\sum_{k=1}^{m+n}\vec{p}_k\big)\Big|_{\vec{q}=0}\,, \tag{F.23}$$

one obtains

$$\begin{aligned}
&\frac{2i\epsilon_{\alpha\beta}}{\varpi}\sum_{k=1}^{m+n} p_k^\alpha \frac{\partial}{\partial p_k^\beta}\tilde{\Gamma}^{(m,n)}_\kappa(\{\mathbf{p}_\ell\}_{1\le\ell\le k-1},\,\omega_k+\varpi,\mathbf{p}_k,\{\mathbf{p}_\ell\}_{k+1\le\ell\le m+n})\\
&= \delta\big(\varpi+\sum_{k=1}^{m+n}\omega_k\big)\delta^d\big(\sum_{k=1}^{m+n}\vec{p}_k\big)\frac{2i\epsilon_{\alpha\beta}}{\varpi}\\
&\times\sum_{k=1}^{m+n-1} p_k^\alpha \frac{\partial}{\partial p_k^\beta}\bar{\Gamma}^{(m,n)}_\kappa(\{\mathbf{p}_\ell\}_{1\le\ell\le k-1},\,\omega_k+\varpi,\vec{p}_k,\{\mathbf{p}_\ell\}_{k+1\le\ell\le m+n-1})\\
&+\frac{2i\epsilon_{\alpha\beta}}{\varpi}\delta\big(\varpi+\sum_{k=1}^{m+n}\omega_k\big)\frac{\partial}{\partial q^\beta}\delta^d\big(\vec{q}+\sum_{k=1}^{m+n}\vec{p}_k\big)\Big|_{\vec{q}=0}\\
&\times\sum_{k=1}^{m+n-1} p_k^\alpha\Big[\bar{\Gamma}^{(m,n)}_\kappa(\{\mathbf{p}_\ell\}_{1\le\ell\le k-1},\,\omega_k+\varpi,\vec{p}_k,\{\mathbf{p}_\ell\}_{k+1\le\ell\le m+n-1})-\bar{\Gamma}^{(m,n)}_\kappa(\{\mathbf{p}_\ell\}_{1\le\ell\le m+n})\Big]\\
&= \delta\big(\varpi+\sum_{k=1}^{m+n}\omega_k\big)\delta^d\big(\sum_{k=1}^{m+n}\vec{p}_k\big)\tilde{\mathcal{R}}(\varpi)\bar{\Gamma}^{(m,n)}_\kappa(\{\mathbf{p}_\ell\}_{1\le\ell\le m+n-1})\\
&+2\delta\big(\varpi+\sum_{k=1}^{m+n}\omega_k\big)\frac{\partial}{\partial q^\beta}\delta^d\big(\vec{q}+\sum_{k=1}^{m+n}\vec{p}_k\big)\Big|_{\vec{q}=0} i\epsilon_{\beta\alpha}\mathcal{D}_\alpha(\varpi)\bar{\Gamma}^{(m,n)}_\kappa(\{\mathbf{p}_\ell\}_{1\le\ell\le m+n})\,.
\end{aligned}\tag{F.24}$$

Thus the second lines of each side cancel each other and we are left with

$$\frac{\partial^2}{\partial q^2}\bar{\Gamma}^{(m+1,n)}_\kappa(\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\le\ell\le m+n-1})\Big|_{\vec{q}=0} = \tilde{\mathcal{R}}(\varpi)\bar{\Gamma}^{(m,n)}_\kappa(\{\mathbf{p}_\ell\}_{1\le\ell\le m+n-1})\,. \tag{F.25}$$

Contrary to the case of Galilean symmetry, this step is not enough to show that the limit $\varpi\to 0$ is well-defined. Indeed, extended Galilean symmetry corresponds to time-gauged space translation and the zero frequency limit corresponds to using time-independent space translation, which are taken care of by going from the $\tilde{\Gamma}^{(m,n)}_\kappa$ to the $\bar{\Gamma}^{(m,n)}_\kappa$. Here, it is necessary to make use of the zero frequency limit of time-gauged rotations, that is time-independent rotations. The corresponding Ward identity reads

$$\epsilon_{\alpha\beta}\sum_{k=1}^{m+n-1} p_k^\alpha\frac{\partial}{\partial p_k^\beta}\bar{\Gamma}^{(m,n)}_\kappa(\{\mathbf{p}_\ell\}_{1\le\ell\le m+n-1}) = 0\,. \tag{F.26}$$

Substracting it from the preceding equation, the final result reads

$$\frac{\partial^2}{\partial q^2}\bar{\Gamma}^{(m+1,n)}_\kappa(\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\le\ell\le m+n-1})\Big|_{\vec{q}=0} = \mathcal{R}(\varpi)\bar{\Gamma}^{(m,n)}_\kappa(\{\mathbf{p}_\ell\}_{1\le\ell\le m+n-1}) \tag{F.27}$$

with

$$\begin{aligned}
&\mathcal{R}(\varpi)F(\{\mathbf{p}_\ell\}_{1\le\ell\le n})\\
&\qquad\equiv 2i\epsilon_{\alpha\beta}\sum_{k=1}^{n} p_k^\alpha\frac{\partial}{\partial p_k^\beta}\left[\frac{F(\{\mathbf{p}_\ell\}_{1\le\ell\le k-1},\,\omega_k+\varpi,\vec{p}_k,\{\mathbf{p}_\ell\}_{k+1\le\ell\le n})-F(\{\mathbf{p}_\ell\}_{1\le\ell\le n})}{\varpi}\right].
\end{aligned}\tag{F.28}$$

As a last note before closing this section, the above subtleties going from the $\tilde{\Gamma}_\kappa^{(m,n)}$ to the $\bar{\Gamma}_\kappa^{(m,n)}$ are also present for symmetries b) and c) of Eq. (5.52) concerning the response field but the derivation is less involved because the r.h.s. is always zero and it will be skipped.

F.2 Form of the flow equation of correlation functions in the large wave-number regime

Let us firt derive the expression (5.82) of the flow equation for a generic generalized n-point connected correlation function $\tilde{G}^{(n)}$ in the regime of large wave-numbers. The flow equation for $G^{(n)}$ is obtained by taking n functional derivatives of (5.58) with respect to the sources j_{i_k}, $1 \leq k \leq n$, which yields

$$\partial_\kappa G^{(n)}_{i_1\dots i_n}[\{\mathbf{x}_\ell\}_{1\leq\ell\leq n};\mathrm{j}] = -\frac{1}{2}\int_{\mathbf{y}_1,\mathbf{y}_2} \partial_\kappa[\mathcal{R}_\kappa]_{ij}(\mathbf{y}_1-\mathbf{y}_2)\left\{ G^{(n+2)}_{iji_1\dots i_n}[\mathbf{y}_1,\mathbf{y}_2,\{\mathbf{x}_\ell\};\mathrm{j}] + \sum_{\substack{(\{i_\ell\}_1,\{i_\ell\}_2)\\ \#1+\#2=n}} G^{(\#1+1)}_{i\{i_\ell\}_1}[\mathbf{y}_1,\{\mathbf{x}_\ell\}_1;\mathrm{j}] G^{(\#2+1)}_{j\{i_\ell\}_2}[\mathbf{y}_2,\{\mathbf{x}_\ell\}_2;\mathrm{j}] \right\}. \tag{F.29}$$

Depending on which field theory we are looking at, the velocity or the stream function one, the indices i_ℓ stand for the sources $\mathrm{j} = \vec{J},\vec{\bar{J}}$ or $\mathrm{j} = J,\bar{J}$ respectively. The notation $(\{i_\ell\}_1,\{i_\ell\}_2)$ indicates all the possible bipartitions of the n indices $\{i_\ell\}_{1\leq\ell\leq n}$, and $(\{\mathbf{x}_\ell\}_1,\{\mathbf{x}_\ell\}_2)$ the corresponding bipartition in coordinates. Finally, $\#1$ and $\#2$ are the cardinals of $\{i_\ell\}_1$ (resp. $\{i_\ell\}_2$). Let us concentrate on the first line of (F.29). One can write

$$\begin{aligned}\int_{\mathbf{y}_1,\mathbf{y}_2} \partial_\kappa[\mathcal{R}_\kappa]_{ij}(\mathbf{y}_1-\mathbf{y}_2)\, G^{(n+2)}_{iji_1\dots i_n}[\mathbf{y}_1,\mathbf{y}_2,\{\mathbf{x}_\ell\};\mathrm{j}] &= \int_{\mathbf{y}_1,\mathbf{y}_2} \partial_\kappa[\mathcal{R}_\kappa]_{ij}(\mathbf{y}_1-\mathbf{y}_2) \\ \times\Bigg[\int_{\mathbf{z}_1,\mathbf{z}_2} G^{(2)}_{ki}[\mathbf{z}_1,\mathbf{y}_1;\mathrm{j}] G^{(2)}_{\ell j}[\mathbf{z}_2,\mathbf{y}_2;\mathrm{j}] \frac{\delta^2}{\delta\Phi_k(\mathbf{z}_1)\delta\Phi_\ell(\mathbf{z}_2)} \\ + \int_{\mathbf{z}} G^{(3)}_{\ell ij}[\mathbf{z},\mathbf{y}_1,\mathbf{y}_2;\mathrm{j}] \frac{\delta}{\delta\Phi_\ell(\mathbf{z})}\Bigg] G^{(n)}_{i_1\dots i_n}[\{\mathbf{x}_\ell\};\mathrm{j}]\,.\end{aligned} \tag{F.30}$$

The derivatives of $G^{(n)}$ with respect to Φ must be understood as acting on $G^{(n)}$ viewed as a diagram constructed from Γ_κ vertices. More precisely, $G^{(n)}$ is the sum of all tree diagrams with vertices the $\Gamma^{(k)}, k \leq n$ and with edges the propagator $G^{(2)}$, the latter satisfying

$$G^{(2)}_{k\ell}[\mathbf{x},\mathbf{y};\mathrm{j}] = \frac{\delta\Phi_k(\mathbf{x})}{\delta\mathrm{j}_\ell(\mathbf{y})} = \left(\frac{\delta\mathrm{j}}{\delta\Phi}\right)^{-1}_{k\ell}(\mathbf{x},\mathbf{y}) = [\Gamma^{(2)}+\mathcal{R}_\kappa]^{-1}_{k\ell}[\mathbf{x},\mathbf{y};\Phi], \tag{F.31}$$

using the property of the Legendre transform (3.48). Furthermore, introducing the differential operator

$$\tilde{\partial}_\kappa \equiv \partial_\kappa R_\kappa \frac{\delta}{\delta R_\kappa} + \partial_\kappa N_\kappa \frac{\delta}{\delta N_\kappa}\,, \tag{F.32}$$

and using the expression (F.31), one has

$$\tilde{\partial}_\kappa G^{(2)}_{k\ell}[\mathbf{z}_1,\mathbf{z}_2;\mathrm{j}] = -\int_{\mathbf{y}_1,\mathbf{y}_2} \partial_\kappa[\mathcal{R}_\kappa]_{ij}(\mathbf{y}_1-\mathbf{y}_2) G^{(2)}_{ki}[\mathbf{z}_1,\mathbf{y}_1;\mathrm{j}] G^{(2)}_{\ell j}[\mathbf{z}_2,\mathbf{y}_2;\mathrm{j}]\,, \tag{F.33}$$

which appears in the first term of the r.h.s. of (F.30). The second term in the r.h.s. of (F.30) vanishes when the sources are set to zero, since it is proportional to the flow of one of the average field. Indeed, the functions $G^{(1)}_i(\mathbf{x})$ are the expectation values of fields. The expression of their flow can be deduced by taking one derivative of (3.76) with respect to a source and setting the sources to zero, which yields

$$\partial_\kappa G^{(1)}_\ell(\mathbf{z}) = -\frac{1}{2}\int_{\mathbf{y}_1,\mathbf{y}_2} \partial_\kappa[\mathcal{R}_\kappa]_{ij}(\mathbf{y}_1-\mathbf{y}_2) G^{(3)}_{\ell ij}(\mathbf{z},\mathbf{y}_1,\mathbf{y}_2)\,, \tag{F.34}$$

omitting additional contribution proportional to $G^{(1)}$. In the velocity formulation, the average fields have to be set to zero to respect isotropy. In the stream function formulation, the average fields are thus constant of space (because they are the primitives of the velocity and response velocity fields). We can thus use the time-gauged Galilean invariance to place ourselves in the comoving frame where they are identically zero. As a consequence, in both formulations, the flow $\partial_\kappa G^{(1)}_\ell(\mathbf{z})$ is zero. By identification, one concludes that the second term in the r.h.s. of Eq. (F.30) vanishes when evaluated at zero fields. Gathering the previous expressions and setting the fields to zero, the flow equation for $G^{(n)}$ may be rewritten as

$$\begin{aligned}
&\partial_\kappa G^{(n)}_{i_1\dots i_n}(\{\mathbf{x}_\ell\}_{1\le\ell\le n}) \\
&= \frac{1}{2}\int_{\mathbf{y}_1,\mathbf{y}_2}\Bigg\{ \tilde{\partial}_\kappa G^{(2)}_{kl}(\mathbf{y}_1,\mathbf{y}_2)\Big[\frac{\delta^2}{\delta\Phi_k(\mathbf{y}_1)\delta\Phi_\ell(\mathbf{y}_2)} G^{(n)}_{i_1\dots i_n}[\{\mathbf{x}_\ell\}_{1\le\ell\le n};\mathrm{j}]\Big]_{\Phi=0} \\
&\quad - \sum_{\substack{(\{i_\ell\}_1,\{i_\ell\}_2)\\ \#1+\#2=n}} G^{(\#1+1)}_{i\{i_\ell\}_1}(\mathbf{y}_1,\{\mathbf{x}_\ell\}_1) G^{(\#2+1)}_{j\{i_\ell\}_2}(\mathbf{y}_2,\{\mathbf{x}_\ell\}_2)\Bigg\}\,.
\end{aligned} \tag{F.35}$$

This yields in Fourier space

$$\begin{aligned}
&\partial_\kappa \tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\}_{1\le\ell\le n}) \\
&= \frac{1}{2}\int_{\mathbf{q}_1,\mathbf{q}_2}\Bigg\{ \tilde{\partial}_\kappa \tilde{G}^{(2)}_{kl}(-\mathbf{q}_1,-\mathbf{q}_2)\Big[\frac{\delta^2}{\delta\Phi_k(\mathbf{q}_1)\delta\Phi_\ell(\mathbf{q}_2)} \tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\};\mathrm{j}]\Big]_{\Phi=0} \\
&\quad - \sum_{\substack{(\{i_\ell\}_1,\{i_\ell\}_2)\\ \#1+\#2=n}} \tilde{G}^{(\#1+1)}_{i\{i_\ell\}_1}(\mathbf{q}_1,\{\mathbf{p}_\ell\}_1)\partial_\kappa[\mathcal{R}_\kappa]_{ij}(-\mathbf{q}_1,-\mathbf{q}_2)\tilde{G}^{(\#2+1)}_{j\{i_\ell\}_2}(\mathbf{q}_2,\{\mathbf{p}_\ell\}_2)\Bigg\}\,,
\end{aligned} \tag{F.36}$$

where in the first line the Fourier transform is meant after the functional derivatives

$$\begin{aligned}
&\Big[\frac{\delta^2}{\delta\Phi_k(\mathbf{q}_1)\delta\Phi_\ell(\mathbf{q}_2)} \tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\};\mathrm{j}]\Big]_{\Phi=0} \\
&\equiv \mathrm{FT}\left(\frac{\delta^2}{\delta\Phi_k(\mathbf{z}_1)\delta\Phi_\ell(\mathbf{z}_2)} G^{(n)}_{i_1\dots i_n}[\{\mathbf{x}_\ell\};\mathrm{j}]\Big|_{\Phi=0}\right)(\mathbf{q}_1,\mathbf{q}_2,\{\mathbf{p}_\ell\})
\end{aligned} \tag{F.37}$$

with $\mathrm{FT}(\dots)$ denoting the Fourier transform. We focus on the flow equation (F.36), and now consider the limit of large wave-numbers, which we define as all external

wave-numbers being large compared to the RG scale $|\vec{p}_\ell| \gg \kappa$ for $1 \leq \ell \leq n$, as well as all possible partial sums being large $\left|\sum_{\ell\in I}\vec{p}_\ell\right| \gg \kappa$, for I a subset of $\{1,\dots n\}$, which means that we exclude exceptional configurations where a partial sum vanishes. The following proof relies on the presence of the derivative of the regulator term $\partial_\kappa[\mathcal{R}_\kappa]$ in the flow equation (F.36). The key properties of this term are that, on the one hand, it rapidly tends to zero for wave-numbers greater that the RG scale, and on the other hand, it ensures the analyticity of all vertex functions at any finite κ. Let us examine the second terms of the r.h.s. of (F.36) in this limit. Using invariance under space-time translation, it can be rewritten as

$$\begin{aligned}
&\int_{\mathbf{q}_1,\mathbf{q}_2}\sum_{\substack{(\{i_\ell\}_1,\{i_\ell\}_2)\\ \#1+\#2=n}} G^{(\#1+1)}_{i\{i_\ell\}_1}(\mathbf{q}_1,\{\mathbf{p}_\ell\}_1)\partial_\kappa[\mathcal{R}_\kappa]_{ij}(-\mathbf{q}_1,-\mathbf{q}_2)G^{(\#2+1)}_{j\{i_\ell\}_2}(\mathbf{q}_2,,\{\mathbf{p}_\ell\}_2)\\
&=(2\pi)^3\delta(\sum_{k=1}^n\omega_k)\delta^2(\sum_{k=1}^n\vec{p}_k)\\
&\times\int_{\mathbf{q}}\sum_{\substack{(\{i_\ell\}_1,\{i_\ell\}_2)\\ \#1+\#2=n}}\bar{G}^{(\#1+1)}_{\{i_\ell\}_1 i}(\{\mathbf{p}_\ell\}_1)\partial_\kappa[\mathcal{R}_\kappa]_{ij}(\sum\{\vec{p}_k\}_1)\bar{G}^{(\#2+1)}_{j\{i_\ell\}_2}(\{\mathbf{p}_\ell\}_2)
\end{aligned}\tag{F.38}$$

where $\sum\{\vec{p}_k\}_1$ is the sum of all the wave-numbers in $\{\vec{p}_k\}_1$. Thus this term is proportional to the derivative of the regulator evaluated at a sum of external wave-numbers $\sum\{\vec{p}_k\}_1 \gg \kappa$. Hence, it is suppressed at least exponentially in the limit of large wave-numbers and can be neglected safely. Finally, only the first term of (F.36) survives in this limit and we obtain Eq. (5.82) of the main text:

$$\partial_\kappa\tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\}_{1\leq\ell\leq n}) = \frac{1}{2}\int_{\mathbf{q}_1,\mathbf{q}_2}\tilde{\partial}_\kappa\tilde{G}^{(2)}_{ij}(-\mathbf{q}_1,-\mathbf{q}_2)\left[\frac{\delta^2}{\delta\Phi_i(\mathbf{q}_1)\delta\Phi_j(\mathbf{q}_2)}\tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\};\mathrm{j}]\right]_{\Phi=0}.\tag{F.39}$$

F.3 Leading order in the velocity formulation

In this appendix, we show that the leading order of the flow equation at unequal times can be closed at large wave-numbers, *i.e.* expressed in terms of $\bar{G}^{(k)}$ with $k \leq n$ only, using the Ward identities associated with the extended Galilean and extended shift identities (5.40) and (5.47), that we reproduce here for convenience:

$$\begin{aligned}
&\tilde{\Gamma}^{(m+1,n)}_{\alpha\alpha_1\dots\alpha_{m+n}}(\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\leq\ell\leq m+n})\Big|_{\vec{q}=0} = \tilde{\mathcal{D}}_\alpha(\varpi)\tilde{\Gamma}^{(m,n)}_{\alpha_1\dots\alpha_{m+n}}(\{\mathbf{p}_\ell\}_{1\leq\ell\leq m+n})\\
&\tilde{\Gamma}^{(m,n+1)}_{\alpha_1\dots\alpha_m\alpha\alpha_{m+1}\dots\alpha_n}(\{\mathbf{p}_\ell\}_{1\leq\ell\leq m},\varpi,\vec{q},\{\mathbf{p}_\ell\}_{1\leq\ell\leq n})\Big|_{\vec{q}=0} = 0
\end{aligned}\tag{F.40}$$

F.3.1 Leading order: two-point function

Let us first perform the explicit calculation in the case of the two-point function. The leading order flow equation reads

$$\begin{aligned}\partial_\kappa \tilde{G}^{(2)}_{v_\alpha v_\beta}(\mathbf{p}_1,\mathbf{p}_2)\Big|_{leading} &= \frac{1}{2}\int_{\mathbf{q}_1,\mathbf{q}_2} \tilde{\partial}_\kappa \tilde{G}^{(2)}_{ij}(-\mathbf{q}_1,-\mathbf{q}_2)\\ &\times \int_{\mathbf{k}_1,\mathbf{k}_2} \tilde{G}^{(2)}_{v_\alpha m}(\mathbf{p}_1,-\mathbf{k}_1)\tilde{G}^{(2)}_{v_\beta n}(\mathbf{p}_2,-\mathbf{k}_2)\Big[-\tilde{\Gamma}^{(4)}_{ijmn}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{k}_1,\mathbf{k}_2)\\ &+ \int_{\mathbf{k}_3,\mathbf{k}_4} \tilde{\Gamma}^{(3)}_{ims}(\mathbf{q}_1,\mathbf{k}_1,\mathbf{k}_3)\tilde{G}^{(2)}_{st}(-\mathbf{k}_3,-\mathbf{k}_4)\tilde{\Gamma}^{(3)}_{jnt}(\mathbf{q}_2,\mathbf{k}_2,\mathbf{k}_4) + (i,\mathbf{q}_1)\leftrightarrow(j,\mathbf{q}_2)\Big]_{\vec{q}_1=\vec{q}_2=0}\end{aligned} \tag{F.41}$$

where the double arrow signifies the permutation of the preceding term. First, looking at the $\tilde{\Gamma}^{(4)}$ term, we have

$$\tilde{\Gamma}^{(4)}_{ijmn}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{k}_1,\mathbf{k}_2)\Big|_{\vec{q}_1=\vec{q}_2=0} = \delta_{iv_\mu}\delta_{jv_\nu}\,\tilde{\mathcal{D}}_\mu(\varpi_1)\tilde{\mathcal{D}}_\nu(\varpi_2)\tilde{\Gamma}^{(2)}_{mn}(\mathbf{k}_1,\mathbf{k}_2)\,, \tag{F.42}$$

using (F.40). Second, doing the same for the one particle-reducible (1-PR) term, it reads

$$\begin{aligned}&\Big[\tilde{\Gamma}^{(3)}_{kms}(\mathbf{q}_1,\mathbf{k}_1,\mathbf{k}_3)\tilde{G}^{(2)}_{st}(-\mathbf{k}_3,-\mathbf{k}_4)\tilde{\Gamma}^{(3)}_{\ell nt}(\mathbf{q}_2,\mathbf{k}_2,\mathbf{k}_4)\Big]_{\vec{q}_1=\vec{q}_2=0}\\ &= \delta_{kv_\mu}\delta_{\ell v_\nu}\,\tilde{\mathcal{D}}_\mu(\varpi_1)\tilde{\Gamma}^{(2)}_{ms}(\mathbf{k}_1,\mathbf{k}_3)\tilde{G}^{(2)}_{st}(-\mathbf{k}_3,-\mathbf{k}_4)\tilde{\mathcal{D}}_\nu(\varpi_2)\tilde{\Gamma}^{(2)}_{nt}(\mathbf{k}_2,\mathbf{k}_4)\,,\end{aligned} \tag{F.43}$$

Plugging back both terms in (F.41), the leading contribution of the flow reads

$$\begin{aligned}\partial_\kappa \tilde{G}^{(2)}_{v_\alpha v_\beta}(\mathbf{p}_1,\mathbf{p}_2)\Big|_{leading} &= \frac{1}{2}\int_{\mathbf{q}_1,\mathbf{q}_2} \tilde{\partial}_\kappa \tilde{G}^{(2)}_{v_\mu v_\nu}(-\mathbf{q}_1,-\mathbf{q}_2)\\ &\int_{\mathbf{k}_1,\mathbf{k}_2} \tilde{G}^{(2)}_{v_\alpha m}(\mathbf{p}_1,-\mathbf{k}_1)\tilde{G}^{(2)}_{v_\beta n}(\mathbf{p}_2,-\mathbf{k}_2)\Big[-\tilde{\mathcal{D}}_\mu(\varpi_1)\tilde{\mathcal{D}}_\nu(\varpi_2)\tilde{\Gamma}^{(2)}_{mn}(\mathbf{k}_1,\mathbf{k}_2)\\ &+ \int_{\mathbf{k}_3,\mathbf{k}_4} \tilde{\mathcal{D}}_\mu(\varpi_1)\tilde{\Gamma}^{(2)}_{ms}(\mathbf{k}_1,\mathbf{k}_3)\tilde{G}^{(2)}_{st}(-\mathbf{k}_3,-\mathbf{k}_4)\tilde{\mathcal{D}}_\nu(\varpi_2)\tilde{\Gamma}^{(2)}_{nt}(\mathbf{k}_2,\mathbf{k}_4) + (\mu,\omega_1)\leftrightarrow(\nu,\omega_2)\Big]\,.\end{aligned} \tag{F.44}$$

Examing further the second term, we insert the following relation

$$\tilde{G}^{(2)}_{st}(-\mathbf{k}_3,-\mathbf{k}_4) = \int_{\mathbf{k}_5,\mathbf{k}_6} \tilde{G}^{(2)}_{su}(-\mathbf{k}_3,-\mathbf{k}_5)\tilde{\Gamma}^{(2)}_{uv}(\mathbf{k}_5,\mathbf{k}_6)\tilde{G}^{(2)}_{vt}(-\mathbf{k}_6,-\mathbf{k}_4)\,, \tag{F.45}$$

thus each $\tilde{\mathcal{D}}(\varpi)\tilde{\Gamma}^{(2)}$ is enclosed between two $\tilde{G}^{(2)}$. Making explicit the operator $\tilde{\mathcal{D}}$, this combination can be rewritten (taking for example the first one)

$$\begin{aligned}
&\int_{\mathbf{k}_1,\mathbf{k}_3} \tilde{G}^{(2)}_{v_\alpha m}(\mathbf{p}_1,-\mathbf{k}_1)\tilde{\mathcal{D}}_\mu(\varpi_1)\tilde{\Gamma}^{(2)}_{ms}(\mathbf{k}_1,\mathbf{k}_3)\tilde{G}^{(2)}_{su}(-\mathbf{k}_3,-\mathbf{k}_5)\\
&\quad= -\int_{\mathbf{k}_1,\mathbf{k}_3} \tilde{G}^{(2)}_{v_\alpha m}(\mathbf{p}_1,-\mathbf{k}_1)\Big[\frac{k_1^\mu}{\varpi_1}\tilde{\Gamma}^{(2)}_{ms}(\nu_1+\varpi_1,\vec{k}_1,\mathbf{k}_3)\\
&\qquad+\frac{k_3^\mu}{\varpi_1}\tilde{\Gamma}^{(2)}_{ms}(\mathbf{k}_1,\nu_3+\varpi_1,\vec{k}_3)\Big]\tilde{G}^{(2)}_{su}(-\mathbf{k}_3,-\mathbf{k}_5)\\
&\quad= -\int_{\mathbf{k}_1}\frac{k_1^\mu}{\varpi}\tilde{G}^{(2)}_{v_\alpha m}(\mathbf{p}_1,-\mathbf{k}_1)\delta_{mu}\delta(\varpi_1+\nu_1-\nu_5)\delta^2(\vec{k}_1-\vec{k}_5)\\
&\qquad+\int_{\mathbf{k}_3}\frac{k_3^\mu}{\varpi}\delta_{v_\alpha s}\delta(\omega_1+\varpi+\nu_3)\delta^2(\vec{p}_1+\vec{k}_3)\tilde{G}^{(2)}_{su}(-\mathbf{k}_3,-\mathbf{k}_5)\\
&\quad= -\frac{k_5^\mu}{\varpi_1}\tilde{G}^{(2)}_{v_\alpha u}(\mathbf{p}_1,-\nu_5+\varpi_1,-\vec{k}_5)+\frac{p_1^\mu}{\varpi_1}\tilde{G}^{(2)}_{v_\alpha u}(\omega_1+\varpi_1,\vec{p}_1,-\vec{k}_5)\\
&\quad= -\tilde{\mathcal{D}}_\mu(\varpi_1)\tilde{G}^{(2)}_{v_\alpha u}(\mathbf{p}_1,-\mathbf{k}_5)\,.
\end{aligned}\tag{F.46}$$

where in the second equality we have used that $\tilde{G}^{(2)}$ and $\tilde{\Gamma}^{(2)}$ are inverse of one another

$$\int_{\mathbf{p}_2}\tilde{G}_{ij}(\mathbf{p}_1,-\mathbf{p}_2)\tilde{\Gamma}_{jk}(\mathbf{p}_2,-\mathbf{p}_3)=\delta_{ik}\delta(\nu_1-\nu_3)\delta^d(\vec{p}_1-\vec{p}_3)\,.\tag{F.47}$$

In fact, we just proved the general property that

$$\begin{aligned}
&\frac{\delta}{\delta\Phi_k(\mathbf{q})}\tilde{G}^{(2)}_{ij}[\mathbf{p}_1,\mathbf{p}_2;\mathrm{j}]_{\varphi=0}\Big|_{\vec{q}=0}\\
&\quad= -\int_{\mathbf{k}_1,\mathbf{k}_2}\tilde{G}^{(2)}_{im}(\mathbf{p}_1,-\mathbf{k}_1)\tilde{\Gamma}^{3)}_{kmn}(\mathbf{q},\mathbf{k}_1,\mathbf{k}_2)\Big|_{\vec{q}=0}\tilde{G}^{(2)}_{nj}(-\mathbf{k}_2,\mathbf{p}_2)\\
&\quad= -\delta_{kv_\mu}\int_{\mathbf{k}_1,\mathbf{k}_2}\tilde{G}^{(2)}_{im}(\mathbf{p}_1,-\mathbf{k}_1)\tilde{\mathcal{D}}_\mu(\varpi)\tilde{\Gamma}^{(2)}_{mn}(\mathbf{k}_1,\mathbf{k}_2)\tilde{G}^{(2)}_{nj}(-\mathbf{k}_2,\mathbf{p}_2)\\
&\quad= \delta_{kv_\mu}\tilde{\mathcal{D}}_\mu(\varpi)\tilde{G}^{(2)}_{ij}(\mathbf{p}_1,\mathbf{p}_2)
\end{aligned}\tag{F.48}$$

which will be useful in the general case. In the case of the flow of the two point function, this allows to write –the integration and summation variables have been renamed for convenience–

$$\begin{aligned}
&\partial_\kappa\tilde{G}^{(2)}_{v_\alpha v_\beta}(\mathbf{p}_1,\mathbf{p}_2)\Big|_{leading}=\frac{1}{2}\int_{\mathbf{q}_1,\mathbf{q}_2}\tilde{\partial}_\kappa\tilde{G}^{(2)}_{v_\mu v_\nu}(-\mathbf{q}_1,-\mathbf{q}_2)\\
&\int_{\mathbf{k}_1,\mathbf{k}_2}\Big[-\tilde{G}^{(2)}_{v_\alpha m}(\mathbf{p}_1,-\mathbf{k}_1)\tilde{G}^{(2)}_{v_\beta n}(\mathbf{p}_2,-\mathbf{k}_2)\tilde{\mathcal{D}}_\mu(\varpi_1)\tilde{\mathcal{D}}_\nu(\varpi_2)\tilde{\Gamma}^{(2)}_{mn}(\mathbf{k}_1,\mathbf{k}_2)\\
&+\tilde{\mathcal{D}}_\mu(\varpi_1)\tilde{G}^{(2)}_{v_\alpha m}(\mathbf{p}_1,-\mathbf{k}_1)\Gamma^{(2)}_{mn}(\mathbf{k}_1,\mathbf{k}_2)\tilde{\mathcal{D}}_\nu(\varpi_2)\tilde{G}^{(2)}_{v_\beta n}(\mathbf{p}_2,-\mathbf{k}_2)+(\mu,\omega_1)\leftrightarrow(\nu,\omega_2)\Big]\,.
\end{aligned}\tag{F.49}$$

Now, making explicit the $\tilde{\mathcal{D}}$ operator, we have on the one hand for the first term in the square bracket

$$
\begin{aligned}
&\int_{\mathbf{k}_1,\mathbf{k}_2} \tilde{G}^{(2)}_{v_\alpha m}(\mathbf{p}_1,-\mathbf{k}_1)\tilde{G}^{(2)}_{v_\beta n}(\mathbf{p}_2,-\mathbf{k}_2)\tilde{\mathcal{D}}_\mu(\varpi_1)\tilde{\mathcal{D}}_\nu(\varpi_2)\tilde{\Gamma}^{(2)}_{mn}(\mathbf{k}_1,\mathbf{k}_2)\\
&= \int_{\mathbf{k}_1,\mathbf{k}_2} \tilde{G}^{(2)}_{v_\alpha m}(\mathbf{p}_1,-\mathbf{k}_1)\tilde{G}^{(2)}_{v_\beta n}(\mathbf{p}_2,-\mathbf{k}_2)\Big[\frac{k_1^\mu k_1^\nu}{\varpi_1\varpi_2}\tilde{\Gamma}^{(2)}_{mn}(\nu_1+\varpi_1+\varpi_2,\vec{k}_1,\mathbf{k}_2)\\
&+\frac{k_1^\mu k_2^\nu}{\varpi_1\varpi_2}\tilde{\Gamma}^{(2)}_{mn}(\nu_1+\varpi_1,\vec{k}_1,\nu_2+\varpi_2,\vec{k}_2)+(\mu,\varpi_1)\leftrightarrow(\nu,\varpi_2)\\
&+\frac{k_2^\mu k_2^\nu}{\varpi_1\varpi_2}\tilde{\Gamma}^{(2)}_{mn}(\mathbf{k}_1,\nu_2+\varpi_1+\varpi_2,\vec{k}_2)\Big]\\
&= \int_{\mathbf{k}_1}\frac{k_1^\mu k_1^\nu}{\varpi_1\varpi_2}\tilde{G}^{(2)}_{v_\alpha m}(\mathbf{p}_1,-\mathbf{k}_1)\delta_{v_\beta m}\delta(\omega_2+\nu_1+\varpi_1+\varpi_2)\delta^2(\vec{p}_2+\vec{k}_1)\\
&+\int_{\mathbf{k}_2}\frac{k_2^\mu k_2^\nu}{\varpi_1\varpi_2}\tilde{G}^{(2)}_{v_\beta n}(\mathbf{p}_2,-\mathbf{k}_2)\delta_{v_\alpha n}\delta(\omega_1+\nu_2+\varpi_1+\varpi_2)\delta^2(\vec{p}_1+\vec{k}_2)\\
&+\int_{\mathbf{k}_1,\mathbf{k}_2}\tilde{G}^{(2)}_{v_\alpha m}(\mathbf{p}_1,-\mathbf{k}_1)\tilde{G}^{(2)}_{v_\beta n}(\mathbf{p}_2,-\mathbf{k}_2)\\
&\times\frac{k_1^\mu k_2^\nu}{\varpi_1\varpi_2}\tilde{\Gamma}^{(2)}_{mn}(\nu_1+\varpi_1,\vec{k}_1,\nu_2+\varpi_2,\vec{k}_2)+(\mu,\varpi_1)\leftrightarrow(\nu,\varpi_2)\\
&= \frac{p_2^\mu p_2^\nu}{\varpi_1\varpi_2}\tilde{G}^{(2)}_{v_\alpha v_\beta}(\mathbf{p}_1,\omega_2+\nu_1+\varpi_1+\varpi_2,\vec{k}_2)+\frac{p_1^\mu p_1^\nu}{\varpi_1\varpi_2}\tilde{G}^{(2)}_{v_\alpha v_\beta}(\omega_1+\nu_1+\varpi_1+\varpi_2,\vec{k}_1,\mathbf{p}_2)\\
&+\int_{\mathbf{k}_1,\mathbf{k}_2}\tilde{G}^{(2)}_{v_\alpha m}(\mathbf{p}_1,-\mathbf{k}_1)\tilde{G}^{(2)}_{v_\beta n}(\mathbf{p}_2,-\mathbf{k}_2)\\
&\times\frac{k_1^\mu k_2^\nu}{\varpi_1\varpi_2}\tilde{\Gamma}^{(2)}_{mn}(\nu_1+\varpi_1,\vec{k}_1,\nu_2+\varpi_2,\vec{k}_2)+(\mu,\varpi_1)\leftrightarrow(\nu,\varpi_2)
\end{aligned}
\tag{F.50}
$$

and on the other hand for the second term

$$
\begin{aligned}
&\int_{\mathbf{k}_1,\mathbf{k}_2}\tilde{\mathcal{D}}_\mu(\varpi_1)\tilde{G}^{(2)}_{v_\alpha m}(\mathbf{p}_1,-\mathbf{k}_1)\Gamma^{(2)}_{mn}(\mathbf{k}_1,\mathbf{k}_2)\tilde{\mathcal{D}}_\nu(\varpi_2)\tilde{G}^{(2)}_{v_\beta n}(\mathbf{p}_2,-\mathbf{k}_2)\\
&=\int_{\mathbf{k}_1,\mathbf{k}_2}\Big[\frac{p_1^\mu}{\varpi_1}\tilde{G}^{(2)}_{v_\alpha m}(\omega_1+\varpi_1,\vec{p}_1,-\mathbf{k}_1)-\frac{k_1^\mu}{\varpi_1}\tilde{G}^{(2)}_{v_\alpha m}(\mathbf{p}_1,-\nu_1+\varpi_1,-\vec{k}_1)\Big]\\
&\times\Gamma^{(2)}_{mn}(\mathbf{k}_1,\mathbf{k}_2)\Big[\frac{p_2^\nu}{\varpi_2}\tilde{G}^{(2)}_{v_\beta n}(\omega_2+\varpi_2,\vec{p}_2,-\mathbf{k}_2)-\frac{k_2^\nu}{\varpi_2}\tilde{G}^{(2)}_{v_\beta n}(\mathbf{p}_2,-\nu_2+\varpi_2,-\vec{k}_2)\Big]\\
&=\frac{p_1^\mu p_2^\nu}{\varpi_1\varpi_2}\tilde{G}^{(2)}_{v_\alpha v_\beta}(\omega_1+\varpi_1,\vec{p}_1,\omega_2+\varpi_2,\vec{p}_2)\\
&+\frac{p_2^\mu p_2^\nu}{\varpi_1\varpi_2}\tilde{G}^{(2)}_{v_\alpha v_\beta}(\mathbf{p}_1,\omega_2+\nu_1+\varpi_1+\varpi_2,\vec{k}_2)+\frac{p_1^\mu p_1^\nu}{\varpi_1\varpi_2}\tilde{G}^{(2)}_{v_\alpha v_\beta}(\omega_1+\nu_1+\varpi_1+\varpi_2,\vec{k}_1,\mathbf{p}_2)\\
&+\int_{\mathbf{k}_1,\mathbf{k}_2}\tilde{G}^{(2)}_{v_\alpha m}(\mathbf{p}_1,-\nu_1+\varpi_1,-\vec{k}_1)\tilde{G}^{(2)}_{v_\beta n}(\mathbf{p}_2,-\nu_2+\varpi_2,-\vec{k}_2)\frac{k_1^\mu k_2^\nu}{\varpi_1\varpi_2}\tilde{\Gamma}^{(2)}_{mn}(\mathbf{k}_1,\mathbf{k}_2)\,.
\end{aligned}
\tag{F.51}
$$

Gathering everything back together, the last lines cancel each other by shifting the frequencies ν_1 and ν_2 and one is left with

$$\begin{aligned}\partial_\kappa \tilde{G}^{(2)}_{v_\alpha v_\beta}(\mathbf{p}_1,\mathbf{p}_2)\Big|_{leading} &= \frac{1}{2}\int_{\mathbf{q}_1,\mathbf{q}_2} \tilde{\partial}_\kappa \tilde{G}^{(2)}_{v_\alpha v_\beta}(-\mathbf{q}_1,-\mathbf{q}_2) \\ &\times \Big[\frac{p_2^\mu p_2^\nu}{\varpi_1\varpi_2}\tilde{G}^{(2)}_{v_\alpha v_\beta}(\mathbf{p}_1,\omega_2+\nu_1+\varpi_1+\varpi_2,\vec{k}_2) + \frac{p_1^\mu p_1^\nu}{\varpi_1\varpi_2}\tilde{G}^{(2)}_{v_\alpha v_\beta}(\omega_1+\nu_1+\varpi_1+\varpi_2,\vec{k}_1,\mathbf{p}_2) \\ &+ \frac{p_2^\mu p_2^\nu}{\varpi_1\varpi_2}\tilde{G}^{(2)}_{v_\alpha v_\beta}(\mathbf{p}_1,\omega_2+\nu_1+\varpi_1+\varpi_2,\vec{k}_2) + \frac{p_1^\mu p_1^\nu}{\varpi_1\varpi_2}\tilde{G}^{(2)}_{v_\alpha v_\beta}(\omega_1+\nu_1+\varpi_1+\varpi_2,\vec{k}_1,\mathbf{p}_2)\Big] \\ &= \frac{1}{2}\int_{\mathbf{q}_1,\mathbf{q}_2} \tilde{\partial}_\kappa \tilde{G}^{(2)}_{v_\mu v_\nu}(-\mathbf{q}_1,-\mathbf{q}_2)\mathcal{D}_\mu(\omega_1)\mathcal{D}_\nu(\omega_2)\tilde{G}^{(2)}_{v_\alpha v_\beta}(\mathbf{p}_1,\mathbf{p}_2)\,. \end{aligned} \tag{F.52}$$

This concludes the derivation for the two-point function case.

F.3.2 Leading order: general case

Going back to the flow of an arbitrary correlation function, the leading order term of the flow equation reads

$$\begin{aligned}\partial_\kappa \tilde{G}^{(n)}_{i_1\dots i_n}(\mathbf{p}_1,\dots,\mathbf{p}_n)\Big|_{leading} &= \frac{1}{2}\int_{\mathbf{q}_1,\mathbf{q}_2} \tilde{\partial}_\kappa \tilde{G}^{(2)}_{ij}(-\mathbf{q}_1,-\mathbf{q}_2) \\ &\times \Big[\frac{\delta^2}{\delta\Phi_i(\mathbf{q}_1)\delta\Phi_j(\mathbf{q}_2)}\tilde{G}^{(n)}_{i_1\dots i_n}[\mathbf{p}_1,\dots,\mathbf{p}_n;\mathrm{j}]\Big]_{\Phi=0}\Big|_{\vec{q}_1=\vec{q}_2=0}\,,\end{aligned} \tag{F.53}$$

Now, we will prove the following property

$$\Big[\frac{\delta^2}{\delta\Phi_i(\mathbf{q}_1)\delta\Phi_j(\mathbf{q}_2)}\tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\}_{1\le\ell\le n};\mathrm{j}]\Big]_{\Phi=0}\Big|_{\vec{q}_1=\vec{q}_2=0} = \delta_{iv_\mu}\delta_{jv_\nu}\tilde{\mathcal{D}}_\mu(\varpi_1)\tilde{\mathcal{D}}_\nu(\varpi_2)\tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\})\,. \tag{F.54}$$

The generalized correlation functions $\tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\})$ can be expressed as the sum over all trees whose edges are the propagators $\tilde{G}^{(2)}$, whose vertices are the vertex functions $\tilde{\Gamma}^{(k)}$ and with external legs whose momenta and indices match the indices of the correlation function: $\{(i_\ell,\mathbf{p}_\ell)\}$. Symbolically,

$$\begin{aligned}\tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\}_{1\le\ell\le n};\mathrm{j}] &= \sum_{\text{trees}} \alpha_{\mathcal{T}}\tilde{\mathcal{T}}^{(n)}_{i_1\cdots i_n}[\{\mathbf{p}_\ell\}] \\ \tilde{\mathcal{T}}^{(n)}_{i_1\cdots i_n}[\{\mathbf{p}_\ell\}] &= \int_{\mathbf{k}_i}\prod_{i=1}^{m}\mathcal{E}_i^{\mathcal{T}}[\{\mathbf{p}_\ell\}_i,\{\mathbf{k}_\ell\}_i]\,,\end{aligned} \tag{F.55}$$

where $\alpha_{\mathcal{T}}$ is a combinatorial factor, the $\mathcal{E}_i^{\mathcal{T}}$ are the vertex functions and propagators entering the composition of the tree $\tilde{\mathcal{T}}$ and the integration is done over all the internal momenta of the diagram. The $\{\mathbf{p}_\ell\}_i$ which are not empty form a partition of the external momenta $\{\mathbf{p}_\ell\}_{1\le\ell\le n}$, and the internal momenta $\{\mathbf{k}_\ell\}_i$ are chosen such that when a propagator is attached to a vertex function, the sum of the momenta of the propagator and of the vertex function at the link is zero. Finally, the internal indices of the theory – here $i_\ell \in \{\vec{v},\bar{\vec{v}}\}$ – have been omitted on $\mathcal{E}_i^{\mathcal{T}}$ but follow

straightforwardly from the partition of momenta. The term in square bracket in the flow equation above is a sum of tree diagrams where the two functional derivatives have been distributed

$$\begin{aligned}
&\frac{\delta^2}{\delta\Phi_k(\mathbf{q}_1)\delta\Phi_\ell(\mathbf{q}_2)}\tilde{\mathcal{T}}^{(n)}_{i_1\cdots i_n}[\{\mathbf{p}_\ell\}]_{\Phi=0}\Big|_{\vec{q}_1=\vec{q}_2=0}\\
&=\Bigg\{\int_{\mathbf{k}_{\rm intern}}\sum_{\substack{i,j\\ i\neq j}}\Big(\prod_{m\neq i,j}\mathcal{E}^{\mathcal{T}}_m(\{\mathbf{p}_\ell\}_m,\{\mathbf{k}_\ell\}_m)\Big)\\
&\times\frac{\delta}{\delta\Phi_k(\mathbf{q}_1)}\mathcal{E}^{\mathcal{T}}_i[\{\mathbf{p}_\ell\}_i,\{\mathbf{k}_\ell\}_i]\Big|_{\Phi=0}\frac{\delta}{\delta\Phi_\ell(\mathbf{q}_2)}\mathcal{E}^{\mathcal{T}}_j[\{\mathbf{p}_\ell\}_j,\{\mathbf{k}_\ell\}_j]\Big|_{\Phi=0}\\
&+\int_{\mathbf{k}_{\rm intern}}\sum_i\Big(\prod_{j\neq i}\mathcal{E}^{\mathcal{T}}_j(\{\mathbf{p}_\ell\}_j,\{\mathbf{k}_\ell\}_j)\Big)\frac{\delta^2}{\delta\Phi_k(\mathbf{q}_1)\delta\Phi_\ell(\mathbf{q}_2)}\mathcal{E}^{\mathcal{T}}_i[\{\mathbf{p}_\ell\}_i,\{\mathbf{k}_\ell\}_i]\Big|_{\Phi=0}\Bigg\}\Big|_{\vec{q}_1=\vec{q}_2=0}.
\end{aligned}\tag{F.56}$$

When acting with a field functional derivative, either the derivative hits on a vertex function, giving

$$\frac{\delta}{\delta\Phi_i(\mathbf{q}_a)}\tilde{\Gamma}^{(k)}_{i_1\ldots i_k}[\{\mathbf{p}_\ell\}_{1\leq\ell\leq k};\Phi]_{\Phi=0}=\tilde{\Gamma}^{(k+1)}_{ii_1\ldots i_k}(\mathbf{q}_a,\{\mathbf{p}_\ell\}_{1\leq\ell\leq k})\tag{F.57}$$

or the derivative hits on a propagator, giving

$$\frac{\delta}{\delta\Phi_i(\mathbf{q}_a)}G^{(2)}_{mn}[\mathbf{p}_1,\mathbf{p}_2;\mathrm{j}]_{\Phi=0}=-\int_{\mathbf{k}_1,\mathbf{k}_2}G^{(2)}_{mu}(\mathbf{p}_1,-\mathbf{k}_1)\tilde{\Gamma}^{(3)}_{iuv}(\mathbf{q}_a,\mathbf{k}_1,\mathbf{k}_2)G^{(2)}_{vn}(-\mathbf{k}_2,\mathbf{p}_2)\,,\tag{F.58}$$

with $a\in\{1,2\}$.

First, let us examine the case of only one functional derivative applied to a generic tree $\tilde{\mathcal{T}}^{(n)}$.

$$\begin{aligned}
&\frac{\delta}{\delta\Phi_i(\mathbf{q}_a)}\tilde{\mathcal{T}}^{(n)}_{i_1\cdots i_n}[\{\mathbf{p}_\ell\}]_{\Phi=0}\Big|_{\vec{q}_a=0}\\
&\qquad=\int_{\mathbf{k}_{\rm intern}}\sum_{i=1}^{m}\left(\prod_{\substack{j=1\\ j\neq i}}^{m}\mathcal{E}^{\mathcal{T}}_j(\{\mathbf{p}_\ell\}_j,\{\mathbf{k}_\ell\}_j)\right)\frac{\delta}{\delta\Phi_i(\mathbf{q}_a)}\mathcal{E}^{\mathcal{T}}_i[\{\mathbf{p}_\ell\}_i,\{\mathbf{k}_\ell\}_i]\,.
\end{aligned}\tag{F.59}$$

giving either

$$\frac{\delta}{\delta\Phi_i(\mathbf{q}_a)}\tilde{\Gamma}^{(k)}_{i_1\ldots i_k}[\{\mathbf{p}_\ell\}_{1\leq\ell\leq k};\Phi]_{\Phi=0}\Big|_{\vec{q}_a=0}=\delta_{iv_\mu}\tilde{\mathcal{D}}_\mu(\varpi_a)\tilde{\Gamma}^{(k)}_{i_1\ldots i_k}(\{\mathbf{p}_\ell\})\tag{F.60}$$

if $\mathcal{E}^{\mathcal{T}}_i$ is a vertex function, or

$$\frac{\delta}{\delta\Phi_i(\mathbf{q}_a)}G^{(2)}_{mn}[\mathbf{p}_1,\mathbf{p}_2;\mathrm{j}]_{\Phi=0}\Big|_{\vec{q}_a=0}=\delta_{iv_\mu}\tilde{\mathcal{D}}_\mu(\varpi_a)\tilde{G}^{(2)}_{mn}(\mathbf{p}_1,\mathbf{p}_2)\,,\tag{F.61}$$

if it is a propagator, using the property (F.48) of the preceding section. This shows that

$$\begin{aligned}
&\frac{\delta}{\delta\Phi_i(\mathbf{q}_a)}\tilde{\mathcal{T}}^{(n)}_{i_1\cdots i_n}[\{\mathbf{p}_\ell\}]_{\Phi=0}\Big|_{\vec{q}_a=0}\\
&\qquad=\delta_{iv_\mu}\int_{\mathbf{k}_{\rm intern}}\sum_{i=1}^{m}\left(\prod_{\substack{j=1\\ j\neq i}}^{m}\mathcal{E}^{\mathcal{T}}_j(\{\mathbf{p}_\ell\}_j,\{\mathbf{k}_\ell\}_j)\right)\tilde{\mathcal{D}}_\mu(\varpi_a)\mathcal{E}^{\mathcal{T}}_i(\{\mathbf{p}_\ell\}_i,\{\mathbf{k}_\ell\}_i)\,.
\end{aligned}\tag{F.62}$$

In other words, the operator $\tilde{\mathcal{D}}_\mu(\varpi_a)$ is distributed on the elements of the tree according to Leibniz rule. In order to reconstruct the operator acting on $\tilde{\mathcal{T}}^{(n)}$, that is only on the external legs $\{\mathbf{p}_\ell\}$, one thus has to show that it is indeed a derivative within this object, that is

$$\tilde{\mathcal{D}}_\mu(\varpi_a)\int_{\mathbf{k}}\mathcal{E}_1^{\mathcal{T}}(\{\mathbf{k}_\ell\}_1,-\mathbf{k})\mathcal{E}_2^{\mathcal{T}}(\mathbf{k},\{\mathbf{k}_\ell\}_2)=\int_{\mathbf{k}}\Big[\tilde{\mathcal{D}}_\mu(\varpi_a)\mathcal{E}_1^{\mathcal{T}}(\{\mathbf{k}_\ell\}_1,-\mathbf{k})\mathcal{E}_2^{\mathcal{T}}(\mathbf{k},\{\mathbf{k}_\ell\}_2) + \mathcal{E}_1^{\mathcal{T}}(\{\mathbf{k}_\ell\}_1,-\mathbf{k})\tilde{\mathcal{D}}_\mu(\varpi_a)\mathcal{E}_2^{\mathcal{T}}(\mathbf{k},\{\mathbf{k}_\ell\}_2)\Big]\,. \tag{F.63}$$

Starting from the r.h.s.,

$$\begin{aligned}
&\int_{\mathbf{k}}\Big[\tilde{\mathcal{D}}_\mu(\varpi_a)\mathcal{E}_i^{\mathcal{T}}(\{\mathbf{k}_\ell\}_{\ell\in I},-\mathbf{k})\mathcal{E}_j^{\mathcal{T}}(\mathbf{k},\{\mathbf{k}_\ell\}_{\ell\in J})+\mathcal{E}_i^{\mathcal{T}}(\{\mathbf{k}_\ell\}_{\ell\in I},-\mathbf{k})\tilde{\mathcal{D}}_\mu(\varpi_a)\mathcal{E}_j^{\mathcal{T}}(\mathbf{k},\{\mathbf{k}_\ell\}_{\ell\in J})\Big]\\
&\quad=-\int_{\mathbf{k}}\Big[\sum_{i\in I}\frac{k_i^\mu}{\varpi_a}\mathcal{E}_i^{\mathcal{T}}(\varpi_i+\omega_a,\vec{k}_i,\{\mathbf{k}_\ell\}_{\ell\in I\setminus i},-\mathbf{k})\mathcal{E}_j^{\mathcal{T}}(\mathbf{k},\{\mathbf{k}_\ell\}_{\ell\in J})\\
&\quad+\sum_{j\in J}\frac{k_j^\mu}{\varpi_a}\mathcal{E}_i^{\mathcal{T}}(\{\mathbf{k}_\ell\}_{\ell\in I},-\mathbf{k})\mathcal{E}_j^{\mathcal{T}}(\mathbf{k},\varpi_j+\omega_a,\vec{k}_j,\{\mathbf{k}_\ell\}_{\ell\in J\setminus j})\\
&\quad+\frac{k^\mu}{\varpi_a}\mathcal{E}_i^{\mathcal{T}}(\{\mathbf{k}_\ell\}_{\ell\in I},-\varpi+\omega_a,-\vec{k})\mathcal{E}_j^{\mathcal{T}}(\mathbf{k},\{\mathbf{k}_\ell\}_{\ell\in J})\\
&\quad-\mathcal{E}_i^{\mathcal{T}}(\{\mathbf{k}_\ell\}_{\ell\in I},-\mathbf{k})\frac{k^\mu}{\varpi_a}\mathcal{E}_j^{\mathcal{T}}(\varpi+\omega_a,\vec{k},\{\mathbf{k}_\ell\}_{\ell\in J})\Big]\,.
\end{aligned}\tag{F.64}$$

Shifting the associated frequency, it is readily shown that the two last terms cancel each other. Thus one is left with the operator $\tilde{\mathcal{D}}_\mu(\varpi_a)$ acting on the external leg of the object $\int_{\mathbf{k}}\mathcal{E}_i\mathcal{E}_j$, proving (F.63). Finally,

$$\begin{aligned}
&\frac{\delta}{\delta\Phi_i(\mathbf{q}_a)}\tilde{\mathcal{T}}^{(n)}_{i_1\cdots i_n}[\{\mathbf{p}_\ell\}]_{\Phi=0}\Big|_{\vec{q}_a=0}\\
&\qquad=\delta_{iv_\mu}\int_{\mathbf{k}_{\rm intern}}\sum_{i=1}^m\left(\prod_{\substack{j=1\\j\neq i}}^m\mathcal{E}_j^{\mathcal{T}}(\{\mathbf{p}_\ell\}_j,\{\mathbf{k}_\ell\}_j)\right)\tilde{\mathcal{D}}_\mu(\varpi_a)\mathcal{E}_i^{\mathcal{T}}(\{\mathbf{p}_\ell\}_i,\{\mathbf{k}_\ell\}_i)\\
&\qquad=\delta_{iv_\mu}\tilde{\mathcal{D}}_\mu(\varpi_a)\int_{\mathbf{k}_{\rm intern}}\prod_{i=1}^m\mathcal{E}_i^{\mathcal{T}}(\{\mathbf{p}_\ell\}_i,\{\mathbf{k}_\ell\}_i)\\
&\qquad=\delta_{iv_\mu}\tilde{\mathcal{D}}_\mu(\varpi_a)\tilde{\mathcal{T}}^{(n)}_{i_1\cdots i_n}(\{\mathbf{p}_\ell\})\,.
\end{aligned}\tag{F.65}$$

To prove (F.54), one still needs to check that the same property follows for two

functional derivatives and their subsequent wave-number derivative. We have

$$\begin{aligned}
&\frac{\delta^2}{\delta\Phi_i(\mathbf{q}_1)\delta\Phi_j(\mathbf{q}_2)}\tilde{\mathcal{T}}^{(n)}_{i_1\cdots i_n}[\{\mathbf{p}_\ell\}]_{\Phi=0}\Big|_{\vec{q}_1=\vec{q}_2=0}\\
&=\int_{\mathbf{k}_{\text{intern}}}\sum_{\substack{k,k'\\k\neq k'}}\Big(\prod_{m\neq k,k'}\mathcal{E}^{\mathcal{T}}_m(\{\mathbf{p}_\ell\}_m,\{\mathbf{k}_\ell\}_m)\Big)\\
&\times\frac{\delta}{\delta\Phi_i(\mathbf{q}_1)}\mathcal{E}^{\mathcal{T}}_k[\{\mathbf{p}_\ell\}_k,\{\mathbf{k}_\ell\}_k]\Big|_{\Phi=0}\frac{\delta}{\delta\Phi_j(\mathbf{q}_2)}\mathcal{E}^{\mathcal{T}}_{k'}[\{\mathbf{p}_\ell\}_{k'},\{\mathbf{k}_\ell\}_{k'}]\Big|_{\Phi=0}\\
&+\int_{\mathbf{k}_{\text{intern}}}\sum_{k}\Big(\prod_{k'\neq k}\mathcal{E}^{\mathcal{T}}_{k'}(\{\mathbf{p}_\ell\}_{k'},\{\mathbf{k}_\ell\}_{k'})\Big)\frac{\delta^2}{\delta\Phi_i(\mathbf{q}_1)\delta\Phi_j(\mathbf{q}_2)}\mathcal{E}^{\mathcal{T}}_k[\{\mathbf{p}_\ell\}_k,\{\mathbf{k}_\ell\}_k]\Big|_{\Phi=0}\Big|_{\vec{q}_1=\vec{q}_2=0}.
\end{aligned}\tag{F.66}$$

At this point, first setting only one wave-number to zero – let us say $\vec{q}_2$ –, the $\Phi_j(\mathbf{q}_2)$ functional derivatives can be replaced by $\delta_{jv_\nu}\tilde{\mathcal{D}}_\nu(\varpi_2)$. This is possible as well in the second term in curly brackets because the object to which it is applied, $\frac{\delta}{\delta\Phi_i(\mathbf{q}_1)}\mathcal{E}^{\mathcal{T}}_k[\{\mathbf{p}_\ell\}_k,\{\mathbf{k}_\ell\}_k]\Big|_{\Phi=0}$ is nothing but a tree with the $\vec{q}_1$ leg amputated and the derivative property of $\tilde{\mathcal{D}}_\mu(\varpi_a)$ (F.63) applies. Thus, one has

$$\begin{aligned}
&\frac{\delta^2}{\delta\Phi_i(\mathbf{q}_1)\delta\Phi_j(\mathbf{q}_2)}\tilde{\mathcal{T}}^{(n)}_{i_1\cdots i_n}[\{\mathbf{p}_\ell\}]_{\Phi=0}\Big|_{\vec{q}_1=\vec{q}_2=0}\\
&=\delta_{jv_\nu}\int_{\mathbf{k}_{\text{intern}}}\sum_{\substack{k,k'\\k\neq k'}}\Big(\prod_{m\neq k,k'}\mathcal{E}^{\mathcal{T}}_m(\{\mathbf{p}_\ell\}_m,\{\mathbf{k}_\ell\}_m)\Big)\\
&\times\frac{\delta}{\delta\Phi_i(\mathbf{q}_1)}\mathcal{E}^{\mathcal{T}}_k[\{\mathbf{p}_\ell\}_k,\{\mathbf{k}_\ell\}_k]\Big|_{\Phi=0}\tilde{\mathcal{D}}_\nu(\varpi_2)\mathcal{E}^{\mathcal{T}}_{k'}(\{\mathbf{p}_\ell\}_{k'},\{\mathbf{k}_\ell\}_{k'})\\
&+\int_{\mathbf{k}_{\text{intern}}}\sum_{k}\Big(\prod_{k'\neq k}\mathcal{E}^{\mathcal{T}}_{k'}(\{\mathbf{p}_\ell\}_{k'},\{\mathbf{k}_\ell\}_{k'})\Big)\tilde{\mathcal{D}}_\nu(\varpi_2)\Big[\frac{\delta}{\delta\Phi_i(\mathbf{q}_1)}\mathcal{E}^{\mathcal{T}}_k[\{\mathbf{p}_\ell\}_k,\{\mathbf{k}_\ell\}_k]\Big|_{\Phi=0}\Big]\Big|_{\vec{q}_1=0}\\
&=\delta_{jv_\nu}\tilde{\mathcal{D}}_\nu(\varpi_2)\Big[\int_{\mathbf{k}_{\text{intern}}}\sum_{k}\Big(\prod_{k'\neq k}\mathcal{E}^{\mathcal{T}}_{k'}(\{\mathbf{p}_\ell\}_{k'},\{\mathbf{k}_\ell\}_{k'})\Big)\frac{\delta}{\delta\Phi_i(\mathbf{q}_1)}\mathcal{E}^{\mathcal{T}}_k[\{\mathbf{p}_\ell\}_k,\{\mathbf{k}_\ell\}_k]\Big|_{\Phi=0}\Big]\Big|_{\vec{q}_1=0}.
\end{aligned}\tag{F.67}$$

Making explicit $\tilde{\mathcal{D}}_\nu(\varpi_2)$,

$$\begin{aligned}
&\frac{\delta^2}{\delta\Phi_i(\mathbf{q}_1)\delta\Phi_j(\mathbf{q}_2)}\tilde{\mathcal{T}}^{(n)}_{i_1\cdots i_n}[\{\mathbf{p}_\ell\}]_{\Phi=0}\Big|_{\vec{q}_1=\vec{q}_2=0}\\
&=-\delta_{jv_\mu}\Bigg\{\frac{q_1^\nu}{\varpi_2}\Big[\int_{\mathbf{k}_{\text{intern}}}\sum_{k}\Big(\prod_{k'\neq k}\mathcal{E}^{\mathcal{T}}_{k'}(\{\mathbf{p}_\ell\}_{k'},\{\mathbf{k}_\ell\}_{k'})\Big)\\
&\times\frac{\delta}{\delta\Phi_i(\varpi_1+\varpi_2,\vec{q}_1)}\mathcal{E}^{\mathcal{T}}_k[\{\mathbf{p}_\ell\}_k,\{\mathbf{k}_\ell\}_k]\Big|_{\Phi=0}\Big]\\
&+\sum_{k=1}^{n}\frac{p_k^\nu}{\varpi_2}\Big[\int_{\mathbf{k}_{\text{intern}}}\sum_{m}\Big(\prod_{m'\neq m}\mathcal{E}^{\mathcal{T}}_{m'}(\{\mathbf{p}_\ell\}_{m'}\backslash k_+,\{\mathbf{k}_\ell\}_{m'})\Big)\\
&\times\frac{\delta}{\delta\Phi_i(\mathbf{q}_1)}\mathcal{E}^{\mathcal{T}}_m[\{\mathbf{p}_\ell\}_m\backslash k_+,\{\mathbf{k}_\ell\}_m]\Big|_{\Phi=0}\Big]\Bigg\}\Bigg|_{\vec{q}_1=0},
\end{aligned}\tag{F.68}$$

with the shorthand notation $\{\mathbf{p}_\ell\}_m \backslash k_+ = \{\nu_k + \omega_2, \vec{p}_k, \mathbf{p}_{\ell\neq k}\}_m$ if $\mathbf{p}_k \in \{\mathbf{p}_\ell\}_m$ and else $\{\mathbf{p}_\ell\}_m \backslash k_+ = \{\mathbf{p}_\ell\}_m$. Setting $\vec{q}_1$ to zero

$$\begin{aligned}
&\frac{\delta^2}{\delta\Phi_k(\mathbf{q}_1)\delta\Phi_\ell(\mathbf{q}_2)}\tilde{\mathcal{T}}^{(n)}_{i_1\cdots i_n}[\{\mathbf{p}_\ell\}]_{\Phi=0}\Big|_{\vec{q}_1=\vec{q}_2=0} \\
&= -\delta_{jv_\nu}\delta_{iv_\mu}\sum_{k=1}^{n}\frac{p_k^\nu}{\varpi_2}\Big[\\
&\int_{\mathbf{k}_{\text{intern}}}\sum_m\Big(\prod_{m'\neq m}\mathcal{E}^{\mathcal{T}}_{m'}(\{\mathbf{p}_\ell\}_{m'}\backslash k_+,\{\mathbf{k}_\ell\}_{m'})\Big)\tilde{\mathcal{D}}_\mu(\varpi_1)\mathcal{E}^{\mathcal{T}}_m(\{\mathbf{p}_\ell\}_m\backslash k_+,\{\mathbf{k}_\ell\}_m)\Big] \\
&= -\delta_{jv_\nu}\delta_{iv_\mu}\sum_{k=1}^{n}\frac{p_k^\nu}{\varpi_2}\Big[\tilde{\mathcal{D}}_\mu(\varpi_1)\int_{\mathbf{k}_{\text{intern}}}\prod_m\mathcal{E}^{\mathcal{T}}_m(\{\mathbf{p}_\ell\}_m\backslash k_+,\{\mathbf{k}_\ell\}_m)\Big] \\
&= \delta_{jv_\nu}\delta_{iv_\mu}\tilde{\mathcal{D}}_\nu(\varpi_2)\tilde{\mathcal{D}}_\mu(\varpi_1)\tilde{\mathcal{T}}^{(n)}_{i_1\cdots i_n}(\{\mathbf{p}_\ell\})\,,
\end{aligned} \tag{F.69}$$

which finishes to prove the property (F.54). Going back to the flow equation (F.98), one obtains the announced result:

$$\begin{aligned}
\partial_\kappa\tilde{G}^{(n)}_{i_1\ldots i_n}(\{\mathbf{p}_\ell\}_{1\leq\ell\leq n}) &= \frac{1}{2}\int_{\mathbf{q}_1,\mathbf{q}_2}\tilde{\partial}_\kappa\tilde{G}^{(2)}_{v_\mu v_\nu}(-\mathbf{q}_1,-\mathbf{q}_2)\tilde{\mathcal{D}}_\mu(\omega_1)\tilde{\mathcal{D}}_\nu(\omega_2)\tilde{G}^{(n)}_{i_1\ldots i_n}(\{\mathbf{p}_\ell\}) \\
&= \frac{1}{2}\int_{\mathbf{q}_1,\mathbf{q}_2}\tilde{\partial}_\kappa\tilde{G}^{(2)}_{v_\mu v_\nu}(-\mathbf{q}_1,-\mathbf{q}_2)\tilde{\mathcal{D}}_\mu(\omega_1)\tilde{\mathcal{D}}_\nu(\omega_2)\tilde{G}^{(n)}_{i_1\ldots i_n}(\{\mathbf{p}_\ell\})\,.
\end{aligned} \tag{F.70}$$

F.4 Solution of the fixed-point equations

The aim of this secion is to derive the solution of the flow equation (5.63) for n-point correlation functions at the fixed point and in the 3-D case. We reproduce it from (Tarpin, Canet, and Wschebor 2018). It is easier to work with the hybrid time-wave-vector representation of the flow equation, which reads

$$\begin{aligned}
&\partial_\kappa G^{(n)}_{\alpha_1\ldots\alpha_n}(t_1,\vec{p}_1,\cdots,t_{n-1},\vec{p}_{n-1}) \\
&= \frac{1}{3}G^{(n)}_{\alpha_1\ldots\alpha_n}(t_1,\vec{p}_1,\cdots,t_{n-1},\vec{p}_{n-1})\sum_{k,\ell}\vec{p}_k\cdot\vec{p}_\ell\int_\varpi J_\kappa(\varpi)\,\frac{e^{i\varpi(t_k-t_\ell)}-e^{i\varpi t_k}-e^{-i\varpi t_\ell}+1}{\varpi^2}\,.
\end{aligned} \tag{F.71}$$

First, let us introduce the effective forcing D_κ and the effective viscosity ν_κ as follows

$$\begin{aligned}
N_\kappa(|\vec{x}-\vec{x}'|) &= \int_{\vec{q}} e^{i\vec{q}\cdot(\vec{x}-\vec{x}')}D_\kappa\tilde{n}(q/\kappa) = D_\kappa\kappa^d n(\kappa|\vec{x}-\vec{x}'|) \\
R_\kappa(|\vec{x}-\vec{x}'|) &= \int_{\vec{q}} e^{i\vec{q}\cdot(\vec{x}-\vec{x}')}\nu_\kappa q^2\tilde{r}(q/\kappa) = \nu_\kappa^d(-\partial^2)r(\kappa|\vec{x}-\vec{x}'|)\,.
\end{aligned} \tag{F.72}$$

As explained in Sec. 3.4.3, let us define the dimensionless fields $\hat{u}_\alpha(\hat{\mathbf{x}},)$ and $\hat{\bar{u}}_\alpha(\hat{\mathbf{x}},)$ as

$$u_\alpha(\mathbf{x}) = V_\kappa\hat{u}_\alpha(\hat{\mathbf{x}})\,,\ \text{and}\ \bar{u}_\alpha(\mathbf{x}) = \bar{V}_\kappa\hat{\bar{u}}_\alpha(\hat{\mathbf{x}})\,, \tag{F.73}$$

where $\hat{\mathbf{x}} = (\kappa\vec{x},\omega_\kappa t)$, such that if they are independent of κ, the EAA and the regulator term $\Delta\mathcal{S}_\kappa$ are independent of κ. Inserting the definitions (F.72) and (F.73)

into the expression of the regulator (5.11), one obtains in a first step

$$V_\kappa = \left(\frac{\omega_\kappa D_\kappa \kappa^d}{\nu_\kappa^2 \kappa^4}\right)^{\frac{1}{2}}, \text{ and } \bar{V}_\kappa = \left(\frac{\omega_\kappa \kappa^d}{D_\kappa}\right)^{\frac{1}{2}}. \tag{F.74}$$

In a second step, let us insert these expressions into the non-renormalized two- and three-point vertex functions of (5.48). Asking that the EAA is independent of κ yields

$$\begin{aligned} \omega_\kappa &= \nu_\kappa \kappa^2 \,, \\ \kappa^{4-d} \nu_\kappa^3 D_\kappa^{-1} &= 1 \,. \end{aligned} \tag{F.75}$$

The first line fixes the scaling of time and the second line (which is the non-renormalization of the Navier-Stokes vertex) fixes ν_κ in terms of D_κ. Up to this point, the results did not depend on the dimension of space. The last ingredients necessary to fix the scaling exponents is the stationarity condition. However, this condition is different in 2- and 3-D (Canet, Delamotte, and Wschebor 2016).

We specify to the 3-D case to write the fixed point solutions. In this case the mean dissipation of energy per unit volume and per unit mass ε has a finite value and is equal to the mean energy injected per unit volume and per unit mass (in 3-D, the damping at large scales R should play no role). Let us calculate ε

$$\begin{aligned} \varepsilon &= \langle f_\alpha(\mathbf{x}) v_\alpha(\mathbf{x}) \rangle \\ &= \int_{\vec{x}'} N_\kappa(|\vec{x} - \vec{x}'|) \langle \bar{v}_\alpha(t, \vec{x}') v_\alpha(t_+, \vec{x}) \rangle \\ &= D_\kappa \kappa^3 \int_{\hat{\vec{x}}} n(|\hat{\vec{x}} - \hat{\vec{x}}'|) \langle \hat{\bar{v}}_\alpha(\hat{t}, \hat{\vec{x}}') v_\alpha(\hat{t}_+, \hat{\vec{x}}) \rangle \\ &= D_\kappa \kappa^3 \gamma^{-1} \,, \end{aligned} \tag{F.76}$$

where γ^{-1} is a numerical factor depending on the precise shape of the forcing. To obtain a κ-independent ε, D_κ and ν_κ are fixed to be

$$D_\kappa = \varepsilon \gamma \kappa^{-3}, \text{ and } \nu_\kappa = (\varepsilon\gamma)^{1/3} \kappa^{-4/3}. \tag{F.77}$$

The dimensionless velocity and response velocity follow as

$$u_\alpha(\mathbf{x}) = \kappa^{-1/3} \epsilon^{1/3} \gamma^{1/2} \hat{u}_\alpha(\hat{\mathbf{x}}), \text{ and } \bar{u}_\alpha(\mathbf{x}) = \kappa^{10/3} \epsilon^{-1/3} \gamma^{-1/2} \hat{\bar{u}}_\alpha(\hat{\mathbf{x}}), \tag{F.78}$$

Let us finally define the dimensionless integral $\hat{J}_s$ through

$$J_\kappa(\varpi) = -\int_{\vec{q}} \tilde{\partial}_s \bar{C}(\varpi, \vec{q}) = \gamma \epsilon^{1/3} \kappa^{-4/3} \hat{J}_s(\hat{\varpi}), \tag{F.79}$$

where $s \equiv \ln(\kappa/\Lambda)$ is the "RG time" introduced in Sec. 3.4.3.

F.4.1 Small time delays

If one defines $t_i \equiv \epsilon \tilde{t}_i$ and let ϵ tend to zero, the integrals in the r.h.s. of Eq. (F.71) are equivalent to $\epsilon^2 I_\kappa \tilde{t}_k \tilde{t}_\ell$, where $I_\kappa \equiv \int_\varpi J_\kappa(\varpi)$:

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \int_\varpi J_\kappa(\varpi) \frac{e^{i\varpi(t_k - t_\ell)} - e^{i\varpi t_k} - e^{-i\varpi t_\ell} + 1}{\varpi^2} = I_\kappa \tilde{t}_k \tilde{t}_\ell \,. \tag{F.80}$$

Furthermore, it was shown in Canet, Delamotte, and Wschebor 2016 that, because of the presence of the regulator, $\hat{J}_\kappa(\varpi)$ is dominated by frequencies of order $\kappa^{2/3}$, and that I_κ is finite. In the limit where all the time delays t_i are small, the flow equation (F.71) simplifies to

$$\partial_\kappa G^{(n)}_{\alpha_1\dots\alpha_n}(t_1,\vec{p}_1,\cdots,t_{n-1},\vec{p}_{n-1}) = \frac{I_\kappa}{3}|\vec{p}_k t_k|^2 G^{(n)}_{\alpha_1\dots\alpha_n}(t_1,\vec{p}_1,\cdots,t_{n-1},\vec{p}_{n-1})\,, \quad \text{(F.81)}$$

(using Einstein convention for repeated indices). In order to find a solution, we define a $(n-1)\times(n-1)$ rotation matrix $\mathcal{R}$, such that $\mathcal{R}_{i1} = \frac{t_i}{\sqrt{t_\ell t_\ell}}$, and introduce new variables $\vec{\rho}_k$ such that $\vec{p}_i = \mathcal{R}_{ij}\vec{\rho}_j$. In particular $\vec{\rho}_1 = \frac{t_k\vec{p}_k}{\sqrt{t_\ell t_\ell}}$ and the flow equation becomes

$$\partial_\kappa G^{(n)}_{\alpha_1\dots\alpha_n}(t_1,\vec{\rho}_1,\cdots,t_{n-1},\vec{\rho}_{n-1}) = \frac{I_\kappa}{3} t_k t_k |\vec{\rho}_1|^2 \bar{G}^{(n)}_{\alpha_1\dots\alpha_n}(t_1,\vec{\rho}_1,\cdots,t_{n-1},\vec{\rho}_{n-1})\,. \quad \text{(F.82)}$$

To study the fixed point of this flow equation, we introduce the dimensionless variables $\hat{\vec{\rho}}_i \equiv \vec{\rho}_i/\kappa$, and $\hat{I}_s \equiv \gamma^{-1}\varepsilon^{-2/3}\kappa^{2/3} I_\kappa$. According to Eq. (F.78), the dimensionless $n-$point function can be defined as

$$\hat{G}^{(n)}_{\alpha_1\dots\alpha_n}(\hat{t}_1,\hat{\vec{\rho}}_1,\cdots,\hat{t}_{n-1},\hat{\vec{\rho}}_{n-1}) \equiv \left(\frac{\kappa^{11/3}}{\gamma\varepsilon^{2/3}}\right)^{\frac{m-\bar{m}}{2}} \kappa^{\frac{3}{2}(m+\bar{m}-2)}\, G^{(n)}_{\alpha_1\dots\alpha_n}(t_1,\vec{\rho}_1,\cdots,t_{n-1},\vec{\rho}_{n-1})\,, \quad \text{(F.83)}$$

where m (resp. $\bar{m}$) is the number of velocity (resp. response velocity) fields in the generalized correlation function $\hat{G}^{(n)}$, with $m+\bar{m}=n$. We note $d_G = 3(m-1) + (m-\bar{m})/3$ the scaling dimension of $G^{(n)}$, and we define $\alpha_s = \gamma\hat{I}_s/2$ (which fixed point value $\alpha_* \equiv \alpha_S$ is the coefficient appearing in (5.69)). The flow equation for $\hat{G}^{(n)}$ then reads

$$\left\{\partial_s - d_G - \hat{\vec{\rho}}_i\cdot\partial_{\hat{\vec{\rho}}_i} + \frac{2}{3}\hat{t}_i\partial_{\hat{t}_i} - \frac{2}{3}\alpha_s\,\hat{t}_k\hat{t}_k|\hat{\vec{\rho}}_1|^2\right\}\hat{G}^{(n)}_{\alpha_1\dots\alpha_n}(\hat{t}_1,\hat{\vec{\rho}}_1,\cdots,\hat{t}_{n-1},\hat{\vec{\rho}}_{n-1}) = 0\,. \quad \text{(F.84)}$$

In the following, hat symbols are omitted to alleviate notation. Let us remark that $\vec{\rho}_i\cdot\partial_{\vec{\rho}_i}$ only acts on the moduli of the vectors $\vec{\rho}_i$, so that if one introduces the polar decomposition $\vec{\rho}_i = \rho_i\vec{n}_i$, one has $\vec{\rho}_i\cdot\partial_{\vec{\rho}_i} = \rho_i\partial_{\rho_i}$. Introducing the scaling variables $y_i = {\rho_1}^{2/3} t_i$, the flow equation further simplifies to

$$\left\{\partial_s - d_G - \rho_i\partial_{\rho_i} - \frac{2}{3}\alpha_s\, y_k y_k \rho_1^{2/3}\right\} G^{(n)}_{\alpha_1\dots\alpha_n}(y_1,\rho_1,\vec{n}_1,\cdots,y_{n-1},\rho_{n-1},\vec{n}_{n-1}) = 0\,. \quad \text{(F.85)}$$

Let us consider RG scales s such that the fixed point is reached: α_s has attained its fixed point value α_*, and the explicit dependence in s (through ∂_s) is zero. Denoting $u_1 \equiv \ln\rho_1$, $u_{i>1} \equiv \ln\rho_1 - \ln\rho_i$, the fixed point equation becomes an ordinary differential equation

$$\left\{-d_G - \partial_{u_1} - \frac{2}{3}\alpha_*\, y_k y_k\, e^{\frac{2}{3}u_1}\right\} G^{(n)}_{\alpha_1\dots\alpha_n}(y_1,u_1,\vec{n}_1,\cdots,y_{n-1},u_{n-1},\vec{n}_{n-1}) = 0\,. \quad \text{(F.86)}$$

The differential equation Eq. (F.86) can be integrated, and yields

$$\begin{aligned}&\log G^{(n)}_{\alpha_1\dots\alpha_n}(y_1,u_1,\vec{n}_1,\cdots,y_{n-1},u_{n-1},\vec{n}_{n-1})\\ &= -\alpha_*\, y_k y_k\, e^{\frac{2}{3}u_1} - d_G u_1 + F^{(n)}_{\alpha_1\dots\alpha_n}(y_1,\vec{n}_1,\cdots,y_{n-1},u_{n-1},\vec{n}_{n-1})\,. \end{aligned} \quad \text{(F.87)}$$

In terms of the original dimensionful variables, one obtains

$$\begin{aligned}
&\log\left[\varepsilon^{\frac{\bar{m}-m}{3}} L^{-d_G} G^{(n)}_{\alpha_1\dots\alpha_n}(t_1,\vec{p}_1,\cdots,t_{n-1},\vec{p}_{n-1})\right] \\
&= -\alpha_S \varepsilon^{2/3} L^{2/3}\, t_k t_k\, \rho_1^2 - d_G \log(\rho_1 L) \\
&+ {F_S^{(n)}}_{\alpha_1\dots\alpha_n}\left(\rho_1^{2/3}\varepsilon^{1/3} t_1, \frac{\vec{\rho}_1}{\rho_1}, \cdots, \rho_1^{2/3}\varepsilon^{1/3} t_{n-1}, \frac{\vec{\rho}_{n-1}}{\rho_1}\right) + \mathcal{O}(p_{\max} L)\,,
\end{aligned} \tag{F.88}$$

where $\vec{\rho}_1 = \frac{t_k \vec{p}_k}{\sqrt{t_\ell t_\ell}}$ and the dimensionless constant γ has been absorbed in the function $F_S^{(n)}$. The error on the solution is bounded by a term of order $p_{\max} L$, where $p_{\max}$ is the maximum of the amplitudes of the $\vec{p}_i$ and of their partial sums. Note that again the Kolmogorov solution, which stems from standard scale invariance, is included explicitly although it is of the same order as the neglected error terms. This part is not calculated exactly since it could receive corrections from the neglected sub-leading terms in the flow equation. The leading term in $\bar{G}^{(n)}$ is a Gaussian in the variable $|\vec{p}_k t_k|$, which explicitly breaks scale invariance. This breaking is related to the sweeping effect, and expression (F.88) provides its exact expression, as a generalization of the Gaussian in tp for the two-point function. The constant α_S is positive, see Canet, Delamotte, and Wschebor 2016. The vectors $\vec{\rho}_i$ are not given explicitly, except for $\vec{\rho}_1$, but they can be constructed for any generalized correlation function.

For instance, in the case of $G^{(3)}$, one can use $\mathcal{R} \propto \left(\begin{smallmatrix} t_1 & -t_2 \\ t_2 & t_1 \end{smallmatrix}\right)$ as the rotation matrix from the $\vec{\rho}_i$ to the $\vec{p}_i$, so that

$$\log\left[\varepsilon^{-1} L^{-7} G^{(3)}_{\alpha\beta\gamma}(t_1,\vec{p}_1,t_2,\vec{p}_2)\right] = -\alpha_S \varepsilon^{2/3} L^{2/3}\, \left|\vec{p}_1 t_1 + \vec{p}_2 t_2\right|^2 + \mathcal{O}(p_{\max} L)\,, \tag{F.89}$$

omitting the Kolmogorov scaling terms. A particular and interesting case corresponds to $t_1 = t_2 = t$. In this case, the expression (F.89) simplifies to

$$\log\left[\varepsilon^{-1} L^{-7} G^{(3)}_{\alpha\beta\gamma}(t,\vec{p}_1,t,\vec{p}_2)\right] = -\alpha_S \varepsilon^{2/3} L^{2/3}\, t^2\, \left|\vec{p}_1 + \vec{p}_2\right|^2 + \mathcal{O}(p_{\max} L)\,. \tag{F.90}$$

This simple prediction could be tested in numerical simulations of NS equation.

F.4.2 Large time delays

In this section, we derive the form of the fixed point solution for the flow equation (5.73) in the limit of large time delays. In this limit, one can reiterate the calculation for the two-point function, writing $J_\kappa(\varpi) = (J_\kappa(\varpi) - J_\kappa(0)) + J_\kappa(0)$. The integrals of the type $\int_\varpi e^{i\varpi t} F(\varpi)$ decay exponentially in t since $F(\varpi) = (J_\kappa(\varpi) - J_\kappa(0))/\varpi^2$ is an analytic function of ϖ, while $\int_\varpi F(\varpi)$ is a constant. Thus, the integral in (5.73) is dominated at large times by

$$\begin{aligned}
&J_\kappa(0) \int_\varpi \frac{e^{i\varpi(t_k - t_\ell)} - e^{i\varpi t_k} - e^{-i\varpi t_\ell} + 1}{\varpi^2} \\
&\qquad = 4 J_\kappa(0) \int_{-\infty}^{\infty} \frac{d\varpi}{2\pi} \sin\left(\frac{\varpi t_k}{2}\right) \sin\left(\frac{\varpi t_\ell}{2}\right) \frac{e^{i\frac{\varpi}{2}(t_k - t_\ell)}}{\varpi^2} \\
&\qquad = \frac{J_\kappa(0)}{2}\Big(|t_k| + |t_\ell| - |t_k - t_\ell|\Big)\,.
\end{aligned} \tag{F.91}$$

The flow equation Eq. (5.73) thus reads in the limit of large time delays $t_i \gg \kappa^{2/3}$

$$\partial_\kappa G^{(n)}_{\alpha_1\dots\alpha_n}(\{t_i,\vec{p}_i\}) = \frac{J_\kappa(0)}{6}\sum_{k,\ell}\vec{p}_k\cdot\vec{p}_\ell\Big(|t_k|+|t_\ell|-|t_k-t_\ell|\Big)\bar{G}^{(n)}_{\alpha_1\dots\alpha_n}(\{t_i\,,\vec{p}_i\})\,. \quad \text{(F.92)}$$

To give an important concrete example, let us focus on the special case where all time differences are equal $t_i \equiv t$ for $i = 1,\cdots,n-1$. In this case, the solution can be simply derived. Introducing as previously a $(n-1)\times(n-1)$ matrix $\mathcal{R}'$, such that $\mathcal{R}'_{i1} = 1$, one defines the variables $\vec{\varrho}_k$ by $\vec{p}_i = \mathcal{R}'_{ij}\vec{\varrho}_j$ with $\vec{\varrho}_1 = \sum_k \vec{p}_k$. The flow equation becomes

$$\partial_\kappa G^{(n)}_{\alpha_1\dots\alpha_n}(t,\vec{\varrho}_1,\cdots,\vec{\varrho}_{n-1}) = \frac{J_\kappa(0)}{3}\,|\vec{\varrho}_1|^2\,|t|\,\bar{G}^{(n)}_{\alpha_1\dots\alpha_n}(t,\vec{\varrho}_1,\cdots,\vec{\varrho}_{n-1})\,. \quad \text{(F.93)}$$

To study the fixed-point, one switches to dimensionless variables as in the previous section and defines $y = \varrho_1^{2/3}t$, to obtain the fixed point equation

$$\Big\{-d_G-\varrho_i\partial_{\varrho_i}-\frac{J_*(0)}{3}\,|\vec{\varrho}_1|^{4/3}\,|y|\Big\}G^{(n)}_{\alpha_1\dots\alpha_n}(y,\varrho_1,\vec{n}_1,\cdots,\varrho_{n-1},\vec{n}_{n-1}) = 0\,, \quad \text{(F.94)}$$

where hat symbols have been dropped. This equation can be integrated introducing $u_1 \equiv \ln\varrho_1$ and $u_{i>1} \equiv \ln\varrho_1 - \ln\varrho_i$. One obtains in terms of the original variables and including the sub-leading contributions corresponding to the Kolmogorov part as previously

$$\begin{aligned}&\log\Big(\varepsilon^{\frac{\bar{m}-m}{3}}L^{-d_G}G^{(n)}_{\alpha_1\dots\alpha_n}(t,\vec{p}_1,\cdots,\vec{p}_{n-1})\\ &= -\alpha_L\varepsilon^{1/3}L^{4/3}\,|t|\,\varrho_1^2 - d_G\log(\varrho_1 L) + {F_L}^{(n)}{}_{\alpha_1\dots\alpha_n}\left(\varrho_1^{2/3}\varepsilon^{1/3}t,\frac{\vec{\varrho}_1}{\varrho_1},\cdots,\frac{\vec{\varrho}_{n-1}}{\varrho_1}\right)+\mathcal{O}(p_{\max}L)\,,\end{aligned} \quad \text{(F.95)}$$

with $\varrho_1 = \sum_k \vec{p}_k$, and $\alpha_L = \gamma\hat{J}_*(0)/4$. As in the previous section, the matrix $\mathcal{R}'$ can be explicitly constructed for each n. Note that in the more general case of $t_i \equiv t$ for $i = 1,\cdots,n'$ and $t_i \equiv 0$ for $i = n'+1,\cdots,n-1$, the above procedure would also lead to a solution with $\mathcal{R}'$ a $n'\times n'$ matrix leaving the p_i for $i = n'+1,\cdots,n-1$ invariant.

As an example, let us specialize to the case of $G^{(3)}$. One can then use $\mathcal{R}' = \left(\begin{smallmatrix}1 & -1\\ 1 & 1\end{smallmatrix}\right)$ as the matrix from the $\vec{\rho}_i$ to the $\vec{p}_i$, so that

$$\begin{aligned}&\log\left(\varepsilon^{-1}L^{-7}G^{(3)}_{\alpha\beta\gamma}(t_1,\vec{p}_1,t_2,\vec{p}_2)\right)\\ &= -\alpha_L\varepsilon^{1/3}L^{4/3}\;|t|\,|\vec{p}_1+\vec{p}_2|^2 - 7\log(|\vec{p}_1+\vec{p}_2|\,L)\\ &+ {F_L}^{(3)}{}_{\alpha\beta\gamma}\left(|\vec{p}_1+\vec{p}_2|^{2/3}\,\varepsilon^{1/3}t,\frac{\vec{p}_1+\vec{p}_2}{|\vec{p}_1+\vec{p}_2|},\frac{\vec{p}_2-\vec{p}_1}{|\vec{p}_1+\vec{p}_2|}\right)+\mathcal{O}(p_{\max}L)\,.\end{aligned} \quad \text{(F.96)}$$

Again, the simple prediction

$$\log\left[\varepsilon^{-1}L^{-7}G^{(3)}_{\alpha\beta\gamma}(t,\vec{p}_1,t,\vec{p}_2)\right] = -\alpha_L\varepsilon^{1/3}L^{4/3}\;|t|\,|\vec{p}_1+\vec{p}_2|^2+\mathcal{O}(p_{\max}L)\,, \quad \text{(F.97)}$$

could be tested in direct numerical simulations of NS equation.

F.5 Next-to-leading order term in the stream function formulation

Let us check that we recover the leading order at unequal time in the stream function formulation. The leading order flow equation reads

$$\begin{aligned}
&\partial_\kappa \tilde{G}^{(n)}_{i_1\dots i_n}(\mathbf{p}_1,\dots,\mathbf{p}_n)\Big|_{leading}\\
&= \frac{1}{2}\int_{\mathbf{q}_1,\mathbf{q}_2} \tilde{\delta}_\kappa \tilde{G}^{(2)}_{ij}(-\mathbf{q}_1,-\mathbf{q}_2)\\
&\times \frac{q_a^\alpha q_b^\beta}{2}\frac{\partial^2}{\partial q_a^\alpha \partial q_b^\beta}\left[\frac{\delta^2}{\delta\Phi_i(\mathbf{q}_1)\delta\Phi_j(\mathbf{q}_2)}\tilde{G}^{(n)}_{i_1\dots i_n}[\mathbf{p}_1,\dots,\mathbf{p}_n;\mathrm{j}]\right]_{\Phi=0}\Bigg|_{\vec{q}_1=\vec{q}_2=0},
\end{aligned} \tag{F.98}$$

The derivation is the same as in Appendix F.3.2, except that after having distributed the field derivatives, we have to distribute the $\vec{q}$-derivatives. Because a vertex function with a wave-number set to zero is zero in the stream function formulation, the terms with $a=b$, which correspond to having two $\vec{q}$-derivatives acting on the same leg of the diagram, are zero. Thus we are left with one derivative hitting on each leg.

$$\begin{aligned}
&\partial_\kappa \tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\}_{1\le\ell\le n})\Big|_{leading}\\
&= \frac{1}{2}\int_{\mathbf{q}_1,\mathbf{q}_2} \tilde{\delta}_\kappa \tilde{G}^{(2)}_{ij}(-\mathbf{q}_1,-\mathbf{q}_2)\\
&\times q_1^\alpha q_2^\beta \frac{\partial^2}{\partial q_1^\alpha \partial q_2^\beta}\left[\frac{\delta^2}{\delta\Phi_i(\mathbf{q}_1)\delta\Phi_j(\mathbf{q}_2)}\tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\};\mathrm{j}]\right]_{\Phi=0}\Bigg|_{\vec{q}_1=\vec{q}_2=0},
\end{aligned} \tag{F.99}$$

At this point, one proves as in the velocity formulation that

$$\frac{\partial}{\partial q_1^\alpha}\left[\frac{\delta}{\delta\Phi_i(\mathbf{q}_1)}\tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\};\mathrm{j}]\right]_{\Phi=0}\Bigg|_{\vec{q}_1=0} = \delta_{i\psi} i\epsilon_{\alpha\mu}\tilde{\mathcal{D}}_\mu(\varpi_a)\tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\};\mathrm{j}) \tag{F.100}$$

using

$$\frac{\partial}{\partial q_a^\alpha}\frac{\delta}{\delta\Phi_i(\mathbf{q}_a)}\mathcal{E}_i^{\mathcal{T}}[\{\mathbf{p}_\ell\}_i,\{\mathbf{k}_\ell\}_i]\Big|_{\Phi=0,\vec{q}_a=0} = \delta_{i\psi} i\epsilon_{\alpha\mu}\tilde{\mathcal{D}}_\mu(\varpi_a)\mathcal{E}_i^{\mathcal{T}}(\{\mathbf{p}_\ell\}_i,\{\mathbf{k}_\ell\}_i) \tag{F.101}$$

where $\mathcal{E}_i^{\mathcal{T}}[\{\mathbf{p}_\ell\}_i,\{\mathbf{k}_\ell\}_i]$ is an element (vertex function or propagator) in the tree composing $\tilde{G}^{(n)}$.

Let us check that it works also for two functional derivatives and their subsequent wave-number derivative. To do this, we first set $\vec{q}_2=0$ to zero, as in Appendix F.3.2.

The calculations follows in the same way, except when distributing the $\vec{q}_1$ derivative:

$$
\begin{aligned}
&\frac{\partial^2}{\partial q_1^\alpha \partial q_2^\beta}\frac{\delta^2}{\delta\Phi_i(\mathbf{q}_1)\delta\Phi_j(\mathbf{q}_2)}\tilde{\mathcal{T}}^{(n)}_{i_1\cdots i_n}[\{\mathbf{p}_\ell\}]_{\Phi=0}\Big|_{\vec{q}_1=\vec{q}_2=0}\\
&= -\delta_{j\psi} i\epsilon_{\beta\mu}\frac{\partial}{\partial q_1^\alpha}\Bigg\{\\
&\times\frac{q_1^\nu}{\varpi_2}\Big[\int_{\mathbf{k}_{\rm intern}}\sum_k\Big(\prod_{k'\neq k}\mathcal{E}^{\mathcal{T}}_{k'}(\{\mathbf{p}_\ell\}_{k'},\{\mathbf{k}_\ell\}_{k'})\Big)\frac{\delta}{\delta\Phi_i(\varpi_1+\varpi_2,\vec{q}_1)}\mathcal{E}^{\mathcal{T}}_k[\{\mathbf{p}_\ell\}_k,\{\mathbf{k}_\ell\}_k]\Big|_{\Phi=0}\Big]\\
&+\sum_{k=1}^n\frac{p_k^\nu}{\varpi_2}\Big[\\
&\times\int_{\mathbf{k}_{\rm intern}}\sum_m\Big(\prod_{m'\neq m}\mathcal{E}^{\mathcal{T}}_{m'}(\{\mathbf{p}_\ell\}_{m'}\backslash k_+,\{\mathbf{k}_\ell\}_{m'})\Big)\frac{\delta}{\delta\Phi_i(\mathbf{q}_1)}\mathcal{E}^{\mathcal{T}}_m[\{\mathbf{p}_\ell\}_m\backslash k_+,\{\mathbf{k}_\ell\}_m]\Big|_{\Phi=0}\Big]\Bigg\}\Big|_{\vec{q}_1=0}\\
&= -\delta_{j\psi} i\epsilon_{\beta\nu}\Bigg\{\\
&\times\frac{\delta_{\alpha\nu}}{\varpi_2}\Big[\int_{\mathbf{k}_{\rm intern}}\sum_k\Big(\prod_{k'\neq k}\mathcal{E}^{\mathcal{T}}_{k'}(\{\mathbf{p}_\ell\}_{k'},\{\mathbf{k}_\ell\}_{k'})\Big)\frac{\delta}{\delta\Phi_i(\varpi_1+\varpi_2,\vec{0})}\mathcal{E}^{\mathcal{T}}_k[\{\mathbf{p}_\ell\}_k,\{\mathbf{k}_\ell\}_k]\Big|_{\Phi=0}\Big]_{\vec{q}_1=0}\\
&+\delta_{i\psi} i\epsilon_{\alpha\mu}\sum_{k=1}^n\frac{p_k^\nu}{\varpi_2}\Big[\\
&\times\int_{\mathbf{k}_{\rm intern}}\sum_m\Big(\prod_{m'\neq m}\mathcal{E}^{\mathcal{T}}_{m'}(\{\mathbf{p}_\ell\}_{m'}\backslash k_+,\{\mathbf{k}_\ell\}_{m'})\Big)\tilde{\mathcal{D}}_\mu(\varpi_1)\mathcal{E}^{\mathcal{T}}_m(\{\mathbf{p}_\ell\}_m\backslash k_+,\{\mathbf{k}_\ell\}_m)\Big]\Bigg\}\\
&= -\delta_{i\psi}\delta_{j\psi} i^2\epsilon_{\alpha\mu}\epsilon_{\beta\nu}\sum_{k=1}^n\frac{p_k^\nu}{\varpi_2}\Big[\tilde{\mathcal{D}}_\mu(\varpi_1)\int_{\mathbf{k}_{\rm intern}}\prod_m\mathcal{E}^{\mathcal{T}}_m(\{\mathbf{p}_\ell\}_m\backslash k_+,\{\mathbf{k}_\ell\}_m)\Big]\\
&= \delta_{i\psi}\delta_{j\psi} i^2\epsilon_{\alpha\mu}\epsilon_{\beta\nu}\tilde{\mathcal{D}}_\nu(\varpi_2)\tilde{\mathcal{D}}_\mu(\varpi_1)\tilde{\mathcal{T}}^{(n)}_{i_1\cdots i_n}(\{\mathbf{p}_\ell\})\,.
\end{aligned}
\tag{F.102}
$$

To go from the second to the third equality, we had to use again that a vertex function with a leg evaluated at zero wave-number is zero. Going back to the flow equation (F.98), one recovers the leading order result of the velocity formulation:

$$
\begin{aligned}
\partial_\kappa\tilde{G}^{(n)}_{i_1\ldots i_n}(\{\mathbf{p}_\ell\}_{1\leq\ell\leq n}) &= \frac{1}{2}\int_{\mathbf{q}_1,\mathbf{q}_2}\tilde{\partial}_\kappa\tilde{G}^{(2)}_{\psi\psi}(-\mathbf{q}_1,-\mathbf{q}_2)i^2q_1^\alpha q_2^\beta\epsilon_{\alpha\mu}\epsilon_{\beta\nu}\tilde{\mathcal{D}}_\mu(\omega_1)\tilde{\mathcal{D}}_\nu(\omega_2)\tilde{G}^{(n)}_{i_1\ldots i_n}(\{\mathbf{p}_\ell\})\\
&= \frac{1}{2}\int_{\mathbf{q}_1,\mathbf{q}_2}\tilde{\partial}_\kappa\tilde{G}^{(2)}_{v_\mu v_\nu}(-\mathbf{q}_1,-\mathbf{q}_2)\tilde{\mathcal{D}}_\mu(\omega_1)\tilde{\mathcal{D}}_\nu(\omega_2)\tilde{G}^{(n)}_{i_1\ldots i_n}(\{\mathbf{p}_\ell\})\,,
\end{aligned}
\tag{F.103}
$$

where we have used the definition of the stream function to reconstruct the velocity correlation:

$$
\tilde{G}^{(n)}_{v_{\alpha_1}\ldots v_{\alpha_n}}(\{\mathbf{p}_\ell\}_{1\leq\ell\leq n}) = \prod_{\ell=1}^n i\epsilon_{\alpha_\ell\beta_\ell}p_\ell^{\beta_\ell}\tilde{G}^{(n)}_{\psi\ldots\psi}(\{\mathbf{p}_\ell\}_{1\leq\ell\leq n})\,. \tag{F.104}
$$

F.6 Next-to-leading order terms at equal times

In this appendix, we will present in more detail the calculation leading to the vanishing of certain terms in the flow of the correlation functions at equal times. The

starting point is the next-to-leading order term of the flow equation for the generalized correlation functions, evaluated at equal times

$$\partial_\kappa \int_{\{\omega_\ell\}} \tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\}_{1\le\ell\le n})\Big|_{NLO} = \frac{1}{2}\int_{\mathbf{q}_1,\mathbf{q}_2} \tilde{\partial}_\kappa \tilde{G}^{(2)}_{ij}(-\mathbf{q}_1,-\mathbf{q}_2) \times \int_{\{\omega_\ell\}} \frac{q_a^\mu q_b^\nu q_c^\rho q_d^\sigma}{4!}\frac{\partial^4}{\partial q_a^\mu \partial q_b^\nu \partial q_c^\rho \partial q_d^\sigma}\left[\frac{\delta^2}{\delta\Phi_i(\mathbf{q}_1)\delta\Phi_j(\mathbf{q}_2)}\tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\};\mathrm{j}]\right]_{\Phi=0}\Bigg|_{\vec{q}_1=\vec{q}_2=0}, \tag{F.105}$$

where as before a, b, c, d take value in $\{1,2\}$. As for the leading order at unequal times, Appendix F.5, the four wave-number derivatives can be classified according to the respective number of $\vec{q}_1$ and $\vec{q}_2$ derivatives. Using the same argument, if the four derivatives are $\vec{q}_1$ (resp. four $\vec{q}_2$), this contribution is zero because the vertex function with the wave-number $\vec{q}_2$ (resp. $\vec{q}_1$) goes to zero when the wave-numbers are set to zero, Thus we have to consider the remaining cases: three $\vec{q}_1$ and one $\vec{q}_2$, one $\vec{q}_1$ and three $\vec{q}_2$, and finally two $\vec{q}_1$ and two $\vec{q}_2$. Let us first treat the cases one and two, which are the same up to a permutation of $\mathbf{q}_1$ and $\mathbf{q}_2$, as well as their respective indices.

F.6.1 Vanishing of the contribution 1-3 and 3-1

Although no Ward identity exists for the third wave-number derivative of the vertex function, the Galilean Ward identity can still be used on the leg with one derivative and the proof of Appendix F.5 for one velocity derivative carries through to show that one obtains a operator $\tilde{\mathcal{D}}$ acting on the external legs of the whole diagram.

$$\partial_\kappa \int_{\{\omega_\ell\}} \tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\}_{1\le\ell\le n})\Big|_{NLO,1-3} = \frac{1}{2}\int_{\mathbf{q}_1,\mathbf{q}_2} \tilde{\partial}_\kappa \tilde{G}^{(2)}_{\psi j}(-\mathbf{q}_1,-\mathbf{q}_2) \times \int_{\{\omega_\ell\}} \frac{q_1^\mu q_2^\nu q_2^\rho q_2^\sigma}{3!}\frac{\partial^3}{\partial q_2^\nu \partial q_2^\rho \partial q_2^\sigma}\left\{ i\epsilon_{\mu\alpha}\tilde{\mathcal{D}}_\alpha(\varpi_1)\Big[\frac{\delta}{\delta\Phi_j(\mathbf{q}_2)}\tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\};\mathrm{j}]\Big]_{\Phi=0}\right\}_{\vec{q}_2=0}, \tag{F.106}$$

Now, distributing the $\vec{q}_2$ derivatives, one obtains two types of terms: either all $\vec{q}_2$ derivatives act on the term in square bracket or one of them acts on the operator $\tilde{\mathcal{D}}$.

$$\begin{aligned}
&\int_{\{\omega_\ell\}} \frac{q_1^\mu q_2^\nu q_2^\rho q_2^\sigma}{3!}\frac{\partial^3}{\partial q_2^\nu \partial q_2^\rho \partial q_2^\sigma}\left\{ i\epsilon_{\mu\alpha}\tilde{\mathcal{D}}_\alpha(\varpi_1)\Big[\frac{\delta}{\delta\Phi_j(\mathbf{q}_2)}\tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\};\mathrm{j}]\Big]_{\Phi=0}\right\}_{\vec{q}_2=0}\\
&= \int_{\{\omega_\ell\}} \frac{q_1^\mu q_2^\nu q_2^\rho q_2^\sigma}{3!} i\epsilon_{\mu\alpha}\tilde{\mathcal{D}}_\alpha(\omega_1)\Big|_{\vec{q}_2=0}\frac{\partial^3}{\partial q_2^\nu \partial q_2^\rho \partial q_2^\sigma}\Big[\frac{\delta}{\delta\Phi_j(\mathbf{q}_2)}\tilde{G}^{(n)}_{i_1\dots i_n}[\mathbf{p}_1,\dots,\mathbf{p}_n;\mathrm{j}]\Big]_{\Phi=0}\Big|_{\vec{q}_2=0}\\
&\quad + \int_{\{\omega_\ell\}} \frac{q_1^\mu q_2^\nu q_2^\rho q_2^\sigma}{2} i\epsilon_{\mu\alpha}\frac{\partial}{\partial q_2^\nu}\tilde{\mathcal{D}}_\alpha(\omega_1)\Big|_{\vec{q}_2=0}\frac{\partial^2}{\partial q_2^\rho \partial q_2^\sigma}\Big[\frac{\delta}{\delta\Phi_j(\mathbf{q}_2)}\tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\};\mathrm{j}]\Big]_{\Phi=0}\Big|_{\vec{q}_2=0}\\
&= \int_{\{\omega_\ell\}} \frac{q_1^\mu q_2^\nu q_2^\rho q_2^\sigma}{2} i\epsilon_{\mu\alpha}\delta_{\nu\alpha}\frac{1}{\omega_1}\frac{\partial^2}{\partial q_2^\rho \partial q_2^\sigma}\Big[\frac{\delta}{\delta\Phi_j(\varpi_2+\varpi_1,\vec{q}_2)}\tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\};\mathrm{j}]\Big]_{\Phi=0}\Big|_{\vec{q}_2=0}
\end{aligned} \tag{F.107}$$

In the first term of the first equality, it is noted explicitly that $\tilde{\mathcal{D}}\big|_{\vec{q}_2=0}$ shifts only the frequencies associated to $\vec{p}_1$ and $\vec{p}_2$, thus this term is zero due to the conservation of wave-number of the object in square bracket and the integration in frequency. The $\vec{q}_2$ derivative on $\tilde{\mathcal{D}}$ selects the frequency shift on the $\mathbf{q}_2$ leg, which does not vanish. However, this term is proportional to $\epsilon_{\mu\nu}q_1^\mu q_2^\nu$. Now, inserting back this term into (F.106), the conservation of wave-number of $G^{(2)}_{ij}(-\mathbf{q}_1,-\mathbf{q}_2)$ gives $\vec{q}_1+\vec{q}_2=0$ thus $\epsilon_{\mu\nu}q_1^\mu q_2^\nu = -\epsilon_{\mu\nu}q_1^\mu q_1^j = 0$ and this term gives no contribution either to the flow equation.

$$\partial_\kappa \int_{\{\omega_\ell\}} \tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\}_{1\le\ell\le n})\Big|_{NLO,1-3} = 0 \tag{F.108}$$

At this point, one is left with

$$\begin{aligned}
&\partial_\kappa \int_{\{\omega_\ell\}} \tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\}_{1\le\ell\le n})\Big|_{NLO} = \partial_\kappa \int_{\{\omega_\ell\}} \tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\}_{1\le\ell\le n})\Big|_{NLO,2-2} \\
&= \frac{1}{2}\int_{\mathbf{q}_1,\mathbf{q}_2} \tilde{\partial}_\kappa \tilde{G}^{(2)}_{ij}(-\mathbf{q}_1,-\mathbf{q}_2) \\
&\quad\times \int_{\{\omega_\ell\}} \frac{q_1^\mu q_1^\nu q_2^\rho q_2^\sigma}{4} \frac{\partial^4}{\partial q_1^\mu \partial q_1^\nu \partial q_2^\rho \partial q_2^\sigma} \Big[\frac{\delta^2}{\delta\Phi_i(\mathbf{q}_1)\delta\Phi_j(\mathbf{q}_2)} \tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\};\mathrm{j}]\Big]_{\Phi=0}\Big|_{\vec{q}_1=\vec{q}_2=0} .
\end{aligned} \tag{F.109}$$

The remaining calculations can be greatly simplified by using the invariance under space translation and rotation in the $\vec{q}_1,\vec{q}_2$ integrals.

$$\begin{aligned}
&\int_{\vec{q}_1,\vec{q}_2} \tilde{\partial}_\kappa \tilde{G}^{(2)}_{ij}(-\mathbf{q}_1,-\mathbf{q}_2)\frac{q_1^\mu q_1^\nu q_2^\rho q_2^\sigma}{4} \\
&= \frac{1}{4}\int_{\vec{q}} \tilde{\partial}_\kappa \tilde{G}^{(2)}_{ij}(-\omega_1,-\omega_2,q^2) q^\mu q^\nu q^\rho q^\sigma \\
&= (\delta_{\mu\nu}\delta_{\rho\sigma}+\delta_{\mu\rho}\delta_{\nu\sigma}+\delta_{\mu\sigma}\delta_{\nu\rho})\tilde{K}_{ij}(\omega_1,\omega_2)\,,
\end{aligned} \tag{F.110}$$

with

$$\tilde{K}_{ij}(\omega_1,\omega_2) \equiv \frac{1}{32}\int_{\vec{q}} \tilde{\partial}_\kappa \tilde{G}^{(2)}_{ij}(-\omega_1,-\omega_2,q^2)(q^2)^2\,. \tag{F.111}$$

Inserting this expression back into (F.109) gives

$$\begin{aligned}
&\partial_\kappa \int_{\{\omega_\ell\}} \tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\}_{1\le\ell\le n})\Big|_{NLO} = \partial_\kappa \int_{\{\omega_\ell\}} \tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\}_{1\le\ell\le n})\Big|_{NLO,2-2} \\
&= \frac{1}{2}\int_{\varpi_1,\varpi_2} \tilde{K}_{ij}(\varpi_1,\varpi_2) \\
&\times \int_{\{\omega_\ell\}} \Big(\frac{\partial^4}{\partial q_1^\mu \partial q_1^\mu \partial q_2^\nu \partial q_2^\nu} + 2\frac{\partial^4}{\partial q_1^\mu \partial q_2^\mu \partial q_1^\nu \partial q_2^\nu}\Big) \Big[\frac{\delta^2}{\delta\Phi_i(\mathbf{q}_1)\delta\Phi_j(\mathbf{q}_2)} \tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\};\mathrm{j}]\Big]_{\Phi=0}\Big|_{\vec{q}_1=\vec{q}_2=0} .
\end{aligned} \tag{F.112}$$

Thus two types of derivatives appear, the uncrossed and the crossed ones, according to whether the $\vec{q}_1$ derivative is contracted with the other $\vec{q}_1$ derivative or with the $\vec{q}_2$ derivative. Concentrating first on the contribution given by the uncrossed one, it

is shown in the next sections first on the example of the two-point function, then in the general case that

$$\begin{aligned}
&\partial_\kappa \tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\}_{1\leq\ell\leq n})\Big|_{uncrossed} \\
&= \frac{1}{2}\int_{\varpi_1,\varpi_2} \tilde{K}_{ij}(\varpi_1,\varpi_2)\frac{\partial^4}{\partial q_1^\mu \partial q_1^\mu \partial q_2^\nu \partial q_2^\nu}\Big[\frac{\delta^2}{\delta\Phi_i(\mathbf{q}_1)\delta\Phi_j(\mathbf{q}_2)}\tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\};\mathrm{j}]\Big]_{\Phi=0}\Big|_{\vec{q}_1=\vec{q}_2=0} \\
&= \frac{1}{2}\int_{\varpi_1,\varpi_2} \tilde{K}_{\psi\psi}(\varpi_1,\varpi_2)\tilde{\mathcal{R}}(\varpi_1)\tilde{\mathcal{R}}(\varpi_2)\tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\})\,. \qquad \text{(F.113)}
\end{aligned}$$

where $\tilde{\mathcal{R}}$ is defined in Eq. (5.54) of the main text.

F.6.2 Uncrossed derivatives contribution for the flow of the two-point function

In this section, it is explicitly shown that the uncrossed contribution to the flow of $G^{(2)}_{\psi\psi}(\mathbf{p}_1,\mathbf{p}_2)$ closes. First, distributing the $\vec{q}$-derivatives, it reads

$$\begin{aligned}
&\partial_\kappa \int_{\omega_1,\omega_2} \tilde{G}^{(2)}_{\psi\psi}(\mathbf{p}_1,\mathbf{p}_2)\Big|_{uncrossed} \\
&= \frac{1}{2}\int_{\varpi_1,\varpi_2} \tilde{K}_{ij}(\varpi_1,\varpi_2)\int_{\omega_1,\omega_2}\frac{\partial^4}{\partial q_1^\mu \partial q_1^\mu \partial q_2^\nu \partial q_2^\nu} \\
&\int_{\mathbf{k}_1,\mathbf{k}_2} \tilde{G}^{(2)}_{\psi m}(\mathbf{p}_1,-\mathbf{k}_1)\tilde{G}^{(2)}_{\psi n}(\mathbf{p}_2,-\mathbf{k}_2)\Big[-\tilde{\Gamma}^{(4)}_{ijmn}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{k}_1,\mathbf{k}_2) \\
&+\int_{\mathbf{k}_3,\mathbf{k}_4}\tilde{\Gamma}^{(3)}_{ims}(\mathbf{q}_1,\mathbf{k}_1,\mathbf{k}_3)\tilde{G}^{(2)}_{st}(-\mathbf{k}_3,-\mathbf{k}_4)\tilde{\Gamma}^{(3)}_{jnt}(\mathbf{q}_2,\mathbf{k}_2,\mathbf{k}_4)+(i,\mathbf{q}_1)\leftrightarrow(j,\mathbf{q}_2)\Big]_{\vec{q}_1=\vec{q}_2=0} \\
&= \frac{1}{2}\int_{\varpi_1,\varpi_2} \tilde{K}_{ij}(\varpi_1,\varpi_2)\int_{\omega_1,\omega_2} \\
&\int_{\mathbf{k}_1,\mathbf{k}_2} \tilde{G}^{(2)}_{\psi m}(\mathbf{p}_1,-\mathbf{k}_1)\tilde{G}^{(2)}_{\psi n}(\mathbf{p}_2,-\mathbf{k}_2)\Big[-\frac{\partial^4}{\partial q_1^\mu \partial q_1^\mu \partial q_2^\nu \partial q_2^\nu}\tilde{\Gamma}^{(4)}_{ijmn}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{k}_1,\mathbf{k}_2) \\
&+\int_{\mathbf{k}_3,\mathbf{k}_4}\frac{\partial^2}{\partial q_1^\mu \partial q_1^\mu}\tilde{\Gamma}^{(3)}_{ims}(\mathbf{q}_1,\mathbf{k}_1,\mathbf{k}_3) \\
&\times\tilde{G}^{(2)}_{st}(-\mathbf{k}_3,-\mathbf{k}_4)\frac{\partial^2}{\partial q_2^\nu \partial q_2^\nu}\tilde{\Gamma}^{(3)}_{jnt}(\mathbf{q}_2,\mathbf{k}_2,\mathbf{k}_4)+(i,\mathbf{q}_1)\leftrightarrow(j,\mathbf{q}_2)\Big]_{\vec{q}_1=\vec{q}_2=0} \\
&= \frac{1}{2}\int_{\varpi_1,\varpi_2} \tilde{K}_{\psi\psi}(\varpi_1,\varpi_2)\int_{\omega_1,\omega_2} \\
&\int_{\mathbf{k}_1,\mathbf{k}_2} \tilde{G}^{(2)}_{\psi m}(\mathbf{p}_1,-\mathbf{k}_1)\tilde{G}^{(2)}_{\psi n}(\mathbf{p}_2,-\mathbf{k}_2)\Big[-\tilde{\mathcal{R}}(\varpi_1)\tilde{\mathcal{R}}(\varpi_2)\tilde{\Gamma}^{(2)}_{mn}(\mathbf{k}_1,\mathbf{k}_2) \\
&+\int_{\mathbf{k}_3,\mathbf{k}_4}\tilde{\mathcal{R}}(\varpi_1)\tilde{\Gamma}^{(2)}_{ms}(\mathbf{k}_1,\mathbf{k}_3)\tilde{G}^{(2)}_{st}(-\mathbf{k}_3,-\mathbf{k}_4)\tilde{\mathcal{R}}(\varpi_2)\tilde{\Gamma}^{(2)}_{nt}(\mathbf{k}_2,\mathbf{k}_4)+(\varpi_1)\leftrightarrow(\varpi_2)\Big]\,.
\end{aligned} \qquad \text{(F.114)}$$

In exactly the same lines as for the operator $\tilde{\mathcal{D}}$, it is possible to show that the terms in square bracket rewrite as the operator $\tilde{\mathcal{R}}$ acting on the external leg of the original diagram. First, examining the second term, using the fact that $\tilde{G}^{(2)}$ and

$\tilde{\Gamma}^{(2)}$ are inverse of one another, the second term of (F.114) can be rewritten as two combinations of $\tilde{G}^{(2)}\tilde{\mathcal{R}}(\varpi_a)\tilde{\Gamma}^{(2)}\tilde{G}^{(2)}$ attached to one another by a $\tilde{\Gamma}^{(2)}$. Looking at one of them, we have

$$\begin{aligned}
&\int_{\mathbf{k}_1,\mathbf{k}_3} \tilde{G}^{(2)}_{\psi m}(\mathbf{p}_1,-\mathbf{k}_1)\tilde{\mathcal{R}}(\varpi_1)\tilde{\Gamma}^{(2)}_{ms}(\mathbf{k}_1,\mathbf{k}_3)\tilde{G}^{(2)}_{su}(-\mathbf{k}_3,-\mathbf{k}_5)\\
&\quad= \frac{2i\epsilon_{\alpha\beta}}{\varpi_1}\int_{\mathbf{k}_1,\mathbf{k}_3} \tilde{G}^{(2)}_{\psi m}(\mathbf{p}_1,-\mathbf{k}_1)\Big[k_1^\alpha\frac{\partial}{\partial k_1^\beta}\tilde{\Gamma}^{(2)}_{ms}(\nu_1+\varpi_1,\vec{k}_1,\mathbf{k}_3)\\
&\qquad+k_3^\alpha\frac{\partial}{\partial k_3^\beta}\tilde{\Gamma}^{(2)}_{ms}(\mathbf{k}_1,\nu_3+\varpi_1,\vec{k}_3)\Big]\tilde{G}^{(2)}_{su}(-\mathbf{k}_3,-\mathbf{k}_5)\\
&\quad= \frac{2i\epsilon_{\alpha\beta}}{\varpi_1}\Big[\int_{\mathbf{k}_1}\tilde{G}^{(2)}_{\psi m}(\mathbf{p}_1,-\mathbf{k}_1)k_1^\alpha\frac{\partial}{\partial k_1^\beta}\delta_{mu}\delta(\varpi_1+\nu_1-\nu_5)\delta^2(\vec{k}_1-\vec{k}_5)\\
&\qquad+\int_{\mathbf{k}_3}\tilde{G}^{(2)}_{su}(-\mathbf{k}_3,-\mathbf{k}_5)k_3^\alpha\frac{\partial}{\partial k_3^\beta}\delta_{\psi s}\delta(\omega_1+\varpi+\nu_3)\delta^2(\vec{p}_1+\vec{k}_3)\Big]\\
&\quad= -k_5^\alpha\frac{\partial}{\partial k_5^\beta}\tilde{G}^{(2)}_{\psi u}(\mathbf{p}_1,-\nu_5+\varpi_1,-\vec{k}_5)-p_1^\alpha\frac{\partial}{\partial p_1^\beta}\tilde{G}^{(2)}_{\psi u}(\omega_1+\varpi_1,\vec{p}_1,-\vec{k}_5)\\
&\quad= -\tilde{\mathcal{R}}(\varpi_1)\tilde{G}^{(2)}_{\psi u}(\mathbf{p}_1,-\mathbf{k}_5)\,,
\end{aligned}\tag{F.115}$$

where in the second equality we have contracted $\tilde{G}^{(2)}$ and $\tilde{\Gamma}^{(2)}$ when possible and in the third we have integrated by parts. In fact, we just proved the equivalent of the property (F.48) of $\tilde{\mathcal{D}}$ for $\tilde{\mathcal{R}}$:

$$\frac{\partial^2}{\partial q_\mu\partial q_\mu}\frac{\delta}{\delta\Phi_k(\mathbf{q})}\tilde{G}^{(2)}_{ij}[\mathbf{p}_1,\mathbf{p}_2;\mathrm{j}]_{\Phi=0}\Big|_{\vec{q}=0}=\tilde{\mathcal{R}}(\varpi)\tilde{G}^{(2)}_{ij}(\mathbf{p}_1,\mathbf{p}_2)\tag{F.116}$$

Inserting back this result into last line into (F.114), it reads

$$\begin{aligned}
&\partial_\kappa\int_{\omega_1,\omega_2}\tilde{G}^{(2)}_{\psi\psi}(\mathbf{p}_1,\mathbf{p}_2)\Big|_{uncrossed}.=\frac{1}{2}\int_{\varpi_1,\varpi_2}K_{\psi\psi}(\varpi_1,\varpi_2)\int_{\omega_1,\omega_2}\int_{\mathbf{k}_1,\mathbf{k}_2}\\
&\Big[-G^{(2)}_{\psi u}(\mathbf{p}_1,-\mathbf{k}_1)G^{(2)}_{\psi v}(\mathbf{p}_2,-\mathbf{k}_2)\tilde{\mathcal{R}}(\varpi_1)\tilde{\mathcal{R}}(\varpi_2)\tilde{\Gamma}^{(2)}_{uv}(\mathbf{k}_1,\mathbf{k}_2)\\
&+\tilde{\mathcal{R}}(\varpi_1)G^{(2)}_{\psi u}(\mathbf{p}_1,-\mathbf{k}_1)\Gamma^{(2)}_{uv}(\mathbf{k}_1,\mathbf{k}_2)\tilde{\mathcal{R}}(\varpi_2)G^{(2)}_{\psi v}(\mathbf{p}_2,-\mathbf{k}_2)+(\varpi_1)\leftrightarrow(\varpi_2)\Big].
\end{aligned}\tag{F.117}$$

In fact, this structure is the same as the one appearing with the operator $\tilde{\mathcal{D}}$ for the leading order at unequal times. It will generalize as well for any correlation functions. To continue the derivation let us examine separately the two terms in

square brackets. On the one hand for the first term,

$$
\begin{aligned}
&\int_{\mathbf{k}_1,\mathbf{k}_2} G^{(2)}_{\psi u}(\mathbf{p}_1,-\mathbf{k}_1)G^{(2)}_{\psi v}(\mathbf{p}_2,-\mathbf{k}_2)\tilde{\mathcal{R}}(\varpi_1)\tilde{\mathcal{R}}(\varpi_2)\tilde{\Gamma}^{(2)}_{uv}(\mathbf{k}_1,\mathbf{k}_2)\\
&= \frac{2i\epsilon_{\alpha\beta}}{\varpi_2}\int_{\mathbf{k}_1,\mathbf{k}_2} G^{(2)}_{\psi u}(\mathbf{p}_1,-\mathbf{k}_1)G^{(2)}_{\psi v}(\mathbf{p}_2,-\mathbf{k}_2)\\
&\quad\times\tilde{\mathcal{R}}(\varpi_1)\Big[k_1^\alpha\frac{\partial}{\partial k_1^\beta}\tilde{\Gamma}^{(2)}_{uv}(\nu_1+\varpi_2,\vec{k}_1,\mathbf{k}_2)+k_2^\alpha\frac{\partial}{\partial k_2^\beta}\tilde{\Gamma}^{(2)}_{uv}(\mathbf{k}_1,\nu_2+\varpi_2,\vec{k}_2)\Big]\\
&= -\frac{4\epsilon_{\alpha\beta}\epsilon_{\rho\sigma}}{\varpi_1\varpi_2}\int_{\mathbf{k}_1,\mathbf{k}_2} G^{(2)}_{\psi u}(\mathbf{p}_1,-\mathbf{k}_1)G^{(2)}_{\psi v}(\mathbf{p}_2,-\mathbf{k}_2)\\
&\quad\times\Bigg\{k_1^\rho\frac{\partial}{\partial k_1^\sigma}\Big[k_1^\alpha\frac{\partial}{\partial k_1^\beta}\tilde{\Gamma}^{(2)}_{uv}(\nu_1+\varpi_1+\varpi_2,\vec{k}_1,\mathbf{k}_2)+k_2^\alpha\frac{\partial}{\partial k_2^\beta}\tilde{\Gamma}^{(2)}_{uv}(\nu_1+\varpi_1,\vec{k}_1,\nu_2+\varpi_2,\vec{k}_2)\Big]\\
&+k_2^\rho\frac{\partial}{\partial k_2^\sigma}\Big[k_1^\alpha\frac{\partial}{\partial k_1^\beta}\tilde{\Gamma}^{(2)}_{uv}(\nu_1+\varpi_2,\vec{k}_1,\nu_2+\varpi_1,\vec{k}_2)+k_2^\alpha\frac{\partial}{\partial k_2^\beta}\tilde{\Gamma}^{(2)}_{uv}(\mathbf{k}_1,\nu_2+\varpi_1+\varpi_2,\vec{k}_2)\Big]\Bigg\}\\
&= -\frac{4\epsilon_{\alpha\beta}\epsilon_{\rho\sigma}}{\varpi_1\varpi_2}\Bigg\{\int_{\vec{k}_1} G^{(2)}_{\psi\psi}(\mathbf{p}_1,\omega_2+\varpi_1+\varpi_2,-\vec{k}_1)k_1^\rho\frac{\partial}{\partial k_1^\sigma}\Big[k_1^\alpha\frac{\partial}{\partial k_1^\beta}\delta^2(\vec{p}_2+\vec{k}_1)\Big]\\
&\quad+\int_{\vec{k}_2} G^{(2)}_{\psi\psi}(\omega_1+\varpi_1+\varpi_2,-\vec{k}_2,\mathbf{p}_2)k_2^\rho\frac{\partial}{\partial k_2^\sigma}\Big[k_2^\alpha\frac{\partial}{\partial k_2^\beta}\delta^2(\vec{p}_1+\vec{k}_2)\Big]\\
&+\int_{\mathbf{k}_1,\mathbf{k}_2} G^{(2)}_{\psi u}(\mathbf{p}_1,-\mathbf{k}_1)G^{(2)}_{\psi v}(\mathbf{p}_2,-\mathbf{k}_2)\\
&\times k_1^\alpha k_2^\rho\frac{\partial^2}{\partial k_1^\beta\partial k_2^\sigma}\tilde{\Gamma}^{(2)}_{uv}(\nu_1+\varpi_2,\vec{k}_1,\nu_2+\varpi_1,\vec{k}_2)+(\varpi_1)\leftrightarrow(\varpi_2)\Bigg\}\\
&= -\frac{4\epsilon_{\alpha\beta}\epsilon_{\rho\sigma}}{\varpi_1\varpi_2}\Bigg\{p_2^\alpha\frac{\partial}{\partial p_2^\beta}\Big[p_2^\rho\frac{\partial}{\partial p_2^\sigma}G^{(2)}_{\psi\psi}(\mathbf{p}_1,\omega_2+\varpi_1+\varpi_2,\vec{p}_2)\Big]\\
&+p_1^\alpha\frac{\partial}{\partial p_1^\beta}\Big[p_1^\rho\frac{\partial}{\partial p_1^\sigma}G^{(2)}_{\psi\psi}(\omega_1+\varpi_1+\varpi_2,\vec{p}_1,\mathbf{p}_2)\Big]\\
&+\int_{\mathbf{k}_1,\mathbf{k}_2} G^{(2)}_{\psi u}(\mathbf{p}_1,-\mathbf{k}_1)G^{(2)}_{\psi v}(\mathbf{p}_2,-\mathbf{k}_2)\\
&\times k_1^\alpha k_2^\rho\frac{\partial^2}{\partial k_1^\beta\partial k_2^\sigma}\tilde{\Gamma}^{(2)}_{uv}(\nu_1+\varpi_2,\vec{k}_1,\nu_2+\varpi_1,\vec{k}_2)+(\varpi_1)\leftrightarrow(\varpi_2)\Bigg\},
\end{aligned}
\tag{F.118}
$$

using integration by parts twice for the first two terms in the last equality. On the other hand for the second term,

$$
\begin{aligned}
&\int_{\mathbf{k}_1,\mathbf{k}_2} \tilde{\mathcal{R}}(\varpi_1) G^{(2)}_{\psi u}(\mathbf{p}_1,-\mathbf{k}_1)\Gamma^{(2)}_{uv}(\mathbf{k}_1,\mathbf{k}_2)\tilde{\mathcal{R}}(\varpi_2) G^{(2)}_{\psi v}(\mathbf{p}_2,-\mathbf{k}_2)\\
&= -\frac{4\epsilon_{\alpha\beta}\epsilon_{\rho\sigma}}{\varpi_1\varpi_2}\int_{\mathbf{k}_1,\mathbf{k}_2}\Big[k_1^\alpha \frac{\partial}{\partial k_1^\beta} G^{(2)}_{\psi u}(\mathbf{p}_1,-\nu_1+\varpi_1,-\vec{k}_1)\\
&+ p_1^\alpha \frac{\partial}{\partial p_1^\beta} G^{(2)}_{\psi u}(\vec{p}_1,\omega_1+\varpi_1,-\mathbf{k}_1)\Big]\tilde{\Gamma}^{(2)}_{uv}(\mathbf{k}_1,\mathbf{k}_2)\\
&\times\Big[k_2^\rho \frac{\partial}{\partial k_2^\sigma} G^{(2)}_{\psi v}(\mathbf{p}_2,-\nu_2+\varpi_2,-\vec{k}_2) + p_2^\rho \frac{\partial}{\partial p_2^\sigma} G^{(2)}_{\psi v}(\omega_2+\varpi_2,\vec{p}_2,-\mathbf{k}_2)\Big]\\
&= -\frac{4\epsilon_{\alpha\beta}\epsilon_{\rho\sigma}}{\varpi_1\varpi_2}\Bigg\{\int_{\mathbf{k}_1,\mathbf{k}_1} k_1^\alpha \frac{\partial}{\partial k_1^\beta} G^{(2)}_{\psi u}(\mathbf{p}_1,-\varpi_1+\varpi_1,-\vec{k}_1)\\
&\times\tilde{\Gamma}^{(2)}_{uv}(\mathbf{k}_1,\mathbf{k}_2) k_2^\rho \frac{\partial}{\partial k_2^\sigma} G^{(2)}_{\psi v}(\mathbf{p}_2,-\varpi_2+\varpi_2,-\vec{k}_2)\\
&+\int_{\vec{k}_1} p_2^\rho \frac{\partial}{\partial p_2^\sigma}\delta^\sigma(\vec{p}_2+\vec{k}_1) k_1^\alpha \frac{\partial}{\partial k_1^\beta} G^{(2)}_{\psi\psi}(\mathbf{p}_1,\omega_2+\varpi_1+\varpi_2,-\vec{k}_1)\\
&+\int_{\vec{k}_2} p_1^\alpha \frac{\partial}{\partial p_1^\beta}\delta^\sigma(\vec{p}_1+\vec{k}_2)\Big[k_2^\rho \frac{\partial}{\partial k_2^\sigma} G^{(2)}_{\psi\psi}(\omega_1+\varpi_1+\varpi_2,-\vec{k}_2,\mathbf{p}_2)\\
&+ p_2^\rho \frac{\partial}{\partial p_2^\sigma} G^{(2)}_{\psi\psi}(\omega_2+\varpi_2,\vec{p}_2,\omega_1+\varpi_1,-\vec{k}_2)\Big]\Bigg\}. \qquad \text{(F.119)}
\end{aligned}
$$

Integrating by parts and shifting the frequencies in the first term in curly bracket and exchanging the $\vec{k}_1$ (resp. $\vec{k}_2$) integral with the $\vec{p}_2$ (resp. $\vec{p}_1$) derivative in the two last terms, one obtains

$$
\begin{aligned}
&\int_{\mathbf{k}_1,\mathbf{k}_2} \tilde{\mathcal{R}}(\varpi_1) G^{(2)}_{\psi u}(\mathbf{p}_1,-\mathbf{k}_1)\Gamma^{(2)}_{uv}(\mathbf{k}_1,\mathbf{k}_2)\tilde{\mathcal{R}}(\varpi_2) G^{(2)}_{\psi v}(\mathbf{p}_2,-\mathbf{k}_2)\\
&= -\frac{4\epsilon_{\alpha\beta}\epsilon_{\rho\sigma}}{\varpi_1\varpi_2}\Bigg\{\int_{\mathbf{k}_1,\mathbf{k}_1} G^{(2)}_{\psi u}(\mathbf{p}_1,-\mathbf{k}_1) k_1^\alpha k_2^\rho \frac{\partial^2}{\partial k_1^\beta \partial k_2^\sigma}\tilde{\Gamma}^{(2)}_{uv}(\varpi_1+\varpi_1,\vec{k}_1,\varpi_2+\varpi_2,\vec{k}_2) G^{(2)}_{\psi v}(\mathbf{p}_2,-\mathbf{k}_2)\\
&+ p_2^\rho \frac{\partial}{\partial p_2^\sigma}\Big[p_2^\alpha \frac{\partial}{\partial p_2^\beta} G^{(2)}_{\psi\psi}(\mathbf{p}_1,\omega_2+\varpi_1+\varpi_2,\vec{p}_2)\Big]\\
&+ p_1^\alpha \frac{\partial}{\partial p_1^\beta}\Big[p_1^\rho \frac{\partial}{\partial p_1^\sigma} G^{(2)}_{\psi\psi}(\omega_1+\varpi_1+\varpi_2,\vec{p}_1,\mathbf{p}_2)\Big] + p_1^\alpha p_2^\rho \frac{\partial^2}{\partial p_1^\beta \partial p_2^\sigma} G^{(2)}_{\psi\psi}(\omega_1+\varpi_2,\vec{p}_1,\omega_2+\varpi_1,\vec{p}_2)\Bigg\}.
\end{aligned}
\qquad \text{(F.120)}
$$

Inserting back (F.118) and (F.120) into (F.117), one obtains finally the expected result

$$\begin{aligned}\partial_\kappa \tilde{G}^{(2)}_{\psi\psi}(\mathbf{p}_1,\mathbf{p}_2)\Big|_{uncrossed} &= \frac{1}{2}\int_{\varpi_1,\varpi_2} K_{\psi\psi}(\varpi_1,\varpi_2)\\ &- \frac{4\epsilon_{\alpha\beta}\epsilon_{\rho\sigma}}{\varpi_1\varpi_2}\Big[p_1^\alpha\frac{\partial}{\partial p_1^\beta}\Big[p_1^\rho\frac{\partial}{\partial p_1^\sigma}G^{(2)}_{\psi\psi}(\omega_1+\varpi_1+\varpi_2,\vec{p}_1,\mathbf{p}_2)\Big]\\ &+ p_2^\rho\frac{\partial}{\partial p_2^\sigma}\Big[p_2^\alpha\frac{\partial}{\partial p_2^\beta}G^{(2)}_{\psi\psi}(\mathbf{p}_1,\omega_2+\varpi_1+\varpi_2,\vec{p}_2)\Big]\\ &+ p_1^\alpha p_2^\rho\frac{\partial^2}{\partial p_1^\beta\partial p_2^\sigma}G^{(2)}_{\psi\psi}(\omega_1+\varpi_2,\vec{p}_1,\omega_2+\varpi_1,\vec{p}_2)+(\varpi_1)\leftrightarrow(\varpi_2)\Big]\\ &= \frac{1}{2}\int_{\varpi_1,\varpi_2} K_{\psi\psi}(\varpi_1,\varpi_2)\tilde{\mathcal{R}}(\varpi_1)\tilde{\mathcal{R}}(\varpi_2)G^{(2)}_{\psi\psi}(\mathbf{p}_1,\mathbf{p}_2)\,. \end{aligned} \tag{F.121}$$

F.6.3 General proof of the closure of the uncrossed derivatives

Let us show that the result of the previous section for $\tilde{G}^{(2)}_{\psi\psi}(\mathbf{p}_1,\mathbf{p}_2)$ is true for any generalized correlation function, *i.e.* that

$$\begin{aligned}&\frac{\partial^4}{\partial q_1^\mu\partial q_1^\mu\partial q_2^\nu\partial q_2^\nu}\Big[\frac{\delta^2}{\delta\Phi_i(\mathbf{q}_1)\delta\Phi_j(\mathbf{q}_2)}\tilde{G}^{(n)}_{i_1\dots i_n}[\{\mathbf{p}_\ell\}_{1\le\ell\le n};\mathrm{j}]\Big]_{\Phi=0}\Big|_{\vec{q}_1=\vec{q}_2=0}\\ &\qquad = \delta_{i\psi}\delta_{j\psi}\tilde{\mathcal{R}}(\varpi_1)\tilde{\mathcal{R}}(\varpi_2)\tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\})\,. \end{aligned} \tag{F.122}$$

Following the same steps as for the leading order at unequal times, let us first examine the action of only one functional derivative and subsequent two wave-number derivatives applied to a tree $\tilde{\mathcal{T}}^{(n)}$ composing $\tilde{G}^{(n)}_{i_1\dots i_n}(\{\mathbf{p}_\ell\})$. Using the property (F.116) demonstrated in the previous section, we obtain readily

$$\begin{aligned}&\frac{\partial^2}{\partial q_a^\mu\partial q_a^\mu}\frac{\delta}{\delta\Phi_i(\mathbf{q}_a)}\tilde{\mathcal{T}}^{(n)}_{i_1\cdots i_n}[\{\mathbf{p}_\ell\}]_{\Phi=0}\Big|_{\vec{q}_a=0}\\ &\qquad = \delta_{i\psi}\int_{\mathbf{k}_{\mathrm{intern}}}\sum_{k=1}^{m}\Bigg(\prod_{\substack{k'=1\\k'\neq k}}^{m}\mathcal{E}^{\mathcal{T}}_{k'}(\{\mathbf{p}_\ell\}_{k'},\{\mathbf{k}_\ell\}_{k'})\Bigg)\tilde{\mathcal{R}}(\varpi_a)\mathcal{E}^{\mathcal{T}}_k(\{\mathbf{p}_\ell\}_k,\{\mathbf{k}_\ell\}_k)\,. \end{aligned} \tag{F.123}$$

Thus, it is enough to show that the operator $\tilde{\mathcal{R}}(\varpi_a)$ enjoys as well the Leibniz property.

$$\begin{aligned}\tilde{\mathcal{R}}(\varpi_a)\int_{\mathbf{k}}\mathcal{E}^{\mathcal{T}}_1(\{\mathbf{k}_\ell\}_1,-\mathbf{k})\mathcal{E}^{\mathcal{T}}_2(\mathbf{k},\{\mathbf{k}_\ell\}_2) &= \int_{\mathbf{k}}\Big[\tilde{\mathcal{R}}(\varpi_a)\mathcal{E}^{\mathcal{T}}_1(\{\mathbf{k}_\ell\}_1,-\mathbf{k})\mathcal{E}^{\mathcal{T}}_2(\mathbf{k},\{\mathbf{k}_\ell\}_2)\\ &\quad+\mathcal{E}^{\mathcal{T}}_1(\{\mathbf{k}_\ell\}_1,-\mathbf{k})\tilde{\mathcal{R}}(\varpi_a)\mathcal{E}^{\mathcal{T}}_2(\mathbf{k},\{\mathbf{k}_\ell\}_2)\Big]\,. \end{aligned} \tag{F.124}$$

Starting from the r.h.s.,

$$\begin{aligned}
&\int_{\mathbf{k}}\Big[\tilde{\mathcal{R}}(\varpi_a)\mathcal{E}_i^{\mathcal{T}}(\{\mathbf{k}_\ell\}_{\ell\in I},-\mathbf{k})\mathcal{E}_j^{\mathcal{T}}(\mathbf{k},\{\mathbf{k}_\ell\}_{\ell\in J})+\mathcal{E}_i^{\mathcal{T}}(\{\mathbf{k}_\ell\}_{\ell\in I},-\mathbf{k})\tilde{\mathcal{R}}(\varpi_a)\mathcal{E}_j^{\mathcal{T}}(\mathbf{k},\{\mathbf{k}_\ell\}_{\ell\in J})\Big]\\
&\quad=\frac{2i\epsilon_{ab}}{\varpi_a}\int_{\mathbf{k}}\Big[\sum_{i\in I}k_i^a\frac{\partial}{\partial k_i^b}\mathcal{E}_i^{\mathcal{T}}(\nu_i+\varpi_a,\vec{k}_i,\{\mathbf{k}_\ell\}_{\ell\in I\setminus i},-\mathbf{k})\mathcal{E}_j^{\mathcal{T}}(\mathbf{k},\{\mathbf{k}_\ell\}_{\ell\in J})\\
&\quad+\sum_{j\in J}k_j^a\frac{\partial}{\partial k_j^b}\mathcal{E}_i^{\mathcal{T}}(\{\mathbf{k}_\ell\}_{\ell\in I},-\mathbf{k})\mathcal{E}_j^{\mathcal{T}}(\mathbf{k},\nu_j+\varpi_a,\vec{k}_j,\{\mathbf{k}_\ell\}_{\ell\in J\setminus j})\\
&\quad+k^a\frac{\partial}{\partial k^b}\mathcal{E}_i^{\mathcal{T}}(\{\mathbf{k}_\ell\}_{\ell\in I},-\nu+\varpi_a,-\vec{k})\mathcal{E}_j^{\mathcal{T}}(\mathbf{k},\{\mathbf{k}_\ell\}_{\ell\in J})\\
&\quad+\mathcal{E}_i^{\mathcal{T}}(\{\mathbf{k}_\ell\}_{\ell\in I},-\mathbf{k})k^a\frac{\partial}{\partial k^b}\mathcal{E}_j^{\mathcal{T}}(\nu+\varpi_a,\vec{k},\{\mathbf{k}_\ell\}_{\ell\in J})\Big]\,.
\end{aligned}\tag{F.125}$$

Integrating by parts in $\vec{k}$ and shifting the associated frequency, it is readily shown that the two last terms cancel each other, proving (F.124) and

$$\frac{\partial^2}{\partial q_a^\mu\partial q_a^\mu}\frac{\delta}{\delta\Phi_i(\mathbf{q}_a)}\tilde{\mathcal{T}}^{(n)}_{i_1\cdots i_n}[\{\mathbf{p}_\ell\}]_{\Phi=0}\Big|_{\vec{q}_a=0}=\tilde{\mathcal{R}}(\omega_a)\tilde{\mathcal{T}}^{(n)}_{i_1\cdots i_n}(\{\mathbf{p}_\ell\})\,.\tag{F.126}$$

To prove (F.122), one still needs to check that the same property follows for two functional derivatives and their subsequent wave-number derivatives. As for the leading order at unequal time, first distributing the two $\vec{q}_2$ derivatives and setting $\vec{q}_2$ to zero, the property (F.124) applies to show that the resulting $\tilde{\mathcal{R}}(\varpi_2)$ can be factorized from the remaining diagram

$$\begin{aligned}
&\frac{\partial^4}{\partial q_1^\mu\partial q_1^\mu\partial q_2^\nu\partial q_2^\nu}\frac{\delta^2}{\delta\Phi_i(\mathbf{q}_1)\delta\Phi_j(\mathbf{q}_2)}\tilde{\mathcal{T}}^{(n)}_{i_1\cdots i_n}[\{\mathbf{p}_\ell\}]_{\Phi=0}\Big|_{\vec{q}_1=\vec{q}_2=0}\\
&=\delta_{j\psi}\frac{\partial^2}{\partial q_1^\mu\partial q_1^\mu}\Bigg\{\\
&\tilde{\mathcal{R}}(\varpi_2)\Big[\int_{\mathbf{k}_{\rm intern}}\sum_k\Big(\prod_{k'\neq k}\mathcal{E}^{\mathcal{T}}_{k'}(\{\mathbf{p}_\ell\}_{k'},\{\mathbf{k}_\ell\}_{k'})\Big)\frac{\delta}{\delta\Phi_i(\mathbf{q}_1)}\mathcal{E}^{\mathcal{T}}_k[\{\mathbf{p}_\ell\}_k,\{\mathbf{k}_\ell\}_k]\Big|_{\Phi=0}\Big]\Bigg\}\Big|_{\vec{q}_1=0}
\end{aligned}\tag{F.127}$$

Making explicit $\tilde{\mathcal{R}}(\varpi_2)$ and distributing the $\vec{q}_1$ derivative gives

$$
\begin{aligned}
&\frac{\partial^4}{\partial q_1^\mu \partial q_1^\mu \partial q_2^\nu \partial q_2^\nu} \frac{\delta^2}{\delta\Phi_i(\mathbf{q}_1)\delta\Phi_j(\mathbf{q}_2)} \tilde{\mathcal{T}}^{(n)}_{i_1\cdots i_n}[\{\mathbf{p}_\ell\}]_{\Phi=0}\Big|_{\vec{q}_1=\vec{q}_2=0} \\
&= \delta_{j\psi} \frac{2i\epsilon_{\rho\sigma}}{\varpi_2} \frac{\partial^2}{\partial q_1^\mu \partial q_1^\mu} \Bigg\{ \\
&q_1^\rho \frac{\partial}{\partial q_1^\sigma} \Big[\int_{\mathbf{k}_{\text{intern}}} \sum_k \Big(\prod_{k'\neq k} \mathcal{E}^{\mathcal{T}}_{k'}(\{\mathbf{p}_\ell\}_{k'}, \{\mathbf{k}_\ell\}_{k'}) \Big) \frac{\delta}{\delta\Phi_i(\varpi_1+\varpi_2, \vec{q}_1)} \mathcal{E}^{\mathcal{T}}_k[\{\mathbf{p}_\ell\}_k, \{\mathbf{k}_\ell\}_k]\Big|_{\Phi=0} \Big] \\
&+ \sum_{k=1}^n p_k^\rho \frac{\partial}{\partial p_k^\sigma} \Big[\\
&\int_{\mathbf{k}_{\text{intern}}} \sum_m \Big(\prod_{m'\neq m} \mathcal{E}^{\mathcal{T}}_{m'}(\{\mathbf{p}_\ell\}_{m'}\backslash k_+, \{\mathbf{k}_\ell\}_{m'}) \Big) \frac{\delta}{\delta\Phi_i(\mathbf{q}_1)} \mathcal{E}^{\mathcal{T}}_m[\{\mathbf{p}_\ell\}_m\backslash k_+, \{\mathbf{k}_\ell\}_m]\Big|_{\Phi=0} \Big] \Bigg\} \Big|_{\vec{q}_1=0} \\
&= \delta_{j\psi} \frac{2i\epsilon_{\rho\sigma}}{\varpi_2} \Bigg\{ 2\delta_{\mu\rho} \frac{\partial^2}{\partial q_1^\mu \partial q_1^\sigma} \Big[\\
&\int_{\mathbf{k}_{\text{intern}}} \sum_k \Big(\prod_{k'\neq k} \mathcal{E}^{\mathcal{T}}_{k'}(\{\mathbf{p}_\ell\}_{k'}, \{\mathbf{k}_\ell\}_{k'}) \Big) \frac{\delta}{\delta\Phi_i(\varpi_1+\varpi_2, \vec{q}_1)} \mathcal{E}^{\mathcal{T}}_k[\{\mathbf{p}_\ell\}_k, \{\mathbf{k}_\ell\}_k]\Big|_{\Phi=0} \Big]_{\vec{q}_1=0} \\
&+ \delta_{i\psi} \sum_{k=1}^n p_k^\rho \frac{\partial}{\partial p_k^\sigma} \Big[\\
&\int_{\mathbf{k}_{\text{intern}}} \sum_m \Big(\prod_{m'\neq m} \mathcal{E}^{\mathcal{T}}_{m'}(\{\mathbf{p}_\ell\}_{m'}\backslash k_+, \{\mathbf{k}_\ell\}_{m'}) \Big) \tilde{\mathcal{R}}(\varpi_1) \mathcal{E}^{\mathcal{T}}_m(\{\mathbf{p}_\ell\}_m\backslash k_+, \{\mathbf{k}_\ell\}_m) \Big] \Bigg\} \\
&= \delta_{i\psi}\delta_{j\psi} \frac{2i\epsilon_{\rho\sigma}}{\varpi_2} \sum_{k=1}^n p_k^\rho \frac{\partial}{\partial p_k^\sigma} \Big[\tilde{\mathcal{R}}(\varpi_1) \int_{\mathbf{k}_{\text{intern}}} \prod_m \mathcal{E}^{\mathcal{T}}_m(\{\mathbf{p}_\ell\}_m\backslash k_+, \{\mathbf{k}_\ell\}_m) \Big] \\
&= \delta_{i\psi}\delta_{j\psi} \tilde{\mathcal{R}}(\varpi_2)\tilde{\mathcal{R}}(\varpi_1) \tilde{\mathcal{T}}^{(n)}_{i_1\cdots i_n}(\{\mathbf{p}_\ell\}) \,, \qquad \text{(F.128)}
\end{aligned}
$$

where the first term in the second equality vanishes by antisymmetry of $\epsilon_{\rho\sigma}$, thus finishing to prove the property (F.122).

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