



HAL
open science

Study of plactic monoids by rewriting methods

Nohra Hage

► **To cite this version:**

Nohra Hage. Study of plactic monoids by rewriting methods. Combinatorics [math.CO]. Université de Lyon, 2016. English. NNT : 2016LYSES065 . tel-02071461

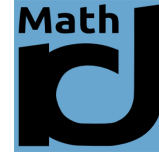
HAL Id: tel-02071461

<https://theses.hal.science/tel-02071461>

Submitted on 18 Mar 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



THESE DE DOCTORAT DE L'UNIVERSITE DE LYON

opérée au sein de

Université Jean Monnet, Saint-Étienne
Institut Camille Jordan, UMR CNRS 5208
École Doctorale Sciences, Ingénierie, Santé : ED SIS 488

Spécialité : **Mathématiques**

Soutenue publiquement le 8 décembre 2016 par

Nohra HAGE

Étude des monoïdes plaxiques par des méthodes de réécriture

Study of plactic monoids by rewriting methods

Devant le Jury composé de:

Nicole BARDY-PANSE	Université de Lorraine	Examinatrice
Stéphane GAUSSENT	Université Jean Monnet, Saint-Étienne	Directeur de thèse
Cédric LECOUBEY	Université François Rabelais, Tours	Rapporteur
Philippe MALBOS	Université Claude Bernard Lyon I	Co-directeur de thèse
Samuel MIMRAM	École Polytechnique	Rapporteur
Nicolas THIÉRY	Université Paris Sud	Examineur



À ma famille, je dédie cette thèse.

Remerciements

A l'issue de ce travail, je profite de ces quelques lignes pour remercier du fond du cœur toutes les personnes qui ont contribué de près ou de loin à sa réussite.

Avant tout, je tiens à remercier chaleureusement Stéphane Gaussent, mon directeur de thèse, qui m'a encadré, encouragé, et avec qui j'ai établi une relation de confiance durant ces trois années de doctorat. Depuis mon premier jour à Saint-Étienne, il était toujours présent pour m'écouter, me conseiller et me diriger dans tous les détails de ma recherche malgré son emploi de temps chargé. Je lui exprime ma plus grande attitude pour ses qualités professionnelles et humaines. Je suis énormément fier d'avoir pu être son premier doctorant.

Je remercie également Philippe Malbos, qui a encadré cette thèse depuis Lyon. Je lui suis très reconnaissant de m'avoir guidé dans mes premiers pas dans le monde de la réécriture. Je voudrais aussi le remercier pour le temps qu'il m'a consacré durant ces années et pour le soutien qu'il m'a accordé tout au long de ce travail. Je n'oublierai jamais les longues heures de travail passées dans son bureau au campus de la Doua et ses conseils avisés.

Je suis très reconnaissant envers Cédric Lecouvey et Samuel Mimram, d'avoir accepté de rapporter ma thèse. Je les remercie d'avoir pris le temps de lire ce manuscrit, pour leurs remarques constructives et pour leurs jugements pertinents. Je remercie également Nicole Bardy-Panse et Nicolas Thiéry d'avoir accepté d'examiner cette thèse. De plus, je remercie sincèrement tous les membres du jury d'avoir fait le voyage pour participer à ma soutenance. C'est un honneur pour moi de présenter mes travaux de thèse devant eux.

Mes sincères remerciements vont aussi aux membres du projet CATHRE pour les différentes conversations fructueuses lors des multiples conférences et des réunions organisées durant ces trois années, et auxquelles j'ai pu présenter mes travaux de recherche.

J'exprime toute ma gratitude aux personnels de l'ICJ, de l'Université Jean Monnet de Saint-Étienne et de l'Université Claude Bernard Lyon 1 de m'avoir accueilli parmi eux et de m'avoir fourni un cadre académique idéal. Je remercie aussi les membres du laboratoire Hubert Curien

de l'Université Jean Monnet pour leur bonne humeur communicative et pour leur contribution à la réalisation d'une partie de mes enseignements en Informatique.

Une pensée aux doctorants qui sont passés par Saint-Étienne et Lyon et qui sont devenus docteurs, en particulier, Imène Boussetouan, Irina Malakhova-Ziablova, Bérénice Delcroix-Oger, Zahraa Salloum, Hanane Kaboul et Hanène Debbiche. Également aux doctorantx Clément Alleaume, Auguste Hébert, Christopher Salinas, Fatima Ziane, Antoine Caradot et Benoit Dejoncheere.

Je remercie encore fortement mes amis, qui directement ou indirectement ont su me soutenir durant mon séjour en France, particulièrement: Sabine Mteirik, Carine EL Khoury, Sarah Mteirik, Amira El Merhie, Elia Tohme, Nahla Salemech Bchara, Basma Mteirik, Mohammad Mteirik, Diane Bou Zogheib, Rachelle El Samrout, Paméla El Hajj, Ziad Hage, Georges Sadaka, Gloria El Hajj, Maria Choufani, Ziad Abboud, Rodrigue Farah, Ghady Abou Rached, Michelle Najjar, Feris Hedded, Zeina Saayfane, Hasan Maatouk, Amal Faddoul Guyot, Antonis Choleridis, Reda Farran, Reda Kassir et Hussein Hammoud.

Finalement, je voudrais exprimer mes profonds remerciements à ma mère, mon père, mes soeurs Nancy et Melissa, mon frère Charbel et toute ma famille au Liban. Malgré leur absence physique, ils étaient toujours présents pour m'encourager, me soutenir et m'aider à surmonter les difficultés. Leurs prières interminables me transmettaient la force dans les moments les plus difficiles.

Une pensée du fond de mon cœur à Saint Charbel, qui par le pouvoir de Dieu était capable de me combler par son amour et son soutien. J'ai toujours senti sa présence à côté de moi et c'est un plus qu'un merci que je veux lui adresser.

Nohra Hage

Résumé

Cette thèse est consacrée à l'étude des monoïdes plaxiques par une nouvelle approche utilisant des méthodes issues de la réécriture. Ces méthodes sont appliquées à des présentations de monoïdes plaxiques décrites en termes de tableaux de Young, de bases cristallines de Kashiwara et de modèle des chemins de Littelmann.

On étudie le problème des syzygies pour la présentation de Knuth des monoïdes plaxiques. En utilisant la procédure de complétion homotopique basée sur les procédures de complétion de Squier et de Knuth–Bendix, on construit des présentations cohérentes de monoïdes plaxiques de type A. Une telle présentation cohérente étend la notion de présentation convergente d'un monoïde par une famille génératrice de syzygies, décrivant toutes les relations entre les relations. On explicite une présentation cohérente finie des monoïdes plaxiques de type A avec les générateurs colonnes. Cependant, cette présentation n'est pas minimale dans le sens que plusieurs de ses générateurs sont superflus. En appliquant la procédure de réduction homotopique, on réduit cette présentation en une présentation cohérente finie qui étend la présentation de Knuth, donnant ainsi toutes les syzygies des relations de Knuth.

D'une manière plus générale, on étudie des présentations de monoïdes plaxiques généralisés du point de vue de la réécriture. On construit des présentations convergentes finies de ces monoïdes en utilisant les chemins de Littelmann. De plus, on étudie ces présentations pour le type C en termes de bases cristallines de Kashiwara. En introduisant les générateurs colonnes admissibles, on construit une présentation convergente finie du monoïde plaxique de type C avec des relations explicites. Cette approche nous permettrait d'étudier le problème des syzygies des présentations de monoïdes plaxiques en tout type.

Mots clés : monoïdes plaxiques, réécriture, problème du mot, problème des syzygies, présentations convergentes, présentations cohérentes, complétion de Squier, complétion de Knuth–Bendix, tableaux de Young, algorithmes d'insertion de Schensted, algorithmes d'insertion de Lecouvey, bases cristallines, modèle des chemins de Littelmann.



Abstract

This thesis focuses on the study of plactic monoids by a new approach using methods issued from rewriting theory. These methods are applied on presentations of plactic monoids given in terms of Young tableaux, Kashiwara's crystal bases and Littelmann path model.

We study the syzygy problem for the Knuth presentation of the plactic monoids. Using the homotopical completion procedure that extends Squier's and Knuth–Bendix's completions procedure, we construct coherent presentations of plactic monoids of type A. Such a coherent presentation extends the notion of a presentation of a monoid by a family of generating syzygies, taking into account all the relations among the relations. We make explicit a finite coherent presentation of plactic monoids of type A with the column generators. However, this presentation is not minimal in the sense that many of its generators are superfluous. After applying the homotopical reduction procedure on this presentation, we reduce it to a finite coherent one that extends the Knuth presentation, giving then all the syzygies of the Knuth relations.

More generally, we deal with presentations of plactic monoids of any type from the rewriting theory perspective. We construct finite convergent presentations for these monoids in a general way using Littelmann paths. Moreover, we study the latter presentations in terms of Kashiwara's crystal graphs for type C. By introducing the admissible column generators, we obtain a finite convergent presentation of the plactic monoid of type C with explicit relations. This approach should allow us to study the syzygy problem for the presentations of plactic monoids for any type.

Keywords : plactic monoids, rewriting theory, word problem, syzygies problem, convergent presentations, coherent presentations, Squier's completion, Knuth–Bendix's completion, Young tableaux, Schensted's insertion algorithm, Lecouvey's insertion algorithm, crystal bases, Littelmann path model.



Table des Matières

Introduction	9
Le monoïde plaxique et ses généralisations	11
Présentations convergentes des monoïdes plaxiques	16
Présentation des résultats	23
Travaux en cours et perspectives	31
Organisation du document	35
General introduction	37
The plactic monoid and its generalisations	39
Convergent presentations of plactic monoids	44
Main results	50
Column presentations of plactic monoids	50
Coherent presentations of plactic monoids of type A	54
Works in progress and perspectives	58
1 Plactic monoids: Young tableaux, crystals and paths	63
1.1 Preliminaries on representation theory	64
1.2 Plactic monoid of type A	68
1.2.1 Schensted's insertion algorithms	68
1.2.2 Longest non-decreasing subsequence of a word	73
1.2.3 The plactic monoid	74
1.3 Crystal graphs and plactic monoids	75
1.3.1 Crystal graphs	75
1.3.2 Crystal graphs for Type A	77
1.3.3 Crystal graphs for type C	79
1.3.4 Crystal plactic monoids	81
1.4 Littelmann paths and plactic monoids	84

TABLE DES MATIÈRES

1.4.1	Lakshmibai-Seshadri's paths	85
1.4.2	Plactic algebra for any semisimple Lie algebra	89
2	Presentations of plactic monoids	93
2.1	Presentations of monoids by 2-polygraphs	94
2.1.1	2-polygraphs	94
2.1.2	Rewriting properties of 2-polygraphs	96
2.1.3	The crystal presentation of the plactic monoid of type A	98
2.2	Tietze transformations and the pre-column presentation	101
2.2.1	Tietze transformations of 2-polygraphs	102
2.2.2	Pre-column presentation of the plactic monoid of type A	103
2.3	The Knuth–Bendix's completion of the Knuth presentation	107
2.3.1	The Knuth–Bendix's completion	107
2.3.2	The computation of $\mathcal{KB}(\text{Knuth}_2(3))$	109
2.3.3	The completion for higher ranks	111
3	Column presentations of plactic monoids	115
3.1	Finite convergent presentation of plactic monoids	116
3.2	Column presentation of the plactic monoid of type A	120
3.2.1	The columns generators	120
3.2.2	The column presentation for type A	121
3.3	Column presentation of the symplectic plactic monoid	124
3.3.1	Symplectic tableaux	125
3.3.2	A bumping algorithm for type C	129
3.3.3	The two two-columns lemmas	133
3.3.4	The column presentation for type C	136
3.4	Column presentation for plactic monoids of types B, D and G_2	142
4	Knuth's coherent presentations of plactic monoids of type A	147
4.1	Coherent column presentations of plactic monoids	148
4.1.1	Coherent presentations of monoids	148
4.1.2	Homotopical completion procedure	152
4.1.3	Coherent column presentation	153
4.2	Knuth's coherent presentations of plactic monoids	162
4.2.1	Homotopical reduction procedure	162
4.2.2	A reduced column presentation	164
4.2.3	Pre-column coherent presentation	166
4.2.4	Knuth's coherent presentation	170
4.2.5	Coherent presentations in small ranks	176

Introduction ¹

Cette thèse porte sur l'étude des monoïdes plaxiques par de nouvelles méthodes issues de la réécriture. Notre approche consiste à étudier les présentations de monoïdes plaxiques du point de vue de la réécriture et, en particulier, à calculer une famille de générateurs de toutes les syzygies de ces présentations. Pour cela, nous orientons les relations de monoïdes plaxiques et nous appliquons une procédure de complétion et de réduction homotopique donnée dans [31] en utilisant des méthodes introduites par Squier, Knuth et Bendix. De cette façon, notre travail constitue une interaction entre l'algèbre et la réécriture.

Les monoïdes plaxiques admettent plusieurs descriptions. En particulier, ils sont reliés aux représentations d'algèbres de Lie semi-simples de dimension finie. On étudie leurs interprétations en termes de tableaux de Young, de bases cristallines de Kashiwara et de chemins de Littelmann. La structure de monoïde plaxique a été introduite par Lascoux et Schützenberger [74] suite aux travaux de Schensted [104] et Knuth [63] sur les propriétés combinatoires des tableaux de Young. En type A, le monoïde plaxique est relié aux représentations de l'algèbre de Lie des matrices carrées de taille n . Il est appelé le *monoïde plaxique de type A* [21, 72, 77]. De plus, la classification des algèbres de Lie semi-simples de dimension finie en plusieurs types entraîne l'existence des monoïdes plaxiques généralisés de mêmes types. Il y a deux approches pour définir ces monoïdes. En effet, ils peuvent être définis cas par cas, en utilisant la théorie des bases cristallines de Kashiwara [83] ou d'une manière générale, grâce à la théorie des chemins de Littelmann [88].

Le *problème des syzygies* consiste à trouver toutes les relations algébriques irréductibles indépendantes entre les générateurs d'un module sur un anneau, [27]. Le problème des syzygies a été aussi étudié du point de vue de la réécriture en trouvant toutes les relations entre les relations de présentations de monoïdes. Une *2-syzygie* pour une présentation d'un monoïde est une relation entre les relations. Le problème des syzygies pour les présentations de monoïdes peut être algorithmiquement résolu grâce aux *systèmes de réécriture convergents*. Les *systèmes de réécriture* sont des présentations orientées de monoïdes formées d'un ensemble de générateurs et

¹Cette introduction est une traduction française du chapitre introductif du manuscrit.

de *règles de réécriture* reliant des mots sur cet ensemble. Un système de réécriture est *terminant* s'il n'admet pas une suite infinie d'étapes de réécriture. Il est *confluent* si pour tous mots u , u' et u'' tel que u se réécrit en u' et u'' , il existe un mot v tel que u' et u'' se réécrivent en v . Un système de réécriture est *convergent* s'il est terminant et confluent. Cela signifie que dans un système de réécriture convergent, toutes règles de réécriture sur un même mot conduisent à un unique mot *irréductible*, c'est-à-dire qui ne peut pas se réécrire en un autre mot.

Récemment, les monoïdes plaxiques ont été étudiés par des méthodes de réécriture, [65, 7, 17, 43, 16, 44]. Dans ce travail, on utilise la structure de *polygraphes* basée sur une interprétation catégorique des systèmes de réécriture, où les générateurs et les règles de réécriture sont représentés par des 1-cellules et des 2-cellules construites sur une unique 0-cellule. On donne une description des 2-szygies de certaines présentations du monoïde plaxique de type A en utilisant la notion de présentation cohérente. Une *présentation cohérente* étend la notion de présentation d'un monoïde par des 3-cellules, décrivant toutes les relations entre les relations. Le calcul de présentations cohérentes d'un monoïde à partir d'une présentation convergente fournit une méthode pour calculer une *résolution polygraphique* du monoïde, c'est-à-dire un remplacement cofibrant du monoïde dans une $(\infty, 1)$ -catégorie libre dont les cellules de dimension supérieure ou égale à 2 sont inversibles, [37]. Une présentation cohérente d'un monoïde constitue les deux premières étapes dans le calcul d'une résolution polygraphique. De plus, une telle étude des relations entre les relations dans un monoïde permet de calculer ses *invariants homologiques*. Les présentations cohérentes sont aussi utiles pour décrire la notion des *actions du monoïde sur les catégories*, [31].

On calcule une présentation cohérente du monoïde plaxique de type A en utilisant la *procédure de complétion et de réduction homotopique* introduite dans [31, 40]. Une telle procédure étend la complétion de Knuth–Bendix [64] en utilisant une méthode introduite par Squier qui calcule une présentation cohérente à partir d'une présentation convergente du monoïde, [112, 31]. La présentation cohérente obtenue pour le monoïde plaxique de type A n'est pas minimale dans le sens que plusieurs de ses générateurs sont superflus. Après plusieurs étapes de *réduction homotopique*, on la réduit en une présentation cohérente minimale ayant les générateurs de Knuth. De cette façon, on donne une méthode algorithmique pour calculer les 2-szygies de la présentation de Knuth du monoïde plaxique de type A.

Afin d'étendre ce résultat aux monoïdes plaxiques généralisés, on construit des présentations convergentes de ces monoïdes. Grâce aux chemins de Littelmann, on construit des présentations convergentes finies de monoïdes plaxiques de tous types. De plus, on étudie ces présentations en termes de graphes cristallins de Kashiwara pour le type C, en utilisant des outils combinatoires tels que les *colonnes admissibles* introduites par Kashiwara et Nakashima, [61].

LE MONOÏDE PLAXIQUE ET SES GÉNÉRALISATIONS

Dans cette section, on présente un panorama historique sur les monoïdes plaxiques. On rappelle la définition du monoïde plaxique de type A et le lien entre ce monoïde, les graphes cristallins de Kashiwara et le modèle des chemins de Littelmann. Une généralisation du monoïde plaxique est alors décrite en termes de cristaux et de chemins de Littelmann.

Le monoïde plaxique de type A et les tableaux. Afin de calculer la longueur de la plus longue sous-suite décroissante d'un mot dans le monoïde libre $[n]^*$ sur l'ensemble ordonné

$$[n] := \{ 1 < \dots < n \},$$

Schensted a donné des algorithmes d'insertion sur les tableaux de Young. Ces tableaux ont été introduits par Young, [124], et ils ont été utilisés pour la première fois par Frobenius afin d'étudier les représentations du groupe symétrique [28]. Les algorithmes de Schensted ont été aussi décrits par Robinson, [103], qui a essayé de donner une preuve de la règle de Littelwood–Richardson. Cette règle décrit d'une manière combinatoire la multiplicité d'un polynôme de Schur dans un produit de polynômes de Schur, c'est-à-dire, la multiplicité d'une représentation irréductible de l'algèbre de Lie générale dans un produit tensoriel de deux représentations irréductibles, voir [123] et [114, Chapter 7]. Avant de présenter les algorithmes de Schensted, nous allons introduire les notions de tableaux et de tableaux standard. Un *tableau* de forme $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{N}^k$, avec $n \geq \lambda_1 \geq \dots \geq \lambda_k \geq 1$, est une collection de cases justifiées à gauche telle que la $i^{\text{ème}}$ ligne contient λ_i cases, pour $i = 1, \dots, k$, et ces cases sont remplies par des éléments de $[n]$ où les entrées sont croissantes dans les lignes de gauche à droite et sont strictement décroissantes dans les colonnes de bas en haut. Un *tableau standard* de forme λ , est un tableau de forme λ dont les entrées sont des entiers de 1 à $\sum_{i=1}^k \lambda_i$, distinctes deux-à-deux. Par exemple, un tableau de forme $(5, 3, 1)$ et un tableau standard de forme $(4, 3, 2)$ sont respectivement

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 & & \\ \hline 3 & & & & \\ \hline \end{array}
 \quad \text{et} \quad
 \begin{array}{|c|c|c|c|} \hline 1 & 3 & 6 & 7 \\ \hline 2 & 5 & 8 & \\ \hline 4 & 9 & & \\ \hline \end{array}
 .$$

Soit T un tableau et x un élément de $[n]$. L'algorithme de Schensted, appelé *insertion par colonne*, insère l'élément x dans le tableau T de la façon suivante. Soit y le plus petit élément de la première colonne de T qui est plus grand ou égal à x . Alors x remplace y dans cette colonne et y est inséré dans la colonne suivante où le processus est répété. Cette procédure termine lorsqu'un élément inséré dans une colonne est supérieur à tous ses éléments. Alors il est placé au bas de cette colonne. Notons qu'il existe un algorithme similaire, appelé *insertion par ligne*, qui est aussi introduit par Schensted. Il consiste à insérer les éléments de $[n]$ dans les lignes d'un tableau au lieu de ses colonnes, voir chapitre 1, section 1.2.1.

Pour tout mot w dans le monoïde libre $[n]^*$, on peut calculer un unique tableau $P(w)$, appelé *tableau de Schensted*, en commençant par le mot vide et en appliquant l'algorithme de Schensted d'une façon itérative sur les éléments de w . Durant le calcul du tableau $P(w)$, un tableau standard $Q(w)$, [104, Lemma 2], est obtenu en mettant successivement un i dans une case à la même place de la case ajoutée en insérant la lettre x_i de w . La bijection $w \mapsto (P(w), Q(w))$, [104, Lemma 3], est appelée la *correspondance de Robinson–Schensted*. Par exemple, les étapes successives pour calculer le tableau $P(1213214)$ et le tableau standard $Q(1213214)$ sont

$$\begin{array}{ccccccc}
 \boxed{1} & \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array} & \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & \\ \hline 3 & & \\ \hline \end{array} & \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & \\ \hline 3 & & \\ \hline 4 & & \\ \hline \end{array} = P(1213214). \\
 \\
 \boxed{1} & \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \\ \hline \end{array} & \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & \\ \hline 4 & & \\ \hline \end{array} & \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & \\ \hline 4 & & \\ \hline 7 & & \\ \hline \end{array} = Q(1213214).
 \end{array}$$

Par ailleurs, on peut définir une relation d'équivalence \sim_{plax} reliant les mots du monoïde $[n]^*$ qui donnent le même tableau de Schensted. En d'autres termes,

$$u \sim_{\text{plax}} v \quad \text{si et seulement si} \quad P(u) = P(v),$$

pour tous mots u et v dans $[n]^*$. De plus, Knuth a montré que la relation d'équivalence \sim_{plax} coïncide avec la congruence engendrée par la famille de *relations de Knuth*, [63, Theorem 6]:

$$\{ yzx = yxz, \text{ pour } x \leq y < z \} \cup \{ xzy = zxy, \text{ pour } x < y \leq z \}. \quad (1)$$

Le *monoïde plaxique de rang n* , noté \mathbf{P}_n , est défini comme le quotient du monoïde libre $[n]^*$ par l'équivalence \sim_{plax} , [74].

Depuis son introduction, plusieurs applications ont montré l'intérêt du monoïde plaxique dans divers domaines de la combinatoire et de la théorie des représentations [72, 106, 29]. En particulier, le monoïde plaxique a été utilisé par Lascoux et Schützenberger afin de donner une description combinatoire des polynômes de Kostka-Foulkes, [73, 74], et une version non-commutative des polynômes de Schubert, [75, 76]. Par ailleurs, le monoïde plaxique a été utilisé par Schützenberger, [105], afin de fournir la première preuve correcte de la règle de Littelwood–Richardson. Notons qu'une autre preuve a été aussi donnée par Thomas, [119].

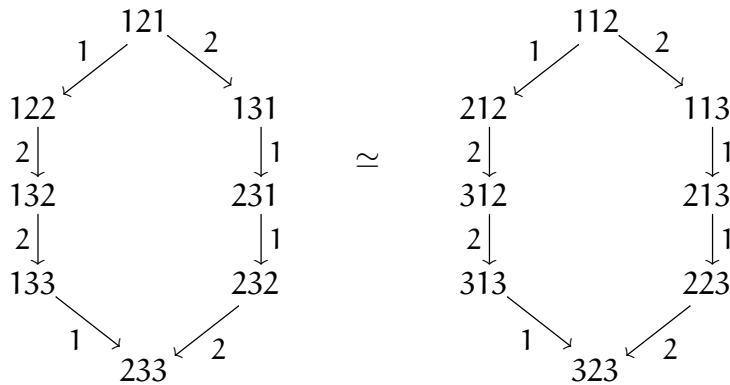
Récemment, plusieurs monoïdes similaires ont été introduits comme le monoïde chinois, [19], le monoïde plaxique décalé, [107], le monoïde hypoplaxique, [99], le monoïde sylvestre, [46], et le monoïde superplaxique, [66, 89]. Ces monoïdes ont aussi plusieurs applications dans divers domaines de la combinatoire et de la théorie des représentations.

Généralisations du monoïde plaxique via les graphes cristallins. Depuis 1990, il y a eu un lien entre la correspondance de Robinson–Schensted et la théorie des bases cristallines de Kashiwara. Avant de présenter ce lien, nous allons introduire le contexte de la théorie des représentations. Une représentation d’une algèbre de Lie est une façon de décrire cette algèbre comme une algèbre de matrices où le crochet de Lie est donné par le commutateur. Depuis leur introduction, les représentations des algèbres de Lie ont eu plusieurs applications dans différents domaines des mathématiques et de la physique. En particulier, certaines représentations des algèbres de Lie semi-simples englobent toutes les particules fondamentales en physique qui sont décrites par le paradigme de la théorie de jauge. De plus, en utilisant les diagrammes de Dynkin, les algèbres de Lie semi-simples de dimension finie sont classifiées en types A, B, C et D et en types exceptionnels E_6 , E_7 , E_8 , F_4 et G_2 . Les types A, B, C et D correspondent respectivement à l’algèbre de Lie spéciale linéaire $\mathfrak{sl}_{n+1}(\mathbb{C})$ (voir chapitre 1, sous-section 1.3.2), l’algèbre de Lie spéciale orthogonale de dimension impaire $\mathfrak{so}_{2n+1}(\mathbb{C})$, l’algèbre de Lie symplectique $\mathfrak{sp}_{2n}(\mathbb{C})$ (voir chapitre 1, sous-section 1.3.3), et l’algèbre de Lie spéciale orthogonale de dimension paire $\mathfrak{so}_{2n}(\mathbb{C})$, voir [49, 30, 10, 47].

Afin d’étudier des solutions des équations de Yang–Baxter classiques qui sont reliées à la mécanique statistique, Jimbo [53] et Drinfeld [25] ont introduit indépendamment en 1985 la notion des groupes quantiques. Un groupe quantique $U_q(\mathfrak{g})$ associé à une algèbre de Lie semi-simple \mathfrak{g} est une déformation de l’algèbre enveloppante universelle de \mathfrak{g} . C’est une algèbre associative sur le corps $\mathbb{C}(q)$ des fonctions rationnelles à coefficients complexes. Quand q tend vers zéro, Kashiwara a montré que les représentations de $U_q(\mathfrak{g})$ possèdent des bases particulières, qu’il a appelé les *bases cristallines*, [57, 58, 59]. Ces bases peuvent être étendues pour tout le q -espace pour obtenir des vraies bases pour les représentations, appelées les *bases globales*. Notons que les bases globales coïncident avec les bases canoniques introduites indépendamment pour les groupes quantiques par Lusztig [92]. Une base cristalline est aussi munie d’une structure de graphe orienté et étiqueté, appelé le *graphe cristallin*, tel que ses flèches sont définies par les *opérateurs de Kashiwara*, voir chapitre 1, section 1.3. Un graphe cristallin peut être aussi décomposé en composantes connexes. Un *isomorphisme de cristaux* entre deux composantes connexes est un isomorphisme de graphes préservant les poids et les étiquettes. Depuis lors, la théorie des bases cristallines est devenue un outil combinatoire très utile pour l’étude des représentations des groupes quantiques et des algèbres de Lie semi-simples.

En utilisant la théorie des groupes quantiques, le premier lien entre la correspondance de Robinson–Schensted et les représentations de l’algèbre de Lie générale $\mathfrak{gl}_n(\mathbb{C})$ des matrices de taille n a été obtenue par Date, Jimbo et Miwa, [21]. Dans le même esprit, le même lien a été obtenu par Lascoux, Leclerc et Thibon en utilisant la théorie des bases cristallines de Kashiwara, [72, 77]. Plus précisément, considérons la *représentation naturelle* $V_n = \mathbb{C}^n$ de l’algèbre de Lie $\mathfrak{gl}_n(\mathbb{C})$. La base canonique de V_n est indexée par l’ensemble ordonné $[n]$. Chaque sommet $x_1 \otimes \dots \otimes x_l$ du graphe cristallin de la représentation $\bigoplus_l V_n^{\otimes l}$ est considéré comme un mot $x_1 \dots x_l$ dans le monoïde libre $[n]^*$. Pour tous mots u et v dans $[n]^*$, on a

$P(u) = P(v)$ si, et seulement si, u et v ont la même place dans leurs composantes connexes isomorphes du graphe cristallin de $\bigoplus_{\downarrow} V_n^{\otimes l}$. De plus, on a $Q(u) = Q(v)$ si, et seulement si, u et v apparaissent dans une même composante connexe du graphe cristallin de $\bigoplus_{\downarrow} V_n^{\otimes l}$. Par conséquent, on obtient que pour tous mots u et v dans le monoïde libre $[n]^*$, $u \sim_{\text{plax}} v$ si, et seulement si, les mots u et v apparaissent à la même place dans leurs composantes connexes isomorphes du graphe cristallin de $\bigoplus_{\downarrow} V_n^{\otimes l}$. Par exemple, les côtés gauches et droites des relations de Knuth (6) correspondent aux sommets des cristaux de l'isomorphisme de cristaux suivant



Comme la théorie des bases cristallines de Kashiwara existe aussi pour les algèbres de Lie semi-simples classiques et l'algèbre de Lie semi-simple exceptionnelle de type G_2 , un monoïde plaxique a été introduit pour chacune de ces algèbres en utilisant une analyse en fonction de chaque cas, [78, 4, 79, 83]. Pour chaque algèbre de Lie semi-simple, on associe un alphabet fini S qui indexe une base de la représentation naturelle V de l'algèbre. On définit une relation d'équivalence \sim_{crys} sur le monoïde libre S^* par: $u \sim_{\text{crys}} v$ si, et seulement si, u et v ont la même place dans leurs composantes connexes isomorphes du graphe cristallin de la représentation $\bigoplus_{\downarrow} V^{\otimes l}$. Pour toute algèbre de Lie semi-simple, le *monoïde plaxique* correspondant est le quotient du monoïde libre S^* par la congruence \sim_{crys} .

Les monoïde plaxiques de types C, B et D correspondent respectivement aux représentations de l'algèbre de Lie symplectique, l'algèbre de Lie spéciale orthogonale de dimension impaire et l'algèbre de Lie spéciale orthogonale de dimension paire. Lecouvey dans [78] et Baker dans [4] ont introduit indépendamment le monoïde plaxique de type C en utilisant la théorie des bases cristallines de Kashiwara et la notion de *colonne admissible* introduite par Kashiwara et Nakashima, [61]. Dans [79], Lecouvey a construit des présentations de monoïdes plaxiques de types B et D en utilisant la notion de colonne admissible généralisée pour ces cas. Lecouvey a aussi introduit le monoïde plaxique de type G_2 en généralisant la notion de colonne admissible pour ce type, [83]. Comme application des présentations de monoïdes plaxiques, Lecouvey a donné une description combinatoire de certains polynômes de Kostka-Foulkes pour les types B, C et D qui proviennent des entrées de la table des caractères des groupes réductifs finis, [82].

Les monoïdes plaxiques via les chemins de Littelmann. Le modèle des chemins de Littelmann établit un pont entre la théorie des monômes standard de Lakshmibai–Sechadri [70, 71] et la théorie des bases cristallines de Kashiwara [57, 58, 59]. Littelmann a construit un modèle combinatoire unifié pour toutes les algèbres de Kac-Moody symétrisables, ce qui lui a permis d’obtenir des formules combinatoires explicites pour les *multiplicités des poids*, la *règle du produit tensoriel* et les *règles des branchements* [85, 87]. Il a introduit des *chemins* linéaires par morceaux reliant l’origine à un poids et une paire d’*opérateurs de racines* pour toute racine simple d’une algèbre de Kac-Moody symétrisable. Les chemins de Littelmann peuvent être encodés dans un graphe orienté et étiqueté tel que ses étiquettes sont données par les opérateurs de racines. De plus, le graphe défini par les opérateurs de racines de Littelmann est isomorphe au graphe cristallin de Kashiwara [55, 60].

Tel que discuté auparavant, les monoïdes plaxiques peuvent être définis cas par cas pour toutes les algèbres de Lie semi-simples classiques et pour l’algèbre de Lie semi-simple exceptionnelle de type G_2 . En utilisant son modèle des chemins, Littelmann a défini d’une manière générale une algèbre plaxique pour toute algèbre de Lie semi-simple, [88]. Par conséquent, il a obtenu des présentations par générateurs et relations de l’algèbre plaxique de type A, B, C, D et G_2 , [88, Theorem C]. Il a aussi introduit la notion de *tableau standard* qui est une généralisation pour tous les types de la notion de tableau définie pour le type A. Les tableaux standard coïncident avec la notion de tableau symplectique pour le type C et de tableau orthogonal pour les types B et D dans le sens de Lecouvey, [83].

Soit \mathfrak{g} une algèbre de Lie semi-simple et P son *réseau de poids*. Un *chemin* est une application $\pi : [0, 1] \rightarrow P \otimes_{\mathbb{Z}} \mathbb{R}$ continue et linéaire par morceaux. On note par

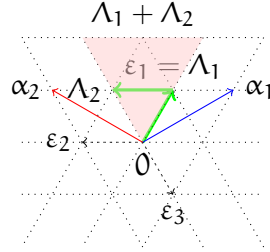
$$\Pi = \left\{ \pi : [0, 1] \rightarrow P \otimes_{\mathbb{Z}} \mathbb{R} \mid \pi(0) = 0 \text{ et } \pi(1) \in P \right\}$$

l’ensemble de tous les chemins de source 0 et de buts appartenant à P . Considérons deux chemins π_1 et π_2 dans Π , la concaténation $\pi_1 \star \pi_2$ est définie par:

$$\pi_1 \star \pi_2(t) := \begin{cases} \pi_1(2t) & \text{pour } 0 \leq t \leq \frac{1}{2}, \\ \pi_1(1) + \pi_2(2t - 1) & \text{pour } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Par exemple, considérons $W = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0 \}$ et soit $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ la base canonique de \mathbb{R}^3 . On note aussi par ε_i la projection de ε_i dans W . On a $P = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$ et $\varepsilon_3 = -\varepsilon_1 - \varepsilon_2$. Les poids $\alpha_1 = \varepsilon_1 - \varepsilon_2$ et $\alpha_2 = \varepsilon_2 - \varepsilon_3$ sont appelés les *racines simples* de l’algèbre de Lie linéaire spéciale \mathfrak{sl}_3 . Les poids $\Lambda_1 = \varepsilon_1$ et $\Lambda_2 = \varepsilon_1 + \varepsilon_2$ sont appelés ses *poids fondamentaux*. Un exemple de *poids dominant* de \mathfrak{sl}_3 est $\Lambda_1 + \Lambda_2$. Considérons les chemins $\pi_1 : t \mapsto t\varepsilon_1$ et $\pi_2 : t \mapsto t\varepsilon_2$, le chemin $\pi_1 \star \pi_2$ est le chemin vert et la *chambre*

dominante est la partie hachurée dans la figure suivante:



Pour toute racine simple α d'une algèbre de Lie \mathfrak{g} , Littelmann a défini les *opérateurs de racines* e_α et f_α de Π à $\Pi \cup \{0\}$ comme suit. Chaque chemin π dans Π est coupé en trois parties $\pi_1 \star \pi_2 \star \pi_3$. Alors le nouveau chemin $e_\alpha(\pi)$ ou $f_\alpha(\pi)$ est égal à 0 ou $\pi_1 \star s_\alpha(\pi_2) \star \pi_3$, où s_α est la réflexion orthogonale par rapport à α , voir chapitre 1, section 1.4.1.

On note par $\mathbb{Z}\Pi$ l'*algèbre des chemins* définie comme le \mathbb{Z} -module libre de base Π tel que le produit est donné par la concaténation des chemins et l'unité est le chemin de source et but 0. Soit \mathcal{A} la sous-algèbre de $\text{End}_{\mathbb{Z}}(\mathbb{Z}\Pi)$ engendrée par les opérateurs de racines f_α et e_α . On note par Π^+ l'ensemble des chemins π tels que leurs images sont contenues dans la chambre dominante et par M_π le \mathcal{A} -module $\mathcal{A}\pi$. Notons que deux \mathcal{A} -modules sont isomorphes si leurs chemins contenus dans la chambre dominante ont le même but.

Soit $\mathbb{Z}\Pi_0$ le \mathcal{A} -sous-module $\mathcal{A}\Pi^+$ de $\mathbb{Z}\Pi$ engendré par les chemins de Π^+ . Pour deux chemins π_1 et π_2 dans $\mathbb{Z}\Pi_0$, on note par π_1^+ et π_2^+ les uniques chemins dans Π^+ tels que π_1 est dans $M_{\pi_1^+}$ et π_2 est dans $M_{\pi_2^+}$. On définit une relation \sim_{path} sur $\mathbb{Z}\Pi_0$ par : $\pi_1 \sim_{\text{path}} \pi_2$ si et seulement si $\pi_1^+(1) = \pi_2^+(1)$ et $\psi(\pi_1) = \pi_2$ sous l'isomorphisme $\psi : M_{\pi_1^+} \rightarrow M_{\pi_2^+}$. Pour toute algèbre de Lie semi-simple \mathfrak{g} , le *monoïde plaxique* correspondant est le quotient du \mathcal{A} -sous-module $\mathbb{Z}\Pi_0$ par la relation d'équivalence \sim_{path} .

Il convient de mentionner que les équivalences \sim_{path} et \sim_{crys} coïncident dans le sens qu'on obtient les mêmes monoïdes plaxiques de types A, B, C, D et G_2 si on les étudie au cas par cas, voir [83] et [88, Theorem C].

PRÉSENTATIONS CONVERGENTES DES MONOÏDES PLAXIQUES

Les systèmes de réécriture et le problème du mot. Les *systèmes de réécriture* sont des présentations orientées de monoïdes formées d'un ensemble de générateurs et de *règles de réécriture* reliant des mots sur cet ensemble. La notion de systèmes de réécriture a été introduite par Thue dans son étude du problème du mot pour les monoïdes finiment présentés, [120]. Le *problème du mot pour un monoïde \mathbf{M}* consiste à trouver un ensemble de générateurs Σ_1 et une procédure décidant si deux éléments dans le monoïde libre Σ_1^* représentent le même élément dans \mathbf{M} . Le problème du mot a été aussi décrit par Dehn pour les groupes finiment présentés, [22]. Un peu plus tard, Post [100] et Markov [94, 95] ont montré indépendamment que le problème du mot est

indécidable. Ensuite, le problème du mot a été considéré dans plusieurs contextes de l'algèbre et de l'informatique théorique. La réécriture apparaît aussi sous différentes formes dans l'algèbre selon l'objet présenté. Elle apparaît pour les algèbres commutatives et les algèbres de Lie avec la notion des bases de Gröbner-Shirshov, [12, 13, 110], pour les algèbres associatives et les opérades, [8, 6, 97, 122, 24] ainsi que pour les termes dans une théorie algébrique, [3, 62, 118], et pour les chaînes de caractères dans un monoïde, [9].

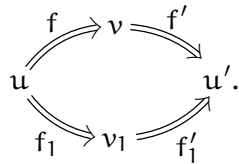
Dans ce travail, une *présentation* (Σ_1, Σ_2) d'un monoïde \mathbf{M} est un système de réécriture tel que le monoïde \mathbf{M} est isomorphe au quotient du monoïde libre Σ_1^* par la relation de congruence engendrée par Σ_2 . Dans la littérature, une présentation d'un monoïde est appelée un *système de réécriture des mots* ou un *système semi-Thue*, [3].

Par exemple, le monoïde plaxique \mathbf{P}_n de rang n est présenté par le système de réécriture $\text{Knuth}_2(n)$, dont l'ensemble de générateurs est $\{1 < \dots < n\}$ et l'ensemble de règles de réécriture est

$$\{ zxy \xrightarrow{\eta_{x,y,z}} xzy \mid 1 \leq x \leq y < z \leq n \} \cup \{ yzx \xrightarrow{\varepsilon_{x,y,z}} yxz \mid 1 \leq x < y \leq z \leq n \}.$$

Les règles de réécriture de $\text{Knuth}_2(n)$ correspondent aux relations de Knuth, [63], avec une orientation compatible avec l'ordre lexicographique induit par l'ordre total sur $\{1 < \dots < n\}$. La présentation $\text{Knuth}_2(n)$ est appelée *présentation de Knuth*.

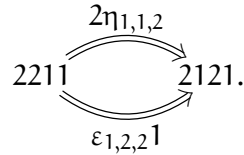
Soit Σ un système de réécriture. Une *étape de réécriture* de Σ est une règle de réécriture de la forme $w\alpha w' : wuw' \Rightarrow wv w'$ où $\alpha : u \Rightarrow v$ est une règle de réécriture dans Σ et w et w' sont des générateurs dans le monoïde libre Σ_1^* . Un *branchement local* de Σ est une paire non-ordonnée (f, f_1) d'étapes de réécriture $f : u \Rightarrow v$ et $f_1 : u \Rightarrow v_1$ de la 2-catégorie libre Σ_2^* avec une source commune u . Un *branchement critique* est un chevauchement d'application de deux étapes de réécriture différentes en un même mot u , où u est de longueur minimale, voir chapitre 2, section 2.1.2. Un branchement (f, f_1) est *confluent* s'il existe des étapes de réécriture f' et f'_1 dans Σ_2^* , comme suit



Un système de réécriture est *terminant* s'il n'existe pas de suite de réécriture infinie et il est *confluent* si pour tous mots u , u' et u'' tel que u se réécrit en u' et u'' , il existe un mot v tel que u' et u'' se réécrivent en v . Il est *convergent* s'il est terminant et confluent. Cela signifie qu'après l'application d'un nombre finie de règles de réécriture, toutes suites de réécriture sur un mot se terminent par un même mot irréductible, appelé une *forme normale*. Un système de réécriture est *localement confluent* si tous ses branchements locaux sont confluents. Selon le lemme des branchements critiques, [39, Theorem 3.1.5.], un système de réécriture est localement confluent si et seulement si ses branchements critiques sont confluents. Selon le lemme de Newman, [39,

Theorem 3.1.6.], pour les systèmes de réécriture terminants, les propriétés de confluence et de confluence locale sont équivalentes.

Par exemple, la présentation de Knuth du monoïde plaxique \mathbf{P}_2 dont les générateurs sont 1 et 2 et les relations de Knuth sont $\varepsilon_{1,2,2} : 221 \Rightarrow 212$ et $\eta_{1,1,2} : 211 \Rightarrow 121$ est convergente. En effet, comme l'ordre lexicographique est monomial, cette présentation est terminante. De plus, elle est confluente car elle admet un unique branchement critique confluent:

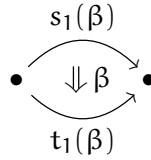


Notons que $2\eta_{1,1,2}$ et $\varepsilon_{1,2,2}1$ sont les étapes de réécriture appliquées sur le mot 2211.

Une façon de résoudre le problème du mot d'un monoïde \mathbf{M} est de trouver une présentation convergente finie (Σ_1, Σ_2) de ce monoïde. En effet, deux éléments dans le monoïde libre Σ_1^* représentent le même élément dans le monoïde \mathbf{M} si, et seulement si, leurs formes normales sont égales dans Σ_1^* . Ainsi, si un monoïde admet une présentation convergente finie alors son problème du mot est décidable. La réciproque de cette implication a été considéré comme un problème ouvert, [51, 52]. Squier a donné une réponse négative à ce problème, [112], en construisant des monoïdes finiment présentés avec un problème du mot décidable et qui ne peuvent pas être présentés par des présentations convergentes finies. Ensuite, Squier a introduit la condition de *type de dérivation fini*, qui est une propriété de finitude homotopique sur le complexe de présentation associé à une présentation d'un monoïde. Il a donné une méthode constructive pour montrer cette propriété de finitude basée sur le calcul des branchements critiques. Squier a montré que la condition de type de dérivation fini est nécessaire pour qu'un monoïde finiment présenté admette une présentation convergente finie, [113].

Les polygraphes de dimension 2. Les *polygraphes* ont été introduits comme des présentations par générateurs et relations de catégories de dimension supérieure. Afin de trouver des présentations pour les 2-catégories, Street a introduit la notion de *computade* qui est définie comme un graphe orienté équipé de bords supplémentaires de dimension 2 reliant ses chemins parallèles, [115, 116]. Un peu plus tard, Power a donné une définition inductive des computades de dimension supérieure, appelés les *n-computades*, [101, 102]. Burroni a introduit indépendamment la notion de *n-polygraphe* en utilisant aussi une définition inductive, [14, 15]. Un 1-polygraphe est un graphe orienté (Σ_0, Σ_1) , donné par un ensemble Σ_0 de 0-cellules, un ensemble Σ_1 de 1-cellules avec deux applications s_0 et t_0 envoyant une 1-cellule x à sa *source* $s_0(x)$ et son *but* $t_0(x)$. Un 1-polygraphe correspond à un système abstrait de réécriture, [3]. Dans le cas des monoïdes, l'ensemble Σ_0 est réduit à un ensemble avec un seul élément \bullet et le 1-polygraphe (Σ_0, Σ_1) s'identifie avec l'ensemble Σ_1 . De plus, un 2-polygraphe est une paire $\Sigma = (\Sigma_1, \Sigma_2)$ formée d'un 1-polygraphe Σ_1 et d'un ensemble de 2-cellules globulaires Σ_2 sur le monoïde libre Σ_1^* équipé d'applications *source* et *but* $s_1, t_1 : \Sigma_2 \rightarrow \Sigma_1^*$ qui vérifient les *relations globulaires*: $s_0s_1 = s_0t_1$

et $t_0s_1 = t_0t_1$. Les 2-cellules de Σ_2 ont la forme globulaire suivante:



Un 2-polygraphe Σ est une *présentation d'un monoïde* \mathbf{M} si le monoïde \mathbf{M} est isomorphe au quotient du monoïde libre Σ_1^* par la relation de congruence engendrée par l'ensemble Σ_2 . De plus, deux 2-polygraphes sont *Tietze équivalents* si les monoïdes qu'ils présentent sont isomorphes. Pour $n \geq 2$, un $(n + 1)$ -polygraphe est donné par un n -polygraphe Σ_n , avec une famille Σ_{n+1} de $(n + 1)$ -cellules additionnelles reliant des n -cellules parallèles de la n -catégorie Σ_n^* engendrée librement par le n -polygraphe Σ_n .

En utilisant l'approche polygraphique, plusieurs travaux ont développé la réécriture en dimension supérieure. De plus, les résultats de Squier ont été présentés dans le langage des polygraphes et des catégories de dimension supérieure, donnant de nouvelles preuves de ces résultats, voir [96, 69, 35, 38, 36, 37, 39].

Complétion de la présentation de Knuth. Une présentation non-convergente d'un monoïde peut être transformée en une présentation convergente du même monoïde en utilisant la *complétion de Knuth–Bendix*. Cette complétion calcule une présentation convergente d'un monoïde à partir d'une présentation terminante en ajoutant itérativement des règles de réécriture, [64]. En d'autres termes, soit Σ une présentation terminante d'un monoïde \mathbf{M} , la complétion de Knuth–Bendix de Σ est le 2-polygraphe obtenu à partir de Σ en examinant chacun de ses branchements critiques et en ajoutant des 2-cellules quand un branchement critique n'est pas confluent. Les 2-cellules ajoutées peuvent aussi créer de nouveaux branchements critiques. Ces branchements sont aussi examinés et de nouvelles 2-cellules peuvent être ajoutées. Finalement, le 2-polygraphe obtenu est une présentation convergente du monoïde \mathbf{M} .

Pour le monoïde \mathbf{P}_3 , la présentation de Knuth $\text{Knuth}_2(3)$ admet les générateurs 1, 2 et 3, et les 8 relations suivantes

$$\begin{aligned} & \{ 211 \xrightarrow{\eta_{1,1,2}} 121, \quad 311 \xrightarrow{\eta_{1,1,3}} 131, \quad 312 \xrightarrow{\eta_{1,2,3}} 132, \quad 322 \xrightarrow{\eta_{2,2,3}} 232 \} \\ & \cup \{ 221 \xrightarrow{\epsilon_{1,1,2}} 212, \quad 231 \xrightarrow{\epsilon_{1,2,3}} 213, \quad 331 \xrightarrow{\epsilon_{1,3,3}} 313, \quad 332 \xrightarrow{\epsilon_{2,3,3}} 323 \}. \end{aligned}$$

En utilisant la complétion de Knuth–Bendix, une présentation convergente finie du monoïde \mathbf{P}_3 est obtenue en ajoutant les relations suivantes

$$32321 \Rightarrow 32132, \quad 32131 \Rightarrow 31321 \quad \text{et} \quad 3212 \Rightarrow 2321$$

à sa présentation de Knuth, voir chapitre 2, section 2.3.2.

Pour $n > 3$, Kubat et Okniński ont montré que la présentation de Knuth du monoïde plaxique \mathbf{P}_n n'admet pas de complétion finie, [65, Theorem 3]:

Théorème 2.3.3.3. *Pour $n > 3$, la présentation de Knuth n'admet pas de complé-
tion finie compatible avec l'ordre lexicographique.*

La présentation colonne. Bokut, Chen, Chen et Li dans [7] et Cain, Gray et Malheiro dans [17] ont construit indépendamment une présentation convergente finie du monoïde plaxique \mathbf{P}_n . Leurs méthodes consistent à ajouter de nouveaux générateurs, appelés les *générateurs colonnes*, à la présentation de Knuth. La nouvelle présentation réécrit deux colonnes qui ne forment pas un tableau en deux autres obtenues après l'application de l'algorithme d'insertion de Schensted. Cette présentation, appelée la *présentation colonne*, est Tietze équivalente à la présentation de Knuth. Cain, Gray et Malheiro ont utilisé l'insertion par ligne dans la définition des 2-cellules de la présentation tandis que Bokut, Chen, Chen et Li ont utilisé l'insertion par colonne. Notons aussi que Bokut, Chen, Chen et Li ont construit dans [7] une présentation convergente infinie du monoïde \mathbf{P}_n en ajoutant l'ensemble infini de lignes sur l'ensemble $[n]$.

Nous considérons la présentation colonne du monoïde \mathbf{P}_n construite par Cain, Gray et Malheiro, [17]. Dans ce cas, une *colonne* est la 1-cellule obtenue en lisant un tableau de forme $(1, \dots, 1)$ de bas en haut. En d'autres termes, une colonne est une 1-cellule strictement décroissante dans le monoïde libre $[n]^*$. Par exemple, la 1-cellule 76431 est une colonne sur l'ensemble [7]. On note par $\text{col}(n)$ l'ensemble des colonnes non-triviales sur l'ensemble $[n]$. Pour chaque 1-cellule u dans $[n]^*$, on note par $\ell(u)$ sa longueur. De plus, le tableau $P(u)$ est calculé en utilisant la procédure d'insertion par ligne.

Soient $u = x_p \dots x_1$ et $v = y_q \dots y_1$ deux colonnes dans $\text{col}(n)$, le tableau $P(uv)$ contient au plus deux colonnes, voir chapitre 3, sous-section 3.2.1. On note par $u \times v$ si la juxtaposition des colonnes u et v ne forme pas un tableau, c'est-à-dire si $p < q$ ou $x_i > y_i$ pour certain $i \leq q$. Dans ce cas, on note par $u \times^1 v$ si le tableau $P(uv)$ contient une colonne et par $u \times^2 v$ s'il contient deux colonnes.

Afin d'obtenir une présentation convergente finie du monoïde plaxique \mathbf{P}_n , on ajoute des générateurs superflus c_u à la présentation $\text{Knuth}_2(n)$, pour toute colonne u dans l'ensemble $\text{col}(n)$. On note par $\text{Col}_1(n)$ l'ensemble des *générateurs colonnes* c_u du monoïde \mathbf{P}_n , pour tout u dans $\text{col}(n)$, et par $\gamma_u : c_{x_p} \dots c_{x_1} \Rightarrow c_u$ les relations correspondantes au rajout de ces générateurs, pour toute colonne $u = x_p \dots x_1$ de longueur supérieure ou égale à 2.

Pour toutes colonnes u et v dans $\text{col}(n)$ telles que $u \times v$, on définit une 2-cellule

$$\alpha_{u,v} : c_u c_v \Rightarrow c_w c_{w'}$$

où $w = uv$ et $c_{w'} = 1$ si $u \times^1 v$, et w et w' sont respectivement les colonnes gauches et droites du tableau $P(uv)$ si $u \times^2 v$.

Considérons le 2-polygraphe $\text{Col}_2(n)$ dont l'ensemble des 1-cellules est $\text{Col}_1(n)$ et l'ensemble des 2-cellules est

$$\{ c_u c_v \xrightarrow{\alpha_{u,v}} c_w c_{w'} \mid u, v \in \text{col}(n) \text{ et } u \times v \}.$$

Comme l'ensemble des générateurs colonnes $\text{Col}_1(n)$ est fini, le 2-polygraphe $\text{Col}_2(n)$ est fini. De plus, les 2-polygraphes $\text{Col}_2(n)$ et $\text{Knuth}_2(n)$ sont Tietze équivalents, voir Chapter 3,

Lemme 3.2.2.2, alors le 2-polygraphe $\text{Col}_2(n)$ est une présentation du monoïde \mathbf{P}_n , appelé la *présentation colonne*. En utilisant certaines propriétés combinatoires des tableaux, on peut montrer que le 2-polygraphe $\text{Col}_2(n)$ est confluente et terminant, voir chapitre 3, section 3.2. Par conséquent, le 2-polygraphe $\text{Col}_2(n)$ est une présentation convergente finie du monoïde \mathbf{P}_n , [17, Theorem 3.4.].

La présentation colonne et l'algorithme de Schensted. L'algorithme d'insertion par ligne de Schensted correspond au chemin de réduction le plus à gauche dans $\text{Col}_2^*(n)$ d'une 1-cellule w à son tableau de Schensted. C'est-à-dire, les chemins de réduction obtenus en appliquant les règles de $\text{Col}_2(n)$ en commençant de la gauche. Par exemple, considérons la 1-cellule $w = 43152452$ dans $[5]^*$. Pour calculer le tableau $P(w)$, on applique les règles consécutives suivantes de $\text{Col}_2(5)$ en commençant en chaque étape de la gauche:

$$\begin{aligned}
 w = 43152452 &\xrightarrow{\alpha_{4,3}} \begin{array}{|c|} \hline 3 \\ \hline 4 \end{array} 1524452 \xrightarrow{\alpha_{43,1}} \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 4 \end{array} 524452 \xrightarrow{\alpha_{5,2}} \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 4 \end{array} \begin{array}{|c|} \hline 2 \\ \hline 5 \end{array} 452 \\
 &\xrightarrow{\alpha_{5,2}} \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 4 \end{array} \begin{array}{|c|} \hline 2 \\ \hline 5 \end{array} 425 \xrightarrow{\alpha_{4,52}} \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 4 \end{array} \begin{array}{|c|} \hline 2 \\ \hline 5 \end{array} \begin{array}{|c|} \hline 2 \\ \hline 4 \end{array} 5 = P(w)
 \end{aligned}$$

Les syzygies et les présentations cohérentes. Le *problème des syzygies* pour les présentations de monoïdes consiste à trouver toutes les relations entre les relations de ces présentations. Plus précisément, soit (Σ_1, Σ_2) une présentation d'un monoïde. Une 2-syzygie pour (Σ_1, Σ_2) est une 2-sphère non-triviale dans la $(2, 1)$ -catégorie libre Σ_2^\top , c'est-à-dire, une paire (f, g) de 2-cellules de Σ_2^\top telle que $s_1(f) = s_1(g)$ et $t_1(f) = t_1(g)$. Une *extension globulaire* de Σ_2^\top est un ensemble Σ_3 équipé d'une application de Σ_3 dans l'ensemble des 2-sphères de Σ_2^\top . Une *famille de 2-syzygies génératrices* est une extension globulaire Σ_3 de Σ_2^\top telle que pour toute 2-sphère γ dans Σ_2^\top il existe une 3-cellule dans la $(3, 1)$ -catégorie libre Σ_3^\top de bordure γ , voir sous-section 4.1.1. Par exemple, pour le monoïde \mathbf{P}_2 , les relations de Knuth $\eta_{1,1,2} : 211 \Rightarrow 121$ et $\varepsilon_{1,2,2} : 221 \Rightarrow 212$ sont reliées par la 2-syzygie suivante

$$\begin{array}{ccc}
 & 2\eta_{1,1,2} & \\
 & \curvearrowright & \\
 2211 & & 2121 \\
 & \curvearrowleft & \\
 & \varepsilon_{1,2,2}1 &
 \end{array}$$

qui donne deux façons de montrer l'égalité $2211 = 2121$ dans le monoïde \mathbf{P}_2 . Pour les rangs supérieurs ou égaux à 3, il est difficile de calculer une famille génératrice de syzygies pour la présentation de Knuth à cause de la complexité combinatoire des relations. Un $(3, 1)$ -polygraphe est une paire (Σ_2, Σ_3) formée d'un 2-polygraphe Σ_2 et d'une extension globulaire Σ_3 de Σ_2^\top . Une *présentation étendue* d'un monoïde est un $(3, 1)$ -polygraphe dont le 2-polygraphe est

une présentation du monoïde. Une *présentation cohérente* d'un monoïde est une présentation étendue (Σ_2, Σ_3) du monoïde telle que l'ensemble Σ_3 est une famille génératrice de 2-syzygies de Σ_2^\top , qu'on appelle une *base d'homotopie*, voir chapitre 4, sous-section 4.1.1.

Le théorème de Squier donne une méthode pour calculer une présentation cohérente d'un monoïde à partir d'une présentation convergente, [112]. Plus précisément, tous les branchements critiques d'une présentation convergente sont confluents. Alors l'adjonction des 3-cellules résultant de ces branchements critiques à la présentation convergente donne lieu à une présentation cohérente du monoïde. Afin de construire des présentations cohérentes des monoïdes plaxiques, on utilise la procédure de *complétion* et de *réduction* homotopique introduite dans [40, 31]. La procédure de *complétion homotopique* étend la complétion de Squier, [113], aux 2-polygraphes terminants en utilisant la complétion de Knuth–Bendix, [64]. Elle examine chacun des branchements critiques d'un 2-polygraphe terminant, ajoute des 2-cellules pour atteindre la confluence et ajoute des 3-cellules pour obtenir une présentation convergente cohérente. Cette présentation cohérente n'est pas minimale en général, alors la procédure de *réduction homotopique* élimine d'une manière cohérente les cellules superflues de cette présentation en utilisant la notion de *partie collapsible* des $(3, 1)$ -polygraphes. En particulier, une méthode pour éliminer certaines 3-cellules superflues est de calculer un *branchement triple critique* afin d'obtenir des relations entre les 3-cellules et alors d'éliminer certaines d'elles par des *transformations de Tietze*, voir chapitre 4, sous-section 4.2.1.

Par exemple, la présentation de Knuth du monoïde plaxique \mathbf{P}_2 admet un unique branchement critique confluent. Alors la présentation $\text{Knuth}_2(2)$ peut être étendue en une présentation cohérente en ajoutant la 3-cellule suivante:

$$\begin{array}{ccc}
 & 2\eta_{1,1,2} & \\
 & \curvearrowright & \\
 2211 & \Downarrow & 2121. \\
 & \curvearrowleft & \\
 & \varepsilon_{1,2,2}1 &
 \end{array} \tag{2}$$

De plus, la présentation colonne du monoïde \mathbf{P}_2 est donnée par les 1-cellules c_1, c_2, c_{21} , et les 2-cellules $\alpha_{2,1} : c_2c_1 \Rightarrow c_{21}$, $\alpha_{1,21} : c_1c_{21} \Rightarrow c_{21}c_1$ et $\alpha_{2,21} : c_2c_{21} \Rightarrow c_{21}c_2$. Cette présentation peut être étendue en une présentation cohérente en ajoutant la 3-cellule suivante:

$$\begin{array}{ccc}
 & \alpha_{2,1}c_{21} \rightarrow c_{21}c_{21} & \\
 & \curvearrowright & \\
 c_2c_1c_{21} & \Downarrow & c_{21}c_{21} \\
 & \curvearrowleft & \\
 & c_2\alpha_{1,21} \rightarrow c_2c_{21}c_1 & \xrightarrow{\alpha_{2,21}c_1} c_{21}c_2c_1 \\
 & & \swarrow c_{21}\alpha_{2,1}
 \end{array} \tag{3}$$

Notons que pour les présentations colonnes des monoïdes $\mathbf{P}_3, \mathbf{P}_4$ et \mathbf{P}_5 on compte respectivement 7, 15 et 31 1-cellules, 22, 115 et 531 2-cellules, 42, 621 et 6893 3-cellules.

PRÉSENTATION DES RÉSULTATS

Présentations colonnes des monoïdes plaxiques

Comme il a déjà été mentionné dans le chapitre 2, section 2.3.2, la présentation de Knuth du monoïde \mathbf{P}_2 est finie et convergente et la présentation de Knuth du monoïde \mathbf{P}_3 se transforme en une présentation convergente en ajoutant trois 2-cellules par la complétion de Knuth–Bendix. De plus, pour $n > 3$, la présentation de Knuth du monoïde plaxique \mathbf{P}_n n’admet pas de complétion finie compatible avec l’ordre lexicographique, [65, Theorem 3]. Les présentations par générateurs et relations construites pour les monoïdes plaxiques pour les types classiques contiennent les relations de Knuth, [88, 78, 79, 83]. Par conséquent, ces présentations n’admettent pas de complétion finie compatible avec l’ordre lexicographique sans le rajout de nouveaux générateurs.

Question. *Est-ce que les monoïdes plaxiques généralisés admettent des présentations convergentes finies?*

La notion de chemin L-S introduite par Littelmann dans [88] permet de répondre positivement à cette question. En effet, un chemin L-S correspond à une colonne pour le type A et à une colonne admissible pour les types C, B, D et G_2 au sens de Lecouvey, voir [83]. Comme pour le type A, cette présentation est aussi appelée *présentation colonne*. La présentation colonne est une présentation des monoïdes plaxiques généralisés, [88, Theorem B]. On étudie cette présentation en utilisant des méthodes de réécriture. On considère le système de réécriture dont l’ensemble des générateurs est l’ensemble fini des chemins L-S. Ce système de réécriture réécrit deux chemins L-S tels que leur concaténation ne forme pas un tableau standard au sens de Littelmann en des chemins L-S formant un tableau standard. En utilisant les formes des tableaux, on montre que cette présentation est convergente et finie.

En traitant les monoïdes plaxiques cas par cas et en utilisant la théorie des bases cristallines de Kashiwara, la présentation colonne peut être aussi construite pour les monoïdes plaxiques de type C, B, D et G_2 . On construit la présentation colonne pour le monoïde plaxique de type C en ajoutant les *générateurs colonnes admissibles*, [43]. La notion de tableau est généralisée à la notion de *tableau symplectique* pour le type C. La partie droite des règles de réécriture de la présentation colonne pour le type C est le résultat de l’insertion de Lecouvey de deux colonnes admissibles. En effectuant une analyse minutieuse des formes des tableaux symplectiques obtenus après cette insertion, on montre que cette présentation est convergente.

Un peu plus tard, Cain, Gray et Malheiro ont construit des présentations colonnes similaires pour les monoïdes plaxiques de type B, C, D et G_2 en ajoutant aussi les générateurs colonnes admissibles, [16]. Les tableaux symplectiques introduits comme une généralisation des tableaux pour le type C sont aussi généralisés pour les autres types, [83]. La présentation colonne des monoïdes plaxiques pour ces types réécrit deux colonnes admissibles qui ne forment pas un tableau (généralisé) en des colonnes admissibles formant un tableau (généralisé). Cain, Gray

et Malheiro ont utilisé les présentations de Lecouvey pour les monoïdes plaxiques alors qu'on utilise les insertions de Lecouvey.

Il convient de noter que la présentation colonne construite d'une manière générale en utilisant les chemins L-S coïncide avec les présentations construites séparément pour les types A, B, C, D et G_2 . De plus, cette présentation couvre les types exceptionnels dont les monoïdes plaxiques correspondants n'admettent pas de présentations explicites par de générateurs et de relations.

Dans les parties ci-après, on va détailler les constructions des présentations colonnes des monoïdes plaxiques. Dans un premier temps, on montre comment ces présentations peuvent être construites en introduisant les chemins L-S. Ensuite, on illustre la présentation colonne du monoïde plaxique de type C en utilisant des outils combinatoires et la théorie des bases cristallines de Kashiwara.

Le présentation colonne généralisée des monoïdes plaxiques. Considérons les monoïdes plaxiques généralisés construits par Littelmann, [88]. Notre objectif est de construire des présentations convergentes finies de ces monoïdes en utilisant les chemins de Lakshmibai-Seshadri. On utilise la notion de tableau standard définie par Littelmann dans [88] comme une généralisation de la notion de tableau pour tout type.

Soit \mathfrak{g} une algèbre de Lie semi-simple et $\Lambda_1, \dots, \Lambda_n$ ses poids fondamentaux. Pour un poids λ dans $P \otimes_{\mathbb{Z}} \mathbb{R}$ et t dans $[0, 1]$, on définit le chemin $\pi_\lambda(t) := t\lambda$ reliant l'origine et λ par une ligne droite. Pour un poids dominant λ , les chemins de Lakshmibai-Seshadri, ou chemins L-S, de forme λ sont des chemins de la forme $f_{\alpha_1} \circ \dots \circ f_{\alpha_s}(\pi_\lambda)$, où $\alpha_1, \dots, \alpha_s$ sont des racines simples de \mathfrak{g} . Un tableau de Young de forme $\lambda = \alpha_1 \Lambda_1 + \dots + \alpha_n \Lambda_n$ est une concaténation

$$\star_{1 \leq i \leq n} \pi_{1, \Lambda_i} \star \dots \star \pi_{\alpha_i, \Lambda_i}$$

où π_{i, Λ_i} est un chemin L-S de forme Λ_i , pour $1 \leq i \leq n$. C'est-à-dire, les premiers α_1 chemins sont de forme Λ_1 , les α_2 suivants sont de forme Λ_2, \dots , les α_n derniers chemins sont de forme Λ_n .

Un tableau de Young de forme $\lambda = \alpha_1 \Lambda_1 + \dots + \alpha_n \Lambda_n$ est un tableau standard de forme λ s'il est de la forme

$$f_{\alpha_1} \circ \dots \circ f_{\alpha_s} \left(\underbrace{(\pi_{\Lambda_1} \star \dots \star \pi_{\Lambda_1})}_{\alpha_1 \text{ fois}} \star \underbrace{(\pi_{\Lambda_2} \star \dots \star \pi_{\Lambda_2})}_{\alpha_2 \text{ fois}} \star \dots \star \underbrace{(\pi_{\Lambda_n} \star \dots \star \pi_{\Lambda_n})}_{\alpha_n \text{ fois}} \right)$$

où $\alpha_1, \dots, \alpha_s$ sont des racines simples de \mathfrak{g} . Notons que les tableaux standard introduits par Littelmann en utilisant les chemins correspondant aux tableaux pour le type A, aux tableaux symplectiques pour le type C et aux tableaux orthogonaux pour les types B et D dans le sens de Lecouvey, [78, 79, 83].

Soit \mathbb{B}_i l'ensemble des chemins L-S de forme Λ_i et $\mathbb{B} = \cup_{i=1}^n \mathbb{B}_i$. Pour tous chemins L-S c_1 et c_2 dans \mathbb{B} tels que $c_1 \star c_2$ n'est pas un tableau standard, on définit la 2-cellule

$$c_1 \cdot c_2 \xrightarrow{\gamma_{c_1, c_2}} T$$

où T est l'unique tableau standard tel que T et $c_1 \star c_2$ sont égaux dans le monoïde plaxique correspondant.

Le 2-polygraphe des chemins, noté par $\text{Path}_2(\mathfrak{n})$, est le 2-polygraphe avec une seule 0-cellule et dont l'ensemble des 1-cellules est \mathbb{B} et l'ensemble des 2-cellules est

$$\text{Path}_2(\mathfrak{n}) = \left\{ c_1.c_2 \xrightarrow{\gamma_{c_1, c_2}} T \mid c_1, c_2 \in \mathbb{B} \text{ et } c_1 \star c_2 \text{ n'est pas un tableau standard} \right\}.$$

Cette présentation est appelée *présentation colonne*. C'est une présentation du monoïde plaxique pour toute algèbre de Lie semi-simple [88, Theorem B]. En utilisant les formes des tableaux, on montre que cette présentation est terminante et confluente. Par conséquent, on obtient le résultat suivant

Théorème 3.1.0.1. *Pour toute algèbre de Lie semi-simple \mathfrak{g} , le 2-polygraphe $\text{Path}_2(\mathfrak{n})$ est une présentation convergente finie du monoïde plaxique de \mathfrak{g} .*

Comme nous le verrons ci-dessous pour le type C, le 2-polygraphe $\text{Path}_2(\mathfrak{n})$ peut être construit en raisonnant cas par cas sur les monoïdes plaxiques de types A, B, C, D et G_2 en utilisant la terminologie des colonnes et des colonnes admissibles.

La présentation colonne du monoïde plaxique de type C. Considérons le monoïde plaxique $\mathbf{P}_n(C)$ de type C et l'ensemble ordonné

$$\mathcal{C}_n = \{1 < 2 < \dots < n < \bar{n} < \dots < \bar{1}\}.$$

Pour une colonne U , on note par $h(U)$ le nombre de ses éléments. Une colonne U est *admissible* si pour $m = 1, \dots, h(U)$, le nombre $N(m)$ des lettres x dans U tel que $x \leq m$ ou $x \geq \bar{m}$ satisfait $N(m) \leq m$. Un mot est dit *admissible* s'il est la lecture de haut en bas d'une colonne admissible. Par exemple, le mot $568\bar{8}\bar{5}$ est admissible sur l'ensemble ordonné \mathcal{C}_8 . On note par $\text{acol}(n)$ l'ensemble de tous les mots admissibles non triviaux dans le monoïde libre \mathcal{C}_n^* . Un *tableau symplectique* est un tableau rempli par les éléments de l'ensemble \mathcal{C}_n et composé par la juxtaposition de colonnes admissibles avec une propriété supplémentaire sur elles, voir chapitre 3, section 3.3.1. En utilisant un algorithme d'insertion, Lecouvey a montré que pour tout mot w dans le monoïde libre \mathcal{C}_n^* , on peut construire un unique tableau symplectique, noté par $P(w)$, [78].

En utilisant la théorie des bases cristallines de Kashiwara, on généralise la présentation colonne de type A. On construit une présentation convergente finie du monoïde $\mathbf{P}_n(C)$ en ajoutant les *générateurs colonnes admissibles*, [43]. La partie droite des règles de réécriture de cette présentation est le résultat de l'insertion de Lecouvey de deux colonnes admissibles. En d'autres mots, le résultat est un tableau symplectique qui consiste en au plus deux colonnes.

Dans le but de construire une présentation convergente finie du monoïde plaxique $\mathbf{P}_n(C)$, on introduit les *générateurs colonnes admissibles*. L'ensemble des générateurs est

$$\text{ACol}_1(\mathfrak{n}) = \left\{ c_u \mid u \in \text{acol}(\mathfrak{n}) \right\},$$

où chaque élément c_u est égal à u dans le monoïde $\mathbf{P}_n(\mathbb{C})$.

Soient u et v les lectures de haut en bas de deux colonnes admissibles U et V respectivement. On note par $U \not\leq V$ si la 1-cellule uv n'est pas la lecture d'un tableau symplectique. C'est-à-dire, quand la juxtaposition de U et V de droite à gauche ne forme pas un tableau symplectique.

Pour définir les 2-cellules de la nouvelle présentation, on a besoin de connaître la forme du tableau symplectique $P(uv)$, pour tous mots admissibles u et v dans $\text{acol}(n)$. On a

Lemme 3.3.3.1. *Soient u et v les lectures de deux colonnes admissibles U et V respectivement. Le tableau symplectique $P(uv)$ consiste en au plus deux colonnes.*

Soient u et v dans $\text{acol}(n)$ tels que leur colonnes admissibles U et V satisfont $U \not\leq V$. On définit la 2-cellule $\alpha_{u,v}$ par

- $c_u c_v \xrightarrow{\alpha_{u,v}} c_w c_{w'}$, où les mots w et w' sont respectivement les lectures de la colonne droite W et la colonne gauche W' de $P(uv)$ si ce tableau symplectique consiste en deux colonnes.
- $c_u c_v \xrightarrow{\alpha_{u,v}} c_w$, où w est la lecture de la colonne W de $P(uv)$ si ce tableau consiste en une seule colonne.
- $c_u c_v \xrightarrow{\alpha_{u,v}} c_\varepsilon$, où ε est le mot vide si $P(uv)$ contient zéro colonne.

Soit $\text{ACol}_2(n)$ le 2-polygraphe dont l'ensemble des 1-cellules est $\text{ACol}_1(n)$ et l'ensemble des 2-cellules contient toutes les 2-cellules $\alpha_{u,v}$, pour tous u et v dans $\text{acol}(n)$ avec $U \not\leq V$.

On montre que le 2-polygraphe $\text{ACol}_2(n)$ est une présentation du monoïde plaxique $\mathbf{P}_n(\mathbb{C})$ de type C, appelée *présentation colonne*, voir Chapter 3, Lemme 3.3.4.5.

Notre objectif est de montrer que le 2-polygraphe $\text{ACol}_2(n)$ est fini et convergent. Il est fini grâce au fait que l'ensemble $\text{ACol}_1(n)$ est fini. De plus, on définit un bon ordre sur le produit des colonnes admissibles tel que les résultats des règles de réécriture décroissent par rapport à cet ordre.

Lemme 3.3.3.2. *Soient u et v les lectures de deux colonnes admissibles U et V respectivement, tels que $U \not\leq V$. Supposons que le tableau $P(uv)$ est un remplissage de deux colonnes admissibles et soit W la colonne droite. Alors la colonne U contient plus d'éléments que W .*

Alors le 2-polygraphe $\text{ACol}_2(n)$ est terminant. La confluence est une conséquence du fait que les tableaux symplectiques forment une section du monoïde $\mathbf{P}_n(\mathbb{C})$. Par conséquent, on obtient

Théorème 3.3.4.6. *Le 2-polygraphe $\text{ACol}_2(n)$ est une présentation convergente finie du monoïde plaxique $\mathbf{P}_n(\mathbb{C})$ de type C.*

Présentations cohérentes des monoïdes plaxiques de type A

Comme il a déjà été mentionné, pour un rang supérieur ou égal à 4, le problème des syzygies pour la présentation de Knuth est difficile à cause de la complexité combinatoire des relations de Knuth. Grâce à la procédure de complétion et de réduction homotopique, on construit une présentation cohérente finie du monoïde plaxique \mathbf{P}_n qui étend la présentation de Knuth. On construit une présentation cohérente finie du monoïde \mathbf{P}_n à partir de sa présentation colonne. Cette présentation n'est pas minimale dans le sens que certains de ses générateurs sont superflus. Après plusieurs étapes de réduction, on obtient une présentation cohérente de \mathbf{P}_n avec les générateurs de Knuth. Par conséquent, on obtient des formes explicites des 2-syzygies de la présentation de Knuth.

Par exemple, la présentation colonne $\text{Col}_2(2)$ du monoïde \mathbf{P}_2 peut être étendue en une présentation cohérente en ajoutant l'unique 3-cellule génératrice (2). En appliquant notre construction sur cette présentation cohérente, on montre que la présentation de Knuth peut être étendue en une présentation cohérente en ajoutant la 3-cellule génératrice (3).

Le diagramme suivant décrit les étapes principales de la construction

$$\begin{array}{ccccc}
 \text{Knuth}_2(n) & \xrightarrow{\text{équivalence de Tietze}} & \text{Col}_2(n) & \xrightarrow{\text{complétion homotopique}} & \text{Col}_3(n) \\
 & & & & \downarrow \text{réduction homotopique} \\
 \text{Knuth}_3(n) & \xleftarrow{\text{équivalence de Tietze}} & \text{PreCol}_3(n) & \xleftarrow{\text{réduction homotopique}} & \overline{\text{Col}}_3(n)
 \end{array}$$

où $\text{Col}_3(n)$ représente la présentation cohérente obtenue à partir de la présentation colonne $\text{Col}_2(n)$, les $(3, 1)$ -polygraphes $\overline{\text{Col}}_3(n)$ et $\text{PreCol}_3(n)$ sont obtenus à partir de $\text{Col}_3(n)$ en appliquant des étapes de réduction homotopique, et $\text{Knuth}_3(n)$ représente la présentation cohérente du monoïde plaxique \mathbf{P}_n qui étend la présentation de Knuth et qui est Tietze équivalente à $\text{PreCol}_3(n)$.

Présentation colonne cohérente. Plus précisément, après la vérification de la confluence de tous les branchements critiques de la présentation colonne $\text{Col}_2(n)$, on montre que tous les diagrammes de confluence de ces branchements ont la forme suivante

$$\begin{array}{ccccc}
 & & c_e c_{e'} c_t & \xrightarrow{c_e \alpha_{e',t}} & c_e c_b c_{b'} & \xrightarrow{\alpha_{e,b} c_{b'}} & & (4) \\
 & \nearrow \alpha_{u,v} c_t & & & \Downarrow \mathcal{X}_{u,v,t} & & & \\
 c_u c_v c_t & & & & & & c_a c_d c_{b'} & \\
 & \searrow c_u \alpha_{v,t} & c_u c_w c_{w'} & \xrightarrow{\alpha_{u,w} c_{w'}} & c_a c_{a'} c_{w'} & \xrightarrow{c_a \alpha_{a',w'}} & &
 \end{array}$$

pour toutes colonnes $u = x_p \dots x_1$, $v = y_q \dots y_1$ et $t = z_l \dots z_1$ telles que uv et vt ne sont pas des tableaux, où e et e' (resp. w et w') représentent les deux colonnes du tableau $P(uv)$ (resp. $P(vt)$)

et a et a' (resp. b et b') représentent les deux colonnes du tableau $P(uw)$ (resp. $P(e't)$) et a , d , b' sont les trois colonnes du tableau $P(uvt)$, qui est une forme normale dans le 2-polygraphe $\text{Col}_2(n)$.

On note par $\text{Col}_3(n)$ la présentation étendue du monoïde \mathbf{P}_n obtenue à partir de $\text{Col}_2(n)$ en ajoutant les 3-cellules de la forme $\mathcal{X}_{u,v,t}$. On obtient le résultat suivant:

Théorème 4.1.3.1. *Pour $n > 0$, le monoïde plaxique \mathbf{P}_n admet $\text{Col}_3(n)$ comme présentation cohérente.*

Ce théorème est montré en vérifiant la confluence des branchements critiques du 2-polygraphe $\text{Col}_2(n)$ dans chacun des cas suivants $u^{\times 1}v^{\times 1}t$, $u^{\times 2}v^{\times 1}t$, $u^{\times 1}v^{\times 2}t$, et $u^{\times 2}v^{\times 2}t$. Notons que les 3-cellules de $\text{Col}_3(n)$ sont de la forme $A_{u,v,t}$ pour $u^{\times 1}v^{\times 1}t$, $B_{u,v,t}$ pour $u^{\times 2}v^{\times 1}t$, $C_{u,v,t}$ pour $u^{\times 1}v^{\times 2}t$ et $D_{u,v,t}$ pour $u^{\times 2}v^{\times 2}t$, voir chapitre 4, sous-section 4.1.

Exemple: le cas du monoïde \mathbf{P}_2 . Considérons la présentation colonne $\text{Col}_2(2)$ du monoïde \mathbf{P}_2 dont les 1-cellules sont c_1 , c_2 et c_{21} et les 2-cellules sont $\alpha_{2,1} : c_2c_1 \Rightarrow c_{21}$, $\alpha_{1,21} : c_1c_{21} \Rightarrow c_{21}c_1$ et $\alpha_{2,21} : c_2c_{21} \Rightarrow c_{21}c_1$. La présentation cohérente $\text{Col}_3(2)$ admet une unique 3-cellule

$$\begin{array}{ccccc}
 & & \alpha_{2,1}c_{21} & \rightarrow & c_{21}c_{21} & \leftarrow & c_{21}\alpha_{2,1} \\
 & & \parallel & & \parallel & & \\
 c_2c_1c_{21} & & & & C'_{2,1,21} & & \\
 & & c_2\alpha_{1,21} & \rightarrow & c_2c_{21}c_1 & \xrightarrow{\alpha_{2,21}c_1} & c_{21}c_2c_1
 \end{array}$$

Une présentation colonne cohérente réduite. La présentation cohérente $\text{Col}_3(n)$ du monoïde plaxique \mathbf{P}_n n'est pas minimale. En utilisant la procédure de réduction homotopique, on la réduit à la présentation $\overline{\text{Col}}_3(n)$ obtenue à partir du 2-polygraphe $\text{Col}_2(n)$ en ajoutant une famille de 3-cellules $\mathcal{X}_{x,v,t}$ de la forme (9), avec x est un élément de $[n]$ de longueur 1, voir chapitre 4, sous-section 4.2.2. Les 3-cellules de $\overline{\text{Col}}_3(n)$ sont de la forme $A_{x,v,t}$, $B_{x,v,t}$, $C_{x,v,t}$ et $D_{x,v,t}$. Par conséquent, on obtient

Proposition 4.2.2.1. *Pour $n > 0$, le monoïde plaxique \mathbf{P}_n admet $\overline{\text{Col}}_3(n)$ comme présentation cohérente.*

Par exemple, comme la première colonne de $c_2c_1c_{21}$ dans la 3-cellule de $\text{Col}_3(2)$ est de longueur 1, les présentations cohérentes $\text{Col}_3(2)$ et $\overline{\text{Col}}_3(2)$ du monoïde \mathbf{P}_2 coïncident.

Présentation pré-colonne cohérente. La deuxième étape de réduction homotopique consiste à réduire la présentation cohérente $\overline{\text{Col}}_3(n)$. Cette réduction, notée par R_{Γ_3} , est donnée par la partie collapsible définie par l'ensemble des 3-cellules de $\overline{\text{Col}}_3(n)$, voir chapitre 4, sous-section 4.2.3. La réduction homotopique R_{Γ_3} élimine de $\overline{\text{Col}}_3(n)$ les 3-cellules de la forme $A_{x,v,t}$, $B_{x,v,t}$ et $C_{x,v,t}$ qui ne sont pas de la forme $C'_{x,v,t}$. La réduction R_{Γ_3} élimine aussi certaines 2-cellules superflues de $\overline{\text{Col}}_3(n)$.

Plus précisément, soit $\text{PreCol}_3(n)$ le $(3, 1)$ -polygraphe dont l'ensemble des 1-cellules est $\text{Col}_1(n)$, l'ensemble des 2-cellules est

$$\{ c_x c_{zy} \xrightarrow{\alpha_{x,zy}} c_{zx} c_y \mid 1 \leq x \leq y < z \leq n \} \cup \{ c_y c_{zx} \xrightarrow{\alpha_{y,zx}} c_{yx} c_z \mid 1 \leq x < y \leq z \leq n \} \\ \cup \{ c_x c_u \xrightarrow{\alpha_{x,u}} c_{xu} \mid xu \in \text{col}(n) \text{ et } 1 \leq x \leq n \},$$

et dont les 3-cellules sont de la forme $R_{\Gamma_3}(C'_{x,v,t})$ où

$$\begin{array}{ccc} & \alpha_{x,v} c_t \rightarrow & c_{xv} c_t \\ c_x c_v c_t & \xrightarrow{\quad} & c_{xv} c_t \\ & \alpha_{x,v} c_t \rightarrow & c_{xv} c_t \\ & \alpha_{x,v} c_t \rightarrow & c_{xv} c_t \\ c_x c_v c_t & \xrightarrow{\alpha_{x,v,t}} & c_x c_w c_{w'} \xrightarrow{\alpha_{x,w} c_{w'}} c_{xv} c_{z_1 \dots z_{q+1}} c_{w'} \end{array}$$

avec $x^{\times 1} v^{\times 2} t$, et de la forme $R_{\Gamma_3}(D_{x,v,t})$ où

$$\begin{array}{ccccc} & \alpha_{x,v} c_t \rightarrow & c_e c_{e'} c_t \xrightarrow{c_e \alpha_{e',t}} & c_e c_b c_{b'} \xrightarrow{\alpha_{e,b} c_{b'}} & c_a c_d c_{b'} \\ c_x c_v c_t & \xrightarrow{\quad} & c_e c_{e'} c_t & \xrightarrow{\quad} & c_a c_d c_{b'} \\ & \alpha_{x,v} c_t \rightarrow & c_e c_{e'} c_t & \xrightarrow{\quad} & c_a c_d c_{b'} \\ & \alpha_{x,v} c_t \rightarrow & c_e c_{e'} c_t & \xrightarrow{\quad} & c_a c_d c_{b'} \\ c_x c_v c_t & \xrightarrow{\alpha_{x,v,t}} & c_x c_w c_{w'} \xrightarrow{\alpha_{x,w} c_{w'}} & c_a c_{a'} c_{w'} \xrightarrow{c_a \alpha_{a',w'}} & c_a c_d c_{b'} \end{array}$$

avec $x^{\times 2} v^{\times 2} t$.

On obtient le résultat suivant

Théorème 4.2.3.2. *Pour $n > 0$, le monoïde plaxique \mathbf{P}_n admet $\text{PreCol}_3(n)$ comme présentation cohérente.*

Exemple: le cas du monoïde \mathbf{P}_2 . La présentation cohérente $\text{PreCol}_3(2)$ du monoïde plaxique \mathbf{P}_2 est donnée par l'ensemble des 1-cellules $\{c_1, c_2, c_{21}\}$, l'ensemble des 2-cellules

$$\{ \alpha_{2,1} : c_2 c_1 \Rightarrow c_{21}, \alpha_{1,21} : c_1 c_{21} \Rightarrow c_{21} c_1, \alpha_{2,21} : c_2 c_{21} \Rightarrow c_{21} c_1 \},$$

et la 3-cellule $C'_{2,1,21}$ suivante:

$$\begin{array}{ccc} & \alpha_{2,1} c_{21} \rightarrow & c_{21} c_{21} \\ c_2 c_1 c_{21} & \xrightarrow{\quad} & c_{21} c_{21} \\ & \alpha_{2,1} c_{21} \rightarrow & c_{21} c_{21} \\ & \alpha_{2,1} c_{21} \rightarrow & c_{21} c_{21} \\ c_2 c_1 c_{21} & \xrightarrow{\alpha_{2,1,21}} & c_2 c_{21} c_1 \xrightarrow{\alpha_{2,21} c_1} c_{21} c_2 c_1 \end{array}$$

Présentation cohérente de Knuth. La dernière étape de la procédure de réduction homotopique consiste à réduire la présentation cohérente $\text{PreCol}_3(n)$ en une présentation cohérente du monoïde \mathbf{P}_n qui étend la présentation de Knuth. On définit une présentation étendue $\text{Knuth}_3(n)$ du monoïde \mathbf{P}_n obtenue à partir de $\text{Knuth}_2(n)$ en ajoutant l'ensemble des 3-cellules suivantes

$$\{\mathcal{R}(C'_{x,v,t}) \mid x^{\times 1}v^{\times 2}t\} \cup \{\mathcal{R}(D_{x,v,t}) \mid x^{\times 2}v^{\times 2}t\},$$

où $\mathcal{R} : \overline{\text{Col}}_3(n)^\top \rightarrow \text{Knuth}_3(n)^\top$ est la transformation de Tietze construite dans la Section 4.2 du chapitre 4. Par conséquent, on obtient le résultat suivant

Théorème 4.2.4.3. *Pour $n > 0$, le monoïde plaxique \mathbf{P}_n admet $\text{Knuth}_3(n)$ comme présentation cohérente.*

Exemple: le cas du monoïde \mathbf{P}_2 . La présentation cohérente $\text{PreCol}_3(2)$ du monoïde \mathbf{P}_2 est réduite à la présentation cohérente dont les 1-cellules sont c_1 et c_2 , les 2-cellules sont $\eta_{1,1,2} : c_2c_1c_1 \Rightarrow c_1c_2c_1$ et $\varepsilon_{1,2,2} : c_2c_2c_1 \Rightarrow c_2c_1c_2$ et la 3-cellule est

$$\begin{array}{ccc} & \xrightarrow{2\eta_{1,1,2}} & \\ & \Downarrow C'' & \\ c_2c_2c_1c_1 & & c_2c_1c_2c_1 \\ & \xleftarrow{\varepsilon_{1,2,2}1} & \end{array}$$

Cette présentation coïncide avec la présentation cohérente $\text{Knuth}_3(2)$, voir sous-section 4.2.4.

Nombre des cellules des présentations cohérentes de \mathbf{P}_n . On note par $\mathcal{KB}(\text{Knuth}_2(n))$ la complétion de Knuth–Bendix de $\text{Knuth}_2(n)$ et par $\text{Knuth}_3^{\text{KB}}(n)$ la complétion de Squier de $\mathcal{KB}(\text{Knuth}_2(n))$. Le tableau suivant présente le nombre des cellules des présentations cohérentes $\text{Knuth}_3^{\text{KB}}(n)$, $\text{Knuth}_3(n)$, $\overline{\text{Col}}_3(n)$ et $\text{Col}_3(n)$ du monoïde \mathbf{P}_n , pour $1 \leq n \leq 10$.

n	$\text{Knuth}_1(n)$	$\text{Col}_1(n)$	$\text{Knuth}_2(n)$	$\mathcal{KB}(\text{Knuth}_2(n))$	$\text{Col}_2(n)$	$\text{Knuth}_3^{\text{KB}}(n)$	$\text{Knuth}_3(n)$	$\overline{\text{Col}}_3(n)$	$\text{Col}_3(n)$
1	1	1	0	0	0	0	0	0	0
2	2	3	2	2	3	1	1	1	1
3	3	7	8	11	22	27	24	34	42
4	4	15	20	∞	115	∞	242	330	621
5	5	31	40	∞	531	∞	1726	2225	6893
6	6	63	70	∞	2317	∞	10273	12635	67635
7	7	127	112	∞	9822	∞	55016	65282	623010
8	8	255	168	∞	40971	∞	275868	318708	5534197
9	9	511	240	∞	169255	∞	1324970	1500465	48052953
10	10	1023	330	∞	694837	∞	6178939	6892325	410881483

TRAVAUX EN COURS ET PERSPECTIVES

En utilisant les résultats et les techniques explorés dans cette thèse, on présente dans cette section, certains projets possibles qui pourraient orienter de futurs travaux de recherche.

Présentations cohérentes et actions sur les catégories. Un monoïde \mathbf{M} peut être vu comme une 2-catégorie avec exactement une 0-cellule \bullet , tel que les 1-cellules et les 2-cellules sont respectivement les éléments de \mathbf{M} et les identités. La catégorie des *actions de \mathbf{M} sur les catégories* est la catégorie $\text{Act}(\mathbf{M})$ des 2-représentations de \mathbf{M} dans la catégorie Cat des catégories. On note par $2\text{Cat}(\mathbf{M}, \text{Cat})$ la sous-catégorie pleine de $\text{Act}(\mathbf{M})$ dont les objets sont les 2-foncteurs. Nous renvoyons le lecteur à [31, Section 5.1] pour une introduction complète sur la catégorie des 2-représentations des 2-catégories. Plus précisément, une action A d'un monoïde \mathbf{M} est donnée par une catégorie $\mathbf{C} = A(\bullet)$, un endofoncteur $A(u) : \mathbf{C} \rightarrow \mathbf{C}$ pour tout u dans \mathbf{M} , un isomorphisme naturel $A_{u,v} : A(u)A(v) \Rightarrow A(uv)$ pour tous éléments u et v de \mathbf{M} , et un isomorphisme naturel $A_\bullet : 1_{\mathbf{C}} \Rightarrow A(1)$ telle que:

i) pour tout triplet (u, v, w) d'éléments du monoïde \mathbf{M} , le diagramme suivant commute

$$\begin{array}{ccc}
 & A_{u,v}A(w) \rightarrow A(uv)A(w) & \xrightarrow{A_{uv,w}} \\
 A(u)A(v)A(w) & & \xrightarrow{\quad} A(uvw) \\
 & \xrightarrow{A(u)A_{v,w}} A(u)A(vw) & \xleftarrow{A_{u,vw}}
 \end{array}
 =$$

ii) pour tout élément u du monoïde \mathbf{M} , les diagrammes suivants commutent

$$\begin{array}{ccc}
 A_\bullet A(u) \rightarrow A(1)A(u) & \xrightarrow{A_{1,u}} & \\
 \downarrow & \text{=} & \downarrow \\
 A(u) & \xrightarrow{\quad} & A(u)
 \end{array}
 \qquad
 \begin{array}{ccc}
 A(u)A_\bullet \rightarrow A(u)A(1) & \xrightarrow{A_{u,1}} & \\
 \downarrow & \text{=} & \downarrow \\
 A(u) & \xrightarrow{\quad} & A(u)
 \end{array}$$

Le résultat suivant relie les présentations cohérentes avec les actions des monoïdes sur les catégories. Soient \mathbf{M} un monoïde et Σ une présentation étendue de \mathbf{M} . Le $(3, 1)$ -polygraphe Σ est une présentation cohérente de \mathbf{M} si, et seulement si, pour toute 2-catégorie \mathcal{C} , il existe une équivalence de catégories

$$\text{Act}(\mathbf{M}) \approx 2\text{Cat}(\Sigma_1^*/\Sigma_2, \mathcal{C})$$

qui est naturelle dans \mathcal{C} , [31, Theorem 5.1.5]. De cette façon, les actions d'un monoïde \mathbf{M} sur les catégories sont à une équivalence près les 2-foncteurs de Σ_1^*/Σ_2 dans Cat .

En utilisant cette description, le Théorème 4.2.4.3 permet de présenter les actions des monoïdes plaxiques sur les catégories de la manière suivante. La catégorie $\text{Act}(\mathbf{P}_n)$ des actions du monoïde plaxique \mathbf{P}_n sur les catégories est équivalente à la catégorie des 2-foncteurs de

la $(2, 1)$ -catégorie $\text{Knuth}_2(\mathfrak{n})^\top$ dans la catégorie Cat des catégories, qui envoie les 3-cellules de $\text{Knuth}_3(\mathfrak{n})$ à des diagrammes commutatifs dans Cat . Une question intéressante est d'étudier les actions du monoïde plaxique sur la catégorie des représentations de dimension finie de l'algèbre de Lie générale ou sur la catégorie \mathcal{O} dont les objets sont certaines représentations de l'algèbre de Lie générale et les morphismes sont des homomorphismes de représentations, [50].

Une implementation de la présentation cohérente de Knuth. Comme il est décrit dans le chapitre 4, section 4.2, on construit une présentation cohérente finie du monoïde plaxique \mathbf{P}_n , qui étend sa présentation de Knuth. En particulier, on montre que ses 3-cellules sont de la forme

$$\{\mathcal{R}(C'_{x,v,t}) \mid x^1 v^{x^2} t\} \cup \{\mathcal{R}(D_{x,v,t}) \mid x^{x^2} v^{x^2} t\},$$

où \mathcal{R} est une transformation de Tietze construite au cours de plusieurs étapes de réduction. On a implémenté un programme en Python donnant le nombre des cellules de la présentation cohérente de Knuth comme présenté dans le tableau ci-dessus. Ce programme consiste à calculer le nombre des sources $s_1(\mathcal{X}_{u,v,t})$ des 3-cellules $\mathcal{X}_{u,v,t}$ de cette présentation.

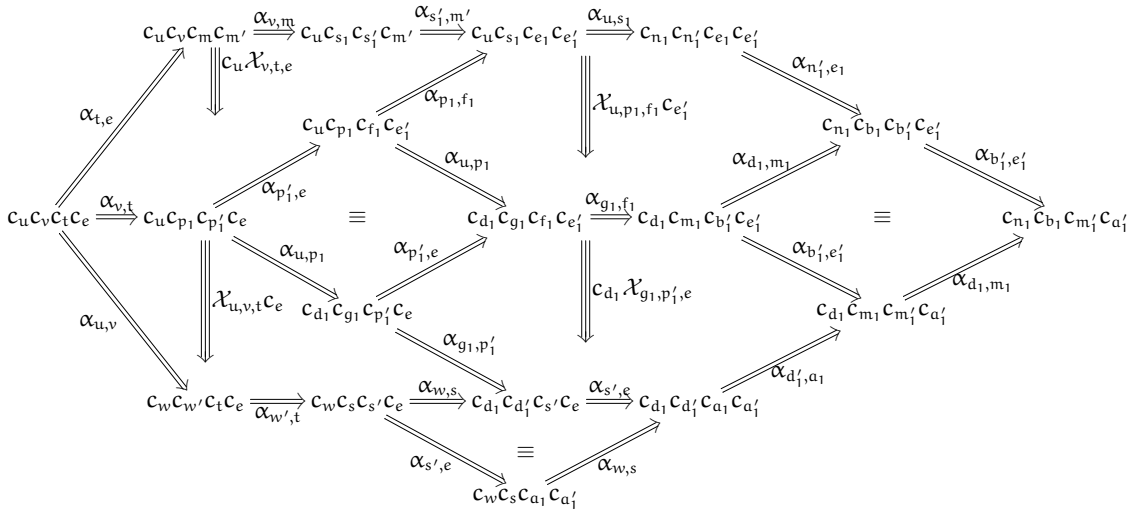
Notons que notre construction permet de donner les formes explicites de ces 3-cellules. Notons aussi qu'après l'application de la transformation de Tietze \mathcal{R} , aucune 3-cellule ne peut se transformer en identité. Nous implémenterions un programme donnant les formes explicites des 3-cellules de la présentation cohérente de Knuth pour les rangs supérieurs. Par exemple, la présentation cohérente de Knuth \mathbf{P}_8 contient 275868 3-cellules. Afin d'obtenir les formes de ces 3-cellules, le programme devrait donner les formes explicites des 3-cellules des présentations $\text{Col}_3(8)$, $\overline{\text{Col}}_3(8)$ et $\text{PreCol}_3(8)$ construites durant l'application de la transformation \mathcal{R} .

Les résolutions d'Anick des monoïdes plaxiques via les chemins de Littelmann. Anick a construit une résolution d'un corps \mathbb{K} vu comme un G -module, où G est une algèbre associative augmentée sur \mathbb{K} , [1, 2]. Il a défini un ensemble de n -chaînes en utilisant la notion des bases de Gröbner–Shirshov. De plus, il a donné une définition inductive des *différentielles* en utilisant une homotopie contractante définie d'une manière inductive. Par la suite, la résolution d'Anick a été généralisée aux cas des opérades et des catégories, [93, 23]. Par ailleurs, la théorie algébrique de Morse discrète introduite indépendamment dans [54, 111] décrit explicitement les différentielles de la résolution d'Anick en utilisant les graphes orientés. Une telle résolution permet de calculer les invariants homologiques du monoïde étudié.

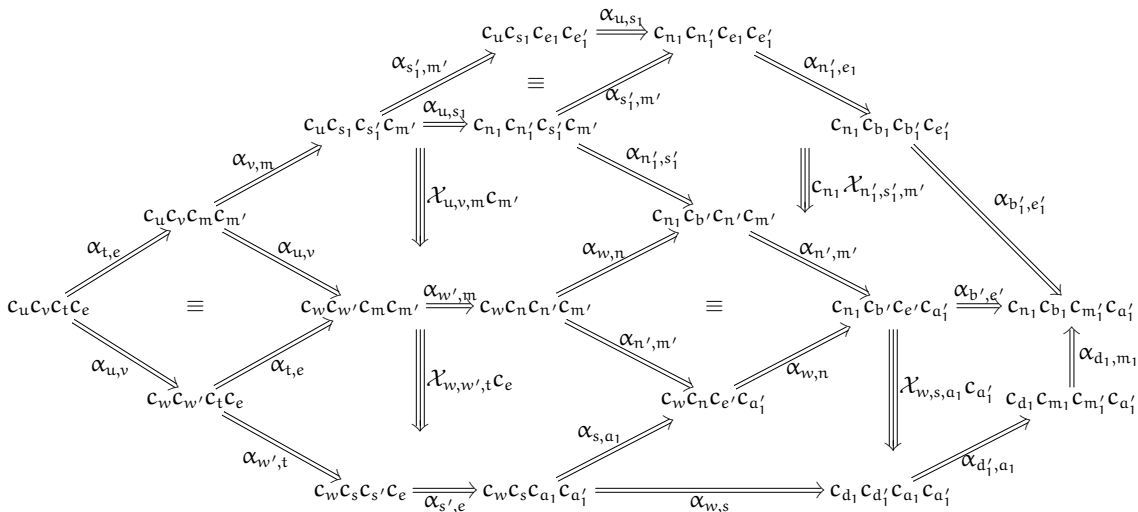
Lopatin a construit une résolution d'Anick du monoïde plaxique \mathbf{P}_n à partir de sa présentation colonne $\text{Col}_2(\mathfrak{n})$ en utilisant la théorie de Morse, [90]. Par conséquent, il a calculé les invariants homologiques du monoïde \mathbf{P}_n . En utilisant la même stratégie, nous devrions ainsi obtenir des résolutions d'Anick pour les monoïdes plaxiques généralisés d'une manière générale à partir de leur présentation colonne avec les chemins L-S. Par conséquent, les invariants homologiques de ces monoïdes pourraient être calculés.

Les syzygies de dimension supérieure pour le monoïde plaxique. Dans [37], Malbos et Guiraud ont montré comment étendre une présentation convergente d'un monoïde en une résolu-

tion polygraphique de ce monoïde. La présentation colonne $\text{Col}_2(n)$ du monoïde \mathbf{P}_n peut alors être étendue en une résolution polygraphique dont les n -cellules, pour tout $n \geq 3$, sont indexées par des $(n - 1)$ branchements de $\text{Col}_2(n)$. On peut expliciter les 4-cellules de cette résolution, qui correspondent aux diagrammes de confluence induits par des branchements triple critiques. C'est-à-dire, pour toutes colonnes u, v, t et e dans $\text{col}(n)$ telles que $u \times v, v \times t$ et $t \times e$, il existe un branchement critique de source $c_u c_v c_t c_e$. En utilisant les mêmes arguments que ceux de la section 4.2.2, on peut montrer que le diagramme de confluence induit par ce branchement triple est représenté par la 3-sphère $\Omega_{u,v,t,e}$ dont la source est la 3-cellule



et le but est la 3-cellule



Dans cette 3-sphère, certaines colonnes peuvent être triviales et alors les 2-cellules α correspondantes sont les identités. Pour faciliter la lecture du diagramme, on a omis le contexte des 2-cellules α .

Plus généralement, nous prévoyons que la n -cellule génératrice de cette résolution aurait la forme du permutohedron de dimension n .

Le monoïde chinois. Le *monoïde chinois* de rang n est le quotient du monoïde libre $[n]^*$ par la relation de congruence engendrée par la famille de relations suivante

$$zyx = zxy = yzx, \text{ pour } x \leq y \leq z. \quad (5)$$

Ce monoïde apparaît dans la classification des *monoïdes de classes* dont la fonction de croissance coïncide avec le monoïde plaxique, [26]. Comme les tableaux de Young jouent un rôle important dans la structure des monoïdes plaxiques, une notion similaire de *tableaux chinois* a été introduite pour le monoïde chinois, [19, Section 2.1]. Un algorithme d'insertion similaire à l'algorithme de Schensted a été aussi introduit donnant lieu à un théorème de cross-section, [19, Sections 2.2]. De plus, une correspondance de Robinson–Schensted a été aussi établie, [19, Sections 5.2]. Récemment, le monoïde chinois a été étudié par des méthodes de réécriture en lui construisant des présentations convergentes finies, [20, 41, 42, 18] et par des méthodes de normalisations quadratiques, [34, Example 5.3.2].

Un futur travail serait de construire une présentation cohérente du monoïde chinois à partir d'une présentation convergente finie. Grâce à la procédure de complétion homotopique, on devrait obtenir un théorème similaire au Théorème 4.1.3.1 pour les monoïdes chinois en prenant en considération que dans ce cas on a besoin d' au plus cinq étapes de réécriture pour obtenir la forme normale alors que dans le cas du monoïde plaxique elle est atteinte après au plus trois étapes. De plus, une telle présentation cohérente pourrait être réduite en utilisant la même stratégie utilisée pour le monoïde plaxique grâce à la procédure de réduction homotopique. Les présentations cohérentes des monoïdes chinois pourraient nous permettre de calculer leurs invariants homologiques et de décrire leurs actions sur les catégories.

Comme présenté dans ce travail, les graphes cristallins et les opérateurs de Kashiwara interagissent avec les tableaux et leurs généralisations et ils fournissent des outils combinatoires pour étudier les monoïdes plaxiques. Un futur travail serait de construire une structure cristalline du monoïde chinois. Les éléments du nouveau graphe cristallin seraient identifiés avec les tableaux chinois et la congruence engendrée par la famille des relations (10) devrait avoir une interprétation en termes d'un isomorphisme de cristaux. Une question est d'étudier l'interaction du graphe cristallin du monoïde plaxique de type A avec celui construit pour le monoïde chinois.

Le monoïde sylvestre. La structure des monoïdes sylvestres apparaît dans l'étude combinatoire de l'algèbre de Loday-Ronco des arbres binaires planaires reliée aux fonctions symétriques non-commutatives et aux fonctions symétriques libres, [46]. Le monoïde sylvestre est construit en utilisant la notion des *arbres binaires de recherche* et en introduisant un algorithme d'insertion similaire à l'insertion de Schensted.

Le *monoïde sylvestre de rang* n est le quotient du monoïde libre $[n]^*$ par la relation de

congruence engendrée par la famille des relations suivantes:

$$cavb = acvb, \text{ tel que } a \leq b < c,$$

où v est un mot dans $[n]^*$. La convergence de la présentation infinie introduite par Hivert, Novelli et Thibon, [46], est montrée par Cain, Gray et Malheiro, [18]. Construire une présentation convergente finie du monoïde sylvestre est considéré comme un problème ouvert. En utilisant une approche similaire à celle utilisée pour la construction de la présentation colonne du monoïde plaxique, on devrait construire une présentation convergente finie du monoïde sylvestre en ajoutant de nouvelles familles d'arbres binaires de recherche. À partir d'une présentation convergente finie du monoïde sylvestre, on devrait aussi donner une méthode algorithmique pour calculer les 2-syzygies de la présentation introduite par Hivert, Novelli et Thibon en construisant des présentations cohérentes de ce monoïde.

ORGANISATION DU DOCUMENT

Le document s'organise de la façon suivante. Le chapitre 1 présente une introduction des monoïdes plaxiques en utilisant les notions de tableaux de Young [124, 28], de graphes cristallins de Kashiwara, [57, 58, 59, 61] et du modèle de chemins de Littelmann [85, 86, 87, 88]. Nous rappelons dans la section 1.1 les définitions et les propriétés des algèbres de Lie semi-simples complexes de dimensions finies. La section 1.2 est consacrée à la définition et aux propriétés du monoïde plaxique de type A. Le matériel de la section 1.3 illustre le lien entre les graphes cristallins de Kashiwara et les monoïdes plaxiques. Nous terminons ce chapitre avec la section 1.4 sur le lien du modèle de chemins de Littelmann avec les monoïdes plaxiques.

On construit dans le chapitre 2 des présentations de monoïdes plaxiques. Après avoir rappelé la définition des 2-polygraphes comme des présentations de monoïdes, nous construisons dans 2.1.3 une présentation convergente finie du monoïde plaxique de type A en utilisant les notions de tableaux de Yamanouchi, des graphes cristallins et des chemins. Nous rappelons dans 2.2 la définition des Tietze transformations entre les 2-polygraphes. Nous construisons dans 2.2.2 une présentation du monoïde plaxique de type A qui est Tietze équivalente à la présentation de Knuth. Nous terminons ce chapitre en rappelant dans 2.3.1 la complétion de Knuth–Bendix qui calcule une présentation convergente d'un monoïde à partir d'une présentation terminante. Nous montrons que la présentation de Knuth du monoïde plaxique de rang 3 admet une complétion finie après l'ajout de 3 relations et qu'à partir du rang 4 cette présentation n'admet pas de complétion finie compatible avec l'ordre lexicographique.

L'objectif du chapitre 3 est de construire des présentations convergentes finies des monoïdes plaxiques. On construit dans la section 3.1 une présentation convergente finie des monoïdes plaxiques généralisés en utilisant les chemins de Lakshmibai–Sehadri introduits par Littelmann. Cette présentation, appelée *présentation colonne*, peut être aussi construite au cas par cas, en

utilisant la théorie des bases cristallines de Kashiwara. Dans la section 3.2, on considère la présentation colonne du monoïde plaxique de type A. On montre comment une telle présentation est construite en utilisant les tableaux de Young et les algorithmes d'insertion de Schensted. On construit dans la section 3.3 la présentation colonne du monoïde plaxique de type C en utilisant les notions de colonnes admissibles et tableaux symplectiques introduits dans 3.3.1. Les règles de réécriture de cette présentation sont définies en utilisant l'algorithme d'insertion de Lecouvey présenté dans 3.3.2. Nous terminons ce chapitre en montrant dans 3.4 comment la présentation colonne peut être construite pour les types B, D et G_2 .

Nous montrons dans le chapitre 4 comment étendre la présentation de Knuth du monoïde plaxique de type A en une présentation cohérente finie. Après avoir rappelé les notions de présentations cohérentes dans 4.1.1 et de procédure de complétion homotopique dans 4.1.2, nous construisons dans 4.1.3 une présentation cohérente du monoïde plaxique de type A à partir de sa présentation colonne. Cette présentation n'est pas minimale. Nous rappelons dans 4.2.1 la procédure de réduction homotopique qui consiste à éliminer des générateurs superflus d'une présentation cohérente. Après plusieurs étapes de réduction présentées dans 4.2.2, 4.2.3 et 4.2.4, on obtient une présentation cohérente finie du monoïde plaxique de type A avec les générateurs de Knuth.

General introduction

This thesis focuses on the study of plactic monoids by new methods issued from rewriting theory. Our approach is to investigate the presentations of plactic monoids from the rewriting theory perspective and in particular to compute a family of generators for all syzygies for these presentations. In order to achieve our goal, we have to orient the presentations of plactic monoids, then a study of the relations among the relations in these monoids is done. Such a study is carried out due to a homotopical completion-reduction procedure introduced in [31] using methods based on Squier's and Knuth–Bendix's completions. In this manner, our work constitutes an interplay between algebra and rewriting theory.

Plactic monoids admit many descriptions. In particular, they are related to the representations of finite-dimensional complex semisimple Lie algebras. We study their interpretations in terms of Young tableaux, Kashiwara's crystal bases and Littelmann path model. The structure of plactic monoids was introduced by Lascoux and Schützenberger [74] after the works of Schensted [104] and Knuth [63] on the combinatorial study of Young tableaux. In type A, the plactic monoid is related to the representations of the Lie algebra of square matrices, and is now called the plactic monoid of type A, [21, 72, 77]. Moreover, the classification of finite-dimensional complex semisimple Lie algebras in different types leads to the existence of generalised plactic monoids of the same types. There are two approaches to define these monoids. Indeed, they can be defined by a case-by-case analysis, using Kashiwara's theory of crystal bases [83] or in a general way thanks to Littelmann path model [88].

The *syzygy problem* consists in finding all independent irreducible algebraic relations among the generators of a module over a ring, [27]. The syzygy problem was studied in the point of view of rewriting theory by finding all the relations between the relations for presentations of monoids. A *2-syzygy* for a presentation of a monoid is a relation amongst relations. The syzygy problem for presentations of monoids can be algorithmically solved thanks to *convergent rewriting systems*. *Rewriting systems* are oriented presentations of monoids consisting of a generating set and *rewriting rules* relating words over this generating set. A rewriting system is *convergent* if it does not admit an infinite sequence of rewritings and it satisfies the confluence

property. The *confluence property* states that for every words u , u' and u'' such that u rewrites into u' and u'' , there exists a word v such that u' and u'' rewrite into v . This means that in a convergent rewriting system any rewritings starting on a same word lead to a unique word that cannot be rewritten into another one.

Lately, plactic monoids were investigated by rewriting methods, [65, 7, 17, 43, 16, 44]. In our study of plactic monoids, we use the structure of *polygraphs* based on a categorical interpretation of rewriting systems, where the generators and the rewriting rules are represented by 1-cells and 2-cells constructed over a unique 0-cell. We give a description of 2-szygies of some presentations of the plactic monoid of type A using the notion of coherent presentations. A *coherent presentation* extends the notion of a presentation of a monoid by 3-cells in a coherent way, taking into account the relations among the relations. The computation of coherent presentations for a monoid starting by a convergent one provides a method to compute a *polygraphic resolution* of the monoid, that is, a categorical cofibrant replacement of the monoid in a free $(\infty, 1)$ -category whose cells of dimension 2 and higher are invertible, [37]. Indeed, a coherent presentation for a monoid constitutes the first two steps in the computation of a polygraphic resolution. In addition, such a study of the relations among the relations in a monoid allows to compute its *homological invariants*. Moreover, coherent presentations are also useful to describe the notion of an *action of the monoid on categories*, [31].

We compute a coherent presentation of the plactic monoid of type A using the *homotopical completion procedure* introduced in [31, 40]. Such a procedure extends the Knuth–Bendix’s completion [64] using a method introduced by Squier that computes a coherent presentation from a convergent one, [112, 31]. The obtained coherent presentation of the plactic monoid of type A is not minimal in the sense that many of its generators are superfluous. After several steps of *homotopical reduction*, we reduce it to a smaller one having Knuth’s generators. In this way, we give an algorithmic method to compute 2-szygies for the Knuth presentation of the plactic monoid of type A.

In order to extend the latter result to plactic monoids of any type, we construct finite convergent presentations for these monoids. Thanks to Littelmann paths, we construct finite convergent presentations of plactic monoids in a general way covering all the types. Moreover, we study the latter presentations in terms of Kashiwara’s crystal graphs for type C, using combinatorial ingredients such as the *admissible columns* introduced by Kashiwara and Nakashima, [61].

THE PLACTIC MONOID AND ITS GENERALISATIONS

In this section, we will give an historical overview of plactic monoids. We recall the definition of the plactic monoid of type A and we present the connection between this monoid, Kashiwara crystal graphs and Littelmann path model. A generalisation of the plactic monoid is then discussed in terms of crystals and Littelmann paths.

The plactic monoid of type A and tableaux. In order to compute the length of the longest non-decreasing subsequence of a given word in the free monoid $[n]^*$ over the ordered set

$$[n] := \{ 1 < \dots < n \},$$

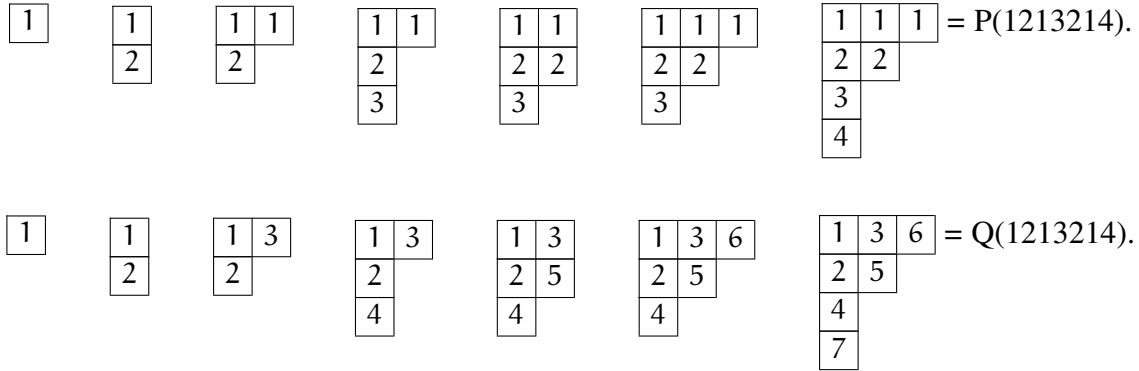
Schensted introduced combinatorial algorithms that deal with Young tableaux. The latter objects were introduced by Young, [124], and they were used for the first time to investigate representations of the symmetric group by Frobenius [28]. Schensted's algorithms were originally found by Robinson, [103], who tried to give a proof of the Littelwood–Richardson rule. This rule describes in a combinatorial way the multiplicity of a Schur polynomial in a product of Schur polynomials, that is, the multiplicity of an irreducible representation of the general Lie algebra in a tensor product of two irreducible representations, see [123] and [114, Chapter 7]. Before presenting Schensted's algorithms, let us introduce the notions of tableaux and standard tableaux. A *tableau* of shape $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{N}^k$, with $n \geq \lambda_1 \geq \dots \geq \lambda_k \geq 1$, is a collection of boxes in a left-justified rows such that the i th row contains λ_i boxes, for $i = 1, \dots, k$, and filled by elements of $[n]$ where the entries weakly increase along each row and strictly increase down each column. A *standard tableau* of shape λ , is a tableau of shape λ where the entries are the numbers from 1 to $\sum_{i=1}^k \lambda_i$, each occurring once. For instance, a tableau of shape $(5, 3, 1)$ and a standard tableau of shape $(4, 3, 2)$ are respectively

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 & & \\ \hline 3 & & & & \\ \hline \end{array}
 \quad \text{and} \quad
 \begin{array}{|c|c|c|c|} \hline 1 & 3 & 6 & 7 \\ \hline 2 & 5 & 8 & \\ \hline 4 & 9 & & \\ \hline \end{array}
 .$$

Consider a tableau T and an element x in the set $[n]$. Schensted's algorithm, called the *column-insertion*, inserts the element x into the tableau T as follows. Let y be the smallest element of the first column of the tableau T that is greater or equal to x . Then x replaces y in the first column and y is bumped into the next column where the process is repeated. This procedure terminates when the element which is bumped is greater than all the elements of the next column. Then it is placed at the bottom of that column. Note that a similar algorithm, called the *row-insertion*, was also introduced by Schensted. It consists in inserting elements of $[n]$ in the rows of a tableau instead of its columns, see Chapter 1, Section 1.2.1.

For every word w in the free monoid $[n]^*$, one computes a unique tableau $P(w)$, called *Schensted's tableau*, by starting with the empty word and iteratively applying Schensted's algorithm to

the letters of w . During the computation of the tableau $P(w)$, a standard tableau $Q(w)$, [104, Lemma 2], is obtained by successively putting i in a box at the same place as the added box when inserting the letter x_i of w . The bijection $w \mapsto (P(w), Q(w))$, [104, Lemma 3], is called the *Robinson–Schensted correspondence*. For instance, the successive steps of the computation of the tableau $P(1213214)$ and the standard tableau $Q(1213214)$ are



Furthermore, one can define an equivalence \sim_{plax} relating words in the free monoid $[n]^*$ yielding the same Schensted’s tableau. That is,

$$u \sim_{\text{plax}} v \quad \text{if and only if} \quad P(u) = P(v),$$

for any words u and v in $[n]^*$. Moreover, Knuth showed that the equivalence \sim_{plax} coincides with the congruence generated by the following family of *Knuth relations*, [63, Theorem 6]:

$$\{ yzx = yxz, \text{ for } x \leq y < z \} \cup \{ xzy = zxy, \text{ for } x < y \leq z \}. \quad (6)$$

The *plactic monoid of rank n* , denoted by \mathbf{P}_n , is defined as the quotient of the free monoid $[n]^*$ by the equivalence \sim_{plax} , [74].

Since its introduction, the plactic monoid has found several applications in algebraic combinatorics and representation theory [72, 106, 29]. In particular, the plactic monoid was used by Lascoux and Schützenberger in order to give a combinatorial description of the Kostka–Foulkes polynomials, [73, 74], and a noncommutative version of the Schubert polynomials, [75, 76]. Furthermore, the plactic monoid was used by Schützenberger, [105], in order to provide the first correct proof of the Littelwood–Richardson rule. Note that another proof of this rule was also independently found by Thomas, [119].

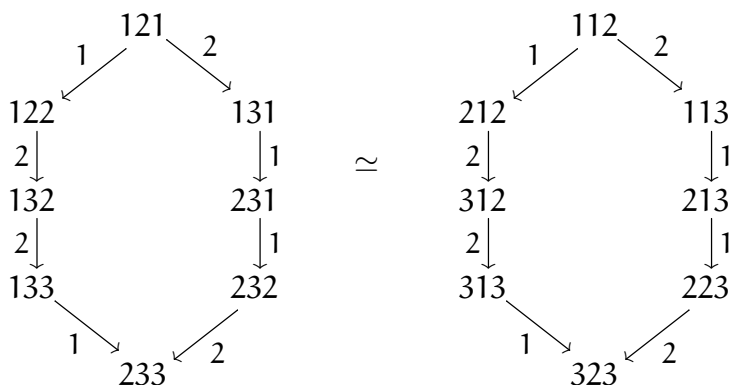
Later, many similar monoids were introduced like the chinese monoid, [19], the shifted plactic monoid, [107], the hypoplactic monoid, [99], the sylvester monoid, [46], and the super plactic monoid, [66, 89]. These monoids have also many applications in combinatorics and representation theory.

Generalisations of the plactic monoid via crystals graphs. Since 1990, there was a very useful relationship between the Robinson–Schensted correspondence and Kashiwara’s theory of crystal bases. Before talking about this relationship, let us introduce the context of the representation theory of Lie algebras. A representation of a Lie algebra is a way to describe this algebra as an algebra of matrices where the Lie bracket is given by the commutator. Since their introduction, the representations of Lie algebras have found interesting applications in several areas of mathematics and physics. In particular, some representations of semisimple Lie algebras encompass all the fundamental particles in physics which are described by the gauge theory paradigm. Moreover, using Dynkin diagrams, the finite complex dimensional semisimple Lie algebras are classified in classical types A, B, C and D and in exceptional ones E_6 , E_7 , E_8 , F_4 and G_2 . The type A, B, C and D corresponds respectively to the *special linear Lie algebra* $\mathfrak{sl}_{n+1}(\mathbb{C})$ (see Chapter 1, Subsection 1.3.2), the *odd-dimensional special orthogonal Lie algebra* $\mathfrak{so}_{2n+1}(\mathbb{C})$, the *symplectic Lie algebra* $\mathfrak{sp}_{2n}(\mathbb{C})$ (see Chapter 1, Subsection 1.3.3), and the *even-dimensional special orthogonal Lie algebra* $\mathfrak{so}_{2n}(\mathbb{C})$, see [49, 30, 10, 47].

In order to study solutions of the classical Yang–Baxter equations that are related to statistical mechanics, Jimbo [53] and Drinfeld [25] introduced independently in 1985 the notion of quantum groups. The *quantum group* $U_q(\mathfrak{g})$ associated to a semisimple Lie algebra \mathfrak{g} is a deformation of the universal enveloping algebra of \mathfrak{g} . It is an associative algebra over the field $\mathbb{C}(q)$ of rational functions with complex coefficients. When q tends to zero, Kashiwara showed that the representations of $U_q(\mathfrak{g})$ have special bases, which he called *crystal bases*, [57, 58, 59]. These bases can be extended to the whole q -space to obtain true bases of the representations, which are called *global bases*. Note that global bases coincide with canonical bases which were independently introduced for quantum groups by Lusztig [92]. A crystal basis was also given a structure of a labelled oriented graph, called the *crystal graph*, with arrows defined by the *Kashiwara operators*, see Chapter 1, Section 1.3. A crystal graph can also be decomposed into connected components and a *crystal isomorphism* between two connected components is a weight-preserving labelled graph isomorphism from one to the other. Since then, the theory of crystal bases has been a very useful combinatoric tool to study the representations of quantum groups and of semisimple Lie algebras.

Using the theory of quantum groups, the first connection between the Robinson–Schensted correspondence and the representations of the general Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ of n by n matrices was obtained by Date, Jimbo and Miwa, [21]. In the same spirit, the same connection was obtained by Lascoux, Leclerc and Thibon using Kashiwara’s theory of crystal bases, [72, 77]. More specifically, consider the *vector representation* $V_n = \mathbb{C}^n$ of the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$. The standard basis of V_n is indexed by the ordered set $[n]$. Each vertex $x_1 \otimes \dots \otimes x_l$ of the crystal graph of the representation $\bigoplus_{\downarrow} V_n^{\otimes l}$ is considered as a word $x_1 \dots x_l$ on the free monoid $[n]^*$. For any words u and v in the free monoid $[n]^*$, we have $P(u) = P(v)$ if, and only if, u and v have the same place in their two isomorphic connected components of the crystal graph of $\bigoplus_{\downarrow} V_n^{\otimes l}$. Moreover, we have $Q(u) = Q(v)$ if, and only if, u and v occur in the same connected component

of the crystal graph of $\bigoplus_{\downarrow} V_n^{\otimes l}$. As a consequence, we obtain that for any words u and v in the free monoid $[n]^*$, $u \sim_{\text{plax}} v$ if, and only if, the words u and v appear in the same place in their isomorphic connected components of the crystal graph of the representation $\bigoplus_{\downarrow} V_n^{\otimes l}$. The plactic monoid \mathbf{P}_n related to the representations of the general Lie algebra $\mathfrak{gl}_n(\mathbb{C})$, which is of type A, is known as the plactic monoid of type A. For instance, the left and right sides of the Knuth relations (6) correspond to the vertices which correspond under the following crystal isomorphism



Since Kashiwara's theory of crystal bases also exists for classical semisimple Lie algebras and for the exceptional one of type G_2 , a plactic monoid was introduced for each of these algebras using a case-by-case analysis, [78, 4, 79, 83]. To each semisimple Lie algebra, one associates a finite alphabet S indexing a basis of the vector representation V of the algebra. Two words u and v in the free monoid S^* are *plactic congruent*, and denoted by $u \sim_{\text{crys}} v$, if they appear in the same place in their isomorphic connected components of the crystal graph of the representation $\bigoplus_{\downarrow} V^{\otimes l}$. For each semisimple Lie algebra, the corresponding *plactic monoid* is defined as the quotient of the free monoid S^* by the congruence \sim_{crys} .

The plactic monoid of type C, B and D corresponds respectively to the representations of the symplectic Lie algebra, the odd-dimensional orthogonal Lie algebra and the even-dimensional orthogonal Lie algebra. Lecouvey in [78] and Baker in [4] introduced independently the plactic monoid of type C using Kashiwara's theory of crystal bases and the notion of *admissible columns* introduced by Kashiwara and Nakashima, [61]. In [79], Lecouvey gave presentations for plactic monoids of types B and D using the notion of admissible columns generalised to these types. Lecouvey also introduced the plactic monoid for type G_2 by generalising the notion of admissible columns to this type, [83]. As an application of the presentations of plactic monoids, Lecouvey gave a combinatorial description of certain Kostka-Foulkes polynomials for type B, C and D which arise as entries of the character table of the finite reductive groups, [82].

Plactic monoids via Littelmann paths. Littelmann path model establishes a bridge between Lakshmibai–Sechadri's theory of standard monomials [70, 71] and Kashiwara's theory of crys-

tal bases [57, 58, 59]. Littelmann gave a unified combinatorial model to all symmetrizable Kac–Moody algebras that allows him to obtain explicit combinatorial formulas for *weight multiplicities*, *tensor product rules* and *branching rules*, [85, 87]. He introduced piecewise-linear *paths* connecting the origin to a weight and a pair of *root operators* for each simple root of a symmetrizable Kac–Moody algebra. Littelmann’s paths can be encoded in a labelled directed graph, with labels given by the roots operators. Moreover, the latter graph defined by Littelmann’s root operators is isomorphic to Kashiwara’s crystal graph [55, 60].

As we have discussed previously, using a case-by-case analysis, plactic monoids can be defined for all classical semisimple Lie algebras and for the exceptional one of type G_2 . Using his path model, Littelmann defined in a general way a plactic algebra for any semisimple Lie algebra, [88]. As a consequence, he gave some presentations by generators and relations of the plactic algebra of types A, B, C, D and G_2 , [88, Theorem C]. Littelmann also introduced the notion of *standard tableaux* as a generalisation of the notion of tableaux defined for type A to any type. Standard tableaux coincide also with the notion of symplectic tableaux for type C and orthogonal tableaux for types B and D in the Lecouvey sense, [83].

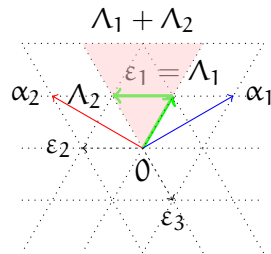
Let \mathfrak{g} be a semisimple Lie algebra and let P be its *weight lattice*. A *path* is a piecewise linear continuous map $\pi : [0, 1] \rightarrow P \otimes_{\mathbb{Z}} \mathbb{R}$. We denote by

$$\Pi = \{ \pi : [0, 1] \rightarrow P \otimes_{\mathbb{Z}} \mathbb{R} \mid \pi(0) = 0 \text{ and } \pi(1) \in P \}$$

the set of all paths with source 0 and whose target belongs to P up to reparametrization. Given two paths π_1 and π_2 in Π , the concatenation $\pi_1 \star \pi_2$ is defined by:

$$\pi_1 \star \pi_2(t) := \begin{cases} \pi_1(2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ \pi_1(1) + \pi_2(2t - 1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

For instance, consider $W = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0 \}$ and let $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ be the canonical basis of \mathbb{R}^3 . We still denote by ε_i the projection of ε_i onto W . We have $P = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$ and $\varepsilon_3 = -\varepsilon_1 - \varepsilon_2$. The weights $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = \varepsilon_2 - \varepsilon_3$ are called the *simple roots* of the special linear algebra \mathfrak{sl}_3 . The weights $\Lambda_1 = \varepsilon_1$ and $\Lambda_2 = \varepsilon_1 + \varepsilon_2$ are called its *fundamental weights*. An example of *dominant weight* of \mathfrak{sl}_3 is $\Lambda_1 + \Lambda_2$. Consider the paths $\pi_1 : t \mapsto t\varepsilon_1$ and $\pi_2 : t \mapsto t\varepsilon_2$, the path $\pi_1 \star \pi_2$ is the green path and the *dominant chamber* is the hatched area in the following figure:



For each simple root α of the Lie algebra \mathfrak{g} , Littelmann defined the *root operators* e_α and f_α from Π to $\Pi \cup \{0\}$ as follows. Every path π in Π is cutted into three parts $\pi_1 \star \pi_2 \star \pi_3$. Then the new path $e_\alpha(\pi)$ or $f_\alpha(\pi)$ is either equal to 0 or $\pi_1 \star s_\alpha(\pi_2) \star \pi_3$, where s_α denotes the orthogonal reflection with respect to the root α , see Chapter 1, Section 1.4.1.

Denote by $\mathbb{Z}\Pi$ the *algebra of paths* defined as the free \mathbb{Z} -module with basis Π whose product is given by the concatenation of paths and where the unity is the path with source and target 0. Let \mathcal{A} be the subalgebra of $\text{End}_{\mathbb{Z}}(\mathbb{Z}\Pi)$ generated by the root operators f_α and e_α . Define Π^+ to be the set of paths π such that the image is contained in the dominant chamber and denote by M_π the \mathcal{A} -module $\mathcal{A}\pi$. Note that two \mathcal{A} -modules are isomorphic if their paths contained in the dominant chamber have the same target.

Let $\mathbb{Z}\Pi_0$ be the \mathcal{A} -submodule $\mathcal{A}\Pi^+$ of $\mathbb{Z}\Pi$ generated by the paths in Π^+ . For two paths π_1 and π_2 in $\mathbb{Z}\Pi_0$, denote by π_1^+ and π_2^+ the unique paths in Π^+ such that π_1 is in $M_{\pi_1^+}$ and π_2 is in $M_{\pi_2^+}$. One can define a relation \sim_{path} on $\mathbb{Z}\Pi_0$ by : $\pi_1 \sim_{\text{path}} \pi_2$ if, and only if, $\pi_1^+(1) = \pi_2^+(1)$ and $\psi(\pi_1) = \pi_2$ under the isomorphism $\psi : M_{\pi_1^+} \rightarrow M_{\pi_2^+}$. For each semisimple Lie algebra \mathfrak{g} , the corresponding *plactic monoid* is defined as the quotient of the \mathcal{A} -submodule $\mathbb{Z}\Pi_0$ by the equivalence \sim_{path} .

It is worth mentioning that the equivalences \sim_{path} and \sim_{crys} coincide in the sense that we obtain the same plactic monoids of types A, B, C, D and G_2 if we treat them by a case-by-case analysis, see [83] and [88, Theorem C].

CONVERGENT PRESENTATIONS OF PLACTIC MONOIDS

Rewriting systems and the word problem. *Rewriting systems* are oriented presentations of monoids consisting of a generating set Σ_1 and a set Σ_2 of *rewriting rules* relating words over this generating set. The notion of *rewriting systems* was introduced by Thue in his study of the word problem for finitely presented monoids, [120]. The *word problem for a monoid \mathbf{M}* consists in finding a generating set Σ_1 and a procedure deciding if two elements of the free monoid Σ_1^* represent the same element in the monoid \mathbf{M} . The word problem was also described for finitely presented groups by Dehn [22]. Much later, Post [100] and Markov [94, 95] showed independently that the word problem is undecidable. Thereafter, the word problem has been considered in many contexts in algebra and computer science. Rewriting theory appeared also in different forms in algebra according to the object to be presented. It appears for commutative algebras and Lie algebras with the notion of Gröbner-Shirshov bases, [12, 13, 110], for associative algebras and operads, [8, 6, 97, 122, 24] as well as for terms in an algebraic theory, [3, 62, 118], strings in a monoid, [9].

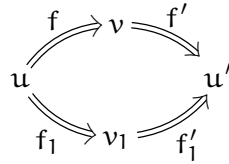
In this work, a *presentation* (Σ_1, Σ_2) of a monoid \mathbf{M} is a rewriting system such that the monoid \mathbf{M} is isomorphic to the quotient of the free monoid Σ_1^* by the congruence generated by Σ_2 . In the literature, a presentation of a monoid is called a *string rewriting system* or a *semi-Thue system*, [3].

For instance, the plactic monoid \mathbf{P}_n of rank n is presented by the rewriting system $\text{Knuth}_2(n)$, whose generating set is $\{1 < \dots < n\}$ and the set of rewriting rules is

$$\{ zxy \xrightarrow{\eta_{x,y,z}} xzy \mid 1 \leq x \leq y < z \leq n \} \cup \{ yzx \xrightarrow{\varepsilon_{x,y,z}} yxz \mid 1 \leq x < y \leq z \leq n \}.$$

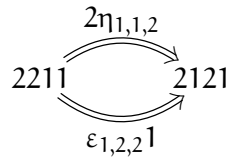
The rewriting rules of $\text{Knuth}_2(n)$ correspond to the Knuth relations, [63], with orientation given by the lexicographic order induced by the total order on $\{1 < \dots < n\}$. The presentation $\text{Knuth}_2(n)$ is called the *Knuth presentation*.

Let Σ be a rewriting system. A *rewriting step* of Σ is a rewriting rule of the form $w\alpha w' : wuw' \Rightarrow wv w'$ where $\alpha : u \Rightarrow v$ is a rewriting rule in Σ and w and w' are generators in the free monoid Σ_1^* . A *local branching* of Σ is a non ordered pair (f, f_1) of rewriting steps $f : u \Rightarrow v$ and $f_1 : u \Rightarrow v_1$ of the free 2-category Σ_2^* with a common source u . A *critical branching* is an overlapping application of two different rewriting steps on the same word u , where u has minimal length, see Chapter 2, Section 2.1.2. A branching (f, f_1) is *confluent* if there exist rewriting steps f' and f'_1 in Σ_2^* , as follows



A rewriting system *terminates* if all the applications of rewriting rules end eventually and it is *confluent* if for every words u, u' and u'' such that u rewrites into u' and u'' , there exists a word v such that u' and u'' rewrite into v . It is *convergent* if it terminates and it is confluent. This means that any rewritings starting on a same word can be extended to end on a same reduced word, called a *normal form*, after applying a finite number of rewriting rules. A rewriting system is *locally confluent* if all of its local branchings are confluent. The critical branching Lemma, [39, Theorem 3.1.5.], states that a rewriting system is locally confluent if and only if all its critical branchings are confluent. Newman's Lemma, [39, Theorem 3.1.6.], states that for terminating rewriting systems, local confluence and confluence are equivalent properties.

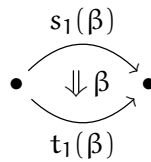
For instance, the Knuth presentation $\text{Knuth}_2(2)$ of the plactic monoid \mathbf{P}_2 with two generators 1 and 2 subject to the Knuth relations $\varepsilon_{1,2,2} : 221 \Rightarrow 212$ and $\eta_{1,1,2} : 211 \Rightarrow 121$ is convergent. Indeed it is terminating, thanks to the fact that the lexicographic order is monomial. On the other hand, the presentation $\text{Knuth}_2(2)$ is confluent since it admits a unique confluent critical branching:



Note that $2\eta_{1,1,2}$ and $\varepsilon_{1,2,2}1$ are the rewriting steps applied on the word 2211.

A way to solve the word problem of a monoid \mathbf{M} is to find a finite convergent presentation (Σ_1, Σ_2) of this monoid. Indeed, two elements in the free monoid Σ_1^* represent the same element in the monoid \mathbf{M} if, and only if, their normal forms are equal in Σ_1^* . In this way, if a monoid admits a finite convergent presentation then it has a decidable word problem. The inverse of the latter implication was considered as an open problem, [51, 52]. Squier gave a negative answer to this question, [112], by constructing finitely presentable monoids with a decidable word problem that cannot be presented by finite convergent presentations. Later, he introduced the condition of *finite derivation type*, which is a homotopical finiteness property on the presentation complex associated to a presentation of a monoid. He gave a constructive way to prove this finiteness property based on the computation of the critical branchings. Squier showed that being of finite derivation type is a necessary condition for a finitely presented monoid to admit a finite convergent presentation [113].

Two-dimensional polygraphs. *Polygraphs* were introduced independently by several authors as presentations by generators and relations of higher-dimensional categories. In order to give presentations of 2-categories, Street introduced the notion of *computad* as a directed graph equipped with extra 2-dimensional directed edges between parallel paths in the graph, [115, 116]. Later on, Power gave an inductive definition of higher dimensional computads, called *n-computads*, [101, 102]. Right after that, Burroni introduced independently the notion of *n-polygraphs* using also an inductive definition, [14, 15]. A *1-polygraph* corresponds to an abstract rewriting system, [3]. More explicitly, a 1-polygraph is a directed graph (Σ_0, Σ_1) , given by a set Σ_0 of 0-cells, a set Σ_1 of 1-cells together with two maps s_0 and t_0 sending a 1-cell x on its *source* $s_0(x)$ and its *target* $t_0(x)$. Since we study monoids, the set Σ_0 is reduced to a set with exactly one element \bullet and the 1-polygraph (Σ_0, Σ_1) is identified to the set Σ_1 . Moreover, a *2-polygraph* is a pair $\Sigma = (\Sigma_1, \Sigma_2)$ made of a 1-polygraph Σ_1 and a set of *globular 2-cells* Σ_2 on the free monoid Σ_1^* equipped with *source* and *target* maps $s_1, t_1 : \Sigma_2 \rightarrow \Sigma_1^*$ that satisfy the *globular relations*: $s_0 s_1 = s_0 t_1$ and $t_0 s_1 = t_0 t_1$. The elements of Σ_2 are called the *rewriting rules* of Σ . They have the following globular form:



A 2-polygraph Σ is a *presentation of a monoid* \mathbf{M} if the monoid \mathbf{M} is isomorphic to the quotient of the free monoid Σ_1^* by the congruence generated by the set Σ_2 . Moreover, two 2-polygraphs are *Tietze equivalent* if their presented monoids are isomorphic. For $n \geq 2$, an $(n+1)$ -polygraph is given by an n -polygraph Σ_n , with a family Σ_{n+1} of additional $(n+1)$ -cells between parallel n -cells of the n -category Σ_n^* free generated by the n -polygraph Σ_n .

Using the polygraphical approach, many works have developed the higher-dimensional rewriting theory. Moreover, Squier's results were also presented in the language of polygraphs

and higher-dimensional categories yielding new proofs of these results, see [96, 69, 35, 38, 36, 37, 39].

Completion of the Knuth presentation. A presentation of a monoid which is not convergent can be transformed into a convergent one that represents the same monoid, by the *Knuth–Bendix’s completion*. This completion computes a convergent presentation of a monoid from a terminating one by iteratively adding rules, [64]. Roughly speaking, let Σ be a terminating presentation of a monoid \mathbf{M} , the Knuth–Bendix’s completion of Σ is the 2-polygraph obtained from Σ by examining its critical branchings one by one and by adding 2-cells when a critical branching is not confluent. The added 2-cells can make new critical branchings that are also examined and then new 2-cells can be added. The resulted 2-polygraph is a convergent presentation of the monoid \mathbf{M} .

For the monoid \mathbf{P}_3 , the Knuth presentation $\text{Knuth}_2(3)$ admits three generators 1, 2 and 3, together with the following 8 relations

$$\begin{aligned} & \{ 211 \xrightarrow{\eta_{1,1,2}} 121, \quad 311 \xrightarrow{\eta_{1,1,3}} 131, \quad 312 \xrightarrow{\eta_{1,2,3}} 132, \quad 322 \xrightarrow{\eta_{2,2,3}} 232 \} \\ & \cup \{ 221 \xrightarrow{\epsilon_{1,1,2}} 212, \quad 231 \xrightarrow{\epsilon_{1,2,3}} 213, \quad 331 \xrightarrow{\epsilon_{1,3,3}} 313, \quad 332 \xrightarrow{\epsilon_{2,3,3}} 323 \}. \end{aligned}$$

A finite convergent presentation of the monoid \mathbf{P}_3 is obtained by adding the following three 2-cells

$$32321 \Rightarrow 32132, \quad 32131 \Rightarrow 31321 \quad \text{and} \quad 3212 \Rightarrow 2321$$

to its Knuth presentation using the Knuth–Bendix’s completion, see Chapter 2, Section 2.3.2.

For $n > 3$, Kubat and Okniński showed that there is no finite completion of the Knuth presentation of the monoid \mathbf{P}_n , [65, Theorem 3]:

Theorem 2.3.3.3. *For $n > 3$, there is no finite completion of the 2-polygraph $\text{Knuth}_2(n)$ compatible with the lexicographic order.*

The column presentation. Bokut, Chen, Chen and Li in [7] and Cain, Gray and Malheiro in [17] constructed a finite convergent presentation of the monoid \mathbf{P}_n with different methods. Both of their methods consist in adding new generators, called the *column generators*, to the Knuth presentation. The new presentation rewrites two adjacent pairs of columns that do not form a tableau to their corresponding tableau form using Schensted’s insertion algorithms. This presentation, called the *column presentation*, is Tietze equivalent to the Knuth presentation. Cain, Gray and Malheiro used the row-insetion algorithm in the definition of the 2-cells of this presentation whereas Bokut, Chen, Chen and Li used the column-insertion procedure. Note also that Bokut, Chen, Chen and Li obtained in [7] an infinite convergent presentation of the monoid \mathbf{P}_n by adding the infinite set of rows over the set $[n]$.

We consider the column presentation for the plactic monoid \mathbf{P}_n constructed by Cain, Gray and Malheiro, [17]. In this case, a *column* is the 1-cell obtained by reading a tableau of

shape $(1, \dots, 1)$ from bottom to top. That is, a column is a decreasing 1-cell in the free monoid $[n]^*$. For instance, the 1-cell 76431 is a column over the set $[7]$. We denote by $\text{col}(n)$ the set of non-empty columns over the set $[n]$. For every 1-cell u in $[n]^*$, we denote by $\ell(u)$ its length. Moreover, the tableau $P(u)$ is computed by the row-insertion procedure.

Consider two columns $u = x_p \dots x_1$ and $v = y_q \dots y_1$ in $\text{col}(n)$, the tableau $P(uv)$ contains at most two columns, see Chapter 3, Subsection 3.2.1. We will denote by $u \times v$ when the juxtaposition of the columns u and v does not form a tableau, that is, if $p < q$ or $x_i > y_i$ for some $i \leq q$. In this case, we denote by $u \times^1 v$ if the tableau $P(uv)$ has one column and by $u \times^2 v$ if the tableau $P(uv)$ has two columns.

In order to obtain a finite convergent presentation of the plactic monoid \mathbf{P}_n , one adds to the presentation $\text{Knuth}_2(n)$ one superfluous generator c_u for any u in the set $\text{col}(n)$. We will denote by $\text{Col}_1(n)$ the set of *column generators* c_u , for u in $\text{col}(n)$, of the monoid \mathbf{P}_n and by $c_{x_p} \dots c_{x_1} \xrightarrow{\gamma_u} c_u$ the defining relations for the column generators $u = x_p \dots x_1$ in $\text{col}(n)$ of length greater than 2.

For every columns u and v in $\text{col}(n)$ such that $u \times v$, we define a 2-cell

$$\alpha_{u,v} : c_u c_v \Rightarrow c_w c_{w'}$$

where $w = uv$ and $c_{w'} = 1$ if $u \times^1 v$, and w and w' are respectively the left and right columns of the tableau $P(uv)$ if $u \times^2 v$.

Let us denote by $\text{Col}_2(n)$ the 2-polygraph whose set of 1-cells is $\text{Col}_1(n)$ and the set of 2-cells is

$$\{ c_u c_v \xrightarrow{\alpha_{u,v}} c_w c_{w'} \mid u, v \in \text{col}(n) \text{ and } u \times v \}.$$

Since the set of column generators $\text{Col}_1(n)$ is finite, we obtain that the 2-polygraph $\text{Col}_2(n)$ is finite. Moreover, the 2-polygraphs $\text{Col}_2(n)$ and $\text{Knuth}_2(n)$ are Tietze equivalent, see Chapter 3, Lemma 3.2.2.2, then the 2-polygraph $\text{Col}_2(n)$ is a presentation of the monoid \mathbf{P}_n . It is called the *column presentation*. Using combinatorial properties of tableaux, one can prove that the 2-polygraph $\text{Col}_2(n)$ is confluent and terminating, see Chapter 3, Section 3.2. As a consequence, for $n > 0$, the 2-polygraph $\text{Col}_2(n)$ is a finite convergent presentation of the monoid \mathbf{P}_n , [17, Theorem 3.4.].

The Column presentation and Schensted's algorithm. The row-insertion introduced by Schensted corresponds to the leftmost reduction path in $\text{Col}_2^*(n)$ from a 1-cell w to its Schensted's tableau, that is, the reduction paths obtained by applying the rules of $\text{Col}_2(n)$ starting from the left. For instance, consider the 1-cell $w = 43152452$ in $[5]^*$. To compute the tableau $P(w)$, one applies the following successive rules of $\text{Col}_2(5)$ starting in each step from the left:

$$w = \begin{array}{cccccccc} 4 & 3 & 1 & 5 & 2 & 4 & 5 & 2 \end{array} \xrightarrow{\alpha_{4,3}} \begin{array}{cccccccc} 3 & 1 & 5 & 2 & 4 & 5 & 2 \\ 4 \end{array} \xrightarrow{\alpha_{43,1}} \begin{array}{cccccccc} 1 & 5 & 2 & 4 & 5 & 2 \\ 3 \\ 4 \end{array} \xrightarrow{\alpha_{5,2}} \begin{array}{cccccccc} 1 & 2 & 4 & 5 & 2 \\ 3 & 5 \\ 4 \end{array}$$

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & 5 & 2 \\ \hline 3 & 5 & & & \\ \hline 4 & & & & \\ \hline \end{array} \xrightarrow{\alpha_{5,2}} \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 2 \\ \hline 3 & 5 & & 5 \\ \hline 4 & & & \\ \hline \end{array} \xrightarrow{\alpha_{4,52}} \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 2 & 5 & \\ \hline 3 & 5 & 4 & & \\ \hline 4 & & & & \\ \hline \end{array} = P(w)$$

Syzygies and coherent presentations. The *syzygy problem* for presentations of monoids consists in finding all the relations amongst the relations for these presentations. More explicitly, let (Σ_1, Σ_2) be a presentation of a monoid. A 2-syzygy for (Σ_1, Σ_2) is a non trivial 2-sphere of the free $(2, 1)$ -category Σ_2^\top , that is, a pair (f, g) of 2-cells of Σ_2^\top such that $s_1(f) = s_1(g)$ and $t_1(f) = t_1(g)$. A *globular extension* of Σ_2^\top is a set Σ_3 equipped with a map from Σ_3 to the set of 2-spheres of Σ_2^\top . A *family of generating 2-syzygies* is a globular extension Σ_3 of Σ_2^\top such that for every 2-sphere γ in Σ_2^\top there exists a 3-cell in the free $(3, 1)$ -category Σ_3^\top with boundary γ , see Subsection 4.1.1. For instance, for the monoid \mathbf{P}_2 , the Knuth relations $\eta_{1,1,2} : 211 \Rightarrow 121$ and $\varepsilon_{1,2,2} : 221 \Rightarrow 212$ are related by the following 2-syzygy

$$\begin{array}{ccc} & 2\eta_{1,1,2} & \\ & \curvearrowright & \\ 2211 & & 2121 \\ & \curvearrowleft & \\ & \varepsilon_{1,2,2}1 & \end{array}$$

that gives two ways to prove the equality $2211 = 2121$ in the monoid \mathbf{P}_2 . For rank greater than 3, it is difficult to compute a family of syzygies of the Knuth presentation due to the combinatorial complexity of the relations. A $(3, 1)$ -polygraph is a pair (Σ_2, Σ_3) made of a 2-polygraph Σ_2 and a globular extension Σ_3 of Σ_2^\top . An *extended presentation* of a monoid is a $(3, 1)$ -polygraph whose 2-polygraph is a presentation of the monoid. A *coherent presentation* of a monoid is an extended presentation (Σ_2, Σ_3) of this monoid such that the set Σ_3 is a family of generating 2-syzygies of Σ_2^\top , that we call a *homotopy basis*, see Chapter 4, Subsection 4.1.1.

Squier's theorem gives a method to compute a coherent presentation of a monoid from a convergent one, [112]. More precisely, all the critical branchings of a convergent presentation are confluent. Then the adjunction of the 3-cells resulting from these critical branchings to the convergent presentation gives rise to a coherent presentation of the monoid. In order to obtain coherent presentations for plactic monoids, we use the *homotopical completion-reduction procedure* introduced in [40, 31]. The *homotopical completion procedure* extends Squier's completion, [113], to terminating 2-polygraphs using the Knuth–Bendix's completion, [64]. It examines the critical branchings of a terminating 2-polygraph one by one, add 2-cells to reach the confluence and add also 3-cells to obtain a convergent coherent presentation. This coherent presentation is not minimal in general, then the *homotopical reduction procedure* coherently eliminates unnecessary superfluous cells from this presentation using the notion of *collapsible parts* of $(3, 1)$ -polygraphs. In particular, a method to eliminate some superfluous 3-cells is to compute the critical triple branchings in order to obtain relations between 3-cells and then eliminate some of them by *Tietze transformations*, see Chapter 4, Subsection 4.2.1.

For instance, the Knuth presentation $\text{Knuth}_2(2)$ of the plactic monoid \mathbf{P}_2 admits a unique confluent critical branching. Thus the presentation $\text{Knuth}_2(2)$ can be extended into a coherent

presentation with the 3-cell:

$$\begin{array}{ccc}
 & \xrightarrow{2\eta_{1,1,2}} & \\
 2211 & \Downarrow & 2121 \\
 & \xleftarrow{\varepsilon_{1,2,2}1} &
 \end{array} \tag{7}$$

Moreover, the column presentation for the monoid \mathbf{P}_2 has generators c_1, c_2, c_{21} , with the rules $\alpha_{2,1} : c_2c_1 \Rightarrow c_{21}$, $\alpha_{1,21} : c_1c_{21} \Rightarrow c_{21}c_1$ and $\alpha_{2,21} : c_2c_{21} \Rightarrow c_{21}c_2$. This presentation has only one critical branching:

$$\begin{array}{ccccc}
 & \xrightarrow{\alpha_{2,1}c_{21}} & c_{21}c_{21} & \xleftarrow{c_{21}\alpha_{2,1}} & \\
 c_2c_1c_{21} & \Downarrow & & \Downarrow & \\
 & \xrightarrow{c_2\alpha_{1,21}} & c_2c_{21}c_1 & \xrightarrow{\alpha_{2,21}c_1} & c_{21}c_2c_1
 \end{array} \tag{8}$$

Note that for column presentations of the monoids $\mathbf{P}_3, \mathbf{P}_4$ and \mathbf{P}_5 we count respectively 7, 15 and 31 generators, 22, 115 and 531 relations, 42, 621 and 6893 3-cells.

MAIN RESULTS

Column presentations of plactic monoids

As discussed in Chapter 2, Section 2.3.2, the Knuth presentation of the plactic monoid \mathbf{P}_2 is finite and convergent and the Knuth presentation of the monoid \mathbf{P}_3 can be completed into a finite convergent one by adding three 2-cells using the Knuth–Bendix’s completion. Moreover, for $n > 3$, there is no finite completion of the Knuth presentation of the plactic monoid \mathbf{P}_n compatible with the lexicographic order, [65, Theorem 3]. The presentations by generators and relations constructed for plactic monoids for the classical types contained the Knuth relations, [88, 78, 79, 83]. Therefore, these presentations do not admit a finite completion compatible without adding new generators.

Question. *Do plactic monoids admit finite convergent presentations?*

In order to answer positively this question, we consider plactic monoids for any semisimple Lie algebra as introduced by Littelmann in [88]. We construct a finite convergent presentation of these monoids in a general way by adding the *L-S paths generators*. An L-S path corresponds to a column for type A, and to an admissible column for types C, B, D and G_2 in the Lecouvey sense, see [83]. As for type A, this presentation is also called the *column presentation*. The column presentation was shown to be a presentation of plactic monoids for any semisimple Lie algebras, [88, Theorem B]. We study this presentation using rewriting methods. For this, we

consider a rewriting system where the generating set contains the finite set of L-S paths. The right-hand sides of rewriting rules are tableaux for any type as defined by Littelmann, [88]. The rewriting system rewrites two L-S paths that their concatenation do not form a (generalised) tableau to their corresponding tableau form. Using the shapes of tableaux, we show that this presentation is finite and convergent.

Using a case-by-case analysis and thanks to Kashiwara's theory of crystal bases, the column presentation can be constructed for plactic monoids of type C, B, D and G_2 . We construct the column presentation of the plactic monoid for type C by adding the *admissible column generators*, [43]. The notion of tableaux is generalised to *symplectic tableaux* for type C. The right-hand side of the rewriting rules of the column presentation for type C is the result of Lecouvey's insertion of an admissible column into another one. By a careful analysis of what shapes of symplectic tableaux can result after this insertion, we show that this presentation is convergent. A bit later, Cain et al. constructed similar column presentations for plactic monoids of type B, C, D and G_2 by also adding the admissible columns generators, [16]. The notion of symplectic tableaux introduced as a generalisation of tableaux for type C has also been generalised to the other types, [83]. The column presentation for plactic monoids of these types rewrite two admissible columns that do not form a tableau (generalised for these types) to the corresponding tableau form. Cain et al. used Lecouvey's presentations for plactic monoids whereas we use Lecouvey's insertions.

It is worth noting that the column presentation constructed in a general way using L-S paths coincide with the ones constructed separately for types A, B, C, D and G_2 . This presentation covers also the exceptional types that their corresponding plactic monoids do not admit explicit presentations with generators and relations.

In the following, we will give an overview on column presentations of plactic monoids. Firstly, we will show how such presentations can be constructed by introducing the L-S paths. Afterwards, we illustrate the column presentation for the plactic monoid of type C using combinatorial tools and by the use of Kashiwara's theory of crystal bases.

The generalised column presentation of plactic monoids. We consider plactic monoids for any semisimple Lie algebras as defined by Littelmann, [88]. Our aim is to construct finite convergent presentations of these monoids using Lakshmibai-Seshadri's paths. We use also the notion of standard tableaux defined by Littelmann in [88] as a generalisation of tableaux for any type.

Let us denote by $\Lambda_1, \dots, \Lambda_n$ the fundamental weights of a semisimple Lie algebra \mathfrak{g} . For λ a weight in $P \otimes_{\mathbb{Z}} \mathbb{R}$ and t in $[0, 1]$, we define $\pi_{\lambda}(t) := t\lambda$ the path that connects the origin with λ by a straight line. For a dominant weight λ , the *Lakshmibai-Seshadri paths*, or *L-S paths* for short, of shape λ are the paths of the form $f_{\alpha_1} \circ \dots \circ f_{\alpha_s}(\pi_{\lambda})$, where $\alpha_1, \dots, \alpha_s$ are simple roots of the semisimple Lie algebra \mathfrak{g} . A *Young tableau of shape* $\lambda = a_1\Lambda_1 + \dots + a_n\Lambda_n$ is a concatenation

$$\star_{1 \leq i \leq n} \pi_{1, \Lambda_i} \star \dots \star \pi_{a_i, \Lambda_i}$$

where π_{i,Λ_i} is an L-S path of shape Λ_i , for $1 \leq i \leq n$. That is, the first α_1 paths are of shape Λ_1 , the next α_2 are of shape Λ_2, \dots , the final α_n paths are of shape Λ_n .

A Young tableau of shape $\lambda = \alpha_1\Lambda_1 + \dots + \alpha_n\Lambda_n$ is a *standard tableau of shape λ* if it is of the form

$$f_{\alpha_1} \circ \dots \circ f_{\alpha_s} \left(\underbrace{(\pi_{\Lambda_1} \star \dots \star \pi_{\Lambda_1})}_{\alpha_1 \text{ times}} \star \underbrace{(\pi_{\Lambda_2} \star \dots \star \pi_{\Lambda_2})}_{\alpha_2 \text{ times}} \star \dots \star \underbrace{(\pi_{\Lambda_n} \star \dots \star \pi_{\Lambda_n})}_{\alpha_n \text{ times}} \right)$$

where $\alpha_1, \dots, \alpha_s$ are simple roots of the semisimple Lie algebra \mathfrak{g} . Note that this notion of standard tableaux introduced by Littelmann using paths correspond to tableaux for type A, symplectic tableaux for type C and orthogonal tableaux for type B and D in the Lecouvey sense, [78, 79, 83].

Let \mathbb{B}_i be the set of L-S paths of shape Λ_i and $\mathbb{B} = \cup_{i=1}^n \mathbb{B}_i$. For every L-S paths c_1 and c_2 in \mathbb{B} such that $c_1 \star c_2$ is not a standard tableau, we define a 2-cell

$$c_1.c_2 \xrightarrow{\gamma_{c_1, c_2}} T$$

where T is the unique standard tableau such that the equality $T = c_1 \star c_2$ holds in the corresponding plactic monoid.

The *2-polygraph of paths*, denoted by $\text{Path}_2(n)$, is the 2-polygraph with only one 0-cell whose set of 1-cells is \mathbb{B} and whose set of 2-cells is

$$\text{Path}_2(n) = \left\{ c_1.c_2 \xrightarrow{\gamma_{c_1, c_2}} T \mid c_1, c_2 \in \mathbb{B} \text{ and } c_1 \star c_2 \text{ is not a standard tableau} \right\}.$$

This presentation is called the *column presentation*. It is a presentation of the plactic monoid for any semisimple Lie algebra [88, Theorem B]. Using the shapes of tableaux, we show that this presentation is terminating and confluent. As a consequence, we obtain the following result

Theorem 3.1.0.1. *For any semisimple Lie algebra \mathfrak{g} , the 2-polygraph $\text{Path}_2(n)$ is a finite convergent presentation of the plactic monoid for \mathfrak{g} .*

As we will see below for type C, the 2-polygraph $\text{Path}_2(n)$ can be constructed by a case-by-case analysis for plactic monoids of types A, B, C, D and G_2 using the terminology of columns and admissible columns.

The column presentation for the plactic monoid of type C. Consider the plactic monoid $\mathbf{P}_n(C)$ of type C and the ordered set

$$\mathcal{C}_n = \{1 < 2 < \dots < n < \bar{n} < \dots < \bar{1}\}.$$

Let us recall some combinatorial ingredients for type C that will be useful afterwards. For a column U , we denote by $h(U)$ the number of its elements. A column U is *admissible* if

for $m = 1, \dots, h(\mathbb{U})$, the number $N(m)$ of letters x in \mathbb{U} such that $x \leq m$ or $x \geq \bar{m}$ satisfies $N(m) \leq m$. A word is called an *admissible column word* if it is the reading of an admissible column from top to bottom. For instance, the word $568\bar{8}\bar{5}$ is an admissible column word over the ordered set \mathcal{C}_8 . We denote by $\text{acol}(\mathfrak{n})$ the set of non-empty admissible column words in the free monoid $\mathcal{C}_\mathfrak{n}^*$. A *symplectic tableau* is a tableau filled by elements of the set $\mathcal{C}_\mathfrak{n}$ and composed by the juxtaposition of admissible columns with an additional property, see Chapter 3, Section 3.3.1. Using his insertion algorithm, Lecouvey showed that for any word w in the free monoid $\mathcal{C}_\mathfrak{n}^*$, one can compute a unique symplectic tableau, denoted by $P(w)$, [78].

Using Kashiwara's theory of crystal bases, we generalized the column presentation for type A. We constructed a finite convergent presentation of the monoid $\mathbf{P}_\mathfrak{n}(C)$ by adding *admissible column generators*, [43]. The right-hand side of the rewriting rules of this presentation is the result of the Lecouvey's insertion of an admissible column into another one. In other words, the result is a symplectic tableau consisting of at most two admissible columns as shown in the following.

In order to give a finite convergent presentation of the plactic monoid $\mathbf{P}_\mathfrak{n}(C)$, one introduces the admissible column generators. The set of generators is

$$\text{ACol}_1(\mathfrak{n}) = \{ c_u \mid u \in \text{acol}(\mathfrak{n}) \},$$

where each element c_u is equal to u in the monoid $\mathbf{P}_\mathfrak{n}(C)$.

Let u and v be the readings from top to bottom of two admissible columns U and V respectively. We will denote by $U \not\leq V$ when the 1-cell uv is not the reading of a symplectic tableau. That is, when the juxtaposition of U and V from right to left does not form a tableau.

To define the 2-cells of the new presentation, we need to know the shape of the symplectic tableau $P(uv)$ for each admissible columns words u and v in $\text{acol}(\mathfrak{n})$. We have

Lemma 3.3.3.1. *Let u and v be the readings of two admissible columns U and V respectively. The symplectic tableau $P(uv)$ consists of at most two columns.*

Let u and v be in $\text{acol}(\mathfrak{n})$ such that their corresponding admissible columns U and V satisfy $U \not\leq V$. We define a 2-cell $\alpha_{u,v}$ by

- $c_u c_v \xrightarrow{\alpha_{u,v}} c_w c_{w'}$, where the words w and w' are respectively the readings of the right and left columns W and W' of $P(uv)$ if this symplectic tableau consists of two columns.
- $c_u c_v \xrightarrow{\alpha_{u,v}} c_w$, where w is the reading of the column W of $P(uv)$ if it consists of one column.
- $c_u c_v \xrightarrow{\alpha_{u,v}} c_\varepsilon$, where ε is the empty word if $P(uv)$ consists of zero column.

Let $\text{ACol}_2(\mathfrak{n})$ be the 2-polygraph whose set of 1-cells is $\text{ACol}_1(\mathfrak{n})$ and whose set of 2-cells contains all the 2-cells $\alpha_{u,v}$ for u and v in $\text{acol}(\mathfrak{n})$ such that $U \not\leq V$. We show that 2-polygraph $\text{ACol}_2(\mathfrak{n})$ is a presentation of the plactic monoid $\mathbf{P}_n(\mathbb{C})$ of type C, called the *column presentation*, see Chapter 3, Lemma 3.3.4.5.

Our aim is to prove that the 2-polygraph $\text{ACol}_2(\mathfrak{n})$ is finite and convergent. It is finite thanks to the fact that the set $\text{ACol}_1(\mathfrak{n})$ is finite. Knowing about the resulting shapes of tableaux allows us to define a well-order on products of columns such that rewriting always decrease with respect to this order.

Lemma 3.3.3.2. *Let u and v be the readings of two admissible columns U and V respectively, such that $U \not\leq V$. Suppose that $P(uv)$ has two columns and let W be the rightmost column. Then the column U contains more elements than W .*

Thus the 2-polygraph $\text{ACol}_2(\mathfrak{n})$ is terminating. The confluence follows from the fact that symplectic tableaux form a cross-section of the monoid $\mathbf{P}_n(\mathbb{C})$. As a consequence, we obtain

Theorem 3.3.4.6. *The 2-polygraph $\text{ACol}_2(\mathfrak{n})$ is a finite convergent presentation of the plactic monoid $\mathbf{P}_n(\mathbb{C})$ of type C.*

Coherent presentations of plactic monoids of type A

As mentioned previously, for rank greater or equal to 4, the syzygies problem for the Knuth presentation is difficult due to the combinatorial complexity for the Knuth relations. Thanks to the homotopical completion-reduction procedure, we construct a finite coherent presentation of the monoid \mathbf{P}_n that extends the Knuth presentation $\text{Knuth}_2(\mathfrak{n})$. In a first step, starting from the column presentation $\text{Col}_2(\mathfrak{n})$, we construct a coherent presentation of the plactic monoid \mathbf{P}_n . This presentation is not minimal in the sense that some of its generators are superfluous. After several steps of reduction, we obtain a coherent presentation of the monoid \mathbf{P}_n with the Knuth generators giving syzygies of the Knuth presentation.

For instance, the column presentation $\text{Col}_2(2)$ of the monoid \mathbf{P}_2 can be extended into a coherent presentation by adding the unique generating 3-cell (7). After applying our construction on this coherent presentation, we show that the Knuth presentation can be extended into a coherent one by adding the generating 3-cell (8).

The following diagram illustrates our construction

$$\begin{array}{ccccc}
 \text{Knuth}_2(\mathfrak{n}) & \xrightarrow{\text{Tietze equivalence}} & \text{Col}_2(\mathfrak{n}) & \xrightarrow{\text{Homotopical completion}} & \text{Col}_3(\mathfrak{n}) \\
 & & & & \downarrow \text{Homotopical reduction} \\
 \text{Knuth}_3(\mathfrak{n}) & \xleftarrow{\text{Tietze equivalence}} & \text{PreCol}_3(\mathfrak{n}) & \xleftarrow{\text{Homotopical reduction}} & \overline{\text{Col}}_3(\mathfrak{n})
 \end{array}$$

where $\text{Col}_3(\mathfrak{n})$ denotes the coherent presentation obtained from the column presentation $\text{Col}_2(\mathfrak{n})$, the $(3, 1)$ -polygraphs $\overline{\text{Col}}_3(\mathfrak{n})$ and $\text{PreCol}_3(\mathfrak{n})$ are obtained from $\text{Col}_3(\mathfrak{n})$ after many steps of homotopical reduction, and $\text{Knuth}_3(\mathfrak{n})$ denotes the coherent presentation of the monoid \mathbf{P}_n that extends the Knuth presentation and that is Tietze equivalent to $\text{PreCol}_3(\mathfrak{n})$.

Coherent column presentation. More precisely, after examining the confluence of the critical branchings of the column presentation $\text{Col}_2(\mathfrak{n})$, we show that all the confluence diagrams of these branchings have the following form

$$\begin{array}{ccccc}
 & & & c_e c_{e'} c_t & \xrightarrow{c_e \alpha_{e',t}} & c_e c_b c_{b'} & & & \\
 & \nearrow \alpha_{u,v} c_t & & & & & \searrow \alpha_{e,b} c_{b'} & & \\
 c_u c_v c_t & & & & \Downarrow \mathcal{X}_{u,v,t} & & & & \\
 & \searrow c_u \alpha_{v,t} & & c_u c_w c_{w'} & \xrightarrow{\alpha_{u,w} c_{w'}} & c_a c_{a'} c_{w'} & \xrightarrow{c_a \alpha_{a',w'}} & c_a c_d c_{b'} & \\
 & & & & & & & &
 \end{array} \tag{9}$$

for any columns $u = x_p \dots x_1$, $v = y_q \dots y_1$ and $t = z_l \dots z_1$ such that uv and vt are not tableaux, where e and e' (resp. w and w') denote the two columns of the tableau $P(uv)$ (resp. $P(vt)$) and a and a' (resp. b and b') denote the two columns of the tableau $P(uw)$ (resp. $P(e't)$) and a , d , b' are the three columns of the tableau $P(uvt)$, which is a normal form in the 2-polygraph $\text{Col}_2(\mathfrak{n})$.

We denote by $\text{Col}_3(\mathfrak{n})$ the extended presentation of the monoid \mathbf{P}_n obtained from $\text{Col}_2(\mathfrak{n})$ by adjunction of 3-cells of the form $\mathcal{X}_{u,v,t}$. We obtain the following result:

Theorem 4.1.3.1. *For $n > 0$, the plactic monoid \mathbf{P}_n admits $\text{Col}_3(\mathfrak{n})$ as a coherent presentation.*

This theorem is proved by examining the confluence of the critical branchings of the 2-polygraph $\text{Col}_2(\mathfrak{n})$ in each of the four cases $u^{\times 1} v^{\times 1} t$, $u^{\times 2} v^{\times 1} t$, $u^{\times 1} v^{\times 2} t$, and $u^{\times 2} v^{\times 2} t$. Note that the 3-cells of $\text{Col}_3(\mathfrak{n})$ are of the form $A_{u,v,t}$ for $u^{\times 1} v^{\times 1} t$, $B_{u,v,t}$ for $u^{\times 2} v^{\times 1} t$, $C_{u,v,t}$ for $u^{\times 1} v^{\times 2} t$ and $D_{u,v,t}$ for $u^{\times 2} v^{\times 2} t$, see Chapter 4, Subsection 4.1.

Example: the case of the monoid \mathbf{P}_2 . Consider the column presentation $\text{Col}_2(2)$ of the monoid \mathbf{P}_2 with 1-cells c_1, c_2 and c_{21} and 2-cells $\alpha_{2,1} : c_2c_1 \Rightarrow c_{21}$, $\alpha_{1,21} : c_1c_{21} \Rightarrow c_{21}c_1$ and $\alpha_{2,21} : c_2c_{21} \Rightarrow c_{21}c_1$. The coherent presentation $\text{Col}_3(2)$ has only one 3-cell

$$\begin{array}{ccc}
 & \xrightarrow{\alpha_{2,1}c_{21}} & c_{21}c_{21} \\
 c_2c_1c_{21} & \xrightarrow{\quad} & \Downarrow C'_{2,1,21} \\
 & \xrightarrow{c_2\alpha_{1,21}} & c_2c_{21}c_1 \xrightarrow{\alpha_{2,21}c_1} c_{21}c_2c_1
 \end{array}$$

A reduced coherent column presentation. The coherent presentation $\text{Col}_3(n)$ of the plactic monoid \mathbf{P}_n is not minimal. Using the homotopical reduction procedure, we reduce it into the presentation $\overline{\text{Col}}_3(n)$ obtained from the 2-polygraph $\text{Col}_2(n)$ by adjunction of one family of 3-cells $\mathcal{X}_{x,v,t}$ of the form (9), but with x is of length 1, see Chapter 4, Subsection 4.2.2. The 3-cells of $\overline{\text{Col}}_3(n)$ are of the form $A_{x,v,t}$, $B_{x,v,t}$, $C_{x,v,t}$ and $D_{x,v,t}$. In this way, we obtain

Proposition 4.2.2.1. *For $n > 0$, the plactic monoid \mathbf{P}_n admits $\overline{\text{Col}}_3(n)$ as a coherent presentation.*

For instance, since the first column in $c_2c_1c_{21}$ in the 3-cell of $\text{Col}_3(2)$ is of length 1, the coherent presentations $\text{Col}_3(2)$ and $\overline{\text{Col}}_3(2)$ of the monoid \mathbf{P}_2 coincide.

Pre-column coherent presentation. In a second step of reduction, we reduce the coherent presentation $\overline{\text{Col}}_3(n)$ into a smaller one. This reduction, denoted by R_{Γ_3} , is given by a collapsible part defined by a set of 3-cells of $\overline{\text{Col}}_3(n)$, see Chapter 4, Subsection 4.2.3. The homotopical reduction R_{Γ_3} eliminates from $\overline{\text{Col}}_3(n)$ the 3-cells of the form $A_{x,v,t}$, $B_{x,v,t}$ and $C_{x,v,t}$, which are not of the form $C'_{x,v,t}$. The reduction R_{Γ_3} also eliminates some superfluous 2-cells from $\overline{\text{Col}}_3(n)$.

More precisely, let $\text{PreCol}_3(n)$ be the $(3, 1)$ -polygraph whose set of 1-cells is $\text{Col}_1(n)$, the set of 2-cells is

$$\begin{aligned}
 & \{ c_x c_{zy} \xrightarrow{\alpha_{x,zy}} c_{zx} c_y \mid 1 \leq x \leq y < z \leq n \} \cup \{ c_y c_{zx} \xrightarrow{\alpha_{y,zx}} c_{yx} c_z \mid 1 \leq x < y \leq z \leq n \} \\
 & \cup \{ c_x c_u \xrightarrow{\alpha_{x,u}} c_{xu} \mid xu \in \text{col}(n) \text{ and } 1 \leq x \leq n \},
 \end{aligned}$$

and whose 3-cells are the form $R_{\Gamma_3}(C'_{x,v,t})$ where

$$\begin{array}{ccc}
 & \xrightarrow{\alpha_{x,v}c_t} & c_{xv}c_t \\
 c_x c_v c_t & \xrightarrow{\quad} & \Downarrow C'_{x,v,t} \\
 & \xrightarrow{c_x \alpha_{v,t}} & c_x c_w c_{w'} \xrightarrow{\alpha_{x,w}c_{w'}} c_{xv} c_{z_1 \dots z_{q+1}} c_{w'}
 \end{array}$$

with $x^{\times 1}v^{\times 2}t$, and of the form $R_{\Gamma_3}(D_{x,v,t})$ where

$$\begin{array}{ccccc}
 & & c_e c_{e'} c_t & \xrightarrow{c_e \alpha_{e',t}} & c_e c_b c_{b'} & \xrightarrow{\alpha_{e,b} c_{b'}} & c_a c_d c_{b'} \\
 & \nearrow \alpha_{x,v} c_t & & & & & \\
 c_x c_v c_t & & & & & & \\
 & \searrow c_x \alpha_{v,t} & & & & & \\
 & & c_x c_w c_{w'} & \xrightarrow{\alpha_{x,w} c_{w'}} & c_a c_{a'} c_{w'} & \xrightarrow{c_a \alpha_{a',w'}} & c_a c_d c_{b'} \\
 & & & \Downarrow D_{x,v,t} & & &
 \end{array}$$

with $x^{\times 2}v^{\times 2}t$.

We obtain the following result

Theorem 4.2.3.2. *For $n > 0$, the plactic monoid \mathbf{P}_n admits $\text{PreCol}_3(n)$ as a coherent presentation.*

Example: the case of the monoid \mathbf{P}_2 . The coherent presentation $\text{PreCol}_3(2)$ of the plactic monoid \mathbf{P}_2 is given by the set of 1-cells $\{c_1, c_2, c_{21}\}$, the set of 2-cells

$$\{ \alpha_{2,1} : c_2 c_1 \Rightarrow c_{21}, \alpha_{1,21} : c_1 c_{21} \Rightarrow c_{21} c_1, \alpha_{2,21} : c_2 c_{21} \Rightarrow c_{21} c_1 \},$$

and the following 3-cell $C'_{2,1,21}$:

$$\begin{array}{ccccc}
 & & c_{21} c_{21} & & \\
 & \nearrow \alpha_{2,1} c_{21} & & \nwarrow c_{21} \alpha_{2,1} & \\
 c_2 c_1 c_{21} & & & & c_{21} c_2 c_1 \\
 & \searrow c_2 \alpha_{1,21} & & \xrightarrow{\alpha_{2,21} c_1} & \\
 & & c_2 c_{21} c_1 & &
 \end{array}$$

Knuth coherent presentation. In a last step of the homotopical reduction procedure, we reduce the coherent presentation $\text{PreCol}_3(n)$ into a coherent presentation of the monoid \mathbf{P}_n whose underlying 2-polygraph is the Knuth presentation $\text{Knuth}_2(n)$. We define an extended presentation $\text{Knuth}_3(n)$ of the monoid \mathbf{P}_n obtained from $\text{Knuth}_2(n)$ by adjunction of the following set of 3-cells

$$\{ \mathcal{R}(C'_{x,v,t}) \mid x^{\times 1}v^{\times 2}t \} \cup \{ \mathcal{R}(D_{x,v,t}) \mid x^{\times 2}v^{\times 2}t \},$$

where $\mathcal{R} : \overline{\text{Col}}_3(n)^\top \rightarrow \text{Knuth}_3(n)^\top$ is a Tietze transformation constructed throughout Chapter 4, Section 4.2. We obtain the following result

Theorem 4.2.4.3. *For $n > 0$, the plactic monoid \mathbf{P}_n admits $\text{Knuth}_3(n)$ as a coherent presentation.*

Example: the case of the monoid \mathbf{P}_2 . The coherent presentation $\text{PreCol}_3(2)$ of the monoid \mathbf{P}_2 is reduced to the presentation with generators c_1 and c_2 subject to the relations $\eta_{1,1,2} : c_2c_1c_1 \Rightarrow c_1c_2c_1$ and $\varepsilon_{1,2,2} : c_2c_2c_1 \Rightarrow c_2c_1c_2$ and the following 3-cell

$$\begin{array}{ccc}
 & \xrightarrow{2\eta_{1,1,2}} & \\
 c_2c_2c_1c_1 & \begin{array}{c} \Downarrow C'' \\ \Downarrow \\ \Downarrow \end{array} & c_2c_1c_2c_1 \\
 & \xleftarrow{\varepsilon_{1,2,2}1} &
 \end{array}$$

which coincides with the Knuth coherent presentation $\text{Knuth}_3(2)$, see Subsection 4.2.4.

Coherent presentations in small ranks. We denote by $\mathcal{KB}(\text{Knuth}_2(n))$ the convergent 2-polygraph obtained from $\text{Knuth}_2(n)$ by the Knuth–Bendix’s completion and by $\text{Knuth}_3^{\text{KB}}(n)$ the Squier completion of $\mathcal{KB}(\text{Knuth}_2(n))$. The following table presents the number of cells of the coherent presentations $\text{Knuth}_3^{\text{KB}}(n)$, $\text{Knuth}_3(n)$, $\overline{\text{Col}}_3(n)$ and $\text{Col}_3(n)$ of the monoid \mathbf{P}_n , for $1 \leq n \leq 10$.

n	$\text{Knuth}_1(n)$	$\text{Col}_1(n)$	$\text{Knuth}_2(n)$	$\mathcal{KB}(\text{Knuth}_2(n))$	$\text{Col}_2(n)$	$\text{Knuth}_3^{\text{KB}}(n)$	$\text{Knuth}_3(n)$	$\overline{\text{Col}}_3(n)$	$\text{Col}_3(n)$
1	1	1	0	0	0	0	0	0	0
2	2	3	2	2	3	1	1	1	1
3	3	7	8	11	22	27	24	34	42
4	4	15	20	∞	115	∞	242	330	621
5	5	31	40	∞	531	∞	1726	2225	6893
6	6	63	70	∞	2317	∞	10273	12635	67635
7	7	127	112	∞	9822	∞	55016	65282	623010
8	8	255	168	∞	40971	∞	275868	318708	5534197
9	9	511	240	∞	169255	∞	1324970	1500465	48052953
10	10	1023	330	∞	694837	∞	6178939	6892325	410881483

WORK IN PROGRESS AND PERSPECTIVES

Using our results and the techniques explored in this thesis, we present in this last section, some possible projects towards which we could orientate further research.

Coherent presentations and actions on categories. A monoid \mathbf{M} can be seen as a 2-category with exactly one 0-cell \bullet , with the elements of the monoid \mathbf{M} as 1-cells and with identity 2-cells only. The category of *actions of \mathbf{M} on categories* is the category $\text{Act}(\mathbf{M})$ of 2-representations of \mathbf{M} in the category Cat of categories. The full subcategory of $\text{Act}(\mathbf{M})$ whose objects are the 2-functors is denoted by $2\text{Cat}(\mathbf{M}, \text{Cat})$. We refer the reader to [31, Section 5.1] for a full introduction on the category of 2-representations of 2-categories. More explicitly, an action A of

the monoid \mathbf{M} is specified by a category $\mathbf{C} = A(\bullet)$, an endofunctor $A(u) : \mathbf{C} \rightarrow \mathbf{C}$ for every u in \mathbf{M} , a natural isomorphism $A_{u,v} : A(u)A(v) \Rightarrow A(uv)$ for every elements u and v of \mathbf{M} , and a natural isomorphism $A_\bullet : 1_{\mathbf{C}} \Rightarrow A(1)$ such that:

i) for every triple (u, v, w) of elements of the monoid \mathbf{M} , the following diagram commutes

$$\begin{array}{ccc}
 & A_{u,v}A(w) \rightarrow A(uv)A(w) & \xrightarrow{A_{uv,w}} \\
 A(u)A(v)A(w) & & A(uvw) \\
 & \xrightarrow{A(u)A_{v,w}} A(u)A(vw) & \xleftarrow{A_{u,vw}}
 \end{array}
 =$$

ii) for every element u of the monoid \mathbf{M} , the following diagrams commute

$$\begin{array}{ccc}
 A_\bullet A(u) \rightarrow A(1)A(u) & \xrightarrow{A_{1,u}} & \\
 \downarrow & = & \downarrow \\
 A(u) & \xrightarrow{\quad\quad\quad} & A(u)
 \end{array}
 \qquad
 \begin{array}{ccc}
 A(u)A_\bullet \rightarrow A(u)A(1) & \xrightarrow{A_{u,1}} & \\
 \downarrow & = & \downarrow \\
 A(u) & \xrightarrow{\quad\quad\quad} & A(u)
 \end{array}$$

The following result relates coherent presentations and the actions of monoids on categories. Let \mathbf{M} be a monoid and let Σ be an extended presentation of \mathbf{M} . The $(3, 1)$ -polygraph Σ is a coherent presentation of \mathbf{M} if, and only if, for every 2-category \mathcal{C} , there is an equivalence of categories

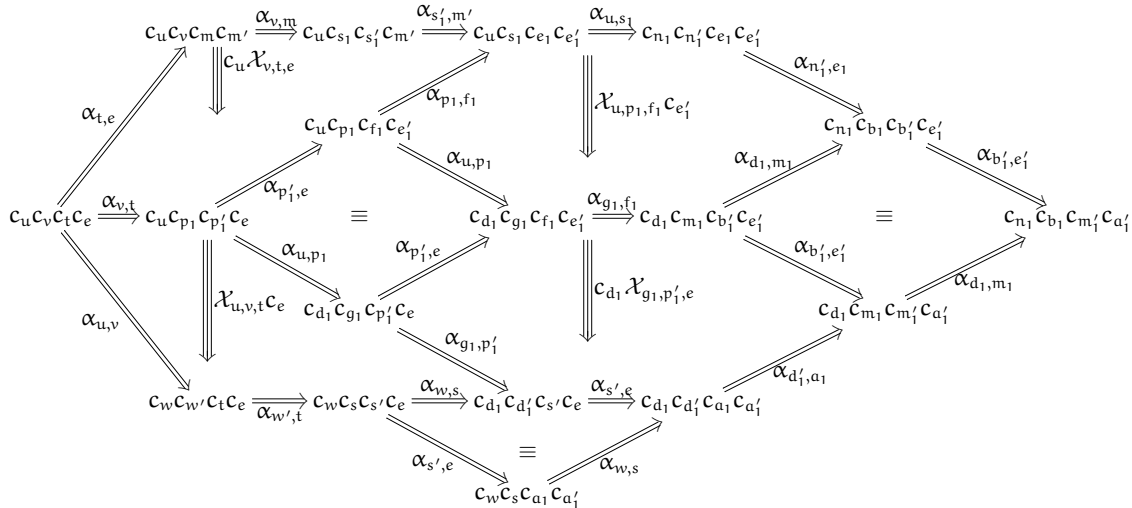
$$\text{Act}(\mathbf{M}) \approx 2\text{Cat}(\Sigma_1^*/\Sigma_2, \mathcal{C})$$

that is natural in \mathcal{C} , [31, Theorem 5.1.5]. In this way, up to equivalence, the actions of a monoid \mathbf{M} on categories are the same as the 2-functors from Σ_1^*/Σ_2 to Cat .

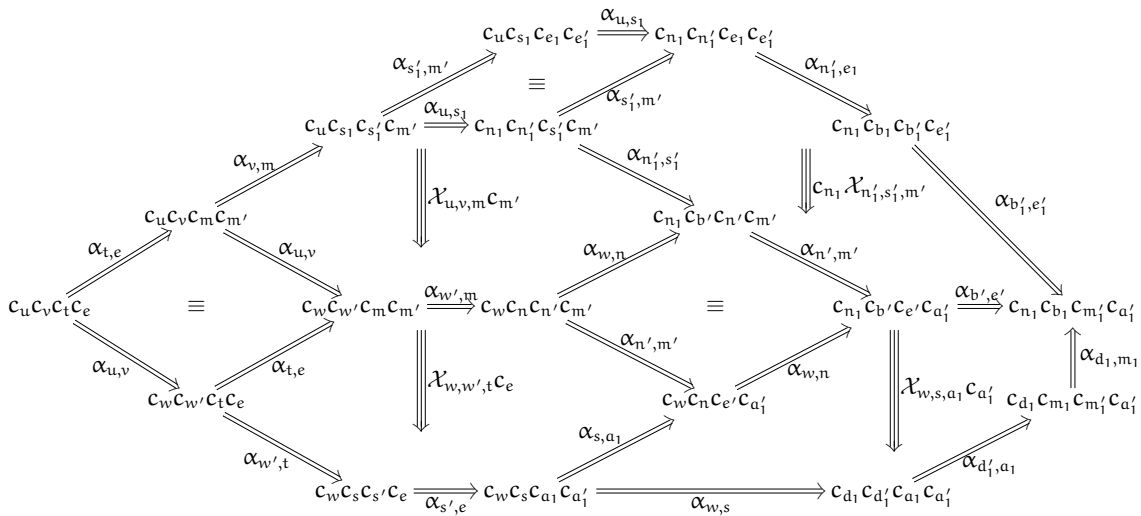
Using this description, Theorem 4.2.4.3 allows to present actions of plactic monoids on categories as follows. The category $\text{Act}(\mathbf{P}_n)$ of actions of the plactic monoid \mathbf{P}_n on categories is equivalent to the category of 2-functors from the $(2, 1)$ -category $\text{Knuth}_2(n)^\top$ to the category Cat of categories, that sends the 3-cells of $\text{Knuth}_3(n)$ to commutative diagrams in Cat . An interesting question is to study the actions of the plactic monoid on the category of finite-dimensional representations of the general Lie algebra or in the category \mathcal{O} whose objects are some representations of the general Lie algebra and morphisms are homomorphisms of representations, [50].

Higher syzygies for the plactic monoid. In [37], the authors show how to extend a convergent presentation of a monoid into a polygraphic resolution of the monoid. The column presentation $\text{Col}_2(n)$ of the monoid \mathbf{P}_n can then be extended into a polygraphic resolution whose n -cells, for every $n \geq 3$, are indexed by $(n - 1)$ -fold branching of $\text{Col}_2(n)$. We can explicit the 4-cells of this resolution, which correspond to the confluence diagrams induced by critical triple branchings. That is, for columns u, v, t and e in $\text{col}(n)$ such that $u \times v, v \times t$ and $t \times e$, there is a critical triple branching with source $c_u c_v c_t c_e$. Using the same arguments of Section 4.2.2,

we can show that the confluence diagram induced by this triple branching is represented by the 3-sphere $\Omega_{u,v,t,e}$ whose the source is the 3-cell



and the target is the 3-cell



In the generating triple confluence, some columns may be empty and thus the indicated 2-cells α may be identities. To facilitate the reading of the diagram, we have omitted the context of the 2-cells α .

More generally, we expect that the generating n -cell of the resolution has the form of the permutohedron of dimension n .

An implementation of the Knuth coherent presentation. As described in Chapter 4, Section 4.2, we construct a finite coherent presentation of the plactic monoid \mathbf{P}_n , whose underlying

2-polygraph is the Knuth presentation. In particular, we show that the 3-cells of this presentation are of the form

$$\{\mathcal{R}(C'_{x,v,t}) \mid x^{\times 1}v^{\times 2}t\} \cup \{\mathcal{R}(D_{x,v,t}) \mid x^{\times 2}v^{\times 2}t\},$$

where \mathcal{R} is a Tietze transformation constructed throughout several steps. We have implemented a program in Python that gives the number of cells of the Knuth coherent presentation as presented in the above table. This program consists in the computation of the number of sources $s_1(\mathcal{X}_{u,v,t})$ of the 3-cells $\mathcal{X}_{u,v,t}$ of this presentation.

Note that our construction allows to give explicit forms for these 3-cells. Note also that after applying the Tietze transformation \mathcal{R} , no 3-cell can be transformed to the identity. We would implement a program that gives the explicit forms of the 3-cells of the Knuth coherent presentation for higher ranks. For instance, the Knuth coherent presentation of the monoid \mathbf{P}_8 contains 275868 3-cells. In order to obtain the forms of these 3-cells, the program should give the explicit forms of the 3-cells of the presentations $\text{Col}_3(8)$, $\overline{\text{Col}}_3(8)$ and $\text{PreCol}_3(8)$ constructed during applying the Tietze transformation \mathcal{R} .

Anick's resolutions of plactic monoids via Littelmann paths. Anick constructed a resolution of a field \mathbb{K} seen as a G -module, where G is an associative augmented algebra over \mathbb{K} , [1, 2]. He defined a set of n -chains using the notion of Gröbner–Shirshov basis. Moreover, he gave an inductive definition of *differentials* using a contracting homotopy defined by an inductive way. Afterwards, Anick's resolution was extended to the cases of operads and categories, [93, 23]. Furthermore, the technique of algebraic discrete Morse theory introduced independently in [54, 111] describes explicitly the differentials of Anick's resolution by a very useful machinery using directed graphs. Such resolution allows to compute the homological invariants of the studied monoid.

Starting from the column presentation $\text{Col}_2(n)$ of the plactic monoid \mathbf{P}_n , Lopatkin constructed an Anick resolution of the monoid \mathbf{P}_n by the use of Morse theory, [90]. As a consequence, he computed the homological invariants of the monoid \mathbf{P}_n . Using the same strategy and starting from the column presentation of plactic monoids for any type constructed using L-S paths, we should obtain Anick's resolutions for these monoids in a general way recovering Lopatkin's work. Homological invariants of plactic monoids for any type could then be computed.

The Chinese monoid. For a totally ordered finite set $[n] = \{1 < \dots < n\}$, the *Chinese monoid* over $[n]$ is the monoid generated by $[n]$ and submitted to the relations

$$zyx = zxy = yzx, \quad \text{for } x \leq y \leq z. \tag{10}$$

This monoid appeared in the classification of *classes monoids* with growth function coinciding with that of the plactic monoid [26]. Since Young tableaux play an important role in the structure of plactic monoids, a similar notion of *Chinese staircases* was found for the Chinese monoid, [19, Section 2.1]. An insertion algorithm similar to Schensted's algorithm was introduced yielding a cross-section theorem, [19, Sections 2.2]. Moreover, a Robinson–Schensted correspondence was

also established, [19, Sections 5.2]. Recently, the Chinese monoid has motivated a wide range of other interesting work in rewriting theory including computing finite convergent presentations [20, 41, 42, 18], and in the theory of quadratic normalisations, [34, Example 5.3.2].

Starting from a convergent presentation of the Chinese monoid, a future work would be to compute coherent presentations for this monoid. Thanks to the homotopical completion procedure, we expect to obtain a result similar to Theorem 4.1.3.1 for Chinese monoids taking into account that we need at most five steps of rewriting to obtain a normal form whereas the normal form in the plactic case is attained in at most three steps. Moreover, such a coherent presentation could be homotopically reduced into a minimal one using the same strategy as in the plactic monoid case. Coherent presentations of the Chinese monoids should allow us to compute their homological invariants and to describe their actions on categories.

As presented in our work, crystal graphs and Kashiwara's operators interact with the notion of tableaux and their generalisations and provide combinatorial tools to study plactic monoids. A future work might be to construct a crystal structure for the Chinese monoid. The elements of the new crystal graph would be identified to Chinese staircases and the congruence generated by the relations (10) would have an interpretation in terms of a crystal isomorphism. A question is to study the interaction of the crystal graph of the plactic monoid of type A and the one constructed for the Chinese monoid.

The sylvester monoid. The structure of sylvester monoids appeared in the combinatorial study of Loday-Ronco's algebra of planar binary trees related to non-commutative symmetric functions and free symmetric functions, [46]. The sylvester monoid is constructed using the notion of *binary search trees* in a similar way as for the plactic monoid by introducing a Schensted-like insertion on these trees.

The *sylvester monoid of rank n* is defined as the quotient of the free monoid over the ordered set $[n]$ by the congruence generated by the family of the following relations:

$$cavb = acvb, \text{ such that } a \leq b < c,$$

where v is a word in the free monoid $[n]^*$. This infinite presentation introduced by Hivert, Novelli and Thibon, [46], was shown to be convergent by Cain, Gray and Malheiro, [18]. Constructing a finite convergent presentation of the sylvester monoid is still an open problem. Using a process similar to the construction of the column presentation of the plactic monoid, one should construct a finite convergent presentation of the sylvester monoid by adding new families of binary trees. Starting from a convergent presentation of the sylvester monoid, one could give an algorithmic method to compute 2-szygies for the presentation introduced by Hivert, Novelli and Thibon by constructing coherent presentations for this monoid.

Plactic monoids: Young tableaux, crystals and paths

Contents

1.1 Preliminaries on representation theory	64
1.2 Plactic monoid of type A	68
1.3 Crystal graphs and plactic monoids	75
1.4 Littelmann paths and plactic monoids	84

In this chapter, we give a full overview on *plactic monoids* using the notions of Young tableaux, [124, 28], Kashiwara’s crystal bases, [57, 58, 59, 61] and Littelmann path model [85, 86, 87, 88].

We begin in Section 1.1 by recalling some definitions and properties about finite-dimensional semisimple Lie algebras that will be crucial afterwards.

We define in Section 1.2 the plactic monoid introduced by Lascoux and Schützenberger and known as the *plactic monoid of type A*. This monoid is presented by the use of Young tableaux and their properties. We explain in 1.2.1 the notion of tableaux and we give in details Schensted’s insertion algorithms with additional examples. We recall in 1.2.2 some properties concerning the longest decreasing and non-decreasing subsequences of a given word over the ordered set $\{1 < \dots < n\}$. We recall in 1.2.3 the definition and some properties of the plactic monoid \mathbf{P}_n of type A and rank n .

The material of Section 1.3 illustrates the connection between Kashiwara’s theory of crystal bases and plactic monoids. For this, we first recall in 1.3.1 the definition of crystal graphs and we give in 1.3.2 the notion of crystal graphs for type A. Equipped with these definitions, we describe the connection between the representations of the special linear Lie algebra and

1.1. Preliminaries on representation theory

the combinatorics of Young tableaux. We explain in 1.3.3 in details the notion of a crystal graph for type C. After we recall in 1.3.4 how the plactic monoid introduced by Lascoux and Schützenberger is related to the representations of the general Lie algebra. Using Kashiwara’s theory of crystal bases, we give the definition of the plactic congruence generalised to other semisimple Lie algebras. Finally, we briefly outline the notion of plactic monoids of type B, C, D and G_2 .

We end this chapter with Section 1.4 on the relationship between Littelmann path model and plactic monoids. The definition of these monoids given in a general way using Littelmann paths coincide in some sense with the one given by a case-by-case analysis using Kashiwara’s crystal graphs. In 1.4.1, we recall the definitions of paths, root operators and Lakshmibai-Seshadri’s paths. After, we present in 1.4.2 the definition and some properties of standard tableaux as defined by Littelmann and we recall the definition of the plactic algebra for any semisimple Lie algebra.

1.1. PRELIMINARIES ON REPRESENTATION THEORY

In this section, we recall the definition and some properties of semisimple Lie algebras. We refer the reader to [49, 30, 10, 47] for a full introduction.

Semisimple Lie algebras. A *Lie algebra* is a vector space \mathfrak{g} over a field \mathbb{K} with a binary operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ called the *Lie bracket*, satisfying the following properties:

- i) *the Jacobi identity*: $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$, for all x, y and z in \mathfrak{g} ,
- ii) *alternativity*: $[x, x] = 0$, for all x in \mathfrak{g} ,
- iii) *anticommutativity*: $[x, y] = -[y, x]$, for all x and y in \mathfrak{g} ,
- iv) *bilinearity*: $[ax + by, z] = a[x, z] + b[y, z]$ and $[z, ax + by] = a[z, x] + b[z, y]$, for all a and b in \mathbb{K} and x, y and z in \mathfrak{g} .

Let \mathfrak{g}_1 and \mathfrak{g}_2 be Lie algebras. A *Lie algebra morphism* of \mathfrak{g}_1 to \mathfrak{g}_2 is a linear map $\varphi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ satisfying

$$[\varphi(x), \varphi(y)] = \varphi([x, y]), \text{ for all } x \text{ and } y \text{ in } \mathfrak{g}_1.$$

If it is also bijective, it is a Lie algebra isomorphism. A subspace \mathfrak{h} of a Lie algebra \mathfrak{g} is a *Lie subalgebra of \mathfrak{g}* if \mathfrak{h} itself is a Lie algebra with the bracket induced from \mathfrak{g} . A Lie subalgebra I of \mathfrak{g} is an *ideal* if the bracket $[x, y]$ is in I , for all x in \mathfrak{g} and y in I .

1.1.0.1. Example. Let V be a vector space over a field \mathbb{K} . Denote by $\text{End}V$ the set of all linear transformations on V . We define the bracket on $\text{End}V$ by

$$[f, g] = fg - gf, \quad \text{for all } f \text{ and } g \text{ in } \text{End}V.$$

1.1. Preliminaries on representation theory

Then, the vector space $\text{End}V$ becomes a Lie algebra, called the *general linear Lie algebra*, and denoted by $\mathfrak{gl}(V)$. Suppose $V = \mathbb{C}^n$, then the general linear Lie algebra is denoted by $\mathfrak{gl}_n(\mathbb{C})$ and the linear endomorphisms of \mathbb{C}^n are identified with the n by n matrices over \mathbb{C} .

A *simple Lie algebra* is a non abelian Lie algebra whose only ideals are 0 and itself. For a Lie algebra \mathfrak{g} , the *derived Lie algebra* is the subalgebra of \mathfrak{g} , denoted by $[\mathfrak{g}, \mathfrak{g}]$, that consists of all Lie brackets of pairs of elements of \mathfrak{g} . The *derived series* is a sequence of derived Lie algebras $D^i\mathfrak{g}$, for $i \geq 0$, defined by induction as follows

$$D^0\mathfrak{g} = \mathfrak{g} \text{ and } D^{i+1}\mathfrak{g} = [D^i\mathfrak{g}, D^i\mathfrak{g}].$$

A Lie algebra \mathfrak{g} is *solvable*, if the derived series eventually arrives at the zero subalgebra.

A Lie algebra \mathfrak{g} is *semisimple*, if there are no non-zero solvable ideals in \mathfrak{g} . Note that a semisimple Lie algebra \mathfrak{g} is a direct sum of simple Lie algebras. In particular, every simple Lie algebra is also semisimple.

1.1.0.2. Example. The only simple Lie algebras over the complex numbers are the following four families of *classical semisimple Lie algebras*:

- i) type A_n corresponding to the *special linear Lie algebra* $\mathfrak{sl}_{n+1}(\mathbb{C})$, as explained in 1.3.2,
 - ii) type B_n corresponding to the *odd-dimensional special orthogonal Lie algebra* $\mathfrak{so}_{2n+1}(\mathbb{C})$,
 - iii) type C_n corresponding to the *symplectic Lie algebra* $\mathfrak{sp}_{2n}(\mathbb{C})$, as explained in 1.3.3,
 - iv) type D_n corresponding to the *even-dimensional special orthogonal Lie algebra* $\mathfrak{so}_{2n}(\mathbb{C})$,
- together with the five *exceptional* ones E_6, E_7, E_8, F_4 , and G_2 .

Representations of Lie algebras. Let \mathfrak{g} be a Lie algebra and let V be a vector space over a field \mathbb{K} . A *representation of \mathfrak{g} on V* is a Lie algebra homomorphism $\psi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$. A vector space V is called an *\mathfrak{g} -module* if there is a bilinear map $\mathfrak{g} \times V \rightarrow V$, denoted by $(x, v) \mapsto x.v$, and satisfying

$$[x, y].v = x.(y.v) - y.(x.v) \quad \text{for } x, y \in \mathfrak{g}, v \in V.$$

A representation $\psi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ of a Lie algebra \mathfrak{g} on a vector space V defines an \mathfrak{g} -module structure on V by

$$x.v = \psi(x)(v) \quad \text{for } x \in \mathfrak{g}, v \in V. \tag{1.1}$$

Conversely, if V is a \mathfrak{g} -module, we obtain a representation $\psi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ defined by (1.1).

1.1.0.3. Example. Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ be the general linear Lie algebra and $V = \mathbb{C}^n$. One defines a map

$$\begin{aligned} \mathfrak{g} \times V &\rightarrow V \\ (x, v) &\mapsto xv \end{aligned}$$

by matrix multiplication. Then V is given an \mathfrak{g} -module structure called the *vector representation* or the *natural representation* of $\mathfrak{gl}_n(\mathbb{C})$.

1.1. Preliminaries on representation theory

Root system. Let \mathfrak{g} be a semisimple Lie algebra and let S be a finite vector space with standard inner product $\langle \cdot, \cdot \rangle$. A *root system* Φ of the Lie algebra \mathfrak{g} is a subset of $S \setminus \{0\}$ satisfying the following:

1. The set Φ is finite and it contains a basis of the vector space S .
2. For roots α and β in Φ , we have $\frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z}$.
3. For a map $s_\alpha : S \rightarrow S$ satisfying $s_\alpha(v) = v - \frac{2\langle \alpha, v \rangle}{\langle \alpha, \alpha \rangle} \alpha$, we have $s_\alpha(\beta) \in \Phi$, for all roots α and β in Φ .
4. If α and $c\alpha$ are in Φ , for some real c , then $c = 1$ or $c = -1$.

Note that a root system is symmetric with respect to the zero vector, we have that $-\alpha$ is in Φ if α is in Φ . Given a root system Φ , one can always choose a subset of *positive roots* Φ^+ satisfying:

- for each root α in Φ , exactly one of the roots α or $-\alpha$ is contained in Φ^+ ,
- for any two distinct roots α and β in Φ^+ such that $\alpha + \beta$ is a root, we have that $\alpha + \beta$ is in Φ^+ .

A positive root α in Φ^+ is *simple*, if it can not be written as $\alpha_1 + \alpha_2$, where α_1 and α_2 are positive roots. The *coroot* of a root α is $\alpha^\vee := \frac{2\alpha}{\langle \alpha, \alpha \rangle}$. Denote by

$$P := \{ v \in S \mid \langle v, \alpha_i^\vee \rangle \in \mathbb{Z}, \text{ for all } i \}$$

the *weight lattice* of the Lie algebra \mathfrak{g} . A *fundamental weight* Λ_i in P satisfies $\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$, for all i and j . The set of *dominant weights* is

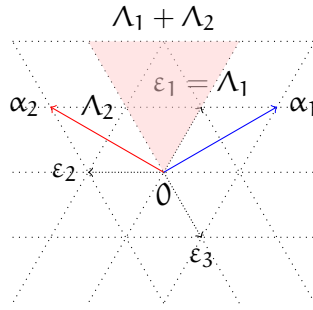
$$P^+ = \{ \lambda \in P \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0, \text{ for all } i \},$$

and the dominant chamber is

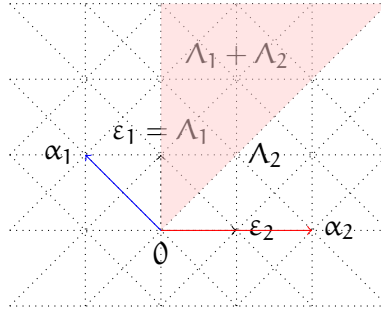
$$P_{\mathbb{R}}^+ = \{ \lambda \in P_{\mathbb{R}} \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0, \text{ for all } i \}.$$

1.1.0.4. Example. Let $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$. Consider $S = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0 \}$ and let $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ be the canonical basis of \mathbb{R}^3 . The simple roots of \mathfrak{g} are $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = \varepsilon_2 - \varepsilon_3$. Its fundamental weights are $\Lambda_1 = \varepsilon_1$ and $\Lambda_2 = \varepsilon_1 + \varepsilon_2$ (we still denote by ε_i the projection of ε_i onto S). An example of dominant weight is $\Lambda_1 + \Lambda_2$. The dominant chamber is the hatched area on the following figure:

1.1. Preliminaries on representation theory



1.1.0.5. Example. Let $\mathfrak{g} = \mathfrak{sp}_4$. Consider $S = \mathbb{R}^2$ with its canonical basis $\{\varepsilon_1, \varepsilon_2\}$. The simple roots are $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = 2\varepsilon_2$. The fundamental weights are $\Lambda_1 = \varepsilon_1$ and $\Lambda_2 = \varepsilon_1 + \varepsilon_2$. An example of dominant weight is $\Lambda_1 + \Lambda_2$. The dominant chamber is the hatched area on the following figure:



Weight module. Let \mathfrak{g} be a Lie algebra. The *lower central series* of \mathfrak{g} is defined inductively as follows: $\mathfrak{g}_{(1)} = \mathfrak{g}$ and $\mathfrak{g}_{(i)} = [\mathfrak{g}, \mathfrak{g}_{(i-1)}]$, for $i > 1$. A Lie algebra is *nilpotent* if its lower central series eventually becomes zero. A *normalizer* of a Lie subalgebra \mathfrak{h} of \mathfrak{g} is the set of elements x in \mathfrak{g} such that $[x, \mathfrak{h}] \subset \mathfrak{h}$. In other words, the normalizer of \mathfrak{h} is the biggest subalgebra of \mathfrak{g} such that \mathfrak{h} is an ideal of it. A subalgebra \mathfrak{h} of \mathfrak{g} is called a *Cartan subalgebra* of \mathfrak{g} if \mathfrak{h} is nilpotent and equal to its own normalizer.

Let \mathfrak{h} be a maximal Cartan subalgebra of a Lie algebra \mathfrak{g} and let V be a representation of \mathfrak{g} . The *weight space* of V with weight $\lambda: \mathfrak{h} \rightarrow \mathbb{K}$ is the subspace

$$V_\lambda := \{ v \in V \mid \forall \xi \in \mathfrak{h}, \quad \xi \cdot v = \lambda(\xi)v \}$$

where $\xi \cdot v$ denotes the action of \mathfrak{h} on V . A *weight of the representation* V is a weight λ such that the corresponding weight space V_λ is non-zero. A *highest weight of the representation* V is a weight λ such that $V_\lambda \neq 0$ but $V_{\lambda+\alpha} = 0$ for all positive root α . If V is the direct sum of its weight spaces

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

then it is called a *weight module*.

1.2. Plactic monoid of type A

For instance, let \mathfrak{g} be a Lie algebra. Define a map $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ by

$$\text{ad}X(Y) = [X, Y],$$

for all X and Y in \mathfrak{g} . The Lie algebra homomorphism ad is called *the adjoint representation* of \mathfrak{g} . It is isomorphic to the following direct sum

$$\bigoplus_{\alpha \in \Phi \cup \{0\}} V_{\alpha}.$$

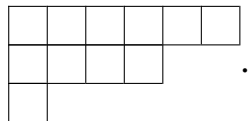
1.2. PLACTIC MONOID OF TYPE A

For a natural number $n > 0$, let denote by $[n]$ the finite set $\{1, 2, \dots, n\}$ totally ordered by $1 < \dots < n$. For every word w in the free monoid $[n]^*$, we denote by $\ell(w)$ the length of w and by $\ell^{\text{nds}}(w)$ the length of its longest non-decreasing subsequence.

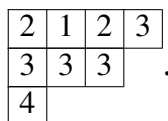
1.2.1. Schensted's insertion algorithms

In order to compute the length of the longest decreasing and non-decreasing subsequences of a given word on the free monoid $[n]^*$, Schensted, [104], introduced two insertion algorithms: the column and the row insertion. These algorithms describe how an element in the set $[n]$ acts on a Young tableau to create a new tableau with one extra entry than before.

Tableaux. A *Young diagram of shape* $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{N}^k$, with $n \geq \lambda_1 \geq \dots \geq \lambda_k \geq 1$, is a collection of boxes in left-justified rows, where the i -th row is of length λ_i , for $i = 1, \dots, k$. For instance, a Young diagram of shape $(6, 4, 1)$ is



A *Young tableau of shape* λ is a Young diagram of the same shape filled with elements of the ordered set $[n]$ where the entries strictly increase down each column. For example, a Young tableau of shape $(4, 3, 1)$ is



A *semistandard tableau of shape* λ , or *tableau* for short, is a Young tableau of shape λ where the entries weakly increase along each row and strictly increase down each column. For instance, a

1.2.1. Schensted's insertion algorithms

tableau of shape $(4,3,1)$ is

1	1	2	3
2	3	3	
3			

A *standard tableau of shape* λ is a tableau of shape λ where the entries are elements from 1 to $\sum_{i=1}^k \lambda_i$, each occurring once. For example, a standard tableau of shape $(5,4,2,1)$ is

1	3	7	10	11
2	4	6	9	
5	8			
12				

Words tableaux. A *row* is a non-decreasing word $x_1 \dots x_k$ in the free monoid $[n]^*$, i.e., with $x_i \leq x_{i+1}$ for $1 \leq i \leq k-1$. A row $x_1 \dots x_k$ *dominates* a row $y_1 \dots y_l$, and we denote by $x_1 \dots x_k \blacktriangleright y_1 \dots y_l$, if $k \leq l$, and $x_i > y_i$, for $1 \leq i \leq k$.

Any word w in the free monoid $[n]^*$ has a unique decomposition as a product of rows of maximal length $w = u_1 \dots u_k$. Such a word w is a *word tableau* if $u_1 \blacktriangleright u_2 \blacktriangleright \dots \blacktriangleright u_k$. It is usual to write words tableaux in a planar form, with the rows placed in order of domination from bottom to top and left-justified as mentioned by Fulton in [29]. In other words, a word tableau admits a tableau as a planar representation.

In the sequel, if there is no confusion we will say a tableau for a word tableau or for its planar form.

1.2.1.1. Example. i) The word $w = 6845562233571112444$ in the free monoid $[8]^*$ is a word tableau whose planar representation is

1	1	1	2	4	4	4
2	2	3	3	5	7	
4	5	5	6			
6	8					

(1.2)

ii) Consider the word $w_1 = 435334$ in the free monoid $[5]^*$. Its planar representation is

3	3	4
3	5	
4		

Since the first row does not dominate the second one, the word w_1 is not a tableau.

iii) Consider the word $w_2 = 23412$ in the free monoid $[4]^*$. Its planar representation is

1	2	
2	3	4

Since it is not a Young diagram, it is not a tableau.

1.2. Plactic monoid of type A

Column readings. The *column reading* of the planar representation of a tableau w constructs a word, denoted by $C(w)$, obtained by reading the planar representation of w columnwise from bottom to top and from left to right. For example, the column reading of the tableau (1.2) is 6421852153163254744.

The *Japanese reading* of the planar representation of a tableau w constructs a word, denoted by $w(w)$, obtained by reading the planar representation of w columnwise from top to bottom and from right to left. For instance, the Japanese reading of the tableau (1.2) is 4474523613512581264.

The column-insertion. The column-insertion procedure, denoted by $x \rightarrow w$, inserts a word x in $[n]$ into a Young tableau w as follows. Let y be the smallest element of the first column of the Young tableau w such that $y \geq x$. Then x replaces y in the first column and y is bumped into the next column where the process is repeated. This procedure terminates when the element which is bumped is greater than all the elements of the next column. Then it is placed at the bottom of that column.

For each word $w = x_1 \dots x_p$, a tableau $P(w)$ can be computed by starting with the empty word, which is a valid tableau, and iteratively applying Schensted's algorithm. That is,

$$P(w) = x_p \rightarrow (x_{p-1} \rightarrow (\dots \rightarrow (x_2 \rightarrow x_1))).$$

1.2.1.2. Example. Consider again the tableau w with planar representation (1.2) and let us compute

$$\boxed{3} \rightarrow \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 4 & 4 & 4 \\ \hline 2 & 2 & 3 & 3 & 5 & 7 & \\ \hline 4 & 5 & 5 & 6 & & & \\ \hline 6 & 8 & & & & & \\ \hline \end{array} .$$

First we begin inserting 3 in the first column,

$$\boxed{3} \rightarrow \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 4 \\ \hline 6 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline 3 & \\ \hline 6 & \\ \hline \end{array}$$

then we insert 4 in the second column,

$$\boxed{4} \rightarrow \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 5 \\ \hline 8 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 5 \\ \hline 2 & \\ \hline 4 & \\ \hline 8 & \\ \hline \end{array}$$

after we insert 5 in the next column,

$$\boxed{5} \rightarrow \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 5 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 5 \\ \hline 3 & \\ \hline 5 & \\ \hline \end{array}$$

1.2.1. Schensted's insertion algorithms

then we insert 5 in the next column,

$$\boxed{5} \rightarrow \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline 6 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 2 & 6 \\ \hline 3 & \\ \hline 5 & \\ \hline \end{array}$$

then we insert 6 in the next column,

$$\boxed{6} \rightarrow \begin{array}{|c|} \hline 4 \\ \hline 5 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 4 \\ \hline 5 \\ \hline 6 \\ \hline \end{array}$$

Hence, we have

$$\boxed{3} \rightarrow \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 4 & 4 & 4 \\ \hline 2 & 2 & 3 & 3 & 5 & 7 & \\ \hline 4 & 5 & 5 & 6 & & & \\ \hline 6 & 8 & & & & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 4 & 4 & 4 \\ \hline 2 & 2 & 3 & 3 & 5 & 7 & \\ \hline 3 & 4 & 5 & 5 & 6 & & \\ \hline 6 & 8 & & & & & \\ \hline \end{array}$$

Robinson-Schensted's correspondence. For each word w in the free monoid $[n]^*$, one can construct a pair $(P(w), Q(w))$ consisting of a tableau $P(w)$ and a standard tableau $Q(w)$ as follows. As previously discussed, for every word $w = x_1 \dots x_p$ in the free monoid $[n]^*$, a tableau $P(w)$ is computed starting with the empty word and iteratively applying Schensted's algorithm. During the computation of the tableau $P(w)$, a standard tableau $Q(w)$, [104, Lemma 2], is obtained by successively putting i in a box at the same place of the added box when inserting the element x_i of w .

The bijection $w \mapsto (P(w), Q(w))$, [104, Lemma 3], between words in $[n]^*$ and pairs consisting of a tableau over $[n]^*$ and a standard tableau of the same shape is called the *Robinson-Schensted correspondence*.

For instance, the successive steps of the computation of the tableau $P(1213214)$ and the standard tableau $Q(1213214)$ are

$$\begin{array}{|c|} \hline 1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & \\ \hline 3 & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & \\ \hline 3 & & \\ \hline 4 & & \\ \hline \end{array} = P(1213214).$$

$$\begin{array}{|c|} \hline 1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & \\ \hline 4 & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & \\ \hline 4 & & \\ \hline 7 & & \\ \hline \end{array} = Q(1213214).$$

1.2. Plactic monoid of type A

The row-insertion. As for the column-insertion, one can compute a tableau from a word w in $[n]^*$ by inserting elements in the rows of tableaux instead of columns. The resulted tableau is also denoted by $P(w)$.

Given a tableau w written as a product of rows of maximal length $w = u_1 \dots u_k$ and y in $[n]$, the row-insertion algorithm, denoted by $w \leftarrow x$, computes the tableau $P(wx)$ as follows:

- If $u_k x$ is a row, the result is $u_1 \dots u_k x$.
- If $u_k x$ is not a row, then suppose $u_k = x_1 \dots x_l$, with x_i in $[n]$ and let j minimal such that $x_j > x$. Then the result is $P(u_1 \dots u_{k-1} x_j) v_k$, where $v_k = x_1 \dots x_{j-1} x x_{j+1} \dots x_l$.

1.2.1.3. Example. Consider again the tableau w with planar representation (1.2) and let us compute $w \leftarrow 3$.

1	1	1	2	4	4	4	←	3
2	2	3	3	5	7			
4	5	5	6					
6	8							

1	1	1	2	3	4	4	←	4
2	2	3	3	5	7			
4	5	5	6					
6	8							

1	1	1	2	3	4	4	←	5
2	2	3	3	4	7			
4	5	5	6					
6	8							

1	1	1	2	3	4	4	←	6
2	2	3	3	4	7			
4	5	5	5					
6	8							

1	1	1	2	3	4	4	←	8
2	2	3	3	4	7			
4	5	5	6					
6	6							

1	1	1	2	3	4	4
2	2	3	3	4	7	
4	5	5	5			
6	6					
8						

= $P(w3)$

Robinson–Schensted’s correspondence. The Robinson–Schensted correspondence can be also constructed using the row-insertion algorithm. For any word $w = x_1 \dots x_l$ in the free monoid $[n]^*$ the tableau $P(w)$ is computed by starting with the empty word, which is a valid tableau, and iteratively applying Schensted’s algorithm. That is,

$$P(w) = (((x_1 \leftarrow x_2) \leftarrow \dots) \leftarrow x_{p-1}) \leftarrow x_p.$$

During the computation of the tableau $P(w)$, the standard tableau $Q(w)$ is obtained by successively putting i in a box at the same place of the added box when inserting the element x_i of w .

1.2.2. Longest non-decreasing subsequence of a word

1.2.1.4. Example. The successive steps of the computation of the tableau $P(1213214)$ are

$$\begin{array}{c} \boxed{1} \leftarrow \boxed{2} \end{array}, \quad
 \begin{array}{c} \boxed{1} \boxed{2} \leftarrow \boxed{1} \end{array}, \quad
 \begin{array}{c} \boxed{1} \boxed{1} \leftarrow \boxed{3} \\ \boxed{2} \end{array}, \quad
 \begin{array}{c} \boxed{1} \boxed{1} \boxed{3} \leftarrow \boxed{2} \\ \boxed{2} \end{array}, \quad
 \begin{array}{c} \boxed{1} \boxed{1} \boxed{2} \leftarrow \boxed{1} \\ \boxed{2} \boxed{3} \end{array}, \quad
 \begin{array}{c} \boxed{1} \boxed{1} \boxed{1} \leftarrow \boxed{4} \\ \boxed{2} \boxed{2} \\ \boxed{3} \end{array}, \quad
 \begin{array}{c} \boxed{1} \boxed{1} \boxed{1} \boxed{4} \\ \boxed{2} \boxed{2} \\ \boxed{3} \end{array}$$

Then

$$P(1213214) = \begin{array}{c} \boxed{1} \boxed{1} \boxed{1} \boxed{4} \\ \boxed{2} \boxed{2} \\ \boxed{3} \end{array} \quad \text{and} \quad
 Q(1213214) = \begin{array}{c} \boxed{1} \boxed{2} \boxed{4} \boxed{7} \\ \boxed{3} \boxed{5} \\ \boxed{6} \end{array}.$$

1.2.2. Longest non-decreasing subsequence of a word

Let w be a permutation in the free monoid $[\mathfrak{n}]^*$, that is a word in which no elements are repeated. The j -th basic subsequence of w consists of the elements which are inserted into the j -th place in the first row of the tableau $P(w)$, where $P(w)$ is computed by the row-insertion procedure. We have the following properties:

1. Each basic subsequence is a decreasing subsequence [104, Lemma 1]. Indeed, by definition of the row-insertion procedure, each element in the j -th basic subsequence replaces a previous larger element.
2. Given any member of the j -th basic subsequence, we can find a member of the $(j - 1)$ st basic subsequence which is smaller and which occurs further to the left in the given sequence [104, Lemma 2]. Indeed, when a given member of the j -th basic subsequence is inserted in the first row, the element in the $(j - 1)$ st place is a member of the $(j - 1)$ st basic subsequence, it is smaller than the inserted one and occurs further to the left in the given sequence.

1.2.2.1. Theorem ([104, Theorem 1]). *Let w be a permutation in $[\mathfrak{n}]^*$. The number of columns in the tableau $P(w)$ is equal to the length of the longest increasing subsequence of w .*

Proof. The number of columns is the same as the number of basic subsequences. By 1., there can be at most one member of each basic subsequence in any increasing subsequence. By 2., we can construct any increasing subsequence with one element from each basic subsequence. \square

Let w_{back} be the word obtained from w by writing its elements backwards. Then the tableau $P(w_{\text{back}})$ is obtained from $P(w)$ by interchanging rows and columns [104, Lemma 7]. Since writing a sequence backwards changes increasing subsequences into decreasing ones, one obtains the following theorem

1.2.2.2. Theorem ([104, Theorem 2]). *Let w be a permutation in $[\mathfrak{n}]^*$. The number of rows in the tableau $P(w)$ is equal to the length of the longest decreasing subsequence of w .*

1.2. Plactic monoid of type A

Furthermore, Schensted showed in Part II of [104] that we can obtain the same properties for words in which no elements are repeated and the ones in which some of the elements are repeated. For this, he considered the following procedure. For any word w in $[n]^*$, suppose the smallest element of w appears p times in it, the next smallest q times, and so on. One replaces the p occurrences of the smallest number by the numbers $1, \dots, p$, the occurrences of the next element by $p + 1, \dots, p + q$, and so on. Then the decreasing subsequences of the two sequences are in one-to-one correspondence, and the increasing subsequences of the new sequence are in one-to-one correspondence with the non-decreasing subsequences of w . Hence, one obtains the following theorem

1.2.2.3. Theorem ([104]). *Let w be a word in the free monoid $[n]^*$. The number of columns of the tableau $P(w)$ is equal to $l^{nds}(w)$. The number of rows of $P(w)$ is equal to the length of the longest decreasing subsequence in w .*

Another proof of this theorem is also given in Section 5.2 in [91, Chapter 6] by proving a more general result of Greene, which gives an interpretation of the lengths of all rows and the heights of all columns of the tableau $P(w)$, for any word w in the free monoid $[n]^*$.

1.2.3. The plactic monoid

The plactic congruence. We will denote by \sim_{plax} the equivalence relation on the free monoid $[n]^*$ defined by

$$u \sim_{\text{plax}} v \text{ if and only if } P(u) = P(v),$$

where the tableaux $P(u)$ and $P(v)$ are respectively computed by the row-insertion procedure.

The *plactic monoid of rank n* , denoted by \mathbf{P}_n , [74], is the quotient of the free monoid $[n]^*$ by the congruence \sim_{plax} . The congruence \equiv on the free monoid $[n]^*$ generated by the *Knuth relations*:

$$\{ zxy \equiv xzy \mid 1 \leq x \leq y < z \leq n \} \cup \{ yzx \equiv yxz \mid 1 \leq x < y \leq z \leq n \}, \quad (1.3)$$

is called the *plactic congruence*. Moreover, the equivalence relation coincides with the plactic congruence, see [63, Theorem 6] or [91, Theorem 5.2.5]. In particular, each plactic class contains exactly one tableau.

In 1.3.4, we will show how the plactic monoid \mathbf{P}_n is related to the representations of the general Lie algebra, which is of type A, and thus called as the *plactic monoid of type A*, [72].

Properties. Any word w in $[n]^*$ is equal to its Schensted's tableau in \mathbf{P}_n , that is, $w = P(w)$ holds in \mathbf{P}_n , [91, Proposition 5.2.3]. Indeed, one proceeds by induction on the length of the word w . By definition of the congruence \equiv , the result is true for $\ell(w) \leq 3$. Suppose that for a word w , we have $P(w) \equiv w$, and let x be an element in $[n]$. One has to show that $P(wx) \equiv wx$. Note that it is sufficient to suppose that w is a row. If wx is a row then $P(wx) = wx$ and

otherwise $P(wx) = yw'$ where y is the leftmost letter in w such that $y > x$, and w' is obtained from w by replacing y by x . Then by a sequence of applications of the relations (1.3), one obtains $wx \equiv uyxv \equiv yuxv = P(wx)$.

In addition, we have that every word w in the free monoid $[n]^*$ is equal to the column reading of the tableau $P(w)$, that is, $w = C(P(w))$ holds in \mathbf{P}_n [91, Problem 5.2.4].

1.2.3.1. Remark. For every words u and v in the free monoid $[n]^*$, if we compute the tableaux $P(u)$ and $P(v)$ by the column-insertion algorithm then the relation \sim_{plac} coincides with the congruence generated by the following *Knuth relations*:

$$\{ zxy = xzy \mid 1 \leq x < y \leq z \leq n \} \cup \{ yzx = yxz \mid 1 \leq x \leq y < z \leq n \}. \quad (1.4)$$

This comes from the fact that, for any word w in the free monoid $[n]^*$, computing the tableau $P(w)$ by the row-insertion algorithm is the same as computing the tableau $P(w_{\text{back}})$ by the column-insertion algorithm. For a word w in $[n]^*$, if the tableau $P(w)$ is computed by the row-insertion algorithm then we read it using the column reading, whereas we use the Japanese reading when it is computed by the column-insertion procedure.

1.3. CRYSTAL GRAPHS AND PLACTIC MONOIDS

In the following section, we will detail the notion of crystal graphs and we show how the plactic monoid introduced by Lascoux and Schützenberger is related to the representations of the general Lie algebra. Using Kashiwara’s theory of crystal bases, we give the definition of the plactic congruence for any semisimple Lie algebra. After, we briefly outline the notion of plactic monoids of type B, C, D and G_2 .

1.3.1. Crystal graphs

Crystals. Let \mathfrak{g} be a semisimple Lie algebra, let P be its weight lattice and let $P^* = \text{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$. Let $\{\alpha_i\}_{i \in I}$ be the simple roots of \mathfrak{g} and $\{h_i\}_{i \in I}$ the corresponding coroots. The two lattices P and P^* are free \mathbb{Z} -modules of rank $\sharp I$. Let $\langle \cdot, \cdot \rangle : P^* \times P \rightarrow \mathbb{Z}$ be the canonical pairing.

A *crystal* is a set B endowed with applications

$$\begin{aligned} \text{wt} : B &\rightarrow P, \\ \varepsilon_i : B &\rightarrow \mathbb{Z} \cup \{-\infty\}, \\ \varphi_i : B &\rightarrow \mathbb{Z} \cup \{-\infty\}, \\ \tilde{e}_i : B &\rightarrow B \cup \{0\}, \\ \tilde{f}_i : B &\rightarrow B \cup \{0\}, \end{aligned}$$

satisfying the following properties:

1.3. Crystal graphs and plactic monoids

1. $\varphi_i(\mathbf{b}) = \varepsilon_i(\mathbf{b}) + \langle h_i, \text{wt}(\mathbf{b}) \rangle$, for any i .
2. If \mathbf{b} is an element in B satisfying $\tilde{e}_i(\mathbf{b}) \neq 0$, then

$$\varepsilon_i(\tilde{e}_i(\mathbf{b})) = \varepsilon_i(\mathbf{b}) - 1, \quad \varphi_i(\tilde{e}_i(\mathbf{b})) = \varphi_i(\mathbf{b}) + 1 \quad \text{and} \quad \text{wt}(\tilde{e}_i(\mathbf{b})) = \text{wt}(\mathbf{b}) + \alpha_i.$$

3. If \mathbf{b} is an element in B satisfying $\tilde{f}_i(\mathbf{b}) \neq 0$, then

$$\varepsilon_i(\tilde{f}_i(\mathbf{b})) = \varepsilon_i(\mathbf{b}) + 1, \quad \varphi_i(\tilde{f}_i(\mathbf{b})) = \varphi_i(\mathbf{b}) - 1 \quad \text{and} \quad \text{wt}(\tilde{f}_i(\mathbf{b})) = \text{wt}(\mathbf{b}) - \alpha_i.$$

4. For two elements \mathbf{b}_1 and \mathbf{b}_2 in B , we have $\mathbf{b}_2 = \tilde{f}_i(\mathbf{b}_1)$ if and only if $\mathbf{b}_1 = \tilde{e}_i(\mathbf{b}_2)$.

5. If \mathbf{b} is an element in B satisfying $\varphi_i(\mathbf{b}) = -\infty$, then $\tilde{e}_i(\mathbf{b}) = \tilde{f}_i(\mathbf{b}) = 0$.

Note that for \mathbf{b} in B , we have $\varepsilon_i(\mathbf{b}) = \max\{k \mid \tilde{e}_i^k(\mathbf{b}) \neq 0\}$ and $\varphi_i(\mathbf{b}) = \max\{k \mid \tilde{f}_i^k(\mathbf{b}) \neq 0\}$, see [59, Section 4].

Tensor product of crystals. The tensor product of two crystals B_1 and B_2 is defined by

$$B_1 \otimes B_2 = \{\mathbf{b}_1 \otimes \mathbf{b}_2 \mid \mathbf{b}_1 \in B_1, \mathbf{b}_2 \in B_2\}.$$

The set $B_1 \otimes B_2$ is endowed with a structure of crystal by defining the action of \tilde{e}_i and \tilde{f}_i on the tensor product by

$$\begin{aligned} \tilde{f}_i(\mathbf{b}_1 \otimes \mathbf{b}_2) &= \begin{cases} \tilde{f}_i(\mathbf{b}_1) \otimes \mathbf{b}_2 & \text{if } \varphi_i(\mathbf{b}_1) > \varepsilon_i(\mathbf{b}_2) \\ \mathbf{b}_1 \otimes \tilde{f}_i(\mathbf{b}_2) & \text{if } \varphi_i(\mathbf{b}_1) \leq \varepsilon_i(\mathbf{b}_2) \end{cases} \\ \tilde{e}_i(\mathbf{b}_1 \otimes \mathbf{b}_2) &= \begin{cases} \mathbf{b}_1 \otimes \tilde{e}_i(\mathbf{b}_2) & \text{if } \varphi_i(\mathbf{b}_1) < \varepsilon_i(\mathbf{b}_2) \\ \tilde{e}_i(\mathbf{b}_1) \otimes \mathbf{b}_2 & \text{if } \varphi_i(\mathbf{b}_1) \geq \varepsilon_i(\mathbf{b}_2) \end{cases} \end{aligned}$$

Crystal graphs. *Crystal graphs* are oriented graphs with labelled arrows. The set of vertices is B and an arrow $\mathbf{a} \xrightarrow{i} \mathbf{b}$ means that $\tilde{f}_i(\mathbf{a}) = \mathbf{b}$ and $\tilde{e}_i(\mathbf{b}) = \mathbf{a}$. For any vertex \mathbf{u} of B , we denote by $B(\mathbf{u})$ the connected component of the crystal graph containing \mathbf{u} . Note that every representation of a semisimple Lie algebra admits a crystal graph.

An *crystal isomorphism* between two connected components $B(\mathbf{u})$ and $B(\mathbf{v})$ is a bijective map $\psi : B(\mathbf{u}) \rightarrow B(\mathbf{v})$ that satisfies the following conditions:

- i) it is *weight-preserving*, that is $\text{wt}(\mathbf{u}) = \text{wt}(\psi(\mathbf{u}))$, for all \mathbf{u} in $B(\mathbf{u})$,
- ii) for all \mathbf{u} and \mathbf{u}' in $B(\mathbf{u})$, if there is an arrow $\mathbf{u} \xrightarrow{i} \mathbf{u}'$, then there is an arrow $\psi(\mathbf{u}) \xrightarrow{i} \psi(\mathbf{u}')$ in $B(\mathbf{v})$.

If there is a crystal isomorphism between two connected components $B(\mathbf{u})$ and $B(\mathbf{v})$, we say simply that $B(\mathbf{u})$ and $B(\mathbf{v})$ are isomorphic.

For a deeper information, we refer the reader to [57, 58, 59, 61].

1.3.2. Crystal graphs for Type A

The special linear Lie algebra. The *special linear Lie algebra* $\mathfrak{sl}_n(\mathbb{C})$ is the Lie algebra of traceless square matrices of rank n . It is a simple Lie algebra of type A_{n-1} . Let $E_{i,j}$ be the n by n elementary matrix having 1 at the (i, j) -entry and 0 elsewhere. Let $I = \{1, 2, \dots, n-1\}$ be an index set, the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ is generated by the elements e_i, f_i and h_i , where

$$e_i = E_{i,i+1}, \quad f_i = E_{i+1,i}, \quad h_i = E_{ii} - E_{i+1,i+1} \quad \text{for } i \in I.$$

Consider the linear functional

$$\begin{aligned} \varepsilon_i : M_{n \times n}(\mathbb{C}) &\rightarrow \mathbb{C} \\ (t_{ij})_{i,j=1,\dots,n} &\mapsto t_{ii}. \end{aligned}$$

The *simple roots* of $\mathfrak{sl}_n(\mathbb{C})$ are given by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $i \in I$. The *fundamental weights* of $\mathfrak{sl}_n(\mathbb{C})$ are $\Lambda_i = \varepsilon_1 + \dots + \varepsilon_i$, for $i \in I$. Note that $\varepsilon_1 + \dots + \varepsilon_n = 0$. In this case, the *weight lattice* is $P = \bigoplus_{i \in I} \mathbb{Z}\Lambda_i$. A *dominant weight* λ of $\mathfrak{sl}_n(\mathbb{C})$ is of the form $\lambda = \lambda_1 \varepsilon_1 + \dots + \lambda_n \varepsilon_n$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$.

Let λ be a dominant weight and let $V(\lambda)$ be the irreducible highest weight module with highest weight λ of $\mathfrak{sl}_n(\mathbb{C})$. We denote by $B(\lambda)$ its crystal graph.

The vector representation. Let $W_n = \mathbb{C}v_1 \oplus \dots \oplus \mathbb{C}v_n$ be a n -dimensional vector space of $\mathfrak{sl}_n(\mathbb{C})$, where v_i 's are the elements of the standard basis. The Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ acts on W_n by matrix multiplication:

$$\begin{aligned} e_i v_j &= \begin{cases} v_i & \text{if } j = i + 1, \\ 0 & \text{otherwise,} \end{cases} \\ f_i v_j &= \begin{cases} v_{i+1} & \text{if } j = i, \\ 0 & \text{otherwise,} \end{cases} \\ h_i v_j &= \begin{cases} v_i & \text{if } j = i, \\ -v_{i+1} & \text{if } j = i + 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The vector space W_n is called the *vector representation* of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$. It is also denoted by $V(\Lambda_1)$ because it is a representation of highest weight Λ_1 . The crystal graph of the vector representation $V(\Lambda_1)$ is

$$B(\Lambda_1) : 1 \xrightarrow{1} 2 \xrightarrow{2} \dots \rightarrow n-1 \xrightarrow{n-1} n.$$

The crystal operators. The free monoid $[n]^*$ over $[n]$ is a crystal, see [61]. Indeed, for a fixed i , the action of the crystal operators \tilde{e}_i and \tilde{f}_i on a word w of the monoid $[n]^*$ is described as follows. First, one considers the word w_i obtained by deleting all symbols other than i and $i+1$ from w . Secondly, we remove adjacent letters $(i, i+1)$, then we obtain a new subword of w . The second step of the process is repeated until it is impossible to remove more letters.

1.3. Crystal graphs and plactic monoids

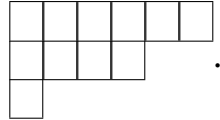
- The word $\tilde{e}_i(w)$ is obtained by replacing in w the rightmost element $i + 1$ of the final subword, by i and the others elements of w stay unchanged. If such $i + 1$ does not exist, then $\tilde{e}_i(w) = 0$.
- The word $\tilde{f}_i(w)$ is obtained by replacing in w the leftmost element i of the final subword, by $i + 1$ and the others elements of w stay unchanged. If such i does not exist, then $\tilde{f}_i(w) = 0$.

1.3.2.1. Example. Let $w = 1322312331112$ be a word on the free monoid $[3]^*$. Let us compute $\tilde{e}_2(w)$ and $\tilde{f}_2(w)$. We have $w_2 = 32232332$. After deleting adjacent letters $(2, 3)$, the first subword of w_2 is 3232 . After repeating this process, the second subword is 32 . We cannot remove new elements from the last subword, then we obtain:

$$\tilde{e}_2(w) = 122231233112 \quad \text{and} \quad \tilde{f}_2(w) = 132231233113.$$

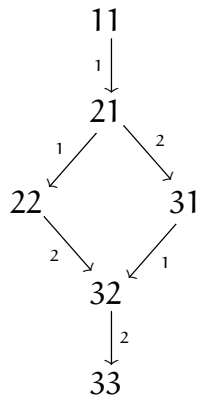
Crystal graphs and Young tableaux. Let $\lambda = \sum_{i=1}^n \lambda_i \Lambda_i$ be the highest weight of an irreducible representation $V(\lambda)$ of $\mathfrak{sl}_n(\mathbb{C})$. For λ , we associate the Young diagram $Y(\lambda)$ containing λ_i columns of height i . This Young diagram is said to have shape λ . The length of its i -th row is $\lambda_i + \dots + \lambda_n$ and the number of its boxes is $\sum_{i=1}^n \lambda_i i$.

For instance, the Young diagram $Y(2\Lambda_1 + 3\Lambda_2 + \Lambda_3)$ is



Let $\mathcal{B}(\lambda)$ be the set of all tableaux of shape λ . If we replace each tableau by its Japanese reading, then the set $\mathcal{B}(\lambda)$ becomes a subset of the free monoid $[n]^*$ that is a crystal as shown above. Hence, we obtain that $\mathcal{B}(\lambda)$ has a crystal structure. By [59, 61], the crystal graph $B(\lambda)$ of $V(\lambda)$ is identified with the set $\mathcal{B}(\lambda)$. Its highest weight vertex is the reading of the tableau of shape λ filled with 1 on the 1st row, 2 on the 2nd row, ... , and n on the n th row.

1.3.2.2. Example. For $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$, the crystal graph of $B(2\Lambda_1)$ is the following



where the vertices are the Japanese readings of all tableaux of shape $2\Lambda_2$.

1.3.3. Crystal graphs for type C

The symplectic Lie algebra. The *symplectic Lie algebra* $\mathfrak{sp}_{2n}(\mathbb{C})$ is the Lie algebra of $2n$ by $2n$ matrices M , for $n > 0$, that satisfy

$$\Omega M + M^T \Omega = 0,$$

where M^T is the transpose of M and $\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

This Lie algebra is a simple Lie algebra of type C_n , for $n \geq 2$. Let $E_{i,j}$ be the $2n$ by $2n$ elementary matrix having 1 at the (i, j) -entry and 0 elsewhere. Then, the Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C})$ is generated by e_i, f_i, h_i , for $i = 1, \dots, n$, where

$$\begin{aligned} e_i &= E_{i,i+1} - E_{n+i+1,n+i}, & f_i &= E_{i+1,i} - E_{n+i,n+i+1}, \\ h_i &= E_{ii} - E_{i+1,i+1} - E_{n+i,n+i} + E_{n+i+1,n+i+1}, \\ e_n &= E_{n,2n}, & f_n &= E_{2n,n}, & h_n &= E_{n,n} - E_{2n,2n} \end{aligned}$$

for $i = 1, \dots, n-1$.

Consider the linear functional

$$\begin{aligned} \varepsilon_i &: \mathbf{M}_{2n \times 2n}(\mathbb{C}) \rightarrow \mathbb{C} \\ & (t_{ij})_{i,j=1,\dots,2n} \mapsto t_{ii}. \end{aligned}$$

The *simple roots* of $\mathfrak{sp}_{2n}(\mathbb{C})$ are $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $i = 1, \dots, n-1$ and $\alpha_n = 2\varepsilon_n$. The *fundamental weights* of $\mathfrak{sp}_{2n}(\mathbb{C})$ are $\Lambda_i = \varepsilon_1 + \dots + \varepsilon_i$, for $i = 1, \dots, n$. In this case, the *weight lattice* of $\mathfrak{sp}_{2n}(\mathbb{C})$ is $P = \bigoplus_i \mathbb{Z}\Lambda_i$.

The vector representation. Let $V_n = \mathbb{C}^{2n}$ be the *vector representation* of $\mathfrak{sp}_{2n}(\mathbb{C})$, this representation is of dimension $2n$ and we index a basis of V_n by the set

$$\mathcal{C}_n = \{1, 2, \dots, n, \bar{n}, \dots, \bar{1}\},$$

totally ordered by $1 < 2 < \dots < n < \bar{n} < \dots < \bar{1}$.

Note that every representation of the Lie algebra \mathfrak{sp}_{2n} admits a crystal graph, [59]. The crystal graph of the vector representation V_n is :

$$1 \xrightarrow{1} 2 \xrightarrow{2} \dots \rightarrow n-1 \xrightarrow{n-1} n \xrightarrow{n} \bar{n} \xrightarrow{n-1} \overline{n-1} \xrightarrow{n-2} \dots \rightarrow \bar{2} \xrightarrow{1} \bar{1}.$$

1.3. Crystal graphs and plactic monoids

The crystal operators. Denote by \mathcal{C}_n^* the free monoid over \mathcal{C}_n . Kashiwara and Nakashima showed in [61] that the free monoid \mathcal{C}_n^* is a crystal and described a process to compute the action of the crystal operators \tilde{e}_i and \tilde{f}_i on a word w of the monoid \mathcal{C}_n^* , for a fixed i . First, one considers the word w_i obtained by deleting all symbols other than $i, i+1, \bar{i+1}$ and \bar{i} from w . One identifies the letters i and $\bar{i+1}$ by the symbol $+$ and the letters $i+1$ and \bar{i} by the symbol $-$. Secondly, we remove the subwords of length 2 in w_i which correspond to the symbol $+-$, *i.e.*, we remove adjacent letters $(i, i+1), (i, \bar{i}), (\bar{i+1}, i+1)$ and $(\bar{i+1}, \bar{i})$. Then we obtain a new subword of w . The second step of the process is repeated until it is impossible to remove more letters. Let r and s be respectively the number of letters corresponding to the symbols $-$ and $+$ in the final subword.

- If $r > 0$ then $\tilde{e}_i(w)$ is obtained by replacing in w the rightmost element with the symbol $-$ of the final subword, by its corresponding element with the symbol $+$, *i.e.*, $i+1$ is transformed into i or \bar{i} into $\bar{i+1}$ or for $i = n$, \bar{n} into n , and the others elements of w stay unchanged. If $r = 0$, then $\tilde{e}_i(w) = 0$.
- If $s > 0$ then $\tilde{f}_i(w)$ is obtained by replacing in w the leftmost element with the symbol $+$ of the final subword, by its corresponding element with the symbol $-$, *i.e.*, i is transformed into $i+1$ or $\bar{i+1}$ into \bar{i} or for $i = n$, n into \bar{n} , and the others elements of w stay unchanged. If $s = 0$, then $\tilde{f}_i(w) = 0$.

1.3.3.1. Example. Consider the word $w = \bar{3}32313\bar{3}233\bar{3}1$. For $i = 2$, we have

$$w_i = \bar{3}3233\bar{3}233\bar{3}.$$

After deleting subwords corresponding to $+-$, the first subword of w_i is $\bar{3}\bar{3}\bar{3}\bar{3}$. After repeating this process, the second subword is $\bar{3}\bar{3}$. We cannot remove new elements from the last subword, then $r = s = 1$. Finally, we obtain :

$$\tilde{e}_2(w) = \bar{3}3231\bar{3}233\bar{3}1 \quad \text{and} \quad \tilde{f}_2(w) = \bar{3}32313\bar{3}233\bar{2}1.$$

Connected components of crystal graphs. Now, we consider tensor products of the vector representation $V_n^{\otimes l}$, for any l and the infinite dimensional representation $\bigoplus_l V_n^{\otimes l}$. The crystal graphs of these representations are respectively denoted by $G_{n,l}$ and G_n . Note that each vertex $x_1 \otimes x_2 \otimes \dots \otimes x_l$ of the crystal graph of $V_n^{\otimes l}$ is identified with the word $x_1 x_2 \dots x_l$ in the monoid \mathcal{C}_n^* . In other words, the vertices of G_n are indexed by the words of \mathcal{C}_n^* and those of $G_{n,l}$ by the words of length l .

In addition, the crystal graph $G_{n,l}$ can be decomposed into connected components. They correspond to the crystal graphs of the irreducible representations occurring in the decomposition of $V_n^{\otimes l}$. In each connected component, there exists a unique vertex w^0 which satisfy the following property:

$$\tilde{e}_i(w^0) = 0, \quad \text{for } i = 1, \dots, n.$$

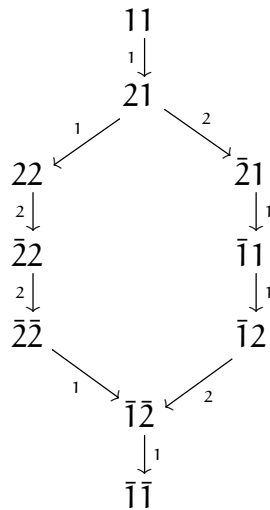
1.3.4. Crystal plactic monoids

This vertex is called the *vertex of highest weight*, and its weight is

$$\text{wt}(w^0) = d_n \Lambda_n + \sum_{i=1}^{n-1} (d_i - d_{i+1}) \Lambda_i,$$

where d_i is the number of letters i in w^0 minus the number of letters \bar{i} .

1.3.3.2. Example. For $n = 2$, the crystal $B(11)$ is presented by :



where the vertices are labelled by words. In this case, the vertex of highest weight is 11 and its weight is $2\Lambda_1$.

1.3.3.3. Lemma ([61]). For any words u and v in C_n^* , the word uv is a vertex of highest weight of a connected component of G_n if, and only if, u is a vertex of highest weight and $\varepsilon_i(v) \leq \varphi_i(u)$ for any $i = 1, \dots, n$.

1.3.4. Crystal plactic monoids

The Robinson-Schensted correspondence has found an interpretation in terms of Kashiwara's theory of crystal bases [21, 72, 77]. In the following, we will show how the plactic monoid introduced by Lascoux and Schützenberger is related to the representations of the general linear Lie algebra $\mathfrak{gl}_n(\mathbb{C})$. Since the theory of Kashiwara crystal bases exists also for the classical simple Lie algebras, a corresponding plactic monoid was constructed for each of these algebras by a case-by-case analysis. We call these monoids, the *crystal plactic monoids*.

1.3. Crystal graphs and plactic monoids

The general linear Lie algebra. Recall from Example 1.1.0.1 that the general linear Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ is the Lie algebra of n by n matrices with the bracket defined by $[X, Y] = XY - YX$, for X and Y in $\mathfrak{gl}_n(\mathbb{C})$. Let $I = \{1, 2, \dots, n-1\}$ be an index set, the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ can be generated by the elements e_i, f_i , for $i \in I$ and E_{jj} for $j = 1, \dots, n$, where

$$e_i = E_{i,i+1}, \quad f_i = E_{i+1,i}, \quad h_i = E_{ii} - E_{i+1,i+1} \quad \text{for } i \in I.$$

The *maximal Cartan subalgebra* for $\mathfrak{gl}_n(\mathbb{C})$ is given by

$$\mathfrak{h} = \mathbb{C}E_{11} \oplus \dots \oplus \mathbb{C}E_{n,n}.$$

For each $i = 1, \dots, n$, one defines a linear map $\varepsilon_i : \mathfrak{h} \rightarrow \mathbb{C}$ by

$$\varepsilon_i(\mathfrak{h}) = \lambda_i, \quad \text{with } \mathfrak{h} = \text{diag}(\lambda_j \mid j = 1, \dots, n) \in \mathfrak{h}.$$

The *simple roots* of $\mathfrak{gl}_n(\mathbb{C})$ are given by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $i = 1, 2, \dots, n-1$. Its *fundamental weights* are $\Lambda_i = \varepsilon_1 + \dots + \varepsilon_i$, for $i = 1, \dots, n$. Its *weight lattice* is $P = \mathbb{Z}\varepsilon_1 \oplus \dots \oplus \mathbb{Z}\varepsilon_n$.

Let $V = \mathbb{C}^n$ be its vector representation as defined in Example 1.1.0.3 and we denote by $[n] = \{1 < \dots < n\}$ the ordered set indexing the standard basis of V .

Plactic monoid of type A. Each vertex $v_1 \otimes \dots \otimes v_l$ of the crystal graph of the representation $\bigoplus_l V^{\otimes l}$ is considered as the word $v_1 \dots v_l$ in the free monoid $[n]^*$. For any words u and v in $[n]^*$, we have, [21]:

- $P(u) = P(v)$ if and only if u and v occur at the same place in their isomorphic connected components of the crystal graph of $\bigoplus_l V^{\otimes l}$,
- $Q(u) = Q(v)$ if and only if u and v occur in the same connected component of this graph.

As a consequence, we obtain that for any words u and v in the free monoid $[n]^*$, the following assertions are equivalent:

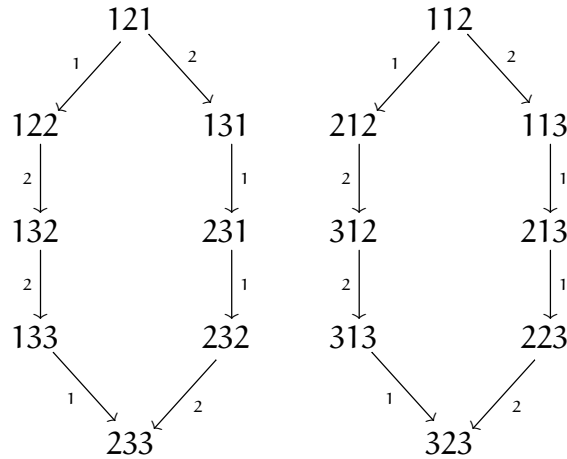
- a) $u \sim_{\text{plax}} v$,
- b) the words u and v appear in the same place in isomorphic connected components of the crystal graph of the representation $\bigoplus_l V^{\otimes l}$,
- c) $P(u) = P(v)$, where $P(u)$ and $P(v)$ are the unique tableau obtained from u and v by Schensted's column-insertion,
- d) one can transform u into v using the Knuth relations:

$$\{ xzy = zxy \mid 1 \leq x < y \leq z \leq n \} \cup \{ yxz = yzx \mid 1 \leq x \leq y < z \leq n \}.$$

1.3.4. Crystal plactic monoids

As we have seen in the previous section, the plactic monoid \mathbf{P}_n of rank n is defined as the quotient of the free monoid $[n]^*$ by the plactic congruence \sim_{plax} . As previously discussed, this congruence is also interpreted in terms of Kashiwara's crystal graphs of the representations of the general Lie algebra. In this way, the plactic monoid \mathbf{P}_n is related to the representations of the general Lie algebra, which is of type A, and called the plactic monoid of type A.

The Knuth relations appear in the crystal isomorphism between $B(121)$ and $B(112)$:



Crystal plactic monoids. The plactic congruence can be generalised for all classical simple Lie algebras as follows. To each classical simple Lie algebra, one associates a finite alphabet S indexing a basis of the vector representation V of the algebra. Two words u and v in the monoid S^* are *plactic congruent*, and denoted by $u \sim_{\text{crys}} v$, if they appear in the same place in their isomorphic connected components of the crystal graph of the representation $\bigoplus_{\lambda} V^{\otimes \lambda}$.

For each classical simple Lie algebra, the corresponding *crystal plactic monoid* is defined as the quotient of the free monoid S^* by the congruence \sim_{crys} .

1.3.4.1. Remark. As we have seen above, the crystal plactic monoid corresponding to the representations of the general Lie algebra is equal to the plactic monoid \mathbf{P}_n . Now if we consider the representations of the special linear Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, the corresponding crystal plactic monoid is equal to the quotient of the free monoid $[n]^*$ by the congruence generated by the following families of relations:

- i) $yzx = yxz$, for $x \leq y < z$ and $xzy = zxy$, for $x < y \leq z$,
- ii) $1 \dots n = \varepsilon$, where ε is the empty word.

The relations i) are the Knuth relations and the relation ii) comes from the fact that for the special linear Lie algebra we have the following equality $\varepsilon_1 + \dots + \varepsilon_n = 0$.

1.4. Littelmann paths and plactic monoids

Plactic monoids of type C, B, D and G_2 . In the literature, the crystal plactic monoid corresponding to the representations of the symplectic Lie algebra, the odd-dimensional orthogonal Lie algebra and the even-dimensional orthogonal Lie algebra is respectively called the plactic monoid of type C, B and D.

Several works have generalised the notions of tableaux for other types, see [5, 117, 84, 86, 88, 108, 32]. Kashiwara and Nakashima was also obtained a generalisation for all classical Lie algebras by introducing the notion of *admissible columns*, see [61].

Lecouvey in [78] and Baker in [4] introduced independently the plactic monoid of type C using Kashiwara's theory of crystal bases and they introduced an insertion procedure similar to the Schensted's one for type A. Let u and v be two words in \mathcal{C}_n^* . The relation \sim_{crys} is defined on the free monoid \mathcal{C}_n^* by : $u \sim_{\text{crys}} v$ if, and only if, $B(u)$ and $B(v)$ are isomorphic and u and v have the same position in their isomorphic connected components $B(u)$ and $B(v)$ of the crystal G_n . In other words, $u \sim_{\text{crys}} v$ if and only if there exist i_1, \dots, i_r such that $u = \tilde{f}_{i_1} \cdots \tilde{f}_{i_r}(u^0)$ and $v = \tilde{f}_{i_1} \cdots \tilde{f}_{i_r}(v^0)$, where u^0 and v^0 are the vertices of highest weight of $B(u)$ and $B(v)$. The quotient

$$\mathbf{P}_n(C) := \mathcal{C}_n^* / \sim_{\text{crys}}$$

is called the *plactic monoid for type C* or the *symplectic plactic monoid*. We will explain in details this monoid in Chapter 3 and we will show how a finite convergent presentation is constructed for it using admissible columns.

In [79], Lecouvey gave presentations for plactic monoids of types B and D. He used the notion of admissible columns and *orthogonal tableaux* introduced by Kashiwara and Nakashima for these types, see [79, Section 3]. Let \mathcal{B}_n and \mathcal{D}_n be respectively the alphabets corresponding to type B and D. Using the same insertion's algorithm described for type C, Lecouvey showed that for any word w in the free monoids \mathcal{B}_n^* and \mathcal{D}_n^* , one can compute a unique orthogonal tableau $P(w)$ that its reading is equal to w in the corresponding plactic monoid.

Note also that a plactic monoid for type G_2 was also introduced by Lecouvey by generalising the notion of admissible columns to this type. He also gave an insertion algorithm similar to the ones for the classical types, see [83].

Numerous applications of plactic monoids have been discovered including a combinatorial description of the Kostka-Foulkes polynomials which arise as entries of the character table of the finite reductive groups, see [80, 82, 81].

1.4. LITTELMANN PATHS AND PLACTIC MONOIDS

In the following section, we recall the definitions and properties of paths, root operators and Lakshmibai-Seshadri's paths. After, we introduce the notion of standard tableaux as defined by Littelmann and we recall the definition of plactic algebra for any semisimple Lie algebra. We refer the reader to [85, 87, 88] for a full introduction.

1.4.1. Lakshmibai-Seshadri's paths

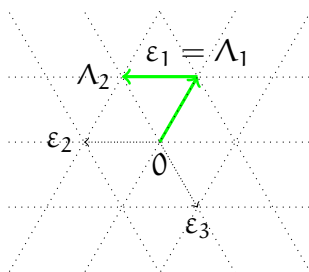
Let us fix a semisimple Lie algebra \mathfrak{g} . Let P be its weight lattice and let $P^* = \text{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$. Let $\{\alpha_i\}_{i \in I}$ be the simple roots of \mathfrak{g} and $\{h_i\}_{i \in I}$ the corresponding coroots. Let $\langle \cdot, \cdot \rangle : P^* \times P \rightarrow \mathbb{Z}$ be the canonical pairing.

Paths. A *path* is a piecewise linear continuous map $\pi : [0, 1] \rightarrow P_{\mathbb{R}}$. We will consider paths up to a reparametrization, that is, the path π is equal to any path $\pi \circ \varphi$, where $\varphi : [0, 1] \rightarrow [0, 1]$ is a piecewise linear non-decreasing surjective continuous map. The *weight* of a path π is defined by $\text{wt}(\pi) := \pi(1)$. For instance, for the Lie algebra $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, the paths $\pi_{\varepsilon_i} : t \mapsto t\varepsilon_i$ are of weight ε_i , for $i = 1, \dots, n$.

Given two paths π_1 and π_2 , the *concatenation* $\pi_1 \star \pi_2$ is defined by:

$$\pi_1 \star \pi_2(t) := \begin{cases} \pi_1(2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ \pi_1(1) + \pi_2(2t - 1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

1.4.1.1. Example. Let $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$. Consider the paths $\pi_1 : t \mapsto t\varepsilon_1$ and $\pi_2 : t \mapsto t\varepsilon_2$. The path $\pi_1 \star \pi_2$ is the green path on the following figure:



We denote by

$$\Pi = \{ \pi : [0, 1] \rightarrow P_{\mathbb{R}} \mid \pi(0) = 0 \text{ and } \pi(1) \in P \}$$

the set of all paths with sources 0 and whose weights lie in P . We will denote by $\theta : [0, 1] \rightarrow P_{\mathbb{R}}$ the trivial path defined by $\theta(t) = 0$, for any $t \in [0, 1]$. For λ in $P_{\mathbb{R}}$, consider the path $\pi_{\lambda} : [0, 1] \rightarrow P_{\mathbb{R}}$ that connects the origin with λ by a straight line, that is $\pi_{\lambda}(t) = t\lambda$, for any $t \in [0, 1]$. The path π_{λ} is in Π if and only if λ is in P .

Denote by $\mathbb{Z}\Pi$ the *algebra of paths* defined as the free \mathbb{Z} -module with basis Π whose product is given by the concatenation of paths and where the unity is the trivial path.

Root operators. For each simple root α , one defines root operators $e_{\alpha}, f_{\alpha} : \Pi \rightarrow \Pi \cup \{0\}$ as follows. Every path π in Π is cutted into three parts $\pi_1 \star \pi_2 \star \pi_3$. Then the new path $e_{\alpha}(\pi)$ or $f_{\alpha}(\pi)$ is either equal to 0 or $\pi_1 \star s_{\alpha}(\pi_2) \star \pi_3$, where s_{α} denotes the simple reflection with respect to the root α .

1.4. Littelmann paths and plactic monoids

More explicitly, consider the function

$$\begin{aligned} h_\alpha : [0, 1] &\rightarrow \mathbb{R} \\ t &\mapsto \langle \pi(t), \alpha^\vee \rangle \end{aligned}$$

Let $\text{Min} := \min(\text{Im}(h_\alpha) \cap \mathbb{Z})$ be the minimum attained by h_α . If $\text{Min} = 0$, define $e_\alpha(\pi) = 0$. If $\text{Min} < 0$, let

$$t_1 = \min\{ t \in [0, 1] \mid h_\alpha(t) = \text{Min} \}$$

and

$$t_0 = \max\{ t < t_1 \mid h_\alpha(t) = \text{Min} + 1 \}.$$

Denote by π_1, π_2 and π_3 the paths defined by

$$\begin{aligned} \pi_1(t) &:= \pi(tt_0) \\ \pi_2(t) &:= \pi(t_0 + t(t_1 - t_0)) - \pi(t_0) \\ \pi_3(t) &:= \pi(t_1 + t(1 - t_1)) - \pi(t_1), \text{ for } t \in [0, 1]. \end{aligned}$$

By definition of the π_i , we have $\pi = \pi_1 \star \pi_2 \star \pi_3$. Then $e_\alpha(\pi) = \pi_1 \star s_\alpha(\pi_2) \star \pi_3$.

Similarly, one can define the operator f_α . Let

$$p = \max\{ t \in [0, 1] \mid h_\alpha(t) = \text{Min} \}.$$

Denote by I the integral part of $h_\alpha(1) - \text{Min}$. If $I = 0$, define $f_\alpha(\pi) = 0$. If $I > 0$, let $x > p$ such that

$$h_\alpha(x) = \text{Min} + 1 \text{ and } \text{Min} < h_\alpha(x) < \text{Min} + 1, \text{ for } p < t < x.$$

Denote by π_1, π_2 and π_3 the paths defined by

$$\begin{aligned} \pi_1(t) &:= \pi(tp) \\ \pi_2(t) &:= \pi(p + t(x - p)) - \pi(p) \\ \pi_3(t) &:= \pi(x + t(1 - x)) - \pi(x), \text{ for } t \in [0, 1]. \end{aligned}$$

By definition of the π_i , we have $\pi = \pi_1 \star \pi_2 \star \pi_3$. Then $f_\alpha(\pi) = \pi_1 \star s_\alpha(\pi_2) \star \pi_3$.

Note that these operators preserve the length of the paths. We have also that if $f_\alpha(\pi) = \pi' \neq 0$ then $e_\alpha(\pi') = \pi \neq 0$ and $\text{wt}(f_\alpha(\pi)) = \text{wt}(e_\alpha(\pi)) - \alpha$.

The algebra generated by the roots operators. For all simple root α , let \mathcal{A} be the subalgebra of $\text{End}_{\mathbb{Z}}(\mathbb{Z}\Pi)$ generated by the root operators f_α and e_α . Define Π^+ to be the set of paths π such that the image is contained in $P_{\mathbb{R}}^+$ and denote by M_π the \mathcal{A} -module $\mathcal{A}\pi$. Let B_π be the \mathbb{Z} -basis $M_\pi \cap \Pi$ of M_π . In other words, let I be the set indexing the set of simple roots of \mathfrak{g} , we have

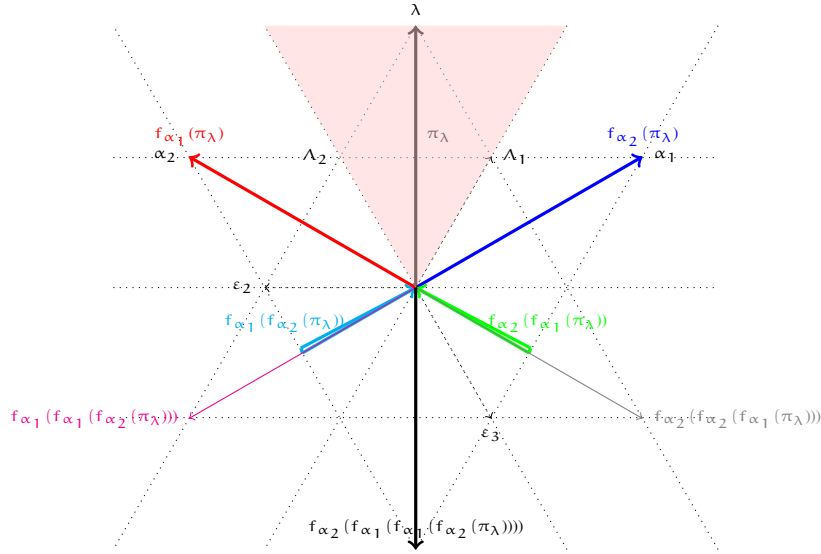
$$B_\pi = \{ f_{\alpha_{i_1}} \circ \dots \circ f_{\alpha_{i_r}}(\pi) \mid \pi \in \Pi^+ \text{ and } i_1, \dots, i_r \in I \}.$$

1.4.1. Lakshmibai-Seshadri's paths

Given two paths π and π' in Π^+ , if $\text{wt}(\pi) = \text{wt}(\pi')$, then the \mathcal{A} -modules M_π and $M_{\pi'}$ are isomorphic. For π in Π^+ , let π' in M_π be an arbitrary path. We have $e_\alpha(\pi') = 0$ if and only if $\pi = \pi'$.

We denote by M_λ the \mathcal{A} -module $\mathcal{A}\pi_\lambda$ generated by the path π_λ . In addition, the \mathbb{Z} -module M_λ has for a basis the set B_{π_λ} consisting of all paths in M_λ .

1.4.1.2. Example. Let $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ and let α_1, α_2 be its simple roots and Λ_1, Λ_2 be its fundamental weights. For $\lambda = \Lambda_1 + \Lambda_2$, consider the path $\pi_\lambda : t \mapsto t\lambda$. Let us compute B_{π_λ} . We have



where

$$\begin{aligned}
 \pi_\lambda &: t \rightarrow t\lambda \\
 f_{\alpha_1}(\pi_\lambda) &=: \pi_2 : t \rightarrow t\alpha_2 \\
 f_{\alpha_2}(\pi_\lambda) &=: \pi_3 : t \rightarrow t\alpha_1 \\
 f_{\alpha_2}(\pi_2) &=: \pi_4 : t \rightarrow \begin{cases} -t\alpha_2 & \text{for } 0 \leq t \leq \frac{1}{2} \\ (t-1)\alpha_2 & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases} \\
 f_{\alpha_1}(\pi_3) &=: \pi_5 : t \rightarrow \begin{cases} -t\alpha_1 & \text{for } 0 \leq t \leq \frac{1}{2} \\ (t-1)\alpha_1 & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases} \\
 f_{\alpha_2}(\pi_4) &=: \pi_6 : t \rightarrow -t\alpha_2 \\
 f_{\alpha_1}(\pi_5) &=: \pi_7 : t \rightarrow -t\alpha_1 \\
 f_{\alpha_2}(\pi_7) = f_{\alpha_1}(\pi_6) &=: \pi_8 : t \rightarrow -t\lambda
 \end{aligned}$$

Hence we obtain $B_{\pi_\lambda} = \{\pi_\lambda, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7, \pi_8\}$.

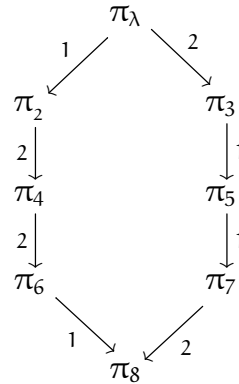
Crystal graphs. For π in Π^+ , let $B(\pi)$ be the oriented graph with set of vertices B_π and an arrow $\pi \xrightarrow{i} \pi'$ means that $f_{\alpha_i}(\pi) = \pi'$ and $e_{\alpha_i}(\pi') = \pi$. We denote by $\mathcal{G}(\Pi^+)$ the union

$$\bigcup_{\pi \in \Pi^+} B(\pi).$$

1.4. Littelmann paths and plactic monoids

The graph $\mathcal{G}(\Pi^+)$ is a crystal graph in the Kashiwara sense, see Subsection 1.3.1.

1.4.1.3. Example. Consider the semisimple Lie algebra $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$. For $\lambda = \Lambda_1 + \Lambda_2$, the elements of B_{π_λ} , obtained in Example 1.4.1.2, can be represented by the following connected component $B(\pi_\lambda)$



of $\mathcal{G}(\Pi^+)$.

L-S paths. For a dominant weight λ , the *Lakshmibai-Seshadri paths*, *L-S paths* for short, of shape λ are the paths π of the form

$$\pi = f_{\alpha_1} \circ \dots \circ f_{\alpha_s}(\pi_\lambda)$$

where $\alpha_1, \dots, \alpha_s$ are simple roots of a semisimple Lie algebra \mathfrak{g} . That is, these paths are all the elements of B_{π_λ} .

Examples. Consider $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ and let Λ_1 and Λ_2 be its fundamental weights. Let $\lambda = \Lambda_1 + \Lambda_2$, the L-S paths of shape λ are all the elements of B_{π_λ} as obtained in Example 1.4.1.2. The L-S paths of shape Λ_2 are the elements of

$$B_{\pi_{\Lambda_2}} = \{ \pi_{\varepsilon_1 + \varepsilon_2}, \pi_{\varepsilon_1 + \varepsilon_3}, \pi_{\varepsilon_2 + \varepsilon_3} \}.$$

For the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, the L-S paths of shape Λ_1 are the paths π_{ε_i} , for $i = 1, \dots, n + 1$.

Examples. For the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$, the L-S paths of shape Λ_1 are the paths π_{ε_i} , for $i = 1, \dots, n$. Consider the Lie algebra $\mathfrak{gl}_3(\mathbb{C})$, the L-S paths of shape Λ_2 are the vertices of the following connected component $B(\pi_{\Lambda_2})$:

$$\begin{array}{c} \pi_{\Lambda_2} \\ \downarrow 2 \\ \pi_{\varepsilon_1 + \varepsilon_3} \\ \downarrow 1 \\ \pi_{\varepsilon_2 + \varepsilon_3} \end{array}$$

1.4.2. Plactic algebra for any semisimple Lie algebra

Standard tableaux. Let $\lambda = \lambda_1 + \dots + \lambda_k$ be a sum of dominant weights. If for all $i = 1, \dots, k$, the path π_i is an L-S path of shape λ_i , then the monomial $\pi_1 \star \dots \star \pi_k$ is called an *L-S monomial of shape* $\underline{\lambda} = (\lambda_1, \dots, \lambda_k)$.

A *Young tableau of shape* $\lambda = a_1\Lambda_1 + \dots + a_n\Lambda_n$ is an L-S monomial

$$\star_{1 \leq i \leq n} \pi_{1, \Lambda_i} \star \dots \star \pi_{a_i, \Lambda_i}$$

where π_{i, Λ_i} is an L-S path of shape Λ_i , for $1 \leq i \leq n$. That is, the first a_1 paths are of shape Λ_1 , the next a_2 are of shape Λ_2, \dots , the final a_n paths are of shape Λ_n .

Let T be a Young tableau of shape $\lambda = a_1\Lambda_1 + \dots + a_n\Lambda_n$. The Young tableau T is called *standard of shape* λ if

$$T \in \mathcal{A} \left(\underbrace{(\pi_{\Lambda_1} \star \dots \star \pi_{\Lambda_1})}_{a_1 \text{ times}} \star \underbrace{(\pi_{\Lambda_2} \star \dots \star \pi_{\Lambda_2})}_{a_2 \text{ times}} \star \dots \star \underbrace{(\pi_{\Lambda_n} \star \dots \star \pi_{\Lambda_n})}_{a_n \text{ times}} \right).$$

The case of type A. For type A_n , consider the ordered set $[n + 1]$. For a dominant weight $\lambda = a_1\Lambda_1 + \dots + a_n\Lambda_n$, set $p_i = a_i + \dots + a_n$. Consider the Young diagram with p_1 boxes in the first row, p_2 boxes in the second row, etc. A standard tableau of shape $\lambda = a_1\Lambda_1 + \dots + a_n\Lambda_n$ and type A is a filling of the boxes of this Young diagram with elements of the set $[n + 1]$ such that the entries are strictly increasing in the column from top to bottom and weakly increasing in the rows from left to right. In other words all the standard tableaux of shape λ and type A are the elements of $B_{\pi_{\Lambda_1} \star \dots \star \pi_{\Lambda_n}}$, where the L-S monomial

$$\underbrace{(\pi_{\Lambda_1} \star \dots \star \pi_{\Lambda_1})}_{a_1 \text{ times}} \star \underbrace{(\pi_{\Lambda_2} \star \dots \star \pi_{\Lambda_2})}_{a_2 \text{ times}} \star \dots \star \underbrace{(\pi_{\Lambda_n} \star \dots \star \pi_{\Lambda_n})}_{a_n \text{ times}}$$

corresponds to the Young tableau with only 1's in the first row, 2's in the second row, ..., n 's in the n -th row. Note that the standard tableaux of type A correspond to tableaux as defined in Section 1.2.

1.4.2.1. Example. In the following two examples, we consider the Lie algebra $\mathfrak{gl}_3(\mathbb{C})$.

- The L-S monomial $\pi_{\varepsilon_2} \star \pi_{\varepsilon_1 + \varepsilon_3} \star \pi_{\varepsilon_1 + \varepsilon_3} \star \pi_{\varepsilon_1 + \varepsilon_2 + \varepsilon_3}$ of shape $\lambda = \Lambda_1 + 2\Lambda_2 + \Lambda_3$ corresponds to the following tableau

1	1	1	2
2	3	3	
3			

In addition, we have that $\pi_{\varepsilon_2} \star \pi_{\varepsilon_1 + \varepsilon_3} \star \pi_{\varepsilon_1 + \varepsilon_3} \star \pi_{\varepsilon_1 + \varepsilon_2 + \varepsilon_3} \in B(\pi_{\Lambda_1} \star \pi_{\Lambda_2} \star \pi_{\Lambda_2} \star \pi_{\Lambda_3})$ with

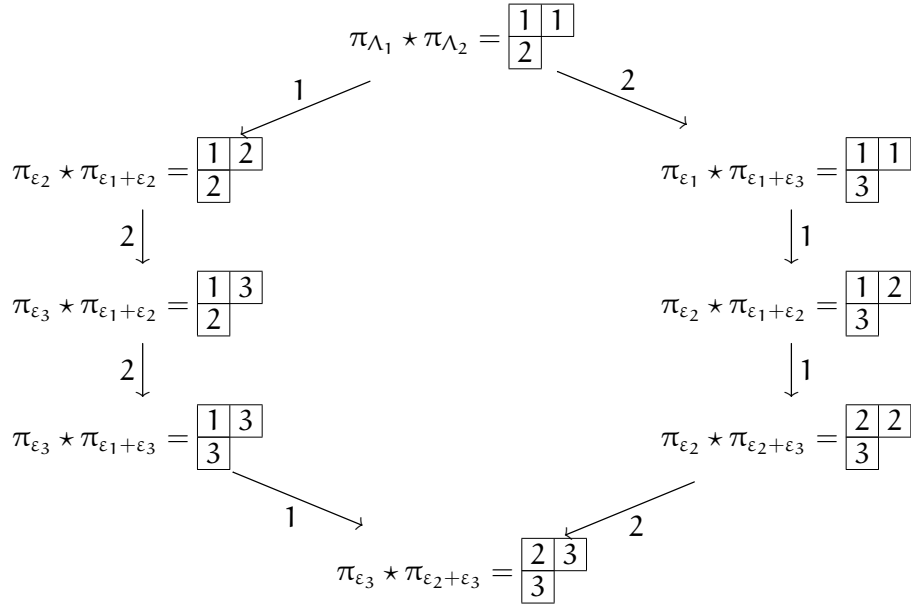
$$\pi_{\varepsilon_2} \star \pi_{\varepsilon_1 + \varepsilon_3} \star \pi_{\varepsilon_1 + \varepsilon_3} \star \pi_{\varepsilon_1 + \varepsilon_2 + \varepsilon_3} = f_{\alpha_2}(f_{\alpha_1}(f_{\alpha_2}(\pi_{\Lambda_1} \star \pi_{\Lambda_2} \star \pi_{\Lambda_2} \star \pi_{\Lambda_3})))$$

1.4. Littelmann paths and plactic monoids

where the path $\pi_{\Lambda_1} \star \pi_{\Lambda_2} \star \pi_{\Lambda_2} \star \pi_{\Lambda_3}$ corresponds to the following tableau

1	1	1	1
2	2	2	
3			

- The tableaux of shape $\Lambda_1 + \Lambda_2$ on the set $[3]$ are the vertices of the following connected component $B(\pi_{\Lambda_1} \star \pi_{\Lambda_2})$



The case of type C. Consider the ordered set $\mathcal{C}_n = \{1 < \dots < n < \bar{n} < \dots < \bar{2} < \bar{1}\}$. For a dominant weight $\lambda = a_1\Lambda_1 + \dots + a_n\Lambda_n$, set $p_1 = a_1 + 2a_2 + \dots + 2a_n$ and for $i \geq 2$ set $p_i = 2a_i + \dots + 2a_n$. Consider the Young diagram with p_1 boxes in the first row, p_2 boxes in the second row, etc. A standard tableau of shape λ and type C is a filling of the boxes of this Young diagram with elements of the set \mathcal{C}_n such that the entries are strictly increasing in the column from top to bottom, but i and \bar{i} are never entries in the same column, and the entries are weakly increasing in the rows from left to right. In addition, for each pair of columns C_{a_1+2j-1} and C_{a_1+2j} , for $j = 1, \dots, a_2 + \dots + a_n$, either these columns are equal or the column C_{a_1+2j} is obtained from C_{a_1+2j-1} by exchanging an even number of times an entry k , $1 \leq k \leq \bar{1}$, in C_{a_1+2j-1} by \bar{k} , see [32, Section 4.2]. Note that the standard tableaux of type C correspond to symplectic tableaux as defined in Chapter 3, Section 3.3.1.

For type C_3 , the following Young tableau is a standard tableau of shape $\Lambda_1 + 2\Lambda_3$

1	1	1	2	$\bar{3}$
2	2	$\bar{3}$	$\bar{3}$	
3	3	2	$\bar{1}$	

1.4.2. Plactic algebra for any semisimple Lie algebra

Columns and L-S paths for type A. Every L-S path of shape Λ_i , for i in $[n]$, is a Young tableau of shape Λ_i consisting of one column. Thus, the L-S paths of shape Λ_i and the columns of length i are in one-to-one correspondence.

1.4.2.2. Example. Consider the Lie algebra $\mathfrak{gl}_3(\mathbb{C})$. The columns of length 2 on the set $[3]$ are the vertices of the following connected component $B(\pi_{\Lambda_2})$:

$$\begin{array}{c}
 \pi_{\Lambda_2} = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \\
 \quad \downarrow 2 \\
 \pi_{\varepsilon_1 + \varepsilon_3} = \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \end{array} \\
 \quad \downarrow 1 \\
 \pi_{\varepsilon_2 + \varepsilon_3} = \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array}
 \end{array}$$

1.4.2.3. Remark. As previously discussed, L-S paths correspond to columns for type A. Similarly, these paths coincide with the notion of admissible columns for type C, B and D in the Lecouvey sense, see Chapter 3, Subsection 3.3.1.

Plactic algebra. Let $\mathbb{Z}\Pi_0$ be the \mathcal{A} -submodule $\mathcal{A}\Pi^+$ of $\mathbb{Z}\Pi$ generated by the paths in Π^+ . For two paths π_1 and π_2 in $\mathbb{Z}\Pi_0$, denote by π_1^+ and π_2^+ the unique paths in Π^+ such that $\pi_1 \in M_{\pi_1^+}$ and $\pi_2 \in M_{\pi_2^+}$. One can define a relation \sim_{path} on $\mathbb{Z}\Pi_0$ by : $\pi_1 \sim_{\text{path}} \pi_2$ if, and only if, $\text{wt}(\pi_1^+) = \text{wt}(\pi_2^+)$ and $\psi(\pi_1) = \pi_2$ under the isomorphism $\psi : M_{\pi_1^+} \rightarrow M_{\pi_2^+}$. The *plactic algebra* for \mathfrak{g} is the quotient

$$\mathbb{Z}\mathcal{P} := \mathbb{Z}\Pi_0 / \sim_{\text{path}} .$$

For $\pi \in \mathbb{Z}\Pi_0$, we denote by $[\pi] \in \mathbb{Z}\mathcal{P}$ its equivalence class. The classes $[m]$ of standard Young tableaux form a basis of the plactic algebra $\mathbb{Z}\mathcal{P}$, see [88, Theorem 7.1].

1.4.2.4. Example ([88, Theorem C]). For type A, consider the alphabet $\{1, \dots, n\}$. The plactic congruence coincides with the congruence generated by the following families of relations on the word algebra $\mathbb{Z}\{1, \dots, n\}$:

- (a) $xzy = zxy$ for $1 \leq x < y \leq z \leq n$.
- (b) $yxz = yzx$ for $1 \leq x \leq y < z \leq n$.
- (c) $12 \dots n$ is the empty word.

1.4. Littelmann paths and plactic monoids

Presentations of plactic monoids

Contents

2.1	Presentations of monoids by 2-polygraphs	94
2.2	Tietze transformations and the pre-column presentation	101
2.3	The Knuth–Bendix’s completion of the Knuth presentation	107

In order to make this document self-contained, we recall in this chapter the basic definitions and properties of presentations of monoids in the language of polygraphs. Examples of presentations of plactic monoids are also presented.

In Section 2.1, we recall the notion of low-dimensional polygraphs. We develop in 2.1.1 the case of 2-polygraphs and we give in 2.1.2 some of their rewriting properties as termination and confluence. After recalling the definition of Yamanouchi path tableaux and using the notions of paths and L-S paths presented in Chapter 1, we construct in 2.1.3 a convergent presentation of the plactic monoid in terms of the crystal isomorphism defined in Chapter 1, Subsection 1.3.4.

In Section 2.2, we recall the notion of Tietze transformations of 2-polygraphs from [31] and we construct in 2.2.2 a presentation of the plactic monoid, called the *pre-column presentation*, that is Tietze equivalent to the Knuth presentation.

In 2.3.1, we recall the Knuth–Bendix’s completion that computes a convergent presentation of a monoid from a terminating one. In 2.3.2, we compute the Knuth–Bendix’s completion of the Knuth presentation of the plactic monoid of type A and rank 3. We recall in 2.3.3 that for higher values of n , the Knuth presentation does not admit a finite completion compatible with the lexicographic order without adding new generators.

For deeper informations, we refer the reader to [37, 36, 39, 40, 31]

2.1. Presentations of monoids by 2-polygraphs

2.1. PRESENTATIONS OF MONOIDS BY 2-POLYGRAPHS

In this section, we briefly recall the notion of two dimensional polygraphs. We present rewriting properties of presentations of monoids in terms of 2-polygraphs and we refer the reader to [39] for a deeper presentation.

2.1.1. 2-polygraphs

1-polygraphs. A 1-polygraph is a directed graph (Σ_0, Σ_1)

$$\Sigma_0 \begin{array}{c} \xleftarrow{s_0} \\ \xleftarrow{t_0} \end{array} \Sigma_1$$

given by a set Σ_0 of 0-cells, a set Σ_1 of 1-cells together with two maps s_0 and t_0 sending a 1-cell x on its *source* $s_0(x)$ and its *target* $t_0(x)$. A 1-polygraph (Σ_0, Σ_1) is *finite* if the sets Σ_0 and Σ_1 are finite. We will denote by Σ_1^* the free category generated by the 1-polygraph (Σ_0, Σ_1) , that is, the category defined as follows:

- i) its objects are the 0-cells in Σ_0 ,
- ii) for any 0-cells p and q , the elements of the homset $\Sigma_1^*(p, q)$ are paths from p to q in the 1-polygraph (Σ_0, Σ_1) , *i.e.*, the finite sequences

$$p \xrightarrow{x_1} p_1 \xrightarrow{x_2} p_2 \xrightarrow{x_3} \dots \xrightarrow{x_{n-1}} p_{n-1} \xrightarrow{x_n} q$$

of 1-cells of Σ_1 ,

- iii) the composition is the concatenation of paths,
- iv) the identity on a 0-cell p is the empty path with source and target p .

The notion of 1-polygraphs correspond to abstract rewriting systems [9].

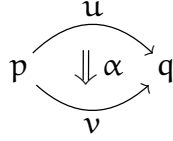
Globular extension. A *globular extension* of the free category Σ_1^* is a set Σ_2 equipped with maps

$$\Sigma_1^* \begin{array}{c} \xleftarrow{s_1} \\ \xleftarrow{t_1} \end{array} \Sigma_2$$

such that, for every α in Σ_2 , the following *globular relations* are satisfied

$$s_0(s_1(\alpha)) = s_0(t_1(\alpha)) \quad \text{and} \quad t_0(s_1(\alpha)) = t_0(t_1(\alpha)).$$

The pair $(s_1(\alpha), t_1(\alpha))$ is called a *1-sphere* in the free category Σ_1^* . A 2-cell α of the globular extension Σ_2 is graphically represented by the following globular shape



that relates parallel 1-cells $s_1(\alpha) = u$ and $t_1(\alpha) = v$ in the free category Σ_1^* .

In this work, we deal with monoids, that is, categories with only one 0-cell. Then the set Σ_0 is reduced to a set with exactly one element denoted \bullet . In this case, the 1-polygraph (Σ_0, Σ_1) will be identified to a set Σ_1 and the free category Σ_1^* will be identified to the free monoid on Σ_1 .

2-polygraphs. Let us introduce the notion of 2-polygraphs that correspond in the literature to string rewriting systems [9]. A *2-polygraph* Σ is a triple $(\Sigma_0, \Sigma_1, \Sigma_2)$, where (Σ_0, Σ_1) is a 1-polygraph and the set Σ_2 is a globular extension of the free category Σ_1^* :

$$\Sigma_0 \begin{array}{c} \xleftarrow{s_0} \\ \xrightarrow{t_0} \end{array} \Sigma_1^* \begin{array}{c} \xleftarrow{s_1} \\ \xrightarrow{t_1} \end{array} \Sigma_2 .$$

The elements of the set Σ_2 are called the *2-cells* of Σ , or the *rewriting rules* of Σ . If there is no possible confusion, we will denote by Σ_2 the set of 2-cells of the 2-polygraph Σ or the 2-polygraph itself.

2-categories. A *2-category* is a category enriched in categories. When two 1-cells, or 2-cells, f and g of a 2-category are i -composable, that is $t_i(f) = s_i(g)$, for $i = 0, 1$, we denote by $f \star_i g$ their i -composite. A *(2, 1)-category* is a category enriched in groupoid, that is a 2-category whose 2-cells are invertible for the 1-composition. We will denote by Σ_2^* (resp. Σ_2^\top) the 2-category (resp. (2, 1)-category) freely generated by the 2-polygraph Σ . For further informations about 2-categories and the construction of the 2-categories Σ_2^* and Σ_2^\top , we refer the reader to [39, Section 2.4.].

Presentations of monoids by 2-polygraphs. The *monoid presented* by a 2-polygraph Σ , denoted by $\bar{\Sigma}$, is defined as the quotient monoid Σ_1^*/Σ_2 , that one gets from the free monoid Σ_1^* by identification of the 1-cells $s_1(\alpha)$ and $t_1(\alpha)$, for every 2-cell α in Σ_2 .

Let \mathbf{M} be a monoid, a presentation of \mathbf{M} is a 2-polygraph Σ whose presented monoid $\bar{\Sigma}$ is isomorphic to \mathbf{M} . Two 2-polygraphs Σ and Υ are *Tietze equivalent* if the presented monoid $\bar{\Sigma}$ and $\bar{\Upsilon}$ are isomorphic.

Example: the plactic monoid of type A. The plactic monoid \mathbf{P}_n of rank n is presented by the 2-polygraph $\text{Knuth}_2(n)$, whose set of 1-cells is the set $[n]$ and the set of 2-cells is

$$\{ zxy \xrightarrow{\eta_{x,y,z}} xzy \mid 1 \leq x \leq y < z \leq n \} \cup \{ yzx \xrightarrow{\varepsilon_{x,y,z}} yxz \mid 1 \leq x < y \leq z \leq n \}. \quad (2.1)$$

2.1. Presentations of monoids by 2-polygraphs

The 2-cells of $\text{Knuth}_2(n)$ correspond to Knuth relations defined in Chapter 1, Section 1.2, with orientation given by the lexicographic order induced by the total order on $[n]$ given by $1 < \dots < n$. The presentation $\text{Knuth}_2(n)$ is called the *Knuth presentation*.

Note that the *lexicographic* order $<_{\text{lex}}$ induced by the total order $<$ on the 1-cells on $[n]$ is defined by

$$\begin{aligned} x_1 \dots x_p <_{\text{lex}} y_1 \dots y_q, & \quad \text{if } p < q, \\ x_1 \dots x_{k-1} x_k \dots x_p <_{\text{lex}} x_1 \dots x_{k-1} y_k \dots y_p, & \quad \text{if } x_k < y_k. \end{aligned}$$

where each x_i and y_j is a 1-cell of $[n]$.

2.1.2. Rewriting properties of 2-polygraphs

Let us fix a 2-polygraph $\Sigma = (\Sigma_1, \Sigma_2)$.

Rewriting sequences. A *rewriting step* of Σ is a 2-cell of the free 2-category Σ_2^* with shape

$$\begin{array}{ccccc} & & \text{u} & & \\ & & \parallel & & \\ x_1 & \xrightarrow{w} & x_2 & \xrightarrow{\alpha} & x_3 & \xrightarrow{w'} & x_4 \\ & & \text{v} & & \end{array}$$

where α is a 2-cell in Σ and w and w' are 1-cells of Σ^* . A *rewriting sequence* of Σ is a finite or infinite sequence of rewriting steps

$$w_1 \Longrightarrow w_2 \Longrightarrow \dots \Longrightarrow w_n \Longrightarrow \dots$$

A 1-cell w rewrites into a 1-cell w' if Σ has a non-empty rewriting sequence from w to w' . A 1-cell w of Σ_1^* is a *normal form* if Σ has no rewriting step with source w . A normal form of w is a 1-cell w' that is a normal form and such that w rewrites into w' .

Termination of 2-polygraphs. We say that Σ *terminates* if it has no infinite rewriting sequences. In this case, every 1-cell has at least one normal form.

A *termination order* on Σ is an order \leq on parallel 1-cells in Σ_1^* that satisfies the following:

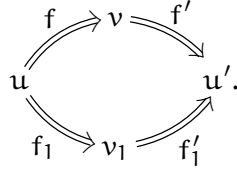
- i) the composition of 1-cells in Σ_1^* is strictly monotone in both arguments,
- ii) every decreasing family $(v_n)_{n \in \mathbb{N}}$ of parallel 1-cells in Σ_1^* is stationary,
- iii) for every 2-cell α in Σ_2 , we have $s(\alpha) > t(\alpha)$.

As a consequence, if Σ admits a termination order, then it terminates. An example of termination order is the lexicographic order defined previously in the example above.

2.1.2. Rewriting properties of 2-polygraphs

Confluence of 2-polygraphs. A *branching* of Σ is a pair (f, f_1) of 2-cells $f : u \Rightarrow v$ and $f_1 : u \Rightarrow v_1$ in Σ_2^* with a common source u . The 1-cell u is the *source* of this branching and the pair (v, v_1) is its *target*. When f and f_1 are rewriting steps, the branching (f, f_1) is called *local*.

A branching (f, f_1) with source u and target (v, v_1) is *confluent* if there exist 2-cells $f' : v \Rightarrow u'$ and $f'_1 : v_1 \Rightarrow u'$ in Σ_2^* , as follows



A 2-polygraph Σ is *confluent* if all of its branchings are confluent. It is *locally confluent* if all of its local branchings are confluent. In a confluent 2-polygraph, every 1-cell admits at most one normal form.

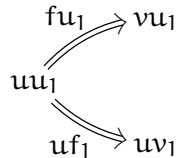
The local confluence does not generally imply confluence. Newman's Lemma, called the *Diamond Lemma*, states that these properties are equivalent for terminating 2-polygraphs [98, Theorem 3].

Convergence of 2-polygraphs. A 2-polygraph Σ is *convergent* if it terminates and it is confluent. Such a Σ is called a *convergent presentation of any monoid isomorphic to $\bar{\Sigma}$* . In that case, every 1-cell w in the free monoid Σ_1^* has a unique normal form, denoted by \widehat{w} , so that we have $w = w'$ in $\bar{\Sigma}$ if, and only if, $\widehat{w} = \widehat{w}'$ holds in Σ_1^* .

Critical branchings. Local branchings belong to one of the following three families:

i) *aspherical* branchings have shape (f, f) with source u and target (v, v) where f is a rewriting step.

ii) *Peiffer* branchings have shape



where $f : u \Rightarrow v$ and $f_1 : u_1 \Rightarrow v_1$ are rewriting steps.

iii) *overlapping* branchings are the remaining local branchings.

Local branchings are ordered by the order \sqsubseteq generated by the relations

$$(f, f_1) \sqsubseteq (ufv, uf_1v)$$

2.1. Presentations of monoids by 2-polygraphs

given for any local branching (f, f_1) and any possible 1-cells u and v of the free monoid Σ_1^* . An overlapping local branching that is minimal for the order \sqsubseteq is called a *critical branching*.

The critical branching Lemma, [39, Theorem 3.1.5.], states that a 2-polygraph is locally confluent if, and only if, all its critical branchings are confluent.

Example: the monoid \mathbf{P}_2 . The Knuth presentation of the monoid \mathbf{P}_2 with two generators 1 and 2 subject to the Knuth relations $\varepsilon_{1,2,2} : 221 \implies 212$ and $\eta_{1,1,2} : 211 \implies 121$ is convergent. Indeed, the lexicographic order being monomial, the 2-polygraph $\text{Knuth}_2(2)$ is terminating. Moreover, this 2-polygraph admits a unique confluent critical branching:

$$\begin{array}{ccc} & 2\eta_{1,1,2} & \\ & \curvearrowright & \\ 2211 & & 2121. \\ & \curvearrowleft & \\ & \varepsilon_{1,2,2}1 & \end{array}$$

2.1.3. The crystal presentation of the plactic monoid of type A

Using the notions of paths, L-S paths and tableaux presented in Chapter 1, Section 1.4, we construct a convergent presentation of the plactic monoid of type A. In this case, the plactic congruence \sim_{path} is defined by the corresponding crystal isomorphism explained in Chapter 1, Subsection 1.4.2. Since we study the plactic monoid of type A, we consider the root system of the general Lie algebra \mathfrak{gl}_n and we denote by \mathbb{F} the set of its fundamental weights.

Paths and words for type A. Any word in the free monoid $P_{\mathbb{R}}^*$ on $P_{\mathbb{R}}$ is a finite sequence of weights. We define a map $P_{\mathbb{R}}^* \longrightarrow \Pi$ sending any word $w = \lambda_1 \dots \lambda_r$, with λ_i in $P_{\mathbb{R}}$, to a path $\pi_w = \pi_{\lambda_1} \star \dots \star \pi_{\lambda_r}$. The path π_w is in Π if and only if $\lambda_1 + \dots + \lambda_r$ is in P . In addition, if we identify every path π_{ε_i} with the integer i , for $1 \leq i \leq n$, then the set of paths $\{\pi_{\varepsilon_i} \mid 1 \leq i \leq n\}$ is identified with the set $[n]$. Hence, for every word $w = x_1 \dots x_r$ in the free monoid $[n]^*$, with x_i in $[n]$, we associate a path $\pi_w = \pi_{\varepsilon_{x_1}} \star \dots \star \pi_{\varepsilon_{x_r}}$. We will denote by Π_W the free monoid over $\{\pi_{\varepsilon_i} \mid 1 \leq i \leq n\}$.

The root operators for type A. Let π_w be a path in Π_W . For each i in $[n]$ and each simple root α_i of \mathfrak{gl}_n , the roots operators defined on the free monoid $[n]^*$ in Chapter 1, Subsection 1.3.2 can be used to define similar root operators on Π_W :

$$e_{\alpha_i}, f_{\alpha_i} : \Pi_W \longrightarrow \Pi_W \cup \{0\}.$$

For instance, consider the path $\pi_w = \pi_{\varepsilon_1} \star \pi_{\varepsilon_2} \star \pi_{\varepsilon_2} \star \pi_{\varepsilon_1} \star \pi_{\varepsilon_3} \star \pi_{\varepsilon_3} \star \pi_{\varepsilon_1}$. Let us compute $f_{\alpha_1}(\pi_w)$ and $e_{\alpha_1}(\pi_w)$. We have $\pi_{w_1} = \pi_{\varepsilon_1} \star \pi_{\varepsilon_2} \star \pi_{\varepsilon_2} \star \pi_{\varepsilon_1} \star \pi_{\varepsilon_1}$. After removing the adjacent paths $(\pi_{\varepsilon_1}, \pi_{\varepsilon_2})$, the final path is $\pi_{\varepsilon_2} \star \pi_{\varepsilon_1} \star \pi_{\varepsilon_1}$. Then $r = 1$ and $s = 2$. Hence we obtain

$$f_{\alpha_1}(\pi_w) = \pi_{\varepsilon_1} \star \pi_{\varepsilon_2} \star \pi_{\varepsilon_2} \star \pi_{\varepsilon_2} \star \pi_{\varepsilon_3} \star \pi_{\varepsilon_3} \star \pi_{\varepsilon_1},$$

2.1.3. The crystal presentation of the plactic monoid of type A

and

$$e_{\alpha_1}(\pi_w) = \pi_{\varepsilon_1} \star \pi_{\varepsilon_2} \star \pi_{\varepsilon_1} \star \pi_{\varepsilon_1} \star \pi_{\varepsilon_3} \star \pi_{\varepsilon_3} \star \pi_{\varepsilon_1}.$$

Yamanouchi path tableau. A *Yamanouchi path* is a path π in Π_W such that any of its left factor path π' satisfies

$$|\pi'|_1 \geq \dots \geq |\pi'|_n$$

where $|\pi'|_i$ denotes the number of occurrences of the path π_{ε_i} in π' . We denote by Π_W^+ the set of all Yamanouchi paths. A path is a Yamanouchi path if and only if it is a path of highest weight, [109, Proposition 2.6.1]. As we have seen in Chapter 1, the highest weight tableau of the connected component containing tableaux of shape $\alpha_1 \Lambda_1 + \dots + \alpha_k \Lambda_k$ has only i 's in the i -th row for $1 \leq i \leq k$. Then this highest weight tableau can be represented by the following Yamanouchi path

$$\underbrace{(\pi_{\varepsilon_1} \star \dots \star \pi_{\varepsilon_1})}_{\alpha_1 \text{ times}} \star \underbrace{(\pi_{\varepsilon_1} \star \pi_{\varepsilon_2}) \star \dots \star (\pi_{\varepsilon_1} \star \pi_{\varepsilon_2})}_{\alpha_2 \text{ times}} \star \dots \star \underbrace{(\pi_{\varepsilon_1} \star \dots \star \pi_{\varepsilon_k}) \star \dots \star (\pi_{\varepsilon_1} \star \dots \star \pi_{\varepsilon_k})}_{\alpha_k \text{ times}}.$$

A Yamanouchi path that represents a tableau is called a *Yamanouchi path tableau*. Yamanouchi paths form a single plactic class whose representative path is a unique Yamanouchi path tableau, [91, Lemma 5.4.7].

Yamanouchi's map. Let us define a map

$$Y: \Pi_W^+ \rightarrow \Pi_W^+$$

that transforms a Yamanouchi path that does not represent a tableau to a Yamanouchi path tableau as follows. Let π_w be a non-Yamanouchi path tableau, then $Y(\pi_w)$ is equal to the path $\pi_{w(\mathbb{T})}$, where $w(\mathbb{T})$ is the Japanese reading of the tableau \mathbb{T} obtained from π_w by putting for every π_{ε_i} in π_w an element i in the i -th row of \mathbb{T} . Note that for every path π_w in Π_W^+ , we have $\text{wt}(\pi_w) = \text{wt}(Y(\pi_w))$.

2.1.3.1. Example. For $n = 3$, the path $\pi_w = \pi_{\varepsilon_1} \star \pi_{\varepsilon_1} \star \pi_{\varepsilon_2} \star \pi_{\varepsilon_3} \star \pi_{\varepsilon_1} \star \pi_{\varepsilon_2} \star \pi_{\varepsilon_1} \star \pi_{\varepsilon_2} \star \pi_{\varepsilon_3}$ is a Yamanouchi path that is not a Yamanouchi path tableau. Moreover, this path can be transformed to the following tableau

$$\mathbb{T} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & \\ \hline 3 & 3 & & \\ \hline \end{array}$$

after replacing each π_{ε_1} in π_w by the element 1 in the first row of \mathbb{T} , each π_{ε_2} in π_w by the element 2 in its second row and each π_{ε_3} in π_w by the element 3 in its third row. Hence we obtain

$$Y(\pi_w) = \pi_{w(\mathbb{T})} = \pi_{\varepsilon_1} \star \pi_{\varepsilon_1} \star \pi_{\varepsilon_2} \star \pi_{\varepsilon_1} \star \pi_{\varepsilon_2} \star \pi_{\varepsilon_3} \star \pi_{\varepsilon_1} \star \pi_{\varepsilon_2} \star \pi_{\varepsilon_3}.$$

2.1. Presentations of monoids by 2-polygraphs

Equivalence on paths. Let π_w and $\pi_{w'}$ be two paths in Π_W . Recall from Chapter 1, Subsection 1.4.2 the definition of the relation \sim_{path} on Π_W by : $\pi_w \sim_{\text{path}} \pi_{w'}$ if, and only if, the two following conditions are satisfied:

- i) the connected components $B(\pi_w)$ and $B(\pi_{w'})$ are isomorphic, that is $\text{wt}(\pi_w^+) = \text{wt}(\pi_{w'}^+)$, where π_w^+ and $\pi_{w'}^+$ are the highest weight paths of $B(\pi_w)$ and $B(\pi_{w'})$.
- ii) π_w and $\pi_{w'}$ have the same position in the components $B(\pi_w)$ and $B(\pi_{w'})$, that is, there exist i_1, \dots, i_r such that $\pi_w = f_{\alpha_{i_1}} \cdots f_{\alpha_{i_r}}(\pi_w^+)$ and $\pi_{w'} = f_{\alpha_{i_1}} \cdots f_{\alpha_{i_r}}(\pi_{w'}^+)$.

2-polygraph of crystals. Let $\text{Crys}_2^0(\mathfrak{n})$ be the 2-polygraph whose set of 1-cells is $\{\pi_{\varepsilon_i} \mid 1 \leq i \leq \mathfrak{n}\}$ and whose set of 2-cells is

$$\left\{ \pi_w \xrightarrow{\vartheta_{\pi_w}} Y(\pi_w) \mid \pi_w \in \Pi_W^+ \right\}.$$

For π_w in Π_W^+ , the path $f_{\alpha_{j_k}} \circ f_{\alpha_{j_{k-1}}} \circ \dots \circ f_{\alpha_{j_1}}(\pi_w)$ will be denoted by $f_{\underline{\alpha}}$, where $\underline{\alpha} = (\alpha_{j_k}, \alpha_{j_{k-1}}, \dots, \alpha_{j_1})$ and for $i = 1, \dots, k$, every j_i is an element of $[\mathfrak{n}]$, α_{j_i} is a simple root and $f_{\alpha_{j_i}}$ is the corresponding root operator.

For $k \geq 0$, let us define the 2-polygraph $\text{Crys}_2^k(\mathfrak{n})$ whose set of 1-cells is $\{\pi_{\varepsilon_i} \mid 1 \leq i \leq \mathfrak{n}\}$ and whose set of 2-cells is

$$\left\{ f_{\underline{\alpha}}(\pi_w) \xrightarrow{\vartheta_{\pi_w}^{\alpha_{j_k}}} f_{\underline{\alpha}}(Y(\pi_w)) \mid \pi_w \in \Pi_W^+ \right\}.$$

Let $\text{Crys}_2(\mathfrak{n})$ be the 2-polygraph whose set of 1-cells is $\{\pi_{\varepsilon_i} \mid 1 \leq i \leq \mathfrak{n}\}$ and whose set of 2-cells is

$$\bigcup_{i \geq 0} \text{Crys}_2^i(\mathfrak{n}).$$

This 2-polygraph is called the *2-polygraph of crystals*. By construction, the monoid presented by the 2-polygraph $\text{Crys}_2(\mathfrak{n})$ is isomorphic to the quotient of Π_W by the equivalence \sim_{path} .

2.1.3.2. Theorem. *For $\mathfrak{n} > 0$, the 2-polygraph $\text{Crys}_2(\mathfrak{n})$ is a convergent presentation of the monoid $\mathbf{P}_{\mathfrak{n}}$.*

Proof. By Chapter 1, Subsection 1.2.3 the equivalence \sim_{plac} coincides with the equivalence \sim_{path} , taking into account that the column reading of Schensted's tableaux obtained by the row-insertion is replaced by the Japanese reading of Schensted's tableaux that are obtained by the column-insertion. Thus, the monoid $\mathbf{P}_{\mathfrak{n}}$ is isomorphic to the quotient of Π_W by the equivalence \sim_{path} . Hence, the 2-polygraph $\text{Crys}_2(\mathfrak{n})$ is a presentation of the monoid $\mathbf{P}_{\mathfrak{n}}$.

Proof of the convergence of the 2-polygraph $\text{Crys}_2(\mathfrak{n})$. The termination is proved by showing that $\text{Crys}_2(\mathfrak{n})$ is compatible with a total order $\preceq_{\mathfrak{n}\text{-weight}}$ defined on the set $\mathbb{F}^{\mathfrak{n}}$ as follows. First, we fix an ordering \prec_{weight} on the set of fundamental weights \mathbb{F} of the Lie algebra $\mathfrak{gl}_{\mathfrak{n}}$ by

$$\Lambda_1 \prec_{\text{weight}} \Lambda_2 \prec_{\text{weight}} \dots \prec_{\text{weight}} \Lambda_{\mathfrak{n}}.$$

2.2. Tietze transformations and the pre-column presentation

Let $\preceq_{n\text{-weight}}$ be the lexicographic order on the set \mathbb{F}^n induced by the order \prec_{weight} , that is, $(\Lambda_{i_1}, \dots, \Lambda_{i_n}) \preceq_{n\text{-weight}} (\Lambda_{i'_1}, \dots, \Lambda_{i'_n})$ if

$$\Lambda_{i_1} \prec_{\text{weight}} \Lambda_{i'_1} \text{ or } [\Lambda_{i_1} = \Lambda_{i'_1} \text{ and } (\Lambda_{i_2}, \dots, \Lambda_{i_n}) \preceq_{n\text{-weight}} (\Lambda_{i'_2}, \dots, \Lambda_{i'_n})],$$

where for every $1 \leq k \leq n$, Λ_{i_k} and $\Lambda_{i'_k}$ are fundamental weights in \mathbb{F} . Then $\preceq_{n\text{-weight}}$ is a well-ordering on the set \mathbb{F}^n . Since the root operators preserve the lengths of paths and the shapes of tableaux, we will suppose that all the paths are Yamanouchi paths. Note also that any path in Π_W has a unique decomposition as an L-S monomial $\pi_1 \star \dots \star \pi_k$ of shape $(\Lambda_{j_1}, \dots, \Lambda_{j_k})$, where the path π_i is an L-S path of maximal shape Λ_{j_i} , for every $1 \leq i \leq k$ and $1 \leq j_i \leq n$. In this way, we will consider this unique decomposition for all the Yamanouchi paths. By construction of the Yamanouchi map Y , every non-Yamanouchi L-S monomial tableau is transformed to a Yamanouchi path tableau by beginning with the concatenation of its L-S paths of shape Λ_1 , after by the concatenation of its paths of shape Λ_2 and so on until the concatenation of its L-S paths of maximal shape with respect to the order \prec_{weight} . Then, for every 2-cell $\vartheta_{\pi_w} : \pi_w \Rightarrow Y(\pi_w)$ in $\text{Crys}_2(n)$, we have $Y(\pi_w) \prec_{n\text{-weight}} \pi_w$. Hence, the 2-polygraph $\text{Crys}_2(n)$ is compatible with the order $\preceq_{n\text{-weight}}$. Hence, rewriting an L-S monomial that is not a Yamanouchi path tableau always decreases it with respect to the order $\preceq_{n\text{-weight}}$. Since every application of a 2-cell in $\text{Crys}_2(n)$ yields a $\preceq_{n\text{-weight}}$ -preceding L-S monomial, it follows that any sequence of rewriting using $\text{Crys}_2(n)$ must terminate.

Let us show that $\text{Crys}_2(n)$ is confluent. Let π_w be a path in Π_W and $\pi_{w'}$, $\pi_{w''}$ be two normal forms such that $\pi_w \Rightarrow \pi_{w'}$ and $\pi_w \Rightarrow \pi_{w''}$. It is sufficient to prove that $\pi_{w'} = \pi_{w''}$. We have that $\pi_{w'}$ is a Yamanouchi path tableau such that $\pi_w \sim_{\text{path}} \pi_{w'}$. Similarly, the path $\pi_{w''}$ is a Yamanouchi path tableau such that $\pi_w \sim_{\text{path}} \pi_{w''}$. Since $\pi_w \sim_{\text{path}} \pi_{w'} \sim_{\text{path}} \pi_{w''}$ and each plactic congruence contains exactly one Yamanouchi path tableau, we obtain that $\pi_{w'} = \pi_{w''}$. Since the 2-polygraph $\text{Crys}_2(n)$ is terminating, and rewriting any non-Yamanouchi path tableau must terminate with a unique normal form, $\text{Crys}_2(n)$ is confluent. \square

2.2. TIETZE TRANSFORMATIONS AND THE PRE-COLUMN PRESENTATION

In this section, we begin by recalling the notions of Tietze transformations of 2-polygraphs as presented in [31, Subsection 2.1.1.]. In a second time, we construct a presentation of the plactic monoid of type A that is Tietze equivalent to the Knuth presentation. This presentation, called the *pre-column presentation*, will be used in Chapter 4 in order to compute a coherent presentation of plactic monoids of type A with the Knuth generators.

2.2. Tietze transformations and the pre-column presentation

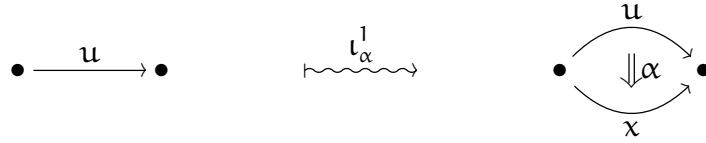
2.2.1. Tietze transformations of 2-polygraphs

Tietze transformations were introduced in group theory in order to transform a presentation of a group into a presentation of the same group by adding or removing generators and rules, [121]. This notion is also defined for 2-polygraphs, [31, Subsection 2.1.1.], as we will see in the following. Let before recall the notion of redundant cells.

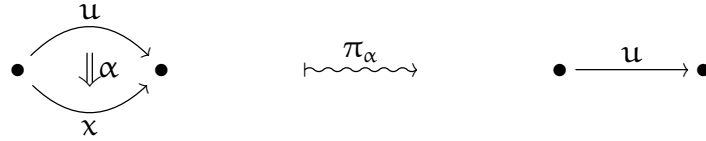
Redundant 1-cells. Let Σ be a 2-polygraph. A 2-cell α of Σ is called *collapsible* if its target is a generating 1-cell of Σ_1 and its source is a 1-cell of the free 1-category generated by the 1-polygraph $\Sigma_1 \setminus \{t_1(\alpha)\}$. If α is collapsible, then its target is called a *redundant cell*.

Elementary Tietze transformations. An *elementary Tietze transformations* of a 2-polygraph Σ is a 2-functor with domain Σ_2^\top that belongs to one of the following four operations:

- i) adjunction $\iota_\alpha^1 : \Sigma_2^\top \rightarrow \Sigma_2^\top[x](\alpha)$ of a redundant 1-cell x with its collapsible 2-cell α :

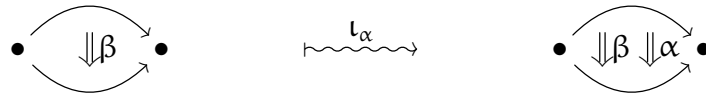


- ii) elimination $\pi_\alpha : \Sigma_2^\top \rightarrow (\Sigma_1 \setminus \{x\}, \Sigma_2 \setminus \{\alpha\})^\top$ of a redundant 1-cell x with its collapsible 2-cell α :

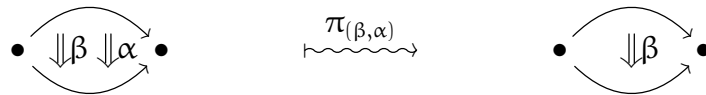


which maps x to u and the 2-cell α on 1_u and being identity on the others cells.

- iii) adjunction $\iota_\alpha : \Sigma_2^\top \rightarrow \Sigma_2^\top(\alpha)$ of a redundant 2-cell α :



- iv) elimination $\pi_{(\beta, \alpha)} : \Sigma_2^\top \rightarrow \Sigma_2^\top/(\beta, \alpha)$ of a redundant 2-cell α :



where $\Sigma_2^\top/(\beta, \alpha)$ is the free $(2, 1)$ -category generated by the 2-polygraph $(\Sigma_1, \Sigma_2 \setminus \{\alpha\})$.

If Σ and Υ are 2-polygraphs, a *Tietze transformation* from Σ to Υ is a 2-functor from Σ^\top to Υ^\top that decomposes into sequence of elementary Tietze transformations. Two 2-polygraphs are Tietze equivalent if, and only if, there exists a Tietze transformation between them [31, Theorem 2.1.3.].

2.2.2. Pre-column presentation of the plactic monoid of type A

Nielsen transformations. Recall the notion of Nielsen transformation from [31, Subsection 2.1.4.]. Given a 2-polygraph Σ and a 2-cell

$$u_1 \xrightarrow{\gamma_1} u \xrightarrow{\gamma} v \xrightarrow{\gamma_2} v_2$$

in Σ_2^\top , the *Nielsen transformation* $\kappa_{\gamma \leftarrow \beta}$ is the Tietze transformation that replaces in the $(2, 1)$ -category Σ_2^\top the 2-cell γ by a 2-cell $\beta : u_1 \Rightarrow v_2$. The transformation $\kappa_{\gamma \leftarrow \beta}$ can be decomposed into the following composition of elementary Tietze transformations:

$$\Sigma_2^\top \xrightarrow{\iota_\beta} \Sigma_2^\top(\beta) \xrightarrow{\pi_{(\gamma_1^- \star_1 \beta \star_1 \gamma_2^-, \gamma)}} \Sigma_2^\top / (\gamma_1^- \star_1 \beta \star_1 \gamma_2^-, \gamma).$$

When γ_2 is an identity, we will denote by $\kappa'_{\gamma \leftarrow \beta}$ the Nielsen transformation which, given a 2-cell $u_1 \xrightarrow{\gamma_1} u \xrightarrow{\gamma} v$ in Σ_2^\top , replaces the 2-cell γ by a 2-cell $\beta : u_1 \Rightarrow v$.

2.2.2. Pre-column presentation of the plactic monoid of type A

Columns. A *column* is a decreasing 1-cell $x_p \dots x_1$ in the free monoid $[n]^*$, *i.e.*, with $x_{i+1} > x_i$, for $1 \leq i \leq p-1$. We will denote by $\text{col}(n)$ the set of non-empty columns in $[n]^*$.

Total orders on columns. We will denote by \preceq_{deglex} the total order on $\text{col}(n)$ defined by $u \preceq_{\text{deglex}} v$ if

$$\ell(u) < \ell(v) \quad \text{or} \quad (\ell(u) = \ell(v) \text{ and } u <_{\text{lex}} v),$$

for all u and v in $\text{col}(n)$, where $<_{\text{lex}}$ denotes the lexicographic order on $[n]^*$ induced by the total order on $[n]$.

We will denote by \preceq_{rev} the total order on $\text{col}(n)$ defined by $u \preceq_{\text{rev}} v$ if

$$\ell(u) > \ell(v) \quad \text{or} \quad (\ell(u) = \ell(v) \text{ and } u <_{\text{lex}} v),$$

for all u and v in $\text{col}(n)$.

Columns as generators. One adds to the presentation $\text{Knuth}_2(n)$ one superfluous generator c_u for any u in $\text{col}(n)$. Let us denote by

$$\text{Col}_1(n) = \{ c_u \mid u \in \text{col}(n) \}$$

the set of *column generators* of the plactic monoid \mathbf{P}_n and by

$$\text{C}_2(n) = \{ c_{x_p \dots x_1} \xrightarrow{\gamma_u} c_u \mid u = x_p \dots x_1 \in \text{col}(n) \text{ with } \ell(u) \geq 2 \}$$

2.2. Tietze transformations and the pre-column presentation

the set of the defining relations for the column generators. In the free monoid $\text{Col}_1(n)^*$, the Knuth relations (2.1) can be written in the following form

$$\{c_z c_x c_y \xrightarrow{\eta_{x,y,z}^c} c_x c_z c_y \mid 1 \leq x \leq y < z \leq n\} \cup \{c_y c_z c_x \xrightarrow{\varepsilon_{x,y,z}^c} c_y c_x c_z \mid 1 \leq x < y \leq z \leq n\}. \quad (2.2)$$

Let denote by $\text{Knuth}_2^c(n)$ the 2-polygraph whose set of 1-cells is $\{c_1, \dots, c_n\}$ and whose set of 2-cells is given by (2.2). By definition, this 2-polygraph is Tietze equivalent to the 2-polygraph $\text{Knuth}_2(n)$. Indeed, the mapping $i \mapsto c_i$, for any i in $[n]$, induces an isomorphism between the two presented monoids. In the sequel, we will identify the 2-polygraphs $\text{Knuth}_2^c(n)$ and $\text{Knuth}_2(n)$ through this mapping.

Let us define the 2-polygraph $\text{Knuth}_2^{cc}(n)$, whose 1-cells are columns and 2-cells are defining relations for columns generators and Knuth relations:

$$\text{Knuth}_2^{cc}(n) := \langle \text{Col}_1(n) \mid C_2(n) \cup \text{Knuth}_2^c(n) \rangle.$$

2.2.2.1. Proposition. *For $n > 0$, the 2-polygraph $\text{Knuth}_2^{cc}(n)$ is a presentation of the monoid \mathbf{P}_n .*

Proof. We have $\text{Col}_1(n) = \{c_1, \dots, c_n\} \cup \{c_u \mid u \in \text{col}(n), \ell(u) \geq 2\}$, thus in order to prove that the 2-polygraphs $\text{Knuth}_2^{cc}(n)$ and $\text{Knuth}_2^c(n)$ are Tietze equivalent, we add to the 2-polygraph $\text{Knuth}_2^c(n)$ all the column generators c_u , for all $u = x_p \dots x_1$ in $\text{col}(n)$ such that $\ell(u) \geq 2$, and the corresponding collapsible 2-cell : $\gamma_u : c_{x_p} \dots c_{x_1} \Rightarrow c_u$. We apply successively a Tietze transformation $\iota_{\gamma_u}^1$, defined in 2.2.1. i), from the bigger column in $\text{col}(n)$ to the smaller one with respect to the order \preceq_{deglex} . The composite

$$T_1 = \iota_{\gamma_1}^1 \circ \dots \circ \iota_{\gamma_{u_i}}^1 \circ \iota_{\gamma_{u_{i+1}}}^1 \circ \dots \circ \iota_{\gamma_{n \dots 1}}^1,$$

with $u_i \preceq_{\text{deglex}} u_{i+1}$, defines a Tietze transformation

$$T_1 : \text{Knuth}_2^c(n)^\top \longrightarrow \text{Knuth}_2^{cc}(n)^\top,$$

which proves that $\text{Knuth}_2^{cc}(n)$ is Tietze equivalent to $\text{Knuth}_2^c(n)$, hence Tietze equivalent to $\text{Knuth}_2(n)$. \square

Pre-column presentation. Let define the 2-polygraph $\text{PreCol}_2(n)$ whose set of 1-cells is $\text{Col}_1(n)$ and the set of 2-cells is

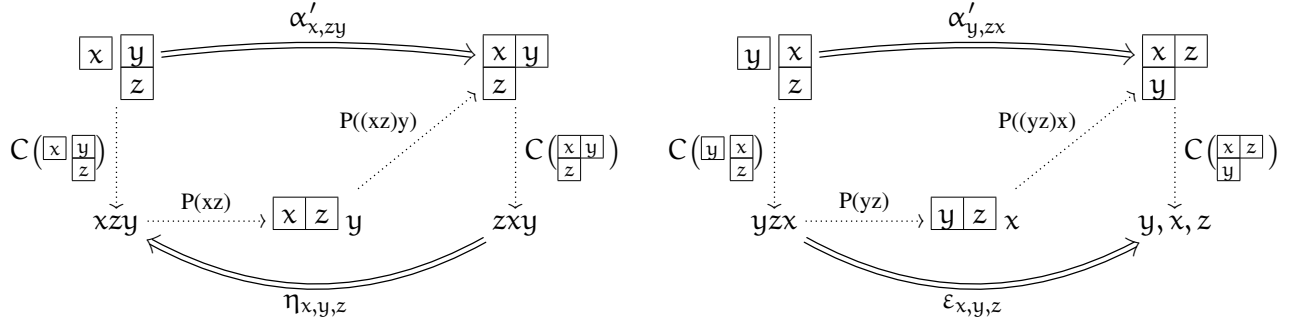
$$\text{PreCol}_2(n) = \text{PC}_2(n) \cup \{c_x c_u \xrightarrow{\alpha'_{x,u}} c_{xu} \mid xu \in \text{col}(n) \text{ and } 1 \leq x \leq n\},$$

where

$$\text{PC}_2(n) = \{c_x c_{zy} \xrightarrow{\alpha'_{x,zy}} c_{zx} c_y \mid 1 \leq x \leq y < z \leq n\} \cup \{c_y c_{zx} \xrightarrow{\alpha'_{y,zx}} c_{yx} c_z \mid 1 \leq x < y \leq z \leq n\}.$$

2.2.2. Pre-column presentation of the plactic monoid of type A

We will see in Lemma 2.2.2.3 that the 2-cells $\alpha'_{x,zy}$ and $\alpha'_{y,zx}$ correspond respectively to the Knuth relations $\eta_{x,y,z}$ for $1 \leq x \leq y < z \leq n$ and $\varepsilon_{x,y,z}$ for $1 \leq x < y \leq z \leq n$. They also correspond to the following Schensted transformations as indicated in the following diagrams:



2.2.2.2. Proposition. For $n > 0$, the 2-polygraph $\text{PreCol}_2(n)$ is a presentation of the monoid \mathbf{P}_n .

The 2-polygraph $\text{PreCol}_2(n)$ is called the *pre-column presentation* of \mathbf{P}_n . The proof of Proposition 2.2.2.2 is given by the following two lemmas.

2.2.2.3. Lemma. The 2-polygraph

$$\text{CPC}_2(n) := \langle \text{Col}_1(n) \mid \text{C}_2(n) \cup \text{PC}_2(n) \rangle$$

is Tietze equivalent to the 2-polygraph $\text{Knuth}_2^{\text{cc}}(n)$.

Proof. For $1 \leq x \leq y < z \leq n$, consider the following critical branching

$$\begin{array}{c} \eta_{x,y,z}^c \rightarrow c_x c_z c_y \xrightarrow{c_x \gamma_{zy}} c_x c_{zy} \\ \swarrow \searrow \\ c_z c_x c_y \xrightarrow{\gamma_{zx}} c_{zx} c_y \end{array}$$

of the 2-polygraph $\text{Knuth}_2^{\text{cc}}(n)$. Let consider the Nielsen transformation

$$\kappa_{\eta_{x,y,z}^c \leftarrow \alpha'_{x,zy}} : \text{Knuth}_2^{\text{cc}}(n)^\top \longrightarrow \text{Knuth}_2^{\text{cc}}(n)^\top / (\eta_{x,y,z}^c \leftarrow \alpha'_{x,zy}),$$

that substitutes the 2-cell $\alpha'_{x,zy} : c_x c_{zy} \Rightarrow c_{zx} c_y$ to the 2-cell $\eta_{x,y,z}^c$, for every $1 \leq x \leq y < z \leq n$.

We denote by $T_{\eta \leftarrow \alpha'}$ the successive applications of the Nielsen transformation $\kappa_{\eta_{x,y,z}^c \leftarrow \alpha'_{x,zy}}$, for every $1 \leq x \leq y < z \leq n$, with respect to the lexicographic order on the triples (x, y, z) induced by the total order on $[n]$.

Similarly, for $1 \leq x < y \leq z \leq n$, consider the following critical branching

$$\begin{array}{c} \varepsilon_{x,y,z}^c \rightarrow c_y c_x c_z \xrightarrow{\gamma_{yx} c_z} c_{yx} c_z \\ \swarrow \searrow \\ c_y c_z c_x \xrightarrow{\gamma_{zx}} c_y c_{zx} \end{array}$$

2.2. Tietze transformations and the pre-column presentation

of the 2-polygraph $\text{Knuth}_2^{\text{cc}}(\mathbf{n})$. Let consider the Nielsen transformation

$$\kappa_{\varepsilon_{x,y,z}^c \leftarrow \alpha'_{y,zx}} : \text{Knuth}_2^{\text{cc}}(\mathbf{n})^\top \longrightarrow \text{Knuth}_2^{\text{cc}}(\mathbf{n})^\top / (\varepsilon_{x,y,z}^c \leftarrow \alpha'_{y,zx}),$$

that substitutes the 2-cell $\alpha'_{y,zx} : c_y c_{zx} \Rightarrow c_y c_z$ to the 2-cell $\varepsilon_{x,y,z}^c$, for every $1 \leq x < y \leq z \leq \mathbf{n}$.

We denote by $T_{\varepsilon \leftarrow \alpha'}$ the successive applications of the Tietze transformation $\kappa_{\varepsilon_{x,y,z}^c \leftarrow \alpha'_{x,zy}}$, for every $1 \leq x < y \leq z \leq \mathbf{n}$, with respect to the lexicographic order on the triples (x, y, z) induced by the total order on $[\mathbf{n}]$.

Let define the composite $T_{\eta, \varepsilon \leftarrow \alpha'} = T_{\eta \leftarrow \alpha'} \circ T_{\varepsilon \leftarrow \alpha'}$, this gives us a Tietze transformation:

$$T_{\eta, \varepsilon \leftarrow \alpha'} : \text{Knuth}_2^{\text{cc}}(\mathbf{n})^\top \longrightarrow \text{CPC}_2(\mathbf{n})^\top.$$

In this way, the 2-polygraphs $\text{Knuth}_2^{\text{cc}}(\mathbf{n})$ and $\text{CPC}_2(\mathbf{n})$ are Tietze equivalent. \square

The following lemma proves that the 2-polygraph $\text{PreCol}_2(\mathbf{n})$ is a presentation of the monoid $\mathbf{P}_\mathbf{n}$.

2.2.2.4. Lemma. *The 2-polygraph $\text{PreCol}_2(\mathbf{n})$ is Tietze equivalent to the 2-polygraph $\text{CPC}_2(\mathbf{n})$.*

Proof. Let $x_p \dots x_1$ be a column with $\ell(x_p \dots x_1) > 2$ and define $\alpha'_{y,x} := \gamma_{yx} : c_y c_x \Rightarrow c_{yx}$, for every $x < y$. Consider the following critical branching

$$\begin{array}{ccc} & c_{x_p} \gamma_{x_{p-1} \dots x_1} & \Rightarrow c_{x_p} c_{x_{p-1} \dots x_1} \\ & \nearrow & \\ c_{x_p} c_{x_{p-1}} \dots c_{x_1} & & \\ & \searrow & \\ & \gamma_{x_p \dots x_1} & \Rightarrow c_{x_p \dots x_1} \end{array}$$

of the 2-polygraph $\text{CPC}_2(\mathbf{n})$ and the following Nielsen transformation

$$\kappa'_{\gamma_{x_p \dots x_1} \leftarrow \alpha'_{x_p, x_{p-1} \dots x_1}} : \text{CPC}_2(\mathbf{n})^\top \longrightarrow \text{CPC}_2(\mathbf{n})^\top / (\gamma_{x_p \dots x_1} \leftarrow \alpha'_{x_p, x_{p-1} \dots x_1}),$$

that substitutes the 2-cell

$$\alpha'_{x_p, x_{p-1} \dots x_1} : c_{x_p} c_{x_{p-1} \dots x_1} \Longrightarrow c_{x_p \dots x_1},$$

to the 2-cell

$$\gamma_{x_p \dots x_1} : c_{x_p} \dots c_{x_1} \Longrightarrow c_{x_p \dots x_1},$$

for each column $x_p \dots x_1$ such that $\ell(x_p \dots x_1) > 2$. Starting from the 2-polygraph $\text{CPC}_2(\mathbf{n})$, we apply successively the Nielsen transformation $\kappa'_{\gamma_{x_p \dots x_1} \leftarrow \alpha'_{x_p, x_{p-1} \dots x_1}}$, for every column $x_p \dots x_1$ such that $\ell(x_p \dots x_1) > 2$, from the bigger to the smaller one with respect to the total order \preceq_{deglex} .

Let us define the composite

$$T_{\gamma \leftarrow \alpha'} = \kappa'_{\gamma_{x_3 x_2 x_1} \leftarrow \alpha'_{x_3, x_2 x_1}} \circ \dots \circ \kappa'_{\gamma_{x_n \dots x_1} \leftarrow \alpha'_{x_n, x_{n-1} \dots x_1}},$$

2.3. The Knuth–Bendix’s completion of the Knuth presentation

with $\chi_3 \chi_2 \chi_1 \preceq_{\text{deglex}} \dots \preceq_{\text{deglex}} \chi_n \dots \chi_1$. This gives us a Tietze transformation:

$$T_{\gamma \leftarrow \alpha'} : \text{CPC}_2(\mathfrak{n})^\top \longrightarrow \text{PreCol}_2(\mathfrak{n})^\top.$$

In this way, we prove that $\text{PreCol}_2(\mathfrak{n})$ is Tietze equivalent to $\text{CPC}_2(\mathfrak{n})$. \square

To resume the construction of this section, we have constructed the following Tietze equivalences:

$$\text{Knuth}_2(\mathfrak{n})^\top \xrightarrow{T_1} \text{Knuth}_2^{\text{cc}}(\mathfrak{n})^\top \xrightarrow{T_{\eta, \varepsilon \leftarrow \alpha'}} \text{CPC}_2(\mathfrak{n})^\top \xrightarrow{T_{\gamma \leftarrow \alpha'}} \text{PreCol}_2(\mathfrak{n})^\top.$$

2.3. THE KNUTH–BENDIX’S COMPLETION OF THE KNUTH PRESENTATION

We recall in 2.3.1 the Knuth–Bendix’s completion that computes convergent presentations of monoids from terminating ones. After that, we show in 2.3.2 that the Knuth presentation of the plactic monoid \mathbf{P}_3 is completed into a convergent one after adding three 2-cells. Finally, we show in 2.3.3 that for higher values of n there is no finite convergent completion of the Knuth presentation of the monoid \mathbf{P}_n without adding new generators. The latter result is also presented in [65].

2.3.1. The Knuth–Bendix’s completion

Recall from [39, Section 3.2.1] the Knuth–Bendix’s completion. Consider a terminating 2-polygraph Σ , with a total termination order \preceq . The *Knuth–Bendix’s completion* of Σ is the 2-polygraph $\mathcal{KB}(\Sigma)$ obtained as follows. One starts with $\mathcal{KB}(\Sigma)$ equal to Σ and with the set \mathcal{CB} of critical branchings of Σ . If the set \mathcal{CB} is empty, then the procedure stops. Otherwise, it picks a branching (f, f_1) with source u and target (v, v_1) from \mathcal{CB} and performs the following operations:

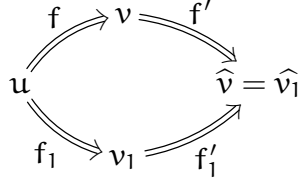
1. It computes 2-cells $f' : v \Rightarrow \widehat{v}$ and $f'_1 : v_1 \Rightarrow \widehat{v}_1$ of $\mathcal{KB}(\Sigma)^*$, where \widehat{v} and \widehat{v}_1 are normal forms for v and v_1 , respectively,

$$\begin{array}{c}
 \begin{array}{ccc}
 & & \widehat{v} \\
 & \nearrow f & \xrightarrow{f'} \\
 u & & v \\
 & \searrow f_1 & \xrightarrow{f'_1} \\
 & & v_1 \\
 & & \xrightarrow{f'_1} \\
 & & \widehat{v}_1
 \end{array}
 \end{array}$$

2. It tests if $\widehat{v} = \widehat{v}_1$ or $\widehat{v} \neq \widehat{v}_1$, then there are three possibilities:

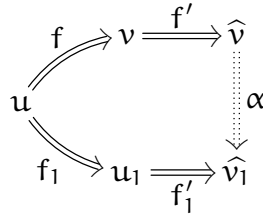
2.3. The Knuth–Bendix’s completion of the Knuth presentation

(a) If $\widehat{v} = \widehat{v}_1$, then the critical branching

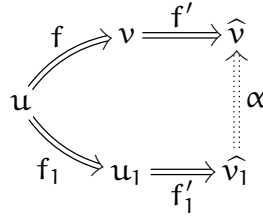


is already confluent and it passes to the next critical branching in \mathcal{CB} .

(b) If $\widehat{v} > \widehat{v}_1$, the procedure adds the 2-cell $\alpha : \widehat{v} \Longrightarrow \widehat{v}_1$ to $\mathcal{KB}(\Sigma)$ and all the new critical branchings created by α to \mathcal{CB} ,



(c) If $\widehat{v} < \widehat{v}_1$, the procedure adds the 2-cell $\alpha : \widehat{v}_1 \Longrightarrow \widehat{v}$ to $\mathcal{KB}(\Sigma)$ and all the new critical branchings created by α to \mathcal{CB} ,



3. It removes the branching (f, f_1) from \mathcal{CB} and restarts from the beginning.

If the procedure stops, then it returns the 2-polygraph $\mathcal{KB}(\Sigma)$. Otherwise, one obtains an increasing sequence of 2-polygraphs, whose limit is the 2-polygraph $\mathcal{KB}(\Sigma)$. Note that if the initial 2-polygraph Σ is convergent, then its Knuth–Bendix’s completion $\mathcal{KB}(\Sigma)$ is Σ itself.

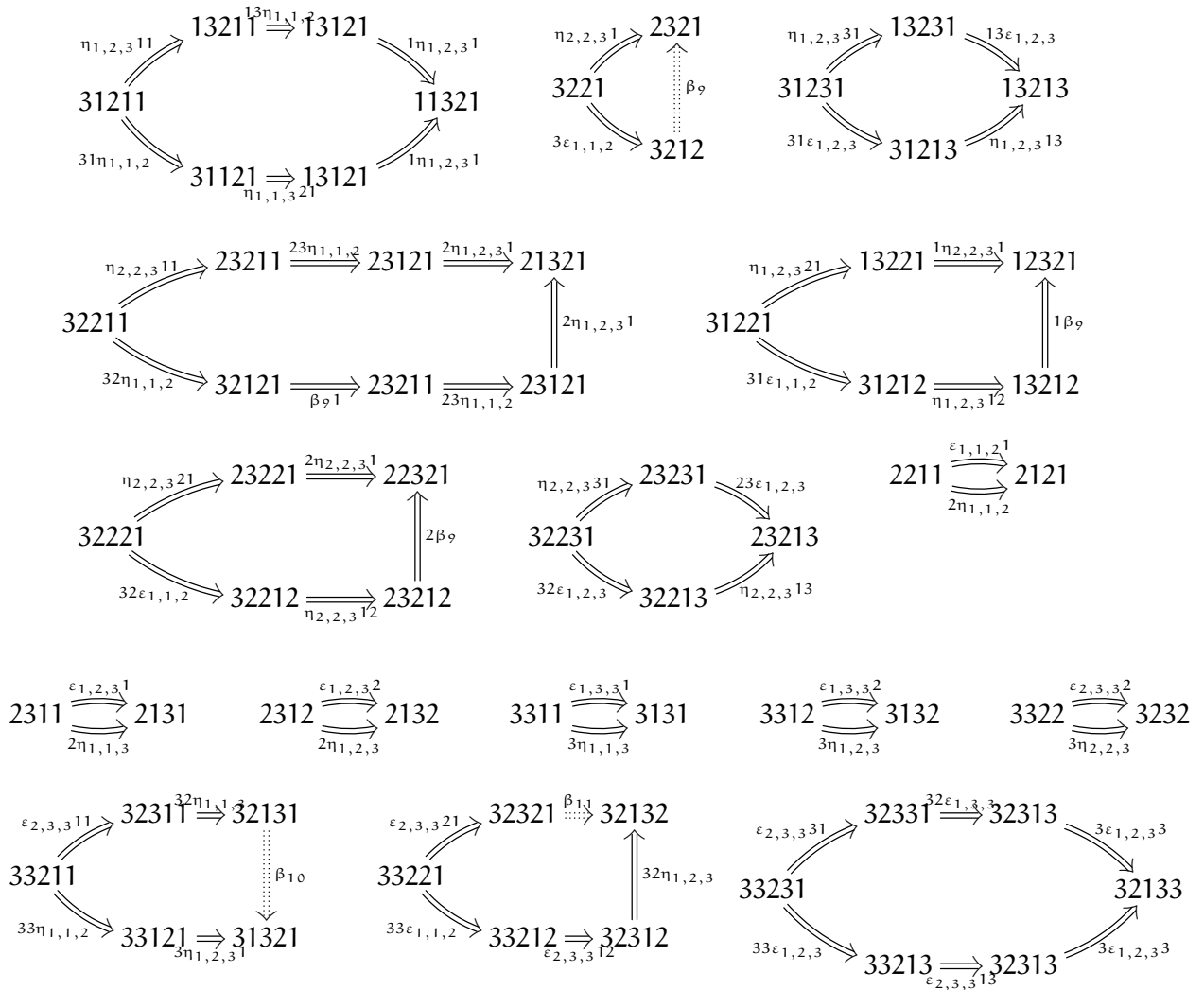
The Knuth–Bendix’s completion $\mathcal{KB}(\Sigma)$ of a 2-polygraph Σ is a convergent presentation of the monoid $\overline{\Sigma}$. In addition, the 2-polygraph $\mathcal{KB}(\Sigma)$ is finite if, and only if, the 2-polygraph Σ is finite and if, and only if, the Knuth–Bendix’s completion procedure halts, see [64, 48].

2.3.2. The computation of $\mathcal{KB}(\text{Knuth}_2(3))$

The monoid \mathbf{P}_3 . For the monoid \mathbf{P}_3 , the Knuth presentation $\text{Knuth}_2(3)$ admits three generators 1, 2 and 3, together with the following 8 relations

$$\begin{aligned} & \{ 211 \xrightarrow{\eta_{1,1,2}} 121, \quad 311 \xrightarrow{\eta_{1,1,3}} 131, \quad 312 \xrightarrow{\eta_{1,2,3}} 132, \quad 322 \xrightarrow{\eta_{2,2,3}} 232 \} \\ & \cup \{ 221 \xrightarrow{\epsilon_{1,1,2}} 212, \quad 231 \xrightarrow{\epsilon_{1,2,3}} 213, \quad 331 \xrightarrow{\epsilon_{1,3,3}} 313, \quad 332 \xrightarrow{\epsilon_{2,3,3}} 323 \}. \end{aligned}$$

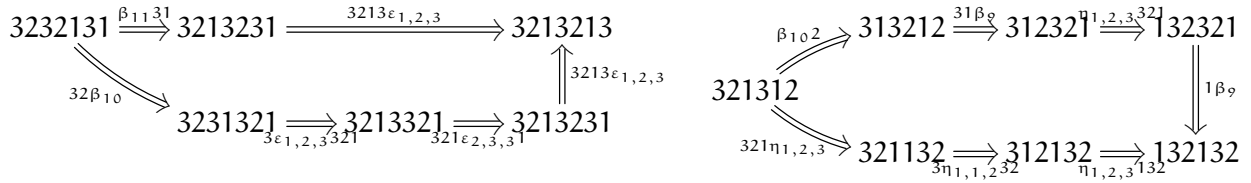
This presentation admits the following critical branchings:



Thus, by the Knuth–Bendix’s completion procedure, one adds the following 2-cells

$$3212 \xrightarrow{\beta_9} 2321, \quad 32131 \xrightarrow{\beta_{10}} 31321, \quad 32321 \xrightarrow{\beta_{11}} 32132.$$

2.3.3. The completion for higher ranks



which are all confluent. Then the Knuth–Bendix’s completion procedure halts. As a consequence, the monoid \mathbf{P}_3 admits a convergent presentation with 3 generators 1, 2 and 3, together with the following 11 relations:

$$\begin{aligned} & \{ 211 \xrightarrow{\eta_{1,1}^2} 121, \quad 311 \xrightarrow{\eta_{1,1}^3} 131, \quad 312 \xrightarrow{\eta_{1,2}^3} 132, \quad 322 \xrightarrow{\eta_{2,2}^3} 232 \} \\ & \cup \{ 221 \xrightarrow{\epsilon_{1,1}^2} 212, \quad 231 \xrightarrow{\epsilon_{1,2}^3} 213, \quad 331 \xrightarrow{\epsilon_{1,3}^3} 313, \quad 332 \xrightarrow{\epsilon_{2,3}^3} 323 \} \\ & \cup \{ 3212 \xrightarrow{\beta_9} 2321, \quad 32131 \xrightarrow{\beta_{10}} 31321, \quad 32321 \xrightarrow{\beta_{11}} 32132 \}. \end{aligned}$$

2.3.3. The completion for higher ranks

In the following, we will show that for higher values of n , the Knuth–Bendix’s completion procedure cannot succeed on the Knuth presentation. For this, we will show that the Knuth presentation $\text{Knuth}_2(4)$ of the monoid \mathbf{P}_4 does not admit a finite completion with the lexicographic order.

The monoid \mathbf{P}_4 . For the monoid \mathbf{P}_4 , the Knuth presentation does not admit a finite completion compatible with the lexicographic order. The proof of this result is given by the following two lemmas. Note that we use the column reading of tableaux defined in 1.2.1 of Chapter 1, and we use Theorem 1.2.2.2 on the length of the longest decreasing subsequence in a word over the set $[n]$.

2.3.3.1. Lemma. *For any $i \geq 1$, the 1-cell $\alpha_i = 323^i 431$ in $[4]^*$ is a normal form for $\mathcal{KB}(\text{Knuth}_2(4))$.*

Proof. It suffices to show that the factors $a = 23^i 431$ and $b = 323^i 43$ of α_i are in normal forms for the 2-polygraph $\mathcal{KB}(\text{Knuth}_2(4))$.

We have that $C(P(a)) = (421)(3)^{i+1}$. Hence the tableau $P(a)$ consists of 3 rows, and all the 1-cells equal to a in \mathbf{P}_4 contain a decreasing subsequence of maximal length 3. The reduction being compatible with the lexicographic order, a cannot be reduced into $4u$, $3u$ or $24u$, for some u in $[4]^*$. Suppose a is reduced into $21u$, for some u in $[4]^*$. In this case, $21u$ doesn’t contain any decreasing subsequences of maximal length 3, then a cannot be reduced into it.

Suppose a is reduced into $1u$, for some u in $[4]^*$. In this case, we have the following possibilities: if a is reduced into $12u$, $132u$, or $142u$, then we can not find any 1-cells that contains a decreasing subsequence of maximal length 3. Else a is reduced into $\alpha_1 = 1343^i 23^{i-j}$, for

2.3. The Knuth–Bendix’s completion of the Knuth presentation

some $j > 0$, $a_2 = 1343^i 2$, $a_3 = 133^j 43^{i-j} 2$, for some $j > 0$, $a_4 = 143^{i+1} 2$, or $a_5 = 1433^j 23^{i-j}$, for some $j > 0$. We have

$$C(P(a_1)) = C(P(a_2)) = C(P(a_3)) = C(P(a_4)) = C(P(a_5)) = (431)(2)(3)^i \neq C(P(a)).$$

Then a cannot be reduced into $1u$, for some u in $[4]^*$. So a minimal 1-cell that represents a in \mathbf{P}_4 should start with 23 and it must be of the form $23^i 431 = a$, then a is a normal form.

We have that $C(P(b)) = (32)(43)(3)^{i-1}$. Hence the tableau $P(b)$ consists of 2 rows, and all the 1-cells equal to b in \mathbf{P}_4 contain a decreasing subsequence of maximal length 2. The reduction being compatible with the lexicographic order, b cannot be reduced into $4u$, $33u$, or $34u$, for some u in $[4]^*$. Suppose b is reduced into $2u$, for some u in $[4]^*$. In this case b can be reduced into $b_1 = 23^{i+1} 43$, $b_2 = 23^j 43^{i-j+2}$, for some $j > 0$, $b_3 = 2343^{i+1}$, or $b_4 = 243^{i+2}$. We have

$$C(P(b_1)) = C(P(b_2)) = C(P(b_3)) = C(P(b_4)) = (42)(3)^{i+2} \neq C(P(b)),$$

then b cannot be reduced into $2u$, for some u in $[4]^*$. So a minimal 1-cell that represents b in \mathbf{P}_4 should start with 32 and it must be of the forms $323^i 43 = b$ or $323^{i+1} 4$, but

$$C(P(323^{i+1} 4)) = (32)(3)^{i+1} 4 \neq C(P(b)),$$

then it must be of the form $323^i 43 = b$, so b is a normal form. \square

2.3.3.2. Lemma. *For any $i \geq 1$, the 1-cell $b_i = 3213^i 43$ in $[4]^*$ is a normal form for $\mathcal{KB}(\text{Knuth}_2(4))$.*

Proof. It is sufficient to show that the factors $e = 213^i 43$ and $d = 3213^i 4$ of b_i are normal forms of $\mathcal{KB}(\text{Knuth}_2(4))$. We have $C(P(e)) = (21)(43)(3)^i$. Hence the tableau $P(e)$ consists of 2 rows and all the 1-cells equal to e in the monoid \mathbf{P}_4 contain a decreasing subsequence of maximal length 2. The reduction being compatible with the lexicographic order, we can not reduce e into $4u$, $3u$, $23u$ or $24u$, for some u in $[4]^*$.

Suppose e is reduced to $1u$, for some u in $[4]^*$. In this case, we have the following possibilities: suppose e is reduced into $e_1 = 123^i 43$, $e_2 = 123^j 43^{i-j+1}$, for some $j > 0$, $e_3 = 1243^{i+1}$, $e_4 = 13^{i+1} 24$, $e_5 = 13^{i+1} 42$, $e_6 = 13^j 23^{i-j+1} 4$, for some $j > 0$, $e_7 = 13^j 423^{i-j+1}$, for some $j > 0$, $e_8 = 13^j 4323^{i-j}$ or $e_9 = 1423^{i+1}$, we have

$$C(P(e_1)) = C(P(e_2)) = C(P(e_3)) = C(P(e_9)) = (41)(2)(3)^{i+1} \neq C(P(e)),$$

$$C(P(e_4)) = C(P(e_5)) = C(P(e_6)) = (31)(2)(3)^i(4) \neq C(P(e)),$$

$$C(P(e_7)) = C(P(e_8)) = (431)(2)(3)^i \neq C(P(e)).$$

Then e cannot be reduced into $1u$, for some u in $[4]^*$. So a minimal 1-cell that represents e in \mathbf{P}_4 should start with 21 and it must be of the forms $213^i 43 = e$ or $213^i 34$, but

$$C(P(213^i 34)) = (21)(3)^{i+1} 4 \neq C(P(e)),$$

2.3.3. The completion for higher ranks

so it must be of the form $213^i43 = e$, then e is a normal form.

We have $C(P(d)) = (321)(3)^i(4)$. Hence the tableau $P(d)$ consists of 3 rows, and all the 1-cells equivalent to d in \mathbf{P}_4 contain a decreasing subsequence of maximal length 3. The reduction being compatible with the lexicographic order, we can not reduce d into $4u$, $33u$ or $34u$, for some u in $[4]^*$. Suppose d is reduced into $31u$, for some u in $[4]^*$. In this case $31u$ does not contain a decreasing subsequence of maximal length 3, then b cannot be reduced into it.

Suppose d is reduced into $1u$, for some u in $[4]^*$. In this case, we have the following possibilities: if d is reduced into $d_1 = 143^{i+1}2$ or $d_2 = 143^j23^{i-j+1}$, for some $j > 0$, we have

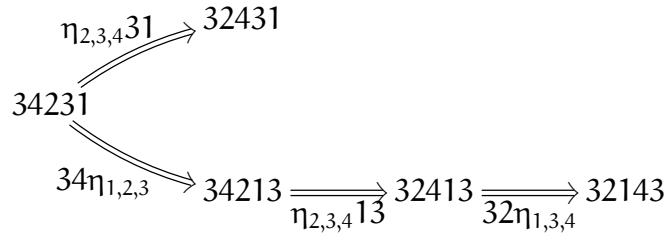
$$C(P(d_1)) = C(P(d_2)) = (431)(2)(3)^{i+1} \neq C(P(d)).$$

Then d cannot be reduced into $1u$, for some u in $[4]^*$. So a minimal 1-cell that represents d in \mathbf{P}_4 should start with 32 and it must be of the forms $3213^i4 = d$, then d is a normal form. \square

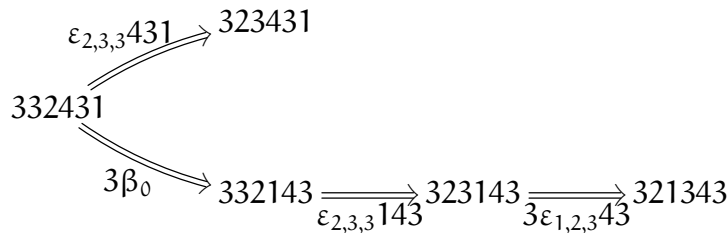
2.3.3.3. Theorem. *For $n > 3$, there is no finite completion of the 2-polygraph $\mathcal{KB}(\text{Knuth}_2(n))$ compatible with the lexicographic order.*

Proof. We prove the result for $n = 4$. Since the relations in $\text{Knuth}_2(4)$ are included in $\text{Knuth}_2(n)$, for $n > 3$, the result is also true for any $n > 3$.

Suppose that the 2-polygraph $\text{Knuth}_2(4)$ admits a finite completion $\mathcal{KB}(\text{Knuth}_2(4))$, compatible with the lexicographic order. After one step of the Knuth–Bendix’s completion procedure, one adds 22 rules. In particular, the rule $\beta_0 : 32431 \Rightarrow 32143$ is added according to the following critical pair:



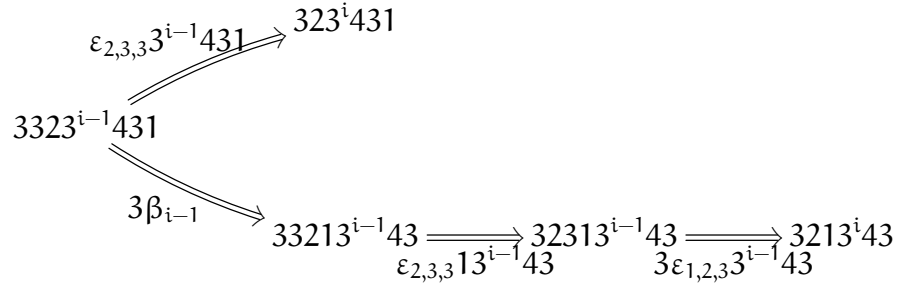
We prove by induction that, the completion add a rule $\beta_i : 323^i431 \Rightarrow 3213^i43$, for any $i \geq 1$. At the second step of the completion procedure, the following critical pair occurs



where, thanks to Lemmas 2.3.3.1 and 2.3.3.2, the 1-cells 323431 and 321343 are in normal form for $\mathcal{KB}(\text{Knuth}_2(4))$. It follows that the rule $\beta_1 : 323431 \Rightarrow 321343$ must be added.

2.3. The Knuth–Bendix’s completion of the Knuth presentation

Suppose now that there exists a rule $\beta_{i-1} : 323^{i-1}431 \Rightarrow 3213^{i-1}43$ added after the i th step of completion. Then after the $(i + 1)$ th step of it, there is the following critical pair:



Thanks to Lemmas 2.3.3.1 and 2.3.3.2, the 1-cells 323^i431 and $3213^{i-1}43$ are in normal forms for $\mathcal{KB}(\text{Knuth}_2(4))$. It follows that the rule $\beta_i : 323^i431 \Rightarrow 3213^i43$ must be added for any $i \geq 1$. It follows that the completion is infinite, making a contradiction and the assertion follows. \square

Column presentations of plactic monoids

Contents

3.1	Finite convergent presentation of plactic monoids	116
3.2	Column presentation of the plactic monoid of type A	120
3.3	Column presentation of the symplectic plactic monoid	124
3.4	Column presentation for plactic monoids of types B, D and G_2	142

The aim of this chapter is to construct finite convergent presentations of plactic monoids using three approaches: Young tableaux, Littelmann paths and Kashiwara’s crystal bases.

We construct in Section 3.1 a finite convergent presentation of plactic monoids for any type using L-S paths. This presentation, called the *column presentation*, can be constructed by a case-by-case analysis using Young tableaux and Kashiwara’s crystal bases.

In section 3.2, we consider the plactic monoid \mathbf{P}_n of type A. The column presentation of this monoid was constructed by Cain, Gray and Malheiro in [17]. We give some properties and examples of this presentation and we show how it is related to Schensted’s insertion algorithm.

We construct in Section 3.3 the column presentation of the symplectic plactic monoid by adding the admissible columns generators introduced by Kashiwara and Nakashima. The right side of the relations is the result of Lecouvey’s insertion of an admissible column into another one, [43]. In 3.3.1, we recall the definitions and some properties of admissible columns and symplectic tableaux. We describe in 3.3.2 the column insertion algorithm for type C introduced by Lecouvey. We study in 3.3.3 the shape of the symplectic tableau obtained after inserting an admissible column into another one. We construct in 3.3.4 the column presentation of the symplectic plactic monoid and we compute this presentation in rank 2.

We end this chapter by showing in 3.4 how the column presentation can be constructed for types B, D and G_2 .

3.1. FINITE CONVERGENT PRESENTATION OF PLACTIC MONOIDS

Since there is no finite completion of the Knuth presentation of the plactic monoid \mathbf{P}_n , for $n \geq 3$, without adding new generators, an open problem was to find a finite convergent presentation of this monoid. Bokut et al. in [7] and Cain et al. in [17] answered positively this question using different methods. Their methods consist in adding column generators to the Knuth presentation, in the spirit of Kapur and Narendran in [56]. Bokut et al. used the column-insertion algorithm whereas Cain et al. used the row-insertion algorithm. We discuss in details the construction of this presentation with additional properties in Section 3.2.

The presentations by generators and rules of plactic monoids for the classical types contain the Knuth relations, see [88, 78, 79, 83] and Subsection 3.3.1 for the presentation of the plactic monoid of type C. Since these presentations contain the Knuth relations, they don't admit a finite completion without adding generators. In this way, a question is to find finite convergent presentations of these monoids. In order to solve this question, we consider plactic monoids for any semisimple Lie algebra. We construct a finite convergent presentation of these monoids, called the *column presentation*, in terms of L-S paths. As discussed in Chapter 1, an L-S path corresponds to a column for type A, and to an admissible column for types C, B, D and G_2 in the Lecouvey sense. Littelmann showed that the column presentation is a presentation of plactic monoids for any semisimple Lie algebra, [88, Theorem B]. We show that this presentation is finite and convergent, [44].

Note that the column presentation of plactic monoids for any semisimple Lie algebra given in this section using L-S paths coincides with the presentations constructed for type A in [17, 7] as presented in Section 3.2, for type C as constructed in Section 3.3 and for types B, D and G_2 in [16] as shown in Section 3.4.

2-polygraph of paths. Let \mathfrak{g} be a semisimple Lie algebra and let $\Lambda_1, \dots, \Lambda_n$ be its fundamental weights. Let \mathbb{B}_i be the set of L-S paths of shape Λ_i and $\mathbb{B} = \cup_{i=1}^n \mathbb{B}_i$. Denote by \mathbb{B}^* the free monoid over \mathbb{B} .

For every L-S paths c_1 and c_2 in \mathbb{B} such that $c_1 \star c_2$ is not a standard tableau, we define a 2-cell

$$c_1.c_2 \xrightarrow{\gamma_{c_1, c_2}} T$$

where T is the unique standard tableau such that $[T] = [c_1 \star c_2]$.

The *2-polygraph of paths*, denoted by $\text{Path}_2(\mathfrak{n})$, is the 2-polygraph with only one 0-cell and whose set of 1-cells is \mathbb{B} and the set of 2-cells is

$$\text{Path}_2(\mathfrak{n}) = \left\{ c_1.c_2 \xrightarrow{\gamma_{c_1, c_2}} T \mid c_1, c_2 \in \mathbb{B} \text{ and } c_1 \star c_2 \text{ is not a standard tableau} \right\}.$$

This presentation is called the *column presentation*. It is a presentation of the plactic monoid for any semisimple Lie algebra [88, Theorem B]. Let us prove that is convergent, [44].

3.1. Finite convergent presentation of plactic monoids

3.1.0.1. Theorem. *For any semisimple Lie algebra \mathfrak{g} , the 2-polygraph $\text{Path}_2(\mathfrak{n})$ is a finite convergent presentation of the plactic monoid for \mathfrak{g} .*

Proof. Every semisimple Lie algebra \mathfrak{g} admits a finite number of fundamental weights, then there is a finite number of L-S paths of shape Λ_i , for $i = 1, \dots, n$. Thus the 2-polygraph $\text{Path}_2(\mathfrak{n})$ is finite.

Prove the termination of the 2-polygraph $\text{Path}_2(\mathfrak{n})$. Consider the partial order \leq on dominant weights defined by

$$\lambda_1 \leq \lambda_2 \quad \text{if and only if} \quad \lambda_2 - \lambda_1 \in \mathbb{N}\Phi^+$$

where λ_1 and λ_2 are dominant weights. That is, $\lambda_2 - \lambda_1$ is a non-negative integral sum of positive roots. Using this order, one can find for each dominant weight, a finite number of dominant weights that are smaller than it, then the partial order \leq is a well-founded order. Let us define an order \preceq on the tableaux of \mathbb{B}^* as follows. For two tableaux m and m' of shape λ and λ' respectively, we have

$$m \preceq m' \quad \text{if and only if} \quad \lambda \leq \lambda'.$$

Let us show that the 2-polygraph $\text{Path}_2(\mathfrak{n})$ is compatible with the order \preceq . We have to prove that if $h \Rightarrow h'$, then $h' \prec h$, where h is not a standard tableau and h' is a standard tableau of shape λ' . There are two cases depending on whether or not h is a tableau. In the proof of Theorem B in [88], Littelmann showed that for every tableau m of shape λ that is not standard, the shape of the standard tableau m' such that $[m] = [m']$ is strictly smaller than λ . Moreover, Littelmann also showed that if we transform an L-S monomial $\pi_\Lambda \star \pi_{\Lambda'}$ of shape $\Lambda + \Lambda'$ that is not a tableau to a standard tableau T . Then the shape of T is strictly smaller than the shape of $\pi_\Lambda \cdot \pi_{\Lambda'}$. Thus, in all cases, we have $h' \prec h$. Hence, rewriting an L-S monomial that is not a standard tableau always decreases it with respect to the order \preceq . Since every application of a 2-cell of $\text{Path}_2(\mathfrak{n})$ yields a \prec -preceding tableau, it follows that any sequence of rewriting using $\text{Path}_2(\mathfrak{n})$ must terminate.

Let us show that the 2-polygraph $\text{Path}_2(\mathfrak{n})$ is confluent. Let $m \in \mathbb{B}^*$ and T, T' be two normal forms such that $m \Rightarrow T$ and $m \Rightarrow T'$. It is sufficient to prove that $T = T'$. Suppose $T = c_1 \star \dots \star c_k$, where the L-S monomial $c_1 \star \dots \star c_k$ is a standard Young tableau such that $[m] = [c_1 \star \dots \star c_k]$. Similarly, $T' = c'_1 \star \dots \star c'_l$ where the L-S monomial $c'_1 \star \dots \star c'_l$ is a standard Young tableau such that $[m] = [c'_1 \star \dots \star c'_l]$. Since $[m] = [c_1 \star \dots \star c_k] = [c'_1 \star \dots \star c'_l]$ and each plactic congruence contains exactly one standard tableau, we have $k = l$ and $c_i = c'_i$, for all $i = 1, \dots, k$. Thus $T = T'$. Since the 2-polygraph $\text{Path}_2(\mathfrak{n})$ is terminating, and rewriting any non-standard tableau must terminate with a unique normal form, $\text{Path}_2(\mathfrak{n})$ is confluent. \square

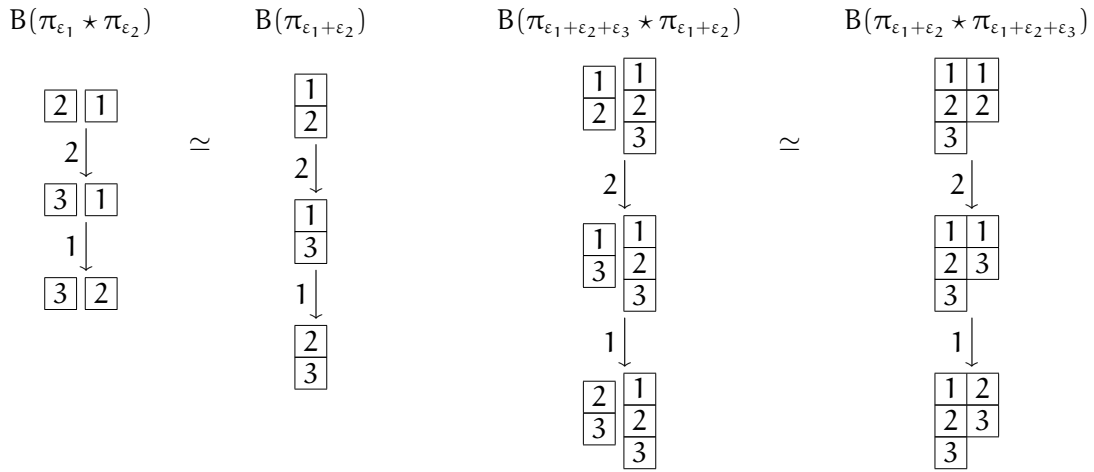
3.1.0.2. Example. In order to ensure consistency with the next section, we construct the column presentation for the case of \mathfrak{gl}_3 instead of \mathfrak{sl}_3 . This presentation can be transformed to the case of \mathfrak{sl}_3 by replacing the path $\pi_{\varepsilon_1 + \varepsilon_2 + \varepsilon_3}$ by the trivial one. Note also that the concatenation of paths in the following crystal isomorphisms corresponds to the Japanese reading of tableaux.

3.1. Finite convergent presentation of plactic monoids

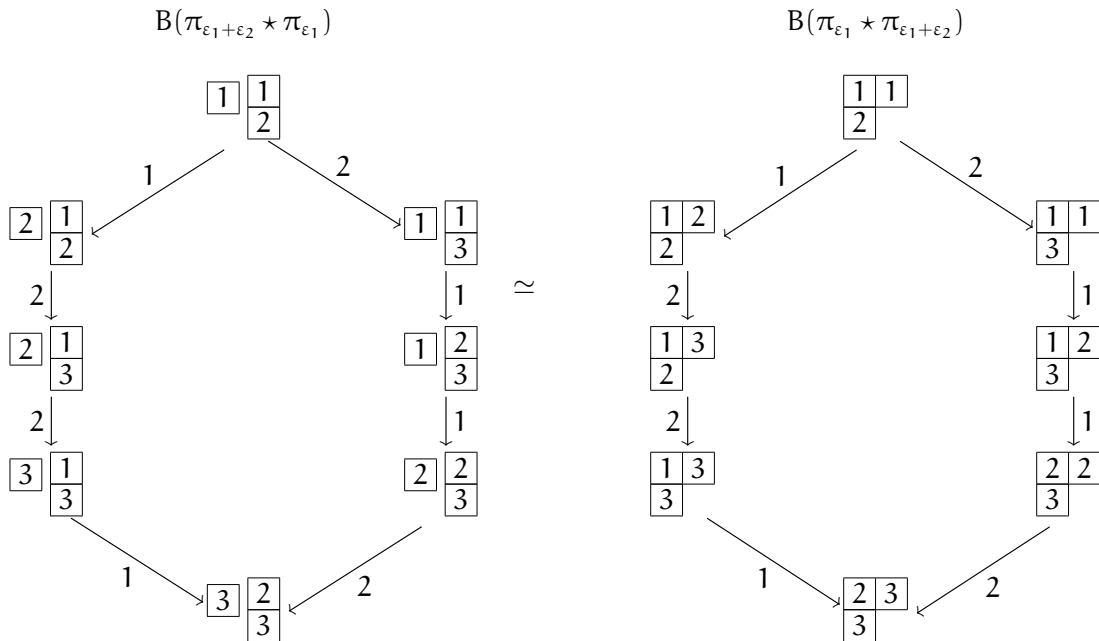
Consider the general Lie algebra \mathfrak{gl}_3 and let us compute the column presentation $\text{Path}_2(3)$ of the monoid \mathbf{P}_3 . The set of 1-cells is

$$\text{Path}_1(3) = \{ \pi_{\varepsilon_1}, \pi_{\varepsilon_2}, \pi_{\varepsilon_3}, \pi_{\varepsilon_1+\varepsilon_2}, \pi_{\varepsilon_1+\varepsilon_3}, \pi_{\varepsilon_2+\varepsilon_3}, \pi_{\varepsilon_1+\varepsilon_2+\varepsilon_3} \}.$$

The left and right sides of the 2-cells of $\text{Path}_2(3)$ are the paths attached to the vertices which correspond under the following crystal isomorphisms



and under the following crystal isomorphism



3.1. Finite convergent presentation of plactic monoids

and under the following crystal isomorphisms

$$\begin{array}{ccc}
 \mathbb{B}(\pi_{\varepsilon_1+\varepsilon_2} \star \pi_{\varepsilon_1+\varepsilon_3}) & & \mathbb{B}(\pi_{\varepsilon_1} \star \pi_{\varepsilon_1+\varepsilon_2+\varepsilon_3}) \\
 \begin{array}{c}
 \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & 2 \\ \hline \end{array} \\
 \downarrow 1 \\
 \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 3 & 2 \\ \hline \end{array} \\
 \downarrow 2 \\
 \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 3 & 3 \\ \hline \end{array}
 \end{array}
 \approx
 \begin{array}{c}
 \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array} \\
 \downarrow 1 \\
 \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array} \\
 \downarrow 2 \\
 \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array}
 \end{array}
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathbb{B}(\pi_{\varepsilon_1} \star \pi_{\varepsilon_1+\varepsilon_2}) & & \mathbb{B}(\pi_{\varepsilon_1+\varepsilon_2+\varepsilon_3}) \\
 \begin{array}{c}
 \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 3 & \\ \hline \end{array} \\
 \approx
 \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}
 \end{array}
 \end{array}$$

and under the following crystal isomorphisms

$$\begin{array}{ccc}
 \mathbb{B}(\pi_{\varepsilon_1+\varepsilon_2+\varepsilon_3} \star \pi_{\varepsilon_1}) & & \mathbb{B}(\pi_{\varepsilon_1} \star \pi_{\varepsilon_1+\varepsilon_2+\varepsilon_3}) \\
 \begin{array}{c}
 \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 3 \\ \hline \end{array} \\
 \downarrow 1 \\
 \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 2 & 3 \\ \hline \end{array} \\
 \downarrow 2 \\
 \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 2 & 3 \\ \hline \end{array}
 \end{array}
 \approx
 \begin{array}{c}
 \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array} \\
 \downarrow 1 \\
 \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array} \\
 \downarrow 2 \\
 \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array}
 \end{array}
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathbb{B}(\pi_{\varepsilon_1+\varepsilon_2} \star \pi_{\varepsilon_3}) & & \mathbb{B}(\pi_{\varepsilon_1+\varepsilon_2+\varepsilon_3}) \\
 \begin{array}{c}
 \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 2 & \\ \hline \end{array} \\
 \approx
 \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}
 \end{array}
 \end{array}$$

3.2. Column presentation of the plactic monoid of type A

3.2. COLUMN PRESENTATION OF THE PLACTIC MONOID OF TYPE A

In this section, we present in details a particular case of Theorem 3.1.0.1 that corresponds to the column presentation of the plactic monoid \mathbf{P}_n of type A. Using combinatorial properties of tableaux, we give some properties and examples of this presentation and we show how it is related to Schensted's insertion algorithm.

3.2.1. The columns generators

Combinatorial properties of two columns tableaux. Recall from Chapter 2, Subsection 2.2.2 that a column is a decreasing 1-cell $x_p \dots x_1$ in the free monoid $[n]^*$. We denote by $\text{col}(n)$ the set of non-empty columns in $[n]^*$. For a word u in $[n]^*$, we denote by $\ell(u)$ the length of u and by $l^{\text{nds}}(u)$ the length of its longest non-decreasing subsequence.

Consider two columns $u = x_p \dots x_1$ and $v = y_q \dots y_1$ in $\text{col}(n)$ of length p and q , respectively. Suppose that the juxtaposition of u and v does not form a tableau. The tableau $P(uv)$, obtained by the row-insertion procedure, contains at most two columns [17, Lemma 3.1]. Indeed, by Chapter 1, Theorem 1.2.2.3, the number of columns of the tableau $P(uv)$ is equal to $l^{\text{nds}}(uv)$. If uv is a column, necessarily its non-decreasing subsequences are each of length equal to one and thus $l^{\text{nds}}(uv) = 1$. Otherwise, if uv is not a column, then $x_1 \leq y_q$. Hence all the non-decreasing subsequences of uv are of length 2.

Furthermore, if the tableau $P(uv)$ contains exactly two columns, the left column contains more elements than u [17, Lemma 3.1]. Indeed, since the juxtaposition of u and v does not form a tableau, we have either $p < q$ or $x_i > y_i$ for some $i \leq q$. In the first case, the column v is a decreasing subsequence of uv containing more elements than the length of u . In the second case, $x_p \dots x_i y_i \dots y_q$ is a decreasing subsequence of uv of length $p + q$ and hence contains more elements than u . In either case, uv contains a decreasing sequence of length greater than u , and so by Theorem 1.2.2.3, the tableau $P(uv)$ contains more rows than there are elements in the column u , and hence the left column of the tableau $P(uv)$ contains more elements than u .

Graphical notations. For every columns $u = x_p \dots x_1$ and $v = y_q \dots y_1$ in $\text{col}(n)$, consider the tableau $P(uv)$ obtained by the row-insertion procedure. In the sequel, we will use graphical notations depending on whether the tableau $P(uv)$ consists in two columns:

- i) we will denote $u \widehat{\vee} v$ if the planar representation of $P(uv)$ is the tableau:

x_1	y_1
\vdots	\vdots
	y_q
x_p	

3.2.2. The column presentation for type A

that is, $p \geq q$ and $x_i \leq y_i$, for any $i \leq q$,

ii) we will denote $u \times v$ in all the other cases, that is, when $p < q$ or $x_i > y_i$ for some $i \leq q$.

In the case ii), we will denote $u \times^1 v$ if the tableau $P(uv)$ has one column and we will denote $u \times^2 v$ if the tableau $P(uv)$ has two columns.

3.2.2. The column presentation for type A

Construction of the presentation. For every 1-cell w in the free monoid $[n]^*$, we compute the tableau $P(w)$ by the row-insertion procedure. Given two columns $u = x_p \dots x_1$ and $v = y_q \dots y_1$ in $\text{col}(n)$, we consider the tableau $P(uv)$. As we have seen previously, the tableau $P(uv)$ contains at most two columns.

For every columns u and v in $\text{col}(n)$ such that $u \times v$, we define a 2-cell

$$\alpha_{u,v} : c_u c_v \Rightarrow c_w c_{w'}$$

where

i) $w = uv$ and w' is the empty column, when $u \times^1 v$,

ii) w and w' are respectively the left and right columns of the tableau $P(uv)$, when $u \times^2 v$.

Let us denote by $\text{Col}_2(n)$ the 2-polygraph whose set of 1-cells is $\text{Col}_1(n)$ and the set of 2-cells is

$$\text{Col}_2(n) = \{ c_u c_v \xrightarrow{\alpha_{u,v}} c_w c_{w'} \mid u, v \in \text{col}(n) \text{ and } u \times v \}. \quad (3.1)$$

Note that the 2-cells of $\text{PreCol}_2(n)$ constructed in Chapter 2, Subsection 2.2.2 correspond to the 2-cells $\alpha_{u,v}$ of $\text{Col}_2(n)$, where $\ell(u) = 1$ and $\ell(v) = 2$. Moreover, we notice that, for any 2-cells $\alpha_{u,v} : c_u c_v \Rightarrow c_w c_{w'}$ of $\text{Col}_2(n)$, there exists a 2-cell in $\text{PreCol}_2(n)^*$ with source $c_u c_v$ and target $c_w c_{w'}$.

Column presentation and Schensted's algorithm. Let us remark that Schensted's Algorithm Chapter 1, Section 1.2.1 that computes a tableau $P(w)$ from a 1-cell w in $[n]^*$, corresponds to the leftmost reduction path in $\text{Col}_2^*(n)$ from the 1-cell w to its normal form $P(w)$, that is, the reduction paths obtained by applying the rules of $\text{Col}_2(n)$ starting from the left. For example, consider the 1-cell $w = 421532435452$ in $[5]^*$. To compute the tableau $P(w)$, one applies the following successive rules of $\text{Col}_2(5)$ starting in each step from the left:

$$w = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline 4 & 2 & 1 & 5 & 3 & 2 & 4 & 3 & 5 & 4 & 5 & 2 \\ \hline \end{array} \xrightarrow{\alpha_{4,2}} \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline 2 & 1 & 5 & 3 & 2 & 4 & 3 & 5 & 4 & 5 & 2 \\ \hline 4 & & & & & & & & & & \\ \hline \end{array} \xrightarrow{\alpha_{42,1}} \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 5 & 3 & 2 & 4 & 3 & 5 & 4 & 5 & 2 \\ \hline 2 & & & & & & & & & & \\ \hline 4 & & & & & & & & & & \\ \hline \end{array}$$

3.2. Column presentation of the plactic monoid of type A

$$\begin{array}{c}
 \begin{array}{|c|c|c|c|c|c|c|c|}
 \hline 1 \\ \hline 2 \\ \hline 4 \\ \hline
 \end{array}
 \begin{array}{|c|c|c|c|c|c|c|c|}
 \hline 5 \\ \hline 3 \\ \hline 2 \\ \hline 4 \\ \hline 3 \\ \hline 5 \\ \hline 4 \\ \hline 5 \\ \hline 2 \\ \hline
 \end{array}
 \xrightarrow{\alpha_{5,3}} (\dots) \xrightarrow{\alpha_{5,4}}
 \begin{array}{|c|c|c|c|c|c|}
 \hline 1 \\ \hline 2 \\ \hline 4 \\ \hline
 \end{array}
 \begin{array}{|c|c|c|c|c|}
 \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline 5 \\ \hline
 \end{array}
 \begin{array}{|c|c|}
 \hline 5 \\ \hline 2 \\ \hline
 \end{array}
 \xrightarrow{\alpha_{5,2}}
 \begin{array}{|c|c|c|c|c|}
 \hline 1 \\ \hline 2 \\ \hline 4 \\ \hline
 \end{array}
 \begin{array}{|c|c|c|c|}
 \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline 5 \\ \hline
 \end{array}
 \begin{array}{|c|c|}
 \hline 2 \\ \hline 5 \\ \hline
 \end{array}
 \xrightarrow{\alpha_{54,52}}
 \begin{array}{|c|c|c|c|}
 \hline 1 \\ \hline 2 \\ \hline 4 \\ \hline
 \end{array}
 \begin{array}{|c|c|c|}
 \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline 5 \\ \hline
 \end{array}
 \begin{array}{|c|c|}
 \hline 2 \\ \hline 4 \\ \hline
 \end{array}
 \begin{array}{|c|}
 \hline 5 \\ \hline
 \end{array}
 \\
 \\
 \begin{array}{|c|c|c|c|c|}
 \hline 1 \\ \hline 2 \\ \hline 4 \\ \hline
 \end{array}
 \begin{array}{|c|c|c|c|}
 \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline 5 \\ \hline
 \end{array}
 \begin{array}{|c|c|}
 \hline 2 \\ \hline 5 \\ \hline
 \end{array}
 \xrightarrow{\alpha_{43,542}}
 \begin{array}{|c|c|c|c|}
 \hline 1 \\ \hline 2 \\ \hline 4 \\ \hline
 \end{array}
 \begin{array}{|c|c|c|}
 \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline 5 \\ \hline
 \end{array}
 \begin{array}{|c|c|}
 \hline 2 \\ \hline 4 \\ \hline
 \end{array}
 \begin{array}{|c|}
 \hline 5 \\ \hline
 \end{array}
 \xrightarrow{\alpha_{532,432}}
 \begin{array}{|c|c|c|c|}
 \hline 1 \\ \hline 2 \\ \hline 4 \\ \hline
 \end{array}
 \begin{array}{|c|c|c|}
 \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline 5 \\ \hline
 \end{array}
 \begin{array}{|c|c|}
 \hline 2 \\ \hline 4 \\ \hline
 \end{array}
 \begin{array}{|c|}
 \hline 5 \\ \hline
 \end{array}
 \xrightarrow{\alpha_{421,5432}}
 \begin{array}{|c|c|c|c|c|}
 \hline 1 \\ \hline 2 \\ \hline 4 \\ \hline
 \end{array}
 \begin{array}{|c|c|c|}
 \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline 5 \\ \hline
 \end{array}
 \begin{array}{|c|c|}
 \hline 2 \\ \hline 4 \\ \hline
 \end{array}
 \begin{array}{|c|}
 \hline 5 \\ \hline
 \end{array}
 = P(w)
 \end{array}$$

In particular, for any columns u and v in $\text{col}(n)$ such that $u \times v$, applying successive rules of $\text{Col}_2(n)$ on uv starting in each step from the left leads to a unique normal form, which is the tableau $P(uv)$.

3.2.2.1. Proposition. *The 2-polygraph $\text{Col}_2(n)$ has the unique normal form property.*

Proof. Consider a 1-cell w in $\text{Col}_1(n)^*$ and let w' and w'' be normal forms of w . Proving the unique normal form property consists in showing that the normal forms w' and w'' are equal. Let T' (resp. T'') be the planar representation of w' (resp. w''). Since w' and w'' are normal forms, they don't contain any subsequences that form sources of 2-cells in $\text{Col}_2(n)$. As a consequence, T' (resp. T'') is a juxtaposition of columns that form a tableau. Hence, the normal forms w' and w'' are tableaux such that the equality $w = w' = w''$ holds in the monoid \mathbf{P}_n . Since each congruence class contains exactly one tableau [91, Theorem 5.2.5], we have that $w' = w''$. \square

3.2.2.2. Proposition. *For $n > 0$, the 2-polygraph $\text{Col}_2(n)$ is a presentation of the monoid \mathbf{P}_n .*

The 2-polygraph $\text{Col}_2(n)$ is called the *column presentation* of the plactic monoid \mathbf{P}_n . Note that, the set of columns being finite, this 2-polygraph is finite.

Proof. Let us prove that the 2-polygraph $\text{Col}_2(n)$ is Tietze equivalent to the 2-polygraph $\text{Knuth}_2^{\text{cc}}(n)$ defined in Chapter 2, Subsection 2.2.2. Any 2-cell in $\text{Knuth}_2^{\text{cc}}(n)$ can be deduced from a 2-cell in $\text{Col}_2(n)$ as follows. For any $1 \leq x \leq y < z \leq n$ (resp. $1 \leq x < y \leq z \leq n$), the 2-cells $\eta_{x,y,z}^c$ (resp. $\varepsilon_{x,y,z}^c$) can be deduced by the following composition

$$\begin{array}{ccc}
 \begin{array}{ccc}
 c_z c_x c_y & \xrightarrow{\eta_{x,y,z}^c} & c_x c_z c_y \\
 \alpha_{z,x} c_y \downarrow & & \downarrow c_x \alpha_{z,y} \\
 c_{zx} c_y & \xleftarrow{\alpha_{x,zy}} & c_x c_{zy}
 \end{array}
 & (\text{resp.}) &
 \begin{array}{ccc}
 c_y c_z c_x & \xrightarrow{\varepsilon_{x,y,z}^c} & c_y c_x c_z \\
 c_y \alpha_{z,x} \downarrow & & \downarrow \alpha_{y,x} c_z \\
 c_y c_{zx} & \xrightarrow{\alpha_{y,zx}} & c_y x c_z
 \end{array}
 \end{array}$$

For any column $x_p \dots x_1$, the 2-cell $\gamma_{x_p \dots x_1}$ can be deduced by the following composition

$$\begin{array}{ccc}
 c_{x_p} \dots c_{x_1} & \xrightarrow{\gamma_{x_p \dots x_1}} & c_{x_p \dots x_1} \\
 \alpha_{x_p, x_{p-1}} c_{x_{p-2}} \dots c_{x_1} \downarrow & & \uparrow \alpha_{x_p \dots x_2, x_1} \\
 c_{x_p x_{p-1}} c_{x_{p-2}} \dots c_{x_1} & \xRightarrow{(\dots)} & c_{x_p \dots x_2} c_{x_1}
 \end{array}$$

3.2.2. The column presentation for type A

As a consequence, if the 1-cells w and w' in $\text{Col}_1(n)^*$ are equal modulo relations in $\text{Knuth}_2^{\text{cc}}(n)$, then they are equal modulo relations in $\text{Col}_2(n)$.

Conversely, if the 1-cells w and w' in $\text{Col}_1(n)^*$ are equal modulo relations in $\text{Col}_2(n)$, by Proposition 3.2.2.1, they have the same normal form with respect to $\text{Col}_2(n)$. Moreover, this normal form is the common tableau of the 1-cells w and w' . It follows that w and w' are in the same plactic congruence class and hence they are equal modulo $\text{Knuth}_2^{\text{cc}}(n)$. \square

Termination of the column presentation. The termination of the 2-polygraph $\text{Col}_2(n)$ can be proved using the terminating order \ll defined on $\text{Col}_1(n)^*$ as follows. For c_{u_i} and c_{v_j} in $\text{Col}_1(n)$, we have $c_{u_1} \dots c_{u_k} \ll c_{v_1} \dots c_{v_l}$, if

$$\begin{cases} k < l & \text{or} \\ k = l & \text{and } \exists i \in \{1, \dots, k\} \text{ such that } u_i \preceq_{\text{rev}} v_i \text{ and } c_{u_j} = c_{v_j} \text{ for any } j < i. \end{cases}$$

where the order \preceq_{rev} on columns is defined in Chapter 2, Subsection 2.2.2. The relation \ll is a well ordering on $\text{Col}_1(n)^*$, which is compatible with rules in $\text{Col}_2(n)$ proving the termination [17, Lemma 3.2]. Indeed, one has to prove that if $pc_{u_i}c_{v_j}q \Rightarrow pc_{w_i}c_{w'_j}q$ then $pc_{w_i}c_{w'_j}q \ll pc_{u_i}c_{v_j}q$, for any $pc_{u_i}c_{v_j}q$ and $pc_{w_i}c_{w'_j}q$ in $\text{Col}_1(n)^*$. If $c_{w'_j} = 1$, then $pc_{u_i}c_{v_j}q$ contains more symbols than $pc_{w_i}q$. Thus $pc_{w_i}q \ll pc_{u_i}c_{v_j}q$. On the other hand, suppose that the tableau $P(uv)$ consists of two columns w and w' . By the combinatorial properties obtained in Section 3.2.1, the column w contains more elements than u . Then, we obtain $w \preceq_{\text{rev}} u$. Hence we have again that $pc_{w_i}c_{w'_j}q \ll pc_{u_i}c_{v_j}q$. Since every application of a rule yield a \ll -preceding 1-cell, it follows that every sequence of rewriting using the 2-cells of $\text{Col}_2(n)$ must terminate.

An other method to prove termination of the 2-polygraph $\text{Col}_2(n)$ will be given in Chapter 4, Remark 4.1.3.6.

Confluence of the column presentation. The column presentation is confluent, [17, Lemma 3.3]. The proof given in [17] consists in showing that the 2-polygraph $\text{Col}_2(n)$ has the unique normal form property. Note that our construction in Chapter 4, Section 4.1 gives an other proof of the confluence of the 2-polygraph $\text{Col}_2(n)$ by showing the confluence of all the critical branchings of the column presentation.

Cardinality of the column presentation. For $m = 1$ and $m = 2$, let denote by $\varkappa(n, m)$ the number of m -cells of the presentation $\text{Col}_2(n)$ of the monoid \mathbf{P}_n . We refer the reader to Chapter 4, Subsection 4.2.5 for the values of number of cells of the 2-polygraph $\text{Col}_2(n)$ for plactic monoids of low-dimensional rank n .

3.2.2.3. Proposition. *For $n > 0$, we have $\varkappa(n, 1) = 2^n - 1$ and*

$$\varkappa(n, 2) = \varkappa(n, 1)^2 - \left(\prod_{1 \leq i < j \leq n} \frac{i+j+1}{i+j-1} - \prod_{1 \leq i < j \leq n} \frac{i+j}{i+j-1} \right).$$

3.3. Column presentation of the symplectic plactic monoid

Proof. The number $\varkappa(n, 1)$ is the sum of the number of columns of length k for any $1 \leq k \leq n$. Moreover, the number of columns of length k is equal to $\binom{n}{k}$. Hence we have

$$\varkappa(n, 1) = \sum_{k=1}^n \binom{n}{k} = 2^n - 1.$$

Denote by $S_{n,q}$ the set of all tableaux with at most q columns and with entries in $[n]$. By Gordon [33], we have

$$|S_{n,q}| = \prod_{1 \leq i \leq j \leq n} \frac{q + i + j - 1}{i + j - 1}.$$

Then, for two columns u and v in $\text{col}(n)$ the number of possibilities of $u \times v$ is $|S_{n,2}| - |S_{n,1}|$. In addition, the number of possibilities of $u \times v$ and $u \widehat{\times} v$ is $\varkappa(n, 1)^2$. Since $\varkappa(n, 2)$ is equal to the number of possibilities of $u \times v$, we have $\varkappa(n, 2) = \varkappa(n, 1)^2 - (|S_{n,2}| - |S_{n,1}|)$. \square

Examples. Let us compute the column presentation of the plactic monoids \mathbf{P}_2 and \mathbf{P}_3 .

1. The column presentation $\text{Col}_2(2)$ of the monoid \mathbf{P}_2 has generators c_1, c_2, c_{21} , together with the rules $\alpha_{2,1} : c_2 c_1 \Rightarrow c_{21}$, $\alpha_{1,21} : c_1 c_{21} \Rightarrow c_{21} c_1$ and $\alpha_{2,21} : c_2 c_{21} \Rightarrow c_{21} c_2$.
2. The column presentation $\text{Col}_2(3)$ of the monoid \mathbf{P}_3 admits 7 generators $c_1, c_2, c_3, c_{32}, c_{31}, c_{21}, c_{321}$ with the following 22 rules

$$\begin{aligned} & c_1 c_{21} \xrightarrow{\alpha_{1,21}} c_{21} c_1, \quad c_1 c_{31} \xrightarrow{\alpha_{1,31}} c_{31} c_1, \quad c_1 c_{32} \xrightarrow{\alpha_{1,32}} c_{31} c_2, \quad c_2 c_{32} \xrightarrow{\alpha_{2,32}} c_{32} c_2, \\ & c_2 c_{31} \xrightarrow{\alpha_{2,31}} c_{21} c_3, \quad c_2 c_1 \xrightarrow{\alpha_{2,1}} c_{21}, \quad c_3 c_{32} \xrightarrow{\alpha_{3,32}} c_{32} c_3, \quad c_3 c_2 \xrightarrow{\alpha_{3,2}} c_{32}, \quad c_{32} c_{321} \xrightarrow{\alpha_{32,321}} c_{321} c_{32}, \\ & c_3 c_{31} \xrightarrow{\alpha_{3,31}} c_{31} c_3, \quad c_3 c_1 \xrightarrow{\alpha_{3,1}} c_{31}, \quad c_3 c_{321} \xrightarrow{\alpha_{3,321}} c_{321} c_3, \quad c_3 c_{21} \xrightarrow{\alpha_{3,21}} c_{321}, \quad c_{31} c_{21} \xrightarrow{\alpha_{31,21}} c_{321} c_1, \\ & c_{32} c_{21} \xrightarrow{\alpha_{32,21}} c_{321} c_2, \quad c_{32} c_1 \xrightarrow{\alpha_{32,1}} c_{321}, \quad c_2 c_{321} \xrightarrow{\alpha_{2,321}} c_{321} c_2, \quad c_1 c_{321} \xrightarrow{\alpha_{1,321}} c_{321} c_1, \\ & c_{21} c_{321} \xrightarrow{\alpha_{21,321}} c_{321} c_{21}, \quad c_{31} c_{321} \xrightarrow{\alpha_{31,321}} c_{321} c_{31}, \quad c_2 c_{21} \xrightarrow{\alpha_{2,21}} c_{21} c_2, \quad c_{32} c_{31} \xrightarrow{\alpha_{32,31}} c_{321} c_3. \end{aligned}$$

We have seen in Example 3.1.0.2 that the previous 22 2-cells appear in crystal isomorphisms.

3.3. COLUMN PRESENTATION OF THE SYMPLECTIC PLACTIC MONOID

In this section, we consider the plactic monoid $\mathbf{P}_n(C)$ of type C. As shown in Subsections 1.3.3 and 1.3.4 of Chapter 1, this monoid is constructed using the crystal graphs of the representations

of the symplectic Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C})$. It is defined as the quotient of the free monoid \mathcal{C}_n^* over the ordered set

$$\mathcal{C}_n = \{1 < 2 < \dots < n < \bar{n} < \dots < \bar{1}\},$$

by the equivalence \sim_{crys} , see Subsection 1.3.4 of Chapter 1.

The aim of this section is to construct a finite convergent presentation of the symplectic plactic monoid using the notion of symplectic tableaux presented in 3.3.1. The generating set of our presentation contains the finite set of admissible columns. The right side of the relations of this presentation is the result of Lecouvey’s insertion of an admissible column into another one. This insertion algorithm is fully presented in 3.3.2. We show in 3.3.4 that this presentation is finite and convergent.

3.3.1. Symplectic tableaux

The notion of admissible columns for type C was introduced as a generalisation of columns of type A, [61, 108, 78]. Using admissible columns, one constructs a generalisation of tableaux for type C, called symplectic tableaux. In the following, we will recall the notions of admissible columns and symplectic tableaux as defined by Lecouvey in [78].

Columns for type C. A *column for type C* is a Young diagram U consisting of one column filled by letters of \mathcal{C}_n strictly increasing from top to bottom. We call the *reading* of a column U the word $w(U)$ obtained by reading the letters of U from top to bottom. The height of a column U is the number of letters in U and denoted by $h(U)$. A word w is a column word if there exists a column U such that $w = w(U)$.

3.3.1.1. Example. The following Young tableau

$$U = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \bar{6} \\ \hline \bar{5} \\ \hline \end{array}$$

is a column. Its reading is $w(U) = 123\bar{6}\bar{5}$.

Admissible columns. In [61], Kashiwara and Nakashima introduced the notion of admissible column. Let $w(U) = x_1 \dots x_{h(U)}$ be the reading of a column U . For us, the column C is *admissible* if for $m = 1, \dots, h(U)$, the number $N(m)$ of letters x in U such that $x \leq m$ or $x \geq \bar{m}$ satisfies $N(m) \leq m$. We will denote by $\text{acol}(n)$ the set of readings of non-empty admissible columns in the free monoid \mathcal{C}_n^* .

Let U be a column and $I = \{x_1 > \dots > x_r\}$ be the set of unbarred letters such that x_i and \bar{x}_i in U , for $i = 1, \dots, r$. The column U can be *split* if there exists a set of unbarred letters $J = \{y_1 > \dots > y_r\}$ containing r elements of \mathcal{C}_n such that :

3.3. Column presentation of the symplectic plactic monoid

- y_1 is the greatest letter of \mathcal{C}_n satisfying $y_1 < x_1$, $y_1 \notin U$ and $\bar{y}_1 \notin U$,
- for $i = 2, \dots, r$, y_i is the greatest letter of \mathcal{C}_n such that $y_i < \min(y_{i-1}, x_i)$, $y_i \notin U$ and $\bar{y}_i \notin U$.

Denote by rU the column obtained by changing in U , \bar{x}_i into \bar{y}_i for each letter x_i in the set I up to reordering. Denote by lU the column obtained by changing in U , x_i into y_i for each letter x_i in the set I up to reordering.

3.3.1.2. Proposition ([108, Section 4]). *A column U is admissible if, and only if, it can be split.*

3.3.1.3. Example. Let $w(U) = 2568\bar{8}\bar{5}\bar{2}$ be the reading of a column U . Then

$$I = \{8 > 5 > 2\}, \quad J = \{7 > 4 > 1\},$$

$$w(rU) = 2568\bar{7}\bar{4}\bar{1} \quad \text{and} \quad w(lU) = 1467\bar{8}\bar{5}\bar{2}.$$

The column U can be split, so that it is an admissible column.

3.3.1.4. Example. Let $w(U') = 2346\bar{6}\bar{3}\bar{2}$ be the reading of a column U' . Then

$$I = \{6, 3, 2\}, \quad y_1 = 5, \quad y_2 = 1$$

and we cannot find an element y_3 of \mathcal{C}_n such that $y_3 < 1$. Thus U' cannot be split.

Using admissible columns, one can construct a tableau whose columns are admissible with an additional property on them. This tableau is called the symplectic tableau. We will recall its definition in our context.

Notations. Let U_1, \dots, U_r be the r columns from left to right of a Young tableau T , then T is denoted by $T = U_1 \dots U_r$.

Let U_1 and U_2 be two admissible columns. Consider the following notation :

- $U_1 \leq U_2$ if $h(U_1) \geq h(U_2)$ and the rows of the tableau $U_1 U_2$ are weakly increasing from left to right.
- $U_1 \preceq U_2$ if $rU_1 \leq lU_2$.

Symplectic tableaux. A tableau $T = U_1 \dots U_r$ is a *symplectic tableau* if we have $U_i \preceq U_{i+1}$ for $i = 1, \dots, r - 1$. The reading of the symplectic tableau T is the word $w(T)$ obtained by reading the columns of T from right to left, that is

$$w(T) = w(U_r)w(U_{r-1}) \dots w(U_1).$$

3.3.1.5. Example. Let us consider the tableau

$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & \bar{3} & \\ \hline 3 & \bar{2} & \\ \hline \end{array}$$

T is a symplectic tableau. Indeed,

- $w(U_1) = 123$, $I_{U_1} = J_{U_1} = \emptyset$ and $w(rU_1) = w(lU_1) = 123$.
- $w(U_2) = 2\bar{3}\bar{2}$, $I_{U_2} = \{2\}$, $J_{U_2} = \{1\}$, $w(rU_2) = 2\bar{3}\bar{1}$ and $w(lU_2) = 1\bar{3}\bar{2}$.
- $w(U_3) = 3$, $I_{U_3} = J_{U_3} = \emptyset$ and $w(rU_3) = w(lU_3) = 3$.

The columns U_1 , U_2 and U_3 can be split, so they are admissible columns. We have $U_1 \preceq U_2 \preceq U_3$, so T is a symplectic tableau and $w(T) = 32\bar{3}\bar{2}123$.

Crystal graphs and symplectic tableaux. In the following remark, we will characterize the crystal graph $B(\lambda)$ of the irreducible module $V(\lambda)$ of highest weight λ as the set of symplectic tableaux of shape λ .

3.3.1.6. Remark. Let $\lambda = \sum_{i=1}^n \lambda_i \Lambda_i$ be a dominant weight. By Theorem 4.5.1 in [61], $B(\lambda)$ coincides with the set of symplectic tableaux of shape λ . More precisely, the readings of these tableaux are the vertices of a connected component of $G_{n,|\lambda|}$ isomorphic to $B(\lambda)$. The highest weight vertex of this component is the reading of the tableau of shape λ filled with 1 on the 1st row, 2 on the 2nd row, ... , and n on the n th row. In particular, the reading of the highest weight vertex of a connected component containing admissible columns of height p is $12 \dots p$.

3.3.1.7. Example. Let us consider the symplectic Lie algebra $\mathfrak{g} = \mathfrak{sp}_6(\mathbb{C})$ of dimension 6 and the symplectic tableau T defined in Example 3.3.1.5. The reading $w(T) = 32\bar{3}\bar{2}123$ of T is a vertex of $B(\Lambda_1 + 2\Lambda_2)$ where its vertex of highest weight is 1123123 . Note that the word 1123123 is the reading of the following tableau

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & \\ \hline 3 & 3 & \\ \hline \end{array}$$

with $w(T) = \tilde{f}_1 \circ \tilde{f}_2 \circ \tilde{f}_3 \circ \tilde{f}_2 \circ \tilde{f}_1 \circ \tilde{f}_2 \circ \tilde{f}_3(1123123)$.

3.3.1.8. Proposition ([78, Proposition 3.1.2]). Every word w in C_n^* admits a unique symplectic tableau T such that $w \sim_{crys} w(T)$.

The unique symplectic tableau T such that $w \sim_{crys} w(T)$ is denoted by $P(w)$.

3.3.2. A bumping algorithm for type C

In [78], Lecouvey introduces an insertion scheme to compute the symplectic tableau $P(w)$ analogous to the Schensted's algorithm for type A. We present in the following sections Lecouvey's algorithms and we refer the reader to [78] for more details.

Let denote by $x \rightarrow T$ the insertion of a letter x in a symplectic tableau T .

Insertion of a letter in an admissible column. Consider a word $w = w(U)x$, where x is a letter and U is an admissible column of height p . We have three cases :

- If w is the reading of an admissible column, then $x \rightarrow U$ is the column obtained by adding a box filled by the letter x at the bottom of U . In this case, the highest weight vertex of $B(w)$ is equal to $1 \dots p(p+1)$.
- If w is a nonadmissible column word such that each strict factor of it is admissible, then $x \rightarrow U$ is the column of reading \tilde{w} obtained from w by applying one relation of type (R_3) , which is uniquely determined [78]. In this case, the highest weight vertex of $B(w)$ is equal to $1 \dots p\bar{p}$.
- If w is not a column word, then $x \rightarrow U$ is obtained by applying relations of type (R_1) or (R_2) to the final subword of length 3 of w . On the resulting word, one continues by applying relations of type (R_1) or (R_2) to the maximal overlapping factor of length 3 to the left and this procedure is repeated until the first factor of length 3 has been operated. The result is the reading of a symplectic tableau consisting of a column U' of height p and a column $\boxed{x'}$, where x' is an element of \mathcal{C}_n . Then

$$x \rightarrow U = U' \boxed{x'} = P(w).$$

In this case, the highest weight vertex of $B(w)$ is equal to $1 \dots p1$.

3.3.2.1. Example. Let us consider the following three examples.

1. Suppose $w(U) = 3\bar{6}\bar{4}$ and $x = \bar{3}$, then

$$\bar{3} \rightarrow \begin{array}{|c|} \hline 3 \\ \hline 6 \\ \hline \bar{6} \\ \hline \bar{4} \\ \hline \end{array} = \begin{array}{|c|} \hline 3 \\ \hline 6 \\ \hline \bar{6} \\ \hline \bar{4} \\ \hline \bar{3} \\ \hline \end{array}.$$

2. Suppose $w(U) = 14\bar{4}\bar{3}$ and $x = \bar{2}$, the word $14\bar{4}\bar{3}\bar{2}$ is a nonadmissible column word such that each strict subword of it is an admissible column word, then we obtain by applying

3.3. Column presentation of the symplectic plactic monoid

relation of type (R_3) ,

$$\bar{2} \rightarrow \begin{array}{|c|} \hline 1 \\ \hline 4 \\ \hline \bar{4} \\ \hline \bar{3} \\ \hline \end{array} = \begin{array}{|c|} \hline 1 \\ \hline \bar{3} \\ \hline \bar{2} \\ \hline \end{array}.$$

3. Suppose $w(U) = 14\bar{4}\bar{3}$ and $\chi = 2$, then the word $14\bar{4}\bar{3}2$ is not a column word. By applying relations of type (R_1) or (R_2) , we obtain:

$$14\bar{4}\bar{3}2 \equiv 14\bar{4}2\bar{3} \equiv 142\bar{4}\bar{3} \equiv 412\bar{4}\bar{3}.$$

Then

$$2 \rightarrow \begin{array}{|c|} \hline 1 \\ \hline 4 \\ \hline \bar{4} \\ \hline \bar{3} \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline \bar{4} & \\ \hline \bar{3} & \\ \hline \end{array}.$$

Insertion of a letter in a symplectic tableau. Let $T = U_1 \dots U_r$ be a symplectic tableau with admissible column U_i , for $i = 1, \dots, r$, and χ be a letter. We have three cases:

- If $w(U_1)\chi$ is an admissible column word, then $\chi \rightarrow T$ is the tableau obtained by adding a box filled by χ on the bottom of U_1 .
- If $w(U_1)\chi$ is a nonadmissible column word whose each strict factor is an admissible column word. Let $w(\widehat{U_1})\chi = y_1 \dots y_s$ be the admissible column word obtained from $w(U_1)\chi$ by applying relation of type (R_3) and $\widehat{T} = U_2 \dots U_r$ be the tableau obtained from T after eliminating the leftmost column U_1 . Then $\chi \rightarrow T$ is obtained by inserting successively the elements of $w(\widehat{U_1})\chi$ in the tableau \widehat{T} . That is,

$$\chi \rightarrow T = y_s \rightarrow (y_{s-1} \rightarrow (\dots y_1 \rightarrow \widehat{T})).$$

Moreover, the insertion of y_1, \dots, y_s in \widehat{T} does not cause a new contraction.

- If $w(U_1)\chi$ is not a column word, then

$$\chi \rightarrow U_1 = \boxed{U'_1 \mid y},$$

where U'_1 is an admissible column of height $h(U_1)$ and y a letter. Then

$$\chi \rightarrow T = U'_1(y \rightarrow U_2 \dots U_r),$$

that is, $\chi \rightarrow T$ is the juxtaposition of U'_1 with the tableau obtained by inserting y in the tableau $U_2 \dots U_r$.

3.3.2.2. **Example.** Consider a symplectic tableau

$$T_1 = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & \bar{3} & \\ \hline 3 & \bar{2} & \\ \hline \end{array}$$

and a letter $x = 1$. Let us compute $x \rightarrow T_1$. First, we begin inserting x in the leftmost column U_1 of T_1 . The word 1231 is not a column word, then by applying at each step (R_1) or (R_2) , we obtain :

$$1231 \equiv 1213 \equiv 1123,$$

so

$$1 \rightarrow \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array}.$$

Then $1 \rightarrow T_1 = U'_1(1 \rightarrow T'_1)$, where

$$U'_1 = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \text{ and } T'_1 = \begin{array}{|c|c|} \hline 2 & 3 \\ \hline \bar{3} & \\ \hline \bar{2} & \\ \hline \end{array}.$$

Similarly, we have $2\bar{3}\bar{2}1 \equiv 2\bar{3}1\bar{2} \equiv 21\bar{3}\bar{2}$, then

$$1 \rightarrow \begin{array}{|c|} \hline 2 \\ \hline \bar{3} \\ \hline \bar{2} \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \bar{3} & \\ \hline \bar{2} & \\ \hline \end{array}.$$

So $1 \rightarrow T_1 = U'_1 U'_2(2 \rightarrow \boxed{3})$, where

$$U'_2 = \begin{array}{|c|} \hline 1 \\ \hline \bar{3} \\ \hline \bar{2} \\ \hline \end{array}.$$

Finally, we have $32 \equiv 32$, then

$$2 \rightarrow \boxed{3} = \begin{array}{|c|c|} \hline 2 & 3 \\ \hline \end{array}.$$

Hence,

$$1 \rightarrow T_1 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & \bar{3} & & \\ \hline 3 & \bar{2} & & \\ \hline \end{array}.$$

3.3. Column presentation of the symplectic plactic monoid

3.3.2.3. Example. Consider a symplectic tableau

$$T_2 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 3 & \\ \hline 3 & \bar{3} & & \\ \hline \end{array}$$

and a letter $\chi = \bar{3}$. Let us compute $\chi \rightarrow T_2$. First, we begin inserting $\chi = \bar{3}$ in the leftmost column U_1 of T_2 . The word $123\bar{3}$ is a nonadmissible column word, that each strict factor is an admissible column word, we have by applying (R_3) ,

$$123\bar{3} \equiv 12,$$

then

$$\widetilde{U}_1 = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \text{ and } \widehat{T}_2 = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 3 & 3 & \\ \hline \bar{3} & & \\ \hline \end{array}.$$

So we have to insert the elements of the column \widetilde{U}_1 in the tableau \widehat{T}_2 .

First, one inserts 1 :

$$13\bar{3}1 \equiv 131\bar{3} \equiv 113\bar{3},$$

then

$$1 \rightarrow \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \bar{3} \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \\ \hline \bar{3} & \\ \hline \end{array}.$$

We have $231 \equiv 213$, then

$$1 \rightarrow \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}.$$

And

$$2 \rightarrow \begin{array}{|c|} \hline 3 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 2 & 3 \\ \hline \end{array}.$$

Hence

$$1 \rightarrow \widehat{T}_2 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 3 & 3 & & \\ \hline \bar{3} & & & \\ \hline \end{array} = \widehat{T}_2'.$$

Secondly, one inserts 2 in the tableau \widehat{T}_2' :

we have $13\bar{3}2 \equiv 132\bar{3} \equiv 312\bar{3}$, then

$$2 \rightarrow \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \bar{3} \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \bar{3} & \\ \hline \end{array}.$$

We have $133 \equiv 313$, then

$$3 \rightarrow \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 & \\ \hline \end{array}.$$

We have $23 \equiv 23$, then

$$3 \rightarrow \begin{array}{|c|} \hline 2 \\ \hline \end{array} = \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array}.$$

Hence,

$$2 \rightarrow \widehat{T}_2' = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 3 & \\ \hline \overline{3} & & & \\ \hline \end{array} = \overline{3} \rightarrow T_2.$$

3.3.2.4. Remark. Consider a word w in \mathcal{C}_n^* . The symplectic tableau $P(w)$ can be computed by starting with the empty word, which is a valid tableau, and iteratively applying the insertion schemes described above. Notice that when w is the reading of a symplectic tableau T , we have $P(w) = T$.

3.3.3. The two two-columns lemmas

Let u and v be the readings of two admissible columns U and V respectively. As we have seen in Subsection 3.3.1, $U \succeq V$ means that the column U can appear to the right of V in a symplectic tableau. Note that $U \not\succeq V$ means that the word uv is not the reading of a symplectic tableau.

3.3.3.1. Lemma. *Let u and v be the readings of two admissible columns U and V respectively. The symplectic tableau $P(uv)$ consists of at most two columns.*

Proof. For $U \succeq V$, the result is trivial. Let $u = x_1 \dots x_p$ and $v = y_1 \dots y_q$ be respectively the readings of two admissible columns U and V of height p and q , such that $U \not\succeq V$. Let $u^0 z_1 \dots z_q$ be the highest weight vertex of the connected component containing uv . We begin inserting the first element y_1 of v in the column U . The shape of the tableau $P(uy_1)$ depends of the connected component containing uy_1 . The highest weight vertex of this component is $u^0 z_1$. By Lemma 1.3.3.3, u^0 is of highest weight and $\varepsilon_i(z_1) \leq \varphi_i(u^0)$, for any $i = 1, \dots, n$. Then we obtain the following cases.

Case 1: $u^0 z_1 = 1 \dots p(p+1)$. In this case, uy_1 is an admissible column word, $z_1 = p+1$ and $\text{wt}(z_1) = \Lambda_{p+1} - \Lambda_p$. Then during the insertion of the letter y_1 in the column U , this column of height p corresponding to the weight Λ_p is transformed into a column of height $p+1$ corresponding to the weight Λ_{p+1} . Its reading is uy_1 . After one continues inserting the others elements y_2, \dots, y_q of the column word v . We know by the definition of an admissible column that every element of this column is strictly larger than its preceding, then we have two cases:

First, suppose that $z_i = p+i$, for $i = 2, \dots, q$. Then $\text{wt}(z_i) = \Lambda_{p+i} - \Lambda_{p+i-1}$ and during the insertion of y_i in the column of reading $uy_1 \dots y_{i-1}$, this column of height $p+i-1$ is turned

3.3. Column presentation of the symplectic plactic monoid

into the column of reading $uy_1 \dots y_i$ and of height $p + i$. Thus uv is an admissible column word and the tableau $P(uv)$ consists of one column uv .

Second, suppose that there exists an element y_k of the column word v such that $uy_1 \dots y_{k-1}y_k$ is a nonadmissible column word whose each strict factor is an admissible column word, then $z_k = \overline{p + k - 1}$ and $\text{wt}(z_k) = \Lambda_{p+k-2} - \Lambda_{p+k-1}$, then during the insertion of y_k in the admissible column of reading $uy_1 \dots y_{k-1}$, this column of height $p + k - 1$ is transformed into a column of height $p + k - 2$. After one continues inserting the remaining elements of v , then one adds those letters in distinct rows in the considered column or one removes some letters from distinct rows of the same column.

Hence, in this case the symplectic tableau $P(uv)$ consists of one column.

Case 2: $u^0 z_1 = 1 \dots p\bar{p}$. In this case, uy_1 is a nonadmissible column word such that each strict factor is an admissible column word. We have $\text{wt}(z_1) = \Lambda_{p-1} - \Lambda_p$, then during the insertion of y_1 in the admissible column U , this column of height p is turned into a column of height $p - 1$. Since the elements of the column V are strictly increasing, one can prove by similar arguments of Case 1, that during the computation of $P((uy_1)y_2 \dots y_q)$, one adds a number of boxes of the considered column in distinct rows and one removes some boxes from distinct rows of the same column. Note also that the column U can be contracted to become empty. Hence, we have in this case that the symplectic tableau $P(uv)$ consists of one column or zero columns.

Case 3: $u^0 z_1 = 1 \dots p1$. In this case, uy_1 is not a column word, then during the insertion of y_1 in the admissible column U , an element appears in a second column. After, one inserts the next element y_2 of the column V in $P(uy_1)$, the highest weight of the connected component containing $w(P(uy_1))y_2$ may be written $w(P(uy_1)^0)z_2$, where $w(P(uy_1)^0)$ is of highest weight and by Lemma 1.3.3.3, we have:

(i) $z_2 = i$ (with $i = p + 1$ or $i = 2$), then its weight is equal to $\Lambda_i - \Lambda_{i-1}$, then during the insertion $y_2 \rightarrow P(uy_1)$ a column of height $i - 1$ is turned into a column of height i . Then one adds a box in the left column or in the right column of $P(uy_1)$.

(ii) $z_2 = \bar{p}$, then its weight is equal to $\Lambda_{p-1} - \Lambda_p$, then during the insertion $y_2 \rightarrow P(uy_1)$, the right column of height p is turned into a column of height $p - 1$.

After we continue inserting the remaining letters of v , and since every element is strictly larger than its preceding, one adds boxes in distinct rows in the right or in the left column and similarly one removes boxes from distinct rows of the considered symplectic tableau. Note also that it is impossible that one of the columns contracts to become empty. Indeed, let u and v be respectively the readings of two admissible columns U and V such that uv is of highest weight. Suppose that after adding k boxes in the right column, one inserts p boxes in the left column to contract it into an empty one. Then in this case we have $u = 1 \dots p$ and $v = 1 \dots k\bar{p}(\overline{p-1}) \dots \bar{1}$. We have in the word v that $N(p) = p + k > p$. So the column V is not admissible, which yields a contradiction.

Hence, the symplectic tableau $P(uv)$ consists of two columns. □

3.3.3.2. Lemma. *Let u and v be the readings of two admissible columns U and V respectively, such that $U \not\geq V$. Suppose that $P(uv)$ has two columns and let W be the rightmost column. Then the column U contains more elements than W .*

Proof. Let $u = x_1 \dots x_p$ and $v = y_1 \dots y_q$ be respectively the readings of two admissible columns U and V of height p and q , such that $U \not\geq V$. Let w and w' be respectively the readings of the right and left column W and W' of $P(uv)$. If the height of U is greater than the height of V , then in all cases we have $h(W) < p$.

Suppose that $q \geq p$ and the columns U and V contain only unbarred letters. Suppose that during the computation of $P(uv)$, we only add boxes by applying relations of type (R_1) . In other words, we compute $P(uv)$ by Schensted's insertion. If $h(W) = p$, then during inserting the first p elements of V , p boxes are added in the second column and they are all filled by elements of U . Since the number of added boxes is equal to the height of U , $w(P(uv)) = uv$. Then $U \geq V$ which yields a contradiction. Hence, $h(W) < p$.

Suppose now that during the computation of $P(uv)$, we only add boxes by applying relations of type (R_1) or (R_2) . By definition of $P(uv)$ we have $w(P(uv)) = ww' \equiv uv$. Then the words uv and ww' occur at the same place in their isomorphic connected components $B(uv)$ and $B(ww')$ of the crystal G_n . Note that all the vertices in a connected component are the readings of tableaux of the same shape. Let $(uv)^0$ and $(ww')^0$ be respectively the highest weight vertices of $B(uv)$ and $B(ww')$. By Remark 3.3.1.6, the word $(ww')^0$ is the reading of a tableau that all its elements are unbarred letters, then $(uv)^0$ and $(ww')^0$ are related by relations of type (R_1) . Hence, as we have seen above, the height of the second column of $P((uv)^0)$ is strictly less than p . Since $(ww')^0$ and ww' are the readings of two symplectic tableaux of the same shape, the length of w is strictly less than p .

Suppose that during the insertion of the first k elements of v , for $k \leq p-1$, into the column U , we add k boxes in a second column. Then

$$P(uy_1 \dots y_k) = \boxed{U_1 \mid U_2},$$

where U_1 contains p elements and U_2 contains the k added boxes. After we insert y_{k+1} in the column U_1 . Suppose that $w(U_1)y_{k+1}$ is a nonadmissible column word such that all of its proper factors are admissible. Let $w(\widetilde{U_1})y_{k+1}$ be the column word obtained from $w(U_1)y_{k+1}$ after applying relation of type (R_3) . Then we insert the elements of $w(\widetilde{U_1})y_{k+1}$ in the column U_2 . This insertion does not cause a new contraction. Then if we obtained two columns, the height of the right one is strictly less than the height of U_2 which is strictly less than p . After we continue inserting the remaining elements of v , and the height of the right column of the final tableau is strictly less than p . \square

3.3. Column presentation of the symplectic plactic monoid

3.3.4. The column presentation for type C

Knuth-like presentation. Consider a presentation of the plactic monoid $\mathbf{P}_n(\mathbb{C})$, by the 2-polygraph $\text{SP}_2(n)$, whose set of 1-cells is \mathcal{C}_n and whose 2-cells correspond to the relations (R_1) , (R_2) and (R_3) oriented with respect to the reverse deglex order, that is

$$\begin{aligned} \text{SP}_2(n) = & \{ xzy \xrightarrow{K_{x,y,z}} zxy \mid x < y \leq z \text{ and } z \neq \bar{x} \} \\ & \cup \{ yxz \xrightarrow{K'_{x,y,z}} yzx \mid x \leq y < z \text{ and } z \neq \bar{x} \} \\ & \cup \{ yx\bar{x} \xrightarrow{\xi_{x,y,\bar{x}}} \overline{y(x-1)(x-1)} \mid x \leq y \leq \bar{x} \text{ and } 1 < x \leq n \} \\ & \cup \{ x\bar{x}y \xrightarrow{\xi'_{x,y,\bar{x}}} \overline{(x-1)(x-1)y} \mid x \leq y \leq \bar{x} \text{ and } 1 < x \leq n \} \\ & \cup \{ w \xrightarrow{\zeta_w} \tilde{w} \mid w \text{ and } \tilde{w} \text{ satisfy the conditions of the relation } (R_3) \}. \end{aligned}$$

The order being monomial, the 2-polygraph $\text{SP}_2(n)$ is terminating.

3.3.4.1. Remark. For $n \geq 4$, the Knuth presentation of the plactic monoid for type A doesn't admit a finite completion compatible with the reverse deglex order. Indeed, by similar arguments used in Chapter 2, Theorem 2.3.3.3, one can show that during the completion one adds an infinity of 2-cells of the form $232^i124 \implies 2342^i12$, for $i > 1$. The 2-polygraph $\text{SP}_2(n)$ contains the Knuth relations for type A and we can not apply relations of type (R_2) and (R_3) on the words 232^i124 and 2342^i12 , for $i > 1$, then the 2-polygraph $\text{SP}_2(n)$ does not also admit a finite completion compatible with the reverse deglex order.

Column generators. In order to give a finite convergent presentation of the plactic monoid $\mathbf{P}_n(\mathbb{C})$, one introduces the admissible column generators. The set of generators is

$$\text{ACol}_1(n) = \{ c_u \mid u \in \text{acol}(n) \},$$

where each symbol c_u represents the element u of $\mathbf{P}_n(\mathbb{C})$. In particular, the word c_x represents the letter x in \mathcal{C}_n , hence the set $\text{ACol}_1(n)$ also generates $\mathbf{P}_n(\mathbb{C})$.

Let $w = x_1 \dots x_{\ell(w)}$ and $\tilde{w} = \tilde{x}_1 \dots \tilde{x}_{\ell(\tilde{w})}$ be two columns such that $w \equiv \tilde{w}$ by a relation of type (R_3) . The 2-cells corresponding to the relations (R_1) , (R_2) and (R_3) can be written in the following form

$$\begin{aligned} \text{SP}_2(n) = & \{ c_x c_z c_y \xrightarrow{C_{K_{x,y,z}}} c_z c_x c_y \mid x < y \leq z \text{ and } z \neq \bar{x} \} \\ & \cup \{ c_y c_x c_z \xrightarrow{C_{K'_{x,y,z}}} c_y c_z c_x \mid x \leq y < z \text{ and } z \neq \bar{x} \} \\ & \cup \{ c_y c_x c_{\bar{x}} \xrightarrow{C_{\xi_{x,y,\bar{x}}}} c_y c_{\overline{(x-1)}} c_{(x-1)} \mid x \leq y \leq \bar{x} \text{ and } 1 < x \leq n \} \\ & \cup \{ c_x c_{\bar{x}} c_y \xrightarrow{C_{\xi'_{x,y,\bar{x}}}} c_{\overline{(x-1)}} c_{(x-1)} c_y \mid x \leq y \leq \bar{x} \text{ and } 1 < x \leq n \} \\ & \cup \{ c_{x_1} \dots c_{x_{\ell(w)}} \xrightarrow{C_{\zeta_w}} c_{\tilde{x}_1} \dots c_{\tilde{x}_{\ell(\tilde{w})}} \mid w \text{ and } \tilde{w} \text{ verify the relation } (R_3) \}. \end{aligned}$$

3.3.4. The column presentation for type C

Let us denote by

$$\mathrm{SP}_2^{c(n)} = \{ c_{y_1} \dots c_{y_k} \xrightarrow{\gamma_u} c_u \mid u = y_1 \dots y_k \in \mathrm{acol}(n) \text{ and } \ell(u) \geq 2 \},$$

the set of 2-cells corresponding to the defining relations for the extra column generators c_u .

Let us define the 2-polygraph $\mathrm{SP}_2^{(n)}$ whose set of 1-cells is $\mathrm{ACol}_1(n)$ and whose set of 2-cells is $\mathrm{SP}_2(n) \cup \mathrm{SP}_2^{c(n)}$. The following lemma shows that the 2-polygraphs $\mathrm{SP}_2(n)$ and $\mathrm{SP}_2^{(n)}$ are Tietze equivalent. As a consequence, the symplectic plactic monoid $\mathbf{P}_n(C)$ is presented by the 2-polygraph $\mathrm{SP}_2^{(n)}$.

3.3.4.2. Lemma. *The 2-polygraphs $\mathrm{SP}_2(n)$ and $\mathrm{SP}_2^{(n)}$ are Tietze equivalent.*

Proof. We have $\mathrm{ACol}_1(n) = \{c_1, \dots, c_n, \dots, c_{\bar{1}}\} \cup \{c_u \mid u \in \mathrm{acol}(n), \ell(u) \geq 2\}$. Thus in order to prove the Tietze equivalence, we add to the 2-polygraph $\mathrm{SP}_2(n)$ all the column generator c_u , for all $u = y_1 \dots y_k$ in $\mathrm{acol}(n)$ such that $\ell(u) \geq 2$, and the corresponding collapsible 2-cell $\gamma_u : c_{y_1} \dots c_{y_k} \Rightarrow c_u$. We apply successively a Tietze transformation $\iota_{\gamma_u}^1$, from the bigger column in $\mathrm{acol}(n)$ to the smaller one with respect to the order $\preceq_{\mathrm{deglex}}$. The composite of these Tietze transformations defines a Tietze equivalence from $\mathrm{SP}_2(n)$ to $\mathrm{SP}_2^{(n)}$. \square

Definition of the new 2-cells. Let u and v be respectively the readings of two non-empty admissible columns U and V . Suppose that $U \not\preceq V$, by Lemma 3.3.3.1 the symplectic tableau $P(uv)$ consists of at most two columns. Define a 2-cell

- $c_u c_v \xrightarrow{\alpha_{u,v}} c_w c_{w'}$, where the words w and w' are respectively the readings of the right and left columns W and W' of $P(uv)$ if this symplectic tableau consists of two columns.
- $c_u c_v \xrightarrow{\alpha_{u,v}} c_w$, where w is the reading of the column W of $P(uv)$ if it consists of one column.
- $c_u c_v \xrightarrow{\alpha_{u,v}} c_\varepsilon$, where ε is the empty word if $P(uv)$ consists of zero columns.

The column presentation. Let $\mathrm{ACol}_2(n)$ be the 2-polygraph whose set of 1-cells is $\mathrm{ACol}_1(n)$ and whose set of 2-cells is

$$\mathrm{ACol}_2(n) = \{ c_u c_v \xrightarrow{\alpha_{u,v}} c_w c_{w'} \mid u, v \in \mathrm{acol}(n) \text{ such that } U \not\preceq V \}.$$

As we will see in Lemma 3.3.4.5, the 2-polygraph $\mathrm{ACol}_2(n)$ is a presentation of the symplectic monoid $\mathbf{P}_n(C)$, called the *column presentation*.

3.3.4.3. Remark. Every rule in $\mathrm{ACol}_2(n)$ holds in the symplectic plactic monoid $\mathbf{P}_n(C)$, indeed,

$$c_u c_v \equiv uv \equiv w(P(uv)) = ww' \equiv c_w c_{w'}.$$

3.3. Column presentation of the symplectic plactic monoid

Well-ordering on ACol_1^* . Let $<$ be the total order on \mathcal{C}_n defined by

$$1 < 2 < \dots < n < \bar{n} < \dots < \bar{1}.$$

Denote by $<_{\text{deg}}$ the deglex order induced by $<$ on the monoid \mathcal{C}_n^* . Let us define an order on ACol_1^* . First, let \sqsubset be the total order on $\text{ACol}_1(n)$ defined by

$$c_u \sqsubset c_v \text{ if } \ell(u) < \ell(v) \text{ or } [\ell(u) = \ell(v) \text{ and } u <_{\text{lex}} v].$$

Secondly, consider the order \prec on $\text{ACol}_1(n)^*$, defined as follows. We have

$$\begin{aligned} c_{u_1} c_{u_2} \dots c_{u_m} \prec c_{v_1} c_{v_2} \dots c_{v_n} & \text{ if } m < n \\ \text{or } (m = n \text{ and there exists } i \text{ such that } c_{u_i} \sqsubset c_{v_i} & \text{ and } \forall j < i, c_{u_j} = c_{v_j}), \end{aligned}$$

where c_{u_i} and c_{v_j} are elements of $\text{ACol}_1(n)$, for $i = 1, \dots, m$ and $j = 1, \dots, n$.

That is, two elements in $\text{ACol}_1(n)^*$ are compared using the number of their symbols. If they have the same number of symbols, we compare them using the total order \sqsubset on the elements of $\text{ACol}_1(n)$ which is induced by the deglex order on the columns words in \mathcal{C}_n^* . Then \prec is a total order on $\text{ACol}_1(n)^*$ and it is a well-ordering.

3.3.4.4. Lemma. *The 2-polygraph $\text{ACol}_2(n)$ is finite.*

Proof. The set $\text{ACol}_1(n)$ is finite thanks to the fact that the admissible columns words in the free monoid \mathcal{C}_n^* have length at most n . Hence, the 2-polygraph $\text{ACol}_2(n)$ is finite. \square

The following lemma shows that the 2-polygraph $\text{ACol}_2(n)$ is a presentation of the symplectic plactic monoid $\mathbf{P}_n(\mathbb{C})$:

3.3.4.5. Lemma. *The 2-polygraphs $\text{SP}_2^{(n)}$ and $\text{ACol}_2(n)$ are Tietze equivalent.*

Proof. Every relation in $\text{SP}_2^{(n)}$ can be deduced from rules in $\text{ACol}_2(n)$ as follows. The 2-cells $c_{\kappa_{x,y,z}}$ for $x < y \leq z$ and $z \neq \bar{x}$, $c_{\kappa'_{x,y,z}}$ for $x \leq y < z$ and $z \neq \bar{x}$, $c_{\xi_{x,y,\bar{x}}}$ and $c_{\xi'_{x,y,\bar{x}}}$ for $x \leq y \leq \bar{x}$ and $1 < x \leq n$ are obtained from rules in $\text{ACol}_2(n)$ according to the following composition

$$\begin{array}{ccc} c_z c_x c_y & \xleftarrow{c_{\kappa_{x,y,z}}} & c_x c_z c_y \\ \Downarrow c_z \alpha_{x,y} & & \Downarrow \alpha_{x,z} c_y \\ c_z c_{xy} & \xleftarrow{\alpha_{xz,y}} & c_{xz} c_y \end{array} \qquad \begin{array}{ccc} c_y c_z c_x & \xleftarrow{c_{\kappa'_{x,y,z}}} & c_y c_x c_z \\ \Downarrow \alpha_{y,z} c_x & & \Downarrow c_y \alpha_{x,z} \\ c_{yz} c_x & \xleftarrow{\alpha_{yz,x}} & c_y c_{xz} \end{array}$$

3.3.4. The column presentation for type C

$$\begin{array}{ccc}
 c_y c_{\bar{x}-1} c_{x-1} \xleftarrow{c_{\xi_{x,y,\bar{x}}}} c_y c_x c_{\bar{x}} & & c_{\bar{x}-1} c_{x-1} c_y \xleftarrow{c_{\xi'_{x,y,\bar{x}}}} c_x c_{\bar{x}} c_y \\
 \Downarrow \alpha_{y,\bar{x}-1} c_{x-1} & & \Downarrow c_{\bar{x}-1} \alpha_{x-1,y} \\
 c_{y(\bar{x}-1)} c_{x-1} \xrightarrow{\alpha_{y(\bar{x}-1),x-1}} c_y c_{x\bar{x}} & & c_{\bar{x}-1} c_{(x-1)y} \xleftarrow{\alpha_{x\bar{x},y}} c_{x\bar{x}} c_y \\
 \Downarrow c_y \alpha_{x,\bar{x}} & & \Downarrow \alpha_{x,\bar{x}} c_y
 \end{array}$$

Let $w = x_1 \dots x_p \dots x_q \dots x_k$ be a nonadmissible column word of length k whose each strict factor is an admissible column word. Let $z = x_p$ be the lowest unbarred letter such that $z = x_p$ and $\bar{z} = x_q$ occur in w and $N(z) = z + 1$. Then the 2-cell c_{ζ_w} is deduced from rules in $\text{ACol}_2(n)$ according to the following composition

$$\begin{array}{ccc}
 c_{x_1} \dots c_{x_p} \dots c_{x_q} \dots c_{x_k} & \xrightarrow{c_{\zeta_w}} & c_{x_1} \dots \widehat{c_{x_p}} \dots \widehat{c_{x_q}} \dots c_{x_k} \\
 \Downarrow \alpha_{x_1,x_2} c_{x_3} \dots c_{x_k} & & \Downarrow \alpha_{x_1,x_2} c_{x_3} \dots c_{x_k} \\
 (\dots) & & (\dots) \\
 \Downarrow \alpha_{x_1 \dots x_{k-2}, x_{k-1}} c_{x_k} & & \Downarrow \alpha_{x_1 \dots \widehat{x_p} \dots \widehat{x_q} \dots x_{k-1}, x_k} \\
 c_{x_1 \dots x_p \dots x_q \dots x_{k-1}} c_{x_k} & \xrightarrow{\alpha_{x_1 \dots x_p \dots x_q \dots x_{k-1}, x_k}} & c_{x_1 \dots \widehat{x_p} \dots \widehat{x_q} \dots x_k}
 \end{array}$$

where the symbol \hat{x} means that x is removed.

In addition, any rules γ_{y_1, \dots, y_k} in $\text{SP}_2^{c(n)}$ can be obtained using those in $\text{ACol}_2(n)$, according to the following composition

$$\begin{array}{ccc}
 c_{y_1} \dots c_{y_k} & \xrightarrow{\gamma_{y_1, \dots, y_k}} & c_{y_1 \dots y_k} \\
 \Downarrow \alpha_{y_1, y_2} c_{y_3} \dots c_{y_k} & & \Downarrow \alpha_{y_1 \dots y_{k-1}, y_k} \\
 c_{y_1 y_2} c_{y_3} \dots c_{y_k} & \xrightarrow{\alpha_{y_1 y_2, y_3} c_{y_4} \dots c_{y_k}} (\dots) \xrightarrow{\alpha_{y_1 \dots y_{k-2}, y_{k-1}} c_{y_k}} & c_{y_1 \dots y_{k-1}} c_{y_k}
 \end{array}$$

As a consequence, if 1-cells w and w' in $\text{ACol}_1(n)^*$ are equal modulo relations in $\text{SP}_2(n) \cup \text{SP}_2^{c(n)}$, then they are equal modulo relations in $\text{ACol}_2(n)$.

Conversely, if 1-cells w and w' in $\text{ACol}_1(n)^*$ are equal modulo relations in $\text{ACol}_2(n)$, by Remark 3.3.4.3, they are equal modulo relations in $\text{SP}_2(n) \cup \text{SP}_2^{c(n)}$. \square

3.3. Column presentation of the symplectic plactic monoid

3.3.4.6. Theorem. *The 2-polygraph $\text{ACol}_2(n)$ is a finite convergent presentation of the symplectic plactic monoid $\mathbf{P}_n(\mathbb{C})$.*

Proof. By Lemma 3.3.4.4, the 2-polygraph $\text{ACol}_2(n)$ is finite. Let us show that it is also convergent. First, in order to prove the termination of $\text{ACol}_2(n)$, we show that if $h \implies h'$ then $h' \prec h$. One finds two cases.

First case: let $h = pc_u c_v q$ and $h' = pc_w q$, with p and q are in $\text{ACol}_1(n)^*$ and c_u, c_v , and c_w are in $\text{ACol}_1(n)$. One remarks that the length of h' is smaller than h , then $h' \prec h$.

Second case: let $h = pc_u c_v q$ and $h' = pc_w c_{w'} q$, with p and q are in $\text{ACol}_1(n)^*$ and c_u, c_v, c_w and $c_{w'}$ are in $\text{ACol}_1(n)$, where w and w' are respectively the readings of the right and left columns of $P(uv)$. One remarks that h and h' have the same length. By Lemma 3.3.3.2 the length of u is strictly larger than the length of w , then $c_w \sqsubset c_u$. Consider $i = \ell(p) + 1$, $c_{u_i} = c_w$ and $c_u = c_{v_i}$ then we have $c_{u_i} \sqsubset c_{v_i}$ and for all $j < i$, $c_{u_j} = c_{v_j}$. Hence $h' \prec h$.

Since every application of a 2-cell of $\text{ACol}_2(n)$ yields a \prec -preceding word, it follows that any sequence of rewriting using $\text{ACol}_2(n)$ must terminate. Hence, the 2-polygraph $\text{ACol}_2(n)$ is terminating.

Let us show the confluence of the 2-polygraph $\text{ACol}_2(n)$. Let $h \in \text{ACol}_1(n)^*$ and h', h'' be two normal forms obtained from h . We have to prove that $h' = h''$. Suppose that $h' = c_{u_k} \dots c_{u_1}$. Since h' is a normal form, the words u_1, \dots, u_k are respectively the readings of k admissible columns U_1, \dots, U_k of a symplectic tableau, i.e, $U_i \preceq U_{i+1}, \forall i$. Then $u_k \dots u_1 = w(T')$, where T' is the unique symplectic tableau such that

$$w(T') = u_k \dots u_1 \equiv h'.$$

Similarly, $h'' = c_{v_1} \dots c_{v_l}$ is a normal form, then there exists a unique symplectic tableau T'' such that

$$w(T'') = v_1 \dots v_l \equiv h''.$$

Since $h \equiv h' \equiv h''$, we have by Theorem 3.3.1.10 that $T' = T''$. Then we have $k = l$ and $u_i = v_i, \forall i = 1, \dots, k$. Thus $h' = h''$.

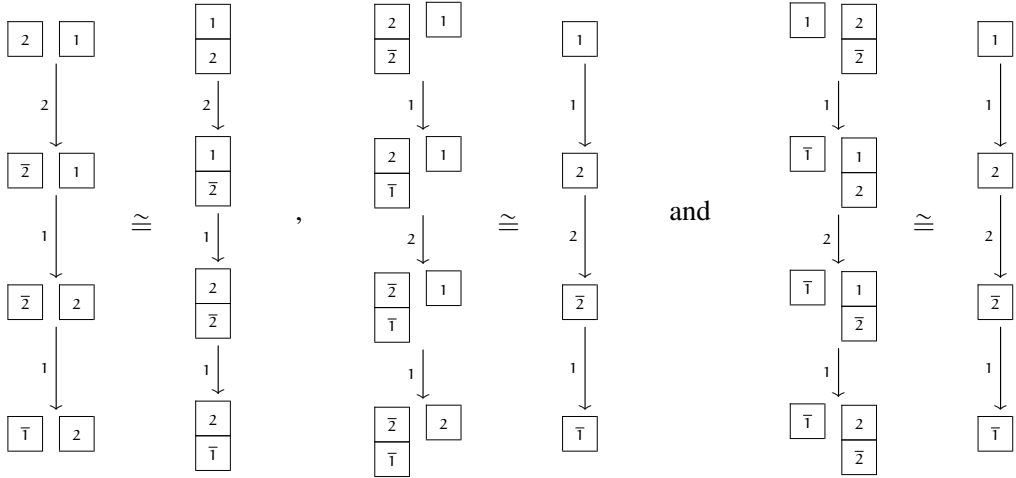
Hence, the 2-polygraph $\text{ACol}_2(n)$ is convergent. \square

Example. Consider the monoid $\mathbf{P}_2(\mathbb{C})$. The admissible columns words over the ordered set $\mathcal{C}_2 = \{1, 2, \bar{2}, \bar{1}\}$ are the readings of the following columns

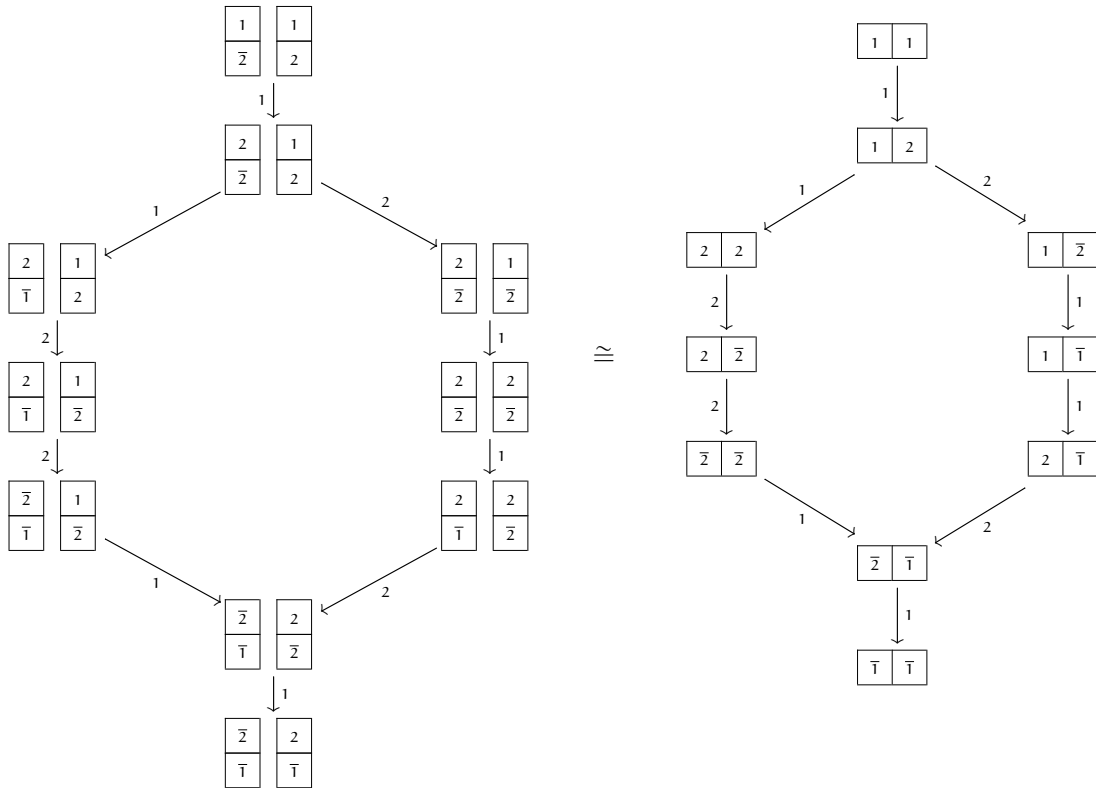
$$\begin{array}{|c|} \hline 1 \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline \end{array}, \begin{array}{|c|} \hline \bar{2} \\ \hline \end{array}, \begin{array}{|c|} \hline \bar{1} \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \bar{2} \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline \bar{2} \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline \bar{1} \\ \hline \end{array}, \begin{array}{|c|} \hline \bar{2} \\ \hline \bar{1} \\ \hline \end{array}.$$

3.3.4. The column presentation for type C

The left and right sides of the 2-cells of the 2-polygraph $\text{ACol}_2(n)$ are the readings of the vertices which correspond under the following crystal isomorphisms

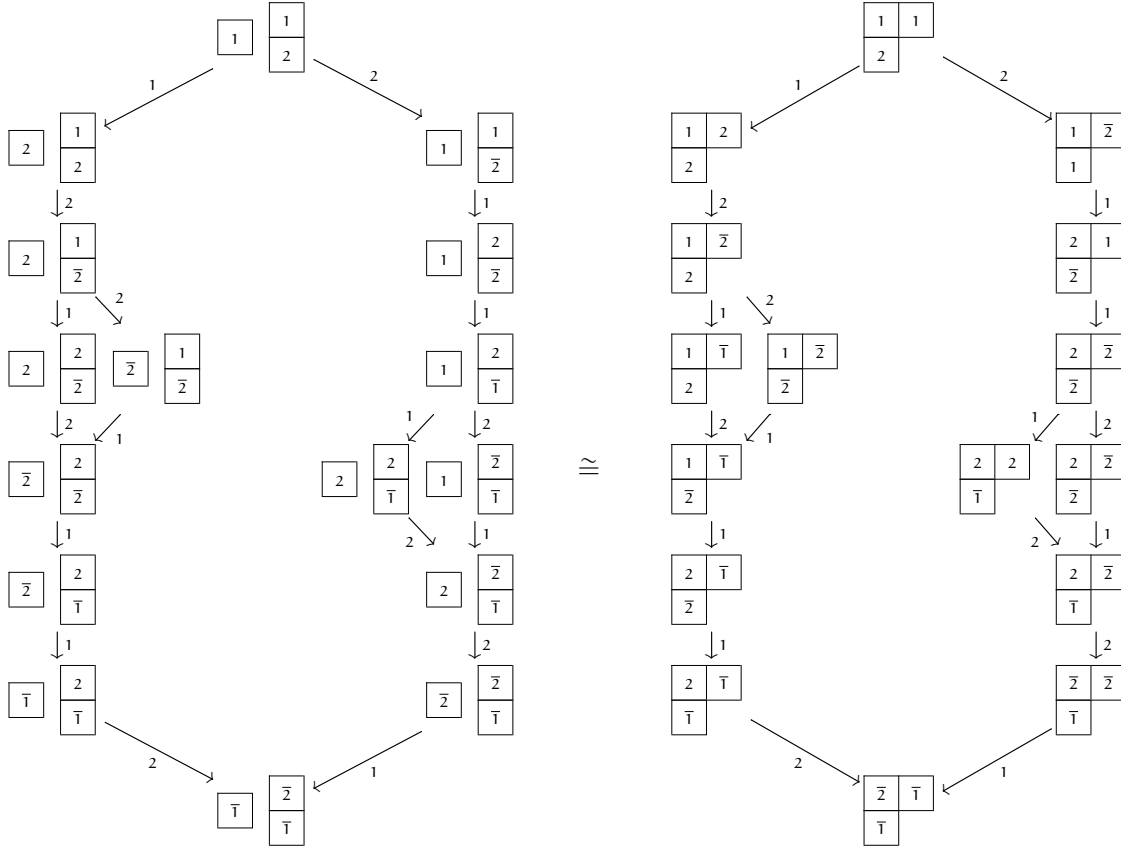


and under the following crystal isomorphisms



3.4. Column presentation for plactic monoids of types B, D and G_2

and under the following crystal isomorphisms



together with the 2-cells $c_1 c_{\bar{1}} \xrightarrow{\alpha_{1, \bar{1}}} c_\varepsilon$ and $c_{12} c_{\bar{2}} \xrightarrow{\alpha_{12, \bar{2}}} c_\varepsilon$.

3.4. COLUMN PRESENTATION FOR PLACTIC MONOIDS OF TYPES B, D AND G_2

For type A, using Schensted's column insertion we can insert a column V into a column U and during this insertion either we add boxes at the bottom of the column U filled by the elements of V or the elements of the column V bump some boxes of U into a new column. Thus we have directly that the result is a tableau consisting of at most two columns where the right one contains fewer elements than U. Note that it is more difficult to prove the later result using the Knuth presentation.

For type C, using Lecouvey's insertion we generalise this construction and we prove the same results in Lemma 3.3.3.1 and Lemma 3.3.3.2 for admissible columns. Using the same strategy as in Subsection 3.3, one can construct finite convergent presentations of plactic monoids for

3.4. Column presentation for plactic monoids of types B, D and G_2

type B and D by introducing admissible column generators. The rewriting system rewrites two admissible columns that do not form an orthogonal tableau to their corresponding orthogonal tableau form. Since Kashiwara's theory of crystal graphs exists for type B and D, one can show that Lemmas 3.3.3.1 and 3.3.3.2 are also true for these types. Hence by this approach, we should obtain the same result as Theorem 3.3.4.6 for plactic monoids of type B and D. Similarly, the notion of Kashiwara's theory exists for type G_2 , then Theorem 3.3.4.6 can be generalised to this type. Hence, using our method and strategy, one can construct the column presentation for plactic monoids of type B, D and G_2 .

The case of type B. For type B, one considers the following totally ordered set

$$\mathcal{B}_n = \{ 1 < 2 < \dots < n < 0 < \bar{n} < \dots < \bar{1} \}.$$

The free monoid \mathcal{B}_n^* over the set \mathcal{B}_n is a crystal and the action of the crystal operators on a word of \mathcal{B}_n^* is similar to the one defined for type C in Subsection 1.3.3. Moreover, Lemma 1.3.3.3 and all combinatorial properties for crystal graphs given in Subsection 1.3.3 for type C can be generalised for type B, see [61, Section 5].

Let us now recall from [79, Subsection 3.1] the notion of admissible columns for type B. A *column for type B* is a Young diagram U consisting of one column filled by letters of \mathcal{B}_n strictly decreasing from bottom to top such that 0 is the unique element that can appear more than once. As for type C, the *reading* of a column U is the word $w(U)$ obtained by reading the letters of U from top to bottom. For instance, the word $1230\bar{2}\bar{1}$ is the reading of a column of type B. The height of a column U is the number of letters in U and denoted by $h(U)$. A word w is a column word if there exists a column U such that $w = w(U)$. For a column U , we denote by $N(z)$ the number of elements x in U such that $x \leq z$ or $x \geq \bar{z}$. An *admissible column* U for type B is a column such that for any $z \leq n$, we have $N(z) \leq z$ and if 0 is in U then $h(U) \leq n$. For instance, for $n = 6$, the word $\bar{5}\bar{6}0654$ is an admissible column word. Note that a column U for type B can be also splitted into a right and a left one in a similar procedure as described in 3.3.1. Splitting columns allows to give a new description of admissible columns for type B, see [79, Subsection 3.1].

Using admissible columns, one can construct a tableau whose columns are admissible with an additional property on them. This tableau is called the orthogonal tableau. The notion of orthogonal tableau is a generalisation for type B of the notions of tableaux for type A and symplectic tableaux for type C, see [79, Subsubsection 3.1.2]. An orthogonal tableau is of highest weight if and only if it has i -th row filled by i , for $i = 1, \dots, n$, [61, Theorem 5.7.1]. Note also that, as for type C, the crystal operators preserve the shapes of orthogonal tableaux. Thus, after applying the crystal operators on an orthogonal tableau of highest weight, one obtains all the orthogonal tableaux of the same shape. For instance, consider the word $0\bar{2}\bar{1}$ obtained by reading the following tableau

$$T = \begin{array}{|c|c|} \hline 2 & 0 \\ \hline \bar{1} & \\ \hline \end{array}$$

3.4. Column presentation for plactic monoids of types B, D and G_2

columnwise from right to left and from top to bottom. We have

$$02\bar{1} = \tilde{f}_2 \circ \tilde{f}_1 \circ \tilde{f}_1 \circ \tilde{f}_1 \circ \tilde{f}_2 \circ \tilde{f}_2(112)$$

where 112 is the reading of the following orthogonal tableau of highest weight

1	1
2	

Hence, we obtain that T is an orthogonal tableau.

By a similar approach as for type C, Lecouvey defined an insertion algorithm that computes an orthogonal tableau $P(w)$ from every word w in the free monoid \mathcal{B}_n^* , see [79, Subsection 3.3]. The *plactic monoid of type B* is isomorphic to the quotient of the free monoid \mathcal{B}_n^* by the equivalence relating words having the same orthogonal tableau after Lecouvey's insertion procedure. Moreover, the latter equivalence coincides with the congruence generated by the five families of relations presented in [79, Definition 3.2.2]. In addition, Lecouvey showed that each plactic congruence contains a unique orthogonal tableau, [79, Theorem 3.2.4].

Lemma 3.3.3.1 can be extended for type B. In other words, let u and v be the readings of two admissible columns U and V respectively. The orthogonal tableau $P(uv)$ consists of at most two columns. Indeed, since the crystal operators preserve the shape of orthogonal tableaux, we suppose that uv is of highest weight. Thus, we obtain by Lemma 1.3.3.3 that the column word u is of highest weight and $\varepsilon_i(v) \leq \varphi_i(u)$, for any $i = 1, \dots, n$. Since u is of highest weight, it can be written on the following form $1 \dots q$, for $q \leq n$. The shape of the tableau $P(uv)$ depends of the first element of the column v . Using Lemma 1.3.3.3, if $u = 1 \dots q$, for $q \leq n - 1$, then the first element of v is $1, q + 1$ or \bar{q} . If $u = 1 \dots n$, then the first element of v is $1, 0$ or \bar{n} . Hence, in all cases, when inserting the first element of v , we obtain an orthogonal tableau that consists of one or two columns. After, one continues inserting the remaining elements of v in the resulted tableau. Since every element of an admissible column is strictly larger than its preceding, one adds boxes in distinct rows in the right or in the left column of the resulted tableau and similarly one removes boxes from distinct rows of the considered tableau. As a consequence, we obtain that the tableau $P(uv)$ consists of zero columns, one column or two columns.

Similarly, one can show that Lemma 3.3.3.2 is also true for type B. Let u and v be the readings of two admissible columns U and V such that the juxtaposition of U and V does not form an orthogonal tableau. Suppose that $P(uv)$ consists of two columns. Then the column U contains more elements than the rightmost column of $P(uv)$. Indeed, since the crystal operators preserve the shape of orthogonal tableaux, the idea is to suppose again that uv is of highest weight. In this case, the tableau $P(uv)$ contains only unbarred letters. By reasoning on which of the five families of relations defined in [79, Definition 3.2.2] are used during the insertion of V into U , and by the same arguments of type C, this lemma is generalised for type B.

Let u and v be respectively the readings of two non-empty admissible columns U and V . Suppose that the juxtaposition of the columns U and V does not form an orthogonal tableau. Define a 2-cell

3.4. Column presentation for plactic monoids of types B, D and G_2

- $c_u c_v \xrightarrow{\alpha_{u,v}} c_w c_{w'}$, where the words w and w' are respectively the readings of the right and left columns W and W' of $P(uv)$ if this orthogonal tableau consists of two columns.
- $c_u c_v \xrightarrow{\alpha_{u,v}} c_w$, where w is the reading of the column W of $P(uv)$ if it consists of one column.
- $c_u c_v \xrightarrow{\alpha_{u,v}} c_\varepsilon$, where ε is the empty word if $P(uv)$ consists of zero columns.

Let $\text{ACol}_2(n)^B$ be the 2-polygraph whose set of 1-cells consists of non-empty admissible columns and the 2-cells are of the form $\alpha_{u,v}$ as defined above. Using the same arguments of Lemma 3.3.4.5 on the five families of relations presented in [79, Definition 3.2.2], one can show that the 2-polygraph $\text{ACol}_2(n)^B$ is a presentation of the plactic monoid of type B. This presentation is called the *column presentation*.

Since the set of admissible columns for type B is finite, the presentation $\text{ACol}_2(n)^B$ is finite. Moreover, the well-founded order defined in 3.3.4 can be also defined on the set of non-empty admissible columns for type B. Using Lemma 3.3.3.2, one can show that the 2-polygraph $\text{ACol}_2(n)^B$ is compatible with this order. In addition, the confluence of $\text{ACol}_2(n)^B$ is shown using the unique normal form property. Indeed, each plactic congruence contains a unique orthogonal tableau. Hence, the 2-polygraph $\text{ACol}_2(n)^B$ is a finite convergent presentation of the plactic monoid of type B. In this way, Theorem 3.1.0.1 is generalised for type B

The construction of Cain, Gray and Malheiro. After our construction of the column presentation of the plactic monoid of type C, similar finite convergent presentations were constructed for plactic monoids of type B, D and G_2 by Cain, Gray and Malheiro, [16]. They used Lecouvey's presentations of these plactic monoids and the notions of admissible columns and tableaux in the Lecouvey sense.

Using Kashiwara's operators for types B, C, D and G_2 and by a case-by-case analysis, they showed that for every two admissible columns such that their juxtaposition does not form a (generalised) tableau, the corresponding (generalised) tableau form consists in at most two columns for type B, C and D, [16, Subsections 5.2, 5.3, 5.4] and in at most three columns for type G_2 , [16, Subsections 5.5]. Moreover, they proved that the first admissible column contains more elements than the right most column of the corresponding tableau, [16, Sections 5]. Using the latter result, they introduced a rewriting system that rewrites two admissible columns to their corresponding (generalised) tableau form. After introducing a well-order on products of columns such that rewriting always decreases with respect to this order, they showed that this rewriting system is terminating, [16, Lemma 6.1]. Using the unique normal form property and the fact that tableaux form a cross section for plactic monoids, they showed that the rewriting system is confluent, [16, Lemma 6.2]. As a consequence, they constructed the column presentation for plactic monoids of type B, D and G_2 , [16, Theorem 6.3].

3.4. Column presentation for plactic monoids of types B, D and G_2

Knuth’s coherent presentations of plactic monoids of type A

Contents

4.1	Coherent column presentations of plactic monoids	148
4.2	Knuth’s coherent presentations of plactic monoids	162

The aim of this chapter is to construct coherent presentations of plactic monoids of type A, [45]. We compute a finite coherent presentation of the plactic monoid from its column presentation that is finite and convergent. Then, in a second time, we show how to reduce this coherent presentation to a Tietze equivalent one having Knuth’s generators.

In 4.1.1, we recall the definition and some properties of coherent presentations of monoids and we present Squier’s completion that extends a convergent presentation of a monoid into a coherent one. We review in 4.1.2 the homotopical completion procedure introduced in [40, 31] that combines the Knuth–Bendix and the Squier’s completions in order to compute a coherent presentation of a monoid starting with a not necessarily convergent presentation. We extend in 4.1.3 the column presentation $\text{Col}_2(n)$ of the plactic monoid \mathbf{P}_n presented in Chapter 3, Section 3.2 into a coherent presentation $\text{Col}_3(n)$.

After recalling in 4.2.1 the homotopical reduction procedure that eliminates superfluous elements from a coherent presentation, we apply three steps of homotopical reduction on the coherent presentation $\text{Col}_3(n)$. In a first step, we apply in 4.2.2 a homotopical reduction on $\text{Col}_3(n)$ with a collapsible part defined by some of the generating triple confluences of the presentation $\text{Col}_2(n)$. Thus, the coherent presentation $\text{Col}_3(n)$ is reduced into a coherent presentation $\overline{\text{Col}}_3(n)$ of the monoid \mathbf{P}_n , whose underlying 2-polygraph is $\text{Col}_2(n)$. In 4.2.3, we reduce the coherent presentation $\overline{\text{Col}}_3(n)$ into a coherent presentation $\text{PreCol}_3(n)$ of the

4.1. Coherent column presentations of plactic monoids

plactic monoid \mathbf{P}_n , whose underlying 2-polygraph is $\text{PreCol}_2(n)$ presented in Chapter 2, Subsection 2.2.2. In a last step, we reduce in 4.2.4 the coherent presentation $\text{PreCol}_3(n)$ into a coherent presentation $\text{Knuth}_3(n)$ of the monoid \mathbf{P}_n whose underlying 2-polygraph is the Knuth presentation $\text{Knuth}_2(n)$. Finally, we give in 4.2.5 the values of number of cells of the coherent presentations for plactic monoids of low-dimensional rank.

4.1. COHERENT COLUMN PRESENTATIONS OF PLACTIC MONOIDS

4.1.1. Coherent presentations of monoids

A coherent presentation of a monoid expresses a system of generators and relations for the monoid extended in a coherent way to present all the relations amongst the relations. In the following, we will recall the notion of coherent presentations and we refer the reader to [40, 39, 31] for a full introduction.

(3, 1)-polygraphs. A *(3, 1)-polygraph* is a pair (Σ_2, Σ_3) made of a 2-polygraph Σ_2 and a set of *globular 3-cells* of the free $(2, 1)$ -category Σ_2^\top equipped with *source* and *target* maps

$$s_2, t_2 : \Sigma_3 \rightarrow \Sigma_2^\top$$

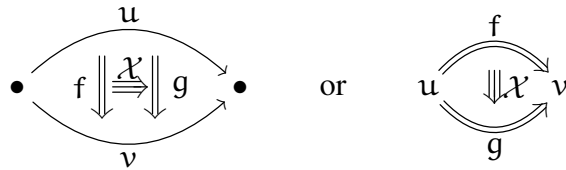
that satisfy the *globular relations*:

$$s_1 s_2 = s_1 t_2 \quad \text{and} \quad t_1 s_2 = t_1 t_2.$$

A $(3, 1)$ -polygraph Σ can be summarised by a diagram representing the cells and the sources of target maps as following

$$\Sigma_0 \begin{array}{c} \xleftarrow{s_0} \\ \xrightarrow{t_0} \end{array} \Sigma_1^* \begin{array}{c} \xleftarrow{s_1} \\ \xrightarrow{t_1} \end{array} \Sigma_2^\top \begin{array}{c} \xleftarrow{s_2} \\ \xrightarrow{t_2} \end{array} \Sigma_3 .$$

An element of the globular extension Σ_3 can be represented by a 3-cell with the following globular shape



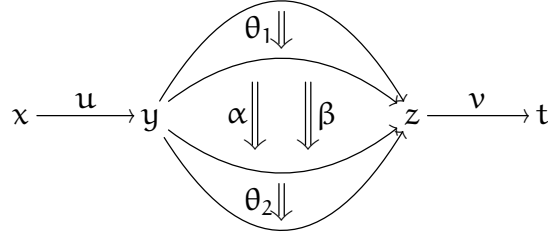
that relates parallel 2-cells f and g in the $(2, 1)$ -category Σ_2^\top .

A pair (f, g) of 2-cells of Σ_2^\top such that $s_1(f) = s_1(g)$ and $t_1(f) = t_1(g)$ is called a *2-sphere* of Σ_2^\top . We will denote by Σ_3^\top the free $(3, 1)$ -category generated by the $(3, 1)$ -polygraph (Σ_2, Σ_3) , see [39, Subsection 4.1.]. A *3-sphere* of the $(3, 1)$ -category Σ_3^\top is a pair (f, g) of 3-cells of Σ_3^\top such that $s_2(f) = s_2(g)$ and $t_2(f) = t_2(g)$.

4.1.1. Coherent presentations of monoids

Coherent presentations of monoids. An *extended presentation* of a monoid \mathbf{M} is a $(3, 1)$ -polygraph Σ whose presented monoid is isomorphic to \mathbf{M} . Explicitly, it is defined by a presentation (Σ_1, Σ_2) of \mathbf{M} , together with a set Σ_3 of globular 3-cells on the free $(2, 1)$ -category Σ_2^\top .

A *homotopy relation* on the $(2, 1)$ -category Σ_2^\top is an equivalence relation \cong on parallel 2-cells of Σ_2^\top , stable under context and composition, that is, for every 1-cells u and v and every 2-cells α, β, θ_1 and θ_2 in Σ_2^\top ,



$\alpha \cong \beta$ implies

$$u \star_0 (\theta_1 \star_1 \alpha \star_1 \theta_2) \star_0 v \cong u \star_0 (\theta_1 \star_1 \beta \star_1 \theta_2) \star_0 v,$$

whenever the composites are possible.

A *coherent presentation* of a monoid \mathbf{M} is an extended presentation Σ of \mathbf{M} such that the smallest homotopy relation containing Σ_3 is the full homotopy relation on Σ_2^\top , that is, for every 2-sphere γ of Σ_2^\top , there exists a 3-cell in Σ_3^\top with boundary γ . In this case, the globular extension Σ_3 is called a *homotopy basis* of the $(2, 1)$ -category Σ_2^\top .

Cofibrant approximations of 2-categories. A 2-category is *cofibrant* if its underlying 1-category is free, [67, 68]. A 2-functor $G : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is called *weak equivalence* if it satisfies the following conditions:

- i) every 0-cell y of \mathcal{C}_2 is equivalent to a 0-cell $G(x)$ for x in \mathcal{C}_1 . That is, there exists 1-cells $u : G(x) \rightarrow y$ and $u' : y \rightarrow G(x)$ and invertible 2-cells $\alpha : u \star_1 u' \implies 1_{G(x)}$ and $\beta : u' \star_1 u \implies 1_y$ in \mathcal{C}_2 ,
- ii) for every 0-cells x and x' in \mathcal{C}_1 , the induced functor $G(x, x') : \mathcal{C}_1(x, x') \rightarrow \mathcal{C}_2(G(x), G(x'))$ is an equivalence of categories.

A *cofibrant approximation* of a 2-category \mathcal{C} is a cofibrant 2-category \mathcal{D} that is weakly equivalent to \mathcal{C} . Let \mathbf{M} be a monoid and let Σ be an extended presentation of \mathbf{M} . The $(3, 1)$ -polygraph Σ is a coherent presentation of \mathbf{M} if, and only if, the $(2, 1)$ -category $\bar{\Sigma}$ presented by Σ is a cofibrant approximation of \mathbf{M} [31, Theorem 1.3.1].

Finiteness properties of monoids. A 2-polygraph Σ is of *finite derivation type* if it is finite and the $(2, 1)$ -category Σ_2^\top has a finite homotopy basis. A monoid \mathbf{M} is of *finite derivation type* (FDT₃) if it admits a finite coherent presentation. The property FDT₃ is a natural extension of the properties of being finitely generated (FDT₁) and finitely presented (FDT₂).

4.1. Coherent column presentations of plactic monoids

Using the notion of polygraphic resolution, one can define the higher-dimensional finite derivation type properties FDT_∞ . Guiraud and Malbos generalised in any dimension the finite derivation type FDT_3 , [37]. A monoid is said to be FDT_∞ if it admits a finite polygraphic resolution. By [37, Corollary 4.5.4], a monoid with a finite convergent presentation is FDT_∞ .

In the homological way, a monoid \mathbf{M} is of *homological type* FP_∞ when there exists a resolution of \mathbf{M} by projective, finitely generated $\mathbb{Z}\mathbf{M}$ -modules. By [37, Corollary 5.4.4], the property FDT_∞ implies the property FP_∞ .

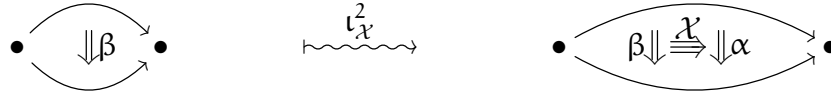
By Chapter 3, Theorem 3.1.0.1, for any semisimple Lie algebra the corresponding plactic monoid admit a finite convergent presentation. Hence, plactic monoids satisfy the homotopical finiteness condition FDT_∞ and the homological finiteness property type FP_∞ .

Tietze transformations of $(3, 1)$ -polygraphs. An equivalence of 2-categories $\mathcal{C} \rightarrow \mathcal{D}$ is a *Tietze equivalence* if the quotient categories $\mathcal{C}_1/\mathcal{C}_2$ and $\mathcal{D}_1/\mathcal{D}_2$ are isomorphic. Two $(3, 1)$ -polygraphs Σ and Υ are *Tietze-equivalent* if the 2-categories Σ_2^\top/Σ_3 and $\Upsilon_2^\top/\Upsilon_3$ they present are Tietze-equivalent. That is, there exists an equivalence of 2-categories $\Sigma_2^\top/\Sigma_3 \rightarrow \Upsilon_2^\top/\Upsilon_3$ and the quotient monoids Σ_1^*/Σ_2 and Υ_1^*/Υ_2 are isomorphic.

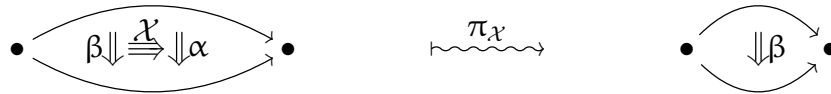
Let Σ be a $(3, 1)$ -polygraph. A 2-cell (rep. 3-cell) γ of Σ is called *collapsible*, [11], if its target is a generating 1-cell (rep. 2-cell) of Σ , and its source is a 1-cell (rep. 2-cell) of the free $(3, 1)$ -category over $\Sigma \setminus \{\mathfrak{t}(\gamma)\}$. If γ is collapsible, then its target is called a *redundant cell*.

The notion of Tietze transformations defined for presentations of groups, [121] and for 2-polygraphs, [31] can be also defined for $(3, 1)$ -polygraphs. Recall from [31, Section 2.1] that an *elementary Tietze transformation* of a $(3, 1)$ -polygraph Σ is a 3-functor with domain Σ_3^\top that belongs to one of the following operations:

- i) adjunction ι_α^1 and elimination π_α of a 2-cell α as described in Chapter 2, Subsection 2.2.1,
- ii) coherent adjunction $\iota_\alpha^2 : \Sigma_3^\top \rightarrow \Sigma_3^\top(\alpha)(\mathcal{X})$ of a redundant 2-cell α with its collapsible 3-cell \mathcal{X} :



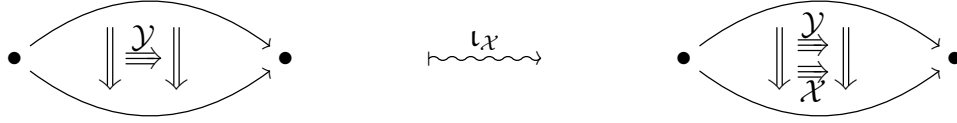
- iii) coherent elimination $\pi_\alpha : \Sigma_3^\top \rightarrow \Sigma_3^\top/\mathcal{X}$ of a redundant 2-cell α with its collapsible 3-cell \mathcal{X} :



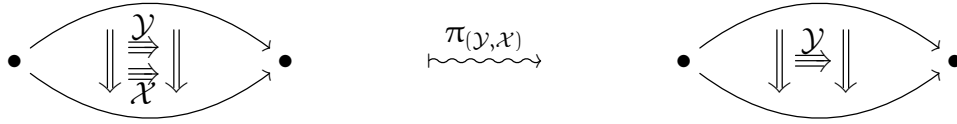
where $\Sigma_3^\top/\mathcal{X}$ is the free $(3, 1)$ -category generated by the $(3, 1)$ -polygraph $(\Sigma_2 \setminus \{\alpha\}, \Sigma_3 \setminus \{\mathcal{X}\})$.

4.1.1. Coherent presentations of monoids

iv) coherent adjunction $\iota_{\mathcal{X}} : \Sigma_3^\top \rightarrow \Sigma_3^\top(\mathcal{X})$ of a redundant 3-cell \mathcal{X} :



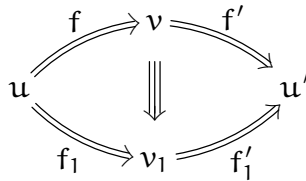
v) coherent elimination $\pi_{(\mathcal{Y}, \mathcal{X})} : \Sigma_3^\top \rightarrow \Sigma_3^\top(\mathcal{Y}, \mathcal{X})$ of a redundant 3-cell \mathcal{X} , that maps \mathcal{X} to \mathcal{Y} :



where $\Sigma_3^\top(\mathcal{Y}, \mathcal{X})$ is the free $(3, 1)$ -category generated by the $(3, 1)$ -polygraph $(\Sigma_2, \Sigma_3 \setminus \{\mathcal{X}\})$.

For $(3, 1)$ -polygraphs Σ and Υ , a *Tietze transformation from Σ to Υ* is a 3-functor from Σ_3^\top to Υ_3^\top that decomposes into a sequence of elementary Tietze transformations. Two $(3, 1)$ -polygraphs are Tietze equivalent if, and only if, there exists a Tietze transformation between them. As a consequence, if Σ is a coherent presentation of a monoid \mathbf{M} and if there exists a Tietze transformations from Σ to Υ , then Υ is a coherent presentation of \mathbf{M} [31, Theorem 2.1.3.].

Squier's completion. For every monoid \mathbf{M} presented by a convergent 2-polygraph Σ , Squier's theorem gives a method to extend Σ into a coherent presentation of \mathbf{M} . Let Σ be a convergent 2-polygraph. A family of *generating confluences of Σ* is a cellular extension of Σ^\top that contains exactly one 3-cell



for every critical branching (f, f_1) of Σ . *Squier's completion of Σ* is the $(3, 1)$ -polygraph obtained from Σ by adjunction of a chosen family of generating confluences of Σ .

Let \mathbf{M} be a monoid presented by a convergent 2-polygraph Σ . Squier's completion of Σ is a coherent presentation of \mathbf{M} [113, Theorem 5.2]. Note that a new proof of this result is given in [39, Theorem 4.3.2] using the polygraphical approach. Furthermore, if a monoid \mathbf{M} admits a finite convergent presentation, then it is of finite derivation type [113, Theorem 5.3].

Example: the plactic monoid \mathbf{P}_2 . The Knuth presentation $\text{Knuth}_2(2)$ of the plactic monoid \mathbf{P}_2 has generators 1 and 2 subject to the Knuth relations $\eta_{1,1,2} : 211 \Rightarrow 121$ and $\varepsilon_{1,2,2} : 221 \Rightarrow 212$.

4.1. Coherent column presentations of plactic monoids

The presentation $\text{Knuth}_2(2)$ of the monoid \mathbf{P}_2 admits a unique confluent critical branching:

$$\begin{array}{ccc}
 & \xrightarrow{2\eta_{1,1,2}} & \\
 2211 & \begin{array}{c} \curvearrowright \\ \Downarrow \\ \curvearrowleft \end{array} & 2121 \\
 & \xleftarrow{\varepsilon_{1,2,2}1} &
 \end{array} \tag{4.1}$$

Thus the presentation $\text{Knuth}_2(2)$ can be extended into a coherent presentation with the 3-cell (4.1).

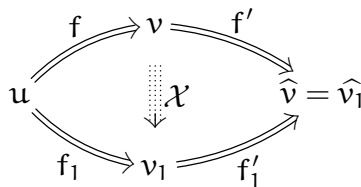
Example: the plactic monoid \mathbf{P}_3 . The plactic monoid \mathbf{P}_3 admits a coherent presentation with 3 generators, 11 relations and 27 3-cells corresponding to the 27 critical branchings explained in Chapter 2, Subsection 2.3.2.

4.1.2. Homotopical completion procedure

Using Squier’s construction, the *homotopical completion procedure* introduced in [31, 40] extends the Knuth–Bendix completion procedure to the case of $(3, 1)$ -polygraphs in order to compute a coherent and convergent presentation from a terminating presentation. The procedure examines the critical branchings one by one, add 2-cells to reach the confluence and add also 3-cells in order to obtain a coherent presentation. If the starting presentation is convergent, the 3-cells induced by confluence diagrams of the critical branchings form a homotopy basis and thus they suffice to obtain a coherent presentation.

Given a terminating 2-polygraph Σ , equipped with a total termination order \leq . For any critical branching (f, f_1) of Σ , there are two possible situations:

- i) If (f, f_1) is confluent, then one adds the dotted 3-cell \mathcal{X} :

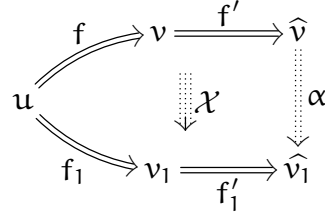


- ii) If (f, f_1) is not confluent, then

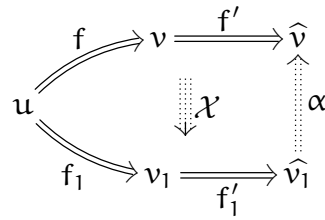
- (a) if $\widehat{v} > \widehat{v}_1$, one performs a Tietze transformation on Σ to coherently add the 2-cell $\alpha : \widehat{v} \Rightarrow \widehat{v}_1$

4.1.3. Coherent column presentation

and the 3-cell \mathcal{X} :



- (b) if $\widehat{v} < \widehat{v}_1$, one performs a Tietze transformation on Σ to coherently add the 2-cell $\alpha : \widehat{v}_1 \Rightarrow \widehat{v}$ and the 3-cell \mathcal{X} :



When adjunction of 2-cells α , new critical branchings can be created. The examination of the confluence of these new critical branchings can give rise to the adjunction of additional 2-cells and 3-cells. This defines an increasing sequence of $(3, 1)$ -polygraphs, where Σ^{n+1} is obtained by completion of the critical branchings of Σ^n :

$$(\Sigma, \emptyset) = \Sigma^0 \subseteq \Sigma^1 \subseteq \dots \subseteq \Sigma^n \subseteq \Sigma^{n+1} \subseteq \dots$$

Hence, the *homotopical completion* of Σ is the $(3, 1)$ -polygraph

$$\mathcal{S}(\Sigma) = \bigcup_{k=0}^{\infty} \Sigma^k.$$

For every terminating presentation Σ of a monoid \mathbf{M} , the homotopical completion $\mathcal{S}(\Sigma)$ of Σ is a coherent convergent presentation of \mathbf{M} [31, Theorem 2.2.5].

4.1.3. Coherent column presentation

Using the homotopical completion procedure described above, we extend the column presentation $\text{Col}_2(n)$ defined in Chapter 3, Section 3.2 into a coherent presentation of the plactic monoid \mathbf{P}_n .

The column presentation. Let us recall from Chapter 3, Section 3.2 the definition of the column presentation. Given two columns u and v in the set of non-empty columns $\text{col}(n)$, we consider the tableau $P(uv)$ computed by the row-insertion algorithm. We denote $u \widehat{v}$ if the planar representation of $P(uv)$ is a tableau and we will denote $u \times v$ in the other cases. In this

4.1. Coherent column presentations of plactic monoids

case, the tableau $P(uv)$ contains at most two columns ; we will denote $u^{\times 1}v$ if the tableau $P(uv)$ has one column and we will denote $u^{\times 2}v$ if the tableau $P(uv)$ has two columns.

For every columns u and v in $\text{col}(n)$ such that $u^{\times}v$, we define a 2-cell

$$\alpha_{u,v} : c_u c_v \Rightarrow c_w c_{w'}$$

where $w = uv$ and $c_{w'} = 1$, if $u^{\times 1}v$, and w and w' are respectively the left and right columns of the tableau $P(uv)$, if $u^{\times 2}v$.

The column presentation of the plactic monoid \mathbf{P}_n is the 2-polygraph $\text{Col}_2(n)$ whose set of 1-cells is

$$\text{Col}_1(n) = \{ c_u \mid u \in \text{col}(n) \}$$

and the 2-cells are the $\alpha_{u,v}$ such that u and v are in $\text{col}(n)$ and we have $u^{\times}v$.

Coherent column presentation. By definition of the rules $\alpha_{u,v}$, the presentation $\text{Col}_2(n)$ has exactly one critical branching of the form

$$\begin{array}{ccc} & \alpha_{u,v} c_t \rightarrow & c_e c_{e'} c_t \\ c_u c_v c_t & \searrow & \\ & c_u \alpha_{v,t} \rightarrow & c_u c_w c_{w'} \end{array} \quad (4.2)$$

for any u, v, t in $\text{col}(n)$ such that $u^{\times}v^{\times}t$, where e and e' (resp. w and w') denote the two columns of the tableau $P(uv)$ (resp. $P(vt)$). We prove in this section that all of these critical branchings are confluent. This gives an alternative proof of the confluence of 2-polygraph $\text{Col}_2(n)$ given in Chapter 3, Subsection 3.2.2. Moreover, we prove that all the confluence diagrams of these branchings are of the following form:

$$\begin{array}{ccccc} & \alpha_{u,v} c_t \rightarrow & c_e c_{e'} c_t & \xrightarrow{c_e \alpha_{e',t}} & c_e c_b c_{b'} & \xrightarrow{\alpha_{e,b} c_{b'}} & c_a c_d c_{b'} \\ c_u c_v c_t & \searrow & & \Downarrow \mathcal{X}_{u,v,t} & & \nearrow & \\ & c_u \alpha_{v,t} \rightarrow & c_u c_w c_{w'} & \xrightarrow{\alpha_{u,w} c_{w'}} & c_a c_{a'} c_{w'} & \xrightarrow{c_a \alpha_{a',w'}} & c_a c_d c_{b'} \end{array} \quad (4.3)$$

where a and a' (resp. b and b') denote the two columns of the tableau $P(uw)$ (resp. $P(e't)$) and a, d, b' are the three columns of the tableau $P(uvt)$, which is a normal form for the 2-polygraph $\text{Col}_2(n)$. Note that in some cases described below, one or further columns e', w', a' and b' can be empty. In those cases some indicated 2-cells α in the confluence diagram correspond to identities.

Let us denote by $\text{Col}_3(n)$ the extended presentation of the monoid \mathbf{P}_n obtained from $\text{Col}_2(n)$ by adjunction of one 3-cell $\mathcal{X}_{u,v,t}$ of the form (4.3), for every columns u, v and t such that $u^{\times}v^{\times}t$.

4.1.3.1. Theorem. *For $n > 0$, the $(3, 1)$ -polygraph $\text{Col}_3(n)$ is a coherent presentation of the monoid \mathbf{P}_n .*

4.1. Coherent column presentations of plactic monoids

vt is a column, in particular $y_{j+1} > z_l$. It follows that $e't$ is a column. Suppose that $i_0 > 1$, then $x_1 \leq y_1$ and the smallest element of e' is y_1 . Since $y_1 > z_l$ by hypothesis, the word $e't$ is a column. Hence, in all cases, $e't$ is a column and there is a 2-cell $\alpha_{e',t} : c_{e't} \Rightarrow c_{e't}$.

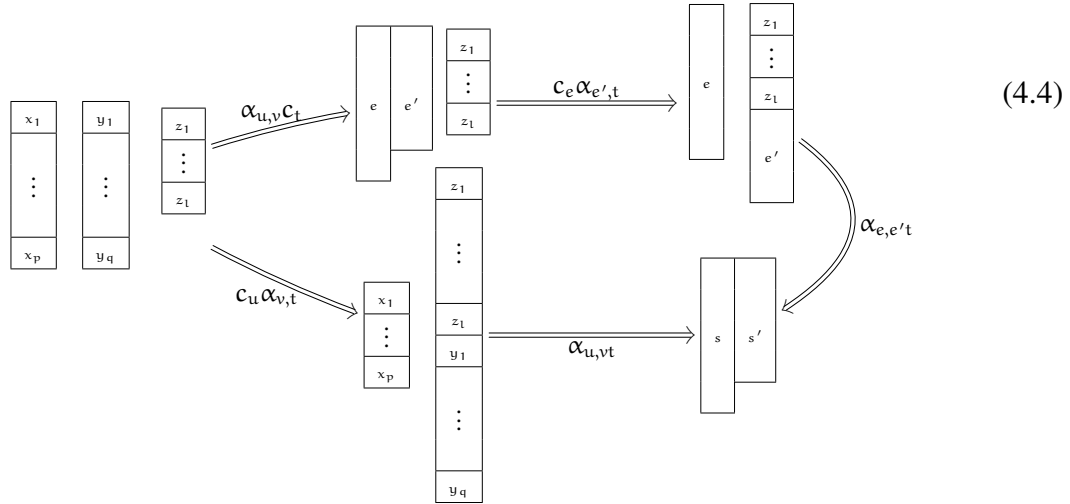
Case 2: $p < q$ and $x_i \leq y_i$ for any $1 \leq i \leq p$.

We have $e = y_q \dots y_{p+1} x_p \dots x_1$ and $e' = y_p \dots y_1$. By hypothesis, $y_1 > z_l$, hence $e't$ is a column and there is a 2-cell $\alpha_{e',t} : c_{e't} \Rightarrow c_{e't}$.

Case 3: $p < q$ and $x_{i_0} > y_{i_0}$ for some $1 \leq i_0 \leq p$.

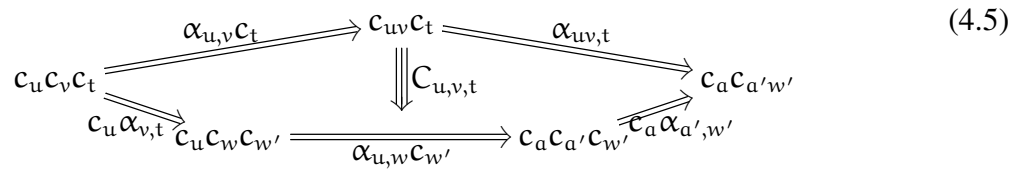
With the same arguments of Case 1, the smallest element of e' is y_1 or y_{j+1} , where y_j is the biggest element of the column v such that $y_j < x_1$. Hence, $e't$ is a column and there is a 2-cell $\alpha_{e',t} : c_{e't} \Rightarrow c_{e't}$.

In case 1, 2 and 3, we have $l^{\text{nds}}(uv) = 2$, hence $l^{\text{nds}}(uvt) = 2$. Thus the tableau $P(uvt)$ consists of two columns, that we denote s and s' and there is a 2-cell $\alpha_{u,vt} : c_u c_{vt} \Rightarrow c_s c_{s'}$. Moreover, to compute the tableau $P(uvt)$, one begins by computing $P(uv)$ and after by introducing the elements of the column t on the tableau $P(uv)$. As $C(P(uv)) = ee'$, we have $P(uvt) = P(P(uv)t) = P(ee't)$. Hence $C(P(ee't)) = ss'$ and there is a 2-cell $\alpha_{e,e't}$ which yields the following confluence diagram



□

4.1.3.4. Lemma. *If $u^{\times 1} v^{\times 2} t$, we have the following confluent critical branching:*



where w and w' (resp. a and a') denote the two columns of the tableau $P(vt)$ (resp. $P(uw)$).

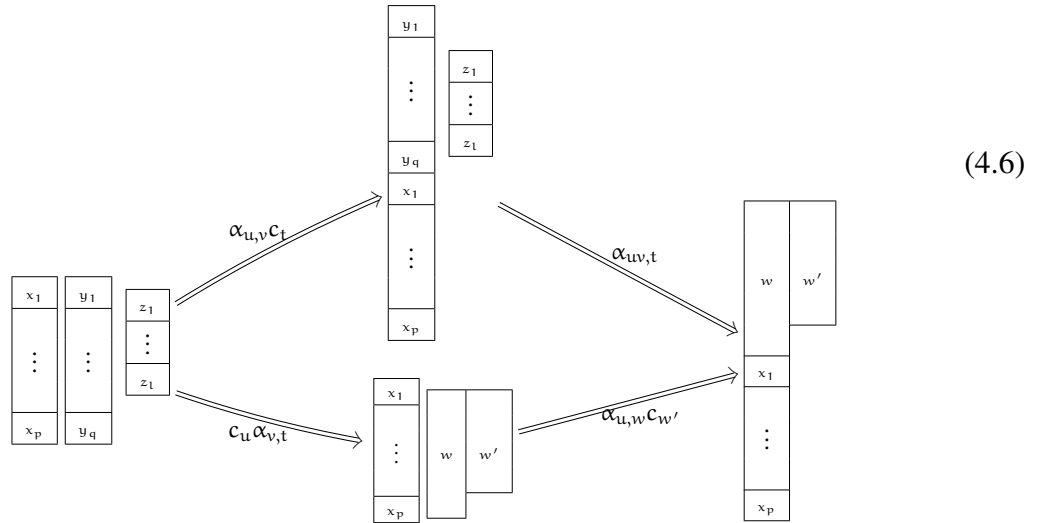
4.1.3. Coherent column presentation

Proof. By hypothesis, uv is a column hence $x_1 > y_q$. Moreover, the tableau $P(vt)$ consists of two columns w and w' , then $l^{nds}(vt) = 2$, hence $y_1 \leq z_l$. We have $v^{\times 2}t$, so that we distinguish the three possible following cases.

Case 1: $q \geq l$ and $y_{i_0} > z_{i_0}$ for some $1 \leq i_0 \leq l$.

Let us denote $w = w_r \dots w_1$ and $w' = w'_{r'} \dots w'_1$. Since $q \geq l$, we have $w_r = y_q$. By hypothesis, $x_1 > y_q$. Then the word uw is a column. As a consequence, there is a 2-cell $\alpha_{u,w} : c_u c_w \Rightarrow c_{uw}$. In addition, the column w appears to the left of w' in the planar representation of the tableau $P(vt)$, that is, $l(w) \geq l(w')$ and $w_i \leq w'_i$ for any $i \leq l(w')$. Then $l(uw) \geq l(w')$. We set $uw = \xi_{l(uw)} \dots \xi_1$ and we have $\xi_i \leq w'_i$ for any $i \leq l(w')$. Then $u\widehat{w}w'$ and $c_{uw}c_{w'}$ is a normal form.

On the other hand, the tableau $P(vt)$ consists of two columns, hence $l^{nds}(vt) = 2$. As a consequence, $l^{nds}(uvt) = 2$ and the tableau $P(uvt)$ consists of two columns. Since $q \geq l$, we have $C(P(uvt)) = uww'$, hence the two columns of $P(uvt)$ are uw and w' . Then there is a 2-cell $\alpha_{uv,t} : c_{uv}c_t \Rightarrow c_{uw}c_{w'}$ which yields the confluence of the critical branching on $c_u c_v c_t$, as follows



Case 2: $q < l$ and $y_i \leq z_i$ for any $i \leq q$.

We have $w = z_l \dots z_{q+1} y_q \dots y_1$ and $w' = z_q \dots z_1$. There are two cases along

$$uw = x_p \dots x_1 z_l \dots z_{q+1} y_q \dots y_1$$

is a column or not.

Case 2. A. If $x_1 > z_l$, then uw is a column. Hence, there is a 2-cell $\alpha_{u,w} : c_u c_w \Rightarrow c_{uw}$. Moreover, using Schensted's algorithm we prove that $C(P(uvt)) = uww'$, it follows that the columns of $P(uvt)$ are uw and w' . Thus there is a 2-cell $\alpha_{uv,t} : c_{uv}c_t \Rightarrow c_{uw}c_{w'}$ which yields the confluence diagram (4.6).

4.1. Coherent column presentations of plactic monoids

Case 2. B. If $x_1 \leq z_1$, then $l^{nds}(uw) = 2$ and $P(uw)$ consists of two columns, that we denote by a and a' . Then there is a 2-cell $\alpha_{u,w} : c_u c_w \Rightarrow c_a c_{a'}$. In addition, by Schensted's algorithm, we deduce that $a' = z_{i_k} \dots z_{i_1}$, with $q+1 \leq i_1 < \dots < i_k \leq l$. We have $a'w' = z_{i_k} \dots z_{i_1} z_q \dots z_1$. Since all the elements of a' are elements of t and bigger than z_q , we have $z_{i_1} > z_q$. It follows that $a'w'$ is a column and there is a 2-cell $\alpha_{a',w'} : c_{a'} c_{w'} \Rightarrow c_{a'w'}$.

In the other hand, we have two cases whether $uv \hat{x} t$ or $uv \hat{t}$. Suppose $uv \hat{x} t$. By Schensted's algorithm, we have $C(P(uvt)) = \alpha a'w'$, showing that the two columns of $P(uvt)$ are a and $a'w'$. Hence there is a 2-cell $\alpha_{uv,t} : c_{uv} c_t \Rightarrow c_a c_{a'w'}$, which yields the confluence of Diagram (4.5). Suppose $uv \hat{t}$. Then we obtain $C(P(uw)) = uvz_1 \dots z_{q+1}$, and $C(P(z_1 \dots z_{q+1}w')) = t$. Hence there is a 2-cell $\alpha_{z_1 \dots z_{q+1},w'}$ yielding the confluence diagram

$$\begin{array}{ccccc}
 & \alpha_{u,v} c_t & \xrightarrow{\quad} & c_{uv} c_t & \xleftarrow{\quad} c_{uv} \alpha_{z_1 \dots z_{q+1}, w'} \\
 c_u c_v c_t & \searrow & & \Downarrow C'_{u,v,t} & \\
 & c_u \alpha_{v,t} & \xrightarrow{\quad} & c_u c_w c_{w'} & \xrightarrow{\quad} c_{uv} c_{z_1 \dots z_{q+1}} c_{w'} \\
 & & & \alpha_{u,w} c_{w'} &
 \end{array}$$

Case 3: $q < l$ and $y_{i_0} > z_{i_0}$ for some $1 \leq i_0 \leq q$.

We compute the columns w and w' of the tableau $P(vt)$. If the biggest element of the column w is y_q , then we obtain the same confluent branching as in Case 1. If the first element of w is z_1 , then one obtains the same confluent critical branchings as in Case 2. \square

4.1.3.5. Lemma. *If $u \hat{x}^2 v \hat{x}^2 t$, we have the following confluent critical branching:*

$$\begin{array}{ccccc}
 & \alpha_{u,v} c_t & \xrightarrow{\quad} & c_e c_{e'} c_t & \xrightarrow{\quad} c_e \alpha_{e',t} & \xrightarrow{\quad} c_e c_b c_{b'} & \xrightarrow{\quad} \alpha_{e,b} c_{b'} & (4.7) \\
 c_u c_v c_t & \searrow & & \Downarrow D_{u,v,t} & & & & \\
 & c_u \alpha_{v,t} & \xrightarrow{\quad} & c_u c_w c_{w'} & \xrightarrow{\quad} c_a c_{a'} c_{w'} & \xrightarrow{\quad} c_a \alpha_{a',w'} & & \\
 & & & \alpha_{u,w} c_{w'} & & & &
 \end{array}$$

where e, e' (resp. w, w') denote the two columns of the tableau $P(uv)$ (resp. $P(vt)$) and a, a' (resp. b, b') denote the two columns of the tableau $P(uw)$ (resp. $P(e't)$).

Proof. By hypothesis, $l^{nds}(uv) = 2$ and $l^{nds}(vt) = 2$, hence $x_1 \leq y_q$ and $y_1 \leq z_1$. In addition, since $u \hat{x}^2 v$, the tableau $P(uw)$ consists of two columns, that we denote by a and a' . Thus there is a 2-cell $\alpha_{u,w} : c_u c_w \Rightarrow c_a c_{a'}$. Moreover, as $u \hat{x}^2 v$ and $v \hat{x}^2 t$, we have

$$((p < q) \text{ or } (x_{i_0} > y_{i_0} \text{ for some } i_0 \leq q)) \quad \text{and} \quad ((q < l) \text{ or } (y_{j_0} > z_{j_0} \text{ for some } j_0 \leq l)),$$

thus we consider the following cases.

Case 1: $p < q < l$ and $y_i \leq z_i$, for all $i \leq q$, and $x_i \leq y_i$, for all $i \leq p$.

4.1.3. Coherent column presentation

We have

$$w = z_1 \dots z_{q+1} y_q \dots y_1, \quad w' = z_q \dots z_1, \quad e = y_q \dots y_{p+1} x_p \dots x_1 \quad \text{and} \quad e' = y_p \dots y_1.$$

Since $z_l \geq y_1$, the tableau $P(e't)$ consists of two columns, that we denote by b and b' . Thus there is a 2-cell $\alpha_{e',t} : c_{e'}c_t \Rightarrow c_b c_{b'}$. In addition, we have

$$b = z_1 \dots z_{p+1} y_p \dots y_1, \quad b' = z_p \dots z_1, \quad a = z_1 \dots z_{q+1} y_q \dots y_{p+1} x_p \dots x_1 \quad \text{and} \quad a' = y_p \dots y_1.$$

Since $z_q \geq y_1$, the tableau $P(a'w')$ consists of two columns, that we denote by d and d' . Thus there is a 2-cell $\alpha_{a',w'} : c_{a'}c_{w'} \Rightarrow c_d c_{d'}$. Since $z_l \geq x_1$, the tableau $P(eb)$ consists of two columns, that we denote by s and s' . Then there is a 2-cell $\alpha_{e,b} : c_e c_b \Rightarrow c_s c_{s'}$. In the other hand, we have $d = z_q \dots z_{p+1} y_p \dots y_1$, $d' = z_p \dots z_1$, $s = z_1 \dots z_{q+1} y_q \dots y_{p+1} x_p \dots x_1$ and $s' = z_q \dots z_{p+1} y_p \dots y_1$. Hence $a = s$, $d = s'$ and $d' = b'$ which yields the confluence diagram (4.7).

Case 2: $\left\{ \begin{array}{l} q < l \text{ and } y_i \leq z_i \text{ for all } i \leq q \\ p \geq q \text{ and } x_{i_0} > y_{i_0} \text{ for some } i_0 \leq q \end{array} \right.$ or $\left\{ \begin{array}{l} q < l \text{ and } y_i \leq z_i \text{ for all } i \leq q \\ p < q \text{ and } x_{i_0} > y_{i_0} \text{ for some } i_0 \leq p \end{array} \right.$

We have $w = z_1 \dots z_{q+1} y_q \dots y_1$ and $w' = z_q \dots z_1$. Using Schensted's algorithm the smallest element of the column a' is an element of v . Since z_q is greater or equal than each element of v , the tableau $P(a'w')$ consists of two columns, that we denote by d and d' .

On the other hand, all the elements of e' are elements of v . Since z_l is bigger than each element of v , the tableau $P(e't)$ consists of two columns, that we denote by b and b' . Thus there is a 2-cell $\alpha_{e',t} : c_{e'}c_t \Rightarrow c_b c_{b'}$. Hence, we consider two cases depending on whether or not $c_e c_b c_{b'}$ is a tableau. Suppose $c_e c_b c_{b'}$ is a tableau. The column e does not contain elements from the column t , then during inserting the column w into the column u , we can only insert some elements of $y_q \dots y_1$ into u and we obtain $a = e$. Since $c_e c_b c_{b'}$ is the unique tableau obtained from $c_u c_v c_t$ and $a = e$, we obtain $C(P(a'w')) = bb'$. As a consequence, there is a 2-cell $\alpha_{a',w'} : c_{a'}c_{w'} \Rightarrow c_b c_{b'}$ yielding the following confluence diagram:

$$\begin{array}{ccc} & & c_e c_{e'} c_t \xrightarrow{c_e \alpha_{e',t}} c_e c_b c_{b'} \\ & \nearrow \alpha_{u,v} c_t & \\ c_u c_v c_t & & \\ & \searrow c_u \alpha_{v,t} & \\ & & c_u c_w c_{w'} \xrightarrow{\alpha_{u,w} c_{w'}} c_a c_{a'} c_{w'} \end{array} \quad \begin{array}{c} \Downarrow D_{u,v,t}^{(1)} \\ \Uparrow c_a \alpha_{a',w'} \end{array} \quad (4.8)$$

Suppose $c_e c_b c_{b'}$ is not a tableau. The first element of the column b is z_l . The smallest element of the column e is either x_1 or y_j , where y_j is the biggest element of the column v such that $y_j < x_1$. By hypothesis the tableau $P(uw)$ consists of two columns, then $x_1 \leq z_l$. In addition, z_l is greater than each element of v then $y_j \leq z_l$. Hence, in all cases, the tableau $P(eb)$ consists of two columns. On the other hand, using Schensted's algorithm, we have $a' = z_{i_k} \dots z_{i_1} y_{j_{k'}} \dots y_{j_1}$ with $q+1 \leq i_1 < \dots < i_k \leq l$, $1 \leq j_1 < \dots < j_{k'} \leq q$ and we have $e' = y_{j_{k'}} \dots y_{j_1}$. In

4.1. Coherent column presentations of plactic monoids

addition, we have $b' = d' = z_{i_{k''}} \dots z_{i_1}$ with $1 \leq i_1 < \dots < i_{k''} \leq q$ and $C(P(eb)) = ad$. Hence there is a 2-cell $\alpha_{e,b} : c_e c_b \Rightarrow c_a c_d$ which yields the confluence diagram (4.7).

Case 3: $\begin{cases} q \geq l \text{ and } y_{i_0} > z_{i_0} \text{ for some } i_0 \leq l \\ p < q \text{ and } x_i \leq y_i \text{ for all } i \leq p \end{cases}$ or $\begin{cases} q < l \text{ and } y_{i_0} > z_{i_0} \text{ for some } i_0 \leq q \\ p < q \text{ and } x_i \leq y_i \text{ for all } i \leq p \end{cases}$

We have $e = y_q \dots y_{p+1} x_p \dots x_1$ and $e' = y_p \dots y_1$. Since $y_1 \leq z_1$, the tableau $P(e't)$ consists of two columns, that we denote by b and b' . The first element of the column b is either z_1 or y_p which are bigger or equal to x_1 , then the tableau $P(eb)$ consists of two columns, that we denote by s and s' . Suppose $l \leq p$. Then by Schensted's insertion algorithm, we obtain $C(P(e't)) = bw'$ and $w = y_q \dots y_{p+1} b$. On the other hand, since $x_p < y_{p+1}$, we have $P(uw) = P(u(y_q \dots y_{p+1} b)) = P(eb)$. Hence there is a 2-cell $\alpha_{e,b} : c_e c_b \Rightarrow c_a c_{a'}$ which yields the confluence diagram:

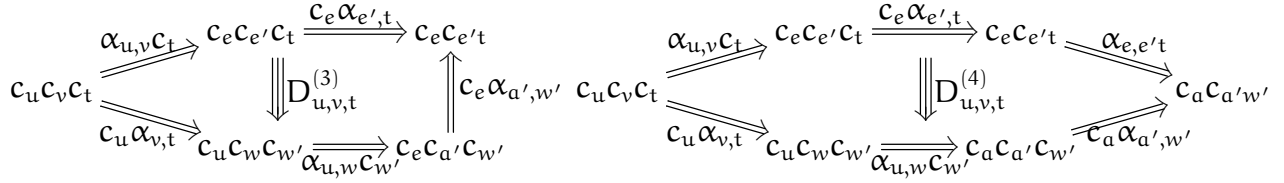
$$\begin{array}{ccccc}
 & & & c_e c_{e'} c_t & \xrightarrow{c_e \alpha_{e',t}} & c_e c_b c_{w'} & \\
 & \nearrow \alpha_{u,v} c_t & & \Downarrow D_{u,v,t}^{(2)} & & \Downarrow \alpha_{e,b} c_{w'} & \\
 c_u c_v c_t & & & & & & \\
 & \searrow c_u \alpha_{v,t} & & c_u c_w c_{w'} & \xrightarrow{\alpha_{u,w} c_{w'}} & c_a c_{a'} c_{w'} & \\
 & & & & & &
 \end{array} \tag{4.9}$$

Suppose $l > p$, then we consider two cases depending on whether or not the first element of the column b is y_p . If this element is y_p , then when computing the tableau $P(vt)$ no element of the column t is inserted in $y_q \dots y_{p+1}$. Hence we have $w = y_q \dots y_{p+1} b$ and $b' = w'$. On the other hand, by Schensted's insertion procedure we have $P(uw) = P(eb)$. Hence, there is a 2-cell $\alpha_{e,b} : c_e c_b \Rightarrow c_a c_{a'}$ which yields the confluence diagram (4.9). Suppose that the first element of the column b is z_1 . Then when computing the tableau $P(vt)$ some elements of the column t are inserted in $y_q \dots y_{p+1}$. In this case, we have that the column w' contains more elements than b' and that $c_s c_{s'} c_{b'}$ is a tableau. Moreover, by Schensted's insertion procedure, we have $a = s$. Since $c_s c_{s'} c_{b'}$ is the unique tableau obtained from $c_u c_v c_t$ and $a = s$, we obtain $C(P(a'w')) = s'b'$. As a consequence, there is a 2-cell $\alpha_{a',w'} : c_{a'} c_{w'} \Rightarrow c_{s'} c_{b'}$ which yields the confluence diagram (4.7).

Case 4: $\begin{cases} q \geq l \text{ and } y_{i_0} > z_{i_0} \text{ for some } i_0 \leq l \\ p \geq q \text{ and } x_{j_0} > y_{j_0} \text{ for some } j_0 \leq q \end{cases}$ or $\begin{cases} q \geq l \text{ and } y_{i_0} > z_{i_0} \text{ for some } i_0 \leq q \\ p < q \text{ and } x_{j_0} > y_{j_0} \text{ for some } j_0 \leq p \end{cases}$
or $\begin{cases} q < l \text{ and } y_{i_0} > z_{i_0} \text{ for some } i_0 \leq q, \\ p \geq q \text{ and } x_{j_0} > y_{j_0} \text{ for some } j_0 \leq q \end{cases}$ or $\begin{cases} q < l \text{ and } y_{i_0} > z_{i_0} \text{ for some } i_0 \leq q \\ p < q \text{ and } x_{j_0} > y_{j_0} \text{ for some } j_0 \leq p \end{cases}$

By Lemma 4.1.3.3, the last term of e' is y_1 or y_{j+1} , where y_j is the biggest element of v such that $y_j < x_1$. Suppose that the last term of e' is y_1 . Since $z_1 \geq y_1$, the tableau $P(e't)$ consists of two columns. Furthermore, if the last term of e' is y_{j+1} , then we consider two cases: $z_1 \geq y_{j+1}$ or $z_1 < y_{j+1}$. Suppose $z_1 < y_{j+1}$, then the tableau $P(e't)$ consists of one column $e't$. We consider two cases depending on whether or not $c_e c_{e't}$ is a tableau. With the same arguments of Case 2, we obtain a confluence diagram of the following forms:

4.1.3. Coherent column presentation



Suppose the tableau $P(e't)$ consists of two columns. Using the same arguments of Case 2 and Case 3, we obtain a confluence diagram of the form $D_{u,v,t}^{(1)}$, $D_{u,v,t}^{(2)}$ or $D_{u,v,t}^{(3)}$. \square

4.1.3.6. Remark. We recall in the following the notion of quadratic normalisations of monoids introduced in [34]. A *normalisation* is a pair (Σ_1, Φ) , where Σ_1 is a set and Φ is a map from the free monoid Σ_1^* to itself, satisfying:

- i) $\ell(\Phi(w)) = \ell(w)$,
- ii) $\ell(w) = 1$ implies $\Phi(w) = w$,
- iii) $\Phi(u\Phi(w)v) = \Phi(uwv)$, for all 1-cells u, v and w in Σ_1^* .

A 1-cell w in Σ_1^* is called *Φ -normal* if it satisfies $\Phi(w) = w$. The normalisation determines a monoid via the defining relation $w = \Phi(w)$. A normalisation (Σ_1, Φ) is *quadratic* if the Φ -normality of a 1-cell in Σ_1^* only depends on its factors of length two and if we can go from a 1-cell w to the 1-cell $\Phi(w)$ in finitely many steps, each of which consists in applying Φ to some factors of length two. The *class* of a quadratic normalisation is a pair (x, y) of positive integers which means that one obtains the normal form after at most x steps when starting from the left and y steps from the right. Let $\bar{\Phi}$ be the restriction of Φ to the set of 1-cells of length two. Every quadratic normalisation (Σ, Φ) gives rise to a quadratic 2-polygraph (Σ_1, Σ_2) , where Σ_2 consists of 2-cells of the form $w \Rightarrow \bar{\Phi}(w)$, with the 1-cell w is of length two. If (Σ_1, Φ) is a quadratic normalisation of class $(3, 3)$, then the associated 2-polygraph (Σ_1, Σ_2) is convergent [34, Proposition 5.1.1].

In the proof of Theorem 4.1.3.1, we don't use the fact that the 2-polygraph $\text{Col}_2(n)$ is convergent. Using the notion of quadratic normalisation of monoids, our construction allows us to give a new proof of the termination of the 2-polygraph $\text{Col}_2(n)$ without considering the combinatorial properties of tableaux. Consider the map $\Phi : \text{Col}_1(n)^* \rightarrow \text{Col}_1(n)^*$ sending a 1-cell in $\text{Col}_1(n)^*$ to its unique corresponding tableau. Then $(\text{Col}_1(n), \Phi)$ is a quadratic normalisation of the monoid \mathbf{P}_n . Using the fact that the 2-polygraph $\text{Col}_2(n)$ has the unique normal form property as proved in Chapter 3, Proposition 3.2.2.1, we show by Theorem 4.1.3.1 that the quadratic normalisation $(\text{Col}_1(n), \Phi)$ is of class $(3, 3)$, that is, one obtains the normal form after at most 3 steps when starting from the left and 3 steps from the right. Hence, we obtain by [34, Proposition 5.1.1], that the corresponding 2-polygraph $\text{Col}_2(n)$ is finite and convergent. As a consequence, we obtain a new proof of the termination of the 2-polygraph $\text{Col}_2(n)$.

4.2. KNUTH'S COHERENT PRESENTATIONS OF PLACTIC MONOIDS

After recalling the homotopical reduction procedure from [31, 40], we apply three steps of homotopical reduction on the $(3, 1)$ -polygraph $\text{Col}_3(n)$. As a first step, we apply in 4.2.2 a homotopical reduction on the $(3, 1)$ -polygraph $\text{Col}_3(n)$ with a collapsible part defined by some of the generating triple confluences of the 2-polygraph $\text{Col}_2(n)$. In this way, we reduce the coherent presentation $\text{Col}_3(n)$ of the monoid \mathbf{P}_n into the coherent presentation $\overline{\text{Col}}_3(n)$ of \mathbf{P}_n , whose underlying 2-polygraph is $\text{Col}_2(n)$ and the 3-cells $\mathcal{X}_{u,v,t}$ are those of $\text{Col}_3(n)$, but with $\ell(u) = 1$. We reduce in 4.2.3 the coherent presentation $\overline{\text{Col}}_3(n)$ into a coherent presentation $\text{PreCol}_3(n)$ of the plactic monoid \mathbf{P}_n , whose underlying 2-polygraph is $\text{PreCol}_2(n)$ defined in Chapter 2, Subsection 2.2.2. This reduction is given by a collapsible part defined by a set of 3-cells of $\overline{\text{Col}}_3(n)$. In a final step, we reduce in 4.2.4 the coherent presentation $\text{PreCol}_3(n)$ into a coherent presentation $\text{Knuth}_3(n)$ of \mathbf{P}_n whose underlying 2-polygraph is $\text{Knuth}_2(n)$.

All these homotopical reductions preserve coherence. That is, the $(3, 1)$ -polygraph $\text{Col}_3(n)$ being a coherent presentation of the monoid \mathbf{P}_n , the $(3, 1)$ -polygraphs $\overline{\text{Col}}_3(n)$, $\text{PreCol}_3(n)$ and $\text{Knuth}_3(n)$ are coherent presentations of \mathbf{P}_n .

4.2.1. Homotopical reduction procedure

Coherent presentations of monoids constructed by the homotopical completion procedure are not minimal in general. The *homotopical reduction procedure* introduced in [31, Section 2.3.] is used to eliminate superfluous cells thanks to the notions of collapsible parts and generating triple confluences that we recall in the following.

Collapsible part of $(3, 1)$ -polygraphs. Let Σ be a $(3, 1)$ -polygraph. A *collapsible part* of Σ is a triple $\Gamma = (\Gamma_2, \Gamma_3, \Gamma_4)$ made of a family Γ_2 of 2-cells of Σ , a family Γ_3 of 3-cells of Σ and a family Γ_4 of 3-spheres of Σ_3^\top , such that the following conditions are satisfied:

- i) every γ of every Γ_k is collapsible, that is, $t_{k-1}(\gamma)$ is in Σ_{k-1} and $s_{k-1}(\gamma)$ does not contain $t_{k-1}(\gamma)$,
- ii) no cell of Γ_2 (resp. Γ_3) is the target of a collapsible 3-cell of Γ_3 (resp. 3-sphere of Γ_4),
- iii) there exists a well-founded order on the cells of Σ such that, for every γ in every Γ_k , $t_{k-1}(\gamma)$ is strictly greater than every generating $(k-1)$ -cell that occurs in the source of γ .

Homotopical reduction procedure. The *homotopical reduction* of the $(3, 1)$ -polygraph Σ with respect to a collapsible part Γ is the Tietze transformation, denoted by R_Γ , from the $(3, 1)$ -category Σ_3^\top to the $(3, 1)$ -category freely generated by the $(3, 1)$ -polygraph obtained from Σ by removing the cells of Γ and all the corresponding redundant cells, and by replacement of the

4.2.1. Homotopical reduction procedure

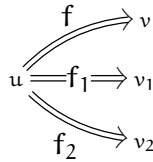
source and target maps of Σ by their compositions with R_Γ . We refer the reader to [31, 2.3.1] for details on the definition of the Tietze transformation R_Γ defined by well-founded induction as follows. For any γ in Γ

$$R_\Gamma(t(\gamma)) = R_\Gamma(s(\gamma)) \quad \text{and} \quad R_\Gamma(\gamma) = 1_{R_\Gamma(s(\gamma))}.$$

In any other cases, the transformation R_Γ acts as an identity.

Triple branchings. Let Σ be a convergent and coherent $(3, 1)$ -polygraph. Its triple critical branchings are used to build a collapsible set of 3-spheres of Σ^\top that contributes to a coherent elimination of 3-cells of Σ by homotopical reduction.

For a 2-polygraph Σ , recall from [37] that a *triple branching* of Σ is a triple (f, f_1, f_2) of 2-cells in Σ_2^* with a common source u , as in the following diagram



A triple branching (f, f_1, f_2) is *local* if f, f_1 and f_2 are rewriting steps. The local triple branchings are classified into the following three families:

- *Peiffer* triple branchings have at least one of their 2-cells that form a Peiffer branching with the other two,
- *aspherical* triple branchings have two of their 2-cells equal,
- *overlap* triple branchings are the remaining local triple branchings.

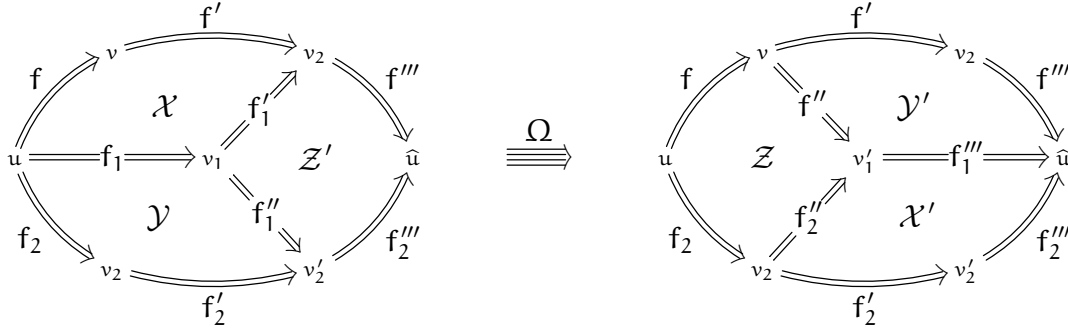
Local triple branchings are ordered by the order \sqsubseteq generated by the relations

$$(f, f_1, f_2) \sqsubseteq (ufv, uf_1v, uf_2v)$$

given for any local triple branching (f, f_1, f_2) and any possible 1-cells u and v of the category Σ_1^* . A overlap triple branching that is minimal for the order \sqsubseteq is called *critical*.

4.2. Knuth's coherent presentations of plactic monoids

Generating triple confluences. If Σ is a coherent and convergent $(3, 1)$ -polygraph, recall from [31, Section 2.3.2] that a *triple generating confluence* of Σ is a 3-sphere



where (f, f_1, f_2) is a triple critical branching of Σ and the other cells are obtained as follows. First, one considers the branching (f, f_1) , the 2-cells f' and f'_1 are obtained using the confluence of (f, f_1) and the coherence is used to get the 3-cell \mathcal{X} . The same procedure is applied on the branchings (f_1, f_2) and (f, f_2) . Then, one considers the branching (f', f'') and using the convergence one obtains the 2-cells f''_1 and f''' with common target \hat{u} , plus the 3-cell \mathcal{Y}' by coherence. The same operation is applied on (f''_2, f'_2) to get \mathcal{X}' . Finally, one builds the 3-cell \mathcal{Z}' to relate the parallel 2-cells $f'_1 \star_1 f'''$ and $f''_1 \star_1 f'_2$.

Homotopical completion-reduction. Fix a terminating 2-polygraph Σ . One applies the homotopical reduction to the homotopical completion $\mathcal{S}(\Sigma)$ by considering a collapsible part Γ of $\mathcal{S}(\Sigma)$ made of some of the generating triple confluences of $\mathcal{S}(\Sigma)$, the 3-cells coherently added with a 2-cell by homotopical completion and some collapsible 2-cells or 3-cells already present in the initial presentation Σ .

The *homotopical completion-reduction* of Σ is the $(3, 1)$ -polygraph

$$\mathcal{R}(\Sigma) = \mathcal{R}_\Gamma(\mathcal{S}(\Sigma))$$

which is obtained from $\mathcal{S}(\Sigma)$ by homotopical reduction with respect to some collapsible part Γ of $\mathcal{S}(\Sigma)$. For every terminating presentation Σ of a monoid \mathbf{M} , the homotopical completion-reduction $\mathcal{R}(\Sigma)$ of Σ is a coherent presentation of \mathbf{M} , [31, Theorem 2.3.4].

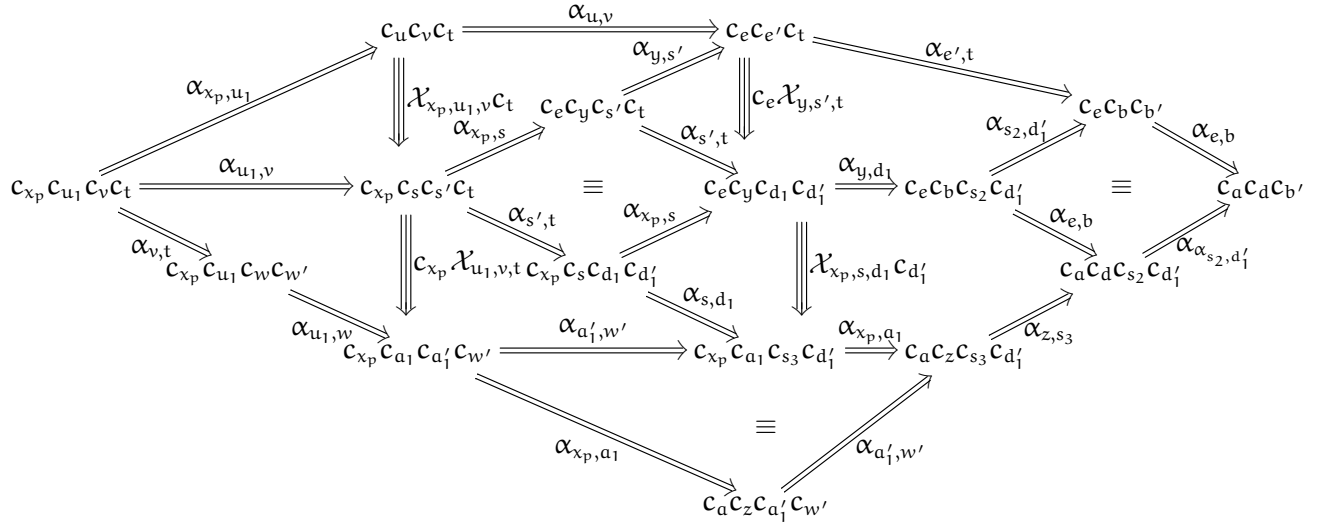
4.2.2. A reduced column presentation

We apply the homotopical reduction procedure in order to reduce the $(3, 1)$ -polygraph $\text{Col}_3(n)$ using the generating triple confluences.

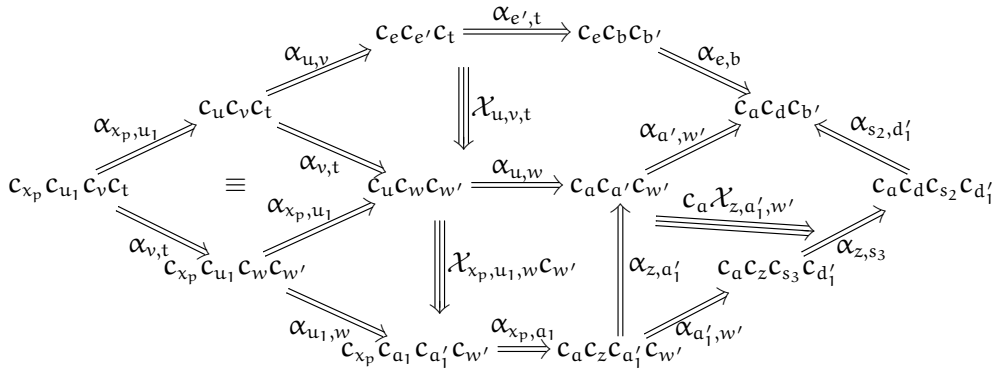
Generating triple confluences of $\text{Col}_2(n)$. Consider the homotopical reduction procedure on the $(3, 1)$ -polygraph $\text{Col}_3(n)$ defined using the collapsible part made of generating triple confluences. By Theorem 4.1.3.1, the family of 3-cells $\mathcal{X}_{u,v,t}$ given in (4.3) and indexed by columns u, v and t in $\text{col}(n)$ such that $u \times v \times t$ forms a homotopy basis of the $(2, 1)$ -category $\text{Col}_2(n)^\top$.

4.2.2. A reduced column presentation

Let us consider such a triple (u, v, t) with $\ell(u) \geq 2$. Let x_p be in $[n]$ such that $u = x_p u_1$ with u_1 in $\text{col}(n)$. There is a critical triple branching with source $c_{x_p} c_{u_1} c_v c_t$. Let us show that the confluence diagram induced by this triple branching is represented by the 3-sphere $\Omega_{x_p, u_1, v, t}$ whose source is the following 3-cell



and whose target is the following 3-cell



In the generating triple confluence, some columns may be empty and thus the indicated 2-cells α may be identities. To facilitate the reading of the diagram, we have omitted the context of the 2-cells α .

Let us explain how the 3-sphere $\Omega_{x_p, u_1, v, t}$ is constructed. We have $x_p \times u_1$ and $u_1 \times w$, thus $\mathcal{X}_{x_p, u_1, w}$ is either of the form $A_{x_p, u_1, w}$ or $C_{x_p, u_1, w}$. Let us denote by a_1 and a_1' the two columns of the tableau $P(u_1 w)$. The 3-cell $\mathcal{X}_{x_p, u_1, w}$ being confluent, we have $C(P(x_p a_1)) = az$ with z in $[n]$ and $C(P(z a_1')) = a'$.

In addition, from $z \times a_1'$ and $a_1' \times w'$, we deduce that $\mathcal{X}_{z, a_1', w'}$ is either of the form $A_{z, a_1', w'}$ or $C_{z, a_1', w'}$.

4.2. Knuth's coherent presentations of plactic monoids

From $x_p^{\times 1} u_1$ and $u_1^{\times} v$, we deduce that $\mathcal{X}_{x_p, u_1, v}$ is either of the form $A_{x_p, u_1, v}$ or $C_{x_p, u_1, v}$. Let us denote by s and s' the two columns of the tableau $P(u_1 v)$. The 3-cell $\mathcal{X}_{x_p, u_1, v}$ being confluent, we obtain that $C(P(x_p s)) = ey$ with y in $[n]$ and $C(P(ys')) = e'$. From $y^{\times 1} s'$ and $s'^{\times} t$, we deduce that $\mathcal{X}_{y, s', t}$ is either of the form $A_{y, s', t}$ or $C_{y, s', t}$. Denote by d_1 and d'_1 the two columns of the tableau $P(s't)$. The 3-cell $\mathcal{X}_{y, s', t}$ being confluent and $C(P(e't)) = bb'$, we have $C(P(yd_1)) = bs_2$ and $C(P(s_2 d'_1)) = b'$. On the other hand, the 3-cell $\mathcal{X}_{u_1, v, t}$ is confluent, then we have $C(P(sd_1)) = a_1 s_3$ and $C(P(a'_1 w')) = s_3 d'_1$. Finally, since the 3-cell $\mathcal{X}_{x_p, s, d_1}$ is confluent, we obtain $C(P(zs_3)) = ds_2$.

Reduced coherent column presentation. Let us define by $\overline{\text{Col}}_3(n)$ the extended presentation of the monoid \mathbf{P}_n obtained from $\text{Col}_2(n)$ by adjunction of one family of 3-cells $\mathcal{X}_{x, v, t}$ of the form (4.3), for every 1-cell x in $[n]$ and columns v and t in $\text{col}(n)$ such that $x^{\times} v^{\times} t$. The following result shows that this reduced presentation is also coherent.

4.2.2.1. Proposition. *For $n > 0$, the $(3, 1)$ -polygraph $\overline{\text{Col}}_3(n)$ is a coherent presentation of the monoid \mathbf{P}_n .*

Proof. Let Γ_4 be the collapsible part made of the family of 3-sphere $\Omega_{x_p, u_1, v, t}$, indexed by x_p in $[n]$ and u_1, v, t in $\text{col}(n)$ such that $u^{\times} v^{\times} t$ and $u = x_p u_1$. On the 3-cells of $\text{Col}_3(n)$ we define a well-founded order \triangleleft by

- i) $A_{u, v, t} \triangleleft C_{u, v, t} \triangleleft B_{u, v, t} \triangleleft D_{u, v, t}$,
- ii) if $\mathcal{X}_{u, v, t} \in \{A_{u, v, t}, B_{u, v, t}, C_{u, v, t}, D_{u, v, t}\}$ and $u' \preceq_{\text{deglex}} u$, then $\mathcal{X}_{u', v', t'} \triangleleft \mathcal{X}_{u, v, t}$,

for any u, v, t in $\text{col}(n)$ such that $u^{\times} v^{\times} t$.

By construction of the 3-sphere $\Omega_{x_p, u_1, v, t}$, its source contains the 3-cell $\mathcal{X}_{u_1, v, t}$ and its target contains the 3-cell $\mathcal{X}_{u, v, t}$ with $\ell(u_1) < \ell(u)$. Up to a Nielsen transformation, the homotopical reduction R_{Γ_4} applied on the $(3, 1)$ -polygraph $\text{Col}_3(n)$ with respect to Γ_4 and the order \triangleleft give us the $(3, 1)$ -polygraph $\overline{\text{Col}}_3(n)$. In this way, the presentation $\overline{\text{Col}}_3(n)$ is a coherent presentation of the plactic monoid \mathbf{P}_n . \square

4.2.3. Pre-column coherent presentation

We reduce the coherent presentation $\overline{\text{Col}}_3(n)$ into a coherent presentation whose underlying 2-polygraph is $\text{PreCol}_2(n)$. This reduction is obtained using the homotopical reduction R_{Γ_3} on the $(3, 1)$ -polygraph $\overline{\text{Col}}_3(n)$ whose collapsible part Γ_3 is defined by

$$\begin{aligned} \Gamma_3 = \{ & A_{x, v, t} \mid x \in [n], v, t \in \text{col}(n) \text{ such that } x^{\times 1} v^{\times 1} t \} \\ & \cup \{ B_{x, v, t} \mid x \in [n], v, t \in \text{col}(n) \text{ such that } x^{\times 2} v^{\times 1} t \} \\ & \cup \{ C_{x, v, t} \mid x \in [n], v, t \in \text{col}(n) \text{ such that } x^{\times 1} v^{\times 2} t \}, \end{aligned}$$

and the well-founded order defined as follows.

4.2.3. Pre-column coherent presentation

A well-founded order on 2-cells. Consider two columns u and v in $\text{col}(n)$ such that $u \times v$. Let denote by $C_r(P(uv))$ the reading of the right column of the tableau $P(uv)$. We define a well-founded order \triangleleft on the 2-cells of $\text{Col}_2(n)$ as follows

$$\alpha_{u',v'} \triangleleft \alpha_{u,v} \quad \text{if} \quad \begin{cases} \ell(uv) > \ell(u'v') & \text{or} \\ \ell(uv) = \ell(u'v') & \text{and} \quad \begin{cases} \ell(u) > \ell(C_r(P(u'v'))) & \text{or} \\ \ell(u) \leq \ell(C_r(P(u'v'))) & \text{and } u' \preceq_{\text{rev}} u \end{cases} \end{cases}$$

for any columns u, v, u' and v' in $\text{col}(n)$ such that $u \times v$ and $u' \times v'$.

The homotopical reduction R_{Γ_3} . Consider the well-founded order \triangleleft on the 2-cells of $\text{Col}_2(n)$ defined above and the well-founded order \triangleleft on 3-cells defined in the proof of Proposition 4.2.2.1. The reduction R_{Γ_3} induced by these orders can be decomposed as follows.

For any x in $[n]$ and columns v, t such that $x^{\times 1} v^{\times 1} t$, we have $\alpha_{x,v} \triangleleft \alpha_{xv,t}$, $\alpha_{v,t} \triangleleft \alpha_{xv,t}$ and $\alpha_{x,vt} \triangleleft \alpha_{xv,t}$. The reduction R_{Γ_3} removes the 2-cell $\alpha_{xv,t}$ together with the following 3-cell:

$$\begin{array}{ccccc} & & \alpha_{x,v}c_t & \xrightarrow{\quad} & c_{xv}c_t & \xrightarrow{\quad} & \alpha_{xv,t} & & \\ & & & & \Downarrow & & & & \\ c_x c_v c_t & \xrightarrow{\quad} & & & A_{x,v,t} & \xrightarrow{\quad} & & & c_{xvt} \\ & & c_x \alpha_{v,t} & \xrightarrow{\quad} & c_x c_{vt} & \xrightarrow{\quad} & \alpha_{x,vt} & & \end{array}$$

By iterating this reduction on the length of the column v , we reduce all the 2-cells $\alpha_{u,v}$ of $\text{Col}_2(n)$ to the following set of 2-cells

$$\{ \alpha_{u,v} \mid \ell(u) \geq 1, \ell(v) \geq 2 \text{ and } u^{\times 2} v \} \cup \{ \alpha_{u,v} \mid \ell(u) = 1, \ell(v) \geq 1 \text{ and } u^{\times 1} v \}. \quad (4.10)$$

For any x in $[n]$ and columns v, t such that $x^{\times 1} v^{\times 2} t$, consider the following 3-cell:

$$\begin{array}{ccccc} & & \alpha_{x,v}c_t & \xrightarrow{\quad} & c_{xv}c_t & \xrightarrow{\quad} & \alpha_{xv,t} & & \\ & & & & \Downarrow & & & & \\ c_x c_v c_t & \xrightarrow{\quad} & & & C_{x,v,t} & \xrightarrow{\quad} & & & c_a c_{a'w'} \\ & & c_x \alpha_{v,t} & \xrightarrow{\quad} & c_x c_w c_{w'} & \xrightarrow{\quad} & \alpha_{x,w} c_{w'} & \xrightarrow{\quad} & c_a c_{a'} c_{w'} & \xrightarrow{\quad} & c_a \alpha_{a',w'} \end{array}$$

where w, w', a and a' are defined in Lemma 4.1.3.4. The 2-cells $\alpha_{x,v}$, $\alpha_{v,t}$, $\alpha_{x,w}$ and $\alpha_{a',w'}$ are smaller than $\alpha_{xv,t}$ for the order \triangleleft . The reduction R_{Γ_3} removes the 2-cell $\alpha_{xv,t}$ together with the 3-cell $C_{x,v,t}$. By iterating this reduction on the length of v , we reduce the set of 2-cells given in (4.10) to the following set:

$$\{ \alpha_{u,v} \mid \ell(u) = 1, \ell(v) \geq 2 \text{ and } u^{\times 2} v \} \cup \{ \alpha_{u,v} \mid \ell(u) = 1, \ell(v) \geq 1 \text{ and } u^{\times 1} v \}. \quad (4.11)$$

4.2. Knuth's coherent presentations of plactic monoids

For any x in $[n]$ and columns v, t such that $x \times^2 v \times^1 t$, consider the following 3-cell:

$$\begin{array}{ccccc}
 & & c_e c_{e',t} & \xrightarrow{c_e \alpha_{e',t}} & c_e c_{e'} & \xrightarrow{\tilde{\alpha}_{e,e',t}} & c_s c_{s'} \\
 & \alpha_{x,v} c_t & \nearrow c_e c_{e',t} & & \searrow c_e c_{e'} & & \\
 c_x c_v c_t & & & \Downarrow B_{x,v,t} & & & \\
 & c_x \alpha_{v,t} & \searrow c_x c_v c_t & & \nearrow c_x c_{vt} & & \\
 & & & & & & c_s c_{s'}
 \end{array}$$

where e, e', s and s' are defined in Lemma 4.1.3.3. Note that $\tilde{\alpha}_{e,e',t}$ is the 2-cell in (4.11) obtained from the 2-cell $\alpha_{e,e',t}$ by the previous step of the homotopical reduction by the 3-cell $C_{x,v,t}$.

Having x in $[n]$, by definition of α we have e' in $[n]$. The 2-cells $\alpha_{x,v}, \alpha_{e',t}, \alpha_{v,t}$ and $\tilde{\alpha}_{e,e',t}$ being smaller than $\alpha_{x,vt}$ for the order \triangleleft , we can remove the 2-cells $\alpha_{x,vt}$ together with the 3-cell $B_{x,v,t}$. By iterating this reduction on the length of the column t , we reduce the set (4.11) to the following set

$$\{ \alpha_{u,v} \mid \ell(u) = 1, \ell(v) = 2 \text{ and } u \times^2 v \} \cup \{ \alpha_{u,v} \mid \ell(u) = 1, \ell(v) \geq 1 \text{ and } u \times^1 v \}. \quad (4.12)$$

Let us recall from Chapter 2, Subsection 2.2.2 that the set of the defining relations for the column generators is

$$C_2(n) = \{ c_{x_p} \dots c_{x_1} \xrightarrow{\gamma_u} c_u \mid u = x_p \dots x_1 \in \text{col}(n) \text{ with } \ell(u) \geq 2 \},$$

and that $\text{PC}_2(n)$ is the cellular extension of $\text{Col}_1^*(n)$ whose set of 2-cells is

$$\{ c_x c_{zy} \xrightarrow{\alpha'_{x,zy}} c_{zx} c_y \mid 1 \leq x \leq y < z \leq n \} \cup \{ c_y c_{zx} \xrightarrow{\alpha'_{y,zx}} c_{yx} c_z \mid 1 \leq x < y \leq z \leq n \}.$$

4.2.3.1. Lemma. *We have*

$$\text{PC}_2(n) = \{ \alpha_{u,v} : c_u c_v \Rightarrow c_w c_{w'} \mid \ell(u) = 1, \ell(v) = 2 \text{ and } u \times^2 v \}.$$

Proof. Consider the 2-cells $\alpha_{u,v}$ in $\text{Col}_2(n)$ such that $\ell(u) = 1, \ell(v) = 2$ and $u \times^2 v$. Suppose that $v = xx'$ with $x > x'$ in $[n]$. Since $u \times^2 v$, we obtain that $u \leq x$. Hence, we have two cases to consider. If $u \leq x'$, then $C(P(uv)) = (xu)x'$. Hence, the 2-cell $\alpha_{u,v}$ is equal to the 2-cell $\alpha'_{u,xx'} : c_u c_{xx'} \Rightarrow c_{xu} c_{x'}$. In the other case, if $x' < u$, then $C(P(uv)) = (ux')x$. Hence the 2-cell $\alpha_{u,v}$ is equal to $\alpha'_{u,xx'} : c_u c_{xx'} \Rightarrow c_{ux'} c_x$. \square

Recall from Chapter 2, Subsection 2.2.2 that the set of 2-cells $\text{PreCol}_2(n)$ is given by

$$\text{PreCol}_2(n) = \text{PC}_2(n) \cup \{ c_x c_u \xrightarrow{\alpha'_{x,u}} c_{xu} \mid xu \in \text{col}(n) \text{ and } 1 \leq x \leq n \}.$$

Thus, by Lemma 4.2.3.1, the set of 2-cells defined in (4.12) is equal to $\text{PreCol}_2(n)$.

4.2.3. Pre-column coherent presentation

Pre-column coherent presentation. The homotopical reduction R_{Γ_3} , defined in 4.2.3, reduces the coherent presentation $\overline{\text{Col}}_3(\mathfrak{n})$ into a coherent presentation of the monoid \mathbf{P}_n . The set of 2-cells of this coherent presentation is given by (4.12), which is $\text{PreCol}_2(\mathfrak{n})$ by Lemma 4.2.3.1.

Let us denote by $\text{PreCol}_3(\mathfrak{n})$ the extended presentation of the monoid \mathbf{P}_n obtained from $\text{PreCol}_2(\mathfrak{n})$ by adjunction of the 3-cell $R_{\Gamma_3}(C'_{x,v,t})$ where

$$\begin{array}{ccccc}
 & \alpha_{x,v}c_t & \rightarrow & c_{xv}c_t & \xleftarrow{c_{xv}\alpha_{z_1\dots z_{q+1},w'}} \\
 c_x c_v c_t & \xrightarrow{\alpha_{x,v}c_t} & & \Downarrow C'_{x,v,t} & \\
 & c_x \alpha_{v,t} & \rightarrow & c_x c_w c_{w'} & \xrightarrow{\alpha_{x,w}c_{w'}} c_{xv} c_{z_1\dots z_{q+1}} c_{w'}
 \end{array}$$

with $x^{\times 1} v^{\times 2} t$, and the 3-cell $R_{\Gamma_3}(D_{x,v,t})$ where

$$\begin{array}{ccccc}
 & \alpha_{x,v}c_t & \rightarrow & c_e c_{e'} c_t & \xrightarrow{c_e \alpha_{e',t}} c_e c_b c_{b'} & \xrightarrow{\alpha_{e,b}c_{b'}} \\
 c_x c_v c_t & \xrightarrow{\alpha_{x,v}c_t} & & \Downarrow D_{x,v,t} & & c_a c_d c_{b'} \\
 & c_x \alpha_{v,t} & \rightarrow & c_x c_w c_{w'} & \xrightarrow{\alpha_{x,w}c_{w'}} c_a c_{a'} c_{w'} & \xrightarrow{c_a \alpha_{a',w'}}
 \end{array}$$

with $x^{\times 2} v^{\times 2} t$.

The homotopical reduction R_{Γ_3} eliminates the 3-cells of $\overline{\text{Col}}_3(\mathfrak{n})$ of the form $A_{x,v,t}$, $B_{x,v,t}$ and $C_{x,v,t}$, which are not of the form $C'_{x,v,t}$. We have then proved the following result.

4.2.3.2. Theorem. *For $n > 0$, the $(3, 1)$ -polygraph $\text{PreCol}_3(\mathfrak{n})$ is a coherent presentation of the monoid \mathbf{P}_n .*

Example: coherent presentation of monoid \mathbf{P}_2 . The Knuth presentation $\text{Knuth}_2(2)$ of the plactic monoid \mathbf{P}_2 has generators 1 and 2 subject to the Knuth relations $\eta_{1,1,2} : 211 \Rightarrow 121$ and $\varepsilon_{1,2,2} : 221 \Rightarrow 212$. This presentation is convergent with only one critical branching with source the 1-cell 2211. This critical branching is confluent:

$$\begin{array}{ccc}
 & 2\eta_{1,1,2} & \\
 2211 & \xrightarrow{\quad} & 2121 \\
 & \Downarrow C'' & \\
 & \varepsilon_{1,2,2}1 &
 \end{array}$$

Following the homotopical completion procedure given in 4.1.2, the 2-polygraph extended by the previous 3-cell is a coherent presentation of the monoid \mathbf{P}_2 .

4.2. Knuth's coherent presentations of plactic monoids

Consider the column presentation $\text{Col}_2(2)$ of the monoid \mathbf{P}_2 with 1-cells c_1, c_2 and c_{21} and 2-cells $\alpha_{2,1}, \alpha_{1,21}$ and $\alpha_{2,21}$. The coherent presentation $\text{Col}_3(2)$ has only one 3-cell

$$\begin{array}{ccccc}
 & \xrightarrow{\alpha_{2,1}c_{21}} & c_{21}c_{21} & \xleftarrow{c_{21}\alpha_{2,1}} & \\
 c_2c_1c_{21} & \searrow & \Downarrow C'_{2,1,21} & \swarrow & \\
 & \xrightarrow{c_2\alpha_{1,21}} & c_2c_{21}c_1 & \xrightarrow{\alpha_{2,21}c_1} & c_{21}c_2c_1
 \end{array}$$

It follows that the $(3, 1)$ -polygraphs $\overline{\text{Col}}_3(2)$ and $\text{Col}_3(2)$ coincide. Moreover, in this case the set Γ_3 is empty and the homotopical reduction R_{Γ_3} is the identity and thus $\text{PreCol}_3(2)$ is also equal to $\text{Col}_3(2)$.

In next section, we will show how to relate the coherent presentations $\text{Col}_3(2)$ and $\langle \text{Knuth}_2(2) \mid C'' \rangle$.

Example: coherent presentation of monoid \mathbf{P}_3 . For the monoid \mathbf{P}_3 , the Knuth presentation has 3 generators and 8 relations. It is not convergent, but it can be completed by adding 3 relations. The obtained presentation has 27 homotopy generators corresponding to the 27 critical branchings.

The column coherent presentation $\text{Col}_3(3)$ of \mathbf{P}_3 has 7 generators, 22 relations and 42 homotopy generators. The coherent presentation $\overline{\text{Col}}_3(3)$ has 7 generators, 22 relations and 34 homotopy generators. After applying the homotopical reduction R_{Γ_3} , the coherent presentation $\text{PreCol}_3(3)$ admits 7 generators, 22 relations and 24 homotopy generators.

We give in 4.2.5 the values of number of cells of the $(3, 1)$ -polygraphs $\overline{\text{Col}}_3(n)$ and $\text{PreCol}_3(n)$ for plactic monoids of rank $n \leq 10$.

4.2.4. Knuth's coherent presentation

We reduce the coherent presentation $\text{PreCol}_3(n)$ into a coherent presentation of the plactic monoid \mathbf{P}_n whose underlying 2-polygraph is $\text{Knuth}_2(n)$ using the Tietze transformations defined in Chapter 2, Subsection 2.2.2. We proceed in three steps developed in this section.

- Step 1.** We apply the inverse of the Tietze transformation $T_{\gamma \leftarrow \alpha'}$, that coherently replaces the 2-cells $\gamma_{x_p \dots x_1}$ by the 2-cells $\alpha'_{x_p, x_{p-1} \dots x_1}$, for each column $x_p \dots x_1$ such that $\ell(x_p \dots x_1) > 2$.
- Step 2.** We apply the inverse of the Tietze transformation $T_{\eta, \varepsilon \leftarrow \alpha'}$, that coherently replaces the 2-cells $\alpha'_{x, zy}$ by $\eta_{x, y, z}^c$ for $1 \leq x \leq y < z \leq n$ and the 2-cells $\alpha'_{y, zx}$ by $\varepsilon_{x, y, z}^c$ for $1 \leq x < y \leq z \leq n$.
- Step 3.** Finally for each column $x_p \dots x_1$, we coherently eliminate the generator $c_{x_p \dots x_1}$ together with the 2-cell $\gamma_{x_p \dots x_1}$ with respect to the order \preceq_{deglex} .

Step 1. The Tietze transformation

$$T_{\gamma \leftarrow \alpha'} : \text{CPC}_2(\mathfrak{n})^\top \rightarrow \text{PreCol}_2(\mathfrak{n})^\top$$

defined in Chapter 2, Lemma 2.2.2.4 substitutes a 2-cell $\alpha'_{x_p, x_{p-1} \dots x_1} : c_{x_p} c_{x_{p-1} \dots x_1} \Rightarrow c_{x_p \dots x_1}$ to the 2-cell $\gamma_{x_p \dots x_1} : c_{x_p} \dots c_{x_1} \Rightarrow c_{x_p \dots x_1}$, for each column $x_p \dots x_1$ such that $\ell(x_p \dots x_1) > 2$, from the bigger column to the smaller one with respect to the total order \preceq_{deglex} .

We consider the inverse of this Tietze transformation $T_{\gamma \leftarrow \alpha'}^{-1} : \text{PreCol}_2(\mathfrak{n})^\top \rightarrow \text{CPC}_2(\mathfrak{n})^\top$ that substitutes the 2-cell $\gamma_{x_p \dots x_1} : c_{x_p} \dots c_{x_1} \Rightarrow c_{x_p \dots x_1}$ to the 2-cell $\alpha'_{x_p, x_{p-1} \dots x_1} : c_{x_p} c_{x_{p-1} \dots x_1} \Rightarrow c_{x_p \dots x_1}$

$$\begin{array}{ccc} c_{x_p} c_{x_{p-1} \dots x_1} & \xrightarrow{\alpha'_{x_p, x_{p-1} \dots x_1}} & c_{x_p \dots x_1} \\ & \swarrow \gamma_{x_{p-1} \dots x_1} \quad \searrow \gamma_{x_p \dots x_1} & \\ c_{x_p} & & c_{x_p} \dots c_{x_1} \end{array}$$

for each column $x_p \dots x_1$ such that $\ell(x_p \dots x_1) > 2$ with respect to the order \preceq_{deglex} .

Let us denote by $\text{CPC}_3(\mathfrak{n})$ the $(3, 1)$ -polygraph whose underlying 2-polygraph is $\text{CPC}_2(\mathfrak{n})$, and the set of 3-cells is defined by

$$\{ T_{\gamma \leftarrow \alpha'}^{-1}(\mathbb{R}_{\Gamma_3}(C'_{x,v,t})) \text{ for } x^1 v^2 t \} \cup \{ T_{\gamma \leftarrow \alpha'}^{-1}(\mathbb{R}_{\Gamma_3}(D_{x,v,t})) \text{ for } x^2 v^2 t \}.$$

In this way, we extend the Tietze transformation $T_{\gamma \leftarrow \alpha'}^{-1}$ into a Tietze transformation between the $(3, 1)$ -polygraphs $\text{PreCol}_3(\mathfrak{n})$ and $\text{CPC}_3(\mathfrak{n})$. The $(3, 1)$ -polygraph $\text{PreCol}_3(\mathfrak{n})$ being a coherent presentation of the monoid \mathbf{P}_n and the Tietze transformation $T_{\gamma \leftarrow \alpha'}^{-1}$ preserves the coherence property, hence we have the following result.

4.2.4.1. Lemma. For $n > 0$, the plactic monoid \mathbf{P}_n admits $\text{CPC}_3(\mathfrak{n})$ as a coherent presentation.

Step 2. The Tietze transformation $T_{\eta, \varepsilon \leftarrow \alpha'}$ from $\text{Knuth}_2^{\text{cc}}(\mathfrak{n})^\top$ into $\text{CPC}_2(\mathfrak{n})^\top$ defined in the proof of Lemma 2.2.2.3 of Chapter 2, replaces the 2-cells $\eta_{x,y,z}^c$ and $\varepsilon_{x,y,z}^c$ in $\text{Knuth}_2^{\text{cc}}(\mathfrak{n})$ by composite of 2-cells in $\text{CPC}_2(\mathfrak{n})$.

Let us consider the inverse of this Tietze transformation

$$T_{\eta, \varepsilon \leftarrow \alpha'}^{-1} : \text{CPC}_2(\mathfrak{n})^\top \longrightarrow \text{Knuth}_2^{\text{cc}}(\mathfrak{n})^\top.$$

making the following transformations. For every $1 \leq x \leq y < z \leq n$, $T_{\eta, \varepsilon \leftarrow \alpha'}^{-1}$ substitutes the 2-cell $\eta_{x,y,z}^c : c_z c_x c_y \Rightarrow c_x c_z c_y$ to the 2-cell $\alpha'_{x,zy}$:

$$\begin{array}{ccc} & c_x \gamma_{zy} & \\ & \xrightarrow{\quad} & c_x c_{zy} \\ c_z c_x c_y & & \\ \searrow \gamma_{zx} c_y & & \swarrow \alpha'_{x,zy} \\ & c_{zx} c_y & \end{array}$$

4.2. Knuth's coherent presentations of plactic monoids

For every $1 \leq x < y \leq z \leq n$, $T_{\eta, \varepsilon \leftarrow \alpha'}^{-1}$ substitutes the 2-cell $\varepsilon_{x,y,z}^c : c_y c_z c_x \Rightarrow c_y c_x c_z$ to the 2-cell $\alpha'_{y,zx}$:

$$\begin{array}{ccc}
 & c_y c_x c_z & \xrightarrow{\gamma_{yx} c_z} & c_{yx} c_z \\
 c_y c_z c_x & \xrightarrow{\gamma_{zx}} & c_y c_{zx} & \xrightarrow{\alpha'_{y,zx}} & c_{yx} c_z
 \end{array}$$

Let us denote by $\text{Knuth}_3^{\text{cc}}(n)$ the $(3, 1)$ -polygraph whose underlying 2-polygraph is $\text{Knuth}_2^{\text{cc}}(n)$ and whose set of 3-cells is

$$\{ T_{\eta, \varepsilon \leftarrow \alpha'}^{-1}(T_{\gamma \leftarrow \alpha'}^{-1}(\mathbb{R}_{\Gamma_3}(C'_{x,v,t}))) \text{ for } x^{\times 1} v^{\times 2} t \} \cup \{ T_{\eta, \varepsilon \leftarrow \alpha'}^{-1}(T_{\gamma \leftarrow \alpha'}^{-1}(\mathbb{R}_{\Gamma_3}(D_{x,v,t}))) \text{ for } x^{\times 2} v^{\times 2} t \}.$$

We extend the Tietze transformation $T_{\eta, \varepsilon \leftarrow \alpha'}^{-1}$ into a Tietze transformation between $(3, 1)$ -polygraphs

$$T_{\eta, \varepsilon \leftarrow \alpha'}^{-1} : \text{CPC}_3(n)^\top \longrightarrow \text{Knuth}_3^{\text{cc}}(n)^\top,$$

where the $(3, 1)$ -polygraph $\text{CPC}_3(n)$ is a coherent presentation of the monoid \mathbf{P}_n and the Tietze transformation $T_{\eta, \varepsilon \leftarrow \alpha'}^{-1}$ preserves the coherence property, hence we have the following result.

4.2.4.2. Lemma. *For $n > 0$, the plactic monoid \mathbf{P}_n admits $\text{Knuth}_3^{\text{cc}}(n)$ as a coherent presentation.*

Step 3. Finally, in order to obtain the Knuth coherent presentation, we perform an homotopical reduction, obtained using the homotopical reduction \mathbb{R}_{Γ_2} on the $(3, 1)$ -polygraph $\text{Knuth}_3^{\text{cc}}(n)$ whose collapsible part Γ_2 is defined by the 2-cells γ_u of $\text{C}_2(n)$ and the well-founded order \preceq_{deglex} . Thus, for every 2-cell $\gamma_{x_p \dots x_1} : c_{x_p} \dots c_{x_1} \Rightarrow c_{x_p \dots x_1}$ in $\text{C}_2(n)$, we eliminate the generator $c_{x_p \dots x_1}$ together with the 2-cell $\gamma_{x_p \dots x_1}$, from the bigger column to the smaller one with respect to the order \preceq_{deglex} .

Knuth coherent presentation. Using the Tietze transformations constructed in the previous sections, we consider the following composite of Tietze transformations

$$\mathcal{R} := \mathbb{R}_{\Gamma_2} \circ T_{\eta, \varepsilon \leftarrow \alpha'}^{-1} \circ T_{\gamma \leftarrow \alpha'}^{-1} \circ \mathbb{R}_{\Gamma_3}$$

defined from $\overline{\text{Col}}_3(n)^\top$ to $\text{Knuth}_3^{\text{cc}}(n)^\top$ as follows. Firstly, the transformation \mathcal{R} eliminates the 3-cells of $\overline{\text{Col}}_3(n)$ of the form $A_{x,v,t}$, $B_{x,v,t}$ and $C_{x,v,t}$ which are not of the form $C'_{x,v,t}$ and reduced its set of 2-cells to $\text{PreCol}_2(n)$. Secondly, this transformation coherently replaces the 2-cells $\gamma_{x_p \dots x_1}$ by the 2-cells $\alpha'_{x_p, x_{p-1} \dots x_1}$, for each column $x_p \dots x_1$ such that $\ell(x_p \dots x_1) > 2$, the 2-cells $\alpha'_{x,yz}$ by $\eta_{x,y,z}^c$ for $1 \leq x \leq y < z \leq n$ and the 2-cells $\alpha'_{y,zx}$ by $\varepsilon_{x,y,z}^c$ for $1 \leq x < y \leq z \leq n$. Finally, for each column $x_p \dots x_1$, the transformation \mathcal{R} eliminates the generator $c_{x_p \dots x_1}$ together with the 2-cell $\gamma_{x_p \dots x_1}$ with respect to the order \preceq_{deglex} .

4.2.4. Knuth's coherent presentation

Let us denote by $\text{Knuth}_3(n)$ the extended presentation of the monoid \mathbf{P}_n obtained from $\text{Knuth}_2(n)$ by adjunction of the following set of 3-cells

$$\{ \mathcal{R}(C'_{x,v,t}) \text{ for } x^{\times 1}v^{\times 2}t \} \cup \{ \mathcal{R}(D_{x,v,t}) \text{ for } x^{\times 2}v^{\times 2}t \},$$

where

$$\begin{array}{c} \alpha_{x,v}c_t \rightarrow c_{xv}c_t \\ \swarrow \quad \searrow \\ c_x c_v c_t \quad \quad \quad c_{xv}c_t \\ \swarrow \quad \searrow \\ c_x \alpha_{v,t} \rightarrow c_x c_w c_w' \xrightarrow{\alpha_{x,w}c_w'} c_{xv}c_{z_1 \dots z_{q+1}} c_{w'} \end{array}$$

$\Downarrow C'_{x,v,t}$

with $x^{\times 1}v^{\times 2}t$, and

$$\begin{array}{c} \alpha_{x,v}c_t \rightarrow c_e c_{e'} c_t \xrightarrow{c_e \alpha_{e',t}} c_e c_b c_{b'} \xrightarrow{\alpha_{e,b}c_{b'}} c_a c_d c_{b'} \\ \swarrow \quad \searrow \\ c_x c_v c_t \quad \quad \quad c_e c_{e'} c_t \quad \quad \quad c_e c_b c_{b'} \\ \swarrow \quad \searrow \\ c_x \alpha_{v,t} \rightarrow c_x c_w c_w' \xrightarrow{\alpha_{x,w}c_w'} c_a c_{a'} c_{w'} \xrightarrow{c_a \alpha_{a',w'}} c_a c_d c_{b'} \end{array}$$

$\Downarrow D_{x,v,t}$

with $x^{\times 2}v^{\times 2}t$. The transformation \mathcal{R} being a composite of Tietze transformations, it follows the following result.

4.2.4.3. Theorem. *For $n > 0$, the $(3, 1)$ -polygraph $\text{Knuth}_3(n)$ is a coherent presentation of the monoid \mathbf{P}_n .*

Example: Knuth's coherent presentation of the monoid \mathbf{P}_2 . We have seen in Example 4.2.3 that the $(3, 1)$ -polygraphs $\text{Col}_3(2)$, $\overline{\text{Col}}_3(2)$ and $\text{PreCol}_3(2)$ are equal. The coherent presentation $\text{PreCol}_3(2)$ is given by

$$\text{PreCol}_1(2) = \{c_1, c_2, c_{21}\}, \quad \text{PreCol}_2(2) = \{\alpha_{2,1}, \alpha_{1,21}, \alpha_{2,21}\}, \quad \text{PreCol}_3(2) = \{C'_{2,1,21}\},$$

where $C'_{2,1,21}$ is the following 3-cell:

$$\begin{array}{c} \alpha_{2,1}c_{21} \rightarrow c_{21}c_{21} \\ \swarrow \quad \searrow \\ c_2 c_1 c_{21} \quad \quad \quad c_{21}c_{21} \\ \swarrow \quad \searrow \\ c_2 \alpha_{1,21} \rightarrow c_2 c_{21} c_1 \xrightarrow{\alpha_{2,21}c_1} c_{21}c_2 c_1 \end{array}$$

$\Downarrow C'_{2,1,21}$

By definition of the 2-cells of $C_2(2)$, we have $\gamma_{21} := \alpha_{2,1}$. Thus we obtain that $T_{\gamma \leftarrow \alpha'}^{-1}(C'_{2,1,21}) = C'_{2,1,21}$ up to replace all the 2-cells $\alpha_{2,1}$ in $C'_{2,1,21}$ by γ_{21} . Hence, the coherent presentation $\text{CPC}_3(2)$ is equal to $\text{PreCol}_3(2)$. In order to compute the 3-cell $T_{\eta, \varepsilon \leftarrow \alpha'}^{-1}(T_{\gamma \leftarrow \alpha'}^{-1}(C'_{2,1,21}))$, the 2-cells $\alpha_{1,21}$

4.2. Knuth's coherent presentations of plactic monoids

and $\alpha_{2,21}$ in $C'_{2,1,21}$ are respectively replaced by the 2-cells $\eta_{1,1,2}^c$ and $\varepsilon_{1,2,2}^c$ as in the following diagram

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & \xrightarrow{\gamma_{21}c_{21}} & c_{21}c_{21} & \\
 & \nearrow^{c_2c_1\gamma_{21}} & & \parallel^{C'_{2,1,21}} & \nwarrow_{c_{21}\gamma_{21}} \\
 c_2c_1c_2c_1 & \xrightarrow{\quad} & c_2c_1c_{21} & & c_{21}c_2c_1 \\
 & \searrow_{c_2\alpha_{1,21}} & \dots & \dots & \dots \\
 & & c_2c_{21}c_1 & \xrightarrow{\alpha_{2,21}c_1} & c_{21}c_2c_1 \\
 & & \parallel^{c_2\gamma_{21}c_1} & & \parallel^{\gamma_{21}c_2c_1} \\
 & & c_2c_2c_1c_1 & \xrightarrow{\varepsilon_{1,2,2}^c} & c_2c_1c_2c_1 \\
 & \nwarrow_{c_2\eta_{1,1,2}^c} & & & \nearrow_{\gamma_{21}c_2c_1}
 \end{array}
 \end{array} \tag{4.13}$$

where the cancel symbol means that the corresponding 2-cell is removed. Hence the coherent presentation $\text{Knuth}_3^c(2)$ of \mathbf{P}_2 has for 1-cells c_1 , c_2 and c_{21} , for 2-cells $\alpha_{2,1}$, $\alpha_{1,21}$ and $\alpha_{2,21}$ and the only 3-cell (4.13).

Let us compute the Knuth coherent presentation $\text{Knuth}_3(2)$. The 3-cell $R_{\Gamma_2}(T_{\eta, \varepsilon \leftarrow \alpha'}^{-1}(T_{\gamma \leftarrow \alpha'}^{-1}(C'_{2,1,21})))$ is obtained from (4.13) by removing the 2-cell γ_{21} together with the 1-cell c_{21} . Thus we obtain the following 3-cell, where the cancel symbol means that the corresponding element is removed,

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & \xrightarrow{\cancel{\gamma_{21}c_{21}}} & \cancel{c_{21}c_{21}} & \\
 & \nearrow^{\cancel{c_2c_1\gamma_{21}}} & & \dots & \nwarrow_{\cancel{c_{21}\gamma_{21}}} \\
 c_2c_1c_2c_1 & \xrightarrow{\quad} & \cancel{c_2c_1c_{21}} & & \cancel{c_{21}c_2c_1} \\
 & \searrow_{\cancel{c_2\alpha_{1,21}}} & \dots & & \dots \\
 & & \cancel{c_2c_{21}c_1} & \xrightarrow{\quad} & \cancel{c_{21}c_2c_1} \\
 & & \parallel^{\cancel{c_2\gamma_{21}c_1}} & & \parallel^{\cancel{\gamma_{21}c_2c_1}} \\
 & & c_2c_2c_1c_1 & \xrightarrow{\varepsilon_{1,2,2}^c} & c_2c_1c_2c_1 \\
 & \nwarrow_{c_2\eta_{1,1,2}^c} & & & \nearrow_{\quad}
 \end{array}
 \end{array}$$

Hence, the Knuth coherent presentation $\text{Knuth}_3(2)$ of the plactic monoid \mathbf{P}_2 has generators c_1 and c_2 subject to the Knuth relations $\eta_{1,1,2}^c : c_2c_1c_1 \Rightarrow c_1c_2c_1$ and $\varepsilon_{1,2,2}^c : c_2c_2c_1 \Rightarrow c_2c_1c_2$ and the following 3-cell

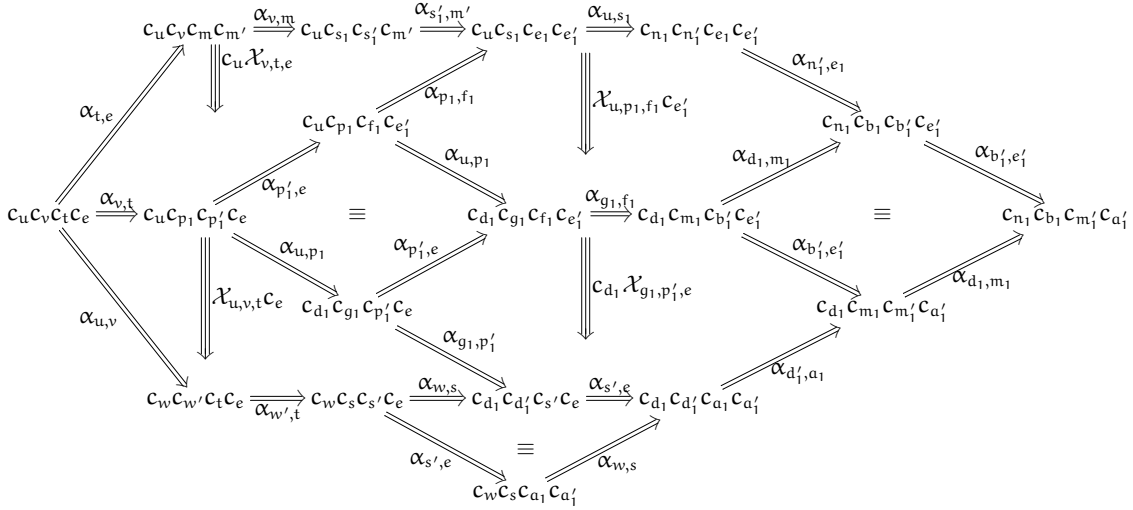
$$\begin{array}{c}
 \begin{array}{ccc}
 & \xrightarrow{2\eta_{1,1,2}^c} & \\
 \curvearrowright & & \curvearrowleft \\
 c_2c_2c_1c_1 & \parallel^{C''} & c_2c_1c_2c_1 \\
 \curvearrowleft & & \curvearrowright \\
 & \xrightarrow{\varepsilon_{1,2,2}^c} &
 \end{array}
 \end{array}$$

In this way, we obtain the Knuth's coherent presentation of the monoid \mathbf{P}_2 that we obtain in Example 4.2.3 as a consequence of the fact that the 2-polygraph $\text{Knuth}_2(2)$ is convergent.

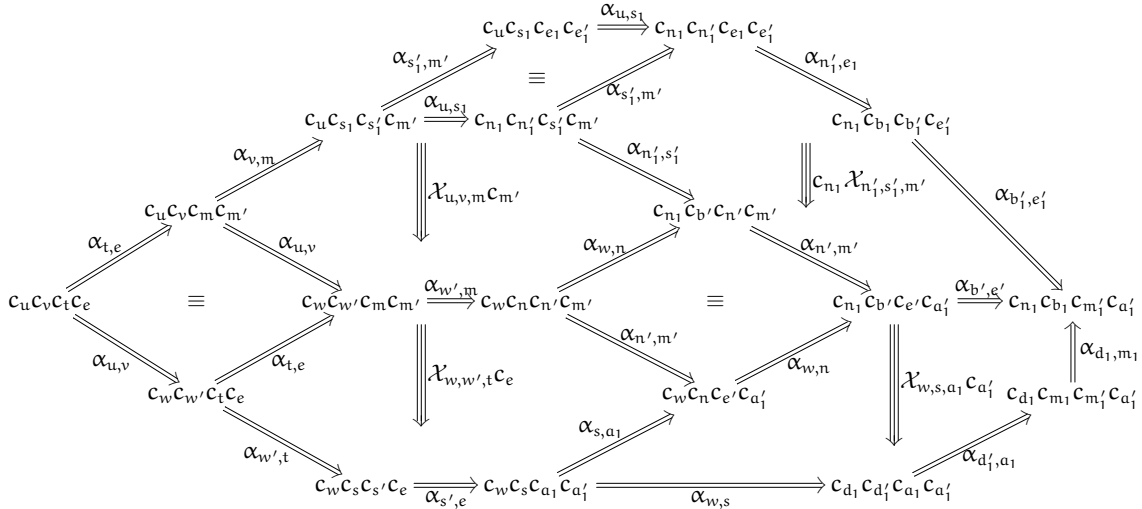
Actions of plactic monoids on categories. In [31], the authors give a description of the category of actions of a monoid on categories in terms of coherent presentations. Using this description, Theorem 4.2.4.3 allows to present actions of plactic monoids on categories as follows. The category $\text{Act}(\mathbf{P}_n)$ of actions of the monoid \mathbf{P}_n on categories is equivalent to the category of 2-functors from the $(2, 1)$ -category $\text{Knuth}_2(n)^\top$ to the category Cat of categories, that sends the 3-cells of $\text{Knuth}_3(n)$ to commutative diagrams in Cat .

4.2.4. Knuth's coherent presentation

Higher syzygies for the plactic monoid. The column presentation $\text{Col}_2(n)$ of the monoid \mathbf{P}_n can then be extended into a polygraphic resolution whose n -cells, for every $n \geq 3$, are indexed by $(n - 1)$ -fold branching of $\text{Col}_2(n)$. We can explicit the 4-cells of this resolution, which correspond to the confluence diagrams induced by critical triple branchings. That is, for columns u , v , t and e in $\text{col}(n)$ such that $u \times v$, $v \times t$ and $t \times e$, there is a critical triple branching with source $c_u c_v c_t c_e$. Using the same arguments of Section 4.2.2, we can show that the confluence diagram induced by this triple branching is represented by the 3-sphere $\Omega_{u,v,t,e}$ whose the source is the 3-cell



and the target is the 3-cell



In the generating triple confluence, some columns may be empty and thus the indicated 2-cells α may be identities. To facilitate the reading of the diagram, we have omitted the context of the

4.2. Knuth's coherent presentations of plactic monoids

2-cells α . More generally, we expect that the generating n -cell of the resolution has the form of the permutohedron of dimension n .

4.2.5. Coherent presentations in small ranks

As we have shown in Chapter 2, Subsection 2.3, the convergent 2-polygraph $\mathcal{KB}(\text{Knuth}_2(3))$ obtained from $\text{Knuth}_2(3)$ by the Knuth–Bendix's completion is finite. For $n \geq 4$, the 2-polygraph $\mathcal{KB}(\text{Knuth}_2(n))$ is infinite. We denote by $\text{Knuth}_3^{\text{KB}}(n)$ the Squier's completion of $\mathcal{KB}(\text{Knuth}_2(n))$. For $n \geq 4$, the polygraph $\mathcal{KB}(\text{Knuth}_2(n))$ having an infinite set of critical branching, the set of 3-cells of $\text{Knuth}_3^{\text{KB}}(n)$ is infinite. However, the $(3, 1)$ -polygraph $\text{Knuth}_3(n)$ constructed in this chapter is a finite coherent convergent presentation of the plactic monoid \mathbf{P}_n .

As shown in Chapter 2, Proposition 3.2.2.3, the number of 1-cells of the 2-polygraph $\text{Col}_2(n)$ is the sum of the number of all the columns over the set $[n]$. Then it is equal to $2^n - 1$. Moreover, the sources of the 2-cells of $\text{Col}_2(n)$ consists in two columns that do not form a tableau. Hence, their number is equal to

$$(2^n - 1)^2 - \left(\prod_{1 \leq i \leq j \leq n} \frac{i+j+1}{i+j-1} - \prod_{1 \leq i \leq j \leq n} \frac{i+j}{i+j-1} \right).$$

The 3-cells of the $(3, 1)$ -polygraph $\text{Col}_3(n)$ are of the form $\mathcal{X}_{u,v,t}$ as presented in 4.3, for every columns u, v and t such that $u \times v \times t$. Then the number of these 3-cells is equal to the number of possibilities $u \times v \times t$, for any columns u, v and t over the set $[n]$. The number of 1-cells and 2-cells in $\text{Col}_3(n)$ and $\overline{\text{Col}}_3(n)$ is the same. The number of 3-cells in $\overline{\text{Col}}_3(n)$ is obtained after eliminating all the 3-cells $\mathcal{X}_{u,v,t}$ with $\ell(u) > 1$ from $\text{Col}_3(n)$. The number of 1-cells of the $(3, 1)$ -polygraph $\text{Knuth}_3(n)$ is equal to n . The number of its 2-cells is equal to the number of possibilities to have zxy for $1 \leq x \leq y < z \leq n$ and yzx for $1 \leq x < y \leq z \leq n$. The number of its 3-cells is obtained after eliminating from $\overline{\text{Col}}_3(n)$ all the 3-cells of the form $A_{x,v,t}$ for $x^1 v^1 t$, $B_{x,v,t}$ for $x^2 v^1 t$ and $C_{x,v,t}$ for $x^1 v^2 t$ which are not of the form $C'_{x,v,t}$. Thus, we obtain a finite set of 3-cells in $\text{Knuth}_3(n)$.

The following table presents the number of cells of the coherent presentations $\text{Knuth}_3(n)$, $\overline{\text{Col}}_3(n)$ and $\text{Col}_3(n)$ of the monoid \mathbf{P}_n , for $1 \leq n \leq 10$.

n	$\text{Knuth}_1(n)$	$\text{Col}_1(n)$	$\text{Knuth}_2(n)$	$\mathcal{KB}(\text{Knuth}_2(n))$	$\text{Col}_2(n)$	$\text{Knuth}_3^{\text{KB}}(n)$	$\text{Knuth}_3(n)$	$\overline{\text{Col}}_3(n)$	$\text{Col}_3(n)$
1	1	1	0	0	0	0	0	0	0
2	2	3	2	2	3	1	1	1	1
3	3	7	8	11	22	27	24	34	42
4	4	15	20	∞	115	∞	242	330	621
5	5	31	40	∞	531	∞	1726	2225	6893
6	6	63	70	∞	2317	∞	10273	12635	67635
7	7	127	112	∞	9822	∞	55016	65282	623010
8	8	255	168	∞	40971	∞	275868	318708	5534197
9	9	511	240	∞	169255	∞	1324970	1500465	48052953
10	10	1023	330	∞	694837	∞	6178939	6892325	410881483

Bibliography

- [1] David J. Anick, *On monomial algebras of finite global dimension*, Trans. Amer. Math. Soc. **291** (1985), no. 1, 291–310. MR 797061 (86k:16002)
- [2] ———, *On the homology of associative algebras*, Trans. Amer. Math. Soc. **296** (1986), no. 2, 641–659.
- [3] Franz Baader and Tobias Nipkow, *Term rewriting and all that*, Cambridge University Press, 1998.
- [4] T. H. Baker, *An insertion scheme for C_n crystals*, Physical combinatorics (Kyoto, 1999), Progr. Math., vol. 191, Birkhäuser Boston, Boston, MA, 2000, pp. 1–48.
- [5] Allan Berele, *A Schensted-type correspondence for the symplectic group*, J. Combin. Theory Ser. A **43** (1986), no. 2, 320–328.
- [6] George M. Bergman, *The diamond lemma for ring theory*, Adv. in Math. **29** (1978), no. 2, 178–218.
- [7] L. A. Bokut, Yuqun Chen, Weiping Chen, and Jing Li, *New approaches to plactic monoid via Gröbner–Shirshov bases*, J. Algebra **423** (2015), 301–317.
- [8] Leonid A. Bokut, *Imbeddings into simple associative algebras*, Algebra i Logika **15** (1976), no. 2, 117–142, 245.
- [9] Ronald Book and Friedrich Otto, *String-rewriting systems*, Texts and Monographs in Computer Science, Springer-Verlag, 1993.
- [10] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*, Actualités Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968.
- [11] Kenneth S. Brown, *The geometry of rewriting systems: a proof of the Anick-Groves-Squier theorem*, Algorithms and classification in combinatorial group theory (Berkeley, CA, 1989), Math. Sci. Res. Inst. Publ., vol. 23, Springer, New York, 1992, pp. 137–163. MR 1230632 (94g:20041)

BIBLIOGRAPHY

- [12] Bruno Buchberger, *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal (An Algorithm for Finding the Basis Elements in the Residue Class Ring Modulo a Zero Dimensional Polynomial Ideal)*, Ph.D. thesis, Mathematical Institute, University of Innsbruck, Austria, 1965, English translation in *J. of Symbolic Computation, Special Issue on Logic, Mathematics, and Computer Science: Interactions*. Vol. 41, Number 3-4, Pages 475–511, 2006.
- [13] ———, *History and basic features of the critical-pair/completion procedure*, *J. Symbolic Comput.* **3** (1987), no. 1-2, 3–38, *Rewriting techniques and applications (Dijon, 1985)*.
- [14] Albert Burroni, *Higher-dimensional word problem*, *Category theory and computer science (Paris, 1991)*, *Lecture Notes in Comput. Sci.*, vol. 530, Springer, Berlin, 1991, pp. 94–105.
- [15] ———, *Higher-dimensional word problems with applications to equational logic*, *Theoret. Comput. Sci.* **115** (1993), no. 1, 43–62, *4th Summer Conference on Category Theory and Computer Science (Paris, 1991)*.
- [16] Alan J. Cain, Robert D. Gray, and António Malheiro, *Crystal monoids & crystal bases: rewriting systems and biautomatic structures for plactic monoids of types A_n , B_n , C_n , D_n , and G_2* , arXiv:1412.7040, 2015.
- [17] ———, *Finite Gröbner–Shirshov bases for Plactic algebras and biautomatic structures for Plactic monoids*, *J. Algebra* **423** (2015), 37–53.
- [18] ———, *Rewriting systems and biautomatic structures for Chinese, hypoplactic, and Sylvester monoids*, *Internat. J. Algebra Comput.* **25** (2015), no. 1-2, 51–80. MR 3325877
- [19] Julien Cassaigne, Marc Espie, Daniel Kroh, Jean-Christophe Novelli, and Florent Hivert, *The Chinese monoid.*, *Int. J. Algebra Comput.* **11** (2001), no. 3, 301–334.
- [20] Yuqun Chen and Jianjun Qiu, *Gröbner-Shirshov basis for the Chinese monoid*, *J. Algebra Appl.* **7** (2008), no. 5, 623–628.
- [21] Etsurō Date, Michio Jimbo, and Tetsuji Miwa, *Representations of $U_q(\mathfrak{gl}(n, \mathbb{C}))$ at $q = 0$ and the Robinson-Shensted [Shensted] correspondence*, *Physics and mathematics of strings*, World Sci. Publ., Teaneck, NJ, 1990, pp. 185–211.
- [22] M. Dehn, *Über die Topologie des dreidimensionalen Raumes*, *Math. Ann.* **69** (1910), no. 1, 137–168.
- [23] Vladimir Dotsenko and Anton Khoroshkin, *Free resolutions via Gröbner bases*, ArXiv e-prints (2009).
- [24] ———, *Gröbner bases for operads*, *Duke Math. J.* **153** (2010), no. 2, 363–396.
- [25] V. G. Drinfeld, *Hopf algebras and the quantum Yang-Baxter equation*, *Dokl. Akad. Nauk SSSR* **283** (1985), no. 5, 1060–1064.

- [26] Gérard Duchamp and Daniel Krob, *Plactic-growth-like monoids*, Words, languages and combinatorics, II (Kyoto, 1992), World Sci. Publ., River Edge, NJ, 1994, pp. 124–142.
- [27] E. Graham Evans and Phillip Griffith, *The syzygy problem*, Ann. of Math. (2) **114** (1981), no. 2, 323–333.
- [28] Ferdinand Georg Frobenius, *Über die charakteristischen Einheiten der symmetrischen Gruppe.*, Berl. Ber. **1903** (1903), 328–358.
- [29] William Fulton, *Young tableaux*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997, With applications to representation theory and geometry.
- [30] William Fulton and Joe Harris, *Representation theory. A first course.*, New York etc.: Springer-Verlag, 1991 (English).
- [31] Stéphane Gaussent, Yves Guiraud, and Philippe Malbos, *Coherent presentations of Artin monoids*, Compos. Math. **151** (2015), no. 5, 957–998.
- [32] Stéphane Gaussent and Peter Littelmann, *One-skeleton galleries, the path model, and a generalization of Macdonald’s formula for Hall-Littlewood polynomials*, Int. Math. Res. Not. IMRN (2012), no. 12, 2649–2707.
- [33] Basil Gordon, *A proof of the Bender-Knuth conjecture*, Pacific J. Math. **108** (1983), no. 1, 99–113.
- [34] Yves Guiraud and Patrick Dehornoy, *Quadratic normalisation in monoids*, arXiv:1504.02717, 2015.
- [35] Yves Guiraud and Philippe Malbos, *Higher-dimensional categories with finite derivation type*, Theory Appl. Categ. **22** (2009), No. 18, 420–478.
- [36] _____, *Coherence in monoidal track categories*, Math. Structures Comput. Sci. **22** (2012), no. 6, 931–969.
- [37] _____, *Higher-dimensional normalisation strategies for acyclicity*, Adv. Math. **231** (2012), no. 3-4, 2294–2351.
- [38] _____, *Identities among relations for higher-dimensional rewriting systems*, OPERADS 2009, Sémin. Congr., vol. 26, Soc. Math. France, Paris, 2013, pp. 145–161.
- [39] _____, *Polygraphs of finite derivation type*, to appear in Math. Structures Comput. Sci., arXiv:1402.2587, 2016.
- [40] Yves Guiraud, Philippe Malbos, and Samuel Mimram, *A homotopical completion procedure with applications to coherence of monoids*, 24th International Conference on Rewriting Techniques and Applications, LIPIcs. Leibniz Int. Proc. Inform., vol. 21, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2013, pp. 223–238.

BIBLIOGRAPHY

- [41] Eylem Güzel Karpuz, *Finite derivation type property on the Chinese monoid*, Appl. Math. Sci. (Ruse) **4** (2010), no. 21-24, 1073–1080.
- [42] ———, *Finite derivation type property on the Chinese monoid*, Appl. Math. Sci. (Ruse) **4** (2010), no. 21-24, 1073–1080.
- [43] Nohra Hage, *Finite convergent presentation of plactic monoid for type C*, Internat. J. Algebra Comput. **25** (2015), no. 8, 1239–1263.
- [44] ———, *Column presentations of plactic monoids*, arXiv:1512.07813, HDRA 2016, Porto, Portugal, 2016.
- [45] Nohra Hage and Philippe Malbos, *Knuth's coherent presentations of plactic monoids for type A*, arXiv:1609.01460, hal-01359677, submitted, 2016.
- [46] F. Hivert, J.-C. Novelli, and J.-Y. Thibon, *The algebra of binary search trees*, Theoret. Comput. Sci. **339** (2005), no. 1, 129–165.
- [47] Jin Hong and Seok-Jin Kang, *Introduction to quantum groups and crystal bases*, Graduate Studies in Mathematics, vol. 42, American Mathematical Society, Providence, RI, 2002. MR 1881971
- [48] Gérard Huet, *A complete proof of correctness of the Knuth-Bendix completion algorithm*, J. Comput. System Sci. **23** (1981), no. 1, 11–21.
- [49] James E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, vol. 9, Springer-Verlag, New York-Berlin, 1978, Second printing, revised.
- [50] ———, *Representations of semisimple Lie algebras in the BGGcategory O*, Graduate Studies in Mathematics, vol. 94, American Mathematical Society, Providence, RI, 2008.
- [51] Matthias Jantzen, *Semi Thue systems and generalized Church-Rosser properties*, Tech. report, Bericht Nr. 92, Fachbereich Informatik, Universität Hamburg, 1982.
- [52] ———, *A note on a special one-rule semi-Thue system*, Inform. Process. Lett. **21** (1985), no. 3, 135–140.
- [53] Michio Jimbo, *A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation*, Lett. Math. Phys. **10** (1985), no. 1, 63–69.
- [54] Michael Jöllenbeck and Volkmar Welker, *Minimal resolutions via algebraic discrete Morse theory*, Mem. Amer. Math. Soc. **197** (2009), no. 923, vi+74.
- [55] Anthony Joseph, *Quantum groups and their primitive ideals*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 29, Springer-Verlag, Berlin, 1995.
- [56] Deepak Kapur and Paliath Narendran, *A finite Thue system with decidable word problem and without equivalent finite canonical system*, Theoret. Comput. Sci. **35** (1985), no. 2-3, 337–344.

-
- [57] Masaki Kashiwara, *Crystallizing the q -analogue of universal enveloping algebras*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), Math. Soc. Japan, Tokyo, 1991, pp. 791–797.
- [58] ———, *Global crystal bases of quantum groups*, Duke Math. J. **69** (1993), no. 2, 455–485.
- [59] ———, *On crystal bases*, Representations of groups (Banff, AB, 1994), CMS Conf. Proc., vol. 16, Amer. Math. Soc., Providence, RI, 1995, pp. 155–197.
- [60] ———, *Similarity of crystal bases*, Lie algebras and their representations (Seoul, 1995), Contemp. Math., vol. 194, Amer. Math. Soc., Providence, RI, 1996, pp. 177–186.
- [61] Masaki Kashiwara and Toshiki Nakashima, *Crystal graphs for representations of the q -analogue of classical Lie algebras*, J. Algebra **165** (1994), no. 2, 295–345.
- [62] Jan Willem Klop, *Term rewriting systems*, Handbook of Logic in Computer Science, vol. 2, Oxford University Press, 1992, pp. 1–117.
- [63] Donald Knuth, *Permutations, matrices, and generalized Young tableaux*, Pacific J. Math. **34** (1970), 709–727.
- [64] Donald Knuth and Peter Bendix, *Simple word problems in universal algebras*, Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967), Pergamon, Oxford, 1970, pp. 263–297.
- [65] Łukasz Kubat and Jan Okniński, *Gröbner-Shirshov bases for plactic algebras*, Algebra Colloq. **21** (2014), no. 4, 591–596.
- [66] Roberto La Scala, Vincenzo Nardoza, and Domenico Senato, *Super RSK-algorithms and super plactic monoid.*, Int. J. Algebra Comput. **16** (2006), no. 2, 377–396.
- [67] Stephen Lack, *A Quillen model structure for 2-categories*, K-Theory **26** (2002), no. 2, 171–205.
- [68] ———, *A Quillen model structure for bicategories*, K-Theory **33** (2004), no. 3, 185–197.
- [69] Yves Lafont, *Algebra and geometry of rewriting*, Appl. Categ. Structures **15** (2007), no. 4, 415–437.
- [70] V. Lakshmibai and C. S. Seshadri, *Geometry of G/P . V*, J. Algebra **100** (1986), no. 2, 462–557.
- [71] ———, *Standard monomial theory*, Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989), Manoj Prakashan, Madras, 1991, pp. 279–322.
- [72] Alain Lascoux, Bernard Leclerc, and Jean-Yves Thibon, *Crystal graphs and q -analogues of weight multiplicities for the root system A_n* , Lett. Math. Phys. **35** (1995), no. 4, 359–374.
- [73] Alain Lascoux and Marcel-Paul Schützenberger, *Sur une conjecture de H. O. Foulkes*, C. R. Acad. Sci. Paris Sér. A-B **286** (1978), no. 7, A323–A324.
- [74] ———, *Le monoïde plaxique*, Noncommutative structures in algebra and geometric combinatorics (Naples, 1978), Quad. “Ricerca Sci.”, vol. 109, CNR, Rome, 1981, pp. 129–156.

BIBLIOGRAPHY

- [75] ———, *Schubert polynomials and the Littlewood-Richardson rule*, Lett. Math. Phys. **10** (1985), no. 2-3, 111–124.
- [76] ———, *Noncommutative Schubert polynomials*, Funktsional. Anal. i Prilozhen. **23** (1989), no. 3, 63–64.
- [77] Bernard Leclerc and Jean-Yves Thibon, *The Robinson-Schensted correspondence, crystal bases, and the quantum straightening at $q = 0$* , Electron. J. Combin. **3** (1996), no. 2, Research Paper 11, approx. 24 pp. (electronic), The Foata Festschrift.
- [78] Cédric Lecouvey, *Schensted-type correspondence, plactic monoid, and jeu de taquin for type C_n* , J. Algebra **247** (2002), no. 2, 295–331.
- [79] ———, *Schensted-type correspondences and plactic monoids for types B_n and D_n* , J. Algebraic Combin. **18** (2003), no. 2, 99–133.
- [80] ———, *Kostka-Foulkes polynomials cyclage graphs and charge statistic for the root system C_n* , J. Algebraic Combin. **21** (2005), no. 2, 203–240.
- [81] ———, *Branching rules, Kostka-Foulkes polynomials and q -multiplicities in tensor product for the root system B_n , C_n and D_n* , Algebr. Represent. Theory **9** (2006), no. 4, 377–402.
- [82] ———, *Combinatorics of crystal graphs and Kostka-Foulkes polynomials for the root systems B_n , C_n and D_n* , European J. Combin. **27** (2006), no. 4, 526–557.
- [83] ———, *Combinatorics of crystal graphs for the root systems of types A_n , B_n , C_n , D_n and G_2* , Combinatorial aspect of integrable systems, MSJ Mem., vol. 17, Math. Soc. Japan, Tokyo, 2007, pp. 11–41.
- [84] Peter Littelmann, *A generalization of the Littlewood-Richardson rule*, J. Algebra **130** (1990), no. 2, 328–368.
- [85] ———, *A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras*, Invent. Math. **116** (1994), no. 1-3, 329–346.
- [86] ———, *Crystal graphs and Young tableaux*, J. Algebra **175** (1995), no. 1, 65–87.
- [87] ———, *Paths and root operators in representation theory*, Ann. of Math. (2) **142** (1995), no. 3, 499–525.
- [88] ———, *A plactic algebra for semisimple Lie algebras*, Adv. Math. **124** (1996), no. 2, 312–331.
- [89] Jean-Louis Loday and Todor Popov, *Parastatistics algebra, Young tableaux and the super plactic monoid.*, Int. J. Geom. Methods Mod. Phys. **5** (2008), no. 8, 1295–1314.
- [90] Viktor Lopatkin, *Cohomology rings of the plactic monoid algebra via a Gröbner-Shirshov basis*, J. Algebra Appl. **15** (2016), no. 5, 1650082, 30.

-
- [91] M. Lothaire, *Algebraic combinatorics on words*, Encyclopedia of Mathematics and its Applications, vol. 90, Cambridge University Press, Cambridge, 2002.
- [92] George Lusztig, *Introduction to quantum groups*, Progress in Mathematics, vol. 110, Birkhäuser Boston, Inc., Boston, MA, 1993.
- [93] Philippe Malbos, *Rewriting systems and hochschild–mitchell homology*, Electr. Notes Theor. Comput. Sci. **81** (2003), 59–72.
- [94] Andrei Markov, *On the impossibility of certain algorithms in the theory of associative systems*, Doklady Akad. Nauk SSSR (N.S.) **55** (1947), 583–586.
- [95] ———, *On the impossibility of certain algorithms in the theory of associative systems. II*, Doklady Akad. Nauk SSSR (N.S.) **58** (1947), 353–356.
- [96] François Métayer, *Resolutions by polygraphs*, Theory Appl. Categ. **11** (2003), No. 7, 148–184.
- [97] Teo Mora, *An introduction to commutative and noncommutative Gröbner bases*, Theoret. Comput. Sci. **134** (1994), no. 1, 131–173, Second International Colloquium on Words, Languages and Combinatorics (Kyoto, 1992).
- [98] Maxwell Newman, *On theories with a combinatorial definition of “equivalence”*, Ann. of Math. (2) **43** (1942), no. 2, 223–243.
- [99] Jean-Christophe Novelli, *On the hypoplactic monoid.*, Discrete Math. **217** (2000), no. 1-3, 315–336.
- [100] Emil L. Post, *Recursive unsolvability of a problem of Thue*, J. Symbolic Logic **12** (1947), 1–11.
- [101] A. J. Power, *A 2-categorical pasting theorem*, J. Algebra **129** (1990), no. 2, 439–445.
- [102] ———, *An n -categorical pasting theorem*, Category theory (Como, 1990), Lecture Notes in Math., vol. 1488, Springer, Berlin, 1991, pp. 326–358.
- [103] Gilbert de Beauregard Robinson, *On the Representations of the Symmetric Group*, Amer. J. Math. **60** (1938), no. 3, 745–760.
- [104] C. Schensted, *Longest increasing and decreasing subsequences*, Canad. J. Math. **13** (1961), 179–191.
- [105] Marcel-Paul Schützenberger, *La correspondance de Robinson*, Combinatoire et représentation du groupe symétrique (Actes Table Ronde CNRS, Univ. Louis-Pasteur Strasbourg, Strasbourg, 1976), Springer, Berlin, 1977, pp. 59–113. Lecture Notes in Math., Vol. 579.
- [106] ———, *Pour le monoïde plaxique*, Math. Inform. Sci. Humaines (1997), no. 140, 5–10.
- [107] Luis Serrano, *The shifted plactic monoid.*, Math. Z. **266** (2010), no. 2, 363–392 (English).
- [108] Jeffrey T. Sheats, *A symplectic jeu de taquin bijection between the tableaux of King and of De Concini*, Trans. Amer. Math. Soc. **351** (1999), no. 9, 3569–3607.

BIBLIOGRAPHY

- [109] Mark Shimozone, *Crystals for dummies*, Notes, URL: www.aimath.org/WWN/kostka/crysdumb.pdf, 2005.
- [110] Anatoly Illarionovich Shirshov, *Some algorithmic problems for Lie algebras.*, Sib. Mat. Zh. **3** (1962), 292–296 (Russian).
- [111] Emil Sköldbberg, *Morse theory from an algebraic viewpoint*, Trans. Amer. Math. Soc. **358** (2006), no. 1, 115–129.
- [112] Craig C. Squier, *Word problems and a homological finiteness condition for monoids*, J. Pure Appl. Algebra **49** (1987), no. 1-2, 201–217.
- [113] Craig C. Squier, Friedrich Otto, and Yuji Kobayashi, *A finiteness condition for rewriting systems*, Theoret. Comput. Sci. **131** (1994), no. 2, 271–294.
- [114] Richard P. Stanley, *Enumerative combinatorics. Vol. 1. With a foreword by Gian-Carlo Rota. Corrected reprint of the 1986 hardback edition.*, corrected reprint of the 1986 hardback edition ed., Cambridge: Cambridge University Press, 1999.
- [115] Ross Street, *Limits indexed by category-valued 2-functors*, J. Pure Appl. Algebra **8** (1976), no. 2, 149–181.
- [116] ———, *The algebra of oriented simplexes*, J. Pure Appl. Algebra **49** (1987), no. 3, 283–335.
- [117] Sheila Sundaram, *Orthogonal tableaux and an insertion algorithm for $SO(2n + 1)$* , J. Combin. Theory Ser. A **53** (1990), no. 2, 239–256.
- [118] Terese, *Term rewriting systems*, Cambridge Tracts in Theoretical Computer Science, vol. 55, Cambridge University Press, 2003.
- [119] Glanffrwd P. Thomas, *On Schensted's construction and the multiplication of Schur functions.*, Adv. Math. **30** (1978), 8–32.
- [120] Axel Thue, *Probleme über Veränderungen von Zeichenreihen nach gegebenen Regeln.*, Kristiania Vidensk. Selsk. Skr. **10** (1914), 493–524.
- [121] Heinrich Tietze, *Über die topologischen Invarianten mehrdimensionaler Mannigfaltigkeiten*, Monatsh. Math. Phys. **19** (1908), no. 1, 1–118.
- [122] Victor A. Ufnarovskij, *Combinatorial and asymptotic methods in algebra*, Algebra, VI, Encyclopaedia Math. Sci., vol. 57, Springer, Berlin, 1995, pp. 1–196.
- [123] Marc A.A. van Leeuwen, *The Littlewood-Richardson rule and related combinatorics.*, Interaction of combinatorics and representation theory, Tokyo: Mathematical Society of Japan, 2001, pp. 95–145.
- [124] Alfred Young, *On Quantitative Substitutional Analysis*, Proc. London Math. Soc. **S2-28**, no. 1, 255.