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The study of cohomogeneity one three dimensional Einstein universe

A thesis submitted in partial fulfillment for the
degree of Doctor of Philosophy

in Mathematics

Laboratoire de Mathématiques d'Avignon, Université d'Avignon et des
pays de Vaucluse

Department of Mathematics, University of Zanjan

Supervisors: Prof. Thierry Barbot and Parviz Ahmadi

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Introduction

Conformal structure

A *semi-Riemannian metric* g on a smooth manifold M is a smooth symmetric nondegenerate $(0, 2)$ -tensor field on M of signature (p, q) . The pair (M, g) is called *Riemannian* (resp. *Lorentzian*) if $p = 0$ (resp. $p = 1$ and $q > 0$). For a semi-Riemannian manifold (M, g) one can consider the *conformal class* $[g]$ which is the set of all semi-Riemannian metrics on M of the form $e^f g$ where f is a smooth function on M . Given such conformal class of metrics is called a *semi-Riemannian conformal structure* on M . A diffeomorphism $\varphi : M \rightarrow M$ is called a *conformal map* if it preserves the conformal structure on M , i.e., for a metric g (hence for any) in the conformal class $[g]$ there exists a smooth function f such that the pullback metric φ^*g coincides with $e^f g$. The group of conformal transformations on $(M, [g])$ is denoted by $\text{Conf}(M, [g])$. When the metric class $[g]$ is understood we simply denote the conformal group by $\text{Conf}(M)$. Indeed, if $\dim M \geq 3$, then $\text{Conf}(M)$ is a Lie group ([31]).

The concept of geodesic in a conformal class $[g]$ is not relevant, since a geodesic γ respect to a metric $g \in [g]$ may fails to be a geodesic respect to other metrics in $[g]$. However, if $[g]$ is an indefinite structure and γ is a lightlike geodesic respect to g , then it is a (unparametrized) geodesic respect to all the metrics in $[g]$. Indeed two indefinite semi-Riemannian metrics on a manifold lie in the same conformal class if and only if they determine the same lightcone cone in the tangent space of every point.

Two semi-Riemannian manifolds (M, g) and (N, h) are said to be *conformally equivalent* if there exists a conformal map φ from M onto N such that φ^*h belongs to the conformal class $[g]$. A semi-Riemannian conformal structure $[g]$ on a manifold M is called *essential* if the conformal group $\text{Conf}(M)$ does not preserves any metric in the class $[g]$. A conformal structure is called *inessential* if it is not essential. Indeed, given an inessential structure $(M, [g])$, the conformal group $\text{Conf}(M)$ coincides with the isometry group of (M, g) , for all $g \in [g]$. Roughly speaking, the conformal group of an essential structure $[g]$ is strictly bigger than the isometry group of any metric $g \in [g]$.

A semi-Riemannian manifold (M, g) of signature (p, q) is called *locally conformally flat* if for every point $p \in M$ there exists an open neighborhood U around p , and a conformal map $f : U \rightarrow \mathbb{R}^{p,q}$.

Riemannian geometry

Let $\mathbb{R}^{1,n+1} = (\mathbb{R}^{n+2}, q)$ be a Lorentzian vector space. The nullcone $\mathfrak{N}^{1,n+1}$ of $\mathbb{R}^{1,n+1}$ is a degenerate hypersurface. The image of nullcone by the natural projection $\mathbb{P} : \mathbb{R}^{1,n+1} \setminus \{0\} \rightarrow \mathbb{RP}^{n+1}$ is homeomorphic to the n -sphere \mathbb{S}^n . The degenerate metric on $\mathfrak{N}^{1,n+1}$ admits a canonical Riemannian conformal structure

$[g]$ on \mathbb{S}^n . The metric class $[g]$ contains the usual round metric on \mathbb{S}^n of radius 1. The Lorentz group $O(1, n + 1)$ leaves the metric class $[g]$. Indeed, $PO(1, n + 1)$ is the conformal group of $(\mathbb{S}^n, [g])$.

The standard Riemannian sphere \mathbb{S}^n is the central object of the Riemannian geometry. It conformally compactifies the Euclidean space

$$\mathbb{S}^n = \mathbb{E}^n \cup \{\infty\}.$$

Theorem 0.1. (*Liouville's Theorem*) Let $U, V \subset \mathbb{E}^n$ ($n \geq 3$) be two connected nonempty open subsets, and $f : U \rightarrow V$ be a conformal map. Then f extends to a global conformal map on \mathbb{E}^n .

The stereographic projection from the Euclidean space \mathbb{E}^n into \mathbb{S}^n is a conformal map. Hence, by Liouville's theorem, one can conclude that, a conformal map between two open non-empty connected subsets of \mathbb{S}^n ($n \geq 3$) is the restriction of an element of $PO(1, n + 1)$. Therefore, every locally conformally flat Riemannian manifold of dimension greater than or equal to 3 admits a $(PO(1, n + 1), \mathbb{S}^n)$ -structure.

The image of the domain $\{q < 0\} \subset \mathbb{R}^{1, n + 1}$ by \mathbb{P} is conformally equivalent to the hyperbolic space $\mathbb{H}^{n + 1}$. The boundary of $\mathbb{H}^{n + 1}$ in $\mathbb{RP}^{n + 1}$ is exactly \mathbb{S}^n . Hence, \mathbb{S}^n can be seen as the conformal boundary in infinity of hyperbolic space.

In [16], J. Ferrand proved that if $[g]$ is an essential Riemannian structure on a manifold M of dimension $n \geq 2$ then $(M, [g])$ is conformally equivalent to the round n -sphere \mathbb{S}^n or the Euclidean space \mathbb{E}^n . However, if the metric g is not definite, there is no such classification of essential structures. Even if we restrict ourselves to the compact manifolds or Lorentzian metrics, there are infinitely many examples of essential structures, see for example [17, 3].

Lorentzian geometry

In contrast with Riemannian geometry and the sphere \mathbb{S}^n , the $(n + 1)$ -dimensional *Einstein universe* $\mathbb{E}in^{1, n}$ ($n \geq 1$) is the Lorentzian analogue. It is a compact manifold equipped with a Lorentzian conformal structure. Indeed, up to a double cover $\widehat{\mathbb{E}in}^{1, n}$ Einstein universe is conformally equivalent to the direct product $(\mathbb{S}^1 \times \mathbb{S}^n, -dt^2 + ds^2)$ where dt^2 and ds^2 are the usual round metrics on the spheres \mathbb{S}^1 and \mathbb{S}^n of radius one, respectively. There is a Lorentzian version of Liouville's Theorem.

Theorem 0.2. (*Liouville's Theorem [18, Theorem 4.4]*) Let $U, V \subset \mathbb{E}in^{1, n}$ be non-empty connected open subsets, and $f : U \rightarrow V$ be a conformal map. Then f extends to a unique global conformal map on $\mathbb{E}in^{1, n}$.

Therefore, every locally conformally flat Lorentzian manifold of dimension $n + 1 \geq 2$ admits a $(\text{Conf}(\mathbb{E}in^{1, n}), \mathbb{E}in^{1, n})$ -structure. In particular, the Lorentzian model spaces of constant sectional curvatures c , namely, the Minkowski space $\mathbb{E}^{1, n}$ for $c = 0$, the Anti de-Sitter space $\text{AdS}^{1, n}$ for $c = -1$, and the de-Sitter space $\text{dS}^{1, n}$ for $c = 1$, all are conformally equivalent to some specific open dense subsets of $\mathbb{E}in^{1, n}$.

V. Pécastaing in [40] proved that, if M is a compact manifold of dimension ≥ 3 equipped with a Lorentzian conformal structure $[g]$, and G is a connected Lie group, locally isomorphic to $SL(2, \mathbb{R})$, which acts on $(M, [g])$ essentially, then $(M, [g])$ is locally conformally flat.

Let $\mathbb{R}^{2, n + 1}$ be the $(n + 3)$ -dimensional real vector space endowed with a quadratic form q of signature $(2, n + 1)$. The nullcone $\mathfrak{N}^{2, n + 1}$ of $\mathbb{R}^{2, n + 1}$ is the set of non-zero vectors $v \in \mathbb{R}^{2, n + 1}$ with $q(v) = 0$. The nullcone $\mathfrak{N}^{2, n + 1}$ is a degenerate hypersurface of $\mathbb{R}^{2, n + 1}$. The $(n + 1)$ -dimensional Einstein universe

$\mathbb{E}in^{1,n}$ is the projectivization of the nullcone $\mathfrak{N}^{2,n+1}$ via $\mathbb{P} : \mathbb{R}^{2,n+1} \setminus \{0\} \rightarrow \mathbb{RP}^{n+2}$. It is a compact submanifold of \mathbb{RP}^{n+2} , and the degenerate metric on $\mathfrak{N}^{2,n+1}$ induces a Lorentzian conformal structure on $\mathbb{E}in^{1,n}$. The group $O(2, n+1)$ -group of the linear isometries of $\mathbb{R}^{2,n+1}$ - acts on Einstein universe conformally. Indeed, one can see the full conformal group $\text{Conf}(\mathbb{E}in^{1,n})$ is $PO(2, n+1)$.

A (inextendable) lightlike geodesic in Einstein universe $\mathbb{E}in^{1,n}$ is called a *photon*. In fact, photons are the images of totally isotropic linear 2-planes in $\mathbb{R}^{2,n+1}$ by \mathbb{P} . The *lightcone* $L(p)$ with vertex $p \in \mathbb{E}in^{1,n}$ is the set of photons through a point p . The lightcone $L(p)$ is a singular degenerate hypersurface. The only singular point is the vertex p and $L(p) \setminus \{p\}$ is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{R}$. The complement of a lightcone $L(p)$ in $\mathbb{E}in^{1,n}$ is an open dense subset called the *Minkowski patch* at p . Every Minkowski patch in $\mathbb{E}in^{1,n}$ is conformally equivalent to the Minkowski space $\mathbb{E}^{1,n}$.

A Linear isometry $\mathbb{R}^{2,n} \rightarrow \mathbb{R}^{2,n+1}$ ($n \geq 2$) embeds naturally a copy of $\mathbb{E}in^{1,n-1}$ in $\mathbb{E}in^{1,n}$ called an *Einstein hypersphere*. The complement of an Einstein hypersphere in $\mathbb{E}in^{1,n}$ is an open dense subset conformally equivalent to the Anti de-Sitter space $\text{AdS}^{1,n}$. On the other hand, a linear isometry $\mathbb{R}^{1,n+1} \rightarrow \mathbb{R}^{2,n+1}$ embeds naturally a copy conformal sphere \mathbb{S}^n in $\mathbb{E}in^{1,n}$ called a *spacelike hypersphere*. The complement of \mathbb{S}^n in $\mathbb{E}in^{1,n}$ is an open subset conformally equivalent to the de-Sitter space $\text{dS}^{1,n}$.

The projectivization of domain $\{q < 0\} \subset \mathbb{R}^{2,n+1}$ via \mathbb{P} is an open subset of \mathbb{RP}^{n+2} , and it is (up to a double cover) conformally equivalent to the model Anti de-Sitter space $\text{AdS}^{1,n+1} = \mathfrak{q}^{-1}(-1) \subset \mathbb{R}^{2,n+1}$. Obviously, the boundary of $\mathbb{P}(q < 0)$ in \mathbb{RP}^{n+2} is $\mathbb{E}in^{1,n}$. Hence, one can consider the $(n+1)$ -dimensional Einstein universe as the conformal boundary of the $(n+2)$ -dimensional Anti de-Sitter space. Indeed, this is the Lorentzian analogy of the hyperbolic space \mathbb{H}^{n+1} and its conformal boundary \mathbb{S}^n .

For $n \geq 2$, the universal covering space $\widetilde{\mathbb{E}in}^{1,n}$ of Einstein universe $\mathbb{E}in^{1,n}$ is conformally equivalent to the direct product $(\mathbb{R} \times \mathbb{S}^n, -dt^2 + ds^2)$ where dt^2 and ds^2 are the canonical Riemannian metrics on \mathbb{R} and \mathbb{S}^n , respectively. The group of conformal transformations of $\widetilde{\mathbb{E}in}^{1,n}$ is locally isomorphic to $\text{Conf}(\mathbb{E}in^{1,n})$ (see [18, Proposition 4.5]). The four-dimensional (simply connected) Einstein universe $\widetilde{\mathbb{E}in}^{1,3}$ was the first cosmological model for our universe proposed by A. Einstein soon after the birth of General Relativity.

The typical example of essential Lorentzian structures is the Minkowski space $\mathbb{E}^{1,n}$. However, Einstein universe $\mathbb{E}in^{1,n}$, its double covering $\widetilde{\mathbb{E}in}^{1,n}$, and its universal covering space $\widehat{\mathbb{E}in}^{1,n}$ are the other examples. On the other hand, the model spaces of non-zero constant sectional curvature, namely, Anti de-Sitter space $\text{AdS}^{1,n}$ and de-Sitter space $\text{dS}^{1,n}$ are examples of inessential structures.

Cohomogeneity one actions: a brief history

Felix Klein is known for his work on the connections between geometry and group theory. By his 1872 Erlangen Program, geometries are classified by their underlying symmetry groups. According to his approach, a geometry is a G -space M , where G is a group of transformations of M . This makes a link between geometry and algebra. The most natural case occurs when the group G acts on M transitively. In this case M is called a *homogeneous G -space*. For instance Euclidean, affine and projective geometries are homogeneous spaces.

One special case of non-transitive actions of transformation groups on manifolds is when the action has an orbit of codimension one, the so called *cohomogeneity one* action. The concept of a cohomogeneity one action on a manifold M was introduced by P.S. Mostert in his 1956 paper [35]. The key hypothesis was the compactness of the acting Lie group in the paper. He assumed that the acting Lie group G

is compact and determined the orbit space up to homeomorphism. More precisely, he proved that by the cohomogeneity one action of a compact Lie group G on a manifold M the orbit space M/G is homeomorphic to \mathbb{R} , \mathbb{S}^1 , $[0, 1]$, or $[0, 1)$. In the general case, in [8] B. Bergery showed that if a Lie group acts on a manifold properly and with cohomogeneity one, then the orbit space M/G is homeomorphic to one of the above spaces. Regarding to this classification, for a point $p \in M$ the orbit $G(p)$ is called *principal* (resp. *singular*) if its corresponding point in the orbit space M/G is an internal (resp. boundary) point.

If G is a compact Lie group which acts on a smooth manifold M , then there exists a complete G -invariant Riemannian metric on M . The compactness assumption is not necessary condition for existence of the metric. A result by D. Alekseevsky in [2] says that, for an arbitrary Lie group G , there is a complete G -invariant Riemannian metric on M if and only if the action of G on M is proper. This theorem provides a link between proper actions and Riemannian G -manifolds.

Cohomogeneity one Riemannian manifolds have been studied by many mathematicians (see, e.g., [1, 8, 21, 22, 23, 33, 35, 36, 39, 41, 43, 44, 45]). The subject is still an active one. The common hypothesis in the theory is that the acting group is closed in the full isometry group of the Riemannian manifold and the action is isometrically. When the metric is indefinite, this assumption in general does not imply that the action is proper, so the study becomes much more complicated. Also, some of the results and techniques of definite metrics fails for indefinite metrics. It is shown in [15] that for a Riemannian Einstein manifold (M, g) the PDE $Ric(g) = \lambda \cdot g$ ($\lambda \in \mathbb{R}$ constant) becomes an ODE if there exists a Lie group $G \subset Iso(M)$ acting properly and with cohomogeneity one.

The natural way to study a cohomogeneity one semi-Riemannian manifold M is to determine the acting group in the full isometry group $Iso(M)$, up to conjugacy, since the actions of two subgroups of $Iso(M)$ one conjugate to the other with an element in $Iso(M)$ admit almost the same orbits in M . This has been done for space forms in some special cases (see [4], [5] and [6]). There is another approximation of acting groups on a manifold: the actions of two Lie subgroups $G, H \subset Iso(M)$ are called *orbitally equivalent* to each other if there exists an element $g \in Iso(M)$ such that for all $p \in M$, the orbits $G(p)$ and $H(g(p))$ coincide. It is obvious that, the action of two subgroup of $Iso(M)$ one conjugate to the other are orbitally equivalent, but the converse is not necessarily true (for example, the action of additive group $\mathbb{R}^{1,n}$ on the Minkowski space $\mathbb{E}^{1,n}$ by translations is orbitally equivalent to the action of $Iso(\mathbb{E}^{1,n})$, but evidently they are not conjugate to each other).

Cohomogeneity one three-dimensional Einstein universe

In this thesis, we propose to study conformal actions of cohomogeneity one on the three-dimensional Einstein universe $\mathbb{E}^{1,2}$. Our strategy in the study is to determine the representation of the acting group in the conformal Lie group of the Einstein universe up to conjugacy. Also, we describe the topology and the causal character of the orbits induced by cohomogeneity one actions in $\mathbb{E}^{1,2}$. As a matter fact, it follows from our study that a 2-dimensional orbit in $\mathbb{E}^{1,2}$ induced by the conformal action of a connected Lie group is homeomorphic to \mathbb{R}^2 , $\mathbb{S}^1 \times \mathbb{R}$, \mathbb{S}^2 , or $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$.

The group of conformal transformations of $\mathbb{E}^{1,2}$ is isomorphic to $O(2, 3)$. The identity component of $O(2, 3)$ is $SO_o(2, 3)$ which acts on $\mathbb{E}^{1,2}$ transitively. Hence, a cohomogeneity one action on $\mathbb{E}^{1,2}$ comes from a proper subgroup of $Conf(\mathbb{E}^{1,2})$.

The proper actions are significant and they deserve to be studied first. Let G be a connected Lie subgroup of $\text{Conf}(\mathbb{E}\text{in}^{1,2})$. Then G acts on $\mathbb{E}\text{in}^{1,2}$ properly if and only if it is compact, since the Einstein universe is compact. Hence, up to conjugacy, G is a subgroup of the connected maximal compact subgroup of $SO_o(2, 3)$, namely $SO(2) \times SO(3)$. This group preserves a 2-dimensional linear subspace of $\mathbb{R}^{2,3}$ of signature $(2, 0)$. In Chapter 2, we show that up to conjugacy there are exactly two cohomogeneity one proper actions on $\mathbb{E}\text{in}^{1,2}$, namely, $SO(3)$ and $SO(2) \times SO(2)$. The action of $SO(3)$ on Einstein universe $\mathbb{E}\text{in}^{1,2}$ admits a codimension one foliation on which each leaf is a spacelike hypersphere. On the other hand, $SO(2) \times SO(2)$ admits a unique 1-dimensional timelike orbit homeomorphic to $\mathbb{R}\mathbb{P}^1$, and on the complement, admits a codimension one foliation one which every leaf is conformally equivalent to the 2-dimensional Einstein universe $\mathbb{E}\text{in}^{1,1}$, and exactly one of them is an Einstein hypersphere.

Rolling out the proper case already discussed above, in the non-proper case, we deal with subgroup of some known geometric groups such as: $(\mathbb{R}^* \times O(1, 2)) \ltimes \mathbb{R}^{1,2}$ group of conformal transformations on the 3-dimensional Minkowski space $\mathbb{E}^{1,2}$, $O(2, 2)$ group of isometries of 3-dimensional Anti de-Sitter space $\text{AdS}^{1,2}$, and $O(1, 3)$ group of isometries of the 3-dimensional de-Sitter space $\text{dS}^{1,2}$. In particular, we consider various representations of *Möbius group* $\text{PSL}(2, \mathbb{R})$ and its subgroups in $O(2, 3)$. In most of the cases, we determine the Lie subgroups acting with cohomogeneity one on $\mathbb{E}\text{in}^{1,2}$ considering their corresponding subalgebras in the Lie algebra $\mathfrak{so}(2, 3) = \text{Lie}(O(2, 3))$.

In order to study the non-proper cohomogeneity one actions we have the following definition.

Definition 0.3. *The action of a Lie subgroup $G \subset \text{Conf}(\mathbb{E}\text{in}^{1,2})$ is called irreducible if G preserves no non-trivial linear subspace of $\mathbb{R}^{2,3}$, and it is called reducible if it is not irreducible.*

Theorem 0.4. [12] *Let G be a connected Lie subgroup of $SO_o(2, n)$ which acts on $\mathbb{R}^{2,n}$ irreducibly. Then G is conjugate to of the following:*

- i: for arbitrary $n \geq 1$: $SO_o(2, n)$.
- ii: for $n = 2p$ even: $U(1, p)$, $SU(1, p)$ or $S^1 \cdot SO_o(1, p)$ if $p > 1$,
- iii: for $n = 3$: $SO_o(1, 2) \subset SO_o(2, 3)$.

Thanks to above theorem, we have the following result about the irreducible cohomogeneity one actions on $\mathbb{E}\text{in}^{1,2}$.

Theorem 0.5. *Let $G \subset \text{Conf}(\mathbb{E}\text{in}^{1,2})$ be a connected Lie subgroup which acts on $\mathbb{E}\text{in}^{1,2}$ irreducibly and with cohomogeneity one. Then G is conjugate to $SO_o(1, 2) \simeq \text{PSL}(2, \mathbb{R})$.*

In Chapter 3, we prove Theorem 0.5. Indeed, this action comes from the known irreducible action of $\text{PSL}(2, \mathbb{R})$ on the vector space $\mathbb{R}_4[X, Y]$ of homogeneous polynomials of degree 4 in two variables X and Y . We show that, this action admits in $\mathbb{E}\text{in}^{1,2}$ three orbits: a 1-dimensional lightlike orbit homeomorphic to $\mathbb{R}\mathbb{P}^1$ which is not a photon, a 2-dimensional degenerate orbit homeomorphic to \mathbb{R}^2 , and an open orbit on which $\text{PSL}(2, \mathbb{R})$ acts freely. Also, we describe the orbits induced by this action on the complement of Einstein universe $\mathbb{E}\text{in}^{1,2}$ in $\mathbb{R}\mathbb{P}^4$.

According to Theorem 0.4, one way to classify the cohomogeneity one reducible actions is to consider the stabilizer of various (in dimension and signature) linear subspaces of $\mathbb{R}^{2,3}$ by the action of $SO_o(2, 3)$. The following theorem shows that considering the actions preserving a 1-dimensional linear subspace of $\mathbb{R}^{2,3}$ is enough (even the proper actions are included in the following).

Theorem 0.6. *Let G be a connected Lie subgroup of $\text{Conf}(\mathbb{E}\text{in}^{1,2})$ which acts on $\mathbb{E}\text{in}^{1,2}$ reducibly and with cohomogeneity one. Then G fixes a point in the projective space \mathbb{RP}^4 .*

By Theorem 0.6 which will be proved in Chapter 7, every reducible cohomogeneity one action G on $\mathbb{E}\text{in}^{1,2}$ preserves a line $\ell \leq \mathbb{R}^{2,3}$. The line ℓ can be lightlike, spacelike or timelike.

The case when G preserves a lightlike line ℓ is the richest one, i.e., actions fixing a point in Einstein universe, which are fully studied in Chapter 4. By the action of $O(2, 3)$, the stabilizer of a point in Einstein universe is isomorphic to the group of conformal transformations of the Minkowski space. More precisely, if G fixes a point $p \in \mathbb{E}\text{in}^{1,2}$, then it preserves the lightcone $L(p)$ and its corresponding Minkowski patch. Hence, the action of G on the Minkowski patch is equivalent to the action of a Lie subgroup of $\text{Conf}(\mathbb{E}^{1,2}) = (\mathbb{R}^* \times O(1, 2)) \ltimes \mathbb{R}^{1,2}$ on Minkowski space $\mathbb{E}^{1,2}$. We apply the adjoint action of $\text{Conf}(\mathbb{E}^{1,2})$ on its Lie algebra $(\mathbb{R} \oplus \mathfrak{so}(1, 2)) \oplus_{\theta} \mathbb{R}^{1,2}$, and then, we determine all the Lie subalgebras \mathfrak{g} of $(\mathbb{R} \oplus \mathfrak{so}(1, 2)) \oplus_{\theta} \mathbb{R}^{1,2}$ with $\dim \mathfrak{g} \geq 2$, up to conjugacy. This leads to the classification of the connected Lie subgroups of $\text{Conf}(\mathbb{E}^{1,2})$ with dimension greater than or equal to 2, up to conjugacy (Theorem 4.6). Actually, there are infinitely many subgroups of $\text{Conf}(\mathbb{E}^{1,2})$ with $\dim \geq 2$, up to conjugacy. All those subgroups act on Einstein universe $\mathbb{E}\text{in}^{1,2}$ with cohomogeneity one except two of them. Even, up to orbit equivalence, still the number of distinguished actions is infinite.

In Chapter 5, we study the actions preserving a non-degenerate line $\ell \leq \mathbb{R}^{2,3}$. Indeed, in this chapter we only consider the cohomogeneity one actions on $\mathbb{E}\text{in}^{1,2}$ with no fixed point in $\mathbb{E}\text{in}^{1,2}$.

- If G preserves a spacelike line ℓ , then it preserves an Einstein hypersphere $\mathbb{E}\text{in}^{1,1} \subset \mathbb{E}\text{in}^{1,2}$, and its complement $\mathbb{E}\text{in}^{1,2} \setminus \mathbb{E}\text{in}^{1,1}$ which is conformally equivalent to de-Sitter space $dS^{1,2}$. Since $\text{AdS}^{1,2}$ is an inessential space, the action of G is conformally equivalent to the natural action of a Lie subgroup of the group of isometries of $\text{AdS}^{1,2}$, namely, $O(2, 2)$. We show that, up to orbit equivalency there are exactly seven connected Lie subgroups which preserve an Einstein hypersphere $\mathbb{E}\text{in}^{1,1}$ in $\mathbb{E}\text{in}^{1,2}$, act on $\mathbb{E}\text{in}^{1,2}$ non-properly, with cohomogeneity one, and with no fixed point in $\mathbb{E}\text{in}^{1,2}$.
- If G preserves a timelike line, then it preserves a spacelike hypersphere $\mathbb{S}^2 \subset \mathbb{E}\text{in}^{1,2}$, and its complement $\mathbb{E}\text{in}^{1,2} \setminus \mathbb{S}^2$ which is conformally equivalent to de-Sitter space $dS^{1,2}$. Again, since $dS^{1,2}$ is an inessential space, the action of G is conformally equivalent to the action a Lie subgroup of the group of isometries of $dS^{1,2}$, namely, $O(1, 3)$. It is shown that, up to conjugacy, there is only one connected Lie subgroup which preserves a spacelike hypersphere in $\mathbb{E}\text{in}^{1,2}$, acts non-properly and with cohomogeneity one, and fixes no point in $\mathbb{E}\text{in}^{1,2}$.

Actions preserving a photon in $\mathbb{E}\text{in}^{1,2}$ are very interesting. In Chapter 6, we prove a theorem which states that a cohomogeneity one action on $\mathbb{E}\text{in}^{1,2}$ which preserves a photon, fixes a point in \mathbb{RP}^4 . This theorem will play a key role in proof of Theorem 0.6 .

Finally, in Chapter 7, we prove Theorem 0.6. This completes the classification of the cohomogeneity one actions on $\mathbb{E}\text{in}^{1,2}$.

Appendix A is devoted to the technical proof of Theorem 4.6, which classifies all the connected Lie subgroups of $\text{Conf}(\mathbb{E}^{1,2})$ of dimension greater than or equal to 2. In Appendix B, we give a list of 1-parameter subgroups of $\text{Conf}(\mathbb{E}^{1,2})$ which are used in Chapter 4, where we will show that any Lie subgroup of $\text{Conf}(\mathbb{E}^{1,2})$ of dimension ≥ 2 contains either one of these 1-parameter subgroups or admits a non-trivial translation part.

Résumé de la thèse

Structure conforme

Une *métrique semi-Riemannienne* g sur une variété lisse M est un champ tensoriel lisse de type $(0, 2)$ qui est symétrique, non dégénéré, de signature constante (p, q) . La paire (M, g) est dite *Riemannienne* (resp. *Lorentzienne*) si $p = 0$ (resp. $p = 1$ et $q > 0$). Pour une variété semi-Riemannienne (M, g) , sa classe conforme, notée $[g]$, est l'ensemble de toutes les métriques pseudo-Riemanniennes sur M de la forme $e^f g$ où f est une fonction lisse sur M . Un difféomorphisme $\varphi : M \rightarrow M$ est une *application conforme* si il préserve la classe conforme $[g]$, i.e., si pour un représentant g (et donc pour tous) la métrique tirée en arrière φ^*g est dans la classe conforme $[g]$. Le groupe des transformations conformes de $(M, [g])$ est noté $\text{Conf}(M, [g])$, ou simplement $\text{Conf}(M)$ lorsqu'aucune confusion n'est à craindre. Si $\dim M \geq 3$, alors $\text{Conf}(M)$ est un groupe de Lie ([31]).

A priori, la notion de géodésique pour une structure conforme n'a pas de sens, cependant, les géodésiques isotropes, i.e. celles dont les vecteurs tangents sont isotropes, ne dépendent pas du représentant de la classe conforme à reparamétrisation près.

Deux variétés semi-Riemanniennes (M, g) et (N, h) sont *conformément équivalentes* si il existe un difféomorphisme φ de M vers N qui soit une application conforme, i.e. tel que φ^*h soit dans la classe conforme $[g]$. La structure conforme $[g]$ est dite *essentielle* si le groupe conforme $\text{Conf}(M)$ ne préserve aucune métrique dans la classe $[g]$. Elle est dite inessentielle sinon.

Une variété semi-Riemannienne (M, g) de signature (p, q) est *localement conformément plate* si pour tout point $p \in M$ il existe un voisinage ouvert U de p et une application conforme de cet ouvert dans un ouvert de $\mathbb{R}^{p,q}$, où $\mathbb{R}^{p,q}$ désigne un espace vectoriel de dimension $p + q$ muni d'une forme quadratique de signature (p, q) .

Le cas de la géométrie Riemannienne

Soit $\mathbb{R}^{1,n+1} = (\mathbb{R}^{n+2}, q)$ un espace vectoriel muni d'une forme Lorentzienne de signature $(1, n)$. Le cône lumière $\mathfrak{N}^{1,n+1}$ de $\mathbb{R}^{1,n+1}$ formé des vecteurs isotropes est une hypersurface singulière dégénérée. Sa projection (une fois l'origine enlevée) dans l'espace projectif $\mathbb{P} : \mathbb{R}^{1,n+1} \setminus \{0\} \rightarrow \mathbb{RP}^{n+1}$ est homéomorphe à la sphère \mathbb{S}^n . La métrique dégénérée sur $\mathfrak{N}^{1,n+1}$ induit une structure conforme Riemannienne essentielle $[g]$ sur \mathbb{S}^n , qui contient la métrique ronde usuelle. Si $n \geq 2$, le groupe $PO(1, n + 1)$ est précisément le groupe des transformations conformes de $(\mathbb{S}^n, [g])$.

La sphère Riemannienne standard \mathbb{S}^n un objet central en géométrie Riemannienne. C'est le compactifié

conforme de l'espace Euclidien

$$\mathbb{S}^n = \mathbb{E}^n \cup \{\infty\}.$$

Théorème 0.1. (Théorème de Liouville) Soient $U, V \subset \mathbb{S}^n$ ($n \geq 3$) deux ouverts connexes non vides, et $f : U \rightarrow V$ une application conforme. Alors f s'étend en une application conforme de \mathbb{S}^n .

Il s'en suit que toute variété conformément plate de dimension supérieure ou égale à 3 admet une $(PO(1, n+1), \mathbb{S}^n)$ -structure.

L'image du domaine $\{q < 0\} \subset \mathbb{R}^{1, n+1}$ par \mathbb{P} est conformément équivalente à l'espace hyperbolique \mathbb{H}^{n+1} . Le bord de \mathbb{H}^{n+1} dans $\mathbb{R}\mathbb{P}^{n+1}$ est précisément \mathbb{S}^n . Ainsi, \mathbb{S}^n peut être vue comme le bord conforme de l'espace hyperbolique.

Dans [16], J. Ferrand a montré que si $[g]$ est une classe conforme essentielle sur une variété M de dimension $n \geq 2$, alors $(M, [g])$ est conformément équivalent à la sphère ronde \mathbb{S}^n ou à l'espace Euclidien \mathbb{E}^n . Cependant, la classification des structures conformes essentielles pour les autres signatures n'est pas complète, même dans le cas des variétés compactes Lorentziennes où il existe de nombreux exemples de telles structures, voir par exemple [17, 3].

Géométrie Lorentzienne

L'analogue Lorentzien de la sphère conforme \mathbb{S}^n est l'univers d'Einstein $\mathbb{E}in^{1, n}$ de dimension $n+1$ ($n \geq 1$) qui, comme dans le cas Riemannien, est le projectivisé dans $\mathbb{R}\mathbb{P}^{n+2}$ du lieu des vecteurs isotropes d'une forme quadratique sur \mathbb{R}^{n+3} , mais cette fois de signature $(2, n+1)$. Son groupe de transformation conforme est alors $PO(2, n+1)$.

Son revêtement double, noté $\widehat{\mathbb{E}in}^{1, n}$, est conformément équivalent au produit $(\mathbb{S}^1 \times \mathbb{S}^n, -dt^2 + ds^2)$ où dt^2 et ds^2 sont les métriques rondes usuelles sur les sphères \mathbb{S}^1 et \mathbb{S}^n de même diamètre 1. L'analogue Lorentzien du Théorème de Liouville reste vrai:

Théorème 0.2. (Théorème de Liouville, cas Lorentzien [18, Theorem 4.4]) Soient $U, V \subset \mathbb{E}in^{1, n}$ deux ouverts connexes non vides, et soit $f : U \rightarrow V$ une application conforme. Alors, si $n \geq 2$, l'application f s'étend de manière unique en une transformation conforme globale de $\mathbb{E}in^{1, n}$.

Ainsi, toute variété Lorentzienne conformément plate de dimension $1+n \geq 3$ admet une $(\text{Conf}(\mathbb{E}in^{1, n}), \mathbb{E}in^{1, n})$ -structure. En particulier, les variétés modèles Lorentziennes à courbure constante c , qui sont l'espace de Minkowski $\mathbb{R}^{1, n}$ pour $c = 0$, l'espace Anti de-Sitter $\text{AdS}^{1, n}$ pour $c = -1$, et l'espace de-Sitter $\text{dS}^{1, n}$ pour $c = 1$, sont chacune conformément équivalente à un domaine ouvert spécifique de $\mathbb{E}in^{1, n}$.

Dans [40] V. Pecastaing montre que, si $(M, [g])$ est une variété Lorentzienne conforme compacte de dimension ≥ 3 dont le groupe conforme contient un sous-groupe de Lie localement isomorphe à $SL(2, \mathbb{R})$ agissant de manière essentielle, alors $(M, [g])$ est conformément plate.

Une géodésique de type lumière (i.e. isotrope) et inextensible de l'univers d'Einstein est appelée *photon*. Ce sont en fait les projections des 2-plans totalement isotropes de $\mathbb{R}^{2, n+1}$ par \mathbb{P} . Le *cône lumière* $L(p)$ de sommet $p \in \mathbb{E}in^{1, n}$ est l'union des photons contenant p . C'est une hypersurface singulière dégénérée. Son unique point singulier est le point p , et $L(p) \setminus \{p\}$ homéomorphe à $\mathbb{S}^{n-1} \times \mathbb{R}$. Le complémentaire de $L(p)$ dans $\mathbb{E}in^{1, n}$ est un ouvert dense conformément équivalent à l'espace de Minkowski $\mathbb{E}^{1, n}$. Il est appelé *carte de Minkowski* au point p .

Toute isométrie linéaire $\mathbb{R}^{2,n} \rightarrow \mathbb{R}^{2,n+1}$ ($n \geq 2$) plonge naturellement $\mathbb{E}in^{1,n-1}$ en une copie dans $\mathbb{E}in^{1,n}$ appelé *hypersphère d'Einstein*. Le complémentaire d'une telle hypersphère est un ouvert dense conformément équivalent à l'espace Anti de-Sitter $AdS^{1,n}$. On peut ainsi aussi considérer l'univers d'Einstein comme étant le bord conforme de l'espace Anti de-Sitter.

De même, toute isométrie linéaire $\mathbb{R}^{1,n+1} \rightarrow \mathbb{R}^{2,n+1}$ induit une réalisation de la sphère conforme \mathbb{S}^n dans $\mathbb{E}in^{1,n}$ appelée *hypersphère de type espace*. Son complément dans $\mathbb{E}in^{1,n}$ est un ouvert dense conformément équivalent à l'espace de-Sitter $dS^{1,n}$.

Pour $n \geq 2$, le revêtement universel $\widetilde{\mathbb{E}in}^{1,n}$ de l'univers d'Einstein $\mathbb{E}in^{1,n}$ est conformément équivalent au produit direct $(\mathbb{R} \times \mathbb{S}^n, -dt^2 + ds^2)$ où dt^2 et ds^2 sont les métriques Riemanniennes usuelles sur \mathbb{R} et \mathbb{S}^n . Le groupe des transformations conformes de $\widetilde{\mathbb{E}in}^{1,n}$ est localement isomorphe à $\text{Conf}(\mathbb{E}in^{1,n})$ (voir [18, Proposition 4.5]).

Les exemples typiques de structures Lorentziennes conformes essentielles sont l'espace de Minkowski $\mathbb{E}^{1,n}$ ainsi que l'univers d'Einstein.

Actions de cohomogénéité un: une histoire brève

Felix Klein est notamment connu pour son travail sur le lien entre géométrie et théorie des groupes. Dans son Programme d'Erlangen (1872), les géométries sont classifiées selon le groupe des isométries sous-jacent. Dans l'esprit de cette approche, une géométrie est un G -espace M . Le cas le plus courant est celui où G agit transitivement sur M . Dans ce cas, M est dite un G -espace *homogène*. Les exemples les plus connus de G -espaces homogènes sont les géométries Euclidienne, affine ou projective.

Un cas qui a attiré l'attention de plusieurs chercheurs est le cas des actions dites de *cohomogénéité un*: c'est celui des actions admettant une orbite de codimension un. Ce concept a été introduit par P.S. Mostert dans son article de 1956 [35]. L'hypothèse clé dans cet article était la compacité prescrite du groupe agissant. Il montre qu'alors l'espace des orbites M/G est homéomorphe à $\mathbb{R}, \mathbb{S}^1, [0, 1]$, ou $[0, 1)$. Dans [8] B. Bergery a montré que ce résultat reste vrai si on remplace l'hypothèse de compacité de G par la l'hypothèse de propreté de l'action. Au vu de ce résultat, l'orbite $G(p)$ d'un point $p \in M$ the orbit est dite *principale* (resp. *singulière*) si le point correspondant dans l'espace des orbites M/G est un point intérieur (resp. un point du bord).

Les variétés Riemanniennes de cohomogénéité un ont été étudiées par plusieurs mathématiciens (voir par exemple [1, 8, 21, 22, 23, 33, 35, 36, 39, 41, 43, 44, 45]). C'est toujours un sujet de recherche actif. L'hypothèse commune dans ces travaux est la fermeture du groupe G considéré dans le groupe d'isométrie de la variété ambiante. Quand la métrique semi-Riemannienne et non plus Riemannienne, l'action n'est en général pas propre, ce qui complique l'étude. De fait, beaucoup de résultats et techniques valables dans le contexte Riemannien ne s'appliquent plus dans le cas semi-Riemannien général. Il est montré dans [15] que pour une variété Riemannienne d'Einstein (M, g) l'équation d'Einstein $Ric(g) = \lambda \cdot g$ ($\lambda \in \mathbb{R}$ constant) devient une équation différentielle ordinaire dans le cas de cohomogénéité un.

Une manière naturelle d'étudier une G -variété Riemannienne (M, g) de cohomogénéité un est de déterminer quels sont les sous-groupes possibles G de $\text{Iso}(M)$ à conjugaison près. Cela a été fait dans le cas de certains espaces modèles (voir [4], [5] et [6]). Une autre notion utile dans le sujet est celle d'*équivalence orbitale* entre deux sous-groupes de Lie $G, H \subset \text{Iso}(M)$ signifiant que les orbites $G(p)$ and $H(g(p))$ pour tout point p peuvent coïncider à conjugaison près. Il est évident que deux sous-groupes

conjugués définissent des actions orbitalement équivalentes, mais l'inverse n'est pas vrai, comme le montre le fait que deux sous-groupes non conjugés peuvent fort bien avoir des actions transitives.

Actions sur l'univers d'Einstein tridimensionnel de cohomogénéité un

Le sujet de cette thèse est l'étude des actions conformes de cohomogénéité un sur l'univers d'Einstein tridimensionnel $\mathbb{E}in^{1,2}$. Notre stratégie est d'établir dans un premier temps quel peut être le groupe de transformations conformes impliqué, à conjugaison près. Nous décrivons aussi la topologie et la nature causale des orbites d'une telle action. On établira, entre autre choses, que les orbites bidimensionnelles d'une telle action sont toujours homéomorphes à \mathbb{R}^2 , $\mathbb{S}^1 \times \mathbb{R}$, \mathbb{S}^2 , ou $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$.

Le groupe des transformations conformes de $\mathbb{E}in^{1,2}$ est isomorphe à $PO(2, 3)$. Sa composante neutre est $SO_o(2, 3)$ qui agit sur $\mathbb{E}in^{1,2}$ de manière transitive - elle est constituée des transformations préservant l'orientation temporelle.

Les actions propres méritent un traitement particulier. Soit G un sous-groupe de Lie connexe de $\text{Conf}(\mathbb{E}in^{1,2})$. Puisque l'univers d'Einstein est compact, G agit proprement si et seulement si il est compact, donc si et seulement si il est, à conjugaison près, contenu dans le sous-groupe compact maximal $SO(2) \times SO(3)$ de $SO_o(2, 3)$. Il doit donc préserver un sous-espace vectoriel de dimension 2 de $\mathbb{R}^{2,3}$ de signature $(2, 0)$. Au Chapitre 2, nous montrons qu'il n'y a, à conjugaison près, que deux telles actions propres, qui sont celles des sous-groupes $SO(3)$ et $SO(2) \times SO(2)$ de $SO(2) \times SO(3)$. Les orbites de la première action sont toutes de codimension un; ce sont plus précisément des hypersphères de type espace. La seconde une orbite de dimension un, qui est une géodésique de type temps dans un ouvert Anti de-Sitter, dont le complémentaire est feuilleté par des hypersphères d'Einstein.

Une fois traité le cas propre, nous supposons désormais que le groupe G est non compact. Un point clé est le théorème suivant:

Définition 0.3. *Une action d'un sous-groupe de Lie $G \subset \text{Conf}(\mathbb{E}in^{1,2})$ est réductible si G préserve un sous-espace vectoriel propre de $\mathbb{R}^{2,3}$. Elle est irréductible si elle n'est pas réductible.*

Théorème 0.4. [12] *Soit G un sous-groupe de Lie connexe de $SO_o(2, n)$ agissant sur $\mathbb{R}^{2,n}$ de manière irréductible. Alors, G est isomorphe à l'un des groupes suivants:*

i: $SO_o(2, n)$.

ii: si $n = 2p$ est pair: $U(1, p)$, $SU(1, p)$ ou $S^1.SO_o(1, p)$ si $p > 1$,

iii: si $n = 3$: $SO_o(1, 2)$.

Il s'en suit, dans notre cas:

Théorème 0.5. *Soit $G \subset \text{Conf}(\mathbb{E}in^{1,2})$ un sous-groupe de Lie connexe agissant sur $\mathbb{E}in^{1,2}$ de manière irréductible et de cohomogénéité un. Alors G est conjugué à $SO_o(1, 2) \simeq \text{PSL}(2, \mathbb{R})$.*

Nous montrons le Théorème 0.5 au Chapitre 3. Ce chapitre est consacré à l'étude du cas irréductible. Il est bien connu que la seule action irréductible de $\text{PSL}(2, \mathbb{R})$ de dimension 5 est celle provenant de l'action canonique sur les polynômes homogènes de degré 4 à deux variables. Nous montrons que cette

action a trois orbites: une orbite de dimension un (qui est un photon), une orbite dégénérée de dimension 2 homéomorphe à \mathbb{R}^2 , et une orbite de dimension 3 sur laquelle $\text{PSL}(2, \mathbb{R})$ agit librement. Nous décrivons aussi les orbites induites sur le complémentaire de $\mathbb{E}\text{in}^{1,2}$ dans l'espace projectif $\mathbb{R}\mathbb{P}^4$. Nous avons publié ces résultats dans l'article [26].

Le dernier cas à considérer est donc celui des actions réductibles. Le point clé de notre étude, qui résulte d'une étude cas par cas selon la dimension et la nature causale du sous-espace vectoriel G -invariant est le suivant:

Théorème 0.6. *Soit $G \subset \text{Conf}(\mathbb{E}\text{in}^{1,2})$ un sous-groupe de Lie connexe de $\text{SO}_o(2, 3)$ agissant sur $\mathbb{R}^{2,3}$ de manière réductible et de cohomogénéité un. Alors, G admet un point fixe global dans l'espace projectif $\mathbb{R}\mathbb{P}^4$.*

La situation la plus intéressante et la plus riche pour établir le Théorème 0.6 est celle où G préserve un plan isotrope dans $\mathbb{R}^{2,3}$, i.e. un photon dans $\mathbb{E}\text{in}^{1,2}$. Elle est traitée au Chapitre 6.

D'après le Théorème 0.6, qui est établi au Chapitre 7, toute action réductible de cohomogénéité un sur $\mathbb{E}\text{in}^{1,2}$ préserve une droite dans $\ell \leq \mathbb{R}^{2,3}$. Cette droite ℓ est soit de type lumière, de type espace, ou de type temps.

Le cas le plus riche est celui où la droite ℓ est isotrope. Il est étudié au Chapitre 4. Dans ce cas, le groupe G préserve un point $p \in \mathbb{E}\text{in}^{1,2}$, son cône lumière $L(p)$ et la carte de Minkowski correspondante. L'action dans cette carte de Minkowski est équivalente à celle d'un sous-groupe de $\text{Conf}(\mathbb{E}^{1,2}) = (\mathbb{R}^* \times O(1, 2)) \times \mathbb{R}^{1,2}$. On étudie l'action adjointe de $\text{Conf}(\mathbb{E}^{1,2})$ sur son algèbre de Lie $(\mathbb{R} \oplus \mathfrak{so}(1, 2)) \oplus_{\theta} \mathbb{R}^{1,2}$, puis déterminons toutes les sous-algèbres de Lie \mathfrak{g} de $(\mathbb{R} \oplus \mathfrak{so}(1, 2)) \oplus_{\theta} \mathbb{R}^{1,2}$ de dimension ≥ 2 et à conjugaison près. Il s'avère qu'il y a une infinité de sous-groupes de $\text{Conf}(\mathbb{E}^{1,2})$ de dimension ≥ 2 à conjugaison près. Tous ces sous-groupes agissent sur $\mathbb{E}\text{in}^{1,2}$ avec cohomogénéité un sauf deux d'entre eux. Même à équivalence orbitale près, il y a une infinité d'actions possibles à distinguer.

Au Chapitre (5), nous étudions le cas où la droite $\ell \leq \mathbb{R}^{2,3}$ n'est pas isotrope, et où G ne fixe aucun point de $\mathbb{E}\text{in}^{1,2}$.

- Si ℓ est de type espace, alors G préserve une hypersphère d'Einstein $\mathbb{E}\text{in}^{1,1} \subset \mathbb{E}\text{in}^{1,2}$ bordant une copie conforme de l'espace Anti de-Sitter $\text{AdS}^{1,2}$. Nous montrons qu'à conjugaison près, il y a exactement 7 sous-groupes de Lie préservant une hypersphère d'Einstein $\mathbb{E}\text{in}^{1,1}$ dans $\mathbb{E}\text{in}^{1,2}$, agissant sur $\mathbb{E}\text{in}^{1,2}$ non-proprement, avec cohomogénéité un et sans point fixe global dans $\mathbb{E}\text{in}^{1,2}$.
- Si ℓ est de type temps, alors G préserve une hypersphère de type espace $\mathbb{S}^2 \subset \mathbb{E}\text{in}^{1,2}$ bordant une copie conforme de l'espace de-Sitter $\text{dS}^{1,2}$. Nous montrons qu'à conjugaison près, il n'y a qu'un seul sous-groupe de Lie préservant une hypersphère de type espace dans $\mathbb{E}\text{in}^{1,2}$, agissant sur $\mathbb{E}\text{in}^{1,2}$ non-proprement, avec cohomogénéité un et sans point fixe global dans $\mathbb{E}\text{in}^{1,2}$.

Finalement, au Chapitre 7, nous montrons le Théorème 0.6. Ceci achève la classification des actions de cohomogénéité un sur $\mathbb{E}\text{in}^{1,2}$.

L'appendice A, assez technique, est dédié à une preuve du Théorème (3.4) qui classe les sous-groupes de Lie connexes de $\text{Conf}(\mathbb{E}^{1,2})$ de dimension supérieure ou égale à 2. Dans l'appendice B, on donne une liste de sous-groupes à un paramètre de $\text{Conf}(\mathbb{E}^{1,2})$ qui est utilisée au Chapitre (3).

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Chapter 1

Preliminaries

We fix the following notation and definitions for the rest of this work.

Let \mathbb{R}^{m+n} be the $(m+n)$ -dimensional real vector space. Assume that $q_{m,n}$ is a quadratic form on \mathbb{R}^{m+n} of signature (m, n) , i.e., in a suitable basis for \mathbb{R}^{m+n} and a vector $v = (v_1, \dots, v_{m+n}) \in \mathbb{R}^{m+n}$ we have:

$$q_{m,n}(v) = -\sum_{i=1}^m v_i^2 + \sum_{j=m+1}^{m+n} v_j^2.$$

We denote by $\mathbb{R}^{m,n}$ the vector space \mathbb{R}^{m+n} equipped with $q_{m,n}$.

Definition 1.1. A linear subspace $V \leq \mathbb{R}^{m,n}$ is said to be of signature (p, q, r) if the restriction of $q_{m,n}$ on V is of signature (p, q, r) , meaning that the maximal totally isotropic subspace has dimension r , and that the maximal definite negative and positive subspaces have dimensions p and q , respectively. If V is nondegenerate (i.e., $r = 0$), we forgot r and simply denote its signature by (p, q) . Also, we call V

- spacelike, if $q \neq 0$ and $p = r = 0$.
- timelike, if $p = 1$ and $q = r = 0$.
- lightlike, if $r = 1$ and $p = q = 0$.
- Lorentzian, if $p = 1, q \neq 0$, and $r = 0$.
- degenerate, if $p + q, r \neq 0$.

A non-zero vector $v \in \mathbb{R}^{m,n}$ is called spacelike (resp. timelike, lightlike) if the value $q_{m,n}(v)$ is positive (resp. negative, zero).

1.1 Group action

In this section, we give some definitions and properties of group action. For more details, we refer to [14]. We use G, H, K , etc. to denote a group, and M to denote a topological space or a manifold.

Definition 1.2. For a manifold M and a Lie group G , a smooth action of G on M , is a group morphism A from G to $\text{Diff}(M)$ the group of diffeomorphisms from M to itself such that the map $G \times M \rightarrow M$ sending (g, x) to $A(g)(x)$ is smooth. For a point $x \in M$ and an element $g \in G$, we denote $A(g)(x)$ by $g(x)$ or gx , when the action is understood.

Definition 1.3. Let A be an action of a Lie group G on a manifold M , and $x \in M$ be an arbitrary point.

- The orbit of x denoted by $G(x)$ is $\{gx : g \in G\}$.
- The stabilizer of x denoted by $Stab_G(x)$ or G_x is $\{g \in G : gx = x\}$.
- The orbit map at point x is $A_x : G \rightarrow$, with $A_x(g) = gx$.
- The stabilizer of a subset $S \subset M$ denoted by $Stab_G(S)$ is $\{g \in G : gx \in S, \forall x \in S\}$.
- The orbit space of the action A denoted by M/G is the set of orbits induced by A .
- A subset $S \subset M$ is called G -invariant, if $G(S) = \bigcup_{x \in S} G(x) \subset S$.

Given a Lie group G we call the connected component of G containing its identity element the identity component of G and denote it by G_\circ .

Definition 1.4. Let G be a Lie group and $H, K \subset G$ be Lie subgroups. The subgroup K is called transversal to H if the Lie subgroup generated by HK has dimension strictly greater than $\max\{\dim H, \dim K\}$.

Let G be a lie group acting on a smooth manifold M . The orbits induced by G in M are immersed submanifolds of M . In fact, for a point $x \in M$, the orbit $G(x)$ is the image of the orbit map A_x . The stabilizer of each point in M is a closed Lie subgroup of G . Also, the orbit space M/G is endowed with quotient topology.

Definition 1.5. Let M be a smooth manifold equipped with a semi-Riemannian metric g and $S \subset M$ an immersed submanifold. Then, S is said to be of signature (p, q, r) if for all $x \in S$ the restriction of the ambient metric g_x on the tangent space $T_x S$ is of signature (p, q, r) .

Let (M, g) be a semi-Riemannian manifold and G a Lie group. If G acts on M smoothly and preserves the metric g , then every orbit has constant signature, meaning that for all $x \in M$ and all $y, z \in G(x)$, the restrictions of the ambient metrics on $T_y G(x)$ and $T_z G(x)$ have the same signature. Thus, we can speak about the signature of an orbit.

Definition 1.6. The action of a Lie group G on a manifold M is called proper if the map $G \times M \rightarrow M \times M$ sending (g, x) to (gx, x) is a proper map, i.e., the preimage of each compact subset of $M \times M$ is a compact subset of $G \times M$. A group action is called non-proper if it is not proper.

Definition 1.7. Let G be a Lie group which acts on a manifold M smoothly. The action of G is called of cohomogeneity one, if G admits a codimension one orbit in M .

1.2 Affine space

In this section we give the definition of affine space and some properties of it. We refer to [9] for additional details.

Definition 1.8. Let V be a real n -dimensional vector space. A non-empty set Ω is said to be an affine space associated to V , if there is a mapping

$$\Omega \times \Omega \rightarrow V, \quad (p, q) \mapsto \overrightarrow{pq},$$

satisfying the following axioms:

(i) for any $p, q, r \in \Omega$, we have $\overrightarrow{pr} = \overrightarrow{pq} + \overrightarrow{qr}$;

(ii) for any $p \in \Omega$ and any $v \in V$, there is one and only one $q \in \Omega$ such that $x = \overrightarrow{pq}$.

The dimension of Ω is defined as the dimension of V , and we call V the underlying vector space of affine space Ω .

Proposition 1.9. ([9] p.p. 34). A non-empty set Ω is an affine space associated to a vector space V if and only if there exists a free and transitive action $\Theta : V \times \Omega \rightarrow \Omega$.

It is convenient to write $q = p + x$ instead of $x = \overrightarrow{pq}$, where "+" denotes the action of V on Ω . Essentially the only difference between Ω and V is that, Ω has not an origin, and thus it is not linear.

Choosing a point $p \in \Omega$, the orbit map $\Theta_p : V \rightarrow \Omega$ induces a vector space structure on Ω with origin p , such that Θ_p becomes an isomorphism. We denote Ω with the vector space structure induced from Θ_p , by Ω_p .

Definition 1.10. For a vector $v \in V$, we call the map $T_v : \Omega \rightarrow \Omega$, $p \mapsto p + v$ a translation on Ω . We denote the set of all translations on Ω by $T(\Omega)$.

In fact, $T(\Omega)$ has a natural vector space structure which makes it isomorphic to V .

Let V and W be real vector spaces. We denote the space of linear maps from V to W by $L(V, W)$.

Definition 1.11. We call a map $f : \Omega \rightarrow \Omega$ an affine morphism if $f \in L(\Omega_p, \Omega_{f(p)})$ for all $p \in \Omega$.

For an affine morphism f , we denote the map $\Theta_{f(p)}^{-1} \circ f \circ \Theta_p \in L(V, V)$ by \overrightarrow{f} . One can see that the map \overrightarrow{f} is independent of p . Furthermore,

$$f(p + v) = f(p) + \overrightarrow{f}(v), \quad \forall p \in \Omega, \forall v \in V. \quad (1.1)$$

We call f an affine isomorphism, if it is a bijective affine morphism. Obviously every translation on Ω is an affine isomorphism.

Lemma 1.12. Let f be an affine morphism on Ω . Then f is a translation on Ω if and only if $\overrightarrow{f} = Id_V$.

Proof. Let f be a translation, which means there exists a vector $v \in V$ such that $f = T_v$. Then, $\overrightarrow{T_v} = Id_V$.

Conversely, let f be an affine morphism and $\overrightarrow{f} = Id_V$. Let $p, q \in \Omega$ be two arbitrary points. There exist unique vectors $w, v \in V$ such that $q = p + v$ and $f(p) = p + w$. Thus by (1.1), we have

$$f(q) = f(p + v) = f(p) + v = p + w + v = q + w.$$

Therefore $f = T_w$. □

For a linear map $B \in L(V, V)$, choosing a point $p \in \Omega$, we can define a map $B_p : \Omega \rightarrow \Omega$ with $B_p(p + v) = p + B(v)$. It is clear that B_p is an affine map and $\overrightarrow{B_p} = B$.

Lemma 1.13. Let f be an affine morphism on Ω and $p \in \Omega$ an arbitrary point. Then, there exist a unique vector $w \in V$ and a linear map $B \in L(V, V)$ such that $f = T_w \circ B_p$.

Proof. There exists a unique vector $w \in V$ such that $f(p) = p + w$. Now, by (1.1), obviously $f = T_w \circ (\overrightarrow{f})_p$. □

The set of all affine morphisms on Ω is a group under composition of maps. We denote it by $\mathbf{A}(\Omega)$. Also, we denote the set of all affine isomorphisms on Ω by $\mathbb{A}\text{ff}(\Omega)$, and call it the *affine group* of Ω . It is a subgroup of $\mathbf{A}(\Omega)$.

Corollary 1.14. *The affine group $\mathbb{A}\text{ff}(\Omega)$ is isomorphic to the semi-direct product $GL(V) \ltimes V$.*

Proof. It follows from Lemma 1.13. □

The following map is a surjective group morphism

$$L : \mathbb{A}\text{ff}(\Omega) \rightarrow GL(V), \quad f \rightarrow \vec{f},$$

and we call it the linear projection. By Lemma 1.12, the kernel of L is $T(\Omega)$. Therefore, the group of translations $T(\Omega)$ is a normal subgroup of $\mathbb{A}\text{ff}(\Omega)$, and so, $\mathbb{A}\text{ff}(\Omega)$ acts on it by conjugation.

The affine group $\mathbb{A}\text{ff}(\Omega)$ acts on Ω transitively, since $T(\Omega) \simeq V$ does. Hence, for an arbitrary point $q \in \Omega$ and its stabilizer $\mathbb{A}\text{ff}(\Omega)_q$, the affine space Ω is isomorphic to the homogeneous space $\mathbb{A}\text{ff}(\Omega)/\mathbb{A}\text{ff}(\Omega)_p$, as sets.

Lemma 1.15. *Let $p \in \Omega$ be an arbitrary point. Then $\mathbb{A}\text{ff}(\Omega) = \mathbb{A}\text{ff}(\Omega)_p \ltimes T(\Omega)$.*

Proof. By Lemma 1.13, $\mathbb{A}\text{ff}(\Omega) = \mathbb{A}\text{ff}(\Omega)_p \cdot T(\Omega)$. Since $T(\Omega)$ is a normal subgroup of $\mathbb{A}\text{ff}(\Omega)$ and $T(\Omega) \cap \mathbb{A}\text{ff}(\Omega)_p = Id_\Omega$, we get $\mathbb{A}\text{ff}(\Omega) = \mathbb{A}\text{ff}(\Omega)_p \ltimes T(\Omega)$. □

Note that, the splitting $\mathbb{A}\text{ff}(\Omega) = \mathbb{A}\text{ff}(\Omega)_p \ltimes T(\Omega)$ depends strongly on the point $p \in \Omega$. Indeed, for two points $p, q \in \Omega$, we have $\mathbb{A}\text{ff}(\Omega)_p = \mathbb{A}\text{ff}(\Omega)_q$ if and only if $p = q$.

Definition 1.16. *An affine d -plane (for $d \leq \dim V$) in Ω through a point $p \in \Omega$, is $p + \Pi$, where $\Pi \leq V$ is a d -dimensional linear plane.*

In fact, an affine d -plane in Ω through p is the orbit induced by the action of a d -dimensional linear subspace of V at p .

1.3 Conformal Structure

Definition 1.17. *Let (M, \mathfrak{g}) be a semi-Riemannian manifold. The conformal structure on M associated to \mathfrak{g} (denoted by $[\mathfrak{g}]$) is the class of metrics conformal to \mathfrak{g} , i.e., metrics of the form $e^f \mathfrak{g}$, for some smooth function $f : M \rightarrow \mathbb{R}$.*

The concept of geodesic is not relevant in a conformal structure $(M, [\mathfrak{g}])$, since a geodesic in M respect to the metric $\mathfrak{g} \in [\mathfrak{g}]$ may fails to be a geodesic respect to the other metrics in the class $[\mathfrak{g}]$. However, if \mathfrak{g} is a indefinite metric, a lightlike geodesic respect to \mathfrak{g} is a lightlike (unparametrized) geodesic respect to the all metrics in the metric class $[\mathfrak{g}]$. Moreover, the cuasal character of a vector tangent to M is invariant in the metric class $[\mathfrak{g}]$. Hence, we can speak about the cuasal character of tangent vectors, curves, or signatures of submanifold etc.. Hence, we have the following definition which is the conformal version of Definition 1.5:

Definition 1.18. *Let M be a smooth manifold equipped with a semi-Riemannian conformal structure $[\mathfrak{g}]$ and $S \subset M$ an immersed submanifold. Then, S is said to be of signature (p, q, r) if for all $x \in S$ the restriction of the ambient metric $[\mathfrak{g}]_x$ on the tangent space $T_x S$ is of signature (p, q, r) .*

For $\dim M \geq 3$, the group of conformal transformations of M is a Lie group ([31]) and we denote it by $\text{Conf}(M)$.

Definition 1.19. A conformal semi-Riemannian manifold $(M, [\mathfrak{g}])$ is said to be essential if the action of its conformal group $\text{Conf}(M)$ preserves no metric in the metric class $[\mathfrak{g}]$. The conformal structure $[\mathfrak{g}]$ is called inessential if its conformal group preserves a metric in the class $[\mathfrak{g}]$.

Observe that, for an inessential conformal structure $(M, [\mathfrak{g}])$ the group of conformal transformations preserves every metric in the metric class $[\mathfrak{g}]$. In other words, the conformal group of an inessential conformal structure is the isometry group of every element in the conformal class.

On the opposite side, for an essential conformal structure $(M, [\mathfrak{g}])$, its conformal group is strictly bigger than the group of isometries of any metric $g \in [\mathfrak{g}]$.

By the conformal action of a Lie group G on a semi-Riemannian conformal structure $(M, [\mathfrak{g}])$, every orbit has constant signature, and so, we can speak about the signature of the orbits induced by G .

Definition 1.20. Let G and H be Lie subgroups of $\text{Conf}(M)$. Then the actions of G and H on M are said to be orbitally-equivalent if there exists a conformal map φ on M such that for all $p \in M$, $\varphi(G(p)) = H(\varphi(p))$.

Two orbitally-equivalent actions on a conformal structure $(M, [\mathfrak{g}])$ give the same information about the orbits. So, it seems considering the conformal actions up to orbit equivalency is a good approximation. However, sometimes finding an equivalent map for two actions is not easy. The following lemma gives a nice tool to distinguish these maps.

Lemma 1.21. Let G and H be connected Lie subgroups of $\text{Conf}(M)$ and φ be a conformal map on M . Then the following statements are equivalent:

(i) G and H are orbitally-equivalent via φ .

(ii) for all $p \in M$,

$$d\varphi_p(T_p G(p)) = T_{\varphi(p)} H(\varphi(p)). \quad (1.2)$$

Proof. (i) \Rightarrow (ii) It is obvious.

(ii) \Rightarrow (i) Assume that φ satisfies (1.2). For an arbitrary point $p \in M$, define the following set

$$A_p = \{q \in G(p) : \varphi(q) \in H(\varphi(p))\}.$$

Evidently, A_p is nonempty. Observe that (1.2) implies, for $q \in A_p$, there exist two neighborhoods $U \subset G(p)$ and $V \subset H(\varphi(p))$ containing q and $\varphi(q)$, respectively, such that $\varphi|_U : U \rightarrow V$ is a diffeomorphism. This shows that A_p is open in $G(p)$. By the same argument, it can be easily seen that the complement of A_p in $G(p)$ is open too. Hence, by connectedness $G(p) = A_p$, and so $\varphi(G(p)) \subset H(\varphi(p))$. Using the same argument, one can show $\varphi^{-1}(H(\varphi(p))) \subset G(p)$. This completes the proof. \square

1.4 Projective Special Linear group

The special linear group $SL(2, \mathbb{R})$ is the group of linear transformations of a 2-dimensional real vector space V with determinant 1. It is a closed connected 3-dimensional simple Lie subgroup of general linear group $GL(V)$.

The projective special linear group $PSL(2, \mathbb{R})$ is the quotient of $SL(2, \mathbb{R})$ by $\mathbb{Z}_2 = \{\pm Id\}$. The group $PSL(2, \mathbb{R})$ acts on the Poincaré half-plane model of hyperbolic plane with conformal boundary in infinity $\overline{\mathbb{H}^2} = \mathbb{H}^2 \cup \partial\mathbb{H}^2$ by *mobius transformation*

$$x \mapsto \frac{ax + b}{cx + d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1.$$

This action preserves the hyperbolic plane \mathbb{H}^2 and its boundary $\partial\mathbb{H}^2 \simeq \mathbb{RP}^1$. Indeed, $PSL(2, \mathbb{R})$ is the group of orientation-preserving isometries of hyperbolic plane, and acts on it transitively. Furthermore it acts on the conformal boundary of hyperbolic plane conformally and transitively.

Definition 1.22. [42, p.p. 141]. Let $[I] \neq [A] \in PSL(2, \mathbb{R})$. Then

- If A has only one eigenvalue, namely 1, then we call it a *parabolic element*. Every parabolic element fixes a unique point on the boundary $\partial\mathbb{H}^2$.
- If A has two distinct real eigenvalues (λ and λ^{-1}), then we call it a *hyperbolic element*. Every hyperbolic element fixes exactly two points on the boundary $\partial\mathbb{H}^2$.
- If A has no real eigenvalues, then we call it an *elliptic element*. Every elliptic element fixes exactly one point in the hyperbolic plane \mathbb{H}^2 .

1.4.1 Connected subgroups

We characterize all the connected Lie subgroups of $PSL(2, \mathbb{R})$ up to conjugacy.

Let $[A], [B] \in PSL(2, \mathbb{R})$ and $[A] \neq [Id]$. Define $Fix([A]) = \{x \in \overline{\mathbb{H}^2} : [A](x) = x\}$. If $[B]$ commutes with $[A]$, then $[B]$ preserves $Fix([A])$ (see [42] p.p. 141-144). Thus, for any connected abelian subgroup $G \subset PSL(2, \mathbb{R})$, $Fix(G) = \{x \in \overline{\mathbb{H}^2} : \forall [A] \in G, [A](x) = x\}$ is a non-empty set. Therefore all the non-trivial elements in a connected abelian subgroup of $PSL(2, \mathbb{R})$ have same fixed point(s).

Definition 1.23. Let $G \subset PSL(2, \mathbb{R})$ be a connected abelian subgroup. We call G

- *parabolic*, if it contains a parabolic element.
- *hyperbolic*, if it contains a hyperbolic element.
- *elliptic*, if it contains an elliptic element.

Since $PSL(2, \mathbb{R})$ acts on \mathbb{H}^2 transitively, the stabilizer of a point $x \in \mathbb{H}^2$ is a 1-dimensional subgroup. Actually, $Stab_{PSL(2, \mathbb{R})}(x)$ is connected (hence it is abelian) and so, it is an elliptic subgroup. It follows that, all the connected elliptic subgroups of $PSL(2, \mathbb{R})$ are conjugate to each other.

Also, $PSL(2, \mathbb{R})$ acts on the boundary $\partial\mathbb{H}^2$, transitively. Thus the stabilizer of each point on the boundary is a 2-dimensional Lie subgroup.

Definition 1.24. The affine subgroup of $PSL(2, \mathbb{R})$ is the stabilizer of $\infty \in \partial\mathbb{H}^2$, and we denote it by $\mathbb{A}ff$.

The affine subgroup acts on $\partial\mathbb{H}^2 \setminus \{\infty\}$ transitively. In fact Aff is isomorphic to the identity component of the group $\text{Aff}(1, \mathbb{R})$ consists of all the affine isomorphisms on the real line $\mathbb{R} \simeq \partial\mathbb{H}^2 \setminus \{\infty\}$. This justifies our notation and terminology for the stabilizer of a point on the boundary.

The affine group Aff is non-abelian and (as the all 2-dimensional Lie groups) solvable. It consists of parabolic and hyperbolic elements. The commutator subgroup of Aff is the only connected 1-dimensional parabolic subgroup of Aff . One can see all the connected 1-dimensional hyperbolic subgroups of Aff are conjugate to each other.

Lemma 1.25. *Every 2-dimensional connected Lie subgroup of $\text{PSL}(2, \mathbb{R})$ is conjugate to Aff .*

Proof. Let $G \subset \text{PSL}(2, \mathbb{R})$ be a 2-dimensional connected Lie subgroup. Since all 2-dimensional Lie groups are solvable (see [25, p.p. 61]), G is either abelian or solvable and non-abelian.

Assume that G is abelian. Then it admits a fixed point in $\overline{\mathbb{H}^2}$. Since the stabilizer of each point in \mathbb{H}^2 is 1-dimensional, G admits a fixed point on the boundary $\partial\mathbb{H}^2$. Thus, up to conjugacy, G is a subgroup of Aff . Since G and Aff both are connected and 2-dimensional, $G = \text{Aff}$ up to conjugacy. But this is a contradiction, since affine group is not abelian. Therefore G is non-abelian.

The commutator subgroup $G' = [G, G]$ is an abelian subgroup since it is 1-dimensional and connected, and so, $\text{Fix}(G') \neq \emptyset$. Let $x_0 \in \text{Fix}(G')$, $[A] \in G$, and $[B] \in G'$. We have

$$[A][B][A]^{-1}[B]^{-1}(x_0) = x_0 \Rightarrow [A](x_0) = [B]([A](x_0)) \Rightarrow [A](x_0) \in \text{Fix}([B]) = \text{Fix}(G').$$

Thus G preserves $\text{Fix}(G')$, and since it is connected, it fixes $\text{Fix}(G')$ pointwisely. Therefore G is the stabilizer of a point in $\partial\mathbb{H}^2$, which implies $G = \text{Aff}$ up to conjugacy. \square

Notation 1.26. *We denote by Y_E the 1-parameter elliptic subgroup of $\text{PSL}(2, \mathbb{R})$ stabilizing $i \in \mathbb{H}^2$. Also, we denote by Y_P the commutator subgroup $[\text{Aff}, \text{Aff}]$ of $\text{Aff} \subset \text{PSL}(2, \mathbb{R})$ which is a 1-parameter parabolic subgroup. Furthermore, we denote by Y_H the 1-parameter hyperbolic subgroup of Aff stabilizing $\{0, \infty\} \subset \partial\mathbb{H}^2$.*

Corollary 1.27. *Let G be a non-trivial connected Lie subgroup of $\text{PSL}(2, \mathbb{R})$. Then G is conjugate to one of the following subgroups.*

$$Y_E, \quad Y_P, \quad Y_H, \quad \text{Aff}_o(1, \mathbb{R}), \quad \text{PSL}(2, \mathbb{R}).$$

Remark 1.28. *The map $\text{SL}(2, \mathbb{R}) \rightarrow \text{PSL}(2, \mathbb{R})$ sending A to $[A]$ is a Lie group double covering. We call an element $A \in \text{SL}(2, \mathbb{R})$ elliptic, parabolic, or hyperbolic if the corresponding element $[A] \in \text{PSL}(2, \mathbb{R})$ is elliptic, parabolic, or hyperbolic, respectively. Also, one can see that the group $\text{SL}(2, \mathbb{R})$ has three distinct 1-parameter subgroup up to conjugacy, and it has a unique (up to conjugacy) 2-dimensional connected Lie subgroup isomorphic to Aff . Hence, we may use the same terminology and notations for the elements and subgroup of $\text{SL}(2, \mathbb{R})$ as we used for those of $\text{PSL}(2, \mathbb{R})$, when there is no ambiguity.*

1.4.2 Morphisms

The projective special linear group $\text{PSL}(2, \mathbb{R})$ is a simple Lie group. Thus, it has no non-trivial normal subgroup.

Lemma 1.29. *Let G be a Lie group with $1 \leq \dim G \leq 2$, and $\varphi : \text{PSL}(2, \mathbb{R}) \rightarrow G$ be a Lie group morphism. Then φ is trivial map.*

Proof. The kernel $\ker \varphi$ is a normal subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Since $\dim \mathrm{PSL}(2, \mathbb{R}) > \dim G$, $\ker \varphi \neq \{[Id]\}$. Since $\mathrm{PSL}(2, \mathbb{R})$ is a simple Lie group, we get $\ker \varphi = \mathrm{PSL}(2, \mathbb{R})$. This completes the proof. \square

Corollary 1.30. *Let $\varphi : \mathrm{PSL}(2, \mathbb{R}) \rightarrow \mathrm{PSL}(2, \mathbb{R})$ be a Lie group morphism. Then either φ is the trivial map or it is an isomorphism.*

1.4.3 Lie algebra

The Lie algebra of projective special Lie group $\mathrm{PSL}(2, \mathbb{R})$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ -the set of traceless 2×2 matrices-. It is a simple Lie algebra, and its corresponding killing form B is a Lorentzian bilinear scalar product.

The set of following matrices is a basis for $\mathfrak{sl}(2, \mathbb{R})$ as a vector space,

$$\mathcal{Y}_E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathcal{Y}_H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathcal{Y}_P = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and we have $[\mathcal{Y}_E, \mathcal{Y}_H] = 2\mathcal{Y}_E - 4\mathcal{Y}_P$, $[\mathcal{Y}_E, \mathcal{Y}_P] = \mathcal{Y}_H$ and $[\mathcal{Y}_H, \mathcal{Y}_P] = 2\mathcal{Y}_P$.

By Corollary (1.27), $\mathfrak{sl}(2, \mathbb{R})$ has exactly four non-trivial proper Lie subalgebras, up to conjugacy; Three 1-dimensional Lie subalgebras generated by \mathcal{Y}_E , \mathcal{Y}_H , and \mathcal{Y}_P , which are the corresponding Lie algebras of the 1-parameter subgroups Y_E , Y_H , and Y_P , respectively. The 2-dimensional Lie algebra generated by $\{\mathcal{Y}_H, \mathcal{Y}_P\}$, is a non-abelian solvable Lie algebra, and we denote it by \mathfrak{aff} . Obviously, \mathfrak{aff} is the corresponding Lie algebra of affine group $\mathbb{A}\mathfrak{ff}$.

Proposition 1.31. *The killing form B on the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ is a scalar product of signature $(1, 2)$. Furthermore, $(\mathfrak{sl}(2, \mathbb{R}), B)$ and $(\mathfrak{sl}(2, \mathbb{R}), -\det)$ are isometric.*

Proof. The set of vectors $\{\mathcal{Y}_E, \mathcal{Y}_H, \mathcal{Y}_E - 2\mathcal{Y}_P\}$ is an orthogonal basis for $\mathfrak{sl}(2, \mathbb{R})$ respect to the killing form B and $-\det$. The vector \mathcal{Y}_E is timelike, and the vectors \mathcal{Y}_H and $\mathcal{Y}_E - 2\mathcal{Y}_P$ are spacelike respect to both B and $-\det$. Since the scalar product spaces $(\mathfrak{sl}(2, \mathbb{R}), B)$ and $(\mathfrak{sl}(2, \mathbb{R}), -\det)$ have the same signatures (i.e., $(1, 2)$), they are isometric. \square

The following map is an explicit linear isometry.

$$(\mathfrak{sl}(2, \mathbb{R}), B) \longrightarrow (\mathfrak{sl}(2, \mathbb{R}), -\det), \quad X \mapsto \frac{1}{\sqrt{8}}X.$$

Lemma 1.32. *The connected components of the group of automorphisms on affine group $\mathbb{A}\mathfrak{ff}$ and the group of automorphisms of affine algebra \mathfrak{aff} are isomorphic to $\mathbb{A}\mathfrak{ff}$.*

Proof. First assume that f is an automorphism on the Lie algebra \mathfrak{aff} . There are constants $a, b, c, d \in \mathbb{R}$ such that $f(\mathcal{Y}_H) = a\mathcal{Y}_H + b\mathcal{Y}_P$ and $f(\mathcal{Y}_P) = c\mathcal{Y}_H + d\mathcal{Y}_P$. Since f preserves the Lie bracket, we have $c = 0$, $a = 1$, and $d \neq 0$. This shows that $\mathrm{Aut}(\mathfrak{aff})$ is a 2-dimensional Lie group. On the other hand, the adjoint action of $\mathbb{A}\mathfrak{ff}$ on \mathfrak{aff} induced a faithful representation from $\mathbb{A}\mathfrak{ff}$ to $\mathrm{Aut}(\mathfrak{aff})$. This proves the lemma. \square

1.4.4 Irreducible representation

In this section we describe the irreducible representation of $\mathrm{PSL}(2, \mathbb{R})$ on a $(n + 1)$ -dimensional vector space.

Let V be real a $(n + 1)$ -dimensional vector space. There is only one irreducible representation of $\mathrm{PSL}(2, \mathbb{R})$ in $GL(V)$. This representation is the natural action of $\mathrm{PSL}(2, \mathbb{R})$ on the vector space $V = \mathbb{R}_n[X, Y]$ of homogeneous polynomials of degree n in two variables X and Y [25]. This action preserves the bilinear form Q_n given in the coordinate system

$$(X^n, X^{n-1}Y, \dots, XY^{n-1}, Y^n)$$

by the matrix

$$\begin{bmatrix} & & & & & \alpha_{n,0} \\ & & & & -\alpha_{n,1} & \\ & & & \ddots & & \\ & & & & & \\ & & (-1)^{n-1}\alpha_{n,n-1} & & & \\ (-1)^n & & & & & \end{bmatrix}$$

where $\alpha_{n,k} = (k!(n - k)!)/n!$ [11].

The bilinear form q_n is anti-symmetric for n odd and symmetric of signature $(n/2, n/2 + 1)$ when n is even. In particular, for $n = 2$, the quadratic form $-2Q_2$ is the discriminant of quadratic polynomials, and this representation gives an isomorphism between $\mathrm{PSL}(2, \mathbb{R})$ and $SO_o(1, 2)$.

For $n = 4$, we have

$$Q_4(a_4X^4 + a_3X^3Y + a_2X^2Y^2 + a_1XY^3 + a_0Y^4) = 2a_4a_0 - \frac{1}{2}a_3a_1 + \frac{1}{6}a_2^2.$$

It is easy to see that the following map is $\mathrm{PSL}(2, \mathbb{R})$ -equivariant.

$$\kappa : \mathbb{R}_2[X, Y] \longrightarrow \mathbb{R}_4[X, Y], \quad F \mapsto F^2.$$

1.5 Minkowski space

In this section, we give the definition of Minkowski space, and we consider some of the properties of this space.

Definition 1.33. *The Minkowski space $\mathbb{E}^{1,n}$ is the affine space with underlying scalar product space $\mathbb{R}^{1,n} = (\mathbb{R}^{n+1}, q_{1,n})$.*

Let Θ denotes the action of $\mathbb{R}^{1,n}$ on $\mathbb{E}^{1,n}$. The topology and differential structure induced by $\mathbb{R}^{1,n}$ via Θ on $\mathbb{E}^{1,n}$, makes it a smooth manifold diffeomorphic to $\mathbb{R}^{1,n}$. The tangent space of each point $p \in \mathbb{E}^{1,n}$ is naturally isomorphic to $\mathbb{R}^{1,n}$, via Θ_p . Therefore, the quadratic form $q_{1,n}$ defines a Lorentzian metric tensor g on $\mathbb{E}^{1,n}$. The Lorentzian manifold $(\mathbb{E}^{1,n}, g)$ is isometric to $\mathbb{R}^{1,n}$, and so, it is a simply-connected geodesically complete, flat Lorentzian manifold.

The geodesics in $\mathbb{E}^{1,n}$ are affine lines $\gamma : \mathbb{R} \rightarrow \mathbb{E}^{1,n}$, $t \mapsto p + tv$, where $p \in \mathbb{E}^{1,n}$ and $v \in \mathbb{R}^{1,n}$ (see [37, p.p. 69]). We call p and v , the start point and the direction of γ , respectively. The causal character of a geodesic is the causal character of its direction.

A homothety on $\mathbb{E}^{1,n}$ (centered at $x_0 \in \mathbb{E}^{1,n}$) is any map conjugate by a translation to scalar multiplication:

$$\mathbb{E}^{1,n} \rightarrow \mathbb{E}^{1,n}, \quad x \mapsto x_0 + r(x - x_0).$$

A conformal map on the Minkowski space $\mathbb{E}^{1,n}$ is a composition of an isometry of $\mathbb{E}^{1,n}$ with a homothety:

$$f : x \rightarrow rA(x) + v.$$

where $A \in O(1, n)$, $r \neq 0$, and $v \in \mathbb{R}^{1,n}$.

Definition 1.34. Let $p \in \mathbb{E}^{1,n}$ and $r \in \mathbb{R}_+^*$. The de Sitter hypersphere of radius r centered at p is defined as

$$S_r(p) = \{p + v \in \mathbb{E}^{1,n} : \mathfrak{q}_{1,n}(v) = r^2\}.$$

The Lorentzian metric on $\mathbb{E}^{1,n}$ restricts to a Lorentzian metric on $S_r(p)$ having constant sectional curvature $1/r^2$. It is geodesically complete and homeomorphic to $\mathbb{S}^n \times \mathbb{R}$. It is a model for de-Sitter space $dS^{1,n-1}$ for $r = 1$.

Definition 1.35. For an arbitrary point $p \in \mathbb{E}^{1,n}$, the affine nullcone centered at p denoted by $L^{aff}(p)$ is the union of the lightlike geodesics through p .

1.5.1 Transformation group

The Lorentz group $O(1, n)$ acts on $\mathbb{R}^{1,n}$ naturally by $\Gamma(A, x) = Ax$. The Poincaré group is the group of isometries on $\mathbb{R}^{1,n}$ and it is isomorphic to the semi-direct product $O(1, n) \ltimes_{\Gamma} \mathbb{R}^{1,n}$. Also, the direct product group $\mathbb{R}^* \times O(1, n)$ acts linearly and conformally on $\mathbb{R}^{1,n}$ by $\Theta((r, A), x) = rAx$. The group of conformal transformations on $\mathbb{R}^{1,n}$ is isomorphic to the semi-direct product $(\mathbb{R}^* \times O(1, n)) \ltimes_{\Theta} \mathbb{R}^{1,n}$.

The group $\text{Conf}(\mathbb{E}^{1,n})$ is a Lie subgroup of $\text{Aff}(\mathbb{E}^{1,n})$. Since $\mathbb{E}^{1,n}$ is isometric to $\mathbb{R}^{1,n}$, we have

$$\text{Iso}(\mathbb{E}^{1,2}) \simeq O(1, n) \ltimes \mathbb{R}^{1,n}, \quad \text{Conf}(\mathbb{E}^{1,2}) \simeq (\mathbb{R}^* \times O(1, n)) \ltimes \mathbb{R}^{1,n}.$$

The multiplication rule in $\text{Conf}(\mathbb{E}^{1,n})$ is

$$(r, A, v), (s, B, w) \in \text{Conf}(\mathbb{E}^{1,n}), \quad (r, A, v)(s, B, w) = (rs, AB, rA(v) + w),$$

and so, the inverse element $(r, A, v)^{-1}$ is $(r^{-1}, A^{-1}, -r^{-1}A^{-1}(v))$.

The Lie algebra of $\text{Conf}(\mathbb{E}^{1,n})$ is isomorphic to the semi-direct sum $\mathfrak{g} = (\mathbb{R} \oplus \mathfrak{so}(1, n)) \oplus_{\theta} \mathbb{R}^{1,n}$, where θ denotes the representation of $\mathbb{R} \oplus \mathfrak{so}(1, n)$ in $gl(\mathbb{R}^{1,n})$ corresponding to the action Θ .

Notation 1.36. For elements $a \in \mathbb{R}$, $V \in \mathfrak{so}(1, n)$, and $v \in \mathbb{R}^{1,n}$, we denote the corresponding element in \mathfrak{g} simply by $a + V + v$.

According to Notation (1.36), the Lie bracket rule on \mathfrak{g} is

$$[a + V + v, b + W + w] = [V, W] + V(w) + aw - W(v) - bv.$$

The adjoint action of $\text{Conf}(\mathbb{E}^{1,n})$ on \mathfrak{g} is as following. For an arbitrary element $(r, A, v) \in \text{Conf}(\mathbb{E}^{1,n})$, we have

$$Ad_{(r,A,v)} : \mathfrak{g} \rightarrow \mathfrak{g}, \quad a + W + w \mapsto a + AWA^{-1} + rA(w) - av - AWA^{-1}(v).$$

1.5.2 Three dimensional Minkowski space

The group of conformal transformations on the 3-dimensional Minkowski space $\mathbb{E}^{1,2}$ is isomorphic to $(\mathbb{R}^* \times O(1,2)) \times \mathbb{R}^{1,2}$ and its connected component is isomorphic to $(\mathbb{R}_+^* \times SO_o(1,2)) \times \mathbb{R}^{1,2}$, where $SO_o(1,2)$ is the group of linear isometries on $\mathbb{R}^{1,2}$ preserving both time and space orientations.

Every element $A \in SO_o(1,2)$ has determinate 1, and because of the dimension, 1 is the common eigenvalue of all the elements in $O(1,2)$.

Lemma 1.37. *Let $A \in O(m,n)$, and $\lambda \in \text{Spec}(A)$. Then $\lambda^{-1} \in \text{Spec}(A)$.*

Proof. Assume that $B = \{e_1, \dots, e_{n+m}\}$ be an orthonormal basis for $\mathbb{R}^{m,n}$ and ε be a signature matrix for $\mathfrak{q}_{m,n}$ respect to B (see [37, p.p. 234]). Then for all $v, w \in \mathbb{R}^{m,n}$, we have $\langle v, w \rangle = \varepsilon v \cdot w$, where $\langle \cdot, \cdot \rangle$ is the usual euclidean inner product respect to the basis B . Now, $A \in O(m,n)$ if and only if

$$\begin{aligned} \forall v, w \in \mathbb{R}^{m,n}, \langle Av, Aw \rangle = \langle v, w \rangle &\iff \forall v, w \in \mathbb{R}^{m,n}, \varepsilon Av \cdot Aw = \varepsilon v \cdot w \\ &\iff \forall v, w \in \mathbb{R}^{m,n}, A^\dagger \varepsilon Av \cdot w = \varepsilon v \cdot w \iff \forall v \in \mathbb{R}^{m,n}, A^\dagger \varepsilon Av = \varepsilon v \\ &\iff A^\dagger \varepsilon A = \varepsilon \iff \varepsilon^{-1} A^\dagger \varepsilon = A^{-1}. \end{aligned}$$

This implies that A^\dagger and A^{-1} are similar, thus $\text{Spec}(A) = \text{Spec}(A^\dagger) = \text{Spec}(A^{-1})$. On the other hand, if $\lambda \in \text{Spec}(A)$, then $\lambda^{-1} \in \text{Spec}(A^{-1})$. Therefore $\lambda^{-1} \in \text{Spec}(A)$. \square

Lemma 1.38. *The Lie group $\text{PSL}(2, \mathbb{R})$ is isomorphic to the Lie groups $SO_o(1,2)$ and $SO_o(2,1)$.*

Proof. By Proposition 1.31, $(\mathfrak{sl}(2, \mathbb{R}), B)$ is a scalar product space of signature $(1,2)$. The adjoint action of $\text{PSL}(2, \mathbb{R})$ on $\mathfrak{sl}(2, \mathbb{R})$ is isometric respect to both B and $-B$, since the killing form is Ad -invariant. Hence,

$$\text{PSL}(2, \mathbb{R}) \simeq SO_o(1,2) \simeq SO_o(2,1).$$

\square

Remark 1.39. *According to Lemma 1.38, we use the same terminology and the same symbols for the objects related to the Lie groups $SO_o(1,2)$ and $SO_o(2,1)$ as we used for those of $\text{PSL}(2, \mathbb{R})$, when there is no ambiguity.*

Definition 1.40. *Let $Id \neq A \in SO_o(1,2)$. Then*

- *If A has only one eigenvalue, namely 1, then we call it a parabolic element. Every parabolic element of $SO_o(1,2)$, preserves a unique lightlike line in $\mathbb{R}^{1,2}$, and its corresponding (orthogonal) lightlike plane.*
- *If A has three distinct real eigenvalues, then we call it a hyperbolic element. Every hyperbolic element in $SO_o(1,2)$ preserves exactly two lightlike lines in $\mathbb{R}^{1,2}$ and their corresponding (orthogonal) lightlike planes.*
- *If A has two non-real eigenvalues, then we call it an elliptic element. Every elliptic element in $SO_o(1,2)$ preserves a timelike line in $\mathbb{R}^{1,2}$ and its corresponding (orthogonal) spacelike plane.*

The set of following matrices is a basis of the Lie algebra $\mathfrak{so}(1, 2) = \text{Lie}(SO_o(1, 2))$ as a vector space

$$\mathcal{Y}_E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \mathcal{Y}_H = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{Y}_P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix},$$

and we have, $[\mathcal{Y}_E, \mathcal{Y}_H] = \mathcal{Y}_E - \mathcal{Y}_P$, $[\mathcal{Y}_E, \mathcal{Y}_P] = \mathcal{Y}_H$ and $[\mathcal{Y}_H, \mathcal{Y}_P] = \mathcal{Y}_P$.

The group $SO_o(1, 2)$ has exactly three distinct 1-parameter subgroups up to conjugacy. We call a 1-parameter subgroup parabolic, hyperbolic, or elliptic, if it contains a parabolic, hyperbolic, or elliptic element. The 1-parameter subgroups generated by \mathcal{Y}_E , \mathcal{Y}_H , and \mathcal{Y}_P are elliptic, hyperbolic, and parabolic respectively and we denote them by Y_E , Y_H , and Y_P , respectively.

Furthermore, $SO_o(1, 2)$ has only one connected 2-dimensional subgroup up to conjugacy. We denote the 2-dimensional Lie subalgebra of $\mathfrak{so}(1, 2)$ generated by $\{\mathcal{Y}_H, \mathcal{Y}_P\}$ by aff . We call the connected Lie subgroup of $SO_o(1, 2)$ corresponding to aff , the affine subgroup and denote it by $\mathbb{A}\text{ff}$. Affine subgroup contains parabolic and hyperbolic elements, and they all preserve a lightlike line in $\mathbb{R}^{1,2}$ and its corresponding (orthogonal) lightlike plane. Also, we have $\mathbb{A}\text{ff} = Y_H \times Y_P$.

Remark 1.41. *Here is a useful model for the conformal group of the 3-dimensional Minkowski space $\mathbb{E}^{1,2}$ which will be used it in Chapter 6. Recall from Section 1.4.3 that, the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ equipped with the quadratic form $-\det$ is a model of Minkowski space. The direct product group $\mathbb{R}^* \times \text{PSL}(2, \mathbb{R})$ acts on $(\mathfrak{sl}(2, \mathbb{R}), -\det)$ linearly and conformally as following*

$$(r, [A]).X := rAXA^{-1}, \quad r \in \mathbb{R}^*, [A] \in \text{PSL}(2, \mathbb{R}), X \in \mathfrak{sl}(2, \mathbb{R}).$$

Now, in this model, the group of conformal transformations of 3-dimensional Minkowski space is isomorphic to $(\mathbb{R}^* \times \text{PSL}(2, \mathbb{R})) \ltimes \mathfrak{sl}(2, \mathbb{R})$ which acts on $\mathfrak{sl}(2, \mathbb{R}) \approx \mathbb{E}^{1,2}$ naturally as following

$$(r, [A], V).X := rAXA^{-1} + V, \quad r \in \mathbb{R}^*, [A] \in \text{PSL}(2, \mathbb{R}), V, X \in \mathfrak{sl}(2, \mathbb{R}).$$

1.6 de-Sitter and Anti de-Sitter space

In this section, we give the definitions of de-Sitter and Anti-de-Sitter space. For more details we refer to [37].

Definition 1.42. *The $(n + 1)$ -dimensional de-Sitter space $dS^{1,n}$ is the hypersurface $\{v \in \mathbb{R}^{1,n+1} : \mathfrak{q}_{1,n+1}(v) = 1\}$ of $\mathbb{R}^{1,n+1} = (\mathbb{R}^{n+2}, \mathfrak{q}_{1,n+1})$ endowed with the Lorentzian metric obtained by restriction of $\mathfrak{q}_{1,n+1}$.*

The de-Sitter space has constant sectional curvature 1. The geodesics in de-Sitter space, are the intersection of $dS^{1,n}$ with 2-planes in $\mathbb{R}^{1,n+1}$. The isometry group of $dS^{1,n}$ is isomorphic to the Lorentz group $O(1, n + 1)$.

Definition 1.43. *The $(n + 1)$ -dimensional Anti de-Sitter space $\text{AdS}^{1,n}$ is the hypersurface $\{v \in \mathbb{R}^{2,n} : \mathfrak{q}_{2,n}(v) = -1\}$ of $\mathbb{R}^{2,n} = (\mathbb{R}^{n+2}, \mathfrak{q}_{2,n})$ endowed with the Lorentzian metric obtained by restriction of $\mathfrak{q}_{2,n}$.*

The Anti de-Sitter space has constant sectional curvature -1 . The geodesics in Anti de-Sitter space, are the intersection of $\text{AdS}^{1,n}$ with 2-planes in $\mathbb{R}^{2,n}$. The isometry group of $\text{AdS}^{1,n}$ is isomorphic to the semi-orthogonal group $O(2, n)$.

Lemma 1.44. *The group $SO_o(2, 2)$ is isomorphic to $(\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}))/\mathbb{Z}_2$.*

Proof. Let $(M(2, \mathbb{R}), -\det)$ denote the vector space consists of 2×2 real matrices endowed with the quadratic form negative determinant of signature $(2, 2)$. The group $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ acts on $M(2, \mathbb{R})$ linearly and isometrically by

$$X \mapsto AXB^{-1}, \quad X \in M(2, \mathbb{R}), (A, B) \in \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \quad (1.3)$$

This induces a surjective representation from $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ to $SO_o(2, 2)$. The kernel of this representation is $\mathbb{Z}_2 = \{\pm(\text{Id}, \text{Id})\}$. This completes the proof. \square

Remark 1.45. *In the semi-Euclidean space $(M(2, \mathbb{R}), -\det)$, the 3-dimensional Anti de-Sitter space $\text{AdS}^{1,2}$ of radius 1 is the level set $-\det^{-1}(-1)$ which coincides with $\text{SL}(2, \mathbb{R}) \subset M(2, \mathbb{R})$.*

1.7 Einstein universe

In this section, we give the definition and some properties of Einstein universe $\mathbb{E}\text{in}^{1,n}$. For more details, we refer to [18], and [7].

Consider the scalar product space $\mathbb{R}^{2,n+1} = (\mathbb{R}^{n+3}, \mathfrak{q}_{2,n+1})$ for $n \geq 1$ and denote by $\langle \cdot, \cdot \rangle$ the bilinear form corresponding to $\mathfrak{q}_{2,n+1}$. The semi-orthogonal group $O(2, n+1)$ is the set of all linear isometries of $\mathbb{R}^{2,n+1}$. The nullcone $\mathfrak{N}^{2,n+1}$ of $\mathbb{R}^{2,n+1}$ is the set of non-zero null (lightlike) vectors in $\mathbb{R}^{2,n+1}$

$$\mathfrak{N}^{2,n+1} = \{v \in \mathbb{R}^{2,n+1} \setminus \{0\} : \mathfrak{q}_{2,n+1}(v) = 0\}.$$

The nullcone is a degenerated hypersurface of $\mathbb{R}^{2,n+1}$ homeomorphic to $\mathbb{S}^n \times (\mathbb{R}^2 \setminus \{0\})$, and it is invariant by the action of $O(2, n+1)$.

Definition 1.46. *The $(n+1)$ -dimensional Einstein universe $\mathbb{E}\text{in}^{1,n}$ is the image of $\mathfrak{N}^{2,n+1}$ under projectivization:*

$$\mathbb{P} : \mathbb{R}^{2,n+1} \setminus \{0\} \longrightarrow \mathbb{R}\mathbb{P}^{n+2}.$$

In the sequel, for notational convenience, we will denote \mathbb{P} as a map from $\mathbb{R}^{2,n+1}$ implicitly assuming that the origin 0 is removed from any subset of $\mathbb{R}^{2,n+1}$ on which we apply \mathbb{P} .

Definition 1.47. *The double covering space $\widehat{\mathbb{E}\text{in}}^{1,n}$ of Einstein universe is the quotient space of the nullcone $\mathfrak{N}^{2,n+1}$ by the action of positive scalar multiplication.*

For many purposes the double covering may be more useful than $\mathbb{E}\text{in}^{1,n}$, itself. Considering the bilinear form $\mathfrak{q}_{2,n+1}$, the nullcone is defined by

$$v_1^2 + v_2^2 = v_3^2 + \cdots + v_{n+3}^2.$$

This common value is always positive. Dividing a vector $v \in \mathfrak{N}^{2,n+1}$ by the positive number $\sqrt{v_1^2 + v_2^2}$, we may assume that

$$v_1^2 + v_2^2 = v_3^2 + \cdots + v_{n+3}^2 = 1,$$

which describes the product $\mathbb{S}^1 \times \mathbb{S}^n$. Thus $\widehat{\mathbb{E}in}^{1,n}$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^n \subset \mathbb{R}^{2,0} \times \mathbb{R}^{0,n+1}$. Scalar multiplication by -1 acts by the antipodal map on both \mathbb{S}^1 and \mathbb{S}^n -factor. Thus $\mathbb{E}in^{1,n} = \widehat{\mathbb{E}in}^{1,n} / \{\pm Id\}$ is deffeomorphic to $(\mathbb{S}^1 \times \mathbb{S}^n) / \{\pm Id\}$.

For an arbitrary open subset $U \subset \mathbb{E}in^{1,n}$, any local section $\sigma : U \rightarrow \mathbb{R}^{2,n+1} \setminus \{0\}$, of the restriction of \mathbb{P} to U determines a pullback of the metric on $\mathbb{R}^{2,n+1}$, to a Lorentzian metric \mathbf{g}_σ on U . For every section $\sigma' : U \rightarrow \mathbb{R}^{2,n+1} \setminus \{0\}$, we have $\sigma' = f\sigma$ for some non-vanishing function $f : U \rightarrow \mathbb{R}$. Then $\mathbf{g}_{\sigma'} = f^2 \mathbf{g}_\sigma$, so the pullbacks are conformally equivalent. Hence the metrics \mathbf{g}_σ altogether define a canonical conformal structure $[\mathbf{g}]$ on $\mathbb{E}in^{1,n}$. Obviously, this structure is $O(2, n + 1)$ -invariant.

The structure $[\mathbf{g}]$ lifts to a conformal Lorentzian structure $[\widehat{\mathbf{g}}]$ on the double cover $\widehat{\mathbb{E}in}^{1,n}$. The double cover $(\widehat{\mathbb{E}in}^{1,n}, [\widehat{\mathbf{g}}])$ is conformally equivalent to $(\mathbb{S}^1 \times \mathbb{S}^n, -d\theta^2 + ds^2)$, where $d\theta^2$ and ds^2 are the usual round metrics on the spheres \mathbb{S}^1 and \mathbb{S}^n of radius one [7].

The group of conformal transformations on $\widehat{\mathbb{E}in}^{1,2}$ is $O(2, n + 1)$. The conformal group of Einstein universe is the projective group $PO(2, n + 1) = O(2, n + 1) / \{\pm Id\}$. The groups $O(2, n + 1)$ and $\mathbb{P}O(2, n + 1)$ act transitively on $\widehat{\mathbb{E}in}^{1,n}$ and $\mathbb{E}in^{1,n}$, respectively, since, $O(2, n + 1)$ acts transitively on the nullcone $\mathfrak{N}^{2,n+1}$.

Definition 1.48. A photon in Einstein universe $\mathbb{E}in^{1,2}$ is the projectivizations of a totally isotropic 2-plane in $\mathbb{R}^{2,n}$. We denote the set of all photons by $\text{Pho}^{1,n}$.

In fact, a photons is a lightlike geodesic in Einstein universe and it is naturally homeomorphic to the real projective line $\mathbb{R}\mathbb{P}^1$.

Definition 1.49. Given any point $[p] \in \mathbb{E}in^{1,n}$ the lightcone $L([p])$ with vertex $[p]$ is the union of all photons containing $[p]$:

$$L([p]) := \{\phi \in \text{Pho}^{1,n} : [p] \in \phi\}.$$

A lightcone $L([p])$ can be equivalently defined as the projectivization of the orthogonal complement $p^\perp \cap \mathfrak{N}^{1,n}$. Furthermore, $L([p])$ is a singular hypersurface, and the only singular point on it is $[p]$. We denote the lightcone with removed vertex by $L([\hat{p}])$ and call it a *vertex-less lightcone*. A vertex-less lightcone is a degenerate hypersurface of $\mathbb{E}in^1$, and it is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{R}$. We also denote a photon $\phi \subset L([p])$ with removed the vertex $[p]$ by $\hat{\phi}$ and call it a *vertex-less photon*.

1.7.1 Conformal compactification of Minkowski space

Consider the scalar product space $\mathbb{R}^{2,n+1}$. Here it will be convenient to use the following scalar product

$$\langle v, w \rangle := -\frac{1}{2}v_1w_2 - \frac{1}{2}v_2w_1 - v_3w_3 + v_4w_4 + \cdots + v_{n+3}w_{n+3}.$$

Let $\langle \cdot, \cdot \rangle_\circ$ denote the bilinear form defined by $\mathfrak{q}_{1,n} = -I_1 \oplus I_n$ on $\mathbb{R}^{1,n}$. Now, consider the following embedding of the Minkowski space $\mathbb{R}^{1,n}$ into $\mathfrak{N}^{2,n+1}$.

$$\mathcal{E} : \mathbb{R}^{1,n} \longrightarrow \mathfrak{N}^{2,n+1} \subset \mathbb{R}^{2,n+1}, \quad x \mapsto \begin{bmatrix} 1 \\ \langle x, x \rangle_\circ \\ x \end{bmatrix}. \quad (1.4)$$

The composition $\Phi := \mathbb{P} \circ \mathcal{E}$, is a conformal embedding, and its image is a dense open subset of $\mathbb{E}in^{1,n}$.

Definition 1.50. Given any point $[p] \in \mathbb{E}in^{1,n}$, the Minkowski patch $Mink([p])$ is the complement of the lightcone $L([p])$ in $\mathbb{E}in^{1,n}$.

Proposition 1.51. For an arbitrary point $[p] \in \mathbb{E}in^{1,n}$, the Minkowski patch $Mink([p])$ is conformally equivalent to the Minkowski space $\mathbb{E}^{1,n}$.

Proof. Since $\mathbb{E}in^{1,n}$ is $\mathbb{P}O(2, n+1)$ -homogeneous, we may assume that $[p]$ has homogeneous coordinate $[0 : 1 : 0 : \dots : 0]$. Now, the conformal map Φ defined above, maps $\mathbb{E}^{1,n}$ to $Mink([0 : 1 : 0 : \dots : 0])$. This completes the proof. \square

Recall from Section 1.5, a geodesic γ through a point x in Minkowski space $\mathbb{E}^{1,n}$ has the form $\gamma(t) = x + tv$ where $x \in \mathbb{E}^{1,n}$ and $v \in \mathbb{R}^{1,n}$. Also, an affine d -plane through a point x in Minkowski space, is $x + \Pi$, where Π is a linear d -plane in underlying vector space $\mathbb{R}^{1,n}$.

Definition 1.52. Let $[p] \in \mathbb{E}in^{1,n}$ and $\gamma(t) = x + tv$ be a geodesic in $Mink([p]) \approx \mathbb{E}^{1,n}$. We call the point $[q] \in L([p])$ the limit point of γ in $L([p]) \subset \mathbb{E}in^{1,n}$, if $\lim_{t \rightarrow \pm\infty} \gamma(t) = [q]$. Similarly, for an affine degenerate hyperplane $\Pi \subset Mink([p])$, we call the point $[q] \in L([p])$ the limit point of Π in $L([p])$ if for all lightlike geodesics $\gamma(t) \subset \Pi$, $\lim_{t \rightarrow \pm\infty} \gamma(t) = [q]$.

Lemma 1.53. Let $[p] \in \mathbb{E}in^{1,n}$ and $[q] \in \widehat{L}([p])$ be an arbitrary point. Then there exists a lightlike geodesic $\gamma(t) = x + tv$ in $Mink([p])$ such that $[q]$ is the limit point of γ . Furthermore, for every lightlike geodesic $\beta(t) = y + tw$ in $Mink([p])$, the limit point of β is $[q]$ if and only if $v = \lambda w$ and $\langle x - \lambda y, v \rangle = 0$ for some $\lambda \in \mathbb{R}^*$.

Proof. Without losing generality, we may assume that $[p]$ has homogeneous coordinate $[0 : 1 : 0 : \dots : 0]$, since, $\mathbb{E}in^{1,n}$ is $\mathbb{P}O(2, n+1)$ -homogeneous. Since $[q] \in L([p])$, the first coordinate of $[q]$ is zero, i.e., $[q] = [0 : u : v]$ for some $u \in \mathbb{R}$ and some null vector $v \in \mathbb{R}^{1,n}$. There exists $x \in \mathbb{E}^{1,n}$ such that $\langle x, v \rangle = \frac{1}{2}u$. Now define $\gamma(t) = x + tv$. The image of γ has homogeneous coordinate

$$\gamma(t) = [1 : \langle x, x \rangle + 2t\langle x, v \rangle + t^2\langle v, v \rangle : x + tv] = [1 : \langle x, x \rangle + tu : x + tv].$$

For $t \neq 0$ we have

$$\gamma(t) = [1/t : \langle x, x \rangle/t + u : x/t + v].$$

and

$$\lim_{t \rightarrow \pm\infty} \gamma(t) = [0 : u : v] = [q].$$

For second part, we have

$$\lim_{t \rightarrow \pm\infty} \beta(t) = [0 : 2\langle y, w \rangle : w].$$

Obviously, $[q]$ is the limit point of β if and only if, $v = \lambda w$ and $\langle x - \lambda y, v \rangle = 0$, for some $\lambda \in \mathbb{R}^*$. \square

Lemma 1.54. Let $[p] \in \mathbb{E}in^{1,n}$ and $[q] \in Mink([p])$ be an arbitrary point. The intersection of the lightcone $L([q])$ with the Minkowski patch $Mink([p])$ is the affine nulcone of $L^{aff}([q])$.

Proof. Assume that $\gamma : \mathbb{R} \rightarrow Mink([p])$ be a lightlike geodesic through $[q]$. Obviously, $\gamma(t) \in L([q]) \cap Mink([p])$, for all $t \in \mathbb{R}$. Hence, $L^{aff}([q]) \subset L([q]) \cap Mink([p])$. On the other hand, suppose that $[q] \neq [x] \in L([q]) \cap Mink([p])$. The intersection of the unique photon ϕ in $L([q])$ through $[x]$ with the Minkowski patch $Mink([p])$ is the unique lightlike geodesic in $L^{aff}([q])$ through $[x] \in Mink([p])$. Thus, $[x] \in L^{aff}([q])$. This proves the lemma. \square

Proposition 1.55. *Let $[q] \in \widehat{L}([p])$ be an arbitrary point. Then there exists a unique affine degenerate hyperplane Π in $Mink([p])$, such that $[q]$ is the limit point of Π .*

Proof. By Lemma 1.53, there exists a lightlike geodesic $\gamma(t) = x + tv$, such that $[q]$ is the limit point of γ . We show that $x + v^\perp$ is the desired affine hyperplane. Let $\beta(t) = y + tw$, be a lightlike geodesic in $x + v^\perp$. There exist $\lambda \in \mathbb{R}^*$ and $u \in v^\perp$ such that $\beta(t) = x + u + t\lambda v$. Consider the limit point of β in $L([p])$.

$$\lim_{t \rightarrow \pm\infty} \beta(t) = [0 : 2\lambda\langle u + x, v \rangle : \lambda v].$$

Since $\langle x + u, v \rangle - \langle x, v \rangle = 0$, Lemma 1.53 implies that $\lim_{t \rightarrow \pm\infty} \beta(t) = [q]$.

For uniqueness, let Π' be an affine lightlike hyperplane with limit point $[q]$. Let $\alpha(t) = z + tw$ be a lightlike geodesic in Π' . Then

$$\lim_{t \rightarrow \pm\infty} \alpha(t) = [0 : 2\langle z, w \rangle : w] = [q] = [0 : 2\langle x, v \rangle : v].$$

Using Lemma 1.53 again we have, $w = \lambda v$ and $\langle x - \lambda z, v \rangle = 0$. Hence $x - \lambda z \in v^\perp$ and so, $z \in x + v^\perp$. Thus $\alpha(t) \subset x + v^\perp$. This completes the proof. \square

Lemma 1.56. *Let $[q_1], [q_2] \in L([\hat{p}])$ be two distinct points. Then $[q_1]$ and $[q_2]$ lie on the same photon in $L([p])$ if and only if their corresponding degenerate affine planes in $Mink([p])$ are parallel.*

Proof. In the setting of Proposition 1.55, we have $[q_1] = [0 : u_1 : v_1]$ and $[q_2] = [0 : u_2 : v_2]$, where $u_1, u_2 \in \mathbb{R}$ are constant numbers and $v_1, v_2 \in \mathbb{R}^{1,2}$ are null vectors. Then, there exist $x_1, x_2 \in Mink([p]) \approx \mathbb{E}^{1,2}$ such that $x_1 + v_1^\perp$ and $x_2 + v_2^\perp$ are the degenerate affine hyperplanes in $Mink([p])$ correspond to $[q_1]$ and $[q_2]$, respectively. The linear 2-plane in $\mathbb{R}^{2,n+1}$ generated by vectors $q_1 = (0, u_1, v_2)$ and $q_2 = (0, u_2, v_2)$ is totally isotropic if and only if $\langle q_1 | q_2 \rangle = 0$ if and only if $\langle v_1, v_2 \rangle = 0$ if and only if $v_1 = \lambda v_2$ if and only if there exists a translation $w \in \mathbb{R}^{1,2}$ such that $x_1 + v_1^\perp + w = x_2 + v_2^\perp$. This completes the proof. \square

Lemma 1.57. [18, p.p. 56] *By the action of $O(2, n)$ on Einstein universe $\mathbb{E}in^{1,n}$, the stabilizer of a point in $\mathbb{E}in^{1,n}$ is isomorphic to $\text{Conf}(\mathbb{E}^{1,n}) \simeq (\mathbb{R}^* \times O(1, n)) \ltimes \mathbb{R}^{1,n}$.*

Let ϕ be a photon in Einstein universe $\mathbb{E}in^{1,n}$. The complement of ϕ in $\mathbb{E}in^{1,n}$ is an open dense subset and we denote it by $\mathbb{E}in_\phi^{1,n}$. It is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^n$. There is a natural codimension 1 foliation \mathcal{F}_ϕ on $\mathbb{E}in_\phi^{1,n}$ on which the leaves are degenerate hypersurfaces diffeomorphic to \mathbb{R}^n . Fixing a point $[p] \in \phi$, the complement of ϕ in the lightcone $L([p])$ is a leaf of \mathcal{F}_ϕ . The other leaves are the degenerate affine hyperplanes in $Mink([p])$ with limit point in ϕ . The group of conformal transformations on $\mathbb{E}in_\phi^{1,n}$ acts on $\mathbb{E}in_\phi^{1,n}$ transitively and it preserves the foliation \mathcal{F}_ϕ . In fact, $\text{Conf}(\mathbb{E}in_\phi^{1,n})$ is the stabilizer of ϕ by the action of $PO(2, n)$ [18].

Definition 1.58. *For an integer n , the $(2n+1)$ -dimensional Heisenberg group $H(2n+1)$ is the unique (up to isomorphism) simply connected Lie group which its center C is a 1-dimensional subgroup isomorphic to \mathbb{R} , and the quotient $H(2n+1)/C$ is isomorphic to the additive group \mathbb{R}^{2n} .*

The Heisenberg group $H(2n + 1)$ is a unipotent group and one can consider it as a subgroup of $GL(n + 2, \mathbb{R})$ as following:

$$\left\{ \begin{bmatrix} 1 & u & a \\ 0 & I_{n \times n} & v^\dagger \\ 0 & 0_{1 \times n} & 1 \end{bmatrix} : u, v \in \mathbb{R}^n, a \in \mathbb{R} \right\}.$$

Lemma 1.59. [18, Lemma 4.15] *By the action of $PO(2, n)$ on Einstein universe $\mathbb{E}in^{1,n}$, the stabilizer of a photon is isomorphic to $(\mathbb{R}^* \times SL(2, \mathbb{R}) \times O(n - 2)) \ltimes H(2n - 3)$ where $H(2n - 3)$ is the $(2n - 3)$ -dimensional Heisenberg group.*

1.7.2 de Sitter and Anti-de Sitter components

In this section, we describe de-Sitter and Anti de-Sitter components in Einstein universe $\mathbb{E}in^{1,n}$.

Definition 1.60. *A spacelike hypersphere in $\mathbb{E}in^{1,n}$ is the one-point compactification of a spacelike affine hyperplane of a Minkowski patch in $\mathbb{E}in^{1,n}$.*

Equivalently, spacelike hyperspheres are projectivizations of $v^\perp \cap \mathfrak{N}^{2,n+1}$ for timelike vectors $v \in \mathbb{R}^{2,n+1}$. It can be easily seen that a spacelike hypersphere in $\mathbb{E}in^{1,n}$ is naturally conformally equivalent to the usual Riemannian round n -sphere \mathbb{S}^n .

Definition 1.61. *A de-Sitter component in $\mathbb{E}in^{1,n}$ is the complement of a spacelike hypersphere $\mathbb{S}^n \subset \mathbb{E}in^{1,n}$. It is homeomorphic to $\mathbb{S}^n \times \mathbb{R}$ and evidently, its conformal boundary is \mathbb{S}^n .*

Lemma 1.62. *A de-Sitter component in $\mathbb{E}in^{1,n}$ is conformally equivalent to the model de-Sitter space $dS^{1,n}$ in $\mathbb{R}^{1,n+1}$ (described in Definition 1.42).*

Proof. Assume that $v \in \mathbb{R}^{2,n+1}$ is a timelike vector. The orthogonal complement subspace v^\perp is of signature $(1, n + 1)$. Let $\{e_1, \dots, e_{n+2}\}$ be an orthonormal basis for v^\perp on which e_1 is a timelike, and $e_{n+3} \in \mathbb{R}v$ be a unit vector. Then $B = \{e_1, \dots, e_{n+2}, e_{n+3}\}$ is an orthonormal basis for $\mathbb{R}^{2,n+1}$. Let $[p] \in \mathbb{E}in^{1,n} \setminus \mathbb{P}(v^\perp)$ be an arbitrary point. There exist unique non-zero vectors $x \in v^\perp$ and $y \in \mathbb{R}v$ such that $p = x + y$. Let (x_1, \dots, x_{n+2}, y) denotes the coordinate of $p = x + y$ respect to the basis B . Then

$$-x_1^2 + x_2^2 + \dots + x_{n+2}^2 = y^2.$$

Since y is non-zero, dividing p by y , we may assume

$$-x_1^2 + x_2^2 + \dots + x_{n+2}^2 = 1,$$

which describes the de-Sitter space $dS^{1,n}$ in $v^\perp \approx \mathbb{R}^{1,n+1}$. □

Definition 1.63. *An Einstein hypersphere is the closure of a de-Sitter hypersphere $S_r([q]) \subset Mink([p])$ (defined in Definition 1.35) in $\mathbb{E}in^{1,n}$, for some $[p] \in \mathbb{E}in^{1,n}$ and $[q] \in Mink([p])$.*

The Einstein hyperspheres are projectivizations of $v^\perp \cap \mathfrak{N}^{2,n+1}$ for spacelike vectors $v \in \mathbb{R}^{2,n+1}$. For $n \geq 2$, every Einstein hypersphere in $\mathbb{E}in^{1,n}$ has a natural structure of $\mathbb{E}in^{1,n-1}$. For $n = 1$, every Einstein hypersphere is homeomorphic to disjoint union of two circles.

Definition 1.64. For $n \geq 2$, an Anti de-Sitter component in $\mathbb{E}in^{1,n}$ is the complement of an Einstein hypersphere $\mathbb{E}in^{1,n-1} \subset \mathbb{E}in^{1,n}$. It is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}^n$ and evidently, its boundary is $\mathbb{E}in^{1,n-1}$.

Lemma 1.65. An Anti de-Sitter component in $\mathbb{E}in^{1,n}$ is conformally equivalent to the model Anti-de Sitter space $AdS^{1,n}$ in $\mathbb{R}^{2,n}$ (described in Definition 1.43).

Proof. Assume that $v \in \mathbb{R}^{2,n+1}$ be a spacelike vector. The orthogonal complement subspace v^\perp is of signature $(2, n)$. Let $\{e_1, e_2, \dots, e_{n+2}\}$ be an orthonormal basis for v^\perp on which e_1 and e_2 are timelike vectors, and $e_{n+3} \in \mathbb{R}v$ be a unit vector. Then $B = \{e_1, \dots, e_{n+2}, e_{n+3}\}$ is an orthonormal basis for $\mathbb{R}^{2,n+1}$. Let $[p] \in \mathbb{E}in^{1,n} \setminus \mathbb{P}(v^\perp)$ be an arbitrary point. There exist unique non-zero vectors $x \in v^\perp$ and $y \in \mathbb{R}v$ such that $p = x + y$. Let (x_1, \dots, x_{n+2}, y) denotes the coordinate of $p = x + y$ respect to the basis B . Then

$$-x_1^2 - x_2^2 + x_3^2 + \dots + x_{n+2}^2 = -y^2.$$

Since y is non-zero, dividing p by y , we may assume

$$-x_1^2 - x_2^2 + x_3^2 + \dots + x_{n+2}^2 = -1,$$

which describes the Anti de-Sitter space $AdS^{1,n}$ in $v^\perp \approx \mathbb{R}^{2,n}$. □

Remark 1.66. It is remarkable that, for an Einstein hypersphere $\mathbb{E}in^{1,n-1} \subset \mathbb{E}in^{1,n}$ and an arbitrary point $p \in \mathbb{E}in^{1,n-1}$, the intersection of $\mathbb{E}in^{1,n-1}$ with $Mink^{1,n}(p)$ (the Minkowski patch at p in Einstein universe $\mathbb{E}in^{1,n}$) is an affine Lorentzian hyperplane Π in the Minkowski patch $Mink(p)$. One can see that, the limit points of the geodesics in Π in the lightcone $L^{1,n}(p)$ (the lightcone of p in $\mathbb{E}in^{1,n}$) determines the lightcone $L^{1,n-1}(p)$ (the lightcone of p in $\mathbb{E}in^{1,n-1}$). This is an alternate description of Einstein hyperspheres in $\mathbb{E}in^{1,n}$ which is the Lorentzian analogue of Definition 1.60.

1.7.3 Two dimensional Einstein universe

The 2-dimensional Einstein universe is diffeomorphic to a 2-torus. Each lightcone $L([p])$ in $\mathbb{E}in^{1,1}$ consists of two photons which intersects at $[p]$. The conformal group of $\mathbb{E}in^{1,1}$ is isomorphic to $\mathbb{P}O(2, 2)$.

Here is a useful model of $\mathbb{E}in^{1,1}$. We use again the vector space $M(2, \mathbb{R})$ equipped with the quadratic form $-\det$ as we used in the proof of Lemma 1.44. In this model, the nullcone $\mathfrak{N}^{2,2}$ of $(M(2, \mathbb{R}), -\det)$ is the set of nonzero singular matrices and the 2-dimensional Einstein universe $\mathbb{E}in^{1,1}$ is the quotient of the nullcone by the action of nonzero scalar multiplication. By Lemma 1.44, the groups $PO_\circ(2, 2)$ and $PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$ are isomorphic.

Every nonzero singular $X \in M(2, \mathbb{R})$ determines two lines in \mathbb{R}^2 : its kernel and its image. Let $([A], [B]) \in PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$ be an arbitrary element. Then $[A]$ preserves $\ker X$ and $[B]$ preserves $\text{Im } X$. Therefore, there is a canonical identification of $\mathbb{E}in^{1,1}$ with $\mathbb{R}P^1 \times \mathbb{R}P^1$ by

$$\mathbb{E}in^{1,1} \rightarrow \mathbb{R}P^1 \times \mathbb{R}P^1, \quad [X] \mapsto [\ker X : \text{Im } X].$$

By this identification, the left factor of $PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$ acts trivially on the left factor $\mathbb{R}P^1$, and the right factor acts trivially on the right factor $\mathbb{R}P^1$.

1.7.4 Three dimensional Einstein universe

The 3-dimensional Einstein universe $\mathbb{E}in^{1,2}$ is conformally equivalent to $(\mathbb{S}^1 \times \mathbb{S}^2)/\{\pm Id\}$ on which $\mathbb{S}^1 \times \mathbb{S}^2$ is equipped with the metric tensor $-d\theta^2 + ds^2$, where $d\theta^2$ and ds^2 are the usual round metrics on the spheres \mathbb{S}^1 and \mathbb{S}^2 of radius one. The group of conformal transformations on $\mathbb{E}in^{1,2}$ is isomorphic to $O(2, 3)$. We denote its identity component by $SO_o(2, 3)$, which is the set of linear isometries on $\mathbb{R}^{2,3}$ preserving both space and time orientations.

Definition 1.67. Let $[p], [q] \in \mathbb{E}in^{1,2}$ be two distinct point and they do not lie on a common photon. Then, the intersection of the lightcones $L([p])$ and $L([q])$ is called an ideal circle.

Lemma 1.68. An ideal circle is projectivized nullcones of a linear subspace of $\mathbb{R}^{2,3}$ of signature $(1, 2)$.

Proof. Let $[p], [q] \in \mathbb{E}in^{1,2}$ and $[q] \in Mink(p)$. Observe that, the intersection of the two degenerate hyperplanes $p^\perp, q^\perp \leq \mathbb{R}^{2,3}$ is a linear subspace of signature $(1, 2)$. Now, the lemma follows easily. \square

Definition 1.69. A timelike circle in $\mathbb{E}in^{1,2}$ is the projectivized nullcone of a linear subspace of $\mathbb{R}^{2,3}$ of signature $(2, 1)$.

Lemma 1.70. The complement of a timelike circle in $\mathbb{E}in^{1,2}$ is conformally equivalent (up to double cover) to $(AdS^{1,1} \times \mathbb{S}^1, d\sigma^2 + d\theta^2)$ where $d\sigma^2$ (resp. $d\theta^2$) is the usual Lorentzian metric on $AdS^{1,1}$ of constant sectional curvature -1 (resp. positive definite metric on \mathbb{S}^1 of constant sectional curvature 1). Furthermore, by the action of $O(2, 3)$, the identity component of the stabilizer of a timelike circle is isomorphic to the direct product $SO_o(2, 1) \times SO(2)$.

Proof. Let $\mathcal{C} \subset \mathbb{E}in^{1,2}$ be a timelike circle and $V \leq \mathbb{R}^{2,3}$ the linear subspace corresponding to \mathcal{C} . The orthogonal complement V^\perp is of signature $(0, 2)$. Choose an orthonormal basis $\{e_1, e_2, e_3\}$ for V where e_3 is spacelike, and an orthonormal basis $\{e_4, e_5\}$ for V^\perp . Then $B = \{e_1, \dots, e_5\}$ is an orthonormal basis of $\mathbb{R}^{2,3}$. Suppose that $[p] \in \mathbb{E}in^{1,2} \setminus \mathcal{C}$ is an arbitrary point, and $p = (p_1, \dots, p_5)$ is an arbitrary representative of $[p]$. We have

$$-p_1^2 - p_2^2 + p_3^2 = -p_4^2 - p_5^2.$$

Observe that $(p_4, p_5) \in V^\perp$ is a non-zero vector. Therefore, dividing p by the positive number $\sqrt{p_4^2 + p_5^2}$, we may assume

$$-p_1^2 - p_2^2 - p_3^2 = -p_4^2 - p_5^2 = -1,$$

which describes the product $AdS^{1,1} \times \mathbb{S}^1 \subset \widehat{\mathbb{E}in}^{1,2}$.

Obviously, the identity component of the stabilizer of V is isomorphic to direct product $SO_o(2, 1) \times SO(2)$. This completes the proof. \square

Remark 1.71. From now on, for a point $[p] \in \mathbb{E}in^{1,2} \subset \mathbb{RP}^4$ and a representative in $\mathbb{R}^{2,3}$ of the class $[p]$, we use the same symbol p when there is no ambiguity.

Chapter 2

Proper actions

In this chapter, we describe the cohomogeneity one proper actions on the 3-dimensional Einstein universe $\mathbb{E}\text{in}^{1,2}$.

Theorem 2.1. *Let $G \subset SO_{\circ}(2, 3)$ be a connected Lie group which acts on $\mathbb{E}\text{in}^{1,2}$ properly and with cohomogeneity one. Then G is conjugate to either $SO(3)$ or $SO(2) \times SO(2)$. Furthermore, the action of $SO(3)$ on $\mathbb{E}\text{in}^{1,2}$ admits a codimension 1 foliation on which each leaf is a spacelike hypersphere. Moreover, the action of $SO(2) \times SO(2)$ on $\mathbb{E}\text{in}^{1,2}$ preserves a timelike circle and acts on it transitively. Also, it admits a codimension 1 foliation on the complement of timelike circle on which each leaf is conformally equivalent to 2-dimensional Einstein universe $\mathbb{E}\text{in}^{1,1}$ and exactly one of them is an Einstein hypersphere.*

Lemma 2.2. *Let G be a Lie group which acts on a compact space X continuously and properly. Then G is compact.*

Proof. Since G acts on X continuously and properly the following map is proper.

$$\pi : G \times X \longrightarrow X \times X, \quad (g, x) \mapsto (gx, x).$$

The space $G \times X$ is compact, since it is the inverse image of the compact space $X \times X$ through the proper map π . This implies that G is compact. \square

It is well-known that every maximal compact subgroup of $SO_{\circ}(2, 3)$ is conjugate to $SO(2) \times SO(3)$ (cf. [27, p.p. 275]). This group preserves a 3-dimensional subspace $V \leq \mathbb{R}^{2,3}$ of signature $(0, 3)$. Also, it preserves the orthogonal complement subspace $V^{\perp} \leq \mathbb{R}^{2,3}$ (which is of signature $(2, 0)$). Considering the action of $SO(2) \times SO(3)$ on the double cover space $\widehat{\mathbb{E}\text{in}}^{1,2} \approx \mathbb{S}^1 \times \mathbb{S}^2$, the $SO(2)$ -factor acts on $\{*\} \times \mathbb{S}^2$ trivially and $SO(3)$ -factor acts on $\mathbb{S}^1 \times \{*\}$ trivially. Obviously, $SO(2) \times SO(3)$ acts on $\widehat{\mathbb{E}\text{in}}^{1,2}$ transitively.

Proposition 2.3. *The group $SO(3) \simeq \text{Iso}_{\circ}(\mathbb{S}^2)$ has no 2-dimensional connected Lie subgroup.*

Proof. First, we show that every 1-parameter subgroup of $SO(3)$ is conjugate to $SO(2)$, the identity component of the stabilizer of a point in \mathbb{S}^2 by the action of $SO(3)$. Let $H \subset SO(3)$ be a 1-parameter subgroup. We show that H admits a fixed point in \mathbb{S}^2 , then the desired result follows, since $SO(3)$ acts on \mathbb{S}^2 transitively. Assume the contrary which is: H admits no fixed point in \mathbb{S}^2 . Therefore, the stabilizer of each point is discrete. By [10, Theorem 11.3.9], H admits a codimension 1 foliation on \mathbb{S}^2 . But this

contradicts the fact that a compact manifold admits a codimension 1 foliation if and only if its Euler characteristic vanishes (see [30]). Hence, H fixes a point in \mathbb{S}^2 .

Assume the contrary statement of the proposition which is: G is a 2-dimensional Lie subgroup of $SO(3)$. In the one hand, by preceding paragraph, every 1-parameter subgroup of G is isomorphic to $SO(2)$. On the other hand, by [38, p.p. 212], G is isomorphic to the 2-torus \mathbb{T}^2 , \mathbb{R}^2 , $SO(2) \times \mathbb{R}$, or affine group $\mathbb{A}ff$. This is a contradiction, since all these groups admit a 1-parameter subgroup isomorphic to \mathbb{R} . \square

Lemma 2.4. *Let G be a proper connected Lie subgroup of $SO(2) \times SO(3)$ with $\dim G \geq 2$. Then, either $G = \{Id\} \times SO(3) \simeq SO(3)$ or G is conjugate to $SO(2) \times SO(2)$.*

Proof. Let P_1 and P_2 denote the projection morphisms from $SO(2) \times SO(3)$ to $SO(2)$ and $SO(3)$, respectively. The group G is a subgroup of $P_1(G) \times P_2(G)$. Proposition 2.3 implies that $\dim P_2(G) \in \{1, 3\}$. If $\dim P_2(G) = 1$, then $P_2(G) = SO(2)$ up to conjugacy and $P_1(G) = SO(2)$. Therefore, $G = SO(2) \times SO(2)$ up to conjugacy.

Now, suppose that $\dim P_2(G) = 3$. Since G is a proper subgroup, $\dim G = 3$. Hence, the differential map dP_2 at the identity element of G is a Lie algebra isomorphism from $\mathfrak{g} = Lie(G)$ to $Lie(SO(3)) = \mathfrak{so}(3)$. So, $f = dP_1 \circ (dP_2)^{-1} : \mathfrak{so}(3) \rightarrow dP_1(\mathfrak{g})$ is a surjective Lie algebra morphism. If $P_1(G) = SO(2)$, then $\ker f$ is a 2-dimensional ideal of $\mathfrak{so}(3)$. This contradicts the simplicity of $\mathfrak{so}(3)$. Hence $P_1(G) = \{Id\}$. Therefore, $G = \{Id\} \times SO(3) \simeq SO(3)$. \square

Proof of Theorem 2.1. In the one hand, since G acts properly, Lemma 2.2 implies that it is compact, since $\mathbb{E}in^{1,2}$ is compact. So, it is a subgroup of $SO(2) \times SO(3)$ up to conjugacy. On the other hand, since G acts with cohomogeneity one, it is a proper subgroup of $SO(2) \times SO(3)$. Hence, by Lemma 2.4, G is conjugate either to $SO(3)$ or $SO(2) \times SO(2)$. We show that both these groups act by cohomogeneity one. There exist a unique $SO(2) \times SO(3)$ -invariant decomposition for $\mathbb{R}^{2,3} = V \oplus V^\perp$ where V is of signature $(2, 0)$ (and hence V^\perp is of signature $(0, 3)$).

Observe that by the action of $SO(3)$ on the double cover space $\widehat{\mathbb{E}in}^{1,2}$, the induced orbit at each point $(x, y) \in \mathbb{S}^1 \times \mathbb{S}^2$ is $\{x\} \times \mathbb{S}^2$. This orbit is the projectivization of $x^\perp \cap \mathfrak{R}^{2,3} \subset \mathbb{R}^{2,3}$ on $\widehat{\mathbb{E}in}^{1,2}$. It is clear that the group $\mathbb{Z}_2 = \{\pm Id\}$ acts on $\{x\} \times \mathbb{S}^2$ trivially, for all $x \in \mathbb{S}^1$. Hence, each orbit induced by $SO(3)$ in $\mathbb{E}in^{1,2}$ is a spacelike hypersphere.

The action of $SO(2)$ on \mathbb{S}^2 admits two antipodal fixed points $\{x_0, -x_0\} \in \mathbb{S}^2 \in \mathbb{P}(V^\perp)$ and acts on $\mathbb{S}^2 \setminus \{x_0, -x_0\}$ freely. Hence, $SO(2) \times SO(2)$ preserves a circle

$$\mathfrak{C} = (\mathbb{S}^1 \times \{x_0, -x_0\})/\mathbb{Z}_2 = \mathbb{P}(V \oplus \mathbb{R}x_0) \cap \mathbb{E}in^{1,2},$$

and acts on it transitively. It is clear that, for arbitrary $y \in \mathbb{S}^1$ and $x \in \mathbb{S}^2 \setminus \{x_0, -x_0\}$, the orbit induced by $SO(2) \times SO(2)$ at $(y, x) \in \mathbb{S}^1 \times \mathbb{S}^2$ is conformally equivalent to $\widehat{\mathbb{E}in}^{1,1}$. Hence, the orbit induced at $[x : y] \in \mathbb{E}in^{1,2}$ is conformally equivalent to $\widehat{\mathbb{E}in}^{1,1}/\mathbb{Z}_2 = \mathbb{E}in^{1,1}$. Since $SO(2) \times SO(2)$ preserves the spacelike line $\mathbb{R}x_0$, it preserves the orthogonal complement subspace $x_0^\perp \leq \mathbb{R}^{2,3}$. Hence the orbit induced at $[p] \in \mathbb{P}(x_0^\perp) \cap \mathbb{E}in^{1,2}$ is an Einstein hypersphere. This completes the proof. \square

Corollary 2.5. *Let $G \subset SO_o(2, 3)$ be a connected Lie subgroup which acts on $\mathbb{E}in^{1,2}$ properly and with cohomogeneity one. Then G admits a fixed point in the projective space $\mathbb{R}P^4$.*

Proof. By Theorem 2.1, G is conjugate either to $SO(3)$ or $SO(2) \times SO(2)$. According to the proof of Theorem 2.1, considering the actions on the double cover space $\widehat{\mathbb{E}in}^{1,2} = \mathbb{S}^1 \times \mathbb{S}^2$, the group $SO(3)$ fixes the \mathbb{S}^1 -factor pointwisely. On the other hand, $SO(2) \times SO(2)$ preserves the spacelike line $\mathbb{R}x_0$. Hence, it fixes $\mathbb{P}(\mathbb{R}x_0) \in \mathbb{RP}^4$. This completes the proof. \square

Chapter 3

The irreducible action of $\mathrm{PSL}(2, \mathbb{R})$

In this chapter, we study the irreducible action of $\mathrm{PSL}(2, \mathbb{R})$ on the 3-dimensional Einstein universe $\mathbb{E}\mathrm{in}^{1,2}$ which is of cohomogeneity one¹.

By Theorem 0.4, up to conjugacy, $\mathrm{PSL}(2, \mathbb{R}) \simeq \mathrm{SO}_o(1, 2)$ is the only connected proper subgroup of $O(2, 3)$ which acts on $\mathbb{R}^{2,3}$ irreducibly. Recall from Section 1.4.4, there is only one irreducible representation of $\mathrm{PSL}(2, \mathbb{R})$ in $GL(\mathbb{R}^5)$. This representation is the natural action of $\mathrm{PSL}(2, \mathbb{R})$ on the vector space $\mathbb{V} = \mathbb{R}_4[X, Y]$ of homogeneous polynomials of degree 4 in two variables X and Y . This action induces three types of orbits in the 4-dimensional projective space $\mathbb{R}\mathbb{P}^4 = \mathbb{P}(\mathbb{V})$: an 1-dimensional orbit, three 2-dimensional orbits, and the orbits which $\mathrm{PSL}(2, \mathbb{R})$ acts on them freely.

Also, this action preserves the quadratic form q of signature $(2, 3)$ given in the coordinate system

$$(X^4, X^3Y, X^2Y^2, XY^3, Y^4),$$

by

$$q(a_4X^4 + a_3X^3Y + a_2X^2Y^2 + a_1XY^3 + a_0Y^4) = 2a_4a_0 - \frac{1}{2}a_1a_3 + \frac{1}{6}a_2^2.$$

This induces a representation $\mathrm{PSL}(2, \mathbb{R}) \rightarrow O(2, 3) \subset \mathrm{PSL}(5, \mathbb{R})$. Consequently, the irreducible action of $\mathrm{PSL}(2, \mathbb{R})$ on $\mathbb{R}^{2,3}$ induces a conformal action on 3-dimensional Einstein universe $\mathbb{E}\mathrm{in}^{1,2}$.

Theorem 3.1. *The irreducible action of $\mathrm{PSL}(2, \mathbb{R})$ on the 3-dimensional Einstein universe $\mathbb{E}\mathrm{in}^{1,2}$ admits three orbits:*

- an 1-dimensional lightlike orbit, i.e. of signature $(0, 0, 1)$,
- a 2-dimensional degenerate orbit i.e. of signature $(0, 1, 1)$,
- an open orbit (hence of signature $(1, 2)$) on which the action is free.

The 1-dimensional orbit is lightlike, homeomorphic to $\mathbb{R}\mathbb{P}^1$, but not a photon. The union of the 1-dimensional orbit and the 2-dimensional orbit is an algebraic surface, whose singular locus is precisely the 1-dimensional orbit. It is the union of all projective lines tangent to the 1-dimensional orbit. Figure 3.1 describes a part of the 1 and 2-dimensional orbits in the Minkowski patch $\mathrm{Mink}([Y^4])$.

¹The results of this chapter are published in [26].

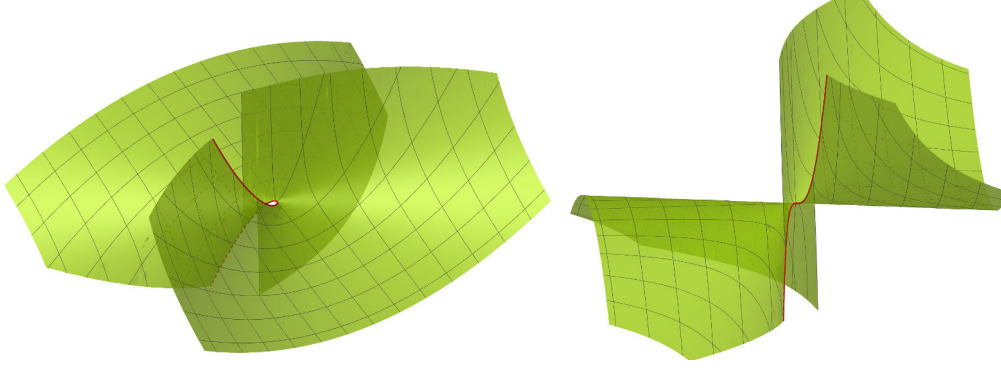


Figure 3.1: Two partial views of the intersection of the 1 and 2-dimensional orbits in Einstein universe with $\text{Mink}([Y^4])$. **Red:** Part of the 1-dimensional orbit in Minkowski patch. **Green:** Part of the 2-dimensional orbit in Minkowski patch.

The complement of the Einstein universe in \mathbb{RP}^4 has two connected components: the 3-dimensional Anti de-Sitter space $\text{AdS}^{1,2}$ and the generalized hyperbolic space $\mathbb{H}^{2,2}$: the first (respectively the second) is the projection of the domain $\mathbb{R}^{2,3}$ defined by $\{q < 0\}$ (respectively $\{q > 0\}$).

We will also describe the actions on Anti de-Sitter space and the generalized hyperbolic space $\mathbb{H}^{2,2}$:

Theorem 3.2. *The orbits of $\text{PSL}(2, \mathbb{R})$ in the Anti de-sitter component $\text{AdS}^{1,3}$ are Lorentzian, i.e. of signature $(1, 2)$. They are the leaves of a codimension 1 foliation. In addition, $\text{PSL}(2, \mathbb{R})$ induces three types of orbits in $\mathbb{H}^{2,2}$: a 2-dimensional spacelike orbit (of signature $(2, 0)$) homeomorphic to the hyperbolic plane \mathbb{H}^2 , a 2-dimensional Lorentzian orbit (i.e., of signature $(1, 1)$) homeomorphic to the de-Sitter space $\text{dS}^{1,1}$, and four kinds of 3-dimensional orbits where the action is free:*

- one-parameter family of orbits of signature $(2, 1)$ consisting of elements with four distinct non-real roots,
- one-parameter family of Lorentzian (i.e. of signature $(1, 2)$) orbits consisting of elements with four distinct real roots,
- two orbits of signature $(1, 1, 1)$,
- one-parameter family of Lorentzian (i.e. of signature $(1, 2)$) orbits consisting of elements with two distinct real roots, and two distinct complex conjugate roots so that the cross-ratio of the four roots has an argument strictly between $-\pi/3$ and $\pi/3$.

Let f be an element in \mathbb{V} . We consider it as a polynomial function from \mathbb{C}^2 into \mathbb{C} . Actually, by specifying $Y = 1$, we consider f as a polynomial of degree at most 4. Such a polynomial is determined, up to a scalar, by its roots z_1, z_2, z_3, z_4 in \mathbb{CP}^1 (some of these roots can be ∞ if f can be divided by Y). It provides a natural identification between $\mathbb{P}(\mathbb{V})$ and the set $\widehat{\mathbb{CP}}_4^1$ made of 4-tuples (up to permutation) (z_1, z_2, z_3, z_4) of \mathbb{CP}^1 such that if some z_i is not in \mathbb{RP}^1 , then its conjugate \bar{z}_i is one of the z_j 's. This identification is $\text{PSL}(2, \mathbb{R})$ -equivariant, where the action of $\text{PSL}(2, \mathbb{R})$ on $\widehat{\mathbb{CP}}_4^1$ is simply the one induced by the diagonal action on $(\mathbb{CP}^1)^4$.

Actually, the complement of \mathbb{RP}^1 in \mathbb{CP}^1 is the union of the upper half-plane model \mathbb{H}^2 of the hyperbolic plane, and the lower half-plane. We can represent every element of $\widehat{\mathbb{CP}}_4^1$ by a 4-tuple (up to permutation) (z_1, z_2, z_3, z_4) such that:

- either every z_i lies in \mathbb{RP}^1 ,
- or z_1, z_2 lies in \mathbb{RP}^1 , z_3 lies in \mathbb{H}^2 and $z_4 = \bar{z}_3$,
- or z_1, z_2 lies in \mathbb{H}^2 and $z_3 = \bar{z}_1, z_4 = \bar{z}_2$.

Theorems 3.1 and 3.2 will follow from the following proposition:

Proposition 3.3. *Let $[f]$ be an element of $\mathbb{P}(\mathbb{V})$. Then:*

- *it lies in $\mathbb{Ein}^{1,2}$ if and only if it has a root of multiplicity at least 3, or two distinct real roots z_1, z_2 , and two complex roots $z_3, z_4 = \bar{z}_3$, with z_3 in \mathbb{H}^2 and such that the argument of the cross-ratio of z_1, z_2, z_3, z_4 is $\pm\pi/3$.*
- *it lies in $\text{AdS}^{1,3}$ if and only if it has two distinct real roots z_1, z_2 , and two complex roots $z_3, z_4 = \bar{z}_3$, with z_3 in \mathbb{H}^2 and such that the argument of the cross-ratio of z_1, z_2, z_3, z_4 has absolute value $> \pi/3$.*
- *it lies in $\mathbb{H}^{2,2}$ if and only if it has no real roots, or four distinct real roots, or a root of multiplicity exactly 2, or it has two distinct real roots z_1, z_2 , and two complex roots $z_3, z_4 = \bar{z}_3$, with z_3 in \mathbb{H}^2 and such that the argument of the cross-ratio of z_1, z_2, z_3, z_4 has absolute value $< \pi/3$.*

Proof. Assume that f has no real root. Hence we are in the situation where z_1, z_2 lie in \mathbb{H}^2 and $z_3 = \bar{z}_1, z_4 = \bar{z}_2$. By applying a suitable element of $\text{PSL}(2, \mathbb{R})$, we can assume $z_1 = i$, and $z_2 = ri$ for some $r > 0$. In other words, f is in the $\text{PSL}(2, \mathbb{R})$ -orbit of $(X^2 + Y^2)(X^2 + r^2Y^2)$. The value of q on this polynomial is $2 \times 1 \times r^2 + \frac{1}{6}(1 + r^2)^2 > 0$, hence $[f]$ lies in $\mathbb{H}^{2,2}$.

Hence we can assume that f admits at least one root in \mathbb{RP}^1 , and by applying a suitable element of $\text{PSL}(2, \mathbb{R})$, one can assume that this root is ∞ , i.e. that f is a multiple of Y .

We first consider the case where this real root has multiplicity at least 2:

$$f = Y^2(aX^2 + bXY + cY^2)$$

Then, $q(f) = \frac{1}{6}a^2$: it follows that if f has a root of multiplicity at least 3, it lies in $\mathbb{Ein}^{1,2}$, and if it has a real root of multiplicity 2, it lies in $\mathbb{H}^{2,2}$.

We assume from now that the real roots of f have multiplicity 1. Assume that all roots are real. Up to $\text{PSL}(2, \mathbb{R})$, one can assume that these roots are $0, 1, r$ and ∞ with $0 < r < 1$.

$$f(X, Y) = XY(X - Y)(X - rY) = X^3Y - (r + 1)X^2Y^2 + rXY^3.$$

Then, $q(f) = -\frac{1}{2}r + \frac{1}{6}(r + 1)^2 = \frac{1}{6}(r^2 - r + 1) > 0$. Therefore f lies in $\mathbb{H}^{2,2}$ once more.

The only remaining case is the case where f has two distinct real roots, and two complex conjugate roots z, \bar{z} with $z \in \mathbb{H}^2$. Up to $\text{PSL}(2, \mathbb{R})$, one can assume that the real roots are $0, \infty$, hence:

$$f(X, Y) = XY(X - zY)(X - \bar{z}Y) = XY(X^2 - 2|z|\cos\theta XY + |z|^2Y^2)$$

where $z = |z|e^{i\theta}$. Then:

$$q(f) = \frac{2|z|^2}{3}(\cos^2\theta - \frac{3}{4}).$$

Hence f lies in $\mathbb{Ein}^{1,2}$ if and only if $\theta = \pi/6$ or $5\pi/6$. The proposition follows easily. \square

Remark 3.4. *F. Fillastre indicated to us that our description of the open orbit in $\mathbb{E}in^{1,2}$ appearing in the first item of Proposition 3.3 has an alternative and more elegant description: this orbit corresponds to polynomials whose roots in $\mathbb{C}\mathbb{P}^1$ are ideal vertices of a regular ideal tetraedra in \mathbb{H}^3 .*

Proof of Theorem 3.1. It follows from Proposition 3.3 that there are precisely three $\mathrm{PSL}(2, \mathbb{R})$ -orbits in $\mathbb{E}in^{1,2}$:

– one orbit \mathcal{N} comprising polynomials with a root of multiplicity 4, i.e. of the form $[(sY - tX)^4]$ with $s, t \in \mathbb{R}$. It is clearly 1-dimensional, and equivariantly homeomorphic to $\mathbb{R}\mathbb{P}^1$ with the usual projective action of $\mathrm{PSL}(2, \mathbb{R})$. Since $\frac{d}{dt}|_{t=0}(Y - tX)^4 = -4XY^3$ is a \mathfrak{q} -null vector, this orbit is lightlike (but cannot be a photon since the action is irreducible).

– one orbit \mathcal{L} comprising polynomials with a real root of multiplicity 3, and another real root. These are the polynomials of the form $[(sY - tX)^3(s'Y - t'X)]$ with $s, t, s', t' \in \mathbb{R}$. It is 2-dimensional, and it is easy to see that it is the union of the projective lines tangent to \mathcal{N} . The vectors tangent to \mathcal{L} induced by the 1-parameter subgroups Y_P and Y_E at $[XY^3] \in \mathcal{L}$ are $v_P = -Y^4$ and $v_E = 3X^2Y^2 + Y^4$. Obviously, v_P is orthogonal to v_E and $v_E + v_P$ is spacelike. Hence \mathcal{L} is of signature $(0, 1, 1)$.

– one open orbit comprising polynomials admitting two distinct real roots and a root in \mathbb{H}^2 such that the argument of the cross-ratio of the four roots is $\pi/3$. The stabilizers of points in this orbit are trivial since an isometry of \mathbb{H}^2 preserving a point in \mathbb{H}^2 and one point in $\partial\mathbb{H}^2$ is necessarily the identity. \square

Proof of Theorem 3.2. According to Proposition 3.3, the polynomials in $\mathrm{AdS}^{1,3}$ have two distinct real roots, and a complex root \mathbb{H}^2 (and its conjugate) such that the argument of the cross-ratio of the four roots has absolute value $> \pi/3$. It follows that the action in $\mathrm{AdS}^{1,3}$ is free, and that the orbits are the level sets of the function θ . Suppose that M is a $\mathrm{PSL}(2, \mathbb{R})$ -orbit in $\mathrm{AdS}^{1,3}$. There exists $r \in \mathbb{R}$ such that $[f] = [Y(X^2 + Y^2)(X - rY)] \in M$. The orbit induced by the 1-parameter subgroup Y_E at $[f]$ is

$$\gamma(t) = [(X^2 + Y^2)((\sin t \cos t - r \sin^2 t)X^2 - (\sin t \cos t + r \cos^2 t)Y^2 + (\cos^2 t - \sin^2 t + 2r \sin t \cos t)XY)].$$

Then $\mathfrak{q}(\frac{d\gamma}{dt}|_{t=0}) = -2 - 2r^2 < 0$. This implies, as for any submanifold of a Lorentzian manifold admitting a timelike vector, that M is Lorentzian, i.e., of signature $(1, 2)$.

The case of $\mathbb{H}^{2,2}$ is the richest one. According to Proposition 3.3 there are four cases to consider:

- *No real roots.* Then f has two complex roots z_1, z_2 in \mathbb{H}^2 (and their conjugates). It corresponds to two orbits: one orbit corresponding to the case $z_1 = z_2$: it is spacelike and has dimension 2. It is the only maximal $\mathrm{PSL}(2, \mathbb{R})$ -invariant surface in $\mathbb{H}^{2,2}$ described in [11, Section 5.3]. The case $z_1 \neq z_2$ provides a one-parameter family of 3-dimensional orbits on which the action is free (the parameter being the hyperbolic distance between z_1 and z_2). One may assume that $z_1 = i$ and $z_2 = ri$ for some $r > 0$. Denote by M the orbit induced by $\mathrm{PSL}(2, \mathbb{R})$ at $[f] = [(X^2 + Y^2)(X^2 + r^2Y^2)]$. The vectors tangent to M at $[f]$ induced by the 1-parameter subgroups Y_H, Y_P and Y_E are:

$$v_H = -4X^4 + 4r^2Y^4, \quad v_P = -4X^3Y - 2(r^2 + 1)XY^3, \quad v_E = 2(r^2 - 1)X^3Y + 2(r^2 - 1)XY^3,$$

respectively. The timelike vector v_H is orthogonal to both v_P and v_E . It is easy to see that the 2-plane generated by $\{v_P, v_E\}$ is of signature $(1, 1)$. Therefore, the tangent space $T_{[f]}M$ is of signature $(2, 1)$.

- *Four distinct real roots.* This case provides a one-parameter family of 3-dimensional orbits on which the action is free - the parameter being the cross-ratio between the roots in $\mathbb{R}\mathbb{P}^1$. Denote

by M the $\mathrm{PSL}(2, \mathbb{R})$ -orbit at $[f] = [XY(X - Y)(X - rY)]$ (here as explained in the proof of Proposition 3.3, we can restrict ourselves to the case $0 < r < 1$). The vectors tangent to M at $[f]$ induced by the 1-parameter subgroups Y_H , Y_P , and Y_E are:

$$\begin{aligned} v_H &= -rY^4 + 2(r+1)XY^3 - 3X^2Y^2, & v_P &= -2X^3Y + 2rXY^3, \\ v_E &= X^4 - rY^4 + 3(r-1)X^2Y^2 + 2(r+1)XY^3 - 2(r+1)X^3Y, \end{aligned}$$

respectively. A vector $x = av_H + bv_P + cv_E$ is orthogonal to v_P if and only if $2ra + b(r+1) + c(r+1)^2 = 0$. Set $a = (b(r+1) + c(r+1)^2) / -2r$ in

$$q(x) = 2ra^2 + \frac{3}{2}b^2 + \left(\frac{7}{2}(r^2+1) - r\right)c^2 + 2(r+1)ab + 2(r+1)^2 + ac(2r^2 - r + 5).$$

Consider $q(x) = 0$ as a quadratic polynomial F in b . Since $0 < r < 1$, the discriminant of F is non-negative and it is positive when $c \neq 0$. Thus, the intersection of the orthogonal complement of the spacelike vector v_P with the tangent space $T_{[f]}M$ is a 2-plane of signature $(1, 1)$. This implies that M is Lorentzian, i.e., of signature $(1, 2)$.

- *A root of multiplicity 2.* Observe that if there is a non-real root of multiplicity 2, when we are in the first "no real root" case. Hence we consider here only the case where the root of multiplicity 2 lies in \mathbb{RP}^1 . Then, we have three subcases to consider:

- two distinct real roots of multiplicity 2: The orbit induced at X^2Y^2 is the image of the $\mathrm{PSL}(2, \mathbb{R})$ -equivariant map

$$dS^{1,1} \subset \mathbb{P}(\mathbb{R}_2[X, Y]) \longrightarrow \mathbb{H}^{2,2}, \quad [L] \mapsto [L^2],$$

where $\mathbb{R}_2[X, Y]$ is the vector space of homogeneous polynomials of degree 2 in two variables X and Y , endowed with discriminant as a $\mathrm{PSL}(2, \mathbb{R})$ -invariant bilinear form of signature $(1, 2)$ [11, Section 5.3]. (Here, L is the projective class of a Lorentzian bilinear form on \mathbb{R}^2). The vectors tangent to the orbit at X^2Y^2 induced by the 1-parameter subgroups Y_P and Y_E are $v_P = -2XY^3$ and $v_E = 2X^3Y - 2XY^3$, respectively. It is easy to see that the 2-plane generated by $\{v_P, v_E\}$ is of signature $(1, 1)$. Hence, the orbit induced at X^2Y^2 is Lorentzian.

- three distinct real roots, one of them being of multiplicity 2: Denote by M the orbit induced by $\mathrm{PSL}(2, \mathbb{R})$ at $[f] = [XY^2(X - Y)]$. The vectors tangent to M at $[f]$ induced by the 1-parameter subgroups Y_H , Y_P and Y_E are:

$$v_H = -2XY^3, \quad v_P = Y^4 - 2XY^3, \quad v_E = Y^4 - X^4 - 2X^2Y^2 + X^3Y - XY^3,$$

respectively. Obviously, the lightlike vector $v_H + v_P$ is orthogonal to $T_{[f]}M$. Therefore, the restriction of the metric on $T_{[f]}M$ is degenerate. It is easy to see that the quotient of $T_{[f]}M$ by the action of the isotropic line $\mathbb{R}(v_H + v_P)$ is of signature $(1, 1)$. Thus, M is of signature $(1, 1, 1)$.

- one real root of multiplicity 2, and one root in \mathbb{H}^2 : Denote by M the orbit induced by $\mathrm{PSL}(2, \mathbb{R})$ at $[f] = [Y^2(X^2 + Y^2)]$. The vectors tangent to M at $[f]$ induced by the 1-parameter subgroups Y_H , Y_P and Y_E are $v_H = 4Y^4$, $v_P = -2XY^3$, and $v_E = 2X^3Y + 2XY^3$, respectively. Obviously, the lightlike vector v_H is orthogonal to $T_{[f]}M$. Therefore, the restriction of the metric on $T_{[f]}M$ is degenerate. It is easy to see that the quotient of $T_{[f]}M$ by the action of the isotropic line $\mathbb{R}(v_H)$ is of signature $(1, 1)$. Thus M is of signature $(1, 1, 1)$.

- *Two distinct real roots, and a complex root in \mathbb{H}^2 (and its conjugate) such that the argument of the cross-ratio of the four roots has absolute value $< \pi/3$. Denote by M the orbit induced by $\text{PSL}(2, \mathbb{R})$ at $[f] = [Y(X^2 + Y^2)(X - rY)]$. The vectors tangent to M at $[f]$ induced by the 1-parameter subgroups Y_H, Y_P and Y_E are:*

$$v_H = -4rY^4 - 2X^3Y + 2XY^3, \quad v_P = -3X^2Y^2 + 2rXY^3 - Y^4, \quad v_E = X^4 - Y^4 - 2rX^3Y - 2rXY^3,$$

respectively. The following set of vectors is an orthogonal basis for $T_{[f]}M$ where the first vector is timelike and the two others are spacelike.

$$\{(7r + 3r^3)v_H + (6 - 2r^2)v_P + (5 + r^2)v_E, 4v_P + v_E, v_H\}.$$

Therefore, M is Lorentzian, i.e., of signature $(1, 2)$. □

Actions on Minkowski patch and lightcone

In this chapter, we study the cohomogeneity one actions on Einstein universe $\mathbb{E}in^{1,2}$ admitting a fixed point in $\mathbb{E}in^{1,2}$. Let $G \subset \text{Conf}(\mathbb{E}in^{1,2})$ be a Lie subgroup fixing a point $p \in \mathbb{E}in^{1,2}$. Then G preserves the lightcone $L(p)$, since G preserves the degenerate hyperplane $p^\perp \leq \mathbb{R}^{2,3}$. Hence, it also preserves the Minkowski patch $Mink(p) \approx \mathbb{E}^{1,2}$. This induces a representation from G into $\text{Conf}(\mathbb{E}^{1,2})$ the group of conformal transformations of Minkowski space $\mathbb{E}^{1,2}$. Therefore, a cohomogeneity one action on $\mathbb{E}in^{1,2}$ fixing a point $p \in \mathbb{E}in^{1,2}$ admits either an open orbit in $L(p)$ or a 2-dimensional orbit in the Minkowski space $Mink(p) \approx \mathbb{E}^{1,2}$.

Recall from Section 1.7.1, by the action of $SO_o(2, 3)$, the identity component of the stabilizer of a point $p \in \mathbb{E}in^{1,2}$ is isomorphic to $\text{Conf}_o(\mathbb{E}^{1,2}) \simeq (\mathbb{R}_+^* \times SO_o(1, 2)) \ltimes \mathbb{R}^{1,2}$. Also, recall from Section 1.2 that, the splitting $(\mathbb{R}_+^* \times SO_o(1, 2)) \ltimes \mathbb{R}^{1,2}$ depends on the unique point $o \in \mathbb{E}^{1,2}$ fixed by $\mathbb{R}_+^* \times SO_o(1, 2)$.

Theorem 4.1. *All the connected Lie subgroups of $\text{Conf}(\mathbb{E}^{1,2}) \simeq (\mathbb{R}_+^* \times SO_o(1, 2)) \ltimes \mathbb{R}^{1,2}$ with $\dim \geq 2$ acts with cohomogeneity one on Einstein universe, except $\mathbb{R}^{1,2}$ and $\mathbb{R}_+^* \ltimes \mathbb{R}^{1,2}$.*

Proof. It follows immediately from Theorem 4.4. □

Remark 4.2. *It is remarkable that choosing $o \in \mathbb{E}^{1,2}$ as the origin of Minkowski space, it becomes a vector space. Moreover, the quadratic form $q_{1,2}$ on $\mathbb{R}^{1,2}$ induces a quadratic form $q := q_{1,2} \circ (\Theta_o)^{-1}$ on $\mathbb{E}^{1,2}$ where Θ_o is the orbit map of the point o . This makes $(\mathbb{E}^{1,2}, o)$ a Lorentzian scalar product space.*

The following maps are Lie group morphisms:

$$\begin{aligned} P_l &: (\mathbb{R}_+^* \times SO_o(1, 2)) \ltimes \mathbb{R}^{1,2} \longrightarrow \mathbb{R}_+^* \times SO_o(1, 2), & (\lambda, A, v) &\mapsto (\lambda, A), \\ P_{li} &: (\mathbb{R}_+^* \times SO_o(1, 2)) \ltimes \mathbb{R}^{1,2} \longrightarrow SO_o(1, 2), & (\lambda, A, v) &\mapsto A, \\ P_h &: (\mathbb{R}_+^* \times SO_o(1, 2)) \ltimes \mathbb{R}^{1,2} \longrightarrow \mathbb{R}_+^*, & (\lambda, A, v) &\mapsto \lambda. \end{aligned}$$

The morphisms P_l , P_{li} , and P_h are called the linear projection, the linear isometry projection and the homothety projects, respectively.

Definition 4.3. *Let G be a Lie subgroup of $\text{Conf}_o(\mathbb{E}^{1,2})$. We call the image of G under P_l , P_{li} , and P_h the linear projection, the linear isometry projection, and the homothety projection of G , respectively. Also, we call the identity component of the kernel of $P_l|_G$ the translation part of G and denote it by $T(G)$.*

Theorem 4.4. *Let G be a Lie subgroup of $\text{Conf}_o(\mathbb{E}^{1,2})$. Then G admits a 2-dimensional orbit in $\mathbb{E}^{1,2}$ if and only if $\dim G \geq 2$ and it satisfies one of the following conditions.*

- *The linear isometry projection $P_i(G)$ is non-trivial.*
- *The linear isometry projection $P_i(G)$ is trivial and the translation part $T(G)$ has dimension less than or equal to 2.*

4.1 Actions on lightcone

According to Proposition 1.55, for an arbitrary point $q \in L(\hat{p})$, there exists a unique affine degenerate plane $\tilde{\Pi}$ in $\mathbb{E}^{1,2} \approx \text{Mink}(p)$ such that q is the limit point of $\tilde{\Pi}$. This induces a one-to-one correspondence between the set of photons in $L(p)$ and the set of degenerate (linear) 2-planes in $\mathbb{R}^{1,2}$: Choosing a photon $\phi \subset L(p)$ there exists a unique degenerate plane $\Pi \leq \mathbb{R}^{1,2}$ such that the corresponding affine degenerate planes with limit points in ϕ are parallel to Π . On the other hand, choosing a degenerate plane Π in $\mathbb{R}^{1,2}$ there exists a unique photon ϕ on which the limit points of the leaves of the foliation induced by Π in $\mathbb{E}^{1,2}$ lie in ϕ (see Lemma 1.56). From now on, for a photon ϕ in the lightcone $L(p)$, we denote the corresponding degenerate plane in $\mathbb{R}^{1,2}$ by Π_ϕ and the foliation induced by Π_ϕ in $\mathbb{E}^{1,2}$ by \mathcal{F}_{Π_ϕ} .

The subgroup $\mathbb{R}_+^* \times SO_o(1, 2)$ fixes a unique point o in Minkowski space $\mathbb{E}^{1,2}$. Thus, every element of $\mathbb{R}_+^* \times SO_o(1, 2)$ maps each affine degenerate plane through o to an affine degenerate plane through o . Hence, this group preserves an ideal circle $S_\infty \simeq \mathbb{RP}^1$ which is the intersection of the lightcones $L(p)$ and $L(o)$.

Suppose λ is a homothety on Minkowski space $\mathbb{E}^{1,2}$ respect to the origin. Obviously, λ preserves every degenerate affine plane through o . Thus, the homothety factor \mathbb{R}_+^* acts on the ideal circle S_∞ trivially. Let ϕ be a photon through p and $\tilde{\Pi}_q$ be the corresponding affine degenerate plane with limit point $q = S_\infty \cap \phi$. For an arbitrary point $u \in \hat{\phi} \setminus [q]$ with corresponding affine degenerate plane Π_u , the homothety λ maps Π_u to a parallel affine degenerate plane above [or below] (depending on the position of Π_u respect to Π_q) Π_q . Thus, the homothety factor \mathbb{R}_+^* maps Π_u to all affine degenerate planes parallel to Π_q above [or below] Π_q . Therefore, \mathbb{R}_+^* acts on the both connected components of $\hat{\phi} \setminus q$, transitively.

Let g be a non-trivial element in $SO_o(1, 2)$.

- If g is an elliptic element, then it preserves no degenerate plane in $\mathbb{R}^{1,2}$, and so, g preserves no photon in $L(p)$. This implies that every elliptic 1-parameter subgroup of $SO_o(1, 2)$ acts on $L(\hat{p})$ freely.
- If g is a parabolic element, then it preserves a unique degenerate plane in $\mathbb{R}^{1,2}$. Consequently, g preserves a unique photon ϕ in $L(p)$ and admits a unique fixed point in the ideal circle S_∞ . Therefore, every parabolic 1-parameter subgroup of $SO_o(1, 2)$ acts on $L(p) \setminus \phi$ freely.
- If g is a hyperbolic element, then it preserves exactly two degenerate planes in $\mathbb{R}^{1,2}$. Hence, g preserves two photons ϕ and ψ in the lightcone $L(p)$ and admits exactly two fixed points in the ideal circle S_∞ . Henceforth, every hyperbolic 1-parameter subgroup of $SO_o(1, 2)$ acts on $L(p) \setminus (\phi \cup \psi)$ freely.

Now, let $v \in \mathbb{R}^{1,2}$ be a translation on Minkowski space $\mathbb{E}^{1,2}$. Assume that $\Pi \leq \mathbb{R}^{1,2}$ be a degenerate linear plane and $\tilde{\Pi}$ be an affine degenerate plane in $\mathbb{E}^{1,2}$ parallel to Π . Then, v maps $\tilde{\Pi}$ to the parallel plane $v + \tilde{\Pi}$. Hence, every translation preserves each photon in $L(p)$ (see Lemma 1.56).

- If v is a timelike vector, then it preserves no affine degenerate plane in $\mathbb{E}^{1,2}$. Hence, a timelike vector admits no fixed point in the vertex-less lightcone $L(\hat{p})$.
- If v is a lightlike vector, then it preserves all the affine degenerate planes in $\mathbb{E}^{1,2}$ parallel to v^\perp . This implies that the set of points in $L(p)$ fixed by a lightlike element is a unique photon.
- If v is a spacelike vector, then it preserves the two degenerate planes $\Pi, \Pi' \leq \mathbb{R}^{1,2}$ directed by the two distinct lightlike directions in the timelike plane v^\perp . Obviously, v preserves all the affine degenerate planes in $\mathbb{E}^{1,2}$ which are parallel to one of Π or Π' . Hence the set of points in $L(p)$ fixed by a spacelike element is the union of two distinct photons.

Let G be a connected Lie subgroup of $\mathbb{R}_+^* \times SO_o(1, 2) \times \mathbb{R}^{1,2}$. The translation part $T(G)$ is a normal subgroup of G , hence G acts on $T(G)$ by conjugation. Therefore, the natural action of the linear isometry projection $P_{li}(G)$ on $\mathbb{R}^{1,2}$ preserves the translation part $T(G) \leq \mathbb{R}^{1,2}$. Furthermore, assume that $T(G)$ fixes a photon ϕ in the lightcone $L([p])$ pointwisely. Thus, $T(G)$ preserves the leaves of the foliation \mathcal{F}_{Π_ϕ} in $\mathbb{E}^{1,2}$. Henceforth, $T(G)$ is a linear subspace of Π_ϕ . This implies that, $T(G)$ is either a degenerate subspace or it is a spacelike line. In the first, obviously $P_{li}(G)$ preserves Π_ϕ . In the later, it is easy to see that $P_{li}(G)$ is a hyperbolic 1-parameter subgroup. Hence, it preserves the two degenerate planes generated by $T(G)$ and one of the null directions in the timelike plane $T(G)^\perp$. We conclude that, if $T(G)$ fixes a photon $\phi \subset L(p)$ pointwisely, then ϕ is invariant by $P_{li}(G)$ and consequently, by G .

Proposition 4.5. *Let G be a connected Lie subgroup of $\mathbb{R}_+^* \times \mathbb{R}^{1,2}$. Then G is conjugate to the semidirect product $P_h(G) \ltimes T(G)$.*

Proof. First assume that $P_h(G)$ is trivial. Then, obviously $G = T(G)$. Now, suppose that $P_h(G) \neq \{1\}$. Let L be a 1-parameter subgroup of G transversal to $T(G)$. Considering the Lie algebra of L , one can see, there exists a unique vector $v \in \mathbb{R}^{1,2}$, such that $L = \{(e^t, (e^t - 1)v) : t \in \mathbb{R}\} \subset \mathbb{R}_+^* \times \mathbb{R}^{1,2}$. Observe that L is conjugate to \mathbb{R}_+^* via $(1, v)$ and $T(G)$ is invariant by this conjugation. Therefore, G is conjugate to $\mathbb{R}_+^* \ltimes T(G)$. \square

Proof of Theorem 4.4. Assume that G admits a 2-dimensional orbit at $q \in \mathbb{E}in^{1,2}$. Obviously $\dim G \geq 2$. If $q \in L(p)$, then $P_{li}(G)$ is nontrivial, since the subgroup $\mathbb{R}_+^* \times \mathbb{R}^{1,2}$ admits no open orbit in $L(p)$. If q belongs to the Minkowski patch $Mink(p) \approx \mathbb{E}^{1,2}$, then $\dim T(G) \leq 2$, since the subgroup $\mathbb{R}^{1,2}$ acts on $\mathbb{E}^{1,2}$ transitively.

Now, we prove the reverse direction. Suppose that $\dim G \geq 2$. First, assume that the linear isometry projection $P_{li}(G)$ is non-trivial. There are some cases:

Case I: The translation part is non-trivial. In this case, the photons which are fixed pointwisely by $T(G)$ (if there exists any) are exactly those which $P_{li}(G)$ preserves them. Hence there exists a photon $\phi \subset L(p)$ such that $T(G)$ acts on $\hat{\phi}$ transitively and G does not preserve it. Let H be a 1-parameter subgroup of G which is transversal to $T(G)$ and its linear isometry projection $P_{li}(H)$ is non-trivial. Since $P_{li}(G) \neq \{Id\}$, such a subgroup exists. Assume that $q \in \hat{\phi}$ is an arbitrary point. Then the vector tangent to $G(q)$ at q induced by L is spacelike, since L does not preserve ϕ . On the other hand, the tangent space

$T_q G(q)$ contains a lightlike vector, since $\hat{\phi}$ is a lightlike curve in $G(q)$ through q . This shows that $T_q G(q)$ is 2-dimensional and so, G admits an open orbit in $L(p)$. *Case II: The translation part is trivial:* There are two subcases.

- ▶ G contains the homothety factor \mathbb{R}_+^* as a subgroup. The subgroup \mathbb{R}_+^* preserves the ideal circle S_∞ and acts on $L(p) \setminus (S_\infty \cup \{p\})$ freely. Let L be a 1-parameter subgroup of G transversal to \mathbb{R}_+^* . Obviously, $P_{li}(L)$ is non-trivial. There exists a photon $\phi \subset L(p)$ which is not invariant by L . For an arbitrary point $q \in \phi \setminus (S_\infty \cup \{p\})$, the vectors tangent to the orbit $G(q)$ at q induced by \mathbb{R}_+^* and L are lightlike and spacelike, respectively. Hence, G admits an open orbit in $L(p)$.
- ▶ *The homothety factor \mathbb{R}_+^* is not a subgroup of G .* In this case, we have $\dim G = \dim P_{li}(G) \in \{2, 3\}$.
 - If $\dim G = 3$, then $P_{li}(G) = SO_o(1, 2)$ -in fact G is isomorphic to $SO_o(1, 2)$ -. The group G is a Levi factor of $(\mathbb{R}_+^* \times SO_o(1, 2)) \ltimes \mathbb{R}^{1,2}$. By the uniqueness of Levi factor we have $G = SO_o(1, 2)$, up to conjugacy (see [29, p.p. 93]). Obviously, $SO_o(1, 2)$ admits a 2-dimensional orbit in $\mathbb{E}^{1,2}$.
 - If $\dim G = 2$, then $P_{li}(G) = \text{Aff}_o(1, \mathbb{R})$, up to conjugacy. Hence, G preserves a unique photon $\phi \subset L(p)$. Let \mathcal{P} be a 1-parameter subgroup of G which its linear isometry projection $P_{li}(\mathcal{P})$ is parabolic. Furthermore, assume that \mathcal{H} is a 1-parameter subgroup of G which is transversal to \mathcal{P} . Obviously, \mathcal{H} has hyperbolic linear isometry projection and it also preserves a photon $\psi \subset L(p)$ different from ϕ . Denote by Π_ψ the degenerate plane in $\mathbb{R}^{1,2}$ correspond to ψ . Observe that \mathcal{P} acts on $L(p) \setminus \phi$ freely, since it preserves only ϕ . Assume that \mathcal{H} induces an open orbit $I \subset \psi$. Then \mathcal{P} maps I to other photons, and therefore, G admits an open orbit in $L(p)$. If \mathcal{H} fixes ψ pointwisely, then it preserves the leaves of the foliation \mathcal{F}_{Π_ψ} induced by Π_ψ in $\mathbb{E}^{1,2}$. Note that, because of the parabolicity of \mathcal{P} , it does not preserve this foliation. Observe that \mathcal{H} admits a 1-dimensional orbit at some point $q \in \mathbb{E}^{1,2}$ included in the leaf $\mathcal{F}_{\Pi_\psi}(q)$, since the action of $\text{Conf}(\mathbb{E}^{1,2})$ is faithful. The vector tangent to $G(q)$ at q induced by \mathcal{P} does not lie in Π_ψ , since \mathcal{P} does not preserve Π_ψ . Hence, G admits a 2-dimensional orbit at q .

Finally, assume that the linear isometry projection $P_{li}(G)$ is trivial, then G is a subgroup of $\mathbb{R}_+^* \times \mathbb{R}^{1,2}$. By Proposition 4.5, G is conjugate to $P_h(G) \ltimes T(G)$. Observe that $T(G)$ admits a $\dim T(G)$ -dimensional foliation in $\mathbb{E}^{1,2}$. If the homothety projection $P_h(G)$ is trivial, then G is a linear 2-plane in $\mathbb{R}^{1,2}$ and so, all the orbits in $\mathbb{E}^{1,2}$ are 2-dimensional. If $P_h(G) = \mathbb{R}_+^*$, then it fixes a unique point x_0 in $\mathbb{E}^{1,2}$. The subgroup \mathbb{R}_+^* preserves exactly one of the leaves induced by $T(G)$; namely the leaf containing x_0 . Now, the result follows easily. \square

4.2 Orbits

In this section we give a complete list of the Lie subgroups of $\text{Conf}_o(\mathbb{E}^{1,2})$ with $\dim G \geq 2$, up to conjugacy. Also, we will describe the orbits induced by the cohomogeneity one actions of the connected Lie subgroups of $\text{Conf}_o(\mathbb{E}^{1,2})$.

Let $B = \{e_1, e_2, e_3\}$ be an orthonormal basis for $\mathbb{R}^{1,2}$, where e_1 is a timelike vector, and let (x, y, z) be the coordinate on $\mathbb{E}^{1,2}$ with origin $o = (0, 0, 0)$ corresponding to B (here o is the unique point

fixed by $\mathbb{R}_+^* \times SO_o(1, 2) \subset \text{Conf}(\mathbb{E}^{1,2})$). Recall from Section 1.5.1, the Lie algebra of $\text{Conf}(\mathbb{E}^{1,2})$ is isomorphic to the semi-direct sum $(\mathbb{R} \oplus \mathfrak{so}(1, 2)) \oplus_{\theta} \mathbb{R}^{1,2}$, where θ is the natural representation of $\mathbb{R} \oplus \mathfrak{so}(1, 2)$ into $\mathfrak{gl}(\mathbb{R}^{1,2})$. Also, recall that, the simple Lie group $SO_o(1, 2) \simeq \text{PSL}(2, \mathbb{R})$ has exactly three 1-dimensional Lie subgroups, up to conjugacy, namely, $Y_E, Y_P,$ and Y_H . Moreover, $SO_o(1, 2)$ has only one 2-dimensional connected Lie subgroup, up to conjugacy, namely, Aff . The set $\{\mathcal{Y}_E, \mathcal{Y}_P, \mathcal{Y}_H\}$ is a basis for the Lie algebra $\mathfrak{so}(1, 2)$ as a vector space, where, $\mathcal{Y}_E, \mathcal{Y}_P,$ and \mathcal{Y}_H are the generators of the corresponding Lie algebras of the subgroups $Y_E, Y_P,$ and Y_H , respectively. For arbitrary elements $\lambda \in \mathbb{R}, X \in \mathfrak{so}(1, 2),$ and $v \in \mathbb{R}^{1,2}$, we denote the corresponding element in $(\mathbb{R} \oplus \mathfrak{so}(1, 2)) \oplus_{\theta} \mathbb{R}^{1,2}$ simply by $\lambda + X + v$, when there is no ambiguity. Furthermore, we denote by $\mathbb{R}(\lambda + X + v)$ the linear subspace of $(\mathbb{R} \oplus \mathfrak{so}(1, 2)) \oplus_{\theta} \mathbb{R}^{1,2}$ generated by the vector $\lambda + X + v$. Also, for a Lie subalgebra $\mathfrak{g} \leq (\mathbb{R} \oplus \mathfrak{so}(1, 2)) \oplus_{\theta} \mathbb{R}^{1,2}$, we denote by $\exp(\mathfrak{g})$ the corresponding connected Lie subgroup of $(\mathbb{R}_+^* \times SO_o(1, 2)) \ltimes \mathbb{R}^{1,2}$.

The following theorem, classifies all the connected Lie subgroups of $\text{Conf}(\mathbb{E}^{1,2})$ with $\dim \geq 2$, up to conjugacy.

Theorem 4.6. *Let $G \subset \text{Conf}_o(\text{Ein}^{1,2})$ be a connected Lie subgroup with $\dim G \geq 2$. Then G is conjugate to one of the subgroups in Tables 4.1-4.8.*

Proof. It follows immediately from Theorem A.1 and Lemma A.5. □

Subgroups with full translation part			
$\mathbb{R}_+^* \ltimes \mathbb{R}^{1,2}$	$(\mathbb{R}_+^* \times SO_o(1, 2)) \ltimes \mathbb{R}^{1,2}$	$SO_o(1, 2) \ltimes \mathbb{R}^{1,2}$	$\exp(\mathbb{R}(a + \mathcal{Y}_E)) \ltimes \mathbb{R}^{1,2}$
$\mathbb{R}^{1,2}$	$(\mathbb{R}_+^* \times \text{Aff}) \ltimes \mathbb{R}^{1,2}$	$\text{Aff} \ltimes \mathbb{R}^{1,2}$	$\exp(\mathbb{R}(a + \mathcal{Y}_P)) \ltimes \mathbb{R}^{1,2}$
$Y_H \ltimes \mathbb{R}^{1,2}$	$(\mathbb{R}_+ \times Y_H) \ltimes \mathbb{R}^{1,2}$	$Y_P \ltimes \mathbb{R}^{1,2}$	$\exp(\mathbb{R}(a + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P) \ltimes \mathbb{R}^{1,2}$
$Y_E \ltimes \mathbb{R}^{1,2}$	$(\mathbb{R}_+ \times Y_P) \ltimes \mathbb{R}^{1,2}$	$(\mathbb{R}_+ \times Y_E) \ltimes \mathbb{R}^{1,2}$	$\exp(\mathbb{R}(a + \mathcal{Y}_H)) \ltimes \mathbb{R}^{1,2}$

Table 4.1: Here $a \in \mathbb{R}^*$ is a constant number.

Subgroups with a Lorentzian plane as the translation part	
$(\mathbb{R}_+^* \times Y_H) \ltimes (\mathbb{R}e_1 \oplus \mathbb{R}e_2)$	$Y_H \ltimes (\mathbb{R}e_1 \oplus \mathbb{R}e_2)$
$\exp(\mathbb{R}(a + \mathcal{Y}_H)) \ltimes (\mathbb{R}e_1 \oplus \mathbb{R}e_2)$	$\exp(\mathbb{R}(\mathcal{Y}_H + e_3)) \ltimes (\mathbb{R}e_1 \oplus \mathbb{R}e_2)$
$\mathbb{R}_+^* \ltimes (\mathbb{R}e_1 \oplus \mathbb{R}e_2)$	$\mathbb{R}e_1 \oplus \mathbb{R}e_2$

Table 4.2: Here $a \in \mathbb{R}^*$ is a constant number.

Subgroups with a spacelike plane as the translation part	
$(\mathbb{R}_+^* \times Y_E) \ltimes (\mathbb{R}e_2 \oplus \mathbb{R}e_3)$	$Y_E \ltimes (\mathbb{R}e_2 \oplus \mathbb{R}e_3)$
$\exp(\mathbb{R}(a + \mathcal{Y}_E)) \ltimes (\mathbb{R}e_2 \oplus \mathbb{R}e_3)$	$\exp(\mathbb{R}(\mathcal{Y}_E + e_1)) \ltimes (\mathbb{R}e_2 \oplus \mathbb{R}e_3)$
$\mathbb{R}_+^* \ltimes (\mathbb{R}e_2 \oplus \mathbb{R}e_3)$	$\mathbb{R}e_2 \oplus \mathbb{R}e_3$

Table 4.3: Here $a \in \mathbb{R}^*$ is a constant number.

By Theorem 4.6, it is not hard to see that, all the subgroups of $\text{Conf}_o(\mathbb{E}^{1,2})$ with $\dim \geq 2$ admit a 2-dimensional orbit in Einstein universe $\text{Ein}^{1,2}$ except the subgroups which are conjugate to $\mathbb{R}^{1,2}$ or $\mathbb{R}_+^* \ltimes \mathbb{R}^{1,2}$.

For an arbitrary point $q \in \mathbb{E}^{1,2}$, there is a natural identification between the tangent space $T_q\mathbb{E}^{1,2}$ and the underlying scalar product space $\mathbb{R}^{1,2}$. Therefore, by the action of a Lie subgroup $G \subset \text{Conf}(\mathbb{E}^{1,2})$,

Subgroups with a degenerate plane as the translation part	
$\text{Aff} \times \Pi_\phi$	$\exp(\mathbb{R}(a + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P) \times \Pi_\phi$
$(\mathbb{R}_+^* \times \text{Aff}) \times \Pi_\phi$	$\exp(\mathbb{R}(a + \mathcal{Y}_P)) \times \Pi_\phi$
$Y_P \times \Pi_\phi$	$\exp(\mathbb{R}(a + \mathcal{Y}_H)) \times \Pi_\phi$
$(\mathbb{R}_+^* \times Y_P) \times \Pi_\phi$	$\exp(\mathbb{R}(1 + \mathcal{Y}_H + e_1) + \mathbb{R}\mathcal{Y}_P) \times \Pi_\phi$
$Y_H \times \Pi_\phi$	$\exp(\mathbb{R}(\mathcal{Y}_P + e_1)) \times \Pi_\phi$
$(\mathbb{R}_+^* \times Y_H) \times \Pi_\phi$	$\exp(\mathbb{R}(1 + \mathcal{Y}_H + e_1)) \times \Pi_\phi$
Π_ϕ	$\exp(\mathbb{R}(2 + \mathcal{Y}_H) + \mathbb{R}(\mathcal{Y}_P + e_1)) \times \Pi_\phi$
$\mathbb{R}_+^* \times \Pi_\phi$	

Table 4.4: Here Π_ϕ denotes the degenerate plane $\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3 \leq \mathbb{R}^{1,2}$, and $a \in \mathbb{R}^*$ is a constant number.

Subgroups with a timelike line as the translation part			
$\mathbb{R}_+^* \times \mathbb{R}e_1$	$(\mathbb{R}_+^* \times Y_E) \times \mathbb{R}e_1$	$Y_E \times \mathbb{R}e_1$	$\exp(\mathbb{R}(a + \mathcal{Y}_E)) \times \mathbb{R}e_1$

Table 4.5: Here $a \in \mathbb{R}^*$ is a constant number.

Subgroups with a spacelike line as the translation part		
$\mathbb{R}_+^* \times \mathbb{R}e_3$	$(\mathbb{R}_+^* \times Y_H) \times \mathbb{R}e_3$	$Y_H \times \mathbb{R}e_3$
$\exp(\mathbb{R}(a + \mathcal{Y}_H)) \times \mathbb{R}e_3$	$\exp(\mathbb{R}(1 + \mathcal{Y}_H + e_1)) \times \mathbb{R}e_3$	$\exp(\mathbb{R}(-1 + \mathcal{Y}_H + e_1)) \times \mathbb{R}e_3$

Table 4.6: Here $a \in \mathbb{R}^*$ is a constant number.

Subgroups with a lightlike line as the translation part	
$(\mathbb{R}_+^* \times \text{Aff}) \times \mathcal{L}$	$\exp(\mathbb{R}(a + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P) \times \mathcal{L}$
$Y_H \times \mathcal{L}$	$\exp(\mathbb{R}(a + \mathcal{Y}_P)) \times \mathcal{L}$
$\text{Aff} \times \mathcal{L}$	$\exp(\mathbb{R}(\mathcal{Y}_H + e_3) + \mathbb{R}\mathcal{Y}_P) \times \mathcal{L}$
$(\mathbb{R}_+^* \times Y_P) \times \mathcal{L}$	$\exp(\mathbb{R}(\mathcal{Y}_P + e_1)) \times \mathcal{L}$
$Y_P \times \mathcal{L}$	$\exp(\mathbb{R}(2 + \mathcal{Y}_H) + \mathbb{R}(\mathcal{Y}_P + e_1)) \times \mathcal{L}$
$(\mathbb{R}_+^* \times Y_H) \times \mathcal{L}$	$\exp(\mathbb{R}(1 + \mathcal{Y}_H + e_1)) \times \mathcal{L}$
$\exp(\mathbb{R}(a + \mathcal{Y}_H)) \times \mathcal{L}$	$\exp(\mathbb{R}(\mathcal{Y}_H + e_3)) \times \mathcal{L}$

Table 4.7: Here \mathcal{L} denotes the lightlike line $\mathbb{R}(e_1 + e_2) \leq \mathbb{R}^{1,2}$ and $a \in \mathbb{R}^*$ is a constant number.

Subgroups with trivial translation part	
$SO_o(1, 2)$	$\mathbb{R}_+^* \times SO_o(1, 2)$
Aff	$\mathbb{R}_+^* \times \text{Aff}$
$\mathbb{R}_+^* \times Y_E$	$\exp(\mathbb{R}(a + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P)$
$\mathbb{R}_+^* \times Y_P$	$\exp(\mathbb{R}(2 + \mathcal{Y}_H) + \mathbb{R}(\mathcal{Y}_P + e_1 - e_2))$
$\mathbb{R}_+^* \times Y_H$	$\exp(\mathbb{R}(-1 + \mathcal{Y}_H + e_1 + e_2) + \mathbb{R}\mathcal{Y}_P)$

Table 4.8: Here $a \in [-1, 1]$ is a constant number.

we may always consider the translation part $T(G)$ as a linear subspace of $T_q G(q)$, since $\mathbb{R}^{1,2}$ acts on $\mathbb{E}^{1,2}$ freely.

We fix some notations here. Denote by Π_ϕ and ϕ the unique degenerate plane $\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3 \leq \mathbb{R}^{1,2}$ invariant by the 1-parameter parabolic subgroup $Y_P \subset SO_o(1, 2)$ and its corresponding photon in $L(p)$, respectively. Furthermore, denote by Π_ψ and ψ the degenerate plane $\mathbb{R}(e_1 - e_2) \oplus \mathbb{R}e_3 \leq \mathbb{R}^{1,2}$ and its corresponding photon in $L(p)$, respectively, which both are invariant by the 1-parameter hyperbolic

subgroup $Y_H \subset SO_o(1, 2)$. Also, for a linear subspace $V \leq \mathbb{R}^{1,2}$, denote by \mathcal{F}_V the foliation induced by V in Minkowski space $\mathbb{E}^{1,2}$.

Remark 4.7. Let $\lambda + X + v$ be an arbitrary element in $(\mathbb{R} \oplus \mathfrak{so}(1, 2)) \oplus_{\theta} \mathbb{R}^{1,2}$ and $q \in \mathbb{E}^{1,2}$ be an arbitrary point. There is an easy way to determine the tangent vector in $T_q \mathbb{E}^{1,2}$ induced by the action of $\exp(\mathbb{R}(\lambda + X + v))$ on $\mathbb{E}^{1,2}$: the vector $\frac{d}{dt}|_{t=0}(\exp(t(\lambda + X + v))(q))$ coincides with $\lambda X(q) + v$ where λ and X act on $\mathbb{E}^{1,2}$ (with origin o) as linear maps.

Now, we are ready to describe the orbits induced by the subgroups indicated in Tables 4.1-4.8.

Subgroups with full translation part

Here, we consider the orbits of a Lie subgroup $G \subset \text{Conf}_o(\mathbb{E}^{1,2})$ with $T(G) = \mathbb{R}^{1,2}$. These groups have been listed in Table (4.1). Obviously, G acts on Minkowski space $\mathbb{E}^{1,2}$ transitively, since it contains $\mathbb{R}^{1,2}$ as a subgroup. Note that the groups $\mathbb{R}^{1,2}$ and $\mathbb{R}_+^* \times \mathbb{R}^{1,2}$ admit 2-dimensional orbit neither in $\mathbb{E}^{1,2}$ nor in the lightcone $L(p)$.

Observe that, the translation part $T(G) = \mathbb{R}^{1,2}$ acts on each vertex-less photon in $L(p)$ transitively.

- If the linear isometry projection $P_{li}(G)$ contains an elliptic element, then G acts on the vertex-less lightcone $L(\hat{p})$ transitively. Hence, the orbits induced in $\mathbb{E}^{1,2}$ by the following groups are exactly the same as the orbits induced by $Y_E \times \mathbb{R}^{1,2}$.

$$\begin{aligned} (\mathbb{R}_+^* \times SO_o(1, 2)) \times \mathbb{R}^{1,2}, & \quad SO_o(1, 2) \times \mathbb{R}^{1,2}, \\ (\mathbb{R}_+ \times Y_E) \times \mathbb{R}^{1,2}, & \quad \exp(\mathbb{R}(a + \mathcal{Y}_E)) \times \mathbb{R}^{1,2}, \quad a \in \mathbb{R}^*. \end{aligned}$$

Therefore, the orbits induced by these five groups are: a fixed point p , a 2-dimensional degenerate (i.e. of signature $(0, 1, 1)$) orbit $L(\hat{p})$, and an open orbit $\mathbb{E}^{1,2} \subset \mathbb{E}^{1,2}$.

- If the linear isometry projection $P_{li}(G)$ is a proper subgroup of $SO_o(1, 2)$ and it contains a parabolic element, then G preserves a unique photon ϕ , and acts on its complement in the lightcone $L(p)$ transitively. Thus all the following groups admit in $\mathbb{E}^{1,2}$ the same orbits as $Y_P \times \mathbb{R}^{1,2}$.

$$\begin{aligned} (\mathbb{R}_+^* \times \text{Aff}_o(1, \mathbb{R})) \times \mathbb{R}^{1,2}, & \quad \text{Aff}_o(1, \mathbb{R}) \times \mathbb{R}^{1,2}, & \quad \exp(\mathbb{R}(a + \mathcal{Y}_P)) \times \mathbb{R}^{1,2}, \\ (\mathbb{R}_+ \times Y_P) \times \mathbb{R}^{1,2}, & & \quad \exp(\mathbb{R}(a + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P) \times \mathbb{R}^{1,2}, \quad a \in \mathbb{R}^* \end{aligned}$$

Therefore, the orbits induced by these six groups are: a fixed point p , a vertex-less photon $\hat{\phi}$, a 2-dimensional degenerate orbit $L(p) \setminus \phi$, and an open orbit $\mathbb{E}^{1,2}$.

- If the linear isometry projection $P_{li}(G)$ is a 1-parameter hyperbolic subgroup of $SO_o(1, 2)$, then G preserves two distinct photons ϕ and ψ , and acts on the two connected components of $L(p) \setminus (\phi \cup \psi)$, transitively. This implies that the following groups admit in $\mathbb{E}^{1,2}$ the same orbits as $Y_H \times \mathbb{R}^{1,2}$.

$$(\mathbb{R}_+ \times Y_H) \times \mathbb{R}^{1,2}, \quad \exp(\mathbb{R}(a + \mathcal{Y}_H)) \times \mathbb{R}^{1,2}, \quad a \in \mathbb{R}^*.$$

Therefore, the orbits induced by these three groups are: a fixed point p , two vertex-less photons $\hat{\phi}$ and $\hat{\psi}$, the two connected components of $L(p) \setminus (\phi \cup \psi)$ which are degenerate orbits, and an open orbit $\mathbb{E}^{1,2}$.

Subgroups with a timelike plane as the translation part

In this section we describe the orbits induced by a connected Lie subgroup $G \subset \text{Conf}_o(\mathbb{E}^{1,2})$ which its translation part is a Lorentzian plane. These groups has been listed in Table (4.2).

Observe that, the translation part $T(G)$ acts on each vertex-less photon in the lightcone $L(p)$ transitively, since the action of a timelike plane does not preserve any degenerate affine plane in $\mathbb{E}^{1,2}$. In this case, the linear isometry projection $P_i(G)$ is either trivial or it is a 1-parameter hyperbolic subgroup of $SO_o(1,2)$. In the first, G preserves every vertex-less photon in $L(p)$. In the later, G preserves two distinct photons ϕ and ψ and acts on the both connected components of the complement $L(p) \setminus (\phi \cup \psi)$ transitively. Also, $T(G)$ induces a codimension 1 foliation $\mathcal{F}_{T(G)}$ in $\mathbb{E}^{1,2}$ on which the leaves are affine Lorentzian planes.

- $G = T(G) = \mathbb{R}e_1 \oplus \mathbb{R}e_2$. The orbits induced in by G are: a fixed point p , the vertex-less photons in $L(p)$, and the leaves of the foliation induced by \mathcal{F}_G in $\mathbb{E}^{1,2} \subset \mathbb{E}in^{1,2}$ which are Lorentzian affine planes.
- $G = \mathbb{R}_+^* \ltimes (\mathbb{R}e_1 \oplus \mathbb{R}e_2)$. The homothety factor \mathbb{R}_+^* preserves the leaf $\mathcal{F}_{T(G)}(o)$. Thus this leaf is a 2-dimensional G -orbit. Furthermore, one can see, G acts on the both connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{T(G)}(o)$ transitively. Thus, the orbits induced by G in $\mathbb{E}in^{1,2}$ are: a fixed point p , the vertex-less photons in $L(p)$, the leaf $\mathcal{F}_{T(G)}(o)$ which is a Lorentzian affine plane, and the two connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{T(G)}(o)$.
- $G = Y_H \ltimes (\mathbb{R}e_1 \oplus \mathbb{R}e_2)$. The 1-parameter hyperbolic subgroup Y_H preserves the leaves of the foliation $\mathcal{F}_{T(G)}$, so does G . Thus, the orbits induced by G are: a fixed point p , two vertex-less photons $\hat{\phi}$ and $\hat{\psi}$, the two connected components of $L(p) \setminus (\phi \cup \psi)$ which are degenerate surfaces, and the leaves of the foliation $\mathcal{F}_{T(G)}$ in $\mathbb{E}^{1,2}$, which are Lorentzian affine planes.
- $G = (\mathbb{R}_+^* \times Y_H) \ltimes (\mathbb{R}e_1 \oplus \mathbb{R}e_2)$. The leaf $\mathcal{F}_{T(G)}(o)$ is a G -orbit, since it is invariant by the action of $\mathbb{R}_+^* \times Y_H$. Also, G acts on the both connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{T(G)}(o)$ transitively. Therefore, the orbits induced by G in $\mathbb{E}in^{1,2}$ are: a fixed point p , the two vertex-less photons $\hat{\phi}$ and $\hat{\psi}$, the two connected components of $L(p) \setminus (\phi \cup \psi)$ which are degenerate surfaces, the leaf $\mathcal{F}_{T(G)}(o)$, and the two connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{T(G)}(o)$.
- $G = \exp(\mathbb{R}(a + \mathcal{Y}_H)) \ltimes (\mathbb{R}e_1 \oplus \mathbb{R}e_2)$, $a \in \mathbb{R}^*$. Since, G is a subgroup of $(\mathbb{R}_+^* \times Y_H) \ltimes (\mathbb{R}e_1 \oplus \mathbb{R}e_2)$, the leaf $\mathcal{F}_{T(G)}(o)$ is a G -orbit. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroup $\exp(\mathbb{R}(a + \mathcal{Y}_H))$ is $v = (ax + y, x + ay, az)$. The set $\{e_1, e_2, v\} \subset T_q G(q)$ is a basis if and only if $q \notin \mathcal{F}_{T(G)}(o)$ if and only if $z \neq 0$. This implies that G acts on the both connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{T(G)}(o)$ transitively. Therefore, G admits the same orbits in $\mathbb{E}in^{1,2}$ as $(\mathbb{R}_+^* \times Y_H) \ltimes (\mathbb{R}e_1 \oplus \mathbb{R}e_2)$.
- $G = \exp(\mathbb{R}(\mathcal{Y}_H + e_3)) \ltimes (\mathbb{R}e_1 \oplus \mathbb{R}e_2)$: For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroup $\exp(\mathbb{R}(\mathcal{Y}_H + e_3))$ is $v = (y, x, 1)$ (see B.10). Obviously, $\{e_1, e_2, v\}$ is a basis of the tangent space $T_q G(q)$. This implies that G acts on $\mathbb{E}^{1,2}$, transitively. Hence, G admits the same orbits in $\mathbb{E}in^{1,2}$ as $Y_H \ltimes \mathbb{R}^{1,2}$.

Remark 4.8. Observe that by the action of $G = (\mathbb{R}_+^* \times Y_H) \times (\mathbb{R}e_1 \oplus \mathbb{R}e_2)$ the union of the leaf $\mathcal{F}_{T(G)}(o)$ (which is a Lorentzian affine 2-plane) and the photons ϕ and ψ is G -invariant. Actually, $\mathcal{F}_{T(G)}(o) \cup \phi \cup \psi$ is an Einstein hypersphere (Definition 1.63). Moreover, G is the unique (up to conjugacy) maximal connected Lie subgroup in $SO_o(2, 3)$ which preserves an Einstein hypersphere $\mathbb{E}in^{1,1} \subset \mathbb{E}in^{1,2}$ and admits a fixed point on it.

Subgroups with a spacelike plane as the translation part

Now, suppose that $G \subset \text{Conf}_o(\mathbb{E}^{1,2})$ is a connected Lie subgroup which its translation part $T(G)$ is a spacelike plane. These groups have been listed in Table (4.3).

Observe that, the translation part $T(G)$ acts on each vertex-less photon in the lightcone $L(p)$ transitively, since the action of a spacelike plane does not preserve any degenerate affine plane in $\mathbb{E}^{1,2}$. In this case the linear isometry projection $P_{li}(G)$ is either trivial or it is a 1-parameter elliptic subgroup of $SO_o(1, 2)$. In the first case, G preserves every vertex-less photon in $L(p)$. In the later, G acts on the vertex-less lightcone $L(\hat{p})$ transitively. Also, $T(G)$ induces a codimension 1 foliation $\mathcal{F}_{T(G)}$ in $\mathbb{E}^{1,2} \subset \mathbb{E}in^{1,2}$ on which the leaves are affine spacelike planes.

- $G = \mathbb{R}e_2 \oplus \mathbb{R}e_3$. The orbits induced by G in $\mathbb{E}in^{1,2}$ are: a fixed point p , the vertex-less photons in $L(p)$, and the leaves of the foliation $\mathcal{F}_{T(G)}$ in $\mathbb{E}^{1,2}$ which are affine spacelike planes.
- $G = \mathbb{R}_+^* \times (\mathbb{R}e_2 \oplus \mathbb{R}e_3)$. The homothety factor \mathbb{R}_+^* preserves the leaf $\mathcal{F}_{T(G)}(o)$. Consequently, the leaf $\mathcal{F}_{T(G)}(o)$ is a G -orbit. Furthermore, G acts on the both connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{T(G)}(o)$ transitively. Therefore, the orbits induced by G in $\mathbb{E}in^{1,2}$ are: a fixed point p , the vertex-less photons in $L(p)$, the leaf $\mathcal{F}_{T(G)}(o)$, and the two connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{T(G)}(o)$.
- $G = Y_E \times (\mathbb{R}e_2 \oplus \mathbb{R}e_3)$. The 1-parameter elliptic subgroup Y_E preserves the leaves of the foliation $\mathcal{F}_{T(G)}$, so does G . Therefore, the orbits induced by G in $\mathbb{E}in^{1,2}$ are: a fixed point p , the vertex-less lightcone $L(\hat{p})$, and the leaves of the foliation $\mathcal{F}_{T(G)}$ in $\mathbb{E}^{1,2}$ which are affine spacelike planes.
- $G = (\mathbb{R}_+^* \times Y_E) \times (\mathbb{R}e_2 \oplus \mathbb{R}e_3)$. The subgroup $\mathbb{R}_+^* \times Y_E$ preserves the leaf $\mathcal{F}_{T(G)}(o)$. Hence, $\mathcal{F}_{T(G)}(o)$ is a G -orbit. Moreover, one can see G acts on the both connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{T(G)}(o)$ transitively. Therefore, the orbits induced by G in $\mathbb{E}in^{1,2}$ are: a fixed point p , the vertex-less lightcone $L(\hat{p})$, the leaf $\mathcal{F}_{T(G)}(o)$ which is a spacelike affine plane, and the two connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{T(G)}(o)$.
- $G = \exp(\mathbb{R}(a + \mathcal{Y}_E)) \times (\mathbb{R}e_2 \oplus \mathbb{R}e_3)$, $a \in \mathbb{R}^*$. The 1-parameter subgroup $H = \exp(\mathbb{R}(a + \mathcal{Y}_E))$ preserves the leaf $\mathcal{F}_{T(G)}(o)$. Hence, $\mathcal{F}_{T(G)}$ is a G -orbit. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G(q)$ at q induced by H is $v = (ax, ay + z, az - y)$. The set $\{e_2, e_3, v\} \subset T_q G(q)$ is a basis if and only if $x \neq 0$ if and only if $q \notin \mathcal{F}_{T(G)}(o)$. This implies that G acts on the both connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{T(G)}(o)$ transitively. Therefore, G admits the same orbits in $\mathbb{E}in^{1,2}$ as $(\mathbb{R}_+^* \times Y_E) \times (\mathbb{R}e_2 \oplus \mathbb{R}e_3)$.
- $G = \exp(\mathbb{R}(\mathcal{Y}_E + e_1)) \times (\mathbb{R}e_2 \oplus \mathbb{R}e_3)$: For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroup $\exp(\mathbb{R}(\mathcal{Y}_E + e_1))$ is $v = (1, z, -y)$ (see B.7). The set $\{e_2, e_3, v\}$ is a basis for the tangent space $T_q G(q)$. This implies that

G acts on $\mathbb{E}^{1,2}$, transitively. Hence, the orbits induced by G in $\mathbb{E}in^{1,2}$ are the same as the orbits induced by $Y_E \times \mathbb{R}^{1,2}$.

Remark 4.9. Observe that by the action of $G = (\mathbb{R}_+^* \times Y_E) \times (\mathbb{R}e_2 \oplus \mathbb{R}e_3)$ the union of the leaf $\mathcal{F}_{T(G)}(o)$ (which is a spacelike affine 2-plane) and the vertex $\{p\}$ is G -invariant. Actually, $\mathcal{F}_{T(G)}(o) \cup \{p\}$ is a spacelike hypersphere (Definition 1.60). Furthermore, G is the unique (up to conjugacy) maximal connected Lie subgroup in $SO_o(2, 3)$ which preserves a spacelike hypersphere $\mathbb{S}^2 \subset \mathbb{E}in^{1,2}$ and admits a fixed point on it.

Subgroups with a degenerate plane as the translation part

Assume that $G \subset \text{Conf}_o(\mathbb{E}^{1,2})$ is a connected Lie subgroup which its translation part $T(G)$ is a degenerate plane, i.e., G belongs to Table (4.4).

The translation part $T(G)$ preserves the leaves of the foliation $\mathcal{F}_{T(G)}$. Hence, $T(G)$ fixes the corresponding photon in $L(p)$, pointwisely. In this case, the linear isometry projection $P_{li}(G)$ is either trivial or it is a subgroup of $\text{Aff} \subset SO_o(1, 2)$, up to conjugacy.

Case I: Suppose that the linear isometry projection $P_{li}(G)$ is trivial.

- $G = \Pi_\phi = \mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3$. The orbits induced by G in $\mathbb{E}in^{1,2}$ are: the points in ϕ , the vertex-less photons in $L(p)$ different from $\hat{\phi}$, and the leaves of the foliation \mathcal{F}_{Π_ϕ} in $\mathbb{E}^{1,2}$ which are degenerate affine planes.
- $G = \mathbb{R}_+^* \times \Pi_\phi$. The homothety factor \mathbb{R}_+^* preserves the leaf $\mathcal{F}_{\Pi_\phi}(o)$. Hence, $\mathcal{F}_{\Pi_\phi}(o)$ is a G -orbit. Thus, G fixes the corresponding point $d \in \hat{\phi}$. On the other hand G acts on the both connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$ transitively. This shows that G acts on the both connected components of $\hat{\phi} \setminus \{d\}$ transitively. Therefore, the orbits induced by G in $\mathbb{E}in^{1,2}$ are: two fixed points p and d , the two connected components of $\hat{\phi} \setminus \{d\}$, the vertex-less photons in $L(p)$ different from $\hat{\phi}$, the leaf $\mathcal{F}_{\Pi_\phi}(o)$, and the two connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$.

Case II: Assume that the linear isometry projection $P_{li}(G)$ is a 1-parameter parabolic subgroup. Then, it can be easily seen that G acts on the degenerate surface $L(p) \setminus \phi$ transitively.

- $G = Y_P \times \Pi_\phi$. The 1-parameter parabolic subgroup Y_P (consequently G) preserves the leaves of the foliation induced by Π_ϕ . Hence, G acts on the photon ϕ trivially. Therefore, the orbits induced by G in $\mathbb{E}in^{1,2}$ are: the points in ϕ , the degenerate surface $L(p) \setminus \phi$, and the leaves of the foliation \mathcal{F}_{Π_ϕ} in $\mathbb{E}^{1,2}$ which are degenerate affine planes.
- $G = (\mathbb{R}_+^* \times Y_P) \times \Pi_\phi$. The subgroup $\mathbb{R}_+^* \times Y_P$ preserves the leaf $\mathcal{F}_{\Pi_\phi}(o)$. Hence, $\mathcal{F}_{\Pi_\phi}(o)$ is a G -orbit. Therefore, G fixes the corresponding point $d \in \hat{\phi}$. Furthermore, G acts on the both connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$ transitively. This implies that G acts on the both connected components of $\hat{\phi} \setminus \{d\}$ transitively. Therefore, the orbits induced by G in $\mathbb{E}in^{1,2}$ are: two fixed points p and d , the two connected components of $\hat{\phi} \setminus \{d\}$, the degenerate surface $L(p) \setminus \phi$, the degenerate affine plane $\mathcal{F}_{\Pi_\phi}(o)$, and the two connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$.
- $G = \exp(\mathbb{R}(a + \mathcal{Y}_P)) \times \Pi_\phi$, $a \in \mathbb{R}^*$. The 1-parameter subgroup $H = \exp(\mathbb{R}(a + \mathcal{Y}_P))$ preserves the leaf $\mathcal{F}_{P_i\phi}(o)$. Therefore, $\mathcal{F}_{\Pi_\phi}(o)$ is a G -orbit. Hence, G fixes the corresponding point $d \in \hat{\phi}$.

For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$ the vector tangent to the orbit $G(q)$ at (q) induced by the 1-parameter subgroup H is $v = (ax + z, ay + z, x - y + az)$. The set $\{(e_1 + e_2), e_3, v\} \subset T_q G(q)$ is a basis if and only if $x \neq y$ if and only if $q \notin \mathcal{F}_{\Pi_\phi}(o)$. Hence, G acts on the both connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$, transitively. Thus, G acts on the both connected components of $\hat{\phi} \setminus \{d\}$ transitively. Observe that the orbits induced by G in $\mathbb{E}^{1,2}$ are exactly the same as the orbits induced by $(\mathbb{R}_+^* \times Y_P) \times \Pi_\phi$.

- $G = \exp(\mathbb{R}(\mathcal{Y}_P + e_1)) \times \Pi_\phi$: For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$ the vector tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroup $\exp(\mathbb{R}(\mathcal{Y}_P + e_1))$ is $v = (z + 1, z, x - y)$ (see B.8). The set $\{(e_1 + e_2), e_3, v\} \subset T_q G(q)$ is a basis. Hence, G acts on $\mathbb{E}^{1,2}$ transitively. This implies that G acts on the vertex-less photon $\hat{\phi}$ transitively. It follows that, the orbits induced by G in $\mathbb{E}^{1,2}$ are exactly the same as the orbits induced by $Y_P \times \mathbb{R}^{1,2}$.

Case III: Suppose that the linear isometry projection is a 1-parameter hyperbolic subgroup of $SO_o(1, 2)$. Then G preserves two distinct photons ϕ and ψ in $L(p)$. Moreover, G acts on the vertex-less photon $\hat{\psi}$ transitively, since $T(G) = \Pi_\phi$ does. Also, G acts on the both connected components of $L(p) \setminus (\phi \cup \psi)$ transitively.

- $G = Y_H \times \Pi_\phi$. The 1-parameter subgroup Y_H preserves the leaf $\mathcal{F}_{\Pi_\phi}(o)$. Thus, $\mathcal{F}_{\Pi_\phi}(o)$ is a 2-dimensional G -orbit. Hence, G fixes the corresponding point $d \in \hat{\phi}$. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G(q)$ at q induced by Y_H is $v = (y, x, z)$. The set $\{(e_1 + e_2), e_3, v\} \subset T_q G(q)$ is a basis if and only if $x \neq y$ if and only if $q \notin \mathcal{F}_{\Pi_\phi}(o)$. Therefore, G acts on the both connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$ transitively, and consequently, it acts on the both connected components of $\hat{\phi} \setminus \{d\}$ transitively. Henceforth, the orbits induced by G in $\mathbb{E}^{1,2}$ are: two fixed points p and d , the two connected components of $\hat{\phi} \setminus \{d\}$, the vertex-less photon $\hat{\psi}$, the two connected components of $L(p) \setminus (\phi \setminus \psi)$, the degenerate affine plane $\mathcal{F}_{\Pi_\phi}(o)$, and the two connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$.
- $G = (\mathbb{R}_+^* \times Y_H) \times \Pi_\phi$. The subgroup $\mathbb{R}_+^* \times Y_H$ preserves the leaf $\mathcal{F}_{\Pi_\phi}(o)$. Thus $\mathcal{F}_{\Pi_\phi}(o)$ is a G -orbit. Hence, G fixes the corresponding point $d \in \hat{\phi}$. On the other hand, G acts on the both connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$ transitively, and consequently, it acts on the both connected components of $\hat{\phi} \setminus \{d\}$ transitively. Observe that the orbits induced by G in $\mathbb{E}^{1,2}$ are exactly the same as the orbits induced by $Y_H \times \Pi_\phi$.
- $G = \exp(\mathbb{R}(a + \mathcal{Y}_H)) \times \Pi_\phi$, $a \in \mathbb{R}^* \setminus \{1\}$. The 1-parameter subgroup $H = \exp(\mathbb{R}(a + \mathcal{Y}_H))$ preserves the leaf $\mathcal{F}_{\Pi_\phi}(o)$. Thus, $\mathcal{F}_{\Pi_\phi}(o)$ is a G -orbit. Henceforth, G fixes the corresponding point $d \in \hat{\phi}$. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G(q)$ at q induced by H is $v = (ax + y, x + ay, az)$. It follows that, the set $\{(e_1 + e_2), e_3, v\} \subset T_q G(q)$ is a basis if and only if $x \neq y$ if and only if $q \notin \mathcal{F}_{\Pi_\phi}(o)$. This implies that, G acts on the both connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$ transitively, and so, it acts on the both connected components of $\hat{\phi} \setminus \{d\}$ transitively too. Observe that the orbits induced by G are exactly the same as the orbits induced by $Y_H \times \Pi_\phi$.
- $G = \exp(\mathbb{R}(1 + \mathcal{Y}_H)) \times \Pi_\phi$. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G(q)$ at q induced by $\exp(\mathbb{R}(1 + \mathcal{Y}_H))$ is $v = (x + y, x + y, z)$. Observe that

$v \in \Pi_\phi \leq T_q G(q)$. Therefore, G preserves the leaves of the foliation \mathcal{F}_{Π_ϕ} . This implies that G acts on ϕ trivially. Hence, the orbits induced by G in $\mathbb{E}in^{1,2}$ are: the points in ϕ , the vertex-less photon $\hat{\psi}$, the two connected components of $L(p) \setminus (\phi \cup \psi)$ which are degenerate surfaces, and the leaves of the foliation \mathcal{F}_{Π_ϕ} in $\mathbb{E}in^{1,2}$ which are degenerate affine planes.

- $G = \exp(\mathbb{R}(1 + \mathcal{Y}_H + e_1)) \times \Pi_\phi$: For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G(q)$ at q induced by $\exp(\mathbb{R}(1 + \mathcal{Y}_H + e_1))$ is $v = (x + y + 1, x + y, z)$ (see B.11). Observe that the set $\{(e_1 + e_2), e_3, v\} \subset T_q G(q)$ is a basis. Therefore, G acts on $\mathbb{E}^{1,2}$ transitively. This implies that G acts on the vertex-less photon $\hat{\phi}$ transitively. Hence, the orbits induced by G in $\mathbb{E}in^{1,2}$ are exactly the same as the orbits induced by $Y_H \times \mathbb{R}^{1,2}$.

Case IV: Finally, assume that the linear isometry projection $P_{li}(G)$ is a 2-dimensional subgroup of $SO_o(1, 2)$, hence $P_{li}(G) = \text{Aff}$ up to conjugacy. Since, $P_{li}(G)$ contains parabolic elements, G acts on $L(p) \setminus \phi$ transitively.

- $G = \text{Aff} \times \Pi_\phi$. The subgroup $\text{Aff} \subset SO_o(1, 2)$ preserves the leaf $\mathcal{F}_{\Pi_\phi}(o)$. Hence, $\mathcal{F}_{\Pi_\phi}(o)$ is a 2-dimensional G -orbit. Thus G fixes the corresponding point $d \in \hat{\phi}$. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroup $Y_H \subset \text{Aff}$ is $v = (y, x, z)$. The set $\{(e_1 + e_2), e_3, v\} \subset T_q G(q)$ is a basis if and only if $x \neq y$ if and only if $q \notin \mathcal{F}_{\Pi_\phi}(o)$. Henceforth, G acts on the both connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$, (and consequently on the both connected components of $\hat{\phi} \setminus \{d\}$) transitively. Observe the orbits induced by G in $\mathbb{E}in^{1,2}$ are exactly the same as the orbits induced by $(\mathbb{R}_+^* \times Y_P) \times \Pi_\phi$.
- $G = (\mathbb{R}_+^* \times \text{Aff}) \times \Pi_\phi$. The subgroup $\mathbb{R}_+^* \times \text{Aff}$ preserves the leaf $\mathcal{F}_{\Pi_\phi}(o)$. Hence, $\mathcal{F}_{\Pi_\phi}(o)$ is a G -orbit, and so, G fixes the corresponding point $d \in \hat{\phi}$. For an arbitrary point $q = (x, y, z)$ the vector tangent to the orbit $G(q)$ at q induced by the homothety factor \mathbb{R}_+^* is $v = (x, y, z)$. The set $\{(e_1 + e_2), e_3, v\}$ is a basis if and only if $x \neq y$ if and only if $q \notin \mathcal{F}_{\Pi_\phi}(o)$. Therefore, G acts on the both connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$ (and consequently on the both connected components of $\hat{\phi} \setminus \{d\}$) transitively. Observe G admits exactly the same orbits in $\mathbb{E}in^{1,2}$ as $(\mathbb{R}_+^* \times Y_P) \times \Pi_\phi$.
- $G = \exp((\mathbb{R}(a + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P)) \times \Pi_\phi$, $a \in \mathbb{R}^* \setminus \{1\}$. The subgroup $\exp((\mathbb{R}(a + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P))$ preserves the leaf $\mathcal{F}_{\Pi_\phi}(o)$. Thus, $\mathcal{F}_{\Pi_\phi}(o)$ is a G -orbit, and so, G fixes the corresponding point $d \in \hat{\phi}$. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroup $\exp(\mathbb{R}(a + \mathcal{Y}_H))$ is $v = (ax + y, x + ay, z)$. The set $\{(e_1 + e_2), e_3, v\} \subset T_q G(q)$ is a basis if and only if $x \neq y$ if and only if $q \notin \mathcal{F}_{\Pi_\phi}(o)$. Therefore, G acts on the both connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$ (and consequently on the both connected components of $\hat{\phi} \setminus \{d\}$) transitively. Observe the orbits induced by G in $\mathbb{E}in^{1,2}$ are exactly the same as the orbits induced by $(\mathbb{R}_+^* \times Y_P) \times \Pi_\phi$.
- $G = \exp((\mathbb{R}(1 + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P)) \times \Pi_\phi$. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vectors tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroups $\exp(\mathbb{R}(1 + \mathcal{Y}_H))$ and Y_P are $v = (x + y, x + y, z)$ and $w = (z, z, x - y)$, respectively. Obviously, $v, w \in \Pi_\phi \leq T_q G(q)$. Therefore, G preserves the leaves of the foliation \mathcal{F}_{Π_ϕ} . This implies that G acts on the photon ϕ trivially. Henceforth, the orbits induced by G in $\mathbb{E}in^{1,2}$ are exactly the same as the orbits induced by $Y_P \times \Pi_\phi$.

- $G = \exp(\mathbb{R}(2 + \mathcal{Y}_H) + \mathbb{R}(\mathcal{Y}_P + e_1)) \times \Pi_\phi$. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroup $\exp(\mathbb{R}(\mathcal{Y}_P + e_1))$ is $v = (z + 1, z, x - y)$ (see B.8). The set $\{(e_1 + e_2), e_3, v\} \subset T_q G(q)$ is a basis. Hence, G acts on $\mathbb{E}^{1,2}$ transitively. This implies that G acts on the vertex-less photon $\hat{\phi}$ transitively as well. Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are exactly the same as the orbits induced by $Y_P \times \mathbb{R}^{1,2}$.
- If $G = \exp(\mathbb{R}(1 + \mathcal{Y}_H + e_1) + \mathbb{R}\mathcal{Y}_P) \times \Pi_\phi$. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G(q)$ at q induced by $\exp(\mathbb{R}(1 + \mathcal{Y}_H))$ is $v = (x + y + 1, x + y, z)$ (see B.11). Observe that the set $\{(e_1 + e_2), e_3, v\} \subset T_q G(q)$ is a basis. Therefore, G acts on $\mathbb{E}^{1,2}$ transitively. This implies that G acts on the vertex-less photon $\hat{\phi}$ transitively. Clearly, G admits the same orbits in $\mathbb{E}^{1,2}$ as $Y_P \times \mathbb{R}^{1,2}$.

Definition 4.10. *The subgroup $\exp(\mathbb{R}(1 + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P) \times \Pi_\phi$ is the identity component of the intersection of the stabilizers of the points in the photon ϕ . In other words, up to conjugacy, it is the unique (up to conjugacy) maximal connected Lie subgroup of $SO_\circ(2, 3)$ which acts on a photon trivially. We denote this group by the special symbol \mathcal{K}_\circ .*

Subgroups with a timelike line as the translation part

Let $G \subset \text{Conf}_\circ(\mathbb{E}^{1,2})$ be a connected Lie subgroup on which its translation part $T(G)$ is a timelike line, i.e., G is an element in Table 4.5.

The translation part $T(G) = \mathbb{R}e_1$ admits a codimension 2 foliation $\mathcal{F}_{\mathbb{R}e_1}$ in $\mathbb{E}^{1,2}$ on which the leaves are affine timelike lines. On the other hand, $\mathbb{R}e_1$ preserves no degenerate affine plane in $\mathbb{E}^{1,2}$, hence, it acts on each vertex-less photon in the lightcone $L(p)$, transitively. In addition, every G -orbit in $\mathbb{E}^{1,2}$ is either timelike (if it is 1-dimensional) or it is Lorentzian (if the dimension > 1), since the timelike vector e_1 is tangent to all the orbits induced in $\mathbb{E}^{1,2}$. In this case, the linear isometry projection $P_{\ell_i}(G)$ is either trivial or it is a 1-parameter elliptic subgroup of $SO_\circ(1, 2)$. In the first case, the orbits induced by G in the lightcone $L(p)$ are the vertex-less photons. In the later, the vertex-less lightcone $L(\hat{p})$ is a G -orbit.

- $G = \mathbb{R}_+^* \times \mathbb{R}e_1$. The homothety factor \mathbb{R}_+^* preserves the leaf $\mathcal{F}_{\mathbb{R}e_1}(o)$. Hence, $\mathcal{F}_{\mathbb{R}e_1}(o)$ is a G -orbit. Also, G preserves every affine Lorentzian plane in $\mathbb{E}^{1,2}$ containing $\mathcal{F}_{\mathbb{R}e_1}(o)$. One can see that, G acts on $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\mathbb{R}e_1}(o)$ freely. Furthermore, every 2-dimensional orbit in $\mathbb{E}^{1,2}$ intersects the spacelike curve $\gamma_\theta = \{(0, \cos \theta, \sin \theta) : \theta \in \mathbb{R}\}$ in a unique point. Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: a fixed point p , the vertex-less photons in $L(p)$, the affine timelike line $\mathcal{F}_{\mathbb{R}e_1}(o)$, and a 1-parameter family of Lorentzian affine half-planes in $\mathbb{E}^{1,2}$ on which G acts freely -the parameter being $q \in \gamma_\theta$.
- $G = Y_E \times \mathbb{R}e_1$. The 1-parameter elliptic subgroup Y_E preserves the leaf $\mathcal{F}_{\mathbb{R}e_1}(o)$. Hence, $\mathcal{F}_{\mathbb{R}e_1}(o)$ is a G -orbit. For an arbitrary point $q \in \mathbb{E}^{1,2} \setminus \mathcal{F}_{\mathbb{R}e_1}(o)$, one can see that the orbit $G(q)$ intersects the affine spacelike half-line $\ell = \{(0, t, 0) : t \in \mathbb{R}_+^*\}$ in a unique point. It follows that G acts on $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\mathbb{R}e_1}(o)$ freely. Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: a fixed point p , a timelike affine line $\mathcal{F}_{\mathbb{R}e_1}(o)$, the vertex-less lightcone $L(\hat{p})$, and a 1-parameter family of Lorentzian orbits on which G acts freely -the parameter being $q \in \ell$.
- $G = (\mathbb{R}_+^* \times Y_E) \times \mathbb{R}e_1$. The subgroup $\mathbb{R}_+^* \times Y_E$ preserves the leaf $\mathcal{F}_{\mathbb{R}e_1}(o)$. Hence, $\mathcal{F}_{\mathbb{R}e_1}(o)$ is a G -orbit. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vectors tangent to the orbit $G(q)$ at q induced

by the 1-parameter subgroups \mathbb{R}_+^* and Y_E are $v = (x, y, z)$ and $w = (0, z, -y)$, respectively. The set $\{e_1, v, w\} \subset T_q G(q)$ is a basis if and only if $y, z \neq 0$ if and only if $q \notin \mathcal{F}_{\mathbb{R}e_1}(o)$. Thus, G acts on $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\mathbb{R}e_1}(o)$ transitively. Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: a fixed point p , the timelike affine line $\mathcal{F}_{\mathbb{R}e_1}(o)$, the vertex-less lightcone $L(\hat{p})$, and an open orbit $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\mathbb{R}e_1}(o)$.

- $G_a = \exp(\mathbb{R}(a + \mathcal{Y}_E)) \times \mathbb{R}e_1$, $a \in \mathbb{R}^*$. The 1-parameter subgroup $\exp(\mathbb{R}(a + \mathcal{Y}_E))$ preserves the leaf $\mathcal{F}_{\mathbb{R}e_1}(o)$. Hence, $\mathcal{F}_{\mathbb{R}e_1}(o)$ is a G_a -orbit. For an arbitrary point $q \in \mathbb{E}^{1,2} \setminus \mathcal{F}_{\mathbb{R}e_1}(o)$ the orbit $G_a(q)$ intersects the spacelike affine half-line $\ell = \{(0, t, 0) : t \in \mathbb{R}_+^*\}$. Now, it can be easily seen that, G_a acts on $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\mathbb{R}e_3}(o)$ freely. Therefore, the orbits induced by G_a in $\mathbb{E}^{1,2}$ are: a fixed point p , a timelike affine line $\mathcal{F}_{\mathbb{R}e_1}(o)$, the vertex-less lightcone $L(\hat{p})$, and a 1-parameter family of Lorentzian orbits in $\mathbb{E}^{1,2}$ on which G_a acts freely -the parameter being $q \in \ell$ -. By Lemma 1.21, the action of G_a on $\mathbb{E}^{1,2}$ is orbitally-equivalent to the action of G_{-a} via the element in $O(1, 2)$ which maps $(x, y, z) \in \mathbb{E}^{1,2}$ to (x, z, y) .

Subgroups with a spacelike line as the translation part

Suppose that $G \subset \text{Conf}_o(\mathbb{E}^{1,2})$ is a connected Lie subgroup with a spacelike line as the translation part. These groups have been listed in Table 4.6.

The translation part $T(G) = \mathbb{R}e_3$ admits a codimension 2 foliation $\mathcal{F}_{\mathbb{R}e_3}$ in $\mathbb{E}^{1,2}$ on which the leaves are affine spacelike lines parallel to $\mathbb{R}e_3$. In the one hand, the translation part fixes two photons $\phi, \psi \subset L(p)$, pointwisely, since $\mathbb{R}e_3$ is contained in the both lightlike planes $\Pi_\phi = \mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3$ and $\Pi_\psi = \mathbb{R}(e_1 - e_2) \oplus \mathbb{R}e_3$. On the other hand, the linear projection $P_l(G)$ preserves both foliations \mathcal{F}_{Π_ϕ} and \mathcal{F}_{Π_ψ} . Hence, the photons ϕ and ψ are invariant by G . Also, $\mathbb{R}e_3$ acts on all the vertex-less photons in the lightcone $L(p)$ different from $\hat{\phi}$ and $\hat{\psi}$ transitively.

In this case, the linear isometry projection is either trivial or it is a 1-parameter hyperbolic subgroup of $SO_o(1, 2)$. In the first, every vertex-less photon in $L(p)$ different from $\hat{\phi}$ and $\hat{\psi}$ is a G -orbit. In the later, G acts on the both connected components of $L(p) \setminus (\phi \cup \psi)$ transitively.

- $G = \mathbb{R}_+^* \times \mathbb{R}e_3$: The homothety factor \mathbb{R}_+^* preserves the leaf $\mathcal{F}_{\mathbb{R}e_3}(o)$. Hence, $\mathcal{F}_{\mathbb{R}e_3}(o)$ is a G -orbit. Also, G preserves every affine plane in $\mathbb{E}^{1,2}$ containing $\mathcal{F}_{\mathbb{R}e_3}(o)$. Specially, G preserves the degenerate leaves $\mathcal{F}_{\Pi_\phi}(o)$ and $\mathcal{F}_{\Pi_\psi}(o)$. This implies that G fixes their corresponding limit points $d_\phi \in \hat{\phi}$ and $d_\psi \in \hat{\psi}$. Clearly, the homothety factor \mathbb{R}_+^* acts on the both connected components of $\hat{\phi} \setminus \{d_\phi\}$ and $\hat{\psi} \setminus \{d_\psi\}$ transitively. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G(q)$ at q induced by homothety factor \mathbb{R}_+^* is $v = (x, y, z)$. Observe that the set $\{v, e_3\} \subset T_q G(q)$ is a basis if and only if $x \neq 0$ or $y \neq 0$ if and only if $q \notin \mathcal{F}_{\mathbb{R}e_3}(o)$. In fact, every 2-dimensional orbit intersects the curve $\gamma(\theta) = \{(\cos \theta, \sin \theta, 0) : 0 \leq \theta \leq 2\pi\}$ in a unique point. A 2-dimensional G -orbit in $\mathbb{E}^{1,2}$ is degenerate if and only if $\theta = \pm\pi/4, \pm 3\pi/4$. Moreover, a 2-dimensional orbit is Lorentzian (resp. spacelike) if and only if $\pi/4 < \theta < 3\pi/4$ or $-\pi/4 < \theta < -3\pi/4$ (resp. $-\pi/4 < \theta < \pi/4$ or $3\pi/4 < \theta < 5\pi/4$). Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: three fixed points p, d_ϕ and d_ψ , the four connected components of $\phi \cup \psi \setminus \{p, d_\phi, d_\psi\}$, the two connected components of $L(p) \setminus (\phi \cup \psi)$, the affine spacelike line $\mathcal{F}_{\mathbb{R}e_3}(o)$, four degenerate orbits which are the degenerate affine planes in $\mathbb{E}^{1,2}$ with $\theta = \pm\pi/4, \pm 3\pi/4$, a 1-parameter family of Lorentzian orbits on which every orbit is a Lorentzian affine half-plane -the parameter being

$\theta \in (\pi/4, 3\pi/4) \cup (-3\pi/4, -\pi/4)$ -, and a 1-parameter family of spacelike orbits on which every orbit is a spacelike affine half-plane -the parameter being $\theta \in (-\pi/4, \pi/4) \cup (3\pi/4, 5\pi/4)$.

- $G = Y_H \times \mathbb{R}e_3$. Every G -orbit in $\mathbb{E}^{1,2}$ intersects either the timelike affine line $\ell_t = \{(t, 0, 0) : t \in \mathbb{R}\}$ or the spacelike affine line $\ell_s = \{(0, t, 0) : t \in \mathbb{R}\}$ or one of the lightlike affine lines $\ell_l = \{(t, t, 0) : t \in \mathbb{R}\}$ and $\ell'_l = \{(t, -t, 0) : t \in \mathbb{R}\}$. Also, G preserves the leaves $\mathcal{F}_{\mathbb{R}e_3}(o)$, $\mathcal{F}_{\Pi_\phi}(o)$, and $\mathcal{F}_{\Pi_\psi}(o)$. Hence, G fixes the corresponding limit points $d_\phi \in \hat{\phi}$ and $d_\psi \in \hat{\psi}$. On the other hand, Y_H (and consequently G) acts on the both connected components of $\hat{\phi} \setminus \{d_\phi\}$ and $\hat{\psi} \setminus \{d_\psi\}$ transitively. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroup Y_H is $v = (y, x, 0)$. The set $B = \{e_3, v\} \subset T_q G(q)$ is a basis if and only if $x \neq 0$ or $y \neq 0$ if and only if $q \notin \mathcal{F}_{\mathbb{R}e_3}(o)$. In fact, one can see, G acts on $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\mathbb{R}e_3}(o)$ freely. Hence, G acts on the four connected components of $(\mathcal{F}_{\Pi_\phi}(o) \cup \mathcal{F}_{\Pi_\psi}(o)) \setminus \mathcal{F}_{\mathbb{R}e_3}(o)$ transitively. Observe, for $q \in \ell_t \setminus \{o\}$ (resp. $q \in \ell_s \setminus \{o\}$) the orbit $G(q)$ is spacelike (resp. Lorentzian). Therefore, the orbits induced by G in $\text{Ein}^{1,2}$ are: three fixed points p, d_ϕ and d_ψ , the four connected components of $(\hat{\phi} \cup \hat{\psi}) \setminus \{d_\phi, d_\psi\}$, the two connected components of $L(p) \setminus (\phi \cup \psi)$, the affine spacelike line $\mathcal{F}_{\mathbb{R}e_3}(o)$, the four connected components of $(\mathcal{F}_{\Pi_\phi}(o) \cup \mathcal{F}_{\Pi_\psi}(o)) \setminus \mathcal{F}_{\mathbb{R}e_3}(o)$ which are degenerate affine half-planes, a 1-parameter family of Lorentzian orbits on which G acts freely -the parameter being $q \in \ell_s \setminus \{o\}$ -, and a 1-parameter family of the spacelike orbits on which G acts freely -the parameter being $q \in \ell_t \setminus \{o\}$ -.
- $G = (\mathbb{R}_+^* \times Y_H) \times \mathbb{R}e_3$. The subgroup $\mathbb{R}_+^* \times Y_H$ preserves the leaf $\mathcal{F}_{\mathbb{R}e_3}(o)$. Hence, $\mathcal{F}_{\mathbb{R}e_3}(o)$ is a G -orbit. Observe that G also preserves the leaves $\mathcal{F}_{\Pi_\phi}(o)$, and $\mathcal{F}_{\Pi_\psi}(o)$. Hence, G fixes the corresponding limit points $d_\phi \in \hat{\phi}$ and $d_\psi \in \hat{\psi}$. On the other hand, the homothety factor \mathbb{R}_+^* (and consequently G) acts on the both connected components of $\hat{\phi} \setminus \{d_\phi\}$ and $\hat{\psi} \setminus \{d_\psi\}$ transitively. The group G acts on the four connected components of $(\mathcal{F}_{\Pi_\phi}(o) \cup \mathcal{F}_{\Pi_\psi}(o)) \setminus \mathcal{F}_{\mathbb{R}e_3}(o)$ transitively, since its subgroup $Y_H \times \mathbb{R}e_3$ does. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vectors tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroups Y_H and \mathbb{R}_+^* are $v = (y, x, 0)$ and $w = (x, y, z)$. The set $\{e_3, v, w\} \subset T_q G(q)$ is a basis if and only if $x \neq \pm y$ if and only if $q \notin (\mathcal{F}_{\Pi_\phi}(o) \cup \mathcal{F}_{\Pi_\psi}(o))$. This implies that G acts on the four connected components of $\mathbb{E}^{1,2} \setminus (\mathcal{F}_{\Pi_\phi}(o) \cup \mathcal{F}_{\Pi_\psi}(o))$ transitively. Therefore, the orbits induced by G in $\text{Ein}^{1,2}$ are: three fixed points p, d_ϕ, d_ψ , the four connected components of $(\hat{\phi} \cup \hat{\psi}) \setminus \{d_\phi, d_\psi\}$, the two connected components of $L(p) \setminus (\phi \cup \psi)$, the spacelike affine line $\mathcal{F}_{\mathbb{R}e_3}(o)$, the four connected components of $(\mathcal{F}_{\Pi_\phi}(o) \cup \mathcal{F}_{\Pi_\psi}(o)) \setminus \mathcal{F}_{\mathbb{R}e_3}(o)$ which are affine degenerate half-planes, and the four connected components of $\text{Ein}^{1,2} \setminus (\mathcal{F}_{\Pi_\phi}(o) \cup \mathcal{F}_{\Pi_\psi}(o))$.
- $G_a = \exp(\mathbb{R}(a + \mathcal{Y}_H)) \times \mathbb{R}e_3$, $a \in \mathbb{R} \setminus \{-1, 0, 1\}$. The 1-parameter subgroup $H_a = \exp(\mathbb{R}(a + \mathcal{Y}_H))$ preserves the leaf $\mathcal{F}_{\mathbb{R}e_3}(o)$. Hence, $\mathcal{F}_{\mathbb{R}e_3}(o)$ is a G_a -orbit. Also, G_a preserves the leaves $\mathcal{F}_{\Pi_\phi}(o)$, and $\mathcal{F}_{\Pi_\psi}(o)$. Hence, G_a fixes the corresponding limit points $d_\phi \in \hat{\phi}$ and $d_\psi \in \hat{\psi}$. Moreover, H_a (and consequently G_a) acts on the connected components of $\hat{\phi} \setminus \{d_\phi\}$ and $\hat{\psi} \setminus \{d_\psi\}$ transitively. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G_a(q)$ at q induced by the 1-parameter subgroup H_a is $v = (ax + y, x + ay, az)$. The set $\{e_3, v\} \subset T_q G_a(q)$ is a basis if and only if $x \neq 0$ or $y \neq 0$ if and only if $q \notin \mathcal{F}_{\mathbb{R}e_3}(o)$. Hence, G_a acts on the four connected components of $(\mathcal{F}_{\Pi_\phi}(o) \cup \mathcal{F}_{\Pi_\psi}(o)) \setminus \mathcal{F}_{\mathbb{R}e_3}(o)$ transitively. For a point $q \in \mathbb{E}^{1,2} \setminus (\mathcal{F}_{\Pi_\phi}(o) \cup \mathcal{F}_{\Pi_\psi}(o))$, it is not hard to see that the orbit $G_a(q)$ intersects either the

spacelike affine line $\ell_s = \{(0, t, 0) : t \in \mathbb{R}\}$ or the timelike affine line $\ell_t = \{(t, 0, 0) : t \in \mathbb{R}\}$. Now, one can see G_a acts on $\mathbb{E}^{1,2} \setminus (\mathcal{F}_{\Pi_\phi}(o) \cup \mathcal{F}_{\Pi_\psi}(o))$ freely. By Lemma 1.21, one can see that the action of G_a on $\mathbb{E}^{1,2}$ is orbitally-equivalent to G_{-a} via the element in $O(1, 2)$ which maps $(x, y, z) \in \mathbb{E}^{1,2}$ to $(-x, y, z)$. Hence, we may restrict ourselves to $a \in (0, \infty) \setminus \{1\}$.

- If $a \in (0, 1)$. the orbit induced at $q \in \ell_t \setminus \{o\}$ (resp. $q \in \ell_s$) is spacelike (resp. Lorentzian).
- If $a \in (1, \infty)$. the orbit induced at $q \in \ell_t \setminus \{o\}$ (resp. $q \in \ell_s$) is Lorentzian (resp. spacelike).

Therefore, the orbits induced by G_a in $\mathbb{E}^{1,2}$ are: three fixed points p, d_ϕ and d_ψ , the four connected components of $(\hat{\phi} \cup \hat{\psi}) \setminus \{d_\phi, d_\psi\}$, the two connected components of $L(p) \setminus (\phi \cup \psi)$, the spacelike affine line $\mathcal{F}_{\mathbb{R}e_3}(o)$, the four connected components of $(\mathcal{F}_{\Pi_\phi}(o) \cup \mathcal{F}_{\Pi_\psi}(o)) \setminus \mathcal{F}_{\mathbb{R}e_3}(o)$ which are degenerate affine half-planes,

- if $a \in (0, 1)$, a 1-parameter family of Lorentzian surfaces in $\mathbb{E}^{1,2}$ on which G_a acts freely -the parameter being $q \in \ell_t \setminus \{o\}$ -, and a 1-parameter family of spacelike surfaces in $\mathbb{E}^{1,2}$ on which G_a acts freely -the parameter being $q \in \ell_s \setminus \{o\}$ -.
 - if $a \in (1, \infty)$, a 1-parameter family of spacelike surfaces in $\mathbb{E}^{1,2}$ on which G_a acts freely -the parameter being $q \in \ell_t \setminus \{o\}$ -, and a 1-parameter family of Lorentzian surfaces in $\mathbb{E}^{1,2}$ on which G_a acts freely -the parameter being $q \in \ell_s \setminus \{o\}$ -.
- $G_1 = \exp(\mathbb{R}(1 + \mathcal{Y}_H)) \times \mathbb{R}e_3$. The 1-parameter subgroup $H_1 = \exp(\mathbb{R}(1 + \mathcal{Y}_H))$ preserves the leaf $\mathcal{F}_{\mathbb{R}e_3}(o)$. Hence, $\mathcal{F}_{\mathbb{R}e_3}(o)$ is a G_1 -orbit. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G_1(q)$ at q induced by H_1 is $v = (x + y, x + y, z)$. Observe that $v \in \Pi_\phi$. This implies that G_1 preserves the leaves of the foliation \mathcal{F}_{Π_ϕ} . Hence, G_1 acts on the photon ϕ trivially (in fact G_1 is a subgroup of \mathcal{K}_o described in Definition 4.10). On the other hand, the vector v belongs to Π_ψ if and only if $x = -y$ if and only if $q \in \mathcal{F}_{\Pi_\psi}(o)$. Hence, G_1 fixes the corresponding limit point $d_\psi \in \hat{\psi}$ and acts on the both connected components of $\hat{\psi} \setminus \{d_\psi\}$ transitively. The set $\{e_3, v\} \subset T_q G_1(q)$ is a basis if and only if $x \neq -y$ if and only if $q \notin \mathcal{F}_{\Pi_\psi}(o)$. Every G_1 -orbit in $\mathbb{E}^{1,2}$ intersects one of the lightlike affine lines $\ell_l = \{(t, -t, 0) : t \in \mathbb{R}\} \subset \mathcal{F}_{\Pi_\psi}(o)$, $\ell_l^+ = \{(t, -t, 1) : t \in \mathbb{R}\}$, or $\ell_l^- = \{(t, -t, -1) : t \in \mathbb{R}\}$. Observe that for $q \in (\ell_l^+ \cup \ell_l^-)$ the orbit $G_1(q)$ is a degenerate affine half-plane. Therefore, the orbits induced by G_1 in $\mathbb{E}^{1,2}$ are: the points in $\phi \cup \{d_\psi\}$, the two connected components of $\hat{\psi} \setminus \{d_\psi\}$, the two connected components of $L(p) \setminus (\phi \cup \psi)$, a 1-parameter family of 1-dimensional spacelike orbits in $\mathbb{E}^{1,2}$ on which every orbit is a leaf $\mathcal{F}_{\mathbb{R}e_3}(q)$ - the parameter being $q \in \ell_l$ -, and a 1-parameter family of degenerate surfaces in $\mathbb{E}^{1,2}$ on which every orbit is a degenerate affine half-plane -the parameter being $q \in (\ell_l^+ \cup \ell_l^-)$ -.
- $G_{-1} = \exp(\mathbb{R}(-1 + \mathcal{Y}_H)) \times \mathbb{R}e_3$. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vectors tangent to the orbit $G_{-1}(q)$ at q induced by the 1-parameter subgroups $\exp(\mathbb{R}(-1 + \mathcal{Y}_H))$ is $v = (-x + y, x - y, z)$. By Lemma 1.21, the action of G_{-1} on $\mathbb{E}^{1,2}$ is orbitally-equivalent to the action of $\exp(\mathbb{R}(1 + \mathcal{Y}_H)) \times \mathbb{R}e_3$ via the element in $O(1, 2)$ which maps $(x, y, z) \in \mathbb{R}^{1,2}$ to $(-x, y, z)$.
- $G = \exp(\mathbb{R}(1 + \mathcal{Y}_H + e_1)) \times \mathbb{R}e_3$. For an arbitrary point $p = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G(q)$ induced by the 1-parameter subgroup $\exp(\mathbb{R}(1 + \mathcal{Y}_H + e_1))$ is $v = (x + y + 1, x + y, z)$ (see B.11). The set $\{e_3, v\}$ is a basis for the tangent space $T_p G(p)$. Thus, all the orbits induced by G

in $\mathbb{E}^{1,2}$ are 2-dimensional. Therefore, the action of G admits a codimension 1 foliation in $\mathbb{E}^{1,2}$. In the one hand, $G(p)$ is degenerate if and only if $x + y = -1/2$. On the other hand, G preserves the affine lightlike plane $\mathcal{F}_{\Pi_\psi}(-1/2, 0, 0)$, and acts on it transitively. Hence, G admits a unique degenerate orbit $\mathcal{F}_{\Pi_\psi}(-1/2, 0, 0)$ in $\mathbb{E}^{1,2}$. Consequently, G admits a unique fixed point d_ψ on $\hat{\psi}$, which is the limit point of $\mathcal{F}_{\Pi_\psi}(-1/2, 0, 0)$, and acts on the both connected components of $\hat{\psi} \setminus \{d_\psi\}$ and on the vertex-less photon $\hat{\phi}$ transitively. Furthermore, for a point $q \in \mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\psi}(-1/2, 0, 0)$ the orbit induced at $G(q)$ intersects either the affine timelike half-line $\ell_t = \{(t, 0, 0) : -\infty < t < -1/2\}$ or the spacelike affine half-line $\ell_s = \{(0, t, 0) : -1/2 < t < \infty\}$ in a unique point. Now, it can be easily seen that G acts on $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\psi}(-1/2, 0, 0)$ freely. Observe that, the orbit induced at $q \in \ell_t$ (resp. $q \in \ell_s$) is Lorentzian (resp. spacelike). Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: two fixed points p and d_ψ , the two connected components of $\hat{\psi} \setminus \{d_\psi\}$, the vertex-less photon $\hat{\phi}$, the two connected components of $L(p) \setminus (\phi \cup \psi)$, the lightlike affine plane $\mathcal{F}_{\Pi_\psi}(-1/2, 0, 0)$, a 1-parameter family of Lorentzian orbits on which G acts freely -the parameter being $q \in \ell_t$ -, and a 1-parameter family of spacelike orbits on which G acts freely -the parameter being $q \in \ell_s$.

- $G = \exp(\mathbb{R}(-1 + \mathcal{Y}_H + e_1)) \times \mathbb{R}e_3$. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroup $\exp(\mathbb{R}(-1 + \mathcal{Y}_H + e_1))$ is $v = (-x + y + 1, x - y, z)$. By Lemma 1.21, the action of G on $\mathbb{E}^{1,2}$ is orbitally equivalent to the action of $\exp(\mathbb{R}(1 + \mathcal{Y}_H + e_1)) \times \mathbb{R}e_3$ via the element in $O(1,2)$ which maps $(x, y, z) \in \mathbb{E}^{1,2}$ to $(-x, y, z)$.

Subgroups with a lightlike line as the translation part

Assume that $G \subset \text{Conf}_o(\mathbb{E}^{1,2})$ is a connected Lie subgroup with a lightlike line as the translation part, i.e., G is an element of Table 4.7.

Throughout, this section we denote by \mathcal{L} the lightlike line $\mathbb{R}(e_1 + e_2) \leq \mathbb{R}^{1,2}$. Also, recall that, the unique lightlike plane in $\mathbb{R}^{1,2}$ invariant by the 1-parameter parabolic subgroup Y_P is $\mathcal{L}^\perp = \mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3$ and is denoted by Π_ϕ .

The translation part $T(G) = \mathcal{L}$ induces a codimension 2 foliation $\mathcal{F}_\mathcal{L}$ in $\mathbb{E}^{1,2}$ on which the leaves are affine lightlike lines parallel to \mathcal{L} . Observe that, the translation part preserves the leaves of the foliation \mathcal{F}_{Π_ϕ} , since $\mathcal{L} \leq \Pi_\phi$. Hence, \mathcal{L} acts on the photon ϕ trivially. Furthermore, \mathcal{L} acts on all the vertex-less photons in $L(p)$ different from $\hat{\phi}$ transitively. On the other hand, all the subgroups listed in Table (4.7) preserve the foliation \mathcal{F}_{Π_ϕ} . Hence, they preserve the photon ϕ . Also, the subgroups with hyperbolic linear projection preserve the foliation \mathcal{F}_{Π_ψ} , and so, these subgroups preserve both photons ϕ and ψ .

Case I: The linear isometry projection trivial. In this case $G = \mathbb{R}_+^* \times \mathcal{L}$. The homothety factor \mathbb{R}_+^* preserves the leaf $\mathcal{F}_\mathcal{L}(o)$. Hence, $\mathcal{F}_\mathcal{L}(o)$ is a G -orbit. Also, G preserves the leaf $\mathcal{F}_{\Pi_\phi}(o)$. This implies that G fixes a point d in $\hat{\phi}$ which is the limit point of $\mathcal{F}_{\Pi_\phi}(o)$. The group G acts on the both connected components of $\hat{\phi} \setminus \{d\}$ transitively. Moreover, one can see, \mathbb{R}_+^* (and consequently G) preserves all the affine planes in $\mathbb{E}^{1,2}$ containing $\mathcal{F}_\mathcal{L}(o)$. Observe that all those affine planes are Lorentzian except $\mathcal{F}_{\Pi_\phi}(o)$. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$ the vector tangent to the orbit $G(q)$ at q induced by \mathbb{R}_+^* is $v = (x, y, z)$. The set $\{(e_1 + e_2) \cdot v\} \subset T_q G(q)$ is a basis if and only if $z \neq 0$ or $x \neq y$ if and only if $q \notin \mathcal{F}_\mathcal{L}(o)$. Thus, G acts on the both connected components of $\mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_\mathcal{L}(o)$ transitively. Furthermore, every G -orbit in $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$ intersects one of the spacelike curves $\gamma_\eta^+ = \{(1, \cos \eta, \sin \eta) : 0 < \eta < 2\pi\}$ or $\gamma_\eta^- = \{(-1, \cos \eta, \sin \eta) : 0 < \eta < 2\pi\}$ in a unique point. Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$

are: two fixed points $\{p\}$ and $\{d\}$, the two connected components of $\hat{\phi} \setminus \{d\}$, the vertex-less photons in $L(p)$ different from $\hat{\phi}$, the affine lightlike line $\mathcal{F}_{\mathcal{L}}(o)$, the two connected components of $\mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$ which are affine degenerate half-planes, and a 1-parameter family of Lorentzian affine half-planes -the parameter being $q \in (\gamma_\eta^+ \cup \gamma_\eta^-)$ -.

Case II: *The linear isometry projection is a 1-parameter parabolic subgroup of $SO_o(1, 2)$.* In this case, G preserves a unique photon in $L(p)$, namely ϕ , and acts on $L(p) \setminus \phi$ transitively.

- $G = Y_P \times \mathcal{L}$. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G(q)$ at q induced by Y_P is $v = (z, z, x - y)$. Since $v \in \Pi_\phi$, G preserves the leaves of the foliation \mathcal{F}_{Π_ϕ} . Actually, G is a subgroup of \mathcal{K}_o (Definition 4.10). Thus, G acts on the photon ϕ trivially. The set $\{(e_1 + e_2), v\}$ is a basis if and only if $x \neq y$, which describes the leaf $\mathcal{F}_{\Pi_\phi}(o)$. Hence, G acts on all the leaves of \mathcal{F}_{Π_ϕ} different from $\mathcal{F}_{\Pi_\phi}(o)$ transitively. Observe that the orbits induced by G in $\mathcal{F}_{\Pi_\phi}(o)$ are lightlike affine lines, namely, they are the leaves of the foliation induced by \mathcal{L} . Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: the points in ϕ , the degenerate surface $L(p) \setminus \phi$, the lightlike affine lines in $\mathcal{F}_{\Pi_\phi}(o)$, and the leaves of the foliation \mathcal{F}_{Π_ϕ} different from $\mathcal{F}_{\Pi_\phi}(o)$, which are degenerate affine planes.
- $G = (\mathbb{R}_+^* \times Y_P) \times \mathcal{L}$. The subgroup $\mathbb{R}_+^* \times Y_P$ preserves the leaves $\mathcal{F}_{\mathcal{L}}(o)$. Hence, $\mathcal{F}_{\mathcal{L}}(o)$ is a G -orbit. Moreover, G preserves the leaf $\mathcal{F}_{\Pi_\phi}(o)$. This implies that G fixes a point $d \in \hat{\phi}$, which is the limit point of $\mathcal{F}_{\Pi_\phi}(o)$. The homothety factor \mathbb{R}_+^* acts on the both connected components of $\hat{\phi} \setminus \{d\}$ transitively. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vectors tangent to the orbit $G(q)$ at q induced by Y_P and \mathbb{R}_+^* are $v = (z, z, x - y)$ and $w = (x, y, z)$, respectively. The set $\{(e_1 + e_2), v, w\} \subset T_q G(q)$ is a basis if and only if $x \neq y$ if and only if $q \notin \mathcal{F}_{\Pi_\phi}(o)$. Thus, G acts on the both connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$ transitively. Observe that for a point $q \in \mathcal{F}_{\Pi_\phi}(o)$ the set $\{(e_1 + e_2), w\} \subset T_q G(q)$ is a basis if and only if $z \neq 0$ if and only if $q \notin \mathcal{F}_{\mathcal{L}}(o)$. Hence, G acts on the both connected components of $\mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$ transitively. Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: two fixed points $\{p\}$ and $\{d\}$, the two connected components of $\hat{\phi} \setminus \{d\}$, the degenerate surface $L(p) \setminus \phi$, the lightlike affine line $\mathcal{F}_{\mathcal{L}}(o)$, the two connected components of $\mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$ which are affine degenerate half-planes, and the two connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$.
- $G_a = \exp(\mathbb{R}(a + \mathcal{Y}_P)) \times \mathcal{L}$, $a \in \mathbb{R}^*$. The 1-parameter subgroup $H_a = \exp(\mathbb{R}(a + \mathcal{Y}_P))$ preserves the leaf $\mathcal{F}_{\mathcal{L}}(o)$. Hence, $\mathcal{F}_{\mathcal{L}}(o)$ is a G_a -orbit. Moreover, G_a preserves the leaf $\mathcal{F}_{\Pi_\phi}(o)$. This implies that G_a fixes a point $d \in \hat{\phi}$, which is the limit point of $\mathcal{F}_{\Pi_\phi}(o)$. One can see that H_a acts on the both connected components of $\hat{\phi} \setminus \{d\}$ transitively. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G_a(q)$ at q induced by the H_a is $v = (ax + z, ay + z, x - y + az)$. The set $\{(e_1 + e_2), v\} \subset T_q G_a(q)$ is a basis if and only if $z \neq 0$ or $x \neq y$ if and only if $q \notin \mathcal{F}_{\mathcal{L}}(o)$. Henceforth, G_a acts on the both connected components of $\mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$ transitively. It is not hard to see that every orbit induced by G_a in $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$ intersects the affine spacelike line $\ell_s = \{(0, t, 0) : t \in \mathbb{R}\}$. Now, it can be easily seen that G_a acts on $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\mathcal{L}}(o)$ freely. For a point $q \in \mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$ the orbit $G_a(q)$ is Lorentzian, since the orthogonal space of the null vector $e_1 + e_2$ in $T_q G(q)$ is \mathcal{L} . Therefore, the orbits induced by G_a in $\mathbb{E}^{1,2}$ are: two fixed points $\{p, d\}$, the two connected components of $\hat{\phi} \setminus \{d\}$, the affine lightlike line $\mathcal{F}_{\mathcal{L}}(o)$, the degenerate surface $L(p) \setminus \phi$, the two connected components of $\mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$ which are degenerate affine

half-planes, and a 1-parameter family of Lorentzian orbits in $\mathbb{E}^{1,2}$ on which G_a acts freely -the parameter being $q \in \ell_s \setminus \{o\}$.

It is remarkable that the action of G_a is orbit equivalent to the action of G_{-a} via the element in $O(1, 2)$ which maps $(x, y, z) \in \mathbb{E}^{1,2}$ to $(x, y, -z)$.

- $G = \exp(\mathbb{R}(\mathcal{Y}_P + e_1)) \times \mathcal{L}$. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G(p)$ at q induced by the 1-parameter subgroup $H = \exp(\mathbb{R}(\mathcal{Y}_P + e_1))$ is $v = (z+1, z, x-y)$ (see B.8). Observe that $v \notin \Pi_\phi$, hence, G preserves no leaf of the foliation \mathcal{F}_{Π_ϕ} , and so, G acts on the vertex-less photon $\hat{\phi}$ transitively. It is clear that the set $\{(e_1 + e_2), v\} \subset T_q G(q)$ is a basis. Thus all the orbits induced by G in $\mathbb{E}^{1,2}$ are 2-dimensional. By applying a suitable element of \mathcal{L} , we may assume $q = (x, 0, z)$. On the other hand, the orbit induced by H at q is

$$H(q) = \left\{ \left(x \left(1 + \frac{1}{2} \right) t^2 + tz + t + \frac{t^3}{6}, \frac{1}{2} t^2 x + tz, tx + z + \frac{t^2}{2} \right) : t \in \mathbb{R} \right\}.$$

Setting $t = -x$, the orbit $H(q)$ intersects the affine degenerate plane $\{x = y\}$. Hence, once more, applying a suitable element of \mathcal{L} , we may assume that $q = (0, 0, z)$. We conclude that every G -orbit in $\mathbb{E}^{1,2}$ intersects the affine spacelike line $\ell = \{(0, 0, t) : t \in \mathbb{R}\}$. Now, it can be easily seen that G acts on $\mathbb{E}^{1,2}$ freely and every G -orbit intersects ℓ in a unique point. The orthogonal space of the null vector $e_1 + e_2$ in $T_p G(p)$ is \mathcal{L} . Hence, all the orbits are Lorentzian. Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: the fixed point p , the vertex-less photon $\hat{\phi}$, the degenerate surface $L(p) \setminus \phi$, and a 1-parameter family of Lorentzian orbits on which G acts freely -the parameter being $q \in \ell$.

Case III: *The linear isometry projection $P_{li}(G)$ is a 1-parameter hyperbolic subgroup of $SO_o(1, 2)$.*

In this case G preserves both foliations \mathcal{F}_{Π_ϕ} and \mathcal{F}_{Π_ψ} , and so, it preserves both photons ϕ and ψ in the lightcone $L(p)$. Also, G acts on the two connected components of $L(p) \setminus (\phi \cup \psi)$ transitively.

- $G = (\mathbb{R}_+^* \times Y_H) \times \mathcal{L}$. The subgroup $\mathbb{R}_+^* \times Y_H$ preserves the leaf $\mathcal{F}_{\mathcal{L}}(o)$. Hence, $\mathcal{F}_{\mathcal{L}}(o)$ is a G -orbit. Moreover, G preserves the leaf $\mathcal{F}_{\Pi_\phi}(o)$. This implies that, G admits a fixed point $d \in \hat{\phi}$, which is the limit point of $\mathcal{F}_{\Pi_\phi}(o)$. The homothety factor \mathbb{R}_+^* acts on the both connected components of $\hat{\phi} \setminus \{d\}$ transitively. Furthermore, G preserves the affine Lorentzian plane $L_0 = \{z = 0\} \subset \mathbb{E}^{1,2}$. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vectors tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroups \mathbb{R}_+^* and Y_H are $v = (x, y, z)$ and $w = (y, x, 0)$, respectively. The set $\{(e_1 + e_2), v, w\} \subset T_q G(q)$ is a basis if and only if $z \neq 0$ and $x \neq y$ if and only if $q \notin (L_0 \cup \mathcal{F}_{\Pi_\phi}(o))$. Thus, G acts on the four connected components of $\mathbb{E}^{1,2} \setminus (L_0 \cup \mathcal{F}_{\Pi_\phi}(o))$ transitively. Also, one can see that G acts on the four connected components of $(L_0 \cup \mathcal{F}_{\Pi_\phi}(o)) \setminus \mathcal{F}_{\mathcal{L}}(o)$ transitively. Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: two fixed points $\{p, d\}$, the two connected components of $\hat{\phi} \setminus \{d\}$, the vertex-less photon $\hat{\psi}$, the affine lightlike line $\mathcal{F}_{\mathcal{L}}(o)$, the two connected components of $L(p) \setminus (\phi \cup \psi)$, the two connected components of $L_0 \setminus \mathcal{F}_{\mathcal{L}}(o)$ which are affine Lorentzian half-planes, the two connected components of $\mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$ which are affine degenerate half-planes, and the four connected components of $\mathbb{E}^{1,2} \setminus (L_0 \cup \mathcal{F}_{\Pi_\phi}(o))$.
- $G = Y_H \times \mathcal{L}$. The 1-parameter subgroup Y_H preserves the leaf $\mathcal{F}_{\mathcal{L}}(o)$. Hence, $\mathcal{F}_{\mathcal{L}}(o)$ is a G -orbit. Moreover, G preserves the leaf $\mathcal{F}_{\Pi_\phi}(o)$. This implies that, G admits a fixed point $d \in \hat{\phi}$, which is the limit point of $\mathcal{F}_{\Pi_\phi}(o)$. The subgroup Y_H acts on the both connected components of $\hat{\phi} \setminus \{d\}$

transitively. Furthermore, G preserves the leaves of the foliation \mathcal{F}_L induced by the Lorentzian plane $L = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \leq \mathbb{R}^{1,2}$. Every G orbit in $\mathbb{E}^{1,2}$ intersects one of the spacelike affine lines $\ell_s = \{(0, t, 0) : t \in \mathbb{R}\} \subset \mathcal{F}_{\Pi_\phi}(o)$, $\ell_s^+ = \{(1, t, 0) : t \in \mathbb{R}\}$, and $\ell_s^- = \{(-1, t, 0) : t \in \mathbb{R}\}$ in a unique point. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G(q)$ at q induced by Y_H is $v = (y, x, 0)$. Observe that, the set $\{(e_1 + e_2, v)\} \subset T_q G(q)$ is a basis if and only if $x \neq y$ if and only if $q \notin \mathcal{F}_{\Pi_\phi}(o)$. Hence, for a point $q \in \ell_s$, the orbit G_q is $\mathcal{F}_{\mathcal{L}}(q)$. For a point $q \in (\ell_s^+ \cup \ell_s^-)$, the orbit G_q is the connected component of $\mathcal{F}_L(q) \setminus \mathcal{F}_{\Pi_\phi}(o)$ which contains q , and it is a Lorentzian affine half plane. Therefore, the orbits induced by the G in $\mathbb{E}^{1,2}$ are: two fixed points $\{p, d\}$, the two connected components of $\hat{\phi} \setminus \{d\}$, the vertex-less photon $\hat{\psi}$, a 1-parameter family of lightlike affine lines in $\mathbb{E}^{1,2}$ -the parameter being $q \in \ell_s^-$, and a 1-parameter family of the 2-dimensional orbits which are Lorentzian affine half-planes -the parameter being $q \in \ell_s^+ \cup \ell_s^-$.

- $G_a = \exp(\mathbb{R}(a + \mathcal{Y}_H)) \times \mathcal{L}$, $a \in \mathbb{R}^* \setminus \{1\}$. The group G_a preserves the orbits induced by $(\mathbb{R}_+^* \times Y_H) \times \mathcal{L}$, since G is a subgroup of $(\mathbb{R}_+^* \times Y_H) \times \mathcal{L}$. In fact, one can see, the orbits induced by G_a in the lightcone $L(p)$ are exactly the same as orbits induced by $(\mathbb{R}_+^* \times Y_H) \times \mathcal{L}$. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroup $\exp(\mathbb{R}(a + \mathcal{Y}_H))$ is $v = (ax + y, x + ay, az)$. The set $\{e_1 + e_2, v\} \subset T_q G(q)$ is a basis if and only if $z \neq 0$ or $x \neq y$ if and only if $p \notin \mathcal{F}_{\mathcal{L}}(o)$. This shows that G acts on the four connected components of $(L_0 \cup \mathcal{F}_{\Pi_\phi}(o)) \setminus \mathcal{F}_{\mathcal{L}}(o)$ transitively. In the one hand, for all $q \in \mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$ the orbit $G(q)$ is Lorentzian, since the orthogonal space of the null vector $e_1 + e_2 \in T_q G(q)$ is \mathcal{L} . On the other hand, the orbit $G(q)$ intersects one of the four timelike affine half-lines $\ell_1^\pm = \{(\pm t, 0, 1) : 0 < t < \infty\}$, $\ell_{-1}^\pm = \{(\pm t, 0, -1) : 0 < t < \infty\}$. Now, it can be easily seen that G acts on $\mathbb{E}^{1,2} \setminus (L_0 \cup \mathcal{F}_{\Pi_\phi}(o))$ freely. Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: two fixed points $\{p, d\}$, the two connected components of $\hat{\phi} \setminus \{d\}$, the vertex-less photon $\hat{\psi}$, the lightlike affine line $\mathcal{F}_{\mathcal{L}}(o)$, the two connected components of $L(p) \setminus (\phi \cup \psi)$, the two connected components of $\mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$ which are affine degenerate half-planes, the two connected components of $L_0 \setminus \mathcal{F}_{\mathcal{L}}(o)$ which are affine Lorentzian half-planes, and a 1-parameter family of Lorentzian orbits on which G acts freely -the parameter being $q \in (\ell_1^\pm \cup \ell_{-1}^\pm)$.

Note that, for all $a \in \mathbb{R}^* \setminus \{1\}$, the action of G_a on $\mathbb{E}^{1,2}$ is orbitally-equivalent to the the action of $G_{1/a}$ via the homothety in $\text{Conf}(\mathbb{E}^{1,2})$ which maps $(x, y, z) \in \mathbb{E}^{1,2}$ to $(x/a, y/a, z/a)$.

- If $G = \exp(\mathbb{R}(1 + \mathcal{Y}_H)) \times \mathcal{L}$. Since, G is a subgroup of \mathcal{K}_o (Definition 4.10), it preserves the leaves of the foliation $\mathcal{F}_{\Pi_\phi}(o)$. Hence, G acts on the photon ϕ trivially. Also, G preserves the leaf $\mathcal{F}_{\mathcal{L}}(o)$, and the affine Lorentzian plane $L_0 = \{z = 0\} \subset \mathbb{E}^{1,2}$. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$ the vector tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroup $\exp(\mathbb{R}(1 + \mathcal{Y}_H))$ is $v = (x + y, x + y, z)$. The set $\{e_1 + e_2, v\} \subset T_q G(q)$ is a basis if and only if $z \neq 0$ if and only if $q \notin L_0$. Thus for all $q \in \mathbb{E}^{1,2}$, G acts on the both connected components of $\mathcal{F}_{\Pi_\phi}(q) \setminus L_0$, transitively. The intersection of a leaf $\mathcal{F}_{\Pi_\phi}(q)$ with L_0 is a leaf of $\mathcal{F}_{\mathcal{L}}$. Indeed, every G -orbit in L_0 (resp. $\mathbb{E}^{1,2} \setminus L_0$) intersects the timelike affine line $\ell = \{(t, 0, 0) : t \in \mathbb{R}\}$ (resp. one of the timelike affine lines $\ell^+ = \{(t, 0, 1) : t \in \mathbb{R}\}$, $\ell^- = \{(t, 0, -1) : t \in \mathbb{R}\}$) in a unique point. Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: the points in ϕ , the vertex-less photon $\hat{\psi}$, a 1-parameter family of 1-dimensional orbits which are lightlike affine lines -the parameter being $q \in \ell^-$, the two connected components of $L(p) \setminus (\phi \cup \psi)$, and a 1-parameter family of 2-dimensional

orbits which are degenerate affine half-planes -the parameter being $q \in \ell^+ \cup \ell^-$.

- $G = \exp(\mathbb{R}(\mathcal{Y}_H + e_3)) \times \mathcal{L}$. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G(p)$ induced by the 1-parameter subgroup $H = \exp(\mathbb{R}(\mathcal{Y}_H + e_3))$ is $v = (y, x, 1)$ (see B.10). It is clear that G preserves the degenerate affine plane $\mathcal{F}_{\Pi_\phi}(o)$. Thus, it fixes the a point $d \in \hat{\phi}$ which is the limit point of $\mathcal{F}_{\Pi_\phi}(o)$. Also, one can see that H acts on the both connected components of $\hat{\phi} \setminus \{d\}$ transitively. The set $\{e_1 + e_2, v\} \subset T_q G(q)$ is a basis. Thus, all the orbits induced by G in $\mathbb{E}^{1,2}$ are 2-dimensional. Every G -orbit in $\mathbb{E}^{1,2}$ intersects the timelike affine line $\ell = \{(t, 0, 0) : t \in \mathbb{R}\}$ in a unique point. Now, one can see that G acts on $\mathbb{E}^{1,2}$ freely. Observe that, for all points $q \in \mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$, the tangent space $T_q G(q)$ is Lorentzian, since the orthogonal space of the lightlike vector $e_1 + e_2$ in $T_p G(p)$ is \mathcal{L} . Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: two fixed points $\{p, d\}$, the two connected components of $\hat{\phi} \setminus \{d\}$, the vertex-less photon $\hat{\psi}$, the two connected components of $L(p) \setminus (\phi \cup \psi)$, and a 1-parameter family of 2-dimensional orbits in $\mathbb{E}^{1,2}$ on which one of the is the degenerate affine plane $\mathcal{F}_{\Pi_\phi}(o)$, and the other orbits are Lorentzian and G acts on them freely -the parameter being $q \in \ell$.
- $\exp(\mathbb{R}(1 + \mathcal{Y}_H + e_1)) \times \mathcal{L}$. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vector tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroup $\exp(\mathbb{R}(1 + \mathcal{Y}_H + e_1))$ is $v = (x+y+1, x+y, z)$ (see B.11). The set $\{e_1 + e_2, v\} \subset T_q G(q)$ is a basis, and so, all the orbits induced by G in $\mathbb{E}^{1,2}$ are 2-dimensional. Observe that, the Lorentzian affine plane $L_0 = \{z = 0\}$ is G -invariant. Hence, G acts on L_0 transitively. Moreover, every G -orbit in $\mathbb{E}^{1,2}$ intersects the spacelike affine lines $\ell = \{(0, 0, t) : t \in \mathbb{R}\}$ in a unique point. Now, it can be easily seen that G acts on $\mathbb{E}^{1,2}$ freely. Furthermore, all the orbits induced by G in $\mathbb{E}^{1,2}$ are Lorentzian, since the orthogonal space of the null vector $e_1 + e_2$ in $T_q G(q)$ is \mathcal{L} . This shows that G acts on the vertex-less photon $\hat{\phi}$ transitively. Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: a fixed point p , the two vertex-less photons $\hat{\phi}$ and $\hat{\psi}$, the two connected components of $L(p) \setminus (\phi \cup \psi)$, and a 1-parameter family of Lorentzian orbits in $\mathbb{E}^{1,2}$ on which G acts freely and on of them is a Lorentzian affine plane -the parameter being $q \in \ell$.
- $\exp(\mathbb{R}(-1 + \mathcal{Y}_H + e_1)) \times \mathcal{L}$. For an arbitrary point $q = (x, y, z) \in \mathbb{R}^{1,2}$, the vector tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroup $H = \exp(\mathbb{R}(-1 + \mathcal{Y}_H + e_1))$ is $v = (-x + y + 1, x - y, -z)$ (see B.12). The set $\{e_1 + e_2, v\} \subset T_q G(q)$ is a basis if and only if $z \neq 0$ and $x - y \neq 1/2$ if and only if $q \notin \mathcal{F}_{\mathcal{L}}(1/2, 0, 0)$. It is not hard to see that G acts on $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\mathcal{L}}(1/2, 0, 0)$ freely. Observe that G preserves the degenerate affine plane $\mathcal{F}_{\Pi_\phi}(1/2, 0, 0)$. Hence, G fixes a point $d \in \hat{\phi}$ which is the limit point of $\mathcal{F}_{\Pi_\phi}(1/2, 0, 0)$. The subgroup H acts on the both connected components of $\hat{\phi} \setminus \{d\}$ transitively. For all $q \in \mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(1/2, 0, 0)$, the orbit $G(q)$ intersects exactly one of the timelike affine lines $\ell^+ = \{(t, 0, 1) : t \in \mathbb{R}\}$, $\ell^- = \{(t, 0, -1) : t \in \mathbb{R}\}$ in a unique point. Furthermore, for $q \in \ell^+ \cup \ell^-$, the orbit $G(q)$ is Lorentzian, since the orthogonal space of the null vector $e_1 + e_2$ in $T_q G(q)$ is \mathcal{L} . Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: two fixed points $\{p, d\}$, the two connected components of $\hat{\phi} \setminus \{d\}$, the vertex-less photon $\hat{\psi}$, the two connected components of $L(p) \setminus (\phi \cup \psi)$, the lightlike affine line $\mathcal{F}_{\mathcal{L}}(1/2, 0, 0)$, the two connected components of $\mathcal{F}_{\Pi_\phi}(1/2, 0, 0) \setminus \mathcal{F}_{\mathcal{L}}(1/2, 0, 0)$ which are degenerate affine half-planes, and a 1-parameter family of Lorentzian orbits in $\mathbb{E}^{1,2}$ on which G acts freely, -the parameter being $q \in \ell^+ \cup \ell^-$.

Case IV: The linear isometry projection $P_i(G)$ is a 2-dimensional subgroup of $SO_o(1, 2)$. In this case, G preserves the foliation \mathcal{F}_{Π_ϕ} , and so it preserves the photon ϕ . Thus, G acts on the degenerate surface $L(p) \setminus \phi$ transitively.

- $G = (\mathbb{R}_+^* \times \text{Aff}) \times \mathcal{L}$. The subgroup $\mathbb{R}_+^* \times \text{Aff}$ preserves the leaf $\mathcal{F}_{\mathcal{L}}(o)$. Hence, $\mathcal{F}_{\mathcal{L}}(o)$ is a G -orbit. Also, G preserves the degenerate affine plane $\mathcal{F}_{\Pi_\phi}(o)$. Thus, G fixes a point $d \in \hat{\phi}$ which is the limit point of $\mathcal{F}_{\Pi_\phi}(o)$. The homothety factor \mathbb{R}_+^* acts on the both connected components of $\hat{\phi} \setminus \{d\}$ transitively. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$ the vectors tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroups \mathbb{R}_+^* and Y_H are $v = (x, y, z)$ and $w = (y, x, 0)$, respectively. Observe that for $q \in \mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$, the set $\{v, w\} \subset T_q G(q)$ is a basis. Hence G acts on the both connected components of $\mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$ transitively. On the other hand, for $q \in \mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$, the set $\{e_1 + e_2, v, w\} \subset T_q G(q)$ is a basis. Thus, G acts on the both connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$ transitively. Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: two fixed points $\{p, d\}$, the two connected components of $\hat{\phi} \setminus \{d\}$, the affine lightlike line $\mathcal{F}_{\mathcal{L}}(o)$, the degenerate surface $L(p) \setminus \phi$, the two connected components of $\mathcal{F}_{\Pi_\phi} \setminus \mathcal{F}_{\mathcal{L}}(o)$ which degenerate affine half-planes, and the two connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$. Note that the orbits induced by G in $\mathbb{E}^{1,2}$ are exactly the same as the orbits induced by $(\mathbb{R}_+^* \times Y_P) \times \mathcal{L}$.
- $G = \text{Aff} \times \mathcal{L}$. The group G preserves the orbits induced by $(\mathbb{R}_+^* \times \text{Aff}) \times \mathcal{L}$. In fact, one can see G and $\mathbb{R}_+^* \times \text{Aff}) \times \mathcal{L}$ admit the same orbits in the lightcone $L(p)$. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$ the vectors tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroups Y_P and Y_H are $v = (z, z, x - y)$ and $w = (y, x, 0)$, respectively. Observe that for all $q \in \mathcal{F}_{\Pi_\phi}(o)$, the leaf $\mathcal{F}_{\mathcal{L}}(q)$ is invariant by G , and it intersects the affine spacelike line $\ell = \{(0, 0, t) : t \in \mathbb{R}\} \subset \mathbb{E}^{1,2}$ in a unique point. The set $\{e_1 + e_2, v, w\} \subset T_q G(q)$ is a basis for all $q \in \mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$. Thus, G acts on the both connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$ transitively. Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: two fixed point $\{p, d\}$, the two connected components of $\hat{\phi} \setminus \{d\}$, the degenerate surface $L(p) \setminus \phi$, a 1-parameter family of 1-dimensional orbits which are lightlike affine lines -the parameter being $q \in \ell$ -, and the two connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$.
- $G = \exp((\mathbb{R}(a + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P)) \times \mathcal{L}$, $a \in \mathbb{R}^* \setminus \{1\}$. Obviously, G preserves the orbits induced by $(\mathbb{R}_+^* \times \text{Aff}) \times \mathcal{L}$. In fact, G and $(\mathbb{R}_+^* \times \text{Aff}) \times \mathcal{L}$ admit the same orbits in the lightcone $L(p)$. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$ the vectors tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroups $\exp(\mathbb{R}(a + Y_H))$ and Y_P are $v = (ax + y, x + ay, az)$ and $w = (z, z, x - y)$, respectively. For a point $q \in \mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$, the set $\{v, w\} \subset T_q G(q)$ is a basis. Hence, G acts on the both connected components of $\mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$ transitively. Also, for a point $q \in \mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$, the set $\{e_1 + e_2, v, w\}$ is a basis. Thus G acts on the both connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$ transitively. Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: two fixed point $\{p, d\}$, the two connected components of $\hat{\phi} \setminus \{d\}$, the degenerate surface $L(p) \setminus \phi$, the affine lightlike line $\mathcal{F}_{\mathcal{L}}(o)$, the two connected components of $\mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$ which are degenerate affine half-planes in $\mathbb{E}^{1,2}$, and the two connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$. Note that the orbits induced by G in $\mathbb{E}^{1,2}$ are exactly the same as the orbits induced by $(\mathbb{R}_+^* \times Y_P) \times \mathcal{L}$.
- $G = \exp((\mathbb{R}(1 + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P)) \times \mathcal{L}$. Since G is a subgroup of \mathcal{K}_o (Definition 4.10), it preserves the leaves of the foliation \mathcal{F}_{Π_ϕ} . Hence, G acts on the photon ϕ trivially. Every leaf of \mathcal{F}_{Π_ϕ} intersects

the lightlike affine line $\ell = \{(t, -t, 0) : t \in \mathbb{R}\}$ in a unique point. It is clear that G preserves the leaf $\mathcal{F}_{\mathcal{L}}(o)$. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$ the vectors tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroups $\exp(\mathbb{R}(1 + Y_H))$ and $Y_P = \exp(\mathbb{R}\mathcal{Y}_P)$ are $v = (x + y, x + y, z)$ and $w = (z, z, x - y)$, respectively. For a point $q \in \mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$, the set $\{e_1 + e_2, v\}$ is a basis. So, G acts on the both connected components of $\mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$ transitively. Furthermore, for a point $q \in \mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$, the set $\{e_1 + e_2, w\} \subset T_q G(q)$ is a basis. This implies that G acts on each leaf $\mathcal{F}_{\Pi_\phi}(q)$ different from $\mathcal{F}_{\Pi_\phi}(o)$ transitively. Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: the point in the photon ϕ , the affine lightlike line $\mathcal{F}_{\mathcal{L}}(o)$, the degenerate surface $L(p) \setminus \phi$, the two connected components of $\mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$ which are affine degenerate half-planes in $\mathbb{E}^{1,2}$, and 1-parameter family of 2-dimensional which are degenerate affine planes -the parameter being $q \in \ell \setminus \{o\}$.

- $G = \exp(\mathbb{R}(2 + \mathcal{Y}_H) + \mathbb{R}(\mathcal{Y}_P + e_1)) \times \mathcal{L}$. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$ the vectors tangent to orbit $G(q)$ at q induced by subgroups $\exp(\mathbb{R}(2 + \mathcal{Y}_H))$ and $\exp(\mathbb{R}(\mathcal{Y}_P + e_1))$ are $v = (2x + y, x + 2y, 2z)$ and $w = (z + 1, z, x - y)$, respectively (see B.8 and B.5). The set $\{e_1 + e_2, v, w\} \subset T_q G(q)$ is a basis if and only if $z \neq (x - y)^2/2$. The set of points in $\mathbb{E}^{1,2}$ with $z = (x - y)^2/2$ is a connected Lorentzian surface S , and G acts on it transitively (in fact it is the orbit induced by the free action of the subgroup $\exp(\mathbb{R}(\mathcal{Y}_P + e_1) \oplus_\theta \mathcal{L})$ at o). Hence, G acts on the both connected components of $\mathbb{E}^{1,2} \setminus S$ transitively. Also, G acts on the vertex-less photon $\hat{\phi}$ transitively, since it preserves no leaf of \mathcal{F}_{Π_ϕ} . Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: a fixed point p , the vertex-less photon $\hat{\phi}$, the degenerate surface $L(p) \setminus \phi$, the Lorentzian surface S which is diffeomorphic to \mathbb{R}^2 , and the two connected components of $\mathbb{E}^{1,2} \setminus S$.
- $G = \exp(\mathbb{R}(\mathcal{Y}_H + e_3) + \mathbb{R}\mathcal{Y}_P) \times \mathcal{L}$. Since G is a subgroup of $\text{Aff}_o(1, \mathbb{R}) \times \Pi_\phi$, it preserves the leaf $\mathcal{F}_{\Pi_\phi}(o)$. Hence, it fixes a point $d \in \hat{\phi}$ which is the limit point of $\mathcal{F}_{\Pi_\phi}(o)$. The 1-parameter subgroup $H = \exp(\mathbb{R}(\mathcal{Y}_H + e_3))$ acts on the both connected components of $\hat{\phi} \setminus \{d\}$ transitively. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$ the vectors tangent to orbit $G(q)$ at q induced by subgroups H and Y_P are $v = (y, x, 1)$ and $w = (z, z, x - y)$, respectively (see B.10). The set $\{e_1 + e_2, v, w\} \subset T_q G(q)$ is a basis if and only if $x \neq y$ if and only if $q \notin \mathcal{F}_{\Pi_\phi}(o)$. Observe that for a point $q \in \mathcal{F}_{\Pi_\phi}(o)$ the set $\{e_1 + e_2, v\} \subset T_q G(q)$ is a basis. Hence, G acts on $\mathcal{F}_{\Pi_\phi}(o)$ transitively. Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: two fixed points $\{p, d\}$, the two connected components of $\hat{\phi} \setminus \{d\}$, the degenerate surface $L(p) \setminus \phi$, the degenerate affine plane $\mathcal{F}_{\Pi_\phi}(o)$, and the two connected components of $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$. Note that the orbits induced by G are exactly the same as orbits induced by $(\mathbb{R}_+^* \times Y_P) \times (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3)$.

Subgroups with trivial translation part

Finally, assume that $G \subset \text{Conf}_o(\mathbb{E}^{1,2})$ is a connected Lie subgroup with trivial translation part $T(G) = \{0\}$. These groups have been listed in Table 4.8.

First, assume that G is a subgroup of $\mathbb{R}_+^* \times SO_o(1, 2)$. This group fixes the origin $o = (0, 0, 0) \in \mathbb{E}^{1,2}$. By Remark 4.2, the Minkowski space $\mathbb{E}^{1,2}$ with origin o is endowed with the quadratic form $\mathfrak{q} := \mathfrak{q}_{1,2} \circ (\Theta_o)^{-1}$, and this makes $\mathbb{E}^{1,2}$ a Lorentzian scalar product space.

The linear group $\mathbb{R}_+^* \times SO_o(1, 2)$ (and all its subgroups) preserves the nullcone centered at o

$$\mathfrak{N}(o) = \{q = (x, y, z) \in \mathbb{E}^{1,2} \setminus \{o\} : \mathfrak{q}(q) = -x^2 + y^2 + z^2 = 0\}.$$

Also, it preserves the three connected components of the complement of the nullcone $\mathfrak{N}(o)$ in $\mathbb{E}^{1,2}$ which are: the domain $\{q > 0\}$ and the two connected components of the domain $\{q < 0\}$. Also, this group preserves the ideal circle $S_\infty = L(p) \cap L(o)$. Observe that, if G is a subgroup of $SO_o(1, 2)$, it also preserves the de-Sitter spaces $dS^{1,1}(r) = q^{-1}(r)$, and the hyperbolic planes $\mathbb{H}^2(r)$ (the connected components of $q^{-1}(-r)$) in $\mathbb{E}^{1,2}$ centered at o with radius $r \in \mathbb{R}_+^*$. Moreover, the group $\mathbb{R}_+^* \times \text{Aff}$ preserves the foliation \mathcal{F}_{Π_ϕ} , and also, preserves the leaves $\mathcal{F}_{\Pi_\phi}(o)$ and $\mathcal{F}_{\mathcal{L}}(o)$.

- $G = \mathbb{R}_+^* \times SO_o(1, 2)$. This group acts on the both components of the nullcone $\mathfrak{N}(o)$, transitively. The complement of $\mathfrak{N}(o) \cup \{o\}$ in $\mathbb{E}^{1,2}$ has three connected components: the domain with $q > 0$ and the two connected components with $q < 0$. One can see that G acts on these components transitively. The group G preserves no foliation in $\mathbb{E}^{1,2}$ induced by a degenerate plane in $\mathbb{R}^{1,2}$. Hence, it acts on the ideal circle S_∞ and the both connected components of $L(p) \setminus (S_\infty \cup \{p\})$ transitively. Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: two fixed point p and o , the ideal circle S_∞ , the two connected components of $L(\hat{p}) \setminus S_\infty$ which are degenerate surfaces, the two connected components of $\mathfrak{N}(o)$, the two connected components of the domain $\{q < 0\}$, and the domain $\{q > 0\}$.
- $G = SO_o(1, 2)$. This group acts on the both connected components of $\mathfrak{N}(o)$ transitively. Also, it preserves the de-Sitter spaces $dS^{1,1}(r)$ and the hyperbolic planes $\mathbb{H}^2(r)$, and acts on them transitively. Indeed, every de-Sitter space $dS^{1,1}(r)$ (resp. hyperbolic plane $\mathbb{H}^2(r)$) intersects the spacelike affine half-line $\ell_s = \{(0, t, 0) : 0 < t < \infty\}$ (resp. the timelike affine line $\ell_t = \{(t, 0, 0) : t \in \mathbb{R}\}$) in a unique point. Observe that G preserves no foliation in $\mathbb{E}^{1,2}$ induced by a degenerate plane in $\mathbb{R}^{1,2}$. Hence, it acts on S_∞ and the both connected components of $L(p) \setminus (S_\infty \cup \{p\})$, transitively. Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: two fixed points $\{p, o\}$, a spacelike curve S_∞ , the two connected components of $L(\hat{p}) \setminus S_\infty$ and the two connected components of $\mathfrak{N}(o)$ which are degenerate surfaces, a 1-parameter family of 2-dimensional Lorentzian orbits on which every orbit is conformally equivalent to $dS^{1,1}$ -the parameter being $q \in \ell_s$ -, and a 1-parameter family of 2-dimensional spacelike orbits on which every orbit is conformally equivalent to \mathbb{H}^2 -the parameter being $q \in \ell_t \setminus \{o\}$.
- $G = \text{Aff}$. This group fixes the point $d = \hat{\phi} \cap S_\infty$ which is the limit point of $\mathcal{F}_{\Pi_\phi}(o)$, and acts on $S_\infty \setminus \{p\}$, the both connected components of $\hat{\phi} \setminus \{d\}$, and the both connected components of $L(p) \setminus (S_\infty \cup \phi)$ transitively. The intersection of the leaf $\mathcal{F}_{\Pi_\phi}(o)$ with the nullcone $\mathfrak{N}(o)$ is a lightlike line, namely, it is $\mathcal{F}_{\mathcal{L}}(o)$. One can see G acts on the both connected components of $\mathcal{F}_{\mathcal{L}}(o) \setminus \{o\}$ and on the both connected components of $\mathfrak{N}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$ transitively. Also, G acts on every hyperbolic plane $\mathbb{H}^2(r)$, transitively. The intersection of the degenerate plane $\mathcal{F}_{\Pi_\phi}(o)$ with a de-Sitter space $dS^{1,1}(r)$ is two disjoint affine lightlike lines. It is not hard to see that G acts on the both connected components of $dS^{1,1}(r) \setminus \mathcal{F}_{\Pi_\phi}(o)$ transitively. Moreover, the 1-parameter parabolic subgroup Y_P (and consequently G) acts on each connected component of $dS^{1,1}(r) \cap \mathcal{F}_{\Pi_\phi}(o)$ transitively. Indeed, every G -orbit in $\mathbb{E}^{1,2}$ intersects one of the affine lines $\ell_t = \{(t, 0, 0) : t \in \mathbb{R}\}$, $\ell_s = \{(0, t, 0) : t \in \mathbb{R}\}$, and $\ell'_s = \{(0, 0, t) : t \in \mathbb{R}\}$. Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are, three fixed points $\{p, o, d\}$, the spacelike curve $S_\infty \setminus \{d\}$, the two connected components of $\hat{\phi} \setminus \{d\}$ and the two connected components of $\mathcal{F}_{\mathcal{L}}(o) \setminus \{o\}$, the two connected components of $L(p) \setminus (S_\infty \cup \phi)$ and the two connected components of $\mathfrak{N}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$,

a 1-parameter family of 1-dimensional orbits on which every orbit is an lightlike affine line -the parameter being $q \in \ell'_s \setminus \{o\}$ -, a 1-parameter family of 2-dimensional spacelike orbits on which every orbit is conformally equivalent to \mathbb{H}^2 -the parameter being $q \in \ell_t \setminus \{o\}$ -, and a 1-parameter family of 2-dimensional Lorentzian orbits on which every orbit is conformally equivalent to a connected component of $dS^{1,1} \setminus \mathcal{F}_{\Pi_\phi}(o)$ -the parameter being $q \in \ell_s \setminus \{o\}$.

- $G = \mathbb{R}_+^* \times \text{Aff}$. The orbits induced by G in the lightcone $L(p)$ is exactly the same as the orbits induced by Aff . The intersection of the leaf $\mathcal{F}_{\Pi_\phi}(o)$ with the nullcone $\mathfrak{N}(o)$ is a lightlike line, namely, it is $\mathcal{F}_{\mathcal{L}}(o)$. The group G acts on the both connected components of $\mathcal{F}_{\mathcal{L}}(o) \setminus \{o\}$ and on the both connected components of $\mathfrak{N}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$ transitively. Observe that G also acts on the both connected components of the domain $\{q < 0\}$ transitively. Moreover, G acts on the both connected components of $\mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$, and on the both connected components of $\mathbb{E}^{1,2} \setminus (\mathfrak{N}(o) \cup \{q < 0\} \cup \mathcal{F}_{\Pi_\phi}(o))$ transitively. Therefore, the orbit by G in $\mathbb{E}^{1,2}$ are: three fixed points $\{p, o, d\}$, the spacelike curve $S_\infty \setminus \{d\}$, the two connected components of $\hat{\phi} \setminus \{d\}$, the two connected components of $\mathcal{F}_{\mathcal{L}}(o) \setminus \{o\}$, the two connected components of $L(p) \setminus (S_\infty \cup \phi)$, the two connected components of $\mathfrak{N}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$, the two connected components of $\mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$, and four open orbits in $\mathbb{E}^{1,2}$.
- $G_a = \exp(\mathbb{R}(a + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P)$, $0 < |a| < 1$. The group G_a admits the same orbits in the lightcone $L(p)$ as Aff . For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vectors tangent to the orbit $G_a(q)$ at q induced by the 1-parameter subgroups $\exp(\mathbb{R}(a + Y_H))$ and Y_P are $v = (ax + y, x + ay, az)$ and $w = (z, z, x - y)$, respectively. Note that, by Lemma 1.21, the action of G_a is orbitally-equivalent to the action G_{-a} via the element in $O(1, 2)$ which maps $(x, y, z) \in \mathbb{E}^{1,2}$ to $(-x, -y, z)$. Hence, we may restrict ourselves to the case $0 < a < 1$. The set $\{v, w\} \subset T_q G_a(q)$ is a basis if and only if $z \neq 0$ and $x \neq y$ if and only if $q \neq \mathcal{F}_{\mathcal{L}}(o)$. Thus, G_a acts on the both connected components of $\mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$, and on the both connected components of $\mathfrak{N}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$ transitively. Also, G_a acts on the both connected components of $\mathcal{F}_{\mathcal{L}}(o)$ transitively. For all $q \in \mathbb{E}^{1,2} \setminus (\mathfrak{N}(o) \cup \mathcal{F}_{\Pi_\phi}(o))$ with $q(q) > 0$ (resp. $q(q) < 0$) the orbit $G_a(q)$ intersects the timelike line $\ell_t = \{(t, 0, 0) : t \in \mathbb{R}\}$ (resp. the spacelike line $\ell_s = \{(0, t, 0) : t \in \mathbb{R}\}$) in a unique point. Now, one can see that G_a acts on $\mathbb{E}^{1,2} \setminus (\mathfrak{N}(o) \cup \mathcal{F}_{\Pi_\phi}(o))$ freely. For a point $q \in \ell_t \setminus \{o\}$ (resp. $q \in \ell_s \setminus \{o\}$), the orbit $G(q)$ is spacelike (resp. Lorentzian). Therefore, the orbits induced by G_a in $\mathbb{E}^{1,2}$ are: three fixed points $\{p, o, d\}$, the spacelike curve $S_\infty \setminus \{d\}$, the two connected components of $\hat{\phi} \setminus \{d\}$, the two connected components of $L(p) \setminus (\phi \cup S_\infty)$, the two connected components of $\mathcal{F}_{\mathcal{L}}(o) \setminus \{o\}$, the two connected components of $\mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$, the two connected components of $\mathfrak{N}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$, a 1-parameter family of 2-dimensional Lorentzian orbits on which G acts freely -the parameter being $q \in \ell_s \setminus \{o\}$ -, and a 1-parameter family of 2-dimensional spacelike orbits on which G acts freely -the parameter being $q \in \ell_t \setminus \{o\}$.
- $G = \exp(\mathbb{R}(1 + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P)$. This group preserves the leaves of the foliation \mathcal{F}_{Π_ϕ} , since it is a subgroup of \mathcal{K}_o (Definition 4.10). Hence, G acts on the photon ϕ trivially. It is not hard to see that G acts on $S_\infty \setminus \phi$ and on the both connected components of $L(p) \setminus (\phi \cup S_\infty)$ transitively. Also, G preserves the lightlike line $\mathcal{F}_{\mathcal{L}}(o)$ and acts on the both connected components of $\mathcal{F}_{\mathcal{L}}(o) \setminus \{o\}$ transitively. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vectors tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroups $\exp(\mathbb{R}(1 + Y_H))$ and Y_P are $v = (x + y, x + y, z)$

and $w = (z, z, x - y)$ respectively. The set $\{u, v\} \subset T_q G(q)$ is a basis if and only if $q \notin \mathfrak{N}(o)$. Hence, G acts on the both connected components of $\mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$ transitively. Every leaf of the foliation \mathcal{F}_{Π_ϕ} intersects the lightlike line $\ell_l = \{(t, -t, 0) : t \in \mathbb{R}\}$ in a unique point. For a point $q \in \ell_l \setminus \{o\}$, the intersection of the leaf $\mathcal{F}_{\Pi_\phi}(q)$ with the nullcone $\mathfrak{N}(o)$ is a spacelike curve γ_q which the 1-parameter subgroup Y_P acts on it transitively. For all $q \in \ell \setminus \{o\}$, G acts on the both connected components of $\mathcal{F}_{\Pi_\phi}(q) \setminus \gamma_q$ transitively. Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: the points in $\phi \cup \{o\}$, the spacelike curve $S_\infty \setminus \phi$, the two connected components of $L(p) \setminus (\phi \cup S_\infty)$, a 1-parameter family of 1-dimensional spacelike orbits γ_q in $\mathbb{E}^{1,2}$ -the parameter being $q \in \ell_l \setminus \{o\}$ -, the two connected components of $\mathcal{F}_{\mathcal{L}}(o) \setminus \{o\}$, the two connected components of $\mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$ which are affine degenerate half-planes, and a 1-parameter family of degenerate orbits which are the connected components of $\mathcal{F}_{\Pi_\phi}(q) \setminus \gamma_q$ -the parameter being $q \in \ell \setminus \{o\}$.

- $G = \exp(\mathbb{R}(-1 + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P)$. This group preserves the leaf $\mathcal{F}_{\Pi_\phi}(o)$, and so it fixed the corresponding limit point $d \in \hat{\phi}$. In the one hand, G acts on the lightlike affine line $\mathcal{F}_{\mathcal{L}}(o) \subset \mathcal{F}_{\Pi_\phi}(o)$ trivially. On the other hand, $\psi = \mathcal{F}_{\mathcal{L}}(o) \cup \{d\}$ is a photon which G acts on it trivially. Therefore, G is a subgroup of \mathcal{K}_o (Definition (4.10)), up to conjugacy.
- $G = \mathbb{R}_+^* \times Y_P$. One can see that G admits the same orbits in the lightcone $L(p)$ as Aff . For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vectors tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroups \mathbb{R}_+^* and Y_P are $w = (x, y, z)$ and $v = (z, z, x - y)$ respectively. The set $\{w, v\} \subset T_q G(q)$ is a basis if and only if $q \notin \mathcal{F}_{\mathcal{L}}(o)$. In fact, G acts on $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\mathcal{L}}(o)$ freely. Thus G acts on the two connected components of $\mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$, and on the both connected components of $\mathfrak{N}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$ transitively. For a point $q \in \mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$, the orbit $G(q)$ intersects the Lorentzian affine plane $\{z = 0\}$. Hence, by applying a suitable element of G we may assume $q = (x, y, 0)$. Now, it can be easily seen that, for a point $q \in \mathbb{E}^{1,2} \setminus (\mathcal{F}_{\Pi_\phi}(o))$ with $q(q) > 0$ (respectively, $q(q) < 0$) the orbit $G(q)$ is spacelike (respectively, Lorentzian) and G acts on it freely. So, the orbits induced by G at the points $q_1 = (1, 0, 0)$ and $q_2 = (-1, 0, 0)$ are Lorentzian. Also, for a point $q \in \mathbb{E}^{1,2} \setminus (\mathfrak{N}(o) \cup \mathcal{F}_{\Pi_\phi}(o) \cup G(q_1) \cup G(q_2))$, the orbit $G(q)$ intersects one of the timelike affine half-lines $\ell_1^\pm = \{(\pm t, \pm 1, 0) : 1 < t < \infty\}$, $\ell_2^\pm = \{(t, \pm 1, 0) : -1 < t < 1\}$ in a unique point. Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: three fixed points $\{p, o, d\}$, the spacelike curve $S_\infty \setminus \{d\}$, the two connected components of $\hat{\phi} \setminus \{d\}$, the two connected components of $L(p) \setminus (\phi \cup S_\infty)$, the two connected components of $\mathcal{F}_{\mathcal{L}}(o) \setminus \{o\}$, the two connected components of $\mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$, the two connected components of $\mathfrak{N}(o) \setminus \mathcal{F}_{\mathcal{L}}(o)$, two 2-dimensional Lorentzian orbits $G(q_1)$ and $G(q_2)$ on which G acts freely, a 1-parameter family of 2-dimensional spacelike orbits on which G acts freely -the parameter being $q \in \ell_2^\pm$ -, and a 1-parameter family of 2-dimensional Lorentzian orbits on which G acts freely -the parameter being $q \in \ell_1^\pm$.
- $G = \mathbb{R}_+^* \times Y_H$. This group preserves also the foliation \mathcal{F}_{Π_ψ} . Observe that G preserves the leaves $\mathcal{F}_{\Pi_\phi}(o)$ and $\mathcal{F}_{\Pi_\psi}(o)$. Hence, G fixes two point $d_\phi = \hat{\phi} \cap S_\infty$ and $d_\psi = \hat{\psi} \cap S_\infty$ which are the limit points of $\mathcal{F}_{\Pi_\phi}(o)$ and $\mathcal{F}_{\Pi_\psi}(o)$ respectively. Furthermore, G acts on the two connected components of $\hat{\phi} \setminus \{d_\phi\}$, the two connected components of $\hat{\psi} \setminus \{d_\psi\}$, the two connected components of $S_\infty \setminus \{d_\phi, d_\psi\}$, and the four connected components of $L(p) \setminus (S_\infty \cup \phi \cup \psi)$ transitively. Moreover, G preserves the timelike plane $L_0 = \{z = 0\} \subset \mathbb{E}^{1,2}$. Also, G preserves the two lightlike lines $\mathcal{F}_{\mathcal{L}}(o)$ and $\ell_0 = \mathcal{F}_{\Pi_\psi}(o) \cap \mathfrak{N}(o)$, and the spacelike line $\ell_s = \mathcal{F}_{\Pi_\phi}(o) \cap \mathcal{F}_{\Pi_\psi}(o)$. The group G acts on the six

connected components of $(\mathcal{F}_{\mathcal{L}}(o) \cup \ell_0 \cup \ell_s) \setminus \{o\}$ transitively. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vectors tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroups \mathbb{R}_+^* and Y_H are $v = (x, y, z)$ and $w = (y, x, 0)$, respectively. The set $\{w, v\} \subset T_q G(q)$ is a basis if and only if $q \notin (\mathcal{F}_{\mathcal{L}}(o) \cup \ell_0 \cup \ell_s)$. Hence, G acts on the four connected components of $\mathfrak{N}(o) \setminus (\mathcal{F}_{\mathcal{L}}(o) \cup \ell_0)$, the eight connected components of $(\mathcal{F}_{\Pi_\phi}(o) \cup \mathcal{F}_{\Pi_\psi}(o)) \setminus (\mathcal{F}_{\mathcal{L}}(o) \cup \ell_0 \cup \ell_s)$, and the four connected components of $L_0 \setminus (\mathcal{F}_{\mathcal{L}}(o) \cup \ell_0)$ transitively. On the other hand, for $q \in \mathbb{E}^{1,2} \setminus L_0$, the orbit $G(q)$ intersects the timelike plane $y = 0$, hence, by applying a suitable element of G we may assume $q = (x, 0, z)$. Now it can be easily seen that, for a point $q \in \mathbb{E}^{1,2} \setminus (\mathfrak{N}(o) \cup L_0 \cup \mathcal{F}_{\Pi_\phi}(o) \cup \mathcal{F}_{\Pi_\psi}(o))$ with $q(q) > 0$ (resp. $q(q) < 0$) the orbit $G(q)$ is spacelike (resp. Lorentzian) and G acts on it freely. So, the orbits induced by G at $q_1 = (1, 0, 0)$ and $q_2 = (-1, 0, 0)$ are Lorentzian. For a point $q \in \mathbb{E}^{1,2} \setminus (L_0 \cup \mathcal{F}_{\Pi_\phi}(o) \cup \mathcal{F}_{\Pi_\psi}(o) \cup \mathfrak{N}(o) \cup G(q_1) \cup G(q_2))$ the orbit $G(q)$ intersects one of the timelike affine half-lines $\ell_1^\pm = \{(\pm t, 0, \pm 1) : 1 < t < \infty\}$, $\ell_2^\pm = \{(t, 0, \pm 1) : -1 < t < 1\}$. Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: four fixed points $\{p, o, d_\phi, d_\psi\}$, the four connected components of $(\hat{\phi} \cup \hat{\psi}) \setminus \{d_\phi, d_\psi\}$, the two connected components of $S_\infty \setminus \{d_\phi, d_\psi\}$, the four connected components of $L(p) \setminus (\phi \cup \psi \cup S_\infty)$, the four connected components of $\mathcal{F}_{\mathcal{L}}(o) \cup \ell_0 \setminus \{o\}$ which are lightlike affine half-lines, the two connected components of $\ell_s \setminus \{o\}$ which are spacelike affine half-lines, the eight connected components of $(\mathcal{F}_{\Pi_\phi} \cup \mathcal{F}_{\Pi_\psi}(o)) \setminus (\mathcal{F}_{\mathcal{L}}(o) \cup \ell_0 \cup \ell_s)$ which are degenerate affine half-planes, the four connected components of $L_0 \setminus (\mathcal{F}_{\mathcal{L}}(o) \cup \ell_0)$ which are Lorentzian affine half-planes, the four connected components of $\mathfrak{N}(o) \setminus (\mathcal{F}_{\mathcal{L}}(o) \cup \ell_0)$ which are degenerate surfaces, two Lorentzian orbits $G(q_1)$ and $G(q_2)$ which G acts freely, a 1-parameter family of 2-dimensional spacelike orbits on which G acts freely -the parameter being $q \in \ell_2^\pm$ -, and a 1-parameter family of Lorentzian orbits on which G acts freely -the parameter being $q \in \ell_1^\pm$.

- $G = \mathbb{R}_+^* \times Y_E$. It is clear that G acts on the ideal circle S_∞ and the both connected components of $L(\hat{p}) \setminus S_\infty$ transitively. Observe that G preserves the timelike line $\ell_t = \{(t, 0, 0) : t \in \mathbb{R}\} \subset \mathbb{E}^{1,2}$ and the spacelike plane $L_0 = \{x = 0\} \subset \mathbb{E}^{1,2}$. Obviously G acts on the both connected components of $\ell_t \setminus \{o\}$ transitively. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vectors tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroups \mathbb{R}_+^* and Y_E are $w = (x, y, z)$ and $v = (0, z, -y)$ respectively. The set $\{w, v\}$ is a basis if and only if $q \notin \ell_t$. Hence, G acts on the both connected components of $\mathfrak{N}(o)$ transitively. Also, $L_0 \setminus \{o\}$ is a G -orbit. Every G -orbit in $\mathbb{E}^{1,2} \setminus (\mathfrak{N}(o) \cup L_0 \cup \ell_t)$ intersects one of the affine timelike half-lines $\ell_1^\pm = \{(t, \pm 1, 0) : 0 < t < 1\}$, $\ell_2^\pm = \{(t, \pm 1, 0) : 1 < t < \infty\}$. Now, one can see that G acts on $\mathbb{E}^{1,2} \setminus \ell_t$ freely. For a point $q \in \ell_1^\pm$ (resp. $q \in \ell_2^\pm$) the orbit $G(q)$ is spacelike (resp. Lorentzian). Therefore, the orbits, induced by G in $\mathbb{E}^{1,2}$ are: two fixed point $\{p, o\}$, the ideal circle S_∞ , the two connected components of $L(\hat{p}) \setminus S_\infty$, the both connected components of $\ell_t \setminus \{o\}$ which are timelike affine half-lines, a 2-dimensional Lorentzian orbit $L_0 \setminus \{o\}$, the two connected components of $\mathfrak{N}(o)$, a 1-parameter family of 2-dimensional Lorentzian orbits on which G acts freely -the parameter being $q \in \ell_2^\pm$ -, a 1-parameter family of 2-dimensional spacelike orbits on which G acts freely -the parameter being $q \in \ell_1^\pm$.

Spacial subgroups with trivial translation part

Here, we describe the orbits of the two subgroups in Table 4.8 which have trivial translation part, but they are not subgroups of the linear group $\mathbb{R}_+^* \times SO_o(1, 2)$, i.e., these groups fix no point in the Minkowski

space $\mathbb{E}^{1,2}$.

► $G = \exp(\mathbb{R}(-1 + \mathcal{Y}_H + e_1 + e_2) + \mathbb{R}\mathcal{Y}_P)$. This group preserves the foliation \mathcal{F}_{Π_ϕ} , since it is a subgroup of $(\mathbb{R}_+^* \times \text{Aff}_o(1, \mathbb{R})) \times \mathbb{R}(e_1 + e_2)$. Also, it preserves the leaf $\mathcal{F}_{\Pi_\phi}(o)$ and the affine lightlike line $\mathcal{F}_\mathcal{L}(o)$. For an arbitrary point $q = (x, y, z) \in \mathbb{E}^{1,2}$, the vectors tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroups $H = \exp(\mathbb{R}(-1 + \mathcal{Y}_H + e_1 + e_2))$ and Y_P are $v = (-x + y + 1, x - y + 1, -z)$ and $w = (z, z, x - y)$, respectively (see B.13). The subgroup H acts on the vertex-less photon $\hat{\psi}$ transitively, since $v \notin \Pi_\psi$. Therefore, G acts on $L(p) \setminus \phi$ transitively. On the other hand, $\mathcal{F}_{\Pi_\phi}(o)$ is the only G -invariant leaf of the foliation \mathcal{F}_{Π_ϕ} . Hence, G fixes its limit point $d \in \hat{\phi}$, and acts on the both connected components of $\hat{\phi} \setminus \{d\}$ transitively. Observe that v is a non-vanishing vector. So, G admits no fixed point in $\mathbb{E}^{1,2}$. Hence, G acts on the lightlike affine line $\mathcal{F}_\mathcal{L}(o)$ transitively. The set $\{u, v\} \subset T_q G(q)$ is a basis if and only if $x \neq y$ and $z \neq 0$ if and only if $q \notin \mathcal{F}_\mathcal{L}(o)$. This implies that G acts on the both connected components of $\mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_\mathcal{L}(o)$ transitively. In the one hand, for $q \in \mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$, the orbit induced by Y_P at q intersects the Lorentzian plane $L_0 = \{z = 0\}$. On the other hand, for a point $q \in L_0$, the orbit $H(q)$ intersects either the timelike affine line $\ell_t = \{(t, 0, 0) : t \in \mathbb{R}\}$ or the spacelike affine line $\ell_s = \{(0, 0, t) : t \in \mathbb{R}\}$. Now, it can be easily seen that G acts on $\mathbb{E}^{1,2} \setminus \mathcal{F}_{\Pi_\phi}(o)$ freely. Also, for $q \in \ell_t$ (resp. $q \in \ell_s$) the orbit $G(q)$ is Lorentzian (resp. spacelike). Therefore, the orbits induced by G in $\mathbb{E}^{1,2}$ are: two fixed points $\{p, d\}$, the two connected components of $\hat{\phi} \setminus \{d\}$, the affine lightlike line $\mathcal{F}_\mathcal{L}(o)$, the degenerate surface $L(p) \setminus \phi$, the two connected components of $\mathcal{F}_{\Pi_\phi}(o) \setminus \mathcal{F}_\mathcal{L}(o)$ which are affine degenerate half-planes, a 1-parameter family of Lorentzian orbits in $\mathbb{E}^{1,2}$ on which G acts freely -the parameter being $q \in \ell_t$ -, and a 1-parameter family of spacelike orbits in $\mathbb{E}^{1,2}$ on which G acts freely -the parameter being $q \in \ell_s$ -.

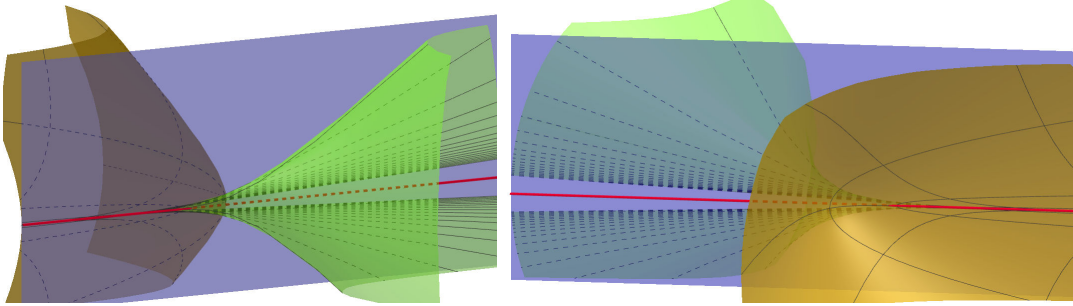


Figure 4.1: Two partial views of the 1-dimensional orbit and four 2-dimensional orbits induced by $\exp(\mathbb{R}(-1 + \mathcal{Y}_H + e_1 + e_2) + \mathbb{R}\mathcal{Y}_P)$ in the Minkowski patch $Mink(p)$. **Red:** Part of the 1-dimensional orbit. **Green:** Part of a 2-dimensional Lorentzian orbit. **Brown:** Part of a 2-dimensional spacelike orbit. **Blue:** Parts of the two 2-dimensional degenerate orbits.

► $G = \exp(\mathbb{R}(2 + \mathcal{Y}_H) + \mathbb{R}(\mathcal{Y}_P + e_1 - e_2))$. This group preserves the foliation \mathcal{F}_{Π_ϕ} . Observe that $G(o)$ is a 1-dimensional lightlike orbit, but it is not a geodesic. Actually, G is the only (up to conjugacy) connected Lie subgroup of $\text{Conf}(\mathbb{E}^{1,2})$ induces a 1-dimensional lightlike orbit in $\mathbb{E}^{1,2}$ which is not a lightlike geodesic. We conclude that, the action of G on $\mathbb{E}^{1,2}$ is conjugate to the action of affine group $\text{Aff} \subset \text{PSL}(2, \mathbb{R})$ within the irreducible action of $\text{PSL}(2, \mathbb{R})$ on Einstein universe described in Chapter 3. Indeed, up to conjugacy, this is the action of the stabilizer of $Y^4 \in \mathcal{N}$ on $\mathbb{E}^{1,2}$ where \mathcal{N} is the 1-dimensional orbit induced by $\text{PSL}(2, \mathbb{R})$. So, we study the orbits induced by this action in the setting of Chapter 3.

Consider (XY^3, X^3Y, X^2Y^2) as a coordinate for the Minkowski patch $Min(Y^4) \subset \mathbb{E}in^{1,2}$. The restriction of the quadratic form Q_4 (Section 1.4.4) on $Mink(Y^4) \approx \mathbb{E}^{1,2}$ with origin $o = (0, 0, 0)$ is

$$q(aXY^3 + bX^3Y + cX^2Y^2) = -\frac{1}{2}ab + \frac{1}{6}c^2.$$

For a point $q = (x, y, z) \in Mink(Y^4)$, the orbit induced by $\mathbb{A}ff \subset \text{PSL}(2, \mathbb{R})$ is

$$\mathbb{A}ff(q) = \{ (xe^{6t} + 3ye^{4t}s^2 - 2ze^{5t}s - 4e^{3t}s^3, ye^{2t} - 4e^t s, ze^{4t} - 3ye^{3t}s + 6e^{2t}s^2) : t, s \in \mathbb{R} \}.$$

The affine group $\mathbb{A}ff$ preserves the foliation \mathcal{F}_{Π_ϕ} on $Mink(Y^4)$ induced by the degenerate plane $\Pi_\phi = (XY^3)^\perp \leq \mathbb{R}^{1,2}$. Hence, it preserves the corresponding photon $\phi \subset L(Y^4)$. On the other hand, one can see, the 1-parameter parabolic subgroup $Y_P \subset \mathbb{A}ff$ preserves no leaf of \mathcal{F}_{Π_ϕ} . Thus, $\mathbb{A}ff$ acts on the vertex-less photon $\hat{\phi} = \phi \setminus \{Y^4\}$ transitively. The 1-parameter hyperbolic subgroup $Y_H \subset \mathbb{A}ff$ preserves the foliation \mathcal{F}_{Π_ψ} where Π_ψ is the degenerate hyperplane $(X^3Y)^\perp \leq \mathbb{R}^{1,2}$. Also, Y_H preserves the leaf $\mathcal{F}_{\Pi_\psi}(o)$, and so, it fixes the corresponding limit point $d \in \hat{\psi} \subset L(Y^4)$. Therefore, $\mathbb{A}ff$ induces a 1-dimensional spacelike orbit in $L(Y^4)$ at d . It is not hard to see that $\mathbb{A}ff$ acts on the both connected components of $L(Y^4) \setminus (\phi \cup \mathbb{A}ff(d))$ transitively.

Now, we describe the orbits induced by $\mathbb{A}ff$ in the Minkowski patch $Mink(Y^4)$. For an arbitrary point $q = (x, y, z) \in Mink(Y^4)$, the vectors tangent to the orbit $G(q)$ at q induced by the 1-parameter subgroups Y_P and Y_H are $v = (-2z, -4, -3y)$ and $w = (6x, 2y, 4z)$, respectively. Observe that v is a non-vanishing vector. Hence, $\mathbb{A}ff$ fixes no point in $Mink(Y^4)$. The orbit $\mathbb{A}ff(q)$ intersects the affine Lorentzain plane $L_0 = \{z = 0\}$ if and only if $3y^2 \geq 8z$. Consequently, the domain $D = \{3y^2 < 8z\} \subset Mink(Y^4)$ is $\mathbb{A}ff$ -invariant.

Case I: Orbits with a representative in L_0 . For a point $q \in L_0$, the orbit $\mathbb{A}ff(q)$ intersects one of the affine lines $\ell_t = \{(t, t, 0) : t \in \mathbb{R}\}$, $\ell_s = \{(t, -t, 0)\}$, $\ell_1 = \{(t, 0, 0) : t \in \mathbb{R}\}$ and $\ell_2 = \{(0, t, 0) : t \in \mathbb{R}\}$. The origin $o = (0, 0, 0)$ is in the intersection of these four lines and the orbit $\mathbb{A}ff(o)$ is the only 1-dimensional orbit in $Mink(Y^4)$ (in fact $\mathbb{A}ff(o) = \mathcal{N} \setminus \{Y^4\}$). One can see, $\mathbb{A}ff$ acts on $Mink(Y^4) \setminus \mathbb{A}ff(o)$ freely.

- If $q \in \ell_t \setminus \{o\}$, then the orbit $\mathbb{A}ff(q)$ is Lorentzian, since the w is a timelike tangent vector.
- If $q \in \ell_s \setminus \{o\}$, then the tangent vector $u = v - w/t$ is orthogonal to the spacelike vector w . The vector u is spacelike (resp. timelike, lightlike) if $|t| > 1$ (resp. $|t| < 1$, $|t| = 1$). Therefore, the orbit $\mathbb{A}ff(q)$ is spacelike (resp. Lorentzian, degenerate) for $|t| > 1$ (resp. $|t| < 1$, $|t| = 1$).
- If $q \in \ell_1 \setminus \{o\}$, then the orbit $\mathbb{A}ff(q)$ is Lorentzian, since both the vectors v and w are lightlike.
- If $q \in \ell_2$, then the orbit $\mathbb{A}ff(o)$ is degenerate, since the lightlike vector w is orthogonal to the tangent space $T_q\mathbb{A}ff(q)$.

Case II: Orbits in the domain $D = \{3y^2 < 8z\}$. For a point $q \in D$, the orbit $\mathbb{A}ff(q)$ intersects the affine line $\ell_3 = \{(-t, t, 1) : t \in \mathbb{R}\}$. Hence, we may assume $q = (-y, y, 1)$. Observe that the tangent vector $v = (-2, -4, -3y) \in T_q\mathbb{A}ff(q)$ is timelike, since $q \in D = \{3y^2 < 8z\}$. Therefore, all the orbits induced by $\mathbb{A}ff$ in D are Lorentzian.

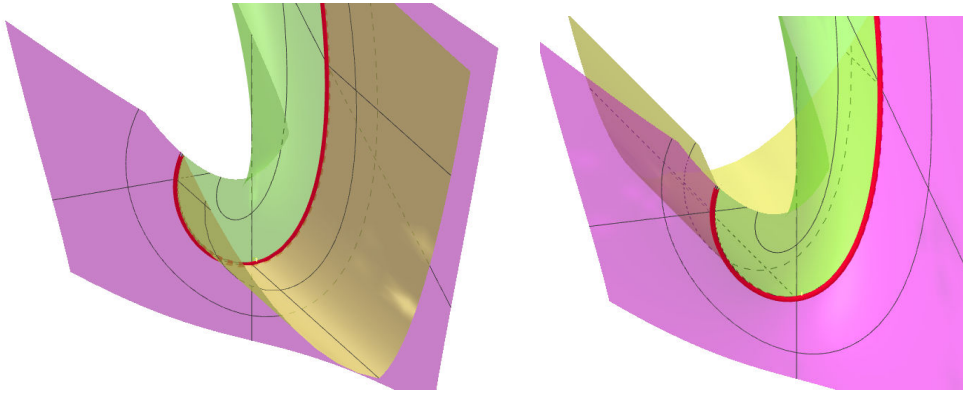


Figure 4.2: Two partial views of the 1-dimensional orbit and three 2-dimensional orbits induced by $\exp(\mathbb{R}(2 + \mathcal{Y}_H) + \mathbb{R}(\mathcal{Y}_P + e_1 - e_2))$ in the Minkowski patch $Mink(p)$. **Red:** Part of the 1-dimensional orbit. **Green:** Part of a 2-dimensional Lorentzian orbit. **Yellow:** Part of a 2-dimensional degenerate orbit. **Purple:** Part of a 2-dimensional spacelike orbit.

Actions on Anti de-sitter and de-Sitter components and their boundaries

In this chapter we consider the cohomogeneity one actions on Einstein universe $\mathbb{E}in^{1,2}$ preserving a spacelike or a timelike direction in $\mathbb{R}^{2,3}$. The first corresponds to the actions preserving an Anti de-Sitter component $AdS^{1,2}$ and its conformal boundary $\partial AdS^{1,2} = \mathbb{E}in^{1,1}$ an Einstein hypersphere. The second corresponds to the actions preserving a de-Sitter component $dS^{1,2}$ and its conformal boundary $\partial dS^{1,2} = \mathbb{S}^2$.

5.1 Actions on Anti de-Sitter component and its boundary

Here, we study cohomogeneity one actions of connected subgroups of $Conf(\mathbb{E}in^{1,2})$ on Einstein universe $\mathbb{E}in^{1,2}$ preserving a 1-dimensional spacelike linear subspace of $\mathbb{R}^{2,3}$.

Let G be a connected Lie subgroup of $SO_o(2, 3)$ admitting a 2-dimensional orbit at $p \in \mathbb{E}in^{1,2}$ and preserving a spacelike line $\ell \leq \mathbb{R}^{2,3}$. Then G preserves the orthogonal complement subspace $\ell^\perp \approx \mathbb{R}^{2,2}$ as well. Obviously, G acts on ℓ trivially, and so, it is a subgroup of $Stab_{SO_o(2,3)}(\ell) \simeq SO_o(2, 2)$. Hence, G preserves an Einstein hypersphere (Definition 1.63), which is a copy of 2-dimensional Einstein universe $\mathbb{E}in^{1,1} \subset \mathbb{E}in^{1,2}$. Also, G preserves the complement of $\mathbb{E}in^{1,1}$ in $\mathbb{E}in^{1,2}$ which by Lemma 1.62 is an Anti de-Sitter component $AdS^{1,2}$. Hence, in this case, the problem reduces to the consideration of the conformal actions with an open orbit in $\mathbb{E}in^{1,1}$ or the isometric actions with a 2-dimensional orbit in $AdS^{1,2}$.

Recall from Remark 1.45, the 3-dimensional Anti de-Sitter space is isometric to $SL(2, \mathbb{R}) \subset M(2, \mathbb{R})$ endowed with the metric induced from $(M(2, \mathbb{R}), -\det)$. Also, the identity component of $Iso(AdS^{1,2})$ is isomorphic to $(SL(2, \mathbb{R}) \times SL(2, \mathbb{R}))/\mathbb{Z}_2$ (Lemma 1.44). Moreover, recall from Remark 1.28, the Lie group $SL(2, \mathbb{R})$ has three 1-parameter subgroups Y_E, Y_P and Y_H and a unique 2-dimensional connected subgroup Aff up to conjugacy.

Notation 5.1. We denote by \mathcal{G}_λ the following 2-dimensional subgroup of $Y_E \times Aff$

$$\left\{ \left(\begin{bmatrix} \cos(\lambda t) & \sin(\lambda t) \\ -\sin(\lambda t) & \cos(\lambda t) \end{bmatrix}, \begin{bmatrix} e^t & s \\ 0 & e^{-t} \end{bmatrix} \right) : t, s \in \mathbb{R} \right\},$$

where $\lambda \in \mathbb{R}_+^*$ is a constant number.

Theorem 5.2. *Let G be a connected Lie subgroup of $\text{Iso}_\circ(\text{AdS}^{1,2}) \simeq (\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}))/\mathbb{Z}_2$ which acts on $\mathbb{E}\text{in}^{1,2}$ with cohomogeneity one. Then either G fixes a point in $\mathbb{E}\text{in}^{1,2}$ or it is conjugate to one of the following groups.*

$$Y_E \times Y_P, \quad Y_E \times Y_H, \quad (Y_E \times Y_E)/\mathbb{Z}_2, \quad \mathcal{G}_\lambda, \quad Y_E \times \text{Aff}, \quad (Y_E \times \text{SL}(2, \mathbb{R}))/\mathbb{Z}_2, \\ \text{graph}(\varphi)/\mathbb{Z}_2, \quad \text{SL}(2, \mathbb{R}) \times Y_H, \quad \text{SL}(2, \mathbb{R}) \times Y_P, \quad \text{SL}(2, \mathbb{R}) \times \text{Aff}, \quad \text{Iso}_\circ(\text{AdS}^{1,2}),$$

where $\lambda \in \mathbb{R}^*$, and $\varphi : \text{SL}(2, \mathbb{R}) \rightarrow \text{SL}(2, \mathbb{R})$ is an isomorphism which is not a conjugation.

The cohomogeneity one isometric actions on 3-dimensional Anti de-Sitter space has been studied by Ahmadi in [6]. We will study some of the actions directly.

The actions which fix a point in Einstein universe $\mathbb{E}\text{in}^{1,2}$ are considered in Chapter 4. Indeed, a subgroup of $SO_\circ(2, 2)$ which admits a fixed point in the boundary $\partial\text{AdS}^{1,2} = \mathbb{E}\text{in}^{1,1}$ is a subgroup of $(\mathbb{R}_+^* \times Y_H) \times (\mathbb{R}e_1 \oplus \mathbb{R}e_2)$ (described in Remark 4.8) up to conjugacy. On the other hand, by the action of $SO_\circ(2, 2)$ on $\text{AdS}^{1,2}$, the stabilizer of a point is a 3-dimensional Lie subgroup isomorphic to $SO_\circ(1, 2)$. In fact, it is conjugate to the Levi factor of $\text{Conf}_\circ(\mathbb{E}^{1,2})$ which has been considered in Section 4.2. Hence, we only discuss on the subgroups on which fix no point in $\mathbb{E}\text{in}^{1,2}$.

The group of conformal transformations on 2-dimensional Einstein universe is $PO(2, 2)$ and its identity component is isomorphic to $\text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$. As we mentioned earlier in Section 1.7.3, there is a canonical $\text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$ -invariant identification of $\mathbb{E}\text{in}^{1,1}$ with $\mathbb{R}\mathbb{P}^1 \times \mathbb{R}\mathbb{P}^1$. According to that, the left factor (resp. the right factor) $\text{PSL}(2, \mathbb{R})$ acts on every photon $\{*\} \times \mathbb{R}\mathbb{P}^1$ (resp. $\mathbb{R}\mathbb{P}^1 \times \{*\}$) trivially.

Note that, the characterization of the connected Lie subgroups of $(\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}))/\mathbb{Z}_2$ is equivalent to characterization of the connected Lie subgroups of $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$. Therefore, first we consider the action of $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ and its connected Lie subgroups on the Einstein universe, then, we obtain the corresponding subgroups in $(\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}))/\mathbb{Z}_2$. For a connected Lie subgroup $G \subset (\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}))/\mathbb{Z}_2$, we denote by \widehat{G} the corresponding connected Lie subgroup in $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$.

Consider the following natural group morphisms

$$P_1 : \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \longrightarrow \text{SL}(2, \mathbb{R}), \quad (g, h) \mapsto g \\ P_2 : \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \longrightarrow \text{SL}(2, \mathbb{R}), \quad (g, h) \mapsto h$$

We call P_1 and P_2 the first and the second projection, respectively. By the action of $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$, the identity component of the stabilizer of a point in $\mathbb{E}\text{in}^{1,1} \approx \mathbb{R}\mathbb{P}^1 \times \mathbb{R}\mathbb{P}^1$ is conjugate to $\text{Aff} \times \text{Aff}$. Therefore, a subgroup $G \subset \text{Conf}_\circ(\mathbb{E}\text{in}^{1,1})$ admits no fixed point in $\mathbb{E}\text{in}^{1,1}$ if and only if the first or the second projection of \widehat{G} contains an elliptic element.

Remark 5.3. *By the action of $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ on $\mathbb{E}\text{in}^{1,1}$ the identity component of the stabilizer of a photon is conjugate to $\text{SL}(2, \mathbb{R}) \times \text{Aff}$ which stabilizes $\phi_0 = \{\infty\} \times \mathbb{R}\mathbb{P}^1$. This group acts on $\text{AdS}^{1,2} \approx \text{SL}(2, \mathbb{R})$ transitively, since its Levi factor $\text{SL}(2, \mathbb{R})$ does. Moreover, the Levi factor $\text{SL}(2, \mathbb{R})$ preserves every photon $\{*\} \times \mathbb{R}\mathbb{P}^1$.*

Remark 5.4. *The inversion map \mathfrak{i} on $\text{SL}(2, \mathbb{R})$ sending A to A^{-1} is an isometry respect to the metric induced by $-\det$. Hence, by Theorem 0.2, \mathfrak{i} extends to a unique global conformal transformation $\tilde{\mathfrak{i}}$*

on $\mathbb{E}in^{1,2}$. The restriction of \tilde{i} on the boundary $\partial\text{AdS}^{1,2} = \mathbb{E}in^{1,1} = \mathbb{RP}^1 \times \mathbb{RP}^1$ sends $([x], [y])$ to $([y], [x])$. On the other hand, the map j on $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ sending an element (A, B) to (B, A) is an Lie group isomorphism. The action of a connected Lie subgroup $H \subset \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ on $\mathbb{E}in^{1,2} = \text{SL}(2, \mathbb{R}) \cup \mathbb{E}in^{1,1}$ is orbitally equivalent to the action of $j(H)$ via \tilde{i} .

By the action of $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ on Anti de-Sitter space $\text{AdS}^{1,2} \approx \text{SL}(2, \mathbb{R})$, the stabilizer of a point is conjugate to the graph of the identity map $\text{Id}_{\text{SL}(2, \mathbb{R})} : \text{SL}(2, \mathbb{R}) \rightarrow \text{SL}(2, \mathbb{R})$ which is the stabilizer of the identity element $I \in \text{SL}(2, \mathbb{R})$

$$\text{Stab}_{\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})}(I) = \text{graph}(\text{id}_{\text{SL}(2, \mathbb{R})}) = \text{diag}(\text{SL}(2, \mathbb{R}), \text{SL}(2, \mathbb{R})) \simeq \text{SL}(2, \mathbb{R}).$$

The action of $G = \text{graph}(\text{id}_{\text{SL}(2, \mathbb{R})})$ admits 2-dimensional orbits in Einstein universe. Indeed, G/\mathbb{Z}_2 is conjugate to $SO_o(1, 2)$, the Levi factor of $\text{Conf}_o(\mathbb{E}^{1,2})$ (see Chapter 4).

Theorem 5.2 follows from the following Proposition.

Proposition 5.5. *Let $\widehat{G} \subset \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ be a connected Lie subgroup with $\dim \widehat{G} \geq 2$. Let \widehat{G} fixes no point in $\mathbb{E}in^{1,2}$ and $P_1(\widehat{G}), P_2(\widehat{G}) \neq \{Id\}$. Then \widehat{G} is conjugate to one of the following subgroups*

$$Y_E \times Y_P, \quad Y_E \times Y_H, \quad Y_E \times Y_E, \quad \mathcal{G}_\lambda, \quad Y_E \times \text{Aff}_o(1, \mathbb{R}), \quad Y_E \times \text{SL}(2, \mathbb{R}), \quad \text{graph}(\varphi), \\ \text{SL}(2, \mathbb{R}) \times Y_H, \quad \text{SL}(2, \mathbb{R}) \times Y_P, \quad \text{SL}(2, \mathbb{R}) \times \text{Aff}_o(1, \mathbb{R}), \quad \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}),$$

where $\lambda \in \mathbb{R}^*$, and $\varphi : \text{SL}(2, \mathbb{R}) \rightarrow \text{SL}(2, \mathbb{R})$ is an isomorphism which is not a conjugation.

Proof. First, suppose that \widehat{G} fixes no point in $\mathbb{E}in^{1,1}$. Therefore, \widehat{G} contains an element (g, h) , such that either g or h is elliptic. By Remark 5.4, we may restrict ourselves to the case g is elliptic, since $j(\widehat{G})$ is orbitally equivalent to \widehat{G} . Hence, according to Corollary 1.27, $P_1(\widehat{G})$ is either conjugate to Y_E or it is $\text{SL}(2, \mathbb{R})$. Denote by \mathfrak{g} the Lie algebra of \widehat{G} and by p_i the differential of P_i at identity element for $i = 1, 2$.

Case I: $P_1(\widehat{G}) = Y_E$. In this case \widehat{G} is a subgroup of $Y_E \times \text{SL}(2, \mathbb{R})$ up to conjugacy.

- If $\dim P_2(\widehat{G}) = 1$, then \widehat{G} is a 2-dimensional subgroup of $Y_E \times P_2(\widehat{G})$ up to conjugacy. Therefore, By Corollary 1.27, \widehat{G} is conjugate to $Y_E \times Y_P, Y_E \times Y_H$, or $Y_E \times Y_E$.
- If $\dim P_2(\widehat{G}) = 2$, then \widehat{G} is a subgroup of $Y_E \times \text{Aff}$ up to conjugacy.
 - If $\dim \widehat{G} = 3$, then $\widehat{G} = Y_E \times \text{Aff}$ up to conjugacy.
 - If $\dim \widehat{G} = 2$. The second projection $p_2 = dP_2$ from \mathfrak{g} to \mathfrak{aff} is a Lie algebra isomorphism. Hence, $f := p_1 \circ p_2^{-1} : \mathfrak{aff} \rightarrow p_1(\mathfrak{h}) = \mathbb{R}\mathcal{Y}_E$ is a surjective Lie algebra morphism. The kernel of f is a 1-dimensional ideal of \mathfrak{aff} , hence $\ker f = \mathbb{R}\mathcal{Y}_P$. This induces an isomorphism from $\mathfrak{aff}/\mathbb{R}\mathcal{Y}_P \simeq \mathbb{R}\mathcal{Y}_H$ to $\mathbb{R}\mathcal{Y}_E$. Thus, there exists a real nonzero number λ , such that $f(t\mathcal{Y}_H + s\mathcal{Y}_P) = \lambda t\mathcal{Y}_E$, for all $t, s \in \mathbb{R}$. This implies that \widehat{G} is conjugate to \mathcal{G}_λ for some $\lambda \in \mathbb{R}^*$.
- If $\dim P_2(\widehat{G}) = 3$, then $P_2(\widehat{G}) = \text{SL}(2, \mathbb{R})$. Assume that $\dim \widehat{G} = 3$. Then the map $p_2 : \mathfrak{g} \rightarrow \mathfrak{sl}(2, \mathbb{R})$ is a Lie algebra isomorphism. Hence, $f := p_1 \circ p_2^{-1} : \mathfrak{sl}(2, \mathbb{R}) \rightarrow p_1(\mathfrak{g})$ is a surjective Lie algebra morphism. But, this contradicts the semi-simplicity of $\mathfrak{sl}(2, \mathbb{R})$, since $\ker f$ is a 2-dimensional ideal of $\mathfrak{sl}(2, \mathbb{R})$. Therefore, $\dim \widehat{G} = 4$ and so, $\widehat{G} = Y_E \times \text{SL}(2, \mathbb{R})$ up to conjugacy.

Case II: $P_1(\widehat{G}) = \text{SL}(2, \mathbb{R})$.

- $\dim P_2(\widehat{G}) = 1$. We claim that $\dim \widehat{G} \neq 3$. On the contrary, assume that $\dim \widehat{G} = 3$. Then $p_1 : \mathfrak{g} \rightarrow \mathfrak{sl}(2, \mathbb{R})$ is an Lie algebra isomorphism. Thus, $f = p_2 \circ p_1^{-1} : \mathfrak{sl}(2, \mathbb{R}) \rightarrow p_2(\mathfrak{g})$ is a surjective Lie algebra morphism. But, this contradicts the simplicity of $\mathfrak{sl}(2, \mathbb{R})$, since $\ker f$ is a 2-dimensional ideal of $\mathfrak{sl}(2, \mathbb{R})$. Thus $\dim \widehat{G} = 4$ and \widehat{G} is conjugate to $\mathrm{SL}(2, \mathbb{R}) \times Y_E$, $\mathrm{SL}(2, \mathbb{R}) \times Y_H$, or $\mathrm{SL}(2, \mathbb{R}) \times Y_P$. By Remark 5.4, $\mathrm{SL}(2, \mathbb{R}) \times Y_E$ is conjugate to $Y_E \times \mathrm{SL}(2, \mathbb{R})$.
- $\dim P_2(\widehat{G}) = 2$. In this case, \widehat{G} is a subgroup of $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{Aff}$ up to conjugacy. We show that $\dim \widehat{G} = 5$. If $\dim \widehat{G} = 3$, then using the same argument as the previous case, $f = p_2 \circ p_1^{-1} : \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{aff}$ is a surjective Lie algebra morphism. But, this contradicts the simplicity of $\mathfrak{sl}(2, \mathbb{R})$, since $\ker f$ is a 1-dimensional ideal of $\mathfrak{sl}(2, \mathbb{R})$. If $\dim \widehat{G} = 4$, then the kernel of $p_1 : \mathfrak{g} \rightarrow \mathfrak{sl}(2, \mathbb{R})$ is a 1-dimensional ideal of $\{0\} \oplus \mathfrak{aff}$, hence $\ker p_1 = \{0\} \oplus \mathbb{R}\mathcal{Y}_P$. In the one hand, $\mathfrak{g}/\ker p_1 \simeq \mathfrak{sl}(2, \mathbb{R})$, thus it is a simple Lie algebra. On the other hand, the map

$$\mathfrak{g}/\ker p_1 \rightarrow \mathfrak{aff}/\mathbb{R}\mathcal{Y}_P, \quad (X, a\mathcal{Y}_H) + \ker p_1 \mapsto p_2(X, a\mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P = a\mathcal{Y}_H + \mathbb{R}\mathcal{Y}_P,$$

where $(X, a\mathcal{Y}_H) \in \mathfrak{g} \leq \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{aff}$, is a surjective Lie algebra morphism. But, this contradicts the simplicity of $\mathfrak{g}/\ker p_1$.

Thus, $\dim \widehat{G} = 5$, and so, $\widehat{G} = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{Aff}$ up to conjugacy.

- $\dim P_2(\widehat{G}) = 3$. In this case $P_2(\widehat{G}) = \mathrm{SL}(2, \mathbb{R})$, and $\dim \widehat{G} \in \{3, 4, 5, 6\}$. We claim that $\dim \widehat{G} \notin \{4, 5\}$, otherwise, $\ker P_1$ is a nontrivial proper normal solvable Lie subgroup of $\{Id\} \times \mathrm{SL}(2, \mathbb{R})$, which contradicts the simplicity of $\mathrm{SL}(2, \mathbb{R})$. Thus $\dim \widehat{G} \in \{3, 6\}$.

If $\dim \widehat{G} = 3$, then $P_1, P_2 : \widehat{G} \rightarrow \mathrm{SL}(2, \mathbb{R})$ are isomorphisms and $\widehat{G} = \mathrm{graph}(\varphi) \simeq \mathrm{SL}(2, \mathbb{R})$ where $\varphi = P_1 \circ P_2^{-1}$. In the one hand, \widehat{G} fixes no point in the boundary $\partial \mathrm{AdS}^{1,2} = \mathbb{E}\mathrm{in}^{1,1}$, since the stabilizer of a point in $\mathbb{E}\mathrm{in}^{1,1}$ is a solvable group conjugate to $\mathrm{Aff} \times \mathrm{Aff}$. On the other hand, φ is a conjugation if and only if \widehat{G} is conjugate to $\mathrm{graph}(id_{\mathrm{SL}(2, \mathbb{R})})$ if and only if it fixes a point in $\mathrm{AdS}^{1,2} \approx \mathrm{SL}(2, \mathbb{R})$ (which should be excluded by assumption).

If $\dim \widehat{G} = 6$, then $\widehat{G} = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$.

□

Lemma 5.6. *Let $\varphi : \mathrm{SL}(2, \mathbb{R}) \rightarrow \mathrm{SL}(2, \mathbb{R})$ be an isomorphism which is not a conjugation, and let $G = \mathrm{graph}(\varphi)$ admit a 2-dimensional orbit in $\mathbb{E}\mathrm{in}^{1,2}$. Then the action of $\mathrm{graph}(\varphi)$ on $(M(2, \mathbb{R}), -\det)$ preserves a unique spacelike line. Therefore, its action on $\mathbb{R}^{2,3}$ is conjugate to the action of $SO_o(2, 1)$ which acts on a 2-dimensional positive definite linear subspace (i.e. of signature $(0, 2)$) of $\mathbb{R}^{2,3}$ trivially.*

Proof. Consider the action of $G = \mathrm{graph}(\varphi)$ on $(M(2, \mathbb{R}), -\det)$. By Theorem 0.4, G preserves a non-trivial linear subspace V of $(M(2, \mathbb{R}), -\det)$. Since it also preserves the orthogonal space V^\perp , it is suffix to focus on $\dim V \leq 2$. We show that V has signature $(0, 1)$. Assume the contrary:

Case I: $\dim V = 1$

- If V is a timelike line, then obviously G fixes two points on $\mathrm{AdS}^{1,2} \approx \mathrm{SL}(2, \mathbb{R})$, namely, $\mathrm{SL}(2, \mathbb{R}) \cap V$, which contradicts the fact that φ is not a conjugation.
- If V is a lightlike line, then G fixes a point in the boundary $\mathbb{E}\mathrm{in}^{1,1}$. This is a contradiction, since the stabilizer of a point in $\mathbb{E}\mathrm{in}^{1,1}$ is a solvable group isomorphic to $\mathrm{Aff} \times \mathrm{Aff}$.

Case II: $\dim V = 2$

- Assume that $\dim V = 2$ and the restriction of $-\det$ on V is definite (positive or negative). Hence, there is a representation $\rho : \mathrm{SL}(2, \mathbb{R}) \rightarrow \mathrm{SO}(2) \times \mathrm{SO}(2)$, since the identity component of the groups of linear isometries on V and V^\perp are isomorphic to $\mathrm{SO}(2)$. By Lemma 1.29, ρ is trivial, a contradiction.
- Suppose that $\dim V = 2$ and the restriction of $-\det$ on V has signature $(1, 1)$, $(1, 0, 1)$, or $(0, 1, 1)$. In all the cases, G preserves a lightlike line. Hence, it admits a fixed point in $\mathbb{E}\mathrm{in}^{1,1}$. Once more, a contradiction.
- If V has signature $(0, 0, 2)$, then it preserves a photon on $\mathbb{E}\mathrm{in}^{1,1}$. Thus, G is conjugate to the Levi factor of $\mathrm{Stab}_{\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})}(\phi) = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{Aff}$ which is $\mathrm{SL}(2, \mathbb{R})$. By Remark 5.3, the Levi factor does not admit a 2-dimensional orbit in $\mathbb{E}\mathrm{in}^{1,2}$. Again, a contradiction.

Henceforth, V is a spacelike line in $M(2, \mathbb{R})$, and G acts on it trivially. We have assumed in very beginning of this section that G preserves a spacelike line in $\mathbb{R}^{2,3}$ which is orthogonal to $M(2, \mathbb{R})$. Thus G acts on a positive definite 2-dimensional linear subspace Π , generated by V and ℓ , of $\mathbb{R}^{2,3}$ trivially. This induces a surjective faithful representation from G to $\mathrm{SO}_o(2, 1)$, the group of linear isometries of Π^\perp . This completes the proof. \square

Lemma 5.7. *The group $\mathrm{SO}_o(2, 1)$ (described in Lemma 5.6) acts on $\mathrm{AdS}^{1,2}$ with cohomogeneity one.*

Proof. According to the proof of Lemma 5.6, V^\perp has signature $(2, 1)$. The intersection of V^\perp with $\mathrm{AdS}^{1,2}$ is a copy of 2-dimensional Anti de-Sitter space $\mathrm{AdS}^{1,1}$, and $\mathrm{SO}_o(2, 1)$ acts on it transitively. This completes the proof. \square

Proof of Theorem 5.2. Assume that, the first projection $P_1(\widehat{G})$ is trivial. Then \widehat{G} is either conjugate to $\mathrm{Aff}_o(1, \mathbb{R})$ or it is $\mathrm{SL}(2, \mathbb{R})$. Observe that, affine group admits a fixed point in $\mathbb{E}\mathrm{in}^{1,1}$. Also, $\mathrm{SL}(2, \mathbb{R})$ acts on $\mathrm{AdS}^{1,2}$ transitively and preserves every photon $\{*\} \times \mathbb{R}\mathbb{P}^1$. Thus, it does not admit a 2-dimensional orbit in $\mathbb{E}\mathrm{in}^{1,2}$. The same happens for the case $P_2(\widehat{G}) = \{Id\}$.

From now on, we assume that $P_1(\widehat{G}), P_2(\widehat{G}) \neq \{Id\}$. Suppose that G admits no fixed point in $\mathbb{E}\mathrm{in}^{1,2}$. Then \widehat{G} is conjugate to one of the subgroups mentioned in Proposition 5.5. It is not hard to see that every subgroup \widehat{G} which is the direct product $P_1(\widehat{G}) \times P_2(\widehat{G})$ admits an open orbit in $\mathbb{E}\mathrm{in}^{1,1} \approx \mathbb{R}\mathbb{P}^1 \times \mathbb{R}\mathbb{P}^1$. On the other hand, by Lemma 5.7, $\mathrm{graph}(\varphi)$ admits a 2-dimensional orbit in $\mathbb{E}\mathrm{in}^{1,2}$.

The remaining case is \mathcal{G}_λ . Observe that \mathcal{G}_λ is a subgroup of $\mathrm{Aff}_o(1, \mathbb{R}) \times Y_E$. One can see $\mathrm{Aff}_o(1, \mathbb{R}) \times Y_E$ acts on $\mathrm{AdS}^{1,2} = \mathrm{SL}(2, \mathbb{R})$ freely. Hence, \mathcal{G}_λ acts on the Anti de-Sitter component $\mathrm{AdS}^{1,2}$ with cohomogeneity one.

This completes the proof. \square

5.2 Actions on de-Sitter component and its boundary

In this section we consider cohomogeneity one actions of connected subgroups of $\mathrm{SO}_o(2, 3)$ on Einstein universe $\mathbb{E}\mathrm{in}^{1,2}$ preserving a 1-dimensional timelike linear subspace of $\mathbb{R}^{2,3}$.

Let G be a connected Lie subgroup of $\mathrm{SO}_o(2, 3)$ admitting a 2-dimensional orbit at $p \in \mathbb{E}\mathrm{in}^{1,2}$ and preserving a timelike line $\ell \leq \mathbb{R}^{2,3}$. Hence G preserves the orthogonal complement subspace $\ell^\perp \approx \mathbb{R}^{1,2}$

as well. The Lie group G acts on ℓ trivially, and it is a subgroup of $Stab_{SO_o(2,3)}(\ell) \simeq SO_o(1, 3)$. Hence, G preserves an spacelike hypersphere (Definition 1.60), which is a copy of the conformal 2-sphere $\mathbb{S}^2 \subset \mathbb{E}in^{1,2}$. Also, G preserves the complement of \mathbb{S}^2 in $\mathbb{E}in^{1,2}$ which by Lemma 1.62 is conformally equivalent to the 3-dimensional de-Sitter space $dS^{1,2}$. Henceforth, in this case, the problem reduces to the consideration of conformal actions with an open orbit in \mathbb{S}^2 or the isometric actions with a 2-dimensional orbit in $dS^{1,2}$.

Theorem 5.8. *Let G be a connected Lie subgroup of $\text{Iso}_o(dS^{1,2}) \simeq SO_o(1, 3)$ which acts on the Einstein universe $\mathbb{E}in^{1,2}$ with cohomogeneity one. Then either G fixes a point in $\mathbb{E}in^{1,2}$ or it is conjugate to $SO(3)$ or it is $\text{Iso}_o(dS^{1,2})$.*

The actions admitting a fixed point in the Einstein universe $\mathbb{E}in^{1,2}$ has been described in Chapter 4. In fact, if a connected Lie subgroup $G \subset SO_o(1, 3)$ fixes a point in the boundary $\partial dS^{1,2} = \mathbb{S}^2$, then it is a subgroup of $(\mathbb{R}_+^* \times Y_E) \times (\mathbb{R}e_2 \oplus \mathbb{R}e_3)$ (described in Remark 4.9) up to conjugacy. On the other hand, by the action of $\text{Iso}_o(dS^{1,2})$ on $dS^{1,2}$ the stabilizer of a point is a 3-dimensional subgroup isomorphic to $SO_o(1, 2)$. In fact, it is conjugate to the Levi factor of $\text{Conf}_o(\mathbb{E}^{1,2})$ which has been considered in Section 4.2. Hence, we only discuss on the subgroups which admit no fixed point $\mathbb{E}in^{1,2}$.

Here is a useful model for the action of $SO_o(1, 3)$ on the de-Sitter component $dS^{1,2}$ and its conformal boundary in $\mathbb{E}in^{1,2}$. Consider the 4-dimensional Lorentzian vector space $\mathbb{R}^{1,3} = (\mathbb{R}^4, \mathfrak{q}_{1,3})$. The de-Sitter space in $\mathbb{R}^{1,3}$ is the level set $\mathfrak{q}_{1,3}^{-1}(1)$. Obviously, $SO_o(1, 3)$ preserves this level set and acts on it isometrically. On the other hand, the image of the nullcone $\mathfrak{N}^{1,3}$ of $\mathbb{R}^{1,3}$ by $\mathbb{P} : \mathbb{R}^{1,3} \setminus \{0\} \rightarrow \mathbb{RP}^3$ is conformally equivalent to the conformal 2-sphere \mathbb{S}^2 . Moreover, $SO_o(1, 3)$ acts on $\mathbb{P}(\mathfrak{N}^{1,3}) \subset \mathbb{RP}^3$ conformally. Indeed, it is the identity component of the group of conformal transformations of \mathbb{S}^2 . We deduce that, the actions of $SO_o(1, 3)$ on the de-Sitter component $dS^{1,2} \subset \mathbb{E}in^{1,2}$ and its conformal boundary \mathbb{S}^2 is conformally equivalent to its actions on the de-Sitter space $\mathfrak{q}_{1,3}^{-1}(1) \subset \mathbb{R}^{1,3}$ and $\mathbb{P}(\mathfrak{N}^{1,3}) \subset \mathbb{RP}^3$, respectively.

The following theorem will play a key rule in the proof of Theorem 5.8.

Theorem 5.9. [13, Theorem 1.1]. *Let G be a connected (non necessarily closed) Lie subgroup of $SO(1, n)$ and assume that the action of G on the Lorentzian space $\mathbb{R}^{1,n}$ is irreducible. Then $G = SO_o(1, n)$.*

Let $G \subset \text{Iso}(dS^{1,3}) \simeq SO_o(1, 3)$ be a connected subgroup which preserves a proper linear subspace V of $\mathbb{R}^{1,3}$. Observe that since G preserves the orthogonal space V^\perp as well, it is suffix to consider the case $\dim V \leq 2$. Furthermore, if V (resp. V^\perp) contains a lightlike vector, then the intersection of $\mathbb{P}(V)$ (resp. $\mathbb{P}(V^\perp)$) with $\mathbb{P}(\mathfrak{N}^{1,3})$ is a discrete subset A consisting of one or two points. Hence, by connectedness, G acts on A trivially. If V is a 1-dimensional timelike subspace, then G is a subgroup of $SO(3)$ -the maximal compact subgroup- up to conjugacy.

Proof of Theorem 5.8. First, assume that G acts on $\mathbb{R}^{1,3}$ irreducibly. Then by Theorem 5.9 $G = SO_o(1, 3)$. Now, suppose that G preserves a proper linear subspace $V \leq \mathbb{R}^{1,3}$.

- If V or V^\perp contains a unique (linear) lightlike line, then G fixes a point in the spacelike hypersphere \mathbb{S}^2 .
- If V is a timelike line, then $G \subset SO(3)$ up to conjugacy. Since G admits a 2-dimensional orbit in $\mathbb{E}in^{1,2}$, $\dim G \geq 2$. By Proposition 2.3, $SO(3)$ has no 2-dimensional Lie subgroup. Hence, $G = SO(3)$ up to conjugacy.

- If V is a spacelike line, then G is a subgroup of $SO_o(1, 2)$ up to conjugacy. Obviously, $SO_o(1, 2)$ admits a fixed point in the de-Sitter component $dS^{1,2}$.

This completes the proof. □

5.3 Orbits

Now, we describe the orbits induced in $\mathbb{E}in^{1,2}$ by the subgroups obtained in Theorem 5.2 and Theorem 5.8.

In order to describe the orbits induced by $SO_o(2, 1)$ we need the following lemma.

First assume that G is a subgroup of $\text{Iso}_o(\text{AdS}^{1,2}) \simeq SO_o(2, 2)$. Note that the subgroup $(Y_E \times Y_E)/\mathbb{Z}_2$ is compact and its orbits has been described in Theorem 2.1. Observe that, every subgroup containing $SL(2, \mathbb{R})$ as the left or right factor acts on the Anti de-Sitter component $\text{AdS}^{1,2}$ transitively. In order to determine the orbits in $\text{AdS}^{1,2}$, let

$$p = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \quad \det(p) = 1$$

be an arbitrary point in $\text{AdS}^{1,2} \approx SL(2, \mathbb{R})$.

- G is one the groups $\text{Iso}_o(\text{AdS}^{1,2})$ or $(SL(2, \mathbb{R}) \times Y_E)/\mathbb{Z}_2$. It is obvious that G acts on the Anti de-Sitter component $\text{AdS}^{1,2}$ transitively, since it contains $SL(2, \mathbb{R})$ as a subgroup. Also, it is easy to see that G acts on the boundary $\partial\text{AdS}^{1,2} = \mathbb{E}in^{1,1}$ transitively.
- G is one of the groups $SL(2, \mathbb{R}) \times \text{Aff}$ or $SL(2, \mathbb{R}) \times Y_P$. Since G contains $SL(2, \mathbb{R})$ as a subgroup, it acts on the Anti de-Sitter component transitively. In the both cases, the right factors Aff and Y_P fix a unique photon $\phi = \{\infty\} \times \mathbb{RP}^1$ pointwisely. Moreover, they act on each lightlike geodesic $\mathbb{RP}^1 \times \{x\} \setminus \phi$ transitively. On the other hand, the left factor $SL(2, \mathbb{R})$ acts on every photon $\{x\} \times \mathbb{RP}^1$ transitively. Hence, G acts on ϕ and $\mathbb{E}in^{1,1} \setminus \phi$ transitively.
- $G = SL(2, \mathbb{R}) \times Y_H$. This group acts on Anti de-Sitter component transitively, since it contains $SL(2, \mathbb{R})$ as a subgroup. The right factor Y_H fixes exactly two photons $\phi = \{\infty\} \times \mathbb{RP}^1$ and $\psi = \{0\} \times \mathbb{RP}^1$ pointwisely. Moreover, it acts on the both connected components of each lightlike geodesic $\mathbb{RP}^1 \times \{x\} \setminus (\phi \cup \psi)$ transitively. On the other hand, the left factor $SL(2, \mathbb{R})$ acts on every photon $\{x\} \times \mathbb{RP}^1$ transitively. Hence, G acts on ϕ, ψ , and the two connected components of $\mathbb{E}in^{1,1} \setminus (\phi \cup \psi)$ transitively.
- $G = Y_E \times \text{Aff}$. The right factor Aff fixes a unique photon $\phi = \{\infty\} \times \mathbb{RP}^1$ pointwisely. Moreover, it acts on each lightlike geodesic $\mathbb{RP}^1 \times \{x\} \setminus \phi$ transitively. On the other hand, the left factor Y_E acts on every photon $\{x\} \times \mathbb{RP}^1$ transitively. Hence, G acts on ϕ and $\mathbb{E}in^{1,1} \setminus \phi$ transitively. Observe that, by the action of G on the Anti de-Sitter component $\text{AdS}^{1,2} = SL(2, \mathbb{R})$ the orbit at $I \in SL(2, \mathbb{R})$ is $SL(2, \mathbb{R})$. Hence, G acts on $\text{AdS}^{1,2}$ transitively.
- $G = \mathcal{G}_\lambda, \lambda \in \mathbb{R}_+^*$. This group preserves the orbit induced by $Y_E \times \text{Aff}$. Denote by H_λ the following 1-parameter subgroup of \mathcal{G}_λ

$$\left\{ \left(\begin{bmatrix} \cos(\lambda t) & \sin(\lambda t) \\ -\sin(\lambda t) & \cos(\lambda t) \end{bmatrix}, \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \right) : t \in \mathbb{R} \right\}.$$

For an arbitrary point $q = (q_0, q_1) = ([x_0 : y_0], [x_1 : y_1]) \in \mathbb{RP}^1 \times \mathbb{RP}^1 = \mathbb{Ein}^{1,1}$ the vectors tangent to the orbit $\mathcal{G}_\lambda(q)$ at q induced by the 1-parameter subgroups H_λ and $\{Id\} \times Y_P$ are

$$v = ((x_0, -y_0), (\lambda y_1, -\lambda x_1)), \quad w = ((y_0, 0), (0, 0)),$$

respectively. Obviously, \mathcal{G}_λ admits no fixed point in $\mathbb{Ein}^{1,1}$. Hence, it acts on the photon $\phi = \{\infty\} \times \mathbb{RP}^1$ transitively. The set $\{v, w\} \subset T_q \mathcal{G}_\lambda(q)$ is a basis if and only if $y_0 \neq 0$ if and only if $q \notin \phi$. This implies that \mathcal{G}_λ acts on $\mathbb{Ein}^{1,1} \setminus \phi$ transitively.

As we mentioned in the previous case, $Y_E \times \text{Aff}$ acts on $\text{AdS}^{1,2}$ freely, so does \mathcal{G}_λ . For an arbitrary point $p \in \text{AdS}^{1,2}$, the vectors tangent to the orbit $\mathcal{G}_\lambda(p)$ at p induced by H_λ and $\{Id\} \times Y_P$ are:

$$v_\lambda = \lambda \begin{bmatrix} -p_{21} & p_{22} \\ p_{11} & -p_{12} \end{bmatrix}, \quad v_P = \begin{bmatrix} 0 & -p_{11} \\ 0 & -p_{21} \end{bmatrix},$$

respectively. It is not hard to see that, the orthogonal space of the null vector v_P in the tangent space $T_p G(p)$ is $\mathbb{R}v_P$. Hence, \mathcal{G}_λ admits a codimension 1 foliation on $\text{AdS}^{1,2}$ where every orbit is Lorentzian and \mathcal{G}_λ acts freely. Note that, for all $\lambda \in \mathbb{R}_+^*$, the orbits induced by \mathcal{G}_λ in $\mathbb{Ein}^{1,2}$ are exactly the same as the orbits induced by \mathcal{G}_1 . In other words, \mathcal{G}_λ is orbit equivalent to \mathcal{G}_1 via the identity map on $\mathbb{Ein}^{1,2}$.

- $G = Y_E \times Y_P$. It is easy to see that G acts on the photon $\phi = \{\infty\} \times \mathbb{RP}^1 \subset \mathbb{Ein}^{1,1}$ and on $\mathbb{Ein}^{1,1} \setminus \phi$ transitively. On the other hand G acts on $\text{AdS}^{1,2}$ freely, since $Y_E \times \text{Aff}_o(1, \mathbb{R})$ does. For an arbitrary point $p \in \text{AdS}^{1,2}$, the vector tangent to the orbit $G(p)$ at p induced by the 1-parameter subgroup $Y_E \times \{Id\}$ is

$$v = \begin{bmatrix} p_{21} & p_{22} \\ -p_{11} & -p_{12} \end{bmatrix}.$$

Observe that v is a timelike vector. Hence, G admits a codimension 1 foliation on $\text{AdS}^{1,2}$ where every orbit is Lorentzian and G acts freely.

- $G = Y_E \times Y_H$. It is easy to see that G acts on the photons $\phi = \{\infty\} \times \mathbb{RP}^1, \psi = \{0\} \times \mathbb{RP}^1 \subset \mathbb{Ein}^{1,1}$, and on the two connected components of $\mathbb{Ein}^{1,1} \setminus (\phi \cup \psi)$ transitively. On the other hand G acts on $\text{AdS}^{1,2}$ freely, since $Y_E \times \text{Aff}_o(1, \mathbb{R})$ does. For an arbitrary point $p \in \text{AdS}^{1,2}$, the vector tangent to the orbit $G(p)$ at p induced by the 1-parameter subgroup $Y_E \times \{Id\}$ is

$$v = \begin{bmatrix} p_{21} & p_{22} \\ -p_{11} & -p_{12} \end{bmatrix}.$$

Observe that v is a timelike vector. Hence, G admits a codimension 1 foliation on $\text{AdS}^{1,2}$ where every orbit is Lorentzian and G acts on freely.

- $G = \text{graph}(\varphi)/\mathbb{Z}_2$. By Lemma 5.6, the action of G on $\mathbb{Ein}^{1,1}$ is conjugate to the action of $SO_o(2, 1)$ which preserves a linear subspace $V \leq \mathbb{R}^{2,3}$ of signature $(2, 1)$, and acts on the orthogonal complement space V^\perp trivially. Observe that G is a subgroup of the stabilizer of a timelike circle \mathfrak{C} (Definition 1.69). By the action of $SO_o(2, 3)$ the stabilizer of \mathfrak{C} is conjugate to $SO_o(2, 1) \times SO(2)$ (Lemma 1.70). Indeed, by the identification in Lemma 1.70 the complement of \mathfrak{C} in $\mathbb{Ein}^{1,2}$ is

conformally equivalent (up to double cover) to the direct product $\text{AdS}^{1,1} \times \mathbb{S}^1 \subset \widehat{\mathbb{E}in}^{1,2}$. By the action of $SO_o(2, 1) \times SO(2)$ on $\text{AdS}^{1,1} \times SO(2)$, the $SO(2, 1)$ -factor (resp. $SO(2)$ -factor) acts on \mathbb{S}^1 -factor (resp. $\text{AdS}^{1,1}$ -factor) trivially. On the other hand, for all $x \in \mathbb{S}^1$, $G \cong SO_o(2, 1)$ acts on $\text{AdS}^{1,1} \times \{x\}$ transitively. Observe that \mathbb{Z}_2 acts on $\text{AdS}^{1,1} \times \{x\}$ trivially, however, the orbits $\text{AdS}^{1,1} \times \{x\}$ and $\text{AdS}^{1,1} \times \{-x\}$ coincide in $\mathbb{E}in^{1,2}$. Hence, the orbits induced by G in $\mathbb{E}in^{1,2}$ are: a timelike circle \mathfrak{C} , and a 1-parameter family of 2-dimensional Lorentzian orbits on which every orbit is conformally equivalent to the 2-dimensional Ant de-Sitter space $\text{AdS}^{1,1}$ -the parameter being $x \in \mathbb{RP}^1$.

Corollary 5.10. *Let G be a connected Lie subgroup of $\text{Iso}_o(\text{AdS}^{1,2})$ which admits a 2-dimensional orbit in Einstein universe $\mathbb{E}in^{1,2}$. Then either G fixes a point in $\mathbb{E}in^{1,2}$ or it is compact or it admits the same orbits in $\mathbb{E}in^{1,2}$ as one of the following groups.*

$$\begin{array}{ccccccc} Y_E \times Y_P, & Y_E \times Y_H, & \mathcal{G}_1, & SO_o(2, 1), \\ SL(2, \mathbb{R}) \times Y_H, & SL(2, \mathbb{R}) \times \text{Aff}, & \text{Iso}_o(\text{AdS}^{1,2}). \end{array}$$

Now, we consider the orbits induced by the subgroups obtained in Theorem 5.8. Indeed, there is only one remaining group, say $\text{Iso}_o(\text{dS}^{1,2})$, since the action of $SO(3)$ has been described in Chapter 2.

Obviously, $\text{Iso}_o(\text{dS}^{1,2}) \simeq SO_o(1, 3)$ acts on the de-Sitter component and its conformal boundary $\partial\text{dS}^{1,2} \approx \mathbb{S}^2$ transitively.

Chapter 6

Actions preserving a photon

In this Chapter, we consider the cohomogeneity one actions on Einstein universe $\mathbb{E}in^{1,2}$ preserving a photon.

Theorem 6.1. *Let ϕ be a photon in Einstein universe $\mathbb{E}in^{1,2} \subset \mathbb{R}P^4$ and $G \subset \text{Conf}(\mathbb{E}in^{1,2})$ a connected Lie subgroup which preserves ϕ . If G acts on $\mathbb{E}in^{1,2}$ with cohomogeneity one, then it fixes a point in the projective space $\mathbb{R}P^4$. Therefore, either G fixes a point in $\mathbb{E}in^{1,2}$ or it preserves an Anti de-Sitter component or a de-Sitter component in $\mathbb{E}in^{1,2}$.*

Let G be a connected Lie subgroup of $SO_o(2, 3)$ stabilizing a photon $\phi \subset \mathbb{E}in^{1,2}$ and admitting a 2-dimensional orbit in $\mathbb{E}in^{1,2}$. Recall from Section 1.7.1 that, the complement of ϕ in $\mathbb{E}in^{1,2}$ is an open homogeneous subset diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^2$. The group of conformal transformations on $\mathbb{E}in_\phi^{1,2} = \mathbb{E}in^{1,2} \setminus \phi$ is $Stab_{O(2,3)}(\phi)$. Furthermore, $\mathbb{E}in_\phi^{1,2}$ admits a codimension 1 foliation \mathcal{F}_ϕ invariant by $\text{Conf}(\mathbb{E}in_\phi^{1,2})$, for which each leaf is a degenerate surface diffeomorphic to \mathbb{R}^2 . More precisely, choosing a point $x_0 \in \phi$, one of the leaves is $L(x_0) \setminus \phi$ and the other leaves are the degenerate affine 2-planes in the Minkowski patch $Mink(x_0)$ with limit point in ϕ . Therefore, we may determine a leaf of \mathcal{F}_ϕ by its limit point $x \in \phi$ and denote it by $\mathcal{F}_\phi(x)$. In other words, for an arbitrary point $x \in \phi$, the leaf $\mathcal{F}_\phi(x)$ is the degenerate surface $L(x) \setminus \phi$.

Recall from Section 1.7.1, by the action of $\text{Conf}(\mathbb{E}^{1,2})$, the stabilizer of a photon in $\mathbb{E}in^{1,2}$ is isomorphic to $(\mathbb{R}^* \times \text{SL}(2, \mathbb{R})) \rtimes H(3)$, where $H(3)$ is the 3-dimensional Heisenberg group. The action of $\text{Conf}(\mathbb{E}in_\phi^{1,2})$ on $\phi \approx \mathbb{R}P^1$ admits a surjective representation

$$\pi : \text{Conf}_o(\mathbb{E}in_\phi^{1,2}) \longrightarrow \text{PSL}(2, \mathbb{R}) \simeq \text{Conf}_o(\mathbb{R}P^1).$$

The kernel $\mathcal{K} = \ker \pi$ has two connected components. The identity component \mathcal{K}_o is conjugate to the Lie subgroup

$$\exp(\mathbb{R}(1 + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P) \rtimes (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3), \quad (6.1)$$

described in Definition 4.10.

Choosing an arbitrary point $x_0 \in \phi$, the group \mathcal{K} splits as the semi-direct product $\mathcal{A} \rtimes \Pi$. where \mathcal{A} is a 2-dimensional Lie subgroup with two connected components which fixes a unique point in the Minkowski patch $Mink(x_0)$ and Π is the unique \mathcal{A} -invariant degenerate 2-plane in $\mathbb{R}^{1,2}$. Note that this

splitting depends strongly on x_0 . Considering $Mink(x_0) \approx \mathbb{E}^{1,2}$ with underlying Lorentzian vector space $(\mathfrak{sl}(2, \mathbb{R}), -\det)$ (described in Remark 1.41), we have

$$\mathcal{K} = \left\{ \left(e^{2t}, \begin{bmatrix} \varepsilon e^t & s \\ 0 & \varepsilon e^{-t} \end{bmatrix}, \begin{bmatrix} u & v \\ 0 & -u \end{bmatrix} \right) : t, s, u, v \in \mathbb{R}, \varepsilon = \pm 1 \right\}. \quad (6.2)$$

Now,

$$\mathcal{A} = \left\{ \left(e^{2t}, \begin{bmatrix} \varepsilon e^t & s \\ 0 & \varepsilon e^{-t} \end{bmatrix}, 0_{2 \times 2} \right) : t, s \in \mathbb{R}, \varepsilon = \pm 1 \right\},$$

and

$$\Pi = \left\{ \left(1, I_{2 \times 2}, \begin{bmatrix} u & v \\ 0 & -u \end{bmatrix} \right) : u, v \in \mathbb{R} \right\}.$$

The subgroup $Y_P \times \Pi \subset \mathcal{K}$ (which in Eq. 6.2 consists of elements in \mathcal{K} with $t = 0$) is the unique connected maximal unipotent subgroup (up to conjugacy) of \mathcal{K} and it is isomorphic to the 3-dimensional Heisenberg group $H(3)$.

The center of Heisenberg group $H(3) \simeq Y_P \times \Pi$ is a 1-dimensional Lie subgroup \mathcal{L} isomorphic to \mathbb{R} . Observe that, \mathcal{L} is the set of lightlike (elements in Π) translations in the Minkowski patch $Mink(x_0)$. More precisely, considering Eq. 6.2, \mathcal{L} is the set of the element in \mathcal{K} with $t = s = u = 0$.

The following proposition gives a powerful tool to prove Theorem 6.1.

Proposition 6.2. *Let G be a connected Lie subgroup of $\text{Conf}(\mathbb{E}in_\phi^{1,2})$ which acts transitively on ϕ and admits a 2-dimensional orbit at $p \in \mathbb{E}in_\phi^{1,2}$. Then the identity component of $G \cap H(3)$ is a subgroup of \mathcal{L} .*

Definition 6.3. *Let $x \in \phi$ be an arbitrary point. A non-trivial element $g \in H(3)$ is called:*

- *a lightlike transformation on $Mink(x)$, if it fixes no point in $\mathbb{E}in_\phi^{1,2}$.*
- *a spacelike transformation on $Mink(x)$, if the set of its fixed points in $\mathbb{E}in_\phi^{1,2}$ is a unique lightlike geodesic included in the leaf $\mathcal{F}_\phi(x) \subset L(x)$.*
- *a parabolic transformation on $Mink(x)$, if the set of its fixed points in $\mathbb{E}in_\phi^{1,2}$ is a unique lightlike geodesic in the Minkowski patch $Mink(x)$.*

Observe that, if $g \in H(3)$ is a lightlike transformation on a Minkowski patch $Mink(x_0)$ (for some $x_0 \in \phi$), then for all $y \in \phi$, it is a lightlike transformation on the Minkowski patch $Mink(y)$. Indeed, one can see, g is a lightlike translation in $Mink(y)$. Considering the splitting $Y_P \times \Pi$, it is easy to see that an element g in $H(3)$ is a lightlike transformation on $Mink(x_0)$ if and only if it is a non-trivial element in \mathcal{L} . This implies that an element $g \in H(3)$ is a lightlike transformation on a (hence any) Minkowski patch $Mink(y)$ ($y \in \phi$) if and only if g is a non-trivial element of the center \mathcal{L} . Thus, we may talk about a lightlike transformation without mentioning a Minkowski patch.

Assume that $g \in H(3)$ is a spacelike transformation on a Minkowski patch $Mink(x_0)$ (for some $x_0 \in \phi$). Again, considering the splitting $Y_P \times \Pi$, it can be easily seen that g is an element in $\Pi \setminus \mathcal{L}$. Observe that for a non-trivial element $g \in \Pi$ the g -invariant subsets of $\mathcal{F}_\phi(x_0) = L(x_0) \setminus \phi$ are included

in the vertex-less photons in the lightcone $L(x_0)$. In other words, elements in Π preserve no spacelike curve in $\mathcal{F}_\phi(x_0)$.

Furthermore, assume that $g \in H(3)$ is a parabolic transformation on a Minkowski patch $Mink(x)$ (for some $x \in \phi$). Denote by γ the lightlike geodesic in $Mink(x)$ which is fixed pointwisely by g . By continuity, g fixes the limit point of γ in the lightcone $L(x)$. Hence, the limit point of γ is contained in ϕ , since g fixes no point in the leaf $\mathcal{F}_\phi(x) = L(x) \setminus \phi$. This shows that γ is contained in a leaf of \mathcal{F}_ϕ . One can see that the 1-parameter subgroup of $H(3)$ generated by g acts on the leaf $\mathcal{F}_\phi(x)$ freely and every orbit is a spacelike curve (i.e. of signature $(0, 1)$).

Lemma 6.4. *Let g be a non-trivial element in $H(3)$. Then there exists a unique point $x_0 \in \phi$ such that g is a spacelike transformation on $Mink(x_0)$ if and only if for all $x \in \phi \setminus \{x_0\}$, g is a parabolic transformation on $Mink(x)$.*

Proof. Assume that g is a spacelike transformation on $Mink(x_0)$. Then the set of points fixed by g in $\mathbb{E}in_\phi^{1,2}$ is a unique lightlike geodesic $\gamma \subset \mathcal{F}_\phi(x_0) = L(x_0) \setminus \phi$. The union of γ with x_0 is a photon ψ . For an arbitrary point $x \in \phi \setminus \{x_0\}$, γ is a lightlike geodesic in the Minkowski patch $Mink(x)$. Hence, g is a parabolic transformation of $Mink(x)$ for all $x \in \phi \setminus \{x_0\}$.

Conversely, assume that g is a parabolic transformation in a Minkowski patch $Mink(x)$ (for some $x \in \phi$). Then the set of points fixed by g in $\mathbb{E}in_\phi^{1,2}$ is a unique lightlike geodesic $\gamma \subset Mink(x)$. Let x_0 denotes the limit point of γ in the lightcone $L(x)$. Obviously, $x_0 \in \phi$. Observe that γ is a vertex-less photon in $\mathcal{F}_\phi(x_0) = L(x_0) \setminus \phi$. Hence, g is a spacelike transformation on $Mink(x_0)$. This completes the proof. \square

Corollary 6.5. *Let $g \in H(3)$ be a non-trivial element. Then, either $g \in \mathcal{L}$ (hence, for all $x \in \phi$, it is a lightlike transformation on the Minkowski patch $Mink(x)$), or there exists a unique point $x_0 \in \phi$ such that g is a spacelike transformation on $Mink(x_0)$ and for all $x \in \phi \setminus \{x_0\}$ it is a parabolic transformation on $Mink(x)$.*

Proof. Fix a point $x_0 \in \phi$ and consider the splitting $H(3) \simeq Y_P \times \Pi$. By some computation, one can see, the composition of a non-trivial element in Y_P and an arbitrary element in Π is a parabolic transformation on $Mink(x_0)$. Now, the corollary follows from Lemma 6.4. \square

Proposition 6.6. *Let ϕ be a photon in $\mathbb{E}in^{1,2}$ and G a connected Lie subgroup of $\text{Conf}(\mathbb{E}in_\phi^{1,2})$ acting transitively on ϕ . Then, for $x \in \phi$ and $p \in \mathcal{F}_\phi(x)$ the orbit induced by G at p is k -dimensional if and only if the orbit induced by $\text{Stab}_G(x)$ at p is $(k - 1)$ -dimensional.*

Proof. In the one hand, since G acts on ϕ transitively, $\dim G = \dim \text{Stab}_G(x) + 1$, for all $x \in \phi$. On the other hand, for $p \in \mathcal{F}_\phi([x])$, $\text{Stab}_G(p)$ is a subgroup of $\text{Stab}_G(x)$, since the action of G preserves the foliation \mathcal{F}_ϕ . More precisely, $\text{Stab}_G(p) = \text{Stab}_{\text{Stab}_G(x)}(p)$. Hence,

$$\begin{aligned} \dim G(p) &= \dim G - \dim \text{Stab}_G(p) \\ &= \dim \text{Stab}_G(x) + 1 - \dim \text{Stab}_{\text{Stab}_G(x)}(p) \\ &= \dim (\text{Stab}_G(x))(p) + 1. \end{aligned}$$

\square

Proof of Proposition 6.2. Assume the contrary, which is: g is an element in the connected component of $G \cap H(3)$ and $g \notin \mathcal{L}$. There exists a point $x_0 \in \phi$ such that $p \in \mathcal{F}_\phi(x_0)$. By Corollary 6.5, g is either a spacelike or parabolic transformation on the Minkowski patch $Mink(x_0)$. Denote by O_p , g^t and γ , the orbit induced by $Stab_G(x_0)$ at p , the 1-parameter subgroup of $G \cap H(3)$ generated by g , and the unique lightlike geodesic in $\mathbb{E}in_\phi^{1,2}$ pointwisely fixed by g , respectively. By Proposition 6.6, the orbit O_p is 1-dimensional. Also, g^t preserves γ , since it is an abelian group.

- If g is a spacelike transformation on $Mink(x_0)$, then $\gamma \subset \mathcal{F}_\phi(x_0) = L(x_0) \setminus \phi$. For an arbitrary point $x \in \phi \setminus \{x_0\}$, there exists $h \in G$ such that $hx_0 = x$, since G acts on ϕ transitively. By Lemma 6.4 g is a parabolic transformation on $Mink(x)$. Hence, g^t acts on $\mathcal{F}_\phi(x)$ freely. Thus, the orbit $C = g^t(hp)$ is an open subset of the orbit induced by $Stab_G(x)$ at hp and it is spacelike (i.e. of signature $(0, 1)$). Hence, the orbit induced by $Stab_G(x)$ at hp is spacelike. Obviously, $h^{-1}(C) \subset \mathcal{F}_\phi(x_0)$ is an open subset of O_p . This is a contradiction, since g preserves no spacelike curve in $\mathcal{F}_\phi(x_0)$.
- If g is a parabolic transformation of $Mink(x_0)$, then the orbit $C = g^t(p)$ is a spacelike (i.e. of signature $(0, 1)$) curve in $\mathcal{F}_\phi(x_0)$. By Lemma 6.4, there exists a unique point $x \in \phi \setminus \{x_0\}$ such that g is a spacelike transformation on the Minkowski patch $Mink(x)$. There exists $h \in G$ such that $hx_0 = x$. The same argument as the previous case shows that $h(C)$ is an open subset of the orbit induced by $Stab_G(x)$ at hp . This contradicts the fact that g preserves no spacelike curve in $\mathcal{F}_\phi(x)$.

This completes the proof. \square

Definition 6.7. A non-trivial element $g \in \mathcal{K}$ is called a hyperbolic-homothety (abbreviation *HH*) - transformation if the set of its fixed points in $\mathbb{E}in_\phi^{1,2}$ is a photon.

The non-trivial elements in the following 1-dimensional subgroup of \mathcal{K} are the obvious examples of *HH*-transformations

$$\mathcal{H} = \left\{ \left(e^{2t}, \begin{bmatrix} \varepsilon e^t & 0 \\ 0 & \varepsilon e^{-t} \end{bmatrix}, 0_{2 \times 2} \right) : t \in \mathbb{R}, \varepsilon = \pm 1 \right\}.$$

Let $g \in \mathcal{K}$ be an *HH*-transformation and $\psi \subset \mathbb{E}in_\phi^{1,2}$ be the unique photon fixed pointwisely by g . The photon ψ intersects every leaf of \mathcal{F}_ϕ in a unique point. More precisely, for an arbitrary point $x \in \phi$, the intersection of ψ with the Minkowski patch $Mink(x)$ is a lightlike geodesic γ . The limit point of γ is contained in $\mathcal{F}_\phi(x) = L(x) \setminus \phi$. Hence, γ intersects every affine degenerate plane in $Mink(x)$ with limit point in ϕ . Assume g^t is the 1-parameter subgroup generated by $g \in \mathcal{K}_\circ$. Observe that g^t preserves ψ , since it is abelian. On the other hand g^t preserves the leaves of the foliation \mathcal{F}_ϕ , hence g^t fixes γ pointwisely. Henceforth, γ is the unique photon fixed pointwisely by every element in g^t .

Lemma 6.8. Let g be a non-trivial element in \mathcal{K} . Then either $g \in H(3)$ or it is an *HH*-transformation.

Proof. Fix a point $x_0 \in \phi$ and consider the splitting Eq. 6.2. Every element in \mathcal{K} with $t = 0$ belongs to $H(3)$. On the other hand, if $t \neq 0$, it is not hard to see that the element is an *HH*-transformation. \square

Lemma 6.9. Let $x_0 \in \phi$ be an arbitrary point. Then every *HH*-transformation in \mathcal{K} preserves a unique \mathcal{L} -invariant affine Lorentzian 2-plane in $Mink(x_0)$.

Proof. Let $g \in \mathcal{K}_\circ$ be an HH -transformation. Denote by ψ the unique photon in $\mathbb{E}\text{in}_\phi^{1,2}$ fixed pointwisely by g . Assume that p is an arbitrary point of the lightlike geodesic $\gamma = \text{Mink}(x_0) \cap \psi$. The orbit induced by the 1-parameter subgroup \mathcal{L} at p is a lightlike geodesic η with limit point in ϕ . The lightlike geodesics γ and η generate a unique affine Lorentzian 2-plan $T \subset \text{Mink}(x_0)$. Obviously, T is invariant by \mathcal{L} . \square

Observe that, a 1-parameter subgroup g^t generated by an HH -transformation $g \in \mathcal{K}_\circ$ preserves the unique g -invariant affine Lorentzian 2-plane in $\text{Mink}(x_0)$, since g^t is abelian.

Notation 6.10. We denote by P the totally isotropic plane in $\mathbb{R}^{2,3}$ corresponding to the photon ϕ . Also, for a Lie subgroup $G \subset \text{Conf}(\mathbb{E}\text{in}_\phi^{1,2})$, we denote by K^G and K_\circ^G the kernel $\ker \pi|_G$ and its identity component, respectively.

Lemma 6.11. Let $x_0 \in \phi$ and G be a 1-parameter subgroup of $\text{Conf}(\mathbb{E}\text{in}_\phi^{1,2})$ with $\dim \pi(G) = 1$ which acts on ϕ transitively. Then the kernel $K^G = \ker \pi|_G$ contains neither a parabolic nor a spacelike transformation of $\text{Mink}(x_0)$.

Proof. Assume the contrary that $g \in K^G$ is a parabolic or spacelike transformation on $\text{Mink}(x_0)$. The element g fixes pointwisely a unique lightlike geodesic $\gamma \subset \mathbb{E}\text{in}_\phi^{1,2}$ contained in a leaf of \mathcal{F}_ϕ . Note that, G preserves γ , since it is abelian. But, this contradicts the fact that every G -orbit in $\mathbb{E}\text{in}_\phi^{1,2}$ intersects every leaf of \mathcal{F}_ϕ . This completes the proof. \square

Lemma 6.12. Let G be a connected Lie subgroup of $\text{Conf}(\mathbb{E}\text{in}_\phi^{1,2})$ with $\dim \pi(G) = 1$ which acts on ϕ transitively and admits a 2-dimensional orbit in $\mathbb{E}\text{in}_\phi^{1,2}$. Then, there exists a 1-parameter subgroup $L \subset G$ transversal to K_\circ^G such that either the kernel $K^L = \ker \pi|_L$ is trivial (hence $L \simeq SO(2)$), or every non-trivial element of K^L is an HH -transformation.

Proof. Let $L \subset G$ be an arbitrary 1-parameter subgroup transversal to K_\circ^G . Obviously, L acts on ϕ transitively. If K^L is trivial, then lemma follows evidently. Otherwise, L is isomorphic to \mathbb{R} and $K^L \simeq \mathbb{Z}$. Assume that $g \in K^L$ is a generator. By Lemma 6.11, g is either an HH -transformation or a lightlike transformation. In the first, lemma follows easily. If g is a lightlike transformation, by Proposition 6.2, there are two possibilities:

- \mathcal{L} is a subgroup of K_\circ^G : There exists a 1-parameter subgroup $L' \subset G$ transversal to K_\circ^G such that it intersects \mathcal{L} only in the identity element, since $\mathcal{L} \subset G$.
- \mathcal{L} is not a subgroup of K_\circ^G : Observe that in this case, K_\circ^G is a 1-parameter subgroup consisting of the identity element and HH -transformations. Also, G is a 2-dimensional connected Lie group and so, by [38, p.p. 212] it is isomorphic to the 2-torus \mathbb{T}^2 , $\mathbb{R} \times SO(2)$, \mathbb{R}^2 , or $\mathbb{A}\text{ff}_\circ(1, \mathbb{R})$. Let $h \in K_\circ^G$ be an arbitrary non-trivial element. Then hg is an HH -transformation by Lemma 6.8. Since the exponential map $\exp : \text{Lie}(G) \rightarrow G$ is surjective, there exists a 1-parameter subgroup L' through hg . In the one hand, $hg \in K^{L'}$, so all the non-trivial elements in $K^{L'}$ are HH -transformations. On the other hand, L' is transversal to K_\circ^G , since $hg \notin K_\circ^G$.

\square

Let P be the totally isotropic 2-plane in $\mathbb{R}^{2,3}$ corresponding to the photon $\phi \subset \mathbb{E}\text{in}_\phi^{1,2}$, and Q be a subspace of $\mathbb{R}^{2,3}$ supplementary to P^\perp . There is a canonical identification between Q and the dual space

P^* . Let $\langle \cdot, \cdot \rangle$ denote the bilinear form on $\mathbb{R}^{2,3}$. The map ϑ sending a vector $v \in Q$ to the functional $\langle v, \cdot \rangle : P \rightarrow \mathbb{R}$ is linear. Also, ϑ is injective: $\vartheta(v) = \vartheta(w)$ implies that $\langle v - w, \cdot \rangle \equiv 0$, since $\langle \cdot, \cdot \rangle$ is non-degenerate, we get $v - w = 0$. Hence, ϑ is an isomorphism.

Furthermore, let $H \subset SO_o(2, 3)$ preserves P and Q be an H -invariant complement for P^\perp in $\mathbb{R}^{2,3}$. In the one hand, the action of H on P induces a representation (not unique) from H to $GL(P^*)$ by duality. On the other hand, the action of H on Q induces a unique representation from H to $GL(P^*)$ on which the isomorphism ϑ is H -equivariant. It is not hard to see that the representation induced via P is conjugate to the one induced via Q .

Let $\psi \subset \mathbb{E}in_\phi^{1,2}$ be a photon and denote by P_ψ its corresponding totally isotropic 2-plane in $\mathbb{R}^{2,3}$. The linear subspace $P \cup P_\psi \leq \mathbb{R}^{2,3}$ is of signature $(2, 2)$. Hence, the union of ϕ and ψ determines a unique Einstein hypersphere $\mathbb{E}in^{1,1} \subset \mathbb{E}in^{1,2}$. Let $x_0 \in \phi$ be an arbitrary point. Then by Remark 1.66, the intersection of $\mathbb{E}in^{1,1}$ with the Minkowski patch $Mink(x_0)$ is a Lorentzian affine 2-plane T . The intersection of $\mathbb{E}in^{1,1}$ with the lightcone $L(x_0)$ is the lightcone of x_0 in $\mathbb{E}in^{1,1}$ consisting of ϕ and another photon ξ which contain the limit points of lightlike geodesics in T . Indeed, $\phi \cup \xi$ is the set of the limit points of lightlike geodesics of every Lorentzian affine 2-plane in $Mink(x_0)$ parallel to T . But, T is the unique such affine plane which contains the lightlike geodesic $\psi \cap Mink(x_0)$.

Proof of Theorem 6.1. First, consider the action of G on ϕ . If G admits a fixed point $[x] \in \phi$, then x is the desired fixed point.

Now, assume that G acts on ϕ transitively. Obviously, $\pi(G)$ is either $PSL(2, \mathbb{R})$ or it is conjugate to $SO(2) \simeq Y_E$. We show that in the both cases, G preserves a line in $\mathbb{R}^{2,3}$.

Case I: $\pi(G) = PSL(2, \mathbb{R})$. Denote by \mathfrak{g} , \mathfrak{k} , and \mathfrak{h} the Lie algebras correspond to G , K^G and $\pi(G)$, respectively. The following short sequence of Lie algebras and Lie algebra morphisms is exact.

$$1 \longrightarrow \mathfrak{k} \hookrightarrow \mathfrak{g} \xrightarrow{d\pi} \mathfrak{h} \longrightarrow 1.$$

One can see $\mathfrak{h} \simeq \mathfrak{sl}(2, \mathbb{R})$ as the Levi factor of \mathfrak{g} , since \mathfrak{k} is the radical solvable ideal of \mathfrak{g} . Henceforth, $PSL(2, \mathbb{R})$ is a subgroup of G , up to finite cover and $G = K^G \cdot PSL(2, \mathbb{R})$. It is clear that G acts on the totally isotropic plane P irreducibly and preserves the orthogonal space P^\perp . By Proposition 6.2, the connected component of the intersection of G with Heisenberg group $H(3)$ is either trivial or it is $\mathcal{L} \simeq \mathbb{R}$. Therefore, K_o^G is either trivial or it is isomorphic to \mathbb{R} or Aff . The subgroup $PSL(2, \mathbb{R}) \subset G$ (up to finite cover) acts on K^G by conjugacy, since K^G is a normal subgroup of G . The simplicity of $PSL(2, \mathbb{R})$, and Lemma 1.32 imply that this action is trivial. Moreover, using the simplicity of $PSL(2, \mathbb{R})$ again, its action on $\mathbb{R}^{2,3}$ splits as the sum of irreducible actions (cf. [28, p.p. 28]).

Suppose that $\mathbb{R}^{2,3} = P \oplus \ell \oplus Q$ is a $PSL(2, \mathbb{R})$ -invariant splitting, where $\ell \leq P^\perp$ is a line supplementary to P in P^\perp and Q is a 2-plane supplementary to P^\perp in $\mathbb{R}^{2,3}$. The canonical identification between Q and P^* shows that $PSL(2, \mathbb{R})$ acts on Q irreducibly. It is not hard to see that ℓ is the only $PSL(2, \mathbb{R})$ -invariant line in $\mathbb{R}^{2,3}$. Since the conjugacy action of $PSL(2, \mathbb{R})$ on K^G is trivial, every element of $PSL(2, \mathbb{R})$ commutes with all the elements of K^G . This implies that K^G preserves ℓ as well. Thus, any element of G preserves ℓ . Henceforth $\mathbb{P}(\ell) \in \mathbb{RP}^4$ is the desired fixed point.

Case II: $\pi(G) = SO(2)$ up to conjugacy. In this case $\dim K^G \geq 1$, since $\dim G \geq 2$. Proposition 6.2 implies that K_o^G is isomorphic to either \mathbb{R} or Aff . We split this case to two subcases: G contains a 1-dimensional compact subgroup (a copy of $SO(2)$), and there is no 1-dimensional compact subgroup in G :

- *G contains a 1-dimensional compact subgroup.* In this case we have $G = K_{\circ}^G \cdot SO(2)$. The group $SO(2) \subset G$ acts on K_{\circ}^G by conjugacy, since K_{\circ}^G is a normal subgroup of G . This action is trivial, since both $\mathbb{R} \simeq \text{Aut}_{\circ}(\mathbb{R})$ and $\mathbb{A}ff \simeq \text{Aut}_{\circ}(\mathbb{A}ff)$ contain no 1-dimensional compact Lie subgroup (see Lemma 1.32). Furthermore, the action of $SO(2)$ on $\mathbb{R}^{2,3}$ splits as the sum of irreducible actions, since it is compact (cf, [25, Proposition 4.36]). Using the same symbols as we used for the previous case, suppose that $\mathbb{R}^{2,3} = P \oplus \ell \oplus Q$ is a $SO(2)$ -invariant splitting. It is easy to see that $SO(2)$ acts irreducibly on Q . Also, ℓ is the only $SO(2)$ -invariant line in $\mathbb{R}^{2,3}$. Since the conjugacy action of $SO(2)$ on K_{\circ}^G is trivial, every element of $SO(2)$ commutes with all the elements of K_{\circ}^G . This implies that K_{\circ}^G preserves ℓ as well. Consequently, all the elements in G preserve ℓ . Henceforth $\mathbb{P}(\ell) \in \mathbb{RP}^4$ is the desired fixed point.
- *G contains no 1-dimensional compact subgroup.* In this case, every 1-parameter subgroup transversal to K_{\circ}^G is isomorphic to \mathbb{R} . By Lemma 6.12, there exists a 1-parameter subgroup $L \subset G$ transversal to K_{\circ}^G such that the kernel $K^L = \ker \pi|_L$ consists of the identity element and HH -transformations. Let $g \in K^L$ be a non-trivial element. Then g fixes a unique photon $\psi \in \mathbb{E}in_{\phi}^{1,2}$ pointwisely. Observe that L preserves ψ since it is abelian. Therefore, L preserves the Einstein hypersphere $\mathbb{E}in^{1,1} \subset \mathbb{E}in^{1,2}$ containing ϕ and ψ . Also, by Lemma 6.9, g preserves a unique \mathcal{L} -invariant affine Lorentzian 2-plane T_g in the Minkowski patch $Mink(x_0)$. In fact, T_g coincides with the intersection of $\mathbb{E}in^{1,1}$ with $Mink(x_0)$. It is suffice to show that K_{\circ}^G preserves T_g . If $K_{\circ}^G = \mathcal{L}$, then obviously $G = K_{\circ}^G \cdot L$ preserves the Einstein hypersphere $\mathbb{E}in^{1,1}$, since T_g is \mathcal{L} -invariant. If K_{\circ}^G contains an HH -transformation h , then by Lemma 6.9, h preserves a unique \mathcal{L} -invariant affine Lorentzian 2-plane T_h . In fact, $T_h = T_g$, since K_{\circ}^G is normal in K^G . Now, it is easy to see that T_g is K_{\circ}^G -invariant. Hence, once more, $G = K_{\circ}^G \cdot L$ preserves $\mathbb{E}in^{1,1}$. Therefore, G preserves the spacelike direction in $\mathbb{R}^{2,3}$ corresponding to $\mathbb{E}in^{1,1}$, and so, it admits a fixed point in the projective space \mathbb{RP}^4 .

□

Chapter 7

Proof of the main theorem

In this chapter we prove Theorem 0.6. This theorem together with Theorem 0.5 complete the classification of the cohomogeneity one action on the three-dimensional Einstein universe $\mathbb{E}\text{in}^{1,2}$.

The following proposition is one element of the proof of Theorem 0.6.

Proposition 7.1. *Let V be a linear subspace of $\mathbb{R}^{2,3}$ of signature $(0, 2)$, and G a connected Lie subgroup of $\text{Stab}_{SO_o(2,3)}(V)$ which acts on $\mathbb{E}\text{in}^{1,2}$ with cohomogeneity one. Then G preserves a 1-dimensional linear subspace of $\mathbb{R}^{2,3}$.*

Proof. Note that $\mathfrak{C} = \mathbb{P}(V^\perp) \cap \mathbb{E}\text{in}^{1,2}$ is a timelike circle (Definition 1.69) and the complement in $\mathbb{E}\text{in}^{1,2} \setminus \mathfrak{C}$ is conformally equivalent (up to double cover) to $\text{AdS}^{1,1} \times \mathbb{S}^1$ (Lemma 1.70). Moreover, by the second part of Lemma 1.70 the stabilizer of \mathfrak{C} is conjugate to $SO_o(2, 1) \times \mathbb{S}^1$, and clearly acts on \mathfrak{C} and $\text{AdS}^{1,1} \times \mathbb{S}^1 \subset \widehat{\mathbb{E}\text{in}^{1,2}}$ transitively. Therefore, G is a proper subgroup of $SO_o(2, 1) \times SO(2)$.

The following projections are group morphisms

$$\begin{aligned} P_1 : SO_o(1, 2) \times SO(2) &\longrightarrow SO_o(2, 1), & (g, h) &\mapsto g \\ P_2 : SO_o(1, 2) \times SO(2) &\longrightarrow SO(2). & (g, h) &\mapsto h. \end{aligned}$$

If $P_1(G) = \{Id\}$ then $\dim G \leq 1$, which is a contradiction, since G admits a 2-dimensional orbit in $\mathbb{E}\text{in}^{1,2}$. On the other hand, if $P_2(G) = \{Id\}$, then G acts on V trivially.

Now, assume that $P_1(G), P_2(G) \neq \{Id\}$. Observe that G is a subgroup of $P_1(G) \times P_2(G)$. It follows from Remark 1.39 and Corollary 1.27 that, G is a subgroup of $Y_E \times SO(2)$ or $\text{Aff} \times SO(2)$, up to conjugacy. We show that both these groups preserve a line in $\mathbb{R}^{2,3}$. Then the result follows easily.

The 1-parameter elliptic subgroup Y_E preserves a unique 1-dimensional spacelike linear subspace $\ell \leq V^\perp$. Since the action of $SO(2)$ -factor on V^\perp is trivial, ℓ is invariant by $Y_E \times SO(2)$.

The affine group Aff preserves a unique lightlike line ℓ in V^\perp . Since the action of $SO(2)$ -factor is trivial on V^\perp , ℓ is invariant by $\text{Aff} \times SO(2)$. □

Proof of Theorem 0.6.

Suppose that V is a G -invariant non-trivial proper linear subspace of $\mathbb{R}^{2,3}$. Denote by $\text{sgn}(V)$ the signature of the restriction of the metric from $\mathbb{R}^{2,3}$ on V . We consider all the possible signatures for V .

If $\dim V = 1$, then obviously, G fixes a point in the projective space $\mathbb{R}\mathbb{P}^4$, namely $\mathbb{P}(V)$. Assume that V is 2-dimensional.

- (I) If $\text{sgn}(V) = (1, 1)$. Then, V contains exactly two distinct lightlike lines. Hence, the intersection of $\mathbb{P}(V)$ with $\mathbb{E}\text{in}^{1,2}$ consists of two points. Since G is connected, it fixes both the points.
- (II) If $\text{sgn}(V) = (2, 0)$. Since G preserves the orthogonal space V^\perp ($\text{sgn}(V^\perp) = (0, 3)$), it is a subgroup of $SO(2) \times SO(3)$ up to conjugacy. By Corollary 2.5, G fixes a point in $\mathbb{R}\mathbb{P}^4$.
- (III) If $\text{sgn}(V) = (0, 1, 1)$ or $(1, 0, 1)$. Then, V contains a unique lightlike line ℓ . Since the action of G on $\mathbb{R}^{2,3}$ is isometric, it preserves ℓ . Hence, G fixes $\mathbb{P}(\ell) \in \mathbb{E}\text{in}^{1,2}$.
- (IV) If $\text{sgn}(V) = (0, 2)$. By Lemma 1.70, G is a proper subgroup of $SO_\circ(2, 1) \times SO(2)$ up to conjugacy. Proposition 7.1 shows that G fixes a point in $\mathbb{R}\mathbb{P}^4$.
- (V) If $\text{sgn}(V) = (0, 0, 2)$. By Theorem 6.1, G fixes a point in $\mathbb{R}\mathbb{P}^4$.

Now, suppose that $\dim V > 2$. Since G preserves V^\perp and $\dim V^\perp \leq 2$, G fixes a point in $\mathbb{R}\mathbb{P}^4$. This completes the proof. \square

We now end this concluding chapter by the following remark.

Remark 7.2. *According to Section 4.2, Section 5.3, and Chapter 3, if G is a connected Lie group which acts on Einstein universe $\mathbb{E}\text{in}^{1,2}$ non-properly and with cohomogeneity one, then there are two distinct orbits $G(p)$ and $G(q)$ such that $G(p)$ accumulates to $G(q)$, i.e., $G(q)$ is in the closure of $G(p)$ in $\mathbb{E}\text{in}^{1,2}$. Hence, every G -invariant open neighborhood in $\mathbb{E}\text{in}^{1,2}$ around $G(q)$ contains $G(p)$ as well. This implies that the orbit space $\mathbb{E}\text{in}^{1,2}/G$ is not Hausdorff. Therefore, the properness of a cohomogeneity one action on $\mathbb{E}\text{in}^{1,2}$ is equivalent to the Hausdorff condition on the orbit space.*

Appendix A

Connected subgroups of $\text{Conf}(\mathbb{E}^{1,2})$ of dimension ≥ 2

In this appendix, we prove Theorem (4.6) by characterizing the Lie subalgebras of $\text{Lie}(\text{Conf}(\mathbb{E}^{1,2}))$ with $\dim \geq 2$.

Recall from Section (1.5.2), the group of conformal transformations on 3-dimensional Minkowski space $\mathbb{E}^{1,2}$ is isomorphic to the semi-direct product $G = (\mathbb{R}^* \times O(1, 2)) \ltimes_{\Theta} \mathbb{R}^{1,2}$, where Θ denotes the natural action of $\mathbb{R}^* \times O(1, 2)$ on $\mathbb{R}^{1,2}$. Hence, the Lie algebra $\text{Lie}(\text{Conf}(\mathbb{E}^{1,2}))$ is isomorphic to the semi-direct sum $\mathfrak{g} = (\mathbb{R} \oplus \mathfrak{so}(1, 2)) \oplus_{\theta} \mathbb{R}^{1,2}$, where θ is the natural representation of $\mathbb{R} \oplus \mathfrak{so}(1, 2)$ in $\mathfrak{gl}(\mathbb{R}^{1,2})$ corresponding to Θ . For elements $a \in \mathbb{R}$, $V \in \mathfrak{so}(1, 2)$ and $w \in \mathbb{R}^{1,2}$, we denote the corresponding element in \mathfrak{g} simply by $a + V + w$ when there is no ambiguity. Also, we denote by $\mathbb{R}(a + V + w)$ the linear subspace of \mathfrak{g} generated by $a + V + w$. The Lie bracket on \mathfrak{g} is

$$[a + V + v, b + W + w] = [V, W] + V(w) + aw - W(v) - bv.$$

The adjoint action of G on its Lie algebra \mathfrak{g} is as following. For an arbitrary element $(r, A, v) \in G$, we have

$$\text{Ad}_{(r,A,v)} : \mathfrak{g} \rightarrow \mathfrak{g}, \quad a + W + w \mapsto a + AWA^{-1} + rA(w) - av - AWA^{-1}(v).$$

Consider the following natural projections which are evidently Lie group morphisms

$$\begin{aligned} P_l &: (\mathbb{R}^* \times O(1, 2)) \ltimes_{\Theta} \mathbb{R}^{1,2} \rightarrow \mathbb{R}^* \times O(1, 2) \\ P_h &: (\mathbb{R}^* \times O(1, 2)) \ltimes_{\Theta} \mathbb{R}^{1,2} \rightarrow \mathbb{R}^* \\ P_{li} &: (\mathbb{R}^* \times O(1, 2)) \ltimes_{\Theta} \mathbb{R}^{1,2} \rightarrow O(1, 2). \end{aligned}$$

For a Lie subgroup $H \subset G$ we denote the identity component of the kernel $\ker P_l|_H$ by $T(H)$ and call it the translation part of H . Also, we call the image of H under the projections P_l , P_h , and P_{li} , the linear projection, the homothety projection, and the linear isometry projection of H , respectively. Obviously $T(H)$ is a linear subspace of $\mathbb{R}^{1,2}$ and it is normal in H . Thus H acts on it by conjugacy. This implies that the translation part $T(H)$ is invariant by the action of the linear projection $P_l(H)$ and the linear isometry projection $P_{li}(H)$.

By differentiating the morphisms P_l , P_h , and P_{li} at the identity element of G we obtain three Lie algebra morphisms

$$\begin{aligned} p_l &= dP_l : (\mathbb{R} \oplus \mathfrak{so}(1, 2)) \oplus_{\theta} \mathbb{R}^{1,2} \rightarrow \mathbb{R} \oplus \mathfrak{so}(1, 2) \\ p_h &= dP_h : (\mathbb{R} \oplus \mathfrak{so}(1, 2)) \oplus_{\theta} \mathbb{R}^{1,2} \rightarrow \mathbb{R} \\ p_{li} &= dP_{li} : (\mathbb{R} \oplus \mathfrak{so}(1, 2)) \oplus_{\theta} \mathbb{R}^{1,2} \rightarrow \mathfrak{so}(1, 2). \end{aligned}$$

which are the corresponding projections in Lie algebra level. For a Lie subalgebra $\mathfrak{h} \leq \mathfrak{g}$, we use the same terminology for $T(\mathfrak{h}) := \ker p_l|_{\mathfrak{h}}$, $p_l(\mathfrak{h})$, $p_h(\mathfrak{h})$, and $p_{li}(\mathfrak{h})$ as we introduced in Lie group level. Observe that $T(\mathfrak{h})$ is invariant by the action of $p_l(\mathfrak{h})$ and $p_{li}(\mathfrak{h})$.

We will characterize the Lie subalgebras of \mathfrak{g} using the adjoint action Ad of G on \mathfrak{g} . The procedure is that, for a Lie subalgebra $\mathfrak{h} \leq \mathfrak{g}$, we take a basis $\{w_i\}$ ($0 \leq i \leq 3$) for $T(\mathfrak{h})$, and expand it to a basis $\{a_j + V_j + v_j, w_i\}$ ($0 \leq j \leq 4$) for \mathfrak{h} . The main problem is that, in general, the subspace $\mathfrak{n} \leq \mathfrak{h}$ generated by $\{a_j + V_j + v_j\}$ is not a Lie algebra. This problem arise from v_j vectors. Restricting Ad to $(\mathbb{R}_+^* \times \{Id\}) \times \mathbb{R}^{1,2}$, it preserves the translation part $T(\mathfrak{h})$. Thus, first we try to *linearize* \mathfrak{n} by $Ad_{((\mathbb{R}_+^* \times \{Id\}) \times \mathbb{R}^{1,2})}$ as more as possible, meaning that: we try to omit the vectors v_j by $Ad_{((\mathbb{R}_+^* \times \{Id\}) \times \mathbb{R}^{1,2})}$ as more as possible, and if it is not possible, we replace v_j with a more suitable vector. Actually, for vectors $a_i + V_i + w_i \in \mathfrak{h}$, we apply $Ad_{(r, Id, x)}$ and obtain

$$Ad_{(r, Id, x)}(a_i + V_i + w_i) = a_i + V_i + rw_i - a_i x - V_i(x).$$

Then, we try to solve the linear systems

$$rw_i - a_i x - V_i(x) \in T(\mathfrak{h}), \quad 0 \leq i \leq \dim p_l(\mathfrak{h}), \quad a_i + V_i + w_i \in \mathfrak{h},$$

simultaneously for x . However, in general, it is not possible to find such solution x , but we may simplify the vectors $rw_i - a_i x - V_i(x) \in \mathbb{R}^{1,2}$ as more as it is possible.

Furthermore, we may use the elements in $O(1, 2)$ which preserve the linear isometry part $p_{li}(\mathfrak{h})$ to obtain a simpler subalgebra.

Consider the scalar product space $\mathbb{R}^{2,3}$. Here it will be convenient to use the scalar product

$$\langle v|w \rangle := -\frac{1}{2}v_1w_2 - \frac{1}{2}v_2w_1 - v_3w_3 + v_4w_4 + v_5w_5$$

Let $\langle -, - \rangle$ denotes the bilinear form defined by $Q = -I_1 \oplus I_n$ on $\mathbb{R}^{1,2}$. Consider the representation of $\text{Conf}(\mathbb{E}^{1,2})$ in $O(2, 3)$

$$(r, A, v) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ \langle v, v \rangle & 1 & 2v^\dagger Q \\ v & 0 & I_3 \end{bmatrix} \begin{bmatrix} r^{-1} & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & A \end{bmatrix}.$$

Actually, this representation maps $\text{Conf}(\mathbb{E}^{1,2})$ surjectively to the stabilizer of the point $p \in \mathbb{E}^{1,2}$ with homogeneous coordinate $[0 : 1 : 0 : 0 : 0]$. Observe that, conjugation by the element $\mathcal{R} := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus I_3 \in O(2, 3)$ leaves the linear subgroup $\mathbb{R}^* \times O(1, 2) \subset \text{Conf}(\mathbb{E}^{1,2})$ invariant and changes the roles of r and r^{-1} . We will use this element in sequel.

Let $\{e_1, e_2, e_3\}$ be an orthonormal basis of $\mathbb{R}^{1,2}$ and e_1 be timelike. The set of following matrices is a basis for $\mathfrak{so}(1, 2)$ as a vector space

$$\mathcal{Y}_E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \mathcal{Y}_H = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{Y}_P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad (\text{A.1})$$

where,

$$\begin{aligned} \exp(\mathbb{R}\mathcal{Y}_E) = Y_E &= \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix} : t \in \mathbb{R} \right\}, \\ \exp(\mathbb{R}\mathcal{Y}_H) = Y_H &= \left\{ \begin{bmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{bmatrix} : t \in \mathbb{R} \right\}, \\ \exp(\mathbb{R}\mathcal{Y}_P) = Y_P &= \left\{ \begin{bmatrix} 1 + \frac{1}{2}t^2 & -\frac{1}{2}t^2 & t \\ \frac{1}{2}t^2 & 1 - \frac{1}{2}t^2 & t \\ t & -t & 1 \end{bmatrix} : t \in \mathbb{R} \right\}. \end{aligned}$$

The group $SO_o(1, 2) \simeq \text{PSL}(2, \mathbb{R})$ has exactly three 1-parameter subgroups up to conjugacy, namely Y_E , Y_H , and Y_P . Also, up to conjugacy, $SO_o(1, 2)$ has only one 2-dimensional Lie subgroup which is $\exp(\mathbb{R}\mathcal{Y}_H + \mathbb{R}\mathcal{Y}_P) = \text{Aff}_o(1, \mathbb{R})$. In Lie algebra level we have

$$[\mathcal{Y}_E, \mathcal{Y}_H] = \mathcal{Y}_E - \mathcal{Y}_P, \quad [\mathcal{Y}_E, \mathcal{Y}_P] = \mathcal{Y}_H, \quad [\mathcal{Y}_H, \mathcal{Y}_P] = \mathcal{Y}_P.$$

We denote the affine subalgebra of $\mathfrak{so}(1, 2)$ by aff .

The elliptic element \mathcal{Y}_E has only one real eigenvalue, namely 0. The hyperbolic element \mathcal{Y}_H has three distinct real eigenvalues $\{-1, 0, 1\}$. The parabolic element \mathcal{Y}_P is nilpotent, and so, 0 is the only eigenvalue for \mathcal{Y}_P . Therefore, we have

$$\begin{aligned} \mathcal{Y}_E(e_1) = \mathcal{Y}_H(e_3) = 0, & \quad -\mathcal{Y}_E(e_2) = \mathcal{Y}_P(e_1) = -\mathcal{Y}_P(e_2) = e_3, \\ \mathcal{Y}_E(e_3) = \mathcal{Y}_H(e_1) = e_2, & \quad \mathcal{Y}_P(e_3) = e_1 + e_2. \end{aligned}$$

We denote by $\lambda = 1 \in \mathbb{R}$, the generator (basis) of the Lie algebra \mathbb{R} .

Let V be a linear subspace of \mathbb{R}^3 and A a linear map on \mathbb{R}^3 . There exists a unique linear map $\bar{A} : \mathbb{R}^3 \rightarrow \mathbb{R}^3/V$ which maps x to $[Ax]$. For simplicity we denote the linear system $\bar{A}x = [v]$ by $Ax - v \in V$, when there is no ambiguity.

Theorem (4.6) follows from the following theorem.

Theorem A.1. *Let $H \subset G$ be a connected Lie subgroup with $\dim H \geq 2$. Then, the Lie algebra $\mathfrak{h} = \text{Lie}(H)$ is either conjugate to a semi-direct sum $\mathfrak{h}_L \oplus_{\Theta} T(\mathfrak{h})$, where \mathfrak{h}_L is a Lie subalgebra of*

$\mathbb{R} \oplus \mathfrak{so}(1, 2)$ or it is conjugate to one of the following Lie algebras.

$$\begin{array}{ll}
(\mathbb{R}(1 + \mathcal{Y}_H + e_1) + \mathbb{R}\mathcal{Y}_P) +_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3), & \mathbb{R}(-1 + \mathcal{Y}_H + e_1) +_{\theta} \mathbb{R}e_3, \\
(\mathbb{R}(\mathcal{Y}_H + e_3) + \mathbb{R}\mathcal{Y}_P) +_{\theta} \mathbb{R}(e_1 + e_2), & (\mathbb{R}(2 + \mathcal{Y}_H) + \mathbb{R}(\mathcal{Y}_P + e_1)) +_{\theta} \mathbb{R}(e_1 + e_2), \\
(\mathbb{R}(2 + \mathcal{Y}_H) + \mathbb{R}(\mathcal{Y}_P + e_1)) +_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3), & \mathbb{R}(2 + \mathcal{Y}_H) + \mathbb{R}(\mathcal{Y}_P + e_1 - e_2), \\
\mathbb{R}(\mathcal{Y}_E + e_1) +_{\theta} (\mathbb{R}e_2 \oplus \mathbb{R}e_3), & \mathbb{R}(\mathcal{Y}_P + e_1) +_{\theta} \mathbb{R}(e_1 + e_2), \\
\mathbb{R}(\mathcal{Y}_P + e_1) +_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3), & \mathbb{R}(\mathcal{Y}_H + e_3) +_{\theta} \mathbb{R}(e_1 + e_2), \\
\mathbb{R}(1 + \mathcal{Y}_H + e_1) +_{\ominus} \mathbb{R}(e_1 + e_2), & \mathbb{R}(1 + \mathcal{Y}_H + e_1) +_{\theta} \mathbb{R}e_3, \\
\mathbb{R}(1 + \mathcal{Y}_H + e_1) +_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3), & \mathbb{R}(\mathcal{Y}_H + e_3) +_{\theta} (\mathbb{R}e_1 \oplus \mathbb{R}e_2).
\end{array}$$

Proof. We prove this theorem by the three following lemmas. \square

Lemma A.2. Let $\mathfrak{h} \leq \mathfrak{g}$ be a Lie subalgebra with $p_{li}(\mathfrak{h}) = \mathfrak{so}(1, 2)$. Then \mathfrak{h} is conjugate to one of the following Lie algebras.

$$\mathfrak{so}(1, 2), \quad \mathbb{R} \oplus \mathfrak{so}(1, 2), \quad \mathfrak{so}(1, 2) \oplus_{\ominus} \mathbb{R}^{1,2}, \quad (\mathbb{R} \oplus \mathfrak{so}(1, 2)) \oplus_{\ominus} \mathbb{R}^{1,2}.$$

Proof. The $\mathfrak{so}(1, 2)$ -invariant subspace of $\mathbb{R}^{1,2}$ are $\{0\}$ and $\mathbb{R}^{1,2}$. Assume that $\dim p_l(\mathfrak{h}) = 4$, then $p_l(\mathfrak{h}) = \mathbb{R} \oplus \mathfrak{so}(1, 2)$. If $T(\mathfrak{h}) = \mathbb{R}^{1,2}$, then clearly $\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{so}(1, 2)) \oplus_{\ominus} \mathbb{R}^{1,2}$.

If $T(\mathfrak{h}) = \{0\}$, then there are four vectors $r, u, v, w \in \mathbb{R}^{1,2}$ such that

$$\{\lambda + u, \mathcal{Y}_E + v, \mathcal{Y}_H + w, \mathcal{Y}_P + r\},$$

is a basis for \mathfrak{h} . Solving the linear system

$$u - \lambda x = 0,$$

we get $x = u$. On the other hand, considering the Lie bracket on \mathfrak{h} , we get

$$\begin{aligned}
[\lambda + u, \mathcal{Y}_E + v] &= v - \mathcal{Y}_E(u) \Rightarrow v - \mathcal{Y}_E(u) = 0 \\
[\lambda + u, \mathcal{Y}_H + w] &= w - \mathcal{Y}_H(u) \Rightarrow w - \mathcal{Y}_H(u) = 0 \\
[\lambda + u, \mathcal{Y}_P + r] &= r - \mathcal{Y}_P(u) \Rightarrow r - \mathcal{Y}_P(u) = 0.
\end{aligned}$$

These three equations lead to

$$v = (0, u_3, -u_2) \quad , \quad w = (u_2, u_1, 0) \quad , \quad r = (u_3, u_3, u_1 - u_2).$$

Now, we have

$$\begin{aligned}
Ad_{(1, Id, u)}(\mathcal{Y}_E + v) &= \mathcal{Y}_E, & Ad_{(1, Id, u)}(\mathcal{Y}_H + w) &= \mathcal{Y}_H, \\
Ad_{(1, Id, u)}(\mathcal{Y}_P + r) &= \mathcal{Y}_P, & Ad_{(1, Id, u)}(\lambda + u) &= \lambda
\end{aligned}$$

Therefore $\mathfrak{h} = \mathbb{R} \oplus \mathfrak{so}(1, 2)$ up to conjugacy.

Now assume that $\dim p_l(\mathfrak{h}) = 3$, then $p_h \circ p_{li}^{-1} : \mathfrak{so}(1, 2) \rightarrow p_h(\mathfrak{h})$ is a surjective Lie algebra morphism. Lemma 1.29 implies that $p_h \circ p_{li}^{-1} \equiv 0$. Hence $p_l(\mathfrak{h}) = \mathfrak{so}(1, 2)$.

If $T(\mathfrak{h}) = \mathbb{R}^{1,2}$, then clearly $\{\mathcal{Y}_E, \mathcal{Y}_H, \mathcal{Y}_P, e_1, e_2, e_3\}$ is a basis for \mathfrak{h} , and so, $\mathfrak{h} = \mathfrak{so}(1, 2) \oplus_{\ominus} \mathbb{R}^{1,2}$.

If $T(\mathfrak{h}) = \{0\}$, there are three vectors $v, u, w \in \mathbb{R}^{1,2}$ such that $\{\mathcal{Y}_E + v, \mathcal{Y}_H + u, \mathcal{Y}_P + w\}$ is a basis for \mathfrak{h} . In the one hand, considering the Lie bracket on \mathfrak{h} , we get

$$\begin{aligned} [\mathcal{Y}_E + v, \mathcal{Y}_H + u] &= \mathcal{Y}_E - \mathcal{Y}_P + \mathcal{Y}_E(u) - \mathcal{Y}_H(v) \Rightarrow \mathcal{Y}_E(u) - \mathcal{Y}_H(v) = v - w \\ [\mathcal{Y}_E + v, \mathcal{Y}_P + w] &= \mathcal{Y}_H + \mathcal{Y}_E(w) - \mathcal{Y}_P(v) \Rightarrow \mathcal{Y}_E(w) - \mathcal{Y}_P(v) = u \\ [\mathcal{Y}_H + u, \mathcal{Y}_P + w] &= \mathcal{Y}_P + \mathcal{Y}_H(w) - \mathcal{Y}_P(u) \Rightarrow \mathcal{Y}_H(w) - \mathcal{Y}_P(u) = w. \end{aligned}$$

These three equations lead to

$$v = (0, v_2, -u_1) \quad , \quad u = (u_1, u_2, 0) \quad , \quad w = (v_2, v_2, u_2 - u_1).$$

On the other hand, solving the linear systems

$$u - \mathcal{Y}_H(x) = 0 \quad \text{and} \quad v - \mathcal{Y}_E(x) = 0,$$

simultaneously we obtain $x = (u_2, u_1, v_2)$. Now, we have

$$\begin{aligned} Ad_{(1, Id, x)}(\mathcal{Y}_E + v) &= \mathcal{Y}_E, & Ad_{(1, Id, x)}(\mathcal{Y}_H + u) &= \mathcal{Y}_H, \\ Ad_{(1, Id, x)}(\mathcal{Y}_P + w) &= \mathcal{Y}_P. \end{aligned}$$

Thus $\mathfrak{h} = \mathfrak{so}(1, 2)$ up to conjugacy. □

Lemma A.3. *Let $\mathfrak{h} \leq \mathfrak{g}$ be a Lie subalgebra with $\dim p_{l_i}(\mathfrak{h}) = 2$. Then either \mathfrak{h} is conjugate to a semi-direct sum $\mathfrak{h}_L \oplus_{\Theta} T(\mathfrak{h})$, where \mathfrak{h}_L is a Lie subalgebra of $\mathbb{R} \oplus \mathfrak{so}(1, 2)$ or it is conjugate to one the of following Lie algebras.*

$$\begin{aligned} (\mathbb{R}(1 + \mathcal{Y}_H + e_1) + \mathbb{R}\mathcal{Y}_P) +_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3), & \quad (\mathbb{R}(2 + \mathcal{Y}_H) + \mathbb{R}(\mathcal{Y}_P + e_1)) +_{\theta} \mathbb{R}(e_1 + e_2) \\ (\mathbb{R}(\mathcal{Y}_H + e_3) + \mathbb{R}\mathcal{Y}_P) +_{\theta} \mathbb{R}(e_1 + e_2), & \quad \mathbb{R}(-1 + \mathcal{Y}_H + e_1 + e_2) + \mathbb{R}\mathcal{Y}_P, \\ (\mathbb{R}(2 + \mathcal{Y}_H) + \mathbb{R}(\mathcal{Y}_P + e_1)) +_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3), & \quad \mathbb{R}(2 + \mathcal{Y}_H) + \mathbb{R}(\mathcal{Y}_P + e_1 - e_2). \end{aligned}$$

Proof. Since every 2-dimensional Lie subalgebras of $\mathfrak{so}(1, 2)$ is conjugated to \mathfrak{aff} , we may assume $p_{l_i}(\mathfrak{h}) = \mathfrak{aff}$. Therefore $p_l(\mathfrak{h})$ is a Lie subalgebra of $\mathbb{R} \oplus \mathfrak{aff}$ up to conjugacy. The \mathfrak{aff} -invariant subspace of $\mathbb{R}^{1,2}$ are $\mathbb{R}^{1,2}$, $\{0\}$, the lightlike line $\mathbb{R}(e_1 + e_2)$, and its corresponding (orthogonal) lightlike plane $\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3$.

Case I: $T(\mathfrak{h}) = \mathbb{R}^{1,2}$. Take $\{e_1, e_2, e_3\}$ as a basis for $T(\mathfrak{h})$.

- If $p_l(\mathfrak{h}) = \mathbb{R} \oplus \mathfrak{aff}$. There are three vectors $u, v, w \in \mathbb{R}^{1,2}$ such that

$$\{e_1, e_2, e_3, \lambda + u, \mathcal{Y}_P + v, \mathcal{Y}_H + w\},$$

is a basis for \mathfrak{h} . Since $T(\mathfrak{h}) = \mathbb{R}^{1,2}$, we have $v, w, u \in T(\mathfrak{h})$, and so, $\{e_1, e_2, e_3, \lambda, \mathcal{Y}_P, \mathcal{Y}_H\}$ is a basis for \mathfrak{h} . Furthermore the subspace generated by $\{\lambda, \mathcal{Y}_H, \mathcal{Y}_P\}$, is a Lie subalgebra of \mathfrak{h} . Therefore \mathfrak{h} is conjugate to the semi-direct sum

$$(\mathbb{R} \oplus (\mathbb{R}\mathcal{Y}_H + \mathbb{R}\mathcal{Y}_P)) \oplus_{\theta} \mathbb{R}^{1,2} = (\mathbb{R} \oplus \mathfrak{aff}) \oplus_{\theta} \mathbb{R}^{1,2}.$$

- If the homothety projection $p_h(\mathfrak{h})$ is trivial. Then $p_{li}(\mathfrak{h}) = \mathfrak{aff}$ and by the same argument as the previous case, we obtain a basis

$$\{e_1, e_2, e_3, \mathcal{Y}_P, \mathcal{Y}_H\},$$

for \mathfrak{h} . Therefore \mathfrak{h} is conjugate to the semi-direct sum

$$(\mathbb{R}\mathcal{Y}_H + \mathbb{R}\mathcal{Y}_P) \oplus_{\theta} \mathbb{R}^{1,2} = \mathfrak{aff} \oplus_{\theta} \mathbb{R}^{1,2}.$$

- If $\dim p_l(\mathfrak{h}) = 2$ and the homothety projection $p_h(\mathfrak{h})$ is not trivial, by the same argument in the two previous cases, there are two constants $a, b \in \mathbb{R}$ such that $a + \mathcal{Y}_H, \mathcal{Y}_P \in \mathfrak{h}$, and

$$\{e_1, e_2, e_3, b + \mathcal{Y}_P, a + \mathcal{Y}_H\},$$

is a basis for \mathfrak{h} . By closeness under the Lie bracket we obtain

$$[a + \mathcal{Y}_H, b + \mathcal{Y}_P] = \mathcal{Y}_P.$$

The above equation implies that $b = 0$, and since the homothety projection is not trivial, we have $a \neq 0$. So

$$\{e_1, e_2, e_3, \mathcal{Y}_P, a + \mathcal{Y}_H\},$$

is a basis for \mathfrak{h} . The vector subspace generated by $\{a + \mathcal{Y}_H, \mathcal{Y}_P\}$ is a Lie subalgebra of \mathfrak{h} . Therefore \mathfrak{h} is conjugate to the semidirect sum

$$(\mathbb{R}(a + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P) \oplus_{\theta} \mathbb{R}^{1,2}.$$

Case II: $T(\mathfrak{h}) = \mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3$.

- $p_l(\mathfrak{h}) = \mathbb{R} \oplus \mathfrak{aff}$. There are three vectors $u, v, w \in \mathbb{R}^{1,2}$ such that

$$\{e_1 + e_2, e_3, \lambda + u, \mathcal{Y}_P + w, \mathcal{Y}_H + v\},$$

is a basis for \mathfrak{h} . Considering the Lie bracket on \mathfrak{h} , we obtain

$$[\lambda + u, \mathcal{Y}_H + v] = v - \mathcal{Y}_H(u), \quad [\lambda + u, \mathcal{Y}_P + w] = w - \mathcal{Y}_P(u).$$

The first equation implies $v - \mathcal{Y}_H(u) = (v_1 - u_2, v_2 - u_1, v_3) \in T(\mathfrak{h})$. Thus, $v_1 - u_2 = v_2 - u_1$. The second equation implies that $w - \mathcal{Y}_P(u) = (w_1 - u_3, w_2 - u_3, w_3 - u_1 + u_2) \in T(\mathfrak{h})$. Hence, $w_1 = w_2$. Now, by applying $Ad_{(1, Id, u)}$ on \mathfrak{h} , we obtain a Lie algebra \mathfrak{h}' which is conjugate to \mathfrak{h} .

$$\begin{aligned} Ad_{(1, Id, u)}(e_1 + e_2) &= e_1 + e_2 & , & & Ad_{(1, Id, u)}(e_3) &= e_3 & , & & Ad_{(1, Id, u)}(\lambda + u) &= \lambda, \\ Ad_{(1, Id, u)}(\mathcal{Y}_H + v) &= \mathcal{Y}_H + v', & v' &= (v_1 - u_2, v_1 - u_2, v_3), \\ Ad_{(1, Id, u)}(\mathcal{Y}_P + w) &= \mathcal{Y}_P + w', & w' &= (w_1 - u_3, w_1 - u_3, w_3 - u_1 + u_2), \end{aligned}$$

Obviously $v', w' \in T(\mathfrak{h}') = T(\mathfrak{h})$, thus $\mathcal{Y}_H, \mathcal{Y}_P \in \mathfrak{h}'$ and

$$\{e_1 + e_2, e_3, \lambda, \mathcal{Y}_H, \mathcal{Y}_P\},$$

is a basis for \mathfrak{h}' . Therefore, \mathfrak{h} is conjugate to the semidirect sum

$$(\mathbb{R} \oplus (\mathbb{R}\mathcal{Y}_H + \mathbb{R}\mathcal{Y}_P)) \oplus_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3) = (\mathbb{R} \oplus \mathfrak{aff}) \oplus_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3).$$

- If the homothety projection $p_h(\mathfrak{h})$ is trivial. Then $p_l(\mathfrak{h}) = \mathfrak{aff}$ up to conjugacy. There are two vectors $u, v \in \mathbb{R}^{1,2}$ such that

$$\{e_1 + e_2, e_3, \mathcal{Y}_P + v, \mathcal{Y}_H + u\},$$

is a basis for \mathfrak{h} . Considering the Lie bracket on \mathfrak{h} , we obtain

$$[\mathcal{Y}_H + u, \mathcal{Y}_P + v] = \mathcal{Y}_P + \mathcal{Y}_H(v) - \mathcal{Y}_P(u).$$

Therefore, there exist $\beta, \alpha \in \mathbb{R}$ such that $\mathcal{Y}_H(v) - \mathcal{Y}_P(u) = v + \beta(e_1 + e_2) + \alpha e_3$. This implies that

$$\begin{aligned} (v_2 - u_3, v_1 - u_3, -u_1 + u_2) &= \mathcal{Y}_H(v) - \mathcal{Y}_P(u) \\ &= v + \beta(e_1 + e_2) + \alpha e_3 \\ &= (v_1 + \beta, v_2 + \beta, v_3 + \alpha) \end{aligned}$$

So, we have $v_1 = v_2$. On the other hand solving the linear systems

$$u - \mathcal{Y}_H(x) \in T(\mathfrak{h}) \quad \text{and} \quad v - \mathcal{Y}_P(x) \in T(\mathfrak{h}),$$

simultaneously, we obtain $x = (u_2, u_1, v_1)$. Now, applying $Ad_{(1, Id, x)}$ on \mathfrak{h} , we obtain a Lie algebra \mathfrak{h}' which is conjugated to \mathfrak{h} .

$$\begin{aligned} Ad_{(1, Id, x)}(e_1 + e_2) &= e_1 + e_2, & Ad_{(1, Id, x)}(e_3) &= e_3, \\ Ad_{(1, Id, x)}(\mathcal{Y}_H + u) &= \mathcal{Y}_H + u', & u' &= (0, 0, u_3), \\ Ad_{(1, Id, x)}(\mathcal{Y}_P + v) &= \mathcal{Y}_P + v', & v' &= (0, 0, v_3 - u_2 + u_1), \end{aligned}$$

Obviously $u', v' \in T(\mathfrak{h}') = T(\mathfrak{h})$, thus $\mathcal{Y}_H, \mathcal{Y}_P \in \mathfrak{h}'$ and

$$\{e_1 + e_2, e_3, \mathcal{Y}_H, \mathcal{Y}_P\},$$

is a basis for \mathfrak{h}' . Therefore, \mathfrak{h} is conjugate to the semi-direct sum

$$(\mathbb{R}\mathcal{Y}_H + \mathbb{R}\mathcal{Y}_P) \oplus_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3) = \mathfrak{aff} \oplus_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3).$$

- If $\dim p_l(\mathfrak{h}) = 2$ and the homothety projection $p_h(\mathfrak{h})$ is not trivial. There are two constants $a, b \in \mathbb{R}$ and two vectors $u, v \in \mathbb{R}^{1,2}$ such that $(a + \mathcal{Y}_H + u), (b + \mathcal{Y}_P + v) \in \mathfrak{h}$, and

$$\{e_1 + e_2, e_3, b + \mathcal{Y}_P + v, a + \mathcal{Y}_H + u\},$$

is a basis for \mathfrak{h} . Considering the Lie bracket on \mathfrak{h} we have

$$[a + \mathcal{Y}_H + u, b + \mathcal{Y}_P + v] = \mathcal{Y}_P + \mathcal{Y}_H(v) + av - \mathcal{Y}_P(u) - bu,$$

which implies that $b = 0$. Since the homothety projection $p_h(\mathfrak{h})$ is not trivial, we have $a \neq 0$. Furthermore, the above equation implies that, there exist $\beta, \eta \in \mathbb{R}$ such that

$$\mathcal{Y}_H(v) + av - \mathcal{Y}_P(u) = v + \beta(e_1 + e_2) + \eta e_3.$$

So,

$$(v_2 + av_1 - u_3, v_1 + av_2 - u_3, av_3 - u_1 + u_2) = \\ \mathcal{Y}_H(v) + av - \mathcal{Y}_P(u) = v + \beta(e_1 + e_2) + \eta e_3 = (v_1 + \beta, v_2 + \beta, v_3 + \eta)$$

This implies that $(2-a)v_1 = (2-a)v_2$. Note that for $a \neq 2$ we have $v_1 = v_2$. For $a \notin \{-1, 0, 1, 2\}$, the vector $x = (\frac{-u_2+au_1}{a^2-1}, \frac{-u_1+au_2}{a^2-1}, \frac{u_3}{a}) \in \mathbb{R}^{1,2}$ is the solution of the linear system

$$u - ax - \mathcal{Y}_H(x) = 0.$$

Therefore, $Ad_{(1,Id,x)}(a + \mathcal{Y}_H + u) = a + \mathcal{Y}_H$. On the the other hand we have,

$$Ad_{(1,Id,x)}(e_1 + e_2) = e_1 + e_2 \quad , \quad Ad_{(1,Id,x)}(e_3) = e_3, \\ Ad_{(1,Id,x)}(\mathcal{Y}_P + v) = \mathcal{Y}_P + v' \quad , \quad v' = (v_1 - \frac{u_3}{a}, v_1 - \frac{u_3}{a}, v_3 + \frac{u_2 - u_1}{a-1}).$$

The Lie algebra $\mathfrak{h}' = Ad_{(1,Id,x)}(\mathfrak{h})$ is conjugated to \mathfrak{h} and obviously $v' \in T(\mathfrak{h}') = T(\mathfrak{h})$. Hence $(a + \mathcal{Y}_H), \mathcal{Y}_P \in \mathfrak{h}$ and so, $\{e_1 + e_2, e_3, a + \mathcal{Y}_H, \mathcal{Y}_P\}$ is a basis for \mathfrak{h}' . Thus, for $a \in \mathbb{R} \setminus \{-1, 0, 1, 2\}$, the lie algebra \mathfrak{h} is conjugate to the semi-direct sum

$$(\mathbb{R}(a + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P) \oplus_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3).$$

For $a = 1$, the linear system

$$u - x - \mathcal{Y}_H(x) \in T(\mathfrak{h}).$$

has a solution if and only if $u_1 = u_2$. Thus, setting $x = (0, u_2, u_3)$ we get

$$Ad_{(1,Id,x)}(e_1 + e_2) = e_1 + e_2 \quad , \quad Ad_{(1,Id,x)}(e_3) = e_3 \\ Ad_{(1,Id,x)}(1 + \mathcal{Y}_H + u) = 1 + \mathcal{Y}_H + u' \quad , \quad u' = (u_1 - u_2, 0, 0) \\ Ad_{(1,Id,x)}(\mathcal{Y}_P + v) = \mathcal{Y}_P + v' \quad , \quad v' = (v_1 - u_3, v_1 - u_3, v_3 + u_2).$$

The Lie algebra $\mathfrak{h}' = Ad_{(1,Id,x)}(\mathfrak{h})$ is conjugate to \mathfrak{h} and clearly $v' \in T(\mathfrak{h}')$. Thus if $u_1 = u_2$, then $\mathcal{Y}_P, 1 + \mathcal{Y}_H \in \mathfrak{h}'$. Therefore, \mathfrak{h} is conjugate to the semi-direct sum

$$(\mathbb{R}(1 + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P) \oplus_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3).$$

If $u_1 \neq u_2$, then \mathfrak{h} is conjugate to the following Lie algebra via $Ad_{(\frac{1}{u_1-u_2}, Id, 0)}$

$$(\mathbb{R}(1 + \mathcal{Y}_H + e_1) + \mathbb{R}\mathcal{Y}_P) +_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3).$$

For $a = -1$, the vector $x = (\frac{-u_1}{2}, \frac{-u_2}{2}, -u_3)$ is a solution for the linear system

$$u + x - \mathcal{Y}_H(x) \in T(\mathfrak{h}).$$

Now, we have

$$Ad_{(1,Id,x)}(e_1 + e_2) = e_1 + e_2 \quad , \quad Ad_{(1,Id,x)}(e_3) = e_3 \\ Ad_{(1,Id,x)}(1 + \mathcal{Y}_H + u) = 1 + \mathcal{Y}_H + u' \quad , \quad u' = (\frac{u_1 + u_2}{2}, \frac{u_1 + u_2}{2}, 0) \\ Ad_{(1,Id,x)}(\mathcal{Y}_P + v) = \mathcal{Y}_P + v' \quad , \quad v' = (v_1 + u_3, v_1 + u_3, v_3 + \frac{u_1 - u_2}{2}).$$

The Lie algebra $\mathfrak{h}' = Ad_{(1, Id, x)}(\mathfrak{h})$ is conjugate to \mathfrak{h} and clearly $u', v' \in T(\mathfrak{h}')$. Thus $\{-1 + \mathcal{Y}_H, \mathcal{Y}_P, e_1 + e_2, e_3\}$ is a basis for \mathfrak{h}' . Therefore, \mathfrak{h} is conjugate to the semi-direct sum

$$(\mathbb{R}(-1 + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P) \oplus_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3).$$

For $a = 2$, the linear system

$$v - \mathcal{Y}_P(x) \in T(\mathfrak{h}),$$

has a solution if and only if $v_1 = v_2$. On the other hand the vector $x = (\frac{-u_2+2u_1}{3}, \frac{-u_1+2u_2}{3}, \frac{u_3}{2})$ is the solution of the linear system

$$u - 2x - \mathcal{Y}_H(x) = 0.$$

Now, we have

$$\begin{aligned} Ad_{(1, Id, x)}(e_1 + e_2) &= e_1 + e_2 \quad , \quad Ad_{(1, Id, x)}(e_3) = e_3 \\ Ad_{(1, Id, x)}(2 + \mathcal{Y}_H + u) &= 2 + \mathcal{Y}_H \quad , \quad Ad_{(1, Id, x)}(\mathcal{Y}_P + v) = \mathcal{Y}_P + v' \end{aligned}$$

for some $v' \in \mathbb{R}^{1,2}$. The vector v' does not belong to $\mathfrak{h}' = Ad_{(1, Id, x)}(\mathfrak{h})$ necessarily. But, subscribing $(v'_2, v'_2, v'_3) \in T(\mathfrak{h}')$ from $\mathcal{Y}_P + v'$, we get $\mathcal{Y}_P + \beta e_1 \in \mathfrak{h}'$ for some $\beta \in \mathbb{R}$. Therefore the set $\{e_1 + e_2, e_3, 2 + \mathcal{Y}_H, \mathcal{Y}_P + \beta e_1\}$ is a basis for \mathfrak{h}' . Hence, in this case, there are two kinds of Lie algebras. If $\beta = 0$, then \mathfrak{h} is conjugate to the semidirect sum

$$(\mathbb{R}(2 + \mathcal{Y}_H) + \mathbb{R}(\mathcal{Y}_P)) \oplus_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3).$$

Otherwise, \mathfrak{h} is conjugate to the following Lie algebra

$$(\mathbb{R}(2 + \mathcal{Y}_H) + \mathbb{R}(\mathcal{Y}_P + \beta e_1)) +_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3).$$

This subalgebra is conjugate to the following subalgebra via $Ad_{(\beta^{-1}, Id, 0)}$

$$(\mathbb{R}(2 + \mathcal{Y}_H) + \mathbb{R}(\mathcal{Y}_P + e_1)) +_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3).$$

Case III: $T(\mathfrak{h}) = \mathbb{R}(e_1 + e_2)$.

- $p_i(\mathfrak{h}) = \mathbb{R} \oplus \text{aff}$. There are three vectors $u, v, w \in \mathbb{R}^{1,2}$ such that

$$\{e_1 + e_2, \lambda + u, \mathcal{Y}_P + v, \mathcal{Y}_H + w\},$$

is a basis for \mathfrak{h} . By considering the Lie bracket

$$\begin{aligned} [\lambda + u, \mathcal{Y}_P + v] &= v - \mathcal{Y}_P(u) = (v_1 - u_3, v_2 - u_3, v_3 - u_1 + u_2) \in T(\mathfrak{h}), \\ [\lambda + u, \mathcal{Y}_H + w] &= w - \mathcal{Y}_H(u) = (w_1 - u_2, w_2 - u_1, w_3) \in T(\mathfrak{h}). \end{aligned}$$

implies that $v_1 = v_2, v_3 = u_1 - u_2, w_3 = 0$, and $w_1 - u_2 = w_2 - u_1$. Now, applying $Ad_{(1, Id, u)}$, we get

$$\begin{aligned} Ad_{(1, Id, u)}(e_1 + e_2) &= e_1 + e_2 \quad , \quad Ad_{(1, Id, u)}(\lambda + u) = \lambda \\ Ad_{(1, Id, u)}(\mathcal{Y}_H + w) &= \mathcal{Y}_H + w', \quad w' = (w_1 - u_2, w_1 - u_2, 0), \\ Ad_{(1, Id, u)}(\mathcal{Y}_P + v) &= \mathcal{Y}_P + v', \quad v' = (v_1 - u_3, v_1 - u_3, 0). \end{aligned}$$

The Lie algebra $\mathfrak{h}' = Ad_{(1, Id, x)}(\mathfrak{h})$ is conjugated to \mathfrak{h} , and obviously, $v', w' \in T(\mathfrak{h}')$, and so, $\mathcal{Y}_H, \mathcal{Y}_P \in \mathfrak{h}'$. Hence $\{e_1 + e_2, \lambda, \mathcal{Y}_H, \mathcal{Y}_P\}$ is a basis for \mathfrak{h}' . Therefore \mathfrak{h} is conjugate to the semidirect sum

$$(\mathbb{R} \oplus (\mathbb{R}\mathcal{Y}_H + \mathbb{R}\mathcal{Y}_P)) \oplus_{\theta} \mathbb{R}(e_1 + e_2) = (\mathbb{R} \oplus \mathfrak{aff}) \oplus_{\theta} \mathbb{R}(e_1 + e_2).$$

- If the homothety projection $p_h(\mathfrak{h})$ is trivial. Then $p_l(\mathfrak{h}) = \mathfrak{aff}$ and there exist two vectors $u, v \in \mathbb{R}^{1,2}$ such that

$$\{e_1 + e_2, \mathcal{Y}_H + u, \mathcal{Y}_P + v\},$$

is a basis for \mathfrak{h} . Considering the Lie bracket

$$[\mathcal{Y}_H + u, \mathcal{Y}_P + v] = \mathcal{Y}_P + \mathcal{Y}_H(v) - \mathcal{Y}_P(u),$$

implies that, there exists $\beta \in \mathbb{R}$ such that $\mathcal{Y}_H(v) - \mathcal{Y}_P(u) = v + \beta(e_1 + e_2)$. Thus, by

$$(v_2 - u_3, v_1 - u_3, v_3 - u_1 + u_2) = \mathcal{Y}_H(v) - \mathcal{Y}_P(u) = (v_1 + \beta, v_2 + \beta, v_3),$$

we have $v_1 = v_2$ and $v_3 = u_1 - u_2$. In the one hand, the vector $x = (u_2, u_1, v_1)$ is the solution of the linear system

$$v - \mathcal{Y}_P(x) = 0.$$

On the other hand, the linear system

$$u - \mathcal{Y}_H(x) \in T(\mathfrak{h}),$$

has a solution if and only if $u_3 = 0$. Therefore, conjugating by $x = (u_2, u_1, v_1)$ we get

$$\begin{aligned} Ad_{(1, Id, x)}(e_1 + e_2) &= e_1 + e_2 \quad , \quad Ad_{(1, Id, x)}(\mathcal{Y}_P + v) = \mathcal{Y}_P \\ Ad_{(1, Id, u)}(\mathcal{Y}_H + u) &= \mathcal{Y}_H + u', \quad u' = (0, 0, u_3). \end{aligned}$$

Therefore, there are two cases: If $u_3 = 0$, then \mathfrak{h} is conjugate to the semi-direct sum

$$(\mathbb{R}\mathcal{Y}_H + \mathbb{R}\mathcal{Y}_P) \oplus_{\theta} \mathbb{R}(e_1 + e_2) = \mathfrak{aff} \oplus_{\theta} \mathbb{R}(e_1 + e_2).$$

If $u_3 \neq 0$, then \mathfrak{h} is conjugate to the following Lie subalgebra via $Ad_{(u_3^{-1}, Id, 0)}$

$$(\mathbb{R}(\mathcal{Y}_H + e_3) + \mathbb{R}\mathcal{Y}_P) +_{\theta} \mathbb{R}(e_1 + e_2).$$

- If $\dim p_l(\mathfrak{h}) = 2$ and the homothety projection $p_h(\mathfrak{h})$ is not trivial. There are two constants $a, b \in \mathbb{R}$ and two vectors $u, v \in \mathbb{R}^{1,2}$ such that $(a + \mathcal{Y}_H + u), (b + \mathcal{Y}_P + v) \in \mathfrak{h}$, and

$$\{e_1 + e_2, b + \mathcal{Y}_P + v, a + \mathcal{Y}_H + u\},$$

is a basis for \mathfrak{h} . Considering the Lie bracket

$$[a + \mathcal{Y}_H + u, b + \mathcal{Y}_P + v] = \mathcal{Y}_P + av + \mathcal{Y}_H(v) - bu - \mathcal{Y}_P(u),$$

implies that $b = 0$, and since the homothety part is nontrivial $a \neq 0$. Furthermore, the above equation implies that, there exist $\beta \in \mathbb{R}$ such that

$$av + \mathcal{Y}_H(v) - \mathcal{Y}_P(u) = v + \beta(e_1 + e_1) \in T(\mathfrak{h}).$$

Thus,

$$(av_1 + v_2 - u_3, av_2 + v_1 - u_3, av_3 - u_1 + u_2) = av + \mathcal{Y}_H(v) - \mathcal{Y}_P(u) = (v_1 + \beta, v_2 + \beta, v_3),$$

implies that $(2 - a)v_1 = (2 - a)v_2$, $(a - 1)v_3 = u_1 - u_2$.

For $a \in \mathbb{R}^* \setminus \{-1, 1, 2\}$, the vector $x = (\frac{-u_2 + au_1}{a^2 - 1}, \frac{-u_1 + au_2}{a^2 - 1}, \frac{u_3}{a})$ is the common solution of the linear systems

$$u - ax - \mathcal{Y}_H(x) = 0 \quad \text{and} \quad v - \mathcal{Y}_P(x) \in T(\mathfrak{h}).$$

Hence, we have

$$\begin{aligned} Ad_{(1, Id, x)}(e_1 + e_2) &= e_1 + e_2, \quad Ad_{(1, Id, x)}(a + \mathcal{Y}_H + u) = a + \mathcal{Y}_H \\ Ad_{(1, Id, x)}(\mathcal{Y}_P + v) &= \mathcal{Y}_P + v', \quad v' = (v_1 - \frac{u_3}{a}, v_1 - \frac{u_3}{a}, 0). \end{aligned}$$

Obviously, \mathfrak{h} is conjugate to the Lie algebra $\mathfrak{h}' = Ad_{(1, Id, x)}(\mathfrak{h})$ and we have $v' \in T(\mathfrak{h}') = T(\mathfrak{h})$. Thus $\{e_1 + e_2, a + \mathcal{Y}_H, \mathcal{Y}_P\}$ is a basis for \mathfrak{h}' . Henceforth, for $a \in \mathbb{R}^* \setminus \{-1, 1, 2\}$, \mathfrak{h} is conjugate to the semi-direct sum

$$(\mathbb{R}(a + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P) \oplus_{\theta} \mathbb{R}(e_1 + e_2).$$

For $a = 1$, the vector $x = (v_3, 0, u_3)$ is a common solution of the linear systems

$$u - x - \mathcal{Y}_H(x) \in T(\mathfrak{h}) \quad \text{and} \quad v - \mathcal{Y}_P(x) \in T(\mathfrak{h}).$$

In fact we have

$$\begin{aligned} Ad_{(1, Id, x)}(e_1 + e_2) &= e_1 + e_2, \\ Ad_{(1, Id, x)}(1 + \mathcal{Y}_H + u) &= 1 + \mathcal{Y}_H + u', \quad u' = (u_1 - v_3, u_1 - v_3, 0), \\ Ad_{(1, Id, x)}(\mathcal{Y}_P + v) &= \mathcal{Y}_P + v', \quad v' = (v_1 - u_3, v_1 - u_3, 0). \end{aligned}$$

The Lie algebra $\mathfrak{h}' = Ad_{(1, Id, x)}(\mathfrak{h})$ is conjugate to \mathfrak{h} and evidently $u', v' \in T(\mathfrak{h}') = T(\mathfrak{h})$. Thus, $\{e_1 + e_2, 1 + \mathcal{Y}_H, \mathcal{Y}_P\}$ is a basis for \mathfrak{h}' . Hence, \mathfrak{h} is conjugate to the semi-direct sum

$$(\mathbb{R}(1 + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P) \oplus_{\theta} \mathbb{R}(e_1 + e_2).$$

For $a = -1$, the vector $x = (\frac{u_2}{2}, \frac{u_1}{2}, -u_3)$ is a common solution of the linear systems

$$u + x - \mathcal{Y}_H(x) \in T(\mathfrak{h}) \quad \text{and} \quad v - \mathcal{Y}_P(x) \in T(\mathfrak{h}).$$

In fact, we have

$$\begin{aligned} Ad_{(1, Id, x)}(e_1 + e_2) &= e_1 + e_2 \\ Ad_{(1, Id, x)}(-1 + \mathcal{Y}_H + u) &= -1 + \mathcal{Y}_H + u', \quad u' = (\frac{u_1 + u_2}{2}, \frac{u_1 + u_2}{2}, 0) \\ Ad_{(1, Id, x)}(\mathcal{Y}_P + v) &= \mathcal{Y}_P + v', \quad v' = (v_1 + u_3, v_1 + u_3, 0). \end{aligned}$$

Obviously, $-1 + \mathcal{Y}_H, \mathcal{Y}_P \in \mathfrak{h}' = Ad_{(1, Id, x)}(\mathfrak{h})$. Hence \mathfrak{h} is conjugate to the semi-direct sum

$$(\mathbb{R}(-1 + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P) \oplus_{\theta} \mathbb{R}(e_1 + e_2),$$

For $a = 2$, the linear system

$$v - \mathcal{Y}_P(x) \in T(\mathfrak{h}),$$

has a solution if and only if $v_1 = v_2$. On the other hand, the vector $x = (\frac{-u_2+2u_1}{3}, \frac{-u_1+2u_2}{3}, \frac{u_3}{2})$ is the solution of the linear system

$$u - 2x - \mathcal{Y}_H(x) = 0.$$

Now, conjugating by x we obtain

$$\begin{aligned} Ad_{(1,Id,x)}(e_1 + e_2) &= e_1 + e_2, & Ad_{(1,Id,x)}(2 + \mathcal{Y}_H + u) &= 2 + \mathcal{Y}_H \\ Ad_{(1,Id,x)}(\mathcal{Y}_P + v) &= \mathcal{Y}_P + v', & v' &= (v_1 - \frac{u_3}{2}, v_2 - \frac{u_3}{2}, 0). \end{aligned}$$

It is easy to see that $\mathcal{Y}_P + v'_1 e_1 \in \mathfrak{h}' = Ad_{(1,Id,x)}(\mathfrak{h})$, and so,

$$\{e_1 + e_2, 2 + \mathcal{Y}_H, \mathcal{Y}_P + v'_1 e_1\}$$

is a basis for \mathfrak{h}' . Hence there are two kinds of subalgebras: If $v'_1 = 0$ then \mathfrak{h} is conjugate to the semi-direct sum

$$(\mathbb{R}(2 + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P) \oplus_{\Theta} \mathbb{R}(e_1 + e_2).$$

Otherwise, \mathfrak{h} is conjugated to the following Lie algebra via $Ad_{(1/v'_1, Id, 0)}$

$$(\mathbb{R}(2 + \mathcal{Y}_H) + \mathbb{R}(\mathcal{Y}_P + e_1)) \oplus_{\Theta} \mathbb{R}(e_1 + e_2).$$

Case IV: $T(\mathfrak{h}) = \{0\}$

- $p_l(\mathfrak{h}) = \mathbb{R} \oplus \mathfrak{aff}$. There are three vectors $u, v, w \in \mathbb{R}^{1,2}$ such that

$$\{\lambda + u, \mathcal{Y}_P + v, \mathcal{Y}_H + w\},$$

is a basis for \mathfrak{h} . Considering Lie bracket, we get

$$[\lambda + u, \mathcal{Y}_P + v] = v - \mathcal{Y}_P(u) = 0, \quad [\lambda + u, \mathcal{Y}_H + w] = w - \mathcal{Y}_H(u) = 0.$$

Thus applying $Ad_{(1,Id,u)}$, we have

$$Ad_{(1,Id,u)}(\lambda + u) = \lambda, \quad Ad_{(1,Id,u)}(\mathcal{Y}_H + w) = \mathcal{Y}_H, \quad Ad_{(1,Id,u)}(\mathcal{Y}_P + v) = \mathcal{Y}_P.$$

Obviously, $\{\lambda, \mathcal{Y}_H, \mathcal{Y}_P\}$ is a basis for $\mathfrak{h}' = Ad_{(1,Id,u)}(\mathfrak{h})$. Therefore, \mathfrak{h} is conjugate to

$$\mathbb{R} \oplus \mathfrak{aff}.$$

- $p_l(\mathfrak{h}) = \mathfrak{aff}$. There are two vectors $u, v \in \mathbb{R}^{1,2}$ such that $\{\mathcal{Y}_H + u, \mathcal{Y}_P + v\}$ is a basis for \mathfrak{h} . Closeness under Lie bracket

$$[\mathcal{Y}_H + u, \mathcal{Y}_P + v] = \mathcal{Y}_P + \mathcal{Y}_H(v) - \mathcal{Y}_P(u) \implies \mathcal{Y}_H(v) - \mathcal{Y}_P(u) = v,$$

implies that

$$v_1 = v_2, \quad u_3 = 0, \quad v_3 = u_2 - u_1.$$

Thus, the vector $x = (u_2, u_1, v_1)$ is the common solution of the linear systems

$$u - \mathcal{Y}_H(x) = 0 \quad \text{and} \quad v - \mathcal{Y}_P(x) = 0.$$

In other words, we have

$$Ad_{(1, Id, x)}(\mathcal{Y}_H + u) = \mathcal{Y}_H \quad , \quad Ad_{1, Id, x}(\mathcal{Y}_P + v) = \mathcal{Y}_P.$$

Hence \mathfrak{h} is conjugate to \mathfrak{aff} .

- If $\dim p_i(\mathfrak{h}) = 2$ and the homothety projection $p_h(\mathfrak{h})$ is nontrivial. There are two constants $a, b \in \mathbb{R}$ and two vectors $u, v \in \mathbb{R}^{1,2}$ such that $\{b + \mathcal{Y}_P + v, a + \mathcal{Y}_H + u\}$ is a basis for \mathfrak{h} . Closeness under the Lie bracket

$$[a + \mathcal{Y}_H + u, b + \mathcal{Y}_P + v] = \mathcal{Y}_P + av + \mathcal{Y}_H(v) - bu - \mathcal{Y}_P(u)$$

implies that $b = 0$, and since the homothety part is nontrivial, $a \neq 0$. Furthermore

$$(2 - a)v_1 = (2 - a)v_2, \quad (1 - a)v_3 = u_2 - u_1, \quad 2u_3 = a(v_1 + v_2).$$

For $a \in \mathbb{R}^* \setminus \{-1, 1, 2\}$, the vector $x = (\frac{-u_2 + au_1}{a^2 - 1}, \frac{-u_1 + au_2}{a^2 - 1}, \frac{u_3}{a})$ is the common solution of the linear systems

$$u - ax - \mathcal{Y}_H(x) = 0 \quad \text{and} \quad v - \mathcal{Y}_P(x) = 0.$$

In other words, we have

$$Ad_{(1, Id, x)}(a + \mathcal{Y}_H + u) = a + \mathcal{Y}_H, \quad Ad_{(1, Id, x)}(\mathcal{Y}_P + v) = \mathcal{Y}_P.$$

Hence for $a \in \mathbb{R}^* \setminus \{-1, 1, 2\}$, the Lie algebra \mathfrak{h} is conjugate to

$$\mathbb{R}(a + \mathcal{Y}_H) + \mathbb{R}(\mathcal{Y}_P).$$

For $a = 1$, the vector $x = (\frac{v_3 + u_2}{2}, \frac{-v_3 + u_2}{2}, v_1)$ is the common solution of the linear systems

$$u - x - \mathcal{Y}_H(x) = 0 \quad \text{and} \quad v - \mathcal{Y}_P(x) = 0.$$

This implies that \mathfrak{h} is conjugate to the semi-direct sum

$$\mathbb{R}(a + \mathcal{Y}_H) + \mathbb{R}(\mathcal{Y}_P).$$

For $a = -1$, the linear system

$$u + x - \mathcal{Y}_H(x) = 0,$$

has a solution if and only if $u_1 = -u_2$. On the other hand, the vector $x = (\frac{u_2}{2}, \frac{u_1}{2}, -u_3)$ is a solution of the linear system

$$v - \mathcal{Y}_P(x) = 0.$$

Now, we have

$$Ad_{(1, Id, x)}(\mathcal{Y}_P + v) = \mathcal{Y}_P \quad , \quad Ad_{(1, Id, x)}(-1 + \mathcal{Y}_H + u) = -1 + \mathcal{Y}_H + \frac{u_1 + u_2}{2}(e_1 + e_2),$$

Therefore, if $u_1 = -u_2$, then \mathfrak{h} is conjugate to

$$\mathbb{R}(-1 + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P.$$

Otherwise, it is conjugate to the following Lie algebra via $Ad_{(\frac{2}{u_1+u_2}, Id, 0)}$

$$\mathbb{R}(-1 + \mathcal{Y}_H + e_1 + e_2) + \mathbb{R}\mathcal{Y}_P.$$

Hence, for all $a \in \mathbb{R}^* \setminus \{2\}$, the Lie algebra \mathfrak{h} is conjugate to $\mathbb{R}(a + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P$.

For $a = 2$ setting $x = (\frac{-u_2+2u_1}{3}, \frac{-u_1+2u_2}{3}, \frac{u_3}{2})$, we get

$$Ad_{(1, Id, x)}(2 + \mathcal{Y}_H + u) = 2 + Y \quad , \quad Ad_{(1, Id, x)}(\mathcal{Y}_P + v) = \mathcal{Y}_P + v'$$

where $v' = (v_1 - v_2)(e_1 - e_2)$. Therefore, if $v_1 = v_2$, then

$$\mathfrak{h} \cong \mathbb{R}(2 + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P.$$

Otherwise, \mathfrak{h} is conjugate to

$$\mathbb{R}(2 + \mathcal{Y}_H) + \mathbb{R}(\mathcal{Y}_P + e_1 - e_2).$$

Note that, for all $a \in \mathbb{R}^*$, the Lie algebra $\mathbb{R}(a + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P$ is conjugate to the Lie algebra $\mathbb{R}(1/a + \mathcal{Y}_H) + \mathbb{R}\mathcal{Y}_P$ via the element $\mathcal{R} := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus I_3 \in O(2, 3)$. \square

Lemma A.4. *Let $\mathfrak{h} \leq \mathfrak{g}$ be a Lie subalgebra with $\dim \mathfrak{h} \geq 2$ and $\dim p_{i_i}(\mathfrak{h}) = 1$. Then either \mathfrak{h} is conjugate to a semi-direct sum $\mathfrak{h}_L \oplus_{\Theta} T(\mathfrak{h})$ where $\mathfrak{h}_L \leq \mathbb{R} \oplus \mathfrak{so}(1, 2)$ is a Lie subalgebra, or it is conjugate to one the of following Lie algebras.*

$$\begin{array}{ll} \mathbb{R}(\mathcal{Y}_E + e_1) +_{\theta} (\mathbb{R}e_2 \oplus \mathbb{R}e_3), & \mathbb{R}(\mathcal{Y}_P + e_1) +_{\theta} \mathbb{R}(e_1 + e_2), \\ \mathbb{R}(\mathcal{Y}_P + e_1) +_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3), & \mathbb{R}(\mathcal{Y}_H + e_3) +_{\theta} \mathbb{R}(e_1 + e_2), \\ \mathbb{R}(1 + \mathcal{Y}_H + e_1) +_{\theta} \mathbb{R}(e_1 + e_2), & \mathbb{R}(1 + \mathcal{Y}_H + e_1) +_{\Theta} \mathbb{R}e_3 \\ \mathbb{R}(1 + \mathcal{Y}_H + e_1) +_{\Theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3), & \mathbb{R}(-1 + \mathcal{Y}_H + e_1) +_{\Theta} \mathbb{R}e_3, \\ \mathbb{R}(\mathcal{Y}_H + e_3) +_{\theta} (\mathbb{R}e_1 \oplus \mathbb{R}e_2). & \end{array}$$

Proof. Observe that, the linear isometry projection $p_{i_i}(\mathfrak{h})$ is generated by \mathcal{Y}_E , \mathcal{Y}_H , or \mathcal{Y}_P up to conjugacy. Notice that, if $T(\mathfrak{h}) = \{0\}$, then $p_h(\mathfrak{h}) = \mathbb{R}$ and for $V = \mathcal{Y}_E$, \mathcal{Y}_H , or \mathcal{Y}_P , there are vectors $u, v \in \mathbb{R}^{1,2}$ such that $\{\lambda + u, V + v\}$ is a basis for \mathfrak{h} . Considering the Lie bracket, we have

$$[\lambda + u, V + v] = v - V(u) \Rightarrow v - V(u) = 0.$$

Hence conjugating by u we have

$$Ad_{(1, Id, u)}(\lambda + u) = \lambda, \quad Ad_{(1, Id, u)}(V + v) = V.$$

Therefore, $\lambda, V \in \mathfrak{h}' = Ad_{(1, Id, u)}(\mathfrak{h})$. Thus, in this case \mathfrak{h} is conjugate to $\mathbb{R} \oplus \mathbb{R}\mathcal{Y}_E$, $\mathbb{R} \oplus \mathbb{R}\mathcal{Y}_H$, or $\mathbb{R} \oplus \mathbb{R}\mathcal{Y}_P$.

Now, assume that $T(\mathfrak{h}) \neq \{0\}$.

Case I: The linear isometry projection $p_{i_i}(\mathfrak{h})$ is elliptic. The \mathcal{Y}_E -invariant subspaces of $\mathbb{R}^{1,2}$ are $\{0\}$, $\mathbb{R}^{1,2}$, the timelike line $\mathbb{R}e_1$, and its corresponding (orthogonal) spacelike plane $\mathbb{R}e_2 \oplus \mathbb{R}e_3$.

- $T(\mathfrak{h}) = \mathbb{R}e_1$.

If $\dim p_l(\mathfrak{h}) = 1$, then there exist a constant $a \in \mathbb{R}$ and a vector $v \in \mathbb{R}^{1,2}$ such that $\{a + \mathcal{Y}_E + v, e_1\}$ is a basis for \mathfrak{h} . For $a = 0$, setting the vector $x = (0, -v_3, v_2)$ is a solution of the linear system

$$v - \mathcal{Y}_E(x) \in T(\mathfrak{h}).$$

In other words, we have

$$Ad_{(1, Id, x)}(\mathcal{Y}_E + v) = \mathcal{Y}_E + v_1 e_1, \quad Ad_{(1, Id, x)}(e_1) = e_1.$$

Thus $\mathcal{Y}_E \in \mathfrak{h}' = Ad_{(1, Id, x)}(\mathfrak{h})$. Therefore, \mathfrak{h} is conjugate to the direct sum

$$\mathbb{R}\mathcal{Y}_E \oplus \mathbb{R}e_1.$$

For $a \neq 0$, the vector $x = (\frac{v_1}{a}, \frac{av_2 - v_3}{a^2 + 1}, \frac{av_3 + v_2}{a^2 + 1})$ is the solution of the linear system

$$v - ax - \mathcal{Y}_E(x) = 0.$$

Hence,

$$Ad_{(1, Id, x)}(a + \mathcal{Y}_E + v) = a + \mathcal{Y}_E, \quad Ad_{(1, Id, x)}(e_1) = e_1.$$

Thus $a + \mathcal{Y}_E \in \mathfrak{h}' = Ad_{(1, Id, x)}(\mathfrak{h})$. Therefore, \mathfrak{h} is conjugate to the semi-direct sum

$$\mathbb{R}(a + \mathcal{Y}_E) \oplus_{\Theta} \mathbb{R}e_1.$$

If $\dim p_l(\mathfrak{h}) = 2$, then there exist two vectors $u, v \in \mathbb{R}^{1,2}$ such that $\{\lambda + u, \mathcal{Y}_E + v, e_1\}$ is a basis for \mathfrak{h} . Considering the Lie bracket we get

$$[\lambda + u, \mathcal{Y}_E + v] = v - \mathcal{Y}_E(u) \in \mathbb{R}e_1 \implies (v_1, v_2 - u_3, v_3 + u_2) = \beta e_1, \text{ for some } \beta \in \mathbb{R}.$$

Observe that u is a solution of the linear system

$$v - \mathcal{Y}_E(u) \in T(\mathfrak{h}).$$

Hence

$$Ad_{(1, Id, u)}(\lambda + u) = \lambda, \quad Ad_{(1, Id, u)}(\mathcal{Y}_E + v) = \mathcal{Y}_E + v_1 e_1, \quad Ad_{(1, Id, u)}(e_1) = e_1.$$

Obviously, $\lambda, \mathcal{Y}_E \in \mathfrak{h}' = Ad_{(1, Id, u)}(\mathfrak{h})$. Therefore \mathfrak{h} is conjugate to the semi-direct sum

$$(\mathbb{R} \oplus \mathbb{R}\mathcal{Y}_E) \oplus_{\Theta} \mathbb{R}e_1.$$

- $T(\mathfrak{h}) = \mathbb{R}e_2 \oplus \mathbb{R}e_3$.

If $\dim L(\mathfrak{h}) = 1$, then there exist a constant $a \in \mathbb{R}$ and a vector $v \in \mathbb{R}^{1,2}$ such that $\{a + \mathcal{Y}_E + v, e_2, e_3\}$ is a basis for \mathfrak{h} .

For $a = 0$, obviously $\mathcal{Y}_E + v_1 e_1 \in \mathfrak{h}$. Therefore, if $v_1 = 0$, then \mathfrak{h} is conjugate to the semidirect sum

$$\mathbb{R}\mathcal{Y}_E \oplus_{\Theta} (\mathbb{R}e_2 \oplus \mathbb{R}e_3).$$

If $v_1 \neq 0$, then \mathfrak{h} is conjugate to the following semidirect sum via $Ad_{(1/v_1, Id, 0)}$

$$\mathbb{R}(\mathcal{Y}_E + e_1) +_{\Theta} (\mathbb{R}e_2 \oplus \mathbb{R}e_3).$$

For $a \neq 0$, the vector $x = (\frac{v_1}{a}, \frac{av_2 - v_3}{a^2 + 1}, \frac{av_3 + v_2}{a^2 + 1})$ is the solution of the linear system

$$v - ax - \mathcal{Y}_E(x) = 0.$$

Hence, $a + \mathcal{Y}_E \in \mathfrak{h}' = Ad_{(1, Id, x)}(\mathfrak{h})$. Therefore \mathfrak{h} is conjugate to the semi-direct sum

$$\mathbb{R}(a + \mathcal{Y}_E) +_{\Theta} (\mathbb{R}e_2 \oplus \mathbb{R}e_3).$$

If $\dim p_l(\mathfrak{h}) = 2$, then there are two vectors $u, v \in \mathbb{R}^{1,2}$ such that $\{\lambda + u, \mathcal{Y}_E + v, e_2, e_3\}$ is a basis for \mathfrak{h} . Considering the Lie bracket

$$[\lambda + u, \mathcal{Y}_E + v] = v - \mathcal{Y}_E(u) \in T(\mathfrak{h}),$$

we get $v_1 = 0$. Observe that u is a solution of the linear system

$$v - \mathcal{Y}_E(u) \in T(\mathfrak{h}).$$

Precisely, we have

$$\begin{aligned} Ad_{(1, Id, u)}(\lambda + u) &= \lambda & , & & Ad_{(1, Id, u)}(\mathcal{Y}_E + v) &= X + (0, v_2 - u_3, v_3 + u_2), \\ Ad_{(1, Id, u)}(e_2) &= e_2 & , & & Ad_{(1, Id, u)}(e_3) &= e_3. \end{aligned}$$

Obviously, $\lambda, \mathcal{Y}_E \in \mathfrak{h}' = Ad_{(1, Id, u)}(\mathfrak{h})$. Therefore \mathfrak{h} is conjugate to the semi-direct sum

$$(\mathbb{R} \oplus \mathbb{R}\mathcal{Y}_E) +_{\Theta} (\mathbb{R}e_2 \oplus \mathbb{R}e_3).$$

- $T(\mathfrak{h}) = \mathbb{R}^{1,2}$.

If $\dim p_l(\mathfrak{h}) = 1$, then there exist a constant $a \in \mathbb{R}$ and a vector $v \in \mathbb{R}^{1,2}$ such that $\{a + \mathcal{Y}_E + v, e_1, e_2, e_3\}$ is a basis for \mathfrak{h} . Obviously, $v \in T(\mathfrak{h})$, and so $a + \mathcal{Y}_E \in \mathfrak{h}$. Therefore \mathfrak{h} is conjugate to the semi-direct sum

$$\mathbb{R}(a + \mathcal{Y}_E) +_{\Theta} \mathbb{R}^{1,2} \text{ for a constant } a \in \mathbb{R}.$$

If $\dim p_l(\mathfrak{h}) = 2$, then there exist two vectors $u, v \in \mathbb{R}^{1,2}$ such that $\{\lambda + u, \mathcal{Y}_E + v, e_1\}$ is a basis for \mathfrak{h} . Clearly $u, v \in T(\mathfrak{h})$, and so $\lambda, \mathcal{Y}_E \in \mathfrak{h}$. Therefore \mathfrak{h} is conjugate to the semi-direct sum

$$(\mathbb{R} \oplus \mathbb{R}\mathcal{Y}_E) +_{\Theta} \mathbb{R}^{1,2}.$$

Case II: The linear isometry projection $p_{li}(\mathfrak{h})$ is parabolic. The \mathcal{Y}_P -invariant subspaces of $\mathbb{R}^{1,2}$ are $\{0\}, \mathbb{R}^{1,2}$, the lightlike line $\mathbb{R}(e_1 + e_2)$, and its corresponding (orthogonal) lightlike plane $\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3$.

- $T(\mathfrak{h}) = \mathbb{R}(e_1 + e_2)$.

- If $\dim p_l(\mathfrak{h}) = 1$, then there exist a constant $a \in \mathbb{R}$ and a vector $v \in \mathbb{R}^{1,2}$ such that $\{a + \mathcal{Y}_P + v, e_1 + e_2\}$ is a basis for \mathfrak{h} .

For $a = 0$, the linear system

$$v - \mathcal{Y}_P(x) \in T(\mathfrak{h}),$$

has a solution if and only if $v_1 = v_2$. Now, setting $x = (0, -v_3, v_2)$, we have

$$Ad_{(1, Id, x)}(\mathcal{Y}_P + v) = \mathcal{Y}_P + (v_1 - v_2)e_1, \quad Ad_{(1, Id, x)}(e_1 + e_2) = e_1 + e_2.$$

Thus if $v_1 = v_2$, then \mathfrak{h} is conjugate to the direct sum

$$\mathbb{R}\mathcal{Y}_P \oplus \mathbb{R}(e_1 + e_2).$$

If $v_1 \neq v_2$, then there exists \mathfrak{h} is conjugate to the following Lie subalgebra via $Ad_{(\frac{1}{v_1 - v_2}, Id, 0)}$

$$\mathbb{R}(\mathcal{Y}_P + e_1) +_{\Theta} \mathbb{R}(e_1 + e_2).$$

For $a \neq 0$, the vector

$$x = \left(\frac{(a^2 + 1)v_1 - av_3 - v_2}{a^3}, \frac{(a^2 - 1)v_2 - av_3 + v_1}{a^3}, \frac{av_3 - v_1 + v_2}{a^2} \right),$$

is the solution of the linear system

$$v - ax - \mathcal{Y}_P(x) = 0.$$

In other words, we have

$$Ad_{(1, Id, x)}(a + \mathcal{Y}_P + v) = a + \mathcal{Y}_P, \quad Ad_{(1, Id, x)}(e_1 + e_2) = e_1 + e_2.$$

Thus $a + \mathcal{Y}_P \in \mathfrak{h}' = Ad_{(1, Id, x)}(\mathfrak{h})$. Therefore \mathfrak{h} is conjugate to the semi-direct sum

$$\mathbb{R}(a + \mathcal{Y}_P) \oplus_{\Theta} \mathbb{R}(e_1 + e_2).$$

- If $\dim p_l(\mathfrak{h}) = 2$, then there exist two vectors $u, v \in \mathbb{R}^{1,2}$ such that $\{\lambda + u, \mathcal{Y}_P + v, e_1 + e_2\}$ is a basis for \mathfrak{h} . Considering the Lie bracket

$$[\lambda + u, \mathcal{Y}_P + v] = v - \mathcal{Y}_P(u) \in T(\mathfrak{h}),$$

we obtain $v_3 = u_1 - u_2$. Observe that conjugating by u we get

$$\begin{aligned} Ad_{(1, Id, u)}(\lambda + u) &= \lambda, & Ad_{(1, Id, u)}(\mathcal{Y}_P + v) &= \mathcal{Y}_P + (v_1 - u_3)(e_1 + e_2), \\ Ad_{(1, Id, u)}(e_1 + e_2) &= e_1 + e_2. \end{aligned}$$

Obviously, $\lambda, \mathcal{Y}_P \in \mathfrak{h}' = Ad_{(1, Id, u)}(\mathfrak{h})$. Therefore \mathfrak{h} is conjugate to the semi-direct sum

$$(\mathbb{R} \oplus \mathbb{R}\mathcal{Y}_P) \oplus_{\Theta} \mathbb{R}(e_1 + e_2).$$

- $T(\mathfrak{h}) = \mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3$.

- If $\dim p_l(\mathfrak{h}) = 1$, then there exist a constant $a \in \mathbb{R}$ and a vector $v \in \mathbb{R}^{1,2}$ such that $\{a + \mathcal{Y}_P + v, e_1 + e_2, e_3\}$ is a basis for \mathfrak{h} .

If $a = 0$, the linear system

$$v - \mathcal{Y}_P(x) \in T(\mathfrak{h}),$$

has a solution if and only if $v_1 = v_2$. Hence, setting $x = (0, -v_3, v_2)$ we have

$$Ad_{(1, Id, x)}(\mathcal{Y}_P + v) = \mathcal{Y}_P + (v_1 - v_2)e_1, \quad Ad_{(1, Id, x)}(e_1 + e_2) = e_1 + e_2, \quad Ad_{(1, Id, x)}(e_3) = e_3.$$

Thus, if $v_1 = v_2$, then \mathfrak{h} is conjugate to the semi-direct sum

$$\mathbb{R}\mathcal{Y}_P \oplus_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3).$$

If $v_1 \neq v_2$, then \mathfrak{h} is conjugate to the following Lie algebra via $Ad_{(\frac{1}{v_1 - v_2}, Id, 0)}$

$$\mathbb{R}(\mathcal{Y}_P + e_1) \oplus_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3).$$

For $a \neq 0$, the vector

$$x = \left(\frac{(a^2 + 1)v_1 - av_3 - v_2}{a^3}, \frac{(a^2 - 1)v_2 - av_3 + v_1}{a^3}, \frac{av_3 - v_1 + v_2}{a^2} \right),$$

is the solution of the linear system

$$v - ax - \mathcal{Y}_P(x).$$

Precisely, we have

$$Ad_{(1, Id, x)}(a + \mathcal{Y}_P + v) = a + \mathcal{Y}_P, \quad Ad_{(1, Id, x)}(e_1 + e_2) = e_1 + e_2, \quad Ad_{(1, Id, x)}(e_3) = e_3.$$

Thus $a + \mathcal{Y}_P \in \mathfrak{h}' = Ad_{(1, Id, x)}(\mathfrak{h})$. Therefore \mathfrak{h} is conjugate to the semi-direct sum

$$\mathbb{R}(a + \mathcal{Y}_P) \oplus_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3).$$

- If $\dim p_l(\mathfrak{h}) = 2$, then there exist two vectors $u, v \in \mathbb{R}^{1,2}$ such that $\{\lambda + u, \mathcal{Y}_P + v, e_1 + e_2, e_3\}$ is a basis for \mathfrak{h} . Considering the Lie bracket

$$[\lambda + u, \mathcal{Y}_P + v] = v - \mathcal{Y}_P(u) \in T(\mathfrak{h}),$$

we have $v_1 = v_2$. Observe u is a solution of the linear system

$$v - \mathcal{Y}_P(x) \in T(\mathfrak{h}).$$

Hence, we have

$$\begin{aligned} Ad_{(1, Id, u)}(\lambda + u) &= \lambda, & Ad_{(1, Id, u)}(e_1 + e_2) &= e_1 + e_2, \\ Ad_{(1, Id, u)}(\mathcal{Y}_P + v) &= \mathcal{Y}_P + (v_1 - u_3)(e_1 + e_2) + (v_3 - u_1 + u_2)e_3. \end{aligned}$$

Obviously, $\lambda, \mathcal{Y}_P \in \mathfrak{h}' = Ad_{(1, Id, u)}(\mathfrak{h})$. Therefore \mathfrak{h} is conjugate to the semi-direct sum

$$(\mathbb{R} \oplus \mathbb{R}\mathcal{Y}_P) \oplus_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3).$$

- $T(\mathfrak{h}) = \mathbb{R}^{1,2}$.

- If $\dim p_l(\mathfrak{h}) = 1$, then there exist a constant $a \in \mathbb{R}$ and a vector $v \in \mathbb{R}^{1,2}$ such that $\{a + \mathcal{Y}_P + v, e_1, e_2, e_3\}$ is a basis for \mathfrak{h} . Obviously, $v \in T(\mathfrak{h})$, and so $a + \mathcal{Y}_P \in \mathfrak{h}$. Therefore \mathfrak{h} is conjugate to the semi-direct sum

$$\mathbb{R}(a + \mathcal{Y}_P) \oplus_{\theta} \mathbb{R}^{1,2} \text{ for a constant } a \in \mathbb{R}.$$

- If $\dim p_l(\mathfrak{h}) = 2$, then there exist two vectors $u, v \in \mathbb{R}^{1,2}$ such that $\{\lambda + u, \mathcal{Y}_P + v, e_1, e_2, e_3\}$ is a basis for \mathfrak{h} . Clearly $u, v \in T(\mathfrak{h})$, and so $\lambda, \mathcal{Y}_P \in \mathfrak{h}$. Therefore \mathfrak{h} is conjugate to the semi-direct sum

$$(\mathbb{R} \oplus \mathbb{R}\mathcal{Y}_P) \oplus_{\theta} \mathbb{R}^{1,2}.$$

Case III: The linear isometry part is hyperbolic.

The \mathcal{Y}_H -invariant subspaces of $\mathbb{R}^{1,2}$ are $\{0\}$, $\mathbb{R}^{1,2}$, the two lightlike lines $\mathbb{R}(e_1 + e_2)$, $\mathbb{R}(e_1 - e_2)$, and their corresponding lightlike planes $\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3$, $\mathbb{R}(e_1 - e_2) \oplus \mathbb{R}e_3$, the spacelike line $\mathbb{R}e_3$, and the timelike plane contains $\mathbb{R}e_1 \oplus \mathbb{R}e_2$, orthogonal to $\mathbb{R}e_3$.

It is remarkable that conjugation by

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in SO_o(1, 2)$$

leaves $\mathbb{R}\mathcal{Y}_H$ invariant, and maps the two lightlike lines $\mathbb{R}(e_1 + e_2)$ and $\mathbb{R}(e_1 - e_2)$, to each other. Consequently, Q maps the two corresponding lightlike planes $\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3$ and $\mathbb{R}(e_1 - e_2) \oplus \mathbb{R}e_3$ to each other. Therefore, Lie subalgebras with translation part $\mathbb{R}(e_1 - e_2)$ (resp. $\mathbb{R}(e_1 - e_2) \oplus \mathbb{R}e_3$), are conjugate to the corresponding Lie subalgebras with translation part $\mathbb{R}(e_1 + e_2)$ (resp. $\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3$).

- $T(\mathfrak{h}) = \mathbb{R}(e_1 + e_2)$.

- If $\dim L(\mathfrak{h}) = 1$, then there exist a constant $a \in \mathbb{R}$ and a vector $v \in \mathbb{R}^{1,2}$ such that $\{a + \mathcal{Y}_H + v, e_1 + e_2\}$ is a basis for \mathfrak{h} .

For $a = 0$, the linear system

$$v - \mathcal{Y}_H(x) \in T(\mathfrak{h}).$$

has a solution if and only if $v_3 = 0$. Hence, setting $x = (v_2, v_1, 0)$, we have

$$Ad_{(1, Id, x)}(\mathcal{Y}_H + v) = \mathcal{Y}_H + v_3 e_3, \quad Ad_{(1, Id, x)}(e_1 + e_2) = e_1 + e_2.$$

Thus if $v_3 = 0$, then \mathfrak{h} is conjugate to the semi-direct sum

$$\mathbb{R}(\mathcal{Y}_H) \oplus_{\theta} \mathbb{R}(e_1 + e_2).$$

If $v_3 \neq 0$, then \mathfrak{h} is conjugate to the following lie subalgebra via $Ad_{(1/v_3, Id, 0)}$

$$\mathbb{R}(\mathcal{Y}_H + e_3) +_{\theta} \mathbb{R}(e_1 + e_2).$$

For $a \in \mathbb{R}^* \setminus \{1, -1\}$, the vector $x = (\frac{-v_2 + av_1}{a^2 - 1}, \frac{-v_1 + av_2}{a^2 - 1}, \frac{v_3}{a})$ is the solution of the linear system

$$v - ax - \mathcal{Y}_H(x) = 0$$

Hence, we have

$$Ad_{(1,Id,x)}(a + \mathcal{Y}_H + v) = a + \mathcal{Y}_H, \quad Ad_{(1,Id,x)}(e_1 + e_2) = e_1 + e_2.$$

Thus $a + \mathcal{Y}_H \in \mathfrak{h}' = Ad_{(1,Id,x)}(\mathfrak{h})$. Therefore \mathfrak{h} is conjugate to the semi-direct sum

$$\mathbb{R}(a + \mathcal{Y}_H) \oplus_{\Theta} \mathbb{R}(e_1 + e_2).$$

For $a = 1$, the linear system

$$v - x - \mathcal{Y}_H(x) \in T(\mathfrak{h}),$$

has a solution if and only if $v_1 = v_2$. Hence, setting $x = (0, v_2, v_3)$, we get

$$Ad_{(1,Id,x)}(1 + \mathcal{Y}_H) = 1 + \mathcal{Y}_H + (v_1 - v_2)e_1, \quad Ad_{(1,Id,x)}(e_1 + e_2) = e_1 + e_2,$$

Therefore, if $v_1 = v_2$, then \mathfrak{h} is conjugate to the semi-direct sum

$$\mathbb{R}(1 + \mathcal{Y}_H) \oplus_{\theta} \mathbb{R}(e_1 + e_2).$$

Otherwise, \mathfrak{h} is conjugate to the following Lie algebra via $Ad_{(\frac{1}{v_1-v_2}, Id, 0)}$

$$\mathbb{R}(1 + \mathcal{Y}_H + e_1) +_{\theta} \mathbb{R}(e_1 + e_2).$$

For $a = -1$, the vector $x = (\frac{v_2}{2}, \frac{v_1}{2}, -v_3)$ is a solution for the linear system

$$v + x - \mathcal{Y}_H(x) \in T(\mathfrak{h}).$$

In other words, we have

$$Ad_{(1,Id,x)}(-1 + \mathcal{Y}_H) = -1 + \mathcal{Y}_H + \frac{v_1 + v_2}{2}(e_1 + e_2), \quad Ad_{(1,Id,x)}(e_1 + e_2) = e_1 + e_2,$$

Obviously, $-1 + \mathcal{Y}_H \in \mathfrak{h}' = Ad_{(1,Id,x)}(\mathfrak{h})$. Therefore, \mathfrak{h} is conjugate to the semi-direct sum

$$\mathbb{R}(-1 + \mathcal{Y}_H) \oplus_{\theta} \mathbb{R}(e_1 + e_2).$$

- If $\dim p_l(\mathfrak{h}) = 2$, then there are two vectors $u, v \in \mathbb{R}^{1,2}$ such that $\{\lambda + u, \mathcal{Y}_H + v, e_1 + e_2\}$ is a basis for \mathfrak{h} . Considering the Lie bracket

$$[\lambda + u, \mathcal{Y}_H + v] = v - \mathcal{Y}_H(u) \in T(\mathfrak{h}).$$

we get $v_3 = 0$ and $v_1 - u_2 = v_2 - u_1$. Observe that u is a solution of the linear system

$$v - \mathcal{Y}_H(x) \in T(\mathfrak{h}),$$

and we have

$$\begin{aligned} Ad_{(1,Id,u)}(\lambda + u) &= \lambda, & Ad_{(1,Id,u)}(\mathcal{Y}_H + v) &= \mathcal{Y}_H + (v_1 - u_2)(e_1 + e_2), \\ Ad_{(1,Id,u)}(e_1 + e_2) &= e_1 + e_2. \end{aligned}$$

Obviously, $\lambda, \mathcal{Y}_H \in \mathfrak{h}' = Ad_{(1,Id,u)}(\mathfrak{h})$. Therefore, \mathfrak{h} is conjugate to the semi-direct sum

$$(\mathbb{R} \oplus \mathbb{R}\mathcal{Y}_H) \oplus_{\theta} \mathbb{R}(e_1 + e_2).$$

• $T(\mathfrak{h}) = \mathbb{R}e_3$.

– If $\dim p_l(\mathfrak{h}) = 1$, then there exist a constant $a \in \mathbb{R}$ and a vector $v \in \mathbb{R}^{1,2}$ such that $\{a + \mathcal{Y}_H + v, e_3\}$ is a basis for \mathfrak{h} .

If $a = 0$, the vector $x = (v_2, v_1, 0)$ is a solution of the linear system

$$v - \mathcal{Y}_H(x) \in T(\mathfrak{h}).$$

In other words, we have

$$Ad_{(1, Id, x)}(\mathcal{Y}_H + v) = \mathcal{Y}_H + v_3 e_3, \quad Ad_{(1, Id, x)}e_3 = e_3.$$

Obviously $\mathcal{Y}_H \in \mathfrak{h}' = Ad_{(1, Id, x)}(\mathfrak{h})$. Therefore, \mathfrak{h} is conjugate to the direct sum

$$\mathbb{R}\mathcal{Y}_H \oplus \mathbb{R}e_3.$$

For $a \in \mathbb{R}^* \setminus \{1, -1\}$, the vector $x = (\frac{-v_2 + av_1}{a^2 - 1}, \frac{-v_1 + av_2}{a^2 - 1}, \frac{v_3}{a})$ is the solution the linear system

$$v - ax - \mathcal{Y}_H(x) = 0,$$

and we have

$$Ad_{(1, Id, x)}(a + \mathcal{Y}_H + v) = a + \mathcal{Y}_H, \quad Ad_{(1, Id, x)}(e_3) = e_3.$$

Thus $a + \mathcal{Y}_H \in \mathfrak{h}' = Ad_{(1, Id, x)}(\mathfrak{h})$. Therefore \mathfrak{h} is conjugate to the semi-direct sum

$$\mathbb{R}(a + \mathcal{Y}_H) \oplus_{\theta} \mathbb{R}e_3.$$

If $a = 1$, the linear system

$$u - x - \mathcal{Y}_H(x) \in T(\mathfrak{h}),$$

has a solution if and only if $v_1 = v_2$. Now, setting $x = (0, v_2, v_3)$, we get

$$Ad_{(1, Id, x)}(1 + \mathcal{Y}_H) = 1 + \mathcal{Y}_H + (v_1 - v_2)e_1, \quad Ad_{(1, Id, x)}(e_3) = e_3,$$

So, if $v_1 = v_2$, \mathfrak{h} is conjugate to the semi-direct sum

$$\mathbb{R}(1 + \mathcal{Y}_H) \oplus_{\theta} \mathbb{R}e_3.$$

Otherwise, it is conjugate to the following Lie algebra via $Ad_{(\frac{1}{v_1 - v_2}, Id, 0)}$

$$\mathbb{R}(1 + \mathcal{Y}_H + e_1) +_{\theta} \mathbb{R}e_3.$$

If $a = -1$, the linear system

$$v + x - \mathcal{Y}_H(x) \in T(\mathfrak{h}),$$

has a solution if and only if $v_1 = -v_2$. Now, setting $x = (0, -v_2, -v_3)$, we get

$$Ad_{(1, Id, x)}(-1 + \mathcal{Y}_H) = -1 + \mathcal{Y}_H + (v_1 + v_2)e_1, \quad Ad_{(1, Id, x)}(e_3) = e_3.$$

So, if $v_1 = -v_2$, \mathfrak{h} is conjugate to the semi-direct sum

$$\mathbb{R}(-1 + \mathcal{Y}_H) \oplus_{\theta} \mathbb{R}e_3.$$

Otherwise, it is conjugate to the following Lie algebra via $Ad_{(\frac{1}{v_1 + v_2}, Id, 0)}$

$$\mathbb{R}(-1 + \mathcal{Y}_H + e_1) +_{\theta} \mathbb{R}e_3.$$

- If $\dim p_l(\mathfrak{h}) = 2$, then there are two vectors $u, v \in \mathbb{R}^{1,2}$ such that $\{\lambda + u, \mathcal{Y}_H + v, e_3\}$ is a basis for \mathfrak{h} . Considering the Lie bracket

$$[\lambda + u, \mathcal{Y}_H + v] = v - \mathcal{Y}_H(u) \in T(\mathfrak{h}),$$

we get, $v_1 = u_2$ and $v_2 = u_1$. Observe that u is a solution of the linear system

$$v - \mathcal{Y}_H(x) \in T(\mathfrak{h}).$$

Precisely, we have

$$\begin{aligned} Ad_{(1,Id,u)}(\lambda + u) &= \lambda, & Ad_{(1,Id,u)}(\mathcal{Y}_H + v) &= \mathcal{Y}_H + v_3 e_3, \\ Ad_{(1,Id,u)}(e_3) &= e_3. \end{aligned}$$

Obviously, $\lambda, \mathcal{Y}_H \in \mathfrak{h}' = Ad_{(1,Id,u)}(\mathfrak{h})$. Therefore, \mathfrak{h} is conjugate to the semi-direct sum

$$(\mathbb{R} \oplus \mathbb{R}Y) \oplus_{\theta} \mathbb{R}e_3.$$

- $T(\mathfrak{h}) = \mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3$.

- If $\dim p_l(\mathfrak{h}) = 1$, then there exist a constant $a \in \mathbb{R}$ and a vector $v \in \mathbb{R}^{1,2}$ such that $\{a + \mathcal{Y}_H + v, e_1 + e_2, e_3\}$ is a basis for \mathfrak{h} .

If $a = 0$, the vector $x = (v_2, v_1, 0)$ is a solution of the linear system

$$v - \mathcal{Y}_H(x) \in T(\mathfrak{h}),$$

and we have

$$Ad_{(1,Id,x)}(\mathcal{Y}_H + v) = \mathcal{Y}_H + v_3 e_3, \quad Ad_{(1,Id,x)}(e_1 + e_2) = e_1 + e_2, \quad Ad_{(1,Id,x)}(e_3) = e_3.$$

Clearly $\mathcal{Y}_H \in \mathfrak{h}' = Ad_{(1,Id,x)}(\mathfrak{h})$. Therefore, \mathfrak{h} is conjugate to the semi-direct sum

$$\mathbb{R}\mathcal{Y}_H \oplus_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3).$$

For $a \in \mathbb{R}^* \setminus \{1, -1\}$, the vector $x = (\frac{-v_2 + av_1}{a^2 - 1}, \frac{-v_1 + av_2}{a^2 - 1}, \frac{v_3}{a})$ is the solution of the linear system

$$v - ax - \mathcal{Y}_H(x) = 0,$$

and we have

$$Ad_{(1,Id,x)}(a + \mathcal{Y}_H + v) = a + \mathcal{Y}_H, \quad Ad_{(1,Id,x)}(e_1 + e_2) = e_1 + e_2, \quad Ad_{(1,Id,x)}(e_3) = e_3.$$

Thus $a + \mathcal{Y}_H \in \mathfrak{h}' = Ad_{(1,Id,x)}(\mathfrak{h})$. Therefore, \mathfrak{h} is conjugate to the semi-direct sum

$$\mathbb{R}(a + \mathcal{Y}_H) \oplus_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3).$$

If $a = 1$, the linear system

$$v - x - \mathcal{Y}_H(x) \in T(\mathfrak{h}),$$

has a solution if and only if $v_1 = v_2$. Now, setting $x = (0, v_2, v_3)$, we get

$$\begin{aligned} Ad_{(1, Id, x)}(e_1 + e_2) &= e_1 + e_2 \quad , \quad Ad_{(1, Id, x)}(e_3) = e_3, \\ Ad_{(1, Id, x)}(1 + \mathcal{Y}_H) &= 1 + \mathcal{Y}_H + (v_1 - v_2)e_1, \end{aligned}$$

Hence, if $v_1 = v_2$, then \mathfrak{h} is conjugate to the semi-direct sum

$$\mathbb{R}(1 + \mathcal{Y}_H) \oplus_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3).$$

Otherwise, it is conjugate to the following Lie algebra via $Ad_{(\frac{1}{v_1 - v_2}, Id, 0)}$

$$\mathbb{R}(1 + \mathcal{Y}_H + e_1) \oplus_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3).$$

If $a = -1$, the vector $x = (\frac{v_2}{2}, \frac{v_1}{2}, -v_3)$ is a solution for the linear system

$$v - x - \mathcal{Y}_H(x) \in T(\mathfrak{h}),$$

and we have

$$\begin{aligned} Ad_{(1, Id, x)}(e_1 + e_2) &= e_1 + e_2 \quad , \quad Ad_{(1, Id, x)}(e_3) = e_3, \\ Ad_{(1, Id, x)}(-1 + \mathcal{Y}_H) &= -1 + \mathcal{Y}_H + \frac{v_1 + v_2}{2}(e_1 + e_2), \end{aligned}$$

Obviously, $-1 + \mathcal{Y}_H \in \mathfrak{h}' = Ad_{(1, Id, x)}(\mathfrak{h})$. Therefore, \mathfrak{h} is conjugate to the semi-direct sum

$$\mathbb{R}(-1 + \mathcal{Y}_H) \oplus_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3).$$

- If $\dim L(\mathfrak{h}) = 2$, then there exist two vectors $u, v \in \mathbb{R}^{1,2}$ such that $\{\lambda + u, \mathcal{Y}_H + v, e_1 + e_2, e_3\}$ is a basis for \mathfrak{h} . Considering the Lie bracket

$$[\lambda + u, \mathcal{Y}_H + v] = v - \mathcal{Y}_H(u) \in T(\mathfrak{h}),$$

we get $v_1 - u_2 = v_2 - u_1$. Hence

$$\begin{aligned} Ad_{(1, Id, u)}(\lambda + u) &= \lambda, & Ad_{(1, Id, u)}(\mathcal{Y}_H + v) &= \mathcal{Y}_H + (v_1 - u_2)(e_1 + e_2) + v_3e_3, \\ Ad_{(1, Id, u)}(e_1 + e_2) &= e_1 + e_2, & Ad_{(1, Id, u)}(e_3) &= e_3. \end{aligned}$$

Obviously, $\lambda, \mathcal{Y}_H \in \mathfrak{h}' = Ad_{(1, Id, u)}(\mathfrak{h})$, and therefore \mathfrak{h} is conjugate to the semidirect sum

$$(\mathbb{R} \oplus \mathbb{R}\mathcal{Y}_H) \oplus_{\Theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3).$$

- $T(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}e_2$.

- If $\dim p_l(\mathfrak{h}) = 1$, then there exist a constant $a \in \mathbb{R}$ and a vector $v \in \mathbb{R}^{1,2}$ such that $\{a + \mathcal{Y}_H + v, e_1, e_2\}$ is a basis for \mathfrak{h} . Obviously, $a + \mathcal{Y}_H + v_3e_3 \in \mathfrak{h}$, hence we may assume $v = (0, 0, v_3)$.

If $a = 0$, the linear system

$$v - \mathcal{Y}_H \in T(\mathfrak{h}),$$

has a solution if and only if $v - 3 = 0$. Therefore, if $v_3 = 0$, \mathfrak{h} is conjugate to the semi-direct sum

$$\mathbb{R}\mathcal{Y}_H \oplus_{\theta} (\mathbb{R}e_1 \oplus \mathbb{R}e_2).$$

Otherwise, \mathfrak{h} is conjugate to the following Lie algebra via $Ad_{(1/v_3, Id, 0)}$

$$\mathbb{R}(\mathcal{Y}_H + e_3) \oplus_{\theta} (\mathbb{R}e_1 \oplus \mathbb{R}e_2).$$

For $a \in \mathbb{R}^*$, the vector $x = (0, 0, \frac{v_3}{a})$ is the solution of the linear system

$$v - ax - \mathcal{Y}_H(x) = 0,$$

and we have

$$Ad_{(1, Id, x)}(a + \mathcal{Y}_H + v_3 e_3) = a + \mathcal{Y}_H, \quad Ad_{(1, Id, x)}(e_1) = e_1, \quad Ad_{(1, Id, x)}(e_2) = e_2.$$

Thus $a + Y \in \mathfrak{h}' = Ad_{(1, Id, x)}(\mathfrak{h})$. Therefore \mathfrak{h} is conjugate to the semi-direct sum

$$\mathbb{R}(a + \mathcal{Y}_H) \oplus_{\theta} (\mathbb{R}e_1 \oplus \mathbb{R}e_2).$$

- If $\dim p_l(\mathfrak{h}) = 2$, then there exist two vectors $u, v \in \mathbb{R}^{1,2}$ such that $\{\lambda + u, \mathcal{Y}_H + v, e_1, e_2\}$ is a basis for \mathfrak{h} . Considering the Lie bracket

$$[\lambda + u, \mathcal{Y}_H + v] = v - \mathcal{Y}_H(u) \in T(\mathfrak{h}),$$

we get $v_3 = 0$. Hence, $\mathcal{Y}_H \in \mathfrak{h}$. On the other hand, we have

$$\begin{aligned} Ad_{(1, Id, u)}(\lambda + u) &= \lambda, & Ad_{(1, Id, u)}(\mathcal{Y}_H) &= \mathcal{Y}_H + (-u_2, -u_1, 0), \\ Ad_{(1, Id, u)}(e_1) &= e_1, & Ad_{(1, Id, u)}(e_2) &= e_2. \end{aligned}$$

Obviously, $\lambda, \mathcal{Y}_H \in \mathfrak{h}' = Ad_{(1, Id, u)}(\mathfrak{h})$, and therefore \mathfrak{h} is conjugate to the semi-direct sum

$$(\mathbb{R} \oplus \mathbb{R}\mathcal{Y}_H) \oplus_{\theta} (\mathbb{R}e_1 \oplus \mathbb{R}e_2).$$

- $T(\mathfrak{h}) = \mathbb{R}^{1,2}$.

If $\dim p_l(\mathfrak{h}) = 1$, then there exist a constant $a \in \mathbb{R}$ and a vector $v \in \mathbb{R}^{1,2}$ such that $\{a + \mathcal{Y}_H + v, e_1, e_2, e_3\}$ is a basis for \mathfrak{h} . Obviously, $v \in T(\mathfrak{h})$, and so $a + Y \in \mathfrak{h}$. Therefore \mathfrak{h} is conjugate to the semi-direct sum

$$\mathbb{R}(a + \mathcal{Y}_H) \oplus_{\theta} \mathbb{R}^{1,2} \text{ for a constant } a \in \mathbb{R}.$$

If $\dim p_l(\mathfrak{h}) = 2$, then there exist two vectors $u, v \in \mathbb{R}^{1,2}$ such that $\{\lambda + u, \mathcal{Y}_H + v, e_1, e_2, e_3\}$ is a basis for \mathfrak{h} . Clearly $u, v \in T(\mathfrak{h})$, and so $\lambda, Y \in \mathfrak{h}$. Therefore \mathfrak{h} is conjugate to the semi-direct sum

$$(\mathbb{R} \oplus \mathbb{R}\mathcal{Y}_H) \oplus_{\theta} \mathbb{R}^{1,2}.$$

□

Lemma A.5. *Let $\mathfrak{h} \leq \mathfrak{g}$ be a Lie subalgebra with $\dim \mathfrak{h} \geq 2$ and its linear isometry projection $p_i(\mathfrak{h})$ be trivial. Then \mathfrak{h} is conjugate to one of the following Lie subalgebras.*

$$\begin{aligned} &\mathbb{R} \oplus_{\theta} \mathbb{R}^{1,2}, \quad \mathbb{R} \oplus_{\theta} (\mathbb{R}e_1 \oplus \mathbb{R}e_2), \quad \mathbb{R} \oplus_{\theta} (\mathbb{R}e_2 \oplus \mathbb{R}e_3), \quad \mathbb{R} \oplus_{\theta} (\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_2), \quad \mathbb{R} \oplus_{\theta} \mathbb{R}e_1, \\ &\mathbb{R} \oplus_{\theta} \mathbb{R}e_2, \quad \mathbb{R} \oplus_{\theta} \mathbb{R}(e_1 + e_2), \quad \mathbb{R}^{1,2}, \quad \mathbb{R}e_1 \oplus \mathbb{R}e_2, \quad \mathbb{R}e_2 \oplus \mathbb{R}e_3, \quad \mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3. \end{aligned}$$

Proof. By Proposition (4.5), \mathfrak{h} is conjugate to $p_h(\mathfrak{h}) \oplus_{\theta} T(\mathfrak{h})$. On the other hand, by the action of $SO_o(1, 2)$ on $\mathbb{R}^{1,2}$, every 2-dimensional (resp. 1-dimensional) linear subspace of $\mathbb{R}^{1,2}$ is conjugate to $\mathbb{R}e_1 \oplus \mathbb{R}e_2$, $\mathbb{R}e_2 \oplus \mathbb{R}e_3$ or $\mathbb{R}(e_1 + e_2) \oplus \mathbb{R}e_3$ (resp. $\mathbb{R}e_1$, $\mathbb{R}e_2$ or $\mathbb{R}(e_1 + e_2)$). Now, the lemma follows easily. \square

Appendix B

Some 1-parameter subgroups of $\text{Conf}(\mathbb{E}^{1,2})$

Here we indicate some 1-parameter subgroups of $\text{Conf}(\mathbb{E}^{1,2})$ which we used in Chapter (4) to determine the orbits induced by some cohomogeneity one actions on $\mathbb{E}in^{1,2}$.

First of all, we list the three 1-parameter subgroups of $SO_o(1, 2) \simeq \text{PSL}(2, \mathbb{R})$.

$$Y_E = \exp(\mathbb{R}\mathcal{Y}_E) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix} : t \in \mathbb{R} \right\}, \quad (\text{B.1})$$

$$Y_H = \exp(\mathbb{R}\mathcal{Y}_H) = \left\{ \begin{bmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{bmatrix} : t \in \mathbb{R} \right\}, \quad (\text{B.2})$$

$$Y_P = \exp(\mathbb{R}\mathcal{Y}_P) = \left\{ \begin{bmatrix} 1 + \frac{1}{2}t^2 & -\frac{1}{2}t^2 & t \\ \frac{1}{2}t^2 & 1 - \frac{1}{2}t^2 & t \\ t & -t & 1 \end{bmatrix} : t \in \mathbb{R} \right\}. \quad (\text{B.3})$$

Now, we list some of the 1-parameter subgroups of $\mathbb{R}_+^* \times SO_o(1, 2)$ on which both homothety and linear isometry projections are non-trivial. Note that, here $a \in \mathbb{R}^*$ is a constant number.

$$\exp(\mathbb{R}(a + \mathcal{Y}_E)) = \left\{ e^{at} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix} : t \in \mathbb{R} \right\}, \quad (\text{B.4})$$

$$\exp(\mathbb{R}(a + \mathcal{Y}_H)) = \left\{ e^{at} \begin{bmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{bmatrix} : t \in \mathbb{R} \right\}, \quad (\text{B.5})$$

$$\exp(\mathbb{R}(a + \mathcal{Y}_P)) = \left\{ e^{at} \begin{bmatrix} 1 + \frac{1}{2}t^2 & -\frac{1}{2}t^2 & t \\ \frac{1}{2}t^2 & 1 - \frac{1}{2}t^2 & t \\ t & -t & 1 \end{bmatrix} : t \in \mathbb{R} \right\}. \quad (\text{B.6})$$

Finally, we list some of the 1-parameter subgroups of $\text{Conf}(\mathbb{E}^{1,2})$ which have trivial translation part but they are not subgroups of the linear subgroup $\mathbb{R}_+^* \times SO_o(1,2)$.

A subgroup with elliptic linear isometry projection

$$\exp(\mathbb{R}(\mathcal{Y}_E + e_1)) = \left\{ \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix}, \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \right) : t \in \mathbb{R} \right\}. \quad (\text{B.7})$$

Subgroups with parabolic linear isometry projection

$$\exp(\mathbb{R}(\mathcal{Y}_P + e_1)) = \left\{ \left(\begin{bmatrix} 1 + \frac{1}{2}t^2 & -\frac{1}{2}t^2 & t \\ \frac{1}{2}t^2 & 1 - \frac{1}{2}t^2 & t \\ t & -t & 1 \end{bmatrix}, \begin{bmatrix} t + \frac{t^3}{6} \\ \frac{t^3}{6} \\ \frac{t^2}{2} \end{bmatrix} \right) : t \in \mathbb{R} \right\}, \quad (\text{B.8})$$

$$\exp(\mathbb{R}(\mathcal{Y}_P + e_1 - e_2)) = \left\{ \left(\begin{bmatrix} 1 + \frac{1}{2}t^2 & -\frac{1}{2}t^2 & t \\ \frac{1}{2}t^2 & 1 - \frac{1}{2}t^2 & t \\ t & -t & 1 \end{bmatrix}, \begin{bmatrix} t + \frac{t^3}{3} \\ -t + \frac{t^3}{3} \\ t^2 \end{bmatrix} \right) : t \in \mathbb{R} \right\}. \quad (\text{B.9})$$

Subgroups with hyperbolic linear isometry projection

$$\exp(\mathbb{R}(\mathcal{Y}_H + e_3)) = \left\{ \left(\begin{bmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} \right) : t \in \mathbb{R} \right\}, \quad (\text{B.10})$$

$$\exp(\mathbb{R}(1 + \mathcal{Y}_H + e_1)) = \left\{ \left(e^t \begin{bmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{e^t \sinh t + t}{2} \\ \frac{e^t \sinh t - t}{2} \\ 0 \end{bmatrix} \right) : t \in \mathbb{R} \right\}, \quad (\text{B.11})$$

$$\exp(\mathbb{R}(-1 + \mathcal{Y}_H + e_1)) = \left\{ \left(e^{-t} \begin{bmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{e^{-t} \sinh t + t}{2} \\ \frac{-e^{-t} \sinh t - t}{2} \\ 0 \end{bmatrix} \right) : t \in \mathbb{R} \right\}, \quad (\text{B.12})$$

$$\exp(\mathbb{R}(-1 + \mathcal{Y}_H + e_1 + e_2)) = \left\{ \left(e^{-t} \begin{bmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} \right) : t \in \mathbb{R} \right\}, \quad (\text{B.13})$$

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