HAL open science

Lyapunov exponents and variations of Hodge structures

Charles Fougeron

▶ To cite this version:

Charles Fougeron. Lyapunov exponents and variations of Hodge structures. General Mathematics [math.GM]. Université Sorbonne Paris Cité, 2017. English. NNT : 2017USPCC160 . tel-02076368

HAL Id: tel-02076368

https://theses.hal.science/tel-02076368v1

Submitted on 22 Mar 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Université Paris Diderot – Paris 7
Université Sorbonne Paris Cité

École doctorale de sciences mathématiques de Paris centre (ED 386)

Thèse de doctorat

Discipline : Mathématiques

présentée par

Charles Fougeron

Exposants de Lyapunov et variations de structures de Hodge

dirigée par Anton Zorich

Soutenue le 29 Juin 2017 devant le jury composé de :

M. Simion Filip	Harvard University	examinateur
M. Giovanni Forni	University of Maryland	président du jury
M. Pascal Hubert	Université d'Aix-Marseille	rapporteur
M. Maxim Kontsevich	IHÉS	examinateur
Mme. Isabelle Liousse	Université Lille 1	examinatrice
M. Martin Möller	Goethe-Universität Frankfurt	rapporteur
M. Carlos Matheus Silva Santos	Université Paris 13	examinateur
M. Carlos Simpson	Université de Nice	examinateur
M. Anton Zorich	Université Paris Diderot	directeur

Université Paris-Diderot
Bâtiment Sophie Germain
Case courrier 7012
8 place Aurélie Nemours
75 205 PARIS Cedex 13

Résumé

Cette thèse est articulée autour de deux thématiques : la première (chapitres 1 à 3) est l'étude des exposants de Lyapunov associés à un fibré plat sur une courbe complexe, et en particulier leur application dans les modèles de wind-tree ainsi que leur lien avec les variations de structures de Hodge quand les fibrés en sont munis. La deuxième (chapitres 4 à 5) traite des surfaces de dilatations, de leurs symétries et de leur dynamique. Dans le chapitre 1, un résultat reliant taux de diffusion d'un modèle de wind-tree à un exposant de Lyapunov d'un espace affine invariant d'une strate de différentielles quadratique est présenté. Ce théorème permet de calculer numériquement ces taux de diffusion pour un grande famille de modèles et d'observer l'influence des la forme des obstacles sur la vitesse du flot. Le chapitre 2 apporte une preuve d'une conjecture sur le comportement des exposants dans des strates à genre fixé avec un grand nombre de pôles dans le cas ou le nombre de zéros est borné. Ce résultat appuie une intuition que le taux de diffusion pour un wind-tree périodique avec un grand nombre d'angles est petit. Enfin dans le chapitre 3 nous considérons des exposants de Lyapunov plus généraux, associés à un fibré plat muni d'une variation de structure de Hodge sur la sphère privée de trois points. Cet exemple venu des équations hypergéométriques mime la structure de fibrés de Hodge sur des espaces de modules paramétrés par la sphère. Nous cherchons à comprendre la relation des exposants avec des grandeurs algébriques, en particulier avec les degrés paraboliques des sous fibrés holomorphes. Dans le chapitre 4 nous considérons les groupes de Veech de surfaces de dilatation et proposons une classification topologique complète de ceux-ci. Ce chapitre est aussi l'occasion de décrire notre intuition de cet objet et de proposer une conjecture sur l'existence de cylindres dans ces surfaces. Dans le chapitre 5 nous décrivons complètement la dynamique d'un exemple de surface de dilatation de genre 2 dans toutes les directions. Nous montrons l'existence et la généricité de directions correspondantes à des cylindres ainsi que l'existence d'une infinité de direction dans lesquels le flot géodésique s'accumule sur des espaces de Cantor.

Mots-clés

géométrie plate, dynamique de Teichmüller, expérimentation numérique, exposant de Lyapunov, variation de structure de Hodge, équation hypergéométrique, surface de translation, surface de dilatation, modèle de windtree.

Lyapunov exponents and variations of Hodge structures

Abstract

This thesis is organized around two main themes : on one hand (chapter 1 to 3) we study the Lyapunov exponents associated to a flat bundle on a complex curve, their application to wind-tree models and links with variation of Hodge structures on the bundle endowed with them. On the other hand (chapter 4 and 5) we introduce dilatation surfaces, and study their symmetries and dynamics. In chapter 1, a result binds diffusion rates of wind-tree models and a Lyapunov exponent of some affine invariant spaces in strata of quadratic differentials. This theorem enables us to compute numerically these diffusion rates for a large familly of models and hence to observe the influence of the shape of the obstacles on the speed of the flow. Chapter 2 is devoted to the proof of a conjecture on Lyapunov exponents behaviour for strata of a given genus and large number of poles when the number of zeros is bounded. It confirms an intuition explained in the previous chapter that diffusion rate on periodic wind-tree models with obstacles with a large number of angles is close to zero. At last, in chapter 3, we consider Lyapunov exponents in the more general setting of flat bundles endowed with a variation of Hodge structure on the sphere minus three points. This example coming from hypergeometric equations mimics the structure of a Hodge bundle on a moduli space parametrized by the sphere. We investigate the relation between these exponents and algebraic numbers like parabolic degrees of holomorphic subbundles. In chapter 4 we consider Veech groups of dilatation surfaces and give a complete topological classification of them. We also convey our intuition of this object and claim a conjecture on the existence of cylinders on each surface. In chapter 5 we describe the dynamics of a genus 2 example in every directions. We show the existence and genericity of directions corresponding to cylinders and we describe an infinite set of directions for which the geodesic flow accumulates on a Cantor set.

Keywords

flat geometry, Teichmüller dynamics, numerical experiment, Lyapunov exponent, variation of Hodge structures, hypergeometric equation, translation surface, dilatation surface, windtree model.

Remerciements

Ces quelques années de thèse ont été très douces et passionnantes pour moi. Je voudrais remercier les mathématicien-ne-s qui les ont rendues si joyeuses.

Tout d'abord Anton Zorich, dont l'humilité et l'enthousiasme m'ont instantanément séduits pour choisir sa direction pour ma thèse. Merci pour cette présence et ce soutient sans failles au cours de ces belles années.

Je suis très reconnaissant à tout-e-s les doctorant-e-s (ou qui l'étaient) avec qui j'ai pu partager et discuter de maths. Ces discussions ont grandement nourris mon intérêt pour la recherche. Tout particulièrement à Selim Ghazouani, Edouard Duryev et Adrien Boulanger avec qui j'ai collaboré, mais aussi à Jeremy Daniel, Elise Goujard, Martin Leguil, Alba Malaga, Simion Filip.

Merci aussi à tou-te-s les chercheur-euse-s qui n'ont jamais été avares de leur temps pour partager leurs visions et leurs idées des objets mathématiques : Maxim Kontsevich, Vincent Delecroix, Pascal Hubert, Julien Grivaux, Erwan Lanneau, Samuel Lelièvre, Alex Eskin, Martin Möller, Carlos Simpson, ...
Ainsi qu'à celle et ceux qui ont accepté de faire partie de mon jury de thèse : Isabelle Liousse, Carlos Matheus et Giovanni Forni.

Merci à tout les personnels de ménage, de bibliothèque, de logistique et de secrétariat qui font la majeure partie du travail invisible de la recherche à l'université et dans les différents lieux de rencontres mathématiques.

Enfin je remercie chaleureusement ma famille et mes ami-e-s.

Introduction

Introduction générale

Dans ce travail de thèse nous étudions différentes propriétés et applications des exposants de Lyapunov sur un fibré plat muni d'une variation de structures de Hodge.

Le premier aspect abordé traite des exposants de Lyapunov associés à une surface de translation. Cette étude est motivée par le calcul du taux de diffusion de modèles de windtree et nous amène à une conjecture de Grivaux-Hubert à laquelle une preuve partielle est apportée.

Dans un deuxième temps, nous étudierons des exposants de Lyapunov associés à des fibrés plats plus généraux induits par des équations hypergéométriques ; nous calculerons le degré parabolique des drapeaux de leur filtration de Hodge et observerons les relations entre ces différents objets.

En collaboration avec A. Boulanger, E. Duryev et S. Ghazouani nous présenterons enfin différentes propriétés de la classe des surfaces de dilatation qui est une structure généralisant les surfaces de translation et ayant des propriétés dynamiques nouvelles. Nous classifions leurs groupes de Veech, l'obstruction à une représentation polygonale, et discutons les différents comportements dynamiques du flot linéaire sur une de ces surfaces.

Exposants de Lyapunov et taux de diffusions

Surfaces de translation

Une surface de translation est une surface fermée épointée dont les changements de cartes sont des translations. Le flot linéaire dans une direction θ est alors

défini dans chaque carte sur ces surfaces comme la géodésique pour la métrique euclidienne parcourue à vitesse 1 dans la direction θ.

De manière équivalente, on peut choisir $P_1, \dots P_n$ des polygones dans le plan qui ne s'autointersectent pas puis recoller leurs cotés deux à deux par translation, avec la règle que deux cotés recollés doivent être de part et d'autre de l'intérieur du polygone correspondant.

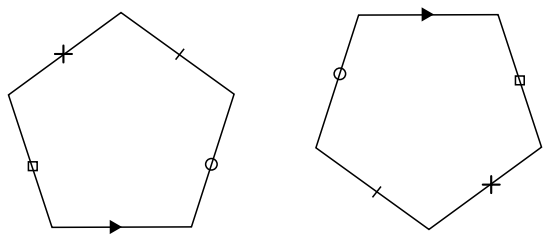


FIGURE 1 – Un élément de $\mathcal{H}(2)$

Lorsqu'on tourne autour d'un des sommets des polygones après identification, on observe que le secteur angulaire est de la forme $2(d_i + 1)\pi$. On note alors

$$\mathcal{H}(d_1, \dots, d_k)$$

l'ensemble des surfaces de translation avec de tels angles.

Il est possible de relâcher la condition sur la position des cotés recollés par rapport à l'intérieur du polygone en recollant les coté égaux et du même coté du polygone par rotation d'angle π puis translation. On dit que ces surfaces ont une structure de *demi-translation*. Les angles des points singulier sont alors de la forme $(m_i + 2)\pi$ et on note l'ensemble des surfaces ayant ces angles,

$$\mathcal{Q}(m_1, \dots, m_k)$$

On appelle ces espaces les strates de l'ensemble des surfaces de (demi-)translation.

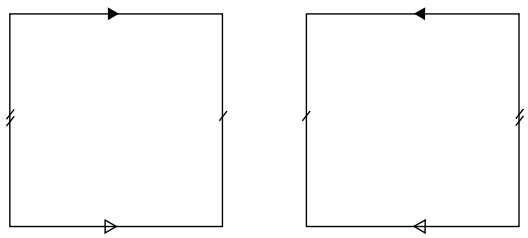


FIGURE 2 – Un élément de $\mathcal{Q}(-1,-1,-1,-1)$

On définit une action des matrices de $\mathrm{GL}(2,\mathbb{R})$ sur ces surfaces en faisant agir la matrice sur chaque coté des polygones comme sur des vecteurs de $\mathbb{R}^2$, et en conservant la combinatoire de recollement. Une famille primordiale est formée pour $t \in \mathbb{R}$ par

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

L'action de cette famille à 1-paramètre sur une surface de translation est appelée son *flot de Teichmüller*. La magie de ce flot est qu'il *renormalise* le flot linéaire sur la surface. En effet, si l'on considère le flot dans la direction verticale, l'action de g_t va contracter sa longueur. On peut donc se ramener à un flot de longueur 1 sur une surface déformée par cette action.

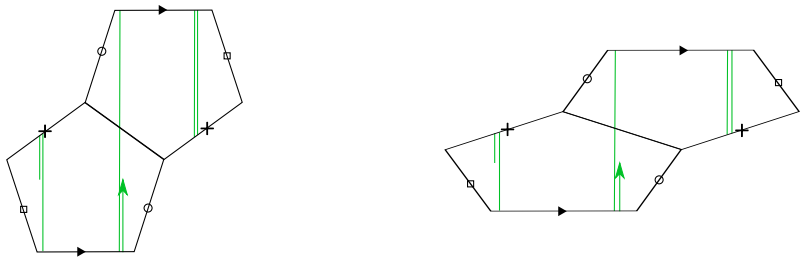


FIGURE 3 – Renormalisation par $g_{3/2}$

Les travaux de Masur et Veech publiés indépendamment en 1982 relient la dynamique du flot de translation sur ces surfaces avec la dynamique d'un flot de renormalisation dans leurs espaces de modules. Ces idées permettent une compréhension nouvelle de la dynamique sur ces surfaces et débouchent sur un des résultats fondateur du domaine,

Theorem 1 (Kerchoff, Masur, Smilie) *Le flot linéaire sur un surface de trans-*

lation est uniquement ergodique dans presque toute direction.

D'après ce théorème on peut définir de manière générique sur une surface de translation le premier retour sur un intervalle transverse au flot linéaire. Cette bijection de l'intervalle est une translation par morceau, on dit que c'est un *échange d'intervalles.*

Exposants de Lyapunov

Il est courant d'associer à un système dynamique des nombres réels qui mesurent sa stabilité : ses exposants de Lyapunov. Ils caractérisent la vitesse de séparation de deux orbites infinitésimalement proches. Dans le cas de la dynamique holomorphe, on considère classiquement la limite

$$\lim_{n\to\infty} |(f^n)'(z)|^{1/n}$$

qui par le théorème de Birkhoff est la même en presque tout points par rapport à une mesure ergodique. Pour les échanges d'intervalles cette dérivée est nulle, la définition est donc vide dans ce cas. L'idée de A. Zorich à la fin des années 90 est de considérer, à la place de la différentielle de l'application, un cocycle induit par un processus de renormalisation : l'induction de Rauzy-Veech. Ces exposants mesurent alors la déviation des sommes de Birkhoff mais aussi le nombre d'intersection asymptotique du flot linéaire avec une classe d'homologie.

Ce cocycle est ensuite interprété par M. Kontsevich comme le transport parallèle des classes d'homologies le long du flot de Teichmüller et est désormais communément désigné par *cocycle de Kontsevich-Zorich.*

Modèle du windtree

Récemment, V. Delecroix, P. Hubert et S. Lelièvre ont présenté une jolie interprétation de ces exposants sur un modèles de *windtree.* Un windtree est une table de billard infinie sur laquelle on place des obstacles. Dans le cas de ces auteurs, on place périodiquement un rectangle, dont un des cotés est horizontal, sur chaque point à coordonnée entière du plan. Sur un tel billard, considérons le déplacement

rectiligne à vitesse unitaire d'une particule partant d'un point p dans une direction θ et qui rebondit de manière élastique sur les obstacles ; on le note $\phi_t^\theta(p)$. Nous attachons à calculer le taux de diffusion de ce déplacement, autrement dit la vitesse à laquelle la particule s'éloigne de son point de départ ; formellement,

$$\limsup_{t\to\infty} \frac{\log ||\phi_t^\theta(p)||}{\log t}$$

Theorem 2 (Delecroix, Hubert, Lelièvre) *Le taux de diffusion pour les windtrees périodiques précédents, dans presque toute direction θ et pour les points p qui ne rencontrent pas les coins des obstacles, est égal à $2/3$.*

Ce taux de diffusion est calculé en démontrant qu'il est égale à l'exposant de Lyapunov principale (i.e. le plus grand) d'un sous espace d'homologie invariant pour le cocycle de Kontsevich-Zorich.

Dans le premier chapitre de cette thèse, nous introduisons une méthode générale pour associer à un windtree l'exposant de Lyapunov principal d'un sous-espace de l'homologie irréductible pour le cocycle de Kontsevich-Zorich. En particulier, cette méthode nous permet d'obtenir le taux de diffusion d'une grande famille de windtree. Soient $Q_1, \ldots, Q_m$ plusieurs polygones dont les angles sont $\pi/2$ ou $3\pi/2$; on place ces obstacles dans un carré sans les superposer, puis on répète le motif $\mathbb{Z}^2$-périodiquement. Si on note p le nombre total d'angles $3\pi/2$ pour tous les polygones,

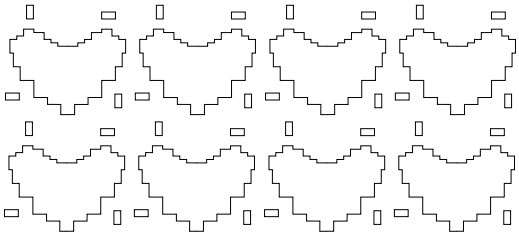


FIGURE 4 – Un exemple d'un tel windtree avec $m = 5$

Theorem 3 *Le taux de diffusion pour ce windtree est égale à l'exposant de Lyapunov principale de $\mathcal{Q}(1^{4m+p}, -1^p)$*

Le cas de l'obstacle rectangle est particulièrement symétrique et cela induit un exposant de Lyapunov rationnel et complètement explicite. Dans le cas générale, on ne s'attend pas à ce que l'exposant de Lyapunov soit rationnel, ni même algébrique, et à ce qu'on aie une quelconque formule ; cependant il est tout à fait possible de l'approcher algorithmiquement. Un partie du temps de cette thèse a d'ailleurs été consacré à développer un programme qui calcule ces exposants de manière très générale. Ainsi nous pouvons observer numériquement le comportement du taux de diffusion de ces windtrees lorsque l'on fait varier m et p.

La première observation est que pour un p fixé, le taux de diffusion lorsqu'on augmente le nombre d'obstacle tend vers $1/2$. C'est le taux de diffusion d'une marche aléatoire ou d'un mouvement brownien.

Montrer ce comportement asymptotique est encore une question ouverte. Les question d'asymptotiques d'exposants de Lyapunov sont à ce jour encore très peu comprise. Le seul résultat qui existe à ce jour est une preuve d'une conjecture de M. Kontsevich et A. Zorich qui date de 1997 qui montre que le deuxième exposant de Lyapunov d'une famille spécifique de surfaces de translation tend vers 1. Cette preuve a été donné par Fei Yu conditionnellement à une autre conjecture qui a été prouvée il y a moins d'un an par A. Eskin, M. Kontsevich, M. Möller et A. Zorich.

Une seconde observation est que pour un nombre d'obstacles fixé, lorsqu'on augmente p, le taux de diffusion tend vers 0. Intuitivement, lorsqu'une particule rebondit près d'un angle $3\pi/2$ sur un obstacle, celle-ci emprunte alors une trajectoire un peu décalée et parallèle à la trajectoire incidente dans un sens opposé. Ainsi, la particule passe beaucoup de temps à revenir sur ses pas lorsque p est grand et taux de diffusion doit être petit.

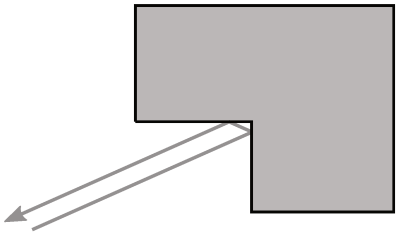


FIGURE 5 – Rebondissement près d'un angles $3\pi/2$

Ceci rentre en résonance avec une conjecture plus générale de J. Grivaux et P. Hubert qui affirme que les exposants de Lyapunov de surfaces de translation à genre fixé dont on augmente le nombre de points d'angles égaux à π tendent vers 0.

Si l'on considère une surface de demi-translation, comme nous l'avons vu ses exposants de Lyapunov mesurent la croissance du nombre d'intersection de son flot linéaire avec les différentes classes d'homologie de la surface. Or intuitivement, lorsque le flot de translation passe à coté d'un point d'angle π, il revient en arrière. Les points d'angle π *ralentissent* le flot linéaire du point de vu du nombre d'intersection. D'où la conjecture de Grivaux-Hubert que ces exposants tendent vers zéro lorsque le nombre de tels points tend vers l'infini pour une surface topologique donnée.

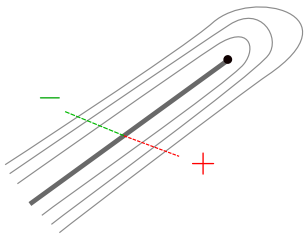


FIGURE 6 – Ralentissement du flot linéaire près d'un angle π

La preuve de cette conjecture dans le cas où un des angles tend vers l'infini est l'objet du deuxième chapitre de cette thèse,

Theorem 4 *Les exposants de Lyapunov des espaces $\mathcal{Q}(m_1, \ldots, m_k)$ pour un genre fixé, tendent vers 0, quand $m_1 \to \infty$.*

Exposants de Lyapunov d'équations hypergéométriques

Les exposants de Lyapunov sur une surface de translation correspondent à ceux induit par le transport parallèle de classes de cohomologie le long du flot de Teichmüller. Ce transport est donné par une connection sur le fibré formé par les groupes de cohomologie au dessus de chaque surface dans l'espace de Teichmül-

ler : c'est le *fibré de Hodge*. À partir de cette interprétation, il est possible de lier la somme des exposants positifs d'une surface de translation avec une grandeur géométrique d'une sous-variété de la strate correspondante. En premier lieu, M. Kontsevich a montré que pour une surface dont l'orbite par l'action de tout le groupe $\mathrm{GL}(2, \mathbb{R})$ est fermé dans la strate, la somme des exposants de Lyapunov est exactement le degré d'un fibré vectoriel holomorphe sur cette orbite.

C'est une propriété tout a fait extraordinaire, car comme nous l'avons noté précédemment les exposants de Lyapunov sont des grandeurs dynamiques et, pris un à un, nous ne connaissons en général aucune structure algébrique sur ceux-ci. Un aspect remarquable de l'argument de M. Kontsevich pour démontrer cela est qu'il s'applique dans un cadre bien plus générale que celui d'une surface de translation.

En effet, l'objet clef de cette propriété est la *variation de structure de Hodge* qui est une filtration (augmentée de quelques propriétés de compatibilité avec la connection) du fibré de Hodge en drapeaux holomorphes. Dans le cas des groupes de cohomologie d'une surfaces complexe donnée, ces drapeaux correspondent à la décomposition de Hodge (i.e. au sous-fibré holomorphe engendré par les 1-formes holomorphes sur la surface). Pour tout fibré plat sur une surface munie d'une métrique hyperbolique, on peut introduire une notion d'exposants de Lyapunov. Ainsi dans le cas ou le fibré est muni d'une variation de structure de Hodge de largeur 1 (avec un seul sous-fibré holomorphe) le résultat de M. Kontsevich s'applique.

La question s'impose alors de savoir si l'on a encore une telle relation dans les cas de variations de structure de Hodge de plus grandes largeurs.

Un article de A. Eskin, M. Kontsevich, M. Möller et A. Zorich que nous avons déjà évoqué montre que la somme partielle des k plus grands exposants de Lyapunov est minoré par le degré parabolique de tout sous-fibré holomorphe de rang k du fibré de Hodge. Mais quand y a-t-il égalité ?

L'*équation hypergéométrique* d'Euler est l'équation différentielle sur le plan

complexe,

$$z(z-1)y'' + [c-(a+b+1)z]y' - aby = 0$$

avec a,b, c des paramètres réels. Les solutions de cette équation forment une nouvelle classe de fonctions spéciales qui contient la plupart des solutions de système de physique classiques. Cette équation est une équation différentielle ordinaire d'ordre 2 à singularités en $0, 1$ et ∞, elle définie donc un fibré vectoriel plat de rang 2 au dessus le $\mathbb{P}^1 - \{0, 1, \infty\}$

Nous considérons les équations hypergéométriques définies sur $\mathbb{P}^1 - \{0, 1, \infty\}$ qui généralisent celle d'Euler à tout rang. Ces équations nous fournissent une large famille de fibrés vectoriels plats munis d'une variation de structure de Hodge sur lesquels tester cette égalité. Dans le troisième chapitre de cette thèse nous calculerons les degrés de sous-fibré holomorphes de ces fibrés plats, et présenterons des simulations numériques sur les exposants de Lyapunov associés. Cette approche dévoile des domaines de paramètres étonnant pour lesquels il y a égalité et amène de nombreuses questions ouvertes sur leur nature.

Surfaces de dilatation

Une *surface de dilatation* est une surface fermée épointée dont les changements de cartes sont des dilatations composées avec des translations.

On donne dès à présent deux exemples de telles surfaces. Le premier est construit comme quotient de $\mathbb{C}^*$ par l'action par multiplication d'un scalaire réel strictement positif. Cet espace est un tore muni d'une structure de dilatation naturelle, on l'appelle le *tore de Hopf.*

Le deuxième donné par une représentation polygonale,

L'action de $\mathrm{GL}(2, \mathbb{R})$ sur ces surfaces reste bien définie, et nous pouvons considérer le sous-groupe de $\mathrm{GL}(2, \mathbb{R})$ qui laisse invariante la structure de dilatation de la surface. Par analogie avec le cas des surfaces de translation, nous appelons ce groupe le groupe de Veech de la surface. Pour une surface de translation, le groupe de Veech est toujours discret dans $\mathrm{SL}(2, \mathbb{R})$, de plus si ce groupe est un

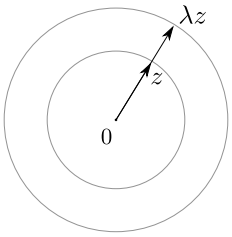


FIGURE 7 – Le tore de Hopf.

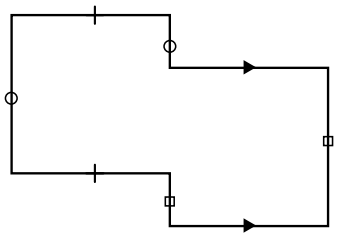


FIGURE 8 – Une surface de dilatation de genre 2.

réseau dans $\mathrm{SL}(2,\mathbb{R})$ on dit que la surface est *surface de Veech*. Veech a démontré une propriété extraordinaire de ces surfaces en utilisant des techniques de renormalisation,

Theorem 5 (Dichotomie de Veech) *Le flot linéaire dans une direction θ d'une surface de Veech satisfait exclusivement l'une de ces deux propriétés,*

1. *Le flot est totalement périodique.*
2. *Le flot est minimal.*

de plus la propriété de minimalité est générique pour la mesure de Lebesgue.

Ce théorème est a mettre en perspective avec le flot linéaire sur le tore $\mathbb{C}/\mathbb{Z} \oplus i\mathbb{Z}$ qui est complètement périodique lorsque θ est rationnel et minimal lorsque θ est irrationnel.

Nous avons étudié dans le quatrième chapitre de cette thèse en collaboration avec E. Duryev et S. Ghazouani, la nature des groupes de Veech pour les surfaces de dilatation.

Theorem 6 *Le groupe de Veech d'une surface de dilatation entre dans l'une de ces trois catégories,*

(a) Angle rationnel (b) Angle irrationnel

1. $\mathrm{GL}(2,\mathbb{R})$ *tout entier.*
2. *Un sous-groupe conjugué à celui des matrices triangulaires supérieures.*
3. *Un sous-groupe discret.*

Observons que l'action de $\mathrm{GL}(2,\mathbb{R})$ commute avec la multiplication par un scalaire, donc la structure du tore de Hopf introduite ci-dessus est fixée par cette action. Cet exemple est représentatif des surfaces de type (1). Nous montrons en effet que les surfaces de dilatation dont le groupe de Veech est $\mathrm{GL}(2,\mathbb{R})$ sont des tores dont la structure de dilatation est induite par l'exponentiation d'une structure plate.

Les surfaces de type (2) sont construites à partir de bouts de plusieurs surfaces de type (1) que l'on découpe puis recolle entre elles le long d'une unique direction. Nous les appelons *surfaces de Hopf*.

Ces deux premiers cas doivent être considéré comme dégénérés ; les géodésiques linéaires reliants deux singularités dans ces surfaces sont toutes dans la même direction pour (2), et dans le cas de (1) n'existent pas. Lorsqu'il existe au moins deux directions pour de tels géodésiques, le groupe de Veech est discret.

Une notion importante se dégage à présent dans la compréhension de la dynamique des surfaces de dilatation,

Definition 1 *Un cylindre affine dans une surface de dilatation est un plongement isométrique maximal d'une section angulaire d'un tore de Hopf.*

L'existence d'un cylindre affine est équivalente à celle d'une géodésique linéaire fermée dans la surface dont le produit de toutes les dilatations lorsqu'on la par-

cours est différent de 1.

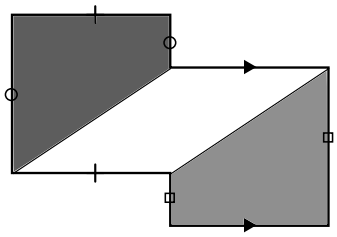


FIGURE 10 – Deux cylindres

En effet nous montrons que l'existence d'un tel cylindre implique que le groupe de Veech d'une surface de dilatation de type (3) ne peut être un réseau.

Comme pour les surfaces de translation, on peut regarder le premier retour du flot linéaire sur un intervalle transverse bien choisi. Cela défini une bijection d'une intervalles affine par morceaux, on appelle de telles applications des *échanges d'intervalles affines*. Des expérimentations numériques sur celles-ci et quelque résultats partiels (citons par exemple le théorème de généricité topologique de I. Liousse) nous poussent à poser la conjecture suivante,

Conjecture 1 *Toute surface de dilatation, en dehors du cas particulier des surfaces de translation, possède un cylindre affine.*

Il semblerait en outre que ce soit le comportement générique dans presque toute direction d'une surface de dilatation. La preuve de cette conjecture est un travail en cours avec S. Ghazouani.

Pour tester cette conjecture nous avons traité avec A. Boulanger et Selim un exemple élémentaire de surface de dilatation de genre 2, donné par la représentation ci-dessous. Avec des techniques d'une part de géométrie hyperbolique et de renormalisation sur les échanges d'intervalles nous avons décrit le comportement dynamique de l'ensemble des directions. C'est l'objet du cinquième chapitre de cette thèse.

Dans un premier temps nous vérifions la conjecture que dans presque toute direction le flot linéaire se retrouve asymptotiquement piégé dans un cylindre affine

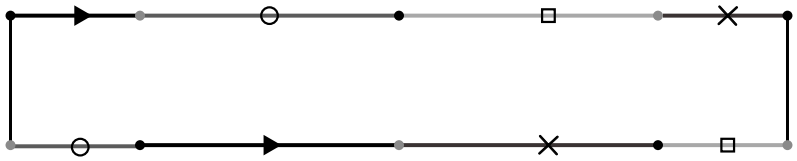


FIGURE 11 – Surface de dilatation étudiée au chapitre 5

et s'accumule sur une courbe fermée. On dit que le flot dans cette direction est *dynamiquement trivial*. Dans les autre directions nous avons trouvé des comportements étonnant ; pour certaines de celle-ci, l'échange d'intervalle affine associé est attiré par un ensemble de Cantor de mesure nulle.

Theorem 7 *Il existe deux Cantor exclus* Λ *et* $\mathcal{H}$ *dans* $[0, 2\pi)$*, tels que,*

- *Pour une direction dans* Λ *le flot linéaire uniquement ergodique sur un sous-ensemble dense et dans son complémentaire, minimal ou périodique.*
- *Pour une direction dans* $\mathcal{H}$ *le flot s'accumule sur un ensemble de Cantor de mesure nulle dans un intervalle transverse.*
- *Pour toute les autres directions, le flot est dynamiquement trivial.*

Cette étude nous permet en plus de calculer le groupe de Veech de cette surface. L'ensemble Λ correspond alors à l'ensemble limite de ce groupe Fuchsien.

Table des matières

Remerciements **7**

Introduction **9**

1 Diffusion rate of windtree models **25**

2 A partial proof for Grivaux-Hubert conjecture **43**

3 Lyapunov exponents and hypergeometric equations **55**

4 Veech group for dilatation surfaces **79**

5 Cascades in the dynamics of affine interval exchange transformations **105**

Chapitre 1

Diffusion rate of windtree models

In this chapter we introduce general methods to compute diffusion rate of a periodic windtree model as a Lyapunov exponent. We deal with the example of periodic windtree model with several aligned obstacle in its fundamental domain with right angles, and highlight its asymptotic properties with numerical experiments.

DIFFUSION RATE OF WINDTREE MODELS

CHARLES FOUGERON

1. Introduction.

The windtree model is a statistical physics example introduce by Paul and Tatiana Ehrenfest in 1912 [EE90]. In this book they introduce a set of light particules that move around but do not interact and a set of massive square particules that do not move but on which the light ones bounce with elastic collision. We classicaly call the light particules the *wind* and the static ones the *trees*. One of the motivation was to understand the cinetic behaviour of such a system. The question asked by the Ehrenfest is, for a generic disposition of square trees orientated in the same direction, does the speed of K light particules equidistributes asymptotically in the 4 possible directions ? A positive answer to this question is provided by [MST16]. Plenty of questions have been studied on this model, in particular for the $\mathbb{Z}^2$-periodic case: [HW80] proved recurrence and abnormal diffusion of the billiard flow for special dimensions of the obstacles; [FU14] exhibites some non ergodic cases of this case; and more recently its diffusion rate was computed in [DHL14].

The argument of [DHL14] relies on a remarkable correspondance between the diffusion rate of an infinite periodic billiard table and the Lyapunov exponent of an associated translation surface. This computation was generalised in [DZ15] to any $\mathbb{Z}^2$-periodic windtree which trees have only right angles and are horizontally and vertically symetric. In every of these cases, the corresponding Lyapunov exponent is computed in some 2 dimensional subbundle of the Hodge bundle. Moreover in all of these cases the Lyapunov exponent is rational and can be computed by some geometric arguments.

In this article, we describe a general strategy to exhibit a Lyapunov exponent of some locus in a stratum which corresponds to the diffusion rate of a periodic windtree models. It relies on two main ingredients: the first one is to find orbit closure of the families of translation surfaces associated to windtree tables; the second one is to find an irreductible subbundle of the Hodge bundle on this locus which top Lyapunov exponent is exactly the diffusion rate. The tools for the first craft are given by recent results of [EMM15], [AEM12] and [Wri14] and are introduces in subsection . For the second one, we show an additional lemma to the work of [CE15] which yields the diffusion rate for any translation surface in a

generic direction.

We apply this method to the case of periodic windtree with several obstacles in its fundamental domain. Pick a family of $n \geq 2$ rectangular obstacles in a square, and repeat this table $\mathbb{Z}^2$-periodically in the plane. We show the following theorem,

Theorem. *The diffusion rate for in almost every such windtree, in almost every direction is equal to the top Lyapunov exponent of* $\mathcal{Q}(1^{4n})$.

If we take more general obstacles, with right angles. Let p be the number of inward right angles in all the obstacles, then we have,

Theorem. *The diffusion rate for almost every such windtree model almost every direction is equal to the top Lyapunov exponent of* $\mathcal{Q}(1^{4n+p}, -1^p)$

In the last section, we discuss the value of these exponents. We present the related Grivaux-Hubert conjecture which was partialy proven in [Fou16] which states that Lyapunov exponents of strata should go to zero with high number of poles. And feature some numerical experiments using a Sage code developped by the author in a collaborative project [D$^+$16]. These experiments yield that the familly we introduced above can approach arbitrarily close any diffusion rate between 1/2 and 0.

2. Translation surfaces

2.1. **Definition.** A *translation surface* is a surface whose change of charts are translations. Such a surface is endowed with a flat metric (the one on $\mathbb{R}^2$) and a canonical direction.

One way to think of these translation surfaces is by gluing sides of a polygon via composition translations. Let P be a polygon with $2k$ edges and let $z_1, \ldots, z_{2k}$ be complex numbers associated to its sides. Assume that $z_i = z_{k+i}$, we glue the sides z_i and z_{k+i} and obtain a flat surface with conical singularities of angle multiple of 2π.

We can define similar structures allowing the change of charts to be also translations composed with $-\mathrm{Id}$. The class of surfaces we obtain are called *half-translation* surfaces.

Using triangulations Veech showed in [Vee93] that this is a general construction with a notion of *pseudo-polygons* (in a much wider class of structures). The $(z_i)_{1 \leq i \leq k}$ are coordinates in the moduli space of such structures, we call them *period coordinates*. We will introduce them as periods of abelian differentials below.

2.1.1. *Differentials and Moduli spaces.* There is a one to one correspondance between compact translation surfaces and Riemann surfaces equipped with a non-zero holomorphic 1-form. As well as between compact half-translation surfaces and Riemann surfaces equipped with quadratic differentials.

Let $g \geq 1$ and α and β be partitions of $2g-2$ and $4g-4$. The strata $\mathcal{H}(\alpha)$ and $\mathcal{Q}(\beta)$ are defined to be the sets of (S, ω) and (S, q) where S is a genus g closed Riemann surface, ω is a holomorphic 1-form on S, q is a quadratic differential on S, and their zeros multiplicities are given by α and β. The conical points in a translation surface correspond to the zero of the differential, and if d is its multiplicity, the angle is equal to $2(d+1)\pi$; for half-translation surfaces $(d+2)\pi$. We will always consider finite cover of these strata to have a manifold structure instead of orbifold, but keep the same notation.

Given a translation surface (S, ω), let $\Sigma \subset S$ be the set of zeros of ω. Pick a basis $\{\xi_1, \ldots, \xi_n\}$ for the relative homology group $H_1(S, \Sigma; \mathbb{Z})$. The map $\Phi : \mathcal{H}(\alpha) \to \mathbb{C}^n$ defined by

$$\Phi(S, \omega) = \left(\int_{\xi_1} \omega, \ldots, \int_{\xi_n} \omega \right)$$

redefines local period coordinates as above.

There is a natural action of $\mathrm{GL}(2, \mathbb{R})$ on connected components of strata coming from linear action of $\mathrm{GL}(2, \mathbb{R})$ on R^2 in charts. For any translation surface in a stratum, its orbit closure via this action is some affine invariant manifold of the stratum, it is defined in local period coordinates by linear equations. They are endowed with a canonical measure supported on these surfaces called affine measures [EM13], [EMM15].

2.1.2. *Translation cover.* To any primitive half-translation surfaces S we associate its *translation cover* $\hat{S}$ corresponding to the subgroup of the fundamental group with holonomy equal to -1. This is a double corver. We endow $\hat{S}$ with the pull-back metric of S, which defines a translation surface structure for $\hat{S}$.

From a differential geometry point of view, we find a double cover of S, on which the quadratic differential q can be written ω^2 where ω is a holomorphic 1-form.

Let (S, q) be a half-translation surface in $\mathcal{Q}(m_1, \ldots m_d)$, $\hat{S}$ its translation cover and $\hat{\Sigma}$ the preimage of its singular points. Following [AEZ16], assume there is a basis $\{a_1, b_1, \ldots, a_g, b_g\}$ of $H_1(S; \mathbb{Z})$ which has trivial linear holonomy (this exists as long as there is a zero with odd multiplicity), and let $\gamma_1, \ldots, \gamma_{d-2}$ be primitive non crossing elements of $H_1(S, \Sigma; \mathbb{Z})$ representing a path from P_i to P_{i+1} where $\{P_1, \ldots, P_d\} = \Sigma$.

Given a saddle connection or an absolute cycle with trivial linear holonomy γ, let γ', γ'' be its 2 lifts in $\hat{S}$ endowed with the orientation inherited from γ. Then we introduce,

$$\hat{\gamma} := \gamma' - \gamma''$$

By definition $\hat{\gamma}$ belongs to $H_1^-(\hat{S}, \hat{\Sigma}; \mathbb{C})$ the -1-eigenspace of the linear automorphism induced by the deck invalution of the double cover.

Proposition 1. *The family* $\{\hat{a}_1, \hat{b}_1, \ldots, \hat{a}_g, \hat{b}_g, \hat{\gamma}_1, \ldots, \hat{\gamma}_{d-2}\}$ *is a basis of* $H_1^-(\hat{S}, \hat{\Sigma}; \mathbb{C})$.

The dual family of $\{\hat{a}_1, \hat{b}_1, \ldots, \hat{a}_g, \hat{b}_g, \hat{\gamma}_1, \ldots, \hat{\gamma}_{d-2}\}$ from a basis of the antiinvariant 1-forms,

$$H^1_-(\hat{S}, \hat{\Sigma}; \mathbb{C}) \subset H^1(\hat{S}, \hat{\Sigma}; \mathbb{C})$$

where the relative cohomology is a local chart for some abelian stratum $\mathcal{H}(\alpha)$. This period is twice the polygonal periods we defined above up to a sign.

This is the basis we will be using to express equations of billiard families.

2.2. **Windtree tables.** Let P be a filled polygon which does not self-intersect: it will stand for the shape of the trees in our infinite forest. Consider an infinite billiard table in the plane $\mathbb{R}^2$ on which we place P periodically centered at each point of a lattice Λ as scatterer such that copies do not overlap. We denote this table by $\mathcal{W}(P, \Lambda)$.

Definition 1. We call $\mathcal{W}(P, \Lambda)$ a Λ-periodic windtree table with obstacle P.

Our purpose here is to understand the billiard flow on this infinite table and its asymptotic speed. We denote the billiard flow by

$$\phi_t^\theta : \mathcal{W}(P, \Lambda) \mapsto \mathcal{W}(P, \Lambda)$$

For $p \in \mathcal{W}(P, \Lambda)$, the point $\phi_t^\theta(p)$ is the position of a particle after time t starting from p in direction θ, which moves in straight lines and bounces on obstacles according to Snell-Descartes law of reflection.

Definition 2. The diffusion rate of a windtree table $\mathcal{W}(P, \Lambda)$, for $d(.,.)$ the Euclidian distance on $\mathbb{R}$, $p \in \mathcal{W}(P, \Lambda)$ and $\theta \in [0, 2\pi)$ is

$$\limsup_{t \to +\infty} \frac{\log d(p, \phi_t^\theta(p))}{\log t}$$

As the billiard table $\mathcal{W}(P, \Lambda)$ is Λ-periodic, we may consider its quotient

$$\mathcal{W}(P, \Lambda) /_\Lambda \simeq \mathbb{R}^2 /_\Lambda - \mathring{P} =: \mathcal{T}_\Lambda(P)$$

which corresponds to playing billiard in a torus.

2.2.1. *Associated flat surface.* A classical idea to study billiard flow is to associate to a table a flat surface on which the linear flow corresponds to the billiard flow. We do so for any $\mathcal{T}(P)$.

Take two copies of $\mathcal{T}(P)$ and glue the two copies of each side of P using an isometry fixing the tangent vector and inverting the normal vector (the axial symmetry along this side). Now when the flow is bouncing in the billiard, the geodesic flow of the flat surface is simply changing of copy in the surface.

The gluing maps are changing orientation, hence in order for this surface to be a flat surface as defined above, we choose as a convention two opposite orientations for the two copies. The change of charts now preserves orientation and is in $\mathrm{Iso}^+(\mathbb{C})$. We denote this flat surface by $S(P,\Lambda) = S$.

For each triple $(p,\theta,t) \in S \times [0,2\pi) \times \mathbb{R}_+$ we define an element $\gamma_t^\theta(p) \in H^1(S;\mathbb{Z})$ as follows. Consider the geodesic segment of lengths t starting from p in the direction θ and close it by a small piece of curve that does not cross h_κ and v_κ. The curve used to close the segment can be chosen to be uniformly bounded.

Let h, v be a horizontal and vertical simple loop in $\mathcal{T}(P)$ that generate the homology of the torus. Let $h^S = h_1 - h_2$ and $v^S = v_1 - v_2$ where h_1, h_2 (resp v_1, v_2) are the two lifts of h (resp. v) in S. And let a $f \in H^1(S;\mathbb{Z}^2)$ be a cocycle dual of (h^S, v^S) with respect to the intersection form.

The proposition below shows that the diffusion rate of a particule in a windtree table $\mathcal{W}(P,\Lambda)$ can be reduced to the study of the pairing of the approximate geodesic flow on S with f.

Proposition 2 (1 in [DHL14])**.** *The diffusion rate of $\phi_t^\theta(\tilde{p})$ is equal to*

$$\limsup_{t\to+\infty} \frac{\log\left|\left\langle f, \gamma_t^\theta(p)\right\rangle\right|}{\log t}$$

when p and $\tilde{p}$ project to the same point on $\mathcal{T}(P)$.

3. Lyapunov exponents

In the previous section, we have seen that the diffusion rate on a windtree table is related to the asymptotic pairing of a cohomology class with a modified linear flow on the associated translation surface. We will consider throughout this section a translation S in some abelian stratum $\mathcal{H}(\alpha)$ which $\mathrm{SL}(2,\mathbb{R})$-orbit closure is the affine invariant subspace $\mathcal{M} \subset \mathcal{H}(\alpha)$ and $\nu_{\mathcal{M}}$ its invariant measure.

The following theorem bounds the Lyapunov exponent with the diffusion rate,

Theorem (2 in [DHL14]). *Let $F_1 \supset F_2 \supset \cdots \supset F_k$ be the Oseledets flag decomposition of the Kontsevich-Zorich cocycle on $\mathcal{M}$, and λ be its top Lyapunov exponent. For every $\nu_{\mathcal{M}}$-Oseledets generic translation surface $S \in \mathcal{H}(\alpha)$, for every point $p \in S$ with infinite forward orbit, for all $f \in F_1 \setminus F_2$,*

$$\limsup_{t \to +\infty} \frac{\log |\langle f, \gamma_t(p) \rangle|}{\log t} = \lambda$$

In [CE15] is proved that any translation surface $S \in \mathcal{H}(\alpha)$ such that its $\mathrm{SL}(2,\mathbb{R})$ orbit closure is $\mathcal{M}$, is Oseledets generic in Lebesgue-almost every direction. In particular they show the following theorem,

Theorem (1.5 in [CE15]). *Fix $S \in \mathcal{H}_1(\alpha)$ and let $\mathcal{M} = \overline{\mathrm{SL}(2,\mathbb{R})S}$ the smallest affine invariant manifold containing S, let V be a $\mathrm{SL}(2,\mathbb{R})$ invariant subbundle of the Hodge bundle which is defined and continuous on $\mathcal{M}$. Let $A_V : \mathrm{SL}(2,\mathbb{R}) \times \mathcal{M} \to V$ denote the restriction of the Kontsevich-Zorich cocycle to V and suppose that A_V is strongly irreducible with respect to the affine measure $\nu_{\mathcal{M}}$ whose support is $\mathcal{M}$. Then, for almost every $\theta \in [0, 2\pi)$,*

$$\lim_{t \to \infty} \frac{\log ||A_V(g_t r_\theta x)||}{\log t} \to \lambda_1$$

where λ_1 is the top Lyapunov exponent of A_V.

A little modification in there argument, which we delay to the annex, enables us to show an additional lemma to this theorem.

Lemma 1. *In the previous theorem, for any $h \in V$ and almost every $\theta \in [0, 2\pi)$*

$$\lim_{t \to \infty} \frac{\log ||A_V(g_t r_\theta x)h||}{\log t} \to \lambda_1$$

This reduces the computation of the diffusion rate of a windtree model to determining irreducible components of the Kontsevich-Zorich cocycle and in which of these is the cohomology class f.

The Simplicity Theorem of [Wri14] implies that the Kontsevich-Zorich cocycle is strongly irreducible on $H^1_-(S;\mathbb{R})$, thus the previous lemma implies the following corollary,

Corollary 1. *Let S be a translation surface which $\mathrm{GL}(2,\mathbb{R})$ orbit is dense in a quadratic stratum, then for every point $p \in S$ with infinite forward orbit, for all $f \in H^1(S;\mathbb{R})$,*

$$\limsup_{t \to +\infty} \frac{\log |\langle f, \gamma_t(p) \rangle|}{\log t} = \lambda_1$$

where λ_1 is the top Lyapunov of the quadratic stratum.

4. Orbit closure

4.1. **Some useful theorems.** In this section we introduce some lemmas resulting from recent breakthrough in the theory [EMM15], [AEM12], [Wri14].

Lemma 2. *Let $\mathcal{B}$ a family of flat surfaces in a fixed stratum which is represented in some period coordinates by a real linear subspace B. Then for Lebesgue almost every $S \in \mathcal{B}$, the orbit closure is an unique $\mathrm{GL}_2(\mathbb{R})$-invariant suborbifold $\mathcal{L}$ of the Teichmlller space. Moreover, in the above period coordinate, $\mathcal{L}$ is a linear subspace L such that $B \subset L$.*

Proof. According to [EMM15] Proposition 2.16 or alternatively [Wri14] Corollary 1.9, There are countably many $\mathrm{GL}_2(\mathbb{R})$-invariant closed orbifolds in each stratum. Thus at least one orbit closure $\mathcal{L}$ of the family $\mathcal{B}$ intersects $\mathcal{B}$ with non-zero Lebesgue measure in $\mathcal{B}$. In period coordinates, if two linear subspace L and B intersect with non-zero Lebesgue measure in B, then $L \subset B$.

Take now $\mathcal{L}_0$ intersection of all $\mathcal{L}$ as above. This intersection, as any $\mathcal{L}$, is a closed $\mathrm{GL}_2(\mathbb{R})$-invariant subset which contains $\mathcal{B}$. Thus the orbit closure of any point of $\mathcal{L}_0$ is contained in $\mathcal{L}_0$. This implies that any $\mathcal{L}$ as above coincide with $\mathcal{L}_0$. Thus for any $S \in \mathcal{B}$ which orbit closure has non-zero measure intersection with $\mathcal{B}$, $\overline{\mathrm{GL}_2(\mathbb{R}) \cdot S} = \mathcal{L}_0$.

Hence for any $S \in \mathcal{B}$ such that $\mathcal{N} := \overline{\mathrm{GL}_2(\mathbb{R}) \cdot S} \neq \mathcal{L}_0$, Lebesgue measure of $\mathcal{N} \cap \mathcal{B}$ is zero. Taking out the countably many such subset of $\mathcal{B}$, the set of remaining points is of full Lebesgue measure and the orbit closure of each of these points is $\mathcal{L}_0$. □

Lemma 3. *Suppose that in period coordinates B contains a $\mathbb{R}$-linear subspace D. Let D_{Re} and D_{Im} be the projections of D to $H^1(X;\mathbb{R})$ and $H^1(X;i\mathbb{R})$. Then $Vect_{\mathbb{C}}(D_{Re}, D_{Im}) \subset L$ i.e. L contains the $\mathbb{C}$-linear span of D_{Re} and D_{Im}.*

Proof. By Lemma 2, we know that $D \subset L$. By [Wri14] we know that the field of definition of such an affine manifold is real, in particular it is the complexification of L_{Re}. □

Let $\mathcal{M}$ be some closed $\mathrm{GL}_2(\mathbb{R})$-invariant orbifold of some stratum $\mathcal{S}$. And let H^1 denote the flat bundle over $\mathcal{M}$ whose fiber over (X, ω) is $H^1(X;\mathbb{C})$ and H^1_{rel} the flat bundle which fiber is $H^1(X, \Sigma; \mathbb{C})$. Let $p : H^1_{rel} \to H^1$ be the natural projection from relative to absolute cohomology. Remark that if $\mathcal{B} \subset \mathcal{M}$, $T(\mathcal{B})$ induces a flat subbundle of H^1_{rel}. According to [EMM15], $T(\mathcal{M})$ also induces a flat subbundle of H^1_{rel}.

Lemma 4. *Suppose that $\mathcal{B} \subset \mathcal{M}$ and that $p(T(\mathcal{B}))$ contains a k-dimensional subbundle on which the intersection form is zero. If the codimension of $T(\mathcal{B})$ in $T(\mathcal{M})$ is less or equal to k, then $\mathcal{L} = \mathcal{M}$.*

In particular, if $p(B)$ is included in an isotropic then $L = M$.

Proof. In period coordinates, by definition, there is a k-dimensional linear subspace $W \subset L \subset M$ such that $B \subset W^{\perp}$. According to [AEM12], the intersection form is non-degenerate on $p(T(\mathcal{L}))$ thus $\dim p(L) = \dim W + \dim W^{\perp} \geq \dim p(B) + k$ thus $\dim L \geq \dim B + k \geq \dim M$. □

4.2. **Periodic windtree with several obstacles.** Choose a layout for n rectangular obstacles in the plane, all oriented in the same horizontal direction. Now repeat $\mathbb{Z}^2$-periodically this pattern in the plane. In other terms, pick a rectangular torus in which you place n horizontal rectangular obstacles. We call $\mathcal{B}_n$ this family of billiards. We investigate its generic $\mathrm{GL}(2,\mathbb{R})$ orbit closure to compute its diffusion rate.
The case of $\mathcal{B}_1$ was done in [DHL14], in which the authors proved that the diffusion rate is $2/3$.

As in 2.2 we associate to each windtree table in $\mathcal{B}_n$ a half-translation surface in $\mathcal{Q}(1^{4n})$. This yields an embedding

$$\mathcal{S} : \mathcal{B}_n \mapsto \mathcal{Q}(1^{4n})$$

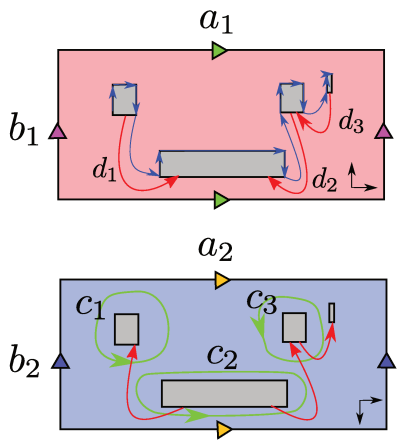
4.2.1. *Orbit closure.* We prove the following Lemma,

Lemma 5. *For Lebesgue almost every windtree table in B_n, $n \geq 2$, its image in $\mathcal{Q}(1^{4n})$ is dense.*

Let $S \in \mathcal{S}(\mathcal{B}_n)$, S is of genus $n+1$ and the stratum $\mathcal{Q}(1^{4n})$ is of dimension $6n$. We consider a_1, b_1, a_2, b_2 simple loops which generate the homology of the two copies of the torus, and take $c_1, d_1, \ldots, c_{n-1}, d_{n-1}$ the loops around the obstacles and between two consecutive obstacles. These generate the absolute homology of S. Now for each obstacle i, start at the lower left corner and browse it clockwise, we denote by $\alpha_i, \beta_i, \alpha_i'$ the three saddle connection we cover until the lower right corner. Let γ_i be a path from the lower right corner of obstacle i to lower left corner of obstacle $i+1$.

These paths form a basis of the relative homology group of S. According to Lemma 1 if we take the hat image of these homology elements besides from $\hat{\alpha}_{n-1}'$ they form a basis of $H_1^-(\hat{S}, \hat{\Sigma}; \mathbb{C})$. This choice of basis yields a choice of fundamental domain for the orientable cover; the first copy with positive orientation glued to the second one flipped along α_{n-1}'.

Now in period coordinates, we see that the image of $\mathcal{S}$ is defined locally by the following equations, where we make the abuse to write the homology class while meaning their period



$$
\begin{aligned}
&\mathrm{Im}(a_1) = 0,\ \mathrm{Re}(b_1) = 0,\ \mathrm{Re}(a_1) = \mathrm{Im}(b_1) \\
&\mathrm{Im}(\beta_i) = 0,\ \mathrm{Re}(\alpha_i) = 0 \text{ for all } 1 \leq i \leq n \\
&d_i = \gamma_i + \overline{\gamma}_i \text{ for all } 1 \leq i \leq n-1
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
&a_1 = -a_2,\ b_1 = b_2 \\
&\alpha_i = \alpha_i' \text{ for all } 1 \leq i \leq n-1 \\
&c_i = 0 \text{ for all } 1 \leq i \leq n-1
\end{aligned} \tag{2}
$$

There are $2n+3$ real equations and $3n-1$ complex equations. The quadratic stratum is of complex dimension $6n$, thus the induced subspace is of real dimension $12n - 2n - 6n - 1 = 4n - 1$. On the other hand for the family of billiards, we have $2n$ variables for the size of each obstacle, $2n-2$ for relative position of the obstacles, and 1 dimensions for the size of the square torus. Thus we have indeed listed all the equations that define our billiard family.

Let $\mathcal{L}$ the generic orbit closure of this family. We introduce two complementary subspaces

$$A = \mathrm{Vect}_{\mathbb{Z}}\left(a_1, b_1, \beta_i, \alpha_i, d_i, \gamma_i\right) \text{ and } B = \mathrm{Vect}_{\mathbb{Z}}\left(a_2, b_2, \alpha_i', c_i\right)$$

where $H_1^-(\hat{S}, \hat{\Sigma}; \mathbb{Z}) = A \oplus B$. Consider π the projection on A parallel to B and let B and B' the subspaces satisfying equations (1), (2) and (2) respectively, where $B \subset B'$.

Remark that $\pi(B') = A$ and $\pi(B)$ is the subspace of A which satisfies equations (1) since they are defined only on vectors of A. We denote by and L the subspace defining the generic $\mathrm{GL}(2, \mathbb{R})$ orbit closure of the two families B' and B. As π is defined on $\mathbb{Z}$ and B is the complexified of a real vector space $\pi(L)$ also. Thus

according to Lemma 3, $\pi(L) = A$ and $B' \subset L$.

We denote by η^* the dual cycle of η in absolute cohomology. Let

$$p(B') \subset \mathrm{Ann}(a_1^* - a_2^*, b_1^* + b_2^*, \alpha_i^* + \alpha_i'^*, c_i^*) = W$$

in order to show that the family is dense in the orbit, it is sufficient to have W an isotropic subspace. Nonetheless, the intersection between $\langle a_1^* - a_2^*, b_1^* + b_2^* \rangle = 2$ is not equal to zero.

We consider

$$v = a_1^* - a_2^* - d_1^* \text{ and } w = b_1^* + b_2^* - 2c_1^*$$

then

$$\langle v, w \rangle = 2 - 2 \langle d_1, c_1 \rangle = 0$$

and

$$p(B') \subset \mathrm{Ann}(v, w, \alpha_i^* + \alpha_i'^*, c_i^*) = W'$$

but now, W' is isotropic, thus according to Lemma 4, L has full dimension and the billiard family is generically dense in the stratum.

This density result together with Lemma 2 and Corollary 1 implies the following,

Theorem 1. *The diffusion rate for in Lebesgue-almost every windtree model B_n, for $n \geq 2$, in Lebesgue-almost every direction is equal to the top Lyapunov exponent of $\mathcal{Q}(1^{4n})$.*

4.2.2. *Obstacles with many right angles.* Consider now a periodic windtree with n obstacles which are horizontal polygons with right angles. For each obstacle i, there are k_i inward right angles and $4 + k_i$ outward. We denote this family by $\mathcal{B}_n(k_1, \ldots, k_n)$.

The associated quadratic differential has simple zeros at the outward right angles and poles at the inward. Thus it has genus $n - 1$ and is in the stratum

$$\mathcal{Q}(1^{4n+p}, -1^p)$$

where $p = \sum k_i$.

To construct a basis of homology of the associated translation surface, we start from the left point of the lowest horizontal side and browse the obstacle boundary clockwise. This yields saddle connections $\alpha_i^1, \beta_i^1, \alpha_i^2, \ldots, \beta_i^{4+k_i-1}, \alpha_i^{4+k_i}$.

The equations in period coordinates are very similar. The equations on a, b, c, d and γ_i are exactly the same, and we have

$$\sum_{j=1}^{4+k_i} \alpha_i^j = 0 \text{ for all } 1 \leq i \leq n-1$$
$$\mathrm{Re}(\alpha_i^j) = 0 \text{ for all } 1 \leq i \leq n \text{ and } 1 \leq j \leq 4+k_i$$
$$\mathrm{Im}(\beta_i^j) = 0 \text{ for all } 1 \leq i \leq n \text{ and } 1 \leq j \leq 4+k_i-1$$

Now the same construction as before implies that the billiards in $\mathcal{B}_n(k_1, \ldots, k_n)$ induce generically quadratic differentials with dense orbits in the stratum. Thus we have the following theorem,

Theorem 2. *The diffusion rate for in Lebesgue-almost every windtree model $B_n(k_1, \ldots, k_n)$ in Lebesgue-almost every direction is equal to the top Lyapunov exponent of $\mathcal{Q}(1^{4n+p}, -1^p)$; for any integers $n \geq 2$, $k_1, \ldots k_n \geq 0$ and $p = \sum k_i$.*

5. Some numerical computations

In Figure 1, we represent a computation of the principal Lyapunov exponent for $\mathcal{Q}(1^{4n})$. We observe that it goes to $1/2$ when $n \to infty$. This enters in resonance with the result of [DZ15], where when we increase the number of right angles of a single obstacle for a periodic windtree, the diffusion rate goes to zero. Intuitively, these angles scatter the linear flow; it follows completely distinct path from on side to the other of the singularity.

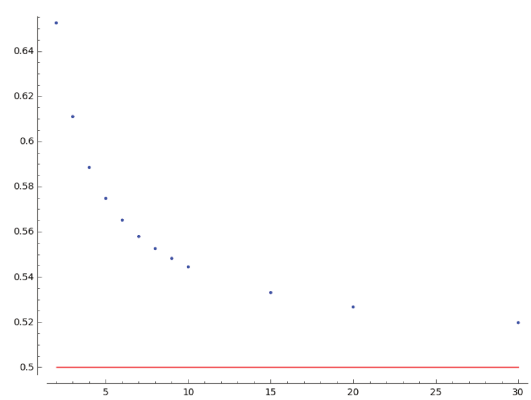


Figure 1. Principal Lyapunov exponent for $\mathcal{Q}(1^{4n})$

In Figure 2, we represent a computation of the principal Lyapunov exponent for $\mathcal{Q}(1^{4n+10}, -1^{10})$. Again when we fix the nomber of $3\pi/2$ angles and increase angles $\pi/2$, the diffusion rate goes to $1/2$.

Finaly, further experiments show that when we fix n and make p go to infinity, the principal Lyapunov exponent is going to zero. The heuristic explaination for this phenomenon is that when a particule bounces close to an angle $3\pi/2$ it

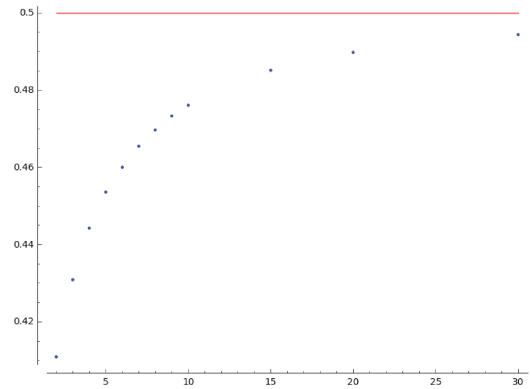


FIGURE 2. Principal Lyapunov exponent for $\mathcal{Q}(1^{4n+10}, -1^{10})$

comes back on its steps slightly shifted. This enter in the frame of a more general Grivaux-Hubert conjecture that we reformulate in [Fou16].

Appendix A. Generic Lyapunov exponent

In this section we follow the proof of [CE15], which endeavors to show that any translation surface is Lyapunov and Birkhoff generic in its orbit closure for almost every direction. We will focus on one of the key results in this article about Lyapunov genericity on a irreducible component for the Kontsevich-Zorich cocycle.

Theorem (1.5 in [CE15])**.** *Fix $x \in \mathcal{H}_1(\alpha)$ and let $\mathcal{M} = \overline{\mathrm{SL}(2,\mathbb{R})x}$ the smallest affine invariant manifold containing x, let V be a $\mathrm{SL}(2,\mathbb{R})$ invariant subbundle of the Hodge bundle which is defined and continuous on $\mathcal{M}$. Let $A_V : \mathrm{SL}(2,\mathbb{R}) \times \mathcal{M} \to V$ denote the restriction of the Kontsevich-Zorich cocycle to V and suppose that A_V is strongly irreducible with respect to the affine measure $\nu_{\mathcal{M}}$ whose support is $\mathcal{M}$. Then, for almost every $\theta \in [0, 2\pi)$,*

$$\lim_{t\to\infty} \frac{\log ||A_V(g_t r_\theta x)||}{\log t} \to \lambda_1$$

where λ_1 is the top Lyapunov exponent of A_V.

Our purpose here is to show the following additional lemma to this theorem.

Lemma 6. *In the previous theorem, for any $h \in V$ and almost every $\theta \in [0, 2\pi)$*

$$\lim_{t\to\infty} \frac{\log ||A_V(g_t r_\theta x)h||}{\log t} \to \lambda_1$$

In [CE15], intuition of the result is provided by analogy with random walks. We start by showing the analog of Lemma 6 for random walks.

A.1. **Random walks.** Let μ be a $\mathrm{SO}(2,\mathbb{R})$-invariant compactly supported measure on $\mathrm{SL}(2,\mathbb{R})$ which is absolutely continuous with respect to Haar measure. A measure ν on $\mathcal{H}_1(\alpha)$ is called μ-stationary if

$$\mu * \nu = \int_{\mathrm{SL}(2,\mathbb{R})} (g_*\nu) d\mu(g) = \nu$$

By a theorem of Furstenberg [Fur63b], [Fur63a], restated in [NZ99][Theorem 1.4], there exist a probability measure ρ on $\mathrm{SL}(2,\mathbb{R})$ such that the map $\nu \to \rho * \nu$ is a bijection between P-ergodic measures and ergodic μ stationary measures. Which are $\mathrm{SL}(2,\mathbb{R})$-invariant affine measures according to [EM13][Theorem 1.4].

This is a first step for an analogy between Teichmüller flow in some affine invariant locus and a random walk with the associated measure.

Let Gr_s denote the Grassmanian of s-dimensional subspaces in the $\mathrm{SL}(2,\mathbb{R})$ invariant subbundle of the Hodge bundle V. Let $\tilde{\mathcal{H}} = \mathcal{H}_1(\alpha) \times \mathrm{Gr}_s$ and $\tilde{\nu}$ be the the μ stationary measure on it; we may write $d\tilde{\nu}(x,U) = d\nu(x)d\eta_x(U)$.

The measure η_x on Gr_s heuristically corresponds to the mean position of any linear subspace carried along the Teichmüller flow using Gauss-Manin connection. Let h be some vector in $V \setminus 0$, then let $I(h) \subset \mathrm{Gr}_s$ be the set of s-dimensional subspaces containing h.

Lemma (C.10 in [EM13])**.** *If the cocycle A_V is strongly irreducible on V, then for almost every $x \in \mathcal{H}_1(\alpha)$ and any vector $h_x \in V$, $\delta_x(I(h_x)) = 0$*

In particular if we consider any Oseledets flag, then this Lemma yields that generically they do not contain a fixed vector h in the case of random walks.

We show a random walk version of Theorem A.

Theorem 3 (Theorem 2.6 of [CE15])**.** *Fix $x \in \mathcal{H}_1(\alpha)$ and let $\mathcal{M} = \overline{\mathrm{SL}(2,\mathbb{R})x}$ the smallest affine invariant manifold containing x, let V be a $\mathrm{SL}(2,\mathbb{R})$ invariant subbundle of the Hodge bundle which is defined and continuous on $\mathcal{M}$. Let $A_V : \mathrm{SL}(2,\mathbb{R}) \times \mathcal{M} \to V$ denote the restriction of the Kontsevich-Zorich cocycle to V and suppose that A_V is strongly irreducible with respect to the affine measure $\nu_{\mathcal{M}}$ whose support is $\mathcal{M}$. Then for a fixed $h \in V$ and for $\mu^{\mathbb{N}}$-almost every $\overline{g} = (g_1, \dots, g_n, \dots)$,*

$$\lim_{n\to\infty} \frac{1}{n} \log ||A_V(g_n \dots g_1, x)h|| \to \lambda_1$$

where λ_1 is the top Lyapunov exponent of A_V.

We have introduced the possibility to use a fixed h for this theorem. The original version is only for the general norm although all the ingredient are present in the proof to show this.

A.2. **Proof Theorem 3.** We fix $\mathcal{M}$ and V as in the theorem. Pick an arbitrary $v_0 \in V$ and let $v_i(\overline{g}) = A_V(g_i \dots g_1, x)v_0$.

Lemma (2.11 and 2.16 in [CE15])**.** *For all $\epsilon > 0$, there exists an integer L such that for every $x \in \mathcal{M}$ almost every $\overline{g}$ we have that all but a set of $\mathbb{N}$ of density 4ϵ is in disjoint blocks $[i+1, i+L]$ so that*

$$\exp(\lambda_1 - \epsilon)^L \leq \frac{||A_v(g_{i+L} \dots g_{i+1}, y)v||}{||v||} \leq \exp(\lambda_1 + \epsilon)^L$$

Let $\overline{g}$ be in the full measure set as above, K be the subset of density 4ϵ and I the set of indices starting the blocks $[i+1, i+L]$.

Then for $n \gg L$,

$$\log ||v_n|| = \sum_{i=1}^{n} \log \frac{||v_i||}{||v_{i-1}||} = \underbrace{\sum_{i \in I \cap [1, n-L]} \log \frac{||v_i||}{||v_{i-1}||}}_{S_1} + \underbrace{\sum_{i \in K} \log \frac{||v_i||}{||v_{i-1}||}}_{S_2} + \underbrace{\sum_{i=n-L}^{n} \log \frac{||v_i||}{||v_{i-1}||}}_{S_3}$$

Let C such that for all g in the support of μ and all $y \in \mathcal{M}$, $||A_V(g, y)|| \leq C$. Then $|S_3| \leq L \log C$, and $|S_2| \leq 4\epsilon n \log C$.

Moreover

$$\frac{|I \cap [1, \dots, n]| \cdot L}{n} \geq 1 - 4\epsilon$$

Hence

$$S_1 \geq |I \cap [1, \dots, n]| \cdot (\lambda_1 - \epsilon)L \geq (1 - 4\epsilon)n(\lambda_1 - \epsilon)$$

For almost all $\overline{g}$ and any $n \gg L$,

$$\frac{1}{n} \log ||v_n|| \geq (1 - 4\epsilon)(\lambda_1 - \epsilon) - 4\epsilon \log C - \frac{L}{n} \log C$$

Since $\epsilon \geq 0$ is arbitrary, we get for all $h \in V$ and almost every $\overline{g}$,

$$\liminf_{n \to \infty} \log ||A_V(g_n \dots g_1, x)h|| \geq \lambda_1$$

With a similar argument we get an upper bound

$$\limsup_{n \to \infty} \log ||A_V(g_n \dots g_1, x)h|| \leq \lambda_1$$

Which implies Theorem 3.

A.3. **Proof of Lemma 6.** According to the sublinear tracking Lemma of [CE15], for almost every $\theta \in [0, 2\pi)$, there exists $\overline{g} = (g_1, \dots, g_n, \dots)$ satisfying Theorem 3 such that we can write

$$g_{\lambda n} r_\theta = \epsilon_n g_n \dots g_1$$

with $\epsilon_n \in \mathrm{SL}(2, \mathbb{R})$ satisfying

$$\lim_{n \to \infty} \frac{1}{n} \log ||\epsilon_n|| = 0$$

By the cocycle relation we have

$$A_V(g_{\lambda n}, r_\theta x) = A_V(\epsilon_n, g_n \dots g_1 x) A_V(g_n \dots g_1, x)$$

There exists $C > 0$ and $N < \infty$ so that for all $g \in \mathrm{SL}(2, \mathbb{R})$ and all $x \in \mathcal{H}_1(\alpha)$, we have

$$||A_V(g, x)|| \leq C||g||^N$$

Hence

$$\log ||A_V(g_{\lambda n}, r_\theta x)h|| = \log ||A_V(g_n \dots g_1, x)||h + o(n)$$

Which shows the Lemma.

References

[AEM12] Artur Avila, Alex Eskin, and Martin Moeller. Symplectic and isometric sl(2,r) invariant subbundles of the hodge bundle, 2012.

[AEZ16] Jayadev S. Athreya, Alex Eskin, and Anton Zorich. Right-angled billiards and volumes of moduli spaces of quadratic differentials on $\mathbb{C}p^1$. *Ann. Sci. Éc. Norm. Supér.*, 49(6), 2016.

[CE15] Jon Chaika and Alex Eskin. Every flat surface is Birkhoff and Oseledets generic in almost every direction. *J. Mod. Dyn.*, 9:1–23, 2015.

[D$^+$16] Vincent Delecroix et al. surface_dynamics, 2016.

[DHL14] Vincent Delecroix, Pascal Hubert, and Samuel Lelièvre. Diffusion for the periodic wind-tree model. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(6):1085–1110, 2014.

[DZ15] Vincent Delecroix and Anton Zorich. Cries and whispers in wind-tree forests, 2015.

[EE90] Paul Ehrenfest and Tatiana Ehrenfest. *The conceptual foundations of the statistical approach in mechanics*. Dover Publications, Inc., New York, english edition, 1990. Translated from the German by Michael J. Moravcsik, With a foreword by M. Kac and G. E. Uhlenbeck.

[EM13] Alex Eskin and Maryam Mirzakhani. Invariant and stationary measures for the sl(2,r) action on moduli space, 2013.

[EMM15] Alex Eskin, Maryam Mirzakhani, and Amir Mohammadi. Isolation, equidistribution, and orbit closures for the $\mathrm{SL}(2,\mathbb{R})$ action on moduli space. *Ann. of Math. (2)*, 182(2):673–721, 2015.

[Fou16] Charles Fougeron. Lyapunov exponents of the hodge bundle over strata of quadratic differentials with large number of poles, 2016.

[FU14] Krzysztof Fraczek and Corinna Ulcigrai. Non-ergodic $\mathbb{Z}$-periodic billiards and infinite translation surfaces. *Invent. Math.*, 197(2):241–298, 2014.

[Fur63a] Harry Furstenberg. Noncommuting random products. *Trans. Amer. Math. Soc.*, 108:377–428, 1963.

[Fur63b] Harry Furstenberg. A Poisson formula for semi-simple Lie groups. *Ann. of Math. (2)*, 77:335–386, 1963.

[HW80] J. Hardy and J. Weber. Diffusion in a periodic wind-tree model. *J. Math. Phys.*, 21(7):1802–1808, 1980.

[MST16] Alba Málaga Sabogal and Serge Troubetzkoy. Ergodicity of the Ehrenfest wind-tree model. *C. R. Math. Acad. Sci. Paris*, 354(10):1032–1036, 2016.

[NZ99] Amos Nevo and Robert J. Zimmer. Homogenous projective factors for actions of semi-simple Lie groups. *Invent. Math.*, 138(2):229–252, 1999.

[Vee93] William A. Veech. Flat surfaces. *Amer. J. Math.*, 115(3):589–689, 1993.

[Wri14] Alex Wright. The field of definition of affine invariant submanifolds of the moduli space of abelian differentials. *Geom. Topol.*, 18(3):1323–1341, 2014.

Chapitre 2

A partial proof for Grivaux-Hubert conjecture

arXiv:1611.07728

Submitted to *Math. Res. Letters.*

Here we show using sheaf algebra a bound on quadratic strata depending on their highest multiplicity and genus. This implies a proof of the fact that a family of Teichmüller curves induced by quadratic differentials of a fixed genus and bounded number of zeros has its Lyapunov exponents going to zero when we increase the number of poles.

LYAPUNOV EXPONENTS OF THE HODGE BUNDLE OVER STRATA OF QUADRATIC DIFFERENTIALS WITH LARGE NUMBER OF POLES

CHARLES FOUGERON

Abstract. We show an upper bound for the sum of positive Lyapunov exponents of any Teichmüller curve in strata of quadratic differentials with at least one zero of large multiplicity. As a corollary it stands for all Teichmüller curves in these strata and $\mathrm{SL}(2,\mathbb{R})$ invariant subspaces defined over $\mathbb{Q}$. This solves Grivaux-Hubert's conjecture about the asymptotics of Lyapunov exponents for strata with large number of poles in the situation when at least one zero has large multiplicity.

1. Introduction.

Lyapunov exponents for translation surfaces have been introduced two decades ago by Zorich in [Zor99] and [Zor97] as a dynamical invariants which describe how associated leaves *wind around the surface*. On any translation surface we can introduce a translation flow which generalizes the linear flow on a flat torus (see [Zor06] for a very nice introduction to the subject). This flow has a most simple local dynamic, it is a parabolic system. Nonetheless the homology of the translation flow presents a nice asymptotic regularity and its deviation to the asymptotic cycle is described by Lyapunov exponents of the Hodge bundle over some $\mathrm{SL}_2(\mathbb{R})$ invariant subspace in the moduli space of curves.

Even if a numerical approximation of these exponents is accessible [D$^+$16], there is a priori no hope for an explicit formula to compute them. Yet a breakthrough of Eskin, Kontsevich and Zorich showed an astonishing formula binding the sum of the Lyapunov exponents to the Siegel-Veech constant of the invariant locus [EKZ14], following-up an insightful observation of [Kon97] that this sum is related to the degree of a Hodge subbundle, the formula was shown latter on in [For02]. This was the starting point to evaluate Lyapunov exponents in certain particularly symmetric cases, for example for square-tiled cyclic covers [FMZ14], [EKZ11] and triangle groups [BM10], and compute explicitly diffusion rate of wind-tree models [DHL14], [DZ15].

Other advances have been made since to estimate Lyapunov exponents for higher genus. In [Yu14], Yu gave a partial proof the conjecture of [KZ97] that the second Lyapunov exponent for hyperelliptic components of strata should go to 1 when genus goes to infinity. His proof was conditional to a conjecture which brought new ideas to find bounds from below to these exponents. His conjecture

has recently been proven in [EKMZ16], see also [DD15] for a insightful foliation point of view. Yu also obtained upper bound for the sum of Lyapunov exponents depending on Weierstrass semi-groups. Yu's idea exposed in [YZ12] and [YZ13] that independently appeared in [CM12] and [CM14] is to use algebraic characterization of sum of Lyapunov exponents and estimate it with homological algebra arguments.

In parallel Grivaux and Hubert remarked that some Teichmüller curves whose Lyapunov exponents are all zero can appear in quadratic strata. Moreover in [GH14] they prove that for this to happen the curve should be in a stratum with at least $\max(2g-2, 2)$ poles. The heuristic explanation they provide with this result is that *poles slow down the linear flow*, since passing by a pole is like doing a U-turn for the linear flow.

Conjecture (Grivaux-Hubert)**.** *Positive Lyapunov exponents associated to a translation surface have a uniform bound depending only on the number of poles of the surface. This bound goes to zero when the number of poles goes to infinity.*

In [CM14] Theorem 8.1, D. Chen an M. Möller obtained a result in this direction for $\mathcal{Q}\left(n, -1^n\right)$ and $\mathcal{Q}\left(n, 1, -1^{n+1}\right)$, showing that these strata are non-varying and computing explicitely the sum of Lyapunov exponents for every Teichmüller curve which is equal to $2/(n+2)$.

Using methods inspired by Yu's homological algebraic methods we obtain a general upper bound with respect to the highest multiplicity of zeros and genus. This proves the conjecture in this case.

Theorem. *For any Teichmüller curve $\mathcal{C}$ in a quadratic stratum $\mathcal{Q}\left(m_1, \ldots, m_k, -1^p\right)$ of genus g where $m_1 \geq m_2 \geq \cdots \geq m_k$, if $m_1 \geq 2g$,*

$$L^+(\mathcal{C}) \leq \frac{(3g-1)g}{m_1+2}$$

Where $L^+(\mathcal{C})$ stands for the sum of its positive Lyapunov exponents.

This bound can be used for more general loci in strata applying the following theorem.

Theorem ([EBW14], Theorem 5 and [EMM15], Theorem 2.3)**.** *Let $\mathcal{N}_n$ be a sequence of affine manifolds and suppose the sequence of affine measure $\nu_{\mathcal{N}_n} \to \nu$. Then ν is a probability measure, and it is an affine measure $\nu_{\mathcal{N}}$ where $\mathcal{N}$ is the smallest submanifold such that there exists n_0 such that $\mathcal{N}_n \subset \mathcal{N}$ for all $n > n_0$. Moreover the Lyapunov exponents of $\nu_{\mathcal{N}_n}$ converge to the Lyapunov exponents of ν.*

As a direct consequence of the two theorems, any $\mathrm{SL}(2, \mathbb{R})$ invariant loci in a stratum with a large multiplicity zero, which can be obtained as the limit of its Teichmüller curves satisfies the bound of the former theorem and Grivaux-Hubert's conjecture. In particular for strata themselves and more generally for affine manifolds defined over $\mathbb{Q}$.

Corollary. *Let $\mathcal{N}$ an affine manifold defined on $\mathbb{Q}$ in a stratum $\mathcal{Q}(m_1, \dots, m_k, -1^p)$ of genus g where $m_1 \geq m_2 \geq \cdots \geq m_k$, if $m_1 \geq 2g$,*

$$L^+(\mathcal{N}) \leq \frac{(3g-1)g}{m_1+2}$$

Where $L^+(\mathcal{C})$ stands for the sum of its positive Lyapunov exponents.

Pascal Hubert pointed out latter on that Remark 2 of [DZ15] gives a counter-example to the conjecture in general with an bounded multiplicity zeros. Taking a cover of a base surface, it exhibits an family of surfaces of genus 1 with arbitrary large number of poles, which first Lyapunov exponent is always equal to 2/3. Thus the conjecture cannot be true in the above generality. Nonetheless broad computer experiments on strata with number of zeros and poles all going to infinity seem to show that the sum of their Lyapunov exponents goes to zero at speed $1/\sqrt{p}$. We state the following conjecture,

Conjecture. *Let Q_n be a sequence of connected components of strata for a fixed genus and p_n be their number of poles going to infinity, then*

$$\lambda_1^+(Q_n) = \mathcal{O}(1/\sqrt{p_n})$$

Aknowledgements. I am very grateful to Fei Yu for very enlightning and friendly discussions in MFO and Beijing. I thank Pascal Hubert and Julien Grivaux who took time to discuss their conjecture and this work in a most pleasant atmosphere in Marseille.

2. Background material.

Strata. A half-translation surface is a pair (X, q) where X is a Riemann surface and a quadratic differential on X with possibly simple poles. If $S(q)$ is the set of zeros and poles of q on X, we can endow $\tilde{X} := X \backslash S(q)$ with charts $\phi_i : U_i \to X$ such that $\phi_i^* q = dz^{\otimes 2}$. In such an atlas, the transition functions are translations composed with $\pm\,\mathrm{Id}$. The quadratic differential induces a flat metric $|q|$ on the open surface $\tilde{X}$. This metric can be extended to the whole surface and would have conical points at $S(q)$ with angles multiples of π. If we fix integers $m_1, \dots, m_k, p$ such that $\sum_{i=1}^k m_i - p = 4g-4$, for some positive integer g, the stratum of half-translation surfaces $\mathcal{Q}(m_1, \dots, m_k, -1^p)$ is the set of half-translation surfaces (X, q) where q has k distinct zeros of multiplicity $m_1, \dots, m_k$ and p simple poles. The projectivized space $P\mathcal{Q}(m_1, \dots, m_k, -1^p)$ is obtained by quotient under the scalar action of $\mathbb{C}^*$ on the quadratic differential.

Teichmüller curves. There is a natural action of $\mathrm{GL}(2, \mathbb{R})$ on every stratum. Take the flat atlas as above, the charts were constructed in such way that transition functions are translations by vectors v_{ij} composed with $\pm\,\mathrm{Id}$. For a matrix $M \in \mathrm{GL}(2, \mathbb{R})$ we define a new surface by multiplying the previous change of charts by M. They will be translations by vectors Mv_{ij} composed with $\pm\,\mathrm{Id}$.

Let $\mathbb{H} = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ denote the Poincaré upper half-plane. For any (X,q) the action of $SL(2,\mathbb{R})$ factors to a map

$$\mathbb{H} \to P\mathcal{Q}\left(m_1,\dots,m_k,-1^p\right)$$

which is an immersion. The image of this map is called a Teichmüller disk. This map also factors through the Veech group which is its stabilizer. If the Veech group Γ of a given half-translation surface is a lattice in $\mathrm{SL}(2,\mathbb{R})$ we say that the surface is a Veech surface and the image of a $\mathbb{H}/\Gamma$ in the projective stratum is called a Teichmüller curve. An other point of view for the Teichmüller curve is to consider the induced algebraic map $\mathcal{C} \to \mathcal{M}_g$. This is the convention used in [Möl13].

By definition a Teichmüller curve is a surface with a hyperbolic structure, eventually with cusps. A main tool for studying these curves is their compactification.

Let $f : \mathcal{X} \to C$ be a smooth family of curves of genus g over the smooth curve C, i.e. a smooth morphism with connected fibers, which are smooth curves of genus g. By definition of the moduli space, such a family induces a map $m : C \to \mathcal{M}_g$. The delicate part in the compactification of a Teichmüller curve is the choice of the surfaces we put at the cusps. Consider the extended map $\overline{m} : \overline{C} \to \overline{\mathcal{M}_g}$, it is not always true that there is a family of stable curve $\overline{f} : \overline{\mathcal{X}} \to C$ which extends f and induces $\overline{m}$, but there is an unramified finite index cover $B \to C$, such that we can define a stable extension on the pullback $\overline{\mathcal{X}} \to \overline{B}$ (see [Möl13]).

The structure sheaf of $\overline{\mathcal{X}}$ will be written $\mathcal{O}$, and the pullback of the dualizing sheaf of $\overline{\mathcal{M}_{g,1}} \to \overline{\mathcal{M}_{g,n}}$, will be referred to as $\omega_{\overline{\mathcal{X}}/\overline{B}}$.

Divisors classes. For a Teichmüller curve generated by a quadratic differential q, there is an associated line bundle $\mathcal{L} \subset f_*\,\omega_{\overline{\mathcal{X}}/\overline{B}}^{\otimes 2}$ on $\overline{B}$ whose fiber over the point corresponding to (X,q) is $\mathbb{C}\cdot q$. This bundle has a maximality property (see [Möl13], [Gou12] and [Wri12]) which implies that $\deg(\mathcal{L}) = \chi$.

We denote by $S_1,\dots,S_k$ and $P_1,\dots,P_p$ sections of $\overline{\mathcal{X}} \to \overline{B}$, which intersect the fibers at each zero with multiplicity $m_1,\dots,m_k$ and at poles. For convenience we also introduce the total divisor $\mathcal{A} := \sum_{i=1}^n m_i Z_i - \sum_{j=1}^p P_i$ and the divisor of zeros $\mathcal{Z} := \sum_{i=1}^n m_i Z_i$. We have the following exact sequence induced by the pullback of the inclusion of $\mathcal{L}$,

$$0 \to f^*\mathcal{L} \to \omega_{\overline{\mathcal{X}}/\overline{B}}^{\otimes 2} \to \mathcal{O}(\mathcal{A}) \to 0$$

This is true for non compactified Teichmüller spaces, and it stays true after compactification since the multiplicities of the zeros of the limit differentials stay the same on stable curves.

According to the adjunction formula,

$$(\omega_{\overline{\mathcal{X}}/\overline{B}} + S_i)_{|S_i} = \omega_{S_i} = 0$$

Which implies $S_i^2 = -\omega_{\overline{\mathcal{X}}/\overline{B}} \cdot S_i$. Moreover $S_i \cdot S_j = 0$ for any $i \neq j$, since two fibers of f have zero intersection. Finally,

$$2S_i^2 = -m_i S_i^2 - \deg \mathcal{L}$$

Which leads to the formula :

$$S_i^2 = \frac{-\chi}{m_i + 2} \tag{1}$$

3. Lyapunov exponents

Lyapunov exponents are dynamical invariants for translation surfaces which describe the behaviour of the translation flow on them. For an introduction to their multiple aspects in translation surfaces see [Zor06]. We will adopt an algebraic point of view to compute them. It is based on the observation of [Kon97] that their sum is related to the degree of a Hodge subbundle on a Teichmüller curve (see [EKZ14] or [For02] for a proof). We will use in the following a more precise formula computed in [CM14] as (20),

$$L^+(\mathcal{C}) = \frac{2}{\chi} \cdot \deg f_* \mathcal{O}(\mathcal{A}) + (6g-6) - \frac{1}{2}\left(\sum_{i=1}^{n} \frac{m_i(m_i+4)}{m_i+2} - 3p\right)$$

Where $L^+(\mathcal{C})$ is the sum of the positive Lyapunov exponents associated to the Teichmüller curve $\mathcal{C}$

By Gauss-Bonnet,

$$\sum_{i=1}^{n} m_i - p = 4g - 4$$

$$6g - 6 + \frac{3}{2}p = \frac{3}{2}\sum_{i=1}^{n} m_i$$

Remark that $m_i(m_i+4) - 3m_i(m_i+2) = -2m_i(m_i+1)$, this leads to the following formula with a form which will be effective for us here :

$$L^+(\mathcal{C}) = \frac{2}{\chi} \cdot \deg f_* \mathcal{O}(\mathcal{A}) + \sum_{i=1}^{n} \frac{m_i(m_i+1)}{m_i+2} \tag{2}$$

This equation relates the sum of Lyapunov exponents for a Teichmüller curve to a term defined as an invariant of the stratum and a term describing the behaviour of the curve at its cusps given by the Chern class of a sheaf.

To estimate the sum of Lyapunov exponents, we are reduced to bound the degree of the sheaf $f_* \mathcal{O}(\mathcal{A})$. This will be done based on the two following lemmas which proof will be given in the next sections.

Lemma 1.

$$\deg f_* \mathcal{O}(\mathcal{A}) = \deg f_* \mathcal{O}(\mathcal{Z})$$

Thus the pole divisors won't interfere in the degree we need to estimate. By a similar method, we will be able to bound the degree while taking out zeros one after the other. We will do so until no zero is left and end up with the canonical sheaf. But first we need to choose an order in which we will remove the zeros.

Let us introduce some notations first. We denote by $\underline{d}$ a collection of non negative integers $(d_1, \dots, d_k)$, $|\underline{d}| := d_1 + \cdots + d_k$ and $\underline{d}\mathcal{Z} := \sum_{i=1}^k d_i Z_i$.

We pick a finite family $\mathcal{F} := \{\underline{d}\}^{i\geq 1}$ satisfying the properties

(1) The first term is $\underline{d}^1 = (m_1, \dots, m_k)$
(2) The last term is $\underline{0}$
(3) If $\underline{d}^i = (d_1, \dots, d_k)$ and $\underline{d}^{i+1} = (d'_1, \dots, d'_k)$, for some n, $d_n = d'_n + 1$ and $d_m = d'_m$ for all other indices.

We use in the following the notations $\delta^i := d_n$, $\mu^i := m_n$, $\Sigma^i := Z_n$ and

$$\mathcal{I}^i := \frac{2}{\chi} \cdot \delta^i Z_n \cdot \underline{d}^i \mathcal{Z} = \frac{2\delta^i}{\mu^i + 2}$$

These properties imply that the indices should vary between $1 \leq i \leq 4g - 4 + p$.

We associate to such a family its gaps $G(\mathcal{F}) = i_1, \dots, i_g$, *i.e.* indices l defined by the property that on a generic fiber F,

$$h^0\left(\underline{d}^l \mathcal{Z}_{|F}\right) = h^0\left(\underline{d}^{l+1} \mathcal{Z}_{|F}\right)$$

According to Weierstrass group theory (see [YZ13] and [ACG11] for an introduction), there will be just g gaps.

Lemma 2. *If $i \in G(\mathcal{F})^c$,*

$$\deg f_* \mathcal{O}\Big(\underline{d}^i \mathcal{Z}\Big) \leq \deg f_* \mathcal{O}\Big(\underline{d}^{i+1} \mathcal{Z}\Big) - \delta^i \frac{\chi}{\mu^i + 2}$$

If $i \in G(\mathcal{F})$,

$$\deg f_* \mathcal{O}\Big(\underline{d}^i \mathcal{Z}\Big) = \deg f_* \mathcal{O}\Big(\underline{d}^{i+1} \mathcal{Z}\Big)$$

By recurrence, we obtain the inequality,

$$\deg f_* \mathcal{O}(\mathcal{A}) \leq \sum_{i=1}^k \frac{m_i(m_i+1)}{2} \cdot \frac{-\chi}{m_i + 2} + \sum_{l \in G(\mathcal{F})} \mathcal{I}^l$$

Using Lemma 1 and formula 2,

$$L^+(\mathcal{C}) \leq \sum_{l \in G(\mathcal{F})} \mathcal{I}^l \tag{3}$$

This will lead to a proof of the Theorem written down in the introduction. Assume we are in the setting of this theorem, we denote the larger zero multiplicity by $m_1 \geq 2g-1$. We pick a family $\mathcal{F}$ such that we first remove all zeros different from the first one in an arbitrary way, and finish by taking off all the multiplicity of the principal one. Remark that for $l \in G(\mathcal{F})$,

- $\delta^l \leq \left|\underline{d}^l\right|$
- $\left|\underline{d}^l\right|$ is decreasing w.r.t. l and is less than $2g-1$
- $\mu^l = m_1$ for every gap

where the two last properties are implied by Riemann-Roch theorem.

Thus,

$$\sum_{l \in G(\mathcal{F})} \mathcal{I}^l \leq \sum_{i=1}^{g} \frac{2(2g-i)}{m_1+2} \leq \frac{(3g-1)g}{m_1+2}$$

Remark. *If we can show that one of the zeros with multiplicity larger than g is not a Weierstrass point generically, the bound becomes $\frac{(g+1)g}{m_1+2}$.*

4. Proof of lemmas

Lemma 1. The main tool in the introduction of this filtration is the structural exact sequence for any divisor D in $\overline{\mathcal{X}}$

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0$$

Assume here that D is one of the divisors P_j. We tensorize this exact sequence by $f^*\mathcal{L} \otimes \mathcal{O}(\mathcal{A}+D)$, where $\mathcal{A}$ is the total divisor as introduced in section 2. The functor f_* implies a long exact sequence,

$$\begin{array}{rcl} 0 & \to & f_*\left(\omega^{\otimes 2}\right) \to \mathcal{L} \otimes f_* \mathcal{O}(\mathcal{A}+D) \to \mathcal{L} \\ & \xrightarrow{\delta} & R^1 f_*\left(\omega^{\otimes 2}\right) \longrightarrow R^1 f_*\left(f^*\mathcal{L} \otimes \mathcal{O}(\mathcal{A}+D)\right) \longrightarrow 0 \end{array}$$

Where we use the fact that f induces an isomorphism between D and $\overline{\mathcal{X}}$, which implies the equalities $f_* \mathcal{O}_D = \mathcal{O}_{\overline{\mathcal{X}}}$ and $\omega^{\otimes 2} = f^*\mathcal{L} \otimes \mathcal{O}(\mathcal{A})$.

Remark that $R^1 f_*\left(\omega^{\otimes 2}\right)$ over each point X of the Teichmüller curve has dimension

$$h^1(X, \omega_X^{\otimes 2}) = h^0(X, \omega_X^{\otimes -1}) = 0$$

by Serre's duality. Hence this sheaf is zero, and so is $R^1 f_*\left(f^*\mathcal{L} \otimes \mathcal{O}(\mathcal{A}+D)\right)$. We end up with a short exact sequence which implies a relation on the degrees,

$$\deg\left(\mathcal{L} \otimes f_* \mathcal{O}(\mathcal{A}+D)\right) = \deg\left(\mathcal{L} \otimes f_* \mathcal{O}(\mathcal{A})\right) + \deg\left(\mathcal{L}\right)$$

We start again with $f^*\mathcal{L} \otimes \mathcal{O}(\mathcal{A}+D)$, as we noticed its image through the derived functor $R^1 f_*$ is zero and so will be the image of $f^*\mathcal{L} \otimes \mathcal{O}(\mathcal{A}+D+D')$

for any other D' divisor picked in the P_j. We do this until all the pole divisors are consumed, and get a formula on degrees implying the divisor without poles $\mathcal{Z}$,

$$\deg\left(\mathcal{L}\otimes f_*\,\mathcal{O}(\mathcal{Z})\right)=\deg\left(\mathcal{L}\otimes f_*\,\mathcal{O}(\mathcal{A})\right)+p\cdot\deg\left(\mathcal{L}\right)$$

To finish, remark that fibers of $f_*\,\mathcal{O}(\mathcal{Z})$ have constant dimension $3g-3+p$ by Riemann-Roch theorem, since the degree of the divisor on any fiber will be $4g-4+p>2g-2$ for $g,p\geq 1$. By Grauert Semicontinuity Theorem (see for example [Har77]) this sheaf will be a vector bundle. And so will be $f_*\,\mathcal{O}(\mathcal{A})$ as it is a subsheaf of the later. Moreover $\operatorname{rk} f_*\,\mathcal{O}\left(\mathcal{Z}\right)=3g-3+p$ and $\operatorname{rk} f_*\,\mathcal{O}\left(\mathcal{Z}\right)=3g-3$.

The previous formula becomes

$$(3g-3+p)\deg\left(\mathcal{L}\right)+\deg\left(f_*\,\mathcal{O}(\mathcal{Z})\right)=(3g-3)\deg\left(\mathcal{L}\right)+\deg\left(f_*\,\mathcal{O}(\mathcal{A})\right)+p\cdot\deg\left(\mathcal{L}\right)$$

And finally,

$$\deg\left(f_*\,\mathcal{O}(\mathcal{Z})\right)=\deg\left(f_*\,\mathcal{O}(\mathcal{A})\right)$$

Lemma 2. Similarly to the previous subsection we tensor the structural exact sequence for the divisor Σ^i by $f_*\,\mathcal{O}\Big(\underline{d}^{\,i+1}\mathcal{Z}\Big)$, we get the following long exact sequence,

$$0\to f_*\,\mathcal{O}\Big(\underline{d}^{\,i+1}\mathcal{Z}\Big)\to f_*\,\mathcal{O}\Big(\underline{d}^{\,i}\mathcal{Z}\Big)\to\mathcal{O}_{\Sigma^i}\Big(\delta^i\Sigma^i\Big)\xrightarrow{\delta}\dots$$

Which induces the formula,

$$\deg f_*\,\mathcal{O}\Big(\underline{d}^{\,i}\mathcal{Z}\Big)-\deg f_*\,\mathcal{O}\Big(\underline{d}^{\,i+1}\mathcal{Z}\Big)=\deg\ker\delta$$

As $\ker\delta$ is a subsheaf of $\mathcal{O}_{\Sigma^i}\big(\delta^i\Sigma^i\big)$ which is locally free on a complex curve, both are locally free and when $\ker\delta$ is not zero, *i.e.* when $i\in G(\mathcal{F})^c$

$$\deg\ker\delta\leq\deg\mathcal{O}_{\Sigma^i}\Big(\delta^i\Sigma^i\Big)=\delta^i\Sigma^i\cdot\Sigma^i$$

Hence by 1,

$$\deg f_*\,\mathcal{O}\Big(\underline{d}^{\,i}\mathcal{Z}\Big)\leq\deg f_*\,\mathcal{O}\Big(\underline{d}^{\,i+1}\mathcal{Z}\Big)-\delta^i\frac{\chi}{\mu^i+2}$$

If $i\in G(\mathcal{F})$, $\ker\delta=0$ thus

$$f_*\,\mathcal{O}\Big(\underline{d}^{\,i}\mathcal{Z}\Big)\simeq f_*\,\mathcal{O}\Big(\underline{d}^{\,i+1}\mathcal{Z}\Big)$$

References

[ACG11] Enrico Arbarello, Maurizio Cornalba, and Pillip A. Griffiths. *Geometry of algebraic curves. Volume II*, volume 268 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011. With a contribution by Joseph Daniel Harris.

[BM10] Irene I. Bouw and Martin Möller. Teichmüller curves, triangle groups, and Lyapunov exponents. *Ann. of Math. (2)*, 172(1):139–185, 2010.

[CM12] Dawei Chen and Martin Möller. Nonvarying sums of Lyapunov exponents of Abelian differentials in low genus. *Geom. Topol.*, 16(4):2427–2479, 2012.

[CM14] Dawei Chen and Martin Möller. Quadratic differentials in low genus: exceptional and non-varying strata. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(2):309–369, 2014.

[D+16] Vincent Delecroix et al. surface_dynamics, 2016, http://www.labri.fr/perso/vdelecro/flatsurf.html.

[DD15] Bertrand Deroin and Romain Dujardin. Lyapunov exponents for surface group representations. *Comm. Math. Phys.*, 340(2):433–469, 2015.

[DHL14] Vincent Delecroix, Pascal Hubert, and Samuel Lelièvre. Diffusion for the periodic wind-tree model. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(6):1085–1110, 2014.

[DZ15] Vincent Delecroix and Anton Zorich. Cries and whispers in wind-tree forests, 2015, arXiv:1502.06405.

[EBW14] Alex Eskin, Christian Bonatti, and Amie Wilkinson. Invariant measures for linear cocycles. *Mathematisches Forschungsinstitut Oberwolfach Reports*, Report No. 15/2014, 'Flat Surfaces and Dynamics on Moduli Space':875–880, 2014.

[EKMZ16] Alex Eskin, Maxim Kontsevich, Martin Moeller, and Anton Zorich. Lower bounds for Lyapunov exponents of flat bundles on curves, 2016, arXiv:1609.01170.

[EKZ11] Alex Eskin, Maxim Kontsevich, and Anton Zorich. Lyapunov spectrum of square-tiled cyclic covers. *J. Mod. Dyn.*, 5(2):319–353, 2011.

[EKZ14] Alex Eskin, Maxim Kontsevich, and Anton Zorich. Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow. *Publ. Math. Inst. Hautes Études Sci.*, 120:207–333, 2014.

[EMM15] Alex Eskin, Maryam Mirzakhani, and Amir Mohammadi. Isolation, equidistribution, and orbit closures for the $\mathrm{SL}(2,\mathbb{R})$ action on moduli space. *Ann. of Math. (2)*, 182(2):673–721, 2015.

[FMZ14] Giovanni Forni, Carlos Matheus, and Anton Zorich. Lyapunov spectrum of invariant subbundles of the Hodge bundle. *Ergodic Theory Dynam. Systems*, 34(2):353–408, 2014.

[For02] Giovanni Forni. Deviation of ergodic averages for area-preserving flows on surfaces of higher genus. *Ann. of Math. (2)*, 155(1):1–103, 2002.

[GH14] Julien Grivaux and Pascal Hubert. Loci in strata of meromorphic quadratic differentials with fully degenerate Lyapunov spectrum. *J. Mod. Dyn.*, 8(1):61–73, 2014.

[Gou12] Elise Goujard. A criterion for being a Teichmüller curve. *Math. Res. Lett.*, 19(4):847–854, 2012.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Kon97] M. Kontsevich. Lyapunov exponents and Hodge theory. In *The mathematical beauty of physics (Saclay, 1996)*, volume 24 of *Adv. Ser. Math. Phys.*, pages 318–332. World Sci. Publ., River Edge, NJ, 1997.

[KZ97] M. Kontsevich and A. Zorich. Lyapunov exponents and Hodge theory, 1997, arXiv:hep-th/9701164.

[Möl13] Martin Möller. Teichmüller curves, mainly from the viewpoint of algebraic geometry. In *Moduli spaces of Riemann surfaces*, volume 20 of *IAS/Park City Math. Ser.*, pages 267–318. Amer. Math. Soc., Providence, RI, 2013.

[Wri12] Alex Wright. Schwarz triangle mappings and Teichmüller curves: abelian square-tiled surfaces. *J. Mod. Dyn.*, 6(3):405–426, 2012.

[Yu14] Fei Yu. Eigenvalues of curvature, Lyapunov exponents and Harder-Narasimhan filtrations, 2014, arXiv:1408.1630.

[YZ12] Fei Yu and Kang Zuo. Weierstrass filtration on Teichmüller curves and Lyapunov exponents: Upper bounds, 2012, arXiv:1209.2733.

[YZ13] Fei Yu and Kang Zuo. Weierstrass filtration on Teichmüller curves and Lyapunov exponents. *J. Mod. Dyn.*, 7(2):209–237, 2013.

[Zor97] Anton Zorich. Deviation for interval exchange transformations. *Ergodic Theory Dynam. Systems*, 17(6):1477–1499, 1997.

[Zor99] Anton Zorich. How do the leaves of a closed 1-form wind around a surface? In *Pseudoperiodic topology*, volume 197 of *Amer. Math. Soc. Transl. Ser. 2*, pages 135–178. Amer. Math. Soc., Providence, RI, 1999.

[Zor06] Anton Zorich. Flat surfaces. In *Frontiers in number theory, physics, and geometry. I*, pages 437–583. Springer, Berlin, 2006.

Chapitre 3

Lyapunov exponents and hypergeometric equations

arXiv:1701.08387

This chapter focuses on variation of Hodge structures on flat bundles over the sphere minus three points induced by hypergeometric differential equations. We compute their parabolic degrees and investigate numerically their relation with Lyapunov exponents.

PARABOLIC DEGREES AND LYAPUNOV EXPONENTS FOR HYPERGEOMETRIC LOCAL SYSTEMS

CHARLES FOUGERON

Abstract. Consider the flat bundle on $\mathbb{P}^1 - \{0, 1, \infty\}$ corresponding to solutions of the hypergeometric differential equation

$$\prod_{i=1}^{n}(\mathrm{D} - \alpha_i) - z\prod_{j=1}^{n}(\mathrm{D} - \beta_j) = 0, \text{ where } \mathrm{D} = z\frac{\mathrm{d}}{\mathrm{d}z}$$

For α_i and β_j real numbers, this bundle is known to underlie a complex polarized variation of Hodge structure. Setting the complete hyperbolic metric on $\mathbb{P}^1 - \{0, 1, \infty\}$, we associate n Lyapunov exponents to this bundle. We study the dependence of these exponents in parameters α_i, β_j through algebraic computations and numerical simulations.

1. Introduction

Oseledets decomposition of flat bundles over an ergodic dynamical system is often referred to as *dynamical variation of Hodge structure*. In the case of Teichmüller dynamics, both Oseledets decomposition and a variation of Hodge structure (VHS) appear. Two decades ago it was observed in [Kon97] that these structures were linked, their invariants are related: the sum of the Lyapunov exponents associated to a Teichmüller curve equals the normalized degree of the Hodge bundle. This formula was studied extensively and extended to strata of abelian and quadratic differentials from then (see [FMZ14], [Kri04], [BM10], [EKZ14]). Soon this link was observed in other settings. In [KM16] it was used as a new invariant to classify hyperbolic structures and distinguish commensurability classes of Deligne-Mostow's non-arithmetic lattices in $\mathrm{PU}(2, 1)$ and $\mathrm{PU}(3, 1)$. In [Fil14] a similar formula was observed for higher weight variation of Hodge structures. The leitmotiv in this work is the study of the relationship between theses two structures in a broad class of examples with arbitrary weight. This family of examples will be given by hypergeometric differential equations which yield a flat bundle endowed with a variation of Hodge structure over the sphere with three punctures. A recent article [EKMZ16] shows that the degrees of holomorphic flags of the Hodge filtration bound by below the sum of Lyapunov exponents. Our investigation will start by computing these degrees and then explore the behaviour of Lyapunov exponents through numerical simulations and their distance to the later lower bounds. This will enable us to bring out some simple algebraic relations

under which there is a conjectural equality.

Hypergeometric equations. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n$ be two disjoint sequences of n real numbers. We define the *hypergeometric differential equation* corresponding to those parameters

$$\prod_{i=1}^{n}(\mathrm{D}-\alpha_i) - z\prod_{j=1}^{n}(\mathrm{D}-\beta_j) = 0, \text{ where } \mathrm{D} = z\frac{\mathrm{d}}{\mathrm{d}z} \tag{1}$$

This equation originates from a large class of special functions called *generalized hypergeometric functions* which satisfy it. For more details about these functions see for example [Yos97].

It is an order n differential equation with three singularities at 0, 1 and ∞ hence the space of solutions is locally a dimension n vector space away from singularities and can be seen in a geometric way as a flat bundle over $\mathbb{P}^1 - \{0, 1, \infty\}$. This flat bundle is completely described by its monodromy matrices around singularities. We will be denoting monodromies associated to simple closed loop going counterclockwise around $0, 1$ and ∞ by M_0, M_1 and M_∞. We get a first relation between these matrices observing that composing the three loops in the same order will give a trivial loop: $M_\infty M_1 M_0 = \mathrm{Id}$. The eigenvalues of M_0 and M_∞ can be expressed with respect to parameters of the *hypergeometric equation* (1) and M_1 has a very specific form as stated in the following proposition.

Proposition 1.1. *For any two sequences of real numbers $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$,*

- *M_0 has eigenvalues $e^{2i\pi\alpha_1}, \ldots, e^{2i\pi\alpha_n}$*
- *M_∞ has eigenvalues $e^{-2i\pi\beta_1}, \ldots, e^{-2i\pi\beta_n}$*
- *M_1 is the identity plus a matrix of rank one*

Proof. See Proposition 2.1 in [Fed15] or alternatively Prop. 3.2 and Theorem 3.5 in [BH89] □

This proposition determines the conjugacy class of the representation associated to the flat bundle $\pi_1\left(\mathbb{P}^1 - \{0, 1, \infty\}\right) \to GL_n(\mathbb{C})$ thanks to the rigidity of hypergeometric equations (see [BH89]). These classes will be computed explicitly in section 3.2.

Lyapunov exponents. We now endow the 3-punctured sphere with its hyperbolic metric. As this metric yields an ergodic geodesic flow, for any integrable norm on the flat bundle we associate to it, using Oseledets theorem, a measurable flag decomposition of the vector bundle and n Lyapunov exponents [KZ03]. These exponents correspond to the growth of the norm of a generic vector in each flag while transporting it along with the flat connection.

According to [EKMZ16] there is a canonical family of integrable norms on the flat bundle associated to the hypergeometric equation which will produce the same flag decomposition and Lyapunov exponents. This family contains the harmonic norm induced by the VHS structure and the norm we will use in our algorithm.

Variation of Hodge structure. Hypergeometric equations on the sphere are well known to be physically rigid (see [BH89] or [Kat96]) and this rigidity together with irreducibility is enough to endow the flat bundle with a (complex) VHS using its associated Higgs bundle structure (see [Fed15] or directly Cor 8.2 in [Sim90]). Using techniques from [Kat96] and [DS13], Fedorov gives in [Fed15] an explicit way to compute the Hodge numbers for the underlying VHS. We extend this computation and give a combinatoric point of view that will be more convenient in the following to express parabolic degrees of the Hodge flag decomposition.

We introduce a canonical way to describe combinatorics of the intertwining of α's and β's on the circle $\mathbb{R}/\mathbb{Z}$. Starting from any eigenvalue, we browse the circle counterclockwise (or in the increasing direction for $\mathbb{R}$) and denote $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ by order of appearance $\eta_1, \eta_2, \ldots, \eta_{2n}$ and define $\tilde{f} : \mathbb{Z} \cap [0, 2n] \mapsto \mathbb{Z}$ recursively by the following properties,

- $\tilde{f}(0) = 0$
- $\tilde{f}(k) = \tilde{f}(k-1) + \begin{cases} 1 & \text{if } \eta_k \text{ is an } \alpha \\ -1 & \text{if } \eta_k \text{ is an } \beta \end{cases}$

Let f be defined on the eigenvalues by $f(\eta_k) = \tilde{f}(k)$. It depends on the choice of starting point up to a shift. For a canonical definition, we shift f such that its minimal value is 0. It is equivalent to starting at the point of minimal value. This defines a unique f which we call the *intertwining diagram* of the equation.

For every integer $1 \leq i \leq n$ we define

$$h_i := \#\{\alpha \mid f(\alpha) = i\} \ = \ \#\{\beta \mid f(\beta) = i - 1\}$$

Then we have the following theorem from [Fed15],

Theorem (Fedorov)**.** *The $h_1, h_2, \ldots, h_n$ are the Hodge numbers of the VHS after an appropriate shifting.*

Remark. *If the α's and β's appear in an alternate order then $f(\alpha) \equiv 1$ and $f(\beta) \equiv 0$ thus there is just one element in the Hodge decomposition and the polarization form is positive definite. In other words the harmonic norm is invariant with respect to the flat connection. This implies that Lyapunov exponents are zero.*

In general, this Hodge structure endows the flat bundle with a pseudo-Hermitian form of signature (p, q) where p is the sum of the even Hodge numbers and q the

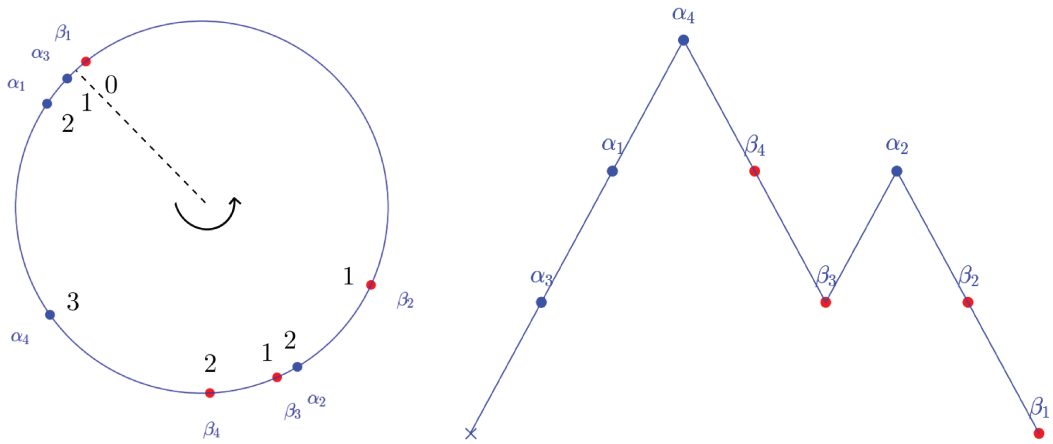


FIGURE 1. Example of computation of f

sum of the odd ones. This gives classically the fact that the Lyapunov spectrum is symmetric with respect to 0 *and that at least* $|p-q|$ *exponents are zero (see Appendix A in* [FMZ14]*).*

Pushing further methods of [Fed15] and [DS13], we compute the parabolic degree of the sub Hodge bundles. This computation was done with the help of computer experiments in Section 4.2 which yielded a conjectural formula for these degrees. Besides the intertwining diagram, another quantity appears. Let us relabel α and β by order of appearance after choosing α_1 such that $f(\alpha_1)=0$, then take the representatives of α and β in $\mathbb{R}$ which are included in $[\alpha_1, \alpha_1+1[$ and define $\gamma := \sum \beta - \sum \alpha$. The formula will depend on the floor value of γ. As $0<\gamma<n$ we have n possible values $0 \leq [\gamma] < n$.

Theorem 1.2. *Let* $1 \leq p \leq n$ *be an integer and* $\mathcal{E}^p$ *the* p*-th graded piece of the Hodge filtration on* $\mathbb{P}^1 - \{0,1,\infty\}$*. We denote by* δ^p *the degree of the Deligne compactification of* $\mathcal{E}^p$ *on the sphere. Then,*

- *if* $p = [\gamma]+1$
$$\deg_{par}(\mathcal{E}^p) = \delta^p + \{\gamma\} + \sum_{f(\alpha)=p} \alpha + \sum_{f(\beta)=p-1} 1-\beta$$
- *otherwise*
$$\deg_{par}(\mathcal{E}^p) = \delta^p + \sum_{f(\alpha)=p} \alpha + \sum_{f(\beta)=p-1} 1-\beta$$
- $$-\delta^p = \#\left\{\beta_i \mid f(\beta_i) = p-1 \text{ and } i \leq n-[\gamma]\right\}$$

Acknowledgement. These experiments were suggested by Maxim Kontsevich, I am very grateful to him for suggesting this problem, sharing his initial experiments, and for his involvement. I thank dearly Jeremy Daniel for his curiosity to

the subject and his answer to my myriad of questions as well as Bertrand Deroin; Anton Zorich for his flawless support and attention, Martin Möller and Roman Fedorov for taking time to explain their understanding of the parabolic degrees and Hodge invariants at MPIM in Bonn. I am also very thankful to Carlos Simpson for his kind answers and encouragements, and to Yuri Manin for pointing out a possible link to cosmology.

2. Degree of Hodge subbundles

2.1. **Variation of Hodge Structure.** We start recalling the definition of a polarized complex variations of Hodge structures (VHS).

A VHS on a curve C consists of a complex flat bundle (E, ∇) together with an hermitian form h and a h-orthogonal decomposition

$$E = \bigoplus_{p \in \mathbb{Z}} \mathcal{E}^p$$

into C^∞-subbundles. We write the induced flag filtrations by $\mathcal{F}^p := \bigoplus_{i \geq p} \mathcal{E}^i$ and $\overline{\mathcal{F}^p} := \bigoplus_{i \leq p} \mathcal{E}^i$. The following conditions are satisfied :

- The decreasing filtration $\mathcal{F}^\bullet$ is holomorphic, and the increasing filtration $\overline{\mathcal{F}}^\bullet$ is anti-holomorphic.
- The connection shifts the grading by at most one, i.e.

$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega^1_{\mathcal{C}} \text{ and } \nabla(\overline{\mathcal{F}^p}) \subset \overline{\mathcal{F}^{p-1}} \otimes \Omega^1_{\mathcal{C}}$$

- h is positive definite on E^p if p is even negative definite if p is odd.

Up to a shift, we can assume that there is a n such that $\mathcal{E}^i = 0$ for $i < 0$ and $i > n$. We call n the width of the VHS.

2.2. **Decomposition of an extended holomorphic bundle.** Let $\mathcal{C}$ be a complex curve, we assume that its boundary set $\Delta := \overline{\mathcal{C}} \backslash \mathcal{C}$ is an union of points. Consider $\mathcal{E}$ an holomorphic bundle on $\overline{\mathcal{C}}$. We introduce structures which will appear on such holomorphic bundle when they are obtained by canonical extension when we compactify $\mathcal{C}$. The first one will take the form of filtrations on each fibers above points of Δ.

Definition 2.1 (Filtration)**.** *A $[0,1)$-filtration on a complex vector space V is a collection of real weights $0 \leq w_1 < w_2 < \cdots < w_n < w_{n+1} = 1$ for some $n \geq 1$ together with a decreasing filtration of sub-vector spaces*

$$G^\bullet : V = V^{\geq w_1} \supsetneq V^{\geq w_2} \supsetneq \cdots \supsetneq V^{\geq w_{n+1}} = V^{\geq 1} = 0$$

The filtration satisfies $V^{\geq \nu} \subset V^{\geq \omega}$ whenever $\nu \geq \omega$ and the previous weights satisfy $V^{\geq w_i + \epsilon} \subsetneq V^{\geq w_i}$ for any $\epsilon > 0$.

We denote the graded vector spaces by $\mathrm{gr}_{w_i} := {}^{V^{\geq w_i}}/_{V^{\geq w_i+\epsilon}}$ *for* ϵ *small. The degree of such a filtration is by definition*

$$\deg(G^\bullet) := \sum_{i=1}^{n} w_i \dim(\mathrm{gr}_{w_i})$$

This leads to the next definition,

Definition 2.2 (Parabolic structure)**.** *A parabolic structure on* $\mathcal{E}$ *with respect to* Δ *is a couple* $(\mathcal{E}, G^\bullet)$ *where* $G^\bullet$ *defines a* $[0,1)$*-filtration* $G^\bullet\mathcal{E}_s$ *on every fiber* $\mathcal{E}_s$ *for any* $s \in \Delta$*.*
A parabolic bundle is a holomorphic bundle endowed with a parabolic structure.
The parabolic degree of $(\mathcal{E}, G^\bullet)$ *is defined to be*

$$\deg_{par}(\mathcal{E}, G^\bullet) := \deg(\mathcal{E}) + \sum_{s\in\Delta} \deg(G^\bullet\mathcal{E}_s)$$

2.3. **Deligne extension.** In the following we consider $\mathbb{V}$ a local system on $\mathbb{P}^1 - \{0, 1, \infty\}$ associated to a monodromy representation with eigenvalues of modulus one. We denote by $\mathcal{V}$ the associated holomorphic vector bundle.

We recall the construction of Deligne's extension of $\mathcal{V}$ which defines a holomorphic bundle on $\overline{\mathcal{C}}$ with a logarithmic flat connection. We describe it on a small pointed disk centered at $s \in D^*$ (the full disk is denoted D). Let ρ be a ray going outward of the singularity, then we can speak of the vector space of flat sections along the ray $L(\rho)$ which has the same rank r as $\mathcal{V}$. As all the $L(\rho)$ are isomorphic, we choose to denote it by V^0. There is a monodromy transformation $T : V^0 \to V^0$ to itself obtained after continuing the solutions. This corresponds to the monodromy matrix in the given representation. For every $\alpha \in [0, 1)$ we define

$$W_\alpha = \{v \in V^0 : (T - \zeta_\alpha)^r v = 0\} \text{ where } \zeta_\alpha = e^{2i\pi\alpha}$$

These vector spaces are non trivial for finitely many $\alpha_i \in [0, 1)$. We define

$$T_\alpha = \zeta_\alpha^{-1} T_{|W_\alpha} \text{ and } N_\alpha = \log T_\alpha$$

Let $q : \mathbb{H} \to D^*, q(z) = e^{2i\pi z}$ be the universal cover of D^*. Choose a basis $v_1, \ldots, v_r$ of V^0 adapted to the generalized eigenspace decomposition $V^0 = \bigoplus_\alpha W_\alpha$. We consider $v_i(z)$ as the pull back of v_i on $\mathbb{H}$. If $v_i \in W_\alpha$, then we define

$$\widetilde{v}_i(z) = \exp(2i\pi\alpha z + zN_\alpha)v_i$$

These sections are equivariant under $z \mapsto z + 1$ hence they give global sections of $\mathcal{V}_{\mathcal{C}}(D^*)$. The Deligne extension of $\mathcal{V}_{\mathcal{C}}$ is the vector bundle whose space of section over D is the $\mathcal{O}_{\mathcal{D}}$-module spanned by $\tilde{v}_1, \ldots, \tilde{v}_r$. This construction naturally gives a filtration on V^0.

In general, we can define various extensions $\mathcal{V}^a \subset \mathcal{V}^{-\infty} \subset j_*\mathcal{V}$ where j is the inclusion $j : D^* \to D$, $\mathcal{V}^\infty$ is the Deligne's meromorphic extension and $\mathcal{V}^a$ (resp. $\mathcal{V}^{>a}$) for $a \in \mathbb{R}$ is the free $\mathcal{O}_{\overline{\mathcal{C}}}$-module on which the residue of ∇ has eigenvalues

α in $[a, a+1)$ (resp. $(a, a+1]$). The bundle $\mathcal{V}^\bullet$ is a filtered vector bundle in the definition of [EKMZ16].

If we have a VHS $F^\bullet$ on $\mathcal{V}$ over $\mathcal{C}$, it induces a filtration of every $\mathcal{V}^a$ simply by taking

$$F^p\mathcal{V}^a := j_* F^p\mathcal{V} \cap \mathcal{V}^a$$

this is a well defined vector bundle thanks to Nilpotent Orbit Theorem ([Sch73] (4.9)).

In general for $\mathcal{C}$, we define on fibers over any singularity $s \in \Delta$, for $a \in (-1, 0]$ and $\lambda = \exp(-2i\pi a)$,

$$\psi_\lambda(\mathcal{V}_s^{-\infty}) = \mathrm{gr}^a_{\mathcal{V}_s} = \mathcal{V}_s^a \Big/ \mathcal{V}_s^{>a}$$

Definition 2.3 (Local Hodge data)**.** *For $a \in [0, 1)$, $\lambda = \exp(2i\pi a)$ and $p \in \mathbb{Z}$, we set for any singularity $s \in \Delta$*

- $\nu^p_\alpha = \dim \mathrm{gr}^p_F \psi_\lambda(\mathcal{V}_s)$ *also written* $h^p\psi_\lambda(\mathcal{V}_s)$
- $h^p(\mathcal{V}) = \sum_\alpha \nu^p_\alpha(\mathcal{V}_s)$

According to Riemann-Hilbert theorem, for any local system with all eigenvalues of the form $\exp(-2\pi i\alpha)$ at the singularities endowed with a trivial filtration we associate a filtered $\mathcal{D}_{\overline{\mathcal{C}}}$-module with residues and jumps both equal to α (see for example synopsis of [Sim90]). Thus the sub $\mathcal{D}_{\overline{\mathcal{C}}}$-module corresponding to the residue α has only one jump of full dimension at α, and

$$\deg_{par}(\mathrm{gr}^p_F \mathcal{V}) = \delta^p(\mathcal{V}) + \sum_{s\in\Delta,\alpha} \alpha\nu^p_\alpha(\mathcal{V}_s) \tag{2}$$

where we choose $\alpha \in [0, 1)$ and where $\delta^p(\mathcal{V})$ is the degree of $j_*\mathcal{V}$.

2.4. **Acceptable metrics and metric extensions.** The above Deligne extension has a geometric interpretation when we endow $\mathcal{C}$ with a acceptable metric K. If V is a holomorphic bundle on $\mathcal{C}$, we define the sheaf $\Xi(V)_\alpha$ on $\mathcal{C} \cup \{s\}$ as follows. The germs of sections of $\Xi(E)_\alpha$ at s are the sections $s(q)$ in j_*V in the neighborhood of s which satisfy a growth condition; for all $\epsilon > 0$ there exists C_ϵ such that

$$|s(q)|_K \leq C_\epsilon |q|^{\alpha-\epsilon}.$$

In general this extension is a filtered vector bundle on which we do not have much information, but the metric is called *acceptable* if it satisfies some extra growth condition on the curvature, and if it induces the above Deligne extensions.

Lemma 2.4 (Theorem 4 [Sim90])**.** *The local system $\mathbb{V}$ with non-expanding cusp monodromies has a metric which is acceptable.*

Proof. For completeness, we reproduce the construction of [EKMZ16]. The idea is to construct locally a nice metric and to patch the local constructions together

with partition of unity. The only delicate choice is for the metric around singularities. We want the basis elements $\tilde{v}_i$ of the α-eigenspace of the Deligne extension to be given the norm of order $|q|^\alpha$ in the local coordinate q around the cusp and to be pairwise orthogonal. Let M be such that $e^{2i\pi M} = T$, where T is the monodromy transformation. Then the hermitian matrix $\exp(\log|q|\overline{M^t}M)$ defines a metric such that the element $\tilde{v}_i$ has norm $|q|^\alpha|\tilde{v}_i|$. □

Corollary 2.5. *When the monodromy representation goes to identity, the parabolic degree goes to zero.*

Proof. In the proof above, it is clear that when $T \to \mathrm{Id}$, $M \to 0$ and thus the metric goes to the standard hermitian metric locally. Thus its curvature goes to zero around singularities and its integral on any subbundle goes to zero. This is its analytic degree, and is equal to the parabolic degree we are considering (Lemma 6.1 [Sim90]). □

2.5. **Local Hodge invariants.** Our purpose in this subsection is to show the following relation on local Hodge invariants:

Theorem 2.6. *The local Hodge invariants for equation (1) are :*

(1) at $z = 0$,

$$\nu^p_{\alpha_m} = \begin{cases} 1 \text{ if } p = f(\alpha_m) \\ 0 \text{ otherwise} \end{cases}$$

(2) at $z = \infty$,

$$\nu^p_{-\beta_m} = \begin{cases} 1 \text{ if } p - 1 = f(\beta_m) \\ 0 \text{ otherwise} \end{cases}$$

(3) at $z = 1$,

$$\nu^p_{\gamma} = \begin{cases} 1 \text{ if } p = [\gamma] + 1 \\ 0 \text{ otherwise} \end{cases}$$

Remark. *Computations of (1) and (2) are done in* [Fed15]. *We give a similar proof with an alternative combinatoric point of view.*

Recall that the monodromy at $z = 1$ has all eigenvalues but one equal to one. The last one is equal to $e^{2i\pi\gamma}$.

2.6. **Computation of local Hodge invariants.** In the following, we denote by M the local system defined by the hypergeometric equation (1) in the introduction. The point at infinity plays a particular role in middle convolution, thus we apply a biholomorphism to the sphere which will send the three singularity points $0, 1, \infty$ to $0, 1, 2$. Hereafter, M will have singularities at $0, 1, 2$.

Similarly $M_{k,j}$ corresponds to the hypergeometric equation where we remove terms in α_k and β_j,

$$\prod_{m\neq k}(D - \alpha_m) - z\prod_{n\neq j}(D - \beta_n) = 0$$

Let $L_{k,j}$ be a flat line bundle above $\mathbb{P}^1 - \{0, 2, \infty\}$ with monodromy $E(\alpha_k)$ at 0, $E(-\beta_j)$ at 2 and $E(\beta_j - \alpha_k)$ at ∞. Similarly $L'_{k,j}$ is defined to have monodromy $E(-\beta_j)$ at 0, $E(\alpha_k)$ at 2 and $E(\beta_j - \alpha_k)$ at ∞.

The two key stones in the proof are Lemma 3.1 in [Fed15] and Theorem 3.1.2 in [DS13] :

Lemma 2.7 (Fedorov)**.** *For any $k, j \in \{1, \dots, n\}$ we have,*

$$M \simeq \mathrm{MC}_{\beta_j - \alpha_k}\left(M_{k,j} \otimes L'_{k,j}\right) \otimes L_{k,j}$$

We modify a little bit the formulation of [DS13], taking $\alpha = 1 - \alpha$. Thus the condition becomes $1 - \alpha \in (0, 1 - \alpha_0] \iff \alpha \in [\alpha_0, 1)$. Which implies the following formulation.

Theorem 2.8 (Dettweiler-Sabbah)**.** *Let $\alpha_0 \in (0, 1)$, for every singular point in Δ, every $\alpha \in [0, 1)$ and any local system M, we have,*

$$\nu^p_\alpha\left(\mathrm{MC}_{\alpha_0}(M)\right) = \begin{cases} \nu^{p-1}_{\alpha-\alpha_0}(M) & \text{if } \alpha \in [0, \alpha_0) \\ \nu^{p}_{\alpha-\alpha_0}(M) & \text{if } \alpha \in [\alpha_0, 1) \end{cases}$$

and,

$$\delta^p\left(\mathrm{MC}_{\alpha_0}(M)\right) = \delta^p(M) + h^p(M) - \sum_{\substack{s \in \Delta \\ \{-\alpha\} \in [0, \alpha_0)}} \nu^{p-1}_{s,\alpha}(M)$$

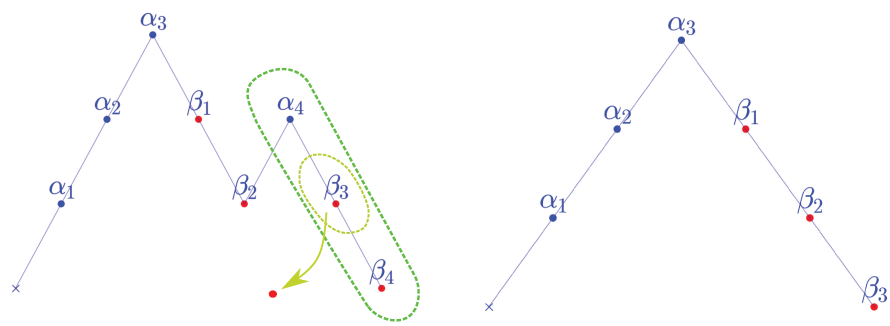
2.6.1. *Recursive argument.* We apply a recursive argument on the dimension of the hypergeometric equation. Let us assume that $n \geq 3$ and that Theorem 2.6 is true for $n - 1$.

For convenience in the demonstration, we change the indices of α and β such that α_i (resp. β_i) is the i-th α (resp. β) we come upon while browsing the circle to construct the function f. For x, y, z in $\mathbb{R}/\mathbb{Z}$ we write $x \prec y \prec z$ if there are three real $\hat{x}, \hat{y}, \hat{z}$ which represent x, y, z such that $x < y < z$ and $z - x < 1$.

We apply Lemma 2.7 with α_k and β_j such that $\alpha_k \leq \beta_j$. Let us describe what happens to the combinatorial function f after we remove these two eigenvalues. We denote by f' the function we obtain.

Removing α_k will make the function decrease by one for the following eigenvalues until we meet β_j, thus for any $\tau \neq \alpha_k, \beta_j$,

$$f(\tau) = \begin{cases} f'(\tau) & \text{if } \tau \prec \alpha_k \prec \beta_j \\ f'(\tau) + 1 & \text{if } \alpha_k \prec \tau \prec \beta_j \end{cases}$$



(A) Graph of f (B) Graph of f'

FIGURE 2. Geometric representation of middle convolution action on f

We apply Theorem 2.8 with $\alpha_0 = \beta_j - \alpha_k$. It yields that we have for all $m \neq k$, at singularity zero,

$$\nu^p_{1+\alpha_m-\alpha_k}\left(M \otimes L^{-1}_{k,j}\right) = \begin{cases} \nu^{p-1}_{\alpha_m-\beta_j}\left(M_{k,j} \otimes L'_{k,j}\right) & \text{if } \{\alpha_m - \alpha_k\} < \beta_j - \alpha_k \\ \nu^p_{\alpha_m-\beta_j}\left(M_{k,j} \otimes L'_{k,j}\right) & \text{otherwise} \end{cases}$$

which can be written in a simpler form

$$\nu^p_{\alpha_m-\alpha_k}\left(M \otimes L^{-1}_{k,j}\right) = \begin{cases} \nu^{p-1}_{\alpha_m-\beta_j}\left(M_{k,j} \otimes L'_{k,j}\right) & \text{if } \alpha_k \prec \alpha_m \prec \beta_j \\ \nu^p_{\alpha_k-\beta_j}\left(M_{k,j} \otimes L'_{k,j}\right) & \text{if } \alpha_m \prec \alpha_k \prec \beta_j \end{cases}$$

In terms of M and $M_{k,j}$,

$$\nu^p_{\alpha_m}(M) = \begin{cases} \nu^{p-1}_{\alpha_m}\left(M_{k,j}\right) & \text{if } \alpha_k \prec \alpha_m \prec \beta_j \\ \nu^p_{\alpha_m}\left(M_{k,j}\right) & \text{if } \alpha_m \prec \alpha_k \prec \beta_j \end{cases}$$

For any integer i, j we denote by $\delta(i, j)$ the function which is 1 when $i = j$ and is zero otherwise.

$$\nu^p_{\alpha_m}(M) = \begin{cases} \delta\left(p-1, f'(\alpha_m)\right) & \text{if } \alpha_k \prec \alpha_m \prec \beta_j \\ \delta\left(p, f'(\alpha_m)\right) & \text{if } \alpha_m \prec \alpha_k \prec \beta_j \end{cases} = \delta\left(p, f(\alpha_m)\right)$$

Similarly for all $m \neq j$, at singularity 2,

$$\nu^p_{-\beta_m}(M) = \begin{cases} \nu^{p-1}_{-\beta_m}\left(M_{k,j}\right) & \text{if } \alpha_k \prec \beta_j \prec \beta_m \\ \nu^p_{-\beta_m}\left(M_{k,j}\right) & \text{if } \alpha_k \prec \beta_m \prec \beta_j \end{cases}$$

$$\nu^p_{\beta_m}(M) = \begin{cases} \delta\left(p-1, f'(\beta_m)\right) & \text{if } \alpha_k \prec \beta_j \prec \beta_m \\ \delta\left(p, f'(\beta_m)\right) & \text{if } \alpha_k \prec \beta_m \prec \beta_j \end{cases} = \delta\left(p, f(\beta_k)\right)$$

And at 1, we set $\tilde{\gamma} := \gamma - \beta_j + \alpha_k = \sum_{m\neq j}\beta_m - \sum_{m\neq k}\alpha_m$,

$$\nu^p_{\gamma}(M) = \begin{cases} \nu^{p-1}_{\tilde{\gamma}}\left(M_{k,j}\right) & \text{if } \{\gamma\} < \beta_j - \alpha_k \\ \nu^{p}_{\tilde{\gamma}}\left(M_{k,j}\right) & \text{otherwise} \end{cases}$$

$$\nu^p_{\gamma}(M) = \begin{cases} \delta\left(p-1, [\tilde{\gamma}]+1\right) & \text{if } \{\gamma\} = \{\tilde{\gamma}\} + \beta_j - \alpha_k - 1 \\ \delta\left(p, [\tilde{\gamma}]+1\right) & \text{if } \{\gamma\} = \{\tilde{\gamma}\} + \beta_j - \alpha_k \end{cases} = \delta\left(p, [\gamma]+1\right)$$

Now if we choose to pick $\beta_n > \alpha_n$ for the computation, we now hodge invariants for all values excepts for α_n and β_n. We will use the computation with for example $\alpha_1 < \beta_1$. From this one we can deduce the invariants at β_n and α_n. Yet, we should keep in mind that the previous computations are always modulo shifting of the VHS. That is why we need to have dimension at least 3, since in this case α_2 will appear in both computations and will show there is no shift in our formulas.

2.6.2. *Initialization for $n = 2$.* We use the computations performed in the previous part for α_2 and β_2. To do so, first remark that the unique (complex polarized) VHS on $M_{2,2}$ is defined by $h^p(M) = \delta(p, 1)$ and the only non-zero local Hodge invariants are

$$\begin{array}{lll} \nu^1_{\alpha_1}(M_{2,2}) & = 1 & \text{at singularity } 0 \\ \nu^1_{-\beta_1}(M_{2,2}) & = 1 & \text{at singularity } \infty \\ \nu^1_{\beta_1-\alpha_1}(M_{2,2}) & = 1 & \text{at singularity } 1 \end{array}$$

Which corresponds to the definition of $\delta(p, f'(\alpha_1))$ for the first two, and to $\delta(p, [\gamma]+1)$ for the last one.

Using the previous subsection, we deduce

$$\begin{array}{lll} \nu^p_{\alpha_1}(M) & = \delta(p, f(\alpha_1)) & \text{at singularity } 0 \\ \nu^p_{-\beta_1}(M) & = \delta(p, f(\alpha_1)) & \text{at singularity } \infty \\ \nu^p_{\gamma}(M) & = \delta(p, [\gamma]+1) & \text{at singularity } 1 \end{array}$$

According to [Fed15], the Hodge numbers of M are

$$h^1 = \begin{cases} 1 & \text{if } \alpha_1 \prec \alpha_2 \prec \beta_1 \prec \beta_2 \\ 2 & \text{if } \alpha_1 \prec \beta_1 \prec \alpha_2 \prec \beta_2 \end{cases}, \qquad h^2 = \begin{cases} 1 & \text{if } \alpha_1 \prec \alpha_2 \prec \beta_1 \prec \beta_2 \\ 0 & \text{if } \alpha_1 \prec \beta_1 \prec \alpha_2 \prec \beta_2 \end{cases}$$

Using the fact that $\sum_\alpha \nu^p_\alpha = h^p$, we can deduce the other Hodge invariants.

$$\begin{array}{ll} \nu^2_{\alpha_2} = 1 & \text{if } \alpha_1 \prec \alpha_2 \prec \beta_1 \prec \beta_2 \\ \nu^1_{\alpha_2} = 1 & \text{if } \alpha_1 \prec \beta_1 \prec \alpha_2 \prec \beta_2 \end{array}$$

We conclude that $\nu^p_{\alpha_2}(M) = \delta(p, f(\alpha_2))$ and similarly $\nu^p_{-\beta_2}(M) = \delta(p, f(\alpha_2))$.

2.7. **Continuity of the parabolic degree.** To compute $\delta^p(V)$ in equation (2), we show in the following Lemma a continuity property which implies that it is constant on a given domain.

Lemma 2.9. *Let $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ be all disjoint, not integers and such that γ also is not an integer. Fix the intertwining diagram of α and β and the value of $[\gamma]$, then for any integer p, $\delta^p(V)$ is constant.*

Proof. The main idea here is that rigidity yields the holomorphic structure to be locally constant for Higgs bundles.

Let L and L' be the flat bundles of solutions to the equation (1) for eigenvalues α, β and α', β' satisfying the above hypothesis. We endow them with a trivial filtration. To L corresponds some Higgs bundle (E, θ) together with a parabolic structure at singularities (see [Sim90]). As L has eigenvalues of modulus one, θ has no residue, moreover its weight filtration is locally the same as the one for the unipotent part of monodromy matrices of L.

Consider now (E', θ') a Higgs bundle with the same holomorphic structure and Higgs form as (E, θ) but a slightly changed parabolic structure for which we keep the initial filtration but modify the parabolic weights α, β to α', β'. The residues stay the same $\mathrm{res}_s(\theta) = \mathrm{res}_s(\theta')$ at every singularity s and the weight filtration also for the unipotent part of the monodromy on any eigenspace, since we only changed the parabolic weights of the filtration. According to Section 7 of [Sim90] this implies that the residues of the local systems L' associated to (E', θ') and L are isomorphic, thus the monodromy matrices of the two local systems are locally isomorphic, and by rigidity of the hypergeometric local systems they are globally isomorphic. Moreover, the VHS on L' induced by its Higgs structure and the pull back on the one on L by the isomorphism are isomorphic. Thus the degrees are equal. As the considered domain is connected this shows that the parabolic degree of the Hodge subbundles is constant. □

Together with Corollary 2.5 this is enough to compute $\delta^p(V)$. We fix an intertwining diagram and a floor value for γ and make the first β's go to 1 and the rest of eigenvalues to 0 while staying in the given domain. At the limit, the parabolic degree is zero and we can deduce $\delta^p(V)$.

3. Algorithm

In this section, we describe the algorithm used to compute the Lyapunov exponents. We start simulating a generic hyperbolic geodesic and following how it winds around the surface, namely the evolution of the homology class of the closed path. Finally we compute the corresponding monodromy matrix after each turn around a cusp.

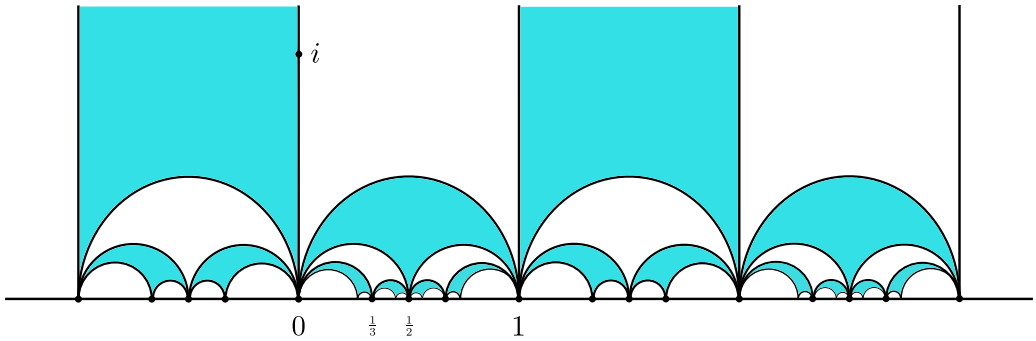


FIGURE 3. The Farey's tessellation

3.1. **Hyperbolic geodesics.** This first question arising to unravel this computation of Lyapunov exponents is how to simulate a generic hyperbolic geodesic. The answer comes from a beautiful theorem proved by Caroline Series in [Ser85] which relates hyperbolic geodesics on the Poincaré half-plane and continued fraction development of real numbers. We follow here the notations of [Dal07] (see part II.4.1).

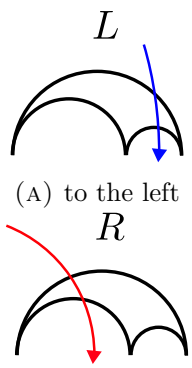
Let us consider the Farey tessellation of $\mathbb{H}$ (see Figure 3). It is invariant with respect to the discrete subgroup of index 3 in $\mathrm{PSL}(\mathbb{Z})$ generated by

$$\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$$

The tessellation yields fundamental domains for the action of this group. The sphere minus three points endowed with its complete hyperbolic metric is a degree two cover of the surface induced by the quotient of $\mathbb{H}$ by this group. This is why we represent the tessellation with two colors : a fundamental domain for the sphere is given by any two adjacent triangles of different colors. Thus it will be easy once we understand the geodesics with respect to this tessellation to see them on the sphere.

Let us consider a geodesic going through i. It lands on the real axis at a positive and a negative real number. The positive real number will be called x, this number determines completely the geodesic since we know two distinct points on it.

We associate to this geodesic a sequence of positive integers. Look at the sequence of hyperbolic triangles the geodesic will cross. For each one of those triangles, the geodesic has two ways to cross them (see Figure 4). Once it enters it, it can leave it crossing either the side of the triangle to its left (a) or to its right (b).



(A) to the left

(B) to the right

FIGURE 4. Two ways to cross an hyperbolic triangle

Remark. *The vertices of hyperbolic triangles are located at rational numbers, so this sequence will be infinite if and only if x is irrational (see* [Dal07] *Lemme 4.2).*

We have now for a generic geodesic an infinite word in two letters L and R associated to a geodesic. For example the word associated to the geodesic in Figure 5, is of the form $LLRRLR\cdots = L^2R^2L^1R\ldots$. We can factorize each of these words and get

$$R^{n_0}L^{n_1}R^{n_2}L^{n_3}\ldots$$

Except for n_0 which can be zero the n_i are positive integers.

Theorem 3.1. *The sequence (n_k) is the continued fraction development of x. In other words,*

$$x = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \ldots}}}$$

The measure induced on the real axis by the measure on $T^1\mathbb{H}$ dominates Lebesgue measure.

See [Dal07] II.4 or [Ser85] for a proof.

Remark. *This theorem states exactly that to study a generic geodesic on the hyperbolic plane, we can consider a Lebesgue generic number in $(0,\infty)$ and compute its continued fraction development.*

To compute Lyapunov exponents of the flat bundle, we need to follow how a generic geodesic winds around the cusps. By the previous theorem we can simulate a generic cutting sequence of a hyperbolic geodesic in $\mathbb{H}$. Our goal now will be to associate to such a sequence a product of monodromy matrices keeping track of its homotopy class.

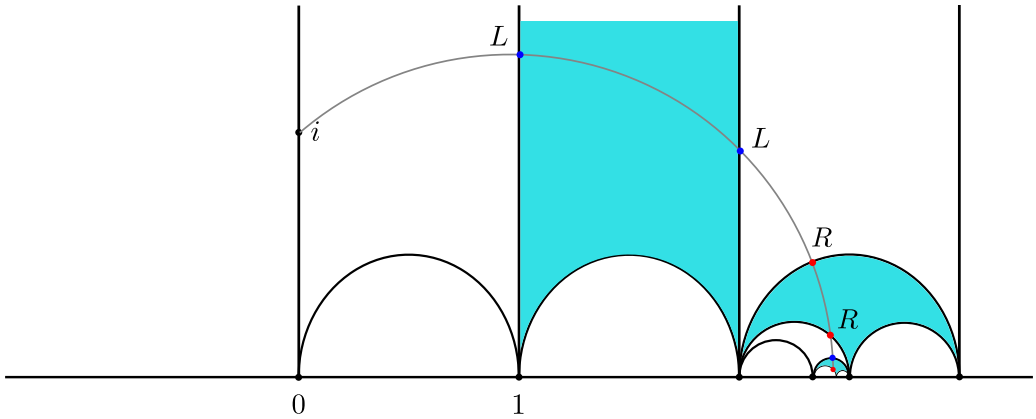


FIGURE 5. Crossings of a given geodesic

Since we will consider universal cover of the sphere minus three points, for convenience we will denote by A, B, C the cusps corresponding in the surface to ∞, 0, 1 respectively and use this latter notation for points in $\mathbb{H}$. Two adjacent hyperbolic triangles of Farey's tessellation, e.g. $0, 1, \infty$ and $1, 2, \infty$ will form a fundamental domain for this surface. All the vertices of the hyperbolic triangles for this tessellation are associated to either A, B or C. To follow how the flow turns around these vertices in the surface, we will need to keep track of orientation. To do so, we color the triangles according to the order of its vertices, when we browse the three vertices counterclockwise if we have $A \to B \to C \to A$ we color the triangle in white (this is the case for $0, 1, \infty$) otherwise we color it in blue (case of $\infty, 2, 1$).

Let us now consider a point P inside the blue triangle, which will be used as a base point for loops around cusps. We choose a homology marking of the surface by denoting the paths going around A, B, C counterclockwise starting and ending at P, a, b, c (see Figure 6). When we concatenate these paths we get $c \cdot b \cdot a = \mathrm{Id}$ and $a^{-1} = c \cdot b$. For monodromy matrices we have the relation

$$M_\infty M_0 M_1 = \mathrm{Id} \tag{3}$$

In our algorithm we will always follow the cutting sequence until we end up in a blue triangle. Then we will apply an isometry that take the fundamental domain we are in to the $(-1, 0, \infty)$ triangle and the edge the flow will cut when going out of the triangle to be the $(0, \infty)$ or $(-1, \infty)$ edge in order to place the cusp we are turning around at ∞. We shall warn the reader here that the corresponding cusp on the surface here at points $-1, 0$ and ∞ may be any of the points A, B, C but their cyclic order will stay unchanged thanks to the orientation. Thus we just need to keep track of the cusp placed at ∞.

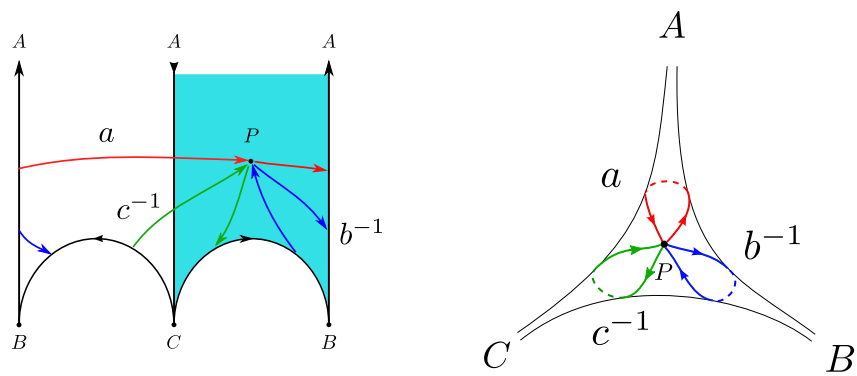
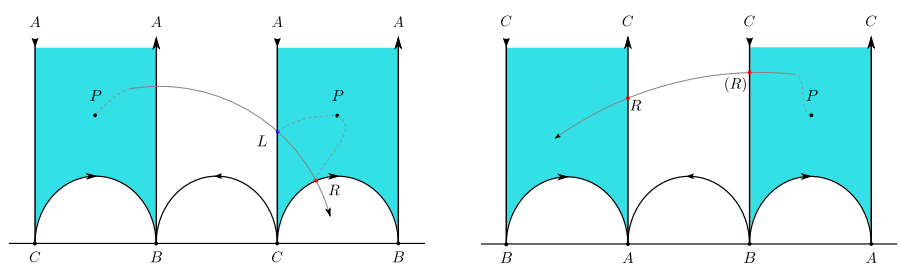


FIGURE 6. Homology marking

When we start with a cutting sequence extracted from the previous theorem we see that the geodesic start by cutting $(0, \infty)$ at i without being counted in the cutting sequence. The first cutting will always be forgotten in the sequence when applying the isometry.

Now remark that when the crossing is a sequence of $2n$ left, we make n turns counterclockwise around the cusp placed at ∞. When it $2n$ right, we make n turn clockwise. It is a little trickier if the geodesic makes an odd number of the same crossing; we need to take one step further from the next term in the sequence of crossings to end up at P (see Figure 7).



(A) Odd number of sections (B) Changing the setting

FIGURE 7. Applying the good orientation preserving isometry

There is a last point to consider, since we want to compare the growth of the harmonic norm with regards to the geodesic flow, need to follow its length. Here the discretized algorithm enables us to follow the type of homotopy it will have, but the length will not correspond a priori to the number of iterations of our algorithm. It is proportional to it by the constant.

3.2. **Monodromy matrices.** In the introduction Proposition 1.1 gave a set of three properties on the monodromy matrices for the hypergeometric differential

equation associated to two distinct sequences of real numbers $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$. We claim that these properties are sufficient to recover the monodromy matrices up to conjugacy.

For convenience we always assume that the $\alpha_1, \dots, \alpha_n$ are disjoint, otherwise the computation becomes way more tedious, and in our computations we will explore generic domains. We choose a basis in which M_0 is diagonal. Property (3) tells us that $M_1 - \mathrm{Id}$ is of rank 1. We can then find two vectors v and w such that $M_1 = \mathrm{Id} + vw^t$.

Since $M_\infty^{-1} = M_0 M_1$ knowing the eigenvalues of M_∞ we can derive the following n equations, for all j,

$$\det\left(M_\infty^{-1} - e^{2i\pi\beta_j}\,\mathrm{Id}\right) = 0$$

We can compute this determinant using the particular form of the matrix and the following lemma.

$$M_0 M_1 - e^{2i\pi\beta_j}\,\mathrm{Id} = (M_0 - e^{2i\pi\beta_j}\,\mathrm{Id}) + (M_0 v)w^t$$

We can conjugate by diagonal matrices so that $M_0 v$ becomes the vector $\mathbf{1}$ which is one on every coordinates. And obtain the equations

$$\det((M_0 - e^{2i\pi\beta_j}\,\mathrm{Id}) + \mathbf{1}w^t) = 0,\ \forall j$$

Lemma. *Let D a diagonal matrix with $d_1, \dots, d_n$ on its diagonal, and x a vector.*

$$\det\left(D + \mathbf{1}x^t\right) = \left(\prod_{i=1}^n d_i\right) \cdot \left(1 + \sum_{i=1}^n x_i/d_i\right)$$

Proof. First consider the case where D is the identity matrix. We know that all the eigenvalues except for one are 1. The determinant will then be the eigenvalue of an eigenvector which image through x^t is not zero. This vector will be $\mathbf{1}$ and its eigenvalue $(1 + \sum_{i=1}^n x_i)$. To finish the proof, just factor each column by d_i in the determinant. □

We obtain

$$\left(\prod_{i=1}^n e^{2i\pi\alpha_i} - e^{2i\pi\beta_j}\right)\left(1 + \sum_{i=1}^n \frac{w_i}{e^{2i\pi\alpha_i} - e^{2i\pi\beta_j}}\right) = 0$$

Corollary. *The vector w satisfies for all j,*

$$\sum_{i=1}^n \frac{w_i}{e^{2i\pi\beta_j} - e^{2i\pi\alpha_i}} = 1$$

We define a matrix $N = \left(\frac{1}{e^{2i\pi\beta_j} - e^{2i\pi\alpha_i}}\right)_{i,j}$ and observe that $w^t N = \mathbf{1}^t$ so $w^t = \mathbf{1}^t N^{-1}$. Hence for a generic setting, we just have to invert N to find the explicit monodromies. And finally we have the expression $M_1 = \mathrm{Id} + M_0^{-1}\mathbf{1}\cdot\mathbf{1}^t N^{-1}$

4. Observations

4.1. **Calabi-Yau families example.** A first family of examples is coming from 14 1-dimensional families of Calabi-Yau varieties of dimension 3. The Gauss-Manin connection for this family on its Hodge bundle gives an example of the hypergeometric family we are considering. The monodromy matrices were computed explicitly in [ES08] and have a specific form parametrized by two integers C and d. We introduce the following monodromy matrices,

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1/2 & 1 & 1 & 0 \\ 1/6 & 1/2 & 1 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & -C/12 & 0 & -d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In the previous notations, $M_0 = T, M_1 = S, M_\infty = (TS)^{-1}$. These matrices satisfy relation (3), $M_\infty M_0 M_1 = \mathrm{Id}$. We see that $M_1 - \mathrm{Id}$ has rank one and eigenvalues of M_0 and M_∞ have modulus one thus correspond to hypergeometric equations. In this setting, T has eigenvalues all equal to one and eigenvalues of $(TS)^{-1}$ are symmetric with respect to zero, we denote them by $\mu_1, \mu_2, -\mu_2, -\mu_1$ where $\mu_1, \mu_2 \geq 0$.

The parabolic degree of the holomorphic Hodge subbundles are given by,

Theorem. [EKMZ16] *Suppose* $0 < \mu_1 \leq \mu_2 \leq 1/2$ *then the degree of the Hodge bundles are*

$$\deg_{par} \mathcal{E}^{3,0} = \mu_1 \quad \textit{and} \quad \deg_{par} \mathcal{E}^{2,1} = \mu_2$$

Thus according to the same article, we know that $2(\mu_1 + \mu_2)$ is a lower bound for the sum of Lyapunov exponents. We call good cases the equality cases and bad cases the cases where there is strict inequality.

There are 14 different couples of values for C and d where the corresponding flat bundle is an actual Hodge bundle over a family of Calabi-Yau varieties. These examples where computed few years ago by M. Kontsevich and were a motivation for this article. We list them in the table below.

To see what happens in a similar setting for more general hypergeometric equations, we vary C, d and compute the corresponding eigenvalues μ_1 and μ_2 as well as the Lyapunov exponents. On Figure 9a we drew a blue point at coordinate (μ_1, μ_2) if the sum of positive Lyapunov exponents are as close to the parabolic degree $2(\mu_1 + \mu_2)$ as the precision we have numerically and we put a red point when this value is outside of the confidence interval.

Note that according to Figure 9a it seems that all points below the line of equation $3\mu_2 = \mu_1 + 1$ are bad cases. In Figure 9b, we represent the distance of the sum of the Lyapunov exponents to the expected formula. We see that this

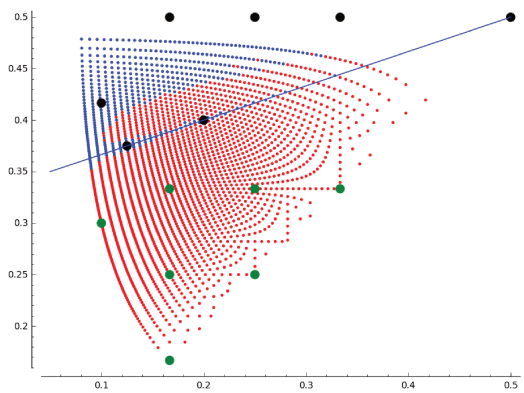
C	d	$\lambda_1 + \lambda_2$	λ_1	μ_1, μ_2
46	1	1	0.97	1/12, 5/12
44	2	1	0.95	1/8, 3/8
52	4	4/3	1.27	1/6, 1/2
50	5	6/5	1.12	1/5, 2/5
56	8	3/2	1.40	1/4, 1/2
60	12	5/3	1.53	1/3, 1/2
64	16	2	1.75	1/2, 1/2

(A) The 7 good cases

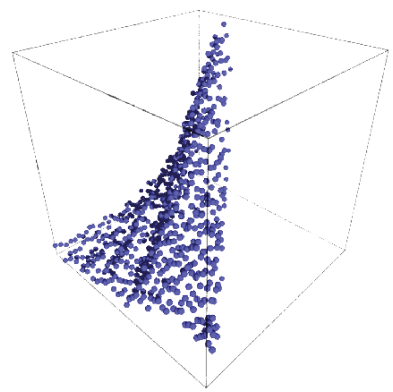
C	d	$\lambda_1 + \lambda_2$	λ_1	μ_1, μ_2
22	1	0.92	0.75	1/6, 1/6
34	1	0.83	0.77	1/10, 3/10
32	2	0.97	0.84	1/6, 1/4
42	3	1.06	0.96	1/6, 1/3
40	4	1.30	1.07	1/4, 1/4
48	6	1.31	1.15	1/4, 1/3
54	9	1.60	1.34	1/3, 1/3

(B) The 7 bad cases

FIGURE 8. Experiments



(A) The good and bad cases



(B) Zoom on the part above the line

FIGURE 9. Experiments

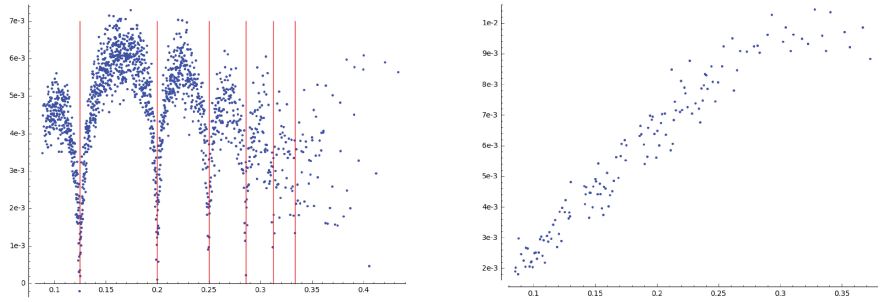
gives a function that oscillates above zero. More precisely, it seems that good cases are outside of some lines passing through $(1/2, 1/2)$.

To push the numerical simulations further, we consider what happens on lines of equation $3\mu_2 = \mu_1 + 1$ (10a) and $48\mu_2 = 10\mu_1 + 19$ (10b) both passing through $(1/2, 1/2)$ and a point corresponding to one of the previous good cases.

We observe that on the graph (10b) there is only one good case which corresponds to $(\mu_1, \mu_2) = (1/10, 3/10)$ in the previous list of good cases. In the graph (10a), there are good cases at points $(\mu_1, \mu_2) = (1/8, 3/8), (2/10, 4/10)$ which were also on the previous list but other points appear such as $(3/12, 5/12)$, $(4/14, 6/14)$, $(5/16, 7/16)$, $(6/18, 8/18)$.

Remark that these all have the form

$$(\frac{k-3}{2k}, \frac{k-1}{2k})$$



(A) $3\mu_2 = \mu_1 + 1$ (B) $48\mu_2 = 10\mu_1 + 19$

FIGURE 10. Lyapunov exponents as a function of μ_1

we checked it numerically for all $4 \leq k \leq 10$, conjecturally this phenomenon appears for every $k \geq 4$.

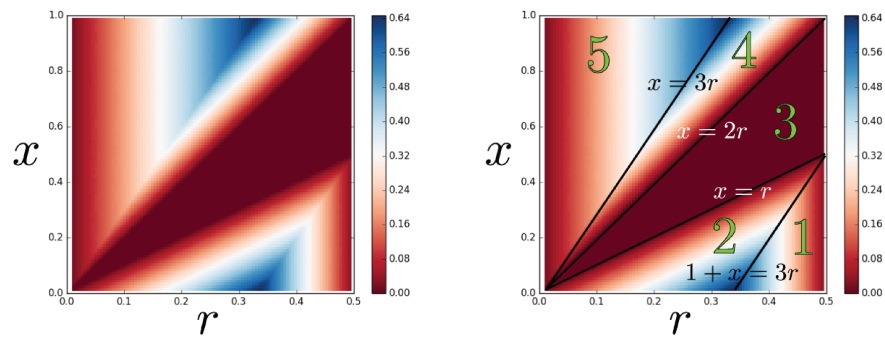
According to [BT14] and [SV14], the 7 good cases correspond to cases where the monodromy group of the hypergeometric local system is of infinite index in $Sp(4, \mathbb{Z})$, which is commonly called *thin*. In the other cases the group is of finite index and is called *thick*. The new conjectural good cases we found by ways of Lyapunov exponents do not seem to have a representation with integers or even rational C and d. A lot of questions arise about these points, for example can we find a number-theoretic interpretation of their equality as in Conjecture 6.5 in [EKMZ16], or is there a specific property on the monodromy group in these cases.

Yuri Manin pointed out a possible interpretation in cosmology of the Lyapunov exponents in our setting, and a probable relation to Kasner exponents using time complexification; compare to [MM14] and [MM15].

4.2. **Examples for $n = 2$.** As we have seen in the introduction the two Lyapunov exponents are symmetric λ_1 and $-\lambda_1$. The sum of the positive Lyapunov exponents is just λ_1. The parameter space we have for these 2-dimensional flat bundles are $\alpha_1, \alpha_2, \beta_1, \beta_2$.

The Lyapunov exponents are invariant through translation of the set of parameters. Indeed, we can consider the bundle with $e^{\delta} M_0$ and $e^{-\delta} M_\infty$ monodromies, it will have the same set of Lyapunov exponents since both scalar will appear with the same frequency and its parameters will be $\alpha_1 + \delta, \ldots, \alpha_h + \delta, \beta_1 + \delta, \ldots, \beta_h + \delta$ hence without loss of generality we can assume $\beta_1 = 0$. Moreover the parameters are given as a set, the order does not matter.

In the following experiments we will consider a set of parameters where the β's will be equidistributed and the α's will be shifted with respect to them. Here we represent the value of the Lyapunov exponent for $\alpha_1 = r, \alpha_2 = 2r, \beta_1 = 0, \beta_2 = x$ and we have by definition $\gamma = x - 3r$.

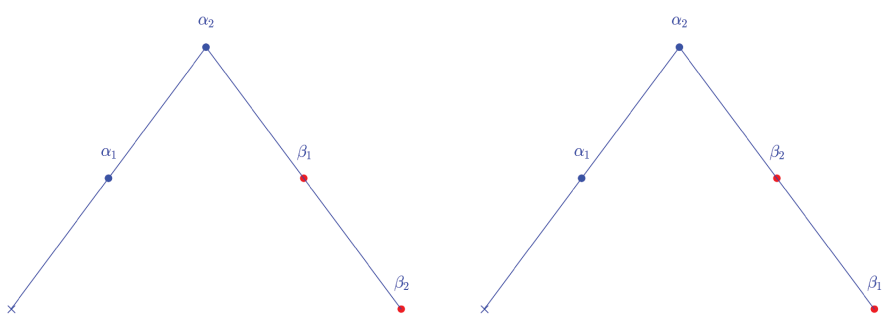


(A) Plotting λ_1 (B) Zones on the graph

FIGURE 11. Experiments

Remark. *We first notice that the zone where the Lyapunov exponent is zero corresponds to the setting where the parameters are alternate and where there is a positive definite bilinear form invariant by the flat connection (see introduction). This will be true whenever the VHS has weight* 0.

Another noticeable fact is that zones are delimited by combinatorics of α *and* β*, and of* $[\gamma]$ *introduced in the introduction.*



(A) Diagram for zones 1 and 2 (B) Diagram for zones 4 and 5

Remark that $[\gamma]$ is 0 in zones $1, 4$, and 1 in zones $2, 5$. In the following table, we give a relation binding λ_1, r, x obtained by linear regression. The other column is the formula for the parabolic degree in the given zone.

Zone	λ_1	$\deg_{par} H^{1,0}$
1	$2(1-2r)$	$-1+\{\gamma\}+\alpha_1+1-\beta_2$
2	$2(r-x)$	$\alpha_1+1-\beta_2$
3	0	0
4	$2(x-2r)$	$-1+\{\gamma\}+\alpha_1+1-\beta_1$
5	$2r$	$\alpha_1+1-\beta_1$

In this case, the VHS is of weight ≤ 1 and thus is in the setting of [Kon97]. In consequence, we have the equality

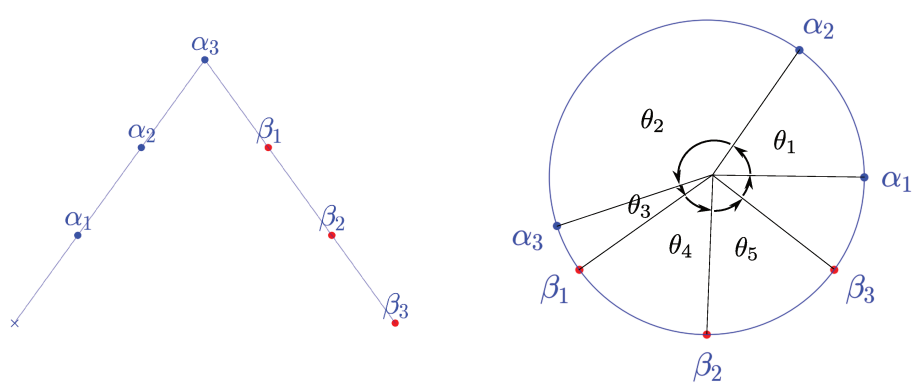
$$\lambda_1 = 2\frac{\deg_{par} \mathcal{E}^1}{\chi(S)}$$

Where $\deg_{par}$ is the parabolic degree of the holomorphic bundle and $\chi(S) = 1$ the Euler characteristic of S.

This is a good test for our algorithm and formula on degree. More generally, for any dimension n, this formula will hold as long as the weight is equal to 1.

4.3. **A peep to weight 2.** Let n be equal to 3. In this case, there will be three Lyapunov exponents $\lambda_1, 0, -\lambda_1$. As explained in the previous subsection, if the weight of the VHS is 0, $\lambda_1 = 0$; if it is 1, λ_1 is equal to twice the parabolic degree of $\mathcal{E}^1$. We consider configurations where the weight is 2. Assume $\alpha_1 = 0$, the only cyclic order in which the VHS is irreducible and of weight 2 is for,

$$0 = \alpha_1 < \alpha_2 < \alpha_3 < \beta_1 < \beta_2 < \beta_3 < 1$$

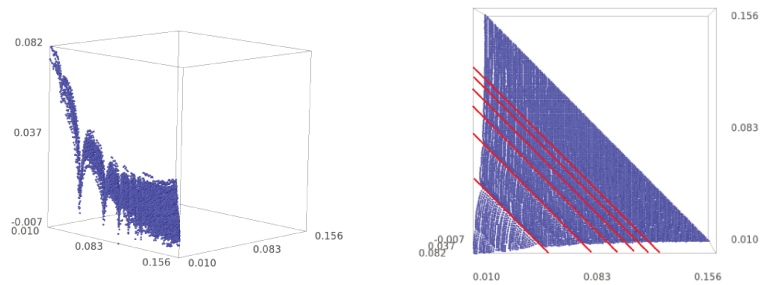


We parametrize these configurations with 5 parameters which will correspond to the distance between two consecutive eigenvalues : $\theta_1 = \alpha_2 - \alpha_1, \theta_2 = \alpha_3 - \alpha_2, \theta_3 = \beta_1 - \alpha_3, \theta_4 = \beta_2 - \beta_1, \theta_5 = \beta_3 - \beta_2$.

Using a Monte-Carlo process, we found some values in this configuration for which there is equality with twice the parabolic degree of $\mathcal{E}^2 \oplus \mathcal{E}^1$. We remarked that several parameter points where there is equality satisfy $\theta_1 = \theta_2$ and $\theta_4 = \theta_5$. This motivated us to consider the 2 dimensional subspace of parameters

$$(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) = (x, x, 1/2, y, y)$$

For these parameters we can observe a remarkable phenomenon; the difference between the Lyapunov exponent and the formula with parabolic degrees depends only on $x + y$. We plot this difference in the Figure below and see that for some values of $x + y$ there is equality.



(A) side (B) top

We computed that for $x + y = 1/10, 1/12, 1/18$ the formula holds.

Remark that

$$\deg_{par} \mathcal{E}^1 \otimes \mathcal{E}^2 = 2(-1 + \alpha_1 + \alpha_2 + 1 - \beta_2 + 1 - \beta_3 + \{\gamma\})$$

Yet $\alpha_1 = 0$, $\alpha_2 = x$, $\beta_2 = 1/2+2x+y$, $\beta_3 = 1/2+2x+2y$ and $\gamma = 3y+3/2+3x > 1$. Hence

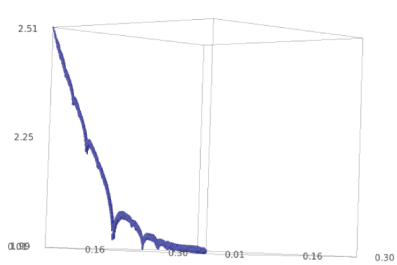
$$\deg_{par} \mathcal{E}^1 \otimes \mathcal{E}^2 = 1$$

There is equality when $\lambda_1 = 1$.
Moreover $\deg_{par} \mathcal{E}^1 = 2(0 + \alpha_1 + 1 - \beta_3) = 1 - 4(x + y) = 3/5, 2/3, 7/9$ in the three previous cases.

We try now another family,

$$(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) = (x, x, \frac{1}{2\sqrt{2}}, y, y)$$

Again we observe that the difference between the Lyapunov exponent and the formula with parabolic degrees depends only on $x + y$.



We observe that the difference goes to zero at $x + y = 0.22978$ which does not seems to be rational as before. Moreover, in this case, $\gamma = 3y + \frac{3}{2\sqrt{2}} + 3x$ and

$$\deg_{par} \mathcal{E}^1 \otimes \mathcal{E}^2 = \frac{1}{\sqrt{2}}$$

and

$$\deg_{par} \mathcal{E}^1 = 2 - \frac{1}{\sqrt{2}} - 4(x+y) = 0.37377$$

References

[BH89] F. Beukers and G. Heckman. Monodromy for the hypergeometric function ${}_nF_{n-1}$. *Invent. Math.*, 95(2):325–354, 1989.

[BM10] Irene I. Bouw and Martin Möller. Teichmüller curves, triangle groups, and Lyapunov exponents. *Ann. of Math. (2)*, 172(1):139–185, 2010.

[BT14] Christopher Brav and Hugh Thomas. Thin monodromy in Sp(4). *Compos. Math.*, 150(3):333–343, 2014.

[Dal07] Françoise Dal'Bo. *Trajectoires géodésiques et horocycliques*. Savoirs Actuels (Les Ulis). [Current Scholarship (Les Ulis)]. EDP Sciences, Les Ulis; CNRS Éditions, Paris, 2007.

[DS13] Michael Dettweiler and Claude Sabbah. Hodge theory of the middle convolution. *Publ. Res. Inst. Math. Sci.*, 49(4):761–800, 2013.

[EKMZ16] Alex Eskin, Maxim Kontsevich, Martin Moeller, and Anton Zorich. Lower bounds for lyapunov exponents of flat bundles on curves, 2016.

[EKZ14] Alex Eskin, Maxim Kontsevich, and Anton Zorich. Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow. *Publ. Math. Inst. Hautes Études Sci.*, 120:207–333, 2014.

[ES08] Christian Van Enckevort and Duco Van Straten. Monodromy calculations of fourth order equations of calabi–yau type. In *in "Mirror Symmetry V", the BIRS Proc. on Calabi–Yau Varieties and Mirror Symmetry, AMS/IP*, 2008.

[Fed15] Roman Fedorov. Variations of hodge structures for hypergeometric differential operators and parabolic higgs bundles, 2015.

[Fil14] Simion Filip. Families of k3 surfaces and lyapunov exponents, 2014.

[FMZ14] Giovanni Forni, Carlos Matheus, and Anton Zorich. Zero Lyapunov exponents of the Hodge bundle. *Comment. Math. Helv.*, 89(2):489–535, 2014.

[Kat96] Nicholas M. Katz. *Rigid local systems*, volume 139 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1996.

[KM16] André Kappes and Martin Möller. Lyapunov spectrum of ball quotients with applications to commensurability questions. *Duke Math. J.*, 165(1):1–66, 2016.

[Kon97] M. Kontsevich. Lyapunov exponents and Hodge theory. In *The mathematical beauty of physics (Saclay, 1996)*, volume 24 of *Adv. Ser. Math. Phys.*, pages 318–332. World Sci. Publ., River Edge, NJ, 1997.

[Kri04] Raphaël Krikorian. Déviations de moyennes ergodiques, flots de teichmüller et cocycle de kontsevich-zorich. *Séminaire Bourbaki*, 46:59–94, 2003-2004.

[KZ03] Maxim Kontsevich and Anton Zorich. Connected components of the moduli spaces of Abelian differentials with prescribed singularities. *Invent. Math.*, 153(3):631–678, 2003.

[MM14] Yuri I. Manin and Matilde Marcolli. Big bang, blowup, and modular curves: Algebraic geometry in cosmology. 2014.

[MM15] Yuri Manin and Matilde Marcolli. Symbolic dynamics, modular curves, and bianchi ix cosmologies, 2015.

[Sch73] Wilfried Schmid. Variation of Hodge structure: the singularities of the period mapping. *Invent. Math.*, 22:211–319, 1973.

[Ser85] Caroline Series. The modular surface and continued fractions. *J. London Math. Soc. (2)*, 31(1):69–80, 1985.

[Sim90] Carlos T. Simpson. Harmonic bundles on noncompact curves. *J. Amer. Math. Soc.*, 3(3):713–770, 1990.

[SV14] Sandip Singh and T. N. Venkataramana. Arithmeticity of certain symplectic hypergeometric groups. *Duke Math. J.*, 163(3):591–617, 2014.

[Yos97] Masaaki Yoshida. *Hypergeometric functions, my love*. Aspects of Mathematics, E32. Friedr. Vieweg & Sohn, Braunschweig, 1997. Modular interpretations of configuration spaces.

Université Paris Diderot 7, 8 place Aurélie Nemours, 75013 Paris, France
E-mail address: charles.fougeron@imj-prg.fr

Chapitre 4

Veech group for dilatation surfaces

arXiv:1609.02130

We introduce some elementary definitions and open questions for dilatation surfaces. We characterize the Veech groups of such surfaces, and introduce through examples conjectures on their generic dynamics. We give also a characterization of such surfaces having a polygonal representation using Veech work on Delaunay triangulations.

DILATATION SURFACES AND THEIR VEECH GROUPS.

EDUARD DURYEV, CHARLES FOUGERON, AND SELIM GHAZOUANI

To the memory of William Veech

Abstract. We introduce a class of objects which we call *'dilatation surfaces'*. These provide families of foliations on surfaces whose dynamics we are interested in. We present and analyze a couple of examples, and we define concepts related to these in order to motivate several questions and open problems. In particular we generalise the notion of Veech group to dilatation surfaces, and we prove a structure result about these Veech groups.

Notations

- Σ_g or only Σ is a compact surface of genus $g \geq 2$;
- $\mathrm{Gl}_2^+(\mathbb{R})$ is the group of 2 by 2 matrices whose determinant is positive;
- $\mathrm{SL}_2(\mathbb{R})$ is the group of 2 by 2 matrices whose determinant is equal to 1;
- $\mathrm{Aff}(\mathbb{C})$ is the one dimensional affine complex group, $\mathbb{C}^* \ltimes \mathbb{C}$;
- $\mathrm{Aff}_{\mathbb{R}_+^*}(\mathbb{C})$ is the subgroup of $\mathrm{Aff}(\mathbb{C})$ of elements whose linear parts are real positive;
- $\mathbb{H}$ is the upper half plane of $\mathbb{C}$;

1. Introduction.

A *translation* structure on a surface is a geometric structure modelled on the complex plane $\mathbb{C}$ with structural group the set of translations. A large part of the interest that these structures have drawn lies in the directional foliations inherited from the standard directional foliations of $\mathbb{C}$ (the latter being invariant under the action of translations). Examples of such structures are polygons whose sides are glued along parallel sides *of same length*, see Figure 1 below.

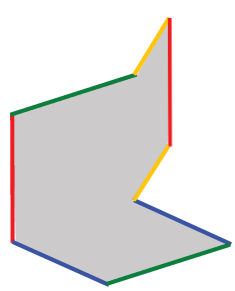


Figure 1. A translation surface of genus 2.

The directional foliation, say in the horizontal direction, can be drawn very explicitly: a leaf is a horizontal line until it meets a side, and continues as the horizontal line starting from the point on the other side to which it is identified. These foliations have been more than extensively studied over the past forty years. They are very closely linked to one dimensional dynamical systems called *interval exchange transformations*, and most of the basic features of these objects (as well as less basic ones !) have long been well understood, see [Zor06] for a broad and clear survey on the subject.

The starting point of this article is the following remark: to define the horizontal foliation discussed in the example above, there is no need to ask for the sides glued together to have same length, but only their being parallel in which case we can glue along affine identifications. In terms of geometric structures, it means that we extend the structural group to all the transformation of the form $z \mapsto az + b$ with $a \in \mathbb{R}_+^*$ and $b \in \mathbb{C}$. Formally, these corresponds to (branched) complex affine structures whose holonomy group lies in the subgroup of $\mathrm{Aff}(\mathbb{C})$ whose linear parts are real positive. A simple example of such a *'dilatation surface'* is given by the gluing below:

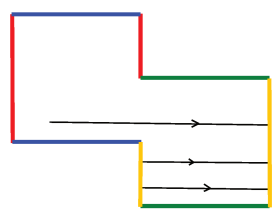


FIGURE 2. A *'dilatation surface'* of genus 2 and a leaf of its horizontal foliation.

A notable feature of these dilatation surfaces is that they present dynamical behaviours of *hyperbolic* type: the directional foliations sometimes have a closed leaf which 'attracts' all the nearby leaves. This is the case of the closed leave drawn in black on Figure 3 below. This situation is in sharp contrast with the case of translation surfaces and promise a very different picture in the dilatation case.

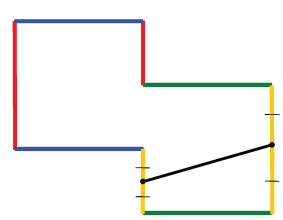


FIGURE 3. A *'hyperbolic'* closed leaf.

It is somewhat surprising to find no systematic study of these *'dilatation surfaces'* in the literature. However, related objects and concepts have kept poping up every now and then, both of geometric and dynamical nature. To our knowledge, their earliest appearance is in the work of Prym on holomorphic 1-forms with values in a flat bundle, see [Pry69]. These provide an algebraico-geometric interpretation of these dilatation surfaces, see also Mandelbaum([Man72, Man73]) and Gunning ([Gun81]). We would also like to mention Veech's remarkable papers [Vee93] and [Vee97] where he investigates moduli spaces of complex affine surfaces with singularities as well as Delaunay partitions for such surfaces. On the dynamical side, the first reference to related questions can be found in Levitt's paper on foliations on surfaces, [Lev82] where he builds an affine interval exchange (AIET) with a wandering interval (these AIETs must be thought of as the one-dimensional reduction of the foliations we are going to consider). It is followed by a serie of works initiated by Camelier and Gutierrez [CG97] and pursued by Bressaud, Hubert and Maass [BHM10], and Marmi, Moussa, and Yoccoz [MMY10]. They generalize a well known construction of Denjoy to build out of a standard IET an AIET having a wandering interval, behaviour which is (conjecturally) highly non-generic. Very striking is that the question of the behavior of a typical AIET has been very little investigated. In this direction, we mention the nice article of Liousse [Lio95] where the author deals with the topological generic behaviour of transversely affine foliations on surfaces.

Contents of the paper and results. After introducing formal definitions as well as a couple of interesting examples of *'dilatation surfaces'*, we prove a structure result about Veech groups of dilatation surfaces.

The Veech group of a dilatation surface Σ is the straightforward generalization of its translation analogue: it is the subgroup of $\mathrm{SL}_2(\mathbb{R})$ made of linear parts of locally affine transformation of Σ. It is a well-known fact that the Veech group of a translation surface is always discrete. This fails to be true in the more general case of dilatation surface, although the examples of surfaces whose Veech group is not discrete are fairly distinguishable. We completely describe the class of surfaces whose Veech group is not discrete. Roughly, those are the surface obtained starting from a ribbon graph and gluing to its edges a finite number of *'dilatation cylinders'*. We call such surfaces **Hopf surfaces**, because they must be thought of as higher genus analogues of Hopf tori, that are quotients $\mathbb{C}^*/_{(z\sim\lambda z)}$ with λ a positive constant different from 1. Precisely we prove

Theorem 1. Let Σ be a dilatation surface of genus ≥ 2. There are two possible cases :

(1) $\mathbf{V}(\Sigma)$ is the subgroup of upper triangular elements of $\mathrm{SL}_2(\mathbb{R})$ and Σ is a Hopf surface.
(2) $\mathbf{V}(\Sigma)$ is discrete.

We also prove the following theorem on the existence of closed geodesics in genus 2 :

Theorem 2. Any dilatation surface of genus 2 has a closed regular geodesic.

The proof is elementary and relies on combinatoric arguments. Nonetheless it is a good motivation for a list of open problems we address in Section 6. We end the article with a short appendix reviewing Veech's results on the geometry of dilatation surfaces contained in the article [Vee97] and the unpublished material [Vee08] that W. Veech kindly shared.

About Bill Veech's contribution. Bill Veech's sudden passing away encouraged us to account for his important contribution to the genesis of the present article. About twenty years ago, he published a very nice paper called *Delaunay partitions* in the journal *Topology* (see [Vee97]), in which he investigated the geometry of complex affine surfaces (of which our 'dilatation surfaces' are particular cases). A remarkable result contained in it is that affine surfaces all have geodesic triangulations in the same way flat surfaces have. We used it extensively when we first started working on dilatation surfaces, overlooking the details of [Vee97]. But at some point, we discovered a family of dilatation surfaces that seemed to be a counter-example to Veech's result and which provides an obstruction for affine surfaces to have a geodesic triangulation. We then decided to contact Bill Veech, who replied to us almost instantly with the most certain kindness. He told us that he realized the existence of the mistake long ago, but since the journal *Topology* no longer existed and that the paper did not draw a lot of attention, he did not bother to write an erratum. However, he shared with us courses notes from 2008 in which he *'fixed the mistake'*. It was a pleasure for us to discover that in these long notes (more than 100p) he completely characterizes the obstruction for the slightly flawed theorem of *Delaunay partitions* to be valid, overcoming serious technical difficulties. We extracted from the notes the Proposition 5 which is somewhat the technical cornerstone of this paper.

A few weeks before his passing away, Bill Veech allowed us to reproduce some of the content of his notes in an appendix to this article. It is a pity he did not live to give his opinion and modify accordingly to his wishes this part of the paper.

Acknowledgements. We are very grateful to Vincent Delecroix, Bertrand Deroin, Pascal Hubert, Erwan Lanneau, Leonid Monid and William Veech for interesting discussions. The third author is grateful to Luc Pirio for introducing him to the paper [Vee97]. The third author acknowledges partial support of ANR Lambda (ANR-13-BS01-0002).

2. Dilatation surfaces.

We give in this section formal definitions of *dilatation surfaces* and several concepts linked to both their geometry and dynamics.

2.1. **Basics.** An affine structure on a complex manifold M of dimension n is an atlas of chart (U_i, φ) with values in $\mathbb{C}^n$ such that the transition maps belong to $\mathrm{Aff}_n(\mathbb{C}) = \mathrm{Gl}_n(\mathbb{C}) \ltimes \mathbb{C}^n$. It is well known that the only compact surfaces (thought

of as 2-dimensional real manifolds) carrying an affine structure are tori. We make the definition of an affine structure less rigid, by allowing a finite number of points where the structure is singular, in order to include the interesting examples mentioned in the introduction:

Definition 1. (1) An **affine surface** $\mathcal{A}$ on Σ is a finite set $\mathcal{S} = \{s_1, \dots, s_n\} \subset \Sigma$ together with an affine structure on $\Sigma \setminus \mathcal{S}$ such that the latter extends to a euclidean cone structure of angle a multiple of 2π at the s_i's.
(2) A **dilatation surface** is an affine surface whose structural group has been restricted to $\mathrm{Aff}_{\mathbb{R}_+^*}(\mathbb{C})$.

The type of singularities we allow results of what seems to be an arbitrary choice. We could have as well allowed singular points to look like affine cones, or the angles to be arbitrary. We will justify our choice very soon. A first important remark is that an affine surface satisfies a *discrete Gauss-Bonnet* equality. If $\mathcal{S} = \{s_1, \dots, s_n\}$ is the set of singular points of a dilatation structure on Σ; and $k_i \geq 2$ is the integer such that the cone angle at s_i is $2k_i\pi$, then

$$\sum_{i=1}^{n} 1 - k_i = \chi(\Sigma) = 2 - 2g$$

From now on and until the end of the paper, we will consider only *dilatation surfaces*.

A general principle with geometric structures is that any object that is defined on the model and is invariant under the transformation group is well defined on the manifolds carrying such a structure. In our case the model is $\mathbb{C}$ with structure group $\mathrm{Aff}_{\mathbb{R}_+^*}(\mathbb{C})$. Among others, angles and (straight) lines are well defined on dilatation surfaces. More striking is the fact that it makes sense to say that the orientation of a line is well defined, and for each angle $\theta \in S^1$ we can define a *foliation* oriented by θ that we denote by $\mathcal{F}_\theta$, whose leaves are exactly the lines oriented by θ.
Finally remark that although the speed of a path is only defined up to a fixed constant, it makes sense to say that a path has constant speed (speed which is not itself well defined), as well as to say that a path has finite or infinite length.

Formally:

- a **geodesic** is an affine immersion of a segment $]a, b[$ (a or b can be $\pm\infty$);
- a **saddle connection** is a geodesic joining two singular points;
- a **leaf** of a directional foliation is a maximal geodesic in the direction of the foliation;
- a **closed geodesic** (or **closed leaf** if the direction of the foliation is unambiguous) is an affine embedding of $\mathbb{R}/\mathbb{Z}$;
- the first return on a little segment orthogonal to such a closed geodesic is a map of the form $x \mapsto \lambda x$ with $\lambda \in \mathbb{R}_+^*$. We say it is **flat** if $\lambda = 1$ and that it is **hyperbolic** otherwise.

2.2. **Cylinders.** These definitions being set, we introduce a first fundamental example, the *Hopf torus*. Consider a real number $\lambda \neq 1$ and identify every two points on $\mathbb{C}^*$ which differ by scalar multiplication by λ. The quotient surface $\mathbb{C}^*/(z \sim \lambda z)$ through this identification is a called a *Hopf torus* and we call λ its *dilatation factor*.

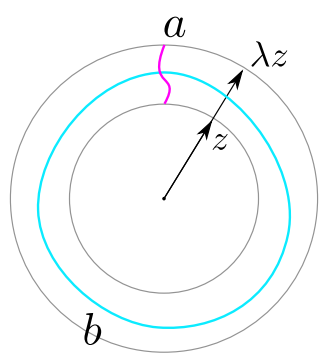


FIGURE 4. Hopf's torus

These provide a 1-parameter family of dilatation structures on the torus. These surfaces have a very specific kind of dynamics. Foliations in all directions have two closed leaves (the one corresponding to the ray from zero in this direction) one of which is attracting and the other repulsive.

Based on this torus, we can construct higher genus examples by gluing two of these tori along a slit in the same direction (see Figure 4). Take an embedded segment along the affine foliation in one direction on one torus, and an other one in the same direction on the second torus. We cut the two surfaces along these segments and identify the upper part of one with the lower part of the other with the corresponding affine map.

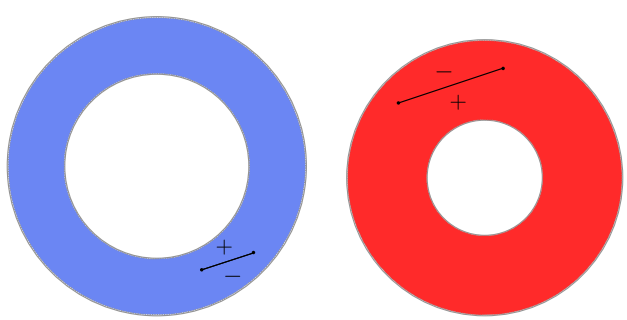


FIGURE 5. The franco-russian slit construction

Another construction based on the Hopf torus is given by considering a finite covers. Denote by a the closed curve of the Hopf torus in direction of the dilation and b the closed curve turning around zero once in the complex plane as in Figure

4. Take the k index subgroup of $\pi_1(T^2)$ generated by a and b^k and consider the associated cover with the induced dilatation structure. It is also a torus which *makes k turns around zero*. We call it a k-Hopf torus. Similarly, a ∞-Hopf cylinder will be the cover associated to the subgroup generated by a.

Remark. *We can also construct this structure as a slit construction along horizontal closed leaves of k different Hopf tori.*

These structures have the remarkable property to be a disjoint union of closed geodesics in different directions. Each of these geodesics is an attractive leaf of the foliation in their direction. This is the property we want to keep track of, that is why we want to consider angular sectors of these tori embedded in a dilatation surfaces. This is the motivation for the following definition:

Definition 2. Consider Σ a dilatation surface. Let C_{θ_2,θ_1} be an angular domain of a ∞-Hopf cylinder between *angles* $\theta_1 \in \mathbb{R}_+^*$ and $\theta_1 < \theta_2$ such that $\theta_2 - \theta_1 = \theta$. A *cylinder* of angle θ is the image of a maximal dilatation embedding of some C_{θ_2,θ_1} in Σ.
We call λ the *dilatation factor* of the cylinder and θ its *angle*.

Remark the isomorphism class of a dilatation cylinder is completely determined by the two numbers θ and λ.

Proposition 1. *Let Σ be a dilatation surface and not a k-Hopf torus. Then the boundary of a maximal cylinder embedded in Σ is a union of saddle connections.*

Proof. Suppose we embedded $C_{0,\theta}$ in the surface Σ. If $\theta = \infty$ we would have a half-infinite cylinder in the surface. But this half-cylinder would have an accumulation point at ∞ in Σ which contradicts its being embedded.

When $\theta < \infty$, there are two reasons why $C_{0,\theta'}$ cannot be embedded for $\theta' > \theta$

(1) The embedding $C_{0,\theta} \to \Sigma$ extends continuously to the boundary of the cylinder in direction θ, if the surface is not a k-Hopf torus the image contains a singular point. The image of the boundary is closed, and it is an union of saddle connections.
(2) The embedding does not extend to the boundary, then there is a geodesic $\gamma : [0, 1) \to C_{0,\theta}$ starting close to the boundary and ending orthogonally in the θ boundary of $C_{0,\theta}$ such that γ has no limit in Σ when approaching 1. Consider an open disk in $C_{0,\theta}$ tangent to the boundary at the point to which γ is ending and centered on γ trajectory. Then Proposition 5 implies that γ starting from the center of the circle is a closed hyperbolic geodesic in Σ. This cannot happen since γ is embedded in Σ

□

Again a cylinder will be the union of closed leaves. The dynamics of a geodesic entering such a cylinder is clear. If the cylinder is of angle less than π and the direction of the flow is not between θ_1 and θ_2 modulo 2π, then it will leave the

cylinder in finite time. Otherwise it will be attracted to the closest closed leaf corresponding to its direction, and be trapped in the cylinder.

As to enter a cylinder we have to cross its border, we see that for cylinder of angle larger than π every geodesic entering the cylinder is also trapped. These *'trap'* cylinders can be ignored when studying dynamics. We can study instead the surface with boundaries where we remove all these cylinders. We will see in the following section that these cylinders are also responsible for degenerate behaviour when trying to triangulate the surface.

Remark. *A degenerate case of dilatation cylinders are flat (or euclidian) cylinders. It is an embedding of the dilatation surface (with boundary)* $C_a = \{z \in \mathbb{C} \mid 0 < \Im(a) < a\}/(z \sim z+1)$. *In this case, the length* a *of the domain of the strip we quotient by* $z \mapsto z+1$ *will be called the modulus of the cylinder.*

2.3. **Triangulations.** An efficient way to build dilatation surfaces is to glue the parallel sides of a (pseudo-)polygon. A surface obtained this way enjoys the property to have a *geodesic triangulation.* It is a triangulation whose edges are geodesic segments and whose set of vertices is exactly the set of singular points. It is natural to wonder if any dilatation surface has such triangulation from which we could easily deduce a polygonal presentation. Remark that the question only makes sense for surfaces of genus $g \geq 2$, for in genus 1 there are no singular points.
Unfortunately, a simple example shows that it is not to be expected in general. The double Hopf torus constructed above cannot have a geodesic triangulation: any geodesic issued from the singular point accumulates on a closed regular geodesic, except for those coming from the slit. This obstruction can be extended to any dilatation surface containing a dilatation cylinder of angle $\geq \pi$: any geodesic entering such a cylinder never exits it which is incompatible with the fact that a triangulated surface deprived of its 1-skeleton is a union of triangles. A remarkable theorem of Veech proves that this obstruction is the only one:

Theorem (Veech, [Vee97, Vee08])**.** *Let* $\mathcal{A}$ *be a dilatation structure which does not contain any dilatation cylinder of angle larger or equal to* π. *Then* $\mathcal{A}$ *admits a geodesic triangulation.*

The exact theorem generalizes a classical construction known as *Delaunay partitions* to a more general class of affine surfaces. We review this construction and more of the material contained in [Vee97, Vee08] in Appendix A.

3. Examples.

3.1. **The two chambers surface.** By gluing the sides of same color of Figure 6 below, we get a genus 2 dilatation surface with a unique singular point of angle 6π. For completely random reasons, we call it the two-chambersurface.

We believe that this example is particularly interesting because it is a first non trivial example for which we can describe the directional foliation in every direction.

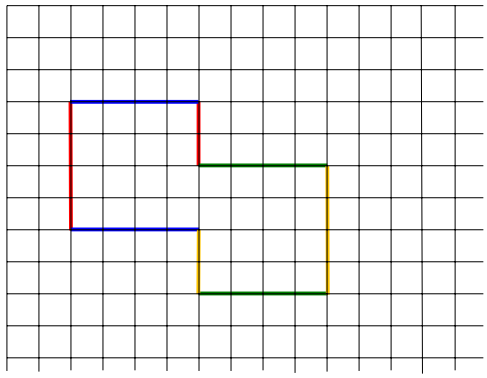


FIGURE 6. The two-chamber surface.

Proposition 2. *Let $\mathcal{F}_\theta$ be the directional foliation oriented by θ on the two-chamber example.*

(1) If $\theta = \pm\frac{\pi}{2}$, then the foliation is completely periodic; the surface decomposes into two euclidean cylinders.
(2) If $\theta = \arctan(n)$ or $\arctan(n + \frac{1}{2})$ for $n \in \mathbb{Z}$ and $\theta \neq \pm\frac{\pi}{2}$, the foliation accumulates on a closed saddle connection.
(3) For any other θ, the foliation accumulates on a hyperbolic closed leave.

Proof. Consider the case when $\theta = \arctan(\frac{1}{4})$. Here the segment linking the middle of the yellow sides projects to a closed leave of the directional foliation of angle $\theta = \arctan(\frac{1}{4})$. Take a little segment transverse to this leaf, the first return map is a dilation of factor $\frac{1}{2}$ (resp 2). This closed leaf is therefore attractive in the sense that every leaf passing close by winds around and accumulate on it.

The other cases are similar, the reader can convince himself by looking for the two closed leafs in the given direction and remark that one will be attractive and the other repulsive. □

3.2. **Affine interval exchange transformations.** We mention in this subsection a construction of Camelier and Gutierrez ([CG97]), improved by Bressaud, Hubert and Maas ([BHM10]), and generalised by Marmi, Moussa and Yoccoz ([MMY10]).

An *affine interval exchange* is a piecewise affine bijective map from $[0, 1]$ in itself. It can be thought of as a generalization of either standard interval exchanges or of piecewise affine homeomorphisms of the circle. To such an AIET (Affine Interval Exchange Transformation), we can associate a dilatation surface which is its *suspension*. It consists in taking a rectangle, identifying two vertical parallel sides in the standard way, and identifying the two horizontal according to the AIET, Figure 7b.

The dynamics of the vertical foliation of such a dilatation surface is exactly the same as the affine interval exchange we started from, for the orbits of the latter are

in correspondence with the leaves of the foliation, singular leaves corresponding to discontinuity points of the AIET.

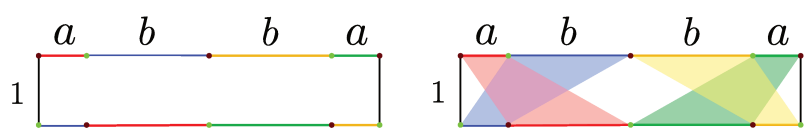
The construction mentioned above brings to light a surprising behaviour for certain affine interval exchanges.

Theorem (Marmi-Moussa-Yoccoz, [MMY10])**.** *For all combinatorics of genus at least* 2*, there exists a uniquely ergodic affine interval exchange whose invariant measure is supported by a Cantor set in* $[0, 1]$*.*

The implication of the theorem for the dilatation surfaces associated is that there are some leaves of the vertical foliation whose closure in the surface is union of leaves which intersects every transverse curve along a Cantor set. This in sharp contrast with both standard interval exchanges and piecewise affine homeomorphisms of the circle.

The construction is quite involved and we will not give any detail. Nonetheless it is worth noticing that in the example we have presented before, we have seen no trace of such a behaviour: the two-chambersurface has a very simple dynamics in every direction.

3.3. **The disco surface.** We build a first surface whose dynamics is *a priori* non-trivial. Choose a, b two positive real numbers, we consider the AIET associated to the permutation $(1, 2)(3, 4)$, and with top and bottom lengths a, b, b, a. Now take the suspension of this AIET with a rectangle of height 1, it defines a dilatation surface which two singularities of angle 4π and genus 2, see Figure 7a. We call it the *disco surface* and denote it by $\mathrm{D}_{a,b}$. Notice that the surface contains at hand several dilatation cylinders. We represent some of them on the following Figure 7b. These cylinders can overlap, and some zones can a priori be without cylinder coverage.



(A) Suspension construction (B) Cylinders

FIGURE 7. The disco surface with a horizontal cylinder

We give an alternative representation of the surface which makes a vertical flat cylinder decomposition appear. To do so we cut out the left part of the surface of width a along a vertical line. We now rescale it by a factor $\frac{b}{a}$ and reglue it on the top b interval. Reproduce the same surgery with the right part of the surface and the new surface is the one drawn on Figure 8

4. VEECH GROUPS.

Given a matrix $M \in \mathrm{Gl}_2^+(\mathbb{R})$ and a dilatation structure $\mathcal{A}$ on Σ, there is a way to create a new dilatation structure by replacing the atlas $(U_i, \varphi_i)_{i \in I}$ by

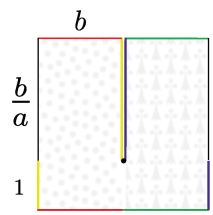


FIGURE 8. An alternative representation of the disco surface

$(U_i, M.\varphi_i)_{i\in I}$. This new dilatation structure is denoted by $M \cdot \mathcal{A}$. A way to put our hands on this operation is to describe it when $\mathcal{A}$ is given by gluing sides of a polygon p. If one sees P embedded in the complex plane $\mathbb{C} \simeq \mathbb{R}^2$, $M \cdot \mathcal{A}$ is the structure one gets after gluing sides of the polygon $M \cdot P$ along the same pattern.

We have defined this way an action of $\mathrm{Gl}_2^+(\mathbb{R})$ on the set of dilatation surfaces which factors through $\mathrm{SL}_2(\mathbb{R})$, since the action of dilation is obviously trivial. If $\mathcal{A}$ is a dilatation structure on Σ, we introduce its Veech group $\mathbf{V}(\mathcal{A})$ which is the stabilizer in $\mathrm{SL}_2(\mathbb{R})$ of $\mathcal{A}$, namely

$$\mathbf{V}(\mathcal{A}) = \{M \in \mathrm{SL}_2(\mathbb{R}) \ \mid M \cdot \mathcal{A} = \mathcal{A}\}$$

The Veech group is the set of real affine symmetries of the considered dilatation surface. It is the direct generalisation of the Veech group in the case of translation surfaces (see [HS06] for a nice introduction to the subject). For example, if $\mathcal{T}$ is a Hopf torus, $\mathbf{V}(\mathcal{T}) = \mathrm{SL}_2(\mathbb{R})$. It is a consequence of the fact that $\mathcal{T} = \mathbb{C}^*/(z \sim \lambda z)$ for a certain $\lambda > 1$, and that the action of $\mathrm{SL}_2(\mathbb{R})$ commutes to $z \mapsto \lambda z$. This fact is in sharp contrast with the case of translation surfaces where the Veech group is known to always be discrete.

4.1. **The Veech group of the two-chambersurface.** We carry on the particular analysis of the examples introduced in the last section, beginning with the two-chambersurface. Our exhaustive understanding of the dynamics of every directional foliation will enable us to describe its Veech group completely.

Proposition 3. *The Veech group of the two-chambersurface is the group generated by the two following matrices*

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Proof. The are only two directions on the two-chambersurface which are completely periodic which are $\frac{\pi}{2}$ and $-\frac{\pi}{2}$. Any element of the Veech group must preserve this set of directions, this is why it must lie in the set of lower triangular matrices. The rotation of angle π belongs to the Veech group since both the polygon that defines the two-chambersurface and the identification are invariant under this rotation. Up to multiplying by the latter, we can always assume that an element of the Veech group fixes the direction $\frac{\pi}{2}$.

We prove now that if a matrix of the form $\begin{pmatrix} \lambda & 0 \\ * & -\lambda^{-1} \end{pmatrix}$ belongs to the Veech group with $\lambda > 0$, then $\lambda = 1$. The two-chambersurface contains two flat cylinders of modulus 1. The image of these two cylinders by the action of such a matrix are two cylinders of modulus 1. So for $\begin{pmatrix} \lambda & 0 \\ * & -\lambda^{-1} \end{pmatrix}$ to belong to the Veech of the two-chambersurface, λ must equal 1.

Now remark that $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is in the Veech group. A simple cut and paste operation proves this fact, see Figure 9 below.

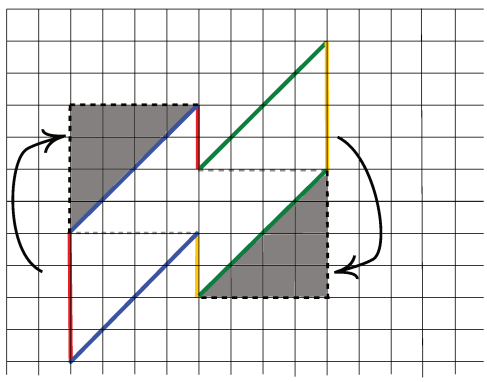


FIGURE 9

To complete the proof of the theorem, remark that the set of vectors of saddle connection must be preserved. This implies that $t = 1$ is the smallest positive number such that $\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)$ belongs to the Veech group of Σ. □

4.2. **The Veech group of the disco surface.** Let us describe some elements of the Veech group of the disco surface. First remark that when we act by the matrix

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

on a vertical cylinder of height 1 and width t, we can rearrange the surface and end up back to the same cylinder. It is exactly a Dehn twist on its core curve. This works also for any cylinder of modulus t. Hence if we have a surface which we can decompose in cylinders of same modulus t in the horizontal direction, the matrix above is in its Veech group.

As we remarked when introducing the disco surface $\mathrm{D}_{a,b}$, they decompose into one cylinder of modulus $2(a+b)$ in horizontal direction (Figure 7b) and into two cylinders of modulus $\frac{1+b/a}{b} = \frac{1}{a} + \frac{1}{b}$ in vertical direction (Figure 8).

As a consequence,

$$\begin{pmatrix} 1 & 2(a+b) \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \frac{1}{a} + \frac{1}{b} & 1 \end{pmatrix} \in \mathbf{V}(\mathrm{D}_{a,b})$$

Remark that these two matrices never generate a lattice in $SL_2(\mathbb{R})$ since it would imply that $2(a+b)(\frac{1}{a}+\frac{1}{b}) \leq 4$ which never happens.

4.3. **Hopf surfaces.** We present in this subsection a general construction of dilatation surfaces whose Veech group is conjugate to

$$\{ \begin{pmatrix} \lambda & t \\ 0 & \lambda^{-1} \end{pmatrix} \mid t \in \mathbb{R} \ and \ \lambda \in \mathbb{R}_+^* \}$$

and we prove that these are the only surfaces whose Veech group is not discrete. In particular this construction includes the Hopf torus and their derivatives introduced in Section 2.2.

Definition 3. A *ribbon graph* is a finite graph with a cyclic ordering of its semi-edges at its vertices.

We can think of a ribbon graph as an embedding of a given graph in a surface, the manifold structure giving the ordering a the vertices. The structure of a tubular neighbourhood of the embedding of the graph completely determines the ribbon graph.
Given a ribbon graph, we can make the following construction: along the boundary components of the infinitesimal thickening of the ribbon graph, we can glue cylinders of angle $k\pi$ respecting the orientation of the foliation to get a dilatation surface. We have to be careful to the factors of the cylinders we glue if we want the singular points to be Euclidean; we give an example from which it will be easy to deduce the general pattern.

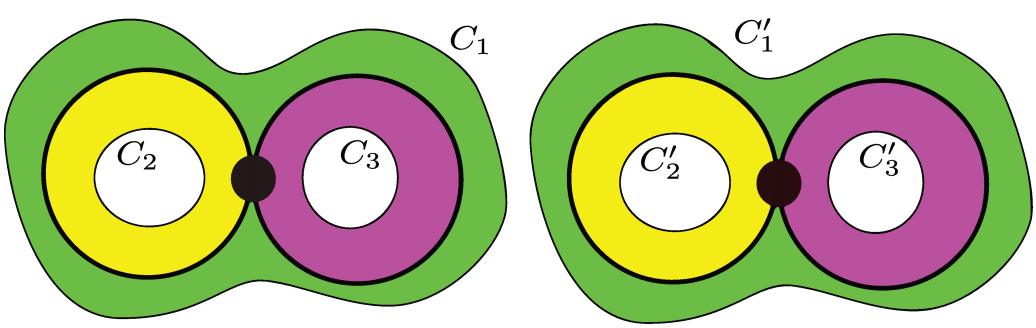


FIGURE 10. A ribbon graph with two vertices.

that we turn into a genus two surface by gluing three cylinders D_i to the boundary components, each joining C_i to C_i' for $i = 1, 2, 3$.

Let λ_i be the dilatation factor of D_i. For the angular points to be euclidean, it is necessary that the product of the factor of the cylinders adjacent at a singular point is trivial. In our case we get for an appropriate choice of orientation:

$$\lambda_1 = \lambda_2 \lambda_3$$

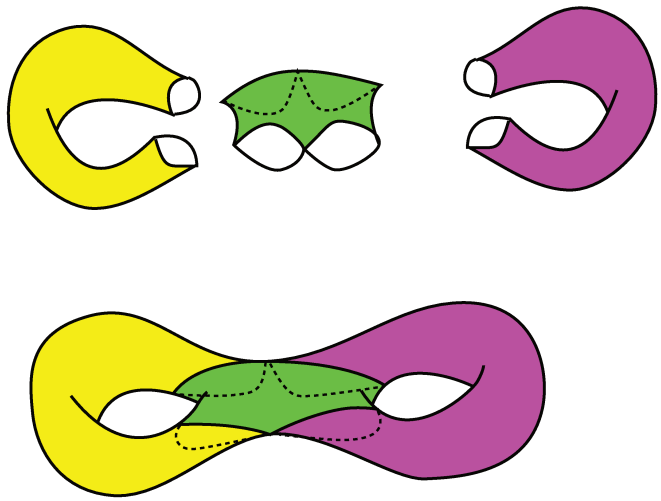


FIGURE 11. A cylinder decomposition of the surface of genus 2.

This constraint straightforwardly generalizes to the general case and is the only obstruction to complete the construction. We call a dilatation surface obtained by this construction a *Hopf surface*, because it generalizes the variations on the Hopf torus explained in Section 2.2.

We will see in the next section that this example corresponds exactly to the case when the Veech group is not discrete and is not $\mathrm{SL}_2\mathbb{R}$

4.4. **Veech group dichotomy.** First we deal with the case of genus 1 with the following lemma,

Lemma 1. *A dilatation torus is the exponential of some flat torus* $\mathbb{C}/\alpha\mathbb{Z} \oplus (\beta + 2ik\pi)\,\mathbb{Z}$ *where* $\alpha, \beta \in \mathbb{R}^+$ *and* $k \in \mathbb{N}$*. Moreover its Veech group is* $\mathrm{SL}_2(\mathbb{R})$

Proof. Consider a dilatation torus. Its developing map goes from $\mathbb{C}$ to $\mathbb{C}$ and its holonomy is commutative. Hence its holonomy is generated either by two translation or two affinel maps with the same fixed point (which we assume to be zero). The former case is to be excluded since we are only considering strictly dilatation surface.

In the second case, we can choose a developing map f which avoids zero, and associate to it the 1-form $\mathrm{d}\log f = \frac{df}{f}$. As the surface has no singularity, the derivative of f is never zero, and the 1-form is invariant with respect to the holonomy. Thus this *a priori* meromorphic form is defined on the torus, has no zeroes and by residue formula no poles, therefore it is holomorphic.

In conclusion, the logarithm form gives a flat structure on the torus that is isomorphic to $\mathbb{C}/\alpha\mathbb{Z} \oplus \tau\mathbb{Z}$ where $\alpha \in \mathbb{R}^+$ and $\tau \in \mathbb{C}^*$. The exponential of this flat torus is the initial dilatation structure, thus $e^\tau \in \mathbb{R}_+^*$ and $\Im(\tau) \equiv 0 \mod 2\pi$.

Any matrix of $\mathrm{SL}_2(\mathbb{R})$ commutes with the scalar multiplications, thus the Veech group of such a surface is the whole $\mathrm{SL}_2(\mathbb{R})$. $\square$

This structure is like taking one α-Hopf torus which we slit at one horizontal closed curve and glue k copies of it. When we glue back the k-th copy to the first one, we apply a β dilation.

Now that we have set aside the peculiar case of genus 1, we prove a classification theorem on dilatation surfaces of higher genus depending on the type of their Veech group.

Theorem 1. *Let Σ be a dilatation surface of genus ≥ 2. There are two possible cases :*

(1) Σ is a Hopf surface and $\mathbf{V}(\Sigma)$ is the subgroup of upper triangular elements of $\mathrm{SL}_2(\mathbb{R})$,

$$\{\begin{pmatrix} \lambda & * \\ 0 & \lambda^{-1} \end{pmatrix} | \lambda \in \mathbb{R}^*\}$$

(2) $\mathbf{V}(\Sigma)$ is discrete.

The end of the section is devoted to proving Theorem 1. To do so, we will distinguish between dilatation surfaces having saddle connections in at least two directions and those who do not. The former enjoy the property that their Veech group is automatically discrete thanks to a classical argument inspired by the case of translation surfaces (see Section 3.1 of [HS06]). The latter will turn out to be Hopf surfaces introduced in section 4.3.

Lemma 2. *Let Σ be a dilatation surface having two saddle connections in different directions. Then $\mathbf{V}(\Sigma)$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$.*

Remark. *As a direct corollary both the two-chambersurface and the disco surface have a discrete Veech group.*

Proof. Consider $\mathbf{V}_0(\Sigma)$ the subgroup of $\mathbf{V}(\Sigma)$ fixing point-wise the set of singular points of Σ. This subgroup has finite index in $\mathbf{V}(\Sigma)$ and its being discrete implies discreteness for $\mathbf{V}(\Sigma)$.
Choose an arbitrary simply connected subset $U \subset \Sigma$ containing all the singularities and an arbitrary developing map of the dilatation structure on U. Thanks to this developing map, we can associate to each oriented saddle connection a vector in $\mathbb{R}^2$. The set of such vectors enjoys two nice properties:

- it is discrete;
- it is invariant under the action of $\mathbf{V}_0(\Sigma)$.

We have made the hypothesis that a pair of those vectors (v_1, v_2) form a basis in $\mathbb{R}^2$. If a_n is a sequence of elements of $\mathbf{V}_0(\Sigma)$ going to identity and A_n the matrices of their action in the normalization induced by U (notice that an element of $\mathbf{V}(\Sigma)$ is an element of the quotient $\mathrm{Gl}_2^+(\mathbb{R})/\mathbb{R}_+^*$ and therefore the matrix of its action depends on normalization both at the source and at the target). $A_n \cdot v_1 \to v_1$ and $A_n \cdot v_2 \to v_2$ but the discreteness of the set of saddle connection vectors implies that for n large enough $A_n \cdot v_1 = v_1$ and $A_n \cdot v_2 = v_2$, and thus $A_n = \mathrm{Id}$. Which proves the discreteness of the Veech group. $\square$

We will now characterize the dilatation surfaces having saddle connections in at most one direction. The two following lemmas will complete the classification.

Lemma 3. *If all the saddle connections of a dilatation surface Σ are in the same direction it is a Hopf surface.*

Proof. We make the assumption that Σ has at least a singular point (if not, Lemma 1 settles the question). We are going to prove that every singular point has a least a saddle connection in every angular sector of angle π. A dilatation surface whose all separatrices in one direction are saddle connection is easily seen to be a Hopf surface, see Subsection 4.3.

Consider the exponential map at a singular point p associated to a local affine normalization and let $r > 0$ be smallest radius such that Δ_r the (open) semi-disk of radius r, bounded below by two horizontal separatices and containing a vertical separatrix, immerses in Σ by means of the exponential map. If $r = \infty$, we would have a maximal affine immersion of $\mathcal{H}$, whose boundary would project to a closed leaf containing p and there would be two saddle connections in the horizontal direction. Otherwise $r < \infty$. There can be two different reasons why $\Delta_{r'}$ does not immerse for $r' > r$:

(1) either the immersion $\Delta_r \longrightarrow \Sigma$ extends continuously to the boundary of Δ_r and the image of this extension contains a singular point. In that case this singular point must be in the unique direction containing saddle connections (say the vertical one) and the vertical separatrix was actually a saddle connection;

(2) or $\Delta_r \longrightarrow \Sigma$ does not extend to the boundary of Δ_r.

We prove that the latter situation cannot occur. Otherwise there would be a geodesic γ issued at p affinely parametrized by $[0, r)$ such that $\gamma(t)$ does not have a limit in Σ when t tends to r. We make the confusion between γ as a subset of Σ and its pre-image in Δ_r. Considering an open disk in Δ_r which is tangent at the boundary to the point in $\partial\Delta_r$ towards which γ is heading and such that its center belongs to γ. By means of the immersion of Δ_r in Σ, it provides an affine immersion of $\mathbb{D}$ satisfying the hypothesis of Proposition 5. γ would therefore project to a closed hyperbolic geodesic, which contradicts its being a separatrix. □

4.5. **The Veech group of a dilatation surface is probably not a lattice.** We start by showing the following lemma,

Lemma 4. *If $\mathbf{V}(\Sigma)$ is discrete it cannot be cocompact.*

Proof. To show this we only need to find a continuous function on $\mathbb{H}/\mathbf{V}(\Sigma)$ that is not bounded. For flat surface the systole does the trick, here we cannot define length of saddle connections but the ratio of the two lengths of saddle connections that intersect. Thus we can define the shortest and the second shortest simple saddle connections starting at a given singular point, and we denote by l and

L their length in one arbitrary chart around the singularity. These two distinct saddle connections exist since we assume $\mathbf{V}(\Sigma)$ discrete according to Theorem 1. Now the ratio L/l is independent from the previous choice of chart. And if we take the minimum value of this ratio on all the singularities, it will be a continuous function on the $\mathbb{H}$ orbit of the surface invariant by the Veech group.

To finish the argument, apply the Teichmüller deformation of the surface with matrices of the form

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

where the horizontal direction will be the direction of the smallest saddle connection with the smallest ratio. This deformation will decrease the length l and increase L as t goes to infinity and thus make this function go to infinity. □

As we saw in Section 2.2, cylinders trap the linear flow in their corresponding angular sector. This behaviour restricts the potential directions for saddle connections around a singularity in the boundary of cylinders and prevent Veech groups from being lattices.

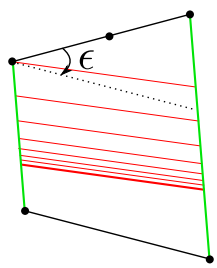


FIGURE 12. Angular section on which leaves are hyperbolic

Consider Σ a dilatation surface endowed with a dilatation cylinder, take any singular point at the border of this cylinder and flow a leaf heading inside the cylinder whose angular direction falls just in-between the two extreme angles of the cylinders. This leaf will be trapped inside the cylinder and accumulate to a close leaf (Figure 12). As a consequence none of these leaves will meet a singular point. There won't be any saddle connection starting from the chosen point in the angular sector define by the cylinder.

This implies the following proposition :

Proposition 4. *If Σ is a dilatation surface with a dilatation cylinder then $\mathbf{V}(\Sigma)$ is not a lattice.*

Proof. Assume for a contradiction that $\mathbf{V}(\Sigma)$ is a lattice. Take a finite index subgroup which stabilizes all the singularities of the surface. As an assumption on Σ there is a dilatation cylinder in the surface, Proposition 1 implies that there

is a singularity in the boundary of the cylinder, and a small angular section in which any separatrix from this singularity will accumulate to a closed leaf. As the subgroup is a lattice it contains a parabolic element and its limit set is the whole border of $\mathbb{H}$. Then we can conjugate this parabolic element to see that there is a dense set of direction in which there is a parabolic. In parabolic directions, the separatrices are all saddle connections. Indeed, if not there would be an accumulation point, and a neighborhood of this point would be crossed infinitely many times by the separatrix on which the parabolic acts as the identity. Hence the parabolic element would act as the identity on the whole neighborhood, which is a contradiction. This shall not be since we showed that there cannot be saddle connection in an open set of directions around the singularity. □

5. Cylinders on genus 2 surfaces.

The purpose of this section is to show the following result

Theorem 2. *Any dilatation surface of genus* 2 *has a cylinder.*

First, remark that Veech's theorem on Delaunay triangulation (see Veech's theorem in 2.3) tells us that if a dilatation surface does not have a triangulation, it must contain a dilatation cylinder (of angle at least π). We can therefore forget about this case and assume that all the surfaces we are looking at have geodesic triangulations.
The theorem is also easy to prove when the surface has two singular points which both must be of angle 4π. Consider a triangle of a Delaunay triangulation of the surface, at least two of its vertices are equal to the same singular point and the side corresponding to these two vertices is a simple closed curve. It must cut the angle of the associated singular point into two angular sectors of respective angle 3π and π. Which implies that it bounds a cylinder on the side of the angle π.

For the remainder of the section, Σ is a surface of genus 2 together with a strictly dilatation structure whose unique singular point of angle 6π is denoted by p. A geodesic triangulation of Σ must have exactly 9 edges and 6 triangles. In this particular case where the triangulation has a unique vertex, each edge defines a simple closed curve, geodesic away from p and cutting the latter into two angular sector. Because each directional foliation is oriented, such an edge must cut the angle 6π into two angles of respective values either 5π, π or 3π, 3π.
The lemma below proves that any such geodesic triangulation has at least one edge cutting the singular point into two angles 5π and π. The theorem is a direct consequence of this lemma.

Lemma 5. *A geodesic triangulation of* Σ *cannot have its* 9 *edges cutting* p *into two sectors of angles* 3π.

Proof. The property on the edges implies that they all intersect with number ± 1 at the only vertex of the triangulation. Since every closed leaf separates the cylinder at p in two angular sectors of angle 3π an oriented leaf can't have it's

in-going and outgoing parts in the same sector cut by any other leaf (the angle would have to be smaller).

We can always see topologically our surface as an octagon which boundary are edges of the triangulation. Take for example the maximal sub-graph of the 1-skeleton of the triangulation such as the complementary of it in the surface is connected and simply-connected, this complement will be the fundamental domain we are looking for. Now the intersection number property tells us that the configuration of the path at the order will be in the setting of Figure 13

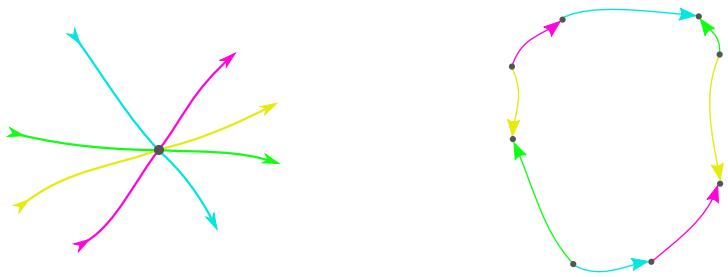


FIGURE 13. Topological setting of the separatrix diagram

Now consider yet another of the 9 edges, as it has to intersect all of the other path with ± 1 the only possibility is when the curves starts between two colors and end up between the two same colors.

Let's add one of these edges as in Figure 14. It is clear now that we won't be able to add any curve with the same property since they can't intersect a curve in other point than p.

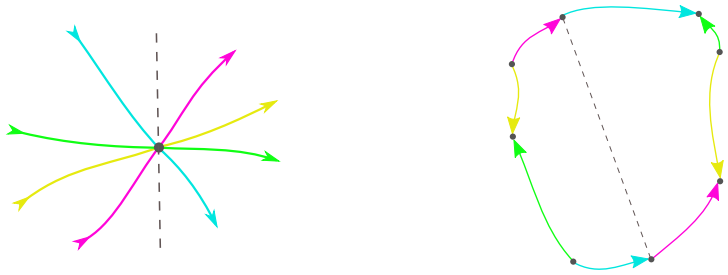


FIGURE 14. Adding one curve

□

This completes the proof of Theorem 2. It is not completely satisfactory: we would like to prove that every dilatation surface contains *strictly dilatation cylinders*.

6. Open problems.

We hope that at this point, most of the problems we are about to suggest seem natural to the reader.

We have so far put our hands on several dynamical behaviours for the directional foliations on our dilatation surfaces. Very often it happens that a finite number of *hyperbolic closed leaves* attract all the others, as it the case for all but one direction on the two-chambersurface. We say in that case that the dynamics is **hyperbolic**. The Camelier-Gutierrez construction discussed in Section 3.2 also proves the existence of directional foliations such that every leaf accumulates to a closed union of leaves which is transversely a *Cantor set*. Finally it is not to be excluded that some directional foliations are minimal (it is actually very easy to build examples of such dilatation surfaces). Because of the conjectural picture we are about to draw, we call both late cases **exceptional**.

We begin our list of open problems by very specific questions concerning the $D_{1,2}$ example. It is a simple and very explicit one, but the dynamical questions that it raises are not straightforwardly answerable.

(1) Does there exist a dense direction on the $D_{1,2}$ example?
(2) Does there exist a 'Cantor like' direction on the $D_{1,2}$ example?
(3) What does the set of hyperbolic directions on the $D_{1,2}$ example look like? Is it dense? Has it full Lebesgue measure?

These might be a good starting point on the way to the general case. Although the combinatorial arguments used in its proof are unlikely to generalize to higher genus, Theorem 2 suggests that systematic dynamical behaviours are to be expected. Is it true that

(4) every dilatation surface has a closed regular geodesic?
(5) every dilatation surface has a closed, regular and hyperbolic geodesic?
(6) the set of hyperbolic direction of every dilatation surface is dense? has full measure?

These questions have natural generalizations to the moduli space of dilatation surfaces together with a directional foliation. It is possible that in this setting some of the aforementioned questions are easier to answer, and that specific surfaces have a very different behaviour from the generic one. We conjecture that the answers to these three questions are positive, with a greater reserve about the full measure one. There are most probably very interesting things to say about the loci of exceptional directions, but making guesses without a clear picture of what happens for hyperbolic ones seems quite dairy. A very natural direction at this point is to investigate the geometrical properties of these dilatation surfaces:

(7) Which are the dilatation surfaces having only finitely many saddle connections?

(8) Is it true that trough a given point on a dilatation surface always passes either a closed geodesic or a saddle connection? It is the case in the two-chamberexample.
(9) What does the set of vectors of saddle connections on a given surface look like?

Proposition 4 most probably prevents the Veech group of a dilatation surface to be a lattice in $\mathrm{SL}_2(\mathbb{R})$. Nonetheless, it seems to be an interesting invariant of these dilatation surfaces.

(10) What kind of Fuchsian groups can appear as Veech groups ?

Finally we want to suggest that a nice source of questions is trying to describe the set of surfaces having specific interesting dynamical/geometric properties. For instance:

(11) Is the set of dilatation surfaces having an exceptional direction dense? generic?
(12) Is the set of dilatation surfaces having *no* exceptional direction dense? generic?
(13) Does there exists a surface having infinitely many *'Cantor like'* directions?

Appendix A. Veech's results on the geometry of affine surfaces.

We review in this appendix results of Veech on the geometry of affine surfaces appearing in [Vee97] and [Vee08]. Note that Veech works in the more general context of affine surfaces with singularities.

For the sake of clarity, we will restrict his results to the case under scrutiny in this paper namely branched affine surface with real positive linear holonomy. The notes [Vee08] remain unpublished and Veech kindly allowed us to reproduce here proofs that are contained in these notes.

A.1. **The property $\mathcal{V}$.** We say an affine surface Σ satisfies *the property* $\mathcal{V}$ if there is no affine immersion of $\mathbb{H}$ in Σ. It is equivalent to ask that Σ has no affine cylinder of angle larger than π.

Theorem (Veech, [Vee08])**.** *An equivalent formulation of the property $\mathcal{V}$ is the following:*

($\mathcal{V}'$) *Every affine immersion of the open unit disk $\mathbb{D} \subset \mathbb{C}$ in Σ extends continuously to a map $\overline{\mathbb{D}} \to \Sigma$.*

We will only give the proof of one sense of the equivalence, namely that the property $\mathcal{V}$ implies the property $\mathcal{V}'$, which will be sufficient for our purpose.

Lemma 6. *Let φ be an affine immersion of the open unit disk $\mathbb{D} \subset \mathbb{C}$ in Σ that does not extend continuously to a map $\overline{\mathbb{D}} \to \Sigma$. Then φ extends to an affine immersion $\mathbb{H} \longrightarrow \Sigma$.*

Proof. Since φ does not extend to $\partial\mathbb{D}$, there exists $z \in \partial D$ such that

$$\lim_{t\to 1} \varphi(tz)$$

does not exist. Let γ be the path $t \mapsto \varphi(tz)$. Let x be an accumulation point of γ. Since $[0,1]$ is connected, we can assume that x is not singular. Let $t_k \to 1$ be an increasing sequence such that $\gamma(t_k) \to x$ and f an affine chart at x, $f : U \mapsto \Delta$ where Δ is the unit disk centered at 0 et U an open set containing x such that $f(x) = 0$. We denote by $\frac{1}{2}U$ the pre-image by f in Σ of the disk centered at 0 or radius $\frac{1}{2}$.

For k large enough, we can assume that $\gamma(t_k)$ belongs to $\frac{1}{2}U$. Denote by V_k the pre-image by in Σ of the disk of radius $\frac{1}{2}$ centered at $\varphi(\gamma(t_k))$. In particular V_k contains x. We claim that the image under φ of D_k the open disk centered at $t_k z \in \mathbb{U}$ tangent to ∂U at z contains V_k. Since φ does not extend at z, there is a closed disk centered a $t_k z$ strictly contained in D_k whose image under φ is not contained in V_k. Since this image is a disk concentric at $t_k z$ it must contain V_k and in particular x. Let w_k be a pre-image of x in D_k.

Let E_k be the largest disk with center w_k to which φ admits analytic continuation. Since $w_k \to z$, the radius of E_k converges to 0. Necessarily $\varphi(E_k)$ contains $U = f^{-1}(\Delta)$ because $\varphi(E_k)$ is a maximal embedded disk of center x. Extending the map $f^{-1} : \Delta \to U \subset \Sigma$ by means of φ defines $F_k : \Delta_k \to \Sigma$. The functions $(F_k)_{k\in\mathbb{N}}$ have the following properties:

- Δ_k is a disk;
- $\Delta \subset \Delta$;
- $\forall \zeta \in \Delta$ we have that $F_k(\zeta) = f^{-1}(\zeta)$;
- $\forall k, k' \in \mathbb{N}$, $F_k = F_{k'}$ on $\Delta_k \cap \Delta_{k'}$;
- the radius of Δ_k tends to $+\infty$, because the radius of E_k tends to 0.

Since all the D_k are connected and $\bigcap_{k\in\mathbb{N}} \Delta_k$ is non empty, the F_k define an affine immersion

$$F : \bigcup_{k\in\mathbb{N}} \Delta_k \longrightarrow \Sigma$$

and $\bigcup_{k\in\mathbb{N}} \Delta_k$ must contain a half-plane since it is the union of disk whose radius tend to infinity whose intersection is not empty. This proves the lemma.

□

Lemma 7. *Any affine immersion $\varphi : \mathbb{H} \longrightarrow \Sigma$ can be extended to $\varphi' : \mathbb{H}' \longrightarrow \Sigma$ such that the latter is invariant by the action by multiplication by a positive real number $\lambda \neq 1$.*

Proof. We show first that such an immersion cannot be one-to-one. Consider the geodesic $\gamma[0,\infty[\to [i, i\infty[\subset \mathbb{H}$. Its image by φ cannot have a limit in Σ for it has infinite (relative) length. Let $x \in \Sigma$ be an accumulation point of this image, and $U \subset \Sigma$ a closed embedded disk with center x such that $\varphi(\gamma)$ is not completely contained in U. Since $\varphi(\gamma)$ is a leaf of a directional foliation on Σ, it must cross U

twice along parallel segment and therefore cannot be injective because the image of horizontal stripes at the first and second crossing will overlap.

There exists then $v \neq w \in \mathbb{H}$ such that $\varphi(z) = \varphi(w)$. There exists then an affine map $z \mapsto \lambda z + a$ with $a \in \mathbb{R}_+^*$ and b in $\mathbb{C}$ such that $w = \lambda v + a$ and for all z in a neighborhood of v

$$\varphi(z) = \varphi(\lambda z + a).$$

The union of the iterated images of $\mathbb{H}$ by $z \mapsto \lambda z + a$ is a half-plane $\mathbb{H}'$ to which φ extends by analytic continuation and on which the above invariance relation holds for all z. If $\lambda = 1$, the image of φ' in Σ is an infinite flat cylinder, which is impossible since Σ is compact; therefore $\lambda \neq 1$. The fixed point of $z \mapsto \lambda z + a$ must lie in $\partial\mathbb{H}'$ and up to an affine transformation we can suppose that $b = 0$.

□

A.2. **Delaunay decompositions and triangulations.** The main result of [Vee97] and [Vee08] combined is establishing the existence of geodesic triangulations for surfaces not verifying the only natural obstruction for this to happen. It is given by the following theorem:

Theorem (Veech)**.** *An affine surface admits a geodesic triangulation if and only if it does not contain an embedded open affine cylinder of angle π. Equivalently, an affine surface admits a geodesic triangulation if and only if it satisfies the property $\mathcal{V}$.*

This theorem is a corollary of the existence of *Delaunay partitions* for affine surfaces that Veech deals with. We explain in the sequel the construction. Let Σ be an affine surface satisfying the property $\mathcal{V}$ and let $x \in \Sigma$ be a regular point. We are going to distinguish points depending on the number of singular points on the boundary of the largest immersed disk at x. We denote this number by $\nu(x)$. The set $\{\nu(x) = 1\}$ is a open dense set in the surface. The special points a those such that $\nu(x) \geq 3$, who form a discrete and therefore finite set in the surface. At these points one can consider the largest embedded disk in the surface and consider in this disk the convex hull of the (at least) three singular point on the boundary. The crucial (but not completely obvious) facts are that:

- this convex hull projects onto an **embedded** convex polygon in the surface;
- the union of such polygons covers the whole surface;
- such polygons only intersects at their boundary;
- an intersection of two such polygons is union of some their (shared) sides;
- the set of vertices of such polygons is exactly the set of singular points;
- the union of the interior of these polygons is exactly the set $\{\nu(x) = 1\}$;
- the union of the interior of their sides is exactly the set $\{\nu(x) = 2\}$.

This decomposition of the surface in convex polygons is called its **Delaunay polygonations**, is unique and only depends on its geometry. A triangulation of each polygon leads to a *geodesic triangulation* of the surface.

The crucial fact (we refer to [Vee97] for details) is that having the property $\mathcal{V}$ implies that maximal affine embeddings of the disk extend to their boundary, which is the technical point that one needs to make sure the construction hinted above can be carried on.

Remark. *The converse of the triangulation theorem is quite easy. Any cylinder of angle at least π behave like a 'trap': any geodesic entering it never escapes. A surface containing such a cylinder therefore cannot be geodesically triangulated, because no edge of the triangulation could enter the cylinder and the complement of the* 1*-skeleton of such a triangulation would not be a union of cells.*

A.3. **Closed geodesics and wild immersions of the disk.** We consider in this subsection a surface Σ which does not satisfy the property $\mathcal{V}$. This implies that there is an affine immersion $\varphi : \mathbb{D} \longrightarrow \Sigma$ which does not extend to $\partial\mathbb{D}$ the boundary of $\mathbb{D}$. This also implies the existence of an affine cylinder and therefore closed hyperbolic geodesics. We give in this subsection a way to localize such a closed hyperbolic geodesic starting from φ. We believe that this result should be attributed to Veech. Even though not stated explicitly in [Vee08] (probably because Veech did not have in mind the dynamical questions we are interested in), the statement is obvious for anyone who has read carefully the sections 23 and 24 of [Vee08].

Proposition 5 (Veech, [Vee08])**.** *Let $\varphi : \mathbb{D} \longrightarrow \Sigma$ be a wild immersion of the disk, i.e. that does not extend to $\partial\mathbb{D}$, and let $z \in \partial\mathbb{D}$ such that the path $\gamma : t \mapsto \varphi(tz)$ does not have a limit in Σ when $t \to 1$. Then $\gamma([0,1))$ is a hyperbolic closed geodesic in Σ .*

Proof. According to both Lemma 6 and 7 φ extends to a maximal $\varphi' : \mathbb{H} \longrightarrow \Sigma$ which is invariant by multiplication by a certain positive number $\lambda \neq 1$. γ in this extension must head toward $0 \in \partial\mathbb{H}$ for otherwise it would have a limit in Σ. Its image is therefore a closed hyperbolic leave because it is invariant by the action of λ. □

References

[BHM10] Xavier Bressaud, Pascal Hubert, and Alejandro Maass. Persistence of wandering intervals in self-similar affine interval exchange transformations. *Ergodic Theory Dynam. Systems*, 30(3):665–686, 2010.

[CG97] Ricardo Camelier and Carlos Gutierrez. Affine interval exchange transformations with wandering intervals. *Ergodic Theory Dynam. Systems*, 17(6):1315–1338, 1997.

[Gun81] R. C. Gunning. Affine and projective structures on Riemann surfaces. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume 97 of *Ann. of Math. Stud.*, pages 225–244. Princeton Univ. Press, Princeton, N.J., 1981.

[HS06] Pascal Hubert and Thomas A Schmidt. Chapter 6 - An Introduction to Veech Surfaces. In B Hasselblatt and A Katok, editors, *Handbook of Dynamical Systems*, volume 1, Part B of *Handbook of Dynamical Systems*, pages 501–526. Elsevier Science, 2006.

[Lev82] Gilbert Levitt. Feuilletages des surfaces. *Ann. Inst. Fourier (Grenoble)*, 32(2):x, 179–217, 1982.

[Lio95] Isabelle Liousse. Dynamique générique des feuilletages transversalement affines des surfaces. *Bull. Soc. Math. France*, 123(4):493–516, 1995.

[Man72] Richard Mandelbaum. Branched structures on Riemann surfaces. *Trans. Amer. Math. Soc.*, 163:261–275, 1972.

[Man73] Richard Mandelbaum. Branched structures and affine and projective bundles on Riemann surfaces. *Trans. Amer. Math. Soc.*, 183:37–58, 1973.

[MMY10] S. Marmi, P. Moussa, and J.-C. Yoccoz. Affine interval exchange maps with a wandering interval. *Proc. Lond. Math. Soc. (3)*, 100(3):639–669, 2010.

[Pry69] F. E. Prym. Zur Integration der gleichzeitigen Differentialgleichungen. *J. Reine Angew. Math.*, 70:354–362, 1869.

[Vee93] William A. Veech. Flat surfaces. *Amer. J. Math.*, 115(3):589–689, 1993.

[Vee97] W. A. Veech. Delaunay partitions. *Topology*, 36(1):1–28, 1997.

[Vee08] W. A. Veech. Informal notes on flat surfaces. *Unpublished course notes*, 2008.

[Zor06] Anton Zorich. Flat surfaces. In *Frontiers in number theory, physics, and geometry. I*, pages 437–583. Springer, Berlin, 2006.

Chapitre 5

Cascades in the dynamics of affine interval exchange transformations

arXiv:1701.02332

Submitted to *Ergodic Theory and Dynamical Systems*

The purpose of this chapter is to describe the dynamics of a dilatation surface in every direction. We derive an infinite family of directions on which the dynamics accumulates to a Cantor set, bringing a new light to Levitt example of such affine interval exchanges.

CASCADES IN THE DYNAMICS OF AFFINE INTERVAL EXCHANGE TRANSFORMATIONS.

ADRIEN BOULANGER, CHARLES FOUGERON, AND SELIM GHAZOUANI

Abstract. We describe in this article the dynamics of a 1-parameter family of affine interval exchange transformations. It amounts to studying the directional foliations of a particular dilatation surface introduced in [DFG], the *Disco surface*. We show that this family displays various dynamical behaviours: it is generically *dynamically trivial*, but for a Cantor set of parameters the leaves of the foliations accumulate to a (transversely) Cantor set. This is done by analysing the dynamics of the Veech group of this surface, combined with an original use of the Rauzy induction in the context of affine interval exchange transformations. Finally, this analysis allows us to completely compute the Veech group of the Disco surface.

1. Introduction.

We consider the map $F : D \longrightarrow D$, where $D = [0, 1[$, defined the following way:

$$\begin{array}{lll}
\text{if } x \in \left[0, \frac{1}{6}\right[& \text{then} & F(x) = \ 2x + \frac{1}{6} \\
\text{if } x \in \left[\frac{1}{6}, \frac{1}{2}\right[& \text{then} & F(x) = \ \frac{1}{2}(x - \frac{1}{6}) \\
\text{if } x \in \left[\frac{1}{2}, \frac{5}{6}\right[& \text{then} & F(x) = \ \frac{1}{2}(x - \frac{1}{2}) + \frac{5}{6} \\
\text{if } x \in \left[\frac{5}{6}, 1\right[& \text{then} & F(x) = \ 2(x - \frac{5}{6}) + \frac{1}{2}
\end{array}$$

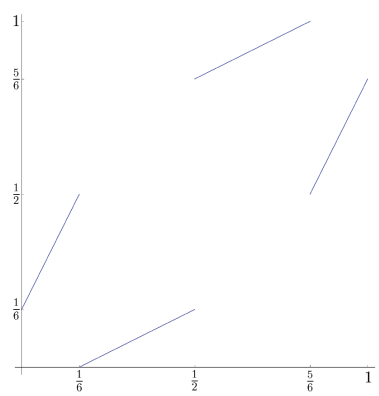


Figure 1. The graph of F.

The map F is an *affine interval exchange transformation* (AIET) and one easily verifies that for all $x \in D$, $F^2(x) = x$. Its dynamical behaviour is therefore as simple as can be. We consider a family $(F_t)_{t\in S^1}$, parametrised by $S^1 = {}^{\mathbb{R}}/_{\mathbb{Z}}$, defined by

$$F_t = F \circ r_t$$

where $r_t : [0,1[\longrightarrow [0,1[$ is the translation by t modulo 1.

This article aims at highlighting that this one-parameter family of AIETs displays rich and various dynamical behaviours. The analysis developed in it, using tools borrowed from the theory of geometric structures on surfaces, leads to the following theorems.

Theorem 1. *For Lebesgue-almost all $t \in S^1$, F_t is dynamically trivial.*

We use the terminology of Liousse (see [Lio95]) who introduced it and proved that this dynamically trivial behaviour is topologically generic for transversely affine foliations on surfaces. As such, this theorem is somewhat a strengthening of Liousse's theorem for this 1-parameter family of AIETs. Indeed, we prove that this genericity is also of measure theoretical nature. We say that F_t is dynamically trivial if there exists two periodic points $x^+, x^- \in D$ of orders $p, q \in \mathbb{N}$ such that

- $(F_t^p)'(x^+) < 1$
- $(F_t^q)'(x^-) > 1$
- for all $z \in D$ which is not in the orbit of x^-, the ω-limit of z is equal to $\mathcal{O}(x^+)$ the orbit of x^+.

Roughly speaking, it means that the map F_t has two periodic orbits, one of which is attracting all the other orbits but the other periodic orbit which is repulsive.

A remarkable feature of this family is that there also are a lot of parameters for which the dynamics has an attracting *exceptional minimal set*. The existence of such a dynamical behavior for AIETs has been known since the work of Levitt [Lev87] (see also [CG97], [BHM10] and [MMY10] for further developments on this construction).

Theorem 2.

- *For all t in a Cantor set of parameters in S^1 there exists a Cantor set $\mathcal{C}_t \subset D$ for which for all $x \in D$, the F_t ω-limit of x is equal to $\mathcal{C}_t$.*
- *There are countably infinitely many parameters $t \in S^1$ for which F_t is completely periodic, that is there exists $p(t) \in \mathbb{N}^*$ such that $F_t^{p(t)} = \mathrm{Id}$.*

The remaining parameters form a Cantor set $\Lambda_\Gamma \subset S^1$ which is the limit set of a Zariski dense subgroup $\Gamma < \mathrm{SL}(2,\mathbb{R})$ of infinite covolume. For the parameters in Λ_Γ, we have,

Theorem 3. *There exists $\mathcal{P}$ and $\mathcal{M}$ subsets of Λ_Σ with dense complementary set, such that,*

- *if $\theta \in \mathcal{P}$ the foliation is decomposed into flat cylinder.*

- *if $\theta \in \mathcal{H}$ the foliation is minimal.*
- *otherwise, the foliation is uniquely ergodic.*

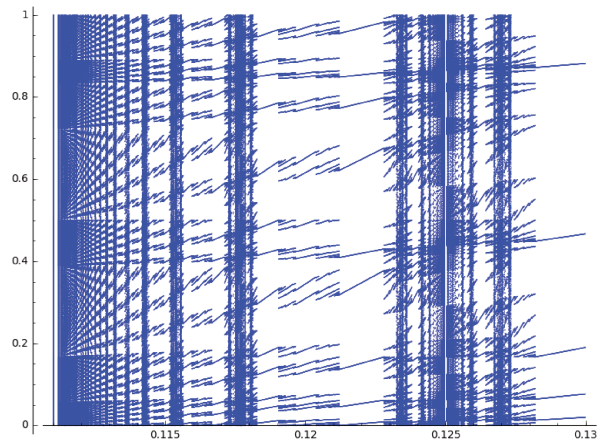


FIGURE 2. The ω-limit of a random point depending on F_t, for $0.11 \leq t \leq 0.13$.

The associated dilatation surface Σ. We associate to the family $(F_t)_{t \in S^1}$ the dilatation surface Σ obtained by the following gluing:

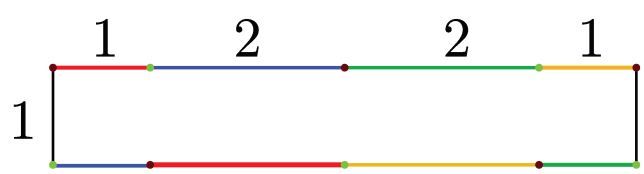


FIGURE 3. The surface Σ.

Those are linked by the fact that the directional foliation in direction θ admits F_t as their first return map on a cross-section, for $t = \frac{6}{\tan\theta}$. In particular they share the exact same dynamical properties and the study of the family F_t reduces to the study of the directional foliations of Σ.

The Veech group of Σ. The major outcome of this change of viewpoint is the appearance of hidden symmetries. Indeed, the surface Σ has a large group of affine symmetries which is called its Veech group and that we denote by $\mathbf{V}_\Sigma$. It is precisely the group of diffeomorphisms which writes in coordinates as an element of $\mathrm{SL}(2, \mathbb{R})$. As a consequence we get that directions θ which are $\mathrm{SL}(2, \mathbb{R})$-equivalent through $\mathbf{V}_\Sigma$ have same dynamical behaviour and this remark allow us to reduce considerably the number of parameters θ (equivalently t) that we need to analyse.

The group $\mathbf{V}_\Sigma$ seen as a subgroup of $\mathrm{SL}(2,\mathbb{R})$ is discrete and contains the following group, see Section 3

$$\Gamma = \langle \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rangle.$$

The projection of Γ on $\mathrm{PSL}(2,\mathbb{R})$ is a discrete group of infinite covolume. It is also a Schottky group of rank 2, and the analysis of its action on $\mathbb{RP}^1$ the set of directions of $\mathbb{R}^2$ leads to the following.

- There is a Cantor set $\Lambda_\Gamma \subset \mathbb{RP}^1$ of zero measure on which Γ acts minimally (Λ_Γ is the limit set of Γ). The directions $\theta \in \Gamma$ are *conjecturally* minimal.
- The action of Γ on $\Omega_\Gamma = \mathbb{RP}^1 \setminus \Lambda_\Gamma$ is properly discontinuous and the quotient is homeomorphic to a circle (Ω_Γ is the discontinuity set of Γ). It allows us to identify a narrow fundamental domain $I \subset \mathbb{RP}^1$ such that the understanding of the dynamics of foliations in directions $\theta \in I$ implies the understanding for the whole Ω_Γ (which is an open set of full measure).

Affine Rauzy-Veech induction. A careful look at the Figure 3 gives that the study of the dynamics of the directional foliations for $\theta \in I$ reduces to the understanding of the dynamics of contracting affine 2-intervals transformations. We use in Section 4 an adapted form of Rauzy-Veech induction to show that

- there is a Cantor set of zero measure of parameters $\theta \in \Omega_\Gamma$ for which the associated foliation accumulates on a (transversely) Cantor set;
- the rest of directions in Ω_Γ are dynamically trivial.

A remarkable corollary of the understanding of the dynamics of the directions in I is the complete description of $\mathbf{V}_\Sigma$:

Theorem 4. *The Veech group of Σ is exactly Γ.*

Acknowledgements. We are grateful to Bertrand Deroin for his encouragements and the interest he has shown in our work. We are also very thankful to Pascal Hubert for having kindly answered the very many questions we asked them, to Nicolas Tholozan for interesting discussions around the proof of Theorem 9 and to Matt Bainbridge for pointing out a mistake in Proposition 1.

2. Dilatation surfaces.

We introduce in this section the various objects and concepts which will play a role in this paper, and recall or prove few structural results about dilatation surfaces.

2.1. **Dilatation surfaces.**

Definition 1. A *dilatation surface* is a surface Σ together with a finite set $S \subset \Sigma$ and an atlas of charts $\mathcal{A} = (U_i, \varphi_i)_{i \in I}$ on $\Sigma \setminus S$ whose charts take values in $\mathbb{C}$ such that

- the transition maps are locally restriction of elements of $\mathrm{Aff}_{\mathbb{R}_+^*}(\mathbb{C}) = \{z \mapsto az + b \mid a \in \mathbb{R}_+^*, b \in \mathbb{C}\}$;
- each point of S has a punctured neighborhood which is affinely equivalent to a punctured neighborhood of the cone point of a euclidean cone of angle a multiple of 2π.

We call an element of the set S a *singularity* of the dilatation surface Σ.

This definition is rather formal, and the picture one has to have in mind is that a dilatation surface is what you get when you take a euclidean polygon, and glue pairs of parallel sides along the unique affine transformation that sends one on the other. An other interesting equivalent formulation of this definition is to say that a dilatation surface if the datum of

- a representation $\rho : \pi_1\Sigma \longrightarrow \mathrm{Aff}_{\mathbb{R}_+^*}(\mathbb{C}) = \{z \mapsto az + b \mid a \in \mathbb{R}_+^*, b \in \mathbb{C}\}$;
- a non constant holomorphic map $\mathrm{dev} : \widetilde{\Sigma} \longrightarrow \mathbb{C}$ which is equivariant with respect to the above representation.

The map dev is called the *developing map* of the dilatation surface and the representation ρ its *holonomy*. One recovers the initial definition by restricting dev to sufficiently small open sets to get the charts of the dilatation structure, the equivariance of dev under the action of ρ ensuring that the transition maps belong to $\mathrm{Aff}_{\mathbb{R}_+^*}(\mathbb{C})$. The singular points are the projection of the critical points of dev. This formalism borrowed from the theory of (G, X)-structures is often convenient to overcome minor technical issues.

2.2. **Foliations and saddle connections.**

Together with a dilatation surface comes a natural family of foliations. Fix an angle $\theta \in S^1$ and consider the trivial foliation of $\mathbb{C}$ by straight lines directed by θ. This foliation being invariant by the action of $\mathrm{Aff}_{\mathbb{R}_+^*}(\mathbb{C})$, it is well defined on $\Sigma \setminus S$ and extends at points of S to a singular foliation on Σ such that its singular type at a point of S is saddle-like.

A *saddle connection* on Σ is a singular leave that goes from a singular point to another, that is which is finite in both directions. The set of saddle connections of a dilatation surface is countable hence so is the set of directions having saddle connections.

2.3. **Cylinders.** If $\theta \in]0, 2\pi[$, the quotient of the angular sector based at 0 in $\mathbb{C}$ by the action by multiplication of $\lambda > 1$ is called a *dilatation cylinder* of angle θ and of multiplier λ which we denote by $C_{\lambda,\theta}$. This definition extends to arbitrarily large values of θ by considering sufficiently large ramified cover of $\mathbb{C}$, see [DFG, Section 2.2] for a precise construction.

A dilatation cylinder is a dilatation surface whose boundary is totally geodesic, homeomorphic to a topological cylinder, with no singular points. For the sake of convenience, we will consider that euclidean cylinders, which are the dilatation sub-surfaces one gets by identifying two opposite sides of a rectangle are a degenerated case of dilatation cylinder. In the sequel when we will refer to a *cylinder* we will mean a dilatation or euclidean cylinder, unless explicitly mentioned.

We say a dilatation surface has a *cylinder in direction* θ if there exists a regular closed leave of the foliation in direction θ. The *cylinder in direction* θ is therefore the cylinder to which this closed leave belongs.

Proposition 1. *A closed dilatation surface of genus g has at most $3g-3$ cylinders in a given direction θ.*

Proof. It suffices to notice that two homotopic simple closed geodesics α and β on a surface Σ which are disjoint must bound a dilatation or a flat cylinder. Using induction on the complexity on a connected complex surface defined by $3g-3+k$, where g is the genus and k the number of boundary components, we see that there are at most $3g-3$ simple closed curves in a topological surface which are not homotopic with each other. Any such set of $3g-3$ curves is a maximal cut system which decomposes the surface in $2g-2$ pair of pants.

Assume that α and β are homotopic then they bound a cylinder C in Σ. This cylinder has totally geodesic boundary. A Euler characteristic argument implies that, since ∂C is totally geodesic, C has no singular point in its interior. Such a cylinder is characterised by the linear holonomy $\lambda \in \mathbb{R}_+^*$ of a generator of its fundamental group that is isomorphic to $\mathbb{Z}$: because of the local rigidity of dilatation structures, the local isomorphism with a cylinder $C_{\lambda,\infty}$ of holonomyλ (if $\lambda \neq 1$) extends to the whole C. This proves that C is affinely isomorphic to $C_{\lambda,\theta}$ for a certain θ or to a flat cylinder if $\lambda = 1$. Moreover, if α and β lie in the same directional foliation and if $\lambda \neq 1$, θ must be a multiple of π.
Thus we can associate a simple closed curve in distinct homotopy classes for each cylinder. □

2.4. **Affine interval exchange transformations.** An *affine interval exchange transformation* (or AIET) is a bijective piecewise continuous map $f : [0, 1] \longrightarrow [0, 1]$ with a finite number of discontinuity points, which is affine on each of its continuity intervals. A convenient way to represent an AIET is to draw on a first copy of $[0, 1]$ its continuity intervals and on a second copy their images.

2.5. **The Veech group of a dilatation surface.** We recall in this paragraph basic definitions. An appendix to this article (see Appendix A) gives further

details on its construction and on the Veech group of the Disco surface (to be formally defined in the next paragraph). Let Σ be a dilatation surface. We say $g \in \mathrm{Diff}^+(\Sigma)$ is an *affine automorphism* of Σ if in coordinates, g writes likes the action of an element of $\mathrm{GL}^+(2,\mathbb{R}) \ltimes \mathbb{R}^2$ with the standard identification $\mathbb{C} \simeq \mathbb{R}^2$. We denote by $\mathrm{Affine}(\Sigma)$ the subgroup of $\mathrm{Diff}^+(\Sigma)$ made of affine automorphisms. The linear part in coordinates of an element of $\mathrm{Affine}(\Sigma)$ is well defined up to multiplication by a constant $\lambda \in \mathbb{R}_+^*$, we have therefore a well defined morphism:

$$\rho : \mathrm{Affine}(\Sigma) \longrightarrow \mathrm{SL}(2,\mathbb{R})$$

which associate to an affine automorphism its normalised linear part. We call ρ the *Fuchsian representation*. Its image $\rho(\mathrm{Affine}(\Sigma))$ is called the *Veech group* of Σ and is denoted by $\mathbf{V}_\Sigma$.

2.6. **The 'Disco' surface.** The surface we are about to define will be the main object of interest of this text. It is the surface obtained after proceeding to the gluing below:

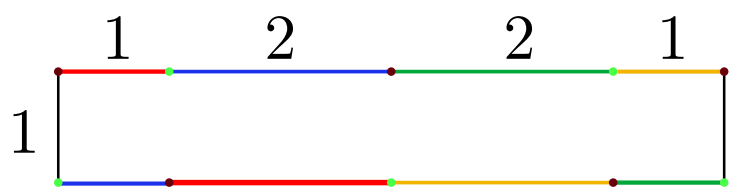


FIGURE 4. The surface Σ.

We call the resulting surface the 'Disco' surface. In the following Σ will denote this particular surface.

- This is a genus 2 dilatation surface, which has two singular points of angle 4π. They correspond to the vertices of the polygon drawn in Figure 4. Green ones project onto one singular point and brown ones project onto the other.
- Line segments joining the middle of the two blue (red, yellow and green respectively) sides project onto four closed simple curves that we call a_1, b_1, a_2 and b_2 respectively. If oriented 'from the bottom to the top', they form a symplectic basis of $\mathrm{H}_1(\Sigma,\mathbb{Z})$.
- With the orientation as above, we have that the linear holonomy of ρ of Σ satisfies $\rho(a_1) = \rho(b_2) = 2$ and $\rho(a_2) = \rho(b_1) = \frac{1}{2}$.

2.7. **About the Veech group of Σ.**

Flat cylinders. A *flat cylinder* is the dilatation surface you get when gluing two opposite sides of a rectangle. The *height* of the cylinder is the length of the sides glued and the*diameter* of the cylinder is the length of the non-glued sides, that is the boundary components of the resulting cylinder. Of course, only the ratio of

these two quantities is actually a well-defined invariant of the flat cylinder, seen as a dilatation surface. More precisely, we define

$$m = \frac{\text{diameter}}{\text{height}}$$

and call this quantity the *modulus* of the associated flat cylinder. The set of flat cylinders (seen as dilatation surfaces) is parametrised by the modulus.
If C is a cylinder of modulus m, there is an element of $f \in \text{Affine}(C)$ which is has the following properties:

- f is the identity on ∂C;
- f acts has a the unique Dehn twist of C;
- the matrix associated to f is $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$, if ∂C is assumed to be in the horizontal direction.

Decomposition in flat cylinders and parabolic elements of the Veech group. We say a dilatation surface Σ has a *decomposition in flat cylinders* in a given direction(say the horizontal one) if there exists a finite number of saddle connections in this direction whose complement in Σ is a union of flat cylinders. If additionally, the flat cylinders of have commensurable moduli, the Veech group of Σ contains the matrix

$$\begin{pmatrix} 1 & m' \\ 0 & 1 \end{pmatrix}$$

where m' is the smallest common multiple of all the moduli of the cylinders appearing in the cylinder decomposition. If the decomposition is in an another direction θ, the Veech group actually contains the conjugate of this matrix by a rotation of angle θ. Moreover, the affine diffeomorphism realising this matrix is a Dehn twist along the multicurve made of all the simple closed curves associated to each of the cylinders of the decomposition.

Calculation of elements of the Veech group of Σ. The above paragraph allows us to bring to light two parabolic elements in $\mathbf{V}_\Sigma$. Indeed, Σ has two cylinder decompositions in the horizontal and vertical direction.

- The decomposition in the horizontal direction has one cylinder of modulus 6, in green in the Figure 16 below.

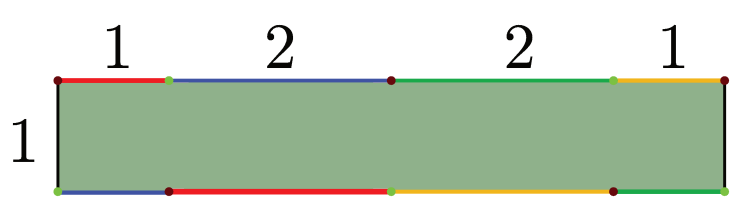


FIGURE 5. Cylinder decomposition in the horizontal direction.

Applying the discussion of the last paragraph gives, we get that the matrix $\begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$ belongs to $\mathbf{V}_\Sigma$.

- The decomposition in the vertical direction has two cylinders, both of modulus $\frac{3}{2}$, in repectively in blue and pink in the Figure 16 below.

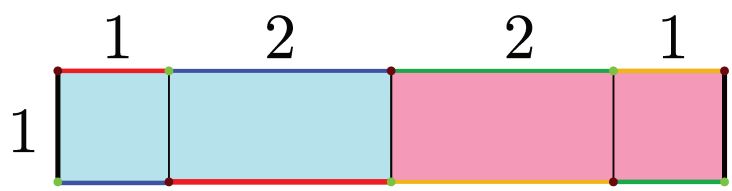


FIGURE 6. Cylinder decomposition in the vertical direction.

Again, we get that the matrix $\begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix}$ belongs to $\mathbf{V}_\Sigma$.

Finally, remark that both the polygon and the gluing pattern we used to build Σ is invariant by the rotation of angle π, which implies that the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is realised by an involution in Affine(Σ). Putting all the pieces together we get the

Proposition 2. *The group*

$$\left\langle \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$$

is a subgroup of $\mathbf{V}_\Sigma$.

2.8. **A word about the directional foliations of Σ.** We have seen in the introduction that we have built the surface Σ in order that its directional foliations capture the dynamical properties of the family of AIETs $\{\varphi_t \mid t \in [0,1]\}$. Precisely, we have that the foliation in direction $\theta \in]0, \frac{\pi}{2}[$ is the suspension of the AIET φ_t for

$$t = \frac{6}{\tan\theta}.$$

3. THE VEECH GROUP OF Σ.

3.1. **The Veech group acting on the set of directional foliations.** We have just seen in the previous section that a dilatation surface comes a family of oriented foliation, called **directional foliation** parametrised by the circle S^1 and denoted by $\mathcal{F}_\theta$. Any affine diffeomorphism of the surface acts on the set of directional foliations. It is locally described by an affine map of $\mathbb{R}^2$:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to A \begin{pmatrix} x \\ y \end{pmatrix} + B$$

where $A \in \mathrm{GL}_2(\mathbb{R})$ and B is a vector of $\mathbb{R}^2$. Such a map sends the θ-directional foliation on the $[A](\theta)$-directional foliation, where $[A]$ is the projective action of the matrix A on the circle S^1. Therefore (as a global object) an element of the Veech group also acts on the set of foliations $(\mathcal{F}_\theta)_{\theta \in S^1}$. Such an affine diffeomorphism conjugates one foliation and her foliations which are image of each other one by an element of the Veech group are $\mathcal{C}^\infty$-conjugate and therefore have the same dynamical behaviour. Recall that the goal of this article is to describe the dynamics of all the foliations $\mathcal{F}_\theta$ of the dilatation surface Σ. The remark above allows us to reduce the study to the quotient of S^1 by the action of Veech group on the set of foliations. We will find an interval (in projective coordinates) which is a fundamental domain for the action of the Veech group on its discontinuity set. The study of the dynamic for parameters θ in this interval will be performed in Section 4.

3.2. **The subgroup Γ.** We found in Section 2 three elements of the Veech group of the surface Σ:

$$\left\langle A = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} \quad , B = \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix}, -\mathrm{Id} \right\rangle \subset \mathbf{V}_\Sigma$$

The presence of the matrix $-\mathrm{Id}$ in $\mathbf{V}_\Sigma$ indicates that directional foliations on the surface Σ are invariant by reversing time. This motivates the study of the Veech group action on $\mathbb{RP}^1 := {}^{S^1}/_{-\mathrm{Id}}$ instead of S^1.

We will often identify $\mathbb{RP}^1$ to the the interval $[-\frac{\pi}{2}, \frac{\pi}{2})$ by using projective coordinates:

$$\begin{array}{ccc} \mathbb{RP}^1 & \to & [-\frac{\pi}{2}, \frac{\pi}{2}] \\ \left[\begin{pmatrix} x \\ y \end{pmatrix}\right] & \mapsto & \arctan\left(\frac{x}{y}\right) \end{array}$$

At the level of the Veech group it means to project it to $\mathrm{PSL}_2(\mathbb{R})$ by the canonical projection π. Let us denote by $\Gamma \subset \pi(\mathbf{V}_\Sigma) \subset \mathrm{PSL}_2(\mathbb{R})$ the group generated by the two elements:

$$\Gamma := \left\langle A = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} \quad , B = \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix} \right\rangle$$

We will study the group Γ as a Fuchsian group, that is a **discrete** group of isometries of the real hyperbolic plane $\mathbb{H}^2$. For the action of a Fuchsian group Φ on $\mathbb{RP}^1$, there are two invariant subsets which we will distinguish:

- one called its *limit set* on which Φ acts minimally and that we will denote by $\Lambda_\Phi \subset \mathbb{RP}^1$;
- the complement of Λ_Φ which is called its *discontinuity set*, on which Φ acts properly and discontinuously and which we will denote by Ω_Φ.

We will give precise definitions in Section 3.4. In restriction to the discontinuity set, one can form the quotient by the action of the group. The topological space Ω_Φ / Φ is a manifold of dimension one: a collection of real lines and circles.
We will show in Proposition 5 that for the group Γ this set is a single circle, and therefore a fundamental domain I for the action of the group Γ can be taken to be an single interval (we will make it explicit: $I = [\arctan(1), \arctan(4)] \subset [0, \pi[\simeq \mathbb{RP}^1$). The dynamic of the directional foliations in the directions θ belonging to the interior of the interval I will be studied in Section 4.

Remark 1. *We will prove in Section 5 that the group Γ is actually equal to the full Veech group of the surface Σ.*

3.3. **The action of the group Γ on $\mathbb{H}$.** Two hyperbolic isometries $A, B \in \mathrm{Isom}_+(\mathbb{H}^2)$ are said to be in **Schottky position** if the following condition holds:

There exists four disjoints domains D_i, $1 \leq i \leq 4$ which satisfy

$$A(D_1^c) = D_2 \qquad\qquad B(D_3^c) = D_4$$

where D_i^c denotes the complementary set of D_i.

A group generated by two elements in Schottky position is also called a **Schottky group**.

Proposition 3. *The group Γ is Schottky. Moreover the surface M_Γ is a three punctured sphere with two cusps and one end of infinite volume.*

Proof. Viewed in the upper half plane model of $\mathbb{H}^2$ the action is easily shown to be Schottky. In fact the action of A *(resp.* B*)* becomes $z \mapsto z + 6$ *(resp.* $z \mapsto \frac{z}{\frac{3z}{2}+1}$*)*. The two matrices are parabolic and fix ∞ and 0 respectively. Moreover, we observe that $A(-3) = 3$ and $B(-1) = 2$. Figure 7 below shows that the two matrices A and B are in Schottky position with associated domains D_i for $1 \leq i \leq 4$.

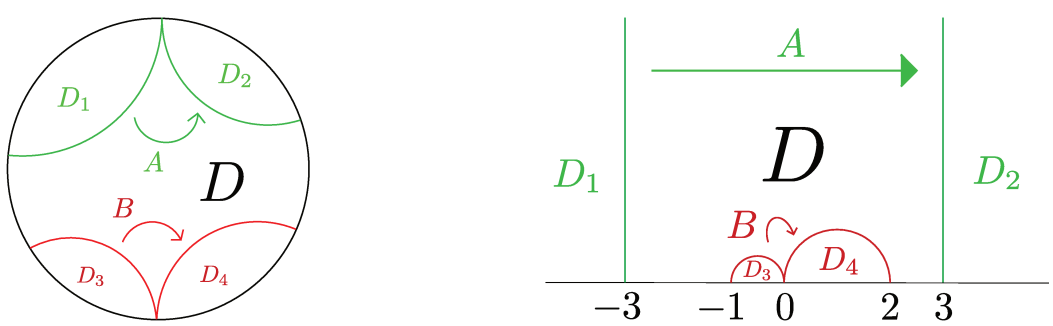


FIGURE 7. A fundamental domain for the action of the group Γ acting on the hyperbolic plane.

The domain D of Figure 7 is a fundamental domain for the action of Γ. The isometry A identifies the two green boundaries of D together and B the two red ones. The quotient surface is homeomorphic to a three punctured sphere. $\square$

3.4. **The limit set and the discontinuity set.** The following notion will play a key role in our analysis of the affine dynamics of the surface Σ:

Definition 2. The **limit set** $\Lambda_\Phi \subset S^1$ of a Fuchsian group Φ is the set of accumulation points in $\mathbb{D} \cup S^1$ of any orbit $\Phi \cdot \{z_0\}$, $z_0 \in \mathbb{D}^2$ where $\mathbb{D} \subset \mathbb{C}$ is the disk model for the hyperbolic plane $\mathbb{H}^2$.

The complementary set of the limit set is the good tool to understand the infinite volume part of such a surface.

Definition 3. The complementary set $\Omega_\Phi := S^1 \backslash \Lambda_\Phi$ is the **set of discontinuity** of the action of Γ on the circle.

The group Φ acts properly and discontinuously on the set of discontinuity one can thus form the quotient space Ω_Φ / Φ which is a manifold of dimension one : a collection of circles and real lines. These sets are very well understood for Schottky groups thanks to the ping-pong Lemma, for further details and developments see [Dal11] chapter 4.

Proposition 4 (ping-pong lemma)**.** *A Schottky group is freely generated by any two elements in Schottky position and its limit set is homeomorphic to a Cantor set.*

The idea of the proof is to code points in the limit set. As it is shown in the following Figure 8 a point in the limit set is uniquely determined by a sequence in the alphabet $\{A, B, A^{-1}, B^{-1}\}$ such that two consecutive elements are not inverse from the other. The first letter of the sequence will determines in which domain D_i the point is, the second one in which sub-domain and so on:
The intersection of all these domains will give the standard construction of a Cantor set by cutting some part of smaller intervals repetitively.
The following theorem will be used in 5 to prove our main Theorem 1

Theorem 5 (Ahlfors, [Ahl66])**.** *A finitely generated Fuchsian group satisfies the following alternative:*

(1) either its limit set is the full circle S^1 ;
(2) or its limit set is of zero Lebesgues measure

In our setting, it is clear that the limit set is not the full circle, thus the theorem implies that the limit set of Γ is of zero Lebesgue measure.

3.5. **The action on the discontinuity set and the fundamental interval.** The following proposition is the ultimate goal of this section.

Proposition 5. *The quotient space*

$$\Omega_\Gamma / \Gamma$$

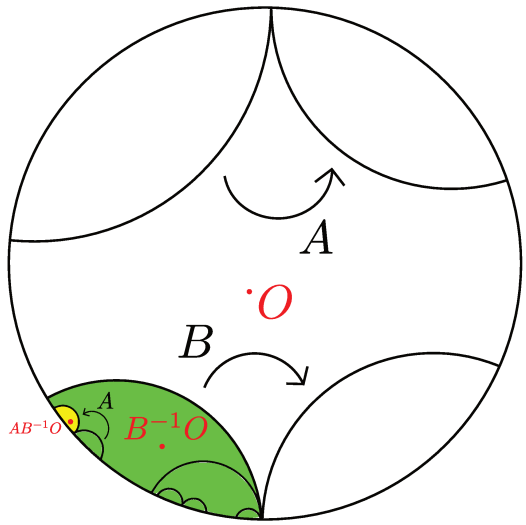


FIGURE 8. In this example the first latter is A and the point as to be on the boundary of the green part. The second one being B^{-1} it has also to lie in the yellow one too.

is a circle. A fundamental domain for the action of Γ *on* Ω_Γ *corresponds to the interval of slopes* $I = [\arctan\left(\frac{1}{4}\right), \arctan(1)]$

Foliations defined by slopes which belong to this precise interval will be studied in section 4. To prove this proposition we will use the associated hyperbolic surface M_Γ and link its geometrical and topological properties to the action of the group Γ on the circle.

The definition of the limit set itself implies that it is invariant by the Fuchsian group. One can therefore seek a geometric interpretation of such a set on the quotient surface. We will consider the smallest convex set (for the hyperbolic metric) which contains all the geodesics which start and end in the limit set Λ_Γ. We denote it by $C(\Lambda_\Gamma)$. Because the group Γ is a group of isometries it preserves $C(\Lambda_\Gamma)$.

Definition 4. The **convex hull** of a hyperbolic surface M_Γ, denoted by $C(M_\Gamma)$ is defined as:

$$C(M_\Gamma) := {}^{C(\Lambda_\Gamma)}/_{\Gamma}$$

This is a subset of the surface M_Γ as a quotient of a Γ-invariant subset of $\mathbb{H}^2$. The convex hull of a Fuchsian group is a surface with geodesic boundary, moreover if the group is finitely generated the convex hull has to be of finite volume. As a remark, a Fuchsian group is a lattice if and only if we have the equality $C(M_\Gamma) = M_\Gamma$. In the special case of the Schottky group Γ the convex hull is a surface whose boundary is a single closed geodesic as it is shown on

Figure 9. For a finitely generated group we will see that we have a one to one correspondence between connected components of the boundary of the convex hull and connected component of the quotient of the discontinuity set by the group.

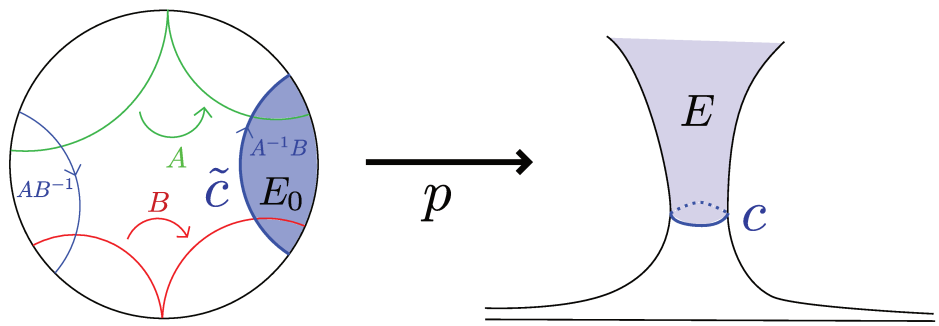


FIGURE 9. The closed geodesic c cuts the surface M_Γ into two pieces. The blue part is the only infinite volume end and its complementary is the convex hull. The choice of a lift $\tilde{c}$ of the geodesic c made on the picture allows us to describe the isometry which translate along $\tilde{c}$ in terms of the generators of the group Γ. Indeed the pairing of the edges of the fundamental domain given by the action of the group Γ shows that the geodesic $\tilde{c}$ is the translation axes of the matrix $A^{-1}B$.

The following lemma is the precise formulation of what we discussed above

Lemma 1. *Let Γ a finitely generated Fuchsian group. Any connected component I_0 of the discontinuity set Ω_Γ is stabilised by a cyclic group generated by an hyperbolic isometry γ_0. Moreover ∂I_0 is composed of the two fixed points of the isometry γ_0.*

We will keep the notations introduced with Figure 9. We start by showing that for any choice of a lift $\tilde{c}$ in the universal cover of a geodesic c in the boundary of the convex hull one can associate an isometry verifying the properties of Lemma 1. c a closed geodesic consisting of a connected component of the boundary of the convex hull of M_Γ. One can choose a lift $\tilde{c}$ of such a geodesic in the universal cover, the geodesic $\tilde{c}$ is the axe of some hyperbolic isometry Φ, whose fixed points are precisely the intersection of $\tilde{c}$ with the circle. As an element of the boundary of C_Γ it cuts the surface M_Γ into two pieces: C_Γ and an end E. One can check that this isometry Φ is exactly the stabiliser of the connected component E_0 of $p^{-1}(E)$ whose boundary is the geodesic $\tilde{c}$. Therefore such an isometry stabilises the connected component of the discontinuity set given by the endpoints of the geodesic $\tilde{c}$. We have shown that given a boundary component of C_Γ one can associate an element (in fact a conjugacy class) of the group Γ which stabilises a connected component of Ω_Γ. We will not show how to associate a geodesic in the boundary of the convex hull to a connected component of the discontinuity set.

However we want to put the emphasis on the fact the assumption that the group is finitely generated will be used here. The key point is the geometric finiteness theorem [Kat92, Theorem 4.6.1] which asserts that any finitely generated group is also **geometrically finite**. It means that the action of such a group admits a polygonal fundamental domain with finitely many edges. It is not difficult to exhibit from such a fundamental domain the desired geodesic by looking at the pairing induced by the group, as it is done in Figure 9 for our Schottky group.

Remark 2. *The proof of Lemma 1 also shows that any point of the boundary of* Ω_Γ *is exactly the set of fixed points of hyperbolic isometries in* Γ*.*

Corollary 1. *Connected components of* ${}^{\Omega_\Gamma}/_{\Gamma}$ *are in one to one correspondence with infinite volume end of the surface* ${}^{\mathbb{H}^2}/_{\Gamma}$*.*

We now have all the materials needed to prove the Proposition 5.

Proof of Proposition 5. Because the surface ${}^{\mathbb{H}^2}/_{\Gamma}$ has only one end of infinite volume the corollary 1 gives immediately that ${}^{\Omega_\Gamma}/_{\Gamma}$is a single circle. Proof of the second part of Lemma 1 consists in a simple matrix computation. Figure 9 gives explicitly the elements of the group Γ which stabilise a connected component of the discontinuity set. We then have to prove:

$$\left[AB^{-1}\begin{pmatrix}1\\1\end{pmatrix}\right] = \left[\begin{pmatrix}4\\1\end{pmatrix}\right]$$

where $[X]$ is the projective class of the vector X. The computation is easy:

$$\begin{aligned} AB^{-1}\begin{pmatrix}1\\1\end{pmatrix} &= \begin{pmatrix}-8 & 6\\ -\frac{3}{2} & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} \\ &= \begin{pmatrix}-2\\ -\frac{1}{2}\end{pmatrix} \end{aligned}$$

□

Remark 3.

(1) *The fact that the boundary of the discontinuity set could be identified with the set of fixed points of hyperbolic isometries is not general. For example the limit set of fundamental group of any hyperbolic compact surface is the full circle, but this group being countable there is only a countable set of fixed points.*
(2) *As a corollary we also have that hyperbolic fixed points are dense in the limit set.*

4. Generic directions and Rauzy induction.

This section aims at describing the dynamics of the Disco surface in an open interval of directions from which we will deduce in the next section its generic dynamics thanks to the study of the group Γ we performed in the previous section.

The boundary of $\mathbb{H}$ is canonically identified with $\mathbb{RP}^1$ through the natural embedding $\mathbb{H} \to \mathbb{CP}^1$. Recall that the action of $\mathrm{PGL}(2, \mathbb{C})$ by Möbius transformations on $\mathbb{C}$ is induced by matrix multiplication on $\mathbb{CP}^1$ after identification with $\mathbb{C}$ by the dilatation chart $z \to [z : 1]$. Thus the action of matrices of the Veech group on the set of directions corresponds to the action of these matrices as homographies on the boundary of $\mathbb{H}$.

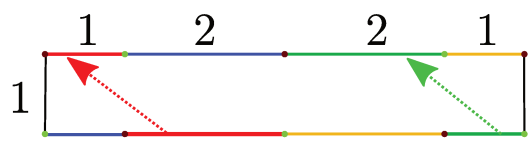


Figure 10. Attractive leaf in red, repulsive in green

We notice straightaway that for directions $[t : 1]$ with t between 1 and 2, there is an obvious attractive leaf of dilatation parameter $1/2$ (see Figure 10). It corresponds to the set of directions of a dilatation cylinder (see Section 2.3). There is also a repulsive closed leaf in this direction. This will always be the case since $-Id$ is in the Veech group, sending attractive closed leaves to repulsive closed leaves.

In the following we will describe dynamics of the directional foliation for t between 2 and 4. According to Section 2.7 the interval of direction $[1, 4]$ is a fundamental domain for the action of Γ on Ω_Γ its discontinuity set. Moreover this discontinuity set has full Lebesgue measure in the set of directions, thus understanding the dynamical behaviour of a typical direction therefore amounts to understanding it for $t \in [1, 4]$. Further discussion on what happens in other directions will be done in the next section.

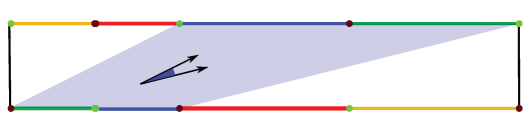


Figure 11. The stable sub-surface and its directions

4.1. **Reduction to an AIET.** The directions for $t \in [2, 4]$ have an appreciable property; they correspond to the directions of a *stable sub-surface* in the Disco

surface represented in Figure 11. Every leaf in the the given angular set of directions that enters the sub-surface will stay trapped in it thereafter. Moreover the foliation in it will have a hyperbolic dynamics. We therefore seek attractive closed leaves in this subset. To do so, take a horizontal interval overlapping exactly this *stable sub-surface* and consider the first return map on it. It has a specific form, which is close to an affine interval exchange, and which we will study in this section.

In the following, an affine interval exchange transformation (AIET) [1] is a piecewise affine function on an interval. For any $m, n \in \mathbb{N}$, let $\mathcal{I}(m,n)$ be the set of AIETs defined on $[0,1]$ with two intervals on which it is affine and such that the image of the left interval is an interval of its length divided by 2^n which rightmost point is 1, and that the image of its right interval is an interval of its length divided by 2^m which leftmost point is 0 (see Figure 12 for such an IET defined on $[0,1]$). When representing an AIET, we will color the intervals on which it is affine in different colors, and represent a second interval on which we color the image of each interval with the corresponding color; this will be sufficient to characterise the map. The geometric representation motivates the fact that we call the former and latter set of intervals the *top* and *bottom intervals*.

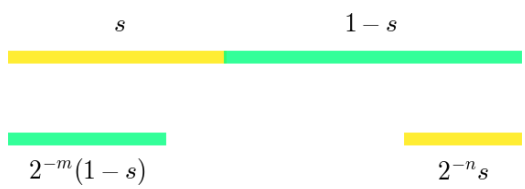


FIGURE 12. Geometric representation of an element of $\mathcal{I}(m,n)$

Remark that the above cross-sections are in $\mathcal{I}(1,1)$. We will study the dynamical behaviour of this family of AIET.

4.2. **Rauzy-Veech induction.** Let T be an AIET and D be its interval of definition. The first return map on a subinterval $D' \subset D$, $T' : D' \to D'$ is defined for every $x \in D'$, as

$$T'(x) = T^{n_0}(x) \text{ where } n_0 = \inf\{n \geq 1 | T^n(x) \in D'\}$$

Since we have no information on the recursivity of an AIET this first return map is a priori not defined on an arbitrary subinterval. Nonetheless generalizing a wonderful idea of Rauzy [Rau79] on IETs, we get a family of subinterval on which this first return map is well defined. Associating an AIET its first return map on this well-chosen smaller interval will be called the Rauzy-Veech induction.

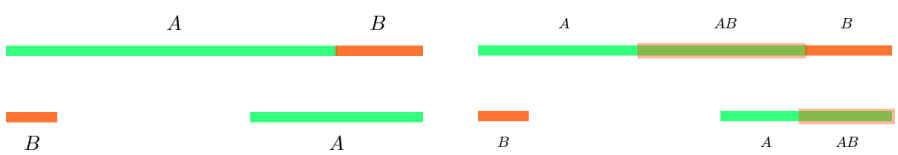
[1] In classical definitions of (affine) interval exchanges the maps are also assumed to be bijective, here they will be injective but not surjective.

The general idea in the choice of this interval is to consider the smallest of the two top and bottom intervals at one end of D (left or right) the interval of definition. We then consider the first return map on D minus this interval.

In the following we describe explicitly the induction for a simple family $\mathcal{I}(m,n)$. A general and rigorous definition of Rauzy-Veech induction here is certainly possible with a lot of interesting questions emerging but is beyond the scope of this article.

Assume now that T is an element of $\mathcal{I}(m,n)$, let $A, B \subset D$ be the left and right top intervals of T, and λ_A, λ_B their length. Several distinct cases can happen,

(1) (a) $B \subset T(A)$ *i.e.* $\lambda_B \leq 2^{-n}\lambda_A$.

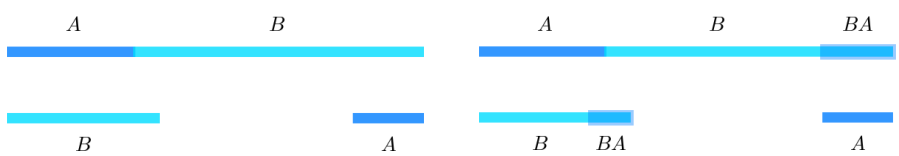


(A) Example of such AIET (B) Right Rauzy-Veech induction

We consider the first return map on $D' = D - B$. $T^{-1}(B)$ of length $2^n\lambda_B$ has no direct image by T in D' but $T(B) \subset D'$. Thus for the first return map, this interval will be sent directly to $T(B)$ dividing its length by 2^{n+m}. We call this a right Rauzy-Veech induction of our AIET. The new AIET is in $\mathcal{I}(m+n,n)$, and its length vector (λ'_A, λ'_B) satisfies

$$\begin{pmatrix} \lambda'_A \\ \lambda'_B \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & -2^n \\ 0 & 2^n \end{pmatrix}}^{R_{m,n}} \begin{pmatrix} \lambda_A \\ \lambda_B \end{pmatrix}$$

(b) If $A \subset T(B)$ *i.e.* $2^{-m}\lambda_B \geq \lambda_A$.



(A) AIET satisfying $A \subset T(B)$ (B) Right Rauzy-Veech induction

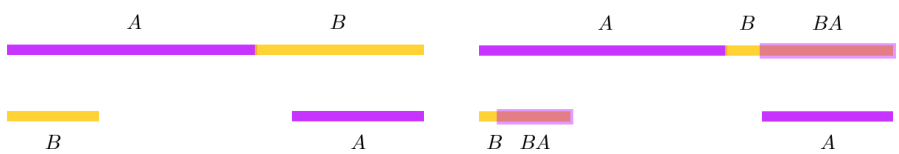
In this case, the right Rauzy-Veech induction will not yield an AIET of the form of the $\mathcal{I}(m,n)$ family therefore we consider the first return map on $D' = D - A$ which we call the left Rauzy-Veech induction

of our AIET. We obtain a new AIET in $\mathcal{I}(m, n+m)$ and its length vector (λ'_A, λ'_B) satisfies,

$$\begin{pmatrix} \lambda'_A \\ \lambda'_B \end{pmatrix} = \overbrace{\begin{pmatrix} 2^m & 0 \\ -2^m & 1 \end{pmatrix}}^{L_{m,n}} \begin{pmatrix} \lambda_A \\ \lambda_B \end{pmatrix}$$

(2) $T(A) \subset B$ *i.e.* $\lambda_B \geq 2^{-n}\lambda_A$ and $T(B) \subset A$ *i.e.* $2^{-m}\lambda_B \leq \lambda_A$.

We consider the first return map on the subinterval $D' = D - T(A)$. Then A has no direct image by T in D' but $T^2(A) \subset T(B) \subset D'$. Thus in the first return map, this interval will be sent directly to $T^2(A)$ dividing its length by 2^{n+m}.



(A) AIET satisfying $T(B) \subset A$ (B) Right Rauzy-Veech induction

Then $T^2(A) \subset A$ thus it has an attractive fixed point of derivative 2^{-n-m}.

Remark 4. *The set of length for which we apply left or right Rauzy-Veech induction in the above trichotomy is exactly the set on which lengths λ'_A and λ'_B implied by the above formulas are both positive.*

More precisely, $0 \leq \lambda_B \leq 2^{-n}\lambda_A \iff R_{m,n} \cdot \begin{pmatrix} \lambda_A \\ \lambda_B \end{pmatrix} \geq 0,$

and $0 \leq 2^{-m}\lambda_B \leq \lambda_A \iff L_{m,n} \cdot \begin{pmatrix} \lambda_A \\ \lambda_B \end{pmatrix} \geq 0.$

This will be useful latter on, to describe the set of parameters which corresponds to the sequence of induction we apply.

The algorithm. We define in what follows an algorithm using the Rauzy induction that will allows us to determine if an element of $\mathcal{I}(1,1)$ has an attractive periodic orbit; and if so the length of its periodic orbit (or equivalently the dilatation coefficient of the associated leaf in Σ).

The algorithm goes the following way:

The entry is an element of $\mathcal{I}(m, n)$,

(1) If the entry is in case (1), perform in case (a) the right Rauzy induction R or in case (b) the left Rauzy induction L to obtain an element of $\mathcal{I}(m+n,n)$ or $\mathcal{I}(m,n+m)$ respectively. Repeat the loop with this new element.

(2) If it is in case (2), it means that the first return map on a well chosen interval has a periodic attractive point of derivative $2^{-m_l-n_l}$. The algorithm stops.

Alongside the procedure comes a sequence of symbols R and L keeping track of whether we have performed the Rauzy induction on the left or on the right at the nth stage. This sequence is finite if and only if the algorithm describe above finishes. An interesting phenomenon will happen for AIET for which the induction never stops, and will be described latter.

4.3. **Directions with attractive closed leaf.** In the directions of Figure 11, we consider the first return map of the directional foliation on the interval given by the two length 1 horizontal interval at the bottom of the rectangle. We have chosen directions such that the first return map is well defined although it is not bijective, and it belongs to $\mathcal{I}(1,1)$. The ratio of the two top intervals' length will vary smoothly between 0 and ∞ depending on the direction we choose. We parametrise this family of AIET by $s \in I := [0,1]$, where $(s, 1-s)$ is the length vector of the element of $\mathcal{I}(1,1)$ we get. Our purpose here is to characterise the subspace $H \subset I$ for which the above algorithm stops, in particular they correspond to AIET with a periodic orbit. The case of $I - H$ will be settled in the next subsection.

We describe for any finite word in the alphabet $\{L, R\}$, $w = w_1 \dots w_{l-1}$, the subset of parameters $H(w) \subset H \subset I$ for which the algorithm stops after the sequence w of Rauzy-Veech inductions.

We associate to w the sequences $n_1 = 1, \dots, n_l$, $m_1 = 1, \dots, m_l$ and $M_1 = Id, \dots, M_l$ defined by the recursive properties,

$$m_{i+1} = \begin{cases} m_i & \text{if } w_i = L \\ n_i + m_i & \text{if } w_i = R \end{cases}, \quad n_{i+1} = \begin{cases} n_i + m_i & \text{if } w_i = L \\ n_i & \text{if } w_i = R \end{cases},$$

$$M_{i+1} = \begin{cases} L_{m_i,n_i} \cdot M_i & \text{if } w_i = L \\ R_{m_i,n_i} \cdot M_i & \text{if } w_i = R \end{cases}$$

Let $s \in I$ such that we can apply Rauzy-Veech inductions corresponding to w to the element of $\mathcal{I}(1,1)$ of lengths $(s, 1-s)$. The induced AIET after all the steps of the induction is in $\mathcal{I}(m_l, n_l)$ and its length vector is

$$M_l \cdot \begin{pmatrix} s \\ 1-s \end{pmatrix} := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} s \\ 1-s \end{pmatrix} = \begin{pmatrix} (a-b)s + b \\ (c-d)s + d \end{pmatrix}$$

Following Remark 4, the property of s that we can apply all the Rauzy-Veech inductions corresponding to w to the initial AIET in $\mathcal{I}(1,1)$ is equivalent to $(a-b)s+b \geq 0$ and $(c-d)s+d \geq 0$. A simple recurrence on M_i shows that it is an integer matrix with $a, d \geq 0$, $b, c \leq 0$, hence $s \in \left[\frac{-b}{a-b}, \frac{d}{d-c}\right] =: I(w)$. $H(w)$ will be the central subinterval of $I(w)$ for which the induced AIET in $\mathcal{I}(m_l, n_l)$ is in case (2).

If we consider the sets

$$H_k := \bigcup_{|w| \leq k} H(w) \quad \text{and} \quad H = \bigcup_k H_k$$

we remark that H has the same construction as the complement of the Cantor triadic set; each H_k is constructed from H_{k-1} by adding a central part of each interval which is a connected component of $I - \bigcup_{j<k} H_j$.

The rest of the subsection aims now at proving the following lemma,

Lemma 2. *$H \subset I$ has full Lebesgue measure.*

Proof. We will prove in the following that for any non-empty word w,

$$\frac{|H(w)|}{|I(w)|} \geq \delta \tag{1}$$

for some $\delta > 0$. Thus at each step k, H_k is at least a δ-proportion larger in Lebesgue measure than H_{k-1}. This implies that the Lebesgue measure

$$\lambda(H) \geq \lambda(H_k) \geq 1 - \delta^k \text{ for any } k$$

□

We now show Inequality 1. Let w be any finite word in the alphabet $\{L, R\}$. For convenience we normalise the interval $I(w)$ for such that it is $[0,1]$. We denote by $(\lambda_A(s), \lambda_B(s))$ the length vector of the AIET induced by the sequence w of Rauzy-Veech inductions. These two lengths are linear functions of s, λ_A is zero at the left end of the interval and λ_B is zero at the right end. As a consequence, these two functions have the form $\lambda_A(s) = \alpha s$ and $\lambda_B(s) = \beta(1-s)$ for $s \in [0,1]$, where α and β are the maximal values of λ_A and λ_B respectively equal to according to the previous computations

$$\alpha = (a-b)\frac{d}{d-c} + b = \frac{ad - bd + bd - bc}{d-c} = \frac{det(M_l)}{d-c}$$

and

$$\beta = (c-d)\frac{-b}{a-b} + d = \frac{-bc + bd + da - db}{a-b} = \frac{det(M_l)}{a-b}$$

We see that $\lambda_A(s) = 2^{-m}\lambda_B(s) \iff 2^m\alpha s = \beta(1-s) \iff s = \frac{\beta}{2^m\alpha+\beta}$ and similarly $\lambda_B(s) = 2^{-n}\lambda_A(s) \iff 2^n\beta(1-s) = \alpha s \iff s = \frac{2^n\beta}{\alpha+2^n\beta}$. Hence

$$\lambda_A(s) \leq 2^{-m}\lambda_B(s) \iff s \in \left[0, \frac{\beta}{2^m\alpha+\beta}\right]$$

and

$$\lambda_B(s) \leq 2^{-n}\lambda_A(s) \iff s \in \left[\frac{\beta}{2^{-n}\alpha+\beta}, 1\right]$$

thus

$$H(w) = \left[\frac{\beta}{2^m\alpha+\beta}, \frac{\beta}{2^{-n}\alpha+\beta}\right].$$

If we denote by $x = \frac{\alpha}{\beta} = \frac{a-b}{d-c}$,

$$\frac{|H(w)|}{|I(w)|} = \frac{1}{1+2^{-n}x} - \frac{1}{1+2^m x}$$

We prove the following technical lemma

Lemma 3. *For any word w in $\{R, L\}$, if $M(w) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have,*

$$2^{-1} \leq \frac{a-b}{d-c} \leq 2$$

This implies directly that

$$\frac{|H(w)|}{|I(w)|} \geq \frac{1}{1+2^{-n+1}} - \frac{1}{1+2^{m-1}}$$

Hence for w not empty, either $n \geq 2$ or $m \geq 2$ thus either

$$\frac{|H(w)|}{|I(w)|} \geq \frac{2}{3} - \frac{1}{2} = \frac{1}{6} \text{ or } \frac{|H(w)|}{|I(w)|} \geq \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Proof. The proof goes by induction on the length of w. Let us assume that $2^{-1} = \frac{a-b}{d-c} = 2$ for some w. We denote by

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = R_{m,n} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-2^n c & b-2^n d \\ 2^n c & 2^n d \end{pmatrix}$$

Thus $\frac{a'-b'}{d'-c'} = 2^{-n}\frac{a-b}{d-c} + 1$ from which the inequality follows.

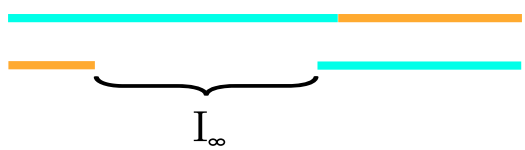
$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = L_{m,n} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2^m a & 2^m b \\ c-2^m a & d-2^m b \end{pmatrix}$$

The inequality is similar to the previous one. □

4.4. **AIET with infinite Rauzy-Veech induction.** We focus in this subsection on what happens for AIETs on which we apply Rauzy-Veech induction infinitely many times. First, remark that if we apply the induction on the same side infinitely many times, the length of the top interval of the corresponding side on the induced AIET is multiplied each time by a positive power of 2, therefore it goes to infinity. Yet the total length of the subinterval is bounded by 1 the length of the definition interval from which we started the induction. Thus the length of the interval has to be zero; this corresponds to the case where there is a saddle connection and it is included in the closed orbit case, since we chose to take $H(w)$ closed.

In consequence, for an IET T with parameter in $I - H$, we apply Rauzy-Veech induction infinitely many times, and the sequence of inductions we apply is not constant after a finite number of steps. Now let as above D be the interval of definition of the given AIET, and $A, B \subset D$ be its two consecutive domain of continuity. Remark that the induction keeps the right end of $T(B)$ and the left end of $T(A)$ unchanged. Moreover the induction divides the length of one of the bottom interval (depending on which Rauzy-Veech induction we apply) by at least two. As a consequence, if we consider I_n to be the open subinterval of D on which we consider the first return map after the n-th induction, the limit of these nested intervals is

$$I_\infty := \bigcap_{n=1}^{\infty} I_n = D - \{T(A) \cup T(B)\}$$



By definition, this interval is disjoint from $T(D)$, and therefore

$$\forall x \in D \text{ and } n \in \mathbb{N},\ T^n(x) \notin I_\infty$$

Moreover, our definition of Rauzy-Veech induction implies that any point outside of the subinterval on which we consider the first return will end up in this subinterval in finite time. Thus

$$\forall x \in D \text{ and } n \in \mathbb{N}, \exists k \in \mathbb{N} \text{ such that } T^k(x) \in I_n$$

This implies that the orbit of any point of D accumulates on ∂I_∞.

Let us introduce the complementary of all the images of I_∞,

$$\Omega := D - \bigcup_{n=0}^{\infty} T^n I_\infty$$

The measure of I_∞ is $1/2$, taking the image by T divides the measure of any interval by two and any iterated image of this set is disjoint, since its image is disjoint

from itself and T is injective. Hence the measure of Ω is $1/2\cdot(1+1/2+1/2^2+\dots) = 1$. As we remarked, the orbit of any point of D accumulates to ∂I_∞ and thus to any image of it, hence to any point of $\partial\Omega$. As $\bigcup_{n=0}^{\infty} T^n I_\infty$ has full measure, Ω has zero Lebesgue measure and thus has empty interior. To conclude, Ω is the limit set of any orbit of T.

Now Ω is closed set with empty interior. Moreover if we take a point in Ω, any neighborhood contains an interval and thus its boundary. Hence no point is isolated, and Ω is a Cantor set. Which leads to the following proposition,

Proposition 6. *In the space of directions* $[1,4]$*, there is a set* H*, which complementary set is a Cantor set of zero measure which satisfies,*

- $\forall\theta \in \overline{H}$ *the foliation* $\mathcal{F}_\theta$ *concentrates to an attractive leaf.*
- $\forall\theta \in \partial H$ *the foliation* $\mathcal{F}_\theta$ *concentrates to a closed saddle connection.*
- $\forall\theta \in [1,4]\setminus\overline{H}$ *the foliation* $\mathcal{F}_\theta$ *concentrates on a stable Cantor set of zero measure in the foliation.*

5. The global picture.

Gathering all materials developed in the previous sections, we prove here the main theorems announced in the introduction.

Proposition 7. *Assume that the foliation* $\mathcal{F}_\theta$ *of* Σ *has a closed attracting leaf* F^+*. Then it has a unique repulsing leaf* F^- *and any leaf which is different from* F^- *and regular accumulates on* F^+*.*

This proposition ensures that in all the cases where we have already found an attracting leaf, the dynamics of the foliation is as simple as can be.

Proof. The image of F^+ by the action of the diffeomorphism of Affine(Σ) whose image by the Fuchsian representation is $-\mathrm{Id}$ is the repulsive leaf announced which we denote by $\mathcal{F}^-$. Let L be a leaf of the **oriented** foliation that accumulates to F^+. We consider the very same leaf but with the reversed orientation. We also assume that L was not issued from a singular point, in which case we consider its ω-limit in Σ which we denote by Q. We want to prove that $Q = F^-$. Assume by contradiction that $Q \neq F^-$.

- Either Q is an attracting closed leaf. In that case the image of Q by the involution of the Veech group is another closed leaf and we get the existence of four different closed attracting leaves. Since the surface Σ does not have cylinders of angle $\geq \pi$ (because Σ is trianglulable, see [DFG]), each of these closed leaves belongs to a different cylinder. This contradicts Proposition 1.
- Or Q is another type of quasiminimal. In that case, any subsurface of Σ containing Q is of genus at least one (see [Lev87, p.91]). On the other hand, Q is disjoint from F^+ and F^- which contradicts the fact that Σ has genus two.

□

We say that a direction having such a dynamical behavior is *dynamically trivial*.

Theorem 6. *The set of dynamically trivial directions in S^1 is open and has full measure.*

Proof. Since $-Id$ belongs to the Veech group of Σ, the foliations $\mathcal{F}_\theta$ and $\mathcal{F}_{-\theta}$ have the same dynamical behavior. We will therefore consider parameters θ in $\mathbb{RP}^1$ instead of in S^1. We denote then by $\mathcal{T} \subset \mathbb{RP}^1$ the set of dynamically trivial directions in $\mathcal{T}$. We have proved in Section 4 that the trace of $\mathcal{T}$ on $J = \{[1 : t] \mid t \in [1, 4]\} \subset \mathbb{RP}^1$ is the complement of a Cantor set and that $\mathcal{T} \cap J$ has full measure.

Also J is a fundamental domain for the action of Γ on Ω_Γ the discontinuity set of Γ. Since $\Gamma < \mathbf{V}_\Sigma$, two directions in $\mathbb{RP}^1$ in the same orbit for the action of θ induce conjugated foliations on Σ and therefore have same dynamical behavior. This implies that $\mathcal{T} \cap \Omega_\Gamma$ is open and has full measure in Ω_Γ. Since Ω_Γ has itself full measure in $\mathbb{RP}^1$, $\mathcal{T}$ has full measure in $\mathbb{RP}^1$. Its openness is a consequence of the stability of dynamically trivial foliations for the $\mathcal{C}^\infty$ topology, see [Lio95] for instance.

□

Relying on a similar argument exploiting in a straightforward manner the action of the Veech group and the depiction of the dynamics made in Section 4, we get the

Theorem 7. *There exists a Cantor set $\mathcal{K} \subset S^1$ such that for all $\theta \in \mathcal{K}$, the foliation $\mathcal{F}_\theta$ accumulates to a set which is transversely a Cantor set of zero Haussdorf measure.*

It remains to describe what kind of dynamics could appears for the direction in the limit set, which is already known to be of null Lebesgue measure.

Theorem 8. *For all $\theta \in \Lambda_\Gamma$ the foliation is not dynamically trivial.*

Proof. The Theorem of Thurston, see the appendix **??**, asserts that an element of the Veech group is pseudo-Anosov (as an element of the mapping class group) if and only if the matrix associated by the Fuchsian representation is hyperbolic. From this, one deduces that the dynamic cannot be trivial along a direction given by an eigenvector of the hyperbolic matrix associated to that pseudo-Anosov. In fact, a foliation along such a direction will be left globally invariant by the pseudo-Anosov diffeomorphism. In particular, it must permute the hyperbolic leaves which is not possible since a pseudo-Anosov cannot fix any finite set of simple curves. We conclude by recalling the Remark 3 which asserts that the limit set of an non-elementary Fuchsian group is the closure of the fixed point of its hyperbolic elements. The set of dynamically non-trivial directions being closed—as the complementary of the the trivial ones, which is open by affine stability—the result follows. □

We believe it is worth pointing out that the method we used to find these *'Cantor like'* directions is new compared to the one used in [CG97], [BHM10] and [MMY10]. Also, our Cantor sets have zero Haussdorf dimension. It would be interesting to compare with the aforementioned examples.

Theorem 9. *The Veech group of* Σ *is exactly* Γ.

Proof. We cut the proof into four steps:

(1) proving that any element in $\mathbf{V}_\Sigma$ preserves Λ_Γ;
(2) proving that Γ has finite index in $\mathbf{V}_\Sigma$;
(3) proving that the group Γ is normal in $\mathbf{V}_\Sigma$;
(4) concluding.

(1) Let us prove the first point. Because of the description of the dynamics of the directional foliations we have achieved, one can show the limit set of the Veech group is the same than the limit set of Γ. If not, there must be a point of $\Lambda_{\mathbf{V}_\Sigma}$ in the fundamental interval I. But since the group $\mathbf{V}_\Sigma$ is non elementary it implies that we have to find in I infinitely many copies of a fundamental domain for the action of Γ on the discontinuity set. In particular infinitely many disjoint intervals corresponding to directions where the foliation has an attracting leave of dilatation parameter 2. But by the study performed in the above section the only sub-interval of I having this property is $]1,2[$.

(2) The second point follows from that the projection

$$\Gamma\backslash\mathbb{H} \longrightarrow \mathbf{V}_\Sigma\backslash\mathbb{H}$$

induces an isometric orbifold covering

$$C(\Gamma\backslash\mathbb{H}) \longrightarrow C(\mathbf{V}_\Sigma\backslash\mathbb{H}).$$

Since $C(\Gamma\backslash\mathbb{H})$ has finite volume and because $[\Gamma : \mathbf{V}_\Sigma] = \frac{\mathrm{vol}(C(\Gamma\backslash\mathbb{H}))}{\mathrm{vol}(C(\mathbf{V}_\Sigma\backslash\mathbb{H}))}$, this ratio must be finite and hence Γ has finite index in $\mathbf{V}_\Sigma$.

(3) Remark that Γ is generated by two parabolic elements A and B and that these define the only two conjugacy class in Γ of parabolic elements. We are going to prove that any element $g \in \mathbf{V}_\Sigma$ normalise both A and B. Since Γ has finite index in $\mathbf{V}_\Sigma$, there exists $n \geq 1$ such that $(gAg^{-1})^n \in \Gamma$. There are but two classes of conjugacy of parabolic elements in Γ which are the ones of A and B. If $n \geq 2$, this implies that $\mathbf{V}_\Sigma$ contains a strict divisor of A, which would make the limit set of $\mathbf{V}_\Sigma$ larger that Λ_Γ (consider the eigenvalues of the matrix AB^{-1} which determine points in the boundary on the limit set, see Lemma 1). Therefore gAg^{-1} belongs to Γ. A similar argument shows that $gBg^{-1} \in \Gamma$ and since A and B generate Γ, g normalises Γ. Hence Γ is normal in $\mathbf{V}_\Sigma$.

(4) Any $g \in \mathbf{V}_\Sigma$ thus acts on the convex hull $C(\Gamma\backslash\mathbb{H})$ of the surface $C(\Gamma\backslash\mathbb{H})$. In particular it has to preserve the boundary of $C(\Gamma\backslash\mathbb{H})$, which is a single geodesic by Proposition 5. At the universal cover it means that g has to fix a lift of the

geodesic c, thus the isometry g permutes two fixed points of a hyperbolic element h of Γ. Two situations can occur:

- g is an elliptic element. His action on $\Gamma\backslash\mathbb{H}$ cannot permute the two cusps because they correspond to two essentially different cylinder decompositions on Σ. It therefore fixes the two cusps and hence must be trivial.
- g is hyperbolic and fixes the two fixed points of h. Moreover, by Lemma 1, it acts on the fundamental interval I and as we discussed above such an action has to be trivial because of our study of the associated directional foliations, the translation length of g is then the same than h. But g is fully determined by its fixed points and its translation length, which shows that $g = h$ and thus $g \in \Gamma$.

Any element of $\mathbf{V}_\Sigma$ therefore belongs to Γ and the theorem is proved.

□

6. Minimality on the limit set

The disco surface has a remarkable property: it can be decomposed in cylinders in the horizontal and vertical directions. Closed translation surfaces with such properties has been widely considered by Thurston in [Thu88] and Veech in [Vee89] to construct a large family of examples of surfaces on which there is a pseudo-Anosov map. More recently McMullen in [McM06] introduced simplifying ideas on this type of surfaces to deal with Prym loci and Hooper in [Hoo15] generalized them to infinite translation surfaces. We use techniques of the latter to show that foliations on the considered dilatation surface in directions Λ_Γ are conjugated to foliations on a translation surface on which it is clearly minimal.

To perform these constructions, we need the horizontal and vertical cylinders to have same moduli, thus we consider a slightly modified disco surface where the vertical side has length 2. Its cylinders all have modulus 3 and its dynamics is conjugated to the one of the previous disco surface.

6.1. Curve systems.

Following [McM06], we first recall Thurston's construction of a translation surface such that its Veech group contains a parabolic element.

Let Σ be a closed surface. A pair of multicurves α, β *bind* a surface Σ it they meet only in transverse double points, and every component of $\Sigma - \alpha \cup \beta$ is a union of polygons with at least 4 sides (running alternatively along α and β).

Indexing the components of $\alpha = \bigcup_{i=1}^{a} \gamma_i$ and $\beta = \bigcup_{i=a+1}^{a+b} \gamma_i$ we consider the symmetric matrix

$$C_{i,j} = |i(\gamma_i, \gamma_j)|$$

We assign a positive weight w_i to each curve γ_i, and let $\mu > 0$ be the largest eigenvalue of the weighted matrix, i.e. its the spectral radius. According to the Perron-Froebenius theorem there is a positive eigenvector for μ, which we denote by h.

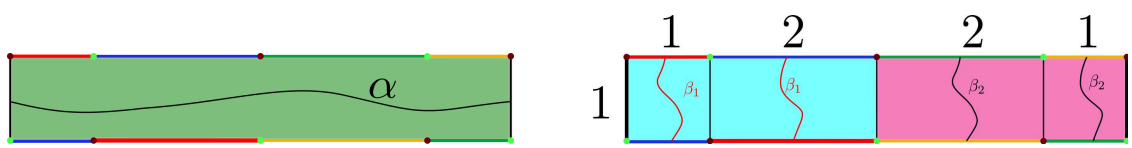
For each intersection point p of γ_i and γ_j with $i < j$, take one rectangle

$$R_p = [0, h_i] \times [0, h_j] \subset \mathbb{C}$$

and glue R_p to R_q whenever p and q are joined by an edge of $\alpha \cup \beta$.

This results to a unique (up to a positive factor) half-translation surface which fully decomposes into horizontal and vertical cylinders X_i of height h_i and of modulus w_i/μ.

In the case of the disco surface, we introduce the following multicurves,



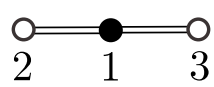
(A) The curve α. (B) The multi-curve $\beta = \beta_1 \cup \beta_2$.

FIGURE 16. Definition of α and β

One checks that these multicurves bind the surface and we have,

$$C = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

This is the incidence matrix for the graph,



Let us construct a translation surface with a decomposition into horizontal and vertical cylinders of same moduli. Consider $w = (1, 1, 1)$. Let μ be the maximal eigenvalue of the matrix $(C_{i,j})$, and h be its unique positive eigenvector (up to a positive scalar). Then,

$$\begin{aligned} \mu h_1 &= 2(h_2 + h_3) \\ \mu h_2 &= 2h_1 \\ \mu h_3 &= 2h_1 \end{aligned}$$

This implies

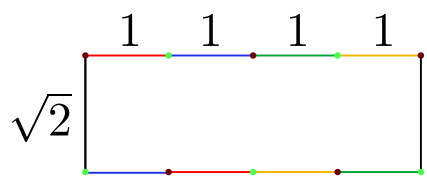
$$\mu h_1 = \frac{8}{\mu} h_1$$

and

$$\mu = 2\sqrt{2}$$

Hence the vector $(\sqrt{2}, 1, 1)$ is an eigenvector for the eigenvalue $2\sqrt{2}$.

This yields the surface Σ_t,



one which the multi-twists τ_α, τ_β are represented by affine automorphisms satisfying,

$$D_{\tau_\alpha} = \begin{pmatrix} 1 & 2\sqrt{2} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad D_{\tau_\beta} = \begin{pmatrix} 1 & 0 \\ 2\sqrt{2} & 1 \end{pmatrix}$$

6.2. **Infinite translation surfaces.** The article [Hoo15] provides with a generalized construction for infinite translations surfaces.

For a given constant $\lambda \geq 2$, we define a representation $\rho_\lambda : G \mapsto \mathrm{SL}(2, \mathbb{R})(2, \mathbb{R})$ where $G = \langle h, v \rangle$ is the free group on two generators, such that

$$\rho_\lambda(h) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho_\lambda(v) = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$$

We say a direction θ is λ-*renormalizable* if the following two statements are satisfied:

- θ lies in the limit set of $\rho_\lambda(G)$.
- θ is not an eigendirection of any matrix $\rho_\lambda(g)$ where g is conjugate in G to an element in the set $\{h, v, v^{-1}h\}$.

If Σ is the disco surface, we can consider $\hat{\Sigma}$ its translation cover which is an infinite translation surfaces which is still decomposed into horizontal and vertical flat cylinders. We represent it in the following Figure 17, where the copy in front of i should thought rescaled by 2^i. We also represent the bipartite graph associated to the combinatorics of its horizontal-vetical cylinder decompositions as in [Hoo15].

Now we check that the infinite translation surface $\widehat{\Sigma}$ is exactly the surface associated to the eigenfunction of eigenvalue 3 defined in Figure 18a. Moreover, if we consider $\widehat{\Sigma}_t$ the pull back of the translation structure of Σ_t on the cover,

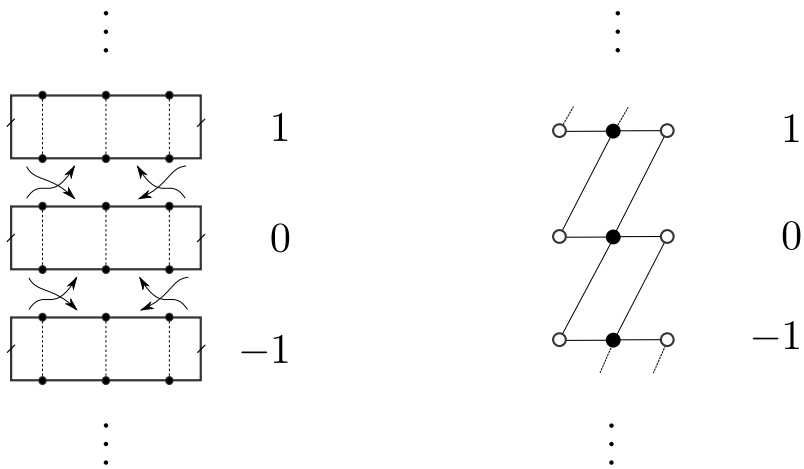
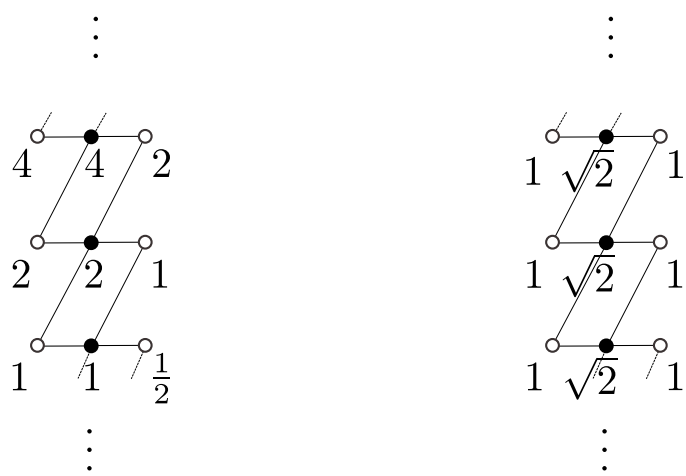


FIGURE 17. Translation cover and associated cylinder graph

we obtain a translation surface associated to the eigenfunction of eigenvalue $2\sqrt{2}$ defined in Figure 18b (This is a eigenfunction since the moduli of the cylinders are all equal).



(A) Eigenfunction corresponding to $\widehat{\Sigma}$ (B) Eigenfunction corresponding to $\widehat{\Sigma}_t$

FIGURE 18. Definition of eigenfunctions

We now use one of the main theorem of [Hoo15],

Theorem (Orbit Equivalence)**.** *On the previous graph, suppose we have two eigenfunctions with eigenvalues λ_1, λ_2 and associated translation surfaces Σ_1, Σ_2. To any λ_1-renormalizable direction θ_1 we can associate bijectively a λ_2-renormalizable direction θ_2 (up to a change of antipodal direction) such that the foliation $\mathcal{F}^1_{\theta_1}$ in Σ_1 is conjugated to the foliation $\mathcal{F}^2_{\theta_2}$ in Σ_2*

Remark that here the image of ρ_3 and $\rho_{2\sqrt{2}}$ are the Veech groups of the associated surfaces. Moreover on Σ_f the Masur criterion implies that any direction θ that is in the limit set of its Veech group is uniquely ergodic. This implies that for every direction that are 3-renormalizable, the foliation in Σ is uniquely ergodic.

To conclude we observe that in Σ, the directions that are eigenvectors for matrices conjugated in G to h or v are parabolic elements of the Veech group. Thus in these directions, the foliation can be decomposed in cylinders (see [DFG]).

The remaining case is for directions which are eigenvectors of a matrix conjugated to $v^{-1}h$ which is hyperbolic. Then in this case, the corresponding first return map is infinitely renormalizable, thus its generalized Rauzy-Veech induction is periodic and its length goes to zero. According to Proposition 7 in [Yoc05], this implies that it is conjugated to a renormalizable interval exchange transformation, which is minimal.

This implies the following,

Theorem 10. *If $\mathcal{P}$ is the orbit of the horizontal and vertical directions and $\mathcal{M}$ the orbit of the eigenvectors directions of $A^{-1}B$ by Γ, for $\theta \in \Lambda_\Sigma$,*

- *if $\theta \in \mathcal{P}$ the foliation is decomposed into flat cylinder.*
- *if $\theta \in \mathcal{H}$ the foliation is minimal.*
- *otherwise, the foliation is uniquely ergodic.*

Appendix A. On the Veech group of Σ.

We aim at discussing the different definitions of the Veech group, and how they relate to each other. Classically there are two ways two define the Veech group of a translation surface: either by saying that it is the group the derivatives of the (real) affine diffeomorphisms of the surface Σ or that it is the stabiliser of Σ for the $\mathrm{SL}(2,\mathbb{R})$-action on a certain moduli space.

It is often convenient to consider the affine diffeomorphism realising a certain element of $\mathrm{SL}(2,\mathbb{R})$ and to know about its topological properties, namely which type of class in the mapping class group of Σ it represents.

In order to extricate the subtleties between these different points of view, we make the following definitions:

(1) $\mathrm{Affine}(\Sigma)$ is the subgroup of $\mathrm{Diff}^+(\Sigma)$ made of affine automorphisms;
(2) $\rho : \mathrm{Affine}(\Sigma) \longrightarrow \mathrm{SL}(2,\mathbb{R})$ is the natural projection obtained the following way: any element $f \in \mathrm{Affine}(\Sigma)$ writes down in coordinate charts as an element of $\mathrm{GL}^+(2,\mathbb{R}) \ltimes \mathbb{R}^2$. Its linear part in $\mathrm{GL}^+(2,\mathbb{R})$ only changes under a different choice of charts by a multiplication by a certain $\lambda \in \mathbb{R}_+^*$. Its class in $\mathrm{GL}^+(2,\mathbb{R})/\mathbb{R}_+^* = \mathrm{SL}(2,\mathbb{R})$ is therefore well defined and is by definition $\rho(f)$. We call ρ the *Fuchsian representation*.
(3) $\tau : \mathrm{Affine}(\Sigma) \longrightarrow \mathrm{MCG}(\Sigma)$ is the natural projection to the mapping class group.

Remark 5. *It is important to understand that the fact that the image of ρ lies in $\mathrm{SL}(2,\mathbb{R})$ is somewhat artificial and that the space it naturally lies in is $\mathrm{GL}^+(2,\mathbb{R})/\mathbb{R}_+^*$. In particular, when an element of the Veech group is looked at in coordinate charts, there is no reason the determinant of its derivative should be equal to 1, however much natural the charts are.*

We first make the following elementary but important remark: the kernel of ρ need not to be trivial. Consider $\widehat{\Sigma}$ any ramified cover of Σ together with the pulled-back affine structure. The Galois group of the cover acts non-trivially as elements of $\mathrm{Affine}(\widehat{\Sigma})$ whose image under ρ are trivial, by definition. Conversely, the quotient of an affine surface Σ by $\mathrm{Ker}(\rho)$ is still a dilatation surface over which Σ is a ramified cover, and whose Fuchsian representation has trivial kernel. The *Veech group* $\mathbf{V}_\Sigma$ of Σ is by definition $\rho(\Sigma) \subset \mathrm{SL}(2,\mathbb{R})$. We also say a dilatation surface is *irreducible* if its Fuchsian representation has trivial kernel or equivalently if it is not a non-trivial ramified cover over another affine surface.

References

[Ahl66] Lars V. Ahlfors. Fundamental polyhedrons and limit point sets of Kleinian groups. *Proc. Nat. Acad. Sci. U.S.A.*, 55:251–254, 1966.

[BHM10] Xavier Bressaud, Pascal Hubert, and Alejandro Maass. Persistence of wandering intervals in self-similar affine interval exchange transformations. *Ergodic Theory Dynam. Systems*, 30(3):665–686, 2010.

[CG97] Ricardo Camelier and Carlos Gutierrez. Affine interval exchange transformations with wandering intervals. *Ergodic Theory Dynam. Systems*, 17(6):1315–1338, 1997.

[Dal11] Francoise Dal'Bo. *Geodesic and horocyclic trajectories*. Universitext. Springer-Verlag London, 1 edition, 2011.

[DFG] Eduard Duryev, Carlos Fougeroc, and Selim Ghazouani. Affine surfaces and their veech groups. *arXiv preprint*, https://arxiv.org/abs/1609.02130.

[Hoo15] W. Patrick Hooper. The invariant measures of some infinite interval exchange maps. *Geom. Topol.*, 19(4):1895–2038, 2015.

[Kat92] Svetlana Katok. *Fuchsian groups*. Chicago lectures in mathematics series. University of Chicago Press, 1 edition, 1992.

[Lev87] Gilbert Levitt. La décomposition dynamique et la différentiabilité des feuilletages des surfaces. *Ann. Inst. Fourier (Grenoble)*, 37(3):85–116, 1987.

[Lio95] Isabelle Liousse. Dynamique générique des feuilletages transversalement affines des surfaces. *Bull. Soc. Math. France*, 123(4):493–516, 1995.

[McM06] Curtis T. McMullen. Prym varieties and Teichmüller curves. *Duke Math. J.*, 133(3):569–590, 2006.

[MMY10] S. Marmi, P. Moussa, and J.-C. Yoccoz. Affine interval exchange maps with a wandering interval. *Proc. Lond. Math. Soc. (3)*, 100(3):639–669, 2010.

[Rau79] Gérard Rauzy. Échanges d'intervalles et transformations induites. *Acta Arith.*, 34(4):315–328, 1979.

[Thu88] William P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 19(2):417–431, 1988.

[Vee89] W. A. Veech. Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards. *Invent. Math.*, 97(3):553–583, 1989.

[Yoc05] Jean-christophe Yoccoz. Échanges d' intervalles. *Cours au Collège de France*, janvier-mars 2005.