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Équations d'évolution stochastiques locales et non locales dans des problèmes de transition de phase

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Introduction

Équations d'évolution stochastiques locales et non locales dans des problèmes de transition de phase

0.1 Introduction en Français

Le but de cette thèse est de développer des méthodes de démonstration d'existence et d'unicité de solutions d'équations et de systèmes d'équations paraboliques stochastiques non linéaires modélisant des phénomènes physiques ou biologiques; dans tout système naturel il se rajoute des incertitudes que nous modélisons ici par des bruits. Ceci explique notre intérêt pour les équations aux dérivées partielles stochastiques. Quand on ajoute un terme stochastique, les méthodes usuelles utilisées pour démontrer l'existence et l'unicité doivent être profondément modifiées; c'est ce qui nous incite à mettre en place de nouvelles méthodes de démonstration qui étendent au cas stochastique les méthodes déterministes habituelles.

0.2 L'équation d'Allen-Cahn

Considérons tout d'abord l'équation d'Allen-Cahn qui est à la base d'une grande classe de problèmes de transition de phase. Cette équation est de la forme:

$$\frac{\partial u}{\partial t} = \Delta u + f(u). \quad (0.2.1)$$

Le premier terme du membre de droite de (0.2.1) représente la diffusion tandis que le deuxième terme est un terme de réaction; on suppose ici que f est la dérivée d'un potentiel à double puits, c'est-à-dire une fonction bistable, comme par exemple $f(u) = u - u^3$, qui admet les trois racines $-1, 0, +1$, et satisfait $f'(\pm 1) < 0$ et $f'(0) > 0$.

0.2.1 Le cas déterministe

Pour le cas déterministe, Martine Marion [18] prouve l'existence, l'unicité et la régularité de la solution d'un problème de réaction-diffusion avec un terme de réaction polynomial assez

général et des conditions aux limites de Dirichlet ou de Neumann homogènes sur un domaine ouvert et borné de \mathbb{R}^d . Elle démontre également l'existence d'un attracteur global.

On se réfère également aux ouvrages de Temam [27] et Robinson [25] pour des études détaillées de propriétés de type systèmes dynamiques de cette équation.

0.2.2 Le cas stochastique

Pour le cas stochastique, nous citerons tout d'abord plusieurs travaux sur l'existence et l'unicité de la solution d'une équation de réaction-diffusion stochastique avec un terme de réaction qui généralise le terme cubique de l'équation d'Allen-Cahn; Peszat [22] applique le théorème de point fixe de Banach avec une notion de solution "mild" d'une équation de réaction-diffusion qui contient un bruit multiplicatif faisant intervenir un processus de Wiener cylindrique, Flandoli [11] démontre l'existence et l'unicité globale de la solution d'une équation de réaction-diffusion stochastique avec un terme de réaction polynomial non linéaire dans un domaine borné et un bruit blanc multiplicatif. Nous mentionnerons également l'article de Cerrai [8] où elle démontre l'existence et l'unicité d'une solution "mild" dans des espaces de fonctions continues pour une classe de problèmes de réaction-diffusion; elle démontre également l'existence d'une mesure invariante. Pour démontrer ces résultats, elle s'appuie sur la théorie des semi-groupes avec des arguments de contraction et de factorisation, en supposant que le terme de réaction est une fonction à croissance polynomiale, localement Lipschitzienne et monotone. Le bruit multiplicatif est blanc en espace et en temps si $d = 1$ et coloré en espace si $d > 1$ et le terme multiplicatif est supposé Lipschitzien. Finalement, nous mentionnerons l'article de Guess [13], où il démontre l'existence et l'unicité de solutions fortes, c'est-à-dire de solutions de régularité spatiale de type H^2 , pour une classe d'équations aux dérivées partielles stochastiques faisant intervenir le sous-différentiel d'une fonction quasi-convexe et un bruit multiplicatif général. Son résultat s'applique à des équations stochastiques de type milieux poreux et p-Laplacien avec également des termes de réaction.

0.2.3 Méthodes de monotonie et compacité pour des approximations de Galerkin

Le point de départ des démonstrations des trois chapitres de la thèse est l'application de la méthode de Galerkin. Il s'agit d'une méthode déjà ancienne utilisée pour les problèmes déterministes et bien adaptée aux problèmes stochastiques. Cette méthode permet de transformer un problème continu (par exemple une équation aux dérivées partielles) en un problème discret. Au lieu de traiter l'équation aux dérivées partielles directement, on l'approche par un système d'équations différentielles ordinaires.

Comme nous faisons l'étude de problèmes à valeur initiale avec des conditions de Neumann homogènes, nous décomposons la solution approchée sur une base de fonctions propres de l'opérateur $-\Delta$ avec une condition aux limites de Neumann homogène, qui sont très régulières car nous supposons que la frontière du domaine est très régulière. On coupe ensuite la série pour approcher les équations aux dérivées partielles par un système fini d'équations différentielles ordinaires. Les résultats d'existence et d'unicité pour les systèmes d'équations différentielles ordinaires impliquent l'existence d'une solution unique de l'approximation de Galerkin sur un petit intervalle de temps $[0, T_n]$. Pour étendre cet intervalle de temps à un intervalle $[0, T]$, avec T arbitraire, nous établissons des estimations a priori pour la solution du système d'équations

différentielles ordinaires.

Ces estimations a priori sont généralement obtenues à l'aide de la formule d'Itô; on remarque que cette formule se réduit à une dérivation en chaîne dans le cas déterministe. Nous nous appuyons maintenant sur trois séries de variables: une variable d'espace, une variable de temps et la variable de probabilité; c'est pourquoi nous utilisons l'espérance des fonctions considérées qui représente l'intégrale sur l'espace de probabilité Ω . Cette espérance est nécessaire pour l'annulation de termes stochastiques, ce qui permet d'obtenir des estimations a priori pour la solution approchée dans des espaces de Sobolev. Nous déduisons ainsi la convergence faible d'une sous-suite de solutions approchées.

Finalement, nous identifions la limite des termes non linéaires. Dans ce but, nous disposons de deux méthodes: la méthode de monotonie et la méthode de compacité, qui ont été utilisées par J.L. Lions [17] et son école pour la résolution de problèmes aux limites non linéaires déterministes. Notre but est d'étendre ces méthodes au cas stochastique pour pouvoir démontrer l'existence de solutions des problèmes stochastiques qui nous intéressent.

(i) *La méthode de monotonie*

La méthode de monotonie stochastique nécessite des estimations a priori "moins fortes" que la méthode de compacité et s'applique à des problèmes en dimension d'espace quelconque. Nous nous basons sur les travaux de Pardoux [19],[20], [21], Prévôt et Röckner [24] et l'article de Krylov et Rozovskii [16]. Ces auteurs étudient des équations aux dérivées partielles stochastiques paraboliques monotones et recherchent des solutions satisfaisant une équation intégrale correspondante de la forme

$$v(t) = v_0 + \int_0^t A(v(s), s)ds + \int_0^t B(v(s), s)dW(s)$$

dans des espaces de Banach. Les coefficients $A(v, s), B(v, s)$ sont généralement des opérateurs non bornés et non linéaires. W est un processus de Wiener à valeur dans un espace de Hilbert. Une condition de monotonie partielle est imposée sur A et B .

Dans cette thèse, nous étudions l'équation d'Allen-Cahn avec un terme non local qui ne vérifie pas la condition de monotonie. Toutefois, la conservation de l'intégrale de la solution nous permet de travailler dans un espace de fonctions d'intégrale nulle et d'éliminer le terme non local dans la plupart des calculs.

(ii) *La méthode de compacité*

Dans le cas d'un bruit multiplicatif, l'intégrale de la solution n'est pas conservée si bien qu'il n'est pas possible d'appliquer la méthode de monotonie. C'est ce qui nous amène à appliquer la méthode de compacité stochastique. Soulignons que les théorèmes habituels de compacité utilisés pour les problèmes déterministes afin d'obtenir la convergence forte de la solution ne peuvent pas s'appliquer pour les problèmes stochastiques à cause de la présence de la variable de probabilité.

De nombreux auteurs ont adopté la méthode de compacité stochastique comme par exemple

Glatt-Holtz, Temam et Wang [14], Debussche, Glatt-Holtz et Temam [9], Breit, Feireisl, Hofmanova [5] et Hofmanova [13].

Cette méthode nécessite des estimations a priori additionnelles notamment pour borner le terme stochastique et la partie déterministe de l'équation dans des espaces de Sobolev fractionnaires. Ces estimations permettent de montrer la "tightness" des mesures de probabilités correspondant aux solutions approchées. C'est ainsi que nous appliquons le théorème de Prokhorov qui assure la convergence faible d'une sous-suite de ces mesures vers une limite. Ensuite nous nous appuyons sur le théorème de Skorokhod pour déduire la convergence presque sûre d'une sous-suite de solutions approchées qui implique par le théorème de Vitali la convergence forte d'une sous-suite de solutions approchées dans une nouvelle base stochastique. Finalement, on démontre l'unicité trajectorielle des solutions martingales qui implique par l'application du théorème de Gyöngy-Krylov l'existence et l'unicité de la solution trajectorielle.

0.3 Plan de la thèse

Cette thèse porte sur l'étude de trois équations :

Une équation de réaction-diffusion non linéaire stochastique non locale avec un bruit additif

$$\frac{\partial u}{\partial t} = \operatorname{div}(A(\nabla u)) + f(u) - \frac{1}{|D|}f(u) + \frac{\partial W(t)}{\partial t}, \quad (0.3.1)$$

qui est étudiée au Chapitre 1, un problème de champ de phase stochastique avec des bruits multiplicatifs dans chacune des équations

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + f(u) + h + \Phi_1(u, h) \frac{\partial W_1(t)}{\partial t}, \\ \frac{\partial [u + h]}{\partial t} = \Delta h + \Phi_2(u, h) \frac{\partial W_2(t)}{\partial t}, \end{cases}$$

qui fait l'objet du Chapitre 2 et une équation de réaction-diffusion stochastique non locale avec un bruit multiplicatif

$$\frac{\partial u}{\partial t} = \Delta u + f(u) - \frac{1}{|D|}f(u) + \Phi(u) \frac{\partial W(t)}{\partial t}, \quad (0.3.2)$$

qui est étudiée au Chapitre 3.

Dans ce qui suit nous discutons à la fois la littérature déterministe et stochastique de chaque équation et faisons un résumé détaillé de tous les résultats. Chaque chapitre a été écrit indépendamment et la même notation peut correspondre à une quantité différente selon le chapitre.

0.3.1 Chapitre 1: Une équation de réaction-diffusion stochastique non locale avec un terme de diffusion non linéaire

Dans ce chapitre, nous démontrons l'existence et l'unicité de la solution d'une équation de réaction-diffusion stochastique non locale avec conservation de l'intégrale de la solution. Rubinstein et Sternberg [26] ont introduit l'équation d'Allen-Cahn non locale avec des conditions

de Neumann homogènes à la frontière du domaine pour modéliser la séparation en phases dans des alliages binaires. La solution représente le paramètre d'ordre.

L'existence et l'unicité de la solution de l'équation d'Allen-Cahn non locale déterministe avec conservation de l'intégrale ont été démontrées par Boussaïd, Hilhorst et Nguyen [4]. Ils étudient également la stabilisation de la solution en temps long. Ils supposent que la solution initiale est bornée et prouvent l'existence de la solution dans un ensemble invariant en s'appuyant sur la méthode de Galerkin et en passant à la limite à l'aide de la méthode de compacité.

L'équation d'Allen-Cahn non locale a été étudiée par Funaki et Yokoyama [12] dans le cas d'un bruit blanc additif régularisé unidimensionnel en temps. Ils démontrent la convergence vers un problème à frontière libre quand le coefficient du terme de réaction et du terme non local tend vers l'infini.

Dans leur article, Antonopoulo, Bates, Blömker et Karali [1] étudient l'équation d'Allen-Cahn stochastique non locale avec un bruit additif dans le but de modéliser le mouvement d'une goutte presque semi-circulaire attachée à la frontière du domaine, qui se déplace vers un point de la frontière de courbure localement maximale. Ils s'intéressent au comportement asymptotique de la solution quand un petit paramètre tend vers zéro. Cependant, ils ne donnent aucune preuve d'existence et d'unicité de la solution; c'est ce qui nous amène à étudier ces questions et à présenter des démonstrations rigoureuses de l'existence et de l'unicité.

Dans ce chapitre nous démontrons l'existence et l'unicité de la solution du problème

$$(P) \quad \begin{cases} \frac{\partial \varphi}{\partial t} = \operatorname{div}(A(\nabla \varphi)) + f(\varphi) - \frac{1}{|D|} \int_D f(\varphi) dx + \frac{\partial W}{\partial t}, & x \in D, \quad t \geq 0, \\ A(\nabla \varphi) \cdot \nu = 0, & x \in \partial D, t \geq 0, \\ \varphi(x, 0) = \varphi_0(x), & x \in D, \end{cases}$$

où D est un domaine ouvert borné de \mathbb{R}^n de frontière ∂D suffisamment régulière, ν est le vecteur normal extérieur à ∂D et la fonction initiale φ_0 est telle que $\varphi_0 \in L^2(D)$. La fonction f est donnée par $f(s) = \sum_{r=0}^{2p-1} b_r s^r$ où $b_{2p-1} < 0, p \geq 2$. De plus, on suppose que $A = \nabla_v \Psi(v) : \mathbb{R}^n \rightarrow \mathbb{R}^n$, où la fonction $\Psi \in C^{1,1}$ est strictement convexe et que A vérifie des conditions de continuité et de monotonie. Finalement, on suppose que la fonction $W = W(x, t)$ est un processus Q-Wiener.

L'idée principale de la démonstration est d'introduire un problème auxiliaire, plus précisément l'équation de chaleur non linéaire stochastique avec des conditions aux limites de Neumann :

$$(P_1) \quad \begin{cases} \frac{\partial W_A}{\partial t} = \operatorname{div}(A(\nabla W_A)) + \frac{\partial W}{\partial t}, & x \in D, \quad t \geq 0, \\ A(\nabla W_A) \cdot \nu = 0, & x \in \partial D, t \geq 0, \\ W_A(x, 0) = 0, & x \in D. \end{cases}$$

On démontre l'existence et l'unicité d'une solution forte au sens de Gess [13], c'est-à-dire une solution de l'équation intégrée en temps qui est de plus satisfaite pour presque tout $x \in D$. On définit ensuite la nouvelle fonction inconnue

$$u(t) := \varphi(t) - W_A(t),$$

dans le but d'éliminer le terme de bruit dans le Problème (P). On déduit que la fonction u satisfait le problème

$$(P_2) \quad \begin{cases} \frac{\partial u}{\partial t} = \operatorname{div}(A(\nabla(u + W_A)) - A(\nabla W_A)) + f(u + W_A) \\ \quad \quad \quad - \frac{1}{|D|} \int_D f(u + W_A) dx, & x \in D, \quad t \geq 0, \\ A(\nabla(u + W_A)) \cdot \nu = 0, & x \in \partial D, t \geq 0, \\ u(x, 0) = \varphi_0(x), & x \in D. \end{cases}$$

que l'on écrit sous forme intégrale afin d'en définir une solution faible.

On est ainsi amené à étudier un problème approché sans terme de bruit explicite, ce qui simplifie considérablement les estimations a priori pour la solution approchée déduite de la méthode de Galerkin. On montre qu'une sous-suite de la solution approchée converge faiblement vers une limite. On applique ensuite la méthode de monotonie pour obtenir des expressions explicites de la limite du terme de diffusion non linéaire et de celle du terme de réaction. On démontre finalement l'unicité de la solution du Problème (P₂), ce qui implique que c'est toute la suite des solutions approchées par la méthode de Galerkin qui converge vers la solution faible unique du Problème (P₂).

0.3.2 Chapitre 2: Existence et unicité de la solution d'un problème de champ de phase avec des bruits multiplicatifs

Dans ce chapitre nous démontrons l'existence et l'unicité de la solution d'un problème de champ de phase avec des bruits multiplicatifs. Les modèles de champ de phase sont utilisés pour décrire des modèles où deux phases distinctes interviennent comme par exemple l'eau et la glace.

Plus précisément nous étudions un modèle de champ de phase initialement proposé par Caginalp [7] auquel sont rajoutées des perturbations stochastiques.

$$(P_3) \quad \begin{cases} d\varphi = (\Delta\varphi + f(\varphi) + h)dt + \Phi_1(\varphi, h)dW_1(t), & \text{dans } D \times (0, T), \\ d[h + \varphi] = \Delta h dt + \Phi_2(\varphi, h)dW_2(t), & \text{dans } D \times (0, T), \\ \frac{\partial \varphi}{\partial \nu} = \frac{\partial h}{\partial \nu} = 0, & \text{sur } \partial D \times (0, T), \\ \varphi(x, 0) = \varphi_0(x), \quad h(x, 0) = h_0(x), & x \in D. \end{cases}$$

Dans ce modèle h est la température et φ est la fonction de phase. L'existence et l'unicité de la solution du problème déterministe ont été prouvées par Brochet, Chen et Hilhorst [6] qui démontrent également l'existence d'un attracteur global et d'un ensemble inertiel. Bertini,

Brassesco, Buttà et Presutti [3] ont considéré le cas de la dimension 1 d'espace et ont démontré l'existence globale et l'unicité de la solution du problème de champ de phase perturbé par un bruit blanc espace-temps additif. Barbu et Da Prato [2] ont prouvé l'existence et l'unicité de la solution en dimension quelconque, dans le cas où les bruits additifs présents dans les deux équations sont induits par des processus de Wiener cylindrique.

Nous démontrons l'existence et l'unicité de la solution d'une formulation intégrale correspondant au Problème (P_3) , où W_1 et W_2 sont des processus Q-Wiener. Dans ce but, nous appliquons la méthode de Galerkin et démontrons des estimations a priori qui bornent uniformément le couple de solutions approchées. Après le passage à la limite, nous identifions la limite du terme de réaction à l'aide de la méthode de monotonie stochastique (voir par exemple [16], [21] et [24]). Finalement, nous démontrons l'unicité du couple de solutions en nous appuyant sur la formule d'Itô.

0.3.3 Chapitre 3: L'équation d'Allen-Cahn non locale stochastique avec un bruit multiplicatif

Dans ce chapitre, nous démontrons l'existence et l'unicité d'une solution trajectorielle forte de l'équation d'Allen-Cahn stochastique en dimensions d'espace $d = 1, \dots, 6$ avec un bruit multiplicatif faisant intervenir un processus Q-Wiener.

Une solution est dite forte au sens des équations aux dérivées partielles si la solution possède une régularité spatiale de type H^2 si bien que l'équation intégrale associée à l'équation aux dérivées partielles est satisfaite presque partout dans D . La solution est dite martingale c'est-à-dire faible au sens des probabilités, quand la base stochastique n'est pas fixée au départ; elle fait alors partie des inconnues du problème. Tandis que la solution est dite trajectorielle ou forte au sens des probabilités quand la base stochastique et le bruit sont fixés au départ.

Nous partons d'un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$ et tout comme aux deux chapitres précédents notre point de départ est l'application de la méthode de Galerkin. On prouve des estimations a priori qui bornent uniformément la solution approchée dans les espaces $L^\infty(0, T; L^2(\Omega; H^1(D))) \cap L^2(\Omega \times (0, T); H^2(D)) \cap L^4(\Omega \times (0, T) \times D)$. Nous nous appuyons sur des résultats de compacité dans des espaces de Sobolev fractionnaires. Ensuite en utilisant ces estimations a priori, nous établissons la "tightness" de collections de mesures de probabilité associées à la solution approchée et nous appliquons le Théorème de Prokhorov pour obtenir la compacité faible. Nous passons ensuite à l'aide du Théorème de Skorohod à une nouvelle base stochastique $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ pour obtenir la convergence presque sûrement de la solution approchée. Nous pouvons ensuite démontrer la convergence forte d'une sous-suite de solutions approchées dans cet espace de probabilité et passer à la limite dans le problème approché. Finalement on démontre l'unicité trajectorielle de la solution martingale qui implique par le Théorème de Gyöngy-Krylov l'existence de solutions sur chaque espace de probabilité donné. On déduit ainsi l'existence et l'unicité de la solution trajectorielle sur l'espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$.

Plusieurs auteurs ont appliqué la méthode de compacité stochastique pour prouver l'existence et l'unicité de la solution de problèmes aux limites pour des équations aux dérivées partielles stochastiques. Hofmanová [13] a utilisé cette méthode pour prouver que le problème de Cauchy associé à une équation aux dérivées partielles parabolique scalaire semi-linéaire dégénérée avec

un bruit induit par un processus de Wiener cylindrique est bien posé, ensuite Debussche, Hofmanova et Vovelle [10] ont étendu ce travail au cas d'une équation aux dérivées partielles parabolique quasilineaire dégénérée stochastique avec un processus de Wiener cylindrique. De plus, Glatt-Holtz, Temam, Wang [14] prouvent par cette méthode l'existence globale de solutions martingales en dimensions 2 et 3 d'espace et l'existence trajectorielle de solutions ainsi que l'unicité trajectorielle globale en dimension 2 pour l'équation de Zakharov-Kuznetsov avec un bruit multiplicatif de type Wiener cylindrique. Debussche, Glatt-Holtz, Temam [9] démontrent l'existence et l'unicité de solutions martingales et trajectorielles locales pour un système d'équations non linéaires primitives stochastiques. Finalement, nous citerons l'article de Pham et Nguyen [23] où ils démontrent l'existence et l'unicité d'une solution trajectorielle localement en temps pour des systèmes d'équations de réaction-diffusion stochastiques.

Local and nonlocal stochastic evolution equations in phase transition problems

0.4 Introduction in English

The aim of this thesis is to develop methods for proving the existence and uniqueness of the solutions of stochastic parabolic nonlinear equations and systems of equations modelling phenomena arising in physics and biology; in every natural system there exist uncertainties which we model by noise. This explains our interest for stochastic partial differential equations.

When we add a stochastic term, the usual methods used to prove the existence and uniqueness should be modified; this is what leads us to study new methods of demonstration that extend to the stochastic case the usual deterministic methods.

0.5 The Allen-Cahn equation

First, we consider the Allen-Cahn equation which is a basic equation in the study of phase transition problems. It is given by:

$$\frac{\partial u}{\partial t} = \Delta u + f(u). \quad (0.5.1)$$

The first term on the right-hand-side of (0.5.1) represents the diffusion while the second term is a reaction term; we suppose that f is the derivative of a double-well potential, and that is a bistable function, for example $f(u) = u - u^3$, which has three zeros $-1, 0, +1$, and satisfies $f'(\pm 1) < 0$ and $f'(0) > 0$.

0.5.1 The deterministic case

For the deterministic case, Martine Marion [18] proved the existence, uniqueness and the regularity of the solution of a reaction-diffusion problem with a general polynomial reaction term and Dirichlet or Neumann boundary conditions in a open bounded domain of \mathbb{R}^d . In addition, she proved the existence of a global attractor.

We also refer to the books of Temam [27] and Robinson [25] for a detailed study of the dynamic system properties for this equation.

0.5.2 The stochastic case

Next we mention several papers on the existence and uniqueness of the solution of a stochastic reaction-diffusion equation with a reaction term which generalizes the cubic term in the Allen-Cahn equation; Peszat [22] applies the Banach fixed point theorem to prove the existence and uniqueness of the mild solution in the case of a reaction-diffusion equation with a multiplicative noise involving a cylindrical Wiener process; Flandoli [11] proves the global existence and uniqueness of the solution of a stochastic reaction-diffusion equation with a polynomial nonlinear reaction term in a bounded domain and a multiplicative white noise. We also mention the

article of Cerrai [8] where she proves the existence and uniqueness of a mild solution in the space of continuous functions for a class of reaction-diffusion problems; she also showed the existence of an invariant measure. To prove those results, she used the theory of semi-groups with contraction and factorization arguments, assuming that the reaction term is a function with polynomial growth, locally Lipschitz-continuous and monotone. The multiplicative noise is white in space and in time if $d = 1$ and colored in space if $d > 1$; the multiplicative term is assumed to be Lipschitz-continuous. Finally, let us mention the article of Guess [13], where he proves the existence and uniqueness of strong solutions, namely solutions with space regularity H^2 , for a class of stochastic partial differential equations involving the subdifferential of a quasi-convex function and a general multiplicative noise. His result applies to stochastic porous medium equations and stochastic p-Laplace equations with reaction terms.

0.5.3 Monotonicity and compactness methods for Galerkin approximations

In the three chapters of this thesis, we apply the Galerkin method. This method has been used for many years for deterministic problems and is well adapted for stochastic problems. It transforms a continuous problem (for example a partial differential equation) into a discrete problem, so that instead of working with the partial differential equation directly, we approximate it by a system of ordinary differential equations.

Since we study initial value problems with homogeneous Neumann conditions, we decompose the approximate solution on a basis of eigenfunctions of the operator $-\Delta$ with homogeneous Neumann boundary conditions, which are smooth since we assume that the boundary of the domain is smooth. We then cut the series to approximate the partial differential equations by a finite system of ordinary differential equations. The starting point for the proofs is the existence of a unique solution for the Galerkin approximate problem on a small time interval $[0, T_n]$. In order to extend this interval to an interval $[0, T]$, with T arbitrary, we establish a priori estimates for the solution of the system of ordinary differential equations.

These a priori estimates are in general obtained using the Itô's formula; we remark that this formula reduces to a chain rule in the deterministic case. We now work with three variables: the space variable, the time variable and the probability variable; this is why we use the expectation of the functions, or in other words their integration on the probability space Ω . This expectation is necessary to cancel stochastic terms and therefore to obtain a priori estimates for the approximate solution. We then deduce the weak convergence of a subsequence of approximate solutions.

Finally, we identify the limit of the nonlinear terms. In order to do so, we can apply two main methods: the monotonicity and the compactness methods, which were used by Lions [17] and his school for the study of deterministic nonlinear boundary value problems. Our goal is to extend these methods to the stochastic case.

(i) *The monotonicity method*

The stochastic monotonicity method uses weaker a priori estimates than the compactness

method and can be applied to problems in arbitrary space dimension. Our work is based on the work of Pardoux [19], [20], [21], Prévôt and Röckner [24] and the article of Krylov and Rozovskii [16]. These authors study stochastic parabolic monotone partial differential equations and search for a solution satisfying a corresponding integral equation of the form:

$$v(t) = v_0 + \int_0^t A(v(s), s)ds + \int_0^t B(v(s), s)dW(s)$$

in Banach spaces. The coefficients $A(v, s)$ and $B(v, s)$ are in general unbounded and nonlinear operators. W is a Wiener process in some Hilbert space. We impose a partial monotonicity condition on A and B .

In this thesis, we study the Allen-Cahn equation with a nonlocal term which does not satisfy the monotonicity condition. However, the space integral of the solution is conserved in time so that we work in spaces of functions with zero integral. This helps us to cancel the nonlocal term in the computations.

(ii) *The compactness method*

In the case of a multiplicative noise, the integral of the solution is not conserved so that the monotonicity method cannot be applied. This is why we apply the stochastic compactness method.

Let us mention that the usual compactness theorems used in the deterministic analysis to deduce the strong convergence of the solutions cannot be applied for stochastic problems because of the additional probability variable. For instance, even if a space X is compact in another space Y , $L^2(\Omega; X)$ is not compact in $L^2(\Omega; Y)$.

Many authors use the compactness method; we can mention for instance the articles of Glatt-Holtz, Temam and Wang [14], Debussche, Glatt-Holtz and Temam [9], Breit, Feireisl, Hofmanova [5] and Hofmanova [13].

In order to apply this method we have to derive additional a priori estimates to bound the stochastic term and the deterministic part of the equation in fractional Sobolev spaces. These a priori estimates are used to prove the tightness of the probability measures corresponding to the approximate solution. Therefore we apply the Prokhorov Theorem to obtain the weak convergence of a subsequence of these measures. Then we apply the Skorokhod Theorem to obtain the almost sure convergence of a subsequence of approximate solutions, which in turn implies by the Vitali Theorem the strong convergence of a subsequence of approximate solutions in a new probability basis. Finally we prove the pathwise uniqueness of the solution which implies by the Gyöngy-Krylov Theorem the existence of the pathwise solution.

0.6 Organization of the thesis

This thesis deals with three main equations:

A stochastic nonlinear nonlocal reaction-diffusion equation with an additive noise

$$\frac{\partial u}{\partial t} = \operatorname{div}(A(\nabla u)) + f(u) - \frac{1}{|D|}f(u) + \frac{\partial W(t)}{\partial t}, \quad (0.6.1)$$

which is studied in Chapter 1, a stochastic phase field problem with multiplicative noise in each equation of the system

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + f(u) + h + \Phi_1(u, h) \frac{\partial W_1(t)}{\partial t}, \\ \frac{\partial [u + h]}{\partial t} = \Delta h + \Phi_2(u, h) \frac{\partial W_2(t)}{\partial t}, \end{cases}$$

which is the subject of Chapter 2 and a stochastic nonlocal reaction-diffusion equation with multiplicative noise

$$\frac{\partial u}{\partial t} = \Delta u + f(u) - \frac{1}{|D|} f(u) + \Phi(u) \frac{\partial W(t)}{\partial t}, \quad (0.6.2)$$

which is studied in Chapter 3.

In what follows we discuss the deterministic and stochastic literature for each type of equations and we summarize the result which we obtain. Each chapter was written independently and the same notation could correspond to different quantities depending on the chapter.

0.6.1 Chapitre 1: A stochastic mass conserved reaction-diffusion equation with nonlinear diffusion

In this chapter, we prove the existence and uniqueness of the solution of a stochastic nonlocal reaction-diffusion equation such that the integral of the solution is conserved. Rubinstein and Sternberg [26] introduced the nonlocal Allen-Cahn equation with homogeneous Neumann boundary conditions to model phase separation in a binary mixture. The solution represents the order parameter.

The existence and uniqueness of the solution of the deterministic mass conserved nonlocal Allen-Cahn equation was proved by Boussaïd, Hilhorst and Nguyen [4]. They studied the stabilization of the solution in large time. They supposed that the initial condition is bounded and proved the existence of the solution in an invariant set using the Galerkin method and passing to the limit by means of compactness arguments.

The nonlocal Allen-Cahn equation was studied by Funaki and Yokoyama [12] in the case of a regularized one-dimensional additive white noise in time. They proved the convergence to a free boundary problem when the coefficient of the reaction term and the nonlocal term tends to infinity.

In their article, Antonopoulou, Bates, Blömker and Karali [1] study the stochastic nonlocal Allen-Cahn equation with an additive noise in order to model the motion of a small almost semicircular droplet attached to domain's boundary and moving towards a point of locally maximum curvature. They are interested in the asymptotic behavior of the solution when a small parameter tends to zero. However, they do not give any proof of existence and uniqueness of the solution; this is our motivation to study these questions and to present rigorous proofs

of existence and uniqueness.

In this chapter we prove the existence and uniqueness of the solution of the problem

$$(P) \quad \begin{cases} \frac{\partial \varphi}{\partial t} = \operatorname{div}(A(\nabla \varphi)) + f(\varphi) - \frac{1}{|D|} \int_D f(\varphi) dx + \frac{\partial W}{\partial t}, & x \in D, \quad t \geq 0, \\ A(\nabla \varphi) \cdot \nu = 0, & x \in \partial D, \quad t \geq 0, \\ \varphi(x, 0) = \varphi_0(x), & x \in D, \end{cases}$$

where D is an open bounded domain of \mathbb{R}^n with sufficiently smooth boundary ∂D , ν is unit outward normal vector to ∂D and the initial function φ_0 is such that $\varphi_0 \in L^2(D)$. The

function f is given by $f(s) = \sum_{r=0}^{2p-1} b_r s^r$ where $b_{2p-1} < 0, p \geq 2$. In addition, we suppose that

$A = \nabla_v \Psi(v) : \mathbb{R}^n \rightarrow \mathbb{R}^n$, where the function $\Psi \in C^{1,1}$ is strictly convex and A satisfies suitable smoothness and monotonicity conditions. Finally, we suppose that the function $W = W(x, t)$ is a Q-Wiener process.

The main idea of the proof is to introduce an auxiliary problem, more precisely the stochastic nonlinear heat equation with homogeneous Neumann boundary conditions:

$$(P_1) \quad \begin{cases} \frac{\partial W_A}{\partial t} = \operatorname{div}(A(\nabla W_A)) + \frac{\partial W}{\partial t}, & x \in D, \quad t \geq 0, \\ A(\nabla W_A) \cdot \nu = 0, & x \in \partial D, \quad t \geq 0, \\ W_A(x, 0) = 0, & x \in D. \end{cases}$$

We prove the existence and uniqueness of a strong solution in the sense of Gess [13], by which we mean a solution of the equation integrated in time which is satisfied for almost every $x \in D$. Next, we define the change of functions

$$u(t) := \varphi(t) - W_A(t),$$

in order to remove the noise term from Problem (P). We deduce that the function u satisfies the problem

$$(P_2) \quad \begin{cases} \frac{\partial u}{\partial t} = \operatorname{div}(A(\nabla(u + W_A)) - A(\nabla W_A)) + f(u + W_A) \\ \quad - \frac{1}{|D|} \int_D f(u + W_A) dx, & x \in D, \quad t \geq 0, \\ A(\nabla(u + W_A)) \cdot \nu = 0, & x \in \partial D, \quad t \geq 0, \\ u(x, 0) = \varphi_0(x), & x \in D, \end{cases}$$

for which we write a weak form and define a weak solution.

This way, we are led to study the approximate problem without any explicit noise term, which simplifies the application of the Galerkin method. We prove that a subsequence of the approximate solutions converges weakly to a limit. Then, we apply the monotonicity method to

obtain explicit expressions of the limit of the nonlinear diffusion term and of the reaction term. Finally, we prove the uniqueness of the solution of Problem (P_2) , which implies that the whole sequence of approximate solutions converges to the unique weak solution of Problem (P_2) .

0.6.2 Chapitre 2: Well-posedness of a stochastic phase field problem with multiplicative noises

In this chapter we prove the existence and uniqueness of the solution of a phase field problem with multiplicative noises. Phase field models are used for instance to describe melting and solidification processes.

More precisely, we study a phase field model proposed by Caginalp [7] to which we add stochastic perturbations

$$(P_3) \quad \begin{cases} d\varphi = (\Delta\varphi + f(\varphi) + h)dt + \Phi_1(\varphi, h)dW_1(t), & \text{in } D \times (0, T), \\ d[h + \varphi] = \Delta h dt + \Phi_2(\varphi, h)dW_2(t), & \text{in } D \times (0, T), \\ \frac{\partial\varphi}{\partial\nu} = \frac{\partial h}{\partial\nu} = 0, & \text{on } \partial D \times (0, T), \\ \varphi(x, 0) = \varphi_0(x), h(x, 0) = h_0(x), & x \in D, \end{cases}$$

In this model h represents the temperature and φ is the phase function. The existence and uniqueness of the solution of the deterministic problem was proved by Brochet, Chen et Hilhorst [6]; they also proved the existence of a global attractor and an inertial set. Bertini, Brassesco, Buttà and Presutti [3] proved the global existence and uniqueness of the solution of the phase field problem perturbed by an additive space-time white noise in one space dimension. Barbu and Da Prato [2] proved the existence and uniqueness of the solution in arbitrary dimension, in the case that an additive noise induced by a cylindrical Wiener process is added to each equation.

We prove the existence and uniqueness of the solution of an integral form corresponding to Problem (P_3) , where W_1 and W_2 are Q-Wiener processes.

In order to obtain this result, we apply the Galerkin method and prove a priori estimates which uniformly bound the approximate solution pair. After passing to the limit, we identify the limit of the reaction term by means of the stochastic monotonicity method (see for example [16], [21] and [24]). Finally, we prove the uniqueness of the solution pair by applying the Itô's formula.

0.6.3 Chapitre 3: On a stochastic nonlocal Allen-Cahn equation with multiplicative noise

In this chapter, we prove the existence and uniqueness of a strong pathwise solution of the stochastic Allen-Cahn equation in space dimensions $d \leq 6$ with a multiplicative noise induced by a Q-Wiener process.

A solution is called strong in the sense of partial differential equations if the corresponding integral equation is satisfied almost everywhere. The solution is called martingale or weak in the probability sense when the stochastic basis is not fixed in advance and is part of the unknowns of the problem, while the solution is called pathwise or strong in the probability sense when the stochastic basis as well as the driving Wiener process are given in advance.

We start with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and as in the previous chapters we apply the Galerkin method. We derive some a priori estimates which bound uniformly the approximate solution in the spaces $L^\infty(0, T; L^2(\Omega; H^1(D))) \cap L^2(\Omega \times (0, T); H^2(D)) \cap L^4(\Omega \times (0, T) \times D)$. We use compactness results in fractional Sobolev spaces. We then deduce from the a priori estimates the tightness of the collection of probability measures associated to the approximate solution and apply the Prokhorov Theorem to prove its weak compactness. Then, by the Skorohod Theorem we pass to a new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ to obtain the almost sure convergence of the approximate solutions. We then prove the strong convergence of a subsequence of approximate solutions in this probability space and pass to the limit in the approximate problem. Finally we prove the pathwise uniqueness of martingale solutions and deduce from the Gyöngy-Krylov Theorem the existence of solutions on every given probability space, that is the existence of a pathwise solution.

Many authors applied the stochastic compactness argument to prove the existence and uniqueness of the solution for initial value problems for partial differential equations. Hofmanová [13] applied this method to prove the well-posedness of the Cauchy problem for a scalar semilinear degenerate parabolic partial differential equation with a multiplicative noise induced by a cylindrical Wiener process, then Debussche, Hofmanová and Vovelle [10] extended this work to the case of a parabolic quasilinear degenerate stochastic partial differential equation with a cylindrical Wiener process.

Glatt-Holtz, Temam, Wang [14] applied this method to the Zakharov-Kuznetsov equation with similar hypotheses on the noise; they proved the global existence of martingale solutions in space dimensions 2 and 3, and the pathwise existence and uniqueness of the solution. Debussche, Glatt-Holtz, Temam [9] proved the existence and uniqueness of local martingale and pathwise solutions for a nonlinear primitive stochastic system of equations. Finally, we mention the work of Pham and Nguyen [23] where they proved the existence and uniqueness of the pathwise solution locally in time for a system of stochastic reaction-diffusion equations.

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A stochastic mass conserved reaction-diffusion equation with nonlinear diffusion

Résumé. Dans ce chapitre, on démontre que le problème à valeur initiale pour une équation de réaction-diffusion stochastique non locale avec un terme de diffusion non linéaire et des conditions de flux nulles aux bords dans un domaine borné ouvert de \mathbb{R}^n avec un bord régulier est bien posé. On suppose que le bruit additif est induit par un Q-mouvement Brownien. Ce chapitre a fait l'objet d'une publication dans le journal "Discrete and Continuous Dynamical Systems".

Abstract. In this chapter, we prove a well-posedness result for an initial boundary value problem for a stochastic nonlocal reaction-diffusion equation with nonlinear diffusion together with a nul-flux boundary condition in an open bounded domain of \mathbb{R}^n with a smooth boundary. We suppose that the additive noise is induced by a Q-Brownian motion. This chapter is published in the journal of "Discrete and Continuous Dynamical Systems".

1.1 Introduction

We study the problem

$$(P) \quad \begin{cases} \frac{\partial \varphi}{\partial t} = \operatorname{div}(A(\nabla \varphi)) + f(\varphi) - \frac{1}{|D|} \int_D f(\varphi) dx + \frac{\partial W}{\partial t}, & x \in D, \quad t \geq 0, \\ A(\nabla \varphi) \cdot \nu = 0, & \text{on } \partial D \times \mathbb{R}^+, \\ \varphi(x, 0) = \varphi_0(x), & x \in D, \end{cases}$$

where:

- D is an open bounded set of \mathbb{R}^n with a smooth boundary ∂D ;
- ν is the outer normal vector to ∂D ;
- The initial function φ_0 is such that $\varphi_0 \in L^2(D)$;
- We suppose that the nonlinear function f is a smooth function which satisfies the following properties:

(F₁) There exist positive constants C_1 and C_2 and a function f_2 such that

$$f(a+b)a \leq -C_1 a^{2p} + f_2(b), \quad |f_2(b)| \leq C_2(b^{2p} + 1), \quad \text{for all } a, b \in \mathbb{R}.$$

(F₂) There exist positive constants C_3 and $\tilde{C}_3(M)$ such that

$$|f(s)| \leq C_3 |s - M|^{2p-1} + \tilde{C}_3(M).$$

(F₃) There exists a positive constant C_4 such that

$$f'(s) \leq C_4.$$

We will check in the Appendix that the function $f(s) = \sum_{r=0}^{2p-1} b_r s^r$ with $b_{2p-1} < 0, p \geq 2$ satisfies the properties (F₁) – (F₃).

- We assume that $A = \nabla_v \Psi(v) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ for some strictly convex function $\Psi \in C^{1,1}$ (i.e. $\Psi(v) \in C^1(\mathbb{R}^n)$ and $\nabla \Psi(v)$ is Lipschitz-continuous) satisfying

$$\begin{cases} A(0) = \nabla \Psi(0) = 0, \Psi(0) = 0 \\ \|D^2 \Psi\|_{L^\infty(\mathbb{R}^n; \mathbb{R}^{n \times n})} \leq c_1, \end{cases} \quad (1.1.1)$$

for some constant $c_1 > 0$. We remark that (1.1.1) implies that

$$|A(a) - A(b)| \leq C|a - b|, \quad (1.1.2)$$

for all $a, b \in \mathbb{R}^n$, where C is a positive constant, and that the strict convexity of Ψ implies that A is strictly monotone, namely there exists a positive constant C_0 such that

$$(A(a) - A(b))(a - b) \geq C_0 |a - b|^2, \quad (1.1.3)$$

for all $a, b \in \mathbb{R}^n$.

We remark that if A is the identity matrix, the nonlinear diffusion operator $-\operatorname{div}(A(\nabla u))$ reduces to the linear operator $-\Delta u$.

- The function $W = W(x, t)$ is a Q-Brownian motion. More precisely, let Q be a nonnegative definite symmetric operator on $L^2(D)$, $\{e_l\}_{l \geq 1}$ be an orthonormal basis in $L^2(D)$ diagonalizing Q , and $\{\lambda_l\}_{l \geq 1}$ be the corresponding eigenvalues, so that

$$Qe_l = \lambda_l e_l$$

for all $l \geq 1$. Since Q is of trace-class, it follows that

$$\operatorname{Tr} Q = \sum_{l=1}^{\infty} \langle Qe_l, e_l \rangle_{L^2(D)} = \sum_{l=1}^{\infty} \lambda_l \leq \Lambda_0, \quad (1.1.4)$$

for some positive constant Λ_0 . We suppose furthermore that $e_l \in H^1(D) \cap L^\infty(D)$ for $l = 1, 2, \dots$ and that there exist positive constants Λ_1 and Λ_2 such that

$$\sum_{l=1}^{\infty} \lambda_l \|e_l\|_{L^\infty(D)}^2 \leq \Lambda_1, \quad (1.1.5)$$

and

$$\sum_{l=1}^{\infty} \lambda_l \|\nabla e_l\|_{L^2(D)}^2 \leq \Lambda_2. \quad (1.1.6)$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration (\mathcal{F}_t) and $\{\beta_l(t)\}_{l \geq 1}$ be a sequence of independent (\mathcal{F}_t) -Brownian motions defined on $(\Omega, \mathcal{F}, \mathbb{P})$; the Q -Wiener process W is defined by

$$W(x, t) = \sum_{l=1}^{\infty} \beta_l(t) Q^{\frac{1}{2}} e_l(x) = \sum_{l=1}^{\infty} \sqrt{\lambda_l} \beta_l(t) e_l(x), \quad (1.1.7)$$

in $L^2(D)$. We recall that a Brownian motion $\beta(t)$ is called an (\mathcal{F}_t) Brownian motion if it is (\mathcal{F}_t) -adapted and the increment $\beta(t) - \beta(s)$ is independent of \mathcal{F}_s for every $0 \leq s < t$.

We define :

$$H = \left\{ v \in L^2(D), \int_D v = 0 \right\}, \quad V = H^1(D) \cap H \quad \text{and} \quad Z = V \cap L^{2p}(D)$$

where $\|\cdot\|$ is the norm corresponding to the space H . We also define $\langle \cdot, \cdot \rangle_{Z^*, Z}$ as the duality product between Z and its dual space $Z^* = V^* + L^{\frac{2p}{2p-1}}(D)$ ([3], p.175).

The corresponding deterministic equation in the case of linear diffusion, when A is the identity matrix, has been introduced by Rubinstein and Sternberg [17] as a model for phase separation in a binary mixture. The well-posedness and the stabilization of the solution for large times for the corresponding Neumann problem were proved by Boussaïd, Hilhorst and Nguyen [4]. They assumed that the initial function was bounded in $L^\infty(D)$ and proved the existence of the solution in an invariant set using a Galerkin approximation together with a compactness method.

The interfacial evolution process corresponding to a second order mass conserved Allen-Cahn equation shares many properties with the fourth order Cahn-Hilliard equation as discussed in [17]. Da Prato and Debussche proved the existence and the uniqueness of the solution of a stochastic Cahn-Hilliard equation in [6] with an additive space-time white noise.

In our work, inspired by this paper, we introduce a nonlinear stochastic heat equation, perform a change of functions in order to maintain a ‘‘deterministic style’’ mass conserved equation by hiding the noise term and prove the existence of the solution in suitable Sobolev spaces similar to those in [6].

Funaki and Yokoyama [8] derive a sharp interface limit for a stochastically perturbed mass conserved Allen-Cahn equation with a sufficiently mild additive noise. This is different from the stochastic term in this chapter which is not smooth.

A singular limit of a rescaled version of Problem (P) with linear diffusion has been studied by Antonopoulou, Bates, Blömker and Karali [1] to model the motion of a droplet. However, they left open the problem of proving the existence and uniqueness of the solution, which we

address here. The problem that we study is more general than the one in [1] since it has a nonlinear diffusion term. The proof is based on a Galerkin method together with a monotonicity argument similar to that used in [14] for a deterministic reaction-diffusion equation, and that in [12] for a stochastic problem.

Our chapter is organised as follows.

In section 1.2 an auxiliary problem is introduced, more precisely the nonlinear stochastic heat equation and a change of function is defined to obtain an equation without the noise term; this simplifies the use of the Galerkin method in section 1.3, which yields uniform bounds for the approximate solution in $L^\infty(0, T; L^2(\Omega \times D))$, $L^2(\Omega \times (0, T); H^1(D))$ and in $L^{2p}(\Omega \times (0, T) \times D)$. We deduce that the approximate weak solution weakly converges along a subsequence to a limit. The main problem is then to identify the limit of the elliptic term and the reaction term, which we do by means of the so-called monotonicity method.

We prove in section 1.4 the uniqueness of the weak solution which in turn implies the convergence of the whole sequence.

Finally, in section 1.5 we return to the study of the nonlinear stochastic heat equation and prove the existence and uniqueness of the solution.

1.2 A preliminary change of functions

We consider the Neumann boundary value problem for the stochastic nonlinear heat equation

$$(P_1) \quad \begin{cases} \frac{\partial W_A}{\partial t} = \operatorname{div}(A(\nabla W_A)) + \frac{\partial W}{\partial t}, & x \in D, \quad t \geq 0, \\ A(\nabla W_A) \cdot \nu = 0, & x \in \partial D, \quad t \geq 0, \\ W_A(x, 0) = 0, & x \in D. \end{cases}$$

Krylov and Rozovskii [12] proved the well-posedness result for classes of problems similar to Problem (P_1) using a definition of solution in the distribution sense, while Gess [10] defines a solution in the sense of $L^2(D)$, namely almost everywhere in D . More precisely, he defines a strong solution as follows (cf. [10], Definition 1.3).

Definition 1.2.1 (Strong solution) *We say that W_A is a strong solution of Problem (P_1) if*

- (i) $W_A \in L^\infty(0, T; L^2(\Omega \times D)) \cap L^2(\Omega \times (0, T); H^1(D))$;
- (ii) $W_A \in L^2(\Omega; C([0, T]; L^2(D)))$;
- (iii) $\operatorname{div}(A(\nabla W_A)) \in L^2(\Omega \times (0, T); L^2(D))$;
- (iv) W_A satisfies a.s. for all $t \in (0, T)$ the problem

$$\begin{cases} W_A(t) = \int_0^t \operatorname{div}(A(\nabla W_A(s))) ds + W(t), \text{ in } L^2(D), \\ A(\nabla W_A(t)) \cdot \nu = 0, \text{ in a suitable sense of trace on } \partial D. \end{cases} \quad (1.2.1)$$

We will show in Section 1.5 the existence and uniqueness of the strong solution W_A of Problem (P_1) . Moreover we will prove that

$$W_A \in L^\infty(0, T; L^q(\Omega \times D)) \quad \text{for all } q \in [2, \infty). \quad (1.2.2)$$

We perform the change of functions

$$u(t) := \varphi(t) - W_A(t);$$

then φ is a solution of (P) if and only if u satisfies:

$$(P_2) \quad \begin{cases} \frac{\partial u}{\partial t} = \operatorname{div}(A(\nabla(u + W_A)) - A(\nabla W_A)) + f(u + W_A) \\ \quad - \frac{1}{|D|} \int_D f(u + W_A) dx, & x \in D, \quad t \geq 0, \\ A(\nabla(u + W_A)) \cdot \nu = 0, & x \in \partial D, t \geq 0, \\ u(x, 0) = \varphi_0(x), & x \in D. \end{cases}$$

We remark that (P_2) has the form of a deterministic problem; however it is stochastic since the random function W_A appears in the parabolic equation for u .

Definition 1.2.2 *We say that u is a solution of Problem (P_2) if :*

- (i) $u \in L^\infty(0, T; L^2(\Omega \times D)) \cap L^2(\Omega \times (0, T); H^1(D)) \cap L^{2p}(\Omega \times (0, T) \times D)$;
 $\operatorname{div}[A\nabla(u + W_A)] \in L^2(\Omega \times (0, T); (H^1(D))')$;
- (ii) u satisfies almost surely the problem, for all $t \in [0, T]$

$$\begin{cases} u(t) = \varphi_0 + \int_0^t \operatorname{div}[A(\nabla(u + W_A)) - A(\nabla W_A)] ds + \int_0^t f(u + W_A) \\ \quad - \int_0^t \frac{1}{|D|} \int_D f(u + W_A) dx ds, & \text{in the sense of distributions,} \\ A(\nabla(u + W_A)) \cdot \nu = 0, & \text{in the sense of distributions on } \partial D \times \mathbb{R}^+. \end{cases} \quad (1.2.3)$$

In order to check the conservation of mass property, namely that

$$\int_D u(x, t) dx = \int_D \varphi_0(x) dx, \quad \text{a.s. for a.e. } t \in \mathbb{R}^+,$$

we recall that $Z^* = V^* + L^{\frac{2p}{2p-1}}(D)$ and take the duality product of (1.2.3) with 1 for a.e. t and ω .

1.3 Existence of a solution of Problem (P_2)

The main result is the following

Theorem 1.3.1 *There exists a unique solution of Problem (P_2) .*

Proof In this subsection we apply the Galerkin method to prove the existence of a solution of Problem (P_2) .

Denote by $0 < \gamma_1 < \gamma_2 \leq \dots \leq \gamma_{\tilde{k}} \leq \dots$ the eigenvalues of the operator $-\Delta$ with homogeneous Neumann boundary conditions, and by $w_{\tilde{k}}, \tilde{k} = 0, \dots$ the corresponding unit eigenfunctions in $L^2(D)$. Note that they are smooth functions.

Lemma 1.3.2 *The functions $\{w_j\}$ are an orthonormal basis of $L^2(D)$ and satisfy :*

$$\int_D w_j w_0 dx = 0 \quad \text{for all } j \neq 0 \quad \text{and} \quad w_0 = \frac{1}{\sqrt{|D|}}.$$

Proof We check below that $\int_D w_j(x) dx = 0$ for all $j \neq 0$. Indeed,

$$\begin{aligned} \int_D w_j dx &= -\frac{1}{\gamma_j} \int_D \Delta w_j dx \\ &= -\frac{1}{\gamma_j} \int_{\partial D} \frac{\partial w_j}{\partial \nu} dx \\ &= 0, \end{aligned} \tag{1.3.1}$$

which implies that $\int_D w_j w_0 dx = 0$ for all $j \neq 0$. Moreover, it is standard that the eigenfunctions corresponding to different eigenvalues are orthogonal. \square

We look for an approximate solution of the form

$$u_m(x, t) - M = \sum_{i=1}^m u_{im}(t) w_i = \sum_{i=1}^m \langle u_m(t), w_i \rangle w_i,$$

where $M = \frac{1}{|D|} \int_D \varphi_0(x) dx$ such that the function $u_m - M$ satisfies the equations

$$\begin{aligned} &\int_D \frac{\partial}{\partial t} (u_m(x, t) - M) w_j dx \\ &= - \int_D [A(\nabla(u_m - M + W_A)) - A(\nabla(W_A))] \nabla w_j dx \\ &+ \int_D f(u_m + W_A) w_j - \frac{1}{|D|} \int_D \left(\int_D f(u_m + W_A) dx \right) w_j dx, \end{aligned} \tag{1.3.2}$$

for all w_j , $j = 1, \dots, m$. We remark that $u_m(x, 0) = M + \sum_{i=1}^m (\varphi_0, w_i) w_i$ converges strongly to φ_0 in $L^2(D)$ as $m \rightarrow \infty$.

Problem (1.3.2) is an initial value problem for a system of m ordinary differential equations with the unknown functions $u_{im}(t)$, $i = 1, \dots, m$ so that it has a unique solution u_m on some interval $(0, T_m)$, $T_m > 0$; in fact the following a priori estimates show that this solution is global in time.

First we remark that the contribution of the nonlocal term vanishes. Indeed for all $j = 1, \dots, m$

$$\begin{aligned} -\frac{1}{|D|} \int_D \left(\int_D f(u_m + W_A(t)) dx \right) w_j dx &= -\frac{1}{|D|} \left(\int_D f(u_m + W_A(t)) dx \right) \times \int_D w_j dx \\ &= 0. \end{aligned}$$

Therefore (1.3.2) reduces to the equation:

$$\int_D \frac{\partial}{\partial t} (u_m(x, t) - M) w_j dx = - \int_D [A(\nabla(u_m - M + W_A)) - A(\nabla(W_A))] \nabla w_j dx$$

$$+ \int_D f(u_m + W_A) w_j dx. \quad (1.3.3)$$

We multiply (1.3.3) by $u_{jm} = u_{jm}(t)$ and sum on $j = 1, \dots, m$:

$$\begin{aligned} & \int_D \frac{\partial}{\partial t} (u_m(x, t) - M)(u_m - M) dx \\ &= - \int_D [A(\nabla(u_m - M + W_A)) - A(\nabla(W_A))] \nabla(u_m - M) dx \\ & \quad + \int_D f(u_m + W_A)(u_m - M) dx. \end{aligned} \quad (1.3.4)$$

Next we apply the monotonicity property of A (1.1.3) to bound the generalized Laplacian term, which yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_D (u_m - M)^2 dx & \leq -C_0 \int_D |\nabla(u_m - M)|^2 dx \\ & \quad + \int_D f(u_m + W_A)(u_m - M) dx. \end{aligned} \quad (1.3.5)$$

Using the property (F_1) we deduce that

$$\begin{aligned} & \int_D f(u_m + W_A(t))(u_m - M) dx \\ &= \int_D f(u_m - M + M + W_A(t))(u_m - M) dx \\ & \leq \int_D [-C_1(u_m - M)^{2p} + C_2((M + W_A)^{2p}(t) + 1)] dx \\ & \leq - \int_D C_1(u_m - M)^{2p} dx + C_2 \int_D |W_A(t)|^{2p} dx + \tilde{C}_2(M)|D|, \end{aligned}$$

which we substitute in (1.3.5) to obtain :

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_D (u_m - M)^2 dx + C_0 \int_D |\nabla(u_m - M)|^2 dx + C_1 \int_D (u_m - M)^{2p} dx \\ & \leq C_2 \int_D |W_A(t)|^{2p} dx + \tilde{C}_2(M)|D|. \end{aligned} \quad (1.3.6)$$

1.3.1 A priori estimates

In what follows, we derive a priori estimates for the function u_m .

Lemma 1.3.3 *There exists a positive constant \mathcal{C} such that*

$$\sup_{t \in [0, T]} \mathbb{E} \int_D (u_m - M)^2 dx \leq \mathcal{C}, \quad (1.3.7)$$

$$\mathbb{E} \int_0^T \int_D |\nabla(u_m - M)|^2 dx dt \leq \mathcal{C}, \quad (1.3.8)$$

$$\mathbb{E} \int_0^T \int_D (u_m - M)^{2p} dx dt \leq \mathcal{C}, \quad (1.3.9)$$

$$\mathbb{E} \int_0^T \int_D (f(u_m + W_A))^{\frac{2p}{2p-1}} dx dt \leq \mathcal{C}, \quad (1.3.10)$$

$$\mathbb{E} \int_0^T \|\operatorname{div} A(\nabla(u_m + W_A))\|_{(H^1(D))}^2 dt \leq \mathcal{C}. \quad (1.3.11)$$

Proof Integrating (1.3.6) from 0 to t and taking the expectation we deduce that for all $t \in [0, T]$

$$\begin{aligned} & \frac{1}{2} \mathbb{E} \int_D (u_m - M)^2(t) dx + C_0 \mathbb{E} \int_0^t \int_D |\nabla(u_m - M)|^2 dx ds \\ & + C_1 \mathbb{E} \int_0^t \int_D (u_m - M)^{2p} dx ds \\ & \leq \frac{1}{2} \int_D (u_m(0) - M)^2 dx + C_2 \mathbb{E} \int_0^t \int_D |W_A(t)|^{2p} dx ds + \tilde{C}_2(M) |D| T \\ & \leq \frac{1}{2} \sum_{i=1}^m |\langle u_0, w_j \rangle|^2 dx + \tilde{C}_2(M) |D| T + c_2 T \\ & \leq \frac{1}{2} \|u_0 - M\|_{L^2(D)}^2 + \tilde{C}_2(M) |D| T + c_2 T \\ & \leq K \end{aligned}$$

where we have used (1.2.2).

We deduce that :

$$\begin{aligned} \mathbb{E} \int_D (u_m - M)^2(t) dx & \leq 2K, \quad \text{for all } t \in [0, T], \\ \mathbb{E} \int_0^T \int_D |\nabla(u_m - M)|^2 dx dt & \leq \frac{K}{C_0}, \\ \mathbb{E} \int_0^T \int_D (u_m - M)^{2p} dx dt & \leq \frac{K}{C_1}. \end{aligned}$$

Therefore $\{u_m\}$ is bounded independently of m in $L^\infty(0, T, L^2(\Omega \times D)) \cap L^2(\Omega \times (0, T); H^1(D)) \cap L^{2p}(\Omega \times (0, T) \times D)$.

Using the property (F₂) we deduce that

$$\begin{aligned} \mathbb{E} \|f(u_m + W_A)\|_{L^{\frac{2p}{2p-1}}((0,T) \times D)}^{\frac{2p}{2p-1}} & = \mathbb{E} \int_0^T \int_D |f(u_m + W_A)|^{\frac{2p}{2p-1}} dx dt \\ & \leq \mathbb{E} \int_0^T \int_D [C_3 |u_m + W_A - M|^{2p-1} + \tilde{C}_3(M)]^{\frac{2p}{2p-1}} dx dt \\ & \leq \mathbb{E} \int_0^T \int_D [C_3 (|u_m - M| + |W_A|)^{2p-1} dx + \tilde{C}_3(M)]^{\frac{2p}{2p-1}} dx dt \\ & \leq 2^{\frac{2p}{2p-1}-1} \mathbb{E} \int_0^T \int_D C_5 [(|u_m - M| + |W_A|)^{2p-1}]^{\frac{2p}{2p-1}} dx dt \\ & \quad + \tilde{C}_5 |D| T \\ & \leq c_3 \mathbb{E} \int_0^T \int_D (|u_m - M|^{2p-1})^{\frac{2p}{2p-1}} dx dt \end{aligned}$$

$$\begin{aligned}
 & + c_3 \mathbb{E} \int_0^T \int_D (|W_A|^{2p-1})^{\frac{2p}{2p-1}} dx dt + \tilde{C}_5 |D| T \\
 \leq & c_3 \mathbb{E} \int_0^T \int_D |u_m - M|^{2p} dx dt \\
 & + c_3 \mathbb{E} \int_0^T \int_D |W_A|^{2p} dx dt + \tilde{C}_5 |D| T \\
 \leq & K_1,
 \end{aligned}$$

by (1.3.9) and (1.2.2), where c_3 is a positive constant.

Finally we show that the elliptic term is bounded in $(H^1(D))'$. We have that

$$\begin{aligned}
 & \mathbb{E} \int_0^T \|\operatorname{div} A(\nabla(u_m + W_A))\|_{(H^1(D))'}^2 \\
 = & \mathbb{E} \int_0^T \left(\sup_{v \in H^1, \|v\|_{H^1} \leq 1} |\langle \operatorname{div}(A(\nabla(u_m + W_A))), v \rangle| \right)^2 \\
 = & \mathbb{E} \int_0^T \left(\sup_{v \in H^1, \|v\|_{H^1} \leq 1} \left| - \int_D A(\nabla(u_m + W_A)) \nabla v \right| \right)^2 \\
 \leq & \mathbb{E} \int_0^T \left\{ \sup_{v \in H^1, \|v\|_{H^1} \leq 1} \left(\int_D |A(\nabla(u_m + W_A))|^2 \right)^{\frac{1}{2}} \left(\int_D |\nabla v|^2 \right)^{\frac{1}{2}} \right\}^2 \\
 \leq & \mathbb{E} \int_0^T \sup_{v \in H^1, \|v\|_{H^1} \leq 1} \int_D |A(\nabla(u_m + W_A))|^2 \int_D \nabla v^2 \\
 \leq & \mathbb{E} \int_0^T \int_D |A(\nabla(u_m + W_A))|^2. \tag{1.3.12}
 \end{aligned}$$

Next we use (1.1.2) and (1.1.1) to estimate the term on the right-hand-side of (1.3.12)

$$\begin{aligned}
 \mathbb{E} \int_0^T \int_D |A(\nabla(u_m + W_A))|^2 & \leq C \mathbb{E} \int_0^T \int_D |\nabla(u_m + W_A)|^2 \\
 & \leq 2C \left(\mathbb{E} \int_0^T \int_D |\nabla u_m|^2 + \mathbb{E} \int_0^T \int_D |\nabla W_A|^2 \right) \\
 & \leq K_2.
 \end{aligned}$$

The last line follows from the a priori estimates and the regularity of the solution of Problem (P_1) . \square

Hence there exist a subsequence which we denote again by $\{u_m - M\}$ and a function $u - M \in L^2(\Omega \times (0, T); V) \cap L^{2p}(\Omega \times (0, T) \times D) \cap L^\infty(0, T; L^2(\Omega \times D))$ such that

$$\begin{aligned}
 u_m - M & \rightharpoonup u - M \quad \text{weakly in } L^2(\Omega \times (0, T); V) & (1.3.13) \\
 & \quad \text{and } L^{2p}(\Omega \times (0, T) \times D)
 \end{aligned}$$

$$u_m - M \rightharpoonup u - M \quad \text{weakly star in } L^\infty(0, T; L^2(\Omega \times D)) \tag{1.3.14}$$

$$f(u_m + W_A) \rightharpoonup \chi \quad \text{weakly in } L^{\frac{2p}{2p-1}}(\Omega \times (0, T) \times D) \tag{1.3.15}$$

$$\operatorname{div}(A(\nabla(u_m + W_A))) \rightharpoonup \Phi \quad \text{weakly in } L^2(\Omega \times (0, T); (H^1)') \quad (1.3.16)$$

as $m \rightarrow \infty$.

Next, we pass to the limit as $m \rightarrow \infty$.

To that purpose we integrate in time the equation (1.3.3) to obtain

$$\begin{aligned} \int_D (u_m(x, t) - M)w_j &= \int_D (u_m(0) - M)w_j \\ &+ \int_0^t \langle \operatorname{div}[A(\nabla(u_m - M + W_A)) - A(\nabla(W_A))], w_j \rangle \\ &+ \int_0^t \int_D f(u_m + W_A)w_j, \quad \text{for all } j = 1, \dots, m. \end{aligned} \quad (1.3.17)$$

Let $y = y(\omega)$ be an arbitrary bounded random variable, and let ψ be an arbitrary bounded function on $(0, T)$. We multiply the equation (1.3.17) by the product $y\psi$, integrate between 0 and T and take the expectation to deduce

$$\begin{aligned} &\mathbb{E} \int_0^T \int_D y\psi(t)(u_m(t) - M)w_j dx dt \\ &= \mathbb{E} \int_0^T \int_D y\psi(t)(u_m(0) - M)w_j dx dt \\ &+ \mathbb{E} \int_0^T y\psi(t) \left\{ \int_0^t \langle \operatorname{div}[A(\nabla(u_m - M + W_A))], w_j \rangle dt \right\} \\ &- \mathbb{E} \int_0^T y\psi(t) \left\{ \int_0^t \langle \operatorname{div}[A(\nabla(W_A))], w_j \rangle dt \right\} \\ &+ \mathbb{E} \int_0^T y\psi(t) \left\{ \int_0^t \int_D f(u_m + W_A)w_j dx ds \right\} dt. \end{aligned} \quad (1.3.18)$$

for all $j=1, \dots, m$.

Next we pass to the limit in (1.3.18); we only give the proof of convergence for the last term using the a priori estimates and Hölder inequality. We have that

$$\begin{aligned} &\left| \psi(t) \mathbb{E} \int_0^t \int_D f(u_m + W_A) y w_j dx ds \right| \\ &\leq \|y\|_{L^\infty(\Omega)} |\psi(t)| \left(\mathbb{E} \int_0^t \int_D |f(u_m + W_A)|^{\frac{2p}{2p-1}} dx ds \right)^{\frac{2p-1}{2p}} \left(\mathbb{E} \int_0^t \int_D |w_j|^{2p} dx ds \right)^{\frac{1}{2p}} \\ &\leq \|y\|_{L^\infty(\Omega)} \|\psi\|_{L^\infty(0, T)} \bar{C}. \end{aligned}$$

This shows that $|\psi(t) \mathbb{E} \int_0^t \int_D f(u_m + W_A) y w_j dx ds|$ is uniformly bounded by a function belonging to $L^1(0, T)$. In addition using (1.3.15) we have that $\psi(t) \mathbb{E} \int_0^t \int_D f(u_m + W_A) y w_j dx ds \rightarrow \psi(t) \mathbb{E} \int_0^t \int_D \chi y w_j dx ds$ for a.e. $t \in (0, T)$. Applying

Lebesgue-dominated convergence theorem we deduce that :

$$\begin{aligned}
 & \lim_{m \rightarrow \infty} \int_0^T \psi(t) dt \mathbb{E} \int_0^t \int_D f(u_m + W_A) y w_j dx ds \\
 &= \int_0^T \lim_{m \rightarrow \infty} \psi(t) \mathbb{E} \int_0^t \int_D f(u_m + W_A) y w_j dx ds dt \\
 &= \int_0^T \psi(t) dt \mathbb{E} \int_0^t \int_D \chi y w_j dx ds \\
 &= \mathbb{E} \int_0^T y \psi(t) dt \left\{ \int_0^t \int_D \chi w_j dx ds \right\}.
 \end{aligned}$$

Performing a similar proof for each term in (1.3.18), we pass to the limit by using Lebesgue-dominated convergence theorem. This yields

$$\begin{aligned}
 & \mathbb{E} \int_0^T \int_D y \psi(t) (u(t) - M) w_j dx dt \\
 &= \mathbb{E} \int_0^T \int_D y \psi(t) (\varphi_0 - M) w_j dx dt \\
 &+ \mathbb{E} \int_0^T y \psi(t) \left\{ \int_0^t \langle \Phi - \operatorname{div} A(\nabla(W_A)), w_j \rangle \right\} \\
 &+ \mathbb{E} \int_0^T y \psi(t) \left\{ \int_0^t \int_D \chi w_j dx ds \right\} dt, \quad \text{for all } j = 1, \dots, m.
 \end{aligned} \tag{1.3.19}$$

We remark that the linear combinations of w_j are dense in $V \cap L^{2p}(D)$, so that

$$\begin{aligned}
 & \mathbb{E} \int_0^T \int_D y \psi(t) (u(t) - M) \tilde{w} dx dt \\
 &= \mathbb{E} \int_0^T \int_D y \psi(t) (\varphi_0 - M) \tilde{w} dx dt \\
 &+ \mathbb{E} \int_0^T y \psi(t) \left\{ \int_0^t \langle \Phi - \operatorname{div} A(\nabla(W_A)), \tilde{w} \rangle ds \right\} dt \\
 &+ \mathbb{E} \int_0^T y \psi(t) \left\{ \int_0^t \int_D \chi \tilde{w} dx ds \right\} dt.
 \end{aligned}$$

for all $\tilde{w} \in V \cap L^{2p}(D)$, $y \in L^\infty(\Omega)$ and $\psi \in L^\infty(0, T)$. This implies that for a.e. $(t, \omega) \in (0, T) \times \Omega$

$$\langle u(t) - M, \tilde{w} \rangle = \langle \varphi_0 - M, \tilde{w} \rangle + \int_0^t \langle \Phi + \chi - \operatorname{div}(A(\nabla W_A)), \tilde{w} \rangle ds \tag{1.3.20}$$

for all $\tilde{w} \in V \cap L^{2p}(D)$.

Lemma 1.3.4 *The function u is such that $u \in C([0, T]; L^2(D))$ a.s.*

Proof

$$Z \subseteq H \subseteq Z^*$$

Since $u - M \in L^2(0, T; Z)$ a.s. and $\frac{du}{dt} \in L^2(0, T; V^*) + L^2(0, T; L^{\frac{2p}{2p-1}}(D)) = L^2(0, T; Z^*)$ a.s., it follows by applying Lemma 1.2 p.260 in [18] that $u - M \in C(0, T; H)$ a.s. \square

It remains to prove that :

$$\langle \Phi + \chi, \tilde{w} \rangle = \langle \operatorname{div}(A(\nabla(u + W_A))) + f(u + W_A(t)), \tilde{w} \rangle \quad \text{for all } \tilde{w} \in V \cap L^{2p}(D).$$

We do so by means of the monotonicity method.

1.3.2 Monotonicity argument

Let w be such that $w - M \in L^2(\Omega \times (0, T); V) \cap L^{2p}(\Omega \times D \times (0, T))$.

Let c be a positive constant which will be fixed later. We define

$$\begin{aligned} \mathcal{O}_m &= \mathbb{E} \left[\int_0^T e^{-cs} \left\{ 2 \langle \operatorname{div} \left(A(\nabla(u_m - M + W_A)) - A(\nabla W_A) \right) \right. \right. \\ &\quad \left. \left. - \operatorname{div} \left(A(\nabla(w - M + W_A)) - A(\nabla W_A) \right), u_m - M - (w - M) \rangle_{Z^*, Z} \right. \right. \\ &\quad \left. \left. + 2 \langle f(u_m + W_A) - f(w + W_A), u_m - M - (w - M) \rangle_{Z^*, Z} \right. \right. \\ &\quad \left. \left. - c \|u_m - M - (w - M)\|^2 \right\} ds \right] \\ &= J_1 + J_2 + J_3, \end{aligned}$$

and prove below the following result

Lemma 1.3.5

$$\mathcal{O}_m \leq 0.$$

Proof First we estimate J_1 and apply (1.1.3)

$$\begin{aligned} J_1 &= \mathbb{E} \int_0^T e^{-cs} \left\{ 2 \langle \operatorname{div} \left(A(\nabla(u_m - M + W_A)) \right) \right. \\ &\quad \left. - \operatorname{div} \left(A(\nabla(w - M + W_A)) \right), u_m - M - (w - M) \rangle_{Z^*, Z} \right\} \\ &= -2 \mathbb{E} \int_0^T e^{-cs} \int_D [A(\nabla(u_m - M + W_A)) - A(\nabla(w - M + W_A))] \\ &\quad [\nabla(u_m - M + W_A) - \nabla(w - M + W_A)] \\ &\leq -2C_0 \mathbb{E} \int_0^T e^{-cs} \|\nabla(u_m - w)\|^2 \\ &\leq 0. \end{aligned}$$

(F₃) and the mean value theorem yield:

$$\begin{aligned} J_2 &= \mathbb{E} \int_0^T e^{-cs} 2 \langle f(u_m + W_A) - f(w + W_A), u_m - w \rangle_{Z^*, Z} ds \\ &\leq \mathbb{E} \int_0^T e^{-cs} 2C_4 \|u_m - w\|^2 ds. \end{aligned}$$

Choosing $c \geq 2C_4$, we conclude the result. \square

We write \mathcal{O}_m in the form $\mathcal{O}_m = \mathcal{O}_m^1 + \mathcal{O}_m^2$ where

$$\begin{aligned} \mathcal{O}_m^1 &= \mathbb{E} \left[\int_0^T e^{-cs} \{ 2 \langle \operatorname{div} (A(\nabla(u_m - M + W_A)) - A(\nabla W_A)), u_m - M \rangle_{Z^*, Z} \right. \\ &\quad \left. + 2 \langle f(u_m + W_A), u_m - M \rangle_{Z^*, Z} - c \|u_m - M\|^2 \} ds \right]. \end{aligned} \quad (1.3.21)$$

We integrate the equation (1.3.3) between 0 and T to obtain

$$\begin{aligned} \int_D (u_m(x, T) - M) w_j &= \int_D (u_m(0) - M) w_j \\ &\quad + \int_0^T \langle \operatorname{div} [A(\nabla(u_m - M + W_A)) - A(\nabla W_A)], w_j \rangle_{Z^*, Z} \\ &\quad + \int_0^T \int_D f(u_m + W_A) w_j, \quad \text{for all } j = 1, \dots, m. \end{aligned} \quad (1.3.22)$$

Next we recall a chain rule formula, which can be viewed as a simplified Itô's formula.

Proposition 1.3.6 *Let X be a real valued function such that*

$$X(t) = X(0) + \int_0^t h(s) ds, \quad 0 \leq s \leq t,$$

and suppose that h is measurable in time such that $h \in L^1(0, T)$. Suppose that the function $F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and its partial derivatives $\frac{\partial F}{\partial t}$ and $\frac{\partial F}{\partial X}$ are continuous on $[0, T] \times \mathbb{R}$. Then for all $t \in [0, T]$

$$F(t, X(t)) = F(0, X(0)) + \int_0^t \frac{\partial F}{\partial t}(s, X(s)) ds + \int_0^t \frac{\partial F}{\partial X}(s, X(s)) h(s) ds. \quad (1.3.23)$$

Applying (1.3.23) to the m equations in (1.3.22) with

$$X_j = \int_D (u_m - M) w_j, \quad j = 1, \dots, m, \quad F(s, q) = e^{-cs} q^2,$$

and $h(s) = \langle \operatorname{div} [A(\nabla(u_m - M + W_A)) - A(\nabla W_A)] + f(u_m + W_A), w_j \rangle_{Z^*, Z}$, we deduce that

$$\begin{aligned} &e^{-cT} \left(\int_D (u_m(x, T) - M) w_j \right)^2 \\ &= \left(\int_D (u_m(0) - M) w_j \right)^2 - c \int_0^T e^{-cs} \left(\int_D (u_m - M) w_j \right)^2 ds \\ &\quad + 2 \int_0^T e^{-cs} \left\{ \int_D (u_m - M) w_j \right\} \langle \operatorname{div} [A(\nabla(u_m - M + W_A)) - A(\nabla W_A)], w_j \rangle \\ &\quad + 2 \int_0^T e^{-cs} \left\{ \int_D (u_m - M) w_j \right\} \langle f(u_m + W_A), w_j \rangle, \quad \text{for all } j = 1, \dots, m. \end{aligned} \quad (1.3.24)$$

In what follows, we will use the identity

Lemma 1.3.7 *Let $F \in Z^*$ and $B_m = \sum_{j=1}^m \langle B_m, w_j \rangle w_j$.*

Then

$$\sum_{j=1}^m \langle F, w_j \rangle \langle B_m, w_j \rangle = \langle F, B_m \rangle. \quad (1.3.25)$$

Proof

$$\begin{aligned} \sum_{j=1}^m \langle F, w_j \rangle \langle B_m, w_j \rangle &= \sum_{j=1}^m \langle F, \langle B_m, w_j \rangle w_j \rangle \\ &= \langle F, \sum_{j=1}^m \langle B_m, w_j \rangle w_j \rangle \\ &= \langle F, B_m \rangle. \end{aligned} \quad \square$$

Summing (1.3.24) on $j = 1, \dots, m$ and applying the identity (1.3.25) yields

$$\begin{aligned} &e^{-cT} \|u_m(T) - M\|^2 \\ &= \|u_m(0) - M\|^2 - c \int_0^T e^{-cs} \|u_m - M\|^2 ds \\ &\quad + 2 \int_0^T e^{-cs} \langle \operatorname{div}[A(\nabla(u_m - M + W_A)) - A(\nabla W_A)], u_m - M \rangle_{Z^*, Z} \\ &\quad + 2 \int_0^T e^{-cs} \langle f(u_m + W_A), u_m - M \rangle_{Z^*, Z}. \end{aligned} \quad (1.3.26)$$

Taking the expectation of the equation (1.3.26) yields

$$\begin{aligned} &\mathbb{E}[e^{-cT} \|u_m(T) - M\|^2] \\ &= \mathbb{E}[\|u_m(0) - M\|^2] - c \mathbb{E}\left[\int_0^T e^{-cs} \|u_m(s) - M\|^2 ds\right] \\ &\quad + 2 \mathbb{E}\left[\int_0^T e^{-cs} \langle \operatorname{div}[A(\nabla(u_m - M + W_A)) - A(\nabla W_A)], u_m - M \rangle_{Z^*, Z} ds\right] \\ &\quad + 2 \mathbb{E}\left[\int_0^T e^{-cs} \langle f(u_m + W_A), u_m - M \rangle_{Z^*, Z} ds\right]. \end{aligned} \quad (1.3.27)$$

It follows from (1.3.21) and (1.3.27) that

$$\mathcal{O}_m^1 = \mathbb{E}[e^{-cT} \|u_m(T) - M\|^2] - \mathbb{E}[\|u_m(0) - M\|^2].$$

From this we obtain

$$\lim_{m \rightarrow \infty} \sup \mathcal{O}_m^1 = \mathbb{E}[e^{-cT} \|u(T) - M\|^2] - \mathbb{E}[\|u(0) - M\|^2] + \delta e^{-cT}, \quad (1.3.28)$$

where

$$\delta = \lim_{m \rightarrow \infty} \sup \mathbb{E}[\|u_m(T) - M\|^2] - \mathbb{E}[\|u(T) - M\|^2] \geq 0.$$

On the other hand, the equation (1.3.20) implies that

$$u(t) - M = \varphi_0 - M + \int_0^t \Phi - \operatorname{div}(A(\nabla W_A)) + \int_0^t \chi, \quad \forall t \in [0, T] \quad (1.3.29)$$

a.s. in $Z^* = V^* + L^{\frac{2p}{2p-1}}(D)$.

Next we recall a second variant of the chain rule formula, which can be viewed as a simplified Itô's formula as in [15] [p.75 Theorem 4.2.5], and involves different function spaces. Consider the Gelfand triple

$$Z \subset H \subset Z^*,$$

where $Z = V \cap L^{2p}(D)$ and Z^* are defined in the introduction.

Proposition 1.3.8 *Let $X \in L^2(0, T; V) \cap L^{2p}(0, T; L^{2p}(D))$ and $Y \in L^2(0, T; V^*) + L^{\frac{2p}{2p-1}}(0, T; L^{\frac{2p}{2p-1}}(D))$ be such that*

$$X(t) := X_0 + \int_0^t Y(s) ds, \quad t \in [0, T].$$

Suppose that the function $F : [0, T] \times Z \rightarrow \mathbb{R}$ and its partial derivatives $\frac{\partial F}{\partial t}$ and $\frac{\partial F}{\partial X}$ are continuous on $[0, T] \times Z$. Then for all $t \in [0, T]$

$$F(t, X(t)) = F(0, X(0)) + \int_0^t \frac{\partial F}{\partial t}(s, X(s)) ds + \int_0^t \langle Y(s), \frac{\partial F}{\partial X}(s, X(s)) \rangle_{Z^*, Z} ds. \quad (1.3.30)$$

Applying Proposition 1.3.8 to the equation (1.3.29), we set $X(t) = u(t) - M$, $F(s, q) = e^{-cs} \|q\|^2$, and $Y(s) = \Phi - \operatorname{div}(A(\nabla W_A)) + \chi$, in (1.3.30) to deduce that

$$\begin{aligned} \mathbb{E}[e^{-cT} \|u(T) - M\|^2] &= \mathbb{E}[\|\varphi_0 - M\|^2] - c\mathbb{E}\left[\int_0^T e^{-cs} \|u(s) - M\|^2 ds\right] \\ &\quad + 2\mathbb{E}\int_0^T e^{-cs} \langle \Phi - \operatorname{div}(A(\nabla W_A)), u - M \rangle_{Z^*, Z} \\ &\quad + 2\mathbb{E}\left[\int_0^T e^{-cs} \langle \chi, u - M \rangle_{Z^*, Z}\right], \end{aligned}$$

which we combine with (1.3.28) to deduce that

$$\begin{aligned} \lim_{m \rightarrow \infty} \sup \mathcal{O}_m^1 &= 2\mathbb{E}\left[\int_0^T e^{-cs} \langle \Phi - \operatorname{div}(A(\nabla W_A)), u - M \rangle_{Z^*, Z}\right] \\ &\quad + 2\mathbb{E}\int_0^T e^{-cs} \langle \chi, u - M \rangle_{Z^*, Z} \\ &\quad - c\mathbb{E}\left[\int_0^T e^{-cs} \|u(s) - M\|^2 ds\right] + \delta e^{-cT}. \end{aligned} \quad (1.3.31)$$

It remains to compute the limit of \mathcal{O}_m^2 :

$$\mathcal{O}_m^2 = \mathcal{O}_m - \mathcal{O}_m^1$$

$$\begin{aligned}
 &= \mathbb{E} \int_0^T e^{-cs} \{ -2 \langle \operatorname{div}[A(\nabla(w - M + W_A)) - A(\nabla W_A)], u_m - M \rangle_{Z^*, Z} \\
 &\quad - 2 \langle \operatorname{div}[A(\nabla(u_m - M + W_A)) - A(\nabla W_A)] \\
 &\quad\quad - \operatorname{div}[A(\nabla(w - M + W_A)) - A(\nabla W_A)], w - M \rangle_{Z^*, Z} \\
 &\quad - 2 \langle f(w + W_A), u_m - M \rangle_{Z^*, Z} - 2 \langle f(u_m + W_A) - f(w + W_A), w - M \rangle_{Z^*, Z} \\
 &\quad - c \|w - M\|^2 + 2c \langle u_m - M, w - M \rangle \} ds.
 \end{aligned}$$

In view of (1.3.13), (1.3.15) and (1.3.16), we deduce that

$$\begin{aligned}
 &\lim_{m \rightarrow \infty} \mathcal{O}_m^2 \\
 &= \mathbb{E} \int_0^T e^{-cs} \{ -2 \langle \operatorname{div}[A(\nabla(w - M + W_A)) - A(\nabla W_A)], u - M \rangle_{Z^*, Z} \\
 &\quad - 2 \langle \Phi - \operatorname{div}(A(\nabla W_A)) - \operatorname{div}[A(\nabla(w - M + W_A)) - A(\nabla W_A)], w - M \rangle_{Z^*, Z} \\
 &\quad - 2 \langle f(w + W_A), u - M \rangle_{Z^*, Z} - 2 \langle \chi - f(w + W_A), w - M \rangle_{Z^*, Z} \\
 &\quad - c \|w - M\|^2 + 2c \langle u - M, w - M \rangle \} ds. \tag{1.3.32}
 \end{aligned}$$

Combining (1.3.31) and (1.3.32), and remembering that $O_m \leq 0$, yields

$$\begin{aligned}
 &\mathbb{E} \int_0^T e^{-cs} \{ 2 \langle \Phi - \operatorname{div}[A(\nabla(w - M + W_A))], u - M - (w - M) \rangle_{Z^*, Z} \\
 &\quad + 2 \langle \chi - f(w + W_A), u - M - (w - M) \rangle_{Z^*, Z} \\
 &\quad - c \|u - M - (w - M)\|^2 \} + \delta e^{-cT} \leq 0.
 \end{aligned}$$

Let $v \in L^2(\Omega \times (0, T); V) \cap L^{2p}(\Omega \times (0, T) \times D)$ be arbitrary and set

$$w - M = u - M - \lambda v, \text{ with } \lambda \in \mathbb{R}_+.$$

We obtain the inequality :

$$\begin{aligned}
 &\mathbb{E} \int_0^T e^{-cs} \{ 2 \langle \Phi - \operatorname{div}[A(\nabla(u - \lambda v - M + W_A))], \lambda v \rangle_{Z^*, Z} \\
 &\quad + 2 \langle \chi - f(u - \lambda v + W_A), \lambda v \rangle_{Z^*, Z} - c \|\lambda v\|^2 \} dt \leq 0.
 \end{aligned}$$

Dividing by λ and letting $\lambda \rightarrow 0$, we find that :

$$\mathbb{E} \int_0^T e^{-cs} \langle \Phi + \chi - \operatorname{div}[A(\nabla(u - M + W_A))] - f(u + W_A), v \rangle_{Z^*, Z} dt \leq 0.$$

Since v is arbitrary, it follows that

$$\mathbb{E} \int_0^T \langle \Phi + \chi, v \rangle_{Z^*, Z} = \mathbb{E} \int_0^T \langle \operatorname{div}[A(\nabla(u - M + W_A))] + f(u + W_A), v \rangle_{Z^*, Z},$$

for all $v \in L^2(\Omega \times (0, T); V) \cap L^{2p}(\Omega \times (0, T) \times D)$, or else

$$\Phi + \chi = \operatorname{div}[A(\nabla(u - M + W_A))] + f(u + W_A) + \theta(t, \omega), \quad (1.3.33)$$

a.s. a.e. in $D \times (0, T)$. Taking the duality product of (1.3.33) with $\tilde{w} \in V \cap L^{2p}(D)$ we obtain that

$$\begin{aligned} \langle \Phi + \chi, \tilde{w} \rangle_{Z^*, Z} &= \langle \operatorname{div}[A(\nabla(u - M + W_A))] + f(u + W_A) + \theta(\omega, t), \tilde{w} \rangle_{Z^*, Z} \\ &= \langle \operatorname{div}[A(\nabla(u - M + W_A))] + f(u + W_A), \tilde{w} \rangle_{Z^*, Z}. \end{aligned} \quad (1.3.34)$$

Substituting (1.3.34) in (1.3.20) we deduce that for a.e. $(t, \omega) \in (0, T) \times \Omega$

$$\begin{aligned} \langle u(t) - M, \tilde{w} \rangle &= \langle \varphi_0 - M, \tilde{w} \rangle + \int_0^t \langle \operatorname{div}[A(\nabla(u - M + W_A))] \\ &\quad + f(u + W_A) - \operatorname{div}(A(\nabla W_A)), \tilde{w} \rangle_{Z^*, Z} ds. \end{aligned} \quad (1.3.35)$$

for all $\tilde{w} \in V \cap L^{2p}(D)$.

This completes the identification of the limit terms by the monotonicity method.

Next, we prove that u satisfies the equation (1.2.3) in Definition 1.2.2. We define

$$\mathcal{V} = H^1(D) \cap L^{2p}(D).$$

The equation (1.3.35) implies that a.s. in $\mathcal{V}^* = (H^1(D))' + L^{\frac{2p}{2p-1}}(D)$

$$\begin{aligned} u(t) &= \varphi_0 + \int_0^t \operatorname{div}[A(\nabla(u - M + W_A))] - \operatorname{div}(A(\nabla W_A)) + \int_0^t f(u + W_A) ds \\ &\quad + \int_0^t \lambda(s) ds, \end{aligned} \quad (1.3.36)$$

for all $t \in [0, T]$.

In order to identify the last term of (1.3.36), we take its duality product $\langle \cdot, \cdot \rangle_{\mathcal{V}^*, \mathcal{V}}$ with 1. Remembering that the equation is mass conserved, we obtain

$$\int_D \int_0^t f(u + W_A) ds dx + \int_0^t \lambda(s) ds |D| = \int_D u(t) dx - \int_D \varphi_0 dx = 0. \quad (1.3.37)$$

Thus,

$$\int_0^t \lambda(s) ds = -\frac{1}{|D|} \int_D \int_0^t f(u + W_A) ds dx,$$

so that also

$$\lambda(t) = -\frac{1}{|D|} \int_D f(u(x, t) + W_A(x, t)) dx.$$

1.4 Uniqueness of the solution of Problem (P_2)

Let ω be given such that two pathwise solutions of Problem (P_2) , $u_1 = u_1(\omega, x, t)$ and $u_2 = u_2(\omega, x, t)$ satisfy

$$\begin{aligned} u_i(\cdot, \cdot, \omega) &\in L^\infty(0, T; L^2(D)) \cap L^2(0, T; H^1(D)) \cap L^{2p}((0, T) \times D), \\ f(u_i + W_A) &\in L^{\frac{2p}{2p-1}}((0, T) \times D), \\ \operatorname{div}(A(\nabla(u_i + W_A))) &\in L^2((0, T); (H^1(D))') \end{aligned}$$

for $i = 1, 2$, and $u_1(\cdot, 0) = u_2(\cdot, 0) = \varphi_0$. Then

$$\begin{aligned} u_1(x, t) &= u_1(x, 0) + \int_0^t \operatorname{div}(A(\nabla(u_1 + W_A))) - \operatorname{div}(A(\nabla W_A)) + \int_0^t f(u_1 + W_A) \\ &\quad - \frac{1}{|D|} \int_0^t \int_D f(u_1 + W_A) dx, \end{aligned}$$

$$\begin{aligned} u_2(x, t) &= u_2(x, 0) + \int_0^t \operatorname{div}(A(\nabla(u_2 + W_A))) - \operatorname{div}(A(\nabla W_A)) + \int_0^t f(u_2 + W_A) \\ &\quad - \frac{1}{|D|} \int_0^t \int_D f(u_2 + W_A) dx, \end{aligned}$$

so that the difference $u_1 - u_2$ satisfies the equation

$$\begin{aligned} u_1(t) - u_2(t) &= \int_0^t \operatorname{div}(A(\nabla(u_1 + W_A)) - A(\nabla(u_2 + W_A))) \\ &\quad + \int_0^t [f(u_1 + W_A) - f(u_2 + W_A)] \\ &\quad - \frac{1}{|D|} \int_0^t \left[\int_D f(u_1 + W_A) - \int_D f(u_2 + W_A) dx \right], \end{aligned}$$

in $L^2((0, T); V^*) + L^{\frac{2p}{2p-1}}((0, T) \times D)$.

We take the duality product of this equation with $u_1 - u_2 \in L^2((0, T); V^*) \cap L^{\frac{2p}{2p-1}}((0, T) \times D)$, to deduce that

$$\begin{aligned} \|u_1 - u_2\|_{L^2(D)}^2 &= 2 \int_0^t \langle \operatorname{div}(A(\nabla(u_1 + W_A)) - A(\nabla(u_2 + W_A))), u_1 - u_2 \rangle_{Z^*, Z} \\ &\quad + 2 \int_0^t \langle f(u_1 + W_A) - f(u_2 + W_A), u_1 - u_2 \rangle_{Z^*, Z} \\ &\quad - 2 \int_0^t \left\langle \frac{1}{|D|} \int_D (f(u_1 + W_A) - f(u_2 + W_A)) dx, u_1 - u_2 \right\rangle_{Z^*, Z} \\ &= -2 \int_0^t \int_D (A(\nabla(u_1 + W_A)) - A(\nabla(u_2 + W_A))) \nabla(u_1 - u_2) \\ &\quad + 2 \int_0^t \int_D (f(u_1 + W_A) - f(u_2 + W_A))(u_1 - u_2) dx \\ &\quad - 2 \frac{1}{|D|} \int_0^t \left[\int_D (f(u_1 + W_A) - f(u_2 + W_A)) dx \int_D (u_1 - u_2) dx \right] \end{aligned}$$

$$\begin{aligned}
 &= -2 \int_0^t \int_D (A(\nabla(u_1 + W_A)) - A(\nabla(u_2 + W_A))) \nabla(u_1 - u_2) \\
 &+ 2 \int_0^t \int_D (f(u_1 + W_A) - f(u_2 + W_A))(u_1 - u_2) dx, \quad (1.4.1)
 \end{aligned}$$

where we remark that since $\int_D u_1(x, t) dx = \int_D u_2(x, t) dx = \int_D \varphi_0(x) dx$, the nonlocal term vanishes.

In view of (1.1.3), (1.4.1) becomes

$$\begin{aligned}
 &\|u_1 - u_2\|_{L^2(D)}^2 \\
 &\leq \int_0^t \int_D (f(u_1 + W_A) - f(u_2 + W_A))(u_1 - u_2) dx dt \\
 &\quad - C_0 \int_0^t \int_D \nabla(u_1 - u_2)^2 dx dt, \quad (1.4.2)
 \end{aligned}$$

for all $t \in (0, T)$. In addition, the property (F_3) implies that

$$(f(u_1 + W_A) - f(u_2 + W_A))(u_1 - u_2) \leq C_4 \int_D (u_1 - u_2)^2. \quad (1.4.3)$$

□

Substituting (1.4.3) in (1.4.2) yields

$$\int_D (u_1 - u_2)^2(x, t) dx \leq C_4 \int_0^t \int_D (u_1 - u_2)^2(x, t) dx \quad \text{for all } t \in (0, T),$$

which in turn implies by Gronwall's Lemma that

$$u_1 = u_2 \quad \text{a.e. in } D \times (0, T).$$

1.5 Existence and uniqueness of the solution of Problem (P_1)

In this section we return to the study of the solution W_A of Problem (P_1) , and derive a priori estimates for a Galerkin approximation in $L^\infty(0, T; L^2(\Omega \times D)) \cap L^2(\Omega \times (0, T); H^1(D)) \cap L^2(\Omega \times (0, T); H^2(D))$ following an idea due to Gess [10]. We are then in a position to show that W_A is also bounded in $L^\infty(0, T; L^q(\Omega \times D))$ for all $q \geq 2$, which is necessary for the proof of Lemma 1.3.3.

We show below a priori estimates, which imply that the elliptic term $\text{div}(A(\nabla W_A))$ is bounded in $L^2(D)$ having in mind that Problem (P_1) is a special case of Problem (4.33) in [10] (see also equation (2.8) in [10]). Whereas Gess concentrates on the special case of the p-Laplacian, we are interested in the uniformly parabolic case, which corresponds to $m = 2$ in [10] p.280-281. We also remark that there are no reaction terms i.e. $f_i = 0$ for all i from 1 to n and that the noise is additive. However, Gess assumes that the nonlinear function Ψ is twice continuously differentiable while we only suppose that $\Psi \in C^{1,1}(\mathbb{R}^n)$.

We prove the following result.

Theorem 1.5.1 *There exists a unique solution of Problem (P_1) .*

Proof To begin with, we approximate the function Ψ by a sufficiently smooth function Ψ^n such that

$$\Psi^n \rightarrow \Psi \text{ in } C^1(\mathbb{R}^n). \quad (1.5.1)$$

and

$$\|D^2\Psi^n\|_{L^\infty(\mathbb{R}^n, \mathbb{R}^{n \times n})} \leq c_1, \quad \nabla\Psi^n(0) = 0,$$

and derive a priori estimates for a Galerkin approximation as in [10] (p. 2363 (2.13)). It turns out that the upper bounds which we find do not depend on n .

We define $W_A^{m,n}$ by

$$W_A^{m,n}(t) = \int_0^t P_m[\operatorname{div}(\nabla\Psi^n(\nabla W_A^{m,n}(s)))]ds + \sum_{l=1}^m P_m(\sqrt{\lambda_l}e_l)\beta_l(t) \quad (1.5.2)$$

a.s., where for $v \in L^2(D)$ $P_m v := \sum_{j=1}^m (\int_D v w_j) w_j$ and $P_m : H^1(D) \rightarrow H_m = \operatorname{span}\{w_1, \dots, w_m\}$, $m \in \mathbb{N}$ is the continuous operator defined by

$$\|a - P_m a\|_{H^1(D)}^2 = \inf_{v \in H_m} \|a - v\|_{H^1(D)}^2, \quad a \in H^1(D) \quad (\text{c.f. [10] p. 2363}).$$

Note that (cf. [5] p.193)

$$\|P_m a\|_{H^1(D)} \leq \|a\|_{H^1(D)}. \quad (1.5.3)$$

and that (cf. [10] Remark 2.3)

$$P_m a \rightarrow a, \quad \text{in } H^1(D) \quad \text{as } m \rightarrow \infty. \quad (1.5.4)$$

This implies in particular that

$$P_m a \rightarrow a, \quad \text{in } L^2(D) \quad \text{as } m \rightarrow \infty. \quad (1.5.5)$$

In addition, we have that (cf. [12] p. 49)

$$\int_D u_m P_m[\operatorname{div}(\nabla\Psi^n(\nabla W_A^{m,n}))] = - \int_D \nabla u_m \nabla\Psi^n(\nabla W_A^{m,n}). \quad (1.5.6)$$

Indeed,

$$\begin{aligned} & \int_D u_m \sum_{j=1}^m \left(\int_D \operatorname{div}(\nabla\Psi^n(\nabla W_A^{m,n})) w_j \right) w_j \\ &= \sum_{j=1}^m \int_D u_m w_j \int_D \operatorname{div}(\nabla\Psi^n(\nabla W_A^{m,n})) w_j \\ &= \int_D \operatorname{div}(\nabla\Psi^n(\nabla W_A^{m,n})) \sum_{j=1}^m \left(\int_D u_m w_j \right) w_j \\ &= \int_D \operatorname{div}(\nabla\Psi^n(\nabla W_A^{m,n})) u_m \\ &= - \int_D \nabla u_m \nabla\Psi^n(\nabla W_A^{m,n}). \end{aligned}$$

Lemma 1.5.2 *There exists a positive constant \mathcal{K} such that*

$$\mathbb{E} \int_0^T \int_D (W_A^{m,n})^2 dx dt \leq \mathcal{K}, \quad (1.5.7)$$

$$\mathbb{E} \int_0^T \int_D |\nabla(W_A^{m,n})|^2 dx dt \leq \mathcal{K}, \quad (1.5.8)$$

$$\mathbb{E} \int_0^T \|P_m \operatorname{div}(\nabla \Psi^n(\nabla(W_A^{m,n})))\|_{L^2(D)}^2 dt \leq \mathcal{K}, \quad (1.5.9)$$

$$\sup_{t \in (0, T)} \mathbb{E} \int_D (W_A^{m,n})^2 dx \leq \mathcal{K}. \quad (1.5.10)$$

Proof We first recall Itô's formula as in [16] p.16-17 which is based on [11] [p.153, Theorem 3.6], and is applicable to systems of stochastic ordinary differential equations .

Lemma 1.5.3 *For a smooth vector function h and an adapted process $(g(t), t \geq 0)$ with $\int_0^T |g(t)| dt < \infty$ almost surely, for all $T > 0$ set*

$$X(t) := \int_0^t g(s) ds + \int_0^t h d\mathcal{W}(s), \quad 0 \leq t \leq T,$$

where h is a vector of components h_l , $l = 1, \dots, m$ and $d\mathcal{W}$ is a vector of components $d\beta_l$, $l = 1, \dots, m$ with β_l a one-dimensional Brownian motion. Then, for F twice continuously differentiable in X and continuously differentiable in t , one has

$$\begin{aligned} F(X(t), t) &= F(X(0), 0) + \int_0^t F_t(X(s), s) ds + \int_0^t F_x(X(s)) g(s) ds \\ &\quad + \int_0^t F_x(X(s)) h d\mathcal{W}(s) + \frac{1}{2} \sum_{l=1}^m \int_0^t F_{xx}(X(s)) h_l^2 ds. \end{aligned} \quad (1.5.11)$$

Next we apply Lemma 1.5.3 to (1.5.2) with $h d\mathcal{W} = \sum_{l=1}^m \mathcal{P}_m \sqrt{\lambda_l} e_l d\beta_l(s)$ and $h_l = \mathcal{P}_m \sqrt{\lambda_l} e_l$, supposing that F does not depend on time and setting

$$\begin{aligned} X(t) &= W_A^{m,n}(t), \\ F(X(t)) &= (X(t))^2, \\ F_x(X(t)) &= 2X(t), \\ F_{xx}(X(t)) &= 2, \\ g(s) &= P_m \operatorname{div}(\nabla \Psi^n(\nabla W_A^{m,n}(s))). \end{aligned}$$

We remark that in this case F does not depend on t . After integrating on D , we obtain almost surely, for all $t \in [0, T]$,

$$\begin{aligned} \int_D W_A^{m,n}(x, t)^2 dx &= 2 \int_0^t \int_D W_A^{m,n} P_m \operatorname{div}(\nabla \Psi^n(\nabla W_A^{m,n}(s))) dx ds \\ &\quad + 2 \sum_{l=1}^m \int_0^t \int_D W_A^{m,n} \mathcal{P}_m \sqrt{\lambda_l} e_l dx d\beta_l(s) \end{aligned}$$

$$+ \int_0^t \sum_{l=1}^m \|\mathcal{P}_m \sqrt{\lambda_l} e_l\|_{L^2(D)}^2 dx ds. \quad (1.5.12)$$

Substituting (1.5.6) into (1.5.12) we obtain,

$$\begin{aligned} & \|W_A^{m,n}(t)\|_{L^2(D)}^2 + 2 \int_0^t \int_D \nabla W_A^{m,n} \nabla \Psi^n (\nabla W_A^{m,n}(s)) dx ds \\ &= 2 \sum_{l=1}^m \int_0^t \int_D W_A^{m,n} \mathcal{P}_m \sqrt{\lambda_l} e_l dx d\beta_l(s) + \int_0^t \sum_{l=1}^m \|\mathcal{P}_m \sqrt{\lambda_l} e_l\|_{L^2(D)}^2 dx ds. \end{aligned} \quad (1.5.13)$$

Taking the expectation, we obtain

$$\begin{aligned} & \mathbb{E} \|W_A^{m,n}(t)\|_{L^2(D)}^2 + 2 \mathbb{E} \int_0^t \int_D \nabla W_A^{m,n} \nabla \Psi^n (\nabla W_A^{m,n}(s)) dx ds \\ &= \mathbb{E} \int_0^t \sum_{l=1}^m \|\mathcal{P}_m \sqrt{\lambda_l} e_l\|_{L^2(D)}^2 dx ds, \end{aligned} \quad (1.5.14)$$

where we have used the fact that $2\mathbb{E}[\sum_{l=1}^m \int_0^t \int_D W_A^{m,n} \sqrt{\lambda_l} e_l dx d\beta_l(s)] = 0$ ([13] Theorem 2.3.4 - p.11).

We deduce from (1.5.3) that

$$\begin{aligned} \sum_{l=1}^m \|\mathcal{P}_m \sqrt{\lambda_l} e_l\|_{L^2(D)}^2 &\leq \sum_{l=1}^m (\|\sqrt{\lambda_l} e_l\|_{L^2(D)}^2 + \|\nabla(\sqrt{\lambda_l} e_l)\|_{L^2(D)}^2) \\ &\leq (\Lambda_0 + \Lambda_2). \end{aligned} \quad (1.5.15)$$

Taking the supremum of equation (1.5.14) and substituting (1.5.15) into (1.5.14) we obtain

$$\sup_{t \in (0, T)} \mathbb{E} \|W_A^{m,n}(t)\|_{L^2(D)}^2 \leq T(\Lambda_0 + \Lambda_2) \leq \mathcal{K}.$$

This completes the proof of (1.5.10).

In order to obtain an H^2 -type estimate for W_A^m , we take the gradient of the equation (1.5.2). For all $x \in D$, we have that

$$\begin{aligned} \nabla W_A^{m,n}(t) &= \int_0^t \nabla \{P_m[\text{div}(\nabla \Psi^n(\nabla W_A^{m,n}))]\} ds + \sum_{l=1}^m \nabla \{P_m[\sqrt{\lambda_l} e_l]\} \beta_l(t) \\ &= \int_0^t \nabla \{P_m[\text{div}(\nabla \Psi^n(\nabla W_A^{m,n}))]\} ds + \sum_{l=1}^m \int_0^t \nabla \{P_m[\sqrt{\lambda_l} e_l]\} d\beta_l(s). \end{aligned} \quad (1.5.16)$$

We fix $x \in D$ and apply below for a second time Itô's formula Lemma 1.5.3 to the integral equation (1.5.16) where in this case $hdW = \sum_{l=1}^m \nabla \{P_m \sqrt{\lambda_l} e_l\} d\beta_l(s)$ and $h_l = \nabla \{P_m \sqrt{\lambda_l} e_l\}$ with:

$$X(t) = \nabla W_A^{m,n}(x, t),$$

$$\begin{aligned}
 F(X(t)) &= \Psi^n(\nabla W_A^{m,n}(x, t)), \\
 F_x(X(t)) &= \nabla \Psi^n(\nabla W_A^{m,n}(x, t)), \\
 F_{xx}(X(t)) &= D^2 \Psi^n(\nabla W_A^{m,n}(x, t)), \text{ and} \\
 g(s) &= \nabla \{P_m \operatorname{div}(\nabla \Psi^n(\nabla W_A^{m,n}(x, s)))\}.
 \end{aligned}$$

After integrating over D , we obtain almost surely, for all $t \in [0, T]$,

$$\begin{aligned}
 & \int_D \Psi^n(\nabla W_A^{m,n}(x, t)) dx \\
 &= \int_0^t \int_D \nabla \Psi^n(\nabla W_A^{m,n}(x, s)) \nabla \{P_m \operatorname{div}(\nabla \Psi^n(\nabla W_A^{m,n}(s)))\} dx ds \\
 & \quad + \sum_{l=1}^m \int_0^t \int_D \nabla \Psi^n(\nabla W_A^{m,n}(x, s)) \nabla \{P_m(\sqrt{\lambda_l} e_l)\} dx d\beta_l(s) \\
 & \quad + \frac{1}{2} \sum_{l=1}^m \int_0^t \int_D D^2 \Psi^n(\nabla W_A^{m,n}(x, s)) |\nabla P_m(\sqrt{\lambda_l} e_l)|^2 dx ds.
 \end{aligned}$$

In view of (1.1.1) and (1.5.6) we have that

$$\begin{aligned}
 & \int_D \Psi^n(\nabla W_A^{m,n}(t)) dx \\
 & \leq - \int_0^t \int_D [P_m \operatorname{div}(\nabla \Psi^n(\nabla W_A^{m,n}(s)))]^2 dx ds \\
 & \quad + \sum_{l=1}^m \int_0^t \int_D \nabla \Psi^n(\nabla W_A^{m,n}(x, s)) \nabla \{P_m(\sqrt{\lambda_l} e_l)\} dx d\beta_l(s) \\
 & \quad + \frac{1}{2} \|D^2 \Psi^n(\nabla W_A^{m,n}(s))\|_{L^\infty(D)} \sum_{l=1}^m \int_0^t \int_D |\nabla P_m(\sqrt{\lambda_l} e_l)|^2 ds \\
 & \leq - \int_0^t \|P_m \operatorname{div}(\nabla \Psi^n(\nabla W_A^{m,n}(s)))\|_{L^2(D)}^2 ds \\
 & \quad + \sum_{l=1}^m \int_0^t \int_D \nabla \Psi^n(\nabla W_A^{m,n}(x, s)) \nabla \{P_m(\sqrt{\lambda_l} e_l)\} dx d\beta_l(s) \\
 & \quad + \frac{c_1}{2} \sum_{k=1}^m \int_0^t \|\nabla P_m \sqrt{\lambda_l} e_l\|_{L^2(D)}^2 ds. \tag{1.5.17}
 \end{aligned}$$

Thus taking the expectation of (1.5.17) and using the fact that $\mathbb{E}[\sum_{l=1}^m \int_0^t \int_D \nabla \Psi^n(\nabla W_A^{m,n}(x, s)) \nabla \{P_m(\sqrt{\lambda_l} e_l)\} dx d\beta_l(s)] = 0$, we obtain

$$\begin{aligned}
 & \mathbb{E} \int_D \Psi^n(\nabla W_A^{m,n}(t)) dx + \mathbb{E} \int_0^t \|P_m \operatorname{div}(\nabla \Psi^n(\nabla W_A^{m,n}(s)))\|_{L^2(D)}^2 ds \\
 & \leq \frac{c_1}{2} \sum_{l=1}^m \mathbb{E} \int_0^t \|\nabla P_m \sqrt{\lambda_l} e_l\|_{L^2(D)}^2 ds. \tag{1.5.18}
 \end{aligned}$$

Adding (1.5.14) and (1.5.18), using (1.5.3), (3.1.4) and (1.1.6) we obtain

$$\mathbb{E} \int_D \Psi^n(\nabla W_A^{m,n}(t)) dx + \mathbb{E} \|W_A^{m,n}\|_{L^2(D)}^2$$

$$\begin{aligned}
 & + \mathbb{E} \int_0^t \|P_m \operatorname{div}(\nabla \Psi^n(\nabla W_A^{m,n}(s)))\|_{L^2(D)}^2 ds \\
 & + 2\mathbb{E} \int_0^t \int_D \nabla W_A^{m,n} \nabla \Psi^n(\nabla W_A^{m,n}(s)) dx ds \\
 \leq & c_0 \mathbb{E} \int_0^t \sum_{l=1}^m \left(\|P_m \sqrt{\lambda_l} e_l\|_{L^2(D)}^2 + \|\nabla P_m \sqrt{\lambda_l} e_l\|_{L^2(D)}^2 \right) \\
 \leq & c_0 \mathbb{E} \int_0^t \sum_{l=1}^m \left(\|\sqrt{\lambda_l} e_l\|_{L^2(D)}^2 + \|\nabla \sqrt{\lambda_l} e_l\|_{L^2(D)}^2 \right) \\
 \leq & c_0 \mathbb{E} \int_0^t \left(\sum_{l=1}^m \lambda_l \|e_l\|_{L^2(D)}^2 + \sum_{l=1}^m \lambda_l \|\nabla e_l\|_{L^2(D)}^2 \right) ds \\
 \leq & c_0 T (\Lambda_0 + \Lambda_2),
 \end{aligned}$$

where $c_0 = \max(1, \frac{c_1}{2})$. In view of (1.1.3) we obtain,

$$\begin{aligned}
 & \mathbb{E} \int_D \Psi^n(\nabla W_A^{m,n}(t)) dx + \mathbb{E} \|W_A^{m,n}(t)\|_{L^2(D)}^2 \\
 & + \mathbb{E} \int_0^t \|P_m[\operatorname{div}(\nabla \Psi^n(\nabla W_A^{m,n}(s)))]\|_{L^2(D)}^2 ds \\
 & + 2C_0 \mathbb{E} \int_0^t \|\nabla W_A^{m,n}(s)\|_{L^2(D)}^2 ds \\
 \leq & c_0 T (\Lambda_0 + \Lambda_2) \leq \mathcal{K},
 \end{aligned}$$

which completes the proof of (1.5.7), (1.5.8) and (1.5.9). \square

Hence there exist a subsequence which we denote again by $W_A^{m,n}$ and a function $W_A \in L^2(\Omega \times (0, T); H^1) \cap L^\infty(0, T; L^2(\Omega \times D))$ such that

$$W_A^{m,n} \rightharpoonup W_A \text{ weakly in } L^2(\Omega \times (0, T); H^1(D)), \quad (1.5.19)$$

$$W_A^{m,n} \rightharpoonup^* W_A \text{ weakly star in } L^\infty(0, T; L^2(\Omega \times D)), \quad (1.5.20)$$

$$P_m \operatorname{div}(\nabla \Psi^n(\nabla W_A^{m,n}(s))) \rightharpoonup \tilde{\Phi} \text{ weakly in } L^2(\Omega \times (0, T); L^2(D)), \quad (1.5.21)$$

as $m, n \rightarrow \infty$.

In addition, one can show the following result.

Lemma 1.5.4

$$\sum_{l=1}^m P_m(\sqrt{\lambda_l} e_l) \beta_l(t) \xrightarrow{m \rightarrow \infty} \sum_{l=1}^{\infty} \sqrt{\lambda_l} e_l \beta_l(t), \text{ in } L^\infty((0, T); L^2(\Omega; L^2(D))). \quad (1.5.22)$$

Proof For all $t \in [0, T]$,

$$\mathbb{E} \int_D \left| \sum_{l=1}^{\infty} \sqrt{\lambda_l} e_l \beta_l(t) - \sum_{l=1}^m P_m(\sqrt{\lambda_l} e_l) \beta_l(t) \right|^2$$

$$\begin{aligned}
 &\leq 2\mathbb{E} \int_D \left| \sum_{l=1}^{\infty} \sqrt{\lambda_l} e_l \beta_l(t) - \sum_{l=1}^m \sqrt{\lambda_l} e_l \beta_l(t) \right|^2 \\
 &+ 2\mathbb{E} \int_D \left| \sum_{l=1}^m \sqrt{\lambda_l} e_l \beta_l(t) - \sum_{l=1}^m \mathcal{P}_m(\sqrt{\lambda_l} e_l) \beta_l(t) \right|^2 \\
 &\leq 2\mathbb{E} \int_D \left| \int_0^t \sum_{l=m+1}^{\infty} \sqrt{\lambda_l} e_l \beta_l(t) \right|^2 \\
 &+ 2\mathbb{E} \int_D \left| \sum_{l=1}^m [\mathcal{P}_m(\sqrt{\lambda_l} e_l) - \sqrt{\lambda_l} e_l] \beta_l(t) \right|^2 \\
 &= W_1 + W_2.
 \end{aligned}$$

By [9] p. 20 we deduce that $W_1 \rightarrow 0$ in $C([0, T])$ as $m \rightarrow \infty$. For W_2 , by the properties of the Brownian motion, we have that

$$\begin{aligned}
 &2 \int_D \mathbb{E} \left| \sum_{l=1}^m [\mathcal{P}_m(\sqrt{\lambda_l} e_l) - \sqrt{\lambda_l} e_l] \beta_l(t) \right|^2 \\
 = &2 \int_D \mathbb{E} \left(\sum_{l=1}^m [\mathcal{P}_m(\sqrt{\lambda_l} e_l) - \sqrt{\lambda_l} e_l]^2 \beta_l^2(t) \right. \\
 &\quad \left. + 2 \sum_{l_1 \neq l_2}^m [\mathcal{P}_m(\sqrt{\lambda_{l_1}} e_{l_1}) - \sqrt{\lambda_{l_1}} e_{l_1}] \beta_{l_1} [\mathcal{P}_m(\sqrt{\lambda_{l_2}} e_{l_2}) - \sqrt{\lambda_{l_2}} e_{l_2}] \beta_{l_2} \right) dx \\
 = &2 \int_D \sum_{l=1}^m [\mathcal{P}_m(\sqrt{\lambda_l} e_l) - \sqrt{\lambda_l} e_l]^2 \mathbb{E}[\beta_l^2(t)] \\
 = &2 \sum_{l=1}^m \|\mathcal{P}_m(\sqrt{\lambda_l} e_l) - \sqrt{\lambda_l} e_l\|_{L^2(D)}^2 t \\
 \leq &2T \sum_{l=1}^{\infty} \|\mathcal{P}_m(\sqrt{\lambda_l} e_l) - \sqrt{\lambda_l} e_l\|_{L^2(D)}^2. \tag{1.5.23}
 \end{aligned}$$

In order to prove that the right-hand side of (1.5.23) tends to zero as $m \rightarrow \infty$, we use (1.5.3) and (1.5.5) to deduce that

$$\begin{aligned}
 &\sum_{l=1}^{\infty} \|\mathcal{P}_m(\sqrt{\lambda_l} e_l) - \sqrt{\lambda_l} e_l\|_{L^2(D)}^2 \\
 = &\sum_{l=1}^K \|\mathcal{P}_m(\sqrt{\lambda_l} e_l) - \sqrt{\lambda_l} e_l\|_{L^2(D)}^2 + \sum_{l=K+1}^{\infty} \|\mathcal{P}_m(\sqrt{\lambda_l} e_l) - \sqrt{\lambda_l} e_l\|_{L^2(D)}^2 \\
 \leq &\sum_{l=1}^K \|\mathcal{P}_m(\sqrt{\lambda_l} e_l) - \sqrt{\lambda_l} e_l\|_{L^2(D)}^2 + 2 \sum_{l=K+1}^{\infty} \|\sqrt{\lambda_l} e_l\|_{H^1(D)}^2 \\
 \leq &\sum_{l=1}^K \|\mathcal{P}_m(\sqrt{\lambda_l} e_l) - \sqrt{\lambda_l} e_l\|_{L^2(D)}^2 + 4 \sum_{l=K+1}^{\infty} (\lambda_l + \lambda_l \|\nabla e_l\|_{L^2(D)}^2)
 \end{aligned}$$

$$\leq P_1 + P_2. \quad (1.5.24)$$

Let $\varepsilon > 0$ be arbitrary. We choose K such that $P_2 \leq \frac{\varepsilon}{2}$. For a fixed K , we choose m sufficiently large such that $P_1 \leq \frac{\varepsilon}{2}$. Therefore,

$$\sum_{l=1}^{\infty} \|\mathcal{P}_m(\sqrt{\lambda_l}e_l) - \sqrt{\lambda_l}e_l\|_{L^2(D)}^2 \leq \varepsilon. \quad (1.5.25)$$

so that $W_2 \rightarrow 0$ in $C([0, T])$ as $m \rightarrow \infty$. \square

Let y be an arbitrary bounded random variable, and let ψ be an arbitrary bounded function on $(0, T)$. Next we multiply the equation (1.5.2) by the product $y\psi$, integrate on D between 0 and T and take the expectation to obtain

$$\begin{aligned} & \mathbb{E} \int_0^T \int_D y\psi(t) W_A^{m,n} w_j dx dt \\ &= \mathbb{E} \int_0^T y\psi(t) \left\{ \int_0^t \langle \mathcal{P}_m(\operatorname{div}[\nabla \Psi^n(\nabla W_A^{m,n})]), w_j \rangle ds \right\} dt \\ &+ \mathbb{E} \int_0^T y\psi(t) \left\{ \int_D \sum_{l=1}^m \mathcal{P}_m(\sqrt{\lambda_l}e_l) \beta_l(t) w_j dx \right\} dt. \end{aligned}$$

Passing to the limit when $m, n \rightarrow \infty$, using (1.5.19)-(1.5.21) and (1.5.22), and remembering that the linear combinations of w_j are dense in $H^1(D)$, yields

$$\begin{aligned} \mathbb{E} \int_0^T \int_D y\psi(t) W_A \tilde{w} dx dt &= \mathbb{E} \int_0^T y\psi(t) \left\{ \int_0^t \langle \tilde{\Phi}, \tilde{w} \rangle ds \right\} dt \\ &+ \mathbb{E} \int_0^T y\psi(t) \left\{ \int_D \sum_{l=1}^{\infty} (\sqrt{\lambda_l}e_l) \beta_l(t) \tilde{w} dx \right\} dt, \end{aligned}$$

for all $\tilde{w} \in H^1(D)$. Therefore, we deduce that

$$W_A(t) = \int_0^t \tilde{\Phi}(s) ds + \sum_{l=1}^{\infty} \sqrt{\lambda_l} e_l \beta_l(t) \quad \text{on } \Omega \times (0, T) \times D. \quad (1.5.26)$$

We will prove below, using again the monotonicity method, that $\tilde{\Phi} = \operatorname{div}(\nabla \Psi(\nabla W_A))$.

1.5.1 Monotonicity argument

Let w be such that $w \in L^2(\Omega \times (0, T); H^1(D))$ and let c be a positive constant.

We define

$$\begin{aligned} \mathcal{O}_{mn} &= \mathbb{E} \left[\int_0^T e^{-cs} \{ 2 \langle \mathcal{P}_m[\operatorname{div}(\nabla \Psi^n(\nabla W_A^{m,n}))], W_A^{m,n} - w \rangle \right. \\ &\quad \left. - c \|W_A^{m,n} - w\|^2 \} ds \right] \\ &= J_1 + J_2. \end{aligned}$$

We will check as before the following result

Lemma 1.5.5

$$O_{mn} \leq 0.$$

Proof Using (1.5.6) and (1.1.3) we have that

$$\begin{aligned} J_1 &= \langle P_m[\operatorname{div}(\nabla\Psi^n(\nabla W_A^{m,n}))] - P_m[\operatorname{div}(\nabla\Psi^n(\nabla w))], W_A^{m,n} - w \rangle \\ &= - \int_D [\nabla\Psi^n(\nabla W_A^{m,n}) - \nabla\Psi^n(\nabla w)] \nabla(W_A^{m,n} - w) \\ &\leq -C_0 \|\nabla(W_A^{m,n} - w)\|_{L^2(D)}^2 \leq 0, \end{aligned} \tag{1.5.27}$$

which completes the proof. \square

We write \mathcal{O}_{mn} in the form $\mathcal{O}_{mn} = \mathcal{O}_{mn}^1 + \mathcal{O}_{mn}^2$ where

$$\mathcal{O}_{mn}^1 = \mathbb{E} \left[\int_0^T e^{-cs} \{ 2 \langle P_m[\operatorname{div}(\nabla\Psi^n(\nabla W_A^{m,n}))], W_A^{m,n} \rangle - c \|W_A^{m,n}\|^2 \} ds \right]. \tag{1.5.28}$$

We apply Itô formula Lemma 1.5.3 on (1.5.2) with $F(X, t) = e^{-ct}(X)^2$ and $F_t = -ce^{-ct}(X)^2$. After integrating on D and taking the expectation, we obtain almost surely, for all $t \in [0, T]$,

$$\begin{aligned} &\mathbb{E}[e^{-cT} \|W_A^{m,n}(x, T)\|_{L^2(D)}^2] \\ &= -c \mathbb{E} \left[\int_0^T e^{-cs} \|W_A^{m,n}(x, s)\|_{L^2(D)}^2 ds \right] \\ &\quad + 2 \mathbb{E} \left[\int_0^T e^{-cs} \int_D W_A^{m,n} P_m[\operatorname{div}(\nabla\Psi^n(\nabla W_A^{m,n}(s)))] dx ds \right] \\ &\quad + 2 \mathbb{E} \left[\sum_{l=1}^m \int_0^T e^{-cs} \int_D W_A^{m,n} \mathcal{P}_m \sqrt{\lambda_l} e_l dx d\beta_l(s) \right] \\ &\quad + \int_0^T e^{-cs} \sum_{l=1}^m \|\mathcal{P}_m \sqrt{\lambda_l} e_l\|_{L^2(D)}^2 dx ds. \end{aligned} \tag{1.5.29}$$

It follows from (1.5.28), (1.5.29) and the fact that

$$\mathbb{E} \left[\sum_{l=1}^m \int_0^T e^{-cs} \int_D W_A^{m,n} \mathcal{P}_m \sqrt{\lambda_l} e_l dx d\beta_l(s) \right] = 0$$

that

$$O_{mn}^1 = \mathbb{E} [e^{-cT} \|W_A^{m,n}(T)\|_{L^2(D)}^2] - \int_0^T e^{-cs} \sum_{l=1}^m \|\mathcal{P}_m \sqrt{\lambda_l} e_l\|_{L^2(D)}^2 ds. \tag{1.5.30}$$

In view of [5] p.193 we have that

$$\begin{aligned} &\left| \int_0^T e^{-cs} \left\{ \sum_{l=1}^m \|\mathcal{P}_m \sqrt{\lambda_l} e_l\|_{L^2(D)}^2 - \sum_{l=1}^{\infty} \|\sqrt{\lambda_l} e_l\|_{L^2(D)}^2 \right\} ds \right| \\ &\leq \int_0^T e^{-cs} \left| \sum_{l=1}^m \|\mathcal{P}_m \sqrt{\lambda_l} e_l\|_{L^2(D)}^2 - \sum_{l=1}^m \|\sqrt{\lambda_l} e_l\|_{L^2(D)}^2 \right| ds \end{aligned}$$

$$\begin{aligned}
 & + \int_0^T e^{-cs} \left| \sum_{l=1}^m \|\sqrt{\lambda_l} e_l\|_{L^2(D)}^2 - \sum_{l=1}^{\infty} \|\sqrt{\lambda_l} e_l\|_{L^2(D)}^2 \right| ds \\
 & \leq \int_0^T e^{-cs} \sum_{l=1}^m \left| \|\mathcal{P}_m \sqrt{\lambda_l} e_l\|_{L^2(D)}^2 - \|\sqrt{\lambda_l} e_l\|_{L^2(D)}^2 \right| ds \\
 & \quad + \int_0^T e^{-cs} \sum_{l=m+1}^{\infty} \|\sqrt{\lambda_l} e_l\|_{L^2(D)}^2 ds \\
 & \leq T \sum_{l=1}^{\infty} \left| \|\mathcal{P}_m \sqrt{\lambda_l} e_l\|_{L^2(D)}^2 - \|\sqrt{\lambda_l} e_l\|_{L^2(D)}^2 \right| + T \sum_{l=m+1}^{\infty} \lambda_l \\
 & \leq T \sum_{l=1}^{\infty} \|\sqrt{\lambda_l} e_l - \mathcal{P}_m \sqrt{\lambda_l} e_l\|_{L^2(D)}^2 + T \sum_{l=m+1}^{\infty} \lambda_l \\
 & \leq \varepsilon,
 \end{aligned}$$

which, in view of (1.5.25) and (3.1.3), tends to zero as $m \rightarrow \infty$. Thus,

$$\lim_{m \rightarrow \infty} \int_0^T e^{-cs} \sum_{l=1}^m \|\mathcal{P}_m \sqrt{\lambda_l} e_l\|_{L^2(D)}^2 ds = \int_0^T e^{-cs} \sum_{l=1}^{\infty} \|\sqrt{\lambda_l} e_l\|_{L^2(D)}^2 ds.$$

Letting m and n tend to infinity in (1.5.30), we deduce that

$$\lim_{m,n \rightarrow \infty} \sup \mathcal{O}_{mn}^1 = \mathbb{E}[e^{-cT} \|W_A(T)\|_{L^2(D)}^2] - \int_0^T e^{-cs} \sum_{l=1}^{\infty} \lambda_l ds + \delta e^{-cT}, \quad (1.5.31)$$

where

$$\delta = \lim_{m,n \rightarrow \infty} \sup \mathbb{E}[\|W_A^{m,n}(T)\|^2] - \mathbb{E}[\|W_A(T)\|^2] \geq 0.$$

On the other hand, the equation (1.5.26) implies that a.s. in $L^2(D)$

$$W_A(t) = \int_0^t \tilde{\Phi}(s) ds + \int_0^t dW(s), \quad \forall t \in [0, T]. \quad (1.5.32)$$

Next we recall a simplified form of the Itô's formula given by [7] (Theorem 4.32 p.106), which will suffice for our purpose. We do so since the Itô's formula given in Lemma 1.5.3 only applies to finite dimensional problems.

Lemma 1.5.6 *Let h be an $L^2(D)$ -valued progressively measurable Bochner integrable process. Consider the following well defined process :*

$$X(t) = \int_0^t h(s) ds + W(t), \quad t \in [0, T].$$

Assume that a function $F : [0, T] \times L^2(D) \rightarrow \mathbb{R}$ and its partial derivatives F_t, F_x, F_{xx} are uniformly continuous on bounded subsets of $[0, T] \times L^2(D)$, and that $F(X(0), 0) = 0$.

Then, a.s., for all $t \in [0, T]$,

$$\begin{aligned} F(X(t), t) &= \int_0^t F_t(X(s), s) ds + \int_0^t \langle F_x(X(s), s), h(s) \rangle_{L^2(D)} ds \\ &\quad + \int_0^t \langle F_x(X(s), s), dW(s) \rangle_{L^2(D)} + \frac{1}{2} \int_0^t \text{Tr}[F_{xx}(X(s), s)Q] ds \end{aligned}$$

where

$$\text{Tr}[F_{xx}(X(s))Q] = \sum_{l=1}^{\infty} \langle F_{xx}(X(s), s)Qe_l, e_l \rangle_{L^2(D)}$$

and

$$\langle u, v \rangle_{L^2(D)} = \int_D u(x)v(x) dx,$$

where we note that $\text{Tr}A = \sum_{l=1}^{\infty} \langle Ae_l, e_l \rangle_{L^2(D)}$ is bounded linear operator on $L^2(D)$.

Applying Lemma 1.5.6 to (1.5.32) with

$$\begin{aligned} X &= W_A, \\ F(X, t) &= e^{-ct} \|X\|_{L^2(D)}^2, \\ F_t(X, t) &= -ce^{-ct} \|X\|_{L^2(D)}^2, \\ F_x(X, t) &= 2e^{-ct} X, \\ h &= \tilde{\Phi}, \\ F_{xx}(X, t) &= 2e^{-ct} I. \end{aligned}$$

After taking the expectation, we deduce that

$$\begin{aligned} \mathbb{E}[e^{-cT} \|W_A\|^2] &= -c\mathbb{E}\left[\int_0^T e^{-cs} \|W_A\|^2 ds\right] + 2\mathbb{E}\int_0^T e^{-cs} \langle \tilde{\Phi}, W_A \rangle \\ &\quad + 2\mathbb{E}\left[\sum_{l=1}^{\infty} \int_0^T e^{-cs} \int_D W_A \sqrt{\lambda_l} e_l dx d\beta_l(s)\right] + \int_0^T e^{-cs} \sum_{l=1}^{\infty} \lambda_l ds, \end{aligned}$$

which we combine with (1.5.31) to deduce that

$$\lim_{m, n \rightarrow \infty} \sup O_{mn}^1 = 2\mathbb{E}\left[\int_0^T e^{-cs} \langle \tilde{\Phi}, W_A \rangle\right] - c\mathbb{E}\left[\int_0^T e^{-cs} \|W_A\|^2 ds\right] + \delta e^{-cT}. \quad (1.5.33)$$

It remains to compute the limit of \mathcal{O}_{mn}^2 :

$$\begin{aligned} &\mathcal{O}_{mn}^2 \\ &= \mathcal{O}_{mn} - \mathcal{O}_{mn}^1 \\ &= \mathbb{E} \int_0^T e^{-cs} \{-2\langle P_m[\text{div}(\nabla \Psi^n(\nabla w))], W_A^{m,n} \rangle - 2\langle P_m(\text{div}(\nabla \Psi^n(\nabla W_A^{m,n}))), w \rangle\} \end{aligned}$$

$$+2\langle P_m[\operatorname{div}(\nabla\Psi^n(\nabla w))], w\rangle - c\|w\|^2 + 2c\langle W_A^{m,n}, w\rangle\}ds.$$

In view of (1.5.19), (1.5.21), using (1.5.1) and (1.5.6) we deduce that

$$\begin{aligned} \lim_{m,n\rightarrow\infty} \mathcal{O}_{mn}^2 &= \mathbb{E} \int_0^T e^{-cs} \{-2\langle \operatorname{div}(\nabla\Psi(\nabla w)), W_A\rangle - 2\langle \check{\Phi} - \operatorname{div}(\nabla\Psi(\nabla w)), w\rangle \\ &\quad - c\|w\|^2 + 2c\langle W_A, w\rangle\}ds. \end{aligned} \quad (1.5.34)$$

Combining (1.5.33) and (1.5.34), and remembering that $O_{mn} \leq 0$, yields

$$\mathbb{E} \int_0^T e^{-cs} \{2\langle \check{\Phi} - \operatorname{div}(\nabla\Psi(\nabla w)), W_A - w\rangle - c\|W_A - w\|^2\}ds + \delta e^{-cT} \leq 0.$$

Let $\tilde{v} \in L^2(\Omega \times (0, T); H^1(D))$ be arbitrary and set

$$w = W_A - \lambda\tilde{v}, \text{ with } \lambda \in \mathbb{R}_+.$$

Dividing by λ and letting $\lambda \rightarrow 0$, we find that :

$$\mathbb{E} \int_0^T e^{-cs} \langle \check{\Phi} - \operatorname{div}(\nabla\Psi(\nabla W_A)), \tilde{v}\rangle dt \leq 0.$$

Since \tilde{v} is arbitrary, it follows that

$$\mathbb{E} \int_0^T \langle \check{\Phi}, \tilde{v}\rangle = \mathbb{E} \int_0^T \langle \operatorname{div}(\nabla\Psi(\nabla W_A)), \tilde{v}\rangle,$$

for all $\tilde{v} \in L^2(\Omega \times (0, T); H^1(D))$, that is

$$\check{\Phi} = \operatorname{div}(\nabla\Psi(\nabla W_A)), \quad (1.5.35)$$

a.s. a.e. in $D \times (0, T)$.

One finally concludes that W_A satisfies Definition 1.2.1.

Next, we prove below the boundedness of W_A in $L^\infty(0, T; L^q(\Omega \times D))$, for all $q \geq 2$. The proof of this result is based on an article by Bauzet, Vallet, Wittbold [2] where a similar result was proved for a convection-diffusion equation with a multiplicative noise on \mathbb{R}^n involving a standard adapted one-dimensional Brownian motion. More precisely, we follow the proof of Proposition A.5 of [2].

Theorem 1.5.7 *Let W_A be a solution of Problem (P_1) ; then $W_A \in L^\infty(0, T; L^q(\Omega \times D))$, for all $q \geq 2$.*

Proof For each positive constant k , denote by $\Phi_k : \mathbb{R} \rightarrow \mathbb{R}$ the function

$$\Phi_k(\xi) = \begin{cases} |\xi|^q, & \text{if } |\xi| < k, \\ \frac{q}{2}(q-1)k^{q-2}\xi^2 - q(q-2)k^{q-1}|\xi| + \left(\frac{q}{2} - 1\right)(q-1)k^q, & \text{if } k \leq |\xi|. \end{cases}$$

Φ_k is a convex C^2 function and Φ'_k is a Lipschitz-continuous function with $\Phi'_k(0)=0$. The function Φ_k satisfies the inequalities $0 \leq \Phi'_k(\xi) \leq c(k)\xi$ and $0 \leq \Phi_k(\xi) = \int_0^\xi \Phi'_k(\zeta)d\zeta \leq \frac{c(k)}{2}\xi^2$ for all $\xi \in \mathbb{R}^+$. This yields in view of Definition 1.2.1 (i) that, $\mathbb{E} \int_D \Phi_k(W_A(x, t))dx \leq \frac{c(k)}{2} \mathbb{E} \int_D W_A^2(x, t)dx \leq \bar{c}(k)$ for a.e. $t \in [0, T]$.

Lemma 1.5.8 (i) One has $0 \leq \Phi''_k(\xi) \leq c_k$ for all $\xi \in \mathbb{R}$ where c_k is a positive constant depending on k .

(ii) One has $0 \leq \Phi''_k(\xi) \leq q(q-1)(1 + \Phi_k(\xi))$, for all $\xi \in \mathbb{R}$.

Proof (i)

$$\Phi''_k(\xi) = \begin{cases} q(q-1)|\xi|^{q-2} & \text{if } 0 \leq |\xi| < k, \\ q(q-1)k^{q-2}, & \text{if } k \leq |\xi|. \end{cases}$$

Thus ,

$$\Phi''_k(\xi) \leq q(q-1)k^{q-2} =: c_k$$

(ii) If $|\xi| < k$, $\Phi''_k(\xi) = q(q-1)|\xi|^{q-2}$,

• if $1 \leq |\xi| < k$, $|\xi|^{q-2} \leq |\xi|^q$ which gives the result.

• if $0 \leq |\xi| < 1$, $|\xi|^q \leq |\xi|^{q-2} < 1$ then $|\xi|^{q-2} < 1 + |\xi|^q$. □

If $|\xi| \geq k$, $\Phi''_k(\xi) = q(q-1)k^{q-2}$ the problem then reduces to prove that

$$H(\xi) = 1 + \frac{q}{2}(q-1)k^{q-2}\xi^2 - q(q-2)k^{q-1}|\xi| + \left(\frac{q}{2} - 1\right)(q-1)k^q - k^{q-2} \geq 0$$

Let us consider the function $H(\xi) = F(\xi) + G$ where $F(\xi) = \frac{q}{2}(q-1)k^{q-2}\xi^2 - q(q-2)k^{q-1}|\xi|$ and $G = \left(\frac{q}{2} - 1\right)(q-1)k^q - k^{q-2} + 1$.

• if $\xi \geq k$, $H'(\xi) = F'(\xi) \geq 0$ and $H(k) \geq 0$ for all $k > 0$, thus $H(\xi) \geq H(k) \geq 0$ for all $\xi \geq k$.

• if $\xi \leq -k$ then $H(-\xi) = F(-\xi) + G \geq 0$. Therefore

$$H(\xi) \geq H(-\xi) \geq 0.$$

Next we apply Lemma 1.5.6 to (1.2.1), supposing that F does not depend on time and setting

$$\begin{aligned} X(t) &= W_A(t), \\ F(X(t)) &= \int_D \Phi_k(X(t))dx, \\ F_x(X(t)) &= \Phi'_k(X(t)), \\ h &= \text{div}(A(\nabla W_A)), \\ F_{xx}(X(t)) &= \Phi''_k(X(t)). \end{aligned}$$

$$\int_D \Phi_k(W_A(t))dx = \int_0^t \langle \text{div}(A(\nabla W_A(s))), \Phi'_k(W_A(s)) \rangle ds$$

$$\begin{aligned}
 & + \int_0^t \int_D \Phi'_k(W_A(s)) dW(s) \\
 & + \frac{1}{2} \sum_{l=1}^{\infty} \int_0^t \int_D \Phi''_k(W_A) \lambda_l(e_l)^2 dx ds \\
 \leq & - \int_0^t \int_D \Phi''_k(W_A(s)) \nabla W_A(s) A(\nabla W_A(s)) ds \\
 & + \int_0^t \int_D \Phi'_k(W_A(s)) dW(s) \\
 & + \frac{1}{2} \sum_{l=1}^{\infty} \lambda_l \|e_l\|_{L^\infty}^2 \int_0^t \int_D \Phi''_k(W_A) dx ds.
 \end{aligned} \tag{1.5.36}$$

Taking the expectation of (1.5.36), and using the fact that $\Phi''_k \geq 0$, we deduce from the fact that $\mathbb{E} \int_0^t \int_D \Phi'_k(W_A) dW(s) = 0$ ([13] Theorem 2.3.4 - p.11), from the coercivity property (1.1.3) and from (1.1.5) that

$$\begin{aligned}
 \mathbb{E} \int_D \Phi_k(W_A(t)) dx & \leq -C_0 \mathbb{E} \int_0^t \int_D \Phi''_k(W_A) |\nabla W_A|^2 \\
 & \quad + \frac{1}{2} \Lambda_1 \mathbb{E} \int_0^t \int_D \Phi''_k(W_A) dx ds.
 \end{aligned}$$

Then using Lemma 1.5.8 (ii) and Gronwall Lemma we obtain, defining $C(q) = \frac{1}{2}q(q-1)$,

$$\begin{aligned}
 \mathbb{E} \int_D \Phi_k(W_A(t)) dx & \leq \frac{1}{2}q(q-1)\Lambda_1 \mathbb{E} \int_0^t \int_D (1 + \Phi_k(W_A)) dx ds \\
 & \leq C(q)\Lambda_1 t |D| + C(q)\Lambda_1 \mathbb{E} \int_0^t \int_D \Phi_k(W_A) dx ds \\
 & \leq C(q)\Lambda_1 t |D| e^{C(q)\Lambda_1 t}.
 \end{aligned}$$

Thus, $\mathbb{E} \int_D \Phi_k(W_A) dx$ is bounded independently of k .

Finally, since $\Phi_k(W_A(x, t))$ converges to $|W_A(x, t)|^q$ for a.e. x and t when k goes to infinity, it follows from Fatou's Lemma that

$$\begin{aligned}
 \mathbb{E} \int_D |W_A(x, t)|^q dx & = \mathbb{E} \int_D \lim_{k \rightarrow \infty} \Phi_k(W_A(x, t)) dx = \mathbb{E} \int_D \liminf_{k \rightarrow \infty} \Phi_k(W_A(x, t)) dx \\
 & \leq \liminf_{k \rightarrow \infty} \mathbb{E} \int_D \Phi_k(W_A(x, t)) dx \\
 & \leq C(q)\Lambda_1 t |D| e^{C(q)\Lambda_1 t}
 \end{aligned}$$

for all $t > 0$.

Therefore, $W_A \in L^\infty(0, T; L^q(\Omega \times D))$ for all $q \geq 2$. \square

1.5.2 Uniqueness of the solution W_A

Let ω be given such that two pathwise solutions of Problem (P_2) , $W_A^1 = W_A^1(\omega, x, t)$ and $W_A^2 = W_A^2(\omega, x, t)$ satisfy

$$\begin{aligned} W_A^i(\cdot, \cdot, \omega) &\in L^\infty(0, T; L^2(D)) \cap L^2(0, T; H^1(D)), \\ \operatorname{div}(A(\nabla W_A^i)) &\in L^2((0, T); L^2(D)), \end{aligned}$$

for $i = 1, 2$. The difference of the two solutions satisfies the equation

$$W_A^1 - W_A^2 = \int_0^t \operatorname{div}\{A(\nabla W_A^1(s)) - A(\nabla W_A^2(s))\} ds \quad (1.5.37)$$

in $L^2((0, T) \times D)$.

We take the duality product of this equation with $W_A^1 - W_A^2 \in L^2((0, T); H^1(D))$. In view of (1.1.3) we obtain

$$\begin{aligned} \|W_A^1 - W_A^2\|_{L^2(D)}^2 &= - \int_0^t [A(\nabla W_A^1) - A(\nabla W_A^2)] \nabla(W_A^1 - W_A^2) \\ &\leq -C_0 \int_0^t \|\nabla(W_A^1 - W_A^2)\|_{L^2(D)}^2, \end{aligned} \quad (1.5.38) \quad \square$$

which in turn implies that

$$W_A^1 = W_A^2 \quad \text{a.e. in } D \times (0, T).$$

1.A Appendix

In this appendix we prove the properties (F_1) , (F_2) and (F_3) for the nonlinear function f .

(F_1) There exist positive constants C_1 and C_2 such that

$$f(a+b)a \leq -C_1 a^{2p} + f_2(b), \quad |f_2(b)| \leq C_2(b^{2p} + 1), \quad \text{for all } a, b \in \mathbb{R}.$$

Proof For simplicity we suppose that $b_j = 1$ for all $j = 0, \dots, 2p-2$ and that $b_{2p-1} = -1$.

$$\begin{aligned} f(a+b) &= \sum_{j=0}^{2p-1} b_j (a+b)^j \\ &= -(a+b)^{2p-1} + (a+b)^{2p-2} + \dots + (a+b)^2 + (a+b) \\ f(a+b)a &= -(a+b)^{2p-1}a + (a+b)^{2p-2}a + \dots + (a+b)^2a + (a+b)a \\ &= L_{2p-1} + L_{2p-2} + \dots L_1. \end{aligned} \quad (1.A.1)$$

We first estimate the term L_{2p-1} .

$$\begin{aligned} L_{2p-1} &= -(a+b)^{2p-1}a \\ &= -a^{2p} - C_{2p-1}^1 a^{2p-1}b - C_{2p-1}^2 a^{2p-2}b^2 - \dots - C_{2p-1}^{2p-3} a^3 b^{2p-3} \end{aligned}$$

$$-C_{2p-1}^{2p-2}a^2b^{2p-2} - ab^{2p-1}, \quad (1.A.2)$$

where $C_n^k = \frac{n!}{k!(n-k)!}$.

Next, we consider the second term on the right-hand-side of (1.A.2).

Using Young's inequality $ab \leq \frac{\varepsilon|a|^t}{t} + \frac{|b|^s}{\varepsilon s}$, with $\frac{1}{t} + \frac{1}{s} = 1$, we obtain

$$C_{2p-1}^1 a^{2p-1} b \leq C_{2p-1}^1 \frac{\varepsilon(2p-1)|a|^{2p}}{2p} + C_{2p-1}^1 \frac{|b|^{2p}}{2p\varepsilon}, \dots$$

For the last term on the right-hand-side of (1.A.2), we obtain

$$ab^{2p-1} \leq \frac{\varepsilon|a|^{2p}}{2p} + \frac{(2p-1)|b|^{2p}}{2p\varepsilon}.$$

Thus,

$$\begin{aligned} L_{2p-1} &= -(a+b)^{2p-1}a \\ &\leq -a^{2p} + C_{2p-1}^1 \frac{\varepsilon(2p-1)|a|^{2p}}{2p} + C_{2p-1}^1 \frac{|b|^{2p}}{2p\varepsilon} + \dots + \frac{\varepsilon|a|^{2p}}{2p} \\ &\quad + \frac{(2p-1)|b|^{2p}}{2p\varepsilon} \\ &\leq (-1 + \varepsilon C'(p))|a|^{2p} + \frac{1}{\varepsilon} C_3(p)|b|^{2p} + \frac{1}{\varepsilon} C_4(p). \end{aligned} \quad (1.A.3)$$

Similarly, we find that.

Lemma 1.A.1

$$L_q \leq \varepsilon C_1(p)|a|^{2p} + \frac{C_3(p)}{\varepsilon}|b|^{2p} + \frac{1}{\varepsilon} C_4(p), \quad (1.A.4)$$

for all $q \in \{1, \dots, 2p-2\}$.

Proof By induction, we first prove that (1.A.4) is true for $q = 1$.

Using Young's inequality, we deduce that

$$\begin{aligned} L_1 &= (a+b)a = a^2 + ab \leq \frac{\varepsilon}{p}|a|^{2p} + \frac{p-1}{\varepsilon p} + \frac{\varepsilon}{2p}|a|^{2p} + \frac{2p-1}{2p\varepsilon}|b|^{\frac{2p}{2p-1}} \\ &\leq \frac{3\varepsilon}{2p}|a|^{2p} + \frac{p-1}{p\varepsilon} + \frac{2p-1}{2p\varepsilon} \left(\frac{1}{(2p-1)}|b|^{2p} + \frac{(2p-2)}{2p-1} \right) \\ &\leq \frac{3\varepsilon}{2p}|a|^{2p} + \frac{1}{2p\varepsilon}|b|^{2p} + \frac{2p-2}{p\varepsilon}. \end{aligned}$$

We suppose that (1.A.4) is true for $q = 2p-3$ and prove that it remains true for $q = 2p-2$:

Using Young's inequality, we obtain

$$L_{2p-2} = (a+b)^{2p-2}a = a^{2p-1} + C_{2p-2}^1 a^{2p-2}b + \dots + C_{2p-2}^{2p-3} a^2 b^{2p-3} + ab^{2p-2}$$

$$\begin{aligned}
 &\leq \frac{\varepsilon(2p-1)}{2p}|a|^{2p} + \frac{1}{2p\varepsilon} + \dots + \frac{\varepsilon}{2p}|a|^{2p} + \frac{2p-1}{2p\varepsilon}|b|^{\frac{(2p-2)2p}{2p-1}} \\
 &\leq \varepsilon C_1(p)|a|^{2p} + \frac{1}{\varepsilon}C_3(p)|b|^{2p} + C_4(p).
 \end{aligned} \tag{1.A.5}$$

□

Combining (1.A.1),(1.A.3) and Lemma 1.A.1 and choosing $\varepsilon < \frac{1}{2(C'(p) + C_1(p))}$ yields

$$\begin{aligned}
 f(a+b)a &\leq (-1 + \varepsilon(C'(p) + C_1(p)))|a|^{2p} \\
 &\quad + \frac{1}{\varepsilon}C_3(p)|b|^{2p} + C_4(p) \\
 &\leq -\frac{1}{2}a^{2p} + C_2(b^{2p} + 1),
 \end{aligned}$$

with $C_2 = \max(\frac{1}{\varepsilon}C_3(p), C_4(p))$.

□

(F₂) **There exists a positive constant C_3 such that**

$$|f(s)| \leq C_3|s - M|^{2p-1} + \tilde{C}_3(M).$$

Proof Again, we suppose that $b_j = 1$ for all $j = 0, \dots, 2p-2$ and that $b_{2p-1} = -1$.

$$f(s) = -s^{2p-1} + s^{2p-2} + \dots + s^2 + s. \tag{1.A.6}$$

We estimate the leading term of (1.A.6)

$$\begin{aligned}
 &|s|^{2p-1} \\
 &= |s - M + M|^{2p-1} \\
 &= |s - M|^{2p-1} + C_{2p-1}^1|s - M|^{2p-2}M + \dots + C_{2p-1}^{2p-2}|s - M|M^{2p-2} + M^{2p-1}.
 \end{aligned}$$

By Young's inequality, there holds

$$\begin{aligned}
 |s - M|^{2p-2}M &\leq \frac{\varepsilon(2p-2)}{2p-1}|s - M|^{2p-1} + \frac{M^{2p-1}}{\varepsilon(2p-1)}, \dots \\
 |s - M|M^{2p-2} &\leq \frac{\varepsilon|s - M|^{2p-1}}{(2p-1)} + \frac{|M|^{2p-1}(2p-2)}{\varepsilon(2p-1)},
 \end{aligned}$$

so that

$$|s|^{2p-1} \leq (1 + \varepsilon C(p))|s - M|^{2p-1} + (1 + \frac{1}{\varepsilon}\tilde{C}(p))|M|^{2p-1}. \tag{1.A.7}$$

Next, we estimate the last term on the right-hand-side of (1.A.6).

It follows from Young's inequality that

$$|s| \leq |s - M| + |M| \leq \frac{\varepsilon}{2p-1} |s - M|^{2p-1} + \frac{2p-2}{\varepsilon(2p-1)} + |M|. \quad (1.A.8) \quad \square$$

Computing all the other terms of (1.A.6) similarly and substituting them in (1.A.6) we obtain

$$|f(s)| \leq C_3 |s - M|^{2p-1} + \tilde{C}_3(M).$$

(F₃) **There exists a positive constant C_4 such that**

$$f'(s) \leq C_4.$$

Proof

$$f'(s) = -(2p-1)s^{2p-2} + (2p-2)s^{2p-3} + \dots + 2s + 1. \quad (1.A.9)$$

By Young's inequality

$$|s|^{2p-3} \leq \frac{\varepsilon(2p-3)}{(2p-2)} |s|^{2p-2} + \frac{1}{(2p-2)\varepsilon}, \dots, \quad (1.A.10)$$

$$|s| \leq \frac{\varepsilon}{2p-2} |s|^{2p-2} + \frac{2p-3}{(2p-2)\varepsilon}. \quad (1.A.11) \quad \square$$

We compute all the other terms similarly, and substitute them in (1.A.9) to obtain

$$f'(s) \leq (-(2p-1) + \varepsilon C(p)) |s|^{2p-2} + \frac{\tilde{C}(p)}{\varepsilon} + 1.$$

Choosing $\varepsilon \leq \frac{2p-1}{2C(p)}$ we conclude that

$$f'(s) \leq C_4.$$

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Well-posedness of a stochastic phase field problem with multiplicative noises

Résumé. Dans ce chapitre, on démontre l'existence et l'unicité de la solution pour un problème de champ de phase stochastique avec un bruit multiplicatif. Le modèle de champ de phase qu'on considère modélise par exemple l'évolution en temps d'une interface entre eau et glace. On considère le cas d'un bruit multiplicatif induit par un Q-mouvement Brownien.

Abstract. In this chapter, we prove the existence and uniqueness of the solution of a stochastic phase field problem with multiplicative noise. The phase field problem that we consider is used to describe melting and solidification processes. We consider here the case of a multiplicative noise induced by a Q-Brownian motion.

2.1 Introduction

We consider the problem

$$(P) \quad \begin{cases} d\varphi = (\Delta\varphi + f(\varphi) + h)dt + \Phi_1(\varphi, h)dW_1(t), & \text{in } D \times (0, T), \\ d[h + \varphi] = \Delta h dt + \Phi_2(\varphi, h)dW_2(t), & \text{in } D \times (0, T), \\ \frac{\partial\varphi}{\partial\nu} = \frac{\partial h}{\partial\nu} = 0, & \text{on } \partial D \times (0, T), \\ \varphi(x, 0) = \varphi_0(x), h(x, 0) = h_0(x), & x \in D, \end{cases}$$

where

- D is an open domain of \mathbb{R}^d with smooth boundary;
- ν is the unit outward normal to ∂D ;
- The nonlinear function f is given by $f(s) = -s^3 + s$. We will use below that

$$f(s)s \leq -\frac{s^4}{2} + \frac{1}{2}, \tag{2.1.1}$$

$$f'(s) \leq C_0 = 1; \tag{2.1.2}$$

- For $i = 1, 2$, the function $W_i = W_i(x, t)$ is a Q-Brownian motion ([8] Definition 2.6 p.20). More precisely, let Q_i be a nonnegative definite symmetric operator on $L^2(D)$ with $\text{Tr } Q_i < +\infty$, $\{e_l\}_{l \geq 1}$ be an orthonormal basis in $L^2(D)$ diagonalizing Q_i , and $\{\lambda_l^i\}_{l \geq 1}$ be the corresponding eigenvalues, so that

$$Q_i e_l = \lambda_l^i e_l,$$

for all $l \geq 1$. Since Q_i is of trace-class,

$$\text{Tr } Q_i = \sum_{l=1}^{\infty} \langle Q_i e_l, e_l \rangle_{L^2(D)} = \sum_{l=1}^{\infty} \lambda_l^i \leq \Lambda_0, \quad (2.1.3)$$

for some positive constant Λ_0 . Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration (\mathcal{F}_t) and $\{\beta_l^i(t)\}_{l \geq 1}$ be a sequence of independent (\mathcal{F}_t) -Brownian motions defined on $(\Omega, \mathcal{F}, \mathbb{P})$; the Q_i -Wiener process W_i in $L^2(D)$ is defined by

$$W_i(x, t) = \sum_{l=1}^{\infty} \beta_l^i(t) Q_i^{\frac{1}{2}} e_l(x) = \sum_{l=1}^{\infty} \beta_l^i(t) \sqrt{\lambda_l^i} e_l(x). \quad (2.1.4)$$

We recall that a Brownian motion $\beta^i(t)$ is called an (\mathcal{F}_t) Brownian motion if it is (\mathcal{F}_t) -adapted and the increment $\beta^i(t) - \beta^i(s)$ is independent of \mathcal{F}_s for every $0 \leq s < t$.

- Let X be a separable real Hilbert space. We denote the family of all linear operators $\Phi_i : H \rightarrow X$ such that $\Phi_i \sqrt{Q_i}$ are Hilbert-Schmidt operators by $\mathcal{L}_{2,Q}(H, X)$ (cf.[7]). If, $\Phi_i \in \mathcal{L}_{2,Q}(H, X)$, the $\mathcal{L}_{2,Q}$ of Φ_i is given by

$$\|\Phi_i\|_{2,Q}^2 := \sum_{l=1}^{\infty} \|\Phi_i \sqrt{Q_i} e_l\|_X^2 < \infty. \quad (2.1.5)$$

Given an X valued predictable process $\Phi_i \in L^2(\Omega; L^2((0, T), \mathcal{L}_{2,Q}(H, X)))$ and setting

$$\Phi_{il}(x, u, \tilde{u}) := \Phi_i(u, \tilde{u}) \sqrt{Q_i} e_l(x) = \Phi_i(u, \tilde{u}) \sqrt{\lambda_l^i} e_l(x), \quad (2.1.6)$$

one may define the stochastic integral as:

$$\int_0^t \Phi_i(s) dW_i(s) = \sum_{l=1}^{\infty} \int_0^t \Phi_{il} d\beta_l^i. \quad (2.1.7)$$

We suppose that there exist positive constants C_1, C_2 and C_3 such that

$$\sum_{l=1}^{\infty} \|\Phi_{il}(x, u, \tilde{u})\|^2 \leq C_1(1 + \|u\|^2 + \|\tilde{u}\|^2), \quad (2.1.8)$$

$$\sum_{l=1}^{\infty} \|\Phi_{il}(x, u, \tilde{u})\|_V^2 \leq C_2(1 + \|u\|_V^2 + \|\tilde{u}\|_V^2), \quad (2.1.9)$$

$$\sum_{l=1}^{\infty} \|\Phi_{il}(x, u_1, \tilde{u}_1) - \Phi_{il}(x, u_2, \tilde{u}_2)\|^2 \leq C_3(\|u_1 - u_2\|^2 + \|\tilde{u}_1 - \tilde{u}_2\|^2), \quad (2.1.10)$$

for all $u, \tilde{u} \in H^1(D)$, all $u_1, u_2, \tilde{u}_1, \tilde{u}_2 \in L^2(D)$ and for $i = 1, 2$.

We will work with the following unknown functions:

$$(\varphi, v) := (\varphi, h + \varphi),$$

so that we are led to the problem,

$$(P_1) \quad \begin{cases} d\varphi = (\Delta\varphi + f(\varphi) - \varphi + v)dt + \Phi_1(\varphi, v - \varphi)dW_1(t), & \text{in } \Omega \times D \times \mathbb{R}^+, \\ dv = (\Delta v - \Delta\varphi)dt + \Phi_2(\varphi, v - \varphi)dW_2(t), & \text{in } \Omega \times D \times \mathbb{R}^+, \\ \frac{\partial\varphi}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, & \text{on } \Omega \times \partial D \times \mathbb{R}^+, \\ \varphi(x, 0) = \varphi_0(x), v(x, 0) = v_0(x), & x \in D, \end{cases}$$

where $v_0 = h_0 + \varphi_0$.

We define

$$H = L^2(D), \quad V = H^1(D), \quad V^* = (H^1(D))' \quad \text{and} \quad Z = V \cap L^4(D),$$

where $\|\cdot\|$, $\|\cdot\|_V$ are the norms corresponding to the space H and V respectively, and $\langle \cdot, \cdot \rangle_{V^*, V}$ is the duality product between V and its dual space V^* . We also define $\langle \cdot, \cdot \rangle_{Z^*, Z}$ as the duality product between Z and its dual space $Z^* = V^* + L^{\frac{4}{3}}(D)$ ([4], p.175).

Phase field models are used to describe for example the phase transition of melting and solidification processes. In this model h is the temperature while φ is the phase function. The existence and uniqueness of the solution of the corresponding deterministic problem was proved by Brochet, Chen and Hilhorst [3]. Bertini, Brassesco, Buttà and Presutti [2], proved the global existence and uniqueness of the solution of a phase-field problem perturbed by an additive one-dimensional space-time white noise. Barbu and Da Prato [1] proved the existence and uniqueness of the solution, where the additive noise in each equation of the system is induced by a cylindrical Wiener process. We are going to prove the existence and uniqueness of the solution of essentially the same stochastic phase-field system, adding to each equation a multiplicative noise induced by a Q -Wiener process. In order to reach our goal, we apply the Galerkin method and derive a priori estimates for the corresponding solutions. Finally, to identify the limit we apply the stochastic monotonicity method as in [11].

The paper is organised as follows. In section 2.2, we apply the usual Galerkin method by decomposing the approximate solution on a basis of eigenfunctions of the operator $-\Delta$ with homogeneous Neumann boundary conditions.

In section 2.3, we derive a priori estimates for the approximate solution in $L^\infty(0, T; L^2(\Omega; H \times H)) \cap L^2(\Omega \times (0, T); V \times V)$ and pass to the limit. The final step is to identify the limit of the reaction and the multiplicative terms which we do by the stochastic monotonicity method in section 2.4. In section 2.5, we prove some extra regularity properties of the solution pair in the case of smoother initial data. Finally, in section 2.6 we prove the uniqueness of the solution pair.

Definition 2.1.1 Let $(\varphi_0, v_0) \in H \times H$ be arbitrary. A function pair (φ, v) is a weak solution of Problem (P) with initial data (φ_0, v_0) if

$$\begin{aligned} (\varphi, v) &\in L^\infty(0, T; L^2(\Omega; H \times H)) \cap L^2(\Omega \times (0, T); V \times V), \\ \varphi &\in L^4(\Omega \times (0, T); L^4(D)), \end{aligned}$$

and

$$(P_2) \quad \begin{cases} \varphi(t) - \varphi_0 = \int_0^t \Delta \varphi(s) ds + \int_0^t f(\varphi(s)) - \int_0^t \varphi(s) ds + \int_0^t v(s) ds \\ \quad + \int_0^t \Phi_1(\varphi(s), v(s) - \varphi(s)) dW_1(s), & \text{in the sense of distributions,} \\ v(t) - v_0 = \int_0^t \Delta v(s) ds - \int_0^t \Delta \varphi(s) ds \\ \quad + \int_0^t \Phi_2(\varphi(s), v(s) - \varphi(s)) dW_2(s), & \text{in the sense of distributions,} \\ \frac{\partial \varphi(x, t)}{\partial \nu} = \frac{\partial v(x, t)}{\partial \nu} = 0, & \text{in the sense of distributions on } \partial D, \end{cases}$$

almost surely for all $t > 0$.

Definition 2.1.2 Let $(\varphi_0, v_0) \in V \times V$ be arbitrary. A function pair (φ, v) is a strong solution of Problem (P) with initial data (φ_0, v_0) if

$$\begin{aligned} (\varphi, v) &\in L^2(\Omega \times (0, T); H^2(D) \times H^2(D)), \\ (\nabla \varphi, \nabla v) &\in L^\infty(0, T; L^2(\Omega \times D) \times L^2(\Omega \times D)), \\ \varphi &\in L^4(\Omega \times (0, T); L^4(D)), \end{aligned}$$

and

$$(P_2) \quad \begin{cases} \varphi(t) - \varphi_0 = \int_0^t \Delta \varphi(s) ds + \int_0^t f(\varphi(s)) - \int_0^t \varphi(s) ds + \int_0^t v(s) ds \\ \quad + \int_0^t \Phi_1(\varphi(s), v(s) - \varphi(s)) dW_1(s), & \text{in } \Omega \times D \times (0, T), \\ v(t) - v_0 = \int_0^t \Delta v(s) ds - \int_0^t \Delta \varphi(s) ds \\ \quad + \int_0^t \Phi_2(\varphi(s), v(s) - \varphi(s)) dW_2(s), & \text{in } \Omega \times D \times (0, T), \\ \frac{\partial \varphi(x, t)}{\partial \nu} = \frac{\partial v(x, t)}{\partial \nu} = 0, & \text{on } \Omega \times \partial D \times (0, T), \end{cases}$$

almost surely for all $t > 0$.

2.2 Galerkin method and main result

Theorem 2.2.1 (i) Let $(\varphi_0, v_0) \in H \times H$ be arbitrary. Then Problem (P) possesses a unique weak solution (φ, v) .

(ii) Let $(\varphi_0, v_0) \in V \times V$ be arbitrary. Then the solution (φ, v) of Problem (P) is a strong solution.

Proof The proof relies on the Galerkin method. We denote by $0 < \gamma_1 < \gamma_2 \leq \dots \leq \gamma_{\tilde{k}} \leq \dots$ the eigenvalues of the operator $-\Delta$ with homogeneous Neumann boundary conditions, and by $w_{\tilde{k}}, \tilde{k} = 1, \dots$ the corresponding unit eigenfunctions. Note that they are smooth functions. For each integer m we look for an approximate solution (φ_m, v_m) of the form:

$$\varphi_m(t) = \sum_{k=0}^m \varphi_{km}(t)w_k, \quad v_m(t) = \sum_{k=0}^m v_{km}(t)w_k, \quad (2.2.1)$$

satisfying

$$\begin{aligned} & \int_D \varphi_m(t)w_j - \int_D \varphi_m(0)w_j + \int_0^t \int_D \nabla \varphi_m \nabla w_j \\ = & \int_0^t \int_D f(\varphi_m)w_j - \int_0^t \int_D \varphi_m w_j + \int_0^t \int_D v_m w_j + \int_0^t \int_D \Phi_1(\varphi_m, v_m - \varphi_m)w_j dW_1^m(s), \end{aligned} \quad (2.2.2)$$

$$\begin{aligned} & \int_D v_m(t)w_j - \int_D v_m(0)w_j + \int_0^t \int_D \nabla v_m \nabla w_j \\ = & \int_0^t \int_D \nabla \varphi_m \nabla w_j + \int_0^t \int_D \Phi_2(\varphi_m, v_m - \varphi_m)w_j dW_2^m(s), \end{aligned} \quad (2.2.3)$$

for $j = 0, \dots, m$, where

$$W_i^m = \tilde{P}_m W_i = \sum_{l=1}^m \beta_l^i(t) \sqrt{\lambda_l^i} e_l(x), \quad (2.2.4)$$

for $i = 1, 2$, where \tilde{P}_m is the projection from $H \rightarrow \tilde{H}_m = \text{span}\{e_1, \dots, e_m\}$ such that $\tilde{P}_m a = \sum_{k=1}^m \left(\int_D a e_k dx \right) e_k$, $a \in H$ and

$$\varphi_m(0) = \varphi_{0m} \rightarrow \varphi_0 \quad \text{in } L^2(D) \quad \text{as } m \rightarrow \infty, \quad (2.2.5)$$

$$v_m(0) = v_{0m} \rightarrow v_0 \quad \text{in } L^2(D) \quad \text{as } m \rightarrow \infty. \quad (2.2.6)$$

□

Problem (2.2.2) - (2.2.6) is an initial value problem for a system of $2m$ stochastic differential equations, so that it has a unique solution $(\varphi_{km}(t), v_{km}(t), k = 0, \dots, m)$ on some time interval $(0, T_m)$, $T_m > 0$ (cf. [15], p.44, Theorem 3.1.1). This solution is such that $\varphi_{km}, v_{km} \in C([0, T])$, $k = 0, \dots, m$; in fact the following a priori estimates show that the solution pair is global in time.

2.3 A priori estimates and passage to the limit

In what follows, we derive a priori estimates for the functions φ_m and v_m .

Lemma 2.3.1 *There exist positive constants K and K_1 such that*

$$\mathbb{E} \int_D \varphi_m^2(t) \leq K \quad \text{for all } t \in [0, T], \quad (2.3.1)$$

$$\mathbb{E} \int_D v_m^2(t) \leq K \quad \text{for all } t \in [0, T], \quad (2.3.2)$$

$$\mathbb{E} \int_0^t \int_D |\nabla \varphi_m|^2 dx ds \leq K, \quad (2.3.3)$$

$$\mathbb{E} \int_0^t \int_D \varphi_m^4 dx ds \leq K, \quad (2.3.4)$$

$$\mathbb{E} \int_0^t \int_D |\nabla v_m|^2 dx ds \leq K, \quad (2.3.5)$$

$$\mathbb{E} \|f(\varphi_m)\|_{L^{\frac{4}{3}}(D \times (0, T))}^{\frac{4}{3}} \leq K_1, \quad (2.3.6)$$

$$\mathbb{E} \int_0^t \|P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m\|_{2, Q}^2 \leq K, \quad (2.3.7)$$

$$\mathbb{E} \int_0^t \|P_m \Phi_2(\varphi_m, v_m - \varphi_m) \tilde{P}_m\|_{2, Q}^2 \leq K. \quad (2.3.8)$$

Proof We multiply equations (2.2.2), (2.2.3) by w_j and sum from $j = 0, \dots, m$ we obtain that

$$\begin{aligned} \varphi_m(t) &= \varphi_{0m} + \int_0^t \Delta \varphi_m(s) ds + \int_0^t P_m f(\varphi_m) ds \\ &\quad - \int_0^t \varphi_m ds + \int_0^t v_m ds \\ &\quad + \int_0^t P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m dW_1(s), \end{aligned} \quad (2.3.9)$$

$$\begin{aligned} v_m(t) &= v_{0m} + \int_0^t \Delta v_m ds - \int_0^t \Delta \varphi_m ds \\ &\quad + \int_0^t P_m \Phi_2(\varphi_m, v_m - \varphi_m) \tilde{P}_m dW_2(s) dx, \end{aligned} \quad (2.3.10)$$

where P_m is the projection from $H \rightarrow H_m = \text{span}\{w_1, \dots, w_m\}$ such that

$$P_m a = \sum_{j=1}^m \left(\int_D a w_j dx \right) w_j, \quad a \in H, \quad (2.3.11)$$

and where we have used the following result

Lemma 2.3.2 $P_m \Delta \varphi_m = \Delta \varphi_m$.

Proof

$$\begin{aligned}
 P_m \Delta \varphi_m &= \sum_{j=0}^m \left(\int_D \Delta \varphi_m w_j \right) w_j = \sum_{j=0}^m \left(\int_D \varphi_m \Delta w_j \right) w_j \\
 &= - \sum_{j=0}^m \left(\int_D \varphi_m \gamma_j w_j \right) w_j \\
 &= - \sum_{j=0}^m \left(\int_D \varphi_m w_j \right) \gamma_j w_j \\
 &= \Delta \left(\sum_{j=0}^m \left(\int_D \varphi_m w_j \right) w_j \right) \\
 &= \Delta \varphi_m. \quad \square
 \end{aligned}$$

In addition note that (cf. [5] p.193),

$$\|P_m a\|_X \leq \|a\|_X, a \in X, \quad (2.3.12)$$

and

$$P_m a \rightarrow a \quad \text{in } X \text{ as } m \rightarrow \infty. \quad (2.3.13)$$

Next we recall an Itô's formula based on [6].

Lemma 2.3.3 *Let X be an E -valued function such that*

$$X(t) = X(0) + \int_0^t h(s) ds + \int_0^t G(s) dW(s), \quad 0 \leq s \leq t,$$

and suppose that h is a E -valued predictable process Bochner integrable on $[0, T]$, a.s., G is an E -valued process stochastically integrable. Suppose that the function $F : [0, T] \times E \rightarrow \mathbb{R}$ and its partial derivatives $\frac{\partial F}{\partial t}$, $\frac{\partial F}{\partial X}$, $\frac{\partial^2 F}{\partial X^2}$ are continuous on $[0, T] \times E$. Then for all $t \in [0, T]$, a.s.

$$\begin{aligned}
 F(t, X(t)) &= F(0, X(0)) + \int_0^t \frac{\partial F}{\partial t}(s, X(s)) ds + \int_0^t \left\langle \frac{\partial F}{\partial X}(s, X(s)), h(s) \right\rangle_E ds \\
 &\quad + \int_0^t \left\langle \frac{\partial F}{\partial X}(s, X(s)), G(s) dW(s) \right\rangle_E \\
 &\quad + \frac{1}{2} \int_0^t \text{Tr} \left[\frac{\partial^2 F}{\partial X^2}(s, X(s)) (G(s) Q^{\frac{1}{2}}) (G(s) Q^{\frac{1}{2}})^* \right] ds, \quad (2.3.14)
 \end{aligned}$$

where [cf. [9]]

$$\begin{aligned}
 &\text{Tr} \left[\frac{\partial^2 F}{\partial X^2}(s, X(s)) (G(s) Q^{\frac{1}{2}}) (G(s) Q^{\frac{1}{2}})^* \right] ds \\
 &= \sum_{k=1}^{\infty} \left\langle \frac{\partial^2 F}{\partial X^2}(s, X(s)) (G(s) Q^{\frac{1}{2}} e_k), (G(s) Q^{\frac{1}{2}} e_k) \right\rangle_E.
 \end{aligned}$$

Applying (2.3.14) to (2.3.9) with $E = \mathbb{R}$, $x \in D$ fixed and with

$$\begin{aligned} X(t) &= \varphi_m(x, t), \\ F(X) &= X^2, \\ \frac{\partial F}{\partial X}(X) &= 2X, \\ \frac{\partial^2 F}{\partial X^2}(X) &= 2, \\ h(s) &= \Delta\varphi_m + P_m f(\varphi_m) - \varphi_m + v_m, \\ G(s) &= P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m. \end{aligned}$$

We remark that here F does not depend on t . We obtain almost surely, for all $t \in [0, T]$ and after integrating on D

$$\begin{aligned} \|\varphi_m(t)\|^2 &= \|\varphi_{0m}\|^2 - 2 \int_0^t \int_D |\nabla\varphi_m|^2 + 2 \int_0^t \int_D f(\varphi_m)\varphi_m dx ds - 2 \int_0^t \int_D \varphi_m^2 dx ds \\ &\quad + 2 \int_0^t \int_D v_m \varphi_m dx ds \\ &\quad + 2 \int_0^t \int_D \varphi_m P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m dW_1(s) dx \\ &\quad + \int_0^t \sum_{l=1}^{\infty} \|P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m \sqrt{\lambda_l^1} e_l\|^2 ds. \end{aligned} \tag{2.3.15}$$

We bound the last term of (2.3.15) using (2.1.8) and (2.3.12)

$$\begin{aligned} \int_0^t \sum_{l=1}^{\infty} \|P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m \sqrt{\lambda_l^1} e_l\|^2 ds &= \int_0^t \sum_{l=1}^{\infty} \|P_m \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l\|^2 ds \\ &\leq \int_0^t \sum_{l=1}^{\infty} \|\Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l\|^2 ds \\ &= \int_0^t \sum_{l=1}^{\infty} \|\Phi_{1l}(x, \varphi_m(x), v_m(x) - \varphi_m(x))\|^2 ds \\ &\leq \int_0^t C_1(1 + \|\varphi_m\|^2 + \|v_m - \varphi_m\|^2) ds \\ &\leq \int_0^t C_1(1 + 3\|\varphi_m\|^2 + 2\|v_m\|^2) ds. \end{aligned}$$

Thus,

$$\begin{aligned} \|\varphi_m(t)\|^2 &\leq \|\varphi_{0m}\|^2 - 2 \int_0^t \|\nabla\varphi_m\|^2 ds - 2 \int_0^t \int_D \varphi_m^4 dx ds \\ &\quad + 2 \int_0^t \int_D v_m \varphi_m dx ds + 2 \int_0^t \int_D \varphi_m P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m dW_1 \\ &\quad + 3C_1 \int_0^t \|\varphi_m\|^2 + 2C_1 \int_0^t \|v_m\|^2 + C_1 T. \end{aligned} \tag{2.3.16}$$

Lemma 2.3.4

$$|ab| \leq \varepsilon a^2 + \frac{1}{2}b^{2p} + C_\varepsilon. \quad (2.3.17)$$

Proof We know that

$$\begin{aligned} |ab| &\leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2 \\ &\leq \varepsilon a^2 + \frac{\eta}{\varepsilon p} b^{2p} + C_\eta, \end{aligned}$$

choosing $\eta = \frac{\varepsilon p}{2}$ we find that

$$|ab| \leq \varepsilon a^2 + \frac{1}{2}b^{2p} + C_\varepsilon. \quad \square$$

In view of the inequality (2.3.17), the equation (2.3.16) becomes:

$$\begin{aligned} \|\varphi_m(t)\|^2 &\leq \|\varphi_{0m}\|^2 - 2 \int_0^t \|\nabla \varphi_m\|^2 ds - 2 \int_0^t \int_D \varphi_m^4 dx ds \\ &\quad + 2\varepsilon \int_0^t \int_D v_m^2 + \int_0^t \int_D \varphi_m^4 dx ds \\ &\quad + C_\varepsilon T |D| + 2 \int_0^t \int_D \varphi_m P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m dW_1 \\ &\quad + 3C_1 \int_0^t \|\varphi_m\|^2 + 2C_1 \int_0^t \|v_m\|^2 + C_1 T \\ &\leq \|\varphi_{0m}\|^2 - 2 \int_0^t \int_D |\nabla \varphi_m|^2 dx ds - \int_0^t \int_D \varphi_m^4 dx ds \\ &\quad + 2\varepsilon \int_0^t \int_D v_m^2 + 3C_1 \int_0^t \|\varphi_m\|^2 + 2C_1 \int_0^t \|v_m\|^2 \\ &\quad + 2 \int_0^t \int_D \varphi_m P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m dW_1 + \tilde{C}_\varepsilon. \end{aligned} \quad (2.3.18)$$

Applying Itô's formula Lemma 2.3.3 for the equation (2.3.10)

$$\begin{aligned} X(s) &= v_m(s), \\ F(X) &= X^2, \\ \frac{\partial F}{\partial X}(X) &= 2X, \\ \frac{\partial^2 F}{\partial X^2}(X) &= 2, \\ h(s) &= \Delta v_m - \Delta \varphi_m, \\ G(s) &= P_m \Phi_2(\varphi_m, v_m - \varphi_m) \tilde{P}_m. \end{aligned}$$

We obtain, after integration on D

$$\|v_m(t)\|^2 = \|v_{0m}\|^2 + 2 \int_0^t \int_D v_m (\Delta v_m - \Delta \varphi_m) dx ds$$

$$\begin{aligned}
 & +2 \int_0^t \int_D v_m P_m \Phi_2(\varphi_m, v_m - \varphi_m) \tilde{P}_m dW_2(s) dx \\
 & + \int_0^t \sum_{k=1}^{\infty} \int_D \left(P_m \Phi_2(\varphi_m, v_m - \varphi_m) \tilde{P}_m \sqrt{\lambda_l^2 e_l} \right)^2 dx ds. \quad (2.3.19)
 \end{aligned}$$

Using (2.1.8) and the same computations for the last term in (2.3.19) we obtain that

$$\begin{aligned}
 \|v_m(t)\|^2 & \leq \|v_{0m}\|^2 - 2 \int_0^t \|\nabla v_m\|^2 + \int_0^t \|\nabla v_m\|^2 + \int_0^t \|\nabla \varphi_m\|^2 \\
 & + 2 \int_0^t \int_D v_m P_m \Phi_2(\varphi_m, v_m - \varphi_m) \tilde{P}_m dW_2 + C_1 T \\
 & + 2C_1 \int_0^t \|v_m\|_{L^2(D)}^2 + 3C_1 \int_0^t \|\varphi_m\|_{L^2(D)}^2. \quad (2.3.20)
 \end{aligned}$$

Taking the expectation of (2.3.18), (2.3.20) and summing both equations we obtain

$$\begin{aligned}
 \mathbb{E}\|\varphi_m(t)\|^2 + \mathbb{E}\|v_m(t)\|^2 & \leq \|\varphi_{0m}\|^2 + \|v_{0m}\|^2 - \mathbb{E} \int_0^t \int_D |\nabla \varphi_m|^2 dx ds \\
 & - \mathbb{E} \int_0^t \int_D \varphi_m^4 dx ds \\
 & - \mathbb{E} \int_0^t \int_D |\nabla v_m|^2 dx ds + 2\varepsilon \mathbb{E} \int_0^t \int_D v_m^2 dx ds \\
 & + 2\mathbb{E} \int_0^t \int_D \varphi_m P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m dW_1(s) dx \\
 & + 2\mathbb{E} \int_0^t \int_D v_m P_m \Phi_2(\varphi_m, v_m - \varphi_m) \tilde{P}_m dW_2 dx \\
 & + 6C_1 \mathbb{E} \int_0^t \|\varphi_m\|^2 ds + 4C_1 \mathbb{E} \int_0^t \|v_m\|^2 ds \\
 & + \tilde{C}_\varepsilon. \quad (2.3.21)
 \end{aligned}$$

Using the facts that (cf. [13] Theorem 2.3.4 - p.11).

$$\mathbb{E} \int_0^t \int_D \varphi_m P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m dW_1(s) dx = 0, \quad (2.3.22)$$

and

$$\mathbb{E} \int_0^t \int_D v_m P_m \Phi_2(\varphi_m, v_m - \varphi_m) \tilde{P}_m dW_2(s) dx = 0, \quad (2.3.23)$$

so that

$$\begin{aligned}
 & \mathbb{E}\|\varphi_m(t)\|^2 + \mathbb{E}\|v_m(t)\|^2 + \mathbb{E} \int_0^t \int_D |\nabla \varphi_m|^2 dx ds + \mathbb{E} \int_0^t \int_D \varphi_m^4 dx ds \\
 & + \mathbb{E} \int_0^t \int_D |\nabla v_m|^2 dx ds
 \end{aligned}$$

$$\begin{aligned}
 &\leq \|\varphi_{0m}\|^2 + \|v_{0m}\|^2 + 6C_1 \mathbb{E} \int_0^t \|\varphi_m\|^2 ds \\
 &\quad + 2(\varepsilon + 2C_1) \mathbb{E} \int_0^t \|v_m\|^2 ds + \tilde{C}_\varepsilon.
 \end{aligned} \tag{2.3.24}$$

Applying Gronwall Lemma yields

$$\mathbb{E} \int_D \varphi_m^2(t) \leq K \quad \text{for all } t \in [0, T], \tag{2.3.25}$$

$$\mathbb{E} \int_D v_m^2(t) \leq K \quad \text{for all } t \in [0, T]. \tag{2.3.26}$$

Substituting (2.3.25) and (2.3.26) into (2.3.24), we obtain

$$\begin{aligned}
 &\mathbb{E} \|\varphi_m(t)\|^2 + \mathbb{E} \|v_m(t)\|^2 + \mathbb{E} \int_0^t \int_D |\nabla \varphi_m|^2 dx ds \\
 &\quad + \mathbb{E} \int_0^t \int_D \varphi_m^4 dx ds + \mathbb{E} \int_0^t \int_D |\nabla v_m|^2 \\
 &\leq \|\varphi_0\|^2 + \|v_0\|^2 + C,
 \end{aligned}$$

where C depends on T , therefore we deduce that

$$\mathbb{E} \int_0^t \int_D |\nabla \varphi_m|^2 dx ds \leq K, \tag{2.3.27}$$

$$\mathbb{E} \int_0^t \int_D \varphi_m^4 \leq K, \tag{2.3.28}$$

$$\mathbb{E} \int_0^t \int_D |\nabla v_m|^2 \leq K. \tag{2.3.29}$$

Next we prove (2.3.6) using the inequality $|a + b|^p \leq C(p)(|a|^p + |b|^p)$, for $p \geq 1$ as well as Young inequality

$$\begin{aligned}
 \mathbb{E} \|f(\varphi_m)\|_{L^{\frac{4}{3}}(D \times (0, T))}^{\frac{4}{3}} &= \mathbb{E} \int_0^T \int_D |f(\varphi_m)|^{\frac{4}{3}} dx dt \\
 &= \mathbb{E} \int_0^T \int_D |\varphi_m - \varphi_m^3|^{\frac{4}{3}} dx dt \\
 &\leq c_{\frac{4}{3}} \mathbb{E} \int_0^T \int_D (|\varphi_m|^{\frac{4}{3}} + |\varphi_m|^4) dx dt \\
 &\leq \tilde{c}_{\frac{4}{3}} \mathbb{E} \int_0^T \int_D |\varphi_m|^4 + \tilde{c}_{\frac{4}{3}} \\
 &\leq K,
 \end{aligned} \tag{2.3.30}$$

where we have used (2.3.4) in Lemma 2.3.1.

In addition, also using (2.1.8), we deduce that

$$\mathbb{E} \int_0^t \|P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m\|_{2, Q}^2 = \mathbb{E} \int_0^t \sum_{l=1}^{\infty} \|P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m \sqrt{\lambda_l^{-1}} e_l\|^2 ds$$

$$\begin{aligned}
 &\leq \mathbb{E} \int_0^t \sum_{l=1}^{\infty} \|P_m \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l\|^2 ds \\
 &\leq \mathbb{E} \int_0^t \sum_{l=1}^{\infty} \|\Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l\|^2 ds \\
 &\leq C_1 \mathbb{E} \int_0^t (1 + 3\|\varphi_m\|^2 + 2\|v_m\|^2) ds \\
 &\leq K.
 \end{aligned} \tag{2.3.31}$$

Similarly, one can show that

$$\begin{aligned}
 &\mathbb{E} \int_0^t \|P_m \Phi_2(\varphi_m, v_m - \varphi_m) \tilde{P}_m\|_{2,Q}^2 \\
 &\leq C_1 \mathbb{E} \int_0^t (1 + 2\|v_m\|^2 + 3\|\varphi_m\|^2) ds \leq K,
 \end{aligned} \tag{2.3.32}$$

where we have used (2.3.1) and (2.3.2) in Lemma 2.3.1. \square

Hence, φ_m and v_m belongs to a bounded set of $L^\infty(0, T; L^2(\Omega \times D)) \cap L^2(\Omega, L^2(0, T; H^1(D)))$. Thus there exist subsequences, which we denoted again by φ_m and v_m , and functions φ, v, χ, Z_1 and Z_2 such that :

$$\varphi_m \rightharpoonup \varphi \text{ weakly in } L^2(\Omega, L^2(0, T; H^1(D))), \tag{2.3.33}$$

$$\varphi_m \rightharpoonup \varphi \text{ weak-star in } L^\infty(0, T; L^2(\Omega \times D)), \tag{2.3.34}$$

$$v_m \rightharpoonup v \text{ weakly in } L^2(\Omega, L^2(0, T; H^1(D))), \tag{2.3.35}$$

$$v_m \rightharpoonup v \text{ weak-star in } L^\infty(0, T; L^2(\Omega \times D)), \tag{2.3.36}$$

$$f(\varphi_m) \rightharpoonup \chi \text{ weakly in } L^{\frac{4}{3}}(\Omega \times (0, T) \times D), \tag{2.3.37}$$

$$P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m \rightharpoonup Z_1 \text{ weakly in } L^2(\Omega \times (0, T); \mathcal{L}_{2,Q}(H; H)), \tag{2.3.38}$$

$$P_m \Phi_2(\varphi_m, v_m - \varphi_m) \tilde{P}_m \rightharpoonup Z_2 \text{ weakly in } L^2(\Omega \times (0, T); \mathcal{L}_{2,Q}(H; H)), \tag{2.3.39}$$

as $m \rightarrow \infty$.

Next, we pass to the limit as $m \rightarrow \infty$.

To that purpose, let $y = y(\omega)$ be an arbitrary bounded random variable, and let ψ be an arbitrary bounded function on $(0, T)$. We multiply the equation (2.3.9) and (2.3.10) by the product $w_j y \psi$, integrate on D between 0 and T , and take the expectation to deduce that

$$\begin{aligned}
 &\mathbb{E} \int_0^T y \psi(t) \int_D \varphi_m(t) w_j dx dt \\
 &= \mathbb{E} \int_0^T y \psi(t) \int_D \varphi_{m0} w_j dx dt \\
 &+ \mathbb{E} \int_0^T y \psi(t) \left\{ \int_0^t \int_D \Delta \varphi_m w_j dx ds \right\} dt \\
 &+ \mathbb{E} \int_0^T y \psi(t) \left\{ \int_0^t \int_D P_m f(\varphi_m) w_j dx ds \right\} dt
 \end{aligned}$$

$$\begin{aligned}
 & -\mathbb{E} \int_0^T y\psi(t) \left\{ \int_0^t \int_D \varphi_m w_j dx ds \right\} dt + \mathbb{E} \int_0^T y\psi(t) \left\{ \int_0^t \int_D v_m w_j dx ds \right\} dt \\
 & + \mathbb{E} \int_0^T y\psi(t) \left\{ \int_D \int_0^t P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m w_j dW_1(s) dx \right\} dt, \tag{2.3.40}
 \end{aligned}$$

for all $j=0, \dots, m$.

$$\begin{aligned}
 & \mathbb{E} \int_0^T y\psi(t) \int_D v_m w_j dx dt \\
 & = \mathbb{E} \int_0^T y\psi(t) \int_D v_{0m} w_j dx dt \\
 & + \mathbb{E} \int_0^T y\psi(t) \left\{ \int_0^t \int_D \Delta v_m w_j dx ds \right\} dt \\
 & - \mathbb{E} \int_0^T y\psi(t) \left\{ \int_0^t \int_D \Delta \varphi_m w_j dx ds \right\} dt \\
 & + \mathbb{E} \int_0^T y\psi(t) \left\{ \int_D \int_0^t P_m \Phi_2(\varphi_m, v_m - \varphi_m) \tilde{P}_m w_j dW_2(s) dx \right\} dt. \tag{2.3.41}
 \end{aligned}$$

Next we pass to the limit in (2.3.40) and (2.3.41); we give the proof of convergence for the reaction term in (2.3.40) using the a priori estimates and Hölder inequality. We remark that $\int_D P_m f(\varphi_m) w_j(x) = \int_D f(\varphi_m) w_j(x) dx$ for all $j = 1, \dots, m$. We have that

$$\begin{aligned}
 & \left| \psi(t) \mathbb{E} \int_0^t \int_D f(\varphi_m) y w_j dx ds \right| \\
 & \leq \|y\|_{L^\infty(\Omega)} |\psi(t)| \left(\mathbb{E} \int_0^t \int_D |f(\varphi_m)|^{\frac{4}{3}} dx ds \right)^{\frac{3}{4}} \left(\mathbb{E} \int_0^t \int_D |w_j|^4 dx ds \right)^{\frac{1}{4}} \\
 & \leq \|y\|_{L^\infty(\Omega)} \|\psi\|_{L^\infty(0,T)} \bar{C}
 \end{aligned}$$

This shows that $|\psi(t) \mathbb{E} \int_0^t \int_D f(\varphi_m) y w_j dx ds|$ is uniformly bounded by a function belonging to $L^1(0, T)$. In addition using (2.3.37) we have that

$\psi(t) \mathbb{E} \int_0^t \int_D f(\varphi_m) y w_j dx ds \rightarrow \psi(t) \mathbb{E} \int_0^t \int_D \chi y w_j dx ds$ for a.e. $t \in (0, T)$. Applying Lebesgue-dominated convergence theorem we deduce that :

$$\begin{aligned}
 & \lim_{m \rightarrow \infty} \int_0^T \psi(t) dt \mathbb{E} \int_0^t \int_D f(\varphi_m) y w_j dx ds \\
 & = \int_0^T \lim_{m \rightarrow \infty} \psi(t) \mathbb{E} \int_0^t \int_D f(\varphi_m) y w_j dx ds dt \\
 & = \int_0^T \psi(t) dt \mathbb{E} \int_0^t \int_D \chi y w_j dx ds \\
 & = \mathbb{E} \int_0^T y \psi(t) dt \left\{ \int_D \int_0^t \chi w_j dx ds \right\}.
 \end{aligned}$$

We perform a similar proof for each term in (2.3.40) and (2.3.41).

It remains to pass to the limit in the stochastic term we know that a bounded linear operator is weakly continuous. [cf. [17] p. 117]

Next we show that the operator \mathcal{T} from $U = L^2(\Omega \times (0, T); \mathcal{L}_{2,Q}(H, H))$ into $\mathcal{U} = L^2(\Omega \times (0, T); L^2(D))$ defined by $: A \rightarrow \int_0^t AdW$ is bounded.

Indeed, we first recall Burkholder-Davis-Gundy inequality

Lemma 2.3.5 (Burkholder-Davis-Gundy inequality) (cf. [14] p.10 and [6])

Given G with the same assumption as in (2.1.5) and (3.1.7). The Burkholder-Davis-Gundy inequality can be written as

$$\mathbb{E} \left(\sup_{t \in (0, T)} \left\| \int_0^t G(s) dW(s) \right\|_X^r \right) \leq \tilde{C}_r \mathbb{E} \left(\int_0^T \|G\|_{2,Q}^2 dt \right)^{\frac{r}{2}}. \quad (2.3.42)$$

Applying the Burkholder inequality with $r = 2, G = A, X = L^2(D)$ we obtain

$$\mathbb{E} \sup_{t \in (0, T)} \left\| \int_0^t AdW(s) \right\|^2 \leq \tilde{C}_2 \mathbb{E} \left(\int_0^T \|A\|_{2,Q}^2 dt \right). \quad (2.3.43)$$

It then follows from (2.3.38) and (2.3.39) that

$$\int_0^t P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m dW_1 \rightharpoonup \int_0^t Z_1 dW_1 \quad \text{weakly in } L^2(\Omega \times (0, T); L^2(D)), \quad (2.3.44)$$

$$\int_0^t P_m \Phi_2(\varphi_m, v_m - \varphi_m) \tilde{P}_m dW_2 \rightharpoonup \int_0^t Z_2 dW_2 \quad \text{weakly in } L^2(\Omega \times (0, T); L^2(D)). \quad (2.3.45)$$

Passing to the limit in all the terms in (2.3.40) and in (2.3.41) yields for all $j = 0, \dots, m$

$$\begin{aligned} \mathbb{E} \int_0^T \int_D y\psi(t) \varphi(t) w_j dx dt &= \mathbb{E} \int_0^T \int_D y\psi(t) \varphi_0 w_j dx dt \\ &+ \mathbb{E} \int_0^T y\psi(t) \int_0^t \langle \Delta \varphi, w_j \rangle_{Z^*, Z} ds dt \\ &+ \mathbb{E} \int_0^T y\psi(t) \left\{ \int_0^t \int_D \chi w_j dx ds \right\} dt \\ &- \mathbb{E} \int_0^T y\psi(t) \left\{ \int_0^t \int_D \varphi w_j dx ds \right\} dt + \mathbb{E} \int_0^T y\psi(t) \left\{ \int_0^t \int_D v w_j dx ds \right\} dt \\ &+ \mathbb{E} \int_0^T y\psi(t) \left\{ \int_D \int_0^t Z_1 dW_1(s) w_j dx \right\} dt. \end{aligned} \quad (2.3.46)$$

And,

$$\begin{aligned} \mathbb{E} \int_0^T y\psi(t) \int_D v w_j dx dt &= \mathbb{E} \int_0^T y\psi(t) \int_D v_0 w_j dx dt \\ &+ \mathbb{E} \int_0^T y\psi(t) \int_0^t \langle \Delta v, w_j \rangle_{V^*, V} ds dt \end{aligned}$$

$$\begin{aligned}
 & -\mathbb{E} \int_0^T y\psi(t) \int_0^t \langle \Delta\varphi, w_j \rangle_{V^*,V} ds dt \\
 & +\mathbb{E} \int_0^T y\psi(t) \left\{ \int_D \int_0^t Z_2 dW_2(s) w_j dx \right\} dt \quad (2.3.47)
 \end{aligned}$$

for all $y \in L^\infty(\Omega)$ and $\psi \in L^\infty(0, T)$.

We remark that the linear combinations of w_j are dense in $V \cap L^4(D)$, so that

$$\begin{aligned}
 \mathbb{E} \int_0^T \int_D y\psi(t) \varphi(t) \tilde{w} dx dt & = \mathbb{E} \int_0^T \int_D y\psi(t) \varphi_0 \tilde{w} dx dt \\
 & +\mathbb{E} \int_0^T y\psi(t) \left\{ \int_0^t \langle \Delta\varphi + \chi, \tilde{w} \rangle_{Z^*,Z} ds \right\} dt \\
 & -\mathbb{E} \int_0^T y\psi(t) \left\{ \int_0^t \int_D \varphi \tilde{w} dx ds \right\} dt \\
 & +\mathbb{E} \int_0^T y\psi(t) \left\{ \int_0^t \int_D v \tilde{w} dx ds \right\} dt \\
 & +\mathbb{E} \int_0^T y\psi(t) \left\{ \int_D \int_0^t Z_1 dW_1(s) \tilde{w} dx \right\} dt.
 \end{aligned}$$

And,

$$\begin{aligned}
 \mathbb{E} \int_0^T y\psi(t) \int_D v \tilde{w} dx dt & = \mathbb{E} \int_0^T y\psi(t) \int_D v_0 \tilde{w} dx dt \\
 & +\mathbb{E} \int_0^T y\psi(t) \int_0^t \langle \Delta v, \tilde{w} \rangle_{V^*,V} ds dt \\
 & -\mathbb{E} \int_0^T y\psi(t) \int_0^t \langle \Delta\varphi, \tilde{w} \rangle_{V^*,V} ds dt \\
 & +\mathbb{E} \int_0^T y\psi(t) \left\{ \int_D \int_0^t Z_2 dW_2(s) \tilde{w} dx \right\} dt \quad (2.3.48)
 \end{aligned}$$

for all $\tilde{w} \in V \cap L^4(D)$, all $\tilde{w} \in V$, $y \in L^\infty(\Omega)$ and $\psi \in L^\infty(0, T)$.

This implies that for a.e. $(t, \omega) \in (0, T) \times \Omega$

$$\begin{aligned}
 \int_D \varphi(t) \tilde{w} dx & = \int_D \varphi_0 \tilde{w} dx + \int_0^t \langle \Delta\varphi + \chi, \tilde{w} \rangle_{Z^*,Z} ds + \int_0^t \int_D (-\varphi + v) \tilde{w} dx ds \\
 & + \int_D \int_0^t Z_1 dW_1 \tilde{w} \quad (2.3.49)
 \end{aligned}$$

$$\int_D v(t) \tilde{w} dx = \int_D v_0 \tilde{w} dx + \int_0^t \int_D \langle \Delta v - \Delta\varphi, \tilde{w} \rangle_{V^*,V} ds + \int_D \int_0^t Z_2 dW_2 \tilde{w} dx \quad (2.3.50)$$

for all $\tilde{w} \in V \cap L^4(D)$ and all $\tilde{w} \in V$.

2.4 Monotonicity argument

In order to prove that the pair (φ, v) is a solution of Problem (P_2) , we still have to identify the functions χ, Z_1 and Z_2 . To that purpose, we apply the stochastic monotonicity method. Let w et ψ be such that $\psi \in L^2(\Omega \times (0, T); V) \cap L^4(\Omega \times D \times (0, T))$ and $w \in L^2(\Omega \times (0, T); V)$.

Let c be a positive constant which will be fixed later. We define

$$\begin{aligned} \mathcal{O}_m &= \mathbb{E} \left[\int_0^T e^{-cs} \{ 2\langle \Delta \varphi_m - \Delta \psi, \varphi_m - \psi \rangle_{Z^*, Z} + 2\langle \Delta v_m - \Delta w, v_m - w \rangle_{V^*, V} \right. \\ &\quad + 2\langle f(\varphi_m) - f(\psi), \varphi_m - \psi \rangle_{Z^*, Z} \\ &\quad - 2\langle \varphi_m - \psi, \varphi_m - \psi \rangle_{Z^*, Z} + 2\langle v_m - w, \varphi_m - \psi \rangle_{Z^*, Z} - 2\langle \Delta \varphi_m - \Delta \psi, v_m - w \rangle_{V^*, V} \\ &\quad - c[\|\varphi_m - \psi\|^2 + \|v_m - w\|^2] + \|P_m(\Phi_1(\varphi_m, v_m - \varphi_m) - \Phi_1(\psi, w - \psi))\tilde{P}_m\|_{2, Q}^2 \\ &\quad \left. + \|P_m(\Phi_2(\varphi_m, v_m - \varphi_m) - \Phi_2(\psi, w - \psi))\tilde{P}_m\|_{2, Q}^2 \right] ds \\ &= J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8, \end{aligned}$$

and prove below the following result

Lemma 2.4.1

$$\mathcal{O}_m \leq 0.$$

Proof First we estimate J_1

$$\begin{aligned} J_1 &= \mathbb{E} \int_0^T e^{-cs} \{ 2\langle \Delta \varphi_m - \Delta \psi, \varphi_m - \psi \rangle_{Z^*, Z} + 2\langle \Delta v_m - \Delta w, v_m - w \rangle_{V^*, V} \} \\ &\leq -2\mathbb{E} \int_0^T e^{-cs} \|\nabla(\varphi_m - \psi)\|^2 ds - 2\mathbb{E} \int_0^T e^{-cs} \|\nabla(v_m - w)\|^2 ds. \end{aligned}$$

Next, (2.1.2) and the mean value theorem yield:

$$\begin{aligned} J_2 &= \mathbb{E} \int_0^T e^{-cs} 2\langle f(\varphi_m) - f(\psi), \varphi_m - \psi \rangle_{Z^*, Z} ds \\ &\leq \mathbb{E} \int_0^T e^{-cs} 2\|\varphi_m - \psi\|^2 ds. \end{aligned}$$

We remark that $J_3 \leq 0$ and estimate J_4 and J_5 in the following way

$$\begin{aligned} J_4 &= 2\mathbb{E} \int_0^T e^{-cs} \langle v_m - w, \varphi_m - \psi \rangle_{Z^*, Z} \\ &\leq \mathbb{E} \int_0^T e^{-cs} \|v_m - w\|_{L^2(D)}^2 ds + \mathbb{E} \int_0^T e^{-cs} \|\varphi_m - \psi\|_{L^2(D)}^2 ds. \end{aligned}$$

$$\begin{aligned} J_5 &= -2\mathbb{E} \int_0^T e^{-cs} \langle \Delta \varphi_m - \Delta \psi, v_m - w \rangle_{V^*, V} \\ &\leq \mathbb{E} \int_0^T e^{-cs} \|\nabla(\varphi_m - \psi)\|_{L^2(D)}^2 ds + \mathbb{E} \int_0^T e^{-cs} \|\nabla(v_m - w)\|_{L^2(D)}^2 ds. \end{aligned}$$

Also $J_6 \leq 0$.

Using (2.1.10) and (2.3.12) we obtain

$$\begin{aligned}
 J_7 &= \mathbb{E} \int_0^T e^{-cs} \|P_m(\Phi_1(\varphi_m, v_m - \varphi_m) - \Phi_1(\psi, w - \psi))\tilde{P}_m\|_{2,Q} \\
 &\leq \mathbb{E} \int_0^T e^{-cs} \sum_{k=1}^{\infty} \|P_m(\Phi_1(\varphi_m, v_m - \varphi_m) - \Phi_1(\psi, w - \psi))\tilde{P}_m(\sqrt{\lambda_k^1}e_k)\|^2 \\
 &\leq \mathbb{E} \int_0^T e^{-cs} \sum_{k=1}^m \|P_m(\Phi_1(\varphi_m, v_m - \varphi_m) - \Phi_1(\psi, w - \psi))\sqrt{\lambda_k^1}e_k\|^2 \\
 &\leq \mathbb{E} \int_0^T e^{-cs} \sum_{k=1}^{\infty} \|P_m(\Phi_{1l}(\varphi_m, v_m - \varphi_m) - \Phi_{1l}(\psi, w - \psi))\|^2 \\
 &\leq \mathbb{E} \int_0^T e^{-cs} \sum_{k=1}^{\infty} \|\Phi_{1l}(\varphi_m, v_m - \varphi_m) - \Phi_{1l}(\psi, w - \psi)\|^2 \\
 &\leq 3C_3\mathbb{E} \int_0^T e^{-cs} \|\varphi_m - \psi\|^2 + 2C_3\mathbb{E} \int_0^T e^{-cs} \|v_m - w\|^2.
 \end{aligned}$$

We deduce in a similar way that

$$\begin{aligned}
 J_8 &= \mathbb{E} \int_0^T e^{-cs} \|P_m(\Phi_2(\varphi_m, v_m - \varphi_m) - \Phi_2(\psi, w - \psi))\tilde{P}_m\|_{2,Q} \\
 &\leq 3C_3\mathbb{E} \int_0^T e^{-cs} \|\varphi_m - \psi\|^2 + 2C_3\mathbb{E} \int_0^T e^{-cs} \|v_m - w\|^2.
 \end{aligned}$$

Combining all the computations and choosing c large enough in J_6 we conclude the result. \square

We write \mathcal{O}_m in the form $\mathcal{O}_m = \mathcal{O}_m^1 + \mathcal{O}_m^2$ where

$$\begin{aligned}
 \mathcal{O}_m^1 &= \mathbb{E} \left[\int_0^T e^{-cs} \{ 2\langle \Delta\varphi_m, \varphi_m \rangle_{Z^*,Z} + 2\langle \Delta v_m, v_m \rangle_{V^*,V} \right. \\
 &\quad + 2\langle f(\varphi_m), \varphi_m \rangle_{Z^*,Z} - 2\langle \varphi_m, \varphi_m \rangle_{Z^*,Z} + 2\langle v_m, \varphi_m \rangle_{Z^*,Z} - 2\langle \Delta\varphi_m, v_m \rangle_{V^*,V} \\
 &\quad - c\|\varphi_m\|^2 - c\|v_m\|^2 + \|P_m\Phi_1(\varphi_m, v_m - \varphi_m)\tilde{P}_m\|_{2,Q}^2 \\
 &\quad \left. + \|P_m\Phi_2(\varphi_m, v_m - \varphi_m)\tilde{P}_m\|_{2,Q}^2 \right] ds, \tag{2.4.1}
 \end{aligned}$$

Applying Itô's formula (2.3.14) to (2.3.9) with $E = \mathbb{R}$, $x \in D$ fixed and

$$\begin{aligned}
 X(t) &= \varphi_m(x, t), \\
 F(t, X) &= e^{-ct} X^2, \\
 \frac{\partial F}{\partial t}(t, X) &= -ce^{-ct} X^2, \\
 \frac{\partial F}{\partial X}(t, X) &= 2e^{-ct} X, \\
 \frac{\partial^2 F}{\partial X^2}(s) &= 2e^{-ct}, \\
 h(s) &= \Delta\varphi_m + P_m f(\varphi_m) - \varphi_m + v_m,
 \end{aligned}$$

$$G(s) = P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m.$$

We obtain almost surely, for all $t \in [0, T]$ and after integrating on D

$$\begin{aligned} e^{-cT} \|\varphi_m(T)\|^2 &= \|\varphi_{0m}\|^2 - c \int_0^T e^{-cs} \|\varphi_m\|^2 ds \\ &\quad - 2 \int_0^T e^{-cs} \|\nabla \varphi_m\|^2 ds + 2 \int_0^T e^{-cs} \int_D f(\varphi_m) \varphi_m ds \\ &\quad - 2 \int_0^T e^{-cs} \|\varphi_m\|^2 ds + 2 \int_0^T e^{-cs} \int_D \varphi_m v_m ds \\ &\quad + 2 \int_0^T e^{-cs} \int_D \varphi_m P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m dW_1(s) dx \\ &\quad + \int_0^T e^{-cs} \sum_{l=1}^{\infty} \|P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m \sqrt{\lambda_l^{-1}} e_l\|^2 ds. \end{aligned} \quad (2.4.2)$$

Similarly, we apply itô's formula Lemma 2.3.3 to (2.3.10). Therefore we obtain,

$$\begin{aligned} e^{-cT} \|v_m(T)\|^2 &= \|v_{0m}\|^2 - c \int_0^T e^{-cs} \|v_m\|^2 ds + 2 \int_0^t e^{-cs} \int_D v_m (\Delta v_m - \Delta \varphi_m) dx ds \\ &\quad + 2 \int_0^T e^{-cs} \int_D v_m P_m \Phi_2(\varphi_m, v_m - \varphi_m) \tilde{P}_m dW_2(s) dx \\ &\quad + \int_0^T e^{-cs} \sum_{l=1}^{\infty} \|P_m \Phi_2(\varphi_m, v_m - \varphi_m) \tilde{P}_m \sqrt{\lambda_l^{-2}} e_l\|^2 ds. \end{aligned} \quad (2.4.3)$$

We take the expectation of each equation (2.4.2), (2.4.3) and we sum up

$$\begin{aligned} &e^{-cT} \mathbb{E} \|\varphi_m(T)\|^2 + e^{-cT} \mathbb{E} \|v_m(T)\|^2 \\ &= \mathbb{E} \|\varphi_{0m}\|^2 + \mathbb{E} \|v_{0m}\|^2 \\ &\quad - c \mathbb{E} \int_0^T e^{-cs} \|\varphi_m\|^2 ds - c \mathbb{E} \int_0^T e^{-cs} \|v_m\|^2 ds \\ &\quad - 2 \mathbb{E} \int_0^T e^{-cs} \|\nabla \varphi_m\|^2 ds - 2 \mathbb{E} \int_0^T e^{-cs} \|\nabla v_m\|^2 ds \\ &\quad + 2 \mathbb{E} \int_0^T e^{-cs} \|\varphi_m\|^2 ds - 2 \mathbb{E} \int_0^T e^{-cs} \|\varphi_m\|_{L^4(D)}^4 ds - 2 \mathbb{E} \int_0^T e^{-cs} \|\varphi_m\|^2 ds \\ &\quad + 2 \mathbb{E} \int_0^T e^{-cs} \int_D \varphi_m v_m ds + 2 \mathbb{E} \int_0^T e^{-cs} \int_D \nabla v_m \nabla \varphi_m dx ds \\ &\quad + \int_0^T e^{-cs} \|P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m\|_{2, Q}^2 ds \\ &\quad + \int_0^T e^{-cs} \|P_m \Phi_2(\varphi_m, v_m - \varphi_m) \tilde{P}_m\|_{2, Q}^2 ds, \end{aligned} \quad (2.4.4)$$

where we have used the fact that :

$$\mathbb{E} \int_0^T e^{-cs} P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m dW_1 = 0 \text{ and } \mathbb{E} \int_0^T e^{-cs} P_m \Phi_2(\varphi_m, v_m - \varphi_m) \tilde{P}_m dW_2 = 0.$$

It follows from (2.4.1) and (2.4.4) that

$$\mathcal{O}_m^1 = \mathbb{E}[e^{-cT} \|\varphi_m(T)\|^2] + \mathbb{E}[e^{-cT} \|v_m(T)\|^2] - \mathbb{E}[\|\varphi_{0m}\|^2] - \mathbb{E}[\|v_{0m}\|^2].$$

From this we obtain

$$\begin{aligned} & \limsup_{m \rightarrow \infty} \mathcal{O}_m^1 \\ &= \mathbb{E}[e^{-cT} \|\varphi(T)\|^2] - \mathbb{E}[\|\varphi(0)\|^2] + \mathbb{E}[e^{-cT} \|v(T)\|^2] - \mathbb{E}[\|v(0)\|^2] + \delta e^{-cT} + \theta e^{-cT}, \end{aligned} \quad (2.4.5)$$

where

$$\begin{aligned} \delta &= \limsup_{m \rightarrow \infty} \mathbb{E}[\|\varphi_m(T)\|^2] - \mathbb{E}[\|\varphi(T)\|^2] \geq 0. \\ \theta &= \limsup_{m \rightarrow \infty} \mathbb{E}[\|v_m(T)\|^2] - \mathbb{E}[\|v(T)\|^2] \geq 0. \end{aligned}$$

On the other hand, the equation (2.3.49) and (2.3.50) imply that

$$\varphi(t) = \varphi_0 + \int_0^t \Delta\varphi + \int_0^t \chi - \int_0^t \varphi + \int_0^t v + \int_0^t Z_1 dW_1, \quad \forall t \in [0, T] \quad (2.4.6)$$

$$v(t) = v_0 + \int_0^t \Delta v - \int_0^t \Delta\varphi + \int_0^t Z_2 dW_2, \quad \forall t \in [0, T] \quad (2.4.7)$$

a.s. in $L^2(D) + L^{\frac{4}{3}}(D)$.

Next we recall an Itô's formula as in [15] [p.75 Theorem 4.2.5], which involves different function spaces. Consider the Gelfand triple

$$Z \subset H \subset Z^*,$$

where $Z = V \cap L^4(D)$ and Z^* are defined in the introduction.

Proposition 2.4.2 *Let $X \in L^2(\Omega \times (0, T); V) \cap L^4(\Omega \times (0, T); L^4(D))$, $Y \in L^2(\Omega \times (0, T); V^*) + L^{\frac{4}{3}}(\Omega \times (0, T); L^{\frac{4}{3}}(D))$ and $G \in L^2(\Omega \times (0, T); \mathcal{L}_{2,Q}(H, H))$ be such that*

$$X(t) := X_0 + \int_0^t Y(s) ds + \int_0^t G(s) dW(s), \quad t \in [0, T].$$

Suppose that the function $F : [0, T] \times Z \rightarrow \mathbb{R}$ and its partial derivatives $\frac{\partial F}{\partial t}$, $\frac{\partial F}{\partial X}$ and $\frac{\partial^2 F}{\partial^2 X}$ are continuous on $[0, T] \times Z$. Then for all $t \in [0, T]$

$$\begin{aligned} F(t, X(t)) &= F(0, X(0)) + \int_0^t \frac{\partial F}{\partial t}(s, X(s)) ds + \int_0^t \langle Y(s), \frac{\partial F}{\partial X}(s, X(s)) \rangle_{Z^*, Z} ds \\ &\quad + \int_0^t \langle \frac{\partial F}{\partial X}(s, X(s)), dW(s) \rangle_{Z^*, Z} \\ &\quad + \frac{1}{2} \int_0^t \text{Tr} \left(\frac{\partial^2 F}{\partial^2 X}(G(s) Q^{\frac{1}{2}}) (G(s) Q^{\frac{1}{2}})^* \right) ds. \end{aligned} \quad (2.4.8)$$

Applying Proposition 2.4.2 to the equation (2.4.6), we set $X(t) = \varphi(t)$, $F(s, X) = e^{-cs}\|X\|^2$, $Y(s) = \Delta\varphi + \chi - \varphi + v$ and $G = Z_1$ in (2.4.8) to deduce that

$$\begin{aligned} \mathbb{E}[e^{-cT}\|\varphi(T)\|^2] &= \mathbb{E}[\|\varphi(0)\|^2] - c\mathbb{E}\left[\int_0^T e^{-cs}\|\varphi(s)\|^2 ds\right] \\ &\quad + 2\mathbb{E}\int_0^T e^{-cs}\langle\Delta\varphi, \varphi\rangle_{Z^*, Z} + 2\mathbb{E}\left[\int_0^T e^{-cs}\langle\chi, \varphi\rangle_{Z^*, Z}\right. \\ &\quad \left. - 2\mathbb{E}\int_0^T e^{-cs}\langle\varphi, \varphi\rangle_{Z^*, Z} + 2\mathbb{E}\left[\int_0^T e^{-cs}\langle v, \varphi\rangle_{Z^*, Z}\right.\right. \\ &\quad \left. \left. + 2\mathbb{E}\int_0^T e^{-cs}\langle\varphi, Z_1 dW_1\rangle_{Z^*, Z}\right] + \mathbb{E}\left[\int_0^T e^{-cs}\|Z_1\|_{2, Q}^2\right], \end{aligned}$$

Similarly, applying Lemma 2.3.3 for (2.4.7) with $E = H$ we obtain

$$\begin{aligned} \mathbb{E}[e^{-cT}\|v(T)\|^2] &= \mathbb{E}[\|v(0)\|^2] - c\mathbb{E}\left[\int_0^T e^{-cs}\|v(s)\|^2 ds\right] \\ &\quad + 2\mathbb{E}\int_0^T e^{-cs}\langle\Delta v, v\rangle_{V^*, V} \\ &\quad - 2\mathbb{E}\left[\int_0^T e^{-cs}\langle\Delta\varphi, v\rangle_{V^*, V}\right. \\ &\quad \left. + 2\mathbb{E}\int_0^T e^{-cs}\langle v, Z_2 dW_2\rangle_{V^*, V}\right] + \mathbb{E}\left[\int_0^T e^{-cs}\|Z_2\|_{2, Q}^2\right], \end{aligned}$$

Summing up both equations and using the facts that $\mathbb{E}\left[\int_0^T e^{-cs}\langle\varphi, Z_1 dW_1\rangle_{Z^*, Z}\right] = 0$ and $\mathbb{E}\left[\int_0^T e^{-cs}\langle v, Z_2 dW_2\rangle_{V^*, V}\right] = 0$, we deduce that

$$\begin{aligned} \mathbb{E}[e^{-cT}\|\varphi(T)\|^2] + \mathbb{E}[e^{-cT}\|v(T)\|^2] &= \mathbb{E}[\|\varphi(0)\|^2] + \mathbb{E}[\|v(0)\|^2] \\ &\quad - c\mathbb{E}\left[\int_0^T e^{-cs}\|\varphi(s)\|^2 ds\right] - c\mathbb{E}\left[\int_0^T e^{-cs}\|v(s)\|^2 ds\right] \\ &\quad + 2\mathbb{E}\int_0^T e^{-cs}\langle\Delta\varphi, \varphi\rangle_{Z^*, Z} + 2\mathbb{E}\int_0^T e^{-cs}\langle\Delta v, v\rangle_{V^*, V} \\ &\quad + 2\mathbb{E}\left[\int_0^T e^{-cs}\langle\chi, \varphi\rangle_{Z^*, Z} - 2\mathbb{E}\left[\int_0^T e^{-cs}\langle\varphi, \varphi\rangle_{Z^*, Z}\right.\right. \\ &\quad \left. \left. + 2\mathbb{E}\int_0^T e^{-cs}\langle v, \varphi\rangle_{Z^*, Z} - 2\mathbb{E}\left[\int_0^T e^{-cs}\langle\Delta\varphi, v\rangle_{V^*, V}\right.\right.\right. \\ &\quad \left. \left. + \mathbb{E}\int_0^T e^{-cs}\|Z_1\|_{2, Q}^2\right] + \mathbb{E}\left[\int_0^T e^{-cs}\|Z_2\|_{2, Q}^2\right], \end{aligned}$$

which we combine with (2.4.5) to deduce that

$$\begin{aligned} \limsup_{m \rightarrow \infty} O_m^1 &= -c\mathbb{E}\left[\int_0^T e^{-cs}\|\varphi(s)\|^2 ds\right] - c\mathbb{E}\left[\int_0^T e^{-cs}\|v(s)\|^2 ds\right] \\ &\quad + 2\mathbb{E}\int_0^T e^{-cs}\langle\Delta\varphi, \varphi\rangle_{Z^*, Z} + 2\mathbb{E}\int_0^T e^{-cs}\langle\Delta v, v\rangle_{V^*, V} \end{aligned}$$

$$\begin{aligned}
 & +2\mathbb{E}\left[\int_0^T e^{-cs}\langle\chi, \varphi\rangle_{Z^*,Z} - 2\mathbb{E}\left[\int_0^T e^{-cs}\langle\varphi, \varphi\rangle_{Z^*,Z}\right.\right. \\
 & \left. +2\mathbb{E}\left[\int_0^T e^{-cs}\langle v, \varphi\rangle_{Z^*,Z} - 2\mathbb{E}\left[\int_0^T e^{-cs}\langle\Delta\varphi, v\rangle_{V^*,V}\right.\right. \\
 & \left. \left. +\mathbb{E}\left[\int_0^T e^{-cs}\|Z_1\|_{2,Q}^2\right] + \mathbb{E}\left[\int_0^T e^{-cs}\|Z_2\|_{2,Q}^2\right] + \delta e^{-cT} + \theta e^{-cT}\right]. \quad (2.4.9)
 \end{aligned}$$

It remains to compute the limit of \mathcal{O}_m^2 :

$$\begin{aligned}
 \mathcal{O}_m^2 & = \mathcal{O}_m - \mathcal{O}_m^1 \\
 & = \mathbb{E} \int_0^T e^{-cs} \{ -2\langle\Delta\psi, \varphi_m - \psi\rangle_{Z^*,Z} - 2\langle\Delta w, v_m - w\rangle_{V^*,V} \\
 & \quad -2\langle\Delta\varphi_m, \psi\rangle_{Z^*,Z} - 2\langle\Delta v_m, w\rangle_{V^*,V} \\
 & \quad -2\langle f(\psi), \varphi_m - \psi\rangle_{Z^*,Z} + 2\langle f(\varphi_m), -\psi\rangle_{Z^*,Z} \\
 & \quad -2\langle -\psi, \varphi_m - \psi\rangle_{Z^*,Z} + 2\langle\varphi_m, \psi\rangle_{Z^*,Z} + 2\langle -w, \varphi_m - \psi\rangle_{Z^*,Z} \\
 & \quad -2\langle v_m, \psi\rangle_{Z^*,Z} - 2\langle -\Delta\psi, v_m - w\rangle_{V^*,V} - 2\langle\Delta\varphi_m, -w\rangle_{V^*,V} \\
 & \quad -c\|\psi\|^2 + 2c\langle\varphi_m, \psi\rangle_{Z^*,Z} - c\|w\|^2 + 2c\langle v_m, w\rangle_{V^*,V} \\
 & \quad +\|P_m\Phi_1(\psi, w - \psi)\tilde{P}_m\|_{2,Q}^2 \\
 & \quad -2\sum_{k=1}^{\infty} \int_D \left(P_m\Phi_1(\varphi_m, v_m - \varphi_m)\tilde{P}_m(\sqrt{\lambda_k^1}e_k) \right) \left(P_m\Phi_1(\psi, w - \psi)\tilde{P}_m(\sqrt{\lambda_k^1}e_k) \right) dx \\
 & \quad +\|P_m\Phi_2(\psi, w - \psi)\tilde{P}_m\|_{2,Q}^2 \\
 & \quad -2\sum_{k=1}^{\infty} \int_D \left(P_m\Phi_2(\varphi_m, v_m - \varphi_m)\tilde{P}_m(\sqrt{\lambda_k^1}e_k) \right) \left(P_m\Phi_2(\psi, w - \psi)\tilde{P}_m(\sqrt{\lambda_k^2}e_k) \right) dx \} ds.
 \end{aligned}$$

Next, we show that

$$\lim_{m \rightarrow \infty} \|\Phi_1(\psi, w - \psi) - P_m\Phi_1(\psi, w - \psi)\tilde{P}_m\|_{2,Q}^2 = 0. \quad (2.4.10)$$

Indeed,

$$\begin{aligned}
 & \|\Phi_1(\psi, w - \psi) - P_m\Phi_1(\psi, w - \psi)\tilde{P}_m\|_{2,Q}^2 \\
 & = \|\Phi_1(\psi, w - \psi) - P_m\Phi_1(\psi, w - \psi)\tilde{P}_m - \Phi_1(\psi, w - \psi)\tilde{P}_m + \Phi_1(\psi, w - \psi)\tilde{P}_m\|_{2,Q}^2 \\
 & \leq 2\|\Phi_1(\psi, w - \psi)\tilde{P}_m - P_m\Phi_1(\psi, w - \psi)\tilde{P}_m\|_{2,Q}^2 \\
 & \quad +2\|P_m\Phi_1(\psi, w - \psi)\tilde{P}_m - \Phi_1(\psi, w - \psi)\tilde{P}_m\|_{2,Q}^2 \\
 & = E_1 + E_2. \quad (2.4.11)
 \end{aligned}$$

As for the first term E_1 , using (2.3.13) and the fact that $\Phi_1(\psi, w - \psi)$ is bounded in $L^2(D)$, we have that

$$\lim_{m \rightarrow \infty} \|\Phi_1(\psi, w - \psi)\tilde{P}_m - \Phi_1(\psi, w - \psi)\|_{\mathcal{L}(H,H)}^2 = 0.$$

where $\|A\|_{\mathcal{L}(H,H)} := \sup_{a \neq 0} \frac{\|Aa\|}{\|a\|}$.

Using a similar argument as in the proof of Theorem 1.3 in [12] we deduce that

$$\lim_{m \rightarrow \infty} \|\Phi_1(\psi, w - \psi)\tilde{P}_m - \Phi_1(\psi, w - \psi)\|_{2,Q}^2 = 0.$$

As for the second term E_2 , we remark that

$$\begin{aligned}
 & \|P_m \Phi_1(\psi, w - \psi) \tilde{P}_m - \Phi_1(\psi, w - \psi) \tilde{P}_m\|_{2,Q}^2 \\
 &= \sum_{k=1}^{\infty} \|P_m \Phi_1(\psi, w - \psi) \tilde{P}_m \sqrt{\lambda_k^1} e_k - \Phi_1(\psi, w - \psi) \tilde{P}_m \sqrt{\lambda_k^1} e_k\|^2 \\
 &= \sum_{k=1}^m \|P_m \Phi_1(\psi, w - \psi) \sqrt{\lambda_k^1} e_k - \Phi_1(\psi, w - \psi) \sqrt{\lambda_k^1} e_k\|^2 \\
 &\leq \sum_{k=1}^{\infty} \|P_m \Phi_1(\psi, w - \psi) \sqrt{\lambda_k^1} e_k - \Phi_1(\psi, w - \psi) \sqrt{\lambda_k^1} e_k\|^2 \\
 &\leq \|P_m \Phi_1(\psi, w - \psi) - \Phi_1(\psi, w - \psi)\|_{2,Q}^2.
 \end{aligned}$$

Using the same arguments as for E_1 we have that

$$\lim_{m \rightarrow \infty} \|P_m \Phi_1(\psi, w - \psi) - \Phi_1(\psi, w - \psi)\|_{2,Q}^2 = 0$$

Thus,

$$\lim_{m \rightarrow \infty} \|P_m \Phi_1(\psi, w - \psi) \tilde{P}_m - \Phi_1(\psi, w - \psi) \tilde{P}_m\|_{2,Q}^2 = 0.$$

In addition, we will show that

$$\lim_{m \rightarrow \infty} |\langle P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m, P_m \Phi_1(\psi, w - \psi) \tilde{P}_m \rangle_{2,Q} - \langle Z_1, \Phi_1(\psi, w - \psi) \rangle_{2,Q}| = 0, \quad (2.4.12)$$

where $\langle A, B \rangle_{2,Q} = \sum_{k=1}^{\infty} \langle A \sqrt{\lambda_k} e_k, B \sqrt{\lambda_k} e_k \rangle$.

Indeed,

$$\begin{aligned}
 & |\langle P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m, P_m \Phi_1(\psi, w - \psi) \tilde{P}_m \rangle_{2,Q} - \langle Z_1, \Phi_1(\psi, w - \psi) \rangle_{2,Q}| \\
 &= |\langle P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m, P_m \Phi_1(\psi, w - \psi) \tilde{P}_m \rangle_{2,Q} \\
 &\quad - \langle P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m, \Phi_1(\psi, w - \psi) \rangle_{2,Q} \\
 &\quad + \langle P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m, \Phi_1(\psi, w - \psi) \rangle_{2,Q} \\
 &\quad - \langle Z_1, \Phi_1(\psi, w - \psi) \rangle_{2,Q}| \\
 &\leq |\langle P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m, P_m \Phi_1(\psi, w - \psi) \tilde{P}_m - \Phi_1(\psi, w - \psi) \rangle_{2,Q}| \\
 &\quad + |\langle P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m, \Phi_1(\psi, w - \psi) \rangle_{2,Q} \\
 &\quad - \langle Z_1, \Phi_1(\psi, w - \psi) \rangle_{2,Q}| \\
 &= F_1 + F_2
 \end{aligned}$$

Using Cauchy-Schwarz inequality and (2.4.10)

$$\begin{aligned}
 & |\langle P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m, P_m \Phi_1(\psi, w - \psi) \tilde{P}_m - \Phi_1(\psi, w - \psi) \rangle_{2,Q}| \\
 &\leq \|P_m \Phi_1(\varphi_m, v_m - \varphi_m) \tilde{P}_m\|_{2,Q} \|P_m \Phi_1(\psi, w - \psi) \tilde{P}_m - \Phi_1(\psi, w - \psi)\|_{2,Q}
 \end{aligned}$$

By (2.4.10), $\lim_{m \rightarrow \infty} \|P_m \Phi_1(\psi, w - \psi) \tilde{P}_m - \Phi_1(\psi, w - \psi)\|_{2,Q} = 0$, thus $\lim_{m \rightarrow \infty} F_1 = 0$.

By (2.3.38), $\lim_{m \rightarrow \infty} F_2 = 0$.

Therefore, (2.4.12) is proved.

In view of (2.3.33)-(2.3.39), (2.4.10) and (2.4.12) we deduce that

$$\begin{aligned}
 \lim_{m \rightarrow \infty} \mathcal{O}_m^2 &= \mathbb{E} \int_0^T e^{-cs} \{ -2\langle \Delta\psi, \varphi - \psi \rangle_{Z^*, Z} - 2\langle \Delta w, v - w \rangle_{V^*, V} \\
 &\quad - 2\langle \Delta\varphi, \psi \rangle_{Z^*, Z} - 2\langle \Delta v, w \rangle_{V^*, V} \\
 &\quad - 2\langle f(\psi), \varphi - \psi \rangle_{Z^*, Z} + 2\langle \chi, -\psi \rangle_{Z^*, Z} \\
 &\quad - 2\langle -\psi, \varphi - \psi \rangle_{Z^*, Z} + 2\langle \varphi, \psi \rangle_{Z^*, Z} + 2\langle -w, \varphi - \psi \rangle_{Z^*, Z} \\
 &\quad - 2\langle v, \psi \rangle_{Z^*, Z} - 2\langle -\Delta\psi, v - w \rangle_{V^*, V} - 2\langle \Delta\varphi, -w \rangle_{V^*, V} \\
 &\quad - c\|\psi\|^2 + 2c\langle \varphi, \psi \rangle_{Z^*, Z} - c\|w\|^2 + 2c\langle v, w \rangle_{V^*, V} \\
 &\quad + \|\Phi_1(\psi, w - \psi)\|_{2, Q}^2 - 2 \sum_{k=1}^{\infty} \int_D Z_1 \sqrt{\lambda_k^1} e_k \Phi_1(\psi, w - \psi) \sqrt{\lambda_k^1} e_k dx \\
 &\quad + \|\Phi_2(\psi, w - \psi)\|_{2, Q}^2 - 2 \sum_{k=1}^{\infty} \int_D Z_2 \sqrt{\lambda_k^2} e_k \Phi_2(\psi, w - \psi) \sqrt{\lambda_k^2} e_k dx \} ds.
 \end{aligned} \tag{2.4.13}$$

Combining (2.4.9) and (2.4.13), and remembering that $O_m \leq 0$, yields

$$\begin{aligned}
 \mathbb{E} \int_0^T e^{-cs} \{ 2\langle \Delta\varphi - \Delta\psi, \varphi - \psi \rangle_{Z^*, Z} + 2\langle \Delta v - \Delta w, v - w \rangle_{V^*, V} \\
 + 2\langle \chi - f(\psi), \varphi - \psi \rangle_{Z^*, Z} - 2\langle \varphi - \psi, \varphi - \psi \rangle_{Z^*, Z} \\
 + 2\langle v - w, \varphi - \psi \rangle_{Z^*, Z} - 2\langle \Delta\varphi - \Delta\psi, v - w \rangle_{V^*, V} \\
 - c\|\varphi - \psi\|^2 - c\|v - w\|^2 \\
 + \|\Phi_1(\psi, w - \psi) - Z_1\|_{2, Q}^2 + \|\Phi_2(\psi, w - \psi) - Z_2\|_{2, Q}^2 \\
 + \delta e^{-cT} + \theta e^{-cT} \leq 0.
 \end{aligned} \tag{2.4.14}$$

Choosing $\psi = \varphi$, $w = v$ we obtain that

$$Z_1 = \Phi_1(\varphi, v - \varphi), \quad Z_2 = \Phi_2(\varphi, v - \varphi), \quad \text{and } \delta = \theta = 0. \tag{2.4.15}$$

a.s. a.e. in $D \times (0, T)$.

Let $a \in L^2(\Omega \times (0, T); V) \cap L^4(\Omega \times (0, T) \times D)$, $b \in L^2(\Omega \times (0, T); V)$ and be arbitrary and set

$$\psi = \varphi - \lambda a, \quad w = v - \lambda b, \quad \text{with } \lambda \in \mathbb{R}_+.$$

Substituting them in (2.4.14) we obtain the following inequality

$$\begin{aligned}
 \mathbb{E} \int_0^T e^{-cs} \{ 2\langle \Delta\varphi - \Delta(\varphi - \lambda a), \lambda a \rangle_{Z^*, Z} + 2\langle \Delta v - \Delta(v - \lambda b), \lambda b \rangle_{V^*, V} \\
 + 2\langle \chi - f(\varphi - \lambda a), \lambda a \rangle_{Z^*, Z} - 2\langle \lambda a, \lambda a \rangle_{Z^*, Z} \\
 + 2\langle \lambda b, \lambda a \rangle_{Z^*, Z} - 2\langle \Delta\varphi - \Delta(\varphi - \lambda a), \lambda b \rangle_{V^*, V}
 \end{aligned}$$

$$-c\|\lambda a\|^2 - c\|\lambda b\|^2 \leq 0.$$

Dividing by λ and letting $\lambda \rightarrow 0$, we find that :

$$\mathbb{E} \int_0^T e^{-cs} \langle \chi - f(\varphi), a \rangle_{Z^*, Z} dt \leq 0.$$

Since a is arbitrary, it follows that

$$\chi = f(\varphi), \tag{2.4.16}$$

a.s. a.e. in $D \times (0, T)$. Substituting (2.4.15) and (2.4.16) in (2.4.6) and (2.4.7) we deduce that for a.e. $(t, \omega) \in (0, T) \times \Omega$

$$\begin{aligned} \varphi(t) &= \varphi_0 + \int_0^t \Delta \varphi ds + \int_0^t f(\varphi) ds - \int_0^t \varphi ds + \int_0^t v ds + \int_0^t \Phi_1(\varphi, v - \varphi) dW_1(s), \\ v(t) &= v_0 + \int_0^t \Delta v ds - \int_0^t \Delta \varphi ds + \int_0^t \Phi_2(\varphi, v - \varphi) dW_2(s), \end{aligned}$$

a.s. in $L^2(D) + L^{\frac{4}{3}}(D)$.

This completes the identification of the limits.

2.5 Extra regularity of the solution pair

Next we prove some extra regularity properties for the solution pair (φ, v) .

Lemma 2.5.1 *Let $(\varphi_0, v_0) \in V \times V$ be arbitrary. The solution (φ, v) is a strong solution of Problem (P_1) (cf. Definition 2.1.2).*

Proof In order to prove the Lemma, we multiply (2.2.2) and (2.2.3) by $\sqrt{\gamma_j}$.

$$\begin{aligned} \int_D \varphi_m(t) \sqrt{\gamma_j} w_j &= \int_D \varphi_m(0) \sqrt{\gamma_j} w_j - \int_0^t \int_D \nabla \varphi_m \nabla w_j \sqrt{\gamma_j} + \int_0^t \int_D f(\varphi_m) \sqrt{\gamma_j} w_j \\ &\quad - \int_0^t \int_D \varphi_m \sqrt{\gamma_j} w_j + \int_0^t \int_D v_m \sqrt{\gamma_j} w_j \\ &\quad + \sum_{l=1}^m \int_0^t \int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l \sqrt{\gamma_j} w_j d\beta_l^1(s) \end{aligned} \tag{2.5.1}$$

$$\begin{aligned} \int_D v_m(t) \sqrt{\gamma_j} w_j &= \int_D v_m(0) \sqrt{\gamma_j} w_j - \int_0^t \int_D \nabla v_m \nabla w_j \sqrt{\gamma_j} + \int_0^t \int_D \nabla \varphi_m \nabla w_j \sqrt{\gamma_j} \\ &\quad + \sum_{l=1}^m \int_0^t \int_D \Phi_2(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^2} e_l \sqrt{\gamma_j} w_j d\beta_l^2(s) \end{aligned} \tag{2.5.2}$$

Next we recall a variant of Itô's formula which is applied to a system of ordinary differential equations

Lemma 2.5.2 [cf.[16]] For a smooth vector function h and an adapted process $(g(t), t \geq 0)$ with $\int_0^T |g(t)| dt < \infty$ almost surely, for all $T > 0$ set

$$X(t) := X(0) + \int_0^t g(s) ds + \int_0^t h d\mathcal{W}(s), \quad 0 \leq t \leq T,$$

where h is a vector of components h_l , $l = 1, \dots, m$ and $d\mathcal{W}$ is a vector of components $d\beta_l$, $l = 1, \dots, m$ with β_l a one-dimensional Brownian motion. Then, for F twice continuously differentiable in X and continuously differentiable in t , one has

$$\begin{aligned} F(X(t), t) &= F(X(0), 0) + \int_0^t F_t(X(s), s) + \int_0^t F_x(X(s))g(s) ds \\ &\quad + \int_0^t F_x(X(s))h d\mathcal{W}(s) + \frac{1}{2} \sum_{l=1}^m \int_0^t F_{xx}(X(s))h_l^2 ds. \end{aligned} \quad (2.5.3)$$

Next we apply Lemma 2.5.2 to (2.5.1) with $h d\mathcal{W} = \sum_{l=1}^m \int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l \sqrt{\gamma_j} w_j dx d\beta_l^1$

and $h_l = \int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l \sqrt{\gamma_j} w_j dx$, supposing that F does not depend on time and setting

$$\begin{aligned} X_j(t) &= \int_D \varphi_m \sqrt{\gamma_j} w_j dx, \\ F(X(t)) &= X(t)^2, \\ \frac{\partial F}{\partial X}(X(t)) &= 2X(t), \\ \frac{\partial^2 F}{\partial X^2}(X(t)) &= 2, \\ g_j(s) &= \int_D \sqrt{\gamma_j} (\Delta \varphi_m + f(\varphi_m) - \varphi_m + v_m) w_j. \end{aligned}$$

We obtain that almost surely, for all $t \in [0, T]$ and all $j = 0, \dots, m$,

$$\begin{aligned} &\left(\int_D \varphi_m(t) \sqrt{\gamma_j} w_j \right)^2 \\ &= \left(\int_D \varphi_{0m} \sqrt{\gamma_j} w_j \right)^2 + 2 \int_0^t \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) \left(\int_D \Delta \varphi_m \sqrt{\gamma_j} w_j \right) \\ &\quad + 2 \int_0^t \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) \left(\int_D f(\varphi_m) \sqrt{\gamma_j} w_j \right) \\ &\quad - 2 \int_0^t \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) \\ &\quad + 2 \int_0^t \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) \left(\int_D v_m \sqrt{\gamma_j} w_j \right) \\ &\quad + 2 \sum_{l=1}^m \int_0^t \left(\int_D \varphi_m \sqrt{\gamma_j} w_j dx \right) \left(\int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l \sqrt{\gamma_j} w_j dx \right) d\beta_l^1(s) \end{aligned}$$

$$+ \sum_{l=1}^m \int_0^t \left(\int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l \sqrt{\gamma_j} w_j \right)^2 dt. \quad (2.5.4)$$

Next we sum (2.5.4) from $j = 0, \dots, m$ and evaluate all the terms

$$\begin{aligned} \sum_{j=0}^m \left(\int_D \varphi_m(t) \sqrt{\gamma_j} w_j \right)^2 &= \sum_{j=0}^m \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) \\ &= \sum_{j=0}^m \int_D \varphi_m \sqrt{\gamma_j} \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) w_j dx \\ &= \sum_{j=0}^m \int_D \varphi_m \left(\int_D \varphi_m w_j \right) \gamma_j w_j dx \\ &= - \sum_{j=0}^m \int_D \varphi_m \left(\int_D \varphi_m w_j \right) \Delta w_j dx \\ &= - \sum_{j=0}^m \int_D \varphi_m \Delta \left(\int_D \varphi_m w_j \right) w_j dx \\ &= - \int_D \varphi_m \Delta \left(\sum_{j=0}^m \left(\int_D \varphi_m w_j \right) w_j dx \right) \\ &= - \int_D \varphi_m \Delta \varphi_m \\ &= \|\nabla \varphi_m(t)\|^2. \end{aligned} \quad (2.5.5)$$

Similarly, one can show that $\sum_{j=0}^m \left(\int_D \varphi_{0m} \sqrt{\gamma_j} w_j \right)^2 = \|\nabla \varphi_{0m}\|^2$.

Next, we consider the elliptic term

$$\begin{aligned} &\sum_{j=0}^m 2 \int_0^t \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) \left(\int_D \Delta \varphi_m \sqrt{\gamma_j} w_j \right) \\ &= \sum_{j=0}^m 2 \int_0^t \int_D \Delta \varphi_m \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) \sqrt{\gamma_j} w_j dx \\ &= - \sum_{j=0}^m 2 \int_0^t \int_D \Delta \varphi_m \left(\int_D \varphi_m w_j \right) \Delta w_j dx \\ &= -2 \int_0^t \int_D \Delta \varphi_m \Delta \left(\sum_{j=0}^m \left(\int_D \varphi_m w_j \right) w_j \right) dx \\ &= -2 \int_0^t \|\Delta \varphi_m\|^2. \end{aligned} \quad (2.5.6)$$

Next, we compute the nonlinear term

$$\sum_{j=0}^m 2 \int_0^t \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) \left(\int_D f(\varphi_m) \sqrt{\gamma_j} w_j \right)$$

$$\begin{aligned}
 &= \sum_{j=0}^m 2 \int_0^t \int_D f(\varphi_m) \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) \sqrt{\gamma_j} w_j dx \\
 &= -2 \int_0^t \int_D f(\varphi_m) \Delta \left(\sum_{j=0}^m \left(\int_D \varphi_m w_j \right) w_j \right) dx \\
 &= -2 \int_0^t \int_D f(\varphi_m) \Delta \varphi_m dx,
 \end{aligned}$$

and remark that by (2.5.5) the fourth and fifth terms on the right-hand-side of (2.5.4) are given by $-2 \int_0^t \|\nabla \varphi_m(s)\|^2 ds + 2 \int_0^t \int_D \nabla \varphi_m \nabla v_m ds dx$.

Next we consider the stochastic term

$$\begin{aligned}
 &2 \sum_{j=0}^m \sum_{l=1}^m \int_0^t \left(\int_D \varphi_m \sqrt{\gamma_j} w_j dx \right) \left(\int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l \sqrt{\gamma_j} w_j dx \right) d\beta_l^1(s) \\
 &= 2 \sum_{j=0}^m \sum_{l=1}^m \int_0^t \int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l \left(\int_D \varphi_m w_j dx \right) \gamma_j w_j dx d\beta_l^1(s) \\
 &= -2 \sum_{l=1}^m \int_0^t \int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l \Delta \left(\sum_{j=0}^m \left(\int_D \varphi_m w_j dx \right) w_j \right) dx d\beta_l^1(s) \\
 &= -2 \sum_{l=1}^m \int_0^t \int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l \Delta \varphi_m dx d\beta_l^1(s).
 \end{aligned} \tag{2.5.7}$$

We deduce that

$$\begin{aligned}
 \|\nabla \varphi_m(t)\|_{L^2(D)}^2 &= \|\nabla \varphi_{0m}\|_{L^2(D)}^2 - 2 \int_0^t \|\Delta \varphi_m\|^2 - 2 \int_0^t \int_D f(\varphi_m) \Delta \varphi_m \\
 &\quad - 2 \int_0^t \int_D \|\nabla \varphi_m\|^2 + 2 \int_0^t \int_D \nabla v_m \nabla \varphi_m \\
 &\quad - 2 \sum_{l=1}^m \int_0^t \int_D \Delta \varphi_m \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l d\beta_l^1(s) \\
 &\quad + \sum_{j=0}^m \sum_{l=1}^m \int_0^t \left(\int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l \sqrt{\gamma_j} w_j \right)^2 dt \\
 &\leq \|\nabla \varphi_{0m}\|^2 - 2 \int_0^t \|\Delta \varphi_m\|_{L^2(D)}^2 + 2 \int_0^t f'(\varphi_m) \|\nabla \varphi_m\|_{L^2(D)}^2 \\
 &\quad - \int_0^t \|\nabla \varphi_m\|^2 + \int_0^t \|\nabla v_m\|^2 \\
 &\quad - 2 \sum_{j=0}^m \int_0^t \int_D \Delta \varphi_m \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l dx d\beta_l^1(s) \\
 &\quad + \sum_{j=0}^m \int_0^t \sum_{l=1}^{\infty} \left(\int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l \sqrt{\gamma_j} w_j \right)^2 dt.
 \end{aligned} \tag{2.5.8}$$

Next we compute the Ito correction term on the right hand side of (2.5.8) using the fact that $\{w_k\}$ are an orthonormal basis of $L^2(D)$ which implies that $\{\nabla w_k\}$ is an orthogonal basis on $L^2(D)$.

$$\begin{aligned}
 & \sum_{j=0}^m \int_0^t \sum_{l=1}^{\infty} \left(\int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l \sqrt{\gamma_j} w_j \right)^2 dt \\
 = & \sum_{j=0}^m \int_0^t \sum_{l=1}^{\infty} \int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l \left(\int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l w_j \right) \gamma_j w_j dx ds \\
 = & \int_0^t \sum_{l=1}^{\infty} \int_D \left[\sum_{i=0}^{\infty} \left(\int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l w_i \right) w_i \right. \\
 & \quad \left. \sum_{j=0}^m \left(\int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l w_j \right) \gamma_j w_j \right] dx ds \\
 = & \int_0^t \sum_{l=1}^{\infty} \int_D \left[\sum_{j=0}^m \left(\int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l w_j \right) w_j \right. \\
 & \quad \left. \sum_{j=0}^m \left(\int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l w_j \right) \gamma_j w_j \right] dx ds \\
 = & - \int_0^t \sum_{l=1}^{\infty} \int_D \left[\sum_{j=0}^m \left(\int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l w_j \right) w_j \right. \\
 & \quad \left. \Delta \left(\sum_{j=0}^m \left(\int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l w_j \right) w_j \right) \right] dx ds \\
 = & \int_0^t \sum_{l=1}^{\infty} \int_D \left[\sum_{j=0}^m \left(\int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l w_j dx \right) \nabla w_j \right]^2 dx ds \\
 = & \int_0^t \sum_{l=1}^{\infty} \sum_{j=0}^m \left(\int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l w_j dx \right)^2 \int_D (\nabla w_j)^2 dx ds \\
 = & - \int_0^t \sum_{l=1}^{\infty} \sum_{j=0}^m \left(\int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l w_j dx \right)^2 \int_D w_j \Delta w_j dx ds \\
 = & \int_0^t \sum_{l=1}^{\infty} \sum_{j=0}^m \gamma_j \left(\int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l w_j dx \right)^2 \int_D (w_j)^2 dx ds \\
 \leq & \int_0^t \sum_{l=1}^{\infty} \sum_{j=0}^{\infty} \gamma_j \left(\int_D \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l w_j dx \right)^2 ds \\
 \leq & \int_0^t \sum_{l=1}^{\infty} \|\nabla \Phi_{1l}(x, \varphi_m, v_m - \varphi_m)\|^2 \\
 \leq & C_2 \int_0^t (1 + 3\|\varphi_m\|_V^2 + 2\|v_m\|_V^2), \tag{2.5.9}
 \end{aligned}$$

where we have used (2.1.9) and [10] p.235.

Substituting into (2.5.8) and using (2.1.2) we obtain

$$\begin{aligned}
 \|\nabla\varphi_m(t)\|^2 &\leq \|\nabla\varphi_{0m}\|^2 - 2\int_0^t \|\Delta\varphi_m\|^2 ds + \int_0^t \|\nabla\varphi_m\|^2 + \int_0^t \|\nabla v_m\|^2 \\
 &\quad - 2\sum_{l=1}^m \int_0^t \int_D \Delta\varphi_m \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l d\beta_l^1(s) \\
 &\quad + C_2 \int_0^t (1 + 3\|\varphi_m\|_V^2 + 2\|v_m\|_V^2). \tag{2.5.10}
 \end{aligned}$$

Thus, taking the expectation yields

$$\begin{aligned}
 \mathbb{E}\|\nabla\varphi_m(t)\|^2 + 2\mathbb{E}\int_0^t \|\Delta\varphi_m\|^2 ds &\leq \|\nabla\varphi_{0m}\|^2 + \mathbb{E}\int_0^t \|\nabla\varphi_m\|^2 + \mathbb{E}\int_0^t \|\nabla v_m\|^2 \\
 &\quad - 2\mathbb{E}\left(\sum_{l=1}^m \int_0^t \int_D \Delta\varphi_m \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l d\beta_l^1(s)\right) \\
 &\quad + C_2 \mathbb{E}\int_0^t (1 + 3\|\varphi_m\|_V^2 + 2\|v_m\|_V^2). \tag{2.5.11}
 \end{aligned}$$

We repeat the same procedure for the equation for v_m by applying Itô's formula to obtain

$$\begin{aligned}
 \|\nabla v_m(t)\|^2 &= \|\nabla v_{0m}\|^2 - 2\int_0^t \int_D \|\Delta v_m\|^2 + 2\int_0^t \int_D \Delta v_m \Delta\varphi_m \\
 &\quad - 2\sum_{l=1}^m \int_0^t \int_D \Delta v_m \Phi_2(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^2} e_l d\beta_l^2(s) \\
 &\quad + \sum_{j=0}^m \sum_{l=1}^m \int_0^t \left(\int_D \Phi_2(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^2} e_l \sqrt{\gamma_j} w_j\right)^2 dt \\
 &\leq \|\nabla v_{0m}\|^2 - 2\int_0^t \|\Delta v_m\|^2 \\
 &\quad + \int_0^t \|\Delta v_m\|^2 + \int_0^t \|\Delta\varphi_m\|^2 \\
 &\quad - 2\sum_{l=1}^m \int_0^t \int_D \Delta v_m \Phi_2(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^2} e_l d\beta_l^2(s) \\
 &\quad + \sum_{j=0}^m \int_0^t \sum_{l=1}^{\infty} \left(\int_D \Phi_2(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^2} e_l \sqrt{\gamma_j} w_j\right)^2 dt. \tag{2.5.12}
 \end{aligned}$$

Finally we remark that a similar computation as in (2.5.9) yields

$$\sum_{j=0}^m \int_0^t \sum_{l=1}^{\infty} \left(\int_D \Phi_2(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^2} e_l \sqrt{\gamma_j} w_j\right)^2 dt$$

$$\leq C_2 \int_0^t (1 + 2\|v_m\|_V^2 + 3\|\varphi_m\|_V^2). \quad (2.5.13)$$

Substituting (2.5.13) into (2.5.12) we obtain

$$\begin{aligned} \|\nabla v_m(t)\|_{L^2(D)}^2 &\leq \|\nabla v_{0m}\|^2 - \int_0^t \|\Delta v_m\|_{L^2(D)}^2 \\ &\quad + \int_0^t \|\Delta \varphi_m\|^2 \\ &\quad - 2 \sum_{l=1}^m \int_0^t \int_D \Delta v_m \Phi_2(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^2} e_l d\beta_l^2(s) \\ &\quad + C_2 \int_0^t (1 + 2\|v_m\|_V^2 + 3\|\varphi_m\|_V^2). \end{aligned} \quad (2.5.14)$$

Taking the expectation we obtain

$$\begin{aligned} &\mathbb{E} \|\nabla v_m(t)\|_{L^2(D)}^2 + \mathbb{E} \int_0^t \|\Delta v_m\|^2 \\ &\leq \|\nabla v_{0m}\|^2 + \mathbb{E} \int_0^t \|\Delta \varphi_m\|^2 \\ &\quad - 2\mathbb{E} \left(\sum_{l=1}^m \int_0^t \int_D \Delta v_m \Phi_2(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^2} e_l d\beta_l^2(s) \right) \\ &\quad + C_2 \mathbb{E} \int_0^t (1 + 2\|v_m\|_V^2 + 3\|\varphi_m\|_V^2). \end{aligned} \quad (2.5.15)$$

Using the facts that $\mathbb{E} \left(\sum_{l=1}^m \int_0^t \int_D \Delta \varphi_m \Phi_1(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^1} e_l dx d\beta_l^1(s) \right) = 0$, and that

$\mathbb{E} \left(\sum_{l=1}^m \int_0^t \int_D \Delta v_m \Phi_2(\varphi_m, v_m - \varphi_m) \sqrt{\lambda_l^2} e_l dx d\beta_l^2(s) \right) = 0$, and summing (2.5.11) and (2.5.15) we deduce that

$$\begin{aligned} &\mathbb{E} \|\nabla \varphi_m(t)\|^2 + \mathbb{E} \|\nabla v_m(t)\|^2 + \mathbb{E} \int_0^t \|\Delta \varphi_m\|^2 + \mathbb{E} \int_0^t \|\Delta v_m\|^2 ds \\ &\leq \|\varphi_{0m}\|_V^2 + \|v_{0m}\|_V^2 + \mathbb{E} \int_0^t \|\nabla \varphi_m\|^2 + \mathbb{E} \int_0^t \|\nabla v_m\|^2 \\ &\quad + 2C_2 \mathbb{E} \int_0^t (1 + 3\|\varphi_m\|_V^2 + 2\|v_m\|_V^2). \end{aligned} \quad (2.5.16)$$

Lemma 2.5.3 *From (2.3.12) we have that $\|\varphi_{0m}\|^2 \leq \|\varphi_0\|^2$, using the fact that $\{w_k\}$ is an orthogonal basis in V and [10] p.235 yields*

$$\begin{aligned} \|\nabla \varphi_{0m}\|^2 &= \int_D \left[\sum_{j=0}^m \left(\int_D \varphi_0 w_j dx \right) \nabla w_j \right]^2 \\ &= \sum_{j=0}^m \left(\int_D \varphi_0 w_j dx \right)^2 \int_D (\nabla w_j)^2 \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=0}^{\infty} \left(\int_D \varphi_0 w_j dx \right)^2 \int_D (\nabla w_j)^2 \\
&\leq \|\nabla \varphi_0\|^2.
\end{aligned}$$

Thus, $\|\varphi_{0m}\|_V^2 \leq \|\varphi_0\|_V^2$. Similarly we have that $\|v_{0m}\|_V^2 \leq \|v_0\|_V^2$.

Using Lemma 2.5.3, (2.3.27), (2.3.29), (2.3.25) and (2.3.26) in (2.5.16) yields

$$\mathbb{E} \|\nabla \varphi_m(t)\|^2 \leq K_1, \quad \text{for all } t \in [0, T], \quad (2.5.17)$$

$$\mathbb{E} \|\nabla v_m(t)\|^2 \leq K_1, \quad \text{for all } t \in [0, T], \quad (2.5.18)$$

$$\mathbb{E} \int_0^T \|\Delta \varphi_m(s)\|^2 ds \leq K_1, \quad (2.5.19)$$

$$\mathbb{E} \int_0^T \|\Delta v_m(s)\|^2 ds \leq K_1. \quad (2.5.20)$$

□

2.6 Uniqueness

We now prove the uniqueness of the weak solution pair (φ, v) .

Suppose that (φ, v) and $(\tilde{\varphi}, \tilde{v})$ are two solution pairs of problem (P_2) such that $\varphi(\cdot, 0) = \tilde{\varphi}(\cdot, 0) = \varphi_0$ and $v(\cdot, 0) = \tilde{v}(\cdot, 0) = v_0$.

We take the difference between the equation for φ and the equation for $\tilde{\varphi}$.

$$\begin{aligned}
(\varphi - \tilde{\varphi})(t) &= \int_0^t \{ \Delta(\varphi - \tilde{\varphi})(s) + (f(\varphi) - f(\tilde{\varphi})) - (\varphi - \tilde{\varphi}) + (v - \tilde{v}) \}(s) ds \\
&\quad + \int_0^t (\Phi_1(\varphi(s), v(s) - \varphi(s)) - \Phi_1(\tilde{\varphi}(s), \tilde{v}(s) - \tilde{\varphi}(s))) dW_1(s).
\end{aligned} \quad (2.6.1)$$

Next, we apply Ito's formula as in Proposition 2.4.2 with

$$\begin{aligned}
X(t) &= \varphi - \tilde{\varphi}, \\
F(X) &= \|X\|^2, \\
\frac{\partial F}{\partial X}(X) &= 2X, \\
\frac{\partial^2 F}{\partial X^2}(X) &= 2I, \\
h(s) &= \Delta(\varphi - \tilde{\varphi}) + (f(\varphi) - f(\tilde{\varphi})) - (\varphi - \tilde{\varphi}) + (v - \tilde{v}), \\
G(s) &= \Phi_1(\varphi(s), v(s) - \varphi(s)) - \Phi_1(\tilde{\varphi}(s), \tilde{v}(s) - \tilde{\varphi}(s)).
\end{aligned}$$

This yields

$$\|(\varphi - \tilde{\varphi})(t)\|^2 = -2 \int_0^t \|\nabla(\varphi - \tilde{\varphi})\|^2 ds$$

$$\begin{aligned}
 & +2 \int_0^t \int_D (\varphi - \tilde{\varphi})(f(\varphi) - f(\tilde{\varphi})) dx ds \\
 & -2 \int_0^t \|\varphi - \tilde{\varphi}\|^2 ds + 2 \int_0^t \int_D (\varphi - \tilde{\varphi})(v - \tilde{v}) dx ds \\
 & +2 \int_0^t \int_D (\varphi - \tilde{\varphi})(\Phi_1(\varphi, v - \varphi) - \Phi_1(\tilde{\varphi}, \tilde{v} - \tilde{\varphi})) dW_1(s) dx \\
 & + \sum_{l=1}^{\infty} \int_0^t \|(\Phi_1(\varphi, v - \varphi) - \Phi_1(\tilde{\varphi}, \tilde{v} - \tilde{\varphi})) \sqrt{\lambda_l^1} e_l\|^2 ds. \tag{2.6.2}
 \end{aligned}$$

Similarly , applying Lemma 2.3.3 with $E = H$ for $v - \tilde{v}$:

$$\begin{aligned}
 \|(v - \tilde{v})(t)\|^2 & = -2 \int_0^t \|\nabla(v - \tilde{v})\|^2 ds + 2 \int_0^t \int_D \nabla(\varphi - \tilde{\varphi}) \nabla(v - \tilde{v}) dx ds \\
 & +2 \int_0^t \int_D (v - \tilde{v})(\Phi_2(\varphi, v - \varphi) - \Phi_2(\tilde{\varphi}, \tilde{v} - \tilde{\varphi})) dW_2(s) \\
 & + \sum_{l=1}^{\infty} \int_0^t \|(\Phi_2(\varphi, v - \varphi) - \Phi_2(\tilde{\varphi}, \tilde{v} - \tilde{\varphi})) \sqrt{\lambda_l^2} e_l\|^2 ds. \tag{2.6.3}
 \end{aligned}$$

Adding up (2.6.2) and (2.6.3) yields

$$\begin{aligned}
 & \|(\varphi - \tilde{\varphi})(t)\|^2 + \|(v - \tilde{v})(t)\|^2 \\
 & \leq -2 \int_0^t \|\nabla(\varphi - \tilde{\varphi})\|^2 ds - 2 \int_0^t \|\nabla(v - \tilde{v})\|^2 ds \\
 & + \int_0^t \|\nabla(\varphi - \tilde{\varphi})\|^2 ds + \int_0^t \|\nabla(v - \tilde{v})\|^2 ds \\
 & + \int_0^t \|\varphi - \tilde{\varphi}\|^2 ds + \int_0^t \|v - \tilde{v}\|^2 ds \\
 & -2 \int_0^t \|\varphi - \tilde{\varphi}\|^2 ds \\
 & +2 \int_0^t \int_D (\varphi - \tilde{\varphi})(f(\varphi) - f(\tilde{\varphi})) dx ds \\
 & + \sum_{l=1}^{\infty} \int_0^t \|\Phi_{1l}(\varphi, v - \varphi) - \Phi_{1l}(\tilde{\varphi}, \tilde{v} - \tilde{\varphi})\|^2 ds \\
 & + \sum_{l=1}^{\infty} \int_0^t \|\Phi_{2l}(\varphi, v - \varphi) - \Phi_{2l}(\tilde{\varphi}, \tilde{v} - \tilde{\varphi})\|^2 ds \\
 & +2 \int_0^t \int_D (\varphi - \tilde{\varphi})(\Phi_1(\varphi, v - \varphi) - \Phi_1(\tilde{\varphi}, \tilde{v} - \tilde{\varphi})) dW_1(s) dx \\
 & +2 \int_0^t \int_D (v - \tilde{v})(\Phi_2(\varphi, v - \varphi) - \Phi_2(\tilde{\varphi}, \tilde{v} - \tilde{\varphi})) dW_2(s) dx. \tag{2.6.4}
 \end{aligned}$$

Neglecting the negative terms using (2.1.2) and (2.1.10)

$$\begin{aligned}
 & \|(\varphi - \tilde{\varphi})(t)\|^2 + \|(v - \tilde{v})(t)\|^2 \\
 & \leq (1 + 3C_3) \int_0^t \|\varphi - \tilde{\varphi}\|^2 ds + (1 + 2C_3) \int_0^t \|v - \tilde{v}\|^2 ds
 \end{aligned}$$

$$\begin{aligned}
& +2 \int_0^t \int_D (\varphi - \tilde{\varphi})(\Phi_1(\varphi, v - \varphi) - \Phi_1(\tilde{\varphi}, \tilde{v} - \tilde{\varphi})) dW_1(s) dx \\
& +2 \int_0^t \int_D (v - \tilde{v})(\Phi_2(\varphi, v - \varphi) - \Phi_2(\tilde{\varphi}, \tilde{v} - \tilde{\varphi})) dW_2(s) dx.
\end{aligned} \tag{2.6.5}$$

Taking the expectation of (2.6.5) we obtain

$$\mathbb{E} \left(\|\varphi - \tilde{\varphi}\|^2 + \|v - \tilde{v}\|^2 \right) \leq \tilde{C}_3 \mathbb{E} \left(\int_0^t \|\varphi - \tilde{\varphi}\|^2 + \|v - \tilde{v}\|^2 ds \right),$$

where we have used the facts that

$$\mathbb{E} \left(\int_0^t \int_D (\varphi - \tilde{\varphi})(\Phi_1(\varphi, v - \varphi) - \Phi_1(\tilde{\varphi}, \tilde{v} - \tilde{\varphi})) dW_1 dx \right) = 0 \text{ and}$$

$$\mathbb{E} \left(\int_0^t \int_D (\varphi - \tilde{\varphi})(\Phi_2(\varphi, v - \varphi) - \Phi_2(\tilde{\varphi}, \tilde{v} - \tilde{\varphi})) dW_2 dx \right) = 0.$$

We deduce from Gronwall's inequality that

$$\varphi = \tilde{\varphi} \text{ and } v = \tilde{v} \quad \text{in } \Omega \times (0, T) \times D.$$

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On a stochastic nonlocal Allen-Cahn equation with a multiplicative noise

Résumé. Dans ce chapitre, on démontre l'existence et l'unicité d'une solution trajectorielle forte pour l'équation d'Allen-Cahn stochastique avec un bruit multiplicatif induit par un processus Q-Wiener en dimensions d'espace $d \in [1, 6]$. Pour prouver ce résultat, nous appliquons la méthode de compacité stochastique.

Abstract. In this chapter we prove the existence and uniqueness of a strong pathwise solution for a stochastic nonlocal Allen-Cahn equation with a multiplicative noise induced by a Q-Wiener process in space dimension $d \in [1, 6]$. In order to prove this result, we apply the stochastic compactness method.

3.1 Introduction

We prove the existence and uniqueness of solution of the following initial value Problem for the stochastic nonlocal Allen-Cahn equation with multiplicative noise

$$(P) \quad \begin{cases} \frac{\partial \varphi}{\partial t} = \Delta \varphi + f(\varphi) - \frac{1}{|D|} \int_D f(\varphi) + \Phi(\varphi) \frac{\partial W}{\partial t}(t), & \text{in } D \times (0, T), \\ \frac{\partial \varphi}{\partial \nu} = 0, & \text{on } \partial D \times (0, T), \\ \varphi(x, 0) = \varphi_0(x), & x \in D, \end{cases}$$

where

- D is an open domain of \mathbb{R}^d with smooth boundary such that $1 \leq d \leq 6$;
- $\varphi_0 \in H^1(D)$ is a given function;
- The nonlinear function f is given by $f(s) = -s^3 + s$. We will use below that

$$f(s)s \leq -\frac{s^4}{2} + \frac{1}{2}, \tag{3.1.1}$$

$$f'(s) \leq C_0 = 1. \tag{3.1.2}$$

We use the notations :

$$H = L^2(D), V = H^1(D), \text{ and } V = (H^1(D))',$$

with $\|\cdot\|$, $\|\cdot\|_V$ the corresponding norms. And we remark that those spaces correspond to a Gelfand triples

$$V \subset H \subset V'$$

where the embeddings from V into H and from H into V' are compact;

- The function $W = W(x, t)$ is a Q -Brownian motion. More precisely, let Q be a nonnegative definite symmetric operator on $L^2(D)$ with $\text{Tr } Q < +\infty$, $\{e_l\}_{l \geq 1}$ be an orthonormal basis in $L^2(D)$ diagonalizing Q_i , and $\{\lambda_l\}_{l \geq 1}$ be the corresponding eigenvalues, so that

$$Qe_l = \lambda_l e_l$$

for all $l \geq 1$. Since Q is of trace-class,

$$\text{Tr } Q = \sum_{l=1}^{\infty} \langle Qe_l, e_l \rangle_{L^2(D)} = \sum_{l=1}^{\infty} \lambda_l \leq \Lambda_0. \quad (3.1.3)$$

for some positive constant Λ_0 . We suppose furthermore that $e_l \in L^\infty(D)$ for $l = 1, 2, \dots$ and that there exists a positive constant Λ_1 such that

$$\sum_{l=1}^{\infty} \lambda_l \|e_l\|_{L^\infty(D)}^2 \leq \Lambda_1, \quad (3.1.4)$$

Let (Ω, \mathcal{F}, P) be a probability space equipped with a filtration (\mathcal{F}_t) and $\{\beta_l(t)\}_{l \geq 1}$ be a sequence of independent (\mathcal{F}_t) -Brownian motions defined on (Ω, \mathcal{F}, P) ; the Q -Wiener process W in $L^2(D)$ is defined by

$$W(x, t) = \sum_{l=1}^{\infty} \beta_l(t) Q^{\frac{1}{2}} e_l(x) = \sum_{l=1}^{\infty} \sqrt{\lambda_l} \beta_l(t) e_l(x) \quad (3.1.5)$$

We recall that a Brownian motion $\beta(t)$ is called an (\mathcal{F}_t) Brownian motion if it is (\mathcal{F}_t) -adapted and the increment $\beta(t) - \beta(s)$ is independent of \mathcal{F}_s for every $0 \leq s < t$;

- Let X be a separable real Hilbert space. We denote the family of all linear operators $\Phi : H \rightarrow X$ such that $\Phi\sqrt{Q}$ are Hilbert-Schmidt operators by $\mathcal{L}_{2,Q}(H, X)$ (cf.[10]). If $\Phi \in \mathcal{L}_{2,Q}(H, X)$, the $\mathcal{L}_{2,Q}$ norm of Φ is given by

$$\|\Phi\|_{2,Q}^2 := \sum_{l=1}^{\infty} \|\Phi\sqrt{Q}e_l\|_X^2 < \infty. \quad (3.1.6)$$

Given an X valued predictable process $\Phi \in L^2(\Omega; L^2((0, T), \mathcal{L}_{2,Q}(H, X)))$ and setting

$$\Phi_l(x, \varphi) := \Phi(\varphi)\sqrt{Q}e_l(x) = \Phi(\varphi)\sqrt{\lambda_l}e_l(x), \quad (3.1.7)$$

One may define the stochastic integral as:

$$\int_0^t \Phi(s) dW(s) = \sum_{l=1}^{\infty} \int_0^t \Phi_l d\beta_l. \quad (3.1.8)$$

We suppose that there exist positive constants C_1 and C_2 such that

$$\sum_{l=1}^{\infty} \|\Phi_l(x, u)\|^2 \leq C_1, \quad (3.1.9)$$

$$\sum_{l=1}^{\infty} \|\Phi_l(x, u)\|_V^2 \leq C_2, \quad (3.1.10)$$

for all $u \in H^1(D) \cap L^4(D)$. Moreover, we also assume the hypotheses

$$\sum_{l=1}^{\infty} \|\Phi_l(x, u)\|_V^p \leq C_3(1 + \|u\|_V^p), \quad (3.1.11)$$

$$\sum_{l=1}^{\infty} \|\Phi_l(x, u_1) - \Phi_l(x, u_2)\|_V^2 \leq C_4 \|u_1 - u_2\|_V^2. \quad (3.1.12)$$

$$\sum_{l=1}^{\infty} \|\Phi_l(x, u_1) - \Phi_l(x, u_2)\|^2 \leq C_5 \|u_1 - u_2\|^2. \quad (3.1.13)$$

In this chapter, we prove the existence of a strong pathwise solution in dimensions up to 6 of an initial value problem for the stochastic nonlocal Allen-Cahn equation with a multiplicative noise, which involves a Q -Wiener process.

We first introduce a Galerkin approximation and establish a priori estimates for the solution. In the stochastic analysis, the classical compactness results of the deterministic theory cannot be applied because we have an additional random variable and even if a space X is compactly embedded in another space Y , it is not the case that $L^2(\Omega; X)$ is compactly embedded in $L^2(\Omega; Y)$; this is why we will use a stochastic compactness method based on fractional Sobolev spaces.

This leads us to establish the tightness of the collection of probability measures associated to the approximate solution and apply the Prokhorov Theorem to deduce their weak compactness. Then we pass to the limit in the Galerkin approximation by using the Skorohod Theorem, which provides almost sure convergence and then strong convergence relative to a new stochastic basis. In this context, the stochastic basis is not specified in the beginning and is viewed as part of the unknowns.

The last step of the proof is to deduce the existence of a pathwise solution, that is solution of the initial problem defined in the initially given stochastic basis. For this purpose, we first show the pathwise uniqueness of martingale solutions and by an application of Gyöngy-Krylov theorem which is an extension to the infinite dimension of the Yamada-Watanabe theorem from finite dimensional stochastic analysis, we obtain the existence of pathwise solutions.

Many authors applied the stochastic compactness argument to prove the existence and uniqueness of the solution for initial value problem for partial differential equations. Hofmanová [13] applied this method to prove the well-posedness of the Cauchy problem for a scalar semilinear degenerate parabolic partial differential equation with a multiplicative noise induced by a cylindrical Wiener process. Glatt-holtz, Temam, Wang [11] proved by this method the global existence of martingale solutions in space dimensions 2 and 3, and the pathwise existence and uniqueness of the solution of the Zakharov-Kuznetsov equation with similar hypotheses on the noise. Debussche, Glatt-Holtz, Temam[4] proved the existence and uniqueness of local martingale and pathwise solutions for a nonlinear primitive stochastic system of equations. Finally, we mention the work of Pham and Nguyen [16] where they proved the existence and uniqueness of the pathwise solution locally in time for a system of stochastic reaction-diffusion equations.

The chapter is organised as follows: In section 3.2, we give the main result and start the proof of the existence of a strong martingale solution by applying the Galerkin method. In section 3.3, we derive a priori estimates for the approximate solution in the spaces $L^\infty(0, T; L^2(\Omega; H^1(D))) \cap L^2(\Omega \times (0, T); H^2(D)) \cap L^4(\Omega \times (0, T) \times D)$. In section 3.4 we prove the tightness of the measures so that we can apply the Prokhorov Theorem and then the Skhorohod Theorem to obtain the strong convergence of a subsequence of approximate solutions to a solution of Problem (P) in a new stochastic basis (section 3.5). Finally, in section 3.6 we prove the pathwise uniqueness of the solution to deduce the existence of the pathwise solution in section 3.7.

3.2 Galerkin approximation and main result

We define the notion of a strong martingale solution.

Definition 3.2.1 *A triple $((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}), \varphi, W)$ is called a strong Martingale solution of Problem (P) with the initial condition φ_0 if*

- (i) $(\Omega, \mathcal{F}, (\mathcal{F}_t); \mathbb{P})$ is a stochastic basis;
- (ii) W is a Q -Wiener process;
- (iii) $\varphi : [0, T] \times \Omega \rightarrow L^2(D)$ is a progressively measurable process such that almost surely $\varphi \in C([0, T]; V')$, $\varphi \in L^2(\Omega \times (0, T); H^2(D)) \cap L^4(\Omega \times (0, T) \times D)$ and $\int_0^t \Phi(\varphi) dW \in L^2(\Omega; L^\infty(0, T; L^2(D)))$;
- (iv) and φ satisfies almost surely

$$(P_2) \quad \begin{cases} \varphi(t) - \varphi_0 = \int_0^t \Delta \varphi(s) ds + \int_0^t f(\varphi(s)) - \frac{1}{|D|} \int_0^t \int_D f(\varphi(s)) \\ \quad + \int_0^t \Phi(\varphi(s)) dW(s), & \text{in } \Omega \times D \times (0, T), \\ \frac{\partial \varphi}{\partial n} = 0, & \text{on } \Omega \times \partial D \times (0, T). \end{cases}$$

We define the notion of a strong pathwise solution.

Definition 3.2.2 Let W be a Q -Wiener process on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. A solution φ is called a strong pathwise solution of Problem (P) with the initial condition φ_0 if

- (i) $\varphi : [0, T] \times \Omega \rightarrow L^2(D)$ is a progressively measurable process such that almost surely $\varphi \in C([0, T]; V')$, $\varphi \in L^2(\Omega \times (0, T); H^2(D)) \cap L^4(\Omega \times (0, T) \times D)$ and $\int_0^t \Phi(\varphi) dW \in L^2(\Omega; L^\infty(0, T; L^2(D)))$;
- (ii) and φ satisfies almost surely

$$(P_3) \quad \begin{cases} \varphi(t) - \varphi_0 = \int_0^t \Delta \varphi(s) ds + \int_0^t f(\varphi(s)) - \frac{1}{|D|} \int_0^t \int_D f(\varphi(s)) \\ \quad + \int_0^t \Phi(\varphi(s)) dW(s), & \text{in } \Omega \times D \times (0, T), \\ \frac{\partial \varphi}{\partial n} = 0, & \text{on } \Omega \times \partial D \times (0, T). \end{cases}$$

Our main result is the following

Theorem 3.2.3 Let $\varphi_0 \in V$ be arbitrary. Then Problem (P) possesses a unique strong martingale solution.

Theorem 3.2.4 Let $\varphi_0 \in V$ be arbitrary. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W)$ be a fixed stochastic basis. Then Problem (P) possesses a unique strong pathwise solution.

Proof The proof relies on the Galerkin method. We denote by $0 < \gamma_1 < \gamma_2 \leq \dots \leq \gamma_k \leq \dots$ the eigenvalues of the operator $-\Delta$ with homogeneous Neumann boundary conditions, and by $w_k, k = 1, \dots$ the corresponding unit eigenfunctions. Note that they are smooth functions.

Lemma 3.2.5 The functions $\{w_j\}$ are an orthonormal basis of $L^2(D)$ and satisfy :

$$\int_D w_j w_0 dx = 0 \quad \text{for all } j \neq 0 \quad \text{and} \quad w_0 = \frac{1}{\sqrt{|D|}}.$$

Proof see paper [5]. □

For each integer m we look for an approximate solution φ_m of the form :

$$\begin{aligned} \varphi_m(x, t) = \sum_{i=0}^m \varphi_{im}(t) w_i &= M_m(t) + \sum_{i=1}^m \varphi_{im}(t) w_i \\ &= M_m(t) + \sum_{i=1}^m \langle \varphi_m(t), w_i \rangle_{L^2(D)} w_i, \end{aligned} \quad (3.2.1)$$

which satisfies the following equation

$$\int_D \varphi_m(t) w_j - \int_D \varphi_m(0) w_j + \int_0^t \int_D \nabla \varphi_m \nabla w_j$$

$$= \int_0^t \int_D f(\varphi_m) w_j - \int_D \frac{1}{|D|} \int_0^t \int_D f(\varphi_m) w_j dx + \int_0^t \int_D \Phi(\varphi_m) w_j dW(s) \quad (3.2.2)$$

for all w_j , $j = 0, \dots, m$.

Note that setting $j = 0$ in (3.2.2) yields

$$\begin{aligned} M_m(t) := \varphi_{0m}(t) w_0 &= \oint \varphi_m(x, t) dx := \frac{1}{|D|} \int_D \varphi_m(x, t) dx \\ &= \frac{1}{|D|} \int_D \varphi_m(0) + \frac{1}{|D|} \int_0^t \int_D \Phi(\varphi_m) dW(s). \end{aligned} \quad (3.2.3)$$

We remark that $\varphi_m(x, 0) = M(0) + \sum_{i=1}^m \langle \varphi_0, w_i \rangle_{L^2(D)} w_i$ converges strongly to φ_0 in $L^2(D)$ as $m \rightarrow \infty$.

And we remark that the contribution of the nonlocal term vanishes for $j = 1, \dots, m$. Indeed for all $j = 1, \dots, m$

$$\begin{aligned} -\frac{1}{|D|} \int_D \left(\int_D f(\varphi_m) dx \right) w_j dx &= -\frac{1}{|D|} \left(\int_D f(\varphi_m) dx \right) \times \int_D w_j dx \\ &= 0. \end{aligned} \quad (3.2.4)$$

Problem (3.2.2) is an initial value problem for a system of m stochastic differential equations, so that it has a unique solution φ_m on some time interval $(0, T_m)$, $T_m > 0$ (cf. [17]); in fact the following a priori estimates show that the solution pair is global in time. \square

3.3 A priori estimates

In what follows, we derive a priori estimates for the functions φ_m .

Lemma 3.3.1 *There exist positive constants K and K_1 such that*

$$\mathbb{E} \int_0^T \int_D |\nabla \varphi_m|^2 dx ds \leq K, \quad (3.3.1)$$

$$\mathbb{E} \int_0^T \int_D \varphi_m^4 dx ds \leq K, \quad (3.3.2)$$

$$\mathbb{E} \sup_{t \in (0, T)} \|\varphi_m(t)\|^2 \leq K, \quad (3.3.3)$$

$$\mathbb{E} \int_0^T \int_D |\Delta \varphi_m|^2 dx ds \leq K_1, \quad (3.3.4)$$

$$\mathbb{E} \|\nabla \varphi_m\|^2 \leq K_1, \quad \text{for all } t \in [0, T]. \quad (3.3.5)$$

$$\mathbb{E} \|f(\varphi_m)\|_{L^{\frac{4}{3}}((0, T) \times D)}^{\frac{4}{3}} \leq K_1, \quad (3.3.6)$$

$$\mathbb{E} \left\| \int_D f(\varphi_m) \right\|_{L^{\frac{4}{3}}((0,T) \times D)}^{\frac{4}{3}} \leq K_1. \quad (3.3.7)$$

Proof Equation (3.2.2) can be re-written as

$$\begin{aligned} \int_D \varphi_m(t) w_j dx &= \int_D \varphi_m(0) w_j dx + \int_0^t \int_D \Delta \varphi_m w_j dx ds + \int_0^t \int_D f(\varphi_m) w_j dx ds \\ &\quad - \frac{1}{|D|} \int_0^t \int_D f(\varphi_m) w_j dx ds + \sum_{l=1}^{\infty} \int_0^t \int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l(x) w_j dx d\beta_l(s). \end{aligned} \quad (3.3.8)$$

for all $j = 0, \dots, m$.

Lemma 3.3.2 [cf. [3]] *Let X be an E -valued function such that*

$$X(t) = X(0) + \int_0^t h(s) ds + \int_0^t G(s) dW(s), \quad 0 \leq s \leq t,$$

which can be re-written as

$$X(t) = X(0) + \int_0^t h(s) ds + \sum_{l=1}^{\infty} \int_0^t G_l(s) d\beta_l(s), \quad s \in [0, t].$$

where $G_l = G \sqrt{\lambda_l} e_l$, and suppose that h is a E -valued predictable process Bochner integrable on $[0, T]$, a.s., G is an E -valued process stochastically integrable and $X(0)$ is an \mathcal{F}_0 -measurable E -valued function. Suppose that the function $F : [0, T] \times E \rightarrow \mathbb{R}$ and its partial derivatives $\frac{\partial F}{\partial t}$, $\frac{\partial F}{\partial X}$, $\frac{\partial^2 F}{\partial X^2}$ are continuous on $[0, T] \times E$. Then for all $t \in [0, T]$, a.s.

$$\begin{aligned} F(t, X(t)) &= F(0, X(0)) + \int_0^t \frac{\partial F}{\partial t}(s, X(s)) ds + \int_0^t \left\langle \frac{\partial F}{\partial X}(s, X(s)), h(s) \right\rangle_E ds \\ &\quad + \int_0^t \left\langle \frac{\partial F}{\partial X}(s, X(s)), G(s) dW(s) \right\rangle_E \\ &\quad + \frac{1}{2} \int_0^t \text{Tr} \left[\frac{\partial^2 F}{\partial X^2}(s, X(s)) (G(s) Q^{\frac{1}{2}}) (G(s) Q^{\frac{1}{2}})^* \right] ds, \\ &= F(0, X(0)) + \int_0^t \frac{\partial F}{\partial t}(s, X(s)) ds + \int_0^t \left\langle \frac{\partial F}{\partial X}(s, X(s)), h(s) \right\rangle_E ds \\ &\quad + \sum_{l=1}^{\infty} \int_0^t \left\langle \frac{\partial F}{\partial X}(s, X(s)), G_l(s) d\beta_l(s) \right\rangle_E \\ &\quad + \frac{1}{2} \int_0^t \text{Tr} \left[\frac{\partial^2 F}{\partial X^2}(s, X(s)) (G(s) Q^{\frac{1}{2}}) (G(s) Q^{\frac{1}{2}})^* \right] ds, \end{aligned}$$

where [cf. [14]]

$$\begin{aligned} &\text{Tr} \left[\frac{\partial^2 F}{\partial X^2}(s, X(s)) (G(s) Q^{\frac{1}{2}}) (G(s) Q^{\frac{1}{2}})^* \right] ds \\ &= \sum_{l=1}^{\infty} \left\langle \frac{\partial^2 F}{\partial X^2}(s, X(s)) (G(s) Q^{\frac{1}{2}} e_l), (G(s) Q^{\frac{1}{2}} e_l) \right\rangle_E. \end{aligned}$$

Next we apply Lemma 3.3.2 to (3.3.8) with $GdW = \sum_{l=1}^{\infty} \int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l w_j dx d\beta_l(s) = \sum_{l=1}^{\infty} G_l d\beta_l(s)$ and $G_l = \int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l w_j dx$, setting

$$\begin{aligned} X_j(t) &= \int_D \varphi_m(t) w_j dx, \\ F(t, X) &= X^2, \\ \frac{\partial F}{\partial X}(t, X) &= 2X, \\ h &= \int_D (\Delta \varphi_m + f(\varphi_m) - \frac{1}{|D|} \int_D f(\varphi_m)) w_j dx, \\ \frac{\partial^2 F}{\partial X^2}(t, X) &= 2. \end{aligned}$$

We obtain almost surely, for all $t \in [0, T]$ and all $j = 0, \dots, m$,

$$\begin{aligned} & \left(\int_D \varphi_m(t) w_j dx \right)^2 \\ &= \left(\int_D \varphi_{0m} w_j \right)^2 \\ &+ 2 \int_0^t \left(\int_D \varphi_m w_j dx \right) \left(\int_D (\Delta \varphi_m + f(\varphi_m) - \frac{1}{|D|} \int_D f(\varphi_m)) w_j dx \right) ds \\ &+ 2 \sum_{l=1}^{\infty} \int_0^t \left(\int_D \varphi_m w_j dx \right) \left(\int_D \Phi(\varphi_m(s)) \sqrt{\lambda_l} e_l w_j dx d\beta_l(s) \right) \\ &+ \int_0^t \sum_{l=1}^{\infty} \left(\int_D \Phi(\varphi_m(s)) \sqrt{\lambda_l} e_l w_j dx \right)^2 ds \\ &= \left(\int_D \varphi_{0m} w_j \right)^2 \\ &+ 2 \int_0^t \left(\int_D \varphi_m w_j dx \right) \left(\int_D (\Delta \varphi_m + f(\varphi_m) - \frac{1}{|D|} \int_D f(\varphi_m)) w_j dx \right) ds \\ &+ 2 \int_0^t \left(\int_D \varphi_m w_j dx \right) \left(\int_D \Phi(\varphi_m(s)) dW(s) w_j dx \right) \\ &+ \int_0^t \sum_{l=1}^{\infty} \left(\int_D \Phi(\varphi_m(s)) \sqrt{\lambda_l} e_l w_j dx \right)^2 ds \end{aligned}$$

Summing from $j = 0$ to $j = m$, we obtain :

$$\begin{aligned} \sum_{j=0}^m \left(\int_D \varphi_m w_j dx \right)^2 &= \sum_{j=0}^m \left(\int_D \varphi_{0m} w_j \right)^2 \\ &+ 2 \sum_{j=0}^m \int_0^t \left(\int_D \varphi_m w_j dx \right) \left(\int_D (\Delta \varphi_m) w_j dx \right) ds \end{aligned}$$

$$\begin{aligned}
 & +2 \int_0^t \left(\int_D \varphi_m w_0 dx \right) \left(\int_D f(\varphi_m) w_0 dx \right) ds \\
 & +2 \sum_{j=1}^m \int_0^t \left(\int_D \varphi_m w_j dx \right) \left(\int_D f(\varphi_m) w_j dx \right) ds \\
 & -2 \int_0^t \left(\int_D \varphi_m w_0 dx \right) \left(\int_D \left(\frac{1}{|D|} \int_D f(\varphi_m) \right) w_0 dx \right) ds \\
 & -2 \sum_{j=1}^m \int_0^t \left(\int_D \varphi_m w_j dx \right) \left(\int_D \left(\frac{1}{|D|} \int_D f(\varphi_m) \right) w_j dx \right) ds \\
 & +2 \sum_{j=0}^m \int_0^t \left(\int_D \varphi_m w_j dx \right) \left(\int_D \Phi(\varphi_m) dW(s) w_j dx \right) \\
 & + \sum_{j=0}^m \int_0^t \sum_{l=1}^{\infty} \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l w_j \right)^2 ds. \tag{3.3.9}
 \end{aligned}$$

In view of (3.2.4), (3.3.9) becomes

$$\begin{aligned}
 \sum_{j=0}^m \left(\int_D \varphi_m w_j dx \right)^2 & = \sum_{j=0}^m \left(\int_D \varphi_{0m} w_j \right)^2 \\
 & +2 \sum_{j=0}^m \int_0^t \left(\int_D \varphi_m w_j dx \right) \left(\int_D (\Delta \varphi_m) w_j dx \right) ds \\
 & +2 \sum_{j=1}^m \int_0^t \left(\int_D \varphi_m w_j dx \right) \left(\int_D f(\varphi_m) w_j dx \right) ds \\
 & +2 \sum_{j=0}^m \int_0^t \left(\int_D \varphi_m w_j dx \right) \left(\int_D \Phi(\varphi_m) dW(s) w_j dx \right) \\
 & + \sum_{j=0}^m \int_0^t \sum_{l=1}^{\infty} \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l w_j \right)^2 ds. \tag{3.3.10}
 \end{aligned}$$

In what follows, we apply the following result which we have proved in [5].

Lemma 3.3.3 *Let $A_m = \sum_{j=0}^m \langle A_m, w_j \rangle w_j$ and $B_m = \sum_{j=0}^m \langle B_m, w_j \rangle w_j$. Then*

$$\sum_{j=0}^m \int_0^t \left(\int_D A_m w_j dx \right) \left(\int_D B_m w_j dx \right) ds = \int_0^t \int_D A_m B_m ds. \tag{3.3.11}$$

We now return to the proof of Lemma 3.3.1, first we consider the third term of the right-hand-side of (3.3.10) which is slightly different from (3.3.11). We have that

$$\begin{aligned}
 & 2 \sum_{j=1}^m \int_0^t \left(\int_D \varphi_m w_j dx \right) \left(\int_D f(\varphi_m) w_j dx \right) ds \\
 & = 2 \int_0^t \left(\int_D f(\varphi_m) \sum_{j=1}^m \left(\int_D \varphi_m w_j dx \right) w_j dx \right) ds
 \end{aligned}$$

$$= 2 \int_0^t \left(\int_D f(\varphi_m)(\varphi_m - M_m(t)) dx \right) ds \quad (3.3.12)$$

where we have also used (3.2.1).

In view of Lemma 3.3.3 and (3.3.12), (3.3.10) becomes

$$\begin{aligned} \|\varphi_m(t)\|^2 &= \|\varphi_{0m}\|^2 + 2 \int_0^t \int_D \varphi_m \Delta \varphi_m + 2 \int_0^t \int_D (\varphi_m - \varphi_m^3)(\varphi_m - M_m(t)) dx ds \\ &\quad + 2 \int_0^t \int_D \varphi_m \Phi(\varphi_m) dx dW \\ &\quad + \sum_{j=0}^m \int_0^t \sum_{l=1}^{\infty} \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l w_j dx \right)^2 ds \\ &\leq \|\varphi_{0m}\|^2 + 2 \int_0^t \int_D \varphi_m \Delta \varphi_m + 2 \int_0^t \int_D \varphi_m^2 - 2 \int_0^t \int_D \varphi_m^4 + \int_0^t \int_D \varphi_m^2 \\ &\quad + \int_0^t \int_D M_m^2 + \int_0^t \int_D \frac{3\varepsilon}{2} \varphi_m^4 + \frac{1}{2\varepsilon} \int_0^t \int_D M_m^4 \\ &\quad + 2 \int_0^t \int_D \varphi_m \Phi(\varphi_m) dx dW \\ &\quad + \sum_{j=0}^m \int_0^t \sum_{l=1}^{\infty} \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l w_j dx \right)^2 ds \\ &\leq \|\varphi_{0m}\|^2 - 2 \int_0^t \|\nabla \varphi_m\|^2 + 3 \int_0^t \|\varphi_m\|^2 \\ &\quad - (2 - \frac{3\varepsilon}{2}) \int_0^t \int_D \varphi_m^4 + (\frac{1}{2\varepsilon} + \frac{1}{2}) |D| \int_0^t |M_m|^4 + \frac{1}{2} |D| T \\ &\quad + 2 \int_0^t \int_D \varphi_m \Phi(\varphi_m) dx dW \\ &\quad + \sum_{j=0}^m \int_0^t \sum_{l=1}^{\infty} \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l w_j dx \right)^2 ds. \end{aligned} \quad (3.3.13)$$

Next we estimate the last term of (3.3.13) using (3.1.9)

$$\begin{aligned} \sum_{j=0}^m \int_0^t \sum_{l=1}^{\infty} \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l w_j dx \right)^2 ds &= \int_0^t \sum_{l=1}^{\infty} \sum_{j=0}^m \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l w_j \right)^2 ds \\ &\leq \int_0^t \sum_{l=1}^{\infty} \|\Phi(\varphi_m) \sqrt{\lambda_l} e_l\|_{L^2(D)}^2 ds \\ &\leq \int_0^t \sum_{l=1}^{\infty} \|\Phi_l(x, \varphi_m(x))\|_{L^2(D)}^2 ds \\ &\leq C_1 T. \end{aligned}$$

Thus,

$$\|\varphi_m(t)\|^2 \leq \|\varphi_{0m}\|^2 - 2 \int_0^t \|\nabla \varphi_m\|^2 + 3 \int_0^t \|\varphi_m\|^2$$

$$\begin{aligned}
 & -(2 - \frac{3\varepsilon}{2}) \int_0^t \int_D \varphi_m^4 + (\frac{1}{2\varepsilon} + \frac{1}{2})|D| \int_0^t |M_m|^4 \\
 & + 2 \int_0^t \int_D \varphi_m \Phi(\varphi_m) dx dW(s) \\
 & + \frac{1}{2}|D|T + C_1T,
 \end{aligned} \tag{3.3.14}$$

for all $t \in (0, T)$.

Taking the expectation of (3.3.14) and using the fact that

$$\mathbb{E} \left[\sum_{l=1}^{\infty} \int_0^t \int_D \Phi(\varphi_m) \varphi_m \sqrt{\lambda_l} e_l dx d\beta_l \right] = 0, \tag{3.3.15}$$

$$\begin{aligned}
 \mathbb{E} \|\varphi_m(t)\|^2 & \leq \|\varphi_{0m}\|^2 - 2\mathbb{E} \int_0^t \|\nabla \varphi_m\|^2 + 3\mathbb{E} \int_0^t \|\varphi_m\|^2 \\
 & - (2 - \frac{3\varepsilon}{2}) \mathbb{E} \int_0^t \int_D \varphi_m^4 + (\frac{1}{2\varepsilon} + \frac{1}{2})|D| \mathbb{E} \int_0^t |M_m|^4 \\
 & + \frac{1}{2}|D|T + C_1T.
 \end{aligned} \tag{3.3.16}$$

In order to bound the left hand side of (3.3.16) term we have to derive bounds on M_m , the latter function satisfies

$$\oint \varphi_m(x, s) dx = \oint \varphi_m(x, 0) dx + \sum_{l=1}^{\infty} \int_0^t \oint (\Phi(\varphi_m(x, s)) \sqrt{\lambda_l} e_l(x)) dx d\beta_l(s) \tag{3.3.17}$$

where : $\oint a dx = \frac{1}{|D|} \int_D a dx$.

Applying Itô's formula Lemma 3.3.2, $x \in D$ fixed and with $E = \mathbb{R}$,

$$\begin{aligned}
 X(t) &= \oint \varphi_m(x, s) dx, \\
 F(t, X) &= X^4, \\
 \frac{\partial F}{\partial X}(t, X) &= 4X^3, \\
 \frac{\partial^2 F}{\partial X^2}(t, X) &= 12X^2, \\
 h &= 0, \\
 G &= \oint (\Phi(\varphi_m(x, s)) \sqrt{\lambda_l} e_l(x)) dx.
 \end{aligned}$$

we obtain

$$\begin{aligned}
 |M_m(t)|^4 &= |\oint \varphi_m(x, 0) dx|^4 + 4 \int_0^t (\oint \varphi_m(x, s) dx)^3 \oint (\Phi(\varphi_m(x, s)) \sqrt{\lambda_l} e_l(x)) dx d\beta_l(s) \\
 &+ 6 \int_0^t \sum_{l=1}^{\infty} (\oint \varphi_m(x, s) dx)^2 (\oint (\Phi(\varphi_m(x, s)) \sqrt{\lambda_l} e_l(x)) dx)^2 ds.
 \end{aligned} \tag{3.3.18}$$

We bound the last term of (3.3.18), also using (3.1.9)

$$\begin{aligned}
 & \int_0^t \sum_{l=1}^{\infty} \left(\int \varphi_m(x, s) dx \right)^2 \left(\int (\Phi(\varphi_m(x, s)) \sqrt{\lambda_l} e_l(x)) dx \right)^2 ds \\
 &= \int_0^t \left(\int \varphi_m(x, s) dx \right)^2 \sum_{l=1}^{\infty} \left(\int (\Phi(\varphi_m(x, s)) \sqrt{\lambda_l} e_l(x)) dx \right)^2 ds \\
 &= \frac{1}{|D|^2} \int_0^t \left(\int \varphi_m(x, s) dx \right)^2 \sum_{l=1}^{\infty} \left[\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l \right]^2 ds \\
 &\leq \frac{1}{|D|^2} \int_0^t \left(\int \varphi_m(x, s) dx \right)^2 \sum_{l=1}^{\infty} \left(\int_D 1^2 \right) \int_D (\Phi(\varphi_m) \sqrt{\lambda_l} e_l)^2 ds \\
 &\leq \frac{1}{|D|} \int_0^t \left(\int \varphi_m(x, s) dx \right)^2 \sum_{l=1}^{\infty} \int_D (\Phi(\varphi_m) \sqrt{\lambda_l} e_l)^2 dx ds \\
 &\leq \frac{1}{|D|} \int_0^t \left(\int \varphi_m(x, s) dx \right)^2 \sum_{l=1}^{\infty} \|\Phi(\varphi_m) \sqrt{\lambda_l} e_l\|^2 ds \\
 &\leq \frac{C_1}{|D|} \int_0^t |M_m(t)|^2 ds \leq \tilde{C}_1 \int_0^t |M_m(t)|^4 ds. \tag{3.3.19}
 \end{aligned}$$

Taking the expectation of (3.3.18)

$$\begin{aligned}
 \mathbb{E}|M_m(t)|^4 &\leq \frac{1}{|D|^3} \left(\int_D \varphi_m(0) \right)^4 + 6\tilde{C}_1 \mathbb{E} \int_0^t |M_m(t)|^4 ds \\
 &\leq \frac{1}{|D|} \|\varphi_m(0)\|^4 + 6\tilde{C}_1 \mathbb{E} \int_0^t |M_m(t)|^4 ds. \tag{3.3.20}
 \end{aligned}$$

By Gronwall's Lemma we deduce that

$$\mathbb{E}|M_m(t)|^4 \leq \tilde{C}, \quad \text{for all } t \in [0, T]. \tag{3.3.21}$$

Substituting (3.3.21) in (3.3.16) we obtain that

$$\begin{aligned}
 & \mathbb{E} \|\varphi_m(t)\|^2 + 2\mathbb{E} \int_0^t \|\nabla \varphi_m\|^2 + \left(2 - \frac{3\varepsilon}{2}\right) \mathbb{E} \int_0^t \int_D \varphi_m^4 \\
 &\leq \|\varphi_{0m}\|^2 + 3\mathbb{E} \int_0^t \|\varphi_m\|^2 \\
 &\quad + \tilde{C} + \frac{1}{2}|D|T + C_1T. \tag{3.3.22}
 \end{aligned}$$

Applying Gronwall Lemma yields

$$\mathbb{E} \int_D \varphi_m^2(t) \leq K \quad \text{for all } t \in [0, T]. \tag{3.3.23}$$

Substituting (3.3.23) into (3.3.22) and choosing $\varepsilon = \frac{1}{3}$ we obtain,

$$\mathbb{E} \|\varphi_m(t)\|^2 + 2\mathbb{E} \int_0^t \int_D |\nabla \varphi_m|^2 dx ds + \frac{3}{2} \mathbb{E} \int_0^t \int_D \varphi_m^4 dx ds$$

$$\leq \|\varphi_0\|^2 + \mathcal{C},$$

where \mathcal{C} depends on T , from which we deduce that

$$\mathbb{E} \int_0^t \int_D |\nabla \varphi_m|^2 dx ds \leq K, \quad (3.3.24)$$

$$\mathbb{E} \int_0^t \int_D \varphi_m^4 \leq K. \quad (3.3.25)$$

Neglecting the negative terms, taking the supremum and then the expectation of (3.3.14) and using (3.3.23) and (3.3.21) we are led to

$$\begin{aligned} \mathbb{E} \sup_{t \in (0, T)} \|\varphi_m(t)\|^2 &\leq \|\varphi_0\|^2 + 2\mathbb{E} \sup_{t \in (0, T)} \left| \sum_{l=1}^{\infty} \int_0^t \int_D \varphi_m \Phi(\varphi_m) \sqrt{\lambda_l} e_l d\beta_l \right| \\ &\quad + C. \end{aligned} \quad (3.3.26)$$

Lemma 3.3.4 (Burkholder-Davis-Gundy inequality) (cf. [16] p.10 and [3])

Let us introduce another separable real space $(X, (\cdot, \cdot)_X)$. We denote the family of all linear operator $G : H \rightarrow X$ such that $G\sqrt{Q}$ are Hilbert Schmidt operators by $\mathcal{L}_{2,Q}(H, X)$. Namely, G satisfies that

$$\|G\|_{2,Q}^2 := \sum_{l=1}^{\infty} \|G\sqrt{Q}e_l\|_X^2 < \infty.$$

Given an X valued predictable process $G \in L^2(\Omega; L^2((0, T), \mathcal{L}_{2,Q}(H, X)))$ and setting $G_l := G\sqrt{Q}e_l$, one may define the stochastic integral as:

$$\int_0^t G(s) dW(s) = \sum_{l=1}^{\infty} \int_0^t G_l d\beta_l.$$

The Burkholder-Davis-Gundy inequality is given by

$$\mathbb{E} \left(\sup_{t \in (0, T)} \left\| \int_0^t G(s) dW(s) \right\|_X^r \right) \leq \tilde{C}_r \mathbb{E} \left(\int_0^T \|G\|_{2,Q}^2 dt \right)^{\frac{r}{2}}, \quad (3.3.27)$$

or else

$$\mathbb{E} \left(\sup_{t \in (0, T)} \left\| \sum_{l=1}^{\infty} \int_0^t G_l d\beta_l \right\|_X^r \right) \leq \tilde{C}_r \mathbb{E} \left(\int_0^T \sum_{l=1}^{\infty} \|G_l\|_X^2 dt \right)^{\frac{r}{2}}. \quad (3.3.28)$$

for all $r \geq 1$, where \tilde{C}_r is a constant only depending on r .

Next we apply the Burkholder inequality with:

$$r = 1, \quad X = \mathbb{R}, \quad G_l = \int_D \varphi_m \Phi(\varphi_m) \sqrt{\lambda_l} e_l dx$$

Also using that, $2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2$ ($\varepsilon > 0$) and $a^2 + b^2 \leq (a+b)^2$ ($a, b > 0$) we can bound the last term in (3.3.26)

$$\begin{aligned}
 & 2\mathbb{E} \left[\sup_{t \in (0, T)} \left| \sum_{l=1}^{\infty} \int_0^t \int_D \varphi_m \Phi(\varphi_m) \sqrt{\lambda_l} e_l d\beta_l \right| \right] \\
 & \leq 2\tilde{C}_1 \mathbb{E} \left(\int_0^T \sum_{l=1}^{\infty} \left| \int_D \varphi_m(t) \Phi(\varphi_m(t)) \sqrt{\lambda_l} e_l dx \right|^2 dt \right)^{\frac{1}{2}} \\
 & \leq 2\tilde{C}_1 \mathbb{E} \left(\int_0^T \sum_{l=1}^{\infty} \|\varphi_m(t)\|^2 \|\Phi(\varphi_m(t)) \sqrt{\lambda_l} e_l\|^2 dt \right)^{\frac{1}{2}} \\
 & \leq 2\tilde{C}_1 \mathbb{E} \left(\sup_{t \in (0, T)} \|\varphi_m(t)\|^2 \int_0^T \sum_{l=1}^{\infty} \|\Phi(\varphi_m(t)) \sqrt{\lambda_l} e_l\|^2 dt \right)^{\frac{1}{2}} \\
 & \leq 2\tilde{C}_1 \mathbb{E} \left[\frac{\varepsilon}{2} \left(\sup_{t \in (0, T)} \|\varphi_m(t)\|^2 \right)^2 + \frac{1}{2\varepsilon} \left(\int_0^T \sum_{l=1}^{\infty} \|\Phi_l(x, \varphi_m(t))\|^2 dt \right)^2 \right]^{\frac{1}{2}} \\
 & \leq \sqrt{2}\tilde{C}_1 \sqrt{\varepsilon} \mathbb{E} \sup_{t \in (0, T)} \|\varphi_m(t)\|^2 + \mathcal{C}(\varepsilon), \tag{3.3.29}
 \end{aligned}$$

where we have used (3.1.9) and (3.1.7).

Substituting (3.3.29) in (3.3.26) yields

$$(1 - \sqrt{2}\tilde{C}_1\sqrt{\varepsilon}) \mathbb{E} \sup_{t \in [0, T]} \|\varphi_m(t)\|^2 \leq \|\varphi_0\|^2 + \tilde{C}(\varepsilon). \tag{3.3.30}$$

Thus,

$$\mathbb{E} \sup_{t \in (0, T)} \|\varphi_m(t)\|^2 \leq \mathcal{K}. \tag{3.3.31}$$

Next we prove (3.3.6); using the inequality $|a + b|^p \leq C(p)(|a|^p + |b|^p)$, for $p \geq 1$ as well as Young inequality, we obtain

$$\begin{aligned}
 \mathbb{E} \|f(\varphi_m)\|_{L^{\frac{4}{3}}(D \times (0, T))}^{\frac{4}{3}} &= \mathbb{E} \int_0^T \int_D |f(\varphi_m)|^{\frac{4}{3}} dx dt \\
 &= \mathbb{E} \int_0^T \int_D |\varphi_m - \varphi_m^3|^{\frac{4}{3}} dx dt \\
 &\leq c_{\frac{4}{3}} \mathbb{E} \int_0^T \int_D (|\varphi_m|^{\frac{4}{3}} + |\varphi_m|^4) dx dt \\
 &\leq \tilde{c}_{\frac{4}{3}} \mathbb{E} \int_0^T \int_D |\varphi_m|^4 + \tilde{c}_{\frac{4}{3}} \\
 &\leq K, \tag{3.3.32}
 \end{aligned}$$

where we have used (3.3.2) in Lemma 3.3.1. In order to prove (3.3.7) we use Hölder inequality and (3.5.15)

$$\mathbb{E} \left\| \int_D f(\varphi_m) \right\|_{L^{\frac{4}{3}}(D \times (0, T))}^{\frac{4}{3}} = \mathbb{E} \int_0^T \int_D \left| \int_D f(\varphi_m) \right|^{\frac{4}{3}} dx dt$$

$$\begin{aligned}
 &\leq \mathbb{E} \int_0^T \int_D \left[\left(\int_D f(\varphi_m)^{\frac{4}{3}} \right)^{\frac{3}{4}} |D|^{\frac{1}{4}} \right]^{\frac{4}{3}} dx dt \\
 &\leq |D|^{\frac{4}{3}} \mathbb{E} \|f(\varphi_m)\|_{L^{\frac{4}{3}}(D \times (0, T))}^{\frac{4}{3}} \\
 &\leq \tilde{K}.
 \end{aligned} \tag{3.3.33}$$

□

Lemma 3.3.5 *Let $p \geq 2$. There exists a positive constant $\mathcal{K} = \mathcal{K}(p)$ such that*

$$\mathbb{E} \left(\sup_{t \in (0, T)} |M_m(t)|^{2p} \right) \leq \mathcal{K}. \tag{3.3.34}$$

Proof we re-write (3.2.3) as :

$$\oint \varphi_m(x, s) dx = \oint \varphi_m(x, 0) dx + \sum_{l=1}^{\infty} \int_0^t \oint (\Phi(\varphi_m(x, s)) \sqrt{\lambda_l} e_l(x)) dx d\beta_l(s).$$

Applying Itô's formula Lemma 3.3.2, $x \in D$ fixed with $E = \mathbb{R}$ and

$$\begin{aligned}
 X(s) &= \oint \varphi_m(x, s) dx, \\
 F(s, X) &= X^2, \\
 \frac{\partial F}{\partial X}(s, X) &= 2X, \\
 \frac{\partial^2 F}{\partial X^2}(s, X) &= 2, \\
 h &= 0, \\
 G_l &= \oint (\Phi(\varphi_m(x, s)) \sqrt{\lambda_l} e_l(x)) dx.
 \end{aligned}$$

Integrating on D , we obtain in view of (3.1.9)

$$\begin{aligned}
 |\oint \varphi_m(x, s) dx|^2 &= |\oint \varphi_m(x, 0) dx|^2 \\
 &\quad + 2 \int_0^t \oint \varphi_m(x, s) dx \oint [\Phi(\varphi_m(x, s)) \sqrt{\lambda_l} e_l(x)] dx d\beta_l(s) \\
 &\quad + \int_0^t \sum_{l=1}^{\infty} |\oint [\Phi(\varphi_m(x, s)) \sqrt{\lambda_l} e_l(x)] dx|^2 \\
 &\leq |\oint \varphi_m(x, 0) dx|^2 \\
 &\quad + 2 \int_0^t \oint \varphi_m(x, s) dx \oint [\Phi(\varphi_m(x, s)) \sqrt{\lambda_l} e_l(x)] dx d\beta_l(s) \\
 &\quad + \frac{1}{|D|} \int_0^t \sum_{l=1}^{\infty} \|\Phi(\varphi_m(x, s)) \sqrt{\lambda_l} e_l\|^2 \\
 &\leq |\oint \varphi_m(x, 0) dx|^2
 \end{aligned}$$

$$\begin{aligned}
 & +2 \int_0^t \oint \varphi_m(x, s) dx \oint [\Phi(\varphi_m(x, s)) \sqrt{\lambda_l} e_l(x)] dx d\beta_l(s) \\
 & +CT. \tag{3.3.35}
 \end{aligned}$$

Next we raise both sides of equation (3.3.35) to the power p , take the supremum on $[0, T]$ and then the expectation.

$$\begin{aligned}
 \mathbb{E} \sup_{t \in [0, T]} \left| \oint \varphi_m(x, s) \right|^{2p} & \leq C(p) \left[\mathbb{E} \left| \oint \varphi_m(x, 0) \right|^{2p} \right. \\
 & \left. + 2^p \mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t \oint \varphi_m(x, s) dx \oint (\Phi(\varphi_m(x, s)) \sqrt{\lambda_l} e_l(x)) dx \right|^p \right. \\
 & \left. + (CT)^p \right]. \tag{3.3.36}
 \end{aligned}$$

Applying the Burkholder inequality (3.3.28) with:

$r = p$, $X = \mathbb{R}$, $G_l = \oint \varphi_m(x, s) dx \oint (\Phi(\varphi_m(x, s)) \sqrt{\lambda_l} e_l(x)) dx$ and using the same computation as before, we can bound the last terms of the right hand side in (3.3.36)

$$\begin{aligned}
 & 2^p \mathbb{E} \left(\sup_{t \in (0, T)} \left| \sum_{l=1}^{\infty} \int_0^t \oint \varphi_m(x, s) dx \oint (\Phi(\varphi_m(x, s)) \sqrt{\lambda_l} e_l(x)) dx d\beta_l \right|^p \right) \\
 & \leq 2^p \tilde{C}_p \mathbb{E} \left(\int_0^T \sum_{l=1}^{\infty} \left| \oint \varphi_m(x, s) dx \oint (\Phi(\varphi_m(x, s)) \sqrt{\lambda_l} e_l(x)) dx \right|^2 dt \right)^{\frac{p}{2}} \\
 & \leq 2^p \tilde{C}_p \mathbb{E} \left(\int_0^T \sum_{l=1}^{\infty} \left| \oint \varphi_m(x, s) dx \right|^2 \left| \oint (\Phi(\varphi_m(x, s)) \sqrt{\lambda_l} e_l(x)) dx \right|^2 \right)^{\frac{p}{2}} \\
 & \leq 2^p \tilde{C}_p \mathbb{E} \left(\int_0^T \left| \oint \varphi_m(x, s) dx \right|^2 \frac{1}{|D|} \sum_{l=1}^{\infty} \|\Phi_l(x, \varphi_m)\|^2 \right)^{\frac{p}{2}} \\
 & \leq 2^p \tilde{C}_p C^{\frac{p}{2}} \mathbb{E} \left(\int_0^T \left| \oint \varphi_m(x, s) dx \right|^2 \right)^{\frac{p}{2}} \\
 & \leq 2^p \tilde{C}_p C^{\frac{p}{2}} \mathbb{E} \left(\frac{1}{2} \int_0^T \left| \oint \varphi_m(x, s) dx \right|^4 + \frac{T}{2} \right)^{\frac{p}{2}} \\
 & \leq \tilde{C}_p \mathbb{E} \left(T^{\frac{p-2}{2}} \int_0^T \left| \oint \varphi_m(x, s) dx \right|^{2p} + T^{\frac{p}{2}} \right) \\
 & \leq \bar{C}_p(T) \mathbb{E} \int_0^T |M_m(t)|^{2p} + C_p(T). \tag{3.3.37}
 \end{aligned}$$

Substituting in (3.3.36) we obtain

$$\mathbb{E} \sup_{t \in (0, T)} |M_m(t)|^{2p} \leq C_p(D) \mathbb{E} \|\varphi_m(0)\|^2 + C(p) \bar{C}_p(T) \mathbb{E} \int_0^T |M_m(t)|^{2p} + \bar{C}_p(T). \tag{3.3.38}$$

By Gronwall Lemma we prove that

$$\mathbb{E} \left(\sup_{t \in (0, T)} |M_m(t)|^{2p} \right) \leq \mathcal{K}. \quad (3.3.39) \quad \square$$

Lemma 3.3.6 *There exists a positive constant \mathcal{K} such that*

$$\mathbb{E} \left(\sup_{t \in (0, T)} \|\varphi_m(t)\|^p \right) \leq \mathcal{K} \quad (3.3.40)$$

for all $p \geq 2$.

Proof Next we neglect the negative terms in (3.3.14), raise both sides to the power $\frac{p}{2}$, take the supremum on $(0, T)$ and then the expectation. We use two inequalities, namely there exists a positive constant $C(p)$ such that $\left(\sum_{i=1}^N |a_i| \right)^p \leq C(p) \sum_{i=1}^N |a_i|^p$ and $\int |ab| dx \leq \left(\int |a|^s \right)^{\frac{1}{s}} \left(\int |a|^q \right)^{\frac{1}{q}}$ with $\frac{1}{s} + \frac{1}{q} = 1$. We deduce that there exists a positive constant $C(p)$ such that

$$\begin{aligned} \mathbb{E} \sup_{t \in (0, T)} \|\varphi_m(t)\|^p &\leq C(p) [\|\varphi_{0m}\|^p + 3^{\frac{p}{2}} T^{\frac{p-2}{2}} \mathbb{E} \int_0^T \|\varphi_m\|^p ds \\ &\quad + T^{\frac{p-2}{2}} \left(\frac{1}{2} + \frac{1}{2\varepsilon} \right)^{\frac{p}{2}} |D|^{\frac{p}{2}} \mathbb{E} \int_0^T |M_m(t)|^{2p} \\ &\quad + 2^{\frac{p}{2}} \mathbb{E} \left(\sup_{t \in (0, T)} \left| \sum_{l=1}^{\infty} \int_0^t \int_D \varphi_m \Phi(\varphi_m) \sqrt{\lambda_l} e_l d\beta_l \right|^{\frac{p}{2}} \right) \\ &\quad + (CT)^{\frac{p}{2}}]. \end{aligned} \quad (3.3.41)$$

Applying the Burkholder inequality (3.3.28) with:

$r = \frac{p}{2}$, $X = \mathbb{R}$, $G_l = \int_D \varphi_m \Phi(\varphi_m) \sqrt{\lambda_l} e_l dx$ and using the same computation as before, we can bound the last terms of the right hand side in (3.3.41)

$$\begin{aligned} &2^{\frac{p}{2}} \mathbb{E} \left(\sup_{t \in (0, T)} \left| \sum_{l=1}^{\infty} \int_0^t \int_D \varphi_m \Phi(\varphi_m) \sqrt{\lambda_l} e_l d\beta_l \right|^{\frac{p}{2}} \right) \\ &= 2^{\frac{p}{2}} \mathbb{E} \left(\sup_{t \in (0, T)} \left| \sum_{l=1}^{\infty} \int_0^t \left\{ \int_D \varphi_m \Phi_l(\varphi_m) \right\} d\beta_l \right|^{\frac{p}{2}} \right) \\ &\leq 2^{\frac{p}{2}} \tilde{C}_p \mathbb{E} \left(\int_0^T \sum_{l=1}^{\infty} \left| \int_D \varphi_m \Phi_l(\varphi_m) \right|^2 dt \right)^{\frac{p}{4}} \\ &\leq 2^{\frac{p}{2}} \tilde{C}_p \mathbb{E} \left(\int_0^T \sum_{l=1}^{\infty} \|\varphi_m\|_{L^2(D)}^2 \|\Phi_l(\varphi_m)\|_{L^2(D)}^2 dt \right)^{\frac{p}{4}} \end{aligned}$$

$$\begin{aligned}
 &\leq 2^{\frac{p}{2}} \tilde{C}_p \mathbb{E} \left(\int_0^T \|\varphi_m\|_{L^2(D)}^2 \sum_{l=1}^{\infty} \|\Phi_l(\varphi_m)\|_{L^2(D)}^2 dt \right)^{\frac{p}{4}} \\
 &\leq 2^{\frac{p}{2}} \tilde{C}_p (C_1)^{\frac{p}{4}} \mathbb{E} \left(\int_0^T \|\varphi_m\|_{L^2(D)}^2 \right)^{\frac{p}{4}} \\
 &\leq 2^{\frac{p}{2}} \tilde{C}_p (C_1)^{\frac{p}{4}} \mathbb{E} \left(\int_0^T \frac{1}{2} \|\varphi_m\|_{L^2(D)}^4 + \frac{T}{2} \right)^{\frac{p}{4}} \\
 &\leq \tilde{C}_p \mathbb{E} \left(T^{\frac{p}{4}} + T^{\frac{p-4}{4}} \int_0^T \|\varphi_m\|^p \right) \\
 &\leq \bar{C}_p(T) \mathbb{E} \int_0^T \|\varphi_m\|^p + C_p(T). \tag{3.3.42}
 \end{aligned}$$

Substituting (3.3.42) in (3.3.41) and using (3.3.39) we obtain

$$\mathbb{E} \sup_{t \in [0, T]} \|\varphi_m(t)\|^p \leq C(p) \|\varphi_{0m}\|^p + \tilde{\mathcal{K}} + \tilde{C}_2 \mathbb{E} \int_0^T \|\varphi_m\|^p ds + \bar{C}_p(T).$$

Applying Gronwall Lemma we deduce that

$$\mathbb{E} \sup_{t \in [0, T]} \|\varphi_m(t)\|^p \leq \mathcal{K}. \quad \square$$

In order to prove the estimates (3.3.4) and (3.3.5), we multiply (3.3.8) by $\sqrt{\gamma_j}$.

$$\begin{aligned}
 \int_D \varphi_m(t) \sqrt{\gamma_j} w_j &= \int_D \varphi_m(0) \sqrt{\gamma_j} w_j - \int_0^t \int_D \nabla \varphi_m \nabla w_j \sqrt{\gamma_j} + \int_0^t \int_D f(\varphi_m) \sqrt{\gamma_j} w_j \\
 &\quad - \frac{1}{|D|} \int_0^t \int_D \left\{ \int_D f(\varphi_m) \right\} \sqrt{\gamma_j} w_j dx ds \\
 &\quad + \sum_{l=1}^{\infty} \int_0^t \int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l \sqrt{\gamma_j} w_j d\beta_l(s).
 \end{aligned}$$

Next we apply Ito's formula Lemma 3.3.2 with $GdW = \sum_{l=1}^{\infty} \int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l \sqrt{\gamma_j} w_j dx d\beta_l(s) = \sum_{l=1}^{\infty} G_l d\beta_l(s)$ and $G_l = \int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l \sqrt{\gamma_j} w_j dx$, setting

$$\begin{aligned}
 X_j(t) &= \int_D \varphi_m \sqrt{\gamma_j} w_j dx, \\
 F(X) &= X^2, \\
 \frac{\partial F}{\partial X}(X) &= 2X, \\
 \frac{\partial F}{\partial X^2}(s) &= 2,
 \end{aligned}$$

$$h_j(s) = \int_D \sqrt{\gamma_j} \left(\Delta \varphi_m + f(\varphi_m) - \frac{1}{|D|} \int_D f(\varphi_m) \right) w_j.$$

We obtain almost surely, for all $t \in [0, T]$ and all $j = 0, \dots, m$,

$$\begin{aligned} & \left(\int_D \varphi_m(t) \sqrt{\gamma_j} w_j dx \right)^2 \\ = & \left(\int_D \varphi_{0m} \sqrt{\gamma_j} w_j dx \right)^2 + 2 \int_0^t \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) \left(\int_D \Delta \varphi_m \sqrt{\gamma_j} w_j \right) \\ & + 2 \int_0^t \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) \left(\int_D f(\varphi_m) \sqrt{\gamma_j} w_j \right) \\ & - \frac{2}{|D|} \int_0^t \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) \left(\int_D \left\{ \int_D f(\varphi_m) \right\} \sqrt{\gamma_j} w_j dx ds \right) \\ & + 2 \sum_{l=1}^{\infty} \int_0^t \left(\int_D \varphi_m \sqrt{\gamma_j} w_j dx \right) \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l \sqrt{\gamma_j} w_j dx \right) d\beta_l(s) \\ & + \sum_{l=1}^{\infty} \int_0^t \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l \sqrt{\gamma_j} w_j \right)^2 dt. \end{aligned} \quad (3.3.43)$$

Next we sum (3.3.43) from $j = 0, \dots, m$.

$$\begin{aligned} & \sum_{j=0}^m \left(\int_D \varphi_m(t) \sqrt{\gamma_j} w_j dx \right)^2 \\ = & \sum_{j=0}^m \left(\int_D \varphi_{0m} \sqrt{\gamma_j} w_j dx \right)^2 \\ & + 2 \sum_{j=0}^m \int_0^t \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) \left(\int_D \Delta \varphi_m \sqrt{\gamma_j} w_j \right) \\ & + 2 \sum_{j=0}^m \int_0^t \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) \left(\int_D f(\varphi_m) \sqrt{\gamma_j} w_j \right) \\ & - \frac{2}{|D|} \sum_{j=0}^m \int_0^t \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) \left(\int_D \left\{ \int_D f(\varphi_m) \right\} \sqrt{\gamma_j} w_j dx ds \right) \\ & + 2 \sum_{j=0}^m \sum_{l=1}^{\infty} \int_0^t \left(\int_D \varphi_m \sqrt{\gamma_j} w_j dx \right) \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l \sqrt{\gamma_j} w_j dx \right) d\beta_l(s) \\ & + \sum_{j=0}^m \sum_{l=1}^{\infty} \int_0^t \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l \sqrt{\gamma_j} w_j \right)^2 dt. \end{aligned} \quad (3.3.44)$$

In what follows we evaluate the terms in (3.3.44)

$$\begin{aligned} \sum_{j=0}^m \left(\int_D \varphi_m(t) \sqrt{\gamma_j} w_j \right)^2 &= \sum_{j=0}^m \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) \\ &= \sum_{j=0}^m \int_D \varphi_m \sqrt{\gamma_j} \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) w_j dx \end{aligned}$$

$$\begin{aligned}
 &= \sum_{j=0}^m \int_D \varphi_m \left(\int_D \varphi_m w_j \right) \gamma_j w_j dx \\
 &= - \sum_{j=0}^m \int_D \varphi_m \left(\int_D \varphi_m w_j \right) \Delta w_j dx \\
 &= - \sum_{j=0}^m \int_D \varphi_m \Delta \left(\int_D \varphi_m w_j \right) w_j dx \\
 &= - \int_D \varphi_m \Delta \left(\sum_{j=1}^m \left(\int_D \varphi_m w_j \right) w_j dx \right) \\
 &= - \int_D \varphi_m \Delta \varphi_m = \|\nabla \varphi_m(t)\|^2.
 \end{aligned}$$

Similarly, one can show that $\sum_{j=0}^m \left(\int_D \varphi_{0m} \sqrt{\gamma_j} w_j \right)^2 = \|\nabla \varphi_{0m}\|^2$.

Next, we compute the elliptic term

$$\begin{aligned}
 &\sum_{j=0}^m 2 \int_0^t \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) \left(\int_D \Delta \varphi_m \sqrt{\gamma_j} w_j \right) \\
 &= \sum_{j=0}^m 2 \int_0^t \int_D \Delta \varphi_m \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) \sqrt{\gamma_j} w_j dx \\
 &= - \sum_{j=0}^m 2 \int_0^t \int_D \Delta \varphi_m \left(\int_D \varphi_m w_j \right) \Delta w_j dx \\
 &= -2 \int_0^t \int_D \Delta \varphi_m \Delta \left(\sum_{j=0}^m \left(\int_D \varphi_m w_j \right) w_j \right) dx \\
 &= -2 \int_0^t \|\Delta \varphi_m\|^2.
 \end{aligned} \tag{3.3.45}$$

Next, we compute the contribution of the nonlinear term

$$\begin{aligned}
 &\sum_{j=1}^m 2 \int_0^t \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) \left(\int_D f(\varphi_m) \sqrt{\gamma_j} w_j \right) \\
 &= \sum_{j=1}^m 2 \int_0^t \int_D f(\varphi_m) \left(\int_D \varphi_m \sqrt{\gamma_j} w_j \right) \sqrt{\gamma_j} w_j dx \\
 &= -2 \int_0^t \int_D f(\varphi_m) \Delta \left(\sum_{j=1}^m \left(\int_D \varphi_m w_j \right) w_j \right) dx \\
 &= -2 \int_0^t \int_D f(\varphi_m) \Delta(\varphi_m - M_m(t)) dx \\
 &= -2 \int_0^t \int_D f(\varphi_m) \Delta \varphi_m dx.
 \end{aligned} \tag{3.3.46}$$

Also we remark that the fourth term on the right-hand-side of (3.3.44) is equal to zero.

Next we consider the stochastic term,

$$\begin{aligned}
 & 2 \sum_{j=0}^m \sum_{l=1}^{\infty} \int_0^t \left(\int_D \varphi_m \sqrt{\gamma_j} w_j dx \right) \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l \sqrt{\gamma_j} w_j dx \right) d\beta_l(s) \\
 &= 2 \sum_{j=0}^m \int_0^t \left(\int_D \varphi_m \sqrt{\gamma_j} w_j dx \right) \left(\int_D \Phi(\varphi_m) \sqrt{\gamma_j} w_j dx dW(s) \right) \\
 &= 2 \sum_{j=0}^m \int_0^t \int_D \Phi(\varphi_m) \left(\int_D \varphi_m w_j dx \right) \gamma_j w_j dx dW(s) \\
 &= -2 \int_0^t \int_D \Phi(\varphi_m) \Delta \left(\sum_{j=0}^m \left(\int_D \varphi_m w_j dx \right) w_j \right) dx dW(s) \\
 &= -2 \int_0^t \int_D \Phi(\varphi_m) \Delta \varphi_m dx dW(s) \\
 &= 2 \sum_{l=1}^{\infty} \int_0^t \int_D \nabla \varphi_m \nabla \{ \Phi(\varphi_m) \sqrt{\lambda_l} e_l \} dx d\beta_l(s). \tag{3.3.47}
 \end{aligned}$$

Substituting the formulas above into (3.3.44) yields

$$\begin{aligned}
 \|\nabla \varphi_m(t)\|^2 &= \|\nabla \varphi_{0m}\|_{L^2(D)}^2 - 2 \int_0^t \|\Delta \varphi_m\|^2 - 2 \int_0^t \int_D f(\varphi_m) \Delta \varphi_m \\
 &\quad - 2 \int_0^t \int_D \Delta \varphi_m \Phi(\varphi_m) dW(s) \\
 &\quad + 2 \sum_{j=0}^m \sum_{l=1}^{\infty} \int_0^t \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l \sqrt{\gamma_j} w_j \right)^2 dt \\
 &\leq \|\nabla \varphi_{0m}\|^2 - 2 \int_0^t \|\Delta \varphi_m\|_{L^2(D)}^2 + 2 \int_0^t f'(\varphi_m) \|\nabla \varphi_m\|_{L^2(D)}^2 \\
 &\quad - 2 \int_0^t \int_D \Delta \varphi_m \Phi(\varphi_m) dx dW(s) \\
 &\quad + \sum_{j=0}^m \sum_{l=1}^{\infty} \int_0^t \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l \sqrt{\gamma_j} w_j \right)^2 ds.
 \end{aligned}$$

We compute the Ito correction term on the right-hand-side of (3.3.48) using the fact that $\{w_k\}$ are an orthonormal basis of $L^2(D)$ and $H^1(D)$.

$$\begin{aligned}
 & \sum_{j=0}^m \int_0^t \sum_{l=1}^{\infty} \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l \sqrt{\gamma_j} w_j \right)^2 dt \\
 &= \sum_{j=0}^m \int_0^t \sum_{l=1}^{\infty} \int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l w_j \right) \gamma_j w_j dx ds
 \end{aligned}$$

$$\begin{aligned}
 &= \int_0^t \sum_{l=1}^{\infty} \int_D \left[\sum_{i=0}^{\infty} \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l w_i \right) w_i \right. \\
 &\quad \left. \sum_{j=0}^m \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l w_j \right) \gamma_j w_j \right] dx ds \\
 &= \int_0^t \sum_{l=1}^{\infty} \int_D \left[\sum_{j=0}^m \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l w_j \right) w_j \right. \\
 &\quad \left. \sum_{j=0}^m \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l w_j \right) \gamma_j w_j \right] dx ds \\
 &= - \int_0^t \sum_{l=1}^{\infty} \int_D \left[\sum_{j=0}^m \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l w_j \right) w_j \right. \\
 &\quad \left. \Delta \left(\sum_{j=0}^m \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l w_j \right) w_j \right) \right] dx ds \\
 &= \int_0^t \sum_{l=1}^{\infty} \int_D \left[\nabla \left\{ \sum_{j=0}^m \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l w_j dx \right) w_j \right\} \right]^2 dx ds \\
 &= \int_0^t \sum_{l=1}^{\infty} \int_D \left[\sum_{j=0}^m \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l w_j dx \right) \nabla w_j \right]^2 dx ds \\
 &= \int_0^t \sum_{l=1}^{\infty} \sum_{j=0}^m \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l w_j dx \right)^2 \int_D (\nabla w_j)^2 dx ds \\
 &= - \int_0^t \sum_{l=1}^{\infty} \sum_{j=0}^m \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l w_j dx \right)^2 \int_D w_j \Delta w_j dx ds \\
 &= \int_0^t \sum_{l=1}^{\infty} \sum_{j=0}^m \gamma_j \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l w_j dx \right)^2 \int_D (w_j)^2 dx ds \\
 &\leq \int_0^t \sum_{l=1}^{\infty} \sum_{j=0}^{\infty} \gamma_j \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l w_j dx \right)^2 ds \\
 &\leq \int_0^t \sum_{l=1}^{\infty} \|\nabla \Phi_l(x, \varphi_m)\|^2 \\
 &\leq C_2 T, \tag{3.3.48}
 \end{aligned}$$

where we have used (3.1.10) and [15] p.235.

Substituting (3.3.48) into (3.3.48) and using (3.1.2) we obtain

$$\begin{aligned}
 \|\nabla \varphi_m(t)\|^2 &\leq \|\nabla \varphi_{0m}\|^2 - 2 \int_0^t \|\Delta \varphi_m(s)\|^2 ds + 2 \int_0^t \|\nabla \varphi_m(s)\|^2 ds \\
 &\quad - 2 \int_0^t \int_D \Delta \varphi_m(s) \Phi(\varphi_m(s)) dW(s) + C_2 T. \tag{3.3.49}
 \end{aligned}$$

Thus, taking the expectation yields

$$\begin{aligned} \mathbb{E}\|\nabla\varphi_m(t)\|^2 + 2\mathbb{E}\int_0^t \|\Delta\varphi_m\|^2 ds &\leq \|\nabla\varphi_{0m}\|^2 + 2\mathbb{E}\int_0^t \|\nabla\varphi_m(s)\|^2 ds \\ &\quad + C_2 T. \end{aligned} \quad (3.3.50)$$

where we used the fact that $\mathbb{E}\left(\int_0^t \int_D \Delta\varphi_m \Phi(\varphi_m) \sqrt{\lambda_l} e_l dx dW(s)\right) = 0$.

Therefore, it follows from Lemma 3.3.1, inequality (3.3.1) that

$$\begin{aligned} \mathbb{E}\|\nabla\varphi_m(t)\|^2 + 2\mathbb{E}\int_0^t \|\Delta\varphi_m\|^2 ds &\leq \|\nabla\varphi_{0m}\|^2 + K \\ &\quad + C_2 T. \end{aligned} \quad (3.3.51)$$

Lemma 3.3.7 *We have that $\|\nabla\varphi_{0m}\|^2 \leq \|\nabla\varphi_0\|^2$.*

Proof Using the fact that $\{w_k\}$ is an orthogonal basis in V and [10] p.235 yields

$$\begin{aligned} \|\nabla\varphi_{0m}\|^2 &= \int_D \left[\sum_{j=0}^m \left(\int_D \varphi_0 w_j dx \right) \nabla w_j \right]^2 \\ &= \int_D \sum_{j=0}^m \left(\int_D \varphi_0 w_j dx \right)^2 (\nabla w_j)^2 \\ &= \sum_{j=0}^m \left(\int_D \varphi_0 w_j dx \right)^2 \int_D (\nabla w_j)^2 \\ &\leq \sum_{j=0}^{\infty} \left(\int_D \varphi_0 w_j dx \right)^2 \int_D (\nabla w_j)^2 \\ &\leq \|\nabla\varphi_0\|^2, \end{aligned}$$

which yields the result of Lemma 3.3.7. \square

It follows from (3.3.51) and Lemma 3.3.7 that

$$\mathbb{E}\|\nabla\varphi_m(t)\|^2 \leq K_1, \quad \text{for all } t \in [0, T], \quad (3.3.52)$$

$$\mathbb{E}\int_0^t \|\Delta\varphi_m(s)\|^2 ds \leq K_1. \quad (3.3.53)$$

Lemma 3.3.8 *There exists a positive constant \tilde{C} such that*

$$\mathbb{E}\left(\sup_{t \in (0, T)} \|\varphi_m(t)\|_V^p\right) \leq \tilde{C}, \quad \text{for all } p \geq 2. \quad (3.3.54)$$

Next we add (3.3.14) and (3.3.49) to obtain

$$\|\varphi_m\|_V^2 \leq \|\varphi_0\|_V^2 + 5 \int_0^t \|\varphi_m\|_V^2 + \left(\frac{1}{2\varepsilon} + \frac{1}{2}\right) |D| \int_0^t |M_m|^4 \quad (3.3.55)$$

$$+2 \int_0^t \int_D \varphi_m \Phi(\varphi_m) dW(s) - 2 \int_0^t \int_D \Delta \varphi_m \Phi(\varphi_m) dW(s) + CT.$$

We raise both sides to the power of $\frac{p}{2}$ and we take the supremum then the expectation.

$$\begin{aligned} \mathbb{E} \sup_{t \in (0, T)} \|\varphi_m(t)\|_V^p &\leq C(p) [\|\varphi_0\|_V^p + T^{\frac{p-2}{2}} 5^{\frac{p}{2}} \mathbb{E} \int_0^T \|\varphi_m\|_V^p \\ &\quad + \mathbb{E} \sup_{t \in (0, T)} |2 \sum_{l=1}^{\infty} \int_0^t \int_D \varphi_m \Phi(\varphi_m) \sqrt{\lambda_l} e_l d\beta_l|^{\frac{p}{2}} \\ &\quad + \mathbb{E} \sup_{t \in (0, T)} |-2 \sum_{l=1}^{\infty} \int_0^t \int_D \Delta \varphi_m \Phi(\varphi_m) \sqrt{\lambda_l} e_l d\beta_l(s)|^{\frac{p}{2}} \\ &\quad + T^{\frac{p-2}{2}} (\frac{1}{2\varepsilon} + \frac{1}{2})^{\frac{p}{2}} |D|^{\frac{p}{2}} \mathbb{E} \int_0^T |M_m|^{2p} \\ &\quad + (CT)^{\frac{p}{2}}]. \end{aligned} \quad (3.3.56)$$

The third term on the right-hand-side of (3.3.56) can be computed as follows also using B urkholder inequality (3.3.28) with $r = \frac{p}{2}$, $X = \mathbb{R}$, $G_l = \int_D \nabla \varphi_m \nabla \{\Phi(\varphi_m) \sqrt{\lambda_l} e_l\}$

$$\begin{aligned} &2^{\frac{p}{2}} \mathbb{E} \left(\sup_{t \in (0, T)} \left| \sum_{l=1}^{\infty} \int_0^t \int_D \nabla \varphi_m \nabla \{\Phi(\varphi_m) \sqrt{\lambda_l} e_l\} d\beta_l \right|^{\frac{p}{2}} \right) \\ &= 2^{\frac{p}{2}} \mathbb{E} \left(\sup_{t \in (0, T)} \left| \sum_{l=1}^{\infty} \int_0^t \int_D \nabla \varphi_m \nabla \Phi_l(\varphi_m) d\beta_l \right|^{\frac{p}{2}} \right) \\ &\leq 2^{\frac{p}{2}} \tilde{C}_p \mathbb{E} \left(\int_0^T \sum_{l=1}^{\infty} \left\{ \int_D \nabla \varphi_m \nabla \Phi_l(\varphi_m) \right\}^2 dt \right)^{\frac{p}{4}} \\ &\leq 2^{\frac{p}{2}} \tilde{C}_p \mathbb{E} \left(\int_0^T \sum_{l=1}^{\infty} \|\nabla \varphi_m\|^2 \|\nabla \Phi_l(\varphi_m)\|^2 dt \right)^{\frac{p}{4}} \\ &\leq 2^{\frac{p}{2}} \tilde{C}_p \mathbb{E} \left(\int_0^T \|\nabla \varphi_m\|^2 \sum_{l=1}^{\infty} \|\nabla \Phi_l(\varphi_m)\|_{L^2(D)}^2 dt \right)^{\frac{p}{4}} \\ &\leq 2^{\frac{p}{2}} \tilde{C}_p \mathbb{E} \left(\int_0^T \|\nabla \varphi_m\|^2 C_2 \right)^{\frac{p}{4}} \\ &\leq 2^{\frac{p}{2}} \tilde{C}_p \mathbb{E} \left(\frac{1}{2} \int_0^T \|\nabla \varphi_m\|^4 + \frac{C_2^2}{2} T \right)^{\frac{p}{4}} \\ &\leq 2^{\frac{p}{2}} \tilde{\tilde{C}}_p \mathbb{E} \left(T^{\frac{p}{4}} C_2^{\frac{p}{2}} + \int_0^T \|\nabla \varphi_m\|^p \right) \\ &\leq 2^{\frac{p}{2}} \tilde{C}_p \mathbb{E} \int_0^T \|\nabla \varphi_m\|^p + C_p(T). \end{aligned} \quad (3.3.57)$$

Substituting (3.3.34), (3.3.42) and (3.3.57) in (3.3.56) we obtain

$$\mathbb{E} \sup_{t \in (0, T)} \|\varphi_m(t)\|_V^p \leq C(p) \|\varphi_0\|_V^p + 2^{\frac{p}{2}} \tilde{C}_p \mathbb{E} \int_0^T \|\varphi_m\|_V^p + \tilde{C}_p(T).$$

By Gronwall Lemma we deduce that

$$\mathbb{E} \|\varphi_m(t)\|_V^p \leq \tilde{C}.$$

This completes the proof of Lemma 3.3.8.

Before proving further estimates, we write below a projected form of Galerkin approximation and recall a standard identity.

To that purpose, we multiply (3.3.8) by w_j and sum from $j = 0, \dots, m$.

$$\begin{aligned} \sum_{j=0}^m \left(\int_D \varphi_m(t) w_j dx \right) w_j &= \sum_{j=0}^m \left(\int_D \varphi_m(0) w_j dx \right) w_j + \sum_{j=0}^m \int_0^t \left(\int_D \Delta \varphi_m w_j dx \right) w_j ds \\ &\quad + \sum_{j=0}^m \int_0^t \left(\int_D f(\varphi_m) w_j dx \right) w_j ds \\ &\quad - \frac{1}{|D|} \sum_{j=0}^m \int_0^t \int_D \left(\int_D f(\varphi_m) dx \right) w_j dx w_j ds \\ &\quad + \sum_{j=0}^m \sum_{l=1}^{\infty} \int_0^t \left(\int_D \Phi(\varphi_m) \sqrt{\lambda_l} e_l(x) w_j dx \right) w_j d\beta_l(s). \end{aligned} \quad (3.3.58)$$

We are led to the following equations

$$\begin{aligned} \varphi_m(t) &= \varphi_{0m} + \int_0^t \Delta \varphi_m + \int_0^t P_m f(\varphi_m) - \frac{1}{|D|} \int_0^t \int_D f(\varphi_m) \\ &\quad + \sum_{l=1}^{\infty} \int_0^t P_m \Phi_l(\varphi_m) d\beta_l(s) \end{aligned} \quad (3.3.59)$$

where P_m is the projection from $H \rightarrow H_m$, and where we have used the following result

Lemma 3.3.9 $P_m \Delta \varphi_m = \Delta \varphi_m$

Proof

$$\begin{aligned} P_m \Delta \varphi_m &= \sum_{j=0}^m \left(\int_D \Delta \varphi_m w_j \right) w_j = \sum_{j=0}^m \left(\int_D \varphi_m \Delta w_j \right) w_j \\ &= - \sum_{j=0}^m \left(\int_D \varphi_m \lambda_j w_j \right) w_j \\ &= - \sum_{j=0}^m \left(\int_D \varphi_m w_j \right) \lambda_j w_j \\ &= \Delta \left(\sum_{j=0}^m \left(\int_D \varphi_m w_j \right) w_j \right) \\ &= \Delta \varphi_m. \end{aligned} \quad \square$$

In what follows we derive further estimates in fractional Sobolev spaces.

Lemma 3.3.10 (*Estimates in fractional Sobolev spaces*) For each $\alpha \in [0, \frac{1}{2})$, $p \geq 2$, there exists a positive constant \tilde{K} such that

$$(i) \quad \mathbb{E} \left(\left\| \int_0^t P_m \Phi(\varphi_m) dW \right\|_{W^{\alpha,p}(0,T;L^2(D))}^p \right) \leq \tilde{K},$$

$$(ii) \quad \mathbb{E} \left(\left\| \varphi_m - \int_0^t P_m \Phi(\varphi_m) dW \right\|_{H^1(0,T;L^2(D))}^2 \right) \leq \tilde{K},$$

where $W^{\alpha,p}(0,T;L^2(D))$ is defined as in Definition 3.A.1 and \tilde{K} depends on the initial data and on p .

Proof Using variants of the Burkholder inequalities (3.3.27) and (3.3.28), for $p \geq 2$, $\alpha \in [0, \frac{1}{2})$ and for all X -value predictable $G \in L^2(\Omega; L^p((0,T); \mathcal{L}_{2,Q}(L^2(D), X)))$ we have [cf. [8] Lemma 2.1 and [16]]

$$\mathbb{E} \left(\left\| \int_0^t G dW \right\|_{W^{\alpha,p}(0,T;X)}^p \right) \leq c_p \mathbb{E} \left(\int_0^T \|G\|_{\mathcal{L}_{2,Q}(H,X)}^p dt \right), \quad (3.3.60)$$

or else

$$\mathbb{E} \left(\left\| \sum_{l=1}^{\infty} \int_0^t G_l d\beta_l \right\|_{W^{\alpha,p}(0,T;X)}^p \right) \leq c_p \mathbb{E} \left(\int_0^T \sum_{l=1}^{\infty} \|G_l\|_X^p dt \right).$$

(i) Applying (3.3.60) with $G = P_m(\Phi(\varphi_m)\sqrt{\lambda_l}e_l) = P_m\Phi_l$, $X = L^2(D)$, using (3.1.11) and Lemma 3.3.8 we obtain

$$\begin{aligned} \mathbb{E} \left(\left\| \int_0^t P_m \Phi(\varphi_m) dW \right\|_{W^{\alpha,p}(0,T;L^2(D))}^p \right) &\leq c_p \mathbb{E} \left(\int_0^T \|P_m \Phi(\varphi_m)\|_{\mathcal{L}_{2,Q}(H,L^2(D))}^p dt \right) \\ &\leq c_p \mathbb{E} \left(\int_0^T \sum_{l=1}^{\infty} \|P_m \Phi(\varphi_m)\sqrt{\lambda_l}e_l\|^p dt \right) \\ &\leq c_p \mathbb{E} \left(\int_0^T \sum_{l=1}^{\infty} \|\Phi_l(\varphi_m)\|_V^p dt \right) \\ &\leq c_p C_3 \mathbb{E} \left(\int_0^T (1 + \|\varphi_m\|_V^p) dt \right) \\ &\leq \tilde{\mathcal{K}}. \end{aligned}$$

where we have used the fact that $\|P_m a\| \leq \|a\|$, $a \in L^2(D)$.

(ii) Using the definition of the $W^{\alpha,p}$ norm (3.A.1) and (3.3.59) we obtain

$$\mathbb{E} \left(\left\| \varphi_m - \int_0^t P_m \Phi(\varphi_m(s)) dW(s) \right\|_{H^1(0,T;L^2(D))}^2 \right)$$

$$\begin{aligned}
&= \mathbb{E} \left(\|\varphi_{0m} + \int_0^t \Delta\varphi_m(s)ds + \int_0^t P_m f(\varphi_m(s))ds \right. \\
&\quad \left. - \frac{1}{|D|} \int_0^t \int_D f(\varphi_m(s))dx ds \right\|_{H^1(0,T;L^2(D))}^2 \\
&= \mathbb{E} \left(\int_0^T \|\varphi_{0m} + \int_0^t \Delta\varphi_m(s)ds + \int_0^t P_m f(\varphi_m(s))ds \right. \\
&\quad \left. - \frac{1}{|D|} \int_0^t \int_D f(\varphi_m(s))dx ds \right\|^2 dt \\
&\quad + \int_0^T \left\| \frac{d}{dt} [\varphi_{0m} + \int_0^t \Delta\varphi_m(s)ds + \int_0^t P_m f(\varphi_m(s))ds \right. \\
&\quad \left. - \frac{1}{|D|} \int_0^t \int_D f(\varphi_m(s))dx ds \right\|^2 dt \\
&= \mathbb{E} \left(\int_0^T \|\varphi_{0m} + \int_0^t \Delta\varphi_m(s)ds + \int_0^t P_m f(\varphi_m(s))ds \right. \\
&\quad \left. - \frac{1}{|D|} \int_0^t \int_D f(\varphi_m(s))dx ds \right\|^2 dt \\
&\quad + \int_0^T \left\| \Delta\varphi_m(t) + P_m f(\varphi_m(t)) - \frac{1}{|D|} \int_D f(\varphi_m(t))dx \right\|^2 dt \\
&\leq c\mathbb{E} \int_0^T \|\varphi_{0m}\|_H^2 dt + c\mathbb{E} \int_0^T t \int_0^t \|\Delta\varphi_m(s) + P_m f(\varphi_m(s)) \\
&\quad - \frac{1}{|D|} \int_D f(\varphi_m(t))dx\|^2 ds dt \\
&\quad + \mathbb{E} \int_0^T \|\Delta\varphi_m(t) + P_m f(\varphi_m(t)) - \frac{1}{|D|} \int_D f(\varphi_m(t))dx\|^2 dt \\
&\leq c\mathbb{E} \int_0^T \|\varphi_{0m}\|_H^2 dt + cT^2 \mathbb{E} \int_0^T \|\Delta\varphi_m(t) + P_m f(\varphi_m(t)) \\
&\quad - \frac{1}{|D|} \int_D f(\varphi_m(t))dx\|^2 dt \\
&\quad + \mathbb{E} \int_0^T \left\| \Delta\varphi_m(t) + P_m f(\varphi_m(t)) - \frac{1}{|D|} \int_D f(\varphi_m(t))dx \right\|^2 dt \\
&\leq \mathbb{E} \left(cT\|\varphi_0\|^2 + (1+cT^2) \int_0^T \|\Delta\varphi_m + P_m f(\varphi_m) - \frac{1}{|D|} \int_D f(\varphi_m(t))dx\|^2 dt \right) \\
&\leq \tilde{c}(T) \mathbb{E} \left(\|\varphi_0\|^2 + \int_0^T \|\Delta\varphi_m\|^2 dt + \int_0^T \|P_m(\varphi_m^3)\|_H^2 dt + \int_0^T \|\varphi_m\|^2 \right. \\
&\quad \left. + \int_0^T \|f(\varphi_m)\|^2 \right) \\
&\leq \tilde{c}(T) \mathbb{E} \left(\|\varphi_0\|^2 + \int_0^T \|\Delta\varphi_m\|^2 dt + \int_0^T \|\varphi_m^3\|^2 dt + \int_0^T \|\varphi_m\|^2 \right. \\
&\quad \left. + 2 \int_0^T \|\varphi_m^3\|^2 dt + 2 \int_0^T \|\varphi_m\|^2 \right)
\end{aligned}$$

$$\leq \tilde{c}(T) \mathbb{E} \left(\|\varphi_0\|^2 + \int_0^T \|\Delta\varphi_m\|^2 dt + 3 \int_0^T \|\varphi_m\|_{L^6(D)}^6 dt + 3 \int_0^T \|\varphi_m\|^2 dt \right) \leq \tilde{K},$$

where we have used the a priori estimates (3.3.3), (3.3.4) in Lemma 3.3.1 and the fact that $H^2(D)$ is embedded in $L^6(D)$ for $d \leq 6$. \square

3.4 Compactness Arguments

We consider the spaces:

$$\begin{aligned} \mathcal{X}_S &= L^2(0, T; V) \cap \mathcal{C}([0, T]; V'), \mathcal{X}_W = \mathcal{C}([0, T]; H), \text{ and} \\ \mathcal{X} &= \mathcal{X}_S \times \mathcal{X}_W. \end{aligned} \quad (3.4.1)$$

We then define the probability measures

$$\mu_{\varphi_m}(\cdot) = \mathbb{P}(\varphi_m \in \cdot), \quad (3.4.2)$$

$$\mu_W^m(\cdot) = \mu_W(\cdot) = \mathbb{P}(W \in \cdot), \quad (3.4.3)$$

namely $\mu_{\varphi_m}(A) = \mathbb{P}(\varphi_m \in A)$, for all $A \subset \mathcal{X}_S$, and $\mu_W(B) = \mathbb{P}(W \in B)$, for all $B \subset \mathcal{X}_W$. This defines a sequence of probability measures

$$\mu^m = \mu_{\varphi_m} \times \mu_W, \quad (3.4.4)$$

on the phase space \mathcal{X} . We are going to show the tightness of μ^m on \mathcal{X} .

Lemma 3.4.1 *The sequence μ^m is tight over \mathcal{X} and hence weakly compact in \mathcal{X} .*

Proof We start by stating compactness results.

Lemma 3.4.2 (i) *For $R > 1$, we define the set*

$$\begin{aligned} B_R^1 &:= \{u \in L^2(0, T; H^2(D)) \cap W^{\frac{1}{5}, 2}(0, T; L^2(D)) : \\ &\quad \|u\|_{L^2(0, T; H^2(D))}^2 + \|u\|_{W^{\frac{1}{5}, 2}(0, T; L^2(D))}^2 \leq R^2\}, \end{aligned}$$

where $W^{\frac{1}{5}, 2}$ is defined as in Definition 3.A.1.

The ball B_R^1 is compact in $L^2(0, T; H^1(D))$.

(ii) *Let $B_R^{2,1}$ and $B_R^{2,2}$ be the balls of radius $R > 1$ in $H^1(0, T; L^2(D))$ and $W^{\alpha, p}(0, T; L^2(D))$ with $p > 2$ and $\alpha p > 1$, respectively. Then*

$$B_{2R}^2 := B_R^{2,1} + B_R^{2,2} \quad \text{is compact in } \mathcal{C}([0, T]; (H^1(D))').$$

(iii) $B_R^1 \cap B_{2R}^2$ is compact in $L^2(0, T; H^1(D)) \cap \mathcal{C}([0, T]; (H^1(D))')$.

Proof (i) By applying Lemma 3.A.2 in the appendix with choosing $\mathcal{E}_0 = H^2(D)$, $\mathcal{E} = H^1(D)$ and $\mathcal{E}_1 = L^2(D)$, we find that the embedding from $L^2(0, T; H^2(D)) \cap W^{\frac{1}{5}, 2}(0, T; L^2(D))$ into $L^2(0, T; H^1(D))$ is compact.

(ii) We first remark that the embedding

$$H^1(0, T; H) \subset \mathcal{C}([0, T]; V') \quad (3.4.5)$$

is compact. Indeed, since

$$H^1(0, T; H) = \{u \in L^2(0, T; H); u_t \in L^2(0, T; H)\},$$

we have that also

$$H^1(0, T; H) = \{u \in L^\infty(0, T; H); u_t \in L^2(0, T; H)\}.$$

We then apply the Aubin-Lions-Simon Theorem II.5.16 as stated by Boyer and Fabrie [2] p.102 with $p = +\infty$ and $r = 2$ and $B_0 = L^2(D)$, and $B_1 = B_2 = (H^1(D))'$, and remark that $H^1(0, T; H) \subset E_{\infty, 2}$ where $E_{\infty, 2}$ corresponds to the space $E_{p, r}$ from [[2], Theorem II.5.16 p.102]. This theorem then implies (3.4.5).

Moreover, for $p > 2$ we choose α such that $\alpha p > 1$. We deduce from Lemma 3.A.3 that the embeddings

$$W^{\alpha, p}(0, T; H) \subset \mathcal{C}([0, T]; V'), \quad (3.4.6)$$

is compact.

Next, we prove that the sum of the two compact balls $B_R^{2,1}$ and $B_R^{2,2}$ is compact in $\mathcal{C}([0, T]; V')$. Indeed, let $\{a_n\} = \{b_n + c_n\}$, with $b_n \in B_R^{2,1}$ and $c_n \in B_R^{2,2}$, be arbitrary and suppose that there exists a positive constant \mathcal{C} such that $\|a_n\|_{B_{2R}^2} \leq \mathcal{C}$, where the norm in B_{2R}^2 is defined as follows: if $A = B + C$, $\|A\|_{B_{2R}^2} = \|B\|_{H^1(0, T; H)} + \|C\|_{W^{\alpha, p}(0, T; H)}$.

Next we show that there exists a subsequence of $\{a_n\}$ and a function $a \in \mathcal{C}([0, T]; V')$ such that $a_n \rightarrow a \in \mathcal{C}([0, T]; V')$ as $n \rightarrow \infty$. Indeed, since $\{b_n\}$ is bounded in $H^1(0, T; H)$, there exists a subsequence $\{b_{n_m}\}$ and a function b such that $b_{n_m} \rightarrow b \in \mathcal{C}([0, T]; V')$ as $n_m \rightarrow \infty$. Similarly, there exists a subsequence of $\{c_{n_m}\}$ which we denote again by $\{c_{n_m}\}$ and a function c such that $c_{n_m} \rightarrow c \in \mathcal{C}([0, T]; V')$ as $n_m \rightarrow \infty$. Thus,

$$a_{n_m} = b_{n_m} + c_{n_m} \rightarrow a = b + c \in \mathcal{C}([0, T]; V') \text{ as } n_m \rightarrow \infty.$$

(iii) The result follows from (i) and (ii). \square

Next we prove Lemma 3.4.1, the proof is done into 4 steps computing the measures of the complements of B_R^1 , B_{2R}^2 and $B_R^1 \cap B_{2R}^2$.

Step 1: We apply Lemma 3.4.2 (i), Lemma 3.3.1, Lemma 3.3.10, (3.A.2) and the Chebychev inequality (Lemma 3.A.4) to deduce that

$$\begin{aligned} \mu_{\varphi_m}((B_R^1)^c) &= \mathbb{P}(\varphi_m \in (B_R^1)^c) \\ &= \mathbb{P}\left(\|\varphi_m\|_{L^2(0, T; H^2(D))}^2 + \|\varphi_m\|_{W^{\frac{1}{5}, 2}(0, T; H)}^2 > R^2\right) \\ &\leq \mathbb{P}\left(\|\varphi_m\|_{L^2(0, T; H^2(D))}^2 > \frac{R^2}{2}\right) + \mathbb{P}\left(\|\varphi_m\|_{W^{\frac{1}{5}, 2}(0, T; H)}^2 > \frac{R^2}{2}\right) \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{2}{R^2} \mathbb{E} \left(\|\varphi_m\|_{L^2(0,T,H^2(D))}^2 \right) + \frac{2}{R^2} \mathbb{E} \left(\|\varphi_m\|_{W^{\frac{1}{5},2}(0,T;H)}^2 \right) \\
 &\leq \frac{2}{R^2} \mathbb{E} \left(\|\varphi_m\|_{L^2(0,T,H^2(D))}^2 \right) + \frac{4}{R^2} \mathbb{E} \left(\left\| \varphi_m - \int_0^t P_m \Phi(\varphi_m) dW \right\|_{W^{\frac{1}{5},2}(0,T;H)}^2 \right) \\
 &\quad + \frac{4}{R^2} \mathbb{E} \left(\left\| \int_0^t P_m \Phi(\varphi_m) dW \right\|_{W^{\frac{1}{5},2}(0,T;H)}^2 \right) \\
 &\leq \frac{2}{R^2} \mathbb{E} \left(\|\varphi_m\|_{L^2(0,T,H^2(D))}^2 \right) + \frac{4C}{R^2} \mathbb{E} \left(\left\| \varphi_m - \int_0^t P_m \Phi(\varphi_m) dW \right\|_{H^1(0,T;H)}^2 \right) \\
 &\quad + \frac{4}{R^2} \mathbb{E} \left(\left\| \int_0^t P_m \Phi(\varphi_m) dW \right\|_{W^{\frac{1}{5},2}(0,T;H)}^2 \right) \\
 &\leq \frac{k}{R^2}. \tag{3.4.7}
 \end{aligned}$$

Step 2: Observe that in view of Lemma 3.4.2 (ii) Lemma 3.3.10 (i) and (iii), we have that for all $R > \max(1, \tilde{K})$

$$\left\{ \varphi_m - \int_0^t P_m \Phi(\varphi_m) dW \in B_R^{2,1} \right\} \cap \left\{ \int_0^t P_m \Phi(\varphi_m) dW \in B_R^{2,2} \right\} \subset \left\{ \varphi_m \in B_{2R}^2 \right\}.$$

By De Morgan's law $(A \cap B)^c = A^c \cup B^c$, the Tchebychev inequality Lemma 3.A.4 and Lemma 3.3.10, we infer that

$$\begin{aligned}
 \mu_{\varphi_m}((B_{2R}^2)^c) &= \mathbb{P}(\varphi_m \in (B_{2R}^2)^c) \\
 &\leq \mathbb{P} \left(\left\| \varphi_m - \int_0^t P_m \Phi(\varphi_m) dW \right\|_{H^1(0,T;H)}^2 > R^2 \right) \\
 &\quad + \mathbb{P} \left(\left\| \int_0^t P_m \Phi(\varphi_m) dW \right\|_{W^{\alpha,p}(0,T;H)}^p > R^p \right) \\
 &\leq \frac{1}{R^2} \mathbb{E} \left(\left\| \varphi_m - \int_0^t P_m \Phi(\varphi_m) dW \right\|_{H^1(0,T;H)}^2 \right) \\
 &\quad + \frac{1}{R^p} \mathbb{E} \left(\left\| \int_0^t P_m \Phi(\varphi_m) dW \right\|_{W^{\alpha,p}(0,T;H)}^p \right) \\
 &\leq \frac{k}{R^2}, \tag{3.4.8}
 \end{aligned}$$

where k is a positive constant which does not depend on m .

Step 3: Using Lemma 3.4.2 (iii), (3.4.7) and (3.4.8)

$$\begin{aligned}
 \mu_{\varphi_m}((B_R^1 \cap B_{2R}^2)^c) &= \mu_{\varphi_m}((B_R^1)^c \cup (B_{2R}^2)^c) \\
 &\leq \mathbb{P}(\varphi_m \in (B_R^1)^c) + \mathbb{P}(\varphi_m \in (B_{2R}^2)^c) \\
 &\leq \frac{2k}{R^2}. \tag{3.4.9}
 \end{aligned}$$

In order to establish the tightness of μ^m over \mathcal{X} , we suppose that $\varepsilon \in (0, 1)$ is arbitrary and define

$$R^\varepsilon = \left(\frac{2\tilde{k}}{1 - \sqrt{1 - \varepsilon}} \right)^{\frac{1}{2}},$$

with $\tilde{k} = \max(k, \frac{1}{2})$, so that $R^\varepsilon \geq 1$.

Substituting $R = R^\varepsilon$ in (3.4.9) and performing straightforward calculations in the resulting inequality, we deduce that

$$\begin{aligned} \mu_{\varphi_m}(Z_\varepsilon^c) &= \mu_{\varphi_m}((B_{R^\varepsilon}^1 \cap B_{2R^\varepsilon}^2)^c) \\ &\leq \frac{2k}{\frac{2\tilde{k}}{1 - \sqrt{1 - \varepsilon}}} \leq 1 - \sqrt{1 - \varepsilon}. \end{aligned}$$

where $Z_\varepsilon = B_{R^\varepsilon}^1 \cap B_{2R^\varepsilon}^2$. Thus,

$$\mu_{\varphi_m}(Z_\varepsilon) = 1 - \mathbb{P}(\varphi_m \in Z_\varepsilon^c) = 1 - \mu_{\varphi_m}(Z_\varepsilon^c) \geq \sqrt{1 - \varepsilon}. \quad (3.4.10)$$

Next we remark that the sequence $\{\mu_W^m\}$ is reduced to one element and is thus weakly compact in \mathcal{X}_W . Hence, as a consequence of Proposition 3.A.6, $\{\mu_W^m\}$ must be tight. Thus there exists a compact set $V_\varepsilon \subset \mathcal{X}_W$ such that

$$\mu_W^m(V_\varepsilon) = \mu_W(V_\varepsilon) \geq 1 - (1 - \sqrt{1 - \varepsilon}) \geq \sqrt{1 - \varepsilon}. \quad (3.4.11)$$

Finally, setting $K_\varepsilon = Z_\varepsilon \times V_\varepsilon$, which is uniformly compact in \mathcal{X} due to Tychonoff's Theorem (Theorem 3.A.7), we deduce from (3.4.10) and (3.4.11) that

$$\mu^m(K_\varepsilon) = \mu_{\varphi_m}(Z_\varepsilon) \cdot \mu_W(V_\varepsilon) \geq 1 - \varepsilon, \quad \text{for all } m. \quad (3.4.12)$$

Since ε is arbitrary the inequality (3.4.12) proves that the sequence μ^m is tight on \mathcal{X} by Definition 3.A.5. \square

3.5 The limit as m_k tends to infinity

From the tightness property and Prokhorov's Theorem (Proposition 3.A.6), there exists a subsequence μ^{m_k} such that $\mu^{m_k} \rightharpoonup \mu$ weakly where μ is a probability measure on \mathcal{X} . We associate the corresponding distribution to the approximate solution of the Galerkin scheme by stating the following proposition:

Proposition 3.5.1 *Given a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$; let φ_m be the sequence defined above or a similar sequence. Then there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, a subsequence $m_k \rightarrow \infty$ and a sequence of \mathcal{X} -valued random variables $(\tilde{\varphi}_{m_k}, \tilde{W}^{m_k})$ such that :*

- (i) *The probability law of $(\tilde{\varphi}_{m_k}, \tilde{W}^{m_k})$ is μ^{m_k} .*
- (ii) *The probability law of $(\tilde{\varphi}, \tilde{W})$ is μ .*
- (iii) *$(\tilde{\varphi}_{m_k}, \tilde{W}^{m_k})$ converges almost surely in the topology of \mathcal{X} to an element $(\tilde{\varphi}, \tilde{W})$ i.e.*

$$\tilde{\varphi}_{m_k} \rightarrow \tilde{\varphi} \quad \text{in } L^2(0, T; V) \cap \mathcal{C}([0, T]; V') \quad \text{a.s.}, \quad (3.5.1)$$

$$\tilde{W}^{m_k} \rightarrow \tilde{W} \quad \text{in } \mathcal{C}([0, T]; H) \quad \text{a.s.} \quad (3.5.2)$$

(iv) Let $\tilde{\mathcal{F}}_t^{m_k} := \sigma(\tilde{W}^{m_k}(s), \tilde{\varphi}_{m_k}, s \leq t)$ which is the union of σ -algebras generated by a random variable $(\tilde{\varphi}_{m_k}, \tilde{W}^{m_k})$, then each \tilde{W}^{m_k} is a Q -Brownian motion process with respect to the filtration $\tilde{\mathcal{F}}_t^{m_k}$.

(v) Each $(\tilde{\varphi}_{m_k}, \tilde{W}^{m_k})$ satisfies $\tilde{\mathbb{P}}$ - a.s.

$$\begin{aligned} \int_D \tilde{\varphi}_{m_k}(t)w_j - \int_D \varphi_{0,m_k}(x)w_j + \int_0^t \int_D \nabla \tilde{\varphi}_{m_k} \nabla w_j &= \int_0^t \int_D f(\tilde{\varphi}_{m_k})w_j \\ &- \frac{1}{|D|} \int_0^t \int_D \left\{ \int_D f(\tilde{\varphi}_{m_k}) \right\} w_j + \int_0^t \int_D \Phi(\tilde{\varphi}_{m_k})w_j d\tilde{W}^{m_k}, \end{aligned} \quad (3.5.3)$$

Let $\tilde{\mathcal{F}}_t := \sigma(\tilde{W}(s), \tilde{\varphi}, s \leq t)$, then $((\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}}), \tilde{\varphi}, \tilde{W})$ is a strong martingale solution of (P).

Proof We have proved in Lemma 3.4.1 that the sequence of measures $(\mu^m)_{m \geq 1}$ associated with the approximation scheme (φ_m, W) is weakly compact in \mathcal{X} . Thus, by a direct application of the Skorohod Theorem Proposition 3.A.8, (i), (ii) and (iii) are proved.

For the proof of (v) we define the following processes

$$\begin{aligned} \mathcal{M}_m(t) &:= \int_D \varphi_m(t)w_j - \int_D \varphi_m(0)w_j + \int_0^t \int_D \nabla \varphi_m \nabla w_j \\ &- \int_0^t \int_D f(\varphi_m)w_j + \frac{1}{|D|} \int_0^t \int_D \left\{ \int_D f(\varphi_m) \right\} w_j dx. \end{aligned}$$

$$\begin{aligned} \tilde{\mathcal{M}}_{m_k}(t) &:= \int_D \tilde{\varphi}_{m_k}(t)w_j - \int_D \varphi_{0,m_k} w_j + \int_0^t \int_D \nabla \tilde{\varphi}_{m_k} \nabla w_j - \int_0^t \int_D f(\tilde{\varphi}_{m_k})w_j \\ &+ \frac{1}{|D|} \int_0^t \int_D \left\{ \int_D f(\tilde{\varphi}_{m_k}) \right\} w_j dx. \end{aligned}$$

Lemma 3.5.2

$$\tilde{\mathcal{M}}_{m_k}(t), \tilde{\mathcal{M}}_{m_k}^2(t) - \sum_{l=1}^{\infty} \int_0^t \left(\int_D \Phi_l(\tilde{\varphi}_{m_k})w_j \right)^2 ds, \tilde{\mathcal{M}}_{m_k}(t)\tilde{\beta}_l(t) - \int_0^t \Phi_l w_j ds$$

are $(\tilde{\mathcal{F}}_t)$ -martingales.

We first show the following result. For a Banach space X we define the restriction operator

$$\rho_t : C([0, T]; X) \rightarrow C([0, t]; X), \quad \rho_t f = f|_{[0, t]}.$$

In order to prove this result, let us fix times, $s, t, \in [0, T]$ such that $s < t$ and let

$$h : \mathcal{X}_S|_{[0, s]} \times \mathcal{X}_W|_{[0, s]} \rightarrow [0, 1]$$

be a continuous function.

Since

$$\mathcal{M}_m(t) = \sum_{l=1}^{\infty} \int_0^t \int_D \Phi_l(\varphi_m) w_j d\beta_l,$$

it follows that the processes

$$\mathcal{M}_m(t), \mathcal{M}_m^2(t) - \sum_{l=1}^{\infty} \int_0^t \left(\int_D \Phi_l w_j \right)^2 ds, \mathcal{M}_m(t) \beta_l(t) - \int_0^t \Phi_l w_j ds$$

are (\mathcal{F}_t) -martingales ([14] p.26) and hence, for all $t > s$, there holds

$$\mathbb{E}[h(\rho_s \varphi_m, \rho_s W)(\mathcal{M}_m(t) - \mathcal{M}_m(s))] = 0, \quad (3.5.4)$$

$$\begin{aligned} \mathbb{E}[h(\rho_s \varphi_m, \rho_s W)(\mathcal{M}_m^2(t) - \sum_{l=1}^{\infty} \int_s^t \left(\int_D \Phi_l(\varphi_m(s)) w_j \right)^2 \\ - \mathcal{M}_m^2(s) + \sum_{l=1}^{\infty} \int_s^t \left(\int_D \Phi_l(\varphi_m(s)) w_j \right)^2)] = 0, \end{aligned} \quad (3.5.5)$$

$$\begin{aligned} \mathbb{E}[h(\rho_s \varphi_m, \rho_s W)(\mathcal{M}_m(t) \beta_l(t) - \int_s^t \int_D \Phi_l(\varphi_m(s)) w_j \\ - \mathcal{M}_m(s) \beta_l(s) + \int_s^t \int_D \Phi_l(\varphi_m(s)) w_j)] = 0. \end{aligned} \quad (3.5.6)$$

Using the equality of laws which means that we have that

$$\tilde{\mathbb{E}}[h(\rho_s \tilde{\varphi}_{m_k}, \rho_s \tilde{W}^{m_k})(\tilde{\mathcal{M}}_{m_k}(t) - \tilde{\mathcal{M}}_{m_k}(s))] = 0, \quad (3.5.7)$$

$$\begin{aligned} \tilde{\mathbb{E}}[h(\rho_s \tilde{\varphi}_{m_k}, \rho_s \tilde{W}^{m_k})(\tilde{\mathcal{M}}_{m_k}^2(t) - \sum_{l=1}^{\infty} \int_s^t \left(\int_D \Phi_l(\tilde{\varphi}_{m_k}(s)) w_j \right)^2 \\ - \tilde{\mathcal{M}}_{m_k}^2(s) + \sum_{l=1}^{\infty} \int_s^t \left(\int_D \Phi_l(\tilde{\varphi}_{m_k}(s)) w_j \right)^2)] = 0, \end{aligned} \quad (3.5.8)$$

$$\begin{aligned} \tilde{\mathbb{E}}[h(\rho_s \tilde{\varphi}_{m_k}, \rho_s \tilde{W}^{m_k})(\tilde{\mathcal{M}}_{m_k}(t) \tilde{\beta}_l - \int_s^t \int_D \Phi_l(\tilde{\varphi}_{m_k}(s)) w_j \\ - \tilde{\mathcal{M}}_{m_k}(s) \tilde{\beta}_l + \int_s^t \int_D \Phi_l(\tilde{\varphi}_{m_k}(s)) w_j)] = 0. \end{aligned} \quad (3.5.9) \quad \square$$

This completes the proof of Lemma 3.5.2.

Thus,

$$\tilde{\mathcal{M}}_{m_k}(t) = \int_0^t \int_D \Phi(\tilde{\varphi}_{m_k}) w_j d\tilde{W}^{m_k}.$$

Next we show that $((\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}}), \tilde{\varphi}, \tilde{W})$ is indeed a strong martingale solution. We will use that all uniform estimates for φ_m are valid for $\tilde{\varphi}_{m_k}$. Hence, $\tilde{\varphi}_{m_k}$ belongs to a bounded set of $L^2(\tilde{\Omega}, L^\infty(0, T; V)) \cap L^2(\tilde{\Omega}, L^2(0, T; H^2(D)))$. Thus there exists a subsequence which we denote again by m_k and a function $\tilde{\varphi} \in L^2(\tilde{\Omega}, L^2(0, T; H^2(D))) \cap L^2(\tilde{\Omega}, L^\infty(0, T; H^1(D)))$ such that :

$$\tilde{\varphi}_{m_k} \rightharpoonup \tilde{\varphi} \text{ weakly in } L^2(\tilde{\Omega}, L^2(0, T; H^2(D))), \quad (3.5.10)$$

$$\tilde{\varphi}_{m_k} \rightharpoonup \tilde{\varphi} \text{ weak-star in } L^2(\tilde{\Omega}, L^\infty(0, T; H^1(D))). \quad (3.5.11)$$

Since from (3.5.1), $\tilde{\varphi}_{m_k} \rightarrow \tilde{\varphi}$ in $L^2(0, T; V)$ a.s., from (3.3.40) we deduce that

$$\mathbb{E} \left[\left(\int_0^T \|\tilde{\varphi}_{m_k}(t)\|_V^2 \right)^2 \right] \leq T^2 \mathbb{E} \sup_{t \in (0, T)} \|\tilde{\varphi}_{m_k}\|_V^4 \leq K.$$

By Lemma 3.A.15 which is deduce from the Vitali theorem (Lemma 3.A.14) we conclude that

$$\tilde{\varphi}_{m_k} \rightarrow \tilde{\varphi} \text{ strongly in } L^2(\tilde{\Omega}; L^2((0, T); V)). \quad (3.5.12)$$

Finally we conclude that there exists a subsequence of $\{\tilde{\varphi}_{m_k}\}$ which we denote again by $\{\tilde{\varphi}_{m_k}\}$ such that

$$\|\tilde{\varphi}_{m_k} - \tilde{\varphi}\|_V^2 \rightarrow 0 \text{ for a.e. } (\omega, t) \in \tilde{\Omega} \times [0, T]. \quad (3.5.13)$$

Let y be an arbitrary bounded random variable, and let ψ be an arbitrary bounded function on $(0, T)$. Next we multiply the equation (3.5.3) by the product $y\psi$ and we integrate from 0 to T , we take the expectation and remark that the linear combinations of w_j are dense in $V \cap L^4(D)$, to obtain

$$\begin{aligned} & \tilde{\mathbb{E}} \int_0^T y\psi(t) \int_D \tilde{\varphi}_{m_k}(t) w - \tilde{\mathbb{E}} \int_0^T y\psi(t) \int_D \varphi_{0, m_k} w \\ &= \tilde{\mathbb{E}} \int_0^T y\psi(t) \int_0^t \int_D \Delta \tilde{\varphi}_{m_k} w + \tilde{\mathbb{E}} \int_0^T y\psi(t) \int_0^t \int_D f(\tilde{\varphi}_{m_k}) w \\ & - \frac{1}{|D|} \tilde{\mathbb{E}} \int_0^T y\psi(t) \int_0^t \int_D \left\{ \int_D f(\tilde{\varphi}_{m_k}) \right\} w + \tilde{\mathbb{E}} \int_0^T y\psi(t) \int_0^t \int_D \Phi(\tilde{\varphi}_{m_k}) w d\tilde{W}^{m_k}, \end{aligned} \quad (3.5.14)$$

for all $w \in V \cap L^4(D)$.

3.5.1 Convergence of the elliptic term

Applying the Hölder inequality, we obtain

$$\begin{aligned} |\tilde{\mathbb{E}} \int_0^T y\psi(t) \int_0^t \langle \Delta \tilde{\varphi}_{m_k} - \Delta \tilde{\varphi}, w \rangle| &\leq |\tilde{\mathbb{E}} \int_0^T y\psi(t) \int_0^t \int_D \nabla w \nabla (\tilde{\varphi}_{m_k} - \tilde{\varphi})| \\ &\leq \|y\|_{L^\infty(\Omega)} \|\psi(t)\|_{L^\infty(0, T)} T \tilde{\mathbb{E}} \int_0^T \int_D |\nabla w \nabla (\tilde{\varphi}_{m_k} - \tilde{\varphi})| \\ &\leq cT \left(\tilde{\mathbb{E}} \int_0^T \int_D (\nabla w)^2 \right)^{\frac{1}{2}} \left(\tilde{\mathbb{E}} \int_0^T \int_D (\nabla (\tilde{\varphi}_{m_k} - \tilde{\varphi}))^2 \right)^{\frac{1}{2}} \\ &\leq \tilde{c}T \left(\tilde{\mathbb{E}} \int_0^T \int_D (\nabla (\tilde{\varphi}_{m_k} - \tilde{\varphi}))^2 \right)^{\frac{1}{2}}, \end{aligned}$$

which in view of (3.5.12), tends to zero as m_k tends to infinity.

3.5.2 Convergence of the reaction term and the nonlocal term

Since by (3.5.12) $\tilde{\varphi}_{m_k} \rightarrow \tilde{\varphi}$ dans $L^2(\tilde{\Omega}; L^2((0, T); V))$, it follows that there exists a subsequence of $\{\tilde{\varphi}_{m_k}\}$ which we denote again by $\{\tilde{\varphi}_{m_k}\}$ such that

$$\tilde{\varphi}_{m_k} \rightarrow \tilde{\varphi} \text{ a.e. in } \tilde{\Omega} \times (0, T) \times D.$$

Moreover it follows from the continuity of f that,

$$f(\tilde{\varphi}_{m_k}) \rightarrow f(\tilde{\varphi}) \text{ a.e. in } \tilde{\Omega} \times (0, T) \times D.$$

In addition, using (3.5.10) and the Sobolev embedding of $H^2(D) \subset L^6(D)$ for $d = 6$,

$$\begin{aligned} \mathbb{E} \|f(\tilde{\varphi}_{m_k})\|_{L^2(D \times (0, T))}^2 &= \mathbb{E} \int_0^T \int_D |f(\tilde{\varphi}_{m_k})|^2 dx dt \\ &= \mathbb{E} \int_0^T \int_D |\tilde{\varphi}_{m_k} - \tilde{\varphi}_{m_k}^3|^2 dx dt \\ &\leq c_2 \mathbb{E} \int_0^T \int_D (|\tilde{\varphi}_{m_k}|^2 + |\tilde{\varphi}_{m_k}|^6) dx dt \\ &\leq c_2 \mathbb{E} \int_0^T \|\tilde{\varphi}_{m_k}\|^2 + c_2 \mathbb{E} \int_0^T \|\tilde{\varphi}_{m_k}\|_{L^6(D)}^6 \\ &\leq K. \end{aligned} \tag{3.5.15}$$

Thus $\{f(\tilde{\varphi}_{m_k})\}_{m_k \in \mathbb{N}}$ is bounded in $L^2(\tilde{\Omega} \times (0, T) \times D)$. Applying Lemma 3.A.15 we deduce that

$$f(\tilde{\varphi}_{m_k}) \rightarrow f(\tilde{\varphi}) \text{ in } L^r(\tilde{\Omega} \times (0, T) \times D), \text{ for all } 1 \leq r < 2.$$

In particular we deduce that

$$f(\tilde{\varphi}_{m_k}) \rightarrow f(\tilde{\varphi}) \text{ in } L^{\frac{4}{3}}(\tilde{\Omega} \times (0, T) \times D). \tag{3.5.16}$$

By Hölder inequality and (3.5.16)

$$\begin{aligned} &\left| \tilde{\mathbb{E}} \int_0^T y \psi(t) \int_0^t \int_D (f(\tilde{\varphi}_{m_k}) - f(\tilde{\varphi})) w \right| \\ &\leq \|y\|_{L^\infty(\Omega)} \|\psi(t)\|_{L^\infty(0, T)} T \left(\tilde{\mathbb{E}} \int_0^T \int_D (f(\tilde{\varphi}_{m_k}) - f(\tilde{\varphi}))^{\frac{4}{3}} \right)^{\frac{3}{4}} \left(\tilde{\mathbb{E}} \int_0^T \int_D w^4 dx ds \right)^{\frac{1}{4}} \\ &\leq c(T) \left(\tilde{\mathbb{E}} \int_0^T \int_D (f(\tilde{\varphi}_{m_k}) - f(\tilde{\varphi}))^{\frac{4}{3}} \right)^{\frac{3}{4}}, \end{aligned} \tag{3.5.17}$$

which tends to zero as $m_k \rightarrow \infty$.

Next we estimate the nonlocal term; to that purpose we apply Hölder inequality and use (3.5.16)

$$\begin{aligned}
 & \left| \tilde{\mathbb{E}} \int_0^T y\psi(t) \int_0^t \int_D \left(\int_D f(\tilde{\varphi}_{m_k}) - \int_D f(\tilde{\varphi}) \right) w dx ds dt \right| \\
 & \leq \|y\|_{L^\infty(\Omega)} \|\psi(t)\|_{L^\infty(0,T)} T \left| \tilde{\mathbb{E}} \int_0^T \int_D \int_D (f(\tilde{\varphi}_{m_k}) - f(\tilde{\varphi})) w dx ds \right| \\
 & \leq c(T) \left(\tilde{\mathbb{E}} \int_0^T \int_D \left| \int_D |f(\tilde{\varphi}_{m_k}) - f(\tilde{\varphi})| \right|^{\frac{4}{3}} \right)^{\frac{3}{4}} \left(\tilde{\mathbb{E}} \int_0^T \int_D w^4 \right)^{\frac{1}{4}} \\
 & \leq c(T) \left(\tilde{\mathbb{E}} \int_0^T \int_D \left| \int_D |f(\tilde{\varphi}_{m_k}) - f(\tilde{\varphi})|^{\frac{4}{3}} \right|^{\frac{3}{4}} \left(\int_D 1^4 \right)^{\frac{1}{4}} \right)^{\frac{3}{4}} \left(\tilde{\mathbb{E}} \int_0^T \int_D w^4 \right)^{\frac{1}{4}} \\
 & \leq c(T) \left(\tilde{\mathbb{E}} \int_0^T \int_D \int_D |f(\tilde{\varphi}_{m_k}) - f(\tilde{\varphi})|^{\frac{4}{3}} |D|^{\frac{1}{3}} \right)^{\frac{3}{4}} \left(\tilde{\mathbb{E}} \int_0^T \int_D w^4 \right)^{\frac{1}{4}} \\
 & \leq \tilde{c}(T) \left(\tilde{\mathbb{E}} \int_0^T \|f(\tilde{\varphi}_{m_k}) - f(\tilde{\varphi})\|_{L^{\frac{4}{3}}(D)}^{\frac{4}{3}} \right)^{\frac{3}{4}} \rightarrow 0, \quad \text{as } m_k \rightarrow \infty, \tag{3.5.18}
 \end{aligned}$$

which tends to zero as $m_k \rightarrow \infty$.

3.5.3 Noise term

Before proving the convergence of the noise term, we recall some basic notions as well as a result due to [4] Lemma 2.1 p. 5.

Definition 3.5.3 *Let X be a Banach space and let $\{Z_n\} \subset X$. We say that a $\{Z_n, n \in \mathbb{N}\}$ converges in probability to Z if*

$$\forall \varepsilon > 0, \lim_{n \rightarrow +\infty} \mathbb{P}(\|Z_n - Z\|_X > \varepsilon) = 0.$$

Lemma 3.5.4 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space, X a separable Hilbert space. Consider a sequence of stochastic bases $\mathcal{S}_n = (\Omega, \mathcal{F}, \{\mathcal{F}_t^n\}_{t \geq 0}, \mathbb{P}, W^n)$, where W^n is a Q -Brownian motion over H with respect to $\{\mathcal{F}_t^n\}$. Assume that $\{G^n\}_{n \geq 0}$ are a collection of X -valued $\{\mathcal{F}_t^n\}$ predictable processes such that $G^n \in L^2(0, T; \mathcal{L}_{2,Q}(H, X))$ a.s. Finally consider $\mathcal{S} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, W)$ and $G \in L^2([0, T], \mathcal{L}_{2,Q}(H, X))$ a.s., which is \mathcal{F}_t predictable. If*

$$G^n \rightarrow G \quad \text{in probability in } L^2(0, T; \mathcal{L}_{2,Q}(H, X)), \tag{3.5.19}$$

$$W^n \rightarrow W \quad \text{in probability in } C([0, T]; H), \tag{3.5.20}$$

then

$$\int_0^t G^n dW^n \rightarrow \int_0^t G dW \quad \text{in probability in } L^2(0, T; X). \tag{3.5.21}$$

From (3.1.12) and (3.5.13) we have that

$$\|\Phi(\tilde{\varphi}_{m_k}) - \Phi(\tilde{\varphi})\|_{\mathcal{L}_{2,Q}(H,V)}^2 \leq C_4 \|\tilde{\varphi}_{m_j} - \tilde{\varphi}\|_V \rightarrow 0 \quad \text{for a.e. } (\omega, t) \in \tilde{\Omega} \times (0, T).$$

(3.5.22)

On the other hand,

$$\begin{aligned} \tilde{\mathbb{E}} \int_0^T \|\Phi(\tilde{\varphi}_{m_k})\|_{\mathcal{L}_{2,Q}(H,V)}^4 dt &\leq C_3 \tilde{\mathbb{E}} \int_0^T (1 + \|\tilde{\varphi}_{m_k}\|_V^4) \\ &\leq \tilde{C}_3 T \end{aligned} \quad (3.5.23)$$

where we have used (3.1.11) and Lemma 3.3.8.

Thus, $\|\Phi(\tilde{\varphi}_{m_k})\|_{\mathcal{L}_{2,Q}(H,V)}$ is bounded in $L^4(\tilde{\Omega} \times (0, T))$. Using Lemma 3.A.15 which is deduced from the Vitali convergence theorem we deduce that,

$$\Phi(\tilde{\varphi}_{m_k}) \rightarrow \Phi(\tilde{\varphi}) \quad \text{in } L^2(\tilde{\Omega}; L^2(0, T; \mathcal{L}_{2,Q}(H, V))). \quad (3.5.24)$$

This implies that the following convergence holds in probability by Proposition 3.A.11 (i),

$$\Phi(\tilde{\varphi}_{m_k}) \rightarrow \Phi(\tilde{\varphi}) \quad \text{in probability in } L^2(0, T; \mathcal{L}_{2,Q}(H, V)). \quad (3.5.25)$$

In addition, by (3.5.2) and the fact that the a.s. convergence implies the convergence in probability we have that

$$\tilde{W}^{m_k} \rightarrow \tilde{W} \quad \text{in probability in } \mathcal{C}([0, T]; H). \quad (3.5.26)$$

Combining (3.5.25), (3.5.26) and applying Lemma 3.5.4 yields :

$$\int_0^t \Phi(\tilde{\varphi}_{m_k}) d\tilde{W}^{m_k} \rightarrow \int_0^t \Phi(\tilde{\varphi}) d\tilde{W}, \quad \text{in probability in } L^2(0, T; V). \quad (3.5.27)$$

This implies by Proposition 3.A.11 (iii) that up to subsequence

$$\int_0^t \Phi(\tilde{\varphi}_{m_k}) d\tilde{W}^{m_k} \rightarrow \int_0^t \Phi(\tilde{\varphi}) d\tilde{W}, \quad \text{a.s. in } L^2(0, T; V). \quad (3.5.28)$$

In addition we have that by the B urkholder inequality (3.3.27) and (3.1.10),

$$\begin{aligned} &\tilde{\mathbb{E}} \left(\sup_{0 \leq t \leq T} \left\| \int_0^t \Phi(\tilde{\varphi}_{m_k}) d\tilde{W}^{m_k} \right\|_V^4 \right) \\ &\leq \tilde{C}_4 \tilde{\mathbb{E}} \left(\int_0^T \|\Phi(\tilde{\varphi}_{m_k})\|_{\mathcal{L}_{2,Q}(H,V)}^2 dt \right)^2 \leq \tilde{C}_4. \end{aligned}$$

Thus, $\left\| \int_0^t \Phi(\tilde{\varphi}_{m_k}) d\tilde{W}^{m_k} \right\|_{L^2(0, T; V)}$ is bounded in $L^4(\tilde{\Omega})$. From Lemma 3.A.15 we infer a strong convergence result

$$\int_0^t \Phi(\tilde{\varphi}_{m_k}) d\tilde{W}_{m_k} \rightarrow \int_0^t \Phi(\tilde{\varphi}) d\tilde{W}, \quad \text{in } L^2(\tilde{\Omega}; L^2(0, T; V)). \quad (3.5.29)$$

Combining (3.5.12), (3.5.17), (3.5.18) and (3.5.29), we pass to the limit in (3.5.14) to deduce that $\tilde{\varphi}$ satisfies the equation

$$\begin{aligned} & \tilde{\mathbb{E}} \int_0^T y\psi(t) \int_D \tilde{\varphi}(t)w - \tilde{\mathbb{E}} \int_0^T y\psi(t) \int_D \varphi_0 w \\ &= \tilde{\mathbb{E}} \int_0^T y\psi(t) \int_0^t \int_D \Delta \tilde{\varphi} w + \tilde{\mathbb{E}} \int_0^T y\psi(t) \int_0^t \int_D f(\tilde{\varphi})w \\ & - \frac{1}{|D|} \tilde{\mathbb{E}} \int_0^T y\psi(t) \int_0^t \int_D \{ \int_D f(\tilde{\varphi}) \} w + \tilde{\mathbb{E}} \int_0^T y\psi(t) \int_0^t \int_D \Phi(\tilde{\varphi})w d\tilde{W}, \end{aligned} \quad (3.5.30)$$

for all $w \in H^1(D) \cap L^4(D)$. This implies for a.e. $(\omega, t) \in \tilde{\Omega} \times (0, T)$ that

$$\begin{aligned} & \int_D \tilde{\varphi}(t)w - \int_D \varphi_0 w \\ &= \int_0^t \int_D \Delta \tilde{\varphi} w + \int_0^t \int_D f(\tilde{\varphi})w - \frac{1}{|D|} \int_0^t \int_D \{ \int_D f(\tilde{\varphi}) \} w \\ & + \int_0^t \int_D \Phi(\tilde{\varphi})w d\tilde{W} \end{aligned} \quad (3.5.31)$$

that is

$$\tilde{\varphi}(t) - \varphi_0 = \int_0^t \Delta \tilde{\varphi} ds + \int_0^t f(\tilde{\varphi}) ds - \frac{1}{|D|} \int_0^t \int_D \{ \int_D f(\tilde{\varphi}) \} + \int_0^t \Phi(\tilde{\varphi}) d\tilde{W} \quad (3.5.32)$$

in $\tilde{\Omega} \times (0, T) \times D$.

This concludes the proof of the main Theorem 3.2.3.

3.6 Pathwise uniqueness

Definition 3.6.1 [14] *We say that pathwise uniqueness holds for Problem (P) if the following holds. Let X_1 and X_2 be two strong martingale solutions of Problem (P) defined on the same stochastic basis with the same Wiener process and assume that $X_1(0) = X_2(0)$ almost surely. Then the processes X_1 and X_2 are indistinguishable, namely*

$$P(X_1(t) = X_2(t), \forall t \geq 0) = 1.$$

Definition 3.6.2 *We say that uniqueness in law holds true for Problem (P) provided: Let $((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}), X, W)$ and $((\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}), \tilde{X}, \tilde{W})$ be two solutions of Problem (P) with the same initial law. Then the two processes X and \tilde{X} have the same law.*

Proposition 3.6.3 (Pathwise uniqueness) *Suppose φ and ψ are martingale solutions of Problem (P) relative to the same stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W)$. We also suppose that $\varphi_0 = \psi_0$. Then φ and ψ are respectively indistinguishable in the sense that*

$$\mathbb{P}(\varphi(t) = \psi(t), \forall t \geq 0) = 1.$$

Proof We take the difference of those two solutions

$$\begin{aligned}\varphi(t) - \psi(t) &= \int_0^t \Delta(\varphi - \psi) ds + \int_0^t (f(\varphi) - f(\psi)) ds \\ &\quad - \frac{1}{|D|} \int_0^t ds \int_D dx (f(\varphi) - f(\psi)) + \int_0^t [\Phi(\varphi) - \Phi(\psi)] dW.\end{aligned}$$

We apply Itô's formula as in Lemma 3.3.2, with

$$\begin{aligned}X(t) &= \varphi - \psi, \\ F(s, X) &= e^{-\int_0^s B(\tau) d\tau} \|X\|^2, \\ \frac{\partial F}{\partial t}(s, X) &= -B(s) e^{-\int_0^s B(\tau) d\tau} \|X\|^2, \\ \frac{\partial F}{\partial X}(s, X) &= 2e^{-\int_0^s B(\tau) d\tau} X, \\ \frac{\partial^2 F}{\partial X^2}(s, X) &= 2e^{-\int_0^s B(\tau) d\tau} I, \\ h(s) &= \Delta(\varphi - \psi) + (f(\varphi) - f(\psi)) - \frac{1}{|D|} \int_D (f(\varphi) - f(\psi)), \\ G(s) &= \Phi(\varphi(s)) - \Phi(\psi(s)),\end{aligned}$$

and where the function B still has to be chosen. Next we take the expectation

$$\begin{aligned}\mathbb{E} \left(e^{-\int_0^t B(s) ds} \|\varphi(t) - \psi(t)\|^2 \right) &= -\mathbb{E} \left(\int_0^t B(s) e^{-\int_0^s B(\tau) d\tau} \|\varphi(s) - \psi(s)\|^2 \right) \\ &\quad - 2\mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \|\nabla(\varphi - \psi)\|^2 \right) \\ &\quad + 2\mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \langle f(\varphi) - f(\psi), \varphi - \psi \rangle \right) \\ &\quad - \frac{2}{|D|} \mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \left\langle \int_D (f(\varphi) - f(\psi)), \varphi - \psi \right\rangle \right) \\ &\quad + 2\mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \langle \varphi - \psi, (\Phi(\varphi) - \Phi(\psi)) dW(s) \rangle \right) \\ &\quad + \mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \|\Phi(\varphi) - \Phi(\psi)\|_{2,Q}^2 \right). \quad (3.6.1)\end{aligned}$$

For the reaction term we have that :

$$2\mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \langle f(\varphi) - f(\psi), \varphi - \psi \rangle \right) \leq 2\mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \|\varphi - \psi\|^2 \right).$$

For the nonlocal term we have that :

$$\begin{aligned}& -\frac{2}{|D|} \mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \left\langle \int_D (f(\varphi) - f(\psi)), \varphi - \psi \right\rangle \right) \\ &= -\frac{2}{|D|} \mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \left\langle \int_D (\varphi - \varphi^3) - (\psi - \psi^3), \varphi - \psi \right\rangle \right)\end{aligned}$$

$$\begin{aligned}
 &= -\frac{2}{|D|} \mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \left(\int_D (\varphi - \psi) \right)^2 \right) \\
 &+ \frac{2}{|D|} \mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \int_D (\varphi^3 - \psi^3) \int_D (\varphi - \psi) \right) \\
 &= -\frac{2}{|D|} \mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \left(\int_D (\varphi - \psi) \right)^2 \right) \\
 &+ \frac{2}{|D|} \mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \left[\int_D (\varphi - \psi) (\varphi^2 + \varphi\psi + \psi^2) \right] \int_D (\varphi - \psi) \right) \\
 &= -\frac{2}{|D|} \mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \left(\int_D (\varphi - \psi) \right)^2 \right) \\
 &+ \frac{2\varepsilon}{|D|} \mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \left(\int_D (\varphi - \psi) \right)^2 \right) \\
 &+ \frac{2C_\varepsilon}{|D|} \mathbb{E} \left(\int_0^T e^{-\int_0^s B(\tau) d\tau} \left[\int_D (\varphi - \psi) (\varphi^2 + \varphi\psi + \psi^2) \right]^2 \right) \\
 &\leq \frac{-2 + 2\varepsilon}{|D|} \mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \left(\int_D (\varphi - \psi) \right)^2 \right) \\
 &+ \frac{2C_\varepsilon}{|D|} \mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \int_D (\varphi - \psi)^2 \int_D (\varphi^2 + \varphi\psi + \psi^2)^2 \right) \\
 &\leq \frac{-2 + 2\varepsilon}{|D|} \mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \left(\int_D (\varphi - \psi) \right)^2 \right) \\
 &+ \tilde{C}_\varepsilon \mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \int_D (\varphi^4 + \psi^4) \|\varphi - \psi\|^2 \right).
 \end{aligned}$$

Moreover it follows in a standard way that $2\mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \langle \Phi(\varphi) - \Phi(\psi), \varphi - \psi \rangle dW(s) \right) = 0$.

In view of (3.1.13) the last term can be computed as follows:

$$\mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \|\Phi(\varphi) - \Phi(\psi)\|_{2,Q}^2 \right) \leq C_5 \mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \|\varphi - \psi\|^2 \right)$$

Combining all the terms in (3.6.1) and choosing ε small enough, we have that

$$\begin{aligned}
 \mathbb{E} e^{-\int_0^t B(s) ds} \|\varphi(t) - \psi(t)\|^2 &\leq -\mathbb{E} \left(\int_0^t B(s) e^{-\int_0^s B(\tau) d\tau} \|\varphi(s) - \psi(s)\|^2 \right) \\
 &+ (2 + C_5) \mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \|\varphi - \psi\|^2 \right) \\
 &+ \tilde{C}_\varepsilon \mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \int_D (\varphi^4 + \psi^4) \|\varphi - \psi\|^2 \right).
 \end{aligned} \tag{3.6.2}$$

Choosing $B(s) = \tilde{C}_\varepsilon \int_D (\varphi^4 + \psi^4)(x, s) dx$ in (3.6.2), we deduce that

$$\mathbb{E} e^{-\int_0^t B(s) ds} \|\varphi(t) - \psi(t)\|^2 \leq (2 + C_5) \mathbb{E} \left(\int_0^t e^{-\int_0^s B(\tau) d\tau} \|\varphi - \psi\|^2 \right).$$

Applying Gronwall Lemma yields

$$\mathbb{E}e^{-\int_0^t B(s)ds} \|\varphi(s) - \psi(s)\|^2 = 0, \quad \text{for all } t \in (0, T),$$

so that for all $t \in (0, T)$ $\varphi(t) = \psi(t)$ a.e. in D almost surely, which gives the pathwise uniqueness. \square

3.7 Existence of a pathwise solution

After proving the existence and pathwise uniqueness of the martingale solution, we will apply the Gyöngy-Krylov Theorem in order to deduce the existence and uniqueness of the pathwise solution φ of Problem (P).

For that purpose, we return to the sequence $\{\varphi_m\}$ of Galerkin approximate solutions relative to the given stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. We argue in a similar manner as for the compactness argument for the existence of martingale solutions by considering the collections of joint distributions $\mu_{\varphi_m, n} := \mu_{\varphi_m} \times \mu_{\varphi_n}$.

We recall the phase space $\mathcal{X} = \mathcal{X}_S \times \mathcal{X}_W$ defined by (3.4.1) and the laws $\mu_{\varphi_m}, \mu_W, \mu^m$ defined by (3.4.2)-(3.4.4). Next we define

$$\mathcal{X}^J = \mathcal{X}_S \times \mathcal{X}_S \times \mathcal{X}_W. \quad (3.7.1)$$

We set

$$\nu^{m, n} := \mu_{\varphi_m} \times \mu_{\varphi_n} \times \mu_W. \quad (3.7.2)$$

Lemma 3.7.1 *The collection $\{\nu^{m, n}\}$ is tight and hence weakly compact on \mathcal{X}^J .*

Proof We follow the same ideas of proof as for Lemma 3.4.1. Let $\varepsilon \in (0, 1)$ be arbitrary. We take B_R^1, B_{2R}^2 as in that proof and we can therefore choose $Z_\varepsilon = B_{R^\varepsilon}^1 \cap B_{2R^\varepsilon}^2, V_\varepsilon$ compact in

\mathcal{X}_S and \mathcal{X}_W respectively and $R^\varepsilon = \left(\frac{2\tilde{k}}{1 - (1 - \varepsilon)^{\frac{1}{3}}} \right)^{\frac{1}{2}}$ so that

$$\mu_{\varphi_m}(Z_\varepsilon) \geq (1 - \varepsilon)^{\frac{1}{3}}, \quad \mu_W(V_\varepsilon) \geq (1 - \varepsilon)^{\frac{1}{3}}. \quad (3.7.3)$$

We then take $\mathcal{A}^\varepsilon = Z_\varepsilon \times Z_\varepsilon \times V_\varepsilon$ which is compact in \mathcal{X}^J . By (3.7.3), we see that

$$\nu^{m, n}(\mathcal{A}^\varepsilon) \geq 1 - \varepsilon.$$

We deduce from Definition 3.A.5 that the collection $\{\nu^{m, n}\}$ is tight so that by Prokhorov's Theorem it is also weakly compact in \mathcal{X}^J . \square

3.7.1 Proof of Theorem 3.2.4

By Lemma 3.7.1, we may extract a subsequence $\{k'\}$ such that $\{\nu^{m_{k'}, n_{k'}}\}$ converges to an element ν' weakly in \mathcal{X}^J . Then applying the Skorohod Theorem, we infer the existence of a new stochastic basis $\tilde{\mathcal{S}} := (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}})$ on which there exists two subsequences of random

functions $\{\hat{\varphi}_{m_{k'}}\}$ and $\{\check{\varphi}_{n_{k'}}\}$ such that $(\hat{\varphi}_{m_{k'}}, \check{\varphi}_{n_{k'}}, \tilde{W}^{k'})$ converges a.s. in \mathcal{X}^J to a triple $(\hat{\varphi}, \check{\varphi}, \tilde{W})$ where $\hat{\varphi}_{m_{k'}}$ satisfies

$$\begin{aligned} \int_D \hat{\varphi}_{m_{k'}}(t)w_j - \int_D \varphi_{0,m_{k'}}(x)w_j + \int_0^t \int_D \nabla \hat{\varphi}_{m_{k'}} \nabla w_j &= \int_0^t \int_D f(\hat{\varphi}_{m_{k'}})w_j \\ &\quad - \frac{1}{|D|} \int_0^t \int_D \left\{ \int_D f(\hat{\varphi}_{m_{k'}}) \right\} w_j + \int_0^t \int_D \Phi(\hat{\varphi}_{m_{k'}})w_j d\tilde{W}^{k'}. \end{aligned}$$

and $\check{\varphi}_{n_{k'}}$ satisfies

$$\begin{aligned} \int_D \check{\varphi}_{n_{k'}}(t)w_j - \int_D \varphi_{0,n_{k'}}(x)w_j + \int_0^t \int_D \nabla \check{\varphi}_{n_{k'}} \nabla w_j &= \int_0^t \int_D f(\check{\varphi}_{n_{k'}})w_j \\ &\quad - \frac{1}{|D|} \int_0^t \int_D \left\{ \int_D f(\check{\varphi}_{n_{k'}}) \right\} w_j + \int_0^t \int_D \Phi(\check{\varphi}_{n_{k'}})w_j d\tilde{W}^{k'}, \end{aligned}$$

In addition, we define the following probability measures

$$\nu^{m_{k'}, n_{k'}}(\cdot) = \tilde{\mathbb{P}}((\hat{\varphi}_{m_{k'}}, \check{\varphi}_{n_{k'}}, \tilde{W}^{k'}) \in \cdot) \quad \text{and} \quad \nu'(\cdot) = \tilde{\mathbb{P}}((\hat{\varphi}, \check{\varphi}, \tilde{W}) \in \cdot). \quad (3.7.4)$$

Note that in particular, $\mu_{\varphi_{m_{k'}, n_{k'}}$ converges weakly to the measure μ_φ defined by:

$$\mu_\varphi(\cdot) = \tilde{\mathbb{P}}((\hat{\varphi}, \check{\varphi}) \in \cdot).$$

Let $\hat{z}_{k'} = (\hat{\varphi}_{m_{k'}}, \tilde{W}^{k'})$, $\check{z}_{k'} = (\check{\varphi}_{n_{k'}}, \tilde{W}^{k'})$ and $\hat{z} = (\hat{\varphi}, \tilde{W})$, $\check{z} = (\check{\varphi}, \tilde{W})$. We remark that both $\hat{\varphi}$ and $\check{\varphi}$ are two martingale solutions of (P) over the same stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}})$. Recalling that $\hat{\varphi}(0) = \check{\varphi}(0) = \varphi_0$, we deduce from the pathwise uniqueness result (Proposition 3.6.3) that $\hat{\varphi} = \check{\varphi}$ in \mathcal{X}_S a.s. In other words,

$$\mu_\varphi(\{(x, y) \in \mathcal{X}_S \times \mathcal{X}_S : x = y\}) = \tilde{\mathbb{P}}(\hat{\varphi} = \check{\varphi} \text{ in } \mathcal{X}_S) = 1.$$

Reasoning as Gyöngy-Krylov [12](proof of Lemma 1.1), we conclude that the sequence of solutions of the Galerkin approximate problem $\{\check{\varphi}_m\}$ is a Cauchy sequence which converges to $\check{\varphi} = \hat{\varphi} = \check{\varphi}$ almost surely in \mathcal{X}_S as $m \rightarrow \infty$.

In view of Gyöngy-Krylov [12](Lemma 1.1 p.144, cf. Proposition 3.A.9), the original sequence $\{\varphi_m\}$ defined on the initial probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ converges to $\varphi := \hat{\varphi} = \check{\varphi}$ in \mathcal{X}_S , i.e.

$$\varphi_m \rightarrow \varphi \text{ a.s. in } L^2(0, T; V) \cap \mathcal{C}([0, T]; V').$$

Finally, by Proposition 3.5.1 we conclude that φ is the unique pathwise solution of Problem (P) .

3.A Appendix

Definition 3.A.1 Suppose that X is a separable Hilbert space. Given $p \geq 2, \alpha \in (0, 1)$, we define the fractional derivative space $W^{\alpha,p}(0, T; X)$ as the Sobolev space of all $u \in L^p(0, T; X)$ such that

$$\int_0^T \int_0^T \frac{|u(t) - u(s)|_X^p}{|t - s|^{1+\alpha p}} dt ds < \infty,$$

endowed with the norm

$$\|u\|_{W^{\alpha,p}(0,T;X)}^p = \int_0^T |u(t)|_X^p dt + \int_0^T \int_0^T \frac{|u(t) - u(s)|_X^p}{|t - s|^{1+\alpha p}} dt ds.$$

We also use the space

$$W^{1,2}(0, T; X) = H^1(0, T; X) = \{u \in L^2(0, T; X) : \frac{du}{dt} \in L^2(0, T; X)\},$$

with its usual norm

$$\|u\|_{H^1(0,T;X)}^2 = \int_0^T |u(s)|_X^2 ds + \int_0^T \left| \frac{du}{dt}(s) \right|_X^2 ds. \quad (3.A.1)$$

Note that for $\alpha \in (0, \frac{1}{2})$,

$$\begin{aligned} H^1(0, T; X) &\subset W^{\alpha,2}(0, T; X) \\ \|u\|_{W^{\alpha,2}(0,T;X)} &\leq C \|u\|_{H^1(0,T;X)}. \end{aligned} \quad (3.A.2)$$

Lemma 3.A.2 (cf.[7] Theorem 4.6 p.91) Let $\mathcal{E}_0 \subset \mathcal{E} \subset \mathcal{E}_1$ be Banach spaces with the injections being continuous and $\mathcal{E}_0, \mathcal{E}_1$ reflexive such that the embedding from \mathcal{E}_0 into \mathcal{E} is compact. Let $p \in (1, \infty)$ and $\alpha \in (0, 1)$ be given. Let \mathcal{Y} be the space

$$\mathcal{Y} := L^p(0, T; \mathcal{E}_0) \cap W^{\alpha,p}(0, T; \mathcal{E}_1),$$

endowed with the natural norm. Then the embedding of \mathcal{Y} in $L^p(0, T; \mathcal{E})$ is compact.

Lemma 3.A.3 (cf.[7] Lemma 4.3 p.91) If $\mathcal{E} \subset \bar{\mathcal{E}}$ are two Banach spaces with \mathcal{E} compactly embedded in $\bar{\mathcal{E}}$ and $p \in (1, \infty), \alpha \in (0, 1]$ are such that $\alpha p > 1$, then the space $W^{\alpha,p}(0, T; \mathcal{E})$ is compactly embedded into $C([0, T]; \bar{\mathcal{E}})$.

Lemma 3.A.4 (Chebyshev's inequality) If X is a random variable and $1 \leq \theta < \infty$, then (cf.[6] p.14)

$$\mathbb{P}(|X| \geq \lambda) \leq \frac{1}{\lambda^\theta} \mathbb{E}(|X|^\theta) \quad \text{for all } \lambda > 0.$$

Definition 3.A.5 Suppose (X, d) is a complete separable metric space with $\mathcal{B}(X)$ its associated Borel σ - algebra. Let $C_b(X)$ be the set of all real-valued continuous bounded functions on X , and let $\text{Pr}(X)$ be the set of all probability measures on $(X, \mathcal{B}(X))$. A collection $\Lambda \subset \text{Pr}(X)$ is tight if for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset X$ such that

$$\mu(K_\varepsilon) \geq 1 - \varepsilon, \quad \text{for all } \mu \in \Lambda.$$

In particular, a sequence $\{\mu_m\}_{m \geq 0} \subset \text{Pr}(X)$ is tight if for every $\varepsilon > 0$ there exist $m_0 > 0$ and a compact set $K_\varepsilon \subset X$ such that

$$\mu_m(K_\varepsilon) \geq 1 - \varepsilon, \quad \text{for all } m \geq m_0.$$

A sequence $\{\mu_m\}_{m \geq 0} \subset \text{Pr}(X)$ converges weakly to a probability measure μ if

$$\int_X f d\mu_n \rightarrow \int_X f d\mu, \quad \text{for all } f \in C_b(X).$$

The tightness property is a criterion for the compactness of probability measures. More precisely, one has the following result.

Proposition 3.A.6 (*Prokhorov's Theorem*) A collection $\Lambda \subset \text{Pr}(X)$ is weakly compact if and only if it is tight.

Theorem 3.A.7 (*Tychonoff's theorem*)

The product of any collection of compact topological spaces is compact with respect to the product topology.

Proposition 3.A.8 (*Skorokhod's Theorem - cf.[1]*) For an arbitrary sequence of probability measures μ_m on X a separable Banach space, equipped with its Borel σ algebra, weakly convergent to a probability measure μ , there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and random variables \tilde{Y}_m, \tilde{Y} with values in X , such that

- (i) The probability law of \tilde{Y}_m is μ_m .
- (ii) The probability law of \tilde{Y} is μ .
- (iii) $\lim_{m \rightarrow \infty} \tilde{Y}_m = \tilde{Y}$, \tilde{P} -a.s.

Proposition 3.A.9 (*Gyöngy-Krylov Theorem*) A sequence of X -valued random variables $\{Y_n\}_{n \geq 0}$ converges in probability if and only if for every sequence of joint probability laws, $\{\mu_{m_k, n_k}\}_{k \geq 0}$ there exists a further subsequence which converges weakly to a probability measure μ such that

$$\mu(\{(x, y) \in X \times X : x = y\}) = 1.$$

Proposition 3.A.10 (*Yamada-Watanabe [18]*) Pathwise uniqueness implies uniqueness in the sense of probability law.

Proposition 3.A.11 The following implications hold

- (i) Let $p > 0$ be arbitrary. Z_n converges to Z in $L^p(\Omega \times (0, T); X)$ implies that Z_n converges to Z in probability in $L^p(0, T; X)$.
- (ii) Z_n converges to Z a.s. in $L^p(0, T; X)$ implies that Z_n converges to Z in probability in $L^p(0, T; X)$.
- (iii) Z_n converges to Z in probability in $L^p(0, T; X)$ implies that, up to a subsequence, Z_n converges to Z a.s. in $L^p(0, T; X)$.

Uniform integrability, Vitali theorem

This part is taken from [14], we refer also to [16] and [9].

Definition 3.A.12 (*Uniform integrability*) Let (A, \mathcal{F}, μ) be a finite measure space. A sequence of functions $\{u_n\} \subset L^1(\mu)$ is called *uniformly integrable* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left| \int_E u_n(x) d\mu(x) \right| < \varepsilon$$

for all $n \in \mathbb{N}$ and $E \in \mathcal{F}$ such that $\mu(E) < \delta$.

We can write the following result [14].

Lemma 3.A.13 Let (A, \mathcal{F}, μ) be a finite measure space. Let a sequence $\{u_n\} \subset L^1(\mu)$ be bounded in $L^p(\mu)$ for some $p > 1$. Then it is uniformly integrable.

In real analysis and measure theory, the Vitali convergence theorem is a generalization of the better-known dominated convergence theorem. It is a strong condition that depends on the uniform integrability. It is useful when a dominating function cannot be found for the sequence of functions in question; when such a function can be found, Lebesgue's dominated convergence theorem follows as a special case of Vitali's theorem.

Lemma 3.A.14 (*Vitali*) Let (A, \mathcal{F}, μ) be a finite measure space. Suppose that $\{u_n\}$ are L^p integrable on a finite measure space, where $1 \leq p < \infty$. Suppose that

- (i) $\{u_n\}$ converges to u a.e. in A ,
- (ii) the sequence $\{|u_n|^p\}$ is uniformly integrable.

Then $u_n \rightarrow u$ in $L^p(\mu)$, i.e.

$$\lim_{n \rightarrow \infty} \int_A |u_n(x) - u(x)|^p d\mu(x) = 0$$

A typical application of Vitali's lemma is provided by the next result.

Lemma 3.A.15 We suppose that

- (i) $u_n \rightarrow u$ a.e. in A ;
- (ii) u_n is uniformly bounded in $L^p(\mu)$, for $p > 1$.

Then

$$u_n \rightarrow u \text{ in } L^r(\mu) \text{ for all } r \in [1, p).$$

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Titre: Équations d'évolution stochastiques locales et non locales dans des problèmes de transition de phase.
Mots clés: Équation de réaction-diffusion non locale stochastique, problème de champ de phase stochastique, méthode de monotonie stochastique, méthode de compacité stochastique.

Résumé: Le but de cette thèse est de développer des méthodes de démonstration d'existence et d'unicité de solutions d'équations d'évolution stochastiques locales ou non locales dans les problèmes de transition de phase. Au chapitre 1, nous étudions un problème à valeur initiale pour une équation de réaction-diffusion stochastique non locale avec des conditions aux limites de Neumann homogènes dans un ouvert borné de \mathbb{R}^n de frontière suffisamment régulière. On considère le cas d'un opérateur elliptique non linéaire assez général et on suppose que le bruit est additif et induit par un processus Q -Wiener. Le problème déterministe modélise la séparation de phases dans des alliages binaires. La démonstration d'existence de la solution du problème stochastique est basée sur un changement de fonction qui fait intervenir la solution de l'équation de la chaleur stochastique avec un terme de diffusion non linéaire. On est ainsi conduit à l'étude d'un problème sans terme de bruit, ce qui facilite l'application de la méthode de monotonie pour identifier la limite des termes non linéaires. Au chapitre 2, nous démontrons l'existence et l'unicité de la solution d'un système de champ de phase stochastique avec des bruits multiplicatifs induits par des processus Q -Wiener. Les problèmes de champ de phase sont utilisés pour décrire des modèles où deux phases distinctes interviennent comme par exemple l'eau et la glace. Dans ce but, nous appliquons la méthode de Galerkin et nous établissons des estimations a priori pour la solution approchée. Nous nous appuyons ensuite sur la méthode de monotonie stochastique pour identifier la limite du terme non linéaire. Finalement, au chapitre 3, nous démontrons l'existence et l'unicité d'une solution trajectorielle en dimension d'espace $d \leq 6$ pour l'équation d'Allen-Cahn non locale stochastique avec un bruit multiplicatif induit par un processus Q -Wiener. La présence d'une variable supplémentaire empêche l'application des théorèmes de compacité usuels utilisés dans les problèmes déterministes. C'est ce qui nous amène à appliquer la méthode de compacité stochastique.

Title: Local and nonlocal stochastic evolution equations in phase transition problems.

Keywords: Stochastic nonlocal reaction-diffusion equation, stochastic phase field problem, stochastic monotonicity method, stochastic compactness method.

Abstract: The aim of this thesis is to develop methods for proving the existence and uniqueness of solutions of local and nonlocal stochastic evolution equations in phase transition problems. In chapter 1, we study an initial value problem for a nonlocal stochastic reaction-diffusion equation with homogeneous Neumann boundary conditions in an open bounded set of \mathbb{R}^n , with a smooth boundary. We consider the case of a general nonlinear elliptic operator and we suppose that the noise is additive and induced by a Q -Wiener process. The deterministic problem with a linear diffusion term is used to model phase separation in a binary mixture. The proof of existence for the stochastic problem is based on a change of function which involves the solution of the stochastic heat equation with a nonlinear diffusion term. We obtain a problem without the noise term. This simplifies the application of the monotonicity method, which we use to identify the limit of the nonlinear terms. In chapter 2, we prove the existence and uniqueness of the solution for a phase field problem with multiplicative noises induced by Q -Wiener processes. This problem models for instance the process of melting and solidification. To that purpose we apply the Galerkin method and derive a priori estimates for the approximate solutions. The last step is to identify the limit of the nonlinear terms which we do by the so-called stochastic monotonicity method. Finally, in chapter 3, we prove the existence and uniqueness of a pathwise solution in space dimension up to 6 for the stochastic nonlocal Allen-Cahn equation with a multiplicative noise induced by a Q -Wiener process. The usual compactness method for deterministic problems cannot be applied in a stochastic context because of the additional probability variable. Therefore, we apply the stochastic compactness method.



