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THÈSE

En vue de l'obtention du

DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

Délivré par : *l'Université Toulouse 3 Paul Sabatier (UT3 Paul Sabatier)*

Présentée et soutenue le *29 Juin 2018* par :

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Flots de Monge-Ampère complexes sur les variétés hermitiennes compactes

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Résumé

Dans cette thèse nous nous intéressons aux flots de Monge-Ampère complexes, à leurs généralisations et à leurs applications géométriques sur les variétés hermitiennes compactes.

Dans les deux premiers chapitres, nous prouvons qu'un flot de Monge-Ampère complexe sur une variété hermitienne compacte peut être exécuté à partir d'une condition initiale arbitraire avec un nombre Lelong nul en tous points. En utilisant cette propriété, nous confirmons une conjecture de Tosatti-Weinkove: le flot de Chern-Ricci effectue une contraction chirurgicale canonique. Enfin, nous étudions une généralisation du flot de Chern-Ricci sur des variétés hermitiennes compactes, le flot de Chern-Ricci tordu. Cette partie a donné lieu à deux publications indépendantes.

Dans le troisième chapitre, une notion de C -sous-solution parabolique est introduite pour les équations paraboliques, étendant la théorie des C -sous-solutions développée récemment par B. Guan et plus spécifiquement G. Székelyhidi pour les équations elliptiques. La théorie parabolique qui en résulte fournit une approche unifiée et pratique pour l'étude de nombreux flots géométriques. Il s'agit ici d'une collaboration avec Duong H. Phong (Université Columbia)

Dans le quatrième chapitre, une approche de viscosité est introduite pour le problème de Dirichlet associé aux équations complexes de type hessienne sur les domaines de \mathbb{C}^n . Les arguments sont modélisés sur la théorie des solutions de viscosité pour les équations réelles de type hessienne développées par Trudinger. En conséquence, nous résolvons le problème de Dirichlet pour les équations de quotient de hessiennes et lagrangiennes spéciales. Nous établissons également des résultats de régularité de base pour les solutions. Il s'agit ici d'une collaboration avec Sławomir Dinew (Université Jagellonne) et Hoang-Son Do (Institut de Mathématiques de Hanoi)

Mots clés

Équations de Monge-Ampère complexes, flot de Kähler-Ricci, flot de Chern-Ricci, variétés hermitiennes.

Abstract

In this thesis we study the complex Monge-Ampère flows, and their generalizations and geometric applications on compact Hermitian manifolds.

In the first two chapters, we prove that a general complex Monge-Ampère flow on a compact Hermitian manifold can be run from an arbitrary initial condition with zero Lelong number at all points. Using this property, we confirm a conjecture of Tosatti-Weinkove: the Chern-Ricci flow performs a *canonical surgical contraction*. Finally, we study a generalization of the Chern-Ricci flow on compact Hermitian manifolds, namely the twisted Chern-Ricci flow. This part gave rise to two independent publications.

In the third chapter, a notion of parabolic C -subsolution is introduced for parabolic non-linear equations, extending the theory of C -subsolutions recently developed by B. Guan and more specifically G. Székelyhidi for elliptic equations. The resulting parabolic theory provides a convenient unified approach for the study of many geometric flows. This part is a joint work with Duong H. Phong (Columbia University)

In the fourth chapter, a viscosity approach is introduced for the Dirichlet problem associated to complex Hessian type equations on domains in \mathbb{C}^n . The arguments are modelled on the theory of viscosity solutions for real Hessian type equations developed by Trudinger. As consequence we solve the Dirichlet problem for the Hessian quotient and special Lagrangian equations. We also establish basic regularity results for the solutions. This part is a joint work with Sławomir Dinew (Jagiellonian University) and Hoang-Son Do (Hanoi Institute of Mathematics).

Keywords

Complex Monge-Ampère equations, Kähler-Ricci flow, Chern-Ricci flow, Hermitian manifolds.

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“Chữ tâm kia mới bằng ba chữ tài”

-Nguyễn Du-

Contents

1	Introduction	5
1.1	Motivation	5
1.1.1	Kähler-Einstein metrics and complex Monge-Ampère equations . . .	5
1.1.2	The Kähler-Ricci flow and complex Monge-Ampère flows	6
1.1.3	Analytic Minimal Model Program	8
1.1.4	Chern-Ricci flow and geometric applications	9
1.1.5	Other geometric flows	10
1.2	Selected results	11
1.2.1	Regularizing properties of Complex-Monge Ampère flows.	11
1.2.2	Geometric contraction of Chern-Ricci flow on Hermitian manifolds .	12
1.2.3	Fully non-linear parabolic equations on compact Hermitian manifolds.	13
1.2.4	A viscosity approach to the Hessian type equations	15
1.3	Warning to the reader	17
2	Regularizing properties of CMAF I	19
2.1	Introduction	19
2.2	Preliminaries and Strategy	21
2.2.1	Plurisubharmonic functions and Lelong number	21
2.2.2	A Laplacian inequality	21
2.2.3	Maximum principle and comparison theorem	22
2.2.4	Evans-Krylov and Schauder estimates for Monge-Ampère flow . . .	24
2.2.5	Monge-Ampère capacity	26
2.2.6	Monge-Ampère energy	27
2.2.7	Reduction to $\frac{\partial F}{\partial s} \geq 0$	27
2.2.8	Strategy of the proof	28
2.3	A priori estimates	29
2.3.1	Bounding φ_t	30
2.3.2	Upper bound for $\dot{\varphi}_t$	32
2.3.3	Bounding the oscillation of φ_t	33
2.3.4	Lower bound for $\dot{\varphi}_t$	33

2.3.5	Bounding the gradient of φ	35
2.3.6	Bounding $\Delta\varphi_t$	40
2.3.7	Higher order estimates	42
2.4	Proof of Theorem A	42
2.4.1	Convergence in L^1	42
2.4.2	Uniform convergence	43
2.4.3	Convergence in capacity	43
2.4.4	Convergence in energy	44
2.5	Uniqueness and stability of solution	45
2.5.1	Uniqueness	45
2.5.2	Quantitative stability estimate	48
2.6	Starting from a nef class	49
3	Regularizing properties of CMAF II	53
3.1	Introduction	53
3.2	Preliminaries	58
3.2.1	Chern-Ricci curvature on Hermitian manifold	58
3.2.2	Plurisubharmonic functions and Lelong number	59
3.3	A priori estimates for complex Monge-Ampère flows	59
3.3.1	Bounds on φ_t and $\dot{\varphi}_t$	60
3.3.2	Bounding the oscillation of φ_t	63
3.3.3	Bounding the gradient of φ	64
3.3.4	Bounding $\Delta\varphi_t$	70
3.3.5	Higher order estimates	74
3.4	Proof of Theorem A and B	75
3.4.1	Convergence in L^1	75
3.4.2	Uniform convergence	76
3.4.3	Uniqueness and stability of solution	76
3.5	Chern-Ricci flow and canonical surgical contraction	77
3.5.1	Continuing the Chern-Ricci flow	78
3.5.2	Backward convergences	79
3.6	Twisted Chern-Ricci flow	86
3.6.1	Maximal existence time for the twisted Chern-Ricci flow	86
3.6.2	Twisted Einstein metric on Hermitian manifolds	87
3.6.3	Convergence of the flow when $c_1^{BC}(X) - \{\eta\} < 0$	88
4	Fully non-linear parabolic equations	93
4.1	Introduction	93
4.2	A Priori Estimates	97
4.2.1	C^0 Estimates	97
4.2.2	C^2 Estimates	99

4.2.3	C^1 Estimates	109
4.2.4	Higher Order Estimates	111
4.3	Proof of Theorems 1 and 2	111
4.4	Applications to Geometric Flows	115
4.4.1	A criterion for subsolutions	115
4.4.2	Székelyhidi's theorem	117
4.4.3	The Kähler-Ricci flow and the Chern-Ricci flow	117
4.4.4	Hessian flows	118
4.4.5	The J flow and quotient Hessian flows	119
4.4.6	Flows with mixed Hessians σ_k	129
4.4.7	Concluding Remarks	130
5	Viscosity approach for Hessian type equations	133
5.1	Introduction	133
5.2	Preliminaries	137
5.2.1	Linear algebra toolkit	137
5.2.2	Viscosity sub(super)-solutions	138
5.2.3	Aleksandrov-Bakelman-Pucci maximum principle	140
5.2.4	Γ -subharmonic functions	141
5.3	Comparison principles	143
5.3.1	A preliminary comparison principle	143
5.3.2	Comparison principle for Hessian type equations	145
5.4	Dirichlet problems	148
5.4.1	Viscosity solutions in Γ -pseudoconvex domains	148
5.4.2	Hölder continuity of Hessian type equations	150
5.5	Viscosity vs. pluripotential solutions	153
5.6	Dirichlet problem for the Lagrangian phase operator	163
6	Projects	165
6.1	Convergence of the Kähler-Ricci flow on manifolds of general type	165
6.2	Viscosity theory for the Hessian type equations	167

Chapter 1

Introduction

1.1 Motivation

1.1.1 Kähler-Einstein metrics and complex Monge-Ampère equations

The study of special Kähler metrics on compact Kähler manifolds, pioneered by Calabi in the 1950's, has been a guiding question in the field ever since. This led to an impressive number of remarkable developments, among which the solution by Yau of the Calabi conjecture in the late 1970's [Yau78] was one of the consummate achievements.

Let us start with a compact complex manifold X equipped with a Hermitian metric $g_{j\bar{k}}$. The associated $(1, 1)$ -form is defined in local coordinates by

$$\omega = ig_{j\bar{k}}dz^j \wedge d\bar{z}^k.$$

We say that the metric g is Kähler if $d\omega = 0$. We define an equivalent relation on closed real 2-forms on X : η' is cohomologous to η if $\eta - \eta'$ is exact, then we denote by $\{\eta\}$ the equivalence class of η . It follows from the $\partial\bar{\partial}$ -lemma that any other Kähler form $\tilde{\omega} \in \{\omega\}$ can be written as

$$\tilde{\omega} = \omega + i\partial\bar{\partial}\varphi.$$

The Ricci curvature of ω is locally defined by

$$\text{Ric}(\omega) = -i\partial\bar{\partial} \log \det(g_{j\bar{k}}).$$

If $\tilde{\omega}$ is another Kähler form on X , then

$$\text{Ric}(\omega) - \text{Ric}(\tilde{\omega}) = -i\partial\bar{\partial} \log \frac{\omega^n}{\tilde{\omega}^n}, \tag{1.1.1}$$

thus $\text{Ric}(\omega)$ and $\text{Ric}(\tilde{\omega})$ are cohomologous. The cohomology class of $\text{Ric}(\omega)$ defines $c_1(X)$, *the first Chern class of X* , which does not depend on ω but only on the complex structure of X .

One of the questions initiated by Calabi was that any representative of $c_1(X)$ is the Ricci form of a (unique) Kähler metric in any given Kähler class. Precisely, let (X, ω) be a Kähler manifold, and \tilde{R} be a $(1, 1)$ -form on X which is cohomologous to $\text{Ric}(\omega)$, can we find another Kähler form $\tilde{\omega} \in \{\omega\}$ such that $\text{Ric}(\tilde{\omega}) = \tilde{R}$? This Calabi conjecture was proved by Yau in 1976 [Yau78] by solving the *complex Monge-Ampère equation*

$$(\omega + i\partial\bar{\partial}\varphi)^n = e^F \omega^n,$$

where F is the unique smooth function satisfying $\tilde{R} = \text{Ric}(\omega) + i\partial\bar{\partial}F$ and $\int_X e^F \omega^n = \int_X \omega^n$. In particular when $c_1(X) = 0$, this shows the existence of *Calabi-Yau metrics* with vanishing Ricci curvature.

Candidates for special metrics on Kähler manifolds are *Kähler-Einstein metrics* that were actually introduced in [Käh33]: a Kähler metric ω is called Kähler-Einstein metric if

$$\text{Ric}(\omega) = \lambda\omega, \tag{1.1.2}$$

with $\lambda \in \mathbb{R}$. A more sophisticated guess for canonical Kähler metrics (in a given Kähler class) is Calabi's theory of extremal metrics which contains the Kähler-Einstein case.

Consider the Kähler-Einstein equation (1.1.2). The case $\lambda = 0$ is the Calabi-Yau case above, and if $\lambda \neq 0$, after scaling, we may assume $\lambda = \pm 1$. In the case $\lambda = \pm 1$, we infer that the equation (1.1.2) can be rewritten as the following complex Monge-Ampère equation

$$(\omega_0 + i\partial\bar{\partial}\varphi)^n = e^{-\lambda\varphi + F} \omega_0^n, \tag{1.1.3}$$

where F satisfies $\text{Ric}(\omega_0) = \lambda\omega_0 + i\partial\bar{\partial}F$, for some fixed Kähler metric $\omega_0 \in \lambda c_1(X)$. It was solved by Aubin and Yau when $\lambda = -1$ using the continuity method. The harder case is when $\lambda = 1$, i.e when X is *Fano*. The obstruction in this case is the absence of the C^0 estimate along the continuity method due to the unfavorable sign in (1.1.3).

A major question is to find a *necessary and sufficient* condition for the existence of the solution in this case. The precise formulation is in terms of an algebro-geometric notion of *K-stability* which is well-known as the Yau-Tian-Donaldson conjecture:

Yau-Tian-Donaldson conjecture ([CDS15a, CDS15b, CDS15c, Tia15, DS16, CSW15, BBJ15]) *A Fano manifold admits a Kähler-Einstein metric if and only if it is K-stable.*

1.1.2 The Kähler-Ricci flow and complex Monge-Ampère flows

In Riemannian geometry, the Ricci flow, first introduced by Hamilton [Ham82], is the evolution equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}. \tag{1.1.4}$$

The Ricci flow has become a key tool in differential geometry. On Kähler manifolds, it can be written as

$$\frac{\partial}{\partial t}\omega(t) = -\text{Ric}(\omega(t)), \quad \omega|_{t=0} = \omega_0. \quad (1.1.5)$$

Its normalized form is the following

$$\frac{\partial}{\partial t}\omega(t) = -\text{Ric}(\omega(t)) + \lambda\omega(t), \quad \omega|_{t=0} = \omega_0, \quad (1.1.6)$$

for $\lambda = 0, \pm 1$, which is the parabolic version of the Kähler-Einstein equation (1.1.2).

Bando observed that the Kähler condition is preserved under the Ricci flow, so it is customary to call the Ricci flow on Kähler manifolds *the Kähler-Ricci flow*. He used an important property of the Kähler-Ricci flow, namely that we can reduce it to a scalar parabolic complex Monge-Ampère equation. Indeed, let $\omega = \omega(t)$ be a solution of the Kähler-Ricci flow (1.1.5). Along the flow, the cohomology class $\{\omega\}$ evolves by

$$\frac{d}{dt}\{\omega\} = -c_1(X), \quad \{\omega(0)\} = \{\omega_0\}. \quad (1.1.7)$$

This ODE has a unique solution $\{\omega(t)\} = \{\omega_0\} - tc_1(X)$. Let θ be a smooth $(1, 1)$ form representing $-c_1(X)$ and Ω be a volume form such that locally we have

$$\Omega = a(z^1, \dots, z^n)idz^1 \wedge d\bar{z}^1 \wedge \dots \wedge idz^n \wedge d\bar{z}^n$$

and

$$i\partial\bar{\partial}\log\Omega := i\partial\bar{\partial}\log a = \theta.$$

Writing $\omega(t) = \omega_0 + t\theta + i\partial\bar{\partial}\varphi$, the Kähler-Ricci flow becomes

$$i\partial\bar{\partial}\log\Omega + i\partial\bar{\partial}\frac{\partial\varphi}{\partial t} = i\partial\bar{\partial}\log\omega(t)^n,$$

hence

$$i\partial\bar{\partial}\left(\frac{\partial\varphi}{\partial t} - \log\frac{(\omega_0 + t\theta + i\partial\bar{\partial}\varphi)^n}{\Omega}\right) = 0.$$

Since X is compact, the maximum principle implies that

$$\frac{\partial\varphi}{\partial t} - \log\frac{(\omega_0 + t\theta + i\partial\bar{\partial}\varphi)^n}{\Omega}$$

is a constant. Therefore we are reduced to study the *complex Monge-Ampère flow*

$$\frac{\partial\varphi}{\partial t} = \log\frac{(\omega_0 + t\theta + i\partial\bar{\partial}\varphi)^n}{\Omega}.$$

Similarly, we can also obtain a complex Monge-Ampère flow corresponding to the normalized Kähler-Ricci flow. In particular, if the normalized Kähler-Ricci flow starts from $\omega_0 \in \lambda c_1(X)$, its scalar equation can be written as

$$\frac{\partial \varphi}{\partial t} = \log \frac{(\omega_0 + i\partial\bar{\partial}\varphi)^n}{\Omega} + \lambda\varphi,$$

where Ω is a smooth volume form satisfying $\omega_0 = -\lambda i\partial\bar{\partial} \log \Omega$ and $\int_X \Omega = \int_X \omega_0^n$.

On a compact Kähler manifold with nonpositive first Chern class, Cao [Cao85] proved that the (normalized) Kähler-Ricci flow converges to a metric which satisfies the Kähler-Einstein equation (1.1.2). The convergence of the Kähler-Ricci flow on Fano manifolds has been studied by many authors (see for example [PS06, PSSW08, PSSW09, PSSW11, Tos10, CS12, Sze10, CSW15] and references therein). Through the work of many authors, the Kähler-Ricci flow became a major tool in Kähler geometry (we refer to [SW13b, Tos18] and references therein).

1.1.3 Analytic Minimal Model Program

In algebraic geometry, the minimal model program (MMP) is part of the birational classification of algebraic varieties. Its goal is to construct a “good” birational model of any complex projective variety. A “good” model is a variety X satisfying either:

- (1) K_X is nef, i.e. $K_X.C \geq 0$ for all curve C . In this case the variety is called a *minimal model*
- (2) There exists a holomorphic map $\pi : X \rightarrow Y$ to a variety Y of lower dimension such that the generic fiber $X_y = \pi^{-1}(y)$ is a Fano manifold (i.e. $c_1(X_y) > 0$). In this case the variety X is called a *Mori (or Fano) fiber space*.

The core of the MMP consists in finding a sequence of birational maps f_1, \dots, f_k and varieties X_1, \dots, X_k with

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \cdots \xrightarrow{f_k} X_k \tag{1.1.8}$$

such that X_k is either a minimal model or Mori fiber space.

Recently, Song and Tian [ST17] have developed a program and studied whether the Kähler-Ricci flow can give a geometric classification of algebraic varieties. They viewed the Kähler-Ricci flow as a metric version of the Minimal Model Program (MMP), each step of the MMP corresponding to a surgery that is used to repair a finite time singularity of the flow and start it over again (we refer to [ST17, SW13a, SW13b, EGZ16, EGZ18] and references therein). We follow [SW13b, ST17] to give a sketchy picture of this program:

Step 1. Starting with a metric ω_0 in the class of a divisor D on a variety X , we consider the solution $\omega(t)$ of the Kähler-Ricci flow on X starting from ω_0 . The flow exists on $[0, T)$ with $T = \sup\{t > 0 \mid D + tK_X \text{ is ample}\}$.

Step 2. If $T = \infty$, then K_X is nef, hence X is a minimal model, and the Kähler-Ricci flow has long time existence. The normalization of the flow should converge to a *canonical generalized Kähler-Einstein metric* (cf. [ST17, EGZ18]) on X as $t \rightarrow \infty$.

Step 3. If $T < \infty$ the Kähler-Ricci flow deforms $(X, \omega(t))$ to (Y, ω_Y) with possibly singular metric ω_Y as $t \rightarrow T$.

- (a) If $\dim X = \dim Y$ and Y may be singular. We consider a weak Kähler-Ricci flow on Y starting from ω_Y . This flow should resolve the singularities of Y and replace Y with another variety X^+ . Then we return to Step 1 with X^+ .
- (b) If $0 < \dim Y < \dim X$, then we return to Step 1 with (Y, ω_Y) .
- (c) If $\dim Y = 0$, X should have $c_1(X) > 0$. Moreover, after normalization, the solution $(X, \omega(t))$ of the Kähler-Ricci flow should deform to (X', ω') where X' is possibly different manifold and ω' is either a Kähler-Einstein metric or a Kähler-Ricci soliton (i.e. $\text{Ric}(\omega') = \omega' + \mathcal{L}_V(\omega')$ for a holomorphic vector field V).

Therefore, the Kähler-Ricci flow should construct a sequence of manifolds X_1, \dots, X_k of the MMP, where X_k is either minimal (as in Step 2) or a Mori fiber space (as in Step 3 (c) or (d)). At the very last step, we expect that the Kähler-Ricci flow converges to a canonical metric. Moreover, one would like to show that the process above is continuous in Gromov–Hausdorff topology: the KRF performs geometric surgeries in Gromov–Hausdorff topology at each singular time and replaces the previous projective variety by a “better” model.

As in the Step 3, this program requires to restart the flow from either a singular variety or a initial data which may not be smooth. Therefore we have to consider an appropriate notion of weak Kähler-Ricci flow on a singular variety. The study of degenerate complex Monge-Ampère flows is therefore indispensable. In [ST17], [SzTo11] the authors succeeded in running a certain complex Monge-Ampère flow from continuous initial data, while [GZ17] is running a simplified flow starting from an initial current with zero Lelong numbers. In [DNL17], the authors also extended the work of [GZ17] for the same flow starting from an initial current with positive Lelong numbers. In this thesis we extend these latter works to deal with general complex Monge-Ampère flows and initial data with zero Lelong numbers.

1.1.4 Chern-Ricci flow and geometric applications

The Chern-Ricci flow on a compact Hermitian manifold (X, ω_0) can be written as

$$\frac{\partial}{\partial t} \omega = -\text{Ric}(\omega), \quad \omega|_{t=0} = \omega_0, \quad (1.1.9)$$

where $\text{Ric}(\omega)$ is the Chern-Ricci form of ω (locally we can define $\text{Ric}(\omega) = -i\partial\bar{\partial}\log\omega^n$). This is a generalization of the Kähler-Ricci flow on Hermitian manifolds. This flow was introduced by Gill [Gil11] and has been further developed by Tosatti and Weinkove in [TW13, TW15]. In particular, the flow can be reduced to a certain complex Monge-Ampère flow just like the Kähler-Ricci flow.

In [TW13, TW15] the authors gave a conjectural picture to see that whether the Chern-Ricci flow will give us a geometric classification of compact Hermitian manifolds. On compact complex surfaces, they proved that the Chern-Ricci flow with non-collapsing finite time singularity can blow down finitely many (-1) curves and continue in a unique way on a new complex surface. They conjectured that the Chern-Ricci flow performs a *canonical surgical contraction* in any dimension: the Chern-Ricci flow with non-collapsing finite time singularity can blow down finitely many exceptional divisor and continue on a new manifold in Gromov-Hausdorff sense. In this thesis, we confirm this conjecture.

1.1.5 Other geometric flows

Besides complex Monge-Ampère flows, a number of geometric flows have been introduced to study the structure of compact complex manifolds. The first example is the J -flow on Kähler manifolds which was introduced by Donaldson in the setting of moments maps and by Chen as the gradient flow of the J -functional appearing in the Mabuchi energy. This flow can be seen as an inverse Monge-Ampère flow. Others flows which preserves the Hermitian property have been proposed by Streets-Tian [StT10, StT11, StT13], Liu-Yang [LY12] and also the Anomaly flow due to Phong-Picard-Zhang [PPZ16b, PPZ16c, PPZ17a] which moreover preserves the conformally balanced condition of Hermitian metrics. In particular, the Anomaly flow appears to be a higher order version of the Kähler-Ricci flow.

It is important to develop some new techniques to study these geometric flows and clarify their structure from the PDE point of view. In a joint work with Duong H. Phong [PT17], we studied fully non-linear parabolic flows which generalize complex Monge-Ampère flows, the J -flow etc. We introduced a notion of parabolic C -subsolutions for parabolic equations, extending the theory of C -subsolutions recently developed by B. Guan and more specifically G. Székelyhidi for elliptic equations. The resulting parabolic theory provides a convenient unified approach for the study of many geometric flows.

Since we eventually want to be able to deal with geometric flows on singular spaces, it is important to study the their degenerate versions and to find some appropriate notion of weak flows. In the case of the complex Monge-Ampère equation, one such approach, known as the viscosity method was invented long ago in the real setting [CIL92], but was only recently introduced for elliptic complex Monge-Ampère equations by Eyssidieux-Guedj-Zeriahi [EGZ11], Wang [Wan12] and Harvey-Lawson [HL09] and for complex Monge-Ampère flows by Eyssidieux-Guedj-Zeriahi [EGZ15b, EGZ16, EGZ18]. In a joint work with Slawomir Dinew and Hoang-Son Do, we introduce a viscosity approach for fully non-linear

elliptic equations on domains in \mathbb{C}^n . The arguments are modelled on the theory of viscosity solutions for real Hessian type equations developed by Trudinger [Tru90]. As consequence we solve the Dirichlet problem for the Hessian quotient and special Lagrangian equations.

1.2 Selected results

1.2.1 Regularizing properties of Complex-Monge Ampère flows.

As mentioned in Section 1.1.3, Song and Tian [ST17] have developed a program viewing the Kähler-Ricci flow as a metric version of the Minimal Model Program (MMP), each step of the MMP corresponding to a surgery that is used to repair a finite time singularity of the flow and start it over again. This requires that the Kähler-Ricci flow can be run from a rough initial data. This reduces to study the smoothing property of certain Monge-Ampère flows. In Section 2 (see also [Tô17]) we extended the results in [ST17, SzTo11, GZ17, DNL17] to deal with the following general Monge-Ampère flow on a Kähler manifold (X, ω) with arbitrary initial data:

$$\frac{\partial \varphi}{\partial t} = \log \frac{(\theta_t + i\partial\bar{\partial}\varphi)^n}{\Omega} - F(t, z, \varphi). \quad (1.2.1)$$

where $(t, z, s) \mapsto F(t, z, s)$ is a smooth function on $[0, T] \times X \times \mathbb{R}$ and $(\theta_t)_{t \in [0, T]}$ is a smooth family of Kähler metrics.

Theorem A. *Let φ_0 be a θ_0 -psh function with zero Lelong number at all points. Let $(t, z, s) \mapsto F(t, z, s)$ be a smooth function on $[0, T] \times X \times \mathbb{R}$ such that $\partial F/\partial s$ and $\partial F/\partial t$ are bounded from below.*

Then, there exists a family of smooth strictly θ_t -psh functions (φ_t) satisfying (1.2.1) in $(0, T] \times X$, with $\varphi_t \rightarrow \varphi_0$ in $L^1(X)$, as $t \searrow 0^+$ and φ_t converges to φ_0 in $C^0(X)$ if φ_0 is continuous. This family is moreover unique if $\partial F/\partial t$ is bounded and $\partial F/\partial s \geq 0$.

Moreover, we also prove the following stability result:

Theorem B. *Let $(t, z, s) \mapsto F(t, z, s)$ be a smooth function on $[0, T] \times X \times \mathbb{R}$ such that $\partial F/\partial s \geq 0$ and $\partial F/\partial t$ are bounded. Let $\varphi_0, \varphi_{0,j}$ be ω -psh functions with zero Lelong number at all points, such that $\varphi_{0,j} \rightarrow \varphi_0$ in $L^1(X)$. Denote by $\varphi_{t,j}$ and φ_t the corresponding solutions of (1.2.1) with initial condition $\varphi_{0,j}$ and φ_0 respectively. Then for each $\varepsilon \in (0, T)$*

$$\varphi_{t,j} \rightarrow \varphi_t \text{ in } C^\infty([\varepsilon, T] \times X) \text{ as } j \rightarrow +\infty.$$

Moreover, if φ_0 and ψ_0 are two continuous initial condition and φ and ψ are two corresponding solutions of the flow (5.2.2), then for any $k \geq 0$, for any $0 < \varepsilon < T$, there exists a positive constant $C(k, \varepsilon)$ depending only on k and ε such that

$$\|\varphi - \psi\|_{C^k([\varepsilon, T] \times X)} \leq C(k, \varepsilon) \|\varphi_0 - \psi_0\|_{L^\infty(X, \omega)}.$$

The same results are also proved for complex Monge-Ampère flows on Hermitian manifolds in [Tô18] (see Section 3). Our techniques are based on pluripotential theory and a priori estimates for Monge-Ampère equations.

1.2.2 Geometric contraction of Chern-Ricci flow on Hermitian manifolds

As explained above, a generalization of the Kähler-Ricci flow on compact Hermitian manifolds is the Chern-Ricci flow

$$\frac{\partial}{\partial t}\omega = -\text{Ric}(\omega), \quad \omega|_{t=0} = \omega_0. \quad (1.2.2)$$

It has been shown in [TW15] that the maximum existence time of smooth solutions is

$$T := \sup\{t \geq 0 \mid \exists \psi \in C^\infty(X) \text{ with } \hat{\omega}_X + i\partial\bar{\partial}\psi > 0\},$$

where $\hat{\omega}_X = \omega_0 + t\chi$, with χ a smooth $(1, 1)$ -form representing $-c_1^{BC}(X)$, the first Bott-Chern class of X .

In [TW15, TW13], assuming the existence of a holomorphic map $\pi : X \rightarrow Y$ contracting a divisor E to $y_0 \in Y$, Tosatti and Weinkove proved that the solution ω_t of (3.5.1) converges, as $t \rightarrow T$, in $C_{loc}^\infty(X \setminus E)$ to a smooth Hermitian metric ω_T on $X \setminus E$. Moreover, there exists a distance function d_T on Y such that (Y, d_T) is a compact metric space and $(X, g(t))$ converges in the Gromov-Hausdorff sense (Y, d_T) as $t \rightarrow T^-$. They conjectured that the Chern-Ricci flow can be started on the new manifold Y with the rough initial data $\pi_*\omega_T$, and we have backward smooth and Gromov-Hausdorff convergence (cf. [TW13, Page 2120]). In [Tô18] (see Section 3) we confirmed this conjecture using the smoothing property of Monge-Ampère flows on Hermitian manifolds proved in the first part of [Tô18]. Precisely, we have

Theorem C. (1) *There exists a smooth maximal solution ω_t of the Chern-Ricci flow on Y for $t \in (T, T_Y)$ with $T < T_Y \leq +\infty$ such that ω_t converges to $\pi_*\omega_T$, as $t \rightarrow T^+$, in $C_{loc}^\infty(Y \setminus \{y_0\})$. Furthermore, ω_t is uniquely determined by ω_0 .*

(2) *The metric space (Y, ω_t) converges to (Y, d_T) as $t \rightarrow T^+$ in the Gromov-Hausdorff sense.*

In a different direction, we introduced a generalization of the Chern-Ricci flow, namely the twisted Chern-Ricci flow

$$\frac{\partial \omega_t}{\partial t} = -\text{Ric}(\omega_t) + \eta, \quad \omega|_{t=0} = \omega_0$$

where ω_0 is a Hermitian metric on X and η is a smooth $(1, 1)$ -form. In particular, if the manifold has negative twisted first Chern class, the twisted Chern-Ricci flow exists for all times and converges to a twisted Einstein metric.

Theorem D. *Suppose $c_1^{BC}(X) - \{\eta\} < 0$. The normalized twisted Chern-Ricci flow smoothly converges to a Hermitian metric $\omega_\infty = \eta - Ric(\Omega) + i\partial\bar{\partial}\varphi_\infty$ which satisfies*

$$Ric(\omega_\infty) = \eta - \omega_\infty.$$

This also give a proof of the existence of a twisted Einstein metric in the twisted Bott-Chern class $c_1^{BC}(X) - \{\eta\}$.

1.2.3 Fully non-linear parabolic equations on compact Hermitian manifolds.

Let (X, α) be a compact Hermitian manifold of dimension n , and $\chi(z)$ be a real $(1, 1)$ -form. If $u \in C^2(X)$, let $A[u]$ be the matrix with entries $A[u]^k_j = \alpha^{k\bar{m}}(\chi_{\bar{m}j} + \partial_j\partial_{\bar{m}}u)$. We consider the fully nonlinear parabolic equation,

$$\partial_t u = F(A[u]) - \psi(z), \quad (1.2.3)$$

where $F(A)$ is a smooth symmetric function $F(A) = f(\lambda[u])$ of the eigenvalues $\lambda_j[u]$, $1 \leq j \leq n$ of $A[u]$, defined on a open symmetric, convex cone $\Gamma \subset \mathbb{R}^n$ with vertex at the origin and containing the positive orthant Γ_n . We shall assume that f satisfies the following conditions:

- (1) $f_i > 0$ for all i , and f is concave.
- (2) $f(\lambda) \rightarrow -\infty$ as $\lambda \rightarrow \partial\Gamma$
- (3) For any $\sigma < \sup_\Gamma f$ and $\lambda \in \Gamma$, we have $\lim_{t \rightarrow \infty} f(t\lambda) > \sigma$.

Fix $T \in (0, \infty]$. The following notion of subsolution is an adaptation to the parabolic case of Székelyhidi's [Sze15] notion in the elliptic case:

Definition 1. *An admissible function $\underline{u} \in C^{2,1}(X \times [0, T])$ is said to be a (parabolic) C -subsolution of (4.1.1), if there exist constants $\delta, K > 0$, so that for any $(z, t) \in X \times [0, T)$, the condition*

$$f(\lambda[\underline{u}(z, t)] + \mu) - \partial_t \underline{u} + \tau = \psi(z), \quad \mu + \delta I \in \Gamma_n, \quad \tau > -\delta \quad (1.2.4)$$

implies that $|\mu| + |\tau| < K$. Here I denotes the vector $(1, \dots, 1)$ of eigenvalues of the identity matrix.

To discuss our results, we need a finer classification of non-linear partial differential operators due to Trudinger [Tru95]. Let Γ_∞ be the projection of Γ_n onto \mathbf{R}^{n-1} ,

$$\Gamma_\infty = \{\lambda' = (\lambda_1, \dots, \lambda_{n-1}); \lambda = (\lambda_1, \dots, \lambda_n) \in \Gamma \text{ for some } \lambda_n\} \quad (1.2.5)$$

and define the function f_∞ on Γ_∞ by

$$f_\infty(\lambda') = \lim_{\lambda_n \rightarrow \infty} f(\lambda', \lambda_n). \quad (1.2.6)$$

It is shown in [Tru95] that, as a consequence of the concavity of f , the limit is either finite for all $\lambda' \in \Gamma_\infty$ or infinite for all $\lambda' \in \Gamma_\infty$. We shall refer to the first case as the *bounded case*, and to the second case as the *unbounded case*. For example, Monge-Ampère flows belong to the unbounded case, while the J -flow and Hessian quotient flows belong to the bounded case. In the unbounded case, any admissible function, and in particular 0 if $\lambda[\chi] \in \Gamma$, is a C -subsolution in both the elliptic and parabolic cases. We have then:

Theorem E. *Consider the flow (1.2.3), and assume that f is in the unbounded case. Then for any admissible initial data u_0 , the flow admits a smooth solution $u(z, t)$ on $[0, \infty)$, and its normalization \tilde{u} defined by*

$$\tilde{u} := u - \frac{1}{V} \int_X u \alpha^n, \quad V = \int_X \alpha^n, \quad (1.2.7)$$

converges in C^∞ to a function \tilde{u}_∞ satisfying the following equation for some constant c ,

$$F(A[\tilde{u}_\infty]) = \psi(z) + c. \quad (1.2.8)$$

The situation is more complicated when f belongs to the bounded case:

Theorem F. *Consider the flow (1.2.3), and assume that it admits a subsolution \underline{u} on $X \times [0, \infty)$, but that f is in the bounded case. Then for any admissible data u_0 , the equation admits a smooth solution $u(z, t)$ on $(0, \infty)$. Let \tilde{u} be the normalization of the solution u , defined as before by (4.1.5). Assume that either one of the following two conditions holds.*

(a) *The initial data and the subsolution satisfy*

$$\partial_t \underline{u} \geq \sup_X (F(A[u_0]) - \psi); \quad (1.2.9)$$

(b) *or there exists a function $h(t)$ with $h'(t) \leq 0$ so that*

$$\sup_X (u(t) - h(t) - \underline{u}(t)) \geq 0 \quad (1.2.10)$$

and the Harnack inequality

$$\sup_X (u(t) - h(t)) \leq -C_1 \inf_X (u(t) - h(t)) + C_2 \quad (1.2.11)$$

holds for some constants $C_1, C_2 > 0$ independent of time.

Then \tilde{u} converges in C^∞ to a function \tilde{u}_∞ satisfying

$$F(A[\tilde{u}_\infty]) = \psi(z) + c. \quad (1.2.12)$$

for some constant c .

The resulting parabolic theory provides a unified approach for the study of many geometric flows, for examples the Kähler-Ricci flow, the J -flow, complex Hessian quotient flows, etc. As an application, we answer a question raised for general $1 \leq \ell < k \leq n$ by Fang-Lai-Ma [FLM11], and extend the solution obtained for $k = n$ by Collins-Székelyhidi [CS17]:

Theorem G. *Assume that (X, α) is a compact Kähler n -manifold, and fix $1 \leq \ell < k \leq n$. Fix a closed $(1, 1)$ -form χ which is k -positive and non-negative constants c_j , and assume that there exists a form $\chi' = \chi + i\partial\bar{\partial}u$ which is a closed k -positive form and satisfies*

$$kc(\chi')^{k-1} \wedge \alpha^{n-k} - \sum_{j=1}^{\ell} jc_j(\chi')^{j-1} \wedge \alpha^{n-j} > 0, \quad (1.2.13)$$

in the sense of positivity of $(n-1, n-1)$ -forms. Here the constant c is given by

$$c[\chi^k][\alpha^{n-k}] = \sum_{j=1}^{\ell} c_j[\chi^j][\alpha^{n-j}].$$

Then the flow

$$\partial_t u = -\frac{\sum_{j=1}^{\ell} c_j \chi_u^j \wedge \alpha^{n-j}}{\chi_u^k \wedge \alpha^{n-k}} + c, \quad u(\cdot, 0) = 0,$$

admits a solution for all time which converges smoothly to a function u_∞ as $t \rightarrow \infty$. The form $\omega = \chi + i\partial\bar{\partial}u_\infty$ is k -positive and satisfies the equation

$$c\omega^k \wedge \alpha^{n-k} = \sum_{j=1}^{\ell} c_j \omega^j \wedge \alpha^{n-j}.$$

1.2.4 A viscosity approach to the Hessian type equations

With Sławomir Dinew and Hoang Son Do [DDT17], we have developed a viscosity theory for general complex Hessian type equations on complex domains, inspired by the theory of viscosity solutions in the real case developed by Trudinger [Tru90].

Let Ω be a bounded domain in \mathbb{C}^n . For any function $u \in C^2(\Omega)$ and $z \in \Omega$, denote by $Hu(z)$ the Hessian matrix of u at z . We consider the Hessian type equation of the form

$$F[u] = \psi(z, u), \quad (1.2.14)$$

where $\psi \in C^0(\Omega \times \mathbb{R}, \mathbb{R}^+)$ and $F[u] = f(\lambda(Hu))$ such that $s \mapsto \psi(\cdot, s)$ is increasing,

Theorem H (Comparison principle). *Let Γ be the ellipticity cone associated to the equation (1.2.14). Assume that the operator $F[u] = f(\lambda(Hu))$ in (1.2.14) satisfies*

$$f \in C^0(\bar{\Gamma}), f > 0 \text{ on } \Gamma, f = 0 \text{ on } \partial\Gamma,$$

and

$$f(\lambda + \mu) \geq f(\lambda), \quad \forall \lambda \in \Gamma, \mu \in \Gamma_n.$$

Assume moreover that either

$$\sum_{i=1}^n \frac{\partial f}{\partial \lambda_i} \lambda_i = \sum_{i=1}^n f_i \lambda_i \geq \nu(f) \text{ in } \Gamma, \text{ and } \inf_{z \in \Omega} \psi(z, \cdot) > 0$$

for some positive increasing function ν , or

f is concave and homogeneous.

Then any bounded subsolution u and supersolution v in Ω to the equation (5.1.3) satisfy

$$\sup_{\Omega} (u - v) \leq \max_{\partial\Omega} \{(u - v)^*, 0\}.$$

One of our main results is the solvability and sharp regularity for viscosity solutions to the Dirichlet problem for a very general class of operators including Hessian quotient type equations.

Theorem I. *The Dirichlet problem*

$$\begin{cases} F[w] = f(\lambda(Hw)) = \psi(z, w(z)) \\ u = \varphi \text{ on } \partial\Omega \end{cases}$$

admits a continuous solution for any bounded Γ -pseudoconvex domain Ω . Under natural growth assumptions on ψ the solution is Hölder continuous for any Hölder continuous boundary data φ .

A large part of the work is devoted to complex Hessian quotient equations in domains in \mathbb{C}^n . One of our goals in this case was to initiate the construction of the undeveloped pluripotential theory associated to such equations. We rely on connections with the corresponding viscosity theory. Our findings yield in particular that the natural domain of definition of these operators is *strictly smaller* than what standard pluripotential theory would predict. We prove the following theorem:

Theorem J. *Assume that $0 < \psi \in C^0(\Omega)$ and $u \in PSH(\Omega) \cap L_{loc}^\infty(\Omega)$ is a viscosity subsolution of $\frac{(dd^c u)^n}{(dd^c u)^{n-k} \wedge \omega^k} = \psi(z)$ in Ω . Then*

$$(dd^c u)^n \geq \psi (dd^c u)^{n-k} \wedge \omega^k$$

and

$$(dd^c u)^k \geq \binom{n}{k}^{-1} \psi \omega^k$$

in the pluripotential sense.

We wish to point out that nonlinear PDEs appear also in geometric problems which are defined over domains in \mathbb{C}^n see for example [CPW17], where a Dirichlet problem for the *special Lagrangian type equation* is studied. We show in Section 5.6 that our method can be applied to solve the Dirichlet problem

$$\begin{cases} F[u] := \sum_{i=1}^n \arctan \lambda_i = h(z), & \text{on } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \quad (1.2.15)$$

where $\lambda_1, \dots, \lambda_n$ is the eigenvalues of the complex Hessian Hu . We can also write $F[u] = f(\lambda(Hu))$. We assume that $\varphi \in C^0(\partial\Omega)$ and $h : \bar{\Omega} \rightarrow [(n-2)\frac{\pi}{2} + \delta, n\frac{\pi}{2}]$ is continuous, for some $\delta > 0$.

Theorem K. *Let Ω is a bounded C^2 domain. Let \underline{u} is an bounded upper semi-continuous function on Ω satisfying $F[\underline{u}] \geq h(z)$ in Ω in the viscosity sense and $\underline{u} = \varphi$ on $\partial\Omega$. Then the Dirichlet problem (1.2.15) admits a unique viscosity solution $u \in C^0(\Omega)$.*

1.3 Warning to the reader

The author has profited from preparing this manuscript to correct some mistakes in [Tô17, Tô18]:

(1) In [Tô17, Theorem A] we need to add a condition on F that $\frac{\partial F}{\partial t}$ is bounded below. This condition is only used in [Tô17, Proposition 2.4].

In the proof of Theorem A in [Tô17], we reduced $\frac{\partial F}{\partial s} \geq -C_0$ to $\frac{\partial F}{\partial s} \geq 0$ by change of variables. However, this reduction does not preserve the condition that $\frac{\partial F}{\partial t}$ is bounded below. Therefore we need to replace [Tô17, Proposition 2.4] by a similar proposition with the hypothesis $\frac{\partial F}{\partial s} \geq -C_0$. In this thesis, these modifications above are given in Theorem A, section 2.1, Chapter 2 and Proposition 2.3.4.

(2) At the beginning of the proof of [Tô17, Proposition 2.4], we need to add a sentence that "Since we consider the interval time $[\varepsilon, T]$, we can assume the flow starting from φ_ε , i.e $\varphi(0, x) = \varphi_\varepsilon$ ". Then we need to change:

- (a) in line 1 of the proof, $[\varepsilon, T] \times X$ to $[0, T - \varepsilon] \times X$
- (b) in line 10 of the proof, $t = \varepsilon$ to $t = 0$
- (c) in line 11 of the proof, $t_0 > \varepsilon$ to $t_0 > 0$.

In this thesis, this modification is given in the proof of Theorem 2.3.4.

(3) At the beginning of the proof of [Tô17, Proposition 2.7], we add a sentence that "Since we deal with the interval time $[\varepsilon, T]$ and the bound on $Osc_X(\varphi_\varepsilon)$ only depends on $\sup_X \varphi_0$ and ε , we can consider the flow starting from φ_ε , i.e $\varphi(0, x) = \varphi_\varepsilon$ ". Then we need to change:

- (a) in line 4 of the proof, $t = \varepsilon$ to $t = 0$
- (b) in line 5 of the proof, $t \in [\varepsilon, T]$ to $t \in [0, T - \varepsilon]$.
- (c) in line 7 of the proof, $[\varepsilon, T] \times X$ with $t_0 > \varepsilon$ to $[0, T - \varepsilon]$ with $t_0 > 0$.

In this thesis, the modification above is given in the proof of Theorem 2.3.7.

(4) In the proof of [Tô18, Proposition], Claim 1, the term with the constant C_4 should be $\frac{C_4}{\beta} \gamma' \circ \varphi \sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}}$ instead of $C_4 \sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}}$. In this thesis, this modification is given in Claim 1, the proof of Proposition 3.3.10.

We would like to thank Nguyen Van Hoang for pointing out the errors (2) and (3) and Tao Zheng for pointing out the error (4) in the original proofs and interesting exchanges.

Chapter 2

Regularizing properties of Complex Monge-Ampère flows

We study the regularizing properties of complex Monge–Ampère flows on a Kähler manifold when the initial data are ω -psh functions with zero Lelong number at all points. We prove that the general Monge–Ampère flow has a solution which is immediately smooth. We also prove the uniqueness and stability of solution.

The results of this chapter can be found in [Tô17].

2.1 Introduction

Let (X, ω) be a compact Kähler manifold of complex dimension n and $\alpha \in H^{1,1}(X, \mathbb{R})$ a Kähler class with $\omega \in \alpha$. Let Ω be a smooth volume form on X . Denote by $(\theta_t)_{t \in [0, T]}$ a family of Kähler forms on X , and assume that $\theta_0 = \omega$. The goal of this note is to prove the regularizing and stability properties of solutions to the following complex Monge-Ampère flow

$$(CMAF) \quad \frac{\partial \varphi_t}{\partial t} = \log \frac{(\theta_t + dd^c \varphi_t)^n}{\Omega} - F(t, z, \varphi_t)$$

where F is a smooth function and $\varphi(0, z) = \varphi_0(z)$ is a ω -plurisubharmonic (ω -psh) function with zero Lelong numbers at all points.

One motivation for studying this Monge-Ampère flow is that the Kähler-Ricci flow can be reduced to a particular case of (CMAF). When $F = F(z)$ and $\theta_t = \omega + t\chi$, where $\chi = \eta - Ric(\omega)$, then (CMAF) is the local potential equation of the twisted Kähler-Ricci flow

$$\frac{\partial \omega_t}{\partial t} = -Ric(\omega_t) + \eta,$$

which was studied recently by several authors.

Running the Kähler-Ricci flow from a rough initial data has been the purpose of several recent works [CD07], [ST17], [SzTo11], [GZ17], [BG13], [DNL17]. In [ST17], [SzTo11] the authors succeeded to run (CMAF) from continuous initial data, while [DNL17] and [GZ17] are running a simplified flow starting from an initial current with zero Lelong numbers. In this note we extend these latter works to deal with general (CMAF) and arbitrary initial data.

A strong motivation for studying (CMAF) with degenerate initial data comes from the Analytic Minimal Model Program introduced by J. Song and G. Tian [ST17], [ST12]. It requires to study the behavior of the Kähler-Ricci flow on mildly singular varieties, and one is naturally lead to study weak solutions of degenerate complex Monge-Ampère flows (when the function F in (CMAF) is not smooth but continuous). Eyssidieux-Guedj-Zeriahi have developed in [EGZ16] a viscosity theory for degenerate complex Monge-Ampère flows which allows in particular to define and study the Kähler-Ricci flow on varieties with canonical singularities.

Our main result is the following:

Theorem A. *Let φ_0 be a ω -psh function with zero Lelong numbers at all points. Let $(t, z, s) \mapsto F(t, z, s)$ be a smooth function on $[0, T] \times X \times \mathbb{R}$ such that $\frac{\partial F}{\partial s}$ and $\frac{\partial F}{\partial t}$ are bounded from below.*

Then there exists a family of smooth strictly θ_t -psh functions (φ_t) satisfying (CMAF) in $(0, T] \times X$, with $\varphi_t \rightarrow \varphi_0$ in $L^1(X)$, as $t \searrow 0^+$. This family is moreover unique if $\frac{\partial F}{\partial s} \geq 0$ and $|\frac{\partial F}{\partial t}| < C'$ for some $C' > 0$.

We further show that

- φ_t converges to φ_0 in $C^0(X)$ if φ_0 is continuous.
- φ_t converges to φ_0 in capacity if φ_0 is merely bounded.
- φ_t converges to φ_0 in energy if $\varphi \in \mathcal{E}^1(X, \omega)$ has finite energy.

Moreover, we also prove the following stability result:

Theorem B. *Let $\varphi_0, \varphi_{0,j}$ be ω -psh functions with zero Lelong number at all points, such that $\varphi_{0,j} \rightarrow \varphi_0$ in $L^1(X)$. Denote by $\varphi_{t,j}$ and φ_j the corresponding solutions of (CMAF) with initial condition $\varphi_{0,j}$ and φ_0 respectively. Then for each $\varepsilon \in (0, T)$*

$$\varphi_{t,j} \rightarrow \varphi_t \text{ in } C^\infty([\varepsilon, T] \times X) \text{ as } j \rightarrow +\infty.$$

Moreover, if φ_0 and ψ_0 are continuous, then for any $k \geq 0$, for any $0 < \varepsilon < T$, there exists a positive constant $C(k, \varepsilon)$ depending only on k and ε such that

$$\|\varphi - \psi\|_{C^k([\varepsilon, T] \times X)} \leq C(k, \varepsilon) \|\varphi_0 - \psi_0\|_{L^\infty(X, \omega)}.$$

We also prove in Section 2.6 that one can run the Monge-Ampère flow from a positive current representing a nef class, generalizing results from [GZ17], [DNL17].

The chapter is organized as follows. In Section 2.2 we recall some analytic tools, and give the strategy of proof of Theorem A. In Section 2.3 we prove various a priori estimates for the regular case. In Section 2.4 we prove Theorem A using the a priori estimates from Section 2.3. In Section 2.5 we prove the uniqueness in Theorem A and Theorem B. In Section 2.6 we show that the Monge-Ampère flow can run from a nef class.

2.2 Preliminaries and Strategy

In this section we recall some analytic tools which will be used in the sequel.

2.2.1 Plurisubharmonic functions and Lelong number

Let (X, ω) be a compact Kähler manifold. We define the following operators:

$$d := \partial + \bar{\partial}, \quad d^c := \frac{1}{2i\pi}(\partial - \bar{\partial}).$$

Definition 2.2.1. We let $PSH(X, \omega)$ denote the set of all ω -plurisubharmonic functions (ω -psh for short), i.e the set of functions $\varphi \in L^1(X, \mathbb{R} \cup \{-\infty\})$ which can be locally written as the sum of a smooth and a plurisubharmonic function, and such that

$$\omega + dd^c\varphi \geq 0$$

in the weak sense of positive currents.

Definition 2.2.2. Let φ be a ω -psh function and $x \in X$. The *Lelong number* of φ at x is

$$\nu(\varphi, x) := \liminf_{z \rightarrow x} \frac{\varphi(z)}{\log|z - x|}.$$

We say φ has a *logarithmic pole of coefficient* γ at x if $\nu(\varphi, x) = \gamma$.

2.2.2 A Laplacian inequality

Let α and ω be $(1, 1)$ -forms on a complex manifold X with $\omega > 0$. Then the trace of α with respect ω is defined as

$$\mathrm{tr}_\omega(\alpha) = n \frac{\alpha \wedge \omega^{n-1}}{\omega^n}.$$

We can diagonalize α with respect to ω at each point of X , with real eigenvalues $\lambda_1 \dots, \lambda_n$ then $\mathrm{tr}_\omega(\alpha) = \sum_j \lambda_j$. The Laplace of a function φ with respect to ω is given by

$$\Delta_\omega \varphi = \mathrm{tr}_\omega(dd^c\varphi).$$

We have the following eigenvalue estimate:

Lemma 2.2.3. *If ω and ω' are two positive (1,1)-forms on a complex manifold X of dimension n , then*

$$\left(\frac{\omega'^n}{\omega^n}\right)^{\frac{1}{n}} \leq \frac{1}{n} \operatorname{tr}_\omega(\omega') \leq \left(\frac{\omega'^n}{\omega^n}\right) (\operatorname{tr}_{\omega'}(\omega))^{n-1}.$$

The next result is a basic tool for establishing second order a priori estimates for complex Monge-Ampère equations.

Proposition 2.2.4 ([Siu87]). *Let ω, ω' be two Kähler forms on a compact complex manifold. If the holomorphic bisectional curvature of ω is bounded below by a constant $B \in \mathbb{R}$ on X , then we have*

$$\Delta_{\omega'} \log \operatorname{tr}_\omega(\omega') \geq -\frac{\operatorname{tr}_\omega \operatorname{Ric}(\omega')}{\operatorname{tr}_\omega(\omega')} + B \operatorname{tr}_{\omega'}(\omega).$$

2.2.3 Maximum principle and comparison theorem

We establish here a slight generalization of the comparison theorem that we will need.

Proposition 2.2.5. *Let $\varphi, \psi \in C^\infty([0, T] \times X)$ be θ_t -psh functions such that*

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &\leq \log \frac{(\theta_t + dd^c \varphi)^n}{\Omega} - F(t, z, \varphi), \\ \frac{\partial \psi}{\partial t} &\geq \log \frac{(\theta_t + dd^c \psi)^n}{\Omega} - F(t, z, \psi), \end{aligned}$$

where $F(t, z, s)$ is a smooth function with $\frac{\partial F}{\partial s} \geq -\lambda$. Then

$$\sup_{[0, T] \times X} (\varphi_t - \psi_t) \leq e^{\lambda T} \max \left\{ \sup_X (\varphi_0 - \psi_0); 0 \right\}. \quad (2.2.1)$$

In particular, if $\varphi_0 \leq \psi_0$, then $\varphi_t \leq \psi_t$.

Proof. We define $u(x, t) = e^{-\lambda t}(\varphi_t - \psi_t)(x) - \varepsilon t \in C^\infty([0, T] \times X)$ where $\varepsilon > 0$ is fixed. Suppose u is maximal at $(t_0, x_0) \in [0, T] \times X$. If $t_0 = 0$ then we have directly the estimate (2.2.1). Assume now $t_0 > 0$, using the maximum principle, we get $\dot{u} \geq 0$ and $dd_x^c u \leq 0$ at (t_0, x_0) , hence

$$-\lambda e^{-\lambda t}(\varphi_t - \psi_t) + e^{-\lambda t}(\dot{\varphi}_t - \dot{\psi}_t) \geq \varepsilon > 0 \text{ and } dd_x^c \varphi_t \leq dd_x^c \psi_t.$$

Observing that at (t_0, x_0)

$$\dot{\varphi} - \dot{\psi} \leq F(t, x, \psi) - F(t, x, \varphi),$$

we infer that

$$0 < F(t, x, \psi) + \lambda \psi - [F(t, x, \varphi) + \lambda \varphi],$$

at (t_0, z_0) . Since $\frac{\partial F}{\partial s} \geq -\lambda$, $F(t, x, s) + \lambda s$ is an increasing function in s , hence $\varphi_{t_0}(x_0) \leq \psi_{t_0}(x_0)$. Thus $u(x, t) \leq u(x_0, t_0) \leq 0$. Letting $\varepsilon \rightarrow 0$, this yields

$$\sup_{[0, T] \times X} (\varphi_t - \psi_t) \leq e^{\lambda T} \max \left\{ \sup_X (\varphi_0 - \psi_0); 0 \right\}.$$

□

The following proposition has been given for the twisted Kähler-Ricci flow by Di Nezza and Lu [DNL17]:

Proposition 2.2.6. *Assume ψ_t a smooth solution of (CMAF) with a smooth initial data ψ_0 and φ_t is a subsolution of (CMAF) with initial data φ_0 which is a ω -psh function with zero Lelong number at all point: i.e $\varphi_t \in C^\infty((0, T] \times X)$ satisfies*

$$\frac{\partial \varphi_t}{\partial t} \leq \log \frac{(\theta_t + dd^c \varphi_t)^n}{\Omega} - F(t, z, \varphi_t),$$

and $\varphi_t \rightarrow \varphi_0$ in $L^1(X)$. Suppose that $\varphi_0 \leq \psi_0$, then $\varphi_t \leq \psi_t$.

Proof. Fix $\epsilon > 0$ and note that $\varphi - \psi$ is a smooth function on $[\epsilon, T] \times X$. It follows from Proposition 2.2.5 that

$$\varphi - \psi \leq e^{\lambda T} \max \left\{ \sup_X (\varphi_\epsilon - \psi_\epsilon); 0 \right\}.$$

By hypothesis $\psi_t \in C^\infty([0, T] \times X)$, hence for any $\delta > 0$ there exists $\epsilon_0 > 0$ such that $\psi_\epsilon(x) \geq \psi_0(x) - \delta$ for all $x \in X$ and $\epsilon \leq \epsilon_0$. This yields

$$\overline{\lim}_{\epsilon \rightarrow 0} \sup_X (\varphi_\epsilon - \psi_\epsilon) \leq \overline{\lim}_{\epsilon \rightarrow 0} \sup_X (\varphi_\epsilon - \psi_0) + \delta.$$

Since φ_t converges to φ_0 in $L^1(X)$ as $t \rightarrow 0$, it follows from Hartogs' Lemma (see for instance [Hör94, Theorem 3.2.13])

$$\overline{\lim}_{\epsilon \rightarrow 0} \sup_X (\varphi_\epsilon - \psi_0) \leq \sup_X (\varphi_0 - \psi_0) \leq 0.$$

Therefore for any $\delta > 0$, we have

$$\overline{\lim}_{\epsilon \rightarrow 0} \sup_X (\varphi_\epsilon - \psi_\epsilon) \leq \delta,$$

hence

$$\overline{\lim}_{\epsilon \rightarrow 0} \sup_X (\varphi_\epsilon - \psi_\epsilon) \leq 0.$$

This implies that $\varphi_t \leq \psi_t$ for all $0 \leq t \leq T$. □

Remark 2.2.7. Note that if φ_0 has some positive Lelong numbers, we cannot expect the existence of a smooth solution φ_t in $(0, T] \times X$ (see [DNL17, Theorem A]).

2.2.4 Evans-Krylov and Schauder estimates for Monge-Ampère flow

The Evans-Krylov and Schauder theorems for nonlinear elliptic equations

$$F(D^2u) = f$$

with F concave, are used to show that bounds on u, D^2u imply $C^{2,\alpha}$ on u for some $\alpha > 0$ and higher order bounds on u . There are also Evans-Krylov estimates for parabolic equations (see [Lie96]), but the precise version we need is as follows

Theorem 2.2.8. *Let $U \Subset \mathbb{C}^n$ be an open subset and $T \in (0, +\infty)$. Suppose that $u \in C^\infty([0, T] \times \bar{U})$ and $(t, x, s) \mapsto f(t, x, s)$ is a function in $C^\infty([0, T] \times \bar{U} \times \mathbb{R})$, satisfy*

$$\frac{\partial u}{\partial t} = \log \det \left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) + f(t, x, u). \quad (2.2.2)$$

In addition, assume that there is a constant $C > 0$ such that

$$\sup_{(0,T) \times U} \left(|u| + \left| \frac{\partial u}{\partial t} \right| + |\nabla u| + |\Delta u| \right) \leq C.$$

Then for any compact $K \Subset U$, for each $\varepsilon > 0$ and $p \in \mathbb{N}$,

$$\|u\|_{C^p([\varepsilon, T] \times K)} \leq C_0.$$

where C_0 only depends on C and $\|f\|_{C^q([0, T] \times \bar{U} \times [-C, C])}$ for some $q \geq p - 2$.

The proof of this theorem follows the arguments of Boucksom-Guedj [BG13, Theorem 4.1.4] where the function f is independent of u .

First of all, we recall the *parabolic α -Hölder norm* of a function f on the cylinder $Q = U \times (0, T)$:

$$\|f\|_{C^\alpha(Q)} := \|f\|_{C^0(Q)} + [f]_{\alpha, Q},$$

where

$$[f]_{\alpha, Q} := \sup_{X, Y \in Q, X \neq Y} \frac{|f(X) - f(Y)|}{\rho^\alpha(X, Y)}$$

is the *α -Hölder seminorm with respect to the parabolic distance*

$$\rho((x, t), (x', t')) = |x - x'| + |t - t'|^{1/2}.$$

For each $k \in \mathbb{N}$, the *$C^{k,\alpha}$ -norm* is defined as

$$\|f\|_{C^{k,\alpha}(Q)} := \sum_{|m|+2j \leq k} \|D_x^m D_t^j f\|_{C^\alpha(Q)}.$$

If $(\omega_t)_{t \in (0, T)}$ is a path of differential forms on U , we can similarly define $[\omega_t]_{\alpha, Q}$ and $\|\omega_t\|_{C^{k,\alpha}(Q)}$, with respect to the flat metric ω_U on U .

The first ingredient in the proof of Theorem 2.2.8 is the Schauder estimates for linear parabolic equations.

Lemma 2.2.9. (*[Kry96, Theorem 8.11.1], [Lie96, Theorem 4.9]*) Let $(\omega_t)_{t \in (0, T)}$ be a smooth path of Kähler metrics on U and ω_U be the flat metric on U . Define $Q = U \times (0, T)$, and assume that $u, g \in C^\infty(Q)$ satisfy

$$\left(\frac{\partial}{\partial t} - \Delta_t - c(t, x) \right) u(t, x) = g(t, x),$$

where Δ_t is the Laplacian with respect to ω_t . Suppose also that there exist $C > 0$ and $0 < \alpha < 1$ such that on Q we have

$$C^{-1}\omega_U \leq \omega_t \leq C\omega_U, \|c\|_{C^\alpha(Q)} \leq C \text{ and } [\omega_t]_{\alpha, Q} \leq C.$$

Then for each $Q' = U' \times (\varepsilon, T)$ with $U' \Subset U$, we can find a constant A only depending on U', ε and C such that

$$\|u\|_{C^{2, \alpha}(Q)} \leq A(\|u\|_{C^0(Q)} + \|g\|_{C^\alpha(Q)}).$$

The second ingredient in the proof Theorem 2.2.8 is the following Evans-Krylov type estimate for complex Monge-Ampère flows.

Lemma 2.2.10. (*[Gil11, Theorem 4.1]*) Suppose $u, g \in C^\infty(Q)$ satisfy

$$\frac{\partial u}{\partial t} = \log \det \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} + g(t, x),$$

and assume also that there exists a constant $C > 0$ such that

$$C^{-1} \leq \left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) \leq C \text{ and } \left| \frac{\partial g}{\partial t} \right| + |dd^c g| \leq C.$$

Then for each $Q' = U' \times (\varepsilon, T)$ with $U' \Subset U$ an open subset and $\varepsilon \in (0, T)$, we can find $A > 0$ and $0 < \alpha < 1$ only depending on U', ε and C such that

$$[dd^c u]_{\alpha, Q'} \leq A.$$

Proof of Theorem 2.2.8. In the sequel of the proof, we say that a constant is under control if it is bounded by the terms of C, ε and $\|f\|_{C^q([0, T] \times \bar{U} \times [-C, C])}$.

Consider the path $\omega_t := dd^c u_t$ of Kähler forms on U . Denote by ω_U the flat metric on U . It follows from (2.2.2) that

$$\omega_t^n = \exp \left(\frac{\partial u}{\partial t} - f \right) \omega_U^n.$$

Since $\frac{\partial u}{\partial t} - f$ is bounded by a constant under control by the assumption, there exists a constant C_1 under control such that $C_1^{-1} \omega_U^n \leq \omega_t^n \leq C_1 \omega_U^n$. It follows from the assumption

that $\text{tr}_{\omega_U} \omega_t$ is bounded. Two latter inequalities imply that $C_2^{-1}\omega_U \leq \omega_t \leq C_2\omega_U$ for some $C_2 > 0$ under control by considering inequalities of eigenvalues. Set $g(t, x) := f(t, x, u)$. Since $C_2^{-1}\omega_U \leq \omega_t \leq C_2\omega_U$ and

$$\sup_{(0,T) \times U} \left(|u| + \left| \frac{\partial u}{\partial t} \right| + |\nabla u| + |\Delta u| \right) \leq C,$$

we get $\left| \frac{\partial g}{\partial t} \right| + |dd^c g| \leq C_3$ with C_3 under control. Apply Lemma 2.2.10 to (2.2.2), we obtain $[dd^c u]_{\alpha, Q}$ is under control for some $0 < \alpha < 1$.

Let D be any first order differential operator with constant coefficients. Differentiating (2.2.2), we get

$$\left(\frac{\partial}{\partial t} - \Delta_t - \frac{\partial f}{\partial s} \right) Du = Df, \quad (2.2.3)$$

with $|u| + |\nabla u| + \left| \frac{\partial u}{\partial t} \right| + |\Delta u|$ and $[dd^c u]_{\alpha, Q}$ are under control, so C^0 norm of Du is under control. Applying the parabolic Schauder estimates (Lemma 2.2.9) to (2.2.3) with $c(t, x) = \frac{\partial f}{\partial s}(t, x, u)$, the $C^{2, \alpha}$ norm of Du is thus under control. Apply D to (2.2.3) we get

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_t - \frac{\partial f}{\partial s} \right) D^2 u &= D^2 f + \frac{\partial(Df)}{\partial s} Du + \sum_{j,k} (D\omega_t^{jk}) \frac{\partial^2 Du}{\partial z_j \partial \bar{z}_k} \\ &\quad + \frac{\partial^2 f}{\partial s^2} |Du|^2 + D \left(\frac{\partial f}{\partial s} \right) Du, \end{aligned}$$

where the parabolic C^α norm of the right-hand side is under control. Thanks to the parabolic Schauder estimates Lemma 2.2.9, the $C^{2, \alpha}$ norm of $D^2 u$ is under control. Iterating this procedure we complete the proof of Theorem 2.2.8. \square

2.2.5 Monge-Ampère capacity

Definition 2.2.11. Let K be a Borel subset of X . We set

$$\text{Cap}_\omega(K) = \sup \left\{ \int_K MA(\varphi); \varphi \in PSH(X, \omega), 0 \leq \varphi \leq 1 \right\}.$$

Then we call Cap_ω is the Monge-Ampère capacity with respect to ω .

Definition 2.2.12. Let $(\varphi_j) \in PSH(X, \omega)$. We say that (φ_j) converges to φ as $j \rightarrow +\infty$ in capacity if for each $\varepsilon > 0$

$$\lim_{j \rightarrow +\infty} \text{Cap}_\omega(|\varphi_j - \varphi| < \varepsilon) = 0.$$

The following Proposition [GZ05, Proposition 3.7] states that decreasing sequences of ω -psh functions converge in capacity.

Theorem 2.2.13. Let $\varphi, \varphi_j \in PSH(X, \omega) \cap L^\infty(X)$ such that (φ_j) decreases to φ , then for each $\varepsilon > 0$

$$\text{Cap}_\omega(\{\varphi_j > \varphi + \varepsilon\}) \rightarrow 0 \quad \text{as } j \rightarrow +\infty.$$

2.2.6 Monge-Ampère energy

The energy of a ω -psh function has been introduced in [GZ07] and further studied in [BBGZ13]. For $\phi \in PSH(X, \omega) \cap L^\infty(X)$, the Aubin-Yau energy functional is

$$E(\phi) := \frac{1}{(n+1)V} \sum_{j=0}^n \int_X \phi (\omega + dd^c \phi)^j \wedge \omega^{n-j},$$

where

$$V := \int_X \omega^n.$$

For any $\phi \in PSH(X, \omega)$, we set

$$E(\phi) := \inf \{E(\psi); \psi \in PSH(X, \omega) \cap L^\infty(X), \phi \leq \psi\}.$$

Definition 2.2.14. We say that $\phi \in PSH(X, \omega)$ has a finite energy if $E(\phi) > -\infty$ and denote by $\mathcal{E}^1(X, \omega)$ the set of all finite energy ω -psh functions.

Let $(\theta_t)_{t \in [0, T]}$ be a family of Kähler metrics on X and Ω be a smooth volume form. We consider the following complex Monge-Ampère flow

$$(CMAF) \quad \begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{(\theta_t + dd^c \varphi)^n}{\Omega} - F(t, z, \varphi), \\ \varphi(0, \cdot) = \varphi_0. \end{cases}$$

We set $\omega_t = \theta_t + dd^c \varphi_t$.

Definition 2.2.15. Suppose φ_t is a solution of (CMAF). The energy for φ_t is

$$E(\varphi_t) := E_{\theta_t}(\varphi_t) := \frac{1}{(n+1)V} \sum_{j=0}^n \int_X \varphi_t (\theta_t + dd^c \varphi_t)^j \wedge \theta_t^{n-j}.$$

In particular, when $\theta_t = \omega$ for all $t \in [0, T]$ we get the Aubin-Yau energy functional.

2.2.7 Reduction to $\frac{\partial F}{\partial s} \geq 0$

We now consider the complex Monge-Ampère flow

$$(CMAF) \quad \frac{\partial \varphi_t}{\partial t} = \log \frac{(\theta_t + dd^c \varphi_t)^n}{\Omega} - F(t, z, \varphi),$$

where $F(t, z, s) \in C^\infty([0, T] \times X \times \mathbb{R}, \mathbb{R})$ with

$$\frac{\partial F}{\partial s} \geq -C,$$

for some $C \geq 0$.

First of all, we observe that it is sufficient to prove Theorem A with F satisfying $F(t, z, s) \in C^\infty([0, T] \times X \times \mathbb{R}, \mathbb{R})$ and $s \mapsto F(t, z, s)$ is non-decreasing. Indeed, assume that φ_t is a solution of (CMAF) with $\partial F/\partial s \geq -C$. By changing variables

$$\phi(t, z) = e^{Bt} \varphi(B^{-1}(1 - e^{-Bt}), z),$$

we get

$$\frac{\partial \phi_t}{\partial t} = \log \frac{(\tilde{\theta}_t + dd^c \phi)^n}{\Omega} - \tilde{F}(t, z, \phi_t),$$

where $\tilde{\theta}_t = e^{Bt} \theta_{\frac{1-e^{-Bt}}{B}}$ and

$$\tilde{F}(t, z, s) = -Bs + Bnt + F(B^{-1}(1 - e^{-Bt}), z, e^{-Bt}s).$$

We thus have

$$\frac{\partial \tilde{F}}{\partial s} = -B + \frac{\partial F}{\partial s} e^{-Bt} \geq -B - Ce^{-Bt}.$$

Choosing $B < 0$ such that $-B - Ce^{-Bt} \geq 0$ or $-Be^{Bt} \geq C$ for all $t \in [0, T]$, we get the desired equation. Note that we can not always choose B for any $T > 0$ because the maximal value of $-Be^{BT}$ is $1/eT$ at $B = -1/T$, but in our case we can assume T is small enough such that $C < 1/eT$. Finally we obtain the equation

$$\frac{\partial \phi_t}{\partial t} = \log \frac{(\tilde{\theta}_t + dd^c \phi)^n}{\Omega} - \tilde{F}(t, z, \phi_t),$$

where $\phi(0, z) = \varphi_0$ and $\partial \tilde{F}/\partial s \geq 0$.

2.2.8 Strategy of the proof

We fix ω a reference Kähler form. Since we are interested in the behavior near 0 of the flow, we can assume that for $0 \leq t \leq T$

$$\frac{\omega}{2} \leq \theta_t \leq 2\omega, \tag{2.2.4}$$

and there exists $\delta > 0$ such that

$$\delta^{-1}\Omega \leq \theta_t^n \leq \delta\Omega, \quad \forall t \in [0, T].$$

We consider the complex Monge-Ampère flow

$$(CMAF) \quad \frac{\partial \varphi_t}{\partial t} = \log \frac{(\theta_t + dd^c \varphi_t)^n}{\Omega} - F(t, z, \varphi_t),$$

where $F(t, z, s) \in C^\infty([0, T] \times X \times \mathbb{R}, \mathbb{R})$ is such that either $\frac{\partial F}{\partial s} \geq 0$ or $-C \leq \frac{\partial F}{\partial s} \leq 0$, for some $C \geq 0$ and $\frac{\partial F}{\partial t}$ is bounded from below. Our first goal is to show the following generalization of [GZ17, DNL17]:

Theorem 2.2.16. *Let φ_0 be a ω -psh function with zero Lelong numbers. There exists a family of smooth strictly θ_t -psh function (φ_t) such that*

$$\frac{\partial \varphi_t}{\partial t} = \log \frac{(\theta_t + dd^c \varphi_t)^n}{\Omega} - F(t, z, \varphi_t)$$

in $(0, T] \times X$, with $\varphi_t \rightarrow \varphi_0$ in $L^1(X)$, as $t \searrow 0^+$. This family is unique if $C = 0$ and $|\frac{\partial F}{\partial t}| < C'$ for some $C' > 0$. Moreover, $\varphi_t \rightarrow \varphi$ in energy if $\varphi \in \mathcal{E}^1(X, \omega)$ and φ_t is uniformly bounded and converges to φ_0 in capacity if $\varphi_0 \in L^\infty(X)$.

The strategy of the proof is as follows:

- We first reduce to the case when $\frac{\partial F}{\partial s} \geq 0$ following Section 2.2.7.
- Approximate φ_0 by a decreasing sequence $(\varphi_{0,j})$ of smooth and strictly ω -psh functions by using the regularization result of Demailly [Dem92, BK07]. There exists unique solutions $\varphi_{t,j} \in PSH(X, \omega) \cap C^\infty(X)$ to the flow above with initial data $\varphi_{0,j}$.
- We then establish various priori estimates which will allow us to pass to the limit as $j \rightarrow \infty$. We prove for each $0 < \varepsilon < T$:
 - (1) $(t, z, j) \mapsto \varphi_{t,j}(z)$ is uniformly bounded on $[\varepsilon, T] \times X \times \mathbb{N}$,
 - (2) $(t, z, j) \mapsto \dot{\varphi}_{t,j}(z)$ is uniformly bounded on $[\varepsilon, T] \times X \times \mathbb{N}$,
 - (3) $(t, z, j) \mapsto \Delta_\omega \varphi_{t,j}(z)$ is uniformly bounded on $[\varepsilon, T] \times X \times \mathbb{N}$.
- Finally, we apply the Evans-Krylov theory and Schauder estimates to show that $\varphi_{t,j} \rightarrow \varphi_t$ in $C^\infty((0, T] \times X)$, as $j \rightarrow +\infty$ such that φ_t satisfies (CMAF). We then check that $\varphi_t \rightarrow \varphi_0$ as $t \rightarrow 0^+$, and also study finer convergence properties:
 - (1) For $\varphi_0 \in L^1(X)$, we show that $\varphi_t \rightarrow \varphi_0$ in $L^1(X)$ as $t \rightarrow 0$.
 - (2) When φ_0 is bounded, we show that $\varphi_t \rightarrow \varphi_0$ in capacity.
 - (3) When $\varphi_0 \in \mathcal{E}^1(X, \omega)$, we show that φ_t converges to φ_0 in energy as $t \rightarrow 0$.

2.3 A priori estimates

In this section we prove various a priori estimates for φ_t which satisfies

$$\frac{\partial \varphi_t}{\partial t} = \log \frac{(\theta_t + dd^c \varphi_t)^n}{\Omega} - F(t, z, \varphi)$$

with a smooth strictly ω -psh initial data φ_0 , where $(t, z, s) \mapsto F(t, z, s) \in C^\infty([0, T] \times X \times \mathbb{R}, \mathbb{R})$ with $\frac{\partial F}{\partial s} \geq 0$. Since we are interested in the behavior near 0 of (CMAF), we can further assume that

$$\theta_t - t\dot{\theta}_t \geq 0 \text{ for } 0 \leq t \leq T. \quad (2.3.1)$$

This assumption will be used to bound the $\dot{\varphi}_t$ from above.

2.3.1 Bounding φ_t

Lemma 2.3.1. *We have*

$$\varphi_t \leq Ct + \max\{\sup \varphi_0, 0\},$$

where $C = -\inf_{z \in X, t \in [0, T]} F(t, x, 0) + n \log \delta$.

Proof. Consider $\psi_t = Ct$, where $C = -\inf_{z \in X, t \in [0, T]} F(t, x, 0) + n \log \delta$. Thus we have

$$\log \frac{(\theta_t + dd^c \psi_t)^n}{\Omega} = \log \frac{\theta_t^n}{\Omega} \leq n \log \delta.$$

Now $F(t, z, \psi_t) \geq F(t, z, 0) \geq \inf_{z \in X, t \in [0, T]} F(t, x, 0)$, since we assume $s \mapsto F(., ., s)$ is increasing. Therefore

$$\frac{\partial \psi_t}{\partial t} \geq \frac{(\theta_t + dd^c \psi_t)^n}{\Omega} - F(t, z, \psi_t).$$

Applying Proposition 2.2.5 for φ_t and ψ_t , we get $\varphi_t \leq Ct + \max\{\sup \varphi_0, 0\}$. \square

We now find a lower bound of φ_t which does not depend on $\inf_X \varphi_0$. First, we assume that $\theta_t \geq \omega + t\chi$, $\forall t \in [0, T]$, for some smooth (1,1)-form χ . Fix $0 < \beta < +\infty$ and $0 < \alpha$ such that

$$\chi + (2\beta - \alpha)\omega \geq 0.$$

It follows from Skoda's integrability theorem [Sko72] that $e^{-2\beta\varphi_0}\omega^n$ is absolutely continuous with density in L^p for some $p > 1$. Therefore Kołodziej's uniform estimate [Kol98] implies that there exists a continuous ω -psh solution u of the equation

$$\alpha^n(\omega + dd^c u)^n = e^{\alpha u - 2\beta\varphi_0}\omega^n,$$

which satisfies

$$\|u\|_{L^\infty(X)} \leq C,$$

where C depends only on $\|e^{-2\beta\varphi_0}\|_{L^p(X)}$, for some $p > 1$.

Remark 2.3.2. Since φ_0 in this section plays the role of smooth approximation $\varphi_{0,j}$ (in Section 1.8) for initial data with zero Lelong numbers, the uniform version of Skoda's integrability theorem [Sko72] yields the uniform bound to $\|e^{-2\beta\varphi_{0,j}}\|_{L^p(X)}$ and $\|u\|_{L^\infty(X)}$. This is where we use the crucial assumption in Theorem 2.2.16 that the initial condition has zero Lelong numbers at all points.

Assume that ϕ_t is solution of the following equation

$$\begin{cases} \frac{\partial \phi_t}{\partial t} = \log \frac{(\omega + t\chi + dd^c \phi)^n}{\omega^n}, \\ \phi(0, \cdot) = \varphi_0. \end{cases}$$

By Lemma 2.9 in [GZ17] we have

$$\phi_t(z) \geq (1 - 2\beta t)\varphi_0(z) + \alpha tu(z) + n(t \log t - t). \quad (2.3.2)$$

Using this we have the following lemma:

Lemma 2.3.3. *For all $z \in X$ and $t \in (0, T]$, we have*

$$\varphi_t(z) \geq \phi_t(z) + At \geq (1 - 2\beta t)\varphi_0(z) + \alpha tu(z) + n(t \log t - t) + At, \quad (2.3.3)$$

where A depends on $\sup_X \varphi_0$ and $u \in C^0(X)$ as defined above satisfies

$$\|u\|_{L^p(X)} \leq C(\omega, \|e^{-2\beta\varphi_0}\|_{L^p(X)}).$$

In particular, there exists $c(t) \geq 0$ such that

$$\varphi_t(z) \geq \varphi_0(z) - c(t),$$

with $c(t) \searrow 0$ as $t \searrow 0$.

Proof. There is $\sigma > 0$ such that $\sigma^{-1}\omega^n \leq \Omega \leq \sigma\omega^n$, so we may assume that

$$\frac{\partial \phi_t}{\partial t} \leq \log \frac{(\omega + t\chi + dd^c \phi_t)^n}{\Omega}.$$

Thanks to Lemma 2.3.1, $\varphi_t \leq C_0$ with $C_0 > 0$ depends on $\sup_X \varphi_0$ and T . As we assume $s \mapsto F(\cdot, \cdot, s)$ is increasing, $F(t, z, \varphi_t) \leq F(t, z, C_0)$. Replacing φ_t by $\varphi_t - At$ and F by $F - A$, where

$$A := \sup_{[0, T] \times X} F(t, z, C_0),$$

we can assume that

$$\sup_{[0, T] \times X} F(t, z, \sup_{[0, T] \times X} \varphi_t) \leq 0.$$

Hence we have

$$\begin{aligned} \frac{\partial \varphi_t}{\partial t} &= \log \frac{(\theta_t + dd^c \varphi_t)^n}{\Omega} - F(t, z, \varphi_t) \\ &\geq \log \frac{(\omega + t\chi + dd^c \varphi_t)^n}{\Omega}, \end{aligned}$$

here we use the assumption $\theta_t \geq \omega + t\chi$, $\forall t \in [0, T]$. Applying the comparison theorem (Proposition 2.2.5) for φ_t and ϕ_t we have $\varphi_t \geq \phi_t$. In general, we get

$$\varphi_t(z) \geq \phi_t(z) + At \geq (1 - 2\beta t)\varphi_0(z) + \alpha tu(z) + n(t \log t - t) + At$$

as required. \square

2.3.2 Upper bound for $\dot{\varphi}_t$

We now prove a crucial estimate which allows us to use the uniform version of Kolodziej's estimates in order to get a bound of the oscillation of φ_t on X .

Proposition 2.3.4. *Suppose $\partial F/\partial s \geq -C_0$, for some $C_0 \geq 0$. Fix $\varepsilon \in (0, T)$. There exists $0 < C = C(\sup_X \varphi_0, \varepsilon, T, C_0)$ such that for all $\varepsilon \leq t \leq T$ and $z \in X$,*

$$\dot{\varphi}_t(z) \leq \frac{-K\varphi_\varepsilon(z) + C}{t} \leq \frac{-K\phi_\varepsilon(z) + C}{t} - A,$$

where ϕ_t and A are as in Lemma 2.3.3 and $K = 1 + C_0T$.

Proof. Since we deal with the interval time $[\varepsilon, T]$, we can assume the flow starting from φ_ε , i.e $\varphi(0, x) = \varphi_\varepsilon$. Consider $G(t, z) = t\dot{\varphi}_t - K\varphi_t - nKt + Bt^2/2$, with $B < \min F'$ and $K = 1 + C_0T$. We obtain

$$\frac{\partial G}{\partial t} = t\ddot{\varphi}_t - (K-1)\dot{\varphi}_t - Kn + Bt = t\Delta_{\omega_t}\dot{\varphi} + t\operatorname{tr}_{\omega_t}\dot{\theta}_t - t\frac{\partial F}{\partial s}\dot{\varphi} - tF' - (K-1)\dot{\varphi}_t - nK + Bt,$$

and

$$\Delta_{\omega_t}G = t\Delta_{\omega_t}\dot{\varphi} - K\Delta_{\omega_t}\varphi_t = t\Delta_{\omega_t}\dot{\varphi} - K(n - \operatorname{tr}_{\omega_t}\theta_t),$$

hence

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_t}\right)G = -\dot{\varphi}\left[t\frac{\partial F}{\partial s} + (K-1)\right] + t\left(B - \frac{\partial F}{\partial t}\right) - \operatorname{tr}_{\omega_t}(K\theta_t - t\dot{\theta}_t).$$

Since we assume that $\theta_t - t\dot{\theta}_t \geq 0$, we get

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_t}\right)G \leq -\dot{\varphi}\left[t\frac{\partial F}{\partial s} + (K-1)\right] + t\left(B - \frac{\partial F}{\partial t}\right).$$

If G attains its maximum at $t = 0$, we have the result. Otherwise, assume that G attains its maximum at (t_0, z_0) with $t_0 > 0$, then using $B < \min F'$ we have at (t_0, z_0)

$$0 \leq \left(\frac{\partial}{\partial t} - \Delta_{\omega_t}\right)G < -\dot{\varphi}\left[t\frac{\partial F}{\partial s} + (K-1)\right].$$

Since $\frac{\partial F}{\partial s} \geq -C_0$ and $K = 1 + C_0T$, the term in the square bracket is positive, we obtain $\dot{\varphi}(t_0, z_0) < 0$ and

$$t\dot{\varphi}_t - K\varphi_t - nt + Bt^2/2 \leq -K\varphi_{t_0}(z_0) - nt_0 + Bt_0^2/2.$$

Using Lemma 2.3.3 we get $\varphi_{t_0} \geq \varphi_\varepsilon - C(\varepsilon)$, hence

$$t\dot{\varphi}_t \leq K\varphi_t - K\varphi_\varepsilon + C_1.$$

It follows from Lemma 2.3.1 that $\varphi_t \leq C_2(\sup \varphi_0, T)$, so

$$\dot{\varphi}_t(x) \leq \frac{-K\varphi_\varepsilon + C}{t},$$

where C depends on $\sup \varphi_0, \varepsilon, T$. Since $\varphi_\varepsilon \geq \phi_\varepsilon + At$ (Lemma 2.3.3), we obtain the desired inequality. \square

2.3.3 Bounding the oscillation of φ_t

Once we get an upper bound for $\dot{\varphi}_t$ as in Proposition 2.3.4, we can bound the oscillation of φ_t by using the uniform version of Kolodziej's estimates. Indeed, observe that φ_t satisfies

$$(\theta_t + dd^c \varphi_t)^n = H_t \Omega,$$

then by Proposition 2.3.4, for any $\varepsilon \in (0, T)$,

$$H_t = \exp(\dot{\varphi}_t + F) \leq \exp\left(\frac{-K\phi_\varepsilon + C}{t} + C'\right)$$

are uniformly in $L^2(\Omega)$ for all $t \in [\varepsilon, T]$ since ϕ_ε is smooth. Thanks to the uniform version of Kolodziej's estimates [Kol98, EGZ08], we infer that the oscillation of φ_t is uniformly bounded:

Theorem 2.3.5. *Fix $0 < t < T$. There exist $C(t) > 0$ independent of $\inf_X \varphi_0$ such that*

$$\text{Osc}_X(\varphi_t) \leq C(t).$$

2.3.4 Lower bound for $\dot{\varphi}_t$

The next result is similar to [ST17, Lemma 3.2] and [GZ17, Proposition 3.3].

Proposition 2.3.6. *There exist constants $A > 0$ and $C = C(A, \text{Osc}_X \varphi_0) > 0$ such that for all $(x, t) \in X \times (0, T]$,*

$$\dot{\varphi} \geq n \log t - A \text{Osc}_X \varphi_0 - C,$$

Proof. We consider $H(t, x) = \dot{\varphi}_t + A\varphi_t - \alpha(t)$, where $\alpha \in C^\infty(\mathbb{R}^+, \mathbb{R})$ will be chosen hereafter. We have

$$\begin{aligned} \frac{\partial H}{\partial t} &= \ddot{\varphi}_t + A\dot{\varphi}_t - \dot{\alpha} \\ &= \Delta_{\omega_t} \dot{\varphi}_t + \text{tr}_{\omega_t} \dot{\theta}_t - F' - \frac{\partial F}{\partial s} \dot{\varphi}_t + A\dot{\varphi}_t - \dot{\alpha}, \end{aligned}$$

and

$$\Delta_{\omega_t} H = \Delta_{\omega_t} \dot{\varphi}_t + A \Delta_{\omega_t} \varphi_t.$$

Therefore, we have

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_{\omega_t}\right) H &= A\dot{\varphi}_t + \text{tr}_{\omega_t} \dot{\theta}_t - A \text{tr}_{\omega_t} (\omega_t - \theta_t) - F' - \dot{\alpha} - \frac{\partial F}{\partial s} \dot{\varphi}_t \\ &= A\dot{\varphi}_t + \text{tr}_{\omega_t} (A\theta_t + \dot{\theta}_t) - An - F' - \dot{\alpha} - \frac{\partial F}{\partial s} \dot{\varphi}_t \\ &= \left(A - \frac{\partial F}{\partial s}\right) \dot{\varphi}_t + \text{tr}_{\omega_t} (A\theta_t + \dot{\theta}_t) - F' - \dot{\alpha} - An. \end{aligned}$$

Now $A\theta_t + \dot{\theta} \geq \omega$ with A sufficiently large, hence

$$\mathrm{tr}_{\omega_t}(A\theta_t + \dot{\theta}) \geq \mathrm{tr}_{\omega_t} \omega.$$

Using the inequality

$$\begin{aligned} \mathrm{tr}_{\omega_t}(\omega) &\geq n \left(\frac{\omega_t^n}{\omega^n} \right)^{-1/n} = n \exp\left(\frac{-1}{n}(\dot{\varphi} + F)\right) \left(\frac{\Omega}{\omega^n} \right)^{-1/n} \\ &\geq \sigma^{-1/n} h_t^{-1/n} \exp\left(-\sup_{[0,T] \times X} F(t, z, C_0)/n\right), \end{aligned}$$

where $h_t = e^{\dot{\varphi}}$ and C_0 depends on $\sup_X \varphi_0$, we have

$$\mathrm{tr}_{\omega_t}(A\theta_t + \dot{\theta}) \geq \frac{h_t^{-1/n}}{C}.$$

In addition, we apply the inequality $\sigma x > \log x - C_\sigma$ for all $x > 0$ with $x = h_t^{-1/n}$ and $\sigma \ll 1$ to obtain $\sigma h_t^{-1/n} = \sigma e^{-\dot{\varphi}/n} > -\dot{\varphi}/n - C_\sigma$. Finally, we can choose A sufficient large and $\sigma > 0$ such that

$$\left(A - \frac{\partial F}{\partial s}\right)\dot{\varphi} + \mathrm{tr}_{\omega_t}(A\theta_t + \dot{\theta}) \geq \frac{h_t^{-1/n}}{C_1} - C'_1.$$

Since $|F'|$ is bounded by some constant $C(\mathrm{Osc}_X \varphi_0) > 0$, we obtain

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_t}\right)H > \frac{h_t^{-1/n}}{C_1} - \alpha'(t) - C_2,$$

where C_2 depends on $\mathrm{Osc}_X \varphi_0$.

We chose α such that $\alpha(0) = -\infty$. This insures that H attains its minimum at (t_0, z_0) with $t_0 > 0$. At (t_0, z_0) we have

$$C_1[C_2 + \alpha'(t_0)] \geq h_{t_0}^{-1/n}(z_0),$$

hence

$$H(t_0, z_0) \geq A\varphi_{t_0}(z_0) - \{n \log[C_2 + \alpha'(t_0)] + \alpha(t_0)\}.$$

From Lemma 2.3.1 we have $\varphi_{t_0} \leq \sup_X \varphi_0 + C'$, hence

$$\dot{\varphi} \geq \alpha(t) - A\mathrm{Osc}_X \varphi_0 - C_3 - \{n \log[C_2 + \alpha'(t_0)] + \alpha(t_0)\}.$$

Choosing $\alpha(t) = n \log t$ we get

$$n \log[C_2 + \alpha'] + \alpha \leq C_4,$$

so obtain the inequality. □

2.3.5 Bounding the gradient of φ

In this section we bound the gradient of φ using the same technique as in [SzTo11, Lemma 4] (which is a parabolic version of Blocki's estimate [Blo09]). In these articles $\theta_t = \omega$ is independent of t . We note that if one is interested in the special case of (twisted) Kähler-Ricci flow, then the gradient estimate is not needed: one can directly obtain in this case a control of the Laplacian by a parabolic version of Yau's celebrated C^2 -estimate (see for instance [GZ17]).

Proposition 2.3.7. *Fix $\varepsilon \in [0, T]$. There exists $C > 0$ depending on $\sup_X \varphi_0$ and ε such that for all $\varepsilon \leq t \leq T$*

$$|\nabla\varphi(z)|_\omega^2 < e^{C/t}.$$

Proof. Since we deal with the interval time $[\varepsilon, T]$ and the bound on $Osc_X(\varphi_\varepsilon)$ only depends on $\sup_X \varphi_0$ and ε , we can consider the flow starting from φ_ε , i.e $\varphi(0, x) = \varphi_\varepsilon$. Define

$$K = t \log |\nabla\varphi|_\omega^2 - \gamma \circ \varphi = t \log \beta - \gamma \circ \varphi,$$

where $\beta = |\nabla\varphi|_\omega^2$ and $\gamma \in C^\infty(\mathbb{R}, \mathbb{R})$ will be chosen hereafter.

If K attains its maximum for $t = 0$, β is bounded in terms of $\sup_X \varphi_0$ and ε , since $|\varphi_t|$ is bounded by a constant depending on $\sup_X \varphi_0$ and ε for all $t \in [0, T - \varepsilon]$ (Lemma 2.3.1 and Lemma 2.3.3).

We now assume that K attains its maximum at (t_0, z_0) in $[0, T - \varepsilon] \times X$ with $t_0 > 0$. Near z_0 we have $\omega = dd^c g$ for some smooth strongly plurisubharmonic g and $\theta_t = dd^c h_t$ for some smooth function h_t , hence $u := h_t + \varphi$ is plurisubharmonic near (t_0, z_0) . We take normal coordinates for ω at z_0 such that

$$g_{i\bar{k}}(z_0) = \delta_{jk} \tag{2.3.4}$$

$$g_{i\bar{k}l}(z_0) = 0 \tag{2.3.5}$$

$$u_{p\bar{q}}(t_0, z_0) \text{ is diagonal,} \tag{2.3.6}$$

here we denote $\alpha_{j\bar{k}} := \frac{\partial^2 \alpha}{\partial z_j \partial \bar{z}_k}$, $\alpha_p := \frac{\partial \alpha}{\partial z_p}$ and $\alpha_{\bar{p}} := \frac{\partial \alpha}{\partial \bar{z}_p}$.

We now compute $K_p, K_{p\bar{p}}$ at (t_0, z_0) in order to use the maximum principle. At (t_0, z_0) we have $K_p = 0$ hence

$$t\beta_p = \beta\gamma' \circ \varphi_p \tag{2.3.7}$$

or

$$\left(\frac{\beta_p}{\beta}\right)^2 = \frac{1}{t^2} (\gamma' \circ \varphi)^2 |\varphi_p|^2.$$

Therefore,

$$\begin{aligned} K_{p\bar{p}} &= t \frac{\beta_{p\bar{p}}\beta - |\beta_p|^2}{\beta^2} - \gamma'' \circ \varphi |\varphi_p|^2 - \gamma' \circ \varphi \varphi_{p\bar{p}} \\ &= t \frac{\beta_{p\bar{p}}}{\beta} - [t^{-1}(\gamma')^2 + \gamma''] |\varphi_p|^2 - \gamma' \varphi_{p\bar{p}}, \end{aligned}$$

where we write γ', γ'' instead of $\gamma' \circ \varphi$ and $\gamma'' \circ \varphi$.

Now we compute $\beta_p, \beta_{p\bar{p}}$ at (t_0, z_0) with $\beta = g^{j\bar{k}} \varphi_j \varphi_{\bar{k}}$ where $(g^{j\bar{k}}) = [(g_{j\bar{k}})^t]^{-1}$. We have

$$\beta_p = g_p^{j\bar{k}} \varphi_j \varphi_{\bar{k}} + g^{j\bar{k}} \varphi_{jp} \varphi_{\bar{k}} + g^{j\bar{k}} \varphi_{jp} \varphi_{\bar{k}p}.$$

At (t_0, z_0) , use (2.3.4), (2.3.5)

$$g_p^{j\bar{k}} = -g^{j\bar{l}} g_{s\bar{l}p} g^{s\bar{k}} = 0,$$

hence

$$\beta_p = \sum_j \varphi_{jp} \varphi_{\bar{j}} + \sum_j \varphi_{p\bar{j}} \varphi_j, \quad (2.3.8)$$

and

$$\beta_{p\bar{p}} = g_{p\bar{p}}^{j\bar{k}} \varphi_j \varphi_{\bar{k}} + 2Re \sum_j \varphi_{p\bar{p}j} \varphi_{\bar{j}} + \sum_j |\varphi_{jp}^2| + \sum_j |\varphi_{j\bar{p}}|^2.$$

Note that

$$R_{i\bar{j}k\bar{l}} = -g_{i\bar{j}k\bar{l}} + g^{s\bar{t}} g_{s\bar{j}k} g_{i\bar{t}l},$$

hence, at (t_0, z_0) $g_{p\bar{p}}^{j\bar{k}} = -g_{j\bar{k}p\bar{p}} = R_{j\bar{k}p\bar{p}}$, and

$$\beta_{p\bar{p}} = R_{j\bar{k}p\bar{p}} \varphi_j \varphi_{\bar{k}} + 2Re \sum_j \varphi_{p\bar{p}j} \varphi_{\bar{j}} + \sum_j |\varphi_{jp}^2| + \sum_j |\varphi_{j\bar{p}}|^2.$$

Now at (t_0, z_0)

$$\Delta_{\omega_{t_0}} K = \sum_{p=1}^n \frac{K_{p\bar{p}}}{u_{p\bar{p}}},$$

hence

$$\begin{aligned} \Delta_{\omega_{t_0}} K &= t \sum_{j,k} \frac{R_{i\bar{j}k\bar{p}\bar{p}} \varphi_j \varphi_{\bar{k}}}{\beta u_{p\bar{p}}} + 2tRe \sum_j \frac{\varphi_{p\bar{p}j} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}} + t \frac{\sum_j |\varphi_{jp}|^2 + |\varphi_{j\bar{p}}|^2}{\beta u_{p\bar{p}}} \\ &\quad - \frac{[t^{-1}(\gamma')^2 + \gamma''] |\varphi_p|^2}{u_{p\bar{p}}} - \frac{\gamma' \varphi_{p\bar{p}}}{u_{p\bar{p}}}. \end{aligned}$$

Since $u_{p\bar{p}} = \varphi_{p\bar{p}} + h_{p\bar{p}}$ near (t_0, z_0) , then at (t_0, z_0)

$$\sum_p \frac{\gamma' \varphi_{p\bar{p}}}{u_{p\bar{p}}} = n\gamma' - \sum_p \frac{\gamma' h_{p\bar{p}}}{u_{p\bar{p}}}.$$

Moreover, assume that the holomorphic bisectional curvature of ω is bounded by a constant $B \in \mathbb{R}$ on X , then at (t_0, z_0)

$$t \sum_{j,k,p} \frac{R_{j\bar{k}p\bar{p}} \varphi_j \varphi_{\bar{k}}}{\beta u_{p\bar{p}}} \geq -Bt \sum_p \frac{1}{u_{p\bar{p}}},$$

therefore

$$\begin{aligned} \Delta_{\omega_{t_0}} K &\geq (\gamma' - tB) \sum_p \frac{1}{u_{p\bar{p}}} + 2t \operatorname{Re} \sum_{j,p} \frac{\varphi_{p\bar{p}j} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}} \\ &\quad + \frac{t}{\beta} \sum_{j,p} \frac{|\varphi_{jp}|^2 + |\varphi_{j\bar{p}}|^2}{\beta u_{p\bar{p}}} - [t^{-1}(\gamma')^2 + \gamma''] \sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}} - n\gamma' + \gamma' \sum_p \frac{th_{p\bar{p}}}{u_{p\bar{p}}}. \end{aligned}$$

By the maximum principle, at (t_0, z_0)

$$0 \leq \left(\frac{\partial}{\partial t} - \Delta_{\omega_t} \right) K$$

hence,

$$\begin{aligned} 0 &\leq \log \beta - \gamma' \dot{\varphi} - (\gamma' - tB) \sum_p \frac{1}{u_{p\bar{p}}} + t \frac{\beta'}{\beta} - 2t \operatorname{Re} \sum_{j,p} \frac{\varphi_{p\bar{p}j} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}} \\ &\quad - \frac{t}{\beta} \sum_{j,p} \frac{|\varphi_{jp}|^2 + |\varphi_{j\bar{p}}|^2}{\beta u_{p\bar{p}}} + [t^{-1}(\gamma')^2 + \gamma''] \sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}} + n\gamma'. \end{aligned} \quad (2.3.9)$$

We will simplify (2.3.9) to get a bound for β at (t_0, z_0) . We now estimate

$$t \frac{\beta'}{\beta} - 2t \operatorname{Re} \sum_{j,p} \frac{\varphi_{p\bar{p}j} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}} \quad \text{and} \quad -\frac{t}{\beta} \sum_{j,p} \frac{|\varphi_{jp}|^2}{\beta u_{p\bar{p}}} + t^{-1}(\gamma')^2 \sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}}.$$

For the first one, we note that near (t_0, z_0)

$$\log \det(u_{p\bar{q}}) = \dot{\varphi} + F(t, z, \varphi) + \log \Omega,$$

hence using

$$\frac{d}{ds} \det A = A^{\bar{j}i} \left(\frac{d}{ds} A_{i\bar{j}} \right) \det A$$

we have at (t_0, z_0)

$$u^{p\bar{p}} u_{p\bar{p}j} = \frac{u_{p\bar{p}j}}{u_{p\bar{p}}} = (\dot{\varphi} + F(t, z, \varphi) + \log \Omega)_j.$$

Therefore

$$\begin{aligned} 2t \operatorname{Re} \sum_{j,p} \frac{\varphi_{p\bar{p}j} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}} &= 2t \operatorname{Re} \sum_{j,p} \frac{(u_{p\bar{p}j} - h_{p\bar{p}j}) \varphi_{\bar{j}}}{\beta u_{p\bar{p}}} \\ &= \frac{2t}{\beta} \operatorname{Re} \sum_j (\dot{\varphi} + F(t, z, \varphi) + \log \Omega)_j \varphi_{\bar{j}} - 2t \operatorname{Re} \sum_{j,p} \frac{h_{p\bar{p}j} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}} \\ &= \frac{2t}{\beta} \operatorname{Re} \sum_j (\dot{\varphi}_j \varphi_{\bar{j}}) + \frac{2t}{\beta} \operatorname{Re} \left((F(t, z, \varphi) + \log \Omega)_j + \frac{\partial F}{\partial s} \varphi_j \right) \varphi_{\bar{j}} \\ &\quad - 2t \operatorname{Re} \sum_{j,p} \frac{h_{p\bar{p}j} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}}. \end{aligned}$$

In addition, at (t_0, z_0)

$$\begin{aligned} t \frac{\beta'}{\beta} &= \frac{t}{\beta} \sum_{j,k} g^{j\bar{k}} (\dot{\varphi}_j \varphi_{\bar{k}} + \varphi_j \dot{\varphi}_{\bar{k}}) \\ &= \frac{2t}{\beta} \operatorname{Re} (\dot{\varphi}_j \varphi_{\bar{j}}), \end{aligned}$$

we infer that

$$\begin{aligned} t \frac{\beta'}{\beta} - 2t \operatorname{Re} \sum_{j,p} \frac{u_{p\bar{p}j} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}} &= -\frac{2t}{\beta} \operatorname{Re} \sum_j (F(t, z, \varphi) + \log \Omega)_j \varphi_{\bar{j}} - \frac{2t}{\beta} \sum_j \frac{\partial F}{\partial s} |\varphi_j|^2 \\ &\quad + 2t \operatorname{Re} \sum_{j,p} \frac{h_{p\bar{p}j} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}}. \end{aligned}$$

We may assume that $\log \beta > 1$ so that

$$\frac{|\varphi_{\bar{j}}|}{\beta} < C$$

By the hypothesis that $\frac{\partial F}{\partial s} \geq 0$ there exists C_1 depends on $\sup |\varphi_0|$ and C_2 depends on h and ε such that

$$t \frac{\beta'}{\beta} - 2t \operatorname{Re} \sum_{j,p} \frac{u_{p\bar{p}j} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}} < C_1 t + C_2 t \sum_p \frac{1}{u_{p\bar{p}}}. \quad (2.3.10)$$

We now estimate

$$-\frac{t}{\beta} \sum_{j,p} \frac{|\varphi_{jp}|^2}{\beta u_{p\bar{p}}} + t^{-1} (\gamma')^2 \sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}}.$$

It follows from (2.3.7) and (3.3.10) that

$$\begin{aligned}\beta_p &= \sum_j \varphi_{jp} \varphi_{\bar{j}} + \sum_j \varphi_{p\bar{j}} \varphi_j, \\ t\beta_p &= \beta \gamma' \varphi_p\end{aligned}$$

then,

$$\sum_j \varphi_{jp} \varphi_{\bar{j}} = (t^{-1} \gamma' \beta - \varphi_{p\bar{p}}) \varphi_p.$$

Hence at (t_0, z_0)

$$\begin{aligned}\frac{t}{\beta} \sum_{j,p} \frac{|\varphi_{jp}|^2}{u_{p\bar{p}}} &\geq \frac{t}{\beta^2} \sum_{j,p} \frac{|\sum \varphi_{jp} \varphi_{\bar{j}}|^2}{u_{p\bar{p}}} = \frac{t}{\beta^2} \sum_p \frac{|t^{-1} \gamma' \beta + 1 - \varphi_{p\bar{p}}|^2 |\varphi_p|^2}{u_{p\bar{p}}} \\ &\geq t^{-1} (\gamma')^2 \sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}} - C_3 \gamma',\end{aligned}\tag{2.3.11}$$

here C_3 depends on h_t for $t \in [\varepsilon, T]$ and we assume $\gamma' > 0$.

We now choose

$$\gamma(s) = As - \frac{1}{A} s^2$$

with A so large that $\gamma' = A - \frac{2}{A} s > 0$ and $\gamma'' = -2/A < 0$ for all $s \leq \sup_{[0,T] \times X} \varphi_t$. From Lemma 2.3.6 we have $\dot{\varphi} \geq C_0 + n \log t$, where C_0 depends on $Ocs_X \varphi_0$. Combining this with (2.3.9), (2.3.10), (2.3.11) we obtain

$$0 \leq -\frac{2}{A} \sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}} - (\gamma' - Bt - C_2 t) \sum_p \frac{1}{u_{p\bar{p}}} + \log \beta + C_4 \gamma' + C_1 t,$$

where C_1, C_2, C_4 depend on $\sup_X |\varphi_0|, h_t$ and ε . If A is chosen sufficiently large, we have a constant $C_5 > 0$ such that

$$\sum_p \frac{1}{u_{p\bar{p}}} + \sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}} \leq C_5 \log \beta,\tag{2.3.12}$$

so we get $(u_{p\bar{p}})^{-1} \leq C_5 \log \beta$ for $1 \leq p \leq n$. From Lemma 2.3.1 and Lemma 2.3.4 we have at (t_0, z_0)

$$\prod_p u_{p\bar{p}} = e^{-\dot{\varphi}_t + F(t, x, \varphi_t)} \leq C_6,$$

where C_6 depends on $\sup_X |\varphi_0|, \varepsilon$. Then we get

$$u_{p\bar{p}} \leq C_6 (C_5 \log \beta)^{n-1},$$

so from (2.3.12) we have

$$\beta = \sum_p |\varphi_p|^2 \leq C_6 (C_5 \log \beta)^n,$$

hence $\log \beta < C_7$ at (t_0, z_0) . This shows that $\beta = |\nabla \varphi(z)|_\omega^2 < e^{C/t}$ for some C depending on $\sup |\varphi_0|$ and ε . \square

2.3.6 Bounding $\Delta\varphi_t$

We now use previous a priori estimates above to get a estimate of $\Delta\varphi$. The estimate on $|\nabla\varphi|_\omega^2$ is needed here, in contrast with [GZ17, DNL17].

Lemma 2.3.8. *For all $z \in X$ and $s, t > 0$ such that $s + t \leq T$,*

$$0 \leq t \log \operatorname{tr}_\omega(\omega_{t+s}) \leq A \operatorname{Osc}_X(\varphi_s) + C + [C - n \log s + A \operatorname{Osc}_X(\varphi_s)]t$$

for some uniform constants $C, A > 0$.

Proof. We define

$$P = t \log \operatorname{tr}_\omega(\omega_{t+s}) - A\varphi_{t+s},$$

and

$$u = \operatorname{tr}_\omega(\omega_{t+s})$$

with $A > 0$ to be chosen latter. We set $\Delta_t := \Delta_{\omega_{t+s}}$. Now,

$$\begin{aligned} \frac{\partial}{\partial t} P &= \log u + t \frac{\dot{u}}{u} - A\dot{\varphi}_{t+s}, \\ \Delta_t P &= t \Delta_t \log u - A \Delta_t \varphi_{t+s} \end{aligned}$$

hence

$$\left(\frac{\partial}{\partial t} - \Delta_t \right) P = \log u + t \frac{\dot{u}}{u} - A\dot{\varphi}_{t+s} - t \Delta_t \log u + A \Delta_t \varphi_{t+s}. \quad (2.3.13)$$

First, we have

$$A \Delta_t \varphi_{t+s} = An - A \operatorname{tr}_{\omega_{t+s}}(\theta_{t+s}) \leq An - \frac{A}{2} \operatorname{tr}_{\omega_{t+s}}(\omega),$$

and by Proposition 2.2.4

$$-t \Delta_t \log u \leq B \operatorname{tr}_{\omega_{t+s}}(\omega) + t \frac{\operatorname{tr}_\omega(\operatorname{Ric}(\omega_{t+s}))}{\operatorname{tr}_\omega(\omega_{t+s})}.$$

Moreover,

$$\begin{aligned} \frac{t\dot{u}}{u} &= \frac{t}{u} \left[\Delta_\omega (\log \omega_{t+s}^n / \omega^n - \log \Omega / \omega^n - F(t, z, \varphi_{t+s})) + \operatorname{tr}_\omega \dot{\theta}_t \right], \\ &= \frac{t}{u} \left[-\operatorname{tr}_\omega(\operatorname{Ric} \omega_{t+s}) + \operatorname{tr}_\omega(\dot{\theta}_t + \operatorname{Ric} \omega) - \Delta_\omega (F(t, z, \varphi) + \log \Omega / \omega^n) \right], \end{aligned}$$

with $u = \operatorname{tr}_\omega(\omega_{t+s})$, and

$$\operatorname{tr}_{\omega_{t+s}}(\omega) \operatorname{tr}_\omega(\omega_{t+s}) \geq n,$$

we get

$$-t \Delta_t \log u + \frac{t\dot{u}}{u} \leq (B + C_1)t \operatorname{tr}_{\omega_{t+s}}(\omega) - t \frac{\Delta_\omega [F(t, z, \varphi) + \log \Omega / \omega^n]}{\operatorname{tr}_\omega(\omega_{t+s})}. \quad (2.3.14)$$

Now

$$\Delta_\omega F(t, z, \varphi_{t+s}) = \Delta_\omega F(z, \cdot) + 2\operatorname{Re} \left[g^{j\bar{k}} \left(\frac{\partial F}{\partial s} \right)_j \varphi_{\bar{k}} \right] + \frac{\partial F}{\partial s} \Delta_\omega \varphi + \frac{\partial^2 F}{\partial s^2} |\nabla \varphi|_\omega^2.$$

So there are constants C_2, C_3, C_4 such that

$$|\Delta_\omega (F(t, z, \varphi_{t+s}) + \log \Omega / \omega^n)| \leq C_2 + C_3 |\nabla \varphi|_\omega^2 + C_4 \operatorname{tr}_\omega \omega_{t+s}.$$

Then we infer

$$-\frac{\Delta_\omega [F(t, z, \varphi) + \log \Omega / \omega^n]}{\operatorname{tr}_\omega(\omega_{t+s})} \leq \frac{1}{n} \operatorname{tr}_{\omega_{t+s}}(\omega) (C_2 + C_3 |\nabla \varphi|_\omega^2) + C_4,$$

so from Lemma 2.3.7 and (2.3.14) we have

$$-t \Delta_t \log u + \frac{t\dot{u}}{u} \leq (B + C_5)t \operatorname{tr}_{\omega_{t+s}}(\omega) + C_6. \quad (2.3.15)$$

From Lemma 2.2.3 and the inequality $(n-1) \log x \leq x + C_n$,

$$\begin{aligned} \log u &= \log \operatorname{tr}_\omega(\omega_{t+s}) \leq \log \left(n \left(\frac{\omega_{t+s}^n}{\omega^n} \right) \operatorname{tr}_{\omega_{t+s}}(\omega)^{n-1} \right) \\ &= \log n + \dot{\varphi}_{t+s} + F(t, z, \varphi) + (n-1) \log \operatorname{tr}_{\omega_{t+s}}(\omega) \\ &\leq \dot{\varphi}_{t+s} + \operatorname{tr}_{\omega_{t+s}}(\omega) + C_7. \end{aligned}$$

It follows from (2.3.13), (2.3.14) and (2.3.15) that

$$\left(\frac{\partial}{\partial t} - \Delta_t \right) P \leq C_8 - (A-1)\dot{\varphi}_{t+s} + [(B+C_5)t + 1 - A/2] \operatorname{tr}_{\omega_{s+t}} \omega.$$

We choose A sufficiently large such that $(B+C_5)t + 1 - A/2 < 0$. Applying Proposition 2.3.6,

$$\left(\frac{\partial}{\partial t} - \Delta_t \right) P \leq C_8 - (A-1)(n \log s - A \operatorname{Osc}_X \varphi_s - C).$$

Now suppose P attains its maximum at (t_0, z_0) . If $t_0 = 0$, we get the desired inequality. Otherwise, at (t_0, z_0)

$$0 \leq \left(\frac{\partial}{\partial t} - \Delta_t \right) P \leq C_8 - (A-1)(n \log s - A \operatorname{Osc}_X \varphi_s - C).$$

Hence we get

$$t \log \operatorname{tr}_\omega(\omega_{t+s}) \leq A \operatorname{Osc}_X(\varphi_s) + C + [C - n \log s + A \operatorname{Osc}_X(\varphi_s)]t.$$

□

Corollary 2.3.9. For all $(t, x) \in (0, T] \times X$

$$0 \leq t \log \operatorname{tr}_\omega(\omega_{t+s}) \leq 2A \operatorname{Osc}_X(\varphi_{t/2}) + C'.$$

2.3.7 Higher order estimates

For the higher order estimates, we can follow Székelyhidi-Tosatti [SzTo11] by bounding

$$S = g_{\varphi}^{i\bar{p}} g_{\varphi}^{q\bar{j}} g_{\varphi}^{k\bar{r}} \varphi_{i\bar{j}k} \varphi_{\bar{p}q\bar{r}} \quad \text{and} \quad |Ric(\omega_t)|_{\omega_t},$$

then using the parabolic Schauder estimates in order to obtain bounds on all higher order derivatives for φ . Besides we can also combine previous estimates with Evans-Krylov and Schauder estimates (Theorem 2.2.8) to get the C^k estimates for all $k \geq 0$.

Theorem 2.3.10. *For each $\varepsilon > 0$ and $k \in \mathbb{N}$, there exists $C_k(\varepsilon)$ such that*

$$\|\varphi\|_{C^k([\varepsilon, T] \times X)} \leq C_k(\varepsilon).$$

2.4 Proof of Theorem A

2.4.1 Convergence in L^1

We approximate φ_0 by a decreasing sequence $\varphi_{0,j}$ of smooth ω -psh functions (using [Dem92] or [BK07]). Denote by $\varphi_{t,j}$ the smooth family of θ_t -psh functions satisfying on $[0, T] \times X$

$$\frac{\partial \varphi_t}{\partial t} = \log \frac{(\theta_t + dd^c \varphi_t)^n}{\Omega} - F(t, z, \varphi)$$

with initial data $\varphi_{0,j}$.

It follows from the comparison principle (Proposition 2.2.5) that $j \mapsto \varphi_{j,t}$ is non-increasing. Therefore we can set

$$\varphi_t(z) := \lim_{j \rightarrow +\infty} \varphi_{t,j}(z).$$

Thanks to Lemma 2.3.3 the function $t \mapsto \sup_X \varphi_{t,j}$ is uniformly bounded, hence φ_t is a well-defined θ_t -psh function. Moreover, it follows from Theorem 2.3.10 that φ_t is also smooth in $(0, T] \times X$ and satisfies

$$\frac{\partial \varphi_t}{\partial t} = \log \frac{(\theta_t + dd^c \varphi_t)^n}{\Omega} - F(t, z, \varphi).$$

Observe that (φ_t) is relatively compact in $L^1(X)$ as $t \rightarrow 0^+$, we now show that $\varphi_t \rightarrow \varphi_0$ in $L^1(X)$ as $t \searrow 0^+$.

First, let φ_{t_k} is a subsequence of (φ_t) such that φ_{t_k} converges to some function ψ in $L^1(X)$ as $t_k \rightarrow 0^+$. By the properties of plurisubharmonic functions, for all $z \in X$

$$\limsup_{t_k \rightarrow 0} \varphi_{t_k}(z) \leq \psi(z),$$

with equality almost everywhere. We infer that for almost every $z \in X$

$$\psi(z) = \limsup_{t_k \rightarrow 0} \varphi_{t_k}(z) \leq \limsup_{t_k \rightarrow 0} \varphi_{t_k, j}(z) = \varphi_{0, j}(z),$$

by continuity of $\varphi_{t, j}$ at $t = 0$. Thus $\psi \leq \varphi_0$ almost everywhere.

Moreover, it follows from Lemma 3.3.3 that

$$\varphi_t(z) \geq (1 - 2\beta t)\varphi_0(z) + \alpha t u(z) + n(t \log t - t) + At,$$

with u continuous, so

$$\varphi_0 \leq \liminf_{t \rightarrow 0} \varphi_t.$$

Since $\psi \leq \varphi_0$ almost everywhere, we get $\psi = \varphi_0$ almost everywhere, so $\varphi_t \rightarrow \varphi_0$ in L^1 .

We next consider some cases in which the initial condition is slightly more regular.

2.4.2 Uniform convergence

If the initial condition φ_0 is continuous then by Proposition 2.2.5 we get $\varphi_t \in C^0([0, T] \times X)$, hence φ_t uniformly converges to φ_0 as $t \rightarrow 0^+$.

2.4.3 Convergence in capacity

When φ_0 is only bounded, we prove this convergence moreover holds in capacity (Definition 2.2.12). It is the strongest convergence we can expect in the bounded case (cf. [GZ05]). First, observe that it is sufficient to prove that $u_t := \varphi_t + c(t)$ converges to φ_0 as $t \rightarrow 0$ in capacity, where $c(t)$ satisfies $\varphi_t + c(t) \geq \varphi_0$ as in Proposition 2.3.3. Since φ_t converges to φ_0 , so does u_t , and we get

$$\limsup_{t \rightarrow 0} u_t \leq \varphi_{0, j},$$

for all $j > 0$, where $(\varphi_{0, j})$ is a family of smooth ω -psh functions decreasing to φ_0 as in Section 2.4.1. It follows from Hartogs' Lemma that for each $j > 0$ and $\varepsilon > 0$, there exists $t_j > 0$ such that

$$u_t \leq \varphi_{0, j} + \varepsilon, \quad \forall 0 \leq t \leq t_j.$$

Therefore

$$\text{Cap}_\omega(\{u_t > \varphi_0 + 2\varepsilon\}) \leq \text{Cap}_\omega(\{\varphi_{0, j} > \varphi_0 + \varepsilon\}),$$

for all $t \leq t_j$. Since $\varphi_{0, j}$ converges to φ_0 in capacity (Proposition 2.2.13), the conclusion follows.

2.4.4 Convergence in energy

Using the same notations as in Section 2.2.6 we get the following monotonicity property of the energy.

Proposition 2.4.1. *Suppose φ_t is a solution of (CMAF) with initial data $\varphi_0 \in \mathcal{E}^1(X, \omega)$. Then there exists a constant $C \geq 0$ such that $t \mapsto E(\varphi_t) + Ct$ is increasing on $[0, T]$.*

Proof. By computation we get

$$\frac{dE(\varphi_t)}{dt} = \frac{1}{V} \int_X \dot{\varphi}_t \omega_t + \frac{1}{(n+1)V} \sum_{j=0}^n \int_X \varphi_t \dot{\theta}_t \wedge [j\theta_t + (n-j)\omega_t] \wedge \omega_t^j \wedge \theta_t^{n-j-1}.$$

For the first term, we use the concavity of the logarithm to get

$$\int_X \dot{\varphi}_t \omega_t^n = \int_X \log\left(\frac{\omega_t^n}{e^F \Omega}\right) \frac{\omega_t^n}{V_t} \geq -\log\left(\frac{\int_X e^{F(t,z,\varphi_t)} \Omega}{V_t}\right) \geq -\log(C_0 \delta)$$

where $F(t, z, \varphi_t) \leq \log C_0$ and

$$V_t := \int_X \omega_t^n = \int_X \theta_t^n \geq \delta^{-1} V.$$

For the second one, there is a constant $A > 0$ such that $\dot{\theta}_t \leq A\theta_t$ for all $0 \leq t \leq T$. We note that

$$\int_X \varphi_t (\theta_t + dd^c \varphi_t)^j \wedge \theta_t^{n-j} \leq \int_X \varphi_t (\theta_t + dd^c \varphi_t)^{j-1} \wedge \theta_t^{n-j+1},$$

hence

$$\frac{dE(\varphi_t)}{dt} \geq -C_1 + C_2 E(\varphi_t),$$

for some $C_1, C_2 > 0$. By Lemma 2.3.3 we have

$$E(\varphi_t) \geq C_3 E(\varphi_0) + C_3 \geq C_4$$

Thus $t \mapsto E(\varphi_t) + Ct$ is increasing on $[0, T]$ for some $C > 0$. \square

Proposition 2.4.2. *If $\varphi_0 \in \mathcal{E}^1(X, \omega)$, then φ_t converges to φ_0 in energy as $t \rightarrow 0$.*

Proof. It follows from Proposition 2.4.1 that φ_t stays in a compact subset of the class $\mathcal{E}^1(X, \omega)$. Let $\psi = \lim_{t_k \rightarrow 0} \varphi_{t_k}$ be a cluster point of (φ_t) as $t \rightarrow 0$. Reasoning as earlier, we have $\psi \leq \varphi_0$. Since the energy $E(\cdot)$ is upper semi-continuous for the weak L^1 -topology (cf. [GZ07]), Proposition 2.4.1 and the monotonicity of Aubin-Yau energy functional yield

$$E(\varphi_0) \leq \lim_{t_k \rightarrow 0} E(\varphi_{t_k}) \leq E(\psi) \leq E(\varphi_0),$$

Therefore $E(\psi) = E(\varphi_0)$, so $\psi = \varphi_0$ and we have $\varphi_t \rightarrow \varphi_0$ in energy. \square

2.5 Uniqueness and stability of solution

We now prove the uniqueness and stability for the complex Monge-Ampère flow

$$(CMAF) \quad \frac{\partial \varphi_t}{\partial t} = \log \frac{(\theta_t + dd^c \varphi_t)^n}{\Omega} - F(t, z, \varphi),$$

where $F(t, z, s) \in C^\infty([0, T] \times X \times \mathbb{R}, \mathbb{R})$ with

$$\frac{\partial F}{\partial s} \geq 0 \quad \text{and} \quad \left| \frac{\partial F}{\partial t} \right| \leq C',$$

for some constant $C' > 0$.

2.5.1 Uniqueness

For the uniqueness and stability of solution we follow the approach of Di Nezza-Lu [DNL17]. The author thanks Eleonora Di Nezza and Hoang Chinh Lu for valuable discussion on the argument in [DNL17, Theorem 5.4].

Suppose φ_t is a solution of

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{(\theta_t + dd^c \varphi)^n}{\Omega} - F(t, z, \varphi), \\ \varphi(0, \cdot) = \varphi_0. \end{cases} \quad (2.5.1)$$

Consider

$$\phi(t, z) = e^{At} \varphi \left((1 - e^{-At})/A, z \right),$$

so $\phi_0 = \varphi_0$. Then

$$\frac{\partial \phi_t}{\partial t} = \log \frac{(\tilde{\theta}_t + dd^c \phi)^n}{\Omega} + A\phi_t - H(t, z, \phi_t), \quad (2.5.2)$$

where

$$\tilde{\theta}_t = e^{At} \theta_{\frac{1-e^{-At}}{A}},$$

and

$$H(t, z, \phi) = Ant + F(A^{-1}(1 - e^{-At}), z, e^{-At} \phi).$$

Since

$$\frac{\partial \tilde{\theta}_t}{\partial t} = Ae^{At} \theta_{\frac{1-e^{-At}}{A}} + \dot{\theta}_{\frac{1-e^{-At}}{A}},$$

we can choose A so large that $\tilde{\theta}_t$ is increasing in t . Observe that the equation (2.5.1) has a unique solution if and only if the equation (2.5.2) has a unique solution. It follows from

Lemma 2.3.3 that

$$\varphi \geq \varphi_0 - c(t),$$

where $c(t) \searrow 0$ as $t \searrow 0$, so for $\phi(t)$:

$$\phi \geq \phi_0 - \alpha(t),$$

with $\alpha(t) \searrow 0$ as $t \searrow 0$.

Theorem 2.5.1. *Suppose ψ and φ are two solutions of (2.5.1) with $\varphi_0 \leq \psi_0$, then $\varphi_t \leq \psi_t$. In particular, the equation (2.5.1) has a unique solution.*

Proof. Thanks to the previous remark, it is sufficient to prove $u \leq v$, where $u(t, z) = e^{At}\varphi\left((1 - e^{-At})/A, z\right)$, and $v(t, z) = e^{At}\psi\left((1 - e^{-At})/A, z\right)$.

Fix $\varepsilon \in (0, T)$, define

$$\tilde{v}(t, z) = v_{t+\varepsilon} + \alpha(\varepsilon)e^{At} + n\varepsilon(e^{At} - 1).$$

then $\tilde{v}_0 \geq v_0 = \psi_0$ and $\tilde{v} \geq v_{t+\varepsilon}$. Since we choose A so large that $\tilde{\theta}_t$ is increasing,

$$\begin{aligned} \frac{\partial \tilde{v}}{\partial t} &= \log \frac{(\tilde{\theta}_{t+\varepsilon} + dd^c v_{t+\varepsilon})^n}{\Omega} + A\tilde{v}_t - H(t, z, v_{t+s}) \\ &\geq \log \frac{(\tilde{\theta}_t + dd^c \tilde{v}_t)^n}{\Omega} + A\tilde{v}_t - H(t, z, v_{t+s}) \end{aligned}$$

Where

$$H(t, z, v_{t+\varepsilon}) = 2Ant - An(t + \varepsilon) + F\left(A^{-1}(1 - e^{-A(t+\varepsilon)}), z, e^{-A(t+\varepsilon)}v_{t+\varepsilon}\right).$$

It follows from the monotonicity of F in the third variable that

$$F\left(A^{-1}(1 - e^{-A(t+\varepsilon)}), z, e^{-A(t+\varepsilon)}v_{t+\varepsilon}\right) \leq F\left(A^{-1}(1 - e^{-A(t+\varepsilon)}), z, e^{-At}\tilde{v}_t\right).$$

By the assumption $|\frac{\partial F}{\partial t}| < C'$, we choose $A > C'$ and get

$$s \mapsto -A(t + s) + F\left(A^{-1}(1 - e^{-A(t+s)}), z, e^{-At}\tilde{v}_t\right)$$

is decreasing. Thus

$$H(t, z, v_{t+\varepsilon}) \leq Ant + F\left(A^{-1}(1 - e^{-At}), z, e^{-At}\tilde{v}_t\right),$$

and

$$\frac{\partial \tilde{v}}{\partial t} \geq \log \frac{(\tilde{\theta}_t + dd^c \tilde{v}_t)^n}{\Omega} + A\tilde{v}_t - H(t, z, \tilde{v}_t).$$

Therefore \tilde{v} is the supersolution of (2.5.2). It follows from Proposition 2.2.6 that $u_t \leq \tilde{v}_t$, $\forall t \in [0, T]$. Letting $\varepsilon \rightarrow 0$, we get $u_t \leq v_t$, so $\varphi_t \leq \psi_t$. \square

Remark 2.5.2. For $\theta_t(x) = \omega(x)$, $\Omega = \omega^n$, $F(t, z, s) = -2|s|^{1/2}$ and $\varphi_0 = 0$, we obtain two distinct solutions to (CMAF), $\varphi_t(z) \equiv 0$ and $\varphi_t(z) = t^2$. Here $\frac{\partial F}{\partial s}$ is negative and F is not smooth along ($s = 0$).

We now prove the following qualitative stability result:

Theorem 2.5.3. *Fix $\varepsilon > 0$. Let $\varphi_{0,j}$ be a sequence of ω -psh functions with zero Lelong number at all points, such that $\varphi_{0,j} \rightarrow \varphi_0$ in $L^1(X)$. Denote by $\varphi_{t,j}$ and φ_j the solutions of (2.5.1) with the initial condition $\varphi_{0,j}$ and φ_0 respectively. Then*

$$\varphi_{t,j} \rightarrow \varphi_t \text{ in } C^\infty([\varepsilon, T] \times X) \text{ as } j \rightarrow +\infty.$$

Proof. Observe that we can use previous techniques in Section 2.3 to obtain estimates of $\varphi_{t,j}$ in $C^k([\varepsilon, T] \times X)$ for all $k \geq 0$. In particular, for the C^0 estimate, we need to have the uniform bound for $H_{t,j} = \exp(\dot{\varphi}_{t,j} + F)$ in order to use the uniform version of Kolodziej's estimates [Kol98, EGZ08]. By Lemma 2.3.4 we have

$$H_{t,j} = \exp(\dot{\varphi}_{t,j} + F) \leq \exp\left(\frac{-\phi_\varepsilon + C}{t} + C'\right),$$

where C, C' depend on $\varepsilon, \sup_X \varphi_{0,j}$. Since $\varphi_{0,j}$ decreases to φ_0 , we have the $\sup_X \varphi_{0,j}$ is uniformly bounded in term of $\sup_X \varphi_0$ for all j , so we can choose C, C' to be independent of j . Hence there is a constant $A(t, \varepsilon)$ depending on t and ε such that $\|H_{t,j}\|_{L^2(X)}$ is uniformly bounded by $A(t, \varepsilon)$ for all $t \in [\varepsilon, T]$.

By the Arzela-Ascoli theorem we can extract a subsequence φ_{j_k} that converges to ϕ_t in $C^\infty([\varepsilon, T] \times X)$. Note that

$$\frac{\partial \phi_t}{\partial t} = \log \frac{(\theta_t + dd^c \phi_t)^n}{\Omega} - F(t, z, \phi_t).$$

We now prove $\phi_t = \varphi_t$. From Lemma 2.3.3 we get

$$\varphi_{t,j_k} \geq (1 - \beta t)\varphi_{0,j_k} - C(t),$$

where $C(t) \searrow 0$ as $t \rightarrow 0$. Let $j_k \rightarrow +\infty$ we get $\phi_t \geq (1 - \beta t)\varphi_0 - C(t)$, hence

$$\liminf_{t \rightarrow 0} \phi_t \geq \varphi_0.$$

It follows from Theorem 2.5.1 that $\phi_t \geq \varphi_t$. For proving $\phi_t \leq \varphi_t$, we consider $\psi_{0,k} = (\sup_{j \geq k} \varphi_{0,j})^*$, hence $\psi_{0,k} \searrow \varphi_0$ by Hartogs theorem. Denote by $\psi_{t,k}$ the solution of (2.5.1) with initial condition $\psi_{0,k}$. It follows from Theorem 2.5.1 that

$$\psi_{t,j} \geq \varphi_{t,j}.$$

Moreover, thanks to the same arguments for proving the existence of a solution in Sections 2 and 3 by using a decreasing approximation of φ_0 , we have that $\psi_{t,j}$ decreases to φ_t . Thus we infer that $\phi_t \leq \varphi_t$ and the proof is complete. \square

2.5.2 Quantitative stability estimate

In this section, we prove the following stability result when the initial condition is continuous.

Theorem 2.5.4. *If $\varphi, \psi \in C^\infty((0, T] \times X)$ are solutions of (CMAF) with continuous initial data φ_0 and ψ_0 , then*

$$\|\varphi - \psi\|_{C^k([\varepsilon, T] \times X)} \leq C(k, \varepsilon) \|\varphi_0 - \psi_0\|_{L^\infty(X)}. \quad (2.5.3)$$

Proof. Step 1. It follows from Demailly's approximation result (cf. [Dem92]) that there exist two sequences $\{\varphi_{0,j}\}, \{\psi_{0,j}\} \subset PSH(X, \omega) \cap C^\infty(X)$ such that

$$\lim_{j \rightarrow \infty} \|\varphi_{0,j} - \varphi_0\|_{L^\infty(X)} = 0 \quad \text{and} \quad \lim_{j \rightarrow \infty} \|\psi_{0,j} - \psi_0\|_{L^\infty(X)} = 0.$$

Denote by $\varphi_{t,j}, \psi_{t,j}$ solution of (CMAF) corresponding to initial data $\varphi_{0,j}, \psi_{0,j}$. Moreover, thanks to Theorem 2.5.3 we obtain

$$\lim_{j \rightarrow \infty} \|\varphi_{j,k} - \varphi_t\|_{C^k([\varepsilon, T] \times X)} = 0 \quad \text{and} \quad \lim_{j \rightarrow \infty} \|\psi_{j,k} - \psi\|_{C^k([\varepsilon, T] \times X)} = 0.$$

Thus it is sufficient to prove (2.5.3) with smooth functions φ_0, ψ_0 .

Step 2. We now assume that φ_0 and ψ_0 are smooth. For each $\lambda \in [0, 1]$, there is a unique solution $\varphi_t^\lambda \in C^\infty((0, T] \times X)$ for the complex Monge-Ampère flow

$$\begin{cases} \frac{\partial \varphi^\lambda}{\partial t} = \log \frac{(\theta_t + dd^c \varphi^\lambda)^n}{\Omega} - F(t, z, \varphi^\lambda), \\ \varphi^\lambda(0, \cdot) = (1 - \lambda)\varphi_0 + \lambda\psi_0. \end{cases} \quad (2.5.4)$$

By the local existence theorem, φ^λ depends smoothly on the parameter λ . We denote by Δ_t^λ the Laplacian with respect to the Kähler form

$$\omega^\lambda := \theta_t + dd^c \varphi^\lambda.$$

Observe that

$$\left(\frac{\partial}{\partial t} - \Delta_t^\lambda \right) \frac{\partial \varphi^\lambda}{\partial \lambda} = - \frac{\partial F}{\partial s} \frac{\partial \varphi^\lambda}{\partial \lambda},$$

so

$$\left(\frac{\partial}{\partial t} - \Delta_t^\lambda \right) u_t^\lambda + g_\lambda(t, z) u_t^\lambda = 0, \quad (2.5.5)$$

where $u_t^\lambda = \frac{\partial \varphi^\lambda}{\partial \lambda}$ and $g_\lambda(t, z) = \frac{\partial F}{\partial s}(t, z, \varphi^\lambda) \geq 0$. Moreover

$$\psi_t - \varphi_t = \int_0^1 u^\lambda d\lambda,$$

thus it is sufficient to show that

$$\|u_t^\lambda\|_{C^k([\varepsilon, T] \times X)} \leq C(k, \varepsilon) \|u_0^\lambda\|_{L^\infty(X)} = C(k, \varepsilon) \|\psi_0 - \varphi_0\|_{L^\infty}.$$

Step 3. It follows from Theorem 2.3.10 that for each $k \geq 0$,

$$\|g_\lambda\|_{C^k([\varepsilon, T] \times X)} \leq C_1(k, \varepsilon) \quad \text{and} \quad \|\omega_t^\lambda\|_{C^k([\varepsilon, T] \times X)} \leq C_2(k, \varepsilon),$$

for all $\lambda \in [0, 1]$. Using the parabolic Schauder estimates [Kry96, Theorem 8.12.1] for the equation (2.5.5) we get

$$\|u_t^\lambda\|_{C^k([\varepsilon, T] \times X)} \leq C(k, \varepsilon) \|u_t^\lambda\|_{L^\infty(X)}.$$

Step 4. Proving

$$\|u_t^\lambda\|_{L^\infty(X)} \leq \|u_0^\lambda\|_{L^\infty(X)}.$$

Indeed, suppose that u^λ attains its maximum at (t_0, z_0) . If $t_0 = 0$, we obtain the desired inequality. Otherwise, by the maximum principle, at (t_0, z_0)

$$0 \leq \left(\frac{\partial}{\partial t} - \Delta_t^\lambda \right) u_t^\lambda = -g_\lambda(t_0, z_0) u_{t_0}^\lambda.$$

Since $g_\lambda \geq 0$, we get

$$u_t^\lambda \leq \max \left\{ 0, \max_X u_0^\lambda \right\}.$$

Similarly, we obtain

$$u_t^\lambda \geq \min \left\{ 0, \min_X u_0^\lambda \right\},$$

hence

$$\|u_t^\lambda\|_{L^\infty(X)} \leq \|u_0^\lambda\|_{L^\infty(X)}.$$

Finally,

$$\|\varphi - \psi\|_{C^k([\varepsilon, T] \times X)} \leq \int_0^1 \|u_t^\lambda\|_{C^k([\varepsilon, T] \times X)} d\lambda \leq C(k, \varepsilon) \|\varphi_0 - \psi_0\|_{L^\infty(X)}.$$

The proof of Theorem B is therefore complete. \square

2.6 Starting from a nef class

Let (X, ω) be a compact Kähler manifold. In [GZ17], the authors proved that the twisted Kähler-Ricci flow can smooth out a positive current T_0 with zero Lelong numbers belonging to a nef class α_0 . At the level of potentials it satisfies the Monge-Ampère flow

$$\frac{\partial \varphi_t}{\partial t} = \log \frac{(\theta_0 + t\omega + dd^c \varphi_t)^n}{\omega^n}, \quad (2.6.1)$$

where θ_0 is a smooth differential closed $(1, 1)$ -form representing a nef class α_0 and $\varphi_0 \in PSH(X, \theta_0)$ is a θ_0 -psh potential for T_0 , i.e. $T_0 = \theta_0 + dd^c\varphi_0$. We prove here this is still true for more general flows we have considered:

Theorem 2.6.1. *Let θ_0 be a smooth closed $(1, 1)$ -form representing a nef class α_0 and φ_0 be a θ_0 -psh function with zero Lelong number at all points. Set $\theta_t := \theta_0 + t\omega$. Then there exists a unique family $(\varphi_t)_{t \in (0, T]}$ of smooth (θ_t) -psh functions satisfying*

$$\frac{\partial \varphi_t}{\partial t} = \log \frac{(\theta_t + dd^c \varphi_t)^n}{\Omega} - F(t, z, \varphi_t), \quad (2.6.2)$$

such that φ_t converges to φ_0 in L^1 .

Proof. First, observe that for $\varepsilon > 0$, $\theta_0 + \varepsilon\omega$ is a Kähler form. Thanks to Theorem A, there exists a family $\varphi_{t,\varepsilon}$ of $(\theta_t + \varepsilon\omega)$ -psh functions satisfying

$$\frac{\partial \varphi_{t,\varepsilon}}{\partial t} = \log \frac{(\theta_t + \varepsilon\omega + dd^c \varphi_{t,\varepsilon})^n}{\Omega} - F(t, z, \varphi_{t,\varepsilon})$$

with initial data φ_0 which is a $(\theta_0 + \varepsilon\omega)$ -psh function with zero Lelong numbers.

First, we prove that $\varphi_{t,\varepsilon}$ is decreasing in ε . Indeed, for any $\varepsilon' > \varepsilon$

$$\begin{aligned} \frac{\partial \varphi_{t,\varepsilon'}}{\partial t} &= \log \frac{(\theta_t + \varepsilon'\omega + dd^c \varphi_{t,\varepsilon'})^n}{\Omega} - F(t, z, \varphi_{t,\varepsilon'}) \\ &\geq \log \frac{(\theta_t + \varepsilon\omega + dd^c \varphi_{t,\varepsilon'})^n}{\Omega} - F(t, z, \varphi_{t,\varepsilon'}) \end{aligned}$$

hence $\varphi_{t,\varepsilon'} \geq \varphi_{t,\varepsilon}$ by the comparison principle (Proposition 2.2.5). Then we consider

$$\varphi_t := \lim_{\varepsilon \rightarrow 0^+} \searrow \varphi_{t,\varepsilon}.$$

We now show that φ_t is bounded below (so it is not $-\infty$). Thanks to [GZ17, Theorem 7.1], there exist a family (ϕ_t) of $(\theta_0 + t\omega)$ -psh functions such that

$$\frac{\partial \phi_t}{\partial t} = \log \frac{(\theta_0 + t\omega + dd^c \phi_t)^n}{\omega^n}$$

There is $\sigma > 0$ such that $\sigma^{-1}\omega^n \leq \Omega \leq \sigma\omega^n$, so we may assume that

$$\frac{\partial \phi_t}{\partial t} \leq \log \frac{(\theta_0 + t\omega + dd^c \phi_t)^n}{\Omega}.$$

Moreover, $\varphi_{t,\varepsilon} \leq C$, where C only depends on $\sup_X \varphi_0$, hence assume that $F(t, z, \varphi_{t,\varepsilon}) \leq A$ for all ε small. Changing variables, we can assume that $F(t, z, \varphi_{t,\varepsilon}) \leq 0$, hence

$$\frac{\partial \varphi_{t,\varepsilon}}{\partial t} \geq \log \frac{(\theta_0 + t\omega + dd^c \varphi_{t,\varepsilon})^n}{\Omega}.$$

Using the comparison principle (Theorem 2.2.5) again, we get $\varphi_{t,\varepsilon} \geq \phi_t$ for all $\varepsilon > 0$ small, so $\varphi_t \geq \phi_t$.

For the essential uniform bound of φ_t , we use the method of Guedj-Zeriahi. For $\delta > 0$, we fix ω_δ a Kähler form such that $\theta_0 + \delta\omega = \omega_\delta + dd^c h_\delta$ for some smooth function h_δ . Our equation can be rewritten, for $t \geq \delta$

$$(\omega_\delta + (t - \delta)\omega + dd^c(\varphi_t + h_\delta))^n = H_t \Omega \quad (2.6.3)$$

where

$$H_t = e^{\dot{\varphi}_t + F(t,x,\varphi_t)}$$

are uniformly in L^2 , since

$$\dot{\varphi}_t \leq \frac{-\phi_\delta + C}{t} + C,$$

for $t \geq \delta$ as in Lemma 2.3.3. Kolodziej's estimates now yields that $\varphi_t + h_\delta$ is uniformly bounded for $t \geq \delta$, so is φ_t .

Now apply the arguments in Section 2.3 to the equation (2.6.3) we obtain the bounds for the time derivative, gradient, Laplacian and higher order derivatives of $\varphi_t + h_\delta$ in $[\delta, T] \times X$. We thus obtain a priori estimates for φ_t which allow us get the existence of solution of (2.6.2) and the convergence to the initial convergence in $L^1(X)$. \square

Chapter 3

Regularizing properties of Complex Monge-Ampère flows on Hermitian manifolds

We prove that a general complex Monge-Ampère flow on a Hermitian manifold can be run from an arbitrary initial condition with zero Lelong number at all points. Using this property, we confirm a conjecture of Tosatti-Weinkove: the Chern-Ricci flow performs a *canonical surgical contraction*. Finally, we study a generalization of the Chern-Ricci flow on compact Hermitian manifolds, namely the twisted Chern-Ricci flow.

The results of this chapter can be found in [Tö18].

3.1 Introduction

Let (X, g, J) be a compact Hermitian manifold of complex dimension n , that is a compact complex manifold such that J is compatible with the Riemannian metric g . Recently a number of geometric flows have been introduced to study the structure of Hermitian manifolds. Some flows which do preserve the Hermitian property have been proposed by Streets-Tian [StT10, StT11, StT13], Liu-Yang [LY12] and also anomaly flows due to Phong-Picard-Zhang [PPZ16b, PPZ16c, PPZ17a] which moreover preserve the conformally balanced condition of Hermitian metrics. Another such flow, namely the Chern-Ricci flow, was introduced by Gill [Gil11] and has been further developed by Tosatti-Weinkove in [TW15]. The Chern-Ricci flow is written as

$$\frac{\partial}{\partial t}\omega = -Ric(\omega), \quad \omega|_{t=0} = \omega_0, \tag{3.1.1}$$

where $Ric(\omega)$ is the *Chern-Ricci form* which is defined locally by

$$Ric(\omega) := -dd^c \log \omega^n := -\frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \log \omega^n.$$

This flow specializes the Kähler-Ricci flow when the initial metric is Kähler. In [TW15, TW13] Tosatti and Weinkove have investigated the flow on arbitrary Hermitian manifolds, notably in complex dimension 2 (see also [TWY15, FTWT16, GS15, Gil13, LV15, Zhe17, Yan16] for more recent works on the Chern-Ricci flow).

For the Kähler case, running the Kähler-Ricci flow (or complex Monge-Ampère flows) from a rough initial data has been studied by several recent works [CD07], [ST17], [SzTo11], [GZ17], [BG13], [DNL17]. In [ST17], [SzTo11] the authors succeeded to run certain complex Monge-Ampère flows from continuous initial data, while [DNL17] and [GZ17] are running a simplified flow starting from an initial current with zero Lelong numbers. Recently, we extended these latter works to deal with general complex Monge-Ampère flows and arbitrary initial condition (cf. [Tô17]). One of the motivations for this problem comes from the Analytic Minimal Model Program proposed by Song-Tian [ST17]. For the Chern-Ricci flow, the same question was asked recently by Tosatti-Weinkove [TW13, TW15] related to the classification of non-Kähler complex surfaces.

Assume that there exists a holomorphic map between compact Hermitian manifolds $\pi : X \rightarrow Y$ blowing down an exceptional divisor E on X to one point $y_0 \in Y$. In addition, assume that there exists a smooth function ρ on X such that

$$\omega_0 - TRic(\omega_0) + dd^c \rho = \pi^* \omega_Y, \quad (3.1.2)$$

with $T < +\infty$. Tosatti and Weinkove proved:

Theorem. ([TW15, TW13]) *The solution ω_t to the Chern-Ricci flow (3.1.1) converges in $C_{loc}^\infty(X \setminus E)$ to a smooth Hermitian metric ω_T on $X \setminus E$.*

Moreover, there exists a distance function d_T on Y such that (Y, d_T) is a compact metric space and $(X, g(t))$ converges in the Gromov-Hausdorff sense to (Y, d_T) as $t \rightarrow T^-$.

Observe that ω_T induces a singular metric ω' on Y which is smooth in $Y \setminus \{y_0\}$. Tosatti and Weinkove conjectured that one can continue the Chern-Ricci flow on Y with initial data ω' . This is an open question in [TW13, Page 2120] in which they conjectured that the Chern-Ricci flow performs a *canonical surgical contraction*:

Conjecture. (Tosatti-Weinkove [TW13, Page 2120])

- (1) *There exists a smooth maximal solution ω_t of the Chern-Ricci on Y for $t \in (T, T_Y)$ with $T < T_Y \leq +\infty$ such that ω_t converges to ω' , as $t \rightarrow T^+$, in $C_{loc}^\infty(Y \setminus \{y_0\})$. Furthermore, ω_t is uniquely determined by ω_0 .*

- (2) *The metric space $(Y, g(t))$ converges to (Y, d_T) as $t \rightarrow T^+$ in the Gromov-Hausdorff sense.*

In this note, we confirm this conjecture^{*}. An essential ingredient of its proof is to prove that the Monge-Ampère flow corresponding to the Chern-Ricci flow can be run from a rough data. By generalizing a result of Székelyhidi-Tosatti [SzTo11], Nie [Nie14] has proved this property for compact Hermitian manifolds of vanishing first Bott-Chern class and continuous initial data. In this chapter, we generalize the previous results of Nie [Nie14] and the author [Tô17] (see Chapter 2) by considering the following complex Monge-Ampère flow:

$$(CMAF) \quad \frac{\partial \varphi_t}{\partial t} = \log \frac{(\theta_t + dd^c \varphi_t)^n}{\Omega} - F(t, x, \varphi_t),$$

where $(\theta_t)_{t \in [0, T]}$ is a family of Hermitian forms with $\theta_0 = \omega$ and F is a smooth function on $\mathbb{R} \times X \times \mathbb{R}$.

Theorem A. *Let φ_0 be a ω -psh function with zero Lelong number at all points. Let $(t, z, s) \mapsto F(t, z, s)$ be a smooth function on $[0, T] \times X \times \mathbb{R}$ such that $\partial F / \partial s \geq 0$ and $\partial F / \partial t$ is bounded from below.*

Then there exists a family of smooth strictly θ_t -psh functions (φ_t) satisfying (CMAF) in $(0, T] \times X$, with $\varphi_t \rightarrow \varphi_0$ in $L^1(X)$, as $t \searrow 0^+$ and φ_t converges to φ_0 in $C^0(X)$ if φ_0 is continuous. This family is moreover unique if $\partial F / \partial t$ is bounded and $\partial F / \partial s \geq 0$.

The following stability result is a straightforward extension of [Tô17, Theorem 4.3, 4.4] (see Theorem 2.5.3 and Theorem 2.5.4).

Theorem B. *Let $\varphi_0, \varphi_{0,j}$ be ω -psh functions with zero Lelong number at all points, such that $\varphi_{0,j} \rightarrow \varphi_0$ in $L^1(X)$. Denote by $\varphi_{t,j}$ and φ_t the corresponding solutions of (CMAF) with initial condition $\varphi_{0,j}$ and φ_0 respectively. Then for each $\varepsilon \in (0, T)$*

$$\varphi_{t,j} \rightarrow \varphi_t \text{ in } C^\infty([\varepsilon, T] \times X) \text{ as } j \rightarrow +\infty.$$

Moreover, if φ_0 and ψ_0 are continuous, then for any $k \geq 0$, for any $0 < \varepsilon < T$, there exists a positive constant $C(k, \varepsilon)$ depending only on k and ε such that

$$\|\varphi - \psi\|_{C^k([\varepsilon, T] \times X)} \leq C(k, \varepsilon) \|\varphi_0 - \psi_0\|_{L^\infty(X, \omega)}.$$

As a consequence of Theorem A and Theorem B, the Chern-Ricci flow on any Hermitian manifold can be run from rough data. Using this result and a method due to Song-Tosatti-Weinkove [SW13a, TW13] we prove the conjecture. The proof is given in Section 3.5.

^{*}After this paper was completed, the author learned that Xiaolan Nie proved the first statement of the conjecture for complex surfaces (cf. [Nie17]). She also proved that the Chern-Ricci flow can be run from a bounded data. The author would like to thank Xiaolan Nie for sending her preprint.

The second purpose of this paper is to study a generalization of the Chern-Ricci flow, namely the *twisted Chern-Ricci flow*:

$$\frac{\partial \omega_t}{\partial t} = -Ric(\omega_t) + \eta, \quad \omega|_{t=0} = \omega_0$$

where $Ric(\omega_t)$ is the Chern-Ricci form of ω_t , ω_0 is a Hermitian metric on X and η is a smooth $(1,1)$ -form. In general, we do not assume η is closed. This flow also generalizes the twisted Kähler-Ricci flow which has been studied recently by several authors (see for instance [CS12, GZ17]).

We show that the twisted Chern-Ricci flow starting from a Hermitian metric ω_0 is equivalent to the following complex Monge-Ampère flow

$$\frac{\partial \varphi}{\partial t} = \log \frac{(\hat{\omega}_t + dd^c \varphi)^n}{\omega_0^n}, \quad (3.1.3)$$

where $\hat{\omega}_t = \omega_0 + t(\eta - Ric(\omega_0))$. We first prove the following, generalizing [TW15, Theorem 1.2]:

Theorem C. *There exists a unique solution to the twisted Chern-Ricci flow on $[0, T)$, where*

$$T := \sup\{t \geq 0 \mid \exists \psi \in C^\infty(X) \text{ with } \hat{\omega}_t + dd^c \psi > 0\}.$$

When the twisted Chern-Ricci flow has a long time solution, it is natural to study its behavior at infinity. When the Bott-Chern class vanishes and $\eta = 0$, Gill has proved that the flow converges to a Chern-Ricci flat Hermitian metric (cf. [Gil11]).

Denote by

$$\{\eta\} := \{\alpha \text{ is a real } (1,1)\text{-form} \mid \exists f \in C^\infty(X) \text{ with } \alpha = \eta + dd^c f\},$$

the equivalence class of η . Suppose that $c_1^{BC}(X) - \{\eta\}$ is negative. Consider the *normalized twisted Chern-Ricci flow*

$$\frac{\partial \omega_t}{\partial t} = -Ric(\omega_t) - \omega_t + \eta. \quad (3.1.4)$$

Then we have the following result for the long time behavior of the flow generalizing [TW15, Theorem 1.7]:

Theorem D. *Suppose $c_1^{BC}(X) - \{\eta\} < 0$. The normalized twisted Chern-Ricci flow smoothly converges to a Hermitian metric $\omega_\infty = \eta - Ric(\Omega) + dd^c \varphi_\infty$ which satisfies*

$$Ric(\omega_\infty) = \eta - \omega_\infty.$$

Observe that ω_∞ satisfies the *twisted Einstein equation*:

$$\text{Ric}(\omega) = \eta - \omega. \quad (3.1.5)$$

We can prove the existence of a unique solution of (3.1.5) using a result of Monge-Ampère equation due to Cherrier [Che87] (see Theorem 3.6.2). Theorem D moreover gives an alternative proof of the existence of the twisted Einstein metric ω_∞ in $-c_1^{BC}(X) + \eta$. This is therefore a generalization of Cao's approach [Cao85] by using Kähler-Ricci flow to prove the existence of Kähler-Einstein metric on Kähler manifold of negative first Chern class. In particular, when $\eta = 0$, we have $c_1^{BC}(X) < 0$ hence we have $c_1(X) < 0$ and X is a Kähler manifold, this is [TW15, Theorem 1.7].

Note that in general, one cannot assume η to be closed, in contrast with the twisted Kähler-Ricci flow. Let us stress also that the limit of the normalized twisted Chern-Ricci flow exists without assuming that the manifold is Kähler (a necessary assumption when studying the long term behavior of the Chern-Ricci flow). Therefore the twisted Chern-Ricci flow is somehow more natural in this context.

As an application of Theorem D, we give an alternative proof of the existence of a unique smooth solution for the following Monge-Ampère equation

$$(\omega + dd^c\varphi)^n = e^\varphi\Omega.$$

We show that the solution is the limit of the potentials of a suitable twisted normalized Chern-Ricci flow. Cherrier [Che87] proved this result by generalizing the elliptic approach of Aubin [Aub78] and [Yau78].

The paper is organized as follows. In Section 3.2, we recall some notations in Hermitian manifolds. In Section 3.3 we prove various *a priori* estimates following our previous work [Tô17]. The main difference is that we will use the recent result of Koloziedj's uniform type estimates for Monge-Ampère on Hermitian manifolds (cf. [DK12, Blo11, Ngu16]) instead of the one on Kähler manifolds to bound the oscillation of the solution. The second arises when estimating the gradient and the Laplacian: we use a special local coordinate system due to Guan-Li [GL10, Lemma 2.1] instead of the usual normal coordinates in Kähler geometry. In Section 3.4 we prove Theorem B and Theorem C. In Section 3.5, we prove the conjecture. In Section 3.6 we define the twisted Chern-Ricci flow and prove the existence of a unique maximal solution using the estimates in Section 3.3. The approach is different from the one for the Chern-Ricci flow due to Tosatti-Weinkove [TW15]. We also show that the twisted Chern-Ricci flow on negative twisted Bott-Chern class smoothly converges to the unique twisted Einstein metric.

3.2 Preliminaries

3.2.1 Chern-Ricci curvature on Hermitian manifold

Let (X, g) be a compact Hermitian manifold of complex dimension n . In local coordinates, g is determined by the $n \times n$ Hermitian matrix $(g_{i\bar{j}}) = g(\partial_i, \partial_{\bar{j}})$. We write $\omega = \sqrt{-1}g_{i\bar{j}}dz_i \wedge d\bar{z}_j$ for its associated $(1, 1)$ -form.

We define the Chern connection ∇ associated to g as follows. If $X = X^j \partial_j$ is a vector field and $\alpha = a_i dz_i$ is a $(1, 0)$ -form then their covariant derivatives have components

$$\nabla_i X^k = \partial_i X^j + \Gamma_{ij}^k X^j, \quad \nabla_i a_j = \partial_i a_j - \Gamma_{ij}^k a_k,$$

where the Christoffel symbols Γ_{ij}^k are given by

$$\Gamma_{ij}^k = g^{\bar{l}k} \partial_i g_{j\bar{l}}.$$

We define the torsion tensors T and \bar{T} of ω as follows

$$\begin{aligned} T &= \sqrt{-1} \partial \omega = \frac{1}{2} T_{ij\bar{k}} dz_i \wedge dz_j \wedge d\bar{z}_k \\ \bar{T} &= \sqrt{-1} \bar{\partial} \omega = \frac{1}{2} \bar{T}_{i\bar{j}k} d\bar{z}_i \wedge d\bar{z}_j \wedge dz_k. \end{aligned}$$

where

$$T_{ij\bar{k}} = \partial_i g_{j\bar{k}} - \partial_j g_{i\bar{k}}, \quad \text{and} \quad \bar{T}_{i\bar{j}k} = \partial_{\bar{j}} g_{k\bar{i}} - \partial_i g_{k\bar{j}}.$$

Then the *torsion tensor* of ω has component

$$T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k = g^{\bar{l}k} T_{ij\bar{l}}.$$

Definition 3.2.1. The *Chern-Ricci curvature* of g is the tensor

$$R_{k\bar{l}}(g) := R_{k\bar{l}}(\omega) := g^{\bar{j}i} R_{k\bar{l}i\bar{j}} = -\partial_k \partial_{\bar{l}} \log \det g,$$

and the *Chern-Ricci form* is

$$Ric(g) := Ric(\omega) := \frac{\sqrt{-1}}{\pi} R_{k\bar{l}}^C dz_k \wedge d\bar{z}_l = -dd^c \log \det g,$$

where

$$d := \partial + \bar{\partial}, \quad d^c := \frac{1}{2i\pi} (\partial - \bar{\partial}).$$

It is a closed real $(1, 1)$ -form and its cohomology class in the *Bott-Chern cohomology group*

$$H_{BC}^{1,1}(X, \mathbb{R}) := \frac{\{\text{closed real } (1,1)\text{-forms}\}}{\{\sqrt{-1} \partial \bar{\partial} \psi, \psi \in C^\infty(X, \mathbb{R})\}}$$

is the *first Bott-Chern class* of X , denoted by $c_1^{BC}(X)$, which is independent of the choice of Hermitian metric g . We also write $R = g^{k\bar{l}} R_{k\bar{l}}$ for the *Chern scalar curvature*.

3.2.2 Plurisubharmonic functions and Lelong number

Let (X, ω) be a compact Hermitian manifold.

Definition 3.2.2. We let $PSH(X, \omega)$ denote the set of all ω -plurisubharmonic functions (ω -psh for short), i.e the set of functions $\varphi \in L^1(X, \mathbb{R} \cup \{-\infty\})$ which can be locally written as the sum of a smooth and a plurisubharmonic function, and such that

$$\omega + dd^c \varphi \geq 0$$

in the weak sense of positive currents.

Definition 3.2.3. Let φ be a ω -psh function and $x \in X$. The *Lelong number* of φ at x is

$$\nu(\varphi, x) := \liminf_{z \rightarrow x} \frac{\varphi(z)}{\log |z - x|}.$$

We say φ has a *logarithmic pole of coefficient* γ at x if $\nu(\varphi, x) = \gamma$.

3.3 A priori estimates for complex Monge-Ampère flows

In this section we prove various *a priori* estimates for φ_t which satisfies

$$\frac{\partial \varphi_t}{\partial t} = \log \frac{(\theta_t + dd^c \varphi_t)^n}{\Omega} - F(t, z, \varphi) \quad (CMAF)$$

with a smooth strictly ω -psh initial data φ_0 , where Ω is a smooth volume form, $(\theta_t)_{t \in [0, T]}$ is a family of Hermitian forms on X and $(t, z, s) \mapsto F(t, z, s)$ is a smooth function on $[0, T] \times X \times \mathbb{R}$ with

$$\frac{\partial F}{\partial s} \geq 0 \text{ and } \frac{\partial F}{\partial t} > B, \quad (3.3.1)$$

for some $B \in \mathbb{R}$.

Since we are interested in the behavior near 0 of (CMAF), we can further assume that

$$\frac{\omega}{2} \leq \theta_t \leq 2\omega \text{ and } \delta^{-1}\Omega \leq \theta_t^n \leq \delta\Omega, \forall t \in [0, T] \text{ for some } \delta > 0, \quad (3.3.2)$$

$$\theta_t - t\dot{\theta}_t \geq 0 \text{ for } 0 \leq t \leq T. \quad (3.3.3)$$

The assumption (3.3.3) will be used to bound $\dot{\varphi}_t$ from above.

3.3.1 Bounds on φ_t and $\dot{\varphi}_t$

As in the Kähler case, the upper bound of φ is a simple consequence of the maximum principle (see [Tô17, Lemma 2.1] or Lemma 2.3.1).

For a lower bound of φ_t , we have

Lemma 3.3.1. *There is a constant $C > 0$ depending only on $\inf_X \varphi_0$ such that,*

$$\varphi_t \geq \inf_X \varphi_0 - Ct, \quad \forall (t, x) \in [0, T] \times X.$$

Proof. Set

$$\psi := \inf_X \varphi_0 - Ct,$$

where C will be chosen hereafter. Since we assume that $2\theta_t \geq \omega$,

$$\theta_t + dd^c \psi \geq \frac{1}{2}\omega.$$

Combine with $\omega^n \geq 2^{-n}\theta_t^n \geq \Omega/(2^n\delta)$, we have

$$\frac{(\theta_t + dd^c \psi)^n}{\Omega} \geq \frac{1}{2^n} \frac{\omega^n}{\Omega} \geq \frac{1}{4^n \delta}.$$

We now choose $C > 0$ satisfying

$$-C + \sup_{[0, T] \times X} F(t, x, \inf_X \varphi_0) \leq \frac{1}{4^n \delta}.$$

hence

$$\frac{\partial \psi_t}{\partial t} \leq \frac{(\theta_t + dd^c \psi)^n}{\Omega} - F(t, x, \psi),$$

It follows from the maximum principle [Tô17, Proposition 1.5] (see Proposition 2.2.5) that

$$\varphi_t \geq \psi_t,$$

as required. □

For another lower bound, we follow the argument in [GZ17], replacing the uniform a priori bound of Kołodziej [Kol98] by its Hermitian version (see for instance [Ngu16, Theorem 2.1]). First, we assume that $\theta_t \geq \omega + t\chi$, $\forall t \in [0, T]$, for some smooth $(1, 1)$ -form χ . Let $0 < \beta < +\infty$ be such that

$$\chi + (2\beta - 1)\omega \geq 0.$$

It follows from Kołodziej's uniform type estimate for Monge-Ampère equation on Hermitian manifolds (cf. [Ngu16, Theorem 2.1]) that there exists a continuous ω -psh solution u of the equation

$$(\omega + dd^c u)^n = e^{u-2\beta\varphi_0} \omega^n,$$

which satisfies

$$\|u\|_{L^\infty(X)} < C,$$

where C only depends on $\|e^{-2\beta\varphi_0}\|_{L^p(X)}$, for some $p > 1$.

Remark 3.3.2. Latter on we will replace φ_0 by smooth approximants $\varphi_{0,j}$ of initial data. Since the latter one has zero Lelong numbers, Skoda's integrability theorem [Sko72] will provide a uniform bound for $\|e^{-2\beta\varphi_0}\|_{L^p(X)}$ and $\|u\|_{L^\infty(X)}$.

Lemma 3.3.3. *For all $z \in X$ and $0 < t < \min(T, (2\beta)^{-1})$, we have*

$$\varphi_t(z) \geq (1 - 2\beta t)\varphi_0(z) + tu(z) + n(t \log t - t) - At, \quad (3.3.4)$$

where A depends on $\sup_X \varphi_0$. In particular, there exists $c(t) \geq 0$ such that

$$\varphi_t(z) \geq \varphi_0(z) - c(t),$$

with $c(t) \searrow 0$ as $t \searrow 0$.

Proof. Set

$$\phi_t := (1 - 2\beta t)\varphi_0 + tu + n(t \log t - t) - At,$$

where $A := \sup_{[0,T] \times X} F(t, z, C_0)$ with $(1 - 2\beta t)\varphi_0 + tu + n(t \log t - t) \leq C_0$ for all $t \in [0, \min(T, (2\beta)^{-1})]$.

By our choice of β and the assumption $\theta_t \geq \omega + t\chi$, $\forall t \in [0, T]$, we have

$$\begin{aligned} \theta_t + dd^c \phi_t &\geq \omega + t\chi + dd^c \phi_t \\ &= (1 - 2\beta t)(\omega + dd^c \varphi_0) + t(\omega + dd^c u) + t[\chi + (2\beta - 1)\omega] \\ &\geq t(\omega + dd^c u) \geq 0. \end{aligned}$$

Moreover

$$(\theta_t + dd^c \phi_t)^n \geq t^n (\omega + dd^c u)^n = e^{\partial_t \phi_t + A} \geq e^{\partial_t \phi_t + F(t, z, \phi_t)},$$

hence ϕ_t is a subsolution to (CMAF). Since $\phi_0 = \varphi_0$ the conclusion follows from the maximum principle [Tô17, Proposition 1.5] (see Proposition 2.2.5). \square

The lower bound for $\dot{\varphi}$ comes from the same argument in [Tô17, Proposition 2.6] (see Proposition 2.3.6):

Proposition 3.3.4. *Assume φ_0 is bounded. There exist constants $A > 0$ and $C = C(A, \text{Osc}_X \varphi_0) > 0$ such that for all $(x, t) \in X \times (0, T]$,*

$$\dot{\varphi} \geq n \log t - A \text{Osc}_X \varphi_0 - C.$$

We now prove a crucial estimate for $\dot{\varphi}_t$ which allows us to use the uniform version of Kolodziej's uniform type estimates in order to get the bound of $\text{Osc}_X \varphi_t$. The proof is the same in [GZ17, Tô17], but we include a proof for the reader's convenience.

Proposition 3.3.5. *There exists $0 < C = C(\sup_X \varphi_0, T)$ such that for all $0 < t \leq T$ and $z \in X$,*

$$\dot{\varphi}_t(z) \leq \frac{-\varphi_0(z) + C}{t}.$$

Proof. We consider $G(t, z) = t\dot{\varphi}_t - \varphi_t - nt + Bt^2/2$, with B is the constant in (3.3.1). We obtain

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_t} \right) G = -t\dot{\varphi} \frac{\partial F}{\partial s} + t \left(B - \frac{\partial F}{\partial t} \right) - \text{tr}_{\omega_t}(\theta_t - t\dot{\theta}_t).$$

Since we assume that $\theta_t - t\dot{\theta}_t \geq 0$ (see (3.3.3)), we get

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_t} \right) G \leq -t\dot{\varphi} \frac{\partial F}{\partial s} + t \left(B - \frac{\partial F}{\partial t} \right).$$

If G attains its maximum at $t = 0$, we have the result. Otherwise, assume that G attains its maximum at (t_0, z_0) with $t_0 > 0$, then at (t_0, z_0) we have

$$0 \leq \left(\frac{\partial}{\partial t} - \Delta_{\omega_t} \right) G < -t_0 \frac{\partial F}{\partial s} \dot{\varphi}.$$

Since $\frac{\partial F}{\partial s} \geq 0$ by the hypothesis, we obtain $\dot{\varphi}(t_0, z_0) < 0$ and

$$t\dot{\varphi}_t - \varphi_t - nt + Bt^2/2 \leq -\varphi_{t_0}(z_0) - nt_0 + Bt_0^2/2.$$

Using Lemma 3.3.3 we get $\varphi_{t_0} \geq \varphi_0 - C_1$, where C_1 only depends on $\sup_X \varphi_0$ and T , hence there is a constant C_2 depending on $\sup_X \varphi_0$ and T such that

$$t\dot{\varphi}_t \leq \varphi_t - \varphi_0 + C_2.$$

Since $\varphi_t \leq C_3(\sup \varphi_0, T)$, so

$$\dot{\varphi}_t(x) \leq \frac{-\varphi_0 + C}{t},$$

where C only depends on $\sup_X \varphi_0$ and T . □

3.3.2 Bounding the oscillation of φ_t

Once we get an upper bound for $\dot{\varphi}_t$ as in Proposition 3.3.5, we can bound the oscillation of φ_t by using the following uniform version of Kolodziej's estimates due to Dinew- Kolodziej [DK12, Theorem 5.2].

Theorem 3.3.6. *Let (X, ω) be a compact Hermitian manifold. Assume $\varphi \in C^2(X)$ is such that $\omega + dd^c\varphi \geq 0$ and*

$$(\omega + dd^c\varphi)^n = f\omega^n.$$

Then for $p > 1$,

$$Osc_X\varphi \leq C,$$

where C only depends on $\omega, p, \|f\|_{L^p(X)}$.

Indeed, observe that φ_t satisfies

$$(\theta_t + dd^c\varphi_t)^n = H_t\Omega,$$

then by Proposition 3.3.5, for any $\varepsilon \in (0, T)$,

$$H_t = \exp(\dot{\varphi}_t + F) \leq \exp\left(\frac{-\varphi_0 + C}{t} + C'\right)$$

for all $t \in [\varepsilon, T]$. Fix $p > 1$ and \mathcal{F} a compact family of ω -psh functions with zero Lelong numbers, and assume that $\varphi_0 \in \mathcal{F}$. It follows from the uniform version of Skoda's integrability theorem (cf. [Sko72, Proposition 7.1] and [Zer01, Theorem 3.1]) that there exists $C_\varepsilon > 0$ such that

$$\|e^{-\psi/t}\|_{L^p(\Omega)} \leq C_\varepsilon,$$

for all $\psi \in \mathcal{F}, t \in [\varepsilon, T]$. We thus write for short $\|H_t\|_{L^p(\Omega)} \leq C(t)$ for some $C(t) > 0$.

Remark 3.3.7. Later on we will replace φ_0 by smooth approximants $\varphi_{0,j}$ of initial data. We can thus apply the previous estimate with $\mathcal{F} = \{\varphi_0\} \cup \{\varphi_{0,j}, j \in \mathbb{N}\}$, where φ_0 is now the initial data. This yields

$$\|H_{t,j}\|_{L^p(\Omega)} \leq C(t).$$

Now, thanks to Theorem 3.3.6, we infer that the oscillation of φ_t is uniformly bounded:

Theorem 3.3.8. *Fix $0 < t \leq T$. There exist $C(t) > 0$ independent of $\inf_X \varphi_0$ such that*

$$Osc_X(\varphi_t) \leq C(t).$$

3.3.3 Bounding the gradient of φ

In this section we bound the gradient of φ using the same technique as in [Tô17] (see also [SzTo11]) which is a parabolic version of Błocki's estimate [Blo09] for Kähler manifolds. In these articles we used the usual normal coordinates in Kähler geometry. For Hermitian manifolds, we need to use the following local coordinate system due to Guan-Li [GL10, Lemma 2.1] (see also [StT11] for a similar argument), which is also essential for our second order estimate. We also refer the reader to [Ha96, Lemma 6] for a gradient estimate for the elliptic Complex Mong-Ampère equation in the Hermitian case without using the local coordinate system. We thank Valentino Tosatti for indicating the reference [Ha96]. We remark that similar arguments of the proof below can be found in [Nie14, Lemma 3.3].

Lemma 3.3.9. *At any point $x \in X$ there exists a local holomorphic coordinate system centered at x such that for all i, j*

$$g_{i\bar{j}}(0) = \delta_{i,j}, \quad \frac{\partial g_{i\bar{i}}}{\partial z_j}(0) = 0. \quad (3.3.5)$$

We now prove

Proposition 3.3.10. *Fix $\varepsilon \in [0, T]$. There exists $C > 0$ depending on $\sup_X \varphi_0$ and ε such that for all $\varepsilon < t \leq T$*

$$|\nabla\varphi(z)|_\omega^2 < e^{C/(t-\varepsilon)}.$$

Proof. Since the bound on $Osc_X \varphi_\varepsilon$ only depends on $\sup_X \varphi_0$ and ε (see Theorem 3.3.8), we can consider the flow starting from φ_ε , i.e $\varphi(0, x) = \varphi_\varepsilon$. Then we need to show that there exists a constant C depending on $Osc_X \varphi_0$ and ε such that

$$|\nabla\varphi(z)|_\omega^2 < e^{C/t},$$

for all $t \in [0, T - \varepsilon]$.

Define

$$K(t, x) = t \log |\nabla\varphi|_\omega^2 - \gamma \circ \varphi = t \log \beta - \gamma \circ \varphi,$$

for $(t, x) \in [0, T - \varepsilon] \times X$ where, $\beta = |\nabla\varphi|_\omega^2$ and $\gamma \in C^\infty(\mathbb{R}, \mathbb{R})$ will be chosen hereafter.

If $K(t, z)$ attains its maximum for $t = 0$, β is bounded in terms of $\sup_X \varphi_0$ and ε , since $|\varphi_t|$ is bounded by a constant depending on $\sup_X \varphi_0$ and ε for all $t \in [0, T - \varepsilon]$ (see Section 3.3.1).

We now assume that $K(t, z)$ attains its maximum at (t_0, z_0) in $[0, T - \varepsilon] \times X$ with $t_0 > 0$. Near z_0 we have $\omega = \sqrt{-1}g_{i\bar{j}}dz_i \wedge d\bar{z}_j$ for some and $\theta_t = \sqrt{-1}h_{i\bar{j}}dz_i \wedge d\bar{z}_j$. We take the local coordinates (3.3.5) for ω at z_0 such that

$$g_{i\bar{k}}(z_0) = \delta_{jk} \quad (3.3.6)$$

$$g_{i\bar{i}l}(z_0) = 0 \quad (3.3.7)$$

$$u_{p\bar{q}}(t_0, z_0) = h_{p\bar{q}} + \varphi_{p\bar{q}} \text{ is diagonal,} \quad (3.3.8)$$

here for convenience we denote in local coordinate, $u_p := \frac{\partial u}{\partial z_p}$, $u_{j\bar{k}} := \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}$, and $g_{i\bar{j}k} := \frac{\partial g_{i\bar{j}}}{\partial z_k}$.

We now compute $K_p, K_{p\bar{p}}$ at (t_0, z_0) in order to use the maximum principle. At (t_0, z_0) we have $K_p = 0$ hence

$$t\beta_p = \beta\gamma' \circ \varphi\varphi_p \quad (3.3.9)$$

or

$$\left(\frac{\beta_p}{\beta}\right)^2 = \frac{1}{t^2}(\gamma' \circ \varphi)^2 |\varphi_p|^2.$$

Therefore,

$$\begin{aligned} K_{p\bar{p}} &= t \frac{\beta_{p\bar{p}}\beta - |\beta_p|^2}{\beta^2} - \gamma'' \circ \varphi |\varphi_p|^2 - \gamma' \circ \varphi\varphi_{p\bar{p}} \\ &= t \frac{\beta_{p\bar{p}}}{\beta} - [t^{-1}(\gamma' \circ \varphi)^2 + \gamma'' \circ \varphi] |\varphi_p|^2 - \gamma' \circ \varphi\varphi_{p\bar{p}}. \end{aligned}$$

Now we compute $\beta_p, \beta_{p\bar{p}}$ at (t_0, z_0) with $\beta = g^{j\bar{k}}\varphi_j\varphi_{\bar{k}}$ where $(g^{j\bar{k}}) = [(g_{j\bar{k}})^t]^{-1}$. We have

$$\beta_p = g_p^{j\bar{k}}\varphi_j\varphi_{\bar{k}} + g^{j\bar{k}}\varphi_{jp}\varphi_{\bar{k}} + g^{j\bar{k}}\varphi_j\varphi_{\bar{k}p}.$$

Since

$$g_p^{j\bar{k}} = -g^{j\bar{l}}g_{s\bar{l}p}g^{s\bar{k}},$$

$$\beta_p = -g^{j\bar{l}}g_{s\bar{l}p}g^{s\bar{k}}\varphi_j\varphi_{\bar{k}} + g^{j\bar{k}}\varphi_{jp}\varphi_{\bar{k}} + g^{j\bar{k}}\varphi_j\varphi_{\bar{k}p}$$

and

$$\begin{aligned} \beta_{p\bar{p}} &= -g_p^{j\bar{l}}g_{s\bar{l}p}g^{s\bar{k}}\varphi_j\varphi_{\bar{k}} - g^{j\bar{l}}g_{s\bar{l}p\bar{p}}g^{s\bar{k}}\varphi_j\varphi_{\bar{k}} - g^{j\bar{l}}g_{s\bar{l}p}g_p^{s\bar{k}}\varphi_j\varphi_{\bar{k}} \\ &\quad - g^{j\bar{l}}g_{s\bar{l}p}g^{s\bar{k}}\varphi_{j\bar{p}}\varphi_{\bar{k}} - g^{j\bar{l}}g_{s\bar{l}p}g^{s\bar{k}}\varphi_j\varphi_{\bar{k}\bar{p}} + g_p^{j\bar{k}}\varphi_{jp}\varphi_{\bar{k}} + g^{j\bar{k}}\varphi_{jp\bar{p}}\varphi_{\bar{k}} \\ &\quad + g^{j\bar{k}}\varphi_{jp}\varphi_{\bar{k}\bar{p}} + g_p^{j\bar{k}}\varphi_j\varphi_{\bar{k}p} + g^{j\bar{k}}\varphi_{j\bar{p}}\varphi_{\bar{k}p} + g^{j\bar{k}}\varphi_j\varphi_{\bar{k}p\bar{p}}. \end{aligned}$$

Therefore, at (t_0, x_0) ,

$$\begin{aligned} g_p^{j\bar{k}} &= -g_{k\bar{j}p}, \\ \beta_p &= -g^{j\bar{l}}g_{s\bar{l}p}g^{s\bar{k}}\varphi_j\varphi_{\bar{k}} + \sum_j \varphi_{jp}\varphi_{\bar{j}} + \sum_j \varphi_{\bar{j}p}\varphi_j, \end{aligned} \quad (3.3.10)$$

and

$$\beta_{p\bar{p}} = -g_{k\bar{j}p\bar{p}}\varphi_j\varphi_{\bar{k}} + 2\operatorname{Re}(\varphi_{jp\bar{p}}\varphi_{\bar{j}}) + |\varphi_{jp} - \sum_k g_{j\bar{k}p}\varphi_k|^2 + |\varphi_{j\bar{p}} - \sum_k g_{k\bar{j}p}\varphi_k|^2$$

Now at (t_0, z_0)

$$\Delta_{\omega_{t_0}} K = \sum_{p=1}^n \frac{K_{p\bar{p}}}{u_{p\bar{p}}} = \frac{t \beta_{p\bar{p}}}{\beta u_{p\bar{p}}} - \frac{[t^{-1}(\gamma' \circ \varphi)^2 + \gamma'' \circ \varphi] |\varphi_p|^2}{u_{p\bar{p}}} - \frac{\gamma' \circ \varphi \varphi_{p\bar{p}}}{u_{p\bar{p}}}.$$

Since $u_{p\bar{p}} = \varphi_{p\bar{p}} + h_{p\bar{p}}$ near (t_0, z_0) , then at (t_0, z_0)

$$\begin{aligned} \sum_p \frac{\gamma' \circ \varphi \varphi_{p\bar{p}}}{u_{p\bar{p}}} &= n\gamma' \circ \varphi - \sum_p \frac{\gamma' \circ \varphi h_{p\bar{p}}}{u_{p\bar{p}}} \\ &\leq n\gamma' \circ \varphi - \lambda \sum_p \frac{\gamma' \circ \varphi}{u_{p\bar{p}}}, \end{aligned}$$

with $\lambda \leq h_{p\bar{p}}$, on $[0, T] \times X$ for all $p = 1, \dots, n$.

Moreover, assume that the holomorphic bisectional curvature of ω is bounded from below by a constant $-B \in \mathbb{R}$ on X , then at (t_0, z_0)

$$t \sum_{j,k,p} \frac{-g_{j\bar{k}p\bar{p}} \varphi_j \varphi_{\bar{k}}}{\beta u_{p\bar{p}}} = t \sum_{j,k,p} \frac{R_{j\bar{k}p\bar{p}} \varphi_j \varphi_{\bar{k}}}{\beta u_{p\bar{p}}} \geq -Bt \sum_p \frac{1}{u_{p\bar{p}}},$$

therefore

$$\begin{aligned} \Delta_{\omega_{t_0}} K &\geq (\lambda\gamma' \circ \varphi - tB) \sum_p \frac{1}{u_{p\bar{p}}} + 2t \operatorname{Re} \sum_{j,p} \frac{\varphi_{p\bar{p}j} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}} \\ &\quad + \frac{t}{\beta} \sum_{j,p} \frac{|\varphi_{jp} - \sum_k g_{k\bar{j}p} \varphi_k|^2 + |\varphi_{j\bar{p}} - \sum_k g_{k\bar{j}p} \varphi_k|^2}{\beta u_{p\bar{p}}} \\ &\quad - [t^{-1}(\gamma' \circ \varphi)^2 + \gamma'' \circ \varphi] \sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}} - n\gamma' \circ \varphi. \end{aligned}$$

By the maximum principle, at (t_0, z_0)

$$0 \leq \left(\frac{\partial}{\partial t} - \Delta_{\omega_t} \right) K$$

hence,

$$\begin{aligned} 0 &\leq \log \beta - \gamma' \circ \varphi \dot{\varphi} - (\lambda\gamma' \circ \varphi - tB) \sum_p \frac{1}{u_{p\bar{p}}} + t \frac{\beta'}{\beta} - 2t \operatorname{Re} \sum_{j,p} \frac{\varphi_{p\bar{p}j} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}} \\ &\quad - \frac{t}{\beta} \sum_{j,p} \frac{|\varphi_{jp} - \sum_k g_{k\bar{j}p} \varphi_k|^2 + |\varphi_{j\bar{p}} - \sum_k g_{k\bar{j}p} \varphi_k|^2}{\beta u_{p\bar{p}}} \\ &\quad + [t^{-1}(\gamma' \circ \varphi)^2 + \gamma'' \circ \varphi] \sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}} + n\gamma' \circ \varphi. \end{aligned} \tag{3.3.11}$$

We will simplify (3.3.11) to get a bound for β at (t_0, z_0) .

Claim 1. *There exist $C_1 > 0$ depending on $\sup |\varphi_0|$ and C_2, C_4 only depending on h and C_3 depending on $\sup |\gamma' \circ \varphi|$ such that*

$$t \frac{\beta'}{\beta} - 2t \operatorname{Re} \sum_{j,p} \frac{u_{p\bar{p}j} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}} < C_1 t + C_2 t \sum_p \frac{1}{u_{p\bar{p}}},$$

and

$$-\frac{t}{\beta} \sum_{j,p} \frac{|\varphi_{jp} - \sum_k g_{k\bar{j}\bar{p}} \varphi_k|^2}{\beta u_{p\bar{p}}} + t^{-1} (\gamma' \circ \varphi)^2 \sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}} \leq C_3 \gamma' \circ \varphi + \frac{C_4}{\beta} \gamma' \circ \varphi \sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}}.$$

Proof of Claim 1. For the first one, we note that near (t_0, z_0)

$$\log \det(u_{p\bar{q}}) = \dot{\varphi} + F(t, z, \varphi) + \log \Omega,$$

hence using

$$\frac{d}{ds} \det A = A^{\bar{j}i} \left(\frac{d}{ds} A_{i\bar{j}} \right) \det A$$

we have at (t_0, z_0)

$$u^{p\bar{p}} u_{p\bar{p}j} = \frac{u_{p\bar{p}j}}{u_{p\bar{p}}} = (\dot{\varphi} + F(t, z, \varphi) + \log \Omega)_j.$$

Therefore

$$\begin{aligned} 2t \operatorname{Re} \sum_{j,p} \frac{\varphi_{p\bar{p}j} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}} &= 2t \operatorname{Re} \sum_{j,p} \frac{(u_{p\bar{p}j} - h_{p\bar{p}j}) \varphi_{\bar{j}}}{\beta u_{p\bar{p}}} \\ &= \frac{2t}{\beta} \operatorname{Re} \sum_j (\dot{\varphi} + F(t, z, \varphi) + \log \Omega)_j \varphi_{\bar{j}} - 2t \operatorname{Re} \sum_{j,p} \frac{h_{p\bar{p}j} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}} \\ &= \frac{2t}{\beta} \operatorname{Re} \sum_j (\dot{\varphi}_j \varphi_{\bar{j}}) + \frac{2t}{\beta} \operatorname{Re} \left((F(t, z, \varphi) + \log \Omega)_j + \frac{\partial F}{\partial s} \varphi_j \right) \varphi_{\bar{j}} \\ &\quad - 2t \operatorname{Re} \sum_{j,p} \frac{h_{p\bar{p}j} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}}. \end{aligned}$$

In addition, at (t_0, z_0)

$$\begin{aligned} t \frac{\beta'}{\beta} &= \frac{t}{\beta} \sum_{j,k} g^{j\bar{k}} (\dot{\varphi}_j \varphi_{\bar{k}} + \varphi_j \dot{\varphi}_{\bar{k}}) \\ &= \frac{2t}{\beta} \operatorname{Re} (\dot{\varphi}_j \varphi_{\bar{j}}), \end{aligned}$$

we infer that

$$\begin{aligned} t \frac{\beta'}{\beta} - 2t \operatorname{Re} \sum_{j,p} \frac{u_{p\bar{p}j} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}} &= -\frac{2t}{\beta} \operatorname{Re} \sum_j \left(F_j(t, z, \varphi) + (\log \Omega)_j \right) \varphi_{\bar{j}} - \frac{2t}{\beta} \sum_j \frac{\partial F}{\partial s} |\varphi_j|^2 \\ &\quad + 2t \operatorname{Re} \sum_{j,p} \frac{h_{p\bar{p}j} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}}. \end{aligned}$$

We may assume that $\log \beta > 1$ so that

$$\frac{|\varphi_{\bar{j}}|}{\beta} < C$$

Since $\frac{\partial F}{\partial s} \geq 0$, there exist $C_1 > 0$ depending on $\sup |\varphi_0|$ and C_2 depending on h such that

$$t \frac{\beta'}{\beta} - 2t \operatorname{Re} \sum_{j,p} \frac{u_{p\bar{p}j} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}} < C_1 t + C_2 t \sum_p \frac{1}{u_{p\bar{p}}}. \quad (3.3.12)$$

We now estimate

$$-\frac{t}{\beta} \sum_{j,p} \frac{|\varphi_{jp} - \sum_k g_{k\bar{j}\bar{p}} \varphi_k|^2}{\beta u_{p\bar{p}}} + t^{-1} (\gamma' \circ \varphi)^2 \sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}}.$$

It follows from (3.3.9) and (3.3.10) that

$$\begin{aligned} \beta_p &= -g_{k\bar{j}\bar{p}} \varphi_j \varphi_{\bar{k}} + g^{j\bar{k}} \varphi_{jp} \varphi_{\bar{k}} + g^{j\bar{k}} \varphi_j \varphi_{\bar{k}p}, \\ t\beta_p &= \beta \gamma' \circ \varphi \varphi_p \end{aligned}$$

then,

$$\sum_j (\varphi_{jp} - \sum_k g_{k\bar{j}\bar{p}} \varphi_k) \varphi_{\bar{j}} = t^{-1} \gamma' \circ \varphi \beta \varphi_p - \sum_j \varphi_{j\bar{p}} \varphi_{\bar{j}} = t^{-1} \gamma' \circ \varphi \beta \varphi_p - u_{p\bar{p}} \varphi_{\bar{p}} + \sum_j h_{j\bar{p}} \varphi_{\bar{j}}$$

Hence at (t_0, z_0) , using $\log \beta > 1$, we have

$$\begin{aligned} \frac{t}{\beta} \sum_{j,p} \frac{|\varphi_{jp} - \sum_k g_{k\bar{j}\bar{p}} \varphi_k|^2}{\beta u_{p\bar{p}}} &\geq \frac{t}{\beta^2} \sum_p \frac{|\sum_j (\varphi_{jp} - \sum_k g_{k\bar{j}\bar{p}} \varphi_k) \varphi_{\bar{j}}|^2}{u_{p\bar{p}}} \\ &= \frac{t}{\beta^2} \sum_p \frac{|t^{-1} \gamma' \circ \varphi \beta \varphi_p - u_{p\bar{p}} \varphi_{\bar{p}} + \sum_j h_{j\bar{p}} \varphi_{\bar{j}}|^2}{u_{p\bar{p}}} \\ &\geq t^{-1} (\gamma' \circ \varphi)^2 \sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}} + \frac{1}{\beta} \gamma' \circ \varphi 2 \operatorname{Re} \sum_{i,p} \frac{h_{j\bar{p}} \varphi_{\bar{j}} \varphi_p}{u_{p\bar{p}}} - 2 \gamma' \circ \varphi - \frac{At}{\beta}, \\ &\geq t^{-1} (\gamma' \circ \varphi)^2 \sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}} - C_3 \gamma' \circ \varphi - \frac{C_4}{\beta} \gamma' \circ \varphi \sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}}, \end{aligned}$$

where C_4 depending on h and C_3 depending on the $\sup |\gamma' \circ \varphi|$ such that

$$-\frac{At}{\beta} \geq (-C_3 + 2)\gamma' \circ \varphi, \forall t \in [0, T - \varepsilon].$$

This completes Claim 1. \square

We now choose

$$\gamma(s) = As - \left(\frac{1}{A} + C_4\right) s^2$$

with A so large that $\gamma' = A - 2\left(\frac{1}{A} + C_4\right)s > 0$ and $\gamma'' = -2/A - 2C_4 < 0$ for all $s \leq \sup_{[0, T] \times X} \varphi_t$. Since

$$\begin{aligned} \gamma'' \circ \varphi + \frac{C_4}{\beta} \gamma' \circ \varphi &= -\frac{2}{A} - 2C_4 + \frac{C_4}{\beta} \left(A - 2\left(\frac{1}{A} + C_4\right) \varphi \right) \\ &\leq -\frac{2}{A} - 2C_4 + \frac{C_4}{\beta} C_5, \end{aligned}$$

where $C_5 > 0$ depending only on $A, \sup_X |\varphi_0|$, we can suppose $\beta > C_5$ (otherwise, we have a bound for β), so

$$\gamma'' \circ \varphi + \frac{C_4}{\beta} \gamma' \circ \varphi \leq -\frac{2}{A}. \quad (3.3.13)$$

From Lemma 3.3.4 we have $\dot{\varphi} \geq C_0 + n \log t$, where C_0 depends on $Ocs_X \varphi_0$. Combining this with (3.3.11), (3.3.13) and Claim 1, we obtain

$$0 \leq -\frac{2}{A} \sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}} - (\lambda \gamma' \circ \varphi - Bt - C_2 t) \sum_p \frac{1}{u_{p\bar{p}}} + \log \beta + C' \gamma' \circ \varphi + C_1 t, \quad (3.3.14)$$

where C_1, C_2, C' depend on $\sup_X |\varphi_0|$ and h . If A is chosen sufficiently large, we have a constant $C_5 > 0$ such that

$$\sum_p \frac{1}{u_{p\bar{p}}} + \sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}} \leq C_5 \log \beta, \quad (3.3.15)$$

since otherwise (3.3.14) implies that β is bounded. So we get $(u_{p\bar{p}})^{-1} \leq C_5 \log \beta$ for $1 \leq p \leq n$. It follows from Lemma 3.3.5 we have at (t_0, z_0)

$$\prod_p u_{p\bar{p}} = e^{-\dot{\varphi}_t + F(t, x, \varphi_t)} \leq C_6,$$

where C_6 depends on $\sup_X |\varphi_0|, \varepsilon$. Then we get

$$u_{p\bar{p}} \leq C_6 (C_5 \log \beta)^{n-1},$$

so from (3.3.15) we have

$$\beta = \sum_p |\varphi_p|^2 \leq C_6 (C_5 \log \beta)^n,$$

hence $\log \beta < C_7$ at (t_0, z_0) . This shows that $\beta = |\nabla \varphi(z)|_\omega^2 < e^{C/t}$ for some C depending on $\sup |\varphi_0|$ and ε . \square

3.3.4 Bounding $\Delta \varphi_t$

We now estimate $\Delta \varphi$. The estimate on $|\nabla \varphi|_\omega^2$ is needed here. The argument follows from [GZ17, T617] but there are difficulties in using this approach because of torsion terms that need to be controlled (see also [TW10a] for similar computation for the elliptic Monge-Ampère equation).

Lemma 3.3.11. *Fix $\varepsilon > 0$. There exist constants A and C only depending on ε and $\sup_X \varphi_0$ such that for all $0 \leq t \leq T - \varepsilon$,*

$$0 \leq t \log \operatorname{tr}_\omega(\omega_{t+\varepsilon}) \leq A \operatorname{Osc}_X(\varphi_\varepsilon) + C + [C - n \log \varepsilon + A \operatorname{Osc}_X(\varphi_\varepsilon)]t.$$

Proof. We first denote by C a uniform constant only depending on ε and $\sup_X \varphi_0$.

Define

$$P = t \log \operatorname{tr}_\omega(\omega_{t+\varepsilon}) - A \varphi_{t+\varepsilon},$$

and

$$u = \operatorname{tr}_\omega(\omega_{t+\varepsilon}),$$

with $A > 0$ to be chosen latter. Set $\Delta_t := \Delta_{\omega_{t+\varepsilon}}$, then

$$\begin{aligned} \frac{\partial}{\partial t} P &= \log u + t \frac{\dot{u}}{u} - A \dot{\varphi}_{t+\varepsilon}, \\ \Delta_t P &= t \Delta_t \log u - A \Delta_t \varphi_{t+\varepsilon} \end{aligned}$$

hence

$$\left(\frac{\partial}{\partial t} - \Delta_t \right) P = \log u + t \frac{\dot{u}}{u} - A \dot{\varphi}_{t+\varepsilon} - t \Delta_t \log u + A \Delta_t \varphi_{t+\varepsilon}. \quad (3.3.16)$$

First, we have

$$A \Delta_t \varphi_{t+\varepsilon} = An - A \operatorname{tr}_{\omega_{t+\varepsilon}}(\theta_{t+\varepsilon}) \leq An - \frac{A}{2} \operatorname{tr}_{\omega_{t+\varepsilon}}(\omega). \quad (3.3.17)$$

Suppose P attains its maximum at (t_0, z_0) . If $t_0 = 0$, we get the desired inequality. We now assume that $P(t, x)$ attains its maximum at (t_0, x_0) with $t_0 > 0$.

It follows from Proposition 3.3.4, Proposition 3.3.5 and Theorem 3.3.8 that $\dot{\varphi}_{t+\varepsilon}$ depends on ε and $\sup_X \varphi_0$, hence

$$\left| \log \frac{\omega_t^n}{\omega^n} \right| \leq C.$$

Combine with the inequality

$$\left(\frac{\tilde{\omega}^n}{\omega^n}\right)^{\frac{1}{n}} \leq \frac{1}{n} \operatorname{tr}_\omega \tilde{\omega} \quad (\text{see Lemma 2.2.3}),$$

we infer that

$$\begin{aligned} \operatorname{tr}_\omega \omega_{t+\varepsilon} &\geq C^{-1}, \quad \operatorname{tr}_{\omega_{t+\varepsilon}} \omega \geq C^{-1}, \\ \operatorname{tr}_\omega \omega_{t+\varepsilon} &\leq C \operatorname{tr}_\omega \omega_{t+\varepsilon} \operatorname{tr}_{\omega_{t+\varepsilon}} \omega, \quad \operatorname{tr}_{\omega_{t+\varepsilon}} \omega \leq C \operatorname{tr}_\omega \omega_{t+\varepsilon} \operatorname{tr}_{\omega_{t+\varepsilon}} \omega. \end{aligned} \quad (3.3.18)$$

Denoting $\tilde{g}(t, x) = g_{t+\varepsilon}(x)$ and using the local coordinate system (3.3.5) at (t_0, x_0) , we have

$$\begin{aligned} \Delta_t \operatorname{tr}_\omega \omega_{t+\varepsilon} &= \tilde{g}^{i\bar{j}} \partial_i \partial_{\bar{j}} (g^{k\bar{l}} \tilde{g}_{k\bar{l}}) \\ &= \sum \tilde{g}^{i\bar{i}} \tilde{g}_{k\bar{k}i\bar{i}} - 2\operatorname{Re} \left(\sum_{i,j,k} g_{j\bar{k}i} \tilde{g}_{k\bar{j}i} \right) + \sum \tilde{g}^{i\bar{i}} g_{j\bar{k}i} g_{k\bar{j}i} \tilde{g}_{k\bar{k}} \\ &\quad + \sum \tilde{g}^{i\bar{i}} g_{k\bar{j}i} g_{j\bar{k}i} \tilde{g}_{k\bar{k}} - \sum \tilde{g}^{i\bar{i}} g_{k\bar{k}i\bar{i}} \tilde{g}_{k\bar{k}} \\ &\geq \sum_{i,k} \tilde{g}^{i\bar{i}} \tilde{g}_{k\bar{k}i\bar{i}} - 2\operatorname{Re} \left(\sum_{i,j,k} \tilde{g}^{i\bar{i}} g_{j\bar{k}i} \tilde{g}_{k\bar{j}i} \right) - C \operatorname{tr}_\omega \omega_{t+\varepsilon} \operatorname{tr}_{\omega_{t+\varepsilon}} \omega, \end{aligned}$$

where the last inequality comes from (3.3.18). Since

$$\Delta_\omega \dot{\varphi} = \operatorname{tr}_\omega (\operatorname{Ric}(\Omega) - \operatorname{Ric}(\omega_{t+\varepsilon})) + \Delta_\omega F(t, z, \varphi_{t+\varepsilon}) \quad (3.3.19)$$

and

$$\operatorname{tr}_\omega \operatorname{Ric}(\omega_{t+\varepsilon}) = \sum_{i,k} \tilde{g}^{i\bar{i}} (-\tilde{g}_{i\bar{i}k\bar{k}} + \tilde{g}^{j\bar{j}} \tilde{g}_{i\bar{j}k} \tilde{g}_{j\bar{i}k}),$$

we have

$$\sum_{i,k} \tilde{g}^{i\bar{i}} \tilde{g}_{i\bar{i}k\bar{k}} = \sum_{i,j,k} \tilde{g}^{i\bar{i}} \tilde{g}^{j\bar{j}} \tilde{g}_{i\bar{j}k} \tilde{g}_{j\bar{i}k} - \operatorname{tr}_\omega \omega_{t+\varepsilon}. \quad (3.3.20)$$

Therefore

$$\begin{aligned} \Delta_t \operatorname{tr}_\omega \omega_{t+\varepsilon} &\geq \sum_{i,j,k} \tilde{g}^{i\bar{i}} \tilde{g}^{j\bar{j}} \tilde{g}_{i\bar{j}k} \tilde{g}_{j\bar{i}k} - \operatorname{tr}_\omega \operatorname{Ric}(\omega_{t+\varepsilon}) \\ &\quad - 2\operatorname{Re} \left(\sum_{i,j,k} \tilde{g}^{i\bar{i}} g_{j\bar{k}i} \tilde{g}_{k\bar{j}i} \right) - C_4 \operatorname{tr}_\omega \omega_{t+\varepsilon} \operatorname{tr}_{\omega_{t+\varepsilon}} \omega \end{aligned}$$

Now we have

$$\begin{aligned} |2\operatorname{Re} \left(\sum_{i,j,k} \tilde{g}^{i\bar{i}} g_{j\bar{k}i} \tilde{g}_{k\bar{j}i} \right)| &\leq \sum_i \sum_{j \neq k} (\tilde{g}^{i\bar{i}} \tilde{g}^{j\bar{j}} \tilde{g}_{i\bar{j}k} \tilde{g}_{j\bar{i}k} + \tilde{g}^{i\bar{i}} \tilde{g}_{j\bar{j}} g_{j\bar{k}} g_{k\bar{j}k}) \\ &\leq \sum_i \sum_{j \neq k} \tilde{g}^{i\bar{i}} \tilde{g}^{j\bar{j}} \tilde{g}_{i\bar{j}k} \tilde{g}_{j\bar{i}k} + C \operatorname{tr}_\omega \omega_{t+\varepsilon} \operatorname{tr}_{\omega_{t+\varepsilon}} \omega. \end{aligned}$$

It follows that

$$\Delta_t \operatorname{tr}_\omega \omega_{t+\varepsilon} \geq \sum_{i,j} \tilde{g}^{i\bar{i}} \tilde{g}^{j\bar{j}} \tilde{g}_{i\bar{j}\bar{j}} \tilde{g}_{j\bar{i}\bar{i}} - \operatorname{tr}_\omega \operatorname{Ric}(\omega_{t+\varepsilon}) - C \operatorname{tr}_\omega \omega_{t+\varepsilon} \operatorname{tr}_{\omega_{t+\varepsilon}} \omega. \quad (3.3.21)$$

We now claim that

$$\begin{aligned} \frac{|\partial \operatorname{tr}_\omega(\omega_{t+\varepsilon})|_{\omega_{t+\varepsilon}}^2}{(\operatorname{tr}_\omega \omega_{t+\varepsilon})^2} &\leq \sum_{i,j} \tilde{g}^{i\bar{i}} \tilde{g}^{j\bar{j}} \tilde{g}_{i\bar{j}\bar{j}} \tilde{g}_{j\bar{i}\bar{i}} + \frac{1}{t} A^2 |\nabla \varphi|^2 \\ &\quad + \left(1 + \frac{1}{t}\right) \frac{C \operatorname{tr}_{\omega_{t+\varepsilon}} \omega}{(\operatorname{tr}_\omega \omega_{t+\varepsilon})^2} + C \operatorname{tr}_{\omega_{t+\varepsilon}} \omega. \end{aligned} \quad (3.3.22)$$

By computation,

$$\begin{aligned} \frac{|\partial \operatorname{tr}_\omega(\omega_{t+\varepsilon})|_{\omega_{t+\varepsilon}}^2}{(\operatorname{tr}_\omega \omega_{t+\varepsilon})^2} &= \frac{1}{(\operatorname{tr}_\omega \omega_{t+\varepsilon})^2} \sum_{i,j,k} \tilde{g}^{i\bar{i}} \tilde{g}_{j\bar{j}i} \tilde{g}_{k\bar{k}\bar{i}} \\ &= \frac{1}{(\operatorname{tr}_\omega \omega_{t+\varepsilon})^2} \sum_{i,j,k} \tilde{g}^{i\bar{i}} (h_{j\bar{j}i} - h_{i\bar{j}j} + \tilde{g}_{i\bar{j}j}) (h_{k\bar{k}\bar{i}} - h_{k\bar{i}\bar{k}} + \tilde{g}_{k\bar{i}\bar{k}}) \\ &= \frac{1}{(\operatorname{tr}_\omega \omega_{t+\varepsilon})^2} \sum_{i,j,k} \tilde{g}^{i\bar{i}} (T_{i\bar{j}\bar{j}} + \tilde{g}_{i\bar{j}j}) (\bar{T}_{i\bar{k}\bar{k}} + \tilde{g}_{k\bar{i}\bar{k}}) \\ &= \frac{1}{(\operatorname{tr}_\omega \omega_{t+\varepsilon})^2} \left(\sum_{i,j,k} \tilde{g}^{i\bar{i}} \tilde{g}_{i\bar{j}\bar{j}} \tilde{g}_{k\bar{i}\bar{k}} + \sum_{i,j,k} \tilde{g}^{i\bar{i}} T_{i\bar{j}\bar{j}} \bar{T}_{i\bar{k}\bar{k}} + 2 \operatorname{Re} \sum_{i,j,k} \tilde{g}^{i\bar{i}} T_{i\bar{j}\bar{j}} \tilde{g}_{k\bar{i}\bar{k}} \right) \end{aligned}$$

where $T_{i\bar{j}\bar{j}} = h_{j\bar{j}i} - h_{i\bar{j}j}$.

It follows from the Cauchy-Schwarz inequality that

$$\frac{1}{\operatorname{tr}_\omega \omega_{t+\varepsilon}} \sum_{i,j,k} \tilde{g}^{i\bar{i}} \tilde{g}_{i\bar{j}\bar{j}} \tilde{g}_{k\bar{i}\bar{k}} \leq \sum_{i,j} \tilde{g}^{i\bar{i}} \tilde{g}^{j\bar{j}} \tilde{g}_{i\bar{j}\bar{j}} \tilde{g}_{j\bar{i}\bar{i}}. \quad (3.3.23)$$

For the second term, we have

$$\sum_{i,j,k} \tilde{g}^{i\bar{i}} T_{i\bar{j}\bar{j}} \bar{T}_{i\bar{k}\bar{k}} \leq C \operatorname{tr}_{\omega_{t+\varepsilon}} \omega.$$

Now at the maximum point (t_0, z_0) , $t_0 > 0$, we have $\nabla P = 0$, hence

$$A \dot{\varphi}_{\bar{i}} = t \frac{u_{\bar{i}}}{u} = \frac{t}{u} \sum_k g_{k\bar{k}\bar{i}}.$$

Since $\tilde{g}_{k\bar{i}\bar{k}} = \tilde{g}_{k\bar{k}\bar{i}} - \bar{T}_{i\bar{k}\bar{k}}$, we have

$$\begin{aligned}
\left| \frac{2}{(\operatorname{tr}_\omega \omega_{t+\varepsilon})^2} \operatorname{Re} \sum_{i,j,k} \tilde{g}^{i\bar{i}} T_{ij\bar{j}} \tilde{g}_{k\bar{k}} \right| &= \frac{2}{(\operatorname{tr}_\omega \omega_{t+\varepsilon})^2} \left| \operatorname{Re} \sum_{i,j,k} \tilde{g}^{i\bar{i}} T_{ij\bar{j}} \tilde{g}_{k\bar{k}} + \operatorname{Re} \sum_{i,j,k} \tilde{g}^{i\bar{i}} T_{ij\bar{j}} \bar{T}_{ik\bar{k}} \right| \\
&\leq \frac{2}{\operatorname{tr}_\omega \omega_{t+\varepsilon}} \left| \frac{A}{t} \operatorname{Re} \sum_{i,j} \tilde{g}^{i\bar{i}} T_{ij\bar{j}} \varphi_{\bar{i}} \right| + C \operatorname{tr}_{\omega_{t+\varepsilon}} \omega \\
&\leq \frac{1}{t} \left(A^2 |\nabla \varphi|_\omega^2 + \frac{C \operatorname{tr}_{\omega_{t+\varepsilon}} \omega}{(\operatorname{tr}_\omega \omega_{t+\varepsilon})^2} \right) + C \operatorname{tr}_{\omega_{t+\varepsilon}} \omega.
\end{aligned}$$

Combining all of these inequalities we obtain (3.3.22).

It now follows from (3.3.21) and (3.3.22) that

$$\Delta_t \log \operatorname{tr}_\omega \omega_{t+\varepsilon} \geq -\frac{\operatorname{tr}_\omega \operatorname{Ric}(\omega_{t+\varepsilon})}{\operatorname{tr}_\omega \omega_{t+\varepsilon}} - \frac{1}{t} A^2 |\nabla \varphi|_\omega^2 - \left(1 + \frac{1}{t}\right) \frac{C \operatorname{tr}_{\omega_{t+\varepsilon}} \omega}{(\operatorname{tr}_\omega \omega_{t+\varepsilon})^2} - C \operatorname{tr}_{\omega_{t+\varepsilon}} \omega.$$

Moreover,

$$\begin{aligned}
\frac{t\dot{u}}{u} &= \frac{t}{u} \left[\Delta_\omega (\log \omega_{t+\varepsilon}^n / \omega^n - \log \Omega / \omega^n - F(t, z, \varphi_{t+\varepsilon})) + \operatorname{tr}_\omega \dot{\theta}_t \right], \\
&= \frac{t}{u} \left[-\operatorname{tr}_\omega (\operatorname{Ric} \omega_{t+\varepsilon}) + \operatorname{tr}_\omega (\dot{\theta}_t + \operatorname{Ric} \omega) - \Delta_\omega (F(t, z, \varphi) + \log \Omega / \omega^n) \right],
\end{aligned}$$

It follows from Proposition 2.3.7, $\operatorname{tr}_{\omega_{t+\varepsilon}} \omega \operatorname{tr}_\omega \omega_{t+\varepsilon} \geq n$ and $\operatorname{tr}_\omega \omega_{t+\varepsilon} \geq C^{-1}$ that

$$-t\Delta_t \log u + \frac{t\dot{u}}{u} \leq Ct \operatorname{tr}_{\omega_{t+\varepsilon}}(\omega) - t \frac{\Delta_\omega [F(t, z, \varphi) + \log \Omega / \omega^n]}{\operatorname{tr}_\omega(\omega_{t+\varepsilon})} + C. \quad (3.3.24)$$

Now

$$\Delta_\omega F(t, z, \varphi_{t+\varepsilon}) = \Delta_\omega F(z, \cdot) + 2\operatorname{Re} \left[g^{j\bar{k}} \left(\frac{\partial F}{\partial s} \right)_j \varphi_{\bar{k}} \right] + \frac{\partial F}{\partial s} \Delta_\omega \varphi + \frac{\partial^2 F}{\partial s^2} |\nabla \varphi|_\omega^2.$$

Therefore

$$|\Delta_\omega (F(t, z, \varphi_{t+\varepsilon}) + \log \Omega / \omega^n)| \leq C + C |\nabla \varphi|_\omega^2 + C \operatorname{tr}_\omega \omega_{t+\varepsilon}.$$

Then we infer

$$-\frac{\Delta_\omega [F(t, z, \varphi) + \log \Omega / \omega^n]}{\operatorname{tr}_\omega(\omega_{t+\varepsilon})} \leq \frac{1}{n} \operatorname{tr}_{\omega_{t+\varepsilon}}(\omega) (C + C |\nabla \varphi|_\omega^2) + C,$$

so from Proposition 2.3.7 and (3.3.24) we have

$$-t\Delta_t \log u + \frac{t\dot{u}}{u} \leq Ct \operatorname{tr}_{\omega_{t+\varepsilon}}(\omega) + C. \quad (3.3.25)$$

Moreover, the inequalities $(n-1)\log x \leq x + C_n$ and

$$\mathrm{tr}_\omega \tilde{\omega} \leq \left(\frac{\tilde{\omega}^n}{\omega^n}\right) (\mathrm{tr}_{\tilde{\omega}} \omega)^{n-1},$$

for any two positive $(1,1)$ -forms ω and $\tilde{\omega}$, imply that

$$\begin{aligned} \log u = \log \mathrm{tr}_\omega(\omega_{t+\varepsilon}) &\leq \log \left(n \left(\frac{\omega_{t+\varepsilon}^n}{\omega^n} \right) \mathrm{tr}_{\omega_{t+\varepsilon}}(\omega)^{n-1} \right) \\ &= \log n + \dot{\varphi}_{t+\varepsilon} + F(t, z, \varphi) + (n-1) \log \mathrm{tr}_{\omega_{t+\varepsilon}}(\omega) \\ &\leq \dot{\varphi}_{t+\varepsilon} + \mathrm{tr}_{\omega_{t+\varepsilon}}(\omega) + C. \end{aligned} \tag{3.3.26}$$

It follows from (3.3.16), (3.3.17), (3.3.25) and (3.3.26) that

$$\left(\frac{\partial}{\partial t} - \Delta_t \right) P \leq C - (A-1)\dot{\varphi}_{t+\varepsilon} + [Ct + 1 - A/2] \mathrm{tr}_{\omega_{t+\varepsilon}} \omega.$$

We choose A sufficiently large such that $Ct + 1 - A/2 < 0$. Applying Proposition 3.3.4,

$$\left(\frac{\partial}{\partial t} - \Delta_t \right) P \leq C - (A-1)(n \log \varepsilon - A \mathrm{Osc}_X \varphi_\varepsilon - C).$$

Now suppose P attains its maximum at (t_0, z_0) . If $t_0 = 0$, we get the desired inequality. Otherwise, at (t_0, z_0)

$$0 \leq \left(\frac{\partial}{\partial t} - \Delta_t \right) P \leq C - (A-1)(n \log \varepsilon - A \mathrm{Osc}_X \varphi_\varepsilon - C).$$

Hence we get

$$t \log \mathrm{tr}_\omega(\omega_{t+\varepsilon}) \leq A \mathrm{Osc}_X(\varphi_\varepsilon) + C + [C - n \log \varepsilon + A \mathrm{Osc}_X(\varphi_\varepsilon)]t,$$

as required. \square

3.3.5 Higher order estimates

For the higher order estimates, one can follow [SzTo11] (see [Nie14] for its version on Hermitian manifolds) by bounding

$$S = g_\varphi^{i\bar{p}} g_\varphi^{q\bar{j}} g_\varphi^{k\bar{r}} \varphi_{i\bar{j}k} \varphi_{p\bar{q}r} \text{ and } |\mathrm{Ric}(\omega_t)|_{\omega_t},$$

then using the parabolic Schauder estimates to obtain higher order estimates for φ . Additionally, we can also combine previous estimates with Evans-Krylov and Schauder estimates [Tô17, Theorem 1.7] (see Theorem 2.2.8) to get the C^k estimates for all $k \geq 0$.

Theorem 3.3.12. *For each $\varepsilon > 0$ and $k \in \mathbb{N}$, there exists $C_k(\varepsilon)$ such that*

$$\|\varphi\|_{C^k([\varepsilon, T] \times X)} \leq C_k(\varepsilon).$$

3.4 Proof of Theorem A and B

We now consider the complex Monge-Ampère flow

$$(CMAF) \quad \frac{\partial \varphi_t}{\partial t} = \log \frac{(\theta_t + dd^c \varphi_t)^n}{\Omega} - F(t, z, \varphi),$$

starting from a ω -psh function φ_0 with zero Lelong numbers at all points, where $F(t, z, s) \in C^\infty([0, T] \times X \times \mathbb{R}, \mathbb{R})$ with $\frac{\partial F}{\partial s} \geq 0$ and $\frac{\partial F}{\partial t}$ is bounded from below.

3.4.1 Convergence in L^1

We first approximate φ_0 by a decreasing sequence $\varphi_{0,j}$ of smooth ω -psh functions (see [BK07]). Denote by $\varphi_{t,j}$ the smooth family of θ_t -psh functions satisfying on $[0, T] \times X$

$$\frac{\partial \varphi_t}{\partial t} = \log \frac{(\theta_t + dd^c \varphi_t)^n}{\Omega} - F(t, z, \varphi)$$

with initial data $\varphi_{0,j}$.

It follows from the maximum principle [Tô17, Proposition 1.5] that $j \mapsto \varphi_{j,t}$ is non-increasing. Therefore we can set

$$\varphi_t(z) := \lim_{j \rightarrow +\infty} \varphi_{t,j}(z).$$

Thanks to Lemma 3.3.3 the function $t \mapsto \sup_X \varphi_{t,j}$ is uniformly bounded, hence φ_t is a well-defined θ_t -psh function. Moreover, it follows from Theorem 3.3.12 that φ_t is also smooth in $(0, T] \times X$ and satisfies

$$\frac{\partial \varphi_t}{\partial t} = \log \frac{(\theta_t + dd^c \varphi_t)^n}{\Omega} - F(t, z, \varphi).$$

Observe that (φ_t) is relatively compact in $L^1(X)$ as $t \rightarrow 0^+$, we now show that $\varphi_t \rightarrow \varphi_0$ in $L^1(X)$ as $t \searrow 0^+$.

First, let φ_{t_k} is a subsequence of (φ_t) such that φ_{t_k} converges to some function ψ in $L^1(X)$ as $t_k \rightarrow 0^+$. By the properties of plurisubharmonic functions, for all $z \in X$

$$\limsup_{t_k \rightarrow 0} \varphi_{t_k}(z) \leq \psi(z),$$

with equality almost everywhere. We infer that for almost every $z \in X$

$$\psi(z) = \limsup_{t_k \rightarrow 0} \varphi_{t_k}(z) \leq \limsup_{t_k \rightarrow 0} \varphi_{t_k,j}(z) = \varphi_{0,j}(z),$$

by continuity of $\varphi_{t,j}$ at $t = 0$. Thus $\psi \leq \varphi_0$ almost everywhere.

Moreover, it follows from Lemma 3.3.3 that

$$\varphi_t(z) \geq (1 - 2\beta t)\varphi_0(z) + tu(z) + n(t \log t - t) + At,$$

with u continuous, so

$$\varphi_0 \leq \liminf_{t \rightarrow 0} \varphi_t.$$

Since $\psi \leq \varphi_0$ almost everywhere, we get $\psi = \varphi_0$ almost everywhere, so $\varphi_t \rightarrow \varphi_0$ in L^1 .

3.4.2 Uniform convergence

If the initial condition φ_0 is continuous then by [Tô17, Proposition 1.5] (see Proposition 2.2.5) we infer that $\varphi_t \in C^0([0, T] \times X)$, hence φ_t uniformly converges to φ_0 as $t \rightarrow 0^+$.

3.4.3 Uniqueness and stability of solution

We now study the uniqueness and stability for the complex Monge-Ampère flow

$$\frac{\partial \varphi_t}{\partial t} = \log \frac{(\theta_t + dd^c \varphi_t)^n}{\Omega} - F(t, z, \varphi), \quad (3.4.1)$$

where $F(t, z, s) \in C^\infty([0, T] \times X \times \mathbb{R}, \mathbb{R})$ satisfies

$$\frac{\partial F}{\partial s} \geq 0 \quad \text{and} \quad \left| \frac{\partial F}{\partial t} \right| \leq C',$$

for some constant $C' > 0$.

The uniqueness of solution follows directly from the same result in the Kähler setting [Tô17] (see Chapter 2)

Theorem 3.4.1. *Suppose ψ and φ are two solutions of (3.4.1) with $\varphi_0 \leq \psi_0$, then $\varphi_t \leq \psi_t$. In particular, the equation (3.4.1) has a unique solution.*

The stability result also comes from the same argument as in [Tô17] (see Chapter 2). The difference is that we use Theorem 3.3.6 instead of the one for Kähler manifolds.

Theorem 3.4.2. *Fix $\varepsilon > 0$. Let $\varphi_{0,j}$ be a sequence of ω -psh functions with zero Lelong number at all points, such that $\varphi_{0,j} \rightarrow \varphi_0$ in $L^1(X)$. Denote by $\varphi_{t,j}$ and φ_j the solutions of (3.4.1) with the initial condition $\varphi_{0,j}$ and φ_0 respectively. Then*

$$\varphi_{t,j} \rightarrow \varphi_t \quad \text{in} \quad C^\infty([\varepsilon, T] \times X) \quad \text{as} \quad j \rightarrow +\infty.$$

Moreover, if $\varphi, \psi \in C^\infty((0, T] \times X)$ are solutions of (CMAF) with continuous initial data φ_0 and ψ_0 , then

$$\|\varphi - \psi\|_{C^k([\varepsilon, T] \times X)} \leq C(k, \varepsilon) \|\varphi_0 - \psi_0\|_{L^\infty(X)}. \quad (3.4.2)$$

Proof. We use the techniques in Section 3.3 to obtain estimates of $\varphi_{t,j}$ in $C^k([\varepsilon, T] \times X)$ for all $k \geq 0$. In particular, for the C^0 estimate, we need to have uniform bound for $H_{t,j} = \exp(\dot{\varphi}_{t,j} + F)$ in order to use Theorem 3.3.6. By Lemma 3.3.5 we have

$$H_{t,j} = \exp(\dot{\varphi}_{t,j} + F) \leq \exp\left(\frac{-\varphi_{0,j} + C}{t} + C'\right),$$

where C, C' depend on $\varepsilon, \sup_X \varphi_{0,j}$. Since $\varphi_{0,j}$ converges to φ_0 in L^1 , we have the $\sup_X \varphi_{0,j}$ is uniformly bounded in term of $\sup_X \varphi_0$ for all j by the Hartogs lemma, so we can choose C, C' independently of j . It follows from [DK01, Theorem 0.2 (2)] that there is a constant $A(t, \varepsilon)$ depending on t and ε such that $\|H_{t,j}\|_{L^2(X)}$ is uniformly bounded by $A(t, \varepsilon)$ for all $t \in [\varepsilon, T]$. The rest of the proof is now similar to [Tô17, Theorem 4.3]. \square

3.5 Chern-Ricci flow and canonical surgical contraction

In this section, we give a proof of the conjecture of Tosatti and Weinkove. Let (X, ω_0) be a Hermitian manifold. Consider the Chern-Ricci flow on X ,

$$\frac{\partial}{\partial t} \omega_t = -Ric(\omega), \quad \omega|_{t=0} = \omega_0. \tag{3.5.1}$$

Denote

$$T := \sup\{t \geq 0 \mid \exists \psi \in C^\infty(X) \text{ with } \hat{\omega}_X + dd^c \psi > 0\},$$

where $\hat{\omega}_X = \omega_0 + t\chi$, with χ is a smooth $(1, 1)$ -form representing $-c_1^{BC}(X)$.

Assume that there exists a holomorphic map between compact Hermitian manifolds $\pi : X \rightarrow Y$ blowing down an exceptional divisor E on X to one point $y_0 \in Y$. In addition, assume that there exists a smooth function ρ on X such that

$$\omega_0 - TRic(\omega_0) + dd^c \rho = \pi^* \omega_Y, \tag{3.5.2}$$

with $T < +\infty$, where ω_Y is a Hermitian metric on Y . In [TW15, TW13], Tosatti and Weinkove proved that the solution ω_t to the Chern-Ricci flow (3.5.1) converges in $C_{loc}^\infty(X \setminus E)$ to a smooth Hermitian metric ω_T on $X \setminus E$. Moreover, there exists a distance function d_T on Y such that (Y, d_T) is a compact metric space and $(X, g(t))$ converges in the Gromov-Hausdorff sense (Y, d_T) as $t \rightarrow T^-$. Denote by ω' the push-down of the current ω_T to Y . They conjectured that:

Conjecture 3.5.1. [TW13, Page 2120]

- (1) *There exists a smooth maximal solution ω_t of the Chern-Ricci flow on Y for $t \in (T, T_Y)$ with $T < T_Y \leq +\infty$ such that ω_t converges to ω' , as $t \rightarrow T^+$, in $C_{loc}^\infty(Y \setminus \{y_0\})$. Furthermore, ω_t is uniquely determined by ω_0 .*

- (2) *The metric space (Y, ω_t) converges to (Y, d_T) as $t \rightarrow T^+$ in the Gromov-Hausdorff sense.*

We now prove this conjecture using Theorem A, Theorem B and some arguments in [SW13a, TW13].

3.5.1 Continuing the Chern-Ricci flow

We prove the first claim in the conjecture showing how to continue the Chern-Ricci flow.

Write $\hat{\omega} = \pi^* \omega_Y = \omega_0 - TRic(\omega_0) + dd^c \rho$. Then there is a positive $(1, 1)$ -current $\omega_T = \hat{\omega} + dd^c \varphi_T$ for some bounded function φ_T . By the same argument in [SW13a, Lemma 5.1] we have

$$\varphi_T|_E = \text{constant}.$$

Hence there exists a bounded function ϕ_T on Y that is smooth on $Y \setminus \{y_0\}$ with $\varphi_T = \pi^* \phi_T$.

We now define a positive current ω' on Y by

$$\omega' = \omega_Y + dd^c \phi_T \geq 0,$$

which is the push-down of the current ω_T to Y and is smooth on $Y \setminus \{y_0\}$. By the same argument in [SW13a, Lemma 5.2] we have $\omega'^n / \omega_Y^n \in L^p(Y)$. It follows from [DK12, Theorem 5.2] that ϕ_T is continuous.

We fix a smooth $(1, 1)$ form $\chi \in -c_1^{BC}(Y)$ and a smooth volume form Ω_Y such that $\chi = dd^c \log \Omega_Y$. Denote

$$T_Y := \sup\{t > T \mid \omega_Y + (t - T)\chi > 0\}.$$

Fix $T' \in (T, T_Y)$, we have:

Theorem 3.5.2. *There is a unique smooth family of Hermitian metrics $(\omega_t)_{T < t \leq T'}$ on Y satisfying the Chern-Ricci flow*

$$\frac{\partial \omega_t}{\partial t} = -Ric(\omega_t), \quad \omega_t|_{t=T} = \omega', \quad (3.5.3)$$

with $\omega_t = \omega_Y + (t - T)\chi + dd^c \phi_t$. Moreover, ϕ_t uniformly converges to ϕ_T as $t \rightarrow T^+$.

Proof. We can rewrite the flow as the following complex Monge-Ampère flow

$$\frac{\partial \phi_t}{\partial t} = \log \frac{(\hat{\omega}_Y + dd^c \phi_t)^n}{\Omega}, \quad \phi|_{t=T} = \phi_T, \quad (3.5.4)$$

where $\hat{\omega}_Y := \omega_Y + (t - T)\chi$ and ϕ_T is continuous.

It follows from Theorem A and Theorem B that there is a unique solution ϕ of (3.5.4) in $C^\infty((T, T'] \times Y)$ such that ϕ_t uniformly converges to ϕ_T as $t \rightarrow T^+$. \square

3.5.2 Backward convergences

Once the Chern-Ricci flow can be run from ω' on Y , we can prove the rest of Conjecture 3.5.1 following the idea in [SW13a, Section 6].

We keep the notation as in [TW15]. Let h be a Hermitian metric on the fibers of the line bundle $[E]$ associated to the divisor E , such that for $\varepsilon_0 > 0$ sufficiently small, we have

$$\pi^*\omega_Y - \varepsilon_0 R_h > 0, \quad \text{where } R_h := -dd^c \log h. \tag{3.5.5}$$

Take s a holomorphic section of $[E]$ vanishing along E to order 1. We fix a coordinate chart U centered at y_0 , which identifies with the unit ball $B \subset \mathbb{C}^n$ via coordinates z_1, \dots, z_n . Then the function $|s|_h^2$ on X is given on $\pi^{-1}(B(0, 1/2))$ by

$$|s|_h^2(x) = |z_1|^2 + \dots + |z_n|^2 := r^2, \quad \text{for } \pi(x) = (z_1, \dots, z_n).$$

Hence, the curvature $R(h)$ of h is given by

$$R(h) := -dd^c \log(|z_1|^2 + \dots + |z_n|^2).$$

The crucial ingredient of the proof of the conjecture is the following proposition:

Proposition 3.5.3. *The solution ω_t of (3.5.3) is in $C^\infty([T, T'] \times Y \setminus \{y_0\})$ and there exists $\eta > 0$ and a uniform constant $C > 0$ such that for $t \in [T, T']$*

- (1) $\omega_t \leq C \frac{\omega_Y}{\pi_* |s|_h^2},$
- (2) $\omega_t \leq C \pi_* \left(\frac{\omega_0}{|s|^{2(1-\eta)}} \right).$

In order to prove this proposition, we use the method in [SW13a] to construct a smooth approximant of the solution ϕ_t of (3.5.4). Denote by f_ε a family of positive smooth functions f_ε on Y such that it has the form

$$f_\varepsilon(z) = (\varepsilon + r^2)^{n-1},$$

on B , hence $f_\varepsilon(z) \rightarrow f(z) = r^{2(n-1)}$ as $\varepsilon \rightarrow 0$. Moreover, there is a smooth volume form Ω_X on X with $\pi^*\Omega_Y = (\pi^*f)\Omega_X$.

Observe that $\hat{\omega}_Y(t) - \frac{\varepsilon}{T}\omega_Y$ is Hermitian on Y for $t \in [T, T']$ if ε is sufficiently small. Therefore

$$\theta^\varepsilon := \pi^*\left(\hat{\omega}_Y - \frac{\varepsilon}{T}\omega_Y\right) + \frac{\varepsilon}{T}\omega_0$$

is Hermitian for $\varepsilon > 0$ sufficiently small.

We denote by ψ_t^ε the unique smooth solution of the following Monge-Ampère flow on X :

$$\begin{aligned} \frac{\partial \psi^\varepsilon}{\partial t} &= \log \frac{(\theta^\varepsilon + dd^c \psi^\varepsilon)^n}{(\pi^* f_\varepsilon) \Omega_X} \\ \psi^\varepsilon|_{t=T} &= \varphi(T - \varepsilon), \end{aligned} \quad (3.5.6)$$

Define Kähler metrics ω^ε on $[T, T'] \times X$ by

$$\omega^\varepsilon = \theta^\varepsilon + dd^c \psi^\varepsilon, \quad (3.5.7)$$

then

$$\frac{\partial \omega^\varepsilon}{\partial t} = -Ric(\omega^\varepsilon) - \eta, \quad (3.5.8)$$

where $\eta = -dd^c \log((\pi^* f_\varepsilon) \Omega_X) + \pi^* \chi = -dd^c \log(\pi^* f_\varepsilon) \Omega_X + dd^c \log((\pi^* f) \Omega_X)$.

We claim that $\pi_* \psi_t^\varepsilon$ converges to the solution ϕ_t of the equation (3.5.4) in $C^\infty([T, T'] \times Y \setminus \{y_0\})$, then $\pi_* \omega^\varepsilon$ smoothly converges to ω_t on $[T, T'] \times Y \setminus \{y_0\}$.

Lemma 3.5.4. *There exists $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ such that on $[T, T'] \times X$ we have*

- (i) $\eta \leq C\omega_0$,
- (ii) $Osc_X \psi^\varepsilon \leq C$;
- (iii) $\frac{(\omega^\varepsilon)^n}{\Omega_X} \leq C$.

Proof. By straightforward calculation, in $\pi^{-1}(B(0, 1/2))$, we have

$$\begin{aligned} \eta &= -dd^c \log((\pi^* f_\varepsilon) \Omega_X) + dd^c \log((\pi^* f) \Omega_X) \\ &\leq (n-1)\pi^* (dd^c \log r^2) \\ &= (n-1) \frac{\sqrt{-1}}{\pi} \pi^* \left(\frac{1}{r^2} \sum_{i,j} \left(\delta_{ij} - \frac{\bar{z}_i z_j}{r^2} dz_i \wedge d\bar{z}_j \right) \right) \\ &\leq C\omega_0, \end{aligned}$$

for some constant $C > 0$. This proves (i). Using the same argument in Section 3.3 (see Theorem 3.3.8) we get (ii). Finally, the estimate (iii) follows from the same proof for the Kähler-Ricci flow (cf. [SW13a, Lemma 6.2]) \square

Two following lemmas are essential to prove Proposition 3.5.3.

Lemma 3.5.5. *There exists $\eta > 0$ and a uniform constant $C > 0$ such that*

$$\omega_t^\varepsilon \leq C \frac{\pi^* \omega_Y}{|s|_h^2}. \quad (3.5.9)$$

Proof. We first denote by $C > 0$ a uniform constant which is independent of ε . Set $\hat{\omega} = \pi^* \omega_Y$ and fix $\delta > 0$ a small constant. Following the same method in [TW13, Lemma 3.4] (see [PS10] for the original idea), we consider

$$H_\delta = \log \operatorname{tr}_{\hat{\omega}} \omega_t^\varepsilon + \log |s|_h^{2(1+\delta)} - A\psi^\varepsilon + \frac{1}{\tilde{\psi}^\varepsilon + C_0},$$

where $\tilde{\psi}^\varepsilon := \psi^\varepsilon - \frac{1+\delta}{A} \log |s|_h^2$ and $C_0 > 0$ satisfies $\tilde{\psi}^\varepsilon + C_0 \geq 1$.

It follows from [TW13, (3.17)] and [SW13a, Lemma 2.4] that

$$\omega_0 \leq \frac{C}{|s|_h^2} \pi^* \omega_Y, \quad (3.5.10)$$

hence $|s|_h^2 \operatorname{tr}_{\hat{\omega}} \omega^\varepsilon \leq C \operatorname{tr}_{\omega_0} \omega^\varepsilon$. Therefore H_δ goes to negative infinity as x tends to E . Suppose that H_δ attains its maximum at $(t_0, x_0) \in [T, T'] \times X \setminus E$. Without loss of generality, we assume that $\operatorname{tr}_{\hat{\omega}} \omega^\varepsilon \geq 1$ at (t_0, x_0) .

The condition (3.5.2) implies that $\pi^* \omega_Y - \omega_0$ is a d -closed form, so that $d\omega_0 = \pi^*(d\omega_Y)$. Therefore we have

$$(T_0)_{j\bar{l}}^p (g_0)_{p\bar{k}} = (2\sqrt{-1} \partial \omega_0)_{j\bar{l}\bar{k}} = (2\sqrt{-1} \pi^* \partial \omega_Y)_{j\bar{l}\bar{k}} = (\pi^* T_Y)_{j\bar{l}}^p (\pi^* \omega_Y)_{p\bar{k}}. \quad (3.5.11)$$

The condition (3.5.2) moreover implies that

$$\omega^\varepsilon(t) = \omega_0 + \beta(t), \quad (3.5.12)$$

where $\beta(t) = (1 - \varepsilon/T)(-TRic(\omega_0) + dd^c \psi) + (t - T)\pi^* \chi$ is a closed $(1, 1)$ -form.

Combining (3.5.12), (3.5.11) and the calculation of [TW15, Proposition 3.1], at (t_0, x_0) we get

$$\left(\frac{\partial}{\partial t} - \Delta_\varepsilon \right) \log \operatorname{tr}_{\hat{\omega}} \omega^\varepsilon \leq \frac{2}{(\operatorname{tr}_{\hat{\omega}} \omega^\varepsilon)^2} \operatorname{Re} \left(g^{k\bar{q}} \hat{T}_{ki}^i \partial_{\bar{q}} \operatorname{tr}_{\hat{\omega}} \omega^\varepsilon \right) + C \operatorname{tr}_{\omega^\varepsilon} \hat{\omega} + \frac{\operatorname{tr}_{\hat{\omega}} \eta}{\operatorname{tr}_{\hat{\omega}} \omega^\varepsilon}, \quad (3.5.13)$$

where $\hat{T} := \pi^* T_Y$.

It follows from Lemma 3.5.4 and (3.5.10) that

$$\frac{\operatorname{tr}_{\hat{\omega}} \eta}{\operatorname{tr}_{\hat{\omega}} \omega^\varepsilon} \leq \frac{C}{|s|_h^2 \operatorname{tr}_{\hat{\omega}} \omega^\varepsilon}.$$

Moreover, we may assume without loss of generality that

$$\frac{C}{|s|_h^2 \operatorname{tr}_{\hat{\omega}} \omega^\varepsilon} \leq C',$$

for some uniform constant C' , since otherwise H_δ is already uniformly bounded. Therefore, we get

$$\frac{\operatorname{tr}_{\omega^\varepsilon} \eta}{\operatorname{tr}_{\hat{\omega}} \omega^\varepsilon} \leq C'. \quad (3.5.14)$$

Since at (t_0, z_0) we have $\nabla H_\delta(t_0, x_0) = 0$,

$$\frac{1}{\operatorname{tr}_{\hat{\omega}} \omega^\varepsilon} \partial_i \operatorname{tr}_{\hat{\omega}} \omega^\varepsilon - A \partial_i \tilde{\psi}^\varepsilon - \frac{1}{(\tilde{\psi}^\varepsilon + C_0)^2} \partial_i \tilde{\psi}^\varepsilon = 0, \quad (3.5.15)$$

hence

$$\begin{aligned} \left| \frac{2}{(\operatorname{tr}_{\hat{\omega}} \omega^\varepsilon)^2} \operatorname{Re} \left(g^{k\bar{q}} \hat{T}_{ki}^i \partial_{\bar{q}} \operatorname{tr}_{\hat{\omega}} \omega^\varepsilon \right) \right| &\leq \left| \frac{2}{\operatorname{tr}_{\hat{\omega}} \omega^\varepsilon} \operatorname{Re} \left(\left(A + \frac{1}{\tilde{\psi}^\varepsilon + C_0} \right) g^{k\bar{q}} \hat{T}_{ki}^i (\partial_{\bar{q}} \tilde{\psi}^\varepsilon) \right) \right| \\ &\leq \frac{|\partial \tilde{\psi}^\varepsilon|_{\omega^\varepsilon}^2}{(\tilde{\psi}^\varepsilon + C_0)^3} + CA^2 (\tilde{\psi}^\varepsilon + C_0)^3 \frac{\operatorname{tr}_{\omega^\varepsilon} \hat{\omega}}{(\operatorname{tr}_{\hat{\omega}} \omega^\varepsilon)^2}. \end{aligned}$$

We also have

$$\left(\frac{\partial}{\partial t} - \Delta_\varepsilon \right) \left(-A \tilde{\psi}^\varepsilon + \frac{1}{\tilde{\psi}^\varepsilon + C_0} \right) = -A \dot{\tilde{\psi}}^\varepsilon + A \Delta_\varepsilon \tilde{\psi}^\varepsilon - \frac{\dot{\tilde{\psi}}^\varepsilon}{(\tilde{\psi}^\varepsilon + C_0)^2} + \frac{\Delta_\varepsilon \tilde{\psi}^\varepsilon}{(\tilde{\psi}^\varepsilon + C_0)^2}.$$

Combining all inequalities above and Lemma 3.5.4 (iii), at (t_0, x_0) , we obtain

$$\begin{aligned} 0 \leq \left(\frac{\partial}{\partial t} - \Delta_\varepsilon \right) H_\delta &\leq -\frac{|\partial \tilde{\psi}^\varepsilon|_{\omega^\varepsilon}^2}{(\tilde{\psi}^\varepsilon + C_0)^3} + C \operatorname{tr}_{\omega^\varepsilon} \hat{\omega} - \left(A + \frac{1}{(\tilde{\psi}^\varepsilon + C_0)^2} \right) \dot{\tilde{\psi}}^\varepsilon \\ &\quad + C' + \left(A + \frac{1}{(\tilde{\psi}^\varepsilon + C_0)^2} \right) \operatorname{tr}_{\omega^\varepsilon} \left(\omega^\varepsilon - \theta^\varepsilon + \frac{1+\delta}{A} R_h \right) \\ &\leq C \operatorname{tr}_{\omega^\varepsilon} \hat{\omega} + (A+1) \log \frac{\Omega_X}{\omega_\varepsilon^n} + (A+1)n \\ &\quad - A \operatorname{tr}_{\omega^\varepsilon} \left(\theta^\varepsilon - \frac{1+\delta}{A} R_h \right) + C. \end{aligned}$$

Since $\pi^* \omega_Y - \varepsilon_0 R_h > 0$, we have

$$\theta^\varepsilon - \frac{1+\delta}{A} R_h \geq c_0 \omega_0$$

for A sufficiently large. Combining with $\operatorname{tr}_{\omega^\varepsilon} \hat{\omega} \leq C \operatorname{tr}_{\omega^\varepsilon} \omega_0$, we can choose A sufficiently large so that at (t_0, x_0)

$$0 \leq -\operatorname{tr}_{\omega^\varepsilon} \omega_0 + C \log \frac{\Omega_X}{(\omega^\varepsilon)^n} + C.$$

Therefore, at (t_0, x_0)

$$\mathrm{tr}_{\omega_0} \omega^\varepsilon \leq \frac{1}{n} (\mathrm{tr}_{\omega^\varepsilon} \omega_0)^{n-1} \frac{(\omega^\varepsilon)^n}{\omega_0^n} \leq C \frac{\omega^\varepsilon}{\Omega_X} \left(\log \frac{\Omega_X}{(\omega^\varepsilon)^n} \right)^{n-1}.$$

Since $\omega^\varepsilon/\Omega_X \leq C$ (Lemma 3.5.4 (iii)) and $x \rightarrow x \log |x|^{n-1}$ is bounded from above for x close to zero, we get

$$\mathrm{tr}_{\omega_0} \omega^\varepsilon \leq C,$$

This implies that H_δ is uniformly bounded from above at its maximum. Hence we obtain the estimate (3.5.9). \square

Lemma 3.5.6. *There exists a uniform $\lambda > 0$ and $C > 0$ such that*

$$\omega_t^\varepsilon \leq \frac{C}{|s|^{2(1-\lambda)}} \omega_0.$$

Proof. Following the method in [TW13, Lemma 3.5] (see also [PS10]), we consider for each $\delta > 0$,

$$H_\delta = \log \mathrm{tr}_{\omega_0} \omega^\varepsilon - A\tilde{\varphi}^\varepsilon + \frac{1}{\tilde{\varphi}^\varepsilon + \tilde{C}} + \frac{1}{(\tilde{\psi}^\varepsilon + \tilde{C})},$$

where $\tilde{\varphi}^\varepsilon := -\log(\mathrm{tr}_{\hat{\omega}} \omega^\varepsilon |s|^{2(1+\delta)}) + A\tilde{\psi}^\varepsilon$ and \tilde{C} is chosen so that $\tilde{\varphi}^\varepsilon + \tilde{C} > 1$ and $\tilde{\psi}^\varepsilon + \tilde{C} > 1$. The constant $A > 0$ will be chosen hereafter. Lemma 3.5.5 and Lemma 3.5.4 (ii) imply that H_δ goes to negative infinity as x tends to E . Hence we can assume that H_δ attains its maximum at $(t_0, x_0) \in [T, T'] \times X \setminus E$. Without loss of generality, let's assume further that $\mathrm{tr}_{\omega_0} \omega^\varepsilon \geq 1$ at (t_0, x_0) .

As in Lemma 3.5.5 we have

$$\left(\frac{\partial}{\partial t} - \Delta_\varepsilon \right) \log \mathrm{tr}_{\hat{\omega}} \omega^\varepsilon \leq \frac{2}{(\mathrm{tr}_{\hat{\omega}} \omega^\varepsilon)^2} \mathrm{Re} \left(g^{k\bar{q}} \hat{T}_{ki}^i \partial_{\bar{q}} \mathrm{tr}_{\hat{\omega}} \omega^\varepsilon \right) + C \mathrm{tr}_{\omega^\varepsilon} \hat{\omega} + \frac{\mathrm{tr}_{\hat{\omega}} \eta}{\mathrm{tr}_{\hat{\omega}} \omega^\varepsilon}.$$

and

$$\left(\frac{\partial}{\partial t} - \Delta_\varepsilon \right) \log \mathrm{tr}_{\omega_0} \omega^\varepsilon \leq \frac{2}{(\mathrm{tr}_{\omega_0} \omega^\varepsilon)^2} \mathrm{Re} \left(g^{k\bar{q}} (T_0)_{ki}^i \partial_{\bar{q}} \mathrm{tr}_{\omega_0} \omega^\varepsilon \right) + C \mathrm{tr}_{\omega^\varepsilon} \omega_0 + \frac{\mathrm{tr}_{\omega_0} \eta}{\mathrm{tr}_{\omega_0} \omega^\varepsilon},$$

It follows from Lemma 3.5.4 and (3.5.10) that

$$\frac{\mathrm{tr}_{\hat{\omega}} \eta}{\mathrm{tr}_{\hat{\omega}} \omega^\varepsilon} \leq \frac{C}{|s|_h^2 \mathrm{tr}_{\hat{\omega}} \omega^\varepsilon},$$

and

$$\frac{\mathrm{tr}_{\omega_0} \eta}{\mathrm{tr}_{\omega_0} \omega^\varepsilon} \leq \frac{C}{\mathrm{tr}_{\omega_0} \omega^\varepsilon} \leq \frac{C}{|s|_h^2 \mathrm{tr}_{\hat{\omega}} \omega^\varepsilon}.$$

Therefore

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta_\varepsilon\right) H_\delta &\leq C_0 \operatorname{tr}_{\omega_0} \omega^\varepsilon + \frac{2}{(\operatorname{tr}_{\omega_0} \omega^\varepsilon)^2} \operatorname{Re} \left(g^{k\bar{q}} (T_0)_{ki}^i \partial_{\bar{q}} \operatorname{tr}_{\omega^\varepsilon} \hat{\omega} \right) + C_0 (A+1) \operatorname{tr}_{\omega^\varepsilon} \hat{\omega} \\
&+ \left(A + \frac{1}{(\tilde{\varphi}^\varepsilon + \tilde{C})^2} \right) \frac{2}{(\operatorname{tr}_{\hat{\omega}} \omega^\varepsilon)^2} \operatorname{Re} \left(g^{k\bar{q}} \hat{T}_{ki}^i \partial_{\bar{q}} \operatorname{tr}_{\omega^\varepsilon} \hat{\omega} \right) \\
&- \left(A \left(A + \frac{1}{(\tilde{\varphi}^\varepsilon + \tilde{C})^2} \right) + \frac{1}{(\tilde{\psi}^\varepsilon + \tilde{C})^2} \right) \dot{\psi}^\varepsilon \\
&+ \left(A \left(A + \frac{1}{(\tilde{\varphi}^\varepsilon + \tilde{C})^2} \right) + \frac{1}{(\tilde{\psi}^\varepsilon + \tilde{C})^2} \right) \operatorname{tr}_{\omega^\varepsilon} \left(\omega^\varepsilon - \theta^\varepsilon + \frac{(1+\delta)R_h}{A} \right) \\
&- \frac{2}{(\tilde{\varphi}^\varepsilon + \tilde{C})^3} |\partial \tilde{\varphi}|_g^2 - \frac{2}{(\tilde{\psi}^\varepsilon + \tilde{C})^3} |\partial \tilde{\psi}^\varepsilon|_g^2 + \frac{CA}{|s|_h^2 \operatorname{tr}_{\hat{\omega}} \omega^\varepsilon}.
\end{aligned}$$

For the last term, we may assume without of generality that

$$\frac{CA}{|s|_h^2 \operatorname{tr}_{\hat{\omega}} \omega^\varepsilon} \leq (\operatorname{tr}_{\omega_0} \omega^\varepsilon)^{1/A},$$

since otherwise H_δ is already uniformly bounded. Using

$$\operatorname{tr}_{\omega_0} \omega^\varepsilon \leq n \left(\frac{(\omega^\varepsilon)^n}{\omega_0^n} \right) (\operatorname{tr}_{\omega^\varepsilon} \omega_0)^{n-1},$$

and $\operatorname{tr}_{\omega_0} \omega^\varepsilon \geq 1$ at (t_0, x_0) , we get

$$\frac{CA}{|s|_h^2 \operatorname{tr}_{\hat{\omega}} \omega^\varepsilon} \leq C_1 \operatorname{tr}_{\omega_0} \omega^\varepsilon, \quad (3.5.16)$$

for $A > n - 1$.

It follows from (3.5.5) that

$$\frac{1}{2} A \theta^\varepsilon - (1 + \delta) R_h \geq c_0 \omega_0,$$

for all A sufficiently large. Therefore we can choose A sufficiently large such that

$$\begin{aligned}
A^2 \theta^\varepsilon - A(1 + \delta) R_h &= \frac{1}{2} A^2 \theta^\varepsilon + A \left(\frac{1}{2} A \theta^\varepsilon - (1 + \delta) R_h \right) \\
&\geq C_0 (A+1) \hat{\omega} + (C_0 + C_1 + 1) \omega_0.
\end{aligned} \quad (3.5.17)$$

Compute at (t_0, x_0) , using (3.5.16), (3.5.17), $\tilde{\psi}^\varepsilon + \tilde{C} \geq 1$ and $\tilde{\varphi}^\varepsilon + \tilde{C} \geq 1$,

$$\begin{aligned} 0 \leq & -\operatorname{tr}_{\omega_0} \omega^\varepsilon + \frac{2}{(\operatorname{tr}_{\omega_0} \omega^\varepsilon)^2} \operatorname{Re} \left(g^{k\bar{q}} (T_0)_{ki}^i \partial_{\bar{q}} \operatorname{tr}_{\omega^\varepsilon} \hat{\omega} \right) \\ & + \left(A + \frac{1}{(\tilde{\varphi}^\varepsilon + \tilde{C})^2} \right) \frac{2}{(\operatorname{tr}_{\hat{\omega}} \omega^\varepsilon)^2} \operatorname{Re} \left(g^{k\bar{q}} \hat{T}_{ki}^i \partial_{\bar{q}} \operatorname{tr}_{\omega^\varepsilon} \hat{\omega} \right) \\ & - B \dot{\psi}^\varepsilon - \frac{2}{(\tilde{\varphi}^\varepsilon + \tilde{C})^3} |\partial \tilde{\varphi}|_g^2 - \frac{2}{(\tilde{\psi}^\varepsilon + \tilde{C})^3} |\partial \tilde{\psi}^\varepsilon|_g^2 + C', \end{aligned}$$

for B is a constant in $[A^2, A^2 + A + 1]$.

By the same argument in [TW13, Lemma 3.5], we get, at (t_0, x_0) ,

$$0 \leq -\frac{1}{4} \operatorname{tr}_{\omega^\varepsilon} \omega_0 - B \log \frac{(\omega^\varepsilon)^n}{\Omega_X} + C'. \quad (3.5.18)$$

As in the proof of Lemma 3.5.5, we infer that $\operatorname{tr}_{\omega_0} \omega^\varepsilon$ is bounded from above at (t_0, x_0) . Therefore, it follows from Lemma 3.5.4 and Lemma 3.5.5 that H_δ is bounded from above uniformly in δ . Let $\delta \rightarrow 0$, we get

$$\log \operatorname{tr}_{\omega_0} \omega^\varepsilon + A \log(|s|_h^2 \operatorname{tr}_{\hat{\omega}} \omega^\varepsilon) \leq C.$$

Since $\operatorname{tr}_{\omega_0} \omega^\varepsilon \leq C \operatorname{tr}_{\hat{\omega}} \omega^\varepsilon$, we have

$$\log(\operatorname{tr}_{\omega_0} \omega^\varepsilon)^{A+1} |s|_h^{2A} \leq C,$$

and the desired inequality follows with $\lambda = 1/(A+1) > 0$. \square

Proof of Proposition 3.5.3. On Y , the function $\phi^\varepsilon := \pi_* \psi_t^\varepsilon$ satisfies

$$\frac{\partial \phi^\varepsilon}{\partial t} = \log \frac{(\hat{\omega}_Y + \frac{\varepsilon}{T}(\omega_Y - \pi_* \omega_0) + dd^c \phi^\varepsilon)^n}{\Omega_Y}, \quad \psi^\varepsilon|_{t=T} = \pi_* \varphi(T - \varepsilon). \quad (3.5.19)$$

Since $\alpha_t^\varepsilon = \hat{\omega}_Y + \frac{\varepsilon}{T}(\omega_Y - \pi_* \omega_0)$ is uniformly equivalent to $\hat{\omega}_Y$ for all $\varepsilon \in [0, \varepsilon_0]$ and $t \in [T, T']$, we can follow the same argument as in Section 3.3 to obtain the C^k -estimates for ϕ_t^ε which are independent of ε , for all $t \in (T, T']$. By Arzela-Ascoli theorem, after extracting a subsequence, we can assume that ϕ^ε converges to $\tilde{\phi}$, as $\varepsilon \rightarrow 0^+$, in $C^\infty([\delta, T'] \times Y)$ for all $\delta \in (T, T')$. Moreover $\tilde{\phi}_t$ uniformly converges to ϕ_T , hence $\tilde{\phi}$ satisfies (3.5.4). Thanks to Theorem 3.4.2, $\tilde{\phi}$ is equal to the solution ϕ of (3.5.4). Using Lemma 3.5.6 and the standard local parabolic theory, we obtain the C^∞ estimates ω^ε on compact sets away from E . Hence ϕ is the smooth solution of 3.5.1 on $[T, T'] \times Y \setminus \{y_0\}$. Finally, Proposition 3.5.3 follows directly from Lemma 3.5.5 and Lemma 3.5.6. \square

Finally, we get the following:

Theorem 3.5.7. *The solution ω_t of (3.5.3) smoothly converges to ω' , as $t \rightarrow T^+$, in $C_{loc}^\infty(Y \setminus \{y_0\})$ and (Y, ω_t) converges in the Gromov-Hausdorff sense to (Y, d_T) as $t \rightarrow T^+$.*

Proof. It follows from the proof of Proposition 3.5.3 that $\phi \in C^\infty([T, T'] \times Y \setminus \{y_0\})$, hence ω_t smoothly converges to $\omega|_{t=T} = \omega'$ in $C_{loc}^\infty(Y \setminus \{y_0\})$.

Denote by d_{ω_t} the metric induced from ω_t and S_r the $2n - 1$ sphere of radius r in B centered at the origin. Then it follows from Lemma 3.5.5 and the argument of [SW13a, Lemma 2.7(i)] that:

(a) There exists a uniform constant C such that

$$\text{diam}_{d_{\omega_t}}(S_r) \leq C, \quad \forall t \in (T, T']. \quad (3.5.20)$$

Following the same argument of [SW13a, Lemma 2.7 (ii)], we have

(b) For any $z \in B(0, \frac{1}{2}) \setminus \{0\}$, the length of a radial path $\gamma(s) = sz$ for $s \in (0, 1]$ with respect to ω_t is uniformly bounded from above by $C|z|^\lambda$, where C is a uniformly constant and λ as in Lemma 3.5.6.

Given (a) and (b), the Gromov-Hausdorff convergence follows exactly as in [SW13a, Section 3]. This completes the proof of Theorem 3.5.7 and Conjecture 3.5.1. \square

3.6 Twisted Chern-Ricci flow

3.6.1 Maximal existence time for the twisted Chern-Ricci flow

Let (X, g) be a compact Hermitian manifold of complex dimension n . We define here the twisted Chern-Ricci flow on X as

$$\frac{\partial \omega_t}{\partial t} = -\text{Ric}(\omega_t) + \eta, \quad \omega|_{t=0} = \omega_0 \quad (3.6.1)$$

where η is a smooth $(1, 1)$ -form. Set $\hat{\omega}_t = \omega_0 + t\eta - t\text{Ric}(\omega_0)$. We now define

$$\begin{aligned} T &:= \sup\{t \geq 0 \mid \exists \psi \in C^\infty(X) \text{ such that } \hat{\omega}_t + dd^c \psi > 0\} \\ &= \sup\{T' \geq 0 \mid \forall t \in [0, T'], \exists \psi \in C^\infty(X) \text{ such that } \hat{\omega}_t + dd^c \psi > 0\}. \end{aligned}$$

We now prove the following theorem generalizing the same result due to Tosatti-Weinkove [TW15, Theorem 1.2]. We remark that our ingredients for the proof come from a priori estimates proved in Section 3.3 which are different from the approach of Tosatti and Weinkove.

Theorem 3.6.1. *There exists a unique maximal solution to the twisted Chern-Ricci flow on $[0, T)$.*

Proof. Fix $T' < T$. We show that there exists a solution of (3.6.1) on $[0, T']$. First we prove that the twisted Chern-Ricci flow is equivalent to a Monge-Ampère flow. Indeed, consider the following Monge-Ampère flow

$$\frac{\partial \varphi}{\partial t} = \log \frac{(\hat{\omega}_t + dd^c \varphi)^n}{\omega_0^n}. \quad (3.6.2)$$

If φ solves (3.6.2) on $[0, T']$ then taking $\omega_t := \hat{\omega}_t + dd^c \varphi$, we get

$$\frac{\partial}{\partial t}(\omega_t - \hat{\omega}_t) = dd^c \log \frac{\omega_t^n}{\omega_0^n},$$

hence

$$\frac{\partial}{\partial t} \omega_t = -Ric(\omega_t) + \eta.$$

Conversely, if ω_t solves (3.6.1) on $[0, T']$, then we get

$$\frac{\partial}{\partial t}(\omega_t - \hat{\omega}_t) = -Ric(\omega_t) + Ric(\omega_0) = dd^c \log \frac{\omega_t^n}{\omega_0^n}.$$

Therefore if φ satisfies

$$\frac{\partial}{\partial t}(\omega_t - \hat{\omega}_t - dd^c \varphi) = 0,$$

so $\omega_t = \hat{\omega}_t + dd^c \varphi$ and φ satisfies (3.6.2).

By the standard parabolic theory [Lie96], there exists a maximal solution of (3.6.2) on some time interval $[0, T_{\max})$ with $0 < T_{\max} \leq \infty$. We may assume without loss of generality that $T_{\max} < T'$. We now show that a solution of (3.6.2) exists beyond T_{\max} . Indeed, the a priori estimates for more general Monge-Ampère flows in Section 3.3 gives us uniform estimates for φ in $[0, T_{\max})$ (see Theorem 3.3.12), so we get a solution on $[0, T_{\max}]$. By the short time existence theory the flow (3.6.2) can go beyond T_{\max} , this gives a contradiction. So the twisted Chern-Ricci flow has a solution in $[0, T)$. Finally, the uniqueness of solution follows from Theorem 3.4.1. \square

3.6.2 Twisted Einstein metric on Hermitian manifolds

We fix a smooth $(1, 1)$ -form η . A solution of the equation

$$Ric(\omega) = \mu\omega + \eta \quad (3.6.3)$$

with $\mu = 1$ or -1 , is called a *twisted Einstein metric*. We recall

$$\{\eta\} := \{\alpha | \exists f \in C^\infty(X) \text{ with } \alpha = \eta + dd^c f\},$$

the equivalence class of η .

In the sequel we study the convergence of the normalized twisted Chern-Ricci flow to a twisted Einstein metric $\omega = \chi + dd^c\varphi \in -(c_1^{BC} - \{\eta\})$ assuming that $c_1^{BC} - \{\eta\} < 0$ and $\mu = -1$. Note that if $c_1^{BC}(X) < 0$ (resp. $c_1^{BC}(X) > 0$) implies that X is a Kähler manifold which admits a Kähler metric in $-c_1(X)$ (resp. in $c_1(X)$). Therefore the positivity of the twisted Bott-Chern class is somehow more natural in our context.

Assume the twisted first Bott-Chern class $\alpha := c_1^{BC}(X) - \{\eta\}$ is negative. We now use a result in elliptic Monge-Ampère equation due to Cherrier [Che87] to prove the existence of twisted Einstein metric. An alternative proof using the convergence of the twisted Chern-Ricci flow will be given in Theorem 3.6.3.

Theorem 3.6.2. *There exists a unique twisted Einstein metric in $-\alpha$ satisfying (3.6.3):*

$$Ric(\omega) = -\omega + \eta. \quad (3.6.4)$$

Proof. Let $\chi = \eta - Ric(\Omega)$ be a Hermitian metric in α , then any Hermitian metric in α can be written as $\omega = \chi + dd^c\varphi$ where φ is smooth strictly and χ -psh. Since

$$\omega - \eta = \chi + dd^c\varphi - \eta = -Ric(\Omega) - dd^c\varphi,$$

we get

$$dd^c \log \frac{\omega^n}{\Omega} = -Ric(\omega) + Ric(\Omega) = dd^c\varphi.$$

Therefore the equation (3.6.4) can be written as the following Monge-Ampère equation

$$(\chi + dd^c\varphi)^n = e^\varphi \Omega \quad (3.6.5)$$

It follows from [Che87] that (3.6.5) admits a unique smooth χ -psh solution, therefore there exists a unique twisted Einstein metric in $-(c_1^{BC}(X) - \{\eta\})$. \square

3.6.3 Convergence of the flow when $c_1^{BC}(X) - \{\eta\} < 0$

We defined the *normalized twisted Chern-Ricci flow* as follows

$$\frac{\partial}{\partial t} \omega_t = -Ric(\omega_t) - \omega_t + \eta, \quad (3.6.6)$$

We have (3.6.6) is equivalent to the following Monge-Ampère flow

$$\frac{\partial \varphi}{\partial t} = \log \frac{(\hat{\omega}_t^n + dd^c\varphi)^n}{\Omega} - \varphi,$$

where $\hat{\omega}_t = e^{-t} + (1 - e^{-t})(\eta - Ric(\Omega))$ and Ω is a fixed smooth volume form on X . Since we assume $c_1^{BC}(X) - \{\eta\}$ is negative, the flow (3.6.6) has a longtime solution. The longtime behavior of (3.6.6) is as follows

Theorem 3.6.3. *Suppose $c_1(X) - \{\eta\} < 0$. Then the normalized twisted Chern-Ricci flow starting from any initial Hermitian metric ω_0 smoothly converges, as $t \rightarrow +\infty$, to a twisted Einstein Hermitian metric $\omega_\infty = \eta - Ric(\Omega) + dd^c\varphi_\infty$ which satisfies*

$$Ric(\omega_\infty) = \eta - \omega_\infty.$$

Proof. We now derive the uniform estimates for the solution φ of the following Monge-Ampère

$$\frac{\partial\varphi}{\partial t} = \log \frac{(\hat{\omega}_t + dd^c\varphi)^n}{\Omega} - \varphi, \quad \varphi|_{t=0} = 0,$$

where $\hat{\omega}_t := e^{-t}\omega_0 + (1 - e^{-t})\chi$, and $\chi = \eta - Ric(\Omega) > 0$.

The C^0 -estimates for φ and $\dot{\varphi}$ follow from the same arguments as in [Cao85, TZ06, Tsu88] for Kähler-Ricci flow (see [TW15] for the same estimates for the Chern-Ricci flow). Moreover, since

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_t}\right)(\varphi + \dot{\varphi} + nt) = \text{tr}_{\omega_t} \chi.$$

and

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_t}\right)(e^t\dot{\varphi}) = -\text{tr}_{\omega_t}(\omega_0 - \chi)$$

therefore

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_t}\right)((e^t - 1)\dot{\varphi} - \varphi - nt) = -\text{tr}_{\omega_t} \omega_0 < 0.$$

The maximum principle follows that $(e^t - 1)\dot{\varphi} - \varphi - nt \leq C$, hence

$$\dot{\varphi} \leq Cte^{-t}. \tag{3.6.7}$$

For the second order estimate, we follow the method of Tosatti and Weinkove [TW15, Lemma 4.1 (iii)] in which they have used a technical trick due to Phong and Sturn [PS10].

Lemma 3.6.4. *There exists uniform constant $C > 1$ such that*

$$\log \text{tr}_{\hat{\omega}}(\omega_t) \leq C.$$

Proof. Since φ is uniformly bounded, we can choose C_0 such that $\varphi + C_0 \geq 1$. Set

$$P = \log \text{tr}_{\hat{\omega}} \omega_t - A\varphi + \frac{1}{\varphi + C_0},$$

where $A > 0$ will be chosen hereafter. The idea of adding the third term in P is due to Phong-Sturn [PS10] and was used in the context of Chern-Ricci flow (cf. [TW15], [TW13], [TWY15]).

Assume without loss of generality that $\text{tr}_{\hat{\omega}} \omega_t \geq 1$ at a maximum point (t_0, x_0) with $t_0 > 0$ of P . It follows from the same calculation in Lemma 3.5.5 that at (t_0, x_0) , we have

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_{\omega_t} \right) \log \text{tr}_{\hat{\omega}} \omega_t &\leq \frac{2}{(\text{tr}_{\hat{\omega}} \omega_t)^2} \text{Re}(\hat{g}^{i\bar{l}} g^{k\bar{q}} \hat{T}_{k\bar{i}l} \partial_{\bar{q}} \text{tr}_{\hat{\omega}} \omega_t) + C \text{tr}_{\omega_t} \hat{\omega} + \frac{\text{tr}_{\hat{\omega}} \eta}{\text{tr}_{\hat{\omega}} \omega_t} \\ &\leq \frac{2}{(\text{tr}_{\hat{\omega}} \omega_t)^2} \text{Re}(\hat{g}^{i\bar{l}} g^{k\bar{q}} \hat{T}_{k\bar{i}l} \partial_{\bar{q}} \text{tr}_{\hat{\omega}} \omega_t) + C \text{tr}_{\omega_t} \hat{\omega} + C_1, \end{aligned}$$

where $C_1 > 0$ satisfies $\eta \leq C_1 \hat{\omega}$.

Now at a maximum point (t_0, x_0) with $t_0 > 0$ we have $\nabla P = 0$, hence

$$\frac{1}{\text{tr}_{\hat{\omega}} \omega_t} \partial_{\bar{i}} \text{tr}_{\hat{\omega}} \omega_t - A \varphi_{\bar{i}} - \frac{\varphi_{\bar{i}}}{(\varphi + C_0)^2} = 0.$$

Therefore

$$\begin{aligned} &\left| \frac{2}{(\text{tr}_{\hat{\omega}} \omega_t)^2} \text{Re}(\hat{g}^{i\bar{l}} g^{k\bar{q}} \hat{T}_{k\bar{i}l} \partial_{\bar{q}} \text{tr}_{\hat{\omega}} \omega_t) \right| \\ &= \left| \frac{2}{(\text{tr}_{\hat{\omega}} \omega_t)^2} \text{Re} \left(\left(A + \frac{1}{(\varphi + C_0)^2} \right) \hat{g}^{i\bar{l}} g^{k\bar{q}} \hat{T}_{k\bar{i}l} \varphi_{\bar{q}} \right) \right| \\ &\leq \frac{CA^2}{(\text{tr}_{\hat{\omega}} \omega_t)^2} (\varphi + C_0)^3 g^{k\bar{q}} \hat{g}^{i\bar{l}} \hat{T}_{k\bar{i}l} \hat{g}^{m\bar{j}} \hat{T}_{q\bar{j}m} + \frac{|\partial \varphi|_g^2}{(\varphi + C_0)^3} \\ &\leq \frac{CA^2 \text{tr}_{\omega_t} \hat{\omega}}{(\text{tr}_{\hat{\omega}} \omega_t)^2} (\varphi + C_0)^3 + \frac{|\partial \varphi|_g^2}{(\varphi + C_0)^3}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_{\omega_t} \right) \left(-A\varphi + \frac{1}{\varphi + C_0} \right) &= -A\dot{\varphi} + A\Delta_{\omega_t} \varphi - \frac{\dot{\varphi}}{(\varphi + C_0)^2} \\ &\quad + \frac{\Delta_{\omega_t} \varphi}{(\varphi + C_0)^2} - \frac{2|\partial \varphi|_g^2}{(\varphi + C_0)^3} \\ &= -\left(A + \frac{1}{(\varphi + C_0)^2} \right) \dot{\varphi} - \frac{2|\partial \varphi|_g^2}{(\varphi + C_0)^3} \\ &\quad + \left(A + \frac{1}{(\varphi + C_0)^2} \right) (n - \text{tr}_{\omega_t} \hat{\omega}). \end{aligned}$$

Combining these inequalities, at (t_0, z_0) we have

$$\begin{aligned} 0 \leq \left(\frac{\partial}{\partial t} - \Delta_{\omega_t} \right) P &\leq \frac{CA^2 \text{tr}_{\omega_t} \hat{\omega}}{(\text{tr}_{\hat{\omega}} \omega_t)^2} + C \text{tr}_{\omega_t} \hat{\omega} - \left(A + \frac{1}{(\varphi + C_0)^2} \right) \dot{\varphi} + C_1 \\ &\quad + \left(A + \frac{1}{(\varphi + C_0)^2} \right) (n - \text{tr}_{\omega_t} \hat{\omega}) - \frac{|\partial \varphi|_g^2}{(\varphi + C_0)^3} \\ &\leq \frac{CA^2 \text{tr}_{\omega_t} \hat{\omega}}{(\text{tr}_{\hat{\omega}} \omega_t)^2} (\varphi + C_0)^3 - C_2 + (C - A) \text{tr}_{\omega_t} \hat{\omega}. \end{aligned}$$

We can choose A sufficiently large such that at the maximum of P either $\text{tr}_{\hat{\omega}} \omega_t \leq A^2(\varphi + C_0)^3$, then we are done, or $\text{tr}_{\hat{\omega}} \omega_t \geq A^2(\varphi + C_0)^3$, and $A \geq 2C$. For the second case, we obtain at the maximum of P , there exists a uniform constant $C_3 > 0$ so that

$$\text{tr}_{\omega_t} \hat{\omega} \leq C_3,$$

Hence combining with the following inequality (see Lemma 2.2.3)

$$\text{tr}_{\hat{\omega}} \omega_t \leq n(\text{tr}_{\omega_t} \hat{\omega})^{n-1} \frac{\omega_t^n}{\hat{\omega}^n},$$

we have

$$\text{tr}_{\hat{\omega}} \omega_t \leq n(\text{tr}_{\omega_t} \hat{\omega})^{n-1} \frac{\omega_t^n}{\hat{\omega}^n} \leq C_4.$$

This implies that P is bounded from above at its maximum, so we complete the proof of the lemma. \square

It follows from Lemma 3.6.4 that ω_t is uniformly equivalent to $\hat{\omega}$ independent of t , hence

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_t} \right) (e^t \dot{\varphi}) = -\text{tr}_{\omega_t}(\omega_0) + \text{tr}_{\omega_t} \chi \geq -C,$$

hence $\dot{\varphi} \geq -C(1+t)e^{-t}$ by the maximum principle. Combining with (3.6.7), we infer that φ converges uniformly exponentially fast to a continuous function φ_∞ . Moreover, by the same argument in Section 3.3, Evans-Krylov and Schauder estimates give us the uniform higher order estimates for φ . Therefore φ_∞ is smooth and φ_t converges to φ_∞ in C^∞ .

Finally, we get the limiting metric $\omega_\infty = \eta - \text{Ric}(\Omega) + dd^c \varphi_\infty$ which satisfies the twisted Einstein equation

$$\text{Ric}(\omega_\infty) = -\omega_\infty + \eta.$$

This proves the existence of a twisted Einstein metric in $-c_1^{BC}(X) + \{\eta\}$. \square

As an application, we prove the existence of a unique solution of the Monge-Ampère equation on Hermitian manifolds. This result was first proved by Cherrier [Che87, Théorème 1, p. 373].

Theorem 3.6.5. *Let (X, ω) be a Hermitian manifold, Ω be a smooth volume form on X . Then there exists a unique smooth ω -psh function φ satisfying*

$$(\omega + dd^c \varphi)^n = e^\varphi \Omega.$$

Proof. Set $\eta = \omega + \text{Ric}(\Omega)$, then we have $c_1^{BC}(X) - \{\eta\} < 0$. It follows from Theorem 3.6.3 that the twisted normalized Chern-Ricci flow

$$\frac{\partial}{\partial t} \omega_t = -\text{Ric}(\omega_t) - \omega_t + \eta$$

admits unique solution which smoothly converges to a twisted Einstein Hermitian metric $\omega_\infty = \eta - Ric(\Omega) + dd^c\varphi_\infty = \omega + dd^c\varphi_\infty$ which satisfies $Ric(\omega_\infty) = -\omega_\infty + \eta = Ric(\Omega) - dd^c\varphi_\infty$. Therefore φ_∞ is a solution of the Monge-Ampère equation

$$(\omega + dd^c\varphi)^n = e^\varphi\Omega.$$

The uniqueness of solution follows from the comparison principle. □

Chapter 4

Fully non-linear parabolic equations on compact Hermitian manifolds

A notion of parabolic C -subsolutions is introduced for parabolic equations, extending the theory of C -subsolutions recently developed by B. Guan and more specifically G. Székelyhidi for elliptic equations. The resulting parabolic theory provides a convenient unified approach for the study of many geometric flows.

The results of this chapter are joint work with Duong H. Phong [PT17].

4.1 Introduction

Subsolutions play an important role in the theory of partial differential equations. Their existence can be viewed as an indication of the absence of any global obstruction. Perhaps more importantly, it can imply crucial a priori estimates, as for example in the Dirichlet problem for the complex Monge-Ampère equation [Spr05, Gua94]. However, for compact manifolds without boundary, it is necessary to extend the notion of subsolution, since the standard notion may be excluded by either the maximum principle or cohomological constraints. Very recently, more flexible and compelling notions of subsolutions have been proposed by Guan [Gua14] and Székelyhidi [Sze15]. In particular, they show that their notions, called C -subsolution in [Sze15], do imply the existence of solutions and estimates for a wide variety of fully non-linear elliptic equations on Hermitian manifolds. It is natural to consider also the parabolic case. This was done by Guan, Shi, and Sui in [GSS15] for the usual notion of subsolution and for the Dirichlet problem. We now carry this out for the more general notion of C -subsolution on compact Hermitian manifolds, adapting the methods of [Gua14] and especially [Sze15]. As we shall see, the resulting parabolic theory

provides a convenient unified approach to the many parabolic equations which have been studied in the literature.

Let (X, α) be a compact Hermitian manifold of dimension n , $\alpha = i \alpha_{\bar{k}j} dz^j \wedge d\bar{z}^k > 0$, and $\chi(z)$ be a real $(1, 1)$ -form,

$$\chi = i \chi_{\bar{k}j}(z) dz^j \wedge d\bar{z}^k.$$

If $u \in C^2(X)$, let $A[u]$ be the matrix with entries $A[u]^k_j = \alpha^{k\bar{m}}(\chi_{\bar{m}j} + \partial_j \partial_{\bar{m}} u)$. We consider the fully nonlinear parabolic equation,

$$\partial_t u = F(A[u]) - \psi(z), \quad (4.1.1)$$

where $F(A)$ is a smooth symmetric function $F(A) = f(\lambda[u])$ of the eigenvalues $\lambda_j[u]$, $1 \leq j \leq n$ of $A[u]$, defined on a open symmetric, convex cone $\Gamma \subset \mathbb{R}^n$ with vertex at the origin and containing the positive orthant Γ_n . We shall assume throughout the paper that f satisfies the following conditions:

- (1) $f_i > 0$ for all i , and f is concave.
- (2) $f(\lambda) \rightarrow -\infty$ as $\lambda \rightarrow \partial\Gamma$
- (3) For any $\sigma < \sup_{\Gamma} f$ and $\lambda \in \Gamma$, we have $\lim_{t \rightarrow \infty} f(t\lambda) > \sigma$.

We shall say that a C^2 function u on X is admissible if the vector of eigenvalues of the corresponding matrix A is in Γ for any $z \in X$. Fix $T \in (0, \infty]$. To alleviate the terminology, we shall also designate by the same adjective functions in $C^{2,1}(X \times [0, T])$ which are admissible for each fixed $t \in [0, T)$. The following notion of subsolution is an adaptation to the parabolic case of Székelyhidi's [Sze15] notion in the elliptic case:

Definition 4.1.1. An admissible function $\underline{u} \in C^{2,1}(X \times [0, T])$ is said to be a (parabolic) C -subsolution of (4.1.1), if there exist constants $\delta, K > 0$, so that for any $(z, t) \in X \times [0, T)$, the condition

$$f(\lambda[\underline{u}(z, t)] + \mu) - \partial_t \underline{u} + \tau = \psi(z), \quad \mu + \delta I \in \Gamma_n, \quad \tau > -\delta \quad (4.1.2)$$

implies that $|\mu| + |\tau| < K$. Here I denotes the vector $(1, \dots, 1)$ of eigenvalues of the identity matrix.

We shall see below (§4.1) that this notion is more general than the classical notion defined by $f(\lambda([\underline{u}])) - \partial_t \underline{u}(z, t) > \psi(z, t)$ and studied by Guan-Shi-Sui [GSS15]. A C -subsolution in the sense of Székelyhidi of the equation $F(A[u]) - \psi = 0$ can be viewed as a parabolic C -subsolution of the equation (4.1.1) which is time-independent. But more generally, to solve the equation $F(A[u]) - \psi = 0$ by say the method of continuity, we must choose a time-dependent deformation of this equation, and we would need then a C -subsolution for each time. The heat equation (4.1.1) and the above notion of parabolic subsolution can be viewed as a canonical choice of deformation.

To discuss our results, we need a finer classification of non-linear partial differential operators due to Trudinger [Tru95]. Let Γ_∞ be the projection of Γ_n onto \mathbf{R}^{n-1} ,

$$\Gamma_\infty = \{\lambda' = (\lambda_1, \dots, \lambda_{n-1}); \lambda = (\lambda_1, \dots, \lambda_n) \in \Gamma \text{ for some } \lambda_n\} \quad (4.1.3)$$

and define the function f_∞ on Γ_∞ by

$$f_\infty(\lambda') = \lim_{\lambda_n \rightarrow \infty} f(\lambda', \lambda_n). \quad (4.1.4)$$

It is shown in [Tru95] that, as a consequence of the concavity of f , the limit is either finite for all $\lambda' \in \Gamma_\infty$ or infinite for all $\lambda' \in \Gamma_\infty$. We shall refer to the first case as the *bounded case*, and to the second case as the *unbounded case*. For example, Monge-Ampère flows belong to the unbounded case, while the J -flow and Hessian quotient flows belong to the bounded case. In the unbounded case, any admissible function, and in particular 0 if $\lambda[\chi] \in \Gamma$, is a C -subsolution in both the elliptic and parabolic cases. We have then:

Theorem 4.1.2. *Consider the flow (4.1.1), and assume that f is in the unbounded case. Then for any admissible initial data u_0 , the flow admits a smooth solution $u(z, t)$ on $[0, \infty)$, and its normalization \tilde{u} defined by*

$$\tilde{u} := u - \frac{1}{V} \int_X u \alpha^n, \quad V = \int_X \alpha^n, \quad (4.1.5)$$

converges in C^∞ to a function \tilde{u}_∞ satisfying the following equation for some constant c ,

$$F(A[\tilde{u}_\infty]) = \psi(z) + c. \quad (4.1.6)$$

The situation is more complicated when f belongs to the bounded case:

Theorem 4.1.3. *Consider the flow (4.1.1), and assume that it admits a subsolution \underline{u} on $X \times [0, \infty)$, but that f is in the bounded case. Then for any admissible data u_0 , the equation admits a smooth solution $u(z, t)$ on $(0, \infty)$. Let \tilde{u} be the normalization of the solution u , defined as before by (4.1.5). Assume that either one of the following two conditions holds.*

(a) *The initial data and the subsolution satisfy*

$$\partial_t \underline{u} \geq \sup_X (F(A[u_0]) - \psi); \quad (4.1.7)$$

(b) *or there exists a function $h(t)$ with $h'(t) \leq 0$ so that*

$$\sup_X (u(t) - h(t) - \underline{u}(t)) \geq 0 \quad (4.1.8)$$

and the Harnack inequality

$$\sup_X (u(t) - h(t)) \leq -C_1 \inf_X (u(t) - h(t)) + C_2 \quad (4.1.9)$$

holds for some constants $C_1, C_2 > 0$ independent of time.

Then \tilde{u} converges in C^∞ to a function \tilde{u}_∞ satisfying (4.1.6) for some constant c .

The essence of the above theorems resides in the a priori estimates which are established in §2. The C^1 and C^2 estimates can be adapted from the corresponding estimates for C -subsolutions in the elliptic case, but the C^0 estimate turns out to be more subtle. Following Blocki [Blo05b] and Székelyhidi [Sze15], we obtain C^0 estimates from the Alexandrov-Bakelman-Pucci (ABP) inequality, using this time a parabolic version of ABP due to K. Tso [Tso85]. However, it turns out that the existence of a C -subsolution gives only partial information on the oscillation of u , and what can actually be estimated has to be formulated with some care, leading to the distinction between the cases of f bounded and unbounded, as well as Theorem 4.1.3.

The conditions (a) and especially (b) in Theorem 2 may seem impractical at first sight since they involve the initial data as well as the long-time behavior of the solution. Nevertheless, as we shall discuss in greater detail in section §4, Theorems 1 and 2 can be successfully applied to a wide range of parabolic flows on Hermitian manifolds previously studied in the literature, including the Kähler-Ricci flow, the Chern-Ricci flow, the J -flow, the Hessian flows, the quotient Hessian flows, and mixed Hessian flows. We illustrate this by deriving in §4 as a corollary of Theorem 2 a convergence theorem for a mixed Hessian flow, which seems new to the best of our knowledge. It answers a question raised for general $1 \leq \ell < k \leq n$ by Fang-Lai-Ma [FLM11] (see also Sun [Sun15a, Sun17a, Sun15b, Sun15c]), and extends the solution obtained for $k = n$ by Collins-Székelyhidi [CS17] and subsequently also by Sun [Sun15c, Sun17c]:

Theorem 4.1.4. *Assume that (X, α) is a compact Kähler n -manifold, and fix $1 \leq \ell < k \leq n$. Fix a closed $(1, 1)$ -form χ which is k -positive and non-negative constants c_j , and assume that there exists a form $\chi' = \chi + i\partial\bar{\partial}u$ which is a closed k -positive form and satisfies*

$$kc(\chi')^{k-1} \wedge \alpha^{n-k} - \sum_{j=1}^{\ell} jc_j(\chi')^{j-1} \wedge \alpha^{n-j} > 0, \quad (4.1.10)$$

in the sense of positivity of $(n-1, n-1)$ -forms. Here the constant c is given by

$$c[\chi^k][\alpha^{n-k}] = \sum_{j=1}^{\ell} c_j[\chi^j][\alpha^{n-j}]. \quad (4.1.11)$$

Then the flow

$$\partial_t u = -\frac{\sum_{j=1}^{\ell} c_j \sigma_j(\lambda(A[u]))}{\sigma_k(\lambda(A[u]))} + c, \quad u(\cdot, 0) = 0, \quad (4.1.12)$$

admits a solution for all time which converges smoothly to a function u_∞ as $t \rightarrow \infty$. The form $\omega = \chi + i\partial\bar{\partial}u_\infty$ is k -positive and satisfies the equation

$$c\omega^k \wedge \alpha^{n-k} = \sum_{j=1}^{\ell} c_j \omega^j \wedge \alpha^{n-j}. \quad (4.1.13)$$

Regarding the condition (a) in Theorem 2, we note that natural geometric flows whose long-time behavior may be very sensitive to the initial data are appearing increasingly frequently in non-Kähler geometry. A prime example is the Anomaly flow, studied in [PPZ17a, PPZ16b, PPZ16c, PPZ17b, FHP17]. Finally, Theorem 4.1.3 will also be seen to imply as a corollary a theorem of Székelyhidi ([Sze15], Proposition 26), and the condition for solvability there will be seen to correspond to condition (a) in Theorem 4.1.3. This suggests in particular that some additional conditions for the convergence of the flow cannot be dispensed with altogether.

4.2 A Priori Estimates

4.2.1 C^0 Estimates

We begin with the C^0 estimates implied by the existence of a C -subsolution for the parabolic flow (4.1.1). One of the key results of [Sze15] was that the existence of a subsolution in the elliptic case implies a uniform bound for the oscillation of the unknown function u . In the parabolic case, we have only the following weaker estimate:

Lemma 4.2.1. *Assume that the equation (4.1.1) admits a parabolic C -solution on $X \times [0, T]$ in the sense of Definition 4.1.1, and that there exists a C^1 function $h(t)$ with $h'(t) \leq 0$ and*

$$\sup_X (u(\cdot, t) - \underline{u}(\cdot, t) - h(t)) \geq 0. \quad (4.2.1)$$

Then there exists a constant C depending only on $\chi, \alpha, \delta, \|u_0\|_{C^0}$, and $\|i\partial\bar{\partial}\underline{u}\|_{L^\infty}$ so that

$$u(\cdot, t) - \underline{u}(\cdot, t) - h(t) \geq -C \quad \text{for all } (z, t) \in X \times [0, T]. \quad (4.2.2)$$

Proof. First, note that by Lemma 4.3.1 proven later in Section 4.3, the function $\partial_t u$ is uniformly bounded for all time by a constant depending only on ψ and the initial data u_0 . Integrating this estimate on $[0, \delta]$ gives a bound for $|u|$ on $X \times [0, \delta]$ depending only on ψ, u_0 and δ . Thus we need only consider the range $t \geq \delta$. Next, the fact that \underline{u} is a parabolic subsolution and the condition that $h'(t) \leq 0$ imply that $\underline{u} + h(t)$ is a parabolic subsolution as well. So it suffices to prove the desired inequality with $h(t) = 0$, as long as the constants involved do not depend on $\partial_t \underline{u}$. Fix now any $T' < T$, and set for each $t, v = u - \underline{u}$, and

$$L = \min_{X \times [0, T']} v = v(z_0, t_0) \quad (4.2.3)$$

for some $(z_0, t_0) \in X \times [0, T']$. We shall show that L can be bounded from below by a constant depending only on the initial data u_0 and independent of T' . We can assume that $t_0 > 0$, otherwise we are already done. Let (z_1, \dots, z_n) be local holomorphic coordinates for X centered at z_0 , $U = \{z; |z| < 1\}$, and define the following function on the set $\mathcal{U} = U \times \{t; -\delta \leq 2(t - t_0) < \delta\}$,

$$w = v + \frac{\delta^2}{4}|z|^2 + |t - t_0|^2, \quad (4.2.4)$$

where $\delta > 0$ is the constant appearing in the definition of subsolutions. Clearly w attains its minimum on \mathcal{U} at (z_0, t_0) , and $w \geq \min_{\mathcal{U}} w + \frac{1}{4}\delta^2$ on the parabolic boundary of \mathcal{U} . We can thus apply the following parabolic version of the Alexandrov-Bakelman-Pucci inequality, due to K. Tso ([Tso85], Proposition 2.1, with the function u there set to $u = -w + \min_{\mathcal{U}} w + \frac{\delta^2}{4}$):

Let \mathcal{U} be the subset of \mathbf{R}^{2n+1} defined above, and let $w : \mathcal{U} \rightarrow \mathbf{R}$ be a smooth function which attains its minimum at $(0, t_0)$, and $w \geq \min_{\mathcal{U}} w + \frac{1}{4}\delta^2$ on the parabolic boundary of \mathcal{U} . Define the set

$$S := \left\{ (x, t) \in \mathcal{U} : \begin{array}{l} w(x, t) \leq w(z_0, t_0) + \frac{1}{4}\delta^2, \quad |D_x w(x, t)| < \frac{\delta^2}{8}, \text{ and} \\ w(y, s) \geq w(x, t) + D_x w(x, t) \cdot (y - x), \forall y \in U, s \leq t \end{array} \right\}. \quad (4.2.5)$$

Then there is a constant $C = C(n) > 0$ so that

$$C\delta^{4n+2} \leq \int_S (-w_t) \det(w_{ij}) dx dt.$$

Returning to the proof of Lemma 4.2.1, we claim that, on the set S , we have

$$|w_t| + \det(D_{jk}^2 w) \leq C \quad (4.2.6)$$

for some constant depending only on δ , and $\|i\partial\bar{\partial}\underline{u}\|_{L^\infty}$. Indeed, let

$$\mu = \lambda[u] - \lambda[\underline{u}], \quad \tau = -\partial_t u + \partial_t \underline{u}. \quad (4.2.7)$$

Along S , we have $D_{ij}^2 w \geq 0$ and $\partial_t w \leq 0$. In terms of μ and τ , this means that $\mu + \delta \mathbf{I} \in \Gamma_n$ and $0 \leq -\partial_t w = \tau - 2(t - t_0) \leq \tau + \delta$. The fact that u is a solution of the equation (4.1.1) can be expressed as

$$f(\lambda[\underline{u}] + \mu) - \partial_t \underline{u} + \tau = \psi(z). \quad (4.2.8)$$

Thus the condition that \underline{u} is a parabolic subsolution implies that $|\mu|$ and $|\tau|$ are bounded uniformly in (z, t) . Since along S , we have $\det(D_{ij}^2 w) \leq 2^n (\det(D_{kj}^2 w))^2$, it follows that both $|w_t|$ and $\det(D_{ij}^2 w)$ are bounded uniformly, as was to be shown.

Next, by the definition of the points (x, t) on S , we have $w(x, t) \leq L + \frac{\delta^2}{4}$. Since we can assume that $|L| > \delta^2$, it follows that $w < 0$ and $|w| \geq \frac{|L|}{2}$ on S . Thus we can write, in view of (4.2.6), for any $p > 0$,

$$C_n \delta^{4n+2} \leq C \int_S dx dt \leq \left(\frac{|L|}{2}\right)^{-p} \int_S |w(x, t)|^p dx dt \leq \left(\frac{|L|}{2}\right)^{-p} \int_{\mathcal{U}} |w(x, t)|^p dx dt. \quad (4.2.9)$$

Next write

$$\begin{aligned} |w| &= -w = -v - \frac{\delta^2}{4}|z|^2 - (t - t_0)^2 \leq -v \\ &\leq -v + \sup_X v \end{aligned} \quad (4.2.10)$$

since $\sup_X v \geq 0$ by the assumption (4.2.1). Since $\lambda[u] \in \Gamma$ and the cone Γ is convex, it follows that $\Delta u \geq -C$ and hence

$$\Delta(v - \sup_X v) = \Delta u - \Delta \underline{u} \geq -A \quad (4.2.11)$$

for some constant A depending only on χ , α , and $\|i\partial\bar{\partial}\underline{u}\|_{L^\infty}$. The Harnack inequality applied to the function $v - \sup_X v$, in the version provided by Proposition 10, [Sze15], implies that

$$\|v - \sup_X v\|_{L^p(X)} \leq C \quad (4.2.12)$$

for C depending only on (X, α) , A , and p . Substituting these bounds into (4.2.9) gives

$$C\delta^{4n+2} \leq \left(\frac{|L|}{2}\right)^{-p} \int_{|t| < \frac{1}{2}\delta} \|\sup_X v - v\|_{L^p(X)}^p dt \leq C'\delta \left(\frac{|L|}{2}\right)^{-p} \quad (4.2.13)$$

from which the desired bound for L follows. Q.E.D.

4.2.2 C^2 Estimates

In this section we prove an estimate for the complex Hessian of u in terms of the gradient. The original strategy goes back to the work of Chou-Wang [CW01], with adaptation to complex Hessian equations by Hou-Ma-Wu [HMW10], and to fully non-linear elliptic equations admitting a C -subsolution by Guan [Gua14] and Székelyhidi [Sze15]. Other adaptations to C^2 estimates can be found in [STW15], [PPZ15], [PPZ16a], [Zhe16]. We follow closely [Sze15].

Lemma 4.2.2. *Assume that the flow (4.1.1) admits a C -subsolution on $X \times [0, T)$. Then we have the following estimate*

$$|i\partial\bar{\partial}u| \leq \tilde{C}(1 + \sup_{X \times [0, T)} |\nabla u|_\alpha^2) \quad (4.2.14)$$

where \tilde{C} depends only on $\|\alpha\|_{C^2}$, $\|\psi\|_{C^2}$, $\|\chi\|_{C^2}$, $\|\tilde{u} - \underline{u}\|_{L^\infty}$, $\|\nabla \underline{u}\|_{L^\infty}$, $\|i\partial\bar{\partial}\underline{u}\|_{L^\infty}$, $\|\partial_t u\|_{L^\infty}$, $\|\partial_t(u - \underline{u})\|_{L^\infty}$, and the dimension n .

Proof. Let $\mathcal{L} = -\partial_t + F^{k\bar{k}}\nabla_k\nabla_{\bar{k}}$. Denote $g = \chi + i\partial\bar{\partial}u$, then $A[u]^{k_j} = \alpha^{k\bar{p}}g_{\bar{p}j}$. We would like to apply the maximum principle to the function

$$G = \log \lambda_1 + \phi(|\nabla u|^2) + \varphi(\tilde{v}) \quad (4.2.15)$$

where $v = u - \underline{u}$, \tilde{v} is the normalization of v , $\lambda_1 : X \rightarrow \mathbf{R}$ is the largest eigenvalue of the matrix $A[u]$ at each point, and the functions ϕ and φ will be specified below. Since the eigenvalues of $A[u]$ may not be distinct, we perturb $A[u]$ following the technique of [Sze15],

Proposition 13. Thus assume that G attains its maximum on $X \times [0, T']$ at some (z_0, t_0) , with $t_0 > 0$. We choose local complex coordinates, so that z_0 corresponds to 0, and $A[u]$ is diagonal at 0 with eigenvalues $\lambda_1 \geq \dots \geq \lambda_n$. Let $B = (B^i_j)$ be a diagonal matrix with $0 = B^1_1 < B^2_2 < \dots < B^n_n$ and small constant entries, and set $\tilde{A} = A - B$. Then at the origin \tilde{A} has eigenvalues $\tilde{\lambda}_1 = \lambda_1$, $\tilde{\lambda}_i = \lambda_i - B^i_i < \tilde{\lambda}_1$ for all $i > 1$.

Since all the eigenvalues of \tilde{A} are distinct, we can define near 0 the following smooth function \tilde{G} ,

$$\tilde{G} = \log \tilde{\lambda}_1 + \phi(|\nabla u|^2) + \varphi(\tilde{v}) \quad (4.2.16)$$

where

$$\phi(t) = -\frac{1}{2} \log\left(1 - \frac{t}{2P}\right), \quad P = \sup_{X \times [0, T']} (|\nabla u|^2 + 1) \quad (4.2.17)$$

and, following [STW15]

$$\varphi(t) = D_1 e^{-D_2 t} \quad (4.2.18)$$

for some large constants D_1, D_2 to be chosen later. Note that

$$\frac{1}{4P} \leq \phi' \leq \frac{1}{2P}, \quad \phi'' = 2(\phi')^2 > 0. \quad (4.2.19)$$

The norm $|\nabla u|^2$ is taken with respect to the fixed Hermitian metric α on X , and we shall compute using covariant derivatives ∇ with respect to α . Since the matrix B^j_m is constant in a neighborhood of 0 and since we are using the Chern unitary connection, we have $\nabla_{\bar{k}} B^j_m = 0$. Our conventions for the curvature and torsion tensors of a Hermitian metric α are as follows,

$$[\nabla_\beta, \nabla_\alpha]V^\gamma = R_{\alpha\beta}{}^\gamma{}_\delta V^\delta + T^\delta{}_{\alpha\beta} \nabla_\delta V^\gamma. \quad (4.2.20)$$

We also set

$$\mathcal{F} = \sum_i f_i(\lambda[u]). \quad (4.2.21)$$

An important observation is that there exists a constant C_1 , depending only on $\|\psi\|_{L^\infty(X)}$ and $\|\partial_t u\|_{L^\infty(X \times [0, T])}$ so that

$$\mathcal{F} \geq C_1. \quad (4.2.22)$$

Indeed it follows from the properties of the cone Γ that $\sum_i f_i(\lambda) \geq C(\sigma)$ for each fixed σ and $\lambda \in \Gamma^\sigma$. When $\lambda = \lambda[u]$, σ must lie in the range of $\partial_t u + \psi$, which is a compact set bounded by $\|\partial_t u\|_{L^\infty(X \times [0, T])} + \|\psi\|_{L^\infty(X)}$, hence our claim.

Estimate of $\mathcal{L}(\log \tilde{\lambda}_1)$

Clearly

$$\mathcal{L} \log \tilde{\lambda}_1 = \frac{1}{\lambda_1} (F^{k\bar{k}} \tilde{\lambda}_{1,\bar{k}k} - \partial_t \tilde{\lambda}_1) - F^{k\bar{k}} \frac{|\tilde{\lambda}_{1,\bar{k}}|^2}{\lambda_1^2}. \quad (4.2.23)$$

We work out the term $F^{k\bar{k}} \tilde{\lambda}_{1,\bar{k}k} - \partial_t \tilde{\lambda}_1$ using the flow. The usual differentiation rules ([Spr05]) readily give

$$\tilde{\lambda}_{1,\bar{k}} = \nabla_{\bar{k}} g_{\bar{1}1} \quad (4.2.24)$$

and

$$\tilde{\lambda}_{1,\bar{k}k} = \nabla_k \nabla_{\bar{k}} g_{\bar{1}1} + \sum_{p>1} \frac{|\nabla_{\bar{k}} g_{\bar{p}1}|^2 + |\nabla_{\bar{k}} g_{\bar{1}p}|^2}{\lambda_1 - \tilde{\lambda}_p} - \sum_{p>1} \frac{\nabla_k B^1_p \nabla_{\bar{k}} g_{\bar{p}1} + \nabla_k B^{p1} \nabla_{\bar{k}} g_{\bar{1}p}}{\lambda_1 - \tilde{\lambda}_p}. \quad (4.2.25)$$

while it follows from the flow that

$$\partial_t \tilde{\lambda}_1 = \partial_t u_{\bar{1}1} = F^{l\bar{k},s\bar{r}} \nabla_{\bar{1}} g_{\bar{k}l} \nabla_1 g_{\bar{r}s} + F^{k\bar{k}} \nabla_1 \nabla_{\bar{1}} g_{\bar{k}k} - \psi_{\bar{1}1}. \quad (4.2.26)$$

Thus

$$\begin{aligned} F^{k\bar{k}} \tilde{\lambda}_{1,\bar{k}k} - \partial_t \tilde{\lambda}_1 &= F^{k\bar{k}} (\nabla_k \nabla_{\bar{k}} g_{\bar{1}1} - \nabla_1 \nabla_{\bar{1}} g_{\bar{k}k}) + F^{l\bar{k},s\bar{r}} \nabla_{\bar{1}} g_{\bar{k}l} \nabla_1 g_{\bar{r}s} - \psi_{\bar{1}1} \\ &\quad + F^{k\bar{k}} \sum_{p>1} \left\{ \frac{|\nabla_{\bar{k}} g_{\bar{p}1}|^2 + |\nabla_{\bar{k}} g_{\bar{1}p}|^2}{\lambda_1 - \tilde{\lambda}_p} - \frac{\nabla_k B^1_p \nabla_{\bar{k}} g_{\bar{p}1} + \nabla_k B^{p1} \nabla_{\bar{k}} g_{\bar{1}p}}{\lambda_1 - \tilde{\lambda}_p} \right\} \end{aligned}$$

A simple computation gives

$$\begin{aligned} \nabla_k \nabla_{\bar{k}} g_{\bar{1}1} - \nabla_1 \nabla_{\bar{1}} g_{\bar{k}k} &= -2\Re(T_{k\bar{1}}^p \nabla_{\bar{k}} g_{\bar{p}1}) + T \star \nabla \chi + R \star \nabla \bar{\nabla} u + T \star T \star \nabla \bar{\nabla} u \\ &\geq -2\Re(T_{k\bar{1}}^p \nabla_{\bar{k}} g_{\bar{p}1}) - C_2(\lambda_1 + 1), \end{aligned} \quad (4.2.27)$$

where C_2 depending only on $\|\alpha\|_{C^2}$ and $\|\chi\|_{C^2}$. We also have

$$\begin{aligned} &\sum_{p>1} \left\{ \frac{|\nabla_{\bar{k}} g_{\bar{p}1}|^2 + |\nabla_{\bar{k}} g_{\bar{1}p}|^2}{\lambda_1 - \tilde{\lambda}_p} - \frac{\nabla_k B^1_p \nabla_{\bar{k}} g_{\bar{p}1} + \nabla_k B^{p1} \nabla_{\bar{k}} g_{\bar{1}p}}{\lambda_1 - \tilde{\lambda}_p} \right\} \\ &\geq \frac{1}{2} \sum_{p>1} \frac{|\nabla_{\bar{k}} g_{\bar{p}1}|^2 + |\nabla_{\bar{k}} g_{\bar{1}p}|^2}{\lambda_1 - \tilde{\lambda}_p} - C_3 \geq \frac{1}{2(n\lambda_1 + 1)} \sum_{p>1} |\nabla_{\bar{k}} g_{\bar{p}1}|^2 + |\nabla_{\bar{k}} g_{\bar{1}p}|^2 - C_3, \end{aligned} \quad (4.2.28)$$

where C_3 only depends on the dimension n , and the second inequality is due to the fact that $(\lambda_1 - \tilde{\lambda}_p)^{-1} \geq (n\lambda_1 + 1)^{-1}$, which follows itself from the fact that $\sum_i \lambda_i \geq 0$ and B was chosen to be small. Thus

$$\begin{aligned} & \nabla_k \nabla_{\bar{k}} g_{\bar{1}1} - \nabla_1 \nabla_{\bar{1}} g_{\bar{k}k} + \sum_{p>1} \left\{ \frac{|\nabla_{\bar{k}} g_{\bar{p}1}|^2 + |\nabla_{\bar{k}} g_{\bar{1}p}|^2}{\lambda_1 - \tilde{\lambda}_p} - \frac{\nabla_k B^1_p \nabla_{\bar{k}} g_{\bar{p}1} + \nabla_k B^p_1 \nabla_{\bar{k}} g_{\bar{1}p}}{\lambda_1 - \tilde{\lambda}_p} \right\} \\ & \geq -2\Re(T_{k\bar{1}}^p \nabla_{\bar{k}} g_{\bar{p}1}) + \frac{1}{2(n\lambda_1 + 1)} \sum_{p>1} |\nabla_{\bar{k}} g_{\bar{p}1}|^2 + |\nabla_{\bar{k}} g_{\bar{1}p}|^2 - C_2(\lambda_1 + 1) - C_3 \\ & \geq -C_4 |\nabla_{\bar{k}} g_{\bar{1}1}| - C_5 \lambda_1 - C_6 \end{aligned} \quad (4.2.29)$$

where we have used the positive terms to absorb all the terms $T_{k\bar{1}}^p \nabla_{\bar{k}} g_{\bar{p}1}$, except for $T_{k\bar{1}}^1 \nabla_{\bar{k}} g_{\bar{1}1}$ and C_4, C_5, C_6 only depend on $\|\alpha\|_{C^2}, \|\chi\|_{C^2}, n$. Altogether,

$$F^{k\bar{k}} \tilde{\lambda}_{1,\bar{k}k} - \partial_t \tilde{\lambda}_1 \geq -C_4 F^{k\bar{k}} |\nabla_{\bar{k}} g_{\bar{1}1}| + F^{l\bar{k},s\bar{r}} \nabla_{\bar{1}} g_{\bar{k}l} \nabla_1 g_{\bar{r}s} - \psi_{\bar{1}1} - C_5 \mathcal{F} \lambda_1 - C_6 \mathcal{F} \quad (4.2.30)$$

and we find

$$\mathcal{L} \log \tilde{\lambda}_1 \geq -F^{k\bar{k}} \frac{|\tilde{\lambda}_{1,\bar{k}k}|^2}{\lambda_1^2} - \frac{1}{\lambda_1} F^{l\bar{k},s\bar{r}} \nabla_{\bar{1}} g_{\bar{k}l} \nabla_1 g_{\bar{r}s} - C_4 \frac{1}{\lambda_1} F^{k\bar{k}} |\nabla_{\bar{k}} g_{\bar{1}1}| - C_7 \mathcal{F} \quad (4.2.31)$$

where we have bounded $\psi_{\bar{1}1}$ by a constant that can be absorbed in $C_6 \mathcal{F} / \lambda_1 \leq C_6 \mathcal{F}$, since $\lambda_1 \geq 1$ by assumption, and \mathcal{F} is bounded below by a constant depending on $\|\psi\|_{L^\infty}$ and $\|\partial_t u\|_{L^\infty}$. The constant C_7 thus only depends on $\|\alpha\|_{C^2}, \|\chi\|_{C^2}, n, \|\partial_t u\|_{L^\infty}$ and $\|\psi\|_{C^2}$. In view of (4.2.24), this can also be rewritten as

$$\mathcal{L} \log \tilde{\lambda}_1 \geq -F^{k\bar{k}} \frac{|\tilde{\lambda}_{1,\bar{k}k}|^2}{\lambda_1^2} - \frac{1}{\lambda_1} F^{l\bar{k},s\bar{r}} \nabla_{\bar{1}} g_{\bar{k}l} \nabla_1 g_{\bar{r}s} - C_4 \frac{1}{\lambda_1} F^{k\bar{k}} |\tilde{\lambda}_{1,\bar{k}k}| - C_7 \mathcal{F}. \quad (4.2.32)$$

Estimate for $\mathcal{L}\phi(|\nabla u|^2)$

Next, a direct calculation gives

$$\begin{aligned} \mathcal{L}\phi(|\nabla u|^2) &= \phi'(F^{q\bar{q}} \nabla_q \nabla_{\bar{q}} - \partial_t) |\nabla u|^2 + \phi'' F^{q\bar{q}} \nabla_q |\nabla u|^2 \nabla_{\bar{q}} |\nabla u|^2 \\ &= \phi' \{ \nabla^j u (F^{q\bar{q}} \nabla_q \nabla_{\bar{q}} - \partial_t) \nabla_j u + \nabla^{\bar{j}} u (F^{q\bar{q}} \nabla_q \nabla_{\bar{q}} - \partial_t) \nabla_{\bar{j}} u \} \\ &\quad + \phi' F^{q\bar{q}} (|\nabla_q \nabla u|^2 + |\nabla_q \bar{\nabla} u|^2) + \phi'' F^{q\bar{q}} \nabla_q |\nabla u|^2 \nabla_{\bar{q}} |\nabla u|^2. \end{aligned} \quad (4.2.33)$$

In view of the flow, we have

$$\nabla_j \partial_t u = F^{k\bar{k}} \nabla_j g_{\bar{k}k} - \psi_j, \quad \nabla_{\bar{j}} \partial_t u = F^{k\bar{k}} \nabla_{\bar{j}} g_{\bar{k}k} - \psi_{\bar{j}}. \quad (4.2.34)$$

It follows that

$$\begin{aligned} (F^{k\bar{k}} \nabla_k \nabla_{\bar{k}} - \partial_t) \nabla_{\bar{j}} u &= F^{k\bar{k}} (\nabla_k \nabla_{\bar{k}} u_{\bar{j}} - \nabla_{\bar{j}} g_{\bar{k}k}) + \psi_{\bar{j}} \\ &= F^{k\bar{k}} (-\nabla_{\bar{j}} \chi_{\bar{k}k} + \overline{T_{kj}^p} \nabla_{\bar{j}} \nabla_k u + R_{\bar{j}k}^{\bar{m}} \nabla_{\bar{k}} \nabla_{\bar{m}} u) + \psi_{\bar{j}} \end{aligned} \quad (4.2.35)$$

and hence, for small ε , there is a constant $C_8 > 0$ depending only on $\varepsilon, \|\chi\|_{C^2}, \|\alpha\|_{C^2}$ and $\|\psi\|_{C^2}$ such that

$$\phi' \nabla^{\bar{j}} u (F^{q\bar{q}} \nabla_q \nabla_{\bar{q}} - \partial_t) \nabla_{\bar{j}} u \geq -C_8 \mathcal{F} - \frac{\varepsilon}{P} F^{q\bar{q}} (|\nabla_q \nabla u|^2 + |\nabla_q \bar{\nabla} u|^2) \quad (4.2.36)$$

since we can assume that $\lambda_1 \gg P = \sup_{X \times [0, T^*]} (|\nabla u|^2 + 1)$ (otherwise the desired estimate $\lambda_1 < CP$ already holds), and $(4P)^{-1} < \phi' < (2P)^{-1}$. Similarly we obtain the same estimate for $\phi' \nabla^j u (F^{q\bar{q}} \nabla_q \nabla_{\bar{q}} - \partial_t) \nabla_j u$. Thus by choosing $\varepsilon = 1/24$, we have

$$\mathcal{L}\phi(|\nabla u|^2) \geq -C_8 \mathcal{F} + \frac{1}{8P} F^{q\bar{q}} (|\nabla_q \nabla u|^2 + |\nabla_q \bar{\nabla} u|^2) + \phi'' F^{q\bar{q}} \nabla_q |\nabla u|^2 \nabla_{\bar{q}} |\nabla u|^2. \quad (4.2.37)$$

Estimate for $\mathcal{L}\tilde{G}$

The evaluation of the remaining term $\mathcal{L}\varphi(\tilde{v})$ is straightforward,

$$\mathcal{L}\varphi(\tilde{v}) = \varphi'(\tilde{v})(F^{k\bar{k}} \nabla_k \nabla_{\bar{k}} \tilde{v} - \partial_t \tilde{v}) + \varphi''(\tilde{v}) F^{k\bar{k}} \nabla_k \tilde{v} \nabla_{\bar{k}} \tilde{v}. \quad (4.2.38)$$

Altogether, we have established the following lower bound for $\mathcal{L}\tilde{G}$,

$$\begin{aligned} \mathcal{L}\tilde{G} &\geq -F^{k\bar{k}} \frac{|\tilde{\lambda}_{1,\bar{k}}|^2}{\lambda_1^2} - \frac{1}{\lambda_1} F^{l\bar{k}, s\bar{r}} \nabla_{\bar{l}} g_{\bar{k}l} \nabla_{\bar{s}} g_{\bar{r}s} - C_4 \frac{1}{\lambda_1} F^{k\bar{k}} |\lambda_{1,\bar{k}}| - C_9 \mathcal{F} \\ &\quad + \frac{1}{8P} F^{q\bar{q}} (|\nabla_q \nabla u|^2 + |\nabla_q \bar{\nabla} u|^2) + \phi'' F^{q\bar{q}} \nabla_q |\nabla u|^2 \nabla_{\bar{q}} |\nabla u|^2 \\ &\quad + \varphi'(\tilde{v})(F^{k\bar{k}} \nabla_k \nabla_{\bar{k}} \tilde{v} - \partial_t \tilde{v}) + \varphi''(\tilde{v}) F^{k\bar{k}} \nabla_k \tilde{v} \nabla_{\bar{k}} \tilde{v}, \end{aligned} \quad (4.2.39)$$

where C_4 and C_9 only depend on $\|\chi\|_{C^2}, \|\alpha\|_{C^2}, \|\psi\|_{C^2}, \|\partial_t u\|_{L^\infty}$ and the dimension n .

For a small $\theta > 0$ to be chosen hereafter, we deal with two following cases.

Case 1: $\theta\lambda_1 \leq -\lambda_n$

In this case, we have $\theta^2 \lambda_1^2 \leq \lambda_n^2$. Thus we can write

$$\begin{aligned} \frac{1}{8P} F^{q\bar{q}} (|\nabla_q \nabla u|^2 + |\nabla_q \bar{\nabla} u|^2) &\geq \frac{F^{n\bar{n}}}{8P} |u_{\bar{n}n}|^2 = \frac{F^{n\bar{n}}}{8P} |\lambda_n - \chi_{\bar{n}n}|^2 \geq \frac{\mathcal{F} \lambda_n^2}{10nP} - \frac{C_{10} \mathcal{F}}{P} \\ &\geq \frac{\theta^2}{10nP} \mathcal{F} \lambda_1^2 - C_{10} \mathcal{F}, \end{aligned} \quad (4.2.40)$$

where C_{10} only depends on $\|\chi\|_{C^2}$. Next, it is convenient to combine the first and third terms in the expression for $\mathcal{L}\tilde{G}$,

$$-F^{k\bar{k}} \frac{|\tilde{\lambda}_{1,\bar{k}}|^2}{\lambda_1^2} - C_4 \frac{1}{\lambda_1} F^{k\bar{k}} |\tilde{\lambda}_{1,\bar{k}}| \geq -\frac{3}{2} F^{k\bar{k}} \frac{|\tilde{\lambda}_{1,\bar{k}}|^2}{\lambda_1^2} - C_{11} \mathcal{F}. \quad (4.2.41)$$

where C_{11} only depends on C_4 .

At a maximum point for \tilde{G} , we have $0 \geq \mathcal{L}\tilde{G}$. Combining the lower bound (4.2.39) for $\mathcal{L}\tilde{G}$ with the preceding inequalities and dropping the second and last terms, which are non-negative, we obtain

$$0 \geq \frac{\theta^2}{10nP} \mathcal{F} \lambda_1^2 - C_{12} \mathcal{F} - \frac{3}{2} F^{k\bar{k}} \frac{|\tilde{\lambda}_{1,\bar{k}}|^2}{\lambda_1^2} + \phi'' F^{q\bar{q}} |\nabla_{\bar{q}} |\nabla u|^2|^2 + \varphi'(\tilde{v})(F^{k\bar{k}} \nabla_k \nabla_{\bar{k}} \tilde{v} - \partial_t \tilde{v}), \quad (4.2.42)$$

where $C_{12} = C_9 + C_{10} + C_{11}$, depending on $\|\chi\|_{C^2}$, $\|\alpha\|_{C^2}$, $\|\psi\|_{C^2}$, $\|\partial_t u\|_{L^\infty}$ and n . Since we are at a critical point of \tilde{G} , we also have $\nabla \tilde{G} = 0$, and hence

$$\frac{\tilde{\lambda}_{1,\bar{k}}}{\lambda_1} + \phi' \nabla_{\bar{k}} |\nabla u|^2 + \varphi' \partial_{\bar{k}} \tilde{v} = 0 \quad (4.2.43)$$

which implies

$$\begin{aligned} \frac{3}{2} F^{k\bar{k}} \left| \frac{\tilde{\lambda}_{1,\bar{k}}}{\lambda_1} \right|^2 &= \frac{3}{2} F^{k\bar{k}} |\phi' \nabla_{\bar{k}} |\nabla u|^2 + \varphi' \partial_{\bar{k}} \tilde{v}|^2 \leq 2F^{k\bar{k}} (\phi')^2 |\nabla_{\bar{k}} |\nabla u|^2|^2 + 4F^{k\bar{k}} (\varphi')^2 |\nabla_{\bar{k}} \tilde{v}|^2 \\ &\leq F^{k\bar{k}} \phi'' |\nabla_{\bar{k}} |\nabla u|^2|^2 + C_{13} \mathcal{F} P, \end{aligned} \quad (4.2.44)$$

where C_{13} depending on $\|\tilde{v}\|_{L^\infty}$ and $\|\nabla \underline{u}\|_{L^\infty}$. Since $\varphi'(\tilde{v})$ is bounded in terms of $\|\tilde{v}\|_{L^\infty}$ and $\|\nabla \underline{u}\|_{L^\infty}$, and $|F^{k\bar{k}} \nabla_k \nabla_{\bar{k}} \tilde{v} - \partial_t \tilde{v}| \leq C_{14} \mathcal{F} \lambda_1 + C_{13}$, where C_{14} depending on $\|\partial_t v\|_{L^\infty}$ and $\|\partial \partial \underline{u}\|_{L^\infty}$, we arrive at

$$0 \geq \frac{\theta^2}{10nP} \mathcal{F} \lambda_1^2 - C_{15} P \mathcal{F}, \quad (4.2.45)$$

where C_{15} depends on $\|\chi\|_{C^2}$, $\|\alpha\|_{C^2}$, n , $\|\psi\|_{C^2}$, $\|\partial \partial \underline{u}\|_{L^\infty}$, $\|\nabla \underline{u}\|_{L^\infty}$, $\|\tilde{v}\|_{L^\infty}$, $\|\partial_t v\|_{L^\infty}$ and $\|\partial_t u\|_{L^\infty}$. This implies the desired estimate $\lambda_1 \leq \tilde{C} P$.

The key estimate provided by subsolutions

In the second case when $\theta \lambda_1 > -\lambda_n$, we need to use the following key property of subsolutions.

Lemma 4.2.3. *Let \underline{u} be a subsolution of the equation (4.1.1) in the sense of Definition 4.1.1 with the pair (δ, K) . Then there exists a constant $C = C(\delta, K)$, so that, if $|\lambda[\underline{u}] - \lambda[\underline{u}]| > K$ with K in Definition 4.1.1, then either*

$$F^{pq}(A[\underline{u}]) (A^p_q[\underline{u}] - A^p_q[u]) - (\partial_t \underline{u} - \partial_t u) > C \mathcal{F} \quad (4.2.46)$$

or we have for any $1 \leq i \leq n$,

$$F^{ii}(A[\underline{u}]) > C \mathcal{F}. \quad (4.2.47)$$

Proof. The proof is an adaptation of the one for the elliptic version [Sze15, Proposition 6](see also [Gua14] for a similar argument). However, because of the time parameter t which may tend to ∞ , we need to produce explicit bounds which are independent of t . As in [Sze15], it suffices to prove that

$$\sum_{i=1}^n f_i(\lambda[u])(\lambda_i[\underline{u}] - \lambda_i[u]) - (\partial_t \underline{u} - \partial_t u) > C\mathcal{F}. \quad (4.2.48)$$

For any $(z_0, t_0) \in X \times [0, T']$, since \underline{u} is a C -subsolution as in Definition 4.1.1, the set

$$A_{z_0, t_0} = \{(w, s) \mid w + \frac{\delta}{2}I \in \bar{\Gamma}_n, s \geq -\delta, f(\lambda[\underline{u}(z_0, t_0)] + w) - \partial_t \underline{u}(z_0, t_0) + s \leq \psi(z_0)\}$$

is compact, and $A_{z_0, t_0} \subset B_{n+1}(0, K)$. For any $(w, s) \in A_{z_0, t_0}$, then the set

$$C_{w, s} = \{v \in \mathbb{R}^n \mid \exists r > 0, w + rv \in -\delta I + \Gamma_n, f(\lambda[\underline{u}(z_0, t_0)] + w + rv) - \partial_t \underline{u}(z_0, t_0) + s = \psi(z_0)\}$$

is a cone with vertex at the origin.

We claim that $C_{w, s}$ is strictly larger than Γ_n . Indeed, for any $v \in \Gamma_n$, we can choose $r > 0$ large enough so that $|w + rv| > K$, then by the definition of C -subsolution, at (z_0, t_0)

$$f(\lambda[\underline{u}] + w + rv) - \partial_t \underline{u} + s > \psi(z_0).$$

Therefore there exist $r' > 0$ such that $f(\lambda[\underline{u}] + w + r'v) - \partial_t \underline{u} + s = \psi(z_0)$, hence $v \in C_{w, s}$. This implies that $\Gamma_n \subset C_{w, s}$. Now, for any pair (i, j) with $i \neq j$ and $i, j = 1, \dots, n$, we choose $v^{(i, j)} := (v_1, \dots, v_n)$ with $v_i = K + \delta$ and $v_j = -\delta/3$ and $v_k = 0$ for $k \neq i, j$, then we have $w + v^{(i, j)} \in -\delta I + \Gamma_n$. By the definition of C -subsolution, we also have, at (z_0, t_0)

$$f(\lambda[\underline{u}] + w + v^{(i, j)}) - \partial_t \underline{u} + s > \psi(z_0),$$

hence $v^{(i, j)} \in C_{w, s}$ for any pair (i, j) .

Denote by $C_{w, s}^*$ the dual cone of $C_{w, s}$,

$$C_{w, s}^* = \{x \in \mathbb{R}^n : \langle x, y \rangle > 0, \forall y \in C_{w, s}\}.$$

We now prove that there is an $\varepsilon > 0$ such that if $x = (x_1, \dots, x_n) \in C_{w, s}^*$ is a unit vector, then $x_i > \varepsilon, \forall i = 1, \dots, n$. First we remark that $x_i > 0, \forall i = 1, n$ since $\Gamma_n \subset C_{w, s}$. Suppose that x_1 is the smallest element between x_i , then $\langle x, v^{(1, j)} \rangle > 0$, implies that $(K + \delta)x_1 \geq \frac{\delta}{3}x_j$, hence $(K + \delta)^2 x_1^2 \geq (\delta^2/9)x_j^2, \forall j = 2, \dots, n$, so $n(K + \delta)^2 x_1^2 \geq \delta^2/9$. Therefore we can choose $\varepsilon = \frac{\delta^2}{9n(K + \delta)^2}$.

Fix $(z_1, t_1) \in X \times [0, T']$ such that at this point $|\lambda[u] - \lambda[\underline{u}]| > K$. Let \mathcal{T} be the tangent plane to $\{(\lambda, \tau) \mid f(\lambda) + \tau = \sigma\}$ at $(\lambda[u(z_1, t_1)], -\partial_t u(z_1, t_1))$. There are two cases:

1) There is some point $(w, s) \in A_{z_1, t_1}$ such that at (z_1, t_1)

$$(\lambda[\underline{u}] + w, -\partial_t \underline{u} + s) \in \mathcal{T},$$

i.e

$$\nabla f(\lambda[u]).(\lambda[\underline{u}] + w - \lambda[u]) + (-\partial_t \underline{u} + s + \partial_t u) = 0. \quad (4.2.49)$$

Now for any $v \in C_{w, s}$, there exist $r > 0$ such that $f(\lambda[\underline{u}] + w + rv) - \partial_t \underline{u} + s = \psi(z)$, this implies that

$$\nabla f(\lambda[u]).(\lambda[\underline{u}] + w + rv - \lambda[u]) + (-\partial_t \underline{u} + s + \partial_t u) > 0,$$

so combing with (4.2.50) we get

$$\nabla f(\lambda[u]).v > 0.$$

It follows that at (z_1, t_1) we have $\nabla f(\lambda[u])(z, t) \in C_{w, s}^*$, so $f_i(\lambda[u]) \geq \varepsilon \nabla f(\lambda[u]), \forall i = 1, \dots, n$, hence

$$f_i(\lambda[u]) > \frac{\varepsilon}{\sqrt{n}} \sum_p f_p(\lambda[u]), \forall i = 1, \dots, n,$$

where

$$\varepsilon = \frac{\delta^2}{9n(K + \delta)^2}.$$

2) Otherwise, we observe that if $A_{z_1, t_1} \neq \emptyset$, then $(w_0, s_0) = (-\delta/2, \dots, -\delta/2, -\delta) \in A_{z_1, t_1}$ and at (z_1, t_1) , $(\lambda[\underline{u}] - w_0, -\underline{u}_t + s_0)$ must lie above \mathcal{T} in the sense that

$$(\nabla f(\lambda[u]), 1).(\lambda[\underline{u}] + w_0 - \lambda[u], -\partial_t \underline{u} + s_0 + \partial_t u) > 0, \text{ at } (z_1, t_1). \quad (4.2.50)$$

Indeed, if it is not the case, using the monotonicity of f we can find $v \in \Gamma_n$ such that $(\lambda[\underline{u}] + w_0 + v, -\partial_t \underline{u} + s_0) \in \mathcal{T}$, so the concavity of $(\lambda, \tau) \mapsto f(\lambda) + \tau$ implies that $(w_0 + v, s_0)$ is in A_{z_1, t_1} and then satisfies the first case, this gives a contradiction. Now it follows from (4.2.50) that at (z_1, t_1)

$$\begin{aligned} (\nabla f(\lambda[u]), 1).(\lambda[\underline{u}] - \lambda[u], -\partial_t \underline{u} + \partial_t u) &\geq -\nabla f(\lambda[u]).w_0 - s_0 \\ &= (\delta/2)\mathcal{F} + \delta \geq (\delta/2)\mathcal{F}, \end{aligned}$$

where $\mathcal{F} = \sum_i f_i(\lambda[u]) > 0$. This means

$$\sum_{i=1}^n f_i(\lambda[u])(\lambda[\underline{u}] - \lambda[u]) - (\partial_t \underline{u} - u_t) > (\delta/2)\mathcal{F} \quad (4.2.51)$$

as required.

Now if $A_{z_1, t_1} = \emptyset$, then at (z_1, t_1)

$$f(\lambda[\underline{u}] + w_0) - \partial_t \underline{u} + s_0 > \psi(z_1),$$

hence we also have that $(\lambda[\underline{u}] + w_0, -\partial_t \underline{u} + s_0)$ lies above \mathcal{T} using the concavity of $(\lambda, \tau) \mapsto f(\lambda) + \tau$. By the same argument above, we also obtain the inequality (4.2.51).

So we get the desired inequalities. Q.E.D.

Case 2: $\theta\lambda_1 > -\lambda_n$

Set

$$I = \{i; F^{i\bar{i}} \geq \theta^{-1}F^{1\bar{1}}\}. \quad (4.2.52)$$

At the maximum point $\partial_{\bar{k}}\tilde{G} = 0$, and we can write

$$\begin{aligned} -\sum_{k \notin I} F^{k\bar{k}} \frac{|\tilde{\lambda}_{1,\bar{k}}|^2}{\lambda_1^2} &= -\sum_{k \notin I} F^{k\bar{k}} |\phi' \nabla_{\bar{k}} |\nabla u|^2 + \phi' \partial_{\bar{k}} \tilde{v}|^2 \\ &\geq -2(\phi')^2 \sum_{k \notin I} F^{k\bar{k}} |\nabla_{\bar{k}} |\nabla u|^2|^2 - 2(\phi')^2 \sum_{k \notin I} F^{k\bar{k}} |\nabla_{\bar{k}} \tilde{v}|^2 \\ &\geq -\phi'' \sum_{k \notin I} F^{k\bar{k}} |\nabla_{\bar{k}} |\nabla u|^2|^2 - 2(\phi')^2 \theta^{-1} F^{1\bar{1}} P - C_{16} \mathcal{F}, \end{aligned} \quad (4.2.53)$$

where C_{16} depends on $\|\nabla u\|_{L^\infty}$ and $\|\tilde{v}\|_{L^\infty}$. On the other hand,

$$-2\theta \sum_{k \in I} F^{k\bar{k}} \frac{|\tilde{\lambda}_{1,\bar{k}}|^2}{\lambda_1^2} \geq -2\theta\phi'' \sum_{k \in I} F^{k\bar{k}} |\nabla_{\bar{k}} |\nabla u|^2|^2 - 4\theta(\phi')^2 \sum_{k \in I} F^{k\bar{k}} |\nabla_{\bar{k}} \tilde{v}|^2. \quad (4.2.54)$$

Choose $0 < \theta \ll 1$ such that $4\theta(\phi')^2 \leq \frac{1}{2}\phi''$. Then (4.2.39) implies that

$$\begin{aligned} 0 &\geq -\frac{1}{\lambda_1} F^{l\bar{k},s\bar{r}} \nabla_{\bar{1}} g_{\bar{k}l} \nabla_{\bar{1}} g_{\bar{r}s} - (1-2\theta) \sum_{k \in I} F^{k\bar{k}} \frac{|\tilde{\lambda}_{1,\bar{k}}|^2}{\lambda_1^2} \\ &\quad - C \frac{1}{\lambda_1} F^{k\bar{k}} |\tilde{\lambda}_{1,\bar{k}}| + \frac{1}{8P} F^{q\bar{q}} (|\nabla_q \nabla u|^2 + |\nabla_q \bar{\nabla} u|^2) \\ &\quad + \frac{1}{2} \phi'' F^{k\bar{k}} |\nabla_{\bar{k}} \tilde{v}|^2 + \phi' (F^{k\bar{k}} \nabla_k \nabla_{\bar{k}} \tilde{v} - \partial_t \tilde{v}) - 2(\phi')^2 \theta^{-1} F^{1\bar{1}} P - C_{17} \mathcal{F}, \end{aligned} \quad (4.2.55)$$

where C_{17} depend on $\|\chi\|_{C^2}$, $\|\alpha\|_{C^2}$, n , $\|\psi\|_{C^2}$, $\|\partial_t u\|_{L^\infty}$, $\|\tilde{v}\|_{L^\infty}$ and $\|\nabla u\|_{L^\infty}$. The concavity of F implies that

$$F^{l\bar{k},s\bar{r}} \nabla_{\bar{1}} g_{\bar{k}l} \nabla_{\bar{1}} g_{\bar{r}s} \leq \sum_{k \in I} \frac{F^{1\bar{1}} - F^{k\bar{k}}}{\lambda_1 - \lambda_k} |\nabla_{\bar{1}} g_{\bar{1}k}|^2 \quad (4.2.56)$$

since $\frac{F^{1\bar{1}} - F^{k\bar{k}}}{\lambda_1 - \lambda_k} \leq 0$. Moreover, for $k \in I$, we have $F^{1\bar{1}} \leq \theta F^{k\bar{k}}$, and the assumption $\theta\lambda_1 \geq -\lambda_n$ yields

$$\frac{1-\theta}{\lambda_1 - \lambda_k} \geq \frac{1-2\theta}{\lambda_1}. \quad (4.2.57)$$

It follows that

$$\sum_{k \in I} \frac{F^{1\bar{1}} - F^{k\bar{k}}}{\lambda_1 - \lambda_k} |\nabla_{\bar{1}} g_{\bar{1}k}|^2 \leq -\sum_{k \in I} \frac{(1-\theta)F^{k\bar{k}}}{\lambda_1 - \lambda_k} |\nabla_{\bar{1}} g_{\bar{1}k}|^2 \leq -\frac{1-2\theta}{\lambda_1} \sum_{k \in I} F^{k\bar{k}} |\nabla_{\bar{1}} g_{\bar{1}k}|^2. \quad (4.2.58)$$

Combining with the previous inequalities, we obtain

$$\begin{aligned}
0 &\geq -(1-2\theta) \sum_{k \in I} F^{k\bar{k}} \frac{|\tilde{\lambda}_{1,\bar{k}}|^2 - |\nabla_1 g_{1\bar{k}}|^2}{\lambda_1^2} - C_{17} \mathcal{F} \\
&\quad - \frac{C_4}{\lambda_1} F^{k\bar{k}} |\tilde{\lambda}_{1,\bar{k}}| + \frac{1}{8P} F^{q\bar{q}} (|\nabla_q \nabla u|^2 + |\nabla_q \bar{\nabla} u|^2) \\
&\quad + \frac{1}{2} \varphi'' F^{k\bar{k}} |\nabla_{\bar{k}} \tilde{v}|^2 + \varphi' (F^{k\bar{k}} \nabla_k \nabla_{\bar{k}} \tilde{v} - \partial_t \tilde{v}) - 2(\varphi')^2 \theta^{-1} F^{1\bar{1}} P. \tag{4.2.59}
\end{aligned}$$

Since $\nabla_1 g_{1\bar{k}} = \tilde{\lambda}_{1,\lambda} + O(\lambda_1)$, we have

$$-(1-2\theta) \sum_{k \in I} F^{k\bar{k}} \frac{|\tilde{\lambda}_{1,\bar{k}}|^2 - |\nabla_1 g_{1\bar{k}}|^2}{\lambda_1^2} \geq -C_{18} \mathcal{F} \tag{4.2.60}$$

where C_{18} depends on $\|\chi\|_{C^2}$ and $\|\alpha\|_{C^2}$. Next, using again the equations for critical points, we can write

$$\begin{aligned}
\frac{C_4}{\lambda_1} F^{k\bar{k}} |\tilde{\lambda}_{1,\bar{k}}| &= \frac{C_4}{\lambda_1} F^{k\bar{k}} |\phi' \nabla_{\bar{k}} |\nabla u|^2 + \varphi' \nabla_{\bar{k}} \tilde{v}| \\
&\leq \frac{1}{2K^{\frac{1}{2}}} \sum F^{k\bar{k}} (|\nabla_{\bar{k}} \nabla_p u| + |\nabla_{\bar{k}} \nabla_{\bar{p}} u|) + C_\varepsilon |\varphi'| F^{k\bar{k}} |\nabla_{\bar{k}} \tilde{v}|^2 + \varepsilon C_{19} |\varphi'| \mathcal{F} + C_{20} \mathcal{F}, \tag{4.2.61}
\end{aligned}$$

where C_{19} and C_{20} depend on C_4 . Accordingly, the previous inequality implies

$$\begin{aligned}
0 &\geq \frac{1}{10K} F^{q\bar{q}} (|\nabla_q \nabla u|^2 + |\nabla_q \bar{\nabla} u|^2) + \frac{1}{2} \varphi'' F^{k\bar{k}} |\nabla_{\bar{k}} \tilde{v}|^2 + \varphi' (F^{k\bar{k}} \nabla_k \nabla_{\bar{k}} \tilde{v} - \partial_t \tilde{v}) \\
&\quad - 2(\varphi')^2 \theta^{-1} F^{1\bar{1}} P - C_\varepsilon |\varphi'| F^{k\bar{k}} |\nabla_{\bar{k}} \tilde{v}|^2 - \varepsilon C_{19} |\varphi'| \mathcal{F} - C_{21} \mathcal{F}, \tag{4.2.62}
\end{aligned}$$

where C_{21} depending only on $\|\chi\|_{C^2}$, $\|\alpha\|_{C^2}$, n , $\|\psi\|_{C^2}$, $\|\partial_t v\|_{C^0}$, $\|\tilde{v}\|_{L^\infty}$, $\|\partial_t u\|_{L^\infty}$ and $\|\nabla u\|_{L^\infty}$. Finally we get

$$\begin{aligned}
0 &\geq F^{1\bar{1}} \left(\frac{\lambda_1^2}{20P} - 2(\varphi')^2 \theta^{-1} P \right) + \left(\frac{1}{2} \varphi'' - C_\varepsilon |\varphi'| \right) F^{k\bar{k}} |\nabla_{\bar{k}} \tilde{v}|^2 \\
&\quad - \varepsilon C_{19} |\varphi'| \mathcal{F} + \varphi' (F^{k\bar{k}} \nabla_k \nabla_{\bar{k}} \tilde{v} - \partial_t \tilde{v}) - C_{21} \mathcal{F}. \tag{4.2.63}
\end{aligned}$$

We now apply Lemma 4.2.3. Fix δ and K as in Definition 4.1.1, if $\lambda_1 > K$, then there are two possibilities:

- Either $F^{k\bar{k}} (\underline{u}_{\bar{k}k} - u_{\bar{k}k}) + (\partial_t u - \partial_t \underline{u}) \geq \kappa \mathcal{F}$, for some κ depending only on δ and K , equivalently,

$$F^{k\bar{k}} \nabla_k \nabla_{\bar{k}} \tilde{v} - \partial_t \tilde{v} - \int_X \partial_t v \alpha^n \leq -\kappa \mathcal{F} + C_{22} \mathcal{F}, \tag{4.2.64}$$

where C_{22} depends on $\|\partial_t v\|_{L^\infty}$. Since $\varphi' < 0$, we find

$$\begin{aligned} 0 \geq & F^{1\bar{1}} \left(\frac{\lambda_1^2}{20P} - 2(\varphi')^2 \theta^{-1} P \right) + \left(\frac{1}{2} \varphi'' - C_\varepsilon |\varphi'| \right) F^{k\bar{k}} |\nabla_{\bar{k}} \tilde{v}|^2 \\ & - C_{23} \mathcal{F} - \varepsilon C_{19} |\varphi'| \mathcal{F} - \varphi' \kappa \mathcal{F} \end{aligned} \quad (4.2.65)$$

with C_{23} depending only on $n, \|\chi\|_{C^2}, \|\alpha\|_{C^2}, \|\psi\|_{C^2}, \|\partial_t v\|_{L^\infty}, \|\tilde{v}\|_{L^\infty}, \|\partial_t u\|_{L^\infty}$ and $\|\nabla \underline{u}\|_{L^\infty}$. We first choose ε small enough so that $\varepsilon C_{19} < \kappa/2$, then D_2 large enough so that $\varphi'' > 2C_\varepsilon |\varphi'|$. We obtain

$$0 \geq F^{1\bar{1}} \left(\frac{\lambda_1^2}{20P} - 2(\varphi')^2 \theta^{-1} P \right) - C_{23} \mathcal{F} - \frac{1}{2} \varphi' \kappa \mathcal{F}. \quad (4.2.66)$$

We now choose D_1 large enough (depending on $\|\tilde{v}\|_{L^\infty}$) so that $-C_{23} - \frac{1}{2} \varphi' \kappa > 0$. Then

$$\frac{\lambda_1^2}{20P} \leq 2(\varphi')^2 \theta^{-1} P \quad (4.2.67)$$

and the desired upper bound for λ_1/P follows.

- Or $F^{1\bar{1}} \geq \kappa \mathcal{F}$. With D_1, D_2 , and θ as above, the inequality (4.2.63) implies

$$0 \geq \kappa \mathcal{F} \left(\frac{\lambda_1^2}{20P} - 2(\varphi')^2 \theta^{-1} P \right) - C_{24} \mathcal{F} - \varphi' F^{k\bar{k}} g_{\bar{k}k}, \quad (4.2.68)$$

with C_{24} depending only on $\|\chi\|_{C^2}, \|\alpha\|_{C^2}, n, \|\psi\|_{C^2}, \|\partial_t v\|_{L^\infty}, \|\tilde{v}\|_{L^\infty}, \|\partial_t u\|_{L^\infty}, \|\nabla \underline{u}\|_{L^\infty}$, and $\|i\partial\bar{\partial}\underline{u}\|_{L^\infty}$. Since $F^{k\bar{k}} g_{\bar{k}k} \leq \mathcal{F} \lambda_1$, we can divide by $\mathcal{F} P$ to get

$$0 \geq \kappa \frac{\lambda_1^2}{20P^2} - C_{25} \left(1 + \frac{1}{P} + \frac{\lambda_1}{P} \right) \quad (4.2.69)$$

with a constant C_{25} depending only on $\|\chi\|_{C^2}, \|\alpha\|_{C^2}, n, \|\psi\|_{C^2}, \|\tilde{v}\|_{L^\infty}, \|\partial_t v\|_{L^\infty}, \|\partial_t u\|_{L^\infty}, \|\nabla \underline{u}\|_{L^\infty}$, and $\|i\partial\bar{\partial}\underline{u}\|_{L^\infty}$. Thus we obtain the desired bound for λ_1/P .

It was pointed out in [Sze15] that, under an extra concavity condition on f , C^2 estimates can be derived directly from C^0 estimates in the elliptic case, using a test function introduced in [PS09]. The same holds in the parabolic case, but we omit a fuller discussion.

4.2.3 C^1 Estimates

The C^1 estimates are also adapted from [Sze15], which reduce the estimates by a blow-up argument to a key Liouville theorem for Hessian equations due to Székelyhidi [Sze15] and Dinew and Kolodziej [DK17].

Lemma 4.2.4. *There exist a constant $C > 0$, depending on $\underline{u}, \|\partial_t u\|_{L^\infty(X \times [0, T])}, \|\tilde{u}\|_{L^\infty(X \times [0, T])}, \|\alpha\|_{C^2}, \chi, \psi$ and the constant \tilde{C} in Lemma 4.2.2 such that*

$$\sup_{X \times [0, T]} |\nabla u|_\alpha^2 \leq C. \quad (4.2.70)$$

Proof. Assume by contradiction that (4.2.70) does not hold. Then there exists a sequence $(x_k, t_k) \in X \times [0, T]$ with $t_k \rightarrow T$ such that

$$\lim_{k \rightarrow \infty} |\nabla u(t_k, x_k)|_\alpha = +\infty.$$

We can assume further that

$$R_k = |\nabla u(x_k, t_k)|_\alpha = \sup_{X \times [0, t_k]} |\nabla u(x, t)|_\alpha, \quad \text{as } k \rightarrow +\infty,$$

and $\lim_{k \rightarrow \infty} x_k = x$.

Using localization, we choose a coordinate chart $\{U, (z_1, \dots, z_n)\}$ centered at x , identifying with the ball $B_2(0) \subset \mathbb{C}^n$ of radius 2 centered at the origin such that $\alpha(0) = \beta$, where $\beta = \sum_j i dz^j \wedge d\bar{z}^j$. We also assume that k is sufficiently large so that $z_k := z(x_k) \in B_1(0)$.

Define the following maps

$$\begin{aligned} \Phi_k : \mathbb{C}^n &\rightarrow \mathbb{C}^n, & \Phi_k(z) &:= R_k^{-1}z + z_k, \\ \tilde{u}_k : B_{R_k}(0) &\rightarrow \mathbb{R}, & \tilde{u}_k(z) &:= \tilde{u}(\Phi_k(z), t_k) = \tilde{u}(R_k^{-1}z + z_k, t_k), \end{aligned}$$

where $\tilde{u} = u - \int_X u \alpha^n$. Then the equation

$$u_t = F(A) - \psi(z),$$

implies that

$$f\left(R_k^2 \lambda[\beta_k^{i\bar{p}}(\chi_{k, \bar{p}j} + \tilde{u}_{k, \bar{p}j})]\right) = \psi(R_k^{-1}z + z_k) + u_t(\Phi_k(z), t_k), \quad (4.2.71)$$

where $\beta_k := R_k^2 \Phi_k^* \alpha$, $\chi_k := \Phi_k^* \chi$. Since $\beta_k \rightarrow \beta$, and $\chi_k(z, t) \rightarrow 0$, in C_{loc}^∞ as $k \rightarrow \infty$, we get

$$\lambda[\beta_k^{i\bar{p}}(\chi_{k, \bar{p}j} + \tilde{u}_{k, \bar{p}j})] = \lambda(\tilde{u}_{k, \bar{j}i}) + O\left(\frac{|z|}{R_k^2}\right). \quad (4.2.72)$$

By the construction, we have

$$\sup_{B_{R_k}(0)} \tilde{u}_k \leq C, \quad \sup_{B_{R_k}(0)} |\nabla \tilde{u}_k| \leq C \quad (4.2.73)$$

where C depending on $\|\tilde{u}\|_{L^\infty}$, and

$$|\nabla \tilde{u}_k|(0) = R_k^{-1} |\nabla u|_\alpha(x_k) = 1.$$

Thanks to Lemma 4.2.2, we also have that

$$\sup_{B_{R_k}(0)} |\partial \bar{\partial} \tilde{u}_k|_\beta \leq C R_k^{-2} \sup_X |\partial \bar{\partial} u(\cdot, t_k)|_\alpha \leq C'. \quad (4.2.74)$$

As the argument in [Sze15, TW17], it follows from (4.2.73), (4.2.74), the elliptic estimates for Δ and the Sobolev embedding that for each given $K \subset \mathbb{C}^n$ compact, $0 < \gamma < 1$ and $p > 1$, there is a constant C such that

$$\|\tilde{u}_k\|_{C^{1,\gamma}(K)} + \|\tilde{u}_k\|_{W^{2,p}(K)} \leq C.$$

Therefore there is a subsequence of \tilde{u}_k converges strongly in $C_{loc}^{1,\gamma}(\mathbb{C}^n)$, and weakly in $W_{loc}^{2,p}(\mathbb{C}^n)$ to a function v with $\sup_{\mathbb{C}^n}(|v| + |\nabla v|) \leq C$ and $\nabla v(0) \neq 0$, in particular v is not constant.

The proof can now be completed exactly as in [Sze15]. The function v is shown to be a Γ -solution in the sense of Székelyhidi [Sze15, Definition 15], and the fact that v is not constant contradicts Székelyhidi's Liouville theorem for Γ -solutions [Sze15, Theorem 20], which is itself based on the Liouville theorem of Dinew and Kolodziej [DK17]. Q.E.D.

4.2.4 Higher Order Estimates

Under the conditions on $f(\lambda)$, the uniform parabolicity of the equation (4.1.1) will follow once we have established an a priori estimate on $\|i\partial\bar{\partial}u\|_{L^\infty}$ and hence an upper bound for the eigenvalues $\lambda[u]$. However, we shall often not have uniform control of $\|u(\cdot, t)\|_{L^\infty}$. Thus we shall require the following version of the Evans-Krylov theorem for uniformly parabolic and concave equations, with the precise dependence of constants spelled out, and which can be proved using the arguments of Trudinger [Tru83], and more particularly Tosatti-Weinkove [TW10a] and Gill [Gil11].

Lemma 4.2.5. *Assume that u is a solution of the equation (4.1.1) on $X \times [0, T)$ and that there exists a constant C_0 with $\|i\partial\bar{\partial}u\|_{L^\infty} \leq C_0$. Then there exist positive constants C and $\gamma \in (0, 1)$ depending only on α, χ, C_0 and $\|\psi\|_{C^2}$ such that*

$$\|i\partial\bar{\partial}u\|_{C^\gamma(X \times [0, T))} \leq C. \quad (4.2.75)$$

Once the C^γ estimate for $i\partial\bar{\partial}u$ has been established, it is well known that a priori estimates of arbitrary order follow by bootstrap, as shown in detail for the Monge-Ampère equation in Yau [Yau78]. We omit reproducing the proofs.

4.3 Proof of Theorems 1 and 2

We begin with the following simple lemma, which follows immediately by differentiating the equation (4.1.1) with respect to t , and applying the maximum principle, which shows that the solution of a linear heat equation at any time can be controlled by its initial value:

Lemma 4.3.1. *Let $u(z, t)$ be a smooth solution of the flow (4.1.1) on any time interval $[0, T)$. Then $\partial_t u$ satisfies the following linear heat equation*

$$\partial_t(\partial_t u) = F^j_k \alpha^{k\bar{m}} \partial_j \partial_{\bar{m}}(\partial_t u) \quad (4.3.1)$$

and we have the following estimate for any $t \in [0, T)$,

$$\min_X (F(A[u_0]) - \psi) \leq \partial_t u(t, \cdot) \leq \max_X F(A[u_0] - \psi) \quad (4.3.2)$$

We can now prove a lemma which provides general sufficient conditions for the convergence of the flow:

Lemma 4.3.2. *Consider the flow (4.1.1). Assume that the equation admits a parabolic C -subsolution $\underline{u} \in C^{2,1}(X \times [0, \infty))$, and that there exists a constant C independent of time so that*

$$\text{osc}_X u(t, \cdot) \leq C. \quad (4.3.3)$$

Then a smooth solution $u(z, t)$ exists for all time, and its normalization \tilde{u} converges in C^∞ to a solution u_∞ of the equation (4.1.6) for some constant c .

In particular, if we assume further that $\|u\|_{L^\infty(X \times [0, \infty))} \leq C$ and for each $t > 0$, there exists $y = y(t) \in X$ such that $\partial_t u(y, t) = 0$, then u converges in C^∞ to a solution u_∞ of the equation (4.1.6) for the constant $c = 0$.

Proof of Lemma 4.3.2. We begin by establishing the existence of the solution for all time. For any fixed $T > 0$, Lemma 4.3.1 shows that $|\partial_t u|$ is uniformly bounded by a constant C . Integrating between 0 and T , we deduce that $|u|$ is uniformly bounded by CT . We can now apply Lemma 4.2.4, 4.2.2, 4.2.5, to conclude that the function u is uniformly bounded in C^k norm (by constants depending on k and T) for arbitrary k . This implies that the solution can be extended beyond T , and since T is arbitrary, that it exists for all time.

Next, we establish the convergence. For this, we adapt the arguments of Cao [Cao85] and especially Gill [Gil11] based on the Harnack inequality.

Since $\text{osc}_X u(t, \cdot)$ is uniformly bounded by assumption, and since $\partial_t u$ is uniformly bounded in view of Lemma 4.3.1, we can apply Lemma 4.2.2 and deduce that the eigenvalues of the matrix $[\chi + i\partial\bar{\partial}u]$ are uniformly bounded over the time interval $[0, \infty)$. The uniform ellipticity of the equation (4.3.5) follows in turn from the properties (1) and (2) of the function $f(\lambda)$. Next set

$$v = \partial_t u + A \quad (4.3.4)$$

for some large constant A so that $v > 0$. The function v satisfies the same heat equation

$$\partial_t v = F^{i\bar{j}} \partial_i \partial_{\bar{j}} v. \quad (4.3.5)$$

Since the equation (4.3.5) is uniformly elliptic, by the differential Harnack inequality proved originally in the Riemannian case by Li and Yau in [LY86], and extended to the Hermitian

case by Gill [Gil11], section 6, it follows that there exist positive constants C_1, C_2, C_3 , depending only on ellipticity bounds, so that for all $0 < t_1 < t_2$, we have

$$\sup_X v(\cdot, t_1) \leq \inf_X v(\cdot, t_2) \left(\frac{t_2}{t_1} \right)^{C_2} \exp \left(\frac{C_3}{t_2 - t_1} + C_1(t_2 - t_1) \right). \quad (4.3.6)$$

The same argument as in Cao [Cao85], section 2, and Gill [Gil11], section 7, shows that this estimate implies the existence of constants C_4 and $\eta > 0$ so that

$$\text{osc}_X v(\cdot, t) \leq C_4 e^{-\eta t} \quad (4.3.7)$$

If we set

$$\tilde{v}(z, t) = v(z, t) - \frac{1}{V} \int_X v \alpha^n = \partial_t u(z, t) - \frac{1}{V} \int_X \partial_t u \alpha^n = \partial_t \tilde{u}, \quad (4.3.8)$$

it follows that

$$|\tilde{v}(z, t)| \leq C_4 e^{-\eta t} \quad (4.3.9)$$

for all $z \in X$. In particular,

$$\partial_t \left(\tilde{u} + \frac{C_4}{\eta} e^{-\eta t} \right) = \tilde{v} - C_4 e^{-\eta t} \leq 0, \quad (4.3.10)$$

and the function $\tilde{u}(z, t) + \frac{C_4}{\eta} e^{-\eta t}$ is decreasing in t . By the assumption (4.3.3), this function is uniformly bounded. Thus it converges to a function $u_\infty(z)$. By the higher order estimates in section §2, the derivatives to any order of \tilde{u} are uniformly bounded, so the convergence of $\tilde{u} + \frac{C_4}{\eta} e^{-\eta t}$ is actually in C^∞ . The function $\tilde{u}(z, t)$ will also converge in C^∞ , to the same limit $u_\infty(z)$. Now the function $\tilde{u}(z, t)$ satisfies the following flow,

$$\partial_t \tilde{u} = F(A[\tilde{u}]) - \psi(z) - \frac{1}{V} \int_X \partial_t u \alpha^n. \quad (4.3.11)$$

Taking limits, we obtain

$$0 = F(A[\tilde{u}_\infty]) - \psi(z) - \lim_{t \rightarrow \infty} \int_X \partial_t u \alpha^n \quad (4.3.12)$$

where the existence of the limit of the integral on the right hand side follows from the equation. Define the constant c as the value of this limit. This implies the first statement in Lemma 4.3.2.

Now we assume that $\|u\|_{L^\infty(X \times [0, \infty))} \leq C$ and for each $t \geq 0$, there exists $y = y(t) \in X$ such that $\partial_t u(y, t) = 0$. By the same argument above, we have

$$\text{osc}_X \partial_t u(\cdot, t) \leq C_4 e^{-\eta t}, \quad (4.3.13)$$

for some $C_4, \eta > 0$. Since for each $t \geq 0$, there exists $y = y(t) \in X$ such that $\partial_t u(y, t) = 0$, we imply that for any $z \in X$,

$$|\partial_t u(z, t)| = |\partial_t u(z, t) - \partial_t u(y, t)| \leq \text{osc}_X \partial_t u(\cdot, t) \leq C_4 e^{-\eta t}. \quad (4.3.14)$$

Therefore by the same argument above, the function $u(z, t) + \frac{C_4}{\eta} e^{-\eta t}$ converges in C^∞ and $\partial_t u$ converges to 0 as $t \rightarrow +\infty$. We thus infer that u converges in C^∞ , to u_∞ satisfying the equation

$$F(A[\tilde{u}_\infty]) = \psi(z). \quad (4.3.15)$$

Lemma 4.3.2 is proved.

Proof of Theorem 4.1.2. Since f is unbounded, the function $\underline{u} = u_0$ is a C -subsolution of the flow. In view of Lemma 4.3.2, it suffices to establish a uniform bound for $\text{osc}_X u(t, \cdot)$. But the flow can be re-expressed as the elliptic equation

$$F(A) = \psi + \partial_t u \quad (4.3.16)$$

where the right hand side $\psi + \partial_t u$ is bounded uniformly in t , since we have seen that $\partial_t u$ is uniformly bounded in t . Furthermore, because f is unbounded, the function $\underline{u} = u_0$ is a C -subsolution of (4.3.16). By the C^0 estimate of [Sze15], the oscillation $\text{osc}_X u(t, \cdot)$ can be bounded for each t by the C^0 norm of the right hand side, and is hence uniformly bounded. Q.E.D.

Proof of Theorem 4.1.3. Again, it suffices to establish a uniform bound in t for $\text{osc}_X u(t, \cdot)$.

Consider first the case (a). In view of Lemma 4.3.1 and the hypothesis, we have

$$\partial_t \underline{u} \geq \partial_t u \quad (4.3.17)$$

on all of $X \times [0, \infty)$. But if we rewrite the flow (4.1.1) as

$$F(A) = \psi + \partial_t u \quad (4.3.18)$$

we see that the condition that \underline{u} be a parabolic C -subsolution for the equation (4.1.1) together with (4.3.17) implies that \underline{u} is a C -subsolution for the equation (4.3.18) in the elliptic sense. We can then apply Székelyhidi's C^0 estimate for the elliptic equation to obtain a uniform bound for $\text{osc}_X u(t, \cdot)$.

Next, we consider the case (b). In this case, the existence of a function $h(t)$ with the indicated properties allows us to apply Lemma 4.2.1, and obtain immediately a lower bound,

$$u - \underline{u} - h(t) \geq -C \quad (4.3.19)$$

for some constant C independent of time. The inequality (4.1.9) implies then a uniform bound for $\text{osc}_X u$.

4.4 Applications to Geometric Flows

Theorems 4.1.2 and 4.1.3 can be applied to many geometric flows. We should stress that they don't provide a completely independent approach, as they themselves are built on many techniques that had been developed to study these flows. Nevertheless, they may provide an attractive uniform approach.

4.4.1 A criterion for subsolutions

In practice, it is easier to verify that a given function \underline{u} on $X \times [0, \infty)$ is a C -subsolution of the equation (4.1.1) using the following lemma rather than the original Definition 4.1.1:

Lemma 4.4.1. *Let \underline{u} be a $C^{2,1}$ admissible function on $X \times [0, \infty)$, with $\|\underline{u}\|_{C^{2,1}(X \times [0, \infty))} < \infty$. Then \underline{u} is a parabolic C -subsolution in the sense of Definition 4.1.1 if and only if there exists a constant $\tilde{\delta} > 0$ independent from (z, t) so that*

$$\lim_{\mu \rightarrow +\infty} f(\lambda[\underline{u}(z, t)] + \mu e_i) - \partial_t \underline{u}(z, t) > \tilde{\delta} + \psi(z) \quad (4.4.1)$$

for each $1 \leq i \leq n$. In particular, if \underline{u} is independent of t , then \underline{u} is a parabolic C -subsolution if and only if

$$\lim_{\mu \rightarrow +\infty} f(\lambda[\underline{u}(z, t)] + \mu e_i) > \psi(z). \quad (4.4.2)$$

Note that there is a similar lemma in the case of subsolutions for elliptic equations (see [Sze15], Remark 8). Here the argument has to be more careful, not just because of the additional time parameter t , but also because the time interval $[0, \infty)$ is not bounded, invalidating certain compactness arguments.

Proof of Lemma 4.4.1. We show first that the condition (4.4.1) implies that \underline{u} is a C -subsolution.

We begin by showing that the condition (4.4.1) implies that there exists $\epsilon_0 > 0$ and $M > 0$, so that for all $\epsilon \leq \epsilon_0$, all $\nu > M$, all (z, t) , and all $1 \leq i \leq n$, we have

$$f(\lambda[\underline{u}(z, t)] - \epsilon I + \nu e_i) - \partial_t \underline{u}(z, t) > \frac{\tilde{\delta}}{4} + \psi(z). \quad (4.4.3)$$

This is because the condition (4.4.1) is equivalent to

$$f_\infty(\lambda'[\underline{u}(z, t)]) - \partial_t \underline{u}(z, t) > \tilde{\delta} + \psi(z). \quad (4.4.4)$$

Now the concavity of $f(\lambda)$ implies the concavity of its limit $f_\infty(\lambda')$ and hence the continuity of $f_\infty(\lambda')$. Furthermore, the set

$$\Lambda = \overline{\{\lambda[\underline{u}(z, t)], \forall (z, t) \in X \times [0, \infty)\}}, \quad (4.4.5)$$

as well as any of its translates by $-\epsilon I$ for a fixed ϵ small enough, is compact in Γ . So are their projections on \mathbf{R}^{n-1} . By the uniform continuity of continuous functions on compact sets, it follows that there exists $\epsilon_0 > 0$ so that

$$f_\infty(\lambda'[\underline{u}(z, t)] - \epsilon I) - \partial_t \underline{u}(z, t) > \frac{\tilde{\delta}}{2} + \psi(z) \quad (4.4.6)$$

for all (z, t) and all $\epsilon \leq \epsilon_0$. But f_∞ is the continuous limit of a sequence of monotone increasing continuous functions

$$f_\infty(\lambda' - \epsilon I) = \lim_{\nu \rightarrow \infty} f(\lambda - \epsilon I + \nu e_i). \quad (4.4.7)$$

By Dini's theorem, the convergence is uniform over any compact subset. Thus there exists $M > 0$ large enough so that $\nu > M$ implies that

$$f(\lambda[\underline{u}(z, t)] - \epsilon I + \nu e_i) > f_\infty(\lambda'[\underline{u}(z, t)] - \epsilon I) - \frac{\tilde{\delta}}{4} \quad (4.4.8)$$

for all (z, t) and all $\epsilon \leq \epsilon_0$. The desired inequality (4.4.3) follows from (4.4.6) and (4.4.8).

Assume now that \underline{u} is not a C -subsolution. Then there exists $\epsilon_m, \nu_m, \tau_m$, with $\epsilon_m \rightarrow 0$, $\nu_m \in -\epsilon_m I + \Gamma_n$, $\tau_m > -\epsilon_m$, and $|\tau_m| + |\nu_m| \rightarrow \infty$, so that

$$f(\lambda[\underline{u}(z_m, t_m)] + \nu_m) - \partial_t \underline{u}(z_m, t_m) + \tau_m = \psi(z_m, t_m). \quad (4.4.9)$$

Set $\nu_m = -\epsilon_m I + \mu_m$, with $\mu_m \in \Gamma_n$. Then we can write

$$\begin{aligned} \tau_m &= -f(\lambda[\underline{u}(z_m, t_m)] - \epsilon_m I + \mu_m) + \partial_t \underline{u}(z_m, t_m) + \psi(z_m, t_m) \\ &\leq -f(\lambda[\underline{u}(z_m, t_m)] - \epsilon_m I) + \partial_t \underline{u}(z_m, t_m) + \psi(z_m, t_m) \end{aligned} \quad (4.4.10)$$

which is bounded by a constant. Thus we must have $|\nu_m|$ tending to $+\infty$, or equivalently, $|\mu_m|$ tending to $+\infty$.

By going to a subsequence, we may assume that there is an index i for which the i -th components μ_m^i of the vector μ_m tend to ∞ as $m \rightarrow \infty$. By the monotonicity of f in each component, we have

$$\begin{aligned} f(\lambda[\underline{u}(z_m, t_m)] - \epsilon_m I + \mu_m^i e_i) - \partial_t \underline{u}(z_m, t_m) &\leq f(\lambda[\underline{u}(z_m, t_m)] - \epsilon_m I + \mu_m) - \partial_t \underline{u}(z_m, t_m) \\ &= f(\lambda[\underline{u}(z_m, t_m)] + \nu_m) - \partial_t \underline{u}(z_m, t_m). \end{aligned}$$

In view of (4.4.3), the left hand side is $\geq \frac{\tilde{\delta}}{4} + \psi(z_m, t_m)$ for μ_m^i large and ϵ_m small enough. On the other hand, the equation (4.4.9) implies that the right hand side is equal to $\psi(z_m, t_m) - \tau_m$. Thus we obtain

$$\frac{\tilde{\delta}}{4} + \psi(z_m, t_m) \leq \psi(z_m, t_m) - \tau_m \leq \psi(z_m, t_m) + \epsilon_m. \quad (4.4.11)$$

Hence $\frac{\delta}{4} \leq \epsilon_m$, which is a contradiction, since $\epsilon_m \rightarrow 0$.

Finally, we show that if \underline{u} is a subsolution, it must satisfy the condition (4.4.1). Assume otherwise. Then there exists an index i and a sequence $\delta_m \rightarrow 0$ and points (z_m, t_m) so that

$$\lim_{\nu \rightarrow \infty} f(\lambda[\underline{u}(z_m, t_m)] + \nu e_i) - \partial_t \underline{u}(z_m, t_m) \leq \delta_m + \psi(z_m). \quad (4.4.12)$$

Since f is increasing in ν , this implies that for any $\nu \in \mathbf{R}_+$, we have

$$f(\lambda[\underline{u}(z_m, t_m)] + \nu e_i) - \partial_t \underline{u}(z_m, t_m) \leq \delta_m + \psi(z_m). \quad (4.4.13)$$

For each $\nu \in \mathbf{R}_+$, define τ_m by the equation

$$f(\lambda[\underline{u}(z_m, t_m)] + \nu e_i) - \partial_t \underline{u}(z_m, t_m) + \tau_m = \psi_m. \quad (4.4.14)$$

The previous inequality means that $\tau_m \geq -\delta_m$, and thus the pair $(\tau_m, \mu = \nu e_i)$ satisfies the equation (4.1.2). Since we can take $\nu \rightarrow +\infty$, this contradicts the defining property of C -subsolutions. The proof of Lemma 4.4.1 is complete.

4.4.2 Székelyhidi's theorem

Theorem 4.1.3 can be applied to provide a proof by parabolic methods of the following theorem originally proved by Székelyhidi [Sze15]:

Corollary 4.4.2. *Let (X, α) be a compact Hermitian manifold, and $f(\lambda)$ be a function satisfying the conditions (1-3) spelled out in §1 and in the bounded case. Let ψ be a smooth function on X . If there exists an admissible function u_0 with $F(A[u_0]) \leq \psi$, and if the equation $F(A[u]) = \psi$ admits a C -subsolution in the sense of [Sze15], then the equation $F(A[u]) = \psi + c$ admits a smooth solution for some constant c .*

Proof of Corollary 4.4.2. It follows from Lemma 4.4.1 that a C -subsolution in the sense of [Sze15] of the elliptic equation $F(A[u]) = \psi$ can be viewed as a time-independent parabolic C -subsolution \underline{u} of the equation (4.1.1). Consider this flow with initial value u_0 . Then

$$\partial_t \underline{u} = 0 \geq F(A[u_0]) - \psi. \quad (4.4.15)$$

Thus condition (a) of Theorem 4.1.3 is satisfied, and the corollary follows.

4.4.3 The Kähler-Ricci flow and the Chern-Ricci flow

On Kähler manifolds (X, α) with $c_1(X) = 0$, the Kähler-Ricci flow is the flow $\dot{g}_{\bar{k}j} = -R_{\bar{k}j}$. For initial data in the Kähler class $[\alpha]$, the evolving metric can be expressed as $g_{\bar{k}j} = \alpha_{\bar{k}j} + \partial_j \partial_{\bar{k}} \varphi$, and the flow is equivalent to the following Monge-Ampère flow,

$$\partial_t \varphi = \log \frac{(\alpha + i\partial\bar{\partial}\varphi)^n}{\alpha^n} - \psi(z) \quad (4.4.16)$$

for a suitable function $\psi(z)$ satisfying the compatibility condition $\int_X e^\psi \alpha^n = \int_X \alpha^n$. The convergence of this flow was proved by Cao [Cao85], thus giving a parabolic proof of Yau's solution of the Calabi conjecture [Yau78]. We can readily derive Cao's result from Theorem 4.1.2:

Corollary 4.4.3. *For any initial data, the normalization $\tilde{\varphi}$ of the flow (4.4.16) converges in C^∞ to a solution of the equation $(\alpha + i\partial\bar{\partial}\varphi)^n = e^\psi \alpha^n$.*

Proof of Corollary 4.4.3. The Monge-Ampère flow (4.4.16) corresponds to the equation (4.1.1) with $\chi = \alpha$, $f(\lambda) = \log \prod_{j=1}^n \lambda_j$, and Γ being the full octant Γ_n . It is straightforward that f satisfies the condition (1-3) in §1. In particular f is in the unbounded case, and Theorem 4.1.2 applies, giving the convergence of the normalizations $\tilde{u}(\cdot, t)$ to a smooth solution of the equation $(\alpha + i\partial\bar{\partial}\varphi)^n = e^{\psi+c} \alpha^n$ for some constant c . Integrating both sides of this equation and using the compatibility condition on ψ , we find that $c = 0$. The corollary is proved.

The generalization of the flow (4.4.16) to the more general set-up of a compact Hermitian manifold (X, α) was introduced by Gill [Gil11]. It is known as the Chern-Ricci flow, with the Chern-Ricci tensor $Ric^C(\omega) = -i\partial\bar{\partial} \log \omega^n$ playing the role of the Ricci tensor in the Kähler-Ricci flow (we refer to [TW13, TW15, TWY15, T618] and references therein). Gill proved the convergence of this flow, thus providing an alternative proof of the generalization of Yau's theorem proved earlier by Tosatti and Weinkove [TW10b]. Generalizations of Yau's theorem had attracted a lot of attention, and many partial results had been obtained before, including those of Cherrier [Che87], Guan-Li [GL10], and others. Theorem 4.1.2 gives immediately another proof of Gill's theorem:

Corollary 4.4.4. *For any initial data, the normalizations $\tilde{\varphi}$ of the Chern-Ricci flow converge in C^∞ to a solution of the equation $(\alpha + i\partial\bar{\partial}\varphi)^n = e^{\psi+c} \alpha^n$, for some constant c .*

We note that there is a rich literature on Monge-Ampère equations, including considerable progress using pluripotential theory. We refer to [Kol98, EGZ09, DP10, GZ17, GZ17b, PSS12, T617, T618, Nie14, Nie17] and references therein.

4.4.4 Hessian flows

Hessian equations, where the Laplacian or the Monge-Ampère determinant of the unknown function u are replaced by the k -th symmetric polynomial of the eigenvalues of the Hessian of u , were introduced by Caffarelli, Nirenberg, and Spruck [CNS85]. More general right hand sides and Kähler versions were considered respectively by Chou and Wang [CW01] and Hou-Ma-Wu [HMW10], who introduced in the process some of the key techniques for C^2 estimates that we discussed in §2. A general existence result on compact Hermitian manifolds was recently obtained by Dinew and Kolodziej [DK17], Sun [Sun17b], and Székelyhidi [Sze15]. See also Zhang [Zha17]. Again, we can derive this theorem as a corollary of Theorem 4.1.2:

Corollary 4.4.5. *Let (X, α) be a compact Hermitian n -dimensional manifold, and let χ be a positive real $(1, 1)$ -form which is k -positive for a given k , $1 \leq k \leq n$. Consider the following parabolic flow for the unknown function u ,*

$$\partial_t u = \log \frac{(\chi + i\partial\bar{\partial}u)^k \wedge \alpha^{n-k}}{\alpha^n} - \psi(z). \quad (4.4.17)$$

Then for any admissible initial data u_0 , the flow admits a solution $u(z, t)$ for all time, and its normalization $\tilde{u}(z, t)$ converge in C^∞ to a function $u_\infty \in C^\infty(X)$ so that $\omega = \chi + i\partial\bar{\partial}u_\infty$ satisfies the following k -Hessian equation,

$$\omega^k \wedge \alpha^{n-k} = e^{\psi+c} \alpha^n. \quad (4.4.18)$$

Proof of Corollary 4.4.5. This is an equation of the form (4.1.1), with $F = f(\lambda) = \log \sigma_k(\lambda)$, defined on the cone

$$\Gamma_k = \{\lambda; \sigma_j(\lambda) > 0, j = 1, \dots, k\}, \quad (4.4.19)$$

where $\binom{n}{k} \sigma_k$ is the k -th symmetric polynomial in the components λ_j , $1 \leq j \leq n$. In our setting,

$$\sigma_k(\lambda[u]) = \frac{(\chi + i\partial\bar{\partial}u)^k \wedge \alpha^{n-k}}{\alpha^n}. \quad (4.4.20)$$

It follows from [Spr05, Corollary 2.4] that $g = \sigma_k^{1/k}$ is concave and $g_i = \frac{\partial g}{\partial \lambda_i} > 0$ on Γ_k , hence $f = \log g$ satisfies the conditions (1-3) mentioned in §1.

The function $\underline{u} = 0$ is a subsolution of (4.4.17) and f is in the unbounded case since for any $\mu = (\mu_1, \dots, \mu_n) \in \Gamma_k$, and any $1 \leq i \leq n$,

$$\lim_{s \rightarrow \infty} \log \sigma_k(\mu_1, \dots, \mu_i + s, \dots, \mu_n) = \infty. \quad (4.4.21)$$

The desired statement follows then from Theorem 4.1.2.

4.4.5 The J flow and quotient Hessian flows

The J -flow on Kähler manifolds was introduced independently by Donaldson [Don99] and Chen [Chen00]. The case $n = 2$ was solved by Weinkove [Wei04, Wei06], and the case of general dimension by Song and Weinkove [SW08], who identified a necessary and sufficient condition for the long-time existence and convergence of the flow as the existence of a Kähler form χ satisfying

$$nc\chi^{n-1} - (n-1)\chi^{n-2} \wedge \omega > 0 \quad (4.4.22)$$

in the sense of positivity of $(n-1, n-1)$ -forms. The constant c is actually determined by cohomology. Their work was subsequently extended to inverse Hessian flows on Kähler

manifolds by Fang, Lai, and Ma [FLM11], and to inverse Hessian flows on Hermitian manifolds by Sun [Sun15a]. These flows are all special cases of quotient Hessian flows on Hermitian manifolds. Their stationary points are given by the corresponding quotient Hessian equations. Our results can be applied to prove the following generalization to quotient Hessian flows of the results of [Wei04, Wei06, FLM11], as well as an alternative proof of a result of Székelyhidi [Sze15, Proposition 22] on the Hessian quotient equations. The flow (4.4.24) below has also been studied recently by Sun [Sun15b] where he obtained a uniform C^0 estimate using Moser iteration. Our proof should be viewed as different from all of these, since its C^0 estimate uses neither Moser iteration nor strict C^2 estimates $\text{Tr}_\alpha \chi_u \leq C e^{u - \inf_X u}$.

Corollary 4.4.6. *Assume that (X, α) is a compact Kähler n -manifold, and fix $1 \leq \ell < k \leq n$. Fix a closed $(1, 1)$ -form χ which is k -positive, and assume that there exists a function u so that the form $\chi' = \chi + i\partial\bar{\partial}u$ is closed k -positive and satisfies*

$$kc(\chi')^{k-1} \wedge \alpha^{n-k} - \ell(\chi')^{\ell-1} \wedge \alpha^{n-\ell} > 0 \quad (4.4.23)$$

in the sense of the positivity of $(n-1, n-1)$ -forms. Here $c = \frac{[\chi^\ell] \cup [\alpha^{n-\ell}]}{[\chi^k] \cup [\alpha^{n-k}]}$. Then for any admissible initial data $u_0 \in C^\infty(X)$, the flow

$$\partial_t u = c - \frac{\chi_u^\ell \wedge \alpha^{n-\ell}}{\chi_u^k \wedge \alpha^{n-k}} \quad (4.4.24)$$

admits a solution u for all time, and it converges to a smooth function u_∞ . The form $\omega = \chi + i\partial\bar{\partial}u_\infty$ is k -positive and satisfies the equation

$$\omega^\ell \wedge \alpha^{n-\ell} = c \omega^k \wedge \alpha^{n-k}. \quad (4.4.25)$$

Proof of Corollary 4.4.6. The flow (4.4.24) is of the form (4.1.1), with

$$f(\lambda) = -\frac{\sigma_\ell(\lambda)}{\sigma_k(\lambda)},$$

defined on the cone

$$\Gamma_k = \{\lambda; \sigma_j(\lambda) > 0, j = 1, \dots, k\}. \quad (4.4.26)$$

By the Maclaurin's inequality (cf. [Spr05]), we have $\sigma_k^{1/k} \leq \sigma_\ell^{1/\ell}$ on Γ_k , hence $f(\lambda) \rightarrow -\infty$ as $\lambda \rightarrow \partial\Gamma_k$. It follows from [Spr05, Theorem 2.16] that the function $g = (\sigma_k/\sigma_\ell)^{\frac{1}{(k-\ell)}}$ satisfies $g_i = \frac{\partial g}{\partial \lambda_i} > 0$, $\forall i = 1, \dots, n$ and g is concave on Γ_k . Therefore $f = -g^{-(k-\ell)}$ satisfies the conditions (1), (2) and (3) spelled out in Section 4.1. Moreover, f is in the bounded case with

$$f_\infty(\lambda') = -\frac{\ell\sigma_{\ell-1}(\lambda')}{k\sigma_{k-1}(\lambda')} \quad \text{where } \lambda' \in \Gamma_\infty = \Gamma_{k-1}.$$

We can assume that $u_0 = 0$ by replacing χ (resp. u and \underline{u}) by $\chi + i\partial\bar{\partial}u_0$ (resp. $u - u_0$ and $\underline{u} - u_0$). The inequality (4.4.23) infers that \underline{u} is a subsolution of the equation (4.4.24). Indeed, for any $(z, t) \in X \times [0, \infty)$, set $\mu = \lambda(B)$, $B^i_j = \alpha^{j\bar{k}}(\chi_{\bar{k}j} + \underline{u}_{\bar{k}j})(z, t)$. Since \underline{u} is independent of t , it follows from Lemma 4.4.1 and the symmetry of f that we just need to show that for any $z \in X$ if $\mu' = (\mu_1, \dots, \mu_{n-1})$ then

$$\lim_{s \rightarrow \infty} f(\mu', \mu_n + s) > -c. \quad (4.4.27)$$

This means

$$f_\infty(\mu') = -\frac{\ell\sigma_{\ell-1}(\mu')}{k\sigma_{k-1}(\mu')} > -c. \quad (4.4.28)$$

As in [Sze15], we restrict to the tangent space of X spanned by the eigenvalues corresponding to μ' . Then on this subspace

$$\sigma_j(\mu') = \frac{\chi^{j-1} \wedge \alpha^{n-j}}{\alpha^{n-1}} \quad (4.4.29)$$

for all j . Thus the preceding inequality is equivalent to

$$kc(\chi')^k \wedge \alpha^{n-k} - \ell(\chi')^{\ell-1} \wedge \alpha^{n-\ell} > 0. \quad (4.4.30)$$

By a priori estimates in Section 2, the solution exists for all times. We now use the second statement in Lemma 4.3.2 to prove the convergence. It suffices to check that u is uniformly bounded in $X \times [0, +\infty)$ and for all $t > 0$, there exists y such that $\partial_t u(y, t) = 0$. The second condition is straightforward since

$$\int_X \partial_t u \chi_u^k \wedge \alpha^{n-k} = 0.$$

For the uniform bound we make use of the following lemma

Lemma 4.4.7. *Let $\phi \in C^\infty(X)$ function and $\{\varphi_s\}_{s \in [0,1]}$ be a path with $\varphi(0) = 0$ and $\varphi(1) = \phi$. Then we have*

$$\int_0^1 \int_X \frac{\partial \varphi}{\partial s} \chi_\varphi^k \wedge \alpha^{n-k} ds = \frac{1}{k+1} \sum_{j=0}^k \int_X \varphi \chi_\varphi^j \wedge \chi^{k-j} \wedge \alpha^{n-k}, \quad (4.4.31)$$

so the left hand side is independent of φ . Therefore we can define the following functional

$$I_k(\phi) = \int_0^1 \int_X \frac{\partial \varphi}{\partial s} \chi_\varphi^k \wedge \alpha^{n-k} ds. \quad (4.4.32)$$

We remark that when $k = n$ and χ is Kähler, this functional is well-known (see for instance [Wei06]). We discuss here the general case.

Proof of Lemma 4.4.7. Observe that

$$\int_0^1 \int_X \frac{\partial \varphi}{\partial s} \chi_\varphi^k \wedge \alpha^{n-k} ds = \sum_{j=1}^k \binom{k}{j} \int_0^1 \int_X \frac{\partial \varphi}{\partial s} (i\partial\bar{\partial}\varphi)^j \wedge \chi^{k-j} \wedge \alpha^{n-k} ds. \quad (4.4.33)$$

For any $j = 0, \dots, k$ we have

$$\begin{aligned} \int_0^1 \int_X \frac{\partial \varphi}{\partial s} (i\partial\bar{\partial}\varphi)^j \wedge \chi^{k-j} \wedge \alpha^{n-k} ds &= \int_0^1 \frac{d}{ds} \left(\int_X \varphi (i\partial\bar{\partial}\varphi)^j \wedge \chi^{k-j} \wedge \alpha^{n-k} \right) ds \\ &\quad - \int_0^1 \int_X \varphi \frac{\partial}{\partial s} \left((i\partial\bar{\partial}\varphi)^j \wedge \chi^{k-j} \wedge \alpha^{n-k} \right) ds \\ &= \int_X \phi (i\partial\bar{\partial}\varphi)^j \wedge \chi^{k-j} \wedge \alpha^{n-k} \\ &\quad - \int_0^1 \int_X \varphi \frac{\partial}{\partial s} \left((i\partial\bar{\partial}\varphi)^j \wedge \chi^{k-j} \wedge \alpha^{n-k} \right) ds \end{aligned} \quad (4.4.34)$$

We also have

$$\begin{aligned} \int_0^1 \int_X \varphi \frac{\partial}{\partial s} \left((i\partial\bar{\partial}\varphi)^j \wedge \chi^{k-j} \wedge \alpha^{n-k} \right) ds &= \int_0^1 \int_X j\varphi \left(i\partial\bar{\partial} \frac{\partial \varphi}{\partial s} \right) \wedge (i\partial\bar{\partial}\varphi)^{j-1} \wedge \chi^{k-j} \wedge \alpha^{n-k} ds \\ &= \int_0^1 \int_X j \frac{\partial \varphi}{\partial s} (i\partial\bar{\partial}\varphi)^j \wedge \chi^{k-j} \wedge \alpha^{n-k} ds, \end{aligned} \quad (4.4.35)$$

here we used in the second identity the integration by parts and the fact that χ and α are closed. Combining (4.4.34) and (4.4.35) yields

$$\int_0^1 \int_X \frac{\partial \varphi}{\partial s} (i\partial\bar{\partial}\varphi)^j \wedge \chi^{k-j} \wedge \alpha^{n-k} ds = \frac{1}{j+1} \int_X \phi (i\partial\bar{\partial}\varphi)^j \wedge \chi^{k-j} \wedge \alpha^{n-k}. \quad (4.4.36)$$

Therefore (4.4.33) implies that

$$\begin{aligned} \int_0^1 \int_X \frac{\partial \varphi}{\partial s} \chi_\varphi^k \wedge \alpha^{n-k} ds &= \sum_{j=1}^k \binom{k}{j} \frac{1}{j+1} \int_X \phi (i\partial\bar{\partial}\varphi)^j \wedge \chi^{k-j} \wedge \alpha^{n-k} \\ &= \sum_{j=1}^k \binom{k}{j} \frac{1}{j+1} \int_X \phi (\chi_\phi - \chi)^j \wedge \chi^{k-j} \wedge \alpha^{n-k} \\ &= \sum_{j=1}^k \binom{k}{j} \frac{1}{j+1} \int_X \sum_{p=0}^j \binom{j}{p} (-1)^{j-p} \phi \chi_\phi^p \wedge \chi^{k-p} \wedge \alpha^{n-k} \\ &= \sum_{p=0}^k \left(\sum_{j=p}^k \binom{k}{j} \frac{1}{j+1} \binom{j}{p} (-1)^{j-p} \right) \int_X \phi \chi_\phi^p \wedge \chi^{k-p} \wedge \alpha^{n-k}. \end{aligned} \quad (4.4.37)$$

By changing $m = j - p$, we get

$$\sum_{j=p}^k \binom{k}{j} \frac{1}{j+1} \binom{j}{p} (-1)^{j-p} = \binom{k}{p} \sum_{m=0}^{k-p} \frac{(-1)^m}{m+p+1} \binom{k-p}{m}. \quad (4.4.38)$$

The right hand side can be computed by

$$\begin{aligned} \binom{k}{p} \sum_{m=0}^{k-p} \frac{(-1)^m}{m+p+1} \binom{k-p}{m} &= \binom{k}{p} \int_0^1 (1-x)^{k-p} x^p dx \\ &= \binom{k}{p} p! \int_0^1 \frac{1}{(k-p+1) \dots k} (1-x)^k dx = \frac{1}{k+1}, \end{aligned}$$

where we used the integration by parts p times in the second identity. Combining this with (4.4.37) and (4.4.38) we get the desired identity (4.4.31). Q.E.D.

We now have for any $t^* > 0$, along the flow

$$I_k(u(t^*)) = \int_0^{t^*} \int_X \frac{\partial u}{\partial t} \chi_u^k \wedge \alpha^{n-k} = \int_0^{t^*} \left(c - \frac{\chi_u^\ell \wedge \alpha^{n-\ell}}{\chi_u^k \wedge \alpha^{n-k}} \right) \chi_u^k \wedge \alpha^{n-k} = 0.$$

As in Weinkove [Wei04, Wei06], there exist $C_1, C_2 > 0$ such that for all $t \in [0, \infty)$,

$$0 \leq \sup_X u(\cdot, t) \leq -C_1 \inf_X u(\cdot, t) + C_2. \quad (4.4.39)$$

Indeed, in view of (4.4.31), $I_k(u) = 0$ along the flow implies that

$$\sum_{j=0}^k \int_X u \chi_u^j \wedge \chi^{k-j} \wedge \alpha^{n-k} = 0, \quad (4.4.40)$$

hence $\sup_X u \geq 0$ and $\inf_X u \leq 0$. For the right inequality in (4.4.39), we remark that there exists a positive constant B such that

$$\alpha^n \leq B \chi^k \wedge \alpha^{n-k}.$$

Therefore combining with (4.4.40) gives

$$\begin{aligned} \int_X u \alpha^n &= \int_X (u - \inf_X u) \alpha^n + \int_X \inf_X u \alpha^n \\ &\leq B \int_X (u - \inf_X u) \chi^k \wedge \alpha^{n-k} + \inf_X u \int_X \alpha^n \\ &= -B \sum_{j=1}^k \int_X u \chi_u^j \wedge \chi^{k-j} \wedge \alpha^{n-k} + \inf_X u \left(\int_X \alpha^n - B \int_X \chi^k \wedge \alpha^{n-k} \right) \\ &= -B \sum_{j=1}^k \int_X \left(u - \inf_X u \right) \chi_u^j \wedge \chi^{k-j} \wedge \alpha^{n-k} + \inf_X u \left(\int_X \alpha^n - B(k+1) \int_X \chi^k \wedge \alpha^{n-k} \right) \\ &\leq \inf_X u \left(\int_X \alpha^n - B(k+1) \int_X \chi^k \wedge \alpha^{n-k} \right) = -C_1 \inf_X u. \end{aligned}$$

Since $\Delta_\alpha u \geq -\text{tr}_\alpha \chi \geq -A$, using the fact that the Green's function $G(\cdot, \cdot)$ of α is bounded from below we infer that

$$\begin{aligned} u(x, t) &= \int_X u \alpha^n - \int_X \Delta_\alpha u(y, t) G(x, y) \alpha^n(y) \\ &\leq -C_1 \inf_X u + C_2. \end{aligned}$$

Hence we obtain the Harnack inequality, $\sup_X u \leq -C_1 \inf_X u + C_2$.

Since we can normalize \underline{u} by $\sup_X \underline{u} = 0$, the left inequality in (4.4.40) implies

$$\sup_X (u(\cdot, t) - \underline{u}(\cdot, t)) \geq 0.$$

It follows from Lemma 4.2.1 that

$$u \geq \underline{u} - C_3$$

for some constant C_3 . This give a lower bound for u since \underline{u} is bounded. The Harnack inequality in (4.4.39) implies then a uniform bound for u . Now the second statement in Lemma 4.3.2 implies the convergence of u . Q.E.D.

A natural generalization of the Hessian quotient flows on Hermitian manifolds is the following flow

$$\partial_t u = \log \frac{\chi_u^k \wedge \alpha^{n-k}}{\chi_u^\ell \wedge \alpha^{n-\ell}} - \psi \quad (4.4.41)$$

where $\psi \in C^\infty(X)$, the admissible cone is Γ_k , $1 \leq \ell < k \leq n$, and $\chi_u = \chi + i\partial\bar{\partial}u$. This flow was introduced by Sun [Sun15a] when $k = n$. We can apply Theorem 4.1.3 to obtain the following result, which is analogous to one of the main results in Sun [Sun15a], and analogous to the results of Song-Weinkove [SW08] and Fang-Lai-Ma [FLM11] for $k = n$:

Corollary 4.4.8. *Let (X, α) be a compact Hermitian manifold and χ be a $(1, 1)$ -form which is k -positive. Assume that there exists a form $\chi' = \chi + i\partial\bar{\partial}\underline{u}$ which is k -positive, and satisfies*

$$k(\chi')^{k-1} \wedge \alpha^{n-k} - e^\psi \ell(\chi')^{\ell-1} \wedge \alpha^{n-\ell} > 0 \quad (4.4.42)$$

in the sense of the positivity of $(n-1, n-1)$ -forms. Assume further that there exists an admissible $u_0 \in C^\infty(X)$ satisfying

$$e^\psi \geq \frac{\chi_{u_0}^k \wedge \alpha^{n-k}}{\chi_{u_0}^\ell \wedge \alpha^{n-\ell}} \quad (4.4.43)$$

Then the flow (4.4.41) admits a smooth solution for all time with initial data u_0 . Furthermore, there exists a unique constant c so that the normalization

$$\tilde{u} = u - \frac{1}{[\alpha^n]} \int_X u \alpha^n \quad (4.4.44)$$

converges in C^∞ to a function u_∞ with $\omega_\infty = \chi + i\partial\bar{\partial}u_\infty$ satisfying

$$\omega_\infty^k \wedge \alpha^{n-k} = e^{\psi+c}\omega_\infty^\ell \wedge \alpha^{n-\ell}. \quad (4.4.45)$$

Proof of Corollary 4.4.8. This equation is of the form (4.1.1), with

$$F(A) = f(\lambda) = \log \frac{\sigma_k(\lambda)}{\sigma_\ell(\lambda)}, \quad \text{with } \lambda = \lambda(A), \quad (4.4.46)$$

defined on Γ_k . As in the proof of Corollary 4.4.6 we also have that f satisfies the conditions (1-3) mentioned in §1. Moreover, f is in the bounded case with

$$f_\infty(\lambda') = \log \frac{k\sigma_{k-1}(\lambda')}{\ell\sigma_{\ell-1}(\lambda')} \quad \text{where } \lambda' \in \Gamma_\infty = \Gamma_{k-1}.$$

It suffices to verify that $\underline{u} = 0$ is a subsolution of the equation (4.4.41). For any $(z, t) \in X \times [0, \infty)$, set $\mu = \lambda(B)$, $B^i_j = \alpha^{j\bar{k}}\chi_{\bar{k}j}(z, t)$. Since \underline{u} is independent of t , Lemma 4.4.1 implies that we just need to show that for any $z \in X$ if $\mu' = (\mu_1, \dots, \mu_{n-1})$,

$$\lim_{s \rightarrow \infty} f(\mu', s) > \psi(z). \quad (4.4.47)$$

This means

$$f_\infty(\mu') = \log \frac{k\sigma_{k-1}(\mu')}{\ell\sigma_{\ell-1}(\mu')} > \psi(z), \quad (4.4.48)$$

where we restrict to the tangent space of X spanned by the eigenvalues corresponding to μ' . As the argument in the proof of Corollary 4.4.6, this inequality is equivalent to

$$k\chi^k \wedge \alpha^{n-k} - \ell e^\psi \chi^{\ell-1} \wedge \alpha^{n-\ell} > 0. \quad (4.4.49)$$

Moreover, the condition (4.4.43) is equivalent to

$$0 = \underline{u} \geq F(A[u_0]) - \psi. \quad (4.4.50)$$

We can now apply Theorem 4.1.3 to complete the proof. Q.E.D

In the case of (X, α) compact Kähler, the condition on ψ can be simplified, and we obtain an alternative proof to the main result of Sun in [Sun17a]. We recently learnt that Sun [Sun17c] also provided independently another proof of [Sun17a] using the same flow as below:

Corollary 4.4.9. *Let (X, α) be Kähler and χ be a k -positive closed $(1, 1)$ -form. Assume that there exists a closed form $\chi' = \chi + i\partial\bar{\partial}u$ which is k -positive, and satisfies*

$$k(\chi')^{k-1} \wedge \alpha^{n-k} - e^\psi \ell(\chi')^{\ell-1} \wedge \alpha^{n-\ell} > 0 \quad (4.4.51)$$

in the sense of the positivity of $(n-1, n-1)$ -forms. Assume further that

$$e^\psi \geq c_{k,\ell} = \frac{[\chi^k] \cup [\chi^{n-k}]}{[\chi^\ell] \cup [\alpha^{n-\ell}]}.$$
 (4.4.52)

Then for any admissible initial data $u_0 \in C^\infty(X)$, the flow (4.4.41) admits a smooth solution for all time. Furthermore, there exists a unique constant c so that the normalization

$$\tilde{u} = u - \frac{1}{[\alpha^n]} \int_X u \alpha^n$$
 (4.4.53)

converges in C^∞ to a function u_∞ with $\omega_\infty = \chi + i\partial\bar{\partial}u_\infty$ satisfying

$$\omega_\infty^k \wedge \alpha^{n-k} = e^{\psi+c} \omega_\infty^\ell \wedge \alpha^{n-\ell}.$$
 (4.4.54)

Proof of Corollary 4.4.8. By the same argument above, the admissible function $\underline{u} \in C^\infty(X)$ with $\sup_X \underline{u} = 0$ satisfying (4.4.51) is a C -subsolution. As explained in the proof of Corollary 4.4.6, we can assume that $u_0 = 0$.

We first observe that along the flow, the functional I_ℓ defined in Lemma 4.4.7 is decreasing. Indeed, using Jensen's inequality and then (4.4.52) we have

$$\begin{aligned} \frac{d}{dt} I_\ell(u) &= \int_X \frac{\partial u}{\partial t} \chi_u^\ell \wedge \alpha^{n-\ell} = \int_X \left(\log \frac{\chi_u^k \wedge \alpha^{n-k}}{\chi_u^\ell \wedge \alpha^{n-\ell}} - \psi \right) \chi_u^\ell \wedge \alpha^{n-\ell} \\ &\leq \log c_{k,\ell} \int_X \chi_u^\ell \wedge \alpha^{n-\ell} - \int_X \psi \chi_u^\ell \wedge \alpha^{n-\ell} \leq 0. \end{aligned}$$
 (4.4.55)

Set

$$\hat{u} := u - h(t), \quad h(t) = \frac{I_\ell(u)}{\int_X \chi^\ell \wedge \alpha^{n-\ell}}.$$
 (4.4.56)

For any $t^* \in [0, \infty)$ we have

$$I_\ell(\hat{u}(t^*)) = \int_0^{t^*} \int_X \frac{\partial \hat{u}}{\partial t} \chi_u^\ell \wedge \alpha^{n-\ell} = \int_0^{t^*} \int_X \left(\frac{\partial u}{\partial t} - \frac{1}{\int_X \chi^\ell \wedge \alpha^{n-\ell}} \frac{d}{dt} I_\ell(u) \right) \chi_u^\ell \wedge \alpha^{n-\ell} = 0.$$

By the same argument in Corollary 4.4.6, we deduce that there exist $C_1, C_2 > 0$ such that

$$0 \leq \sup_X \hat{u}(\cdot, t) \leq -C_1 \inf_X \hat{u}(\cdot, t) + C_2,$$
 (4.4.57)

for all $t \in [0, \infty)$. By our choice, $\sup_X \underline{u} = 0$, and (4.4.57) implies that

$$\sup_X (u - h(t) - \underline{u}) = \sup_X (\hat{u} - \underline{u}) \geq 0, \quad \forall t \geq 0.$$

Since $I_\ell(u)$ is decreasing along the flow, we also have $h'(t) \leq 0$. Theorem 4.1.3 now gives us the required result. Q.E.D.

Similarly, we can consider the flow (4.1.1) with

$$\partial_t u = - \left(\frac{\chi_u^\ell \wedge \alpha^{n-\ell}}{\chi_u^k \wedge \alpha^{n-k}} \right)^{\frac{1}{k-\ell}} + \psi(z), \quad u(z, 0) = 0, \quad (4.4.58)$$

where $1 \leq \ell < k \leq n$. When (X, α) is Kähler, ψ is constant and $k = n$, this is the inverse Hessian flow studied by Fang-Lai-Ma [FLM11]. We can apply Theorem 4.1.3 to obtain another corollary which is analogous to the main result of Fang-Lai-Ma [FLM11].

Corollary 4.4.10. *Let (X, α) , and χ as in Corollary 4.4.8. Assume further that $\psi \in C^\infty(X, \mathbb{R}^+)$ and there exists a smooth function \underline{u} with $\chi' = \chi + i\partial\bar{\partial}\underline{u}$ a k -positive $(1, 1)$ -form which satisfies*

$${}_k\psi^{k-\ell}(\chi')^{k-1} \wedge \alpha^{n-k} - \ell(\chi')^{n-\ell-1} \wedge \alpha^{n-\ell} > 0 \quad (4.4.59)$$

in the sense of positivity of $(n-1, n-1)$ forms, and

$$\psi^{k-\ell} \leq \frac{\chi^\ell \wedge \alpha^{n-\ell}}{\chi^k \wedge \alpha^{n-k}}. \quad (4.4.60)$$

Then the flow (4.4.58) exists for all time, and there is a unique constant c so that the normalized function \tilde{u} converges to a function u_∞ with $\omega = \chi + i\partial\bar{\partial}u_\infty$ a k -positive form satisfying the equation

$$\omega^{n-\ell} \wedge \alpha^{n-\ell} = (\psi + c)^{k-\ell} \omega^k \wedge \alpha^{n-k}. \quad (4.4.61)$$

In particular, if (X, α) is Kähler, we assume further that χ is closed, then the condition (4.4.60) can be simplified as

$$\psi^{k-\ell} \leq c_{\ell, k} = \frac{[\chi^\ell] \cup [\alpha^{n-\ell}]}{[\chi^k] \cup [\alpha^{n-k}]}. \quad (4.4.62)$$

Proof of Corollary 4.4.10. This equation is of the form (4.1.1), with

$$F(A) = f(\lambda) = - \left(\frac{\sigma_\ell(\lambda)}{\sigma_k(\lambda)} \right)^{\frac{1}{k-\ell}}, \quad \text{with } \lambda = \lambda(A), \quad (4.4.63)$$

defined on Γ_k . As in Corollary 4.4.6, it follows from the Maclaurin's inequality, the monotonicity and concavity of $g = (\sigma_k/\sigma_\ell)^{\frac{1}{k-\ell}}$ (cf. [Spr05]) that f satisfies the conditions (1-3) spelled out in §1. Moreover, f is in the bounded case with

$$f_\infty(\lambda') = - \left(\frac{\ell\sigma_{\ell-1}(\lambda')}{k\sigma_{k-1}(\lambda')} \right)^{\frac{1}{k-\ell}} \quad \text{where } \lambda' \in \Gamma_\infty = \Gamma_{k-1}.$$

In addition, as the same argument in previous corollaries, the condition (4.4.59) is equivalent to that $\underline{u} = 0$ is a C -subsolution for (4.4.58). Moreover, the condition (4.4.60) implies that

$$0 = \underline{u} \geq F(A[0]) + \psi. \quad (4.4.64)$$

We can now apply Theorem 4.1.3 to get the first result.

Next, assume that (X, α) is Kähler and χ is closed. As in Corollary 4.4.9 and [FLM11], the functional I_ℓ (see Lemma 4.4.7) is decreasing along the flow. Indeed, using (4.4.62),

$$\begin{aligned} \frac{d}{dt} I_\ell(u) &= \int_X \frac{\partial u}{\partial t} \chi_u^\ell \wedge \alpha^{n-\ell} = \int_X \left(- \left(\frac{\sigma_\ell(\lambda)}{\sigma_k(\lambda)} \right)^{\frac{1}{k-\ell}} + \psi \right) \chi_u^\ell \wedge \alpha^{n-\ell} \\ &\leq - \int_X \left(\frac{\sigma_\ell(\lambda)}{\sigma_k(\lambda)} \right)^{\frac{1}{k-\ell}} \chi_u^\ell \wedge \alpha^{n-\ell} + c_{\ell,k}^{\frac{1}{k-\ell}} \int_X \chi_u^\ell \wedge \alpha^{n-\ell}. \end{aligned} \quad (4.4.65)$$

Using the Hölder inequality, we get

$$\begin{aligned} \int_X \chi_u^\ell \wedge \alpha^{n-\ell} &= \int_X \sigma_\ell \alpha^n = \int_X \left(\frac{\sigma_\ell}{\sigma_k^{\frac{1}{1/(k-\ell+1)}}} \right) \sigma_k^{\frac{1}{k-\ell+1}} \alpha^n \\ &\leq \left[\int_X \left(\frac{\sigma_\ell}{\sigma_k^{\frac{1}{1/(k-\ell+1)}}} \right)^{\frac{k-\ell+1}{k-\ell}} \alpha^n \right]^{\frac{k-\ell}{k-\ell+1}} \left(\int_X \sigma_k \alpha^n \right)^{\frac{1}{k-\ell+1}} \\ &= \left[\int_X \left(\frac{\sigma_\ell(\lambda)}{\sigma_k(\lambda)} \right)^{\frac{1}{k-\ell}} \chi_u^\ell \wedge \alpha^{n-\ell} \right]^{\frac{k-\ell}{k-\ell+1}} \left(\int_X \chi_u^k \wedge \alpha^{n-k} \right)^{\frac{1}{k-\ell+1}} \\ &= \left[\int_X \left(\frac{\sigma_\ell(\lambda)}{\sigma_k(\lambda)} \right)^{\frac{1}{k-\ell}} \chi_u^\ell \wedge \alpha^{n-\ell} \right]^{\frac{k-\ell}{k-\ell+1}} c_{\ell,k}^{\frac{-1}{k-\ell+1}} \left(\int_X \chi_u^\ell \wedge \alpha^{n-\ell} \right)^{\frac{1}{k-\ell+1}}. \end{aligned}$$

This implies that

$$c_{\ell,k}^{\frac{1}{k-\ell}} \int_X \chi_u^\ell \wedge \alpha^{n-\ell} \leq \int_X \left(\frac{\sigma_\ell(\lambda)}{\sigma_k(\lambda)} \right)^{\frac{1}{k-\ell}} \chi_u^\ell \wedge \alpha^{n-\ell},$$

hence $dI_\ell(u)/dt \leq 0$.

For the rest of the proof, we follow the argument in Corollary 4.4.9, starting from the fact that $I_\ell(\hat{u}) = 0$ where

$$\hat{u} = u - \frac{I_\ell(u)}{\int_X \chi_u^\ell \wedge \alpha^{n-\ell}}.$$

Then we obtain the Harnack inequality

$$0 \leq \sup_X \hat{u}(\cdot, t) \leq -C_1 \inf_X \hat{u}(\cdot, t) + C_2, \quad (4.4.66)$$

for some constants $C_1, C_2 > 0$. Finally, Theorem 4.1.3 gives us the last claim. Q.E.D.

4.4.6 Flows with mixed Hessians σ_k

Our method can be applied to solve other equations containing many terms of σ_k . We illustrate this with the equation

$$\sum_{j=1}^{\ell} c_j \chi_u^j \wedge \alpha^{n-j} = c \chi_u^k \wedge \alpha^{n-k} \quad (4.4.67)$$

on a Kähler manifold (X, α) , where $1 \leq \ell < k \leq n$, $c_j \geq 0$ are given non-negative constants, and $c \geq 0$ is determined by c_j by integrating the equation over X .

When $k = n$, It was conjectured by Fang-Lai-Ma [FLM11] that this equation is solvable assuming that

$$nc\chi^{n-1} - \sum_{k=1}^{n-1} kc_k\chi^{k-1} \wedge \alpha^{n-k} > 0,$$

for some closed k -positive form $\chi' = \chi + i\partial\bar{\partial}v$. This conjecture was solved recently by Collins-Székelyhidi [CS17] using the continuity method. An alternative proof by flow methods is in Sun [Sun15c]. Theorem 4.1.4 stated earlier in Section 4.1 is an existence result for more general equations (4.4.67) using the flow (4.1.12). In particular, it gives a parabolic proof of a generalization of the conjecture due to Fang-Lai-Ma [FLM11, Conjecture 5.1]. We also remark that the flow (4.1.12) was mentioned in Sun [Sun15a], but no result given there, to the best of our understanding.

Proof of Theorem 4.1.4. This equation is of the form (4.1.1), with

$$F(A) = f(\lambda) = -\frac{\sum_{j=1}^{\ell} c_j \sigma_j(\lambda)}{\sigma_k(\lambda)} + c,$$

defined on the cone Γ_k . As in the proof of Corollary 4.4.6, for any $j = 1, \dots, \ell$, the function $-\sigma_j/\sigma_k$ on Γ_k satisfies the conditions (1-3) in §1, so does f . We also have that f is in the bounded case with

$$f_{\infty}(\lambda') = -\frac{\sum_{j=1}^{\ell} jc_j\sigma_{j-1}(\lambda)}{k\sigma_{k-1}(\lambda)} \quad \text{where } \lambda' \in \Gamma_{\infty} = \Gamma_k.$$

Suppose $\chi' = \chi + i\partial\bar{\partial}\underline{u}$ with $\sup_X \underline{u} = 0$ satisfies

$$kc(\chi')^{k-1} \wedge \alpha^{n-k} - \sum_{j=1}^{\ell} jc_j(\chi')^{j-1} \wedge \alpha^{n-j} > 0.$$

By the same argument in Corollary 4.4.6, this is equivalent to that \underline{u} is a C -subsolution of (4.1.12). Observe that for all $t^* > 0$,

$$\begin{aligned} I_k(u(t^*)) &= \int_0^{t^*} \int_X \frac{\partial u}{\partial t} \chi_u^k \wedge \alpha^{n-k} = \int_0^{t^*} \int_X \left(c - \frac{\sum_{j=1}^{\ell} c_j \sigma_j(\lambda)}{\sigma_k(\lambda)} \right) \chi_u^k \wedge \alpha^{n-k} \\ &= \int_0^{t^*} \left(c \int_X \chi_u^k \wedge \alpha^{n-k} - \sum_{j=1}^{\ell} c_j \int_X \chi_u^j \wedge \alpha^{n-j} \right) = 0. \end{aligned} \quad (4.4.68)$$

Therefore Lemma 4.4.7 implies that

$$\sum_{j=0}^k \int_X u \chi_u^j \wedge \chi^{k-j} \wedge \alpha^{n-k} = 0.$$

Therefore we can obtain the Harnack inequality as in Corollary 4.4.6:

$$0 \leq \sup_X u(\cdot, t) \leq -C_1 \inf_X u(\cdot, t) + C_2, \quad (4.4.69)$$

and $\inf_X u < 0$, for some positive constants C_1, C_2 . Lemma 4.2.1 then gives a uniform bound for u . Since

$$\int_X \partial_t u \chi_u^k \wedge \alpha^{n-k} = 0,$$

for any $t > 0$, there exists $y = y(t)$ such that $\partial_t u(y, t) = 0$. The rest of the proof is the same to the proof of Corollary 4.4.6 where we used Lemma 4.3.2 to imply the convergence of the flow. Q.E.D.

We observe that equations mixing several Hessians seem to appear increasingly frequently in complex geometry. A recent example of particular interest is the Fu-Yau equation [FY08, FY07, PPZ15, PPZ17c] and its corresponding geometric flows [PPZ16b].

4.4.7 Concluding Remarks

We conclude with a few open questions.

It has been conjectured by Lejmi and Székelyhidi [LS15] that conditions of the form (4.4.22) and their generalizations can be interpreted as geometric stability conditions. This conjecture has been proved in the case of the J -flow on toric varieties by Collins and Székelyhidi [CS17]. Presumably there should be similar interpretations in terms of stability of the conditions formulated in the previous section. A discussion of stability conditions for constant scalar curvature Kähler metrics can be found in [PS09].

It would also be very helpful to have a suitable geometric interpretation of conditions such as the one on the initial data u_0 . Geometric flows whose behavior may behave very

differently depending on the initial data include the anomaly flows studied in [PPZ17a], [PPZ17b], [FHP17].

For many geometric applications, it would be desirable to extend the theory of subsolutions to allow the forms χ and ψ to depend on time as well as on u and ∇u .

Chapter 5

A viscosity approach to the Dirichlet problem for degenerate complex Hessian type equations

In this chapter, a viscosity approach is introduced for the Dirichlet problem associated to complex Hessian type equations on domains in \mathbb{C}^n . The arguments are modelled on the theory of viscosity solutions for real Hessian type equations developed by Trudinger [Tru90]. As consequence we solve the Dirichlet problem for the Hessian quotient and special Lagrangian equations. We also establish basic regularity results for the solutions.

The results of this chapter are joint work with Sławomir Dinew and Hoang-Son Do [DDT17].

5.1 Introduction

Partial differential equations play pivotal role in modern complex geometric analysis. Their applications typically involve a geometric problem which can be reduced to the solvability of an associated equation. This solvability can be deduced by various methods yet most of the basic approaches exploit a priori estimates for suitably defined weak solutions. Thus although geometers work in the smooth category, the associated weak theory plays an important role.

One of the most successful such theories is the pluripotential theory associated to the complex Monge-Ampère equation developed by Bedford and Taylor [BT76, BT82], Kołodziej [Kol98], Guedj and Zeriahi [GZ05] and many others. Roughly speaking pluripotential theory allows to define $(i\partial\bar{\partial}u)^k$ as a measure valued positive closed differential form (i.e. a closed positive current) for any locally bounded plurisubharmonic function which in turn allows to deal with non smooth weak solutions of Monge-Ampère equations. Unfortunately the pluripotential approach is applicable only for a limited class of nonlinear operators, such

as the m -Hessian equations- see [DK17, Lu13].

Some of the most important examples on nonlinear operators for which pluripotential tools do not seem to apply directly are the complex Hessian quotient operators. These are not only interesting for themselves but also appear in interesting geometrical problems. One such example is the Donaldson equation that we describe below.

Given a compact Kähler manifold (X, ω) equipped with another Kähler form χ one seeks a Kähler form $\tilde{\chi}$ cohomologous to χ such that

$$\omega \wedge \tilde{\chi}^{n-1} = c\tilde{\chi}^n \quad (5.1.1)$$

with the constant c dependent only on the cohomology classes of χ and ω .

In [Don99] Donaldson introduced this equation in order to study the properness of the Mabuchi functional. Its parabolic version known as the J -flow was introduced independently by Donaldson [Don99] and Chen [Chen00] and investigated afterwards by Song and Weinkove [Wei04, Wei06],[SW08]. It is known that the equation (5.1.1) is not always solvable. It was shown in [SW08] that a necessary and sufficient condition for the solvability of (5.1.1) is that there exists a metric χ' in $[\chi]$, the Kähler class of χ , satisfying

$$(nc\chi' - (n-1)\omega) \wedge \chi'^{n-2} > 0, \quad (5.1.2)$$

in the sense of $(n-1, n-1)$ forms. A conjecture of Lejmi and Székelyhidi [LS15] predicts that the solvability is linked to positivity of certain integrals which can be viewed as geometric stability conditions. It was also proved that, in general, these positivity conditions are equivalent to the existence of C -subsolutions introduced by Székelyhidi [Sze15]. They are also equivalent to the existence of parabolic C -subsolutions for the corresponding flows (cf. [PT17]). It would be helpful to study the boundary case when we only have nonnegativity conditions (see [FLSW14] for Donaldson equation on surfaces). It is expected that in this boundary case the equation admits suitably defined singular solutions which are smooth except on some analytic set. This has been confirmed in complex dimension two in [FLSW14] but the proof cannot be generalized to higher dimensions. In fact a major part of the problem is to develop the associated theory of weak solutions for the given Hessian quotient equation. An essential problem in applying some version of pluripotential theory for this equation is that one has to define the quotient of two measure valued operators.

In order to circumvent this difficulty one can look for possibly different theory of weak solutions. One such approach, known as the viscosity method was invented long ago in the real setting [CIL92], but was only recently introduced for complex Monge-Ampère equations by Eyssidieux-Guedj-Zeriahi [EGZ11], Wang [Wan12] and Harvey-Lawson [HL09].

In the current note we initiate the viscosity theory for general complex nonlinear elliptic PDEs. As the manifold case is much harder we focus only on the local theory i.e. we deal with functions defined over domains in \mathbb{C}^n . Precisely, let $\Omega \subset \mathbb{C}^n$ be a bounded domain, we consider the equation

$$F[u] := f(\lambda(Hu)) = \psi(x, u), \quad (5.1.3)$$

where $\lambda(Hu)$ denotes the vector of the eigenvalues of the complex Hessian Hu of the real valued function u and $\psi : \Omega \times \mathbb{R} \rightarrow \mathbb{R}_+$ is a given nonnegative function which is weakly increasing in the second variable. We wish to point out that nonlinear PDEs appear also in geometric problems which are defined over domains in \mathbb{C}^n - see for example [CPW17], where a Dirichlet problem for the *special Lagrangian type equation* is studied. These are the equations defined for a given function h by

$$F[u] := \sum_{i=1}^n \arctan \lambda_i = h(z),$$

with λ_i denoting the eigenvalues of the Hessian of u at z . In the real case the special Lagrangian equations were introduced by Harvey and Lawson [HL82] in the study of *calibrated geometries*. More precisely the graphs of gradients of the solutions correspond to calibrated minimal submanifolds. We show in Section 5.6 that our method can be applied to solve the Dirichlet problem for the special degenerate Lagrangian type equation.

In our investigations we heavily rely on the corresponding real theory developed by Trudinger in [Tru90]. Some of our results can be seen as complex analogues of the real results that can be found there. In particular we have focused on various comparison principles in Section 5.3. Our first major result can be summarized as follows (we refer to the next section for the definitions of the objects involved):

Theorem 5.1.1 (Comparison principle). *Let Γ be the ellipticity cone associated to the equation (5.1.3). Assume that the operator $F[u] = f(\lambda(Hu))$ in (5.1.3) satisfies*

$$f \in C^0(\bar{\Gamma}), f > 0 \text{ on } \Gamma, f = 0 \text{ on } \partial\Gamma,$$

and

$$f(\lambda + \mu) \geq f(\lambda), \forall \lambda \in \Gamma, \mu \in \Gamma_n.$$

Assume moreover that either

$$\sum_{i=1}^n \frac{\partial f}{\partial \lambda_i} \lambda_i = \sum_{i=1}^n f_i \lambda_i \geq \nu(f) \text{ in } \Gamma, \text{ and } \inf_{z \in \Omega} \psi(z, \cdot) > 0$$

for some positive increasing function ν , or

$$f \text{ is concave and homogeneous.}$$

Then any bounded subsolution u and supersolution v in Ω to the equation (5.1.3) satisfy

$$\sup_{\Omega} (u - v) \leq \max_{\partial\Omega} \{(u - v)^*, 0\}.$$

We use later on this seemingly technical result to study existence, uniqueness and regularity of the associated Dirichlet problems. One of our main result is the solvability and sharp regularity for viscosity solutions to the Dirichlet problem for a very general class of operators including Hessian quotient type equations.

Theorem 5.1.2. *The Dirichlet problem*

$$\begin{cases} F[w] = f(\lambda(Hw)) = \psi(z, w(z)) \\ u = \varphi \text{ on } \partial\Omega \end{cases}$$

admits a continuous solution for any bounded Γ -pseudoconvex domain Ω . Under natural growth assumptions on ψ the solution is Hölder continuous for any Hölder continuous boundary data φ .

Another interesting topic is the comparison between viscosity and pluripotential theory whenever the latter can be reasonably defined. A guiding principle for us is the basic observation made by Eyssidieux, Guedj and Zeriahi [EGZ11] that plurisubharmonic functions correspond to viscosity subsolutions to the complex Monge-Ampère equation. We prove several analogous results for general complex nonlinear operators. It has to be stressed that the notion of a supersolution, which does not appear in pluripotential theory, is a very subtle one for nonlinear elliptic PDEs and several alternative definitions are possible. We in particular compare these and introduce a notion of supersolution that unifies the previously known approaches.

A large part of the note is devoted to complex Hessian quotient equations in domains in \mathbb{C}^n . One of our goals in this case was to initiate the construction of the undeveloped pluripotential theory associated to such equations. We rely on connections with the corresponding viscosity theory. Our findings yield in particular that the natural domain of definition of these operators is *strictly smaller* than what standard pluripotential theory would predict. We prove the following theorem:

Theorem 5.1.3. *Assume that $0 < \psi \in C^0(\Omega)$ and $u \in PSH(\Omega) \cap L_{loc}^\infty(\Omega)$ is a viscosity subsolution of $\frac{(dd^c u)^n}{(dd^c u)^{n-k} \wedge \omega^k} = \psi(z)$ in Ω . Then*

$$(dd^c u)^n \geq \psi (dd^c u)^{n-k} \wedge \omega^k$$

and

$$(dd^c u)^k \geq \binom{n}{k}^{-1} \psi \omega^k$$

in the pluripotential sense.

We guess that this observation, rather obvious in the case of smooth functions, will play an important role in the resolution of the issue caused by the division of measures.

The chapter is organized as follows: in the next section we collect the basic notions from linear algebra, viscosity and pluripotential theory. Then we investigate the various notions of supersolutions in [EGZ11] and [Lu13] and compare them with the complex analogue of Trudinger's supersolutions. Section 3 is devoted to the proof of a very general

comparison principle. Then in Section 4 we restrict our attention to operators depending on the eigenvalues of the complex Hessian matrix of the unknown function. We show existence and uniqueness of viscosity solutions under fairly mild conditions. One subsection is devoted to the regularity of these weak solutions. Using classical methods due to Walsh [Wal68] (see also [BT76]) we show the optimal Hölder regularity for sufficiently regular data. Section 5 is devoted to comparisons between viscosity and pluripotential subsolutions and supersolutions. Finally in Section 5.6 we solve the Dirichlet problem for the Lagrangian phase operator.

5.2 Preliminaries

In this section we collect the notation and the basic results and definitions that will be used throughout the note.

5.2.1 Linear algebra toolkit

We begin by introducing the notion of an admissible cone that will be used throughout the note:

Definition 5.2.1. A cone Γ in \mathbb{R}^n with vertex at the origin is called admissible if:

- (1) Γ is open and convex, $\Gamma \neq \mathbb{R}^n$;
- (2) Γ is symmetric i.e. if $x = (x_1, \dots, x_n) \in \Gamma$ then for any permutation of indices $i = (i_1, \dots, i_n)$ the vector $(x_{i_1}, \dots, x_{i_n})$ also belongs to Γ ;
- (3) $\Gamma_n \subset \Gamma$, where $\Gamma_n := \{x \in \mathbb{R}^n \mid x_i > 0, i = 1, \dots, n\}$.

From the very definition it follows that Γ_n is an admissible cone. Other examples involve the Γ_k cones that we describe below:

Consider the m -th elementary symmetric polynomial defined by

$$\sigma_m(x) = \sum_{1 \leq j_1 < \dots < j_m \leq n} x_{j_1} x_{j_2} \dots x_{j_m}.$$

We shall use also the normalized version

$$S_m(x) := \binom{n}{m}^{-1} \sigma_m.$$

Definition 5.2.2. For any $m = 1, \dots, n$, the positive cone Γ_m of vectors $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is defined by

$$\Gamma_m = \{x \in \mathbb{R}^n \mid \sigma_1(x) > 0, \dots, \sigma_m(x) > 0\}. \quad (5.2.1)$$

It is obvious that these cones are open and symmetric with respect to a permutation of the x_i 's. It is a nontrivial but classical fact that Γ_m is also convex.

Exploiting the symmetry of Γ it is possible to discuss Γ positivity for Hermitian matrices:

Definition 5.2.3. A Hermitian $n \times n$ matrix A is called Γ positive (respectively Γ -semi positive) if the vector of eigenvalues $\lambda(A) := (\lambda_1(A), \dots, \lambda_n(A))$ belongs to Γ (resp. to the Euclidean closure $\bar{\Gamma}$ of Γ). The definition is independent of the ordering of the eigenvalues.

Finally one can define, following [Li04], the notion of Γ -admissible and Γ -subharmonic functions through the following definitions:

Definition 5.2.4. A C^2 function u defined on a domain $\Omega \subset \mathbb{C}$ is called Γ -admissible if for any $z \in \Omega$ the complex Hessian $Hu(z) := [\frac{\partial^2}{\partial z_j \partial \bar{z}_k}]_{j,k=1}^n$ is Γ -positive.

In particular, if Γ is an admissible cone, then $\Gamma \subset \Gamma_1$ (see [CNS85]), hence we have the following corollary:

Corollary 5.2.5. Any Γ -admissible function is subharmonic.

Definition 5.2.6. An upper semicontinuous function v defined on a domain $\Omega \subset \mathbb{C}^n$ is called Γ -subharmonic if near any $z \in \Omega$ it can be written as a decreasing limit of local Γ -admissible functions.

We refer to [HL09] for a detailed discussion and potential theoretic properties of general Γ -subharmonic functions.

5.2.2 Viscosity sub(super)-solutions

Let Ω be a bounded domain in \mathbb{C}^n . Consider the following equation:

$$F[u] := F(x, u, Du, Hu) = 0, \quad \text{on } \Omega, \quad (5.2.2)$$

where $Du = (\partial_{z_1} u, \dots, \partial_{z_n} u)$, $Hu = (u_{j\bar{k}})$ is the Hessian matrix of u and F is continuous on $\Omega \times \mathbb{R} \times \mathbb{C}^n \times \mathcal{H}^n$. The operator F is called *degenerate elliptic* at a point (z, s, p, M) if

$$F(z, s, p, M + N) \geq F(z, s, p, M) \quad \text{for all } N \geq 0, N \in \mathcal{H}^n, \quad (5.2.3)$$

where \mathcal{H}^n is the set of Hermitian matrices of size $n \times n$. We remark that in our case $F(z, s, p, M)$ is not necessarily degenerate elliptic everywhere on $\Omega \times \mathbb{R} \times \mathbb{C}^n \times \mathcal{H}^n$. Motivated by the paper of Trudinger [Tru90] we pose the following definition:

Definition 5.2.7. A function $u \in L^\infty(\Omega)$ is a viscosity subsolution of (5.2.2) if it is upper semi-continuous in Ω and for any $z_0 \in \Omega$, and any C^2 smooth function q defined in some neighbourhood of z_0 and satisfying $u \leq q$, $u(z_0) = q(z_0)$, the inequality

$$F[q](z_0) \geq 0 \quad (5.2.4)$$

holds. We also say that $F[u] \geq 0$ in the viscosity sense and q is an upper (differential) test for u at z_0 .

A function $v \in L^\infty(\Omega)$ is a viscosity supersolution of equation (5.2.2) if it is lower semi-continuous and there are no points $z_0 \in \Omega$ and functions C^2 smooth functions defined locally around z_0 , such that $v \geq q$ in Ω , $v(z_0) = q(z_0)$ and

$$\inf_{N \geq 0} F(z_0, q(z_0), Dq(z_0), N + H\psi(z_0)) > 0. \quad (5.2.5)$$

We also say that $F[u] \leq 0$ in the viscosity sense and q is a lower (differential) test for u at z_0 .

For fixed $(z, s, p) \in \Omega \times \mathbb{R} \times \mathbb{C}^n$ the set of all Hermitian matrices M , such that F is degenerate elliptic at (z, s, p, M) is called the *ellipticity set* $\mathcal{A}(z, s, p)$ for the data (z, s, p) . Note that the ellipticity set has the property that

$$\mathcal{A}(z, s, p) + \Gamma_n \subset \mathcal{A}(z, s, p),$$

but it may not be a cone. Throughout the note we shall however focus on the situation when the ellipticity set is a cone which is moreover constant for all the possible data sets. We then define the *ellipticity cone* associated to the operator F which is modelled on the notion of a subequation coined by Harvey and Lawson in [HL09] :

Definition 5.2.8. An operator $F(z, s, p, M)$ has an ellipticity cone Γ if for any M in the ellipticity set the vector $\lambda(M)$ of the eigenvalues of M belongs to the closure $\bar{\Gamma}$ of Γ . Furthermore Γ is the minimal cone with such properties.

Throughout the note we consider only the situation when Γ is an admissible cone in the sense of Definition 5.2.1. We shall make also the following additional assumption (compare with Condition (2) in Subsection 5.4.1):

$$\forall \lambda \in \partial\Gamma, \forall (z, s, p) \in \Omega \times \mathbb{R} \times \mathbb{C}^n \quad F(z, s, p, \lambda) \leq 0. \quad (5.2.6)$$

This condition arises naturally whenever one seeks solutions to

$$F(z, u(z), Du(z), Hu(z)) = 0$$

with pointwise Hessian eigenvalues in Γ (recall that F increases in the Γ_n directions).

It is evident that in Definition 5.2.7 the notion of a supersolution is different and substantially more difficult than the notion of a subsolution. The reason for this is that there is no analog for the role of the positive cone Γ_n from the case of subsolutions in the supersolutions' case. As an illustration we recall that while any plurisubharmonic function is a subsolution for $F(u) := \det(H(u)) = 0$ (see [EGZ11]) it is far from being true that all supersolutions can be written as the negative of a plurisubharmonic function.

Below we also give another notion of a supersolution that was coined in [EGZ11] for the Monge-Ampère equation (see also [Lu13] for the case of m -Hessian operator). It can be generalized for all operators admitting an elliptic admissible cone:

Definition 2. A lower semicontinuous function u is said to be a supersolution for the operator $F(z, s, p, M)$ with the associated ellipticity cone Γ iff for any $z_0 \in \Omega$ and every lower differential test q at z_0 for which $\lambda(Hq(z_0)) \in \bar{\Gamma}$ one has

$$F(z, q(z_0), Dq(z_0), Hq(z_0)) \leq 0.$$

Note that in the definition we limit the differential tests only to those for which $\lambda(Hq(z_0)) \in \bar{\Gamma}$.

The next proposition shows that under the assumption (5.2.6) the definition above coincides with the one from Definition 5.2.7.

Proposition 5.2.9. *Suppose that the operator $F(z, s, p, M)$ satisfies (5.2.6). Then a lower semicontinuous function u defined on a domain Ω is a supersolution for $F(z, s, p, M) = 0$ in the sense of Definition 2 if and only if it is a supersolution in the sense of Definition 5.2.7.*

Proof. Suppose first that u is a supersolution in the sense of Definition 2. Fix any z_0 in Ω and q a lower differential test for u at z_0 . If $\lambda(Hq(z_0)) \in \Gamma$ then

$$F(z, q(z_0), Dq(z_0), Hq(z_0)) \leq 0,$$

hence taking $N = 0$ in Definition 5.2.7 we see that the condition is fulfilled. If $\lambda(Hq(z_0))$ fails to be in Γ then there is a positive definite matrix N and a positive number t such that $\lambda(Hq(z_0) + tN) \in \partial\Gamma$. But this implies that $F(z, q(z_0), Dq(z_0), Hq(z_0) + tN) \leq 0$ which fulfills the condition in Definition 5.2.7 again.

Suppose now that u is a supersolution in the sense of Definition 5.2.7. Again choose z_0 in Ω and q a lower differential test for u at z_0 . We can assume that $\lambda(Hq(z_0))$ is in $\bar{\Gamma}$, for otherwise such a differential test cannot be applied in Definition 2. But then by ellipticity

$$F(z, q(z_0), Dq(z_0), Hq(z_0)) \leq F(z, q(z_0), Dq(z_0), Hq(z_0) + N), \forall N \geq 0, N \in \mathcal{H}^n.$$

The infimum over N for the right hand side is non positive by definition which implies

$$F(z, q(z_0), Dq(z_0), Hq(z_0)) \leq 0$$

which was to be proved. □

5.2.3 Aleksandrov-Bakelman-Pucci maximum principle

In this section, we recall a variant of Aleksandrov-Bakelman-Pucci (ABP) maximum principle following [Jen88]. We first recall the following definition (cf. [Jen88]):

Definition 5.2.10. Let Ω be a bounded domain in \mathbb{R}^n centered at the origin, and $u \in C(\bar{\Omega})$. We define

$$E_\delta = \{x \in \Omega \mid \text{for some } p \in \overline{B(0, \delta)}, u(z) \leq u(x) + p \cdot (z - x), \forall z \in \Omega\}.$$

Then we have the following lemma due to Jensen [Jen88] which will be used in the proof of Lemma 5.3.1. Recall that a function u is said to be semi-convex if $u + k|z|^2$ is convex for a sufficiently large constant k .

Lemma 5.2.11. *Let $u \in C(\bar{\Omega})$ be semi-convex for some constant $k > 0$. If u has an interior maximum and $\sup_{\Omega} u - \sup_{\partial\Omega} u = \delta_0 d > 0$, where $d = \text{diam}(\Omega)$. Then there is a constant $C = C(n, k) > 0$ such that*

$$|E_{\delta}| \geq C\delta^n, \quad \text{for all } \delta \in (0, \delta_0). \quad (5.2.7)$$

Proof. As in Jensen [Jen88], by regularization, we can reduce to the case when $u \in C^2(\Omega)$. Now, suppose that u has an interior maximum at x_0 and

$$\delta_0 = \frac{\sup_{\Omega} u - \sup_{\partial\Omega} u}{d} = \frac{u(x_0) - \sup_{\partial\Omega} u}{d},$$

where $d = \text{diam}(\Omega)$.

We now prove that for $\delta < \delta_0$ we have $B(0, \delta) \subset Du(E_{\delta})$. Indeed, for any $p \in B(0, \delta)$, consider the hyperplane $\ell_p(x) = h + \langle p, x \rangle$ where $h = \sup_{y \in \Omega} (u(y) - \langle p, y \rangle)$. Then we have $u(x) \leq \ell_p(x)$ on Ω and $u(x_1) = \ell_p(x_1)$ for some $x_1 \in \bar{\Omega}$. If we can prove that $x_1 \in \Omega$, then $Du(x_1) = p$, so $B(0, \delta) \subset Du(E_{\delta})$. Suppose by contradiction that $x_1 \in \partial\Omega$, then

$$\begin{aligned} \sup_{\Omega} u &= u(x_0) \\ &\leq \ell_p(x_1) + \langle p, x_0 - x_1 \rangle \\ &= u(x_1) + \langle p, x_0 - x_1 \rangle \leq \sup_{\partial\Omega} u + \delta d < \sup_{\partial\Omega} u + \delta_0 d = \sup_{\Omega} u, \end{aligned}$$

hence we get a contradiction.

Next, as we have proved that $B(0, \delta) \subset Du(E_{\delta})$, then by comparing volumes, we infer that

$$c(n)\delta^n \leq \int_{E_{\delta}} |\det(D^2u)|. \quad (5.2.8)$$

Since u is semi-convex with the constant $k > 0$ and $D^2u \leq 0$ in E_{δ} , we have $|\det(D^2u)| \leq k^n$. It follows that $|E_{\delta}| \geq c(n)k^{-n}\delta^n$. \square

5.2.4 Γ -subharmonic functions

We have defined Γ subharmonic functions as limits of admissible ones. Below we present the alternative viscosity and pluripotential points of view:

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. Denote $\omega = dd^c|z|^2$, where $d := i(\bar{\partial} + \partial)$ and $d^c := \frac{i}{2\pi}(\bar{\partial} - \partial)$ so that $dd^c = \frac{i}{\pi}\partial\bar{\partial}$. Let $\Gamma \subsetneq \mathbb{R}^n$ be an admissible cone as in Definition 5.2.1. We first recall the definition of k -subharmonic function:

Definition 3. We call a function $u \in C^2(\Omega)$ is k -subharmonic if for any $z \in \Omega$, the Hessian matrix $(u_{i\bar{j}})$ has eigenvalues forming a vector in the closure of the cone Γ_k .

Following the ideas of Bedford-Taylor [BT82], Blocki [Blo05a] introduced the pluripotential definition of the k -sh function.

Definition 5.2.12. Let u be subharmonic function on a domain $\Omega \subset \mathbb{C}^n$. Then u is called k -subharmonic (k -sh for short) if for any collection of C^2 -smooth k -sh functions v_1, \dots, v_{k-1} , the inequality

$$dd^c u \wedge dd^c v_1 \wedge \dots \wedge dd^c v_{k-1} \wedge \omega^{n-k} \geq 0$$

holds in the weak sense of currents.

For a general cone Γ , we have the following definition in the spirit of viscosity theory:

Definition 5.2.13. An upper semicontinuous function u is called Γ -subharmonic (resp. strictly Γ -subharmonic) if for any $z \in \Omega$, and any upper test function q of u at z , we have

$$\lambda(Hq(z)) \in \bar{\Gamma} \quad (\text{resp. } \lambda(Hq(z)) \in \Gamma).$$

By definition, if u is a Γ -subharmonic function, it is a Γ -subsolution in the sense of Székelyhidi [Sze15]. In particular, when $\Gamma = \Gamma_k$ for $k = 1, \dots, n$, u is a viscosity subsolution of the equation

$$S_k(\lambda(Hu)) = 0,$$

where

$$S_k(\lambda(Hu)) = \frac{(dd^c u)^k \wedge \omega^{n-k}}{\omega^n}.$$

Then it follows from [EGZ11, Lu13] that u is a k -subharmonic function on Ω , hence u is a subharmonic function if $k = 1$ and a plurisubharmonic function if $k = n$.

We also have the following definition generalizing the pseudoconvex domains (see also [Li04] for similar definition for smooth domains):

Definition 5.2.14. Let Ω be a bounded domain in \mathbb{C}^n , we say that Ω is a Γ -pseudoconvex domain if there is a constant $C_\Omega > 0$ depending only on Ω so that $-d(z) + C_\Omega d^2(z)$ is Γ -subharmonic on $\partial\Omega$, where $d(z) := \text{dist}(z, \partial\Omega)$.

We recall the following lemma which was proved in [Li04, Theorem 3.1].

Lemma 5.2.15. *Let Ω be bounded domain in \mathbb{C}^n with C^2 smooth boundary. Let $\rho \in C^2(\bar{\Omega})$ be a defining function of Ω so that $\lambda(H\rho) \in \Gamma$ on $\partial\Omega$. Then there exists a defining function $\bar{\rho} \in C^2(\bar{\Omega})$ for Ω such that $\lambda(H\bar{\rho}) \in \Gamma$ on $\bar{\Omega}$.*

Finally we wish to recall the survey article [Zer13] where the Reader may find a thorough discussion of the viscosity theory associated to complex Monge-Ampère type equations.

5.3 Comparison principles

Comparison principles are basic tools in pluripotential theory- we refer to [Kol98, GZ17b] for a thorough discussion of these inequalities. In viscosity theory one compares sub- and supersolutions to the same equation. It is a crucial observation (cf. [EGZ11]) that even though supersolutions may fail to have nice pluripotential properties a version of the comparison principle holds for the complex Monge-Ampère equation. In this section we discuss under what assumptions such comparison principles hold for general operators.

5.3.1 A preliminary comparison principle

Let Ω be a bounded domain in \mathbb{C}^n . In this subsection we prove a comparison principle for viscosity solutions of the following equation:

$$F[u] := F(x, u, Du, Hu) = 0. \quad (5.3.1)$$

It is well known that mere ellipticity is insufficient to guarantee comparison type result. Hence we add some natural structural conditions for the equation (5.3.1).

First of all we assume that F is decreasing in the s variable, namely

$$\forall r > 0 \quad F(z, s, p, M) - F(z, s + r, p, M) \geq 0. \quad (5.3.2)$$

This is a natural assumption in the theory (see [Zer13]) as it yields an inequality in the "right" direction for the maximum principle.

Next we assume certain continuity property with respect to the z and p variables:

$$|F(z_1, s, p_1, M) - F(z_2, s, p_2, M)| \leq \alpha_z(|z_1 - z_2|) + \alpha_p(|p_1 - p_2|), \quad (5.3.3)$$

for all $z_1, z_2 \in \Omega$, $\sigma \in \mathbb{R}$, $p_1, p_2 \in \mathbb{C}^n$, $M \in \mathcal{H}^n$. Here α_z and α_p are certain moduli of continuity i.e. increasing functions defined for nonnegative reals which tend to zero as the parameter decreases to zero.

We can now state the following general comparison principle for the equation (5.3.1).

Lemma 5.3.1. *Suppose $u \in L^\infty(\overline{\Omega})$ (resp. $v \in L^\infty(\overline{\Omega})$) satisfies $F[u] \geq \delta$ (resp. $F[v] \leq 0$) in Ω in the viscosity sense for some $\delta > 0$. Then*

$$\sup_{\Omega} (u - v) \leq \max_{\partial\Omega} \{(u - v)^*, 0\}, \quad (5.3.4)$$

with $*$ denoting the standard upper semicontinuous regularization.

Proof. The idea comes from [Tru90]. We use Jensen's approximation (cf. [Jen88]) for u, v which is defined by

$$\begin{aligned} u^\varepsilon(z) &= \sup_{z' \in \Omega} \left\{ u(z') - \frac{C_0}{\varepsilon} |z' - z|^2 \right\}, \\ v_\varepsilon(z) &= \inf_{z' \in \Omega} \left\{ v(z') + \frac{C_0}{\varepsilon} |z' - z|^2 \right\}, \end{aligned} \quad (5.3.5)$$

where $\varepsilon > 0$ and $C_0 = \max\{\text{osc}_\Omega u, \text{osc}_\Omega v\}$ with $\text{osc}(u) = \sup_\Omega u - \inf_\Omega u$. Then the supremum and infimum in (5.3.5) are achieved at points $z^*, z_* \in \Omega$ with $|z - z^*|, |z - z_*| < \varepsilon$ provided that $z \in \Omega_\varepsilon = \{z \in \Omega \mid \text{dist}(z, \partial\Omega) > \varepsilon\}$. It follows from [CC95] (see also [Wan12] for an adaption in the complex case) that u^ε (resp. v_ε) is Lipschitz and semi-convex (resp. semi-concave) in Ω_ε , with

$$|Du^\varepsilon|, |Dv_\varepsilon| \leq \frac{2C_0}{\varepsilon}, \quad Hu^\varepsilon, -Hv_\varepsilon \geq -\frac{2C_0}{\varepsilon^2} \text{Id}, \quad (5.3.6)$$

whenever these derivatives are well defined.

Exploiting the definition of viscosity subsolution one can show that u^ε satisfies

$$F(z^*, u^\varepsilon(z), Du^\varepsilon(z), Hu^\varepsilon(z)) \geq \delta \quad (5.3.7)$$

in the viscosity sense for all $z \in \Omega_\varepsilon$. Indeed, let q be an upper test of u^ε at z_0 , then the function

$$\tilde{q}(z) := q(z + z_0 - z_0^*) + \frac{1}{\varepsilon} |z_0 - z_0^*|^2$$

is an upper test for u at z_0^* . Therefore we get (5.3.7) as u is a viscosity subsolution. This also implies that

$$F(z^*, u^\varepsilon(z), Du^\varepsilon(z), N + Hu^\varepsilon(z)) \geq \delta, \quad (5.3.8)$$

in the viscosity sense for any fixed matrix $N \geq 0$. Since any locally semi-convex (semi-concave) function is twice differentiable almost everywhere by Aleksandroff's theorem, we infer that for almost all $z \in \Omega_\varepsilon$, F is degenerate elliptic at $(z^*, u^\varepsilon(z), Du^\varepsilon(z), Hu^\varepsilon(z))$ and

$$F(z^*, u^\varepsilon(z), Du^\varepsilon(z), N + Hu^\varepsilon(z)) \geq \delta, \quad (5.3.9)$$

for all $N \in \mathcal{H}^n$ such that $N \geq 0$.

We assume by contradiction that $\sup_\Omega (u - v) = u(z_0) - v(z_0) = a > 0$ for some $z_0 \in \Omega$. For any ε sufficiently small the function $w_\varepsilon := u^\varepsilon - v_\varepsilon$ has a positive maximum on Ω_ε at some point $z_\varepsilon \in \Omega_\varepsilon$ such that $z_\varepsilon \rightarrow z_0$ as $\varepsilon \rightarrow 0$. So we can choose $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, $w_\varepsilon := u^\varepsilon - v_\varepsilon$ has a positive maximum on Ω_ε at some point $z_\varepsilon \in \Omega$ with $d(z_\varepsilon, \partial\Omega) > \varepsilon_0$. Applying the ABP maximum principle (Lemma 5.2.11), for the function w_ε on Ω_{ε_0} and for any $\lambda > 0$ sufficiently small, there exist a set $E_\lambda \subset \Omega_{\varepsilon_0}$ containing z_ε with $|E_\lambda| \geq c\lambda^n$, where c is $c(n)\varepsilon^{2n}$, such that $|Dw_\varepsilon| \leq \lambda$ and $Hw_\varepsilon \leq 0$ almost everywhere in E_λ . Since $w_\varepsilon(z_\varepsilon) > 0$, we can choose λ small enough such that $w_\varepsilon \geq 0$ in E_λ . The condition (5.3.2) and the fact that F is degenerate elliptic at $(z^*, u^\varepsilon(z), Du^\varepsilon(z), Hu^\varepsilon(z))$ for almost all $z \in E_\lambda$, imply that

$$F(z^*, u^\varepsilon(z), Du^\varepsilon(z), N + Hu^\varepsilon(z)) \leq F(z^*, v_\varepsilon(z), Dv_\varepsilon(z), N + Hv_\varepsilon(z)). \quad (5.3.10)$$

Using (5.3.3) and the fact that $|D(u^\varepsilon - v_\varepsilon)| \leq \lambda$, we get

$$F(z^*, v_\varepsilon(z), Dv_\varepsilon(z), N + Hv_\varepsilon(z)) \leq F(z^*, v_\varepsilon(z), Dv_\varepsilon(z), N + Hv_\varepsilon(z)) + \alpha_p(\lambda).$$

Combining with (5.3.3), (5.3.9), (5.3.10) and $|z^* - z_*| < \varepsilon$ that for almost all $z \in E_\lambda$,

$$F(z_*, v_\varepsilon(z), Dv_\varepsilon(z), N + Hv_\varepsilon(z)) \geq \delta - \alpha_z(\varepsilon) - \alpha_p(\lambda). \quad (5.3.11)$$

By taking λ , and then ε sufficiently small and using the fact that v_ε is twice differentiable almost everywhere on Ω , we can find at a fixed point $z_1 \in E_\lambda$ a lower test q of v at z_1 such that

$$F(z_0, q(z_0), Dq(z_0), N + Hq(z_0)) \geq \frac{1}{2}\delta, \quad (5.3.12)$$

for all $N \geq 0$. This contradicts the definition of viscosity supersolution. Therefore we get (5.3.4). \square

Remark. By assuming more properties of F , it is possible to obtain $\delta = 0$ in the previous result. This is the case for the Monge-Ampère equation. Otherwise we need to adjust function u to achieve a strict inequality in order to use Lemma 5.3.1.

5.3.2 Comparison principle for Hessian type equations

We now consider the Hessian type equation of the form

$$F[u] = \psi(z, u), \quad (5.3.13)$$

where $\psi \in C^0(\Omega \times \mathbb{R})$ and $F[u] = f(\lambda(Hu))$ such that

$$s \mapsto \psi(\cdot, s) \text{ is weakly increasing,} \quad (5.3.14)$$

$$f \in C^0(\bar{\Gamma}), f > 0 \text{ on } \Gamma, f = 0 \text{ on } \partial\Gamma, \quad (5.3.15)$$

and

$$f(\lambda + \mu) \geq f(\lambda), \quad \forall \lambda \in \Gamma, \mu \in \Gamma_n. \quad (5.3.16)$$

First, in order to use Lemma 5.3.1, we extend f continuously on \mathbb{R}^n by taking $f(\lambda) = 0$ for all $\lambda \in \mathbb{R}^n \setminus \Gamma$. For a δ independent comparison principle we need more assumptions on F . Similarly to [Tru90], we can assume that the operator $F[u] = f(\lambda(Hu))$ satisfies

$$\sum_{i=1}^n \frac{\partial f}{\partial \lambda_i} \lambda_i = \sum_{i=1}^n f_i \lambda_i \geq \nu(f) \text{ in } \Gamma, \quad \inf_{z \in \Omega} \psi(z, \cdot) > 0 \quad (5.3.17)$$

for some positive increasing function ν .

This condition is satisfied for example in the case of the complex Hessian equations $F[u] := \sigma_k(\lambda(Hu))$, $k \in \{1, \dots, n\}$.

We also study a *new condition* namely

$$f \text{ is concave and homogeneous,} \quad (5.3.18)$$

i.e $f(t\lambda) = tf(\lambda), \forall t \in \mathbb{R}^+$.

Theorem 5.3.2. *Let $u, v \in L^\infty(\bar{\Omega})$ be viscosity subsolution and supersolution of equation (5.3.13) in Ω . Assume that either f satisfies either (5.3.17) or (5.3.18). Then*

$$\sup_{\Omega} (u - v) \leq \max_{\partial\Omega} \{(u - v)^*, 0\}. \quad (5.3.19)$$

Proof. Assume first that f satisfies (5.3.17). Then following [Tru90], we set for any $t \in (1, 2)$,

$$u_t(z) = tu(z) - C(t - 1),$$

where $C = \sup_{\Omega} u$. Therefore we have $u_t(z) \leq u(z)$ on Ω for all $t \in (1, 2)$. Then for any $z_0 \in \Omega$ and an upper test function $q_t(z)$ of u_t at z_0 we have $q(z) := t^{-1}q_t(z) - C(t^{-1} - 1)$ is also an upper test for u at z_0 . Set $\lambda = \lambda[q](z_0)$, then $\lambda[q_t](z_0) = t\lambda$ and $q(z_0) \geq q_t(z_0)$. We also recall that the function $s \mapsto f(s\lambda)$ is increasing on \mathbb{R}^+ (by (5.3.17)) and $f(\lambda) \geq \psi(z, u(z_0))$ since q is an upper test for u at z_0 . It follows that at z_0 ,

$$\begin{aligned} F[q_t] &= f(\lambda[q_t]) = f(t\lambda) \\ &\geq f(\lambda) + (t - 1) \sum \lambda_i f_i(t^* \lambda) \\ &\geq \psi(z_0, q(z_0)) + (t - 1) \sum \lambda_i f_i(t^* \lambda) \\ &\geq \psi(z_0, q_t(z_0)) + \frac{t - 1}{2} \nu(\inf_{\Omega} \psi(z, \inf_{\Omega} u)) \end{aligned}$$

for $1 \leq t^* < t$, sufficiently close to 1. Therefore we have for some $\delta > 0$

$$F[u_t] \geq \psi(z, u_t) + \delta,$$

in the viscosity sense in Ω . Thus the inequality (5.3.19) follows from Lemma 5.3.1.

Next, consider the second case when f is concave and homogeneous. Suppose, without loss of generality, that $0 \in \Omega$. We set

$$u_\tau(z) = u(z) + \tau(|z|^2 - R),$$

where $R = \text{diam}(\Omega)$. Then for any $q_\tau \in C^2(\Omega)$ such that $q_\tau \geq u_\tau$ near z_0 and $q_\tau(z_0) = u_\tau(z_0)$, we have $q = q_\tau - \tau(|z|^2 - R) \geq q_\tau$, and q is also an upper test for u at z_0 . Therefore, we have at z_0 ,

$$\begin{aligned} F[q_\tau] &= 2^d f\left(\frac{\lambda(Hq) + \tau \mathbf{1}}{2}\right) \\ &\geq f(\lambda(Hq)) + f(\tau \mathbf{1}) \\ &\geq \psi(z_0, q_\tau) + \delta. \end{aligned} \quad (5.3.20)$$

Therefore $F[u_\tau] \geq \psi + \delta$ in the viscosity sense. Applying Lemma 5.3.1 we get (5.3.19). \square

By definition, we have the following properties of sub(super)-solutions. Their proofs follow in a straightforward way from [CIL92, Proposition 4.3].

Lemma 5.3.3. (a) Let $\{u_j\}$ be viscosity subsolutions of (5.3.13) in Ω , which are uniformly bounded from above. Then $(\limsup_{\Omega} u_j)^*$ is also a viscosity subsolution of (5.3.13) in Ω .
 (b) Let $\{v_j\}$ be viscosity supersolutions of (5.3.13) in Ω , which are uniformly bounded from below. Then $(\liminf_{\Omega} v_j)_*$ is also a viscosity supersolution of (5.3.13) in Ω .

Now using Perron's method (see for instance [CIL92]), we obtain the next result:

Lemma 5.3.4. Let $\underline{u}, \bar{u} \in L^\infty(\Omega)$ are a subsolution and a supersolution of (5.3.13) on Ω . Suppose that $\underline{u}_*(z) = \bar{u}^*(z)$ on the boundary of Ω . Then the function

$$u := \sup\{v \in L^\infty(\Omega) \cap USC(\Omega) : v \text{ is a subsolution of (5.3.13), } \underline{u} \leq v \leq \bar{u}\}$$

satisfies $u \in C^0(\Omega)$ and

$$F[u] = \psi(x, u) \quad \text{in } \Omega,$$

is the viscosity sense.

Proof. It is straightforward that u^* is a viscosity subsolution of (5.3.13). We next prove that u_* is a supersolution of (5.3.13). Assume by contradiction that u_* is not a supersolution of (5.3.13), then there exists a point $z_0 \in \Omega$ and a lower differential test q for u_* at z_0 such that

$$F[q](z_0) > \psi(z_0, q(z_0)). \quad (5.3.21)$$

Set $\tilde{q}(z) = q(z) + b - a|z - z_0|^2$, where $b = (ar^2)/6$ with $a, r > 0$ small enough so that $F[\tilde{q}] \geq \psi(x, \tilde{q})$ for all $|z - z_0| \leq r$. Since $u_* \geq q$ for $|z - z_0| \leq r$, we get $u^* \geq u_* > \tilde{q}$ for $r/2 \leq |z - z_0| < r$. Then the function

$$w(z) = \begin{cases} \max\{u^*(z), \tilde{q}(z)\} & \text{if } |z - z_0| \leq r, \\ u^*(z) & \text{otherwise} \end{cases}$$

is a viscosity subsolution of (5.3.13). By choosing a sequence $z_n \rightarrow z_0$ so that $u(z_n) \rightarrow u_*(z_0)$, we have $\tilde{q}(z_n) \rightarrow u_*(z_0) + b$. Therefore, for n sufficiently large, we have $w(z_n) > u(z_n)$ and this contradicts the definition of u . Thus we have u_* is also a supersolution. Then it follows from Theorem 5.3.2 and $\underline{u}_*(z) = \bar{u}^*(z)$ for $z \in \partial\Omega$ that $u^* \leq u_*$ on Ω , hence $u = u_* = u^*$. \square

5.4 Dirichlet problems

5.4.1 Viscosity solutions in Γ -pseudoconvex domains

Let $\Omega \subset \mathbb{C}^n$ be a C^2 bounded domain. In this section, we study the following Dirichlet problem

$$\begin{cases} F[u] = f(\lambda(Hu)) = \psi(x, u) \text{ on } \Omega \\ u = \varphi \text{ on } \partial\Omega, \end{cases} \quad (5.4.1)$$

where $\varphi \in C^0(\partial\Omega)$ and $\psi \in C^0(\bar{\Omega} \times \mathbb{R})$ such that $\psi > 0$ and

$$s \mapsto \psi(\cdot, s) \text{ is weakly increasing.}$$

Let $\Gamma \subsetneq \mathbb{R}^n$ be an admissible cone. We assume further that $f \in C^0(\bar{\Gamma})$ satisfies:

- (1) f is concave and $f(\lambda + \mu) \geq f(\lambda)$, $\forall \lambda \in \Gamma, \mu \in \Gamma_n$.
- (2) $\sup_{\partial\Gamma} f = 0$, and $f > 0$ in Γ .
- (3) f is homogeneous on Γ .

We remark that, the condition ((2)) and ((3)) imply that for any $\lambda \in \Gamma$ we have

$$\lim_{t \rightarrow \infty} f(t\lambda) = +\infty. \quad (5.4.2)$$

We now can solve the equation (5.4.1) in the viscosity sense:

Theorem 5.4.1. *Let Ω be a C^2 bounded Γ -pseudoconvex domain in \mathbb{C}^n . The the Dirichlet problem*

$$f(\lambda[u]) = \psi(x, u) \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega.$$

admits a unique admissible solution $u \in C^0(\bar{\Omega})$.

In particular, we have a L^∞ bound for u which only depends on $\|\varphi\|_{L^\infty}$ and $\|\psi(x, C)\|_{L^\infty}$ and Ω , where C is a constant depending on Ω .

Proof. By Lemma 5.2.15, there is a defining function $\rho \in C^2(\bar{\Omega})$ for Ω such that $\lambda(H\rho) \in \Gamma$ on $\bar{\Omega}$. The C^2 -smoothness of the boundary implies the existence of a harmonic function h on Ω for arbitrary given continuous boundary data φ . Set

$$\underline{u} = (A_1\rho + h) + A_2\rho,$$

where $A_1 > 0$ is chosen so that $A_1\rho + h$ is admissible and A_2 will be chosen later.

By the concavity of f and (5.4.2), for A_2 sufficiently large we get

$$\begin{aligned} f(\lambda[\underline{u}]) &\geq \frac{1}{2}f(2\lambda[A_1\rho + h]) + \frac{1}{2}f(2A_2\lambda[\rho]) \\ &\geq \max_{\bar{\Omega}} \psi(x, h) \geq \psi(x, \underline{u}). \end{aligned}$$

Therefore \underline{u} is a subsolution of (5.4.1).

Since h is harmonic, for each $z \in \Omega$ there is a Hermitian matrix $N \geq 0$ so that $\lambda(N + H(h)(z)) \in \partial\Gamma$. But then $f(\lambda(N + H(h)(z))) = 0$. Therefore, $\bar{v} := h$ is a supersolution of (5.4.1).

Finally, the existence of solution follows from Perron's method. We set

$$u := \sup\{w \text{ is subsolution of (5.4.1) on } \Omega, \underline{u} \leq w \leq \bar{v}\}.$$

As in the argument from Lemma 5.3.4 we have u^* (resp. u_*) is a subsolution (resp. supersolution) of (5.4.1). It follows from the comparison principle (Theorem 5.3.2) that

$$u^*(z) - u_*(z) \leq \limsup_{w \rightarrow \partial\Omega} (u^* - u_*)^+(w).$$

Since \underline{u} and \bar{v} are continuous and $\underline{u} = \bar{v} = \varphi$ on $\partial\Omega$ we infer that $u^* \leq u_*$ on Ω and $u^* = u_*$ on $\partial\Omega$. Therefore $u = u^* = u_*$ is a viscosity solution of (5.4.1). The uniqueness follows from the comparison principle (Theorem 5.3.2). \square

As a corollary of Theorem 5.4.1, we solve the following Dirichlet problem for Hessian quotient equations

$$\begin{cases} S_{k,\ell}(\lambda(Hu)) := \frac{S_k}{S_\ell}(\lambda(Hu)) = \psi(x, u) & \text{on } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}, \quad (5.4.3)$$

where $\Omega \subset \mathbb{C}^n$ be a smooth bounded Γ_k -pseudoconvex domain, $1 \leq \ell < k \leq n$ and

$$S_k(\lambda(Hu)) = \frac{(dd^c u)^k \wedge \omega^{n-k}}{\omega^n}.$$

Note that the operator $S_{k,\ell}^{1/(k-\ell)}$ is concave and homogeneous (see [Spr05]).

Corollary 5.4.2. *The Dirichlet problem (5.4.3) admits a unique viscosity solution $u \in C^0(\bar{\Omega})$ for any continuous data φ .*

We also remark that a viscosity subsolution is always a Γ -subharmonic function.

Lemma 5.4.3. *Any viscosity subsolution of the equation $f(\lambda(Hu)) = \psi(z, u)$ is a Γ -subharmonic function. In particular, if u is a viscosity subsolution of the equation*

$$S_{k,\ell}(\lambda(Hu)) = \psi(z, u), \quad (5.4.4)$$

then u is k -subharmonic.

Proof. Let $z_0 \in \Omega$ and $q \in C_{loc}^2(\{z_0\})$, such that $u - q$ attains its maximum at z_0 and $u(z_0) = q(z_0)$. By definition we have

$$f(\lambda(Hq)(z_0)) > 0.$$

Observe that for any semi-positive Hermitian matrix N , the function

$$q_N(z) := q(z) + \langle N(z - z_0), z - z_0 \rangle$$

is also an upper test function for u at z_0 . By the definition of viscosity subsolutions we have

$$f(\lambda(H\tilde{q})(z_0)) > 0. \tag{5.4.5}$$

Suppose that $\lambda(Hq)(z_0) \notin \bar{\Gamma}$. Then we can find $N \geq 0$ so that $\lambda(H\tilde{q})(z_0) \in \partial\Gamma$, so $f(\lambda(H\tilde{q})(z_0)) = 0$ by the condition (3) above, this contradicts to (5.4.5). Hence we always have $\lambda[q](z_0) \geq 0$, and so u is Γ -subharmonic. \square

5.4.2 Hölder continuity of Hessian type equations

In this subsection, we study the Hölder continuity of the viscosity solution obtained in Section 5.4.1 to the Dirichlet problem

$$\begin{cases} F[u] = f(\lambda(Hu)) = \psi(x, u) \text{ on } \Omega \\ u = \varphi \text{ on } \partial\Omega, \end{cases} \tag{5.4.6}$$

where f, φ and ψ satisfy the conditions spelled out in the previous subsection. We prove the following result:

Theorem 5.4.4. *Let Ω be a strictly Γ pseudoconvex domain. Let u be the viscosity solution of (5.4.6). Suppose that $\varphi \in C^{2\alpha}(\partial\Omega)$ for some $\alpha \in (0, 1)$. If additionally $\psi(z, s)$ satisfies*

- (1) $|\psi(z, s)| \leq M_1(s)$ for some L_{loc}^∞ function M_1 ;
- (2) $|\psi(z, s) - \psi(w, s)| \leq M_2(s)|z - w|^\alpha$ for some L_{loc}^∞ function M_2 ;

Then $u \in C^\alpha(\bar{\Omega})$.

Remark. Classical examples (see [BT76]) show that the claimed regularity cannot be improved. Conditions 1 and 2 can be regarded as a weak growth conditions and seem to be optimal. If ψ does not depend on the second variable then these conditions mean that ψ is globally bounded and contained in C^α .

Proof. The proof relies on the classical idea of Walsh-[Wal68]. Similar argument was used by Bedford and Taylor- [BT76] who dealt with the complex Monge-Ampère operator. We shall apply a small adjustment in the construction of the local barriers which is due to Charabati [Cha16].

Suppose for definiteness that $0 \in \Omega$. Assume without loss of generality that the Γ -subharmonic function $\rho = -\text{dist}(z, \partial\Omega) + C_\Omega \text{dist}(z, \partial\Omega)^2$ satisfies $F(\rho) \geq 2$ (multiply ρ by a constant if necessary and exploit the homogeneity of F). Recall ρ vanishes on $\partial\Omega$. As $\partial\Omega \in C^2$ we know that $\rho \in C^2$ near the boundary. Then it is easy to find a continuation of ρ in the interior of Ω (still denoted by ρ), so that ρ is Γ -subharmonic and satisfies $F(\rho) \geq 1$.

Fix $\xi \in \partial\Omega$. There is a uniform $C \gg 1$ (dependent on Ω , but independent on ξ) such that the function

$$g_\xi(z) := C\rho(z) - |z - \xi|^2$$

is Γ -sh. In particular $g_\xi \leq 0$ in $\bar{\Omega}$.

By definition there is a constant \tilde{C} , such that for any $z \in \partial\Omega$

$$\varphi(z) \geq \varphi(\xi) - \tilde{C}|z - \xi|^{2\alpha}.$$

Consider the function $h_\xi(z) := -\tilde{C}(-g_\xi(z))^\alpha$. Then

$$H(h_\xi(z)) \geq \tilde{C}\alpha(1 - \alpha)(-g_\xi(z))^{\alpha-2}H(g_\xi(z)), \quad (5.4.7)$$

where $\lambda(H(g_\xi(z))) \in \Gamma$, thus h_ξ is Γ -subharmonic.

Observe that

$$h_\xi(z) \leq -\tilde{C}|z - \xi|^{2\alpha} \leq \varphi(z) - \varphi(\xi).$$

Thus $h_\xi(z) + \varphi(\xi)$ are local boundary barriers constructed following the method of Charabati from [Cha16] (in the paper [BT76], where the Monge-Ampère case was considered, h_ξ was simply chosen as $-(x_n)^\alpha$ in a suitable coordinate system, but this is not possible in the general case).

At this stage we recall that u is bounded a priori by Theorem 5.4.1. Hence we know that for some uniform constant A one has $F[u] \leq A$ in the viscosity sense.

From the gathered information one can produce a global barrier for u in a standard way (see [BT76]). Indeed, consider the function $\tilde{h}(z) := \sup_\xi \{ah_\xi(z) + \varphi(\xi)\}$ for a large but uniform constant a . Using the balayage procedure it is easy to show that $F(\tilde{h}(z)) \geq A$ in the viscosity sense once a is taken large enough. Thus \tilde{h} majorizes u by the comparison principle and so is a global barrier for u matching the boundary data given by φ . By construction \tilde{h} is globally α -Hölder continuous.

Note on the other hand that u is subharmonic as $\Gamma \subset \Gamma_1$, thus the harmonic extension u_φ of φ in Ω majorizes u from above. Recall that u_φ is α -Hölder continuous by classical elliptic regularity.

Coupling the information for both the lower and the upper barrier one obtains

$$\forall z \in \bar{\Omega}, \forall \xi \in \partial\Omega \quad |u(z) - u(\xi)| \leq K|z - \xi|^\alpha \quad (5.4.8)$$

Denote by K_1 the quantity $K_2 \text{diam}^2(\Omega) \max\{1, f(\mathbf{1})\} + K$, where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$ is the vector of the eigenvalues of the identity matrix, while $K_2 := \tilde{C}f(\mathbf{1})^{-1}$ and finally \tilde{C} is the α -Lipschitz constant of ψ . Consider for a small vector $\tau \in \mathbb{C}^n$ the function

$$v(z) := u(z + \tau) + K_2|\tau|^\alpha|z|^2 - K_1|\tau|^\alpha$$

defined over $\Omega_\tau := \{z \in \Omega \mid z + \tau \in \Omega\}$.

It is easy to see by using the barriers that if $z + \tau \in \partial\Omega$ or $z \in \partial\Omega$ then

$$v(z) \leq u(z) + K|\tau|^\alpha + K_2 \text{diam}^2 \Omega |\tau|^\alpha - K_1|\tau|^\alpha \leq u(z).$$

We now claim that $v(z) \leq u(z)$ in Ω_τ . By the previous inequality this holds on $\partial(\Omega_\tau)$. Suppose the claim is false and consider the open subdomain U of Ω_τ defined by $U_\tau = \{z \in \Omega_\tau \mid v(z) > u(z)\}$.

We will now prove that v is a subsolution to $F[u] = f(\lambda(Hu)) = \psi(z, u(z))$ in U . To this end pick a point z_0 and an upper differential test q for v at z_0 . Observe then that $\tilde{q}(z) := q(z) - K_2|\tau|^\alpha|z|^2 - K_1|\tau|^\alpha$ is then an upper differential test for $u(\tau + \cdot)$ at the point z_0 . Hence

$$\begin{aligned} F[q(z_0)] &= f(\lambda(H\tilde{q}(z_0)) + K_2|\tau|^\alpha \mathbf{1}) \\ &\geq f(\lambda(H\tilde{q}(z_0)) + K_2|\tau|^\alpha f(\mathbf{1})) \\ &\geq \psi(z_0 + \tau, u(z_0 + \tau)) + K_2|\tau|^\alpha f(\mathbf{1}), \end{aligned}$$

where we have used the concavity and homogeneity of f in the first inequality and the fact that \tilde{q} is an upper differential test for $u(\tau + \cdot)$ for the second one.

Next

$$\begin{aligned} &\psi(z_0 + \tau, u(z_0 + \tau)) + K_2|\tau|^\alpha f(\mathbf{1}) \\ &\geq \psi(z_0 + \tau, u(z_0 + \tau) + K_2|\tau|^\alpha|z_0|^2 - K_1|\tau|^\alpha) + K_2|\tau|^\alpha f(\mathbf{1}) \\ &= \psi(z_0 + \tau, v(z_0)) + K_2|\tau|^\alpha f(\mathbf{1}) \\ &\geq \psi(z_0 + \tau, u(z_0)) + K_2|\tau|^\alpha f(\mathbf{1}), \end{aligned}$$

where we have exploited twice the monotonicity of ψ with respect to the second variable (and the fact that $z_0 \in U_\tau$).

Exploiting now the Hölder continuity of ψ with respect to the first variable we obtain

$$\psi(z_0 + \tau, u(z_0 + \tau)) + K_2|\tau|^\alpha f(\mathbf{1}) \geq \psi(z_0 + \tau, u(z_0)) + K_2|\tau|^\alpha f(\mathbf{1}) \geq \psi(z_0, u(z_0)).$$

This proves that $F[q(z_0)] \geq \psi(z_0, u(z_0))$ and hence $F[v(z)] \geq \psi(z, v(z))$ in the viscosity sense.

Thus over U_τ , v is subsolution and u is a solution, which implies by comparison principle that $v \leq u$ there, a contradiction unless the set U_τ is empty.

We have thus proven that

$$\forall z \in \Omega_\tau \quad u(z + \tau) + K_2|\tau|^\alpha|z|^2 - K_1|\tau|^\alpha \leq u(z),$$

which implies the claimed α - Hölder continuity. \square

5.5 Viscosity vs. pluripotential solutions

Let Ω be a bounded smooth strictly pseudoconvex domain in \mathbb{C}^n . Let $0 < \psi \in C(\bar{\Omega} \times \mathbb{R})$ be a continuous function non-decreasing in the last variable. In this section, we study the relations between viscosity concepts with respect to the inverse σ_k equations

$$\frac{(dd^c u)^n}{(dd^c u)^{n-k} \wedge \omega^k} = \psi(z, u) \quad \text{in } \Omega, \quad (5.5.1)$$

and pluripotential concepts with respect to the equation

$$(dd^c u)^n = \psi(z, u)(dd^c u)^{n-k} \wedge \omega^k \quad \text{in } \Omega. \quad (5.5.2)$$

For the regular case, the following result was shown in [GS15]:

Theorem 5.5.1 (Guan-Sun). *Let $0 < h \in C^\infty(\bar{\Omega})$ and $\varphi \in C^\infty(\partial\Omega)$. Then, there exists a smooth strictly plurisubharmonic function u in $\bar{\Omega}$ such that*

$$\frac{(dd^c u)^n}{(dd^c u)^{n-k} \wedge \omega^k} = h(z) \quad \text{in } \Omega, \quad u = \varphi \quad \text{in } \partial\Omega. \quad (5.5.3)$$

Note that the function u in Theorem 5.5.1 is a viscosity solution of (5.5.1) in the case when $\psi(z, u) = h(z)$. Using Theorem 5.5.1, we obtain

Proposition 5.5.2. *If $u \in C(\bar{\Omega}) \cap PSH(\Omega)$ is a viscosity solution of (5.5.1) then there exists a sequence of smooth plurisubharmonic functions u_j in Ω such that u_j is decreasing to u and the function $\frac{(dd^c u_j)^n}{(dd^c u_j)^{n-k} \wedge \omega^k}$ converges uniformly to $\psi(z, u)$ as $j \rightarrow \infty$. In particular, u is a solution of (5.5.2) in the pluripotential sense.*

Proof. Let $\varphi_j \in C^\infty(\partial\Omega)$ and $0 < \psi_j \in C^\infty(\bar{\Omega})$ be sequences of smooth functions such that $\varphi_j \searrow \varphi$ and $\psi_j \nearrow \psi(z, u)$ as $j \rightarrow \infty$. Then, by Theorem 5.5.1, for any $j = 1, 2, \dots$, there exists a smooth strictly plurisubharmonic function u_j in $\bar{\Omega}$ such that

$$\frac{(dd^c u_j)^n}{(dd^c u_j)^{n-k} \wedge \omega^k} = \psi_j(z) \quad \text{in } \Omega, \quad u_j = \varphi_j \quad \text{in } \partial\Omega. \quad (5.5.4)$$

By the comparison principle, we have

$$u_1 \geq u_2 \geq \dots \geq u_j \geq \dots \geq u.$$

Let $C > \sup_{\Omega} |z|^2$. By the homogeneity and the concavity of $S_{n, n-k}^{1/k}$, we have

$$\frac{(dd^c(u_j + \epsilon|z|^2))^n}{(dd^c(u_j + \epsilon|z|^2))^{n-k} \wedge \omega^k} \geq \frac{(dd^c u_j)^n}{(dd^c u_j)^{n-k} \wedge \omega^k} + \epsilon^k.$$

Then, by the comparison principle, for any $\epsilon > 0$, there exists $N > 0$ such that

$$u_j + \epsilon(|z|^2 - C) \leq u,$$

for any $j > N$. Hence, u_j is decreasing to u as $j \rightarrow \infty$. \square

Observe that a continuous solution of (5.5.2) in the pluripotential sense may not be a viscosity solution of (5.5.1). For example, if a continuous plurisubharmonic function $u : \Omega \rightarrow \mathbb{R}$ depends only on $n - k - 1$ variables then u is a solution of (5.5.2) in the pluripotential sense but u is not a viscosity solution of (5.5.1). Moreover, by Theorem 5.5.6, we know that a viscosity solution of (5.5.1) has to satisfy $(dd^c u)^k \geq a\omega^k$ for some $a > 0$. The following question is natural:

Question 5.5.3. *If $u \in PSH(\Omega) \cap C(\bar{\Omega})$ satisfies (5.5.2) in the pluripotential sense and*

$$(dd^c u)^k \geq a\omega^k \tag{5.5.5}$$

for some $a > 0$, does u satisfy (5.5.1) in the viscosity sense?

At the end of this section, we will give the answer to a special case of this question. Now, we consider the relation between viscosity subsolutions of (5.5.1) and *pluripotential subsolutions* of (5.5.2). Recall that according to the definition in subsection 2.1 for any $n \times n$ complex matrix A and $k \in \{1, \dots, n\}$, $S_k(A)$ denotes the coefficient with respect to t^{n-k} of the polynomial $\binom{n}{k}^{-1} \det(A + t\text{Id}_n)$.

Next we prove the following technical result:

Lemma 5.5.4. *Assume that A, B are $n \times n$ complex matrices and $k \in \{1, \dots, n\}$. Then*

$$S_k(AA^*)S_k(BB^*) \geq |S_k(AB^*)|^2.$$

Proof. Denote by a_1, \dots, a_n and b_1, \dots, b_n , respectively, the row vectors of A and B . Then

$$S_k(AA^*) = \left(\binom{n}{k}\right)^{-1} \sum_{\#J=k} \det(\langle a_p, a_q \rangle)_{p,q \in J},$$

$$S_k(BB^*) = \left(\binom{n}{k}\right)^{-1} \sum_{\#J=k} \det(\langle b_p, b_q \rangle)_{p,q \in J},$$

and

$$S_k(AB^*) = \left(\binom{n}{k}\right)^{-1} \sum_{\#J=k} \det(\langle a_p, b_q \rangle)_{p,q \in J}.$$

We will show that, for any $J = \{p_1, \dots, p_k\}$ with $1 \leq p_1 < \dots < p_k \leq n$,

$$\det(\langle a_p, a_q \rangle)_{p,q \in J} \cdot \det(\langle b_p, b_q \rangle)_{p,q \in J} \geq |\det(\langle a_p, b_q \rangle)_{p,q \in J}|^2. \tag{5.5.6}$$

Indeed, if either $\{a_{p_1}, \dots, a_{p_k}\}$ or $\{b_{p_1}, \dots, b_{p_k}\}$ are linearly dependent then both sides of (5.5.6) are equal to 0. Otherwise, exploiting the Gram-Schmidt process, we can assume

that $\{a_{p_1}, \dots, a_{p_k}\}$ and $\{b_{p_1}, \dots, b_{p_k}\}$ are orthogonal systems (observe that the quantities in question do not change during the orthogonalization process). Next normalizing the vectors a_{p_j} and b_{p_j} , $j = 1, \dots, n$ to unit length both sides change by the same factor. Hence it suffices to prove the statement for two collections of orthonormal bases.

Under this assumption we have

$$(\langle a_p, a_q \rangle)_{p,q \in J} = (\langle b_p, b_q \rangle)_{p,q \in J} = Id_k. \quad (5.5.7)$$

Let $M = (\langle a_p, b_q \rangle)_{p,q \in J}$. Then MM^* is semi-positive Hermitian matrix, and

$$\begin{aligned} \text{Tr}(MM^*) &= \sum_{l=1}^k \sum_{j=1}^k |\langle b_{p_j}, a_{p_l} \rangle|^2 \\ &= \sum_{j=1}^k \langle b_{p_j}, \sum_{l=1}^k \langle b_{p_j}, a_{p_l} \rangle a_{p_l} \rangle \\ &\leq \sum_{j=1}^k \|b_{p_j}\|^2 = k. \end{aligned}$$

Therefore, $|\det(M)| = \sqrt{\det(MM^*)} \leq 1$, hence we obtain (5.5.6). Finally, using (5.5.6) and the Cauchy-Schwarz inequality, we infer that

$$S_k(AA^*)S_k(BB^*) \geq |S_k(AB^*)|^2,$$

as required. □

For any $n \times n$ Hermitian matrix $A = (a_{j\bar{\ell}})$, we denote

$$\omega_A = \sum_{j,\ell=1}^n a_{j\bar{\ell}} \frac{i}{\pi} dz_j \wedge d\bar{z}_\ell,$$

and

$$\mathcal{B}(A, k) := \{B \in \mathcal{H}_+^n \mid \frac{\omega_B^k \wedge \omega_A^{n-k}}{\omega^n} = 1\},$$

where $k = 1, 2, \dots, n$.

Theorem 5.5.5. *Let $u \in PSH(\Omega) \cap L_{loc}^\infty(\Omega)$ and $0 < g \in C(\Omega)$. Then the following are equivalent:*

- (i) $\frac{(dd^c u)^n}{(dd^c u)^{n-k} \wedge \omega^k} \geq g^k(z)$ in the viscosity sense.
- (ii) For all $B \in \mathcal{B}(Id, n - k)$,

$$(dd^c u)^k \wedge \omega_{B^2}^{n-k} \geq g^k(z) \omega^n,$$

in viscosity sense.

(iii) For any open set $U \Subset \Omega$, there are smooth plurisubharmonic functions u_ϵ and functions $0 < g^\epsilon \in C^\infty(U)$ such that u_ϵ are decreasing to u and g^ϵ converge uniformly to g as $\epsilon \searrow 0$, and

$$(dd^c u_\epsilon) \wedge \omega_{A_1} \wedge \dots \wedge \omega_{A_{k-1}} \wedge \omega_{B^2}^{n-k} \geq g^\epsilon \omega^n, \quad (5.5.8)$$

pointwise in U for any $B \in \mathcal{B}(Id, n-k)$ and $A_1, \dots, A_{k-1} \in \mathcal{B}(B^2, k)$.

(iv) For any open set $U \Subset \Omega$, there are smooth strictly plurisubharmonic functions u_ϵ and functions $0 < g^\epsilon \in C^\infty(U)$ such that the sequence u_ϵ is decreasing to u and the sequence g^ϵ converges uniformly to g as $\epsilon \searrow 0$, and

$$\frac{(dd^c u_\epsilon)^n}{(dd^c u_\epsilon)^{n-k} \wedge \omega^k} \geq (g^\epsilon)^k, \quad (5.5.9)$$

pointwise in U for any $B \in \mathcal{B}(Id, n-k)$.

Proof. ($iv \Rightarrow i$) is obvious. It remains to show ($i \Rightarrow ii \Rightarrow iii \Rightarrow iv$).

($i \Rightarrow ii$) Assume that $q \in C^2$ is an upper test for u from at $z_0 \in \Omega$. Then q is strictly plurisubharmonic in a neighborhood of z_0 and

$$\frac{(dd^c q)^n}{(dd^c q)^{n-k} \wedge \omega^k} \geq g^k,$$

at z_0 .

By using Lemma 5.5.4 for \sqrt{Hq} and $(\sqrt{Hq})^{-1}B$, we have

$$\frac{(dd^c q)^{n-k} \wedge \omega^k}{(dd^c q)^n} \frac{(dd^c q)^k \wedge \omega_{B^2}^{n-k}}{\omega^n} = \frac{(dd^c q)^{n-k} \wedge \omega^k}{\omega^n} \frac{(dd^c q)^k \wedge \omega_{B^2}^{n-k}}{(dd^c q)^n} \geq \left(\frac{\omega_B^{n-k} \wedge \omega^k}{\omega^n} \right)^2,$$

for any $B \in \mathcal{H}_+^n$, (observe that $S_{n-k}(CC^*) = \frac{(dd^c q)^k \wedge \omega_{B^2}^{n-k}}{(dd^c q)^n}$ and $S_{n-k}(\sqrt{Hq}C^*) = \frac{\omega_B^{n-k} \wedge \omega^k}{\omega^n}$ for $C = (\sqrt{Hq})^{-1}B$.)

Then, for any $B \in \mathcal{B}(Id, n-k)$ we have

$$(dd^c q)^k \wedge \omega_{B^2}^{n-k} \geq g^k \omega^n,$$

at z_0 . Hence

$$(dd^c u)^k \wedge \omega_{B^2}^{n-k} \geq g^k \omega^n,$$

in the viscosity sense.

(ii \Rightarrow iii) Assume that $q \in C^2$ touches u from above at $z_0 \in \Omega$. Then, for any $B \in \mathcal{B}(Id, n-k)$,

$$(dd^c q)^k \wedge \omega_{B^2}^{n-k} \geq g^k \omega^n,$$

at z_0 . By the same arguments as in [Lu13], we have

$$(dd^c q) \wedge \omega_{A_1} \wedge \dots \wedge \omega_{A_{k-1}} \wedge \omega_{B^2}^{n-k} \geq g \omega^n,$$

for any $B \in \mathcal{B}(Id, n-k)$, $A_1, \dots, A_{k-1} \in \mathcal{B}(B^2, k)$. Hence

$$(dd^c u) \wedge \omega_{A_1} \wedge \dots \wedge \omega_{A_{k-1}} \wedge \omega_{B^2}^{n-k} \geq g \omega^n, \quad (5.5.10)$$

in the viscosity sense for any $B \in \mathcal{B}(Id, n-k)$, $A_1, \dots, A_{k-1} \in \mathcal{B}(B^2, k)$.

Let g_j be a sequence of smooth functions in Ω such that $g_j \nearrow g$ as $j \rightarrow \infty$. Then

$$(dd^c u) \wedge \omega_{A_1} \wedge \dots \wedge \omega_{A_{k-1}} \wedge \omega_{B^2}^{n-k} \geq g_j \omega^n, \quad (5.5.11)$$

in the viscosity sense for any $j \in \mathbb{N}$, $B \in \mathcal{B}(Id, n-k)$ and $A_1, \dots, A_{k-1} \in \mathcal{B}(B^2, k)$. By the same arguments as in [EGZ11] (the proof of Proposition 1.5), u satisfies (5.5.11) in the sense of positive Radon measures. Using convolution to regularize u and setting $u_\epsilon = u * \rho_\epsilon$, we see that u_ϵ is smooth strictly plurisubharmonic and

$$(dd^c u_\epsilon) \wedge \omega_{A_1} \wedge \dots \wedge \omega_{A_{k-1}} \wedge \omega_{B^2}^{n-k} \geq (g_j)_\epsilon \omega^n,$$

pointwise in Ω_ϵ . Choosing $g^\epsilon := (g_{[1/\epsilon]})_\epsilon$, we obtain (5.5.8).

(iii \Rightarrow iv) At $z_0 \in \Omega_\epsilon$, choosing

$$B = \frac{Hu_\epsilon(z_0)}{(S_{n-k}(Hu_\epsilon(z_0)))^{1/(n-k)}}$$

and

$$A_1 = A_2 = \dots = A_{k-1} = \left(\frac{(dd^c u_\epsilon(z_0))^k \wedge \omega_{B^2}^{n-k}}{\omega^n} \right)^{-1/k} Hu_\epsilon(z_0),$$

we get,

$$\begin{aligned} g^\epsilon &\leq \left(\frac{(dd^c u_\epsilon(z_0))^k \wedge \omega_{B^2}^{n-k}}{\omega^n} \right)^{1/k} \\ &= \left(\frac{(dd^c u_\epsilon(z_0))^n}{\omega^n} \frac{1}{S_{n-k}(Hu_\epsilon(z_0))} \right)^{1/k} \\ &= \left(\frac{(dd^c u_\epsilon(z_0))^n}{\omega^n} \frac{\omega^n}{(dd^c u_\epsilon)^{n-k} \wedge \omega^k} \right)^{1/k} \\ &= \left(\frac{(dd^c u_\epsilon)^n}{(dd^c u_\epsilon)^{n-k} \wedge \omega^k} \right)^{1/k}, \end{aligned}$$

pointwise in Ω_ϵ . Then

$$\frac{(dd^c u_\epsilon)^n}{(dd^c u_\epsilon)^{n-k} \wedge \omega^k} \geq (g^\epsilon)^k.$$

The proof is completed. \square

As a consequence, our result implies that a viscosity subsolution is a pluripotential subsolution.

Theorem 5.5.6. *Assume that $\psi(z, s) = \psi(z)$ with $\psi \in C^0(\Omega)$ and $u \in PSH(\Omega) \cap L_{loc}^\infty(\Omega)$ is a viscosity subsolution of (5.5.1). Then*

$$(dd^c u)^n \geq \psi (dd^c u)^{n-k} \wedge \omega^k, \quad (5.5.12)$$

and

$$(dd^c u)^k \geq \binom{n}{k}^{-1} \psi \omega^k, \quad (5.5.13)$$

in the pluripotential sense. If u is continuous then the conclusion still holds in the case where ψ depends on both variables.

Proof. By Theorem 5.5.5, for any open set $U \Subset \Omega$, there are strictly plurisubharmonic functions $u_\epsilon \in C^\infty(U)$ and functions $0 < h^\epsilon \in C^\infty(U)$ such that u_ϵ is decreasing to u and h^ϵ converges uniformly to ψ as $\epsilon \searrow 0$, and

$$\frac{(dd^c u_\epsilon)^n}{(dd^c u_\epsilon)^{n-k} \wedge \omega^k} \geq h^\epsilon, \quad (5.5.14)$$

pointwise in U . Choosing $B = \text{Id}_n$ and letting $\epsilon \rightarrow 0$, we obtain (5.5.12).

It also follows from Theorem 5.5.5 that we can choose u_ϵ and h^ϵ so that

$$(dd^c u_\epsilon)^k \wedge \omega_{B^2}^{n-k} \geq h^\epsilon \omega^n, \quad (5.5.15)$$

pointwise in U for any $B \in \mathcal{B}(\text{Id}, n-k)$. Fix $z_0 \in U$ and $0 < \epsilon \ll 1$. We can choose complex coordinates so that $Hu_\epsilon(z_0) = \text{diag}(\lambda_1, \dots, \lambda_n)$, where $0 \leq \lambda_1 \leq \dots \leq \lambda_n$. Choosing

$$B = \binom{n}{k}^{1/(n-k)} \text{diag}(0, \dots, \underbrace{0}_{k\text{-th}}, 1, \dots, 1),$$

we get

$$\lambda_1 \dots \lambda_k \geq \binom{n}{k}^{-1} h^\epsilon.$$

Then

$$(dd^c u_\epsilon)^k \geq \binom{n}{k}^{-1} h^\epsilon \omega^k,$$

pointwise in U . Letting $\epsilon \rightarrow 0$, we obtain (5.5.13). \square

Remark Note that for strictly positive ψ (5.5.13) implies that the *natural* space of functions to consider for the Hessian quotient problem (5.5.1) is *not* the space of bounded plurisubharmonic functions but a considerably smaller one.

By assuming an additional conditions, we can also prove that a pluripotential subsolution is a viscosity one.

Proposition 5.5.7. *Assume that $\psi(z, s) = \psi(z) > 0$ with $\psi \in C^0(\Omega)$ and u is a local bounded plurisubharmonic function in Ω satisfying*

$$(dd^c u)^k \geq \psi \omega^k,$$

in the pluripotential sense. Then

$$\frac{(dd^c u)^n}{(dd^c u)^{n-k} \wedge \omega^k} \geq \psi,$$

in the viscosity sense.

Proof. By the assumption, for any $A \in \mathcal{H}_+^n$,

$$(dd^c u)^k \wedge \omega_A^{n-k} \geq \psi \omega^k \wedge \omega_A^{n-k}, \quad (5.5.16)$$

in the pluripotential sense. By [Lu13], (5.5.16) also holds in the viscosity sense. If $A = B^2$ for some $B \in \mathcal{B}(\text{Id}, n - k)$ then, by using Lemma 5.5.4, we have

$$\omega^k \wedge \omega_{B^2}^{n-k} \geq \left(\frac{\omega_B^{n-k} \wedge \omega^k}{\omega^n} \right)^2 \omega^n = \omega^n.$$

Then

$$(dd^c u)^k \wedge \omega_{B^2}^{n-k} \geq \psi \omega^n,$$

in the viscosity sense, for any $B \in \mathcal{B}(\text{Id}, n - k)$. Applying Theorem 5.5.5, we obtain

$$\frac{(dd^c u)^n}{(dd^c u)^{n-k} \wedge \omega^k} \geq \psi,$$

in the viscosity sense. \square

We now discuss the notion of a *supersolution*. By the same argument as in [GLZ17], (relying on the Berman's idea from [Ber13]) we obtain the following relation between viscosity supersolutions of (5.5.1) and pluripotential supersolutions of (5.5.2):

Proposition 5.5.8. *Let $u \in PSH(\Omega) \cap C(\bar{\Omega})$ be a viscosity supersolution of (5.5.1). Then there exists an increasing sequence of strictly psh functions $u_j \in C^\infty(\bar{\Omega})$ such that u_j converges in capacity to u as $j \rightarrow \infty$, and*

$$\frac{(dd^c u_j)^n}{(dd^c u_j)^{n-k} \wedge \omega^k} \leq \psi(z, u),$$

pointwise in Ω . In particular,

$$(dd^c u)^n \leq \psi(z, u)(dd^c u)^{n-k} \wedge \omega^k,$$

in the pluripotential sense.

If there exists $a > 0$ such that $(dd^c u)^k \geq a\omega^k$ then u_j can be chosen such that

$$\frac{(dd^c u_j)^n}{(dd^c u_j)^{n-k} \wedge \omega^k} \geq b,$$

pointwise in Ω for some $b > 0$.

For the definition of convergence in capacity, we refer to [GZ17b] and references therein.

Proof. Denote $\varphi = u|_{\partial\Omega}$ and $g(z) = \psi(z, u(z))$. Then, for any $j \geq 1$, there exists a unique viscosity solution v_j of

$$\begin{cases} \frac{(dd^c v_j)^n}{(dd^c v_j)^{n-k} \wedge \omega^k} = e^{j(v_j - u)} g(z) & \text{in } \Omega, \\ v_j = \varphi & \text{in } \partial\Omega. \end{cases} \quad (5.5.17)$$

Applying the comparison principle to the equation

$$\frac{(dd^c v)^n}{(dd^c v)^{n-k} \wedge \omega^k} = e^{j(v-u)} g(z),$$

we get $u \geq v_j$ and $v_{j+1} \geq v_j$ for any $j \geq 1$.

Note that, by Proposition 5.5.2,

$$(dd^c v_j)^n = e^{j(v_j - u)} g(z) (dd^c v_j)^{n-k} \wedge \omega^k,$$

in the pluripotential sense. For any $h \in PSH(\Omega)$ such that $-1 \leq h \leq 0$, we have,

$$\begin{aligned} \epsilon^n \int_{\{v_j < u - 2\epsilon\}} (dd^c h)^n &\leq \int_{\{v_j < u + \epsilon h - \epsilon\}} (dd^c(u + \epsilon h))^n \\ &\leq \int_{\{v_j < u + \epsilon h - \epsilon\}} (dd^c v_j)^n \\ &\leq \int_{\{v_j < u - \epsilon\}} e^{j(v_j - u)} g(z) (dd^c v_j)^{n-k} \wedge \omega^k \\ &\leq e^{-j\epsilon} \int_{\{v_1 < u - \epsilon\}} g(z) (dd^c v_j)^{n-k} \wedge \omega^k \\ &\leq C e^{-j\epsilon}, \end{aligned}$$

where $C > 0$ is independent of j . The last inequality holds by the Chern-Levine-Nirenberg inequalities (cf. [GZ17b]). This implies that v_j converges to u in capacity.

If there exists $a > 0$ such that $(dd^c u)^k \geq a\omega^k$ then, by Proposition 5.5.7,

$$\frac{(dd^c u)^n}{(dd^c u)^{n-k} \wedge \omega^k} \geq a,$$

in the viscosity sense. Choosing $M \gg 1$ such that $e^{-M} \sup_{\Omega} g < a$, we get

$$\frac{(dd^c v_j)^n}{(dd^c v_j)^{n-k} \wedge \omega^k} \leq a e^{j(v_j - u) + M}.$$

Applying the comparison principle to the equation

$$\frac{(dd^c v)^n}{(dd^c v)^{n-k} \wedge \omega^k} = a e^{j(v-u)},$$

we get $v_j + \frac{M}{j} \geq u$ for any $j \geq 1$. Then

$$\frac{(dd^c v_j)^n}{(dd^c v_j)^{n-k} \wedge \omega^k} = e^{j(v_j - u)} g(z) \geq e^{-M} g(z),$$

for any $j \geq 1$. Hence, by Theorem 5.5.6,

$$(dd^c v_j)^k \geq \binom{n}{k}^{-1} e^{-M} g(z) \geq \binom{n}{k}^{-1} e^{-M} \min_{\Omega} g,$$

for any $j \geq 1$.

Now, by Proposition 5.5.2, for any j we can choose a strictly plurisubharmonic function $u_j \in C^\infty(\bar{\Omega})$, such that

$$v_j - \frac{1}{2^j} \leq u_j \leq v_j - \frac{1}{2^{j+1}}$$

and

$$-\frac{1}{2^j} \leq \frac{(dd^c u_j)^n}{(dd^c u_j)^{n-k} \wedge \omega^k} - e^{j(v_j - u)} g(z) \leq 0.$$

It is easy to see that u_j satisfies the required properties. \square

The next result gives the answer to a special case of Question 5.5.3:

Theorem 5.5.9. *Let $u \in PSH(\Omega) \cap C(\Omega)$ such that*

$$\frac{(dd^c u)^n}{(dd^c u)^{n-k} \wedge \omega^k} \leq \psi(z, u), \tag{5.5.18}$$

in the viscosity sense and

$$(dd^c u)^n \geq \psi(z, u)(dd^c u)^{n-k} \wedge \omega^k, \quad (5.5.19)$$

in the pluripotential sense. If there exists $a > 0$ such that $(dd^c u)^k \geq a\omega^k$ then u is a viscosity solution of the equation

$$\frac{(dd^c u)^n}{(dd^c u)^{n-k} \wedge \omega^k} = \psi(z, u). \quad (5.5.20)$$

Proof. It remains to show that u is a viscosity subsolution of (5.5.20) in any smooth strictly pseudoconvex domain $U \Subset \Omega$.

Let V be a smooth strictly pseudoconvex domain such that $U \Subset V \Subset \Omega$. By Proposition 5.5.8, there exists an increasing sequence of strictly plurisubharmonic functions $u_j \in C^\infty(\bar{V})$, such that u_j converges in capacity to u as $j \rightarrow \infty$, and

$$b \leq \frac{(dd^c u_j)^n}{(dd^c u_j)^{n-k} \wedge \omega^k} \leq \psi(z, u),$$

pointwise in V , where $b > 0$. By Theorem 5.5.6, we have $(dd^c u_j)^k \geq \binom{n}{k}^{-1} b \omega^k$. Then, there exists $C > 0$ such that

$$(dd^c u_j)^{n-k} \wedge \omega^k \geq \frac{1}{\psi(z, u)} (dd^c u_j)^n \geq C \omega^n.$$

Denote

$$f_j(z) := \frac{(dd^c u_j)^n}{(dd^c u_j)^{n-k} \wedge \omega^k}.$$

Then $f_j(z) \leq \psi(z, u)$ for any $z \in V$, and $(\psi - f_j)(dd^c u_j)^{n-k} \wedge \omega^k \geq C(\psi - f_j)\omega^n$ converges weakly to 0. Hence f_j converges in Lebesgue measure to ψ in V as $j \rightarrow \infty$.

Now, by Theorem 5.5.5, we have

$$(dd^c u_j) \wedge \omega_{A_1} \wedge \dots \wedge \omega_{A_{k-1}} \wedge \omega_{B^2}^{n-k} \geq (f_j)^{1/k} \omega^n,$$

pointwise in V for any $B \in \mathcal{B}(Id, n-k)$ and $A_1, \dots, A_{k-1} \in \mathcal{B}(B^2, k)$. Letting $j \rightarrow \infty$, we get

$$(dd^c u) \wedge \omega_{A_1} \wedge \dots \wedge \omega_{A_{k-1}} \wedge \omega_{B^2}^{n-k} \geq \psi^{1/k} \omega^n,$$

in the sense of Radon measures. It follows from [Lu13] that

$$(dd^c u)^k \wedge \omega_{B^2}^{n-k} \geq \psi^{1/k} \omega^n,$$

in the viscosity sense. Using Theorem 5.5.5, we get that u is a viscosity subsolution of (5.5.20) in U . The proof is completed. \square

5.6 Dirichlet problem for the Lagrangian phase operator

In this section, we prove the existence of unique viscosity solution to the Dirichlet problem for the Lagrangian phase operator. The existence and uniqueness of the smooth version was obtained recently by Collins-Picard-Wu [CPW17]. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. Consider the Dirichlet problem

$$\begin{cases} F[u] := \sum_{i=1}^n \arctan \lambda_i = h(z), & \text{on } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \quad (5.6.1)$$

where $\lambda_1, \dots, \lambda_n$ is the eigenvalues of the complex Hessian Hu . We can also write $F[u] = f(\lambda(Hu))$. We assume that $\varphi \in C^0(\partial\Omega)$ and $h : \bar{\Omega} \rightarrow [(n-2)\frac{\pi}{2} + \delta, n\frac{\pi}{2}]$ is continuous, for some $\delta > 0$.

The Lagrangian phase operator F in (5.6.1) arises in geometry and mathematical physics. We refer to [CPW17, HL82, JY17, CJY15, Yua06, WY13, WY14] and references therein for the details.

Since $h \geq (n-2)\frac{\pi}{2}$, this case is called the *supercritical phase* following [Yua06, JY17, CJY15, CPW17]. Recall first the following properties (cf. [Yua06, WY14, CPW17]);

Lemma 5.6.1. *Suppose $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ satisfying $\sum_i \arctan \lambda_i \geq (n-2)\frac{\pi}{2} + \delta$ for some $\delta > 0$. Then we have*

- (1) $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} > 0$ and $|\lambda_n| \leq \lambda_{n-1}$,
- (2) $\sum_i \lambda_i \geq 0$, and $\lambda_n \geq -C(\delta)$,
- (3) $\sum \lambda_i^{-1} \leq -\tan(\delta)$ when $\lambda_n < 0$.
- (4) for any $\sigma \in ((n-2)\frac{\pi}{2}, n\frac{\pi}{2})$, the set $\Gamma^\sigma := \{\lambda \in \mathbb{R}^n \mid \sum_i \arctan \lambda_i > \sigma\}$ is a convex set and $\partial\Gamma^\sigma$ is a smooth convex hypersurface.

It follows from Lemma 5.6.1 that the function f can be defined on a cone Γ satisfying $\Gamma_n \subset \Gamma \subset \Gamma_1$. We also remark that if $h \geq (n-1)\frac{\pi}{2}$, then F is concave while F have concave level sets if $(n-2)\frac{\pi}{2}h \leq (n-1)\frac{\pi}{2}$, but in general F may not be concave (cf. [CPW17]). Therefore we can not apply Theorem 5.3.2 directly. Fortunately, we still have a comparison principle for the Lagrangian operator using Lemma 5.3.1.

Lemma 5.6.2. *Let $u, v \in L^\infty(\bar{\Omega})$ be viscosity subsolution and supersolution of equation $F[u] = f(\lambda(Hu)) = h$ on Ω . Then*

$$\sup_{\Omega} (u - v) \leq \max_{\partial\Omega} \{(u - v)^*, 0\}. \quad (5.6.2)$$

Proof. We first define $\epsilon > 0$ by $\max_{\bar{\Omega}} h = n\frac{\pi}{2} - \epsilon$. Now for any $0 < \tau \leq \epsilon/2$, set $u_\tau = u + \tau|z|^2$. Let q_τ be any upper test for u_τ at any point $z_0 \in \Omega$, then $q = q_\tau - \tau|z|^2$ is also an upper test for u at z_0 . By the definition we have

$$F[q](z_0) = \sum_{i=1}^n \arctan \lambda_i(z_0) \geq h(z_0),$$

where $\lambda(z_0) = \lambda(Hq(z_0))$. We also have

$$F[q_\tau](z_0) = \sum_{i=1}^n \arctan(\lambda_i(z_0) + \tau). \quad (5.6.3)$$

Next, if $F[q](z_0) \geq n\frac{\pi}{2} - \frac{\epsilon}{2}$, then $F[q](z_0) \geq h(z_0) + \frac{\epsilon}{2}$ hence

$$F[q_\tau](z_0) \geq h(z_0) + \frac{\epsilon}{2}. \quad (5.6.4)$$

Conversely, if $F[q](z_0) < n\frac{\pi}{2} - \frac{\epsilon}{2}$, this implies that $\arctan(\lambda_n(z_0)) \leq \frac{\pi}{2} - \frac{\epsilon}{2n}$. Combining with Lemma 5.6.1 (2), we get $-C(\delta) \leq \lambda_n(z_0) \leq C(\epsilon)$. Using the Mean value theorem, there exists $\hat{\lambda}_n \in (\lambda_n(z_0), \lambda_n(z_0) + \tau)$ such that

$$\arctan(\lambda_n(z_0) + \tau) - \arctan \lambda_n(z_0) = \frac{1}{1 + \hat{\lambda}_n^2} \tau \geq C(\delta, \epsilon, \tau) > 0.$$

It follows that

$$F[q_\tau](z_0) \geq F[q](z_0) + C(\delta, \epsilon, \tau) \geq h(z_0) + C(\delta, \epsilon, \tau). \quad (5.6.5)$$

Combing with (5.6.4) yields

$$F[q_\tau](z_0) \geq h(z_0) + C,$$

where $C > 0$ depending only on δ, ϵ, τ . We thus infer that u_τ satisfies $F[u_\tau] \geq h(z) + C$ in the viscosity sense. Therefore applying Lemma 5.3.1 to u_τ and v , then let $\tau \rightarrow 0$, we obtain the desired inequality. \square

Theorem 5.6.3. *Let Ω is a bounded C^2 domain. Let \underline{u} is an bounded upper semi-continuous function on Ω satisfying $F[\underline{u}] \geq h(z)$ in Ω in the viscosity sense and $\underline{u} = \varphi$ on $\partial\Omega$. Then the Dirichlet problem 5.6.1 admits a unique viscosity solution $u \in C^0(\Omega)$.*

Proof. It suffices to find a viscosity supersolution \bar{u} for the equation $F[u] = h(z)$, satisfying $\bar{u} = \varphi$ on $\partial\Omega$. The C^2 -boundary implies the existence of a harmonic function ϕ on Ω for arbitrary given continuous boundary data φ . Since $\sum_i \lambda_i(H\phi) = 0$, it follows from Lemma 5.6.1 that we have $F[\phi] < (n-2)\frac{\pi}{2} + \delta \leq h$, hence ϕ is a supersolution for 5.6.1. The rest of the proof is similar to the one of Theorem 5.4.1, by using Lemma 5.6.2. \square

Chapter 6

Projects

One of my future projects is to study the geometric convergence (in the Gromov-Hausdorff sense) of the Kähler-Ricci flow on normal Kähler spaces. Most of the convergence results obtained so far stay away from the singularities themselves, proving C^∞ convergence on compact sets away from the singularities. Understanding the global behavior of the flow in the Gromov-Hausdorff topology is a fundamental and very challenging problem.

Another project is to study other geometric flows on compact Hermitian manifolds such as the Chern-Ricci flow, the Anomaly flow. It is also important to study their degenerate versions and their geometric convergence in the Gromov-Hausdorff sense.

I explain below in some detail two related and more specialized projects.

6.1 Convergence of the Kähler-Ricci flow on manifolds of general type

We wish to study the normalized Kähler-Ricci flow on a projective variety of general type X , whose canonical bundle K_X is big but not nef:

$$\begin{cases} \frac{\partial \omega_t}{\partial t} = -Ric(\omega_t) - \omega_t, \\ \omega|_{t=0} = \omega_0, \end{cases} \quad (6.1.1)$$

Let $T < +\infty$ be the maximal existence time of the flow, then the limiting class of the flow is

$$\{\alpha_T\} = \lim_{t \rightarrow T} \{\omega(t)\} = e^{-T} \{\omega_0\} - (1 - e^{-T})c_1(X).$$

Now at the maximal existence time T , the class α_T is big and nef. However, for $t > T$, α_t is not nef but big, thus we can not continue the flow in the classical sense. In the Analytic Minimal Model Program by Song-Tian [ST17, ST12], they have tried to repair a finite time singularity of the flow and start it over again. But in a different point of view, it was asked

by Feldman-Ilmanen-Knopf [FIK03, Question 8, Section 10] whether one can define and construct weak solutions of Kähler-Ricci flow after the maximal existence time for smooth solutions. In this project we are trying to answer the question and also study the weak convergence of the flow.

The key ingredient is the construction of weak solution for degenerate complex Monge-Ampère flows. Degenerate complex elliptic Monge-Ampère equation on compact Kähler manifold have recently been studied intensively using tools from pluripotential theory following the pioneering work of Bedford and Taylor in the local case [BT76, BT82, Kol98, GZ05, GZ07, BEGZ10]. A complementary viscosity approach has been developed only recently in [EGZ11, EGZ15a, HL09, Wan12]. The similar theory for the parabolic case however has been developed in [EGZ15b, EGZ16].

In the first part of the project, we are developing a viscosity theory for degenerate complex Monge-Ampère flows in big cohomology classes adapting the one in [EGZ15b, EGZ16]. Let $(\theta_t)_{t \in [0, T]}$ be a family of smooth closed $(1, 1)$ -forms such that $\{\theta_t\}$ is big and $\text{Amp}(\theta_t)$ contains $\Omega := \text{Amp}(\alpha)$ for all $t \in [0, T]$, where α is a fixed big class and $T \in (0, \infty)$ is fixed. We first study the degenerate complex Monge-Ampère flow

$$(\theta_t + dd^c \varphi_t)^n = e^{\partial_t \varphi_t + F(t, x, \varphi_t)} \mu \quad \text{on } X_T := [0, T] \times X, \quad (6.1.2)$$

starting from $\varphi(0, t) = \varphi_0$ a θ_0 -psh function with minimal singularities which is continuous in $\text{Amp}(\alpha)$, where

- $F(t, x, r)$ is a continuous in $[0, T] \times X$ and non decreasing in r .
- $\mu(x) \geq 0$ is a bounded continuous volume form on X ,

We have proved a viscosity comparison principle for this problem:

Theorem A. ([Tô]) *Let φ (resp. ψ) be a viscosity subsolution (resp. a supersolution) to (6.1.2) with the initial condition φ_0 is a θ_0 -psh function with minimal singularities which is continuous in $\text{Amp}(\alpha)$. Assume that $\partial_t \varphi$ is locally bounded in $\text{Amp}(\alpha)$. Then*

$$\varphi(t, x) \leq \psi(t, x) \quad \text{in } [0, T] \times \text{Amp}(\alpha).$$

By adding some condition of $(\theta)_{t \in [0, T]}$, we can remove the condition on $\partial_t \varphi$:

Theorem B. ([Tô]) *Suppose that there exists a smooth positive function $f : [0, T] \rightarrow \mathbb{R}$ such that $\chi_t := f(t)\theta_t$ is monotone. Then the comparison principle in Theorem A is also true without assuming the condition on $\partial_t \varphi$.*

We can see that the monotone condition in Theorem B is natural in studying the normalized Kähler-Ricci flow on Kähler manifolds of general type. Indeed, in this case the

evolving class along the flow is $\theta_t = e^{-t}\omega_0 + (1 - e^{-t})\theta$, where θ is a $(1, 1)$ form representing $-c_1(X)$.

As a first application of the comparison principle, we study the Cauchy problem

$$(CP_1) \quad \begin{cases} (\theta + dd^c \varphi_t)^n = e^{\partial_t \varphi_t + \varphi_t} \mu \\ \varphi(0, x) = \varphi_0, \end{cases}$$

where φ_0 is an θ -psh function with minimal singularities which is continuous in $\text{Amp}(\alpha)$.

We prove the existence of viscosity subsolution and supersolution to (CP_1) and construct barriers at each point $\{0\} \times \text{Amp}(\alpha)$. We then use the Perron's method to show the existence of a unique viscosity solution to the corresponding Cauchy problem.

Corollary C. ([Tô]) *The exists a unique viscosity solution to (CP_1) in $[0, T) \times \text{Amp}(\alpha)$. Moreover, the flow asymptotically recovers the solution of the corresponding elliptic Monge-Ampère equation.*

The second part of the project is to study the normalized Kähler-Ricci flow on a projective variety of general type X . We first use the viscosity theory above to construct the weak flow through the singularities. This gives an answer to the previous question by Feldman-Ilmanen-Knopf. In addition, it follows from [BEGZ10, EGZ09] that there exists a unique singular non-negatively curved metric on K_X satisfying the Kähler-Einstein equation. The following question is motivated by the smooth convergence of the Kähler-Ricci flow on Kähler manifolds with K_X is ample (cf. [Cao85]):

Question D. *Can we run the (normalized) Kähler-Ricci flow through the maximal existence time for smooth solutions in a weak sense with long time existence and does this weak flow converge to the singular Kähler-Einstein metric constructed in [BEGZ10]?*

In particular, Corollary C gives an affirmative answer when the normalized Kähler-Ricci flow starting from an initial data with minimal singularities in $c_1(X)$. In general, by the viscosity method above, we can prove the long time existence of viscosity solution to the flow by constructing sub/super solutions and barriers. The difficulty now comes from the convergence of the flow. Using the singular Kähler-Einstein metric constructed in [BEGZ10], we are trying to construct compatible sub/super solutions in order to show the convergence to this metric in $\text{Amp}(K_X)$.

6.2 Viscosity theory for the Hessian type equations on compact complex manifolds

A viscosity theory has been developed in [DDT17] for Hessian type equations on complex domains (see Section 5). With S. Dinew and H-S. Do, we are trying to adapt this construction to compact complex manifolds.

Let (X, ω) be a compact Hermitian manifold of dimension n , and $\chi(z)$ be a real $(1, 1)$ -form. If $u \in C^2(X)$, let $A[u]$ be the matrix with entries $A[u]^k_j = \omega^{k\bar{m}}(\chi_{\bar{m}j} + \partial_j \partial_{\bar{m}} u)$. We consider the fully nonlinear parabolic equation,

$$F(A[u]) = \psi(z, u), \quad (6.2.1)$$

where $F(A)$ is a smooth symmetric function $F(A) = f(\lambda[u])$ of the eigenvalues $\lambda_j[u]$, $1 \leq j \leq n$ of $A[u]$, defined on a open symmetric, convex cone $\Gamma \subset \mathbb{R}^n$ with vertex at the origin and containing the positive orthant Γ_n . We are adapting the local comparison principle to prove the following comparison principle on this case:

(Comparison principle) *Let Γ be the ellipticity cone associated to the equation (6.2.1). Assume that the operator $F(A[u]) = f(\lambda[u])$ in (6.2.1) satisfies*

$$f \in C^0(\bar{\Gamma}), f > 0 \text{ on } \Gamma, f = 0 \text{ on } \partial\Gamma,$$

and

$$f(\lambda + \mu) \geq f(\lambda), \quad \forall \lambda \in \Gamma, \mu \in \Gamma_n.$$

Assume moreover that either

$$\sum_{i=1}^n \frac{\partial f}{\partial \lambda_i} \lambda_i = \sum_{i=1}^n f_i \lambda_i \geq \nu(f) \text{ in } \Gamma$$

for some positive increasing function ν , or

f is concave and homogeneous.

Then any bounded subsolution u and supersolution v to the equation (6.2.1) satisfy

$$u \leq v \text{ on } X.$$

Once we obtain the comparison theorem for the sub/super-solutions, we can solve some degenerate geometric equations such as the Donaldson equation, special Lagrangian type equation, and also answer several open questions on the degenerate J flow in the boundary case (cf. [SW13a, FLSW14]) on compact Kähler manifolds.

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