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présentée par

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Le problème de Cauchy pour les systèmes quasi-linéaires faiblement hyperboliques ou non-hyperboliques en régularité Gevrey

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Résumé

Nous considérons dans cette thèse le problème de Cauchy pour des systèmes d'EDP quasilinéaires, du premier ordre. Dans le cas initialement elliptique, c'est-à-dire un spectre non-réel pour le symbole principal du système à $t = 0$, nous prouvons un résultat d'instabilité au sens d'Hadamard. La preuve est basée sur la construction d'une famille de solutions présentant une croissance exponentielle en temps et fréquence. Cette famille invalide la régularité Hölder du flot, partant d'espaces de Gevrey vers L^2. Nous prouvons un résultat analogue pour différents cas de transition de l'hyperbolique vers l'elliptique, avec une restriction possible sur l'indice Gevrey pour lequel l'instabilité est observée. Dans un second temps, nous considérons le cas faiblement hyperbolique et semilinéaire. Grâce à des estimations d'énergie dans les espaces de Gevrey et à la construction d'un symétriseur adapté, nous prouvons le caractère localement bien-posé pour un tel système. Pour ce faire, nous utilisons et démontrons aussi un résultat d'action d'opérateurs pseudo-différentiels dont le symbole possède une régularité Gevrey dans la variable d'espace.

Mots clés: problème de Cauchy, quasilinéaire, Gevrey, bien-posé, mal-posé, non hyperbolique, faiblement hyperbolique.

Abstract

We consider the Cauchy problem for first-order, quasilinear systems of PDEs. In the initially elliptic case, that is when the principal symbol of the system has nonreal spectrum at time $t = 0$, we prove an instability result in the sense of Hadamard. The proof is based on the construction of a family of exact solutions which exhib an exponential growth, both in time and frequency. That family leads to a defect of Hölder regularity of the flow, starting from evrey spaces to L^2 space. We prove analogous results for some cases of transition from hyperbolicity to ellipticity, with a potential restriction on the Gevrey index for which we may observe the instability. In a second time, we consider weakly hyperbolic systems. Thanks to an energy estimate in Gevrey spaces and the construction of a suitable symetriser, we prove local well-posedness for such a system. In doing so we use and prove a result on actions of pseudo-differential operators whose symbols have Gevrey regularity in the spatial variable.

Keywords: Cauchy problem, quasilinear, Gevrey, ill-posedness, well-posedness, non hyperbolic, weakly hyperbolic.

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Chapter 1

Introduction

1.1 Préliminaire sur le problème de Cauchy en régularité Gevrey

Dans cette thèse, nous nous sommes intéressés au problème de Cauchy pour des systèmes d'équations aux dérivées partielles quasi-linéaires. Nous considérons

$$\partial_t u = \sum_{j=1}^{d} A_j(t,x,u)\partial_{x_j} u + A_0(t,x)u + f(t,x,u) \quad \text{avec} \quad u(0,x) = u_0(x). \tag{1.1.1}$$

Ici x est dans $\mathbb{R}^d$, u et $f(u)$ sont des vecteurs de $\mathbb{R}^N$ avec $N \geq 2$, et les A_j et A_0 sont des fonctions à valeurs dans $\mathbb{R}^{N\times N}$. Nous nous intéressons au problème de Cauchy dans les temps futurs, donc pour $t \geq 0$, pour lequel se posent plusieurs questions. La première étant: pour toute condition initiale u_0 donnée, peut-on prouver l'existence d'une solution du système (1.1.1), au moins pour des temps petits ? La seconde, qui en découle: pour deux conditions initiales relativement proches, pourrait-on montrer que les solutions qui en découlent sont encore proches, en un certain sens (et là encore, pour des temps petits)?

Ces deux questions sont vastes, et vagues. Sans plus d'informations sur le problème (1.1.1), y répondre semble une gageure. Il nous semble que ces informations peuvent se répartir en trois grandes lignes:

1. La *structure* du système (1.1.1): que connait-on du spectre du symbole principal $\xi \mapsto \sum_j A_j(t,x,u)\xi_j$, à (t,x,u) fixé ?

2. La *régularité* des coefficients du système, à savoir les A_j et le terme source f. Mais aussi la régularité attendue de la condition initiale u_0 et de la solution u. Dans cette thèse, nous travaillerons avec des coefficients à régularité élevée.

3. Le caractère *linéaire ou non-linéaire* du système, qui *grosso modo* pour (1.1.1) se répartit en trois classes: 1) linéarité du système, les A_j et f ne dépendent pas de la solution u; 2) semi-linéarité, la source f seule dépend de u ; 3) quasi-linéarité, où les A_j et f dépendent de u.

Dans cette thèse, nous avons considéré plusieurs situations pour (1.1.1), qui correspondent à plusieurs types de structure pour le spectre du symbole principal. Ces différentes situations sont *stables ou instables par perturbations*, que ce soit par des termes linéaires (notamment par le terme linéaire A_0, d'ordre 0) ; par des termes non-linéaires, via $f(u)$; ou par perturbation de la donnée initiale du problème de Cauchy. Nous avons étudié le cas strictement elliptique, qui est stable vis-à-vis de ces trois formes de perturbations, et les cas faiblement hyperbolique ou elliptique, qui ne sont pas stables spectralement.

Pour chacune de ces situations, nous avons étudié la *stabilité du système* (1.1.1), au sens du caractère bien-posé du problème de Cauchy, localement en temps. Dans le cas initialement elliptique, nous avons étendu la preuve de l'instabilité donnée par Métivier dans [Mét05] dans un cadre Sobolev, au cadre Gevrey. La difficulté vient, en particulier, du *temps d'observation de l'instabilité dans les espaces de Gevrey:* la différence des ordres de grandeur - logarithmique en fréquence en Sobolev contre puissance négative en fréquence en Gevrey - impose un traitement plus fin des termes linéaires et non-linéaires. Nous avons ensuite considéré le cas de certaines transitions de l'hyperbolique vers l'elliptique, pour lesquelles nous avons étendu le précédent travail, toujours en régularité Gevrey. Ces situations sont instables spectralement, et leur caractère faiblement elliptique implique un temps de transition plus long. Pour des temps courts, le système n'est pas assez éloigné du domaine d'hyperbolicité pour que la nature elliptique du système, dans les temps futurs et plus longs, apparaisse. Nous avons dû ainsi développer les techniques et l'analyse développées dans [Mét05], et surmonter la double difficulté de décrire certaines transitions faiblement elliptiques en régularité Gevrey.

Dans certains cas faiblement hyperboliques, et donc spectralement instables, nous avons montré le caractère bien-posé, localement en temps, du problème de Cauchy dans les espaces de Gevrey. L'instabilité spectrale de telles situations impose une étude précise en régularité Gevrey, régularité "naturelle" des systèmes faiblement hyperboliques. Ceci est connu depuis l'article précurseur de Colombini, Janelli et Spagnolo [CJS83], dont nous avons poursuivi l'étude. Les travaux suivants de Colombini et Nishitani dans [CN07] ont pu étendre l'analyse au cas de systèmes non-strictement hyperboliques, mais loin de transitions vers l'elliptique - situations plus instables encore, et que nous étudions au Chapitre 4 de la présente thèse. Pour ce faire, nous avons construit un symétriseur adapté à la classe de systèmes étudiés, développé (dans le Chapitre 5) une nouvelle ingénierie technique autour des opérateurs pseudo-différentiels dans les espaces de Gevrey, et finalement prouvé une estimation d'énergie Gevrey qui prouve le caractère bien-posé localement en temps pour tout indice Gevrey supérieur à $1/2$, limite heuristique de ce type de systèmes faiblement hyperboliques.

Nous allons étudier dans la suite ces trois points, de manière informelle et dans le cas où les A_j sont constants, avec pour but de dégager à chaque fois les points importants pour la suite. Nous pouvons aussi nous attendre à ce que les trois points sus-mentionnés ne soient pas indépendants, au contraire.

1.1.1 Étude du symbole principal, avec coefficients constants

Supposons ici que les A_j sont des matrices constantes, indépendantes des variables (t, x, u). Supposons aussi que les termes linéaire A_0 et non-linéaire f sont nuls. Nous nous focal-

isons donc sur la partie linéaire d'ordre 1 du système, avec une régularité des coefficients optimale, en un sens. En utilisant la transformée de Fourier dans la variable x, nous pouvons écrire (1.1.1) comme

$$\partial_t v = \sum_j A_j i\xi_j v \quad \text{avec} \quad v(0) = v_0 \tag{1.1.2}$$

et où nous avons posé $v(t,\xi) = \widehat{u}(t,\xi)$ et $v_0(\xi) = \widehat{u}_0(\xi)$. Alors, le système (1.1.2) se résoud en

$$v(t,\xi) = e^{itA(\xi)} v_0(\xi) \tag{1.1.3}$$

où $A(\xi) = \sum_j A_j \xi_j$ est *le symbole principal* du système (1.1.2), et est une matrice de $\mathbb{R}^{N\times N}$ homogène de degré 1 en ξ. La question est maintenant d'étudier l'exponentielle matricielle $e^{itA(\xi)}$, qui est le *propagateur* de l'équation. Ceci dépend notamment du spectre de A.

1. Cas strictement hyperbolique: si le spectre de A est réel et séparé pour tout ξ, la matrice A est diagonalisable et il vient $\left|e^{itA(\xi)}\right| \approx 1$ pour tout t et ξ.

2. Cas (faiblement) hyperbolique: si le spectre de A est réel mais des croisements de valeurs propres apparaissent quand ξ varie, l'exponentielle matricielle croit polynomialement en ξ et t: $\left|e^{itA(\xi)}\right| \lesssim (t|\xi|)^{(m-1)}$. Notons que la puissance $m \geq 2$ est égale à la taille du plus grand des blocs de Jordan des valeurs propres qui se croisent.

3. Cas elliptique: si le spectre n'est pas réel, alors on doit s'attendre à une croissance exponentielle comme $\left|e^{itA(\xi)}\right| \lesssim e^{ct|\xi|}$.

En revenant à l'expression (1.1.3) de la solution (côté Fourier) de (1.1.2), nous pouvons nous intéresser à la question de la régularité de v: la régularité de v_0 est modifiée par l'action du propagateur $e^{itA(\xi)}$.

Une première classe de régularité est celle des espaces de Sobolev. On note d'abord $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, puis on considère H^s, pour s réel, l'ensemble des fonctions $v(\xi)$ telles que $\langle\xi\rangle^s v(\xi)$ est dans L^2. Cet espace est muni d'une norme: $|v|_{H^s} = |\langle\cdot\rangle^s v|_{L^2}$. Alors, dans le cas strictement hyperbolique 1 décrit au-dessus, si v_0 est dans H^s pour un certain s, la solution $v(t)$ de (1.1.2) l'est aussi et vérifie $|v(t)|_{H^s} \approx |v_0|_{H^s}$: le propagateur $e^{itA(\xi)}$ agit de H^s dans lui-même, de façon continue.

Dans le cas faiblement hypebolique 2 décrit ci-dessus, l'action du propagateur diminue la régularité de v_0. En effet, si v_0 est toujours dans H^s, alors ici $v(t)$ est dans H^{s-m}, avec comme norme $|v(t)|_{H^{s-m}} \lesssim |v_0|_{H^s}$.

En revanche, dans le cas elliptique 3, l'action du propagateur impose une perte exponentielle en fréquence et en temps à la régularité initiale. Si v_0 est dans H^s, même avec s très grand, instantanément $v(t)$ n'est plus dans aucun Sobolev. Précisément, et comme nous le verrons plus en détails dans la Section 1.1.4, la perte $e^{ct|\xi|}$ est typique d'une perte de régularité analytique.

Ainsi, nous avons vu que la *structure* même du système (1.1.1) (caractérisée ici par le spectre de $A(\xi)$) impose une contrainte sur la *régularité* attendue de la solution.

1.1.2 Stabilité du spectre par perturbation linéaire d'ordre 0

On considère maintenant que A_0 est une matrice constante non nulle, et on étudie donc le système linéaire

$$\partial_t v = (iA(\xi) + A_0)\, v \quad \text{avec} \quad v(0) = v_0. \tag{1.1.4}$$

Comme pour (1.1.2), ce système se résout en $v(t) = e^{t(iA(\xi)+A_0)}v_0$ et il s'agit donc de décrire le propagateur $e^{t(iA(\xi)+A_0)}$ dans chacune des trois situations décrites ci-dessus. Sont-elles stables par ajout de ce terme linéaire A_0 ?

Les cas strictement hyperbolique 1 et strictement elliptique 3 sont stables par perturbations, par continuité des valeurs propres de $\xi \mapsto A(\xi)$ (voir par exemple [Kat66] ou [Tex17]). Ainsi, dans le cas strictement hyperbolique, le propagateur vérifie encore $\left|e^{t(iA(\xi)+A_0)}\right| \approx 1$ pour tout temps t petit et ξ. Dans le cas elliptique 3, le propagateur satisfait $\left|e^{t(iA(\xi)+A_0)}\right| \lesssim e^{ct|\xi|}$ pour tout ξ et petit temps t.

Au contraire, le cas faiblement hyperbolique 2 n'est pas stable par perturbation. Par exemple, si pour une certaine fréquence ξ_0 le symbole principal est

$$A(\xi_0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

qui a donc 0 comme valeur propre double, la perturbation

$$A_0 = \begin{pmatrix} 0 & 0 \\ a^2 & 0 \end{pmatrix}$$

entraine un glissement du spectre imaginaire pur de $iA(\xi_0)$ vers $\mathbb{R}$, le spectre de $iA(\xi_0)+A_0$ étant $\{\pm(1+i)a/\sqrt{2}\}$. On peut quantifier ce glissement, en donnant une borne sur la partie réelle des valeurs propres de $iA(\xi) + A_0$. Supposons que pour $\xi_0 \neq 0$, la matrice $A(\xi_0)$ a une valeur propre λ_0 de multiplicité $m \geq 2$. Par homogénéité du symbole principal $A(\xi)$, on peut écrire

$$iA(\xi) + A_0 = |\xi| \left(iA(\omega) + |\xi|^{-1}A_0\right)$$

où $\omega = |\xi|^{-1}\xi$ pour $\xi \neq 0$. Alors, en considérant $|\xi|$ grand, avec ξ dans un voisinage conique de ξ_0, le terme $|\xi|^{-1}A_0$ est une petite perturbation de $iA(\omega)$. En $\omega = \omega_0 = |\xi_0|^{-1}\xi_0$, la matrice $iA(\omega_0)$ a une valeur propre $\lambda_0 = \lambda(\omega_0, 0)$ de multiplicité $m \geq 2$. Ainsi (voir par exemple la Proposition 8 dans [Tex17]), cette valeur propre engendre donc une branche de valeurs propres $\lambda(\omega_0, |\xi|^{-1})$ de $iA(\omega_0) + |\xi|^{-1}A_0$, et qui vérifie de plus

$$\left|\lambda(\omega_0, |\xi|^{-1}) - \lambda_0\right| \lesssim |\xi|^{-1/m}.$$

Cette estimation se transpose aux valeurs propres $\lambda(\xi)$ de $iA(\xi) + A_0$, et on obtient

$$|\lambda(\xi) - \lambda(\xi_0)| \lesssim |\xi|^{1-1/m}.$$

On peut en déduire alors l'estimation suivante pour le propagateur, au voisinage de $\xi = \xi_0$:

$$\left|e^{t(iA(\xi)+A_0)}\right| \lesssim e^{ct\langle\xi\rangle^{(m-1)/m}}. \tag{1.1.5}$$

Cette estimation pour le propagateur est typique de la régularité Gevrey, comme nous le verrons plus bas dans la Section 1.1.4.

1.1.3 Influence des termes non-linéaires

On considère à présent, toujours dans le cas d'un symbole principal à coefficients constants et sans terme linéaire A_0, le système

$$\partial_t v = \sum_j A_j i\xi_j v + \widetilde{f}(t,\xi,v) \quad \text{avec} \quad v(0) = v_0 \tag{1.1.6}$$

où $\widetilde{f}(v)$ est un terme source a priori non-linéaire en v. En utilisant le propagateur défini à la Section précédente et la formule de Duhamel, on peut réécrire le système (1.1.6) en l'équation de point fixe

$$v(t,\xi) = e^{itA(\xi)}v_0(\xi) + \int_0^t e^{i(t-t')A(\xi)}\widetilde{f}\left(t',\xi,v(t')\right)dt'. \tag{1.1.7}$$

Dans les cas strictement hyperbolique 1 et faiblement hyperbolique 2, nous avons vu que le système (1.1.2) avait une solution $v(t)$, respectivement dans H^s et H^{s-m} si v_0 est dans H^s. La question est maintenant d'étendre ce résultat au cas d'un terme source non nul et potentiellement non-linéaire.

Pour ce faire, il faut étudier l'opérateur $v \mapsto \int_0^t e^{i(t-t')A(\xi)}\widetilde{f}(t',\xi,v(t'))\,dt'$ dans les espaces de Sobolev. Si cet opérateur est une contraction, le théorème du point fixe de Banach suffirait à montrer l'existence de solutions pour (1.1.6). Une première question se pose: cet opérateur est-il bien défini dans les espaces de Sobolev ? Si v est dans H^s, $\widetilde{f}(v)$ est-il encore dans H^s ?

Dans le cas strictement hyperbolique 1, la question revient à demander que H^s, l'espace dans lequel vit la solution du problème linéaire, est une algèbre, et munie potentiellement d'une norme d'algèbre. Ceci est réalisé pour $s > d/2$, où d est la dimension de la variable d'espace x. C'est ainsi une première obstruction imposée par un terme source non-linéaire: la régularité attendue de la solution doit être assez grande.

Dans le cas faiblement hyperbolique 2, le problème est plus profond. L'opérateur $v \mapsto \int_0^t e^{i(t-t')A(\xi)}\widetilde{f}(t',\xi,v(t'))\,dt'$ est défini de H^s dans H^{s-m}. Pour pouvoir espérer utiliser un théorème de point fixe, il faudrait donc se placer dans un espace de très faible régularité, à savoir $H^{-\infty} = \cup_{s\in\mathbb{R}} H^s$.

Ainsi, après avoir étudié la perte de régularité "infinie" du cas elliptique 3, nous voyons ici une autre obstruction apportée par la structure du système, cette fois-ci sur le terme source et son caractère linéaire ou non. En particulier, la perte polynomiale du propagateur dans le cas faiblement hyperbolique 2 indique que les espaces de Sobolev (qui décrivent une décroissance polynomiale en fréquence) sont trop limités pour étudier (1.1.1) dans ce cas. Nous allons donc introduire dans la Section suivante les espaces de Gevrey, espaces qui nous suivront tout au long de cette thèse.

1.1.4 Régularité Gevrey

Dans le cas elliptique 3 traité précédemment, nous avons vu que le propagateur $e^{itA(\xi)}$ pouvait avoir une croissance exponentielle du type $e^{ct|\xi|}$. On définit alors

$$\mathcal{G}_c^1 = \left\{ v \in L^2 \,:\, e^{c\langle\xi\rangle}v(\xi) \in L^2 \right\}$$

en posant $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$. On note que, pour de grandes fréquences ξ, on a $\langle\xi\rangle \sim |\xi|$: les deux ne diffèrent qu'aux basses fréquences, mais l'utilisation de $\langle\cdot\rangle$ en place de $|\xi|$ apporte de meilleures propriétés aux espaces définis, comme nous le verrons par la suite.

En utilisant ce nouvel espace, dans le cas elliptique 3, on peut montrer que si $v_0 \in \mathcal{G}^1_{c_0}$ pour un certain $c_0 > 0$, alors le problème de Cauchy (1.1.2) admet une solution $v(t)$ dans $\mathcal{G}^1_{c_0-c_1t}$ pour un certain $c_1 > 0$. Même, l'espace $\mathcal{G}^1_c$ étant une algèbre, on peut étendre ce résultat au problème de Cauchy (1.1.6).

Ce résultat est une forme particulière du *théorème de Cauchy-Kovalevskaya* qui concerne la résolution du problème de Cauchy (1.1.1) dans le cas de coefficients et d'une donnée initiale *analytiques*. En effet, si $u(x)$ est une fonction dont la transformée de Fourier $\widehat{u}(\xi)$ est dans $\mathcal{G}^1_c$, alors u est une fonction analytique: elle vérifie

$$|\partial_x^\alpha u(x)| \leq C_K R^{|\alpha|}\alpha!$$

sur tout compact K de $\mathbb{R}^d$, avec C_K une constante dépendant uniquement de K, et $R \approx c^{-1}$ correspond à l'inverse du rayon de convergence.

On peut étendre la définition de $\mathcal{G}^1_c$. Pour $\sigma \in (0,1)$ et $c > 0$, on pose

$$\mathcal{G}^\sigma_c = \left\{ v \in L^2 \,:\, e^{c\langle\xi\rangle^\sigma} v(\xi) \in L^2 \right\}.$$

Ces espaces sont appelés espaces de Gevrey, en l'honneur de Maurice Gevrey [Gev18]. Ces espaces ont notamment la bonne propriété d'être des algèbre de Banach pour la norme

$$|v|_{\sigma,c} = \left| e^{c\langle\cdot\rangle^\sigma} v \right|_{L^2}. \tag{1.1.8}$$

Dans le cas analytique $\mathcal{G}^1_c$, le nombre c est le rayon de convergence. Dans le cas général $\mathcal{G}^\sigma_c$, on appelle donc c le rayon Gevrey. On appelle aussi σ l'indice Gevrey.

De plus, l'exponentielle en fréquence permet de contrôler tout polynôme. En effet, pour tout $m \geq 2$, on a l'inégalité

$$|\xi|^{m-1} e^{-\tau\langle\xi\rangle^\sigma} \leq \tau^{-(m-1)/\sigma}$$

uniformément en $\xi \in \mathbb{R}^d$. On considère alors un rayon Gevrey $c_0 - ct$, ce qui permet d'obtenir

$$\begin{aligned} e^{(c_0-ct)\langle\xi\rangle^\sigma} \left| e^{i(t-t')A(\xi)} f(t',\xi) \right| &\leq e^{(c_0-ct)\langle\xi\rangle^\sigma} \left((t-t')|\xi|\right)^{m-1} \left| f(t',\xi) \right| \\ &\leq \left((t-t')|\xi|\right)^{m-1} e^{-c(t-t')\langle\xi\rangle^\sigma} \left| e^{(c_0-ct')\langle\xi\rangle^\sigma} f(t',\xi) \right| \\ &\lesssim_c \frac{1}{(t-t')^{(m-1)(1/\sigma-1)}} \left| f(t') \right|_{\sigma,c(t')}. \end{aligned}$$

On utilise ensuite l'intégration en temps de $t' = 0$ à $t' = t$ du terme source. Pour cela, il faut que la borne ci-dessus soit intégrable: en particulier, il faut avoir $(m-1)(1/\sigma-1) < 1$ ce qui correspond à la borne

$$\sigma > \frac{m-1}{m} = 1 - \frac{1}{m} \tag{1.1.9}$$

de l'indice Gevrey. Ainsi, nous venons de montrer que, dans le cas faiblement hyperbolique 2, la croissance polynomiale en temps et fréquence du propagateur peut être contrôlée dans les espaces de Gevrey $\mathcal{G}_c^\sigma$, à condition que σ vérifie la borne (1.1.9).

Nous avons vu aussi, dans la Section 1.1.2, que le cas faiblement hyperbolique était instable par perturbation d'un terme linéaire A_0, d'ordre 0. Nous avons ainsi montré que le propagateur vérifiait l'estimation (1.1.5). Cette croissance sous-exponentielle du propagateur est typiquement Gevrey, et peut donc être contrôlé dans les espaces $\mathcal{G}_c^\sigma$, si la condition (1.1.9) sur l'indice Gevrey est vérifié.

1.1.5 Instabilité

Nous avons vu plus haut dans les Sections 1.1.1 à 1.1.4 que le probleme de Cauchy, pour certains triplets, est bien posé. On peut se demander si on peut décrire de manière plus précise l'obstruction au caractère bien-posé de (1.1.6) pour les autres triplets.

Nous allons ici étudier *la régularité du flot* de (1.1.2) dans le cas elliptique 3, c'est-à-dire considérer l'application Φ qui, à une condition initiale v_0, associe la solution $v(t) =: \Phi(t; v_0)$ qui en découle. D'après la discussion précédente, cette application est bien définie de $\mathcal{G}^1$ dans lui-même. Pour étudier la continuité de Φ, on peut choisir alors comme espace de départ $\mathcal{G}^1_{c_0}$ associé à la norme Sobolev H^m, pour $m \geq 0$ potentiellement grand, ou associé à la norme Gevrey $|\cdot|_{\sigma,\tilde{c}}$ définie par (1.1.8), avec $\tilde{c}$ potentiellement petit ; et comme espace d'arrivée $\mathcal{G}^1_{c_0-ct}$ associé à la norme $|\cdot|_{L^2}$. Dans ce cadre topologique, le flot Φ est-il encore continu ? C'est-à-dire, peut-on montrer

$$|\Phi(t; v_0)|_{L^2} \lesssim |v_0|_{H^m} \tag{1.1.10}$$

ou

$$|\Phi(t; v_0)|_{L^2} \lesssim |v_0|_{\sigma,\tilde{c}} \tag{1.1.11}$$

pour tout v_0 dans une petite boule de $\mathcal{G}^1_{c_0}$, uniformément en temps dans un petit intervalle?

Pour ce faire, nous allons constuire une famille de conditions intiales $v_0(\xi\,;\varepsilon)$ dans $\mathcal{G}^1_{c_0}$, avec ε un petit paramètre. Dans le cas elliptique 3, par hypothèse il existe $\xi_0 \in \mathbb{S}^{d-1}$ tel que $A(\xi_0)$ possède un vecteur propre unitaire v_+ associé à une valeur propre $-i\gamma_0$ avec $\gamma_0 > 0$. Alors, on pose

$$v_0(\xi\,;\varepsilon) = e^{-M(\varepsilon)} v_+ \delta\left(\xi = \xi_0/\varepsilon\right)$$

où δ est la fonction delta de Dirac. Notons que, puisque $v(t,\xi) = \widehat{u}(t,\xi)$, cela correspond à $u_0(x\,;\varepsilon) = e^{-M(\varepsilon)} v_+ e^{ix\cdot\xi_0/\varepsilon}$. Le poids $M(\varepsilon) > 0$ est choisi de sorte que la norme Sobolev ou Gevrey de v_0 soit petite. Ainsi, en norme Sobolev, on a

$$|v_0(\cdot\,;\varepsilon)|_{H^m} \approx \varepsilon^{-m} e^{-M(\varepsilon)}$$

et donc on pose $M(\varepsilon) = m'|\ln(\varepsilon)|$ avec $m' > m$. En norme Gevrey, d'après la définition (1.1.8) on a

$$\begin{aligned} |v_0(\cdot\,;\varepsilon)|_{\sigma,\tilde{c}} &= \left| e^{\tilde{c}\langle\xi\rangle^\sigma} e^{-M(\varepsilon)} v_+ \delta(\xi = \xi_0/\varepsilon) \right|_{L^2_\xi} \\ &= e^{-M(\varepsilon)} e^{\tilde{c}\langle\xi_0/\varepsilon\rangle^\sigma}. \end{aligned}$$

Comme $\xi_0 \in \mathbb{S}^{d-1}$ et que $\langle \varepsilon^{-1}\xi_0 \rangle \sim \varepsilon^{-1}$ à la limite $\varepsilon \to 0$, on obtient

$$|v_0(\cdot\,;\varepsilon)|_{\sigma,\tilde{c}} \approx e^{-M(\varepsilon)} e^{\tilde{c}\varepsilon^{-\sigma}}$$

et donc on pose $M(\varepsilon) = c'\varepsilon^{-\sigma}$ pour $c' > \tilde{c}$.

En utilisant la formule (1.1.3), on a donc une famille de solutions $v(\varepsilon)$qui découlent de $v_0(\cdot\,;\varepsilon)$, et qui vérifient la croissance exponentielle

$$|v(t,\xi\,;\varepsilon)| \approx e^{-M(\varepsilon)} e^{t\gamma_0/\varepsilon}. \tag{1.1.12}$$

Afin de mettre en contradiction (1.1.10), on calcule

$$\frac{|v(t,\cdot\,;\varepsilon)|_{L^2}}{|v_0(\cdot\,;\varepsilon)|_{H^m}} \approx \varepsilon^m e^{t\gamma_0/\varepsilon}.$$

Ainsi, pour $t_\varepsilon = m'\gamma_0^{-1}\varepsilon|\ln(\varepsilon)|$, on a

$$\frac{|v(t_\varepsilon,\cdot\,;\varepsilon)|_{L^2}}{|v_0(\cdot\,;\varepsilon)|_{H^m}} \approx \varepsilon^{m-m'}$$

et donc

$$\lim_{\varepsilon\to 0} \frac{|v(t_\varepsilon,\cdot\,;\varepsilon)|_{L^2}}{|v_0(\cdot\,;\varepsilon)|_{H^m}} = \lim_{\varepsilon\to 0} \varepsilon^{m-m'} = +\infty \tag{1.1.13}$$

car $m' > m$, alors que le temps t_ε tend vers 0. L'inégalité (1.1.10) est donc mise en défaut par une famille de petites (au sens de la norme Sobolev) conditions initiales, en un temps $t_\varepsilon \approx m\varepsilon|\ln(\varepsilon)|$ très court: c'est le *temps d'observation de l'instabilité* de (1.1.2) dans l'espace de Sobolev H^m, et l'instabilité est générée par une famille de conditions initiales oscillant fortement.

De la même façon, on peut montrer aussi l'instabilité de (1.1.2) dans l'espace de Gevrey $\mathcal{G}^\sigma_{\tilde{c}}$. En reprenant le calcul précédent, on obtient

$$\frac{|v(t,\cdot\,;\varepsilon)|_{L^2}}{|v_0(\cdot\,;\varepsilon)|_{\sigma,\tilde{c}}} \approx e^{-\tilde{c}\varepsilon^{-\sigma}} e^{t\gamma_0/\varepsilon}.$$

En posant ici comme temps d'observation de l'instabilité Gevrey $t_\varepsilon = c'\gamma_0^{-1}\varepsilon^{1-\sigma}$, on a donc

$$\lim_{\varepsilon\to 0} \frac{|v(t_\varepsilon,\cdot\,;\varepsilon)|_{L^2}}{|v_0(\cdot\,;\varepsilon)|_{\sigma,\tilde{c}}} = \lim_{\varepsilon\to 0} e^{(c'-\tilde{c})\varepsilon^{-\sigma}} = +\infty \tag{1.1.14}$$

alors qu'ici encore, $t_\varepsilon \to 0$ quand $\varepsilon \to 0$. L'inégalité (1.1.11) est donc elle aussi mise en défaut. En comparaison, le temps d'instabilité Sobolev est beaucoup plus court que le temps d'instabilité gevrey. En effet, $t_\varepsilon/\varepsilon$ est logarithmique en ε dans le cadre Sobolev, alors qu'il est une puissance négative de ε dans le cadre Gevrey. Cette différence est essentielle pour la suite.

Nous allons maintenant passer en revue les résultats de cette thèse, qui reposent en grande partie sur les quelques idées développées dans cette introduction. Dans la Section 1.2, nous allons poursuivre la discussion ci-dessus, en étendant cette idée de preuve

d'instabilité dans un cadre plus général, et dans des situations plus variées: ellipticité initiale, mais aussi transitions de l'hyperbolique vers l'elliptique. Ces travaux sont les deux articles [Mor16a] et [Mor16b], qui forment les Chapitres 2 et 3 de cette thèse. Ces travaux portent sur des systèmes quasi-linéaires. Nous décrirons aussi les perspectives d'un papier à venir [Mor17b].

Ensuite, nous poursuivons par l'introduction 1.3, nous décrirons nos travaux concernant le caractère bien-posé en Gevrey d'une certaine classe de systèmes faiblement hyperboliques. Ces résultats forment le Chapitre 4 de cette thèse, et correspondent au papier [Mor17a]. Ces résultats portent sur des systèmes semi-linéaires, voir la Remarque 1.4.1.

Enfin, dans la dernière partie 1.4 de cette introduction, qui décrit plusieurs résultats portant sur les espaces de Gevrey, et notamment l'action des opérateurs pseudo-différentiels à symbole dans une certaine classe décrivant une régularité Gevrey. Ces résultats feront l'objet d'un futur article [Mor17c], et forment le Chapitre 5 de cette thèse.

1.2 Caractère mal-posé pour un système non-hyperbolique

1.2.1 Présentation

On considère dans la suite le problème de Cauchy (1.1.1), avec les $A_j(t,x,u)$ et $f(t,x,u)$ analytiques localement autour de $(0,x_0,u_0) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^N$, et qui ne dépendent pas trivialement de u (on parle alors de système quasi-linéaire). Dans la continuité de la discussion précédente dans la Section 1.1.5, nous considérons que le système n'est pas hyperbolique, au sens où le symbole principal

$$A(t,x,u,\xi) = \sum_j A_j(t,x,u)\xi_j$$

n'a pas un spectre réel. Dans la suite, nous allons considérer deux cas:

• Le cas initialement elliptique, c'est-à-dire qu'il existe $(x_0,u_0,\xi_0) \in \mathbb{R}^d\times\mathbb{R}^N\times\mathbb{R}^d$ tel que $A(0,x_0,u_0,\xi_0)$ n'a pas un spectre réel. La matrice $A(0,x_0,u_0,\xi_0)$ étant à coefficients réels, elle a alors au moins un couple de valeurs propres non-réelles conjuguées. On note $\pm i\gamma_0$ la partie imaginaire d'un tel couple, avec $\gamma_0 > 0$. Ce cas est traité dans le Chapitre 2.

• Le cas d'une transition de l'hyperbolique vers l'elliptique: le symbole $A(0,x,u,\xi)$ a un spectre réel au voisinage d'un point $(x_0,u_0,\xi_0) \in \mathbb{R}^d\times\mathbb{R}^N\times\mathbb{R}^d$, mais $A(t,x,u,\xi)$ n'a plus un spectre réel dès que $t > 0$, localement autour de (x_0,u_0,ξ_0). Cette situation est traitée dans le Chapitre 3.

Le cas initialement elliptique a été d'abord étudié dans les espaces de Sobolev dans [Mét05], dans lequel le problème de Cauchy (1.1.1) est prouvé être Hölder mal-posé dans tous les espaces de Sobolev H^m. Le caractère Hölder bien-posé dans H^m est défini ici au sens suivant: il existe $\alpha \in (0,1]$ et $r_0 > 0$ tels que toute condition initiale h_1 et h_2 dans $H^m(B_{r_0}(x_0))$ engendre une solution u_1 et u_2, respectivement, de (1.1.1), et qui vérifient

$$|u_2 - u_1|_{L^2(\Omega)} \lesssim |h_2 - h_1|^{\alpha}_{H^m(B_{r_0}(x_0))}$$

pour un certain domaine Ω de $\mathbb{R}_t\times\mathbb{R}^d_x$. Le caractère mal-posé démontré dans [Mét05] repose en partie sur un théorème de Cauchy-Kovalevskaya en temps long, idée que nous

avons décrite dans la Section 1.1.5 précédente, et que nous reprenons et développons dans deux directions:

- nous passons des espaces de Sobolev aux espaces de Gevrey, ce qui entraîne un changement d'échelle important du "temps long".
- Nous adaptons la preuve à certains cas de transitions de l'hyperbolique vers l'elliptique.

Le cas des transitions de l'hyperbolique vers l'elliptique a d'abord été considéré dans l'article [LMX10], dans le cadre des équations scalaires quasi-linéaires. Un exemple typique est $\partial_t u + it\partial_x u = u^2$, avec $u \in \mathbb{C}$. Il y est prouvé un résultat d'instabilité des solutions analytiques de telles équations vis-à-vis de perturbations C^∞. L'article [LNT17] élargit le résultat de [LMX10] au cadre des systèmes quasi-linéaires, instables au sens d'Hadamard dans les espaces de Sobolev.

Dans les deux cas, il est à noter que les preuves de [LMX10] et [LNT17] reposent en partie sur une représentation des solutions, que ce soit via la méthode des caractéristiques pour les équations scalaires dans [LMX10], ou via une méthode d'approximation des flots pseudo-différentiels introduite par Texier dans [Tex16].

Avant de développer, dans les Section 1.2.3 à 1.2.5, les idées et méthodes utilisées dans les Chapitres 2 et 3, nous allons décrire dans la Section 1.2.2 les différentes situations considérées, en étudiant les formes normales des cas initialement elliptique et des transitions de l'hyperbolique vers l'elliptique.

1.2.2 Formes normales

Le cas initialement elliptique

On considère ici le cas initialement elliptique décrit plus haut, et on note

$$A_0 = A(0, x_0, u_0, \xi_0).$$

À un changement de base près, on peut écrire A_0 par blocs

$$A_0 = \begin{pmatrix} A_0^{(0)} & 0 \\ 0 & A_0^{(1)} \end{pmatrix}$$

où $A_0^{(1)}$ est une matrice $(N-2) \times (N-2)$, et

$$A_0^{(0)} = \mu_0 \mathrm{Id} + \begin{pmatrix} 0 & \gamma_0 \\ -\gamma_0 & 0 \end{pmatrix} \tag{1.2.1}$$

avec $\gamma_0 > 0$ et μ_0 est la partie réelle du couple de valeurs propres conjuguées. C'est la *forme normale* du cas initialement elliptique, et on peut supposer que γ_0 est la plus grande des parties imaginaires des valeurs propres de A_0. Cette description correspond à la Section 2.2.2 dans le Chapitre 2.

Dans le prolongement de la discussion de la Section (1.1.5), on considère l'équation (1.1.2) avec $\xi = \xi_0/\varepsilon$ pour $\varepsilon > 0$ un petit paramètre, c'est-à-dire le problème de Cauchy

$$\partial_t v(t\,;\varepsilon) = \varepsilon^{-1} i A_0^{(0)} v(t\,;\varepsilon) \quad \text{avec} \quad v(0\,;\varepsilon) = v_0(\varepsilon) \tag{1.2.2}$$

pour une famille de conditions initiales $v_0(\varepsilon)$ définies par

$$v_0(\varepsilon) = e^{-M(\varepsilon)} v_+ \quad \text{avec} \quad v_+ = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

On rappelle que le poids $M(\varepsilon)$ mesure la petitesse des conditions initiales dans la régularité étudiée: en régularité Sobolev H^m, on a $M(\varepsilon) = m'|\ln(\varepsilon)|$ avec $m' > m$; en régularité Gevrey $\mathcal{G}_c^\sigma$, on a $M(\varepsilon) = c'\varepsilon^{-\sigma}$ avec $c' > c$.

Les solutions issues de ces conditions initiales sont alors $v(t\,;\varepsilon) = e^{-M(\varepsilon)} e^{it\mu_0/\varepsilon} e^{t\gamma_0/\varepsilon} v_+$, et vérifient la même croissance exponentielle (1.1.12), comme décrit dans la Section 1.1.5. On remarque les fortes oscillations $e^{it\mu_0/\varepsilon}$ introduites par la partie réelle μ_0 des valeurs propres de $A_0^{(0)}$. Ces oscillations des coefficients de Fourier ne posent pas de problème dans l'analyse, car de module 1. On peut donc poursuivre l'analyse de la Section 1.1.5, et le temps d'observation de l'instabilité est donc $t_\varepsilon = \gamma_0^{-1}\varepsilon M(\varepsilon)$. On peut donc espérer, heuristiquement, prouver l'instabilité en utilisant soit (1.1.13), soit (1.1.14).

Transitions de l'hyperbolique vers l'elliptique

Au contraire du cas initialement elliptique, le cas d'une transition recouvre plus de possibilités, et il n'y a pas qu'une forme normale. On considère dans cette thèse essentiellement deux cas, en suivant en grande partie le travail de [LNT17] et la Section 3.2 du Chapitre 3. Pour simplifier la discussion, nous considérons ici $N = 2$.

- On considère d'abord le cas d'une transition lissement diagonalisable, avec

$$\underline{A}^{\mathrm{S}}(t) = \mu_0 \mathrm{Id} + \begin{pmatrix} 0 & t \\ -\gamma_0^2 t & 0 \end{pmatrix} \tag{1.2.3}$$

où μ_0 est réel. La matrice $\underline{A}^{\mathrm{S}}(t)$ a pour valeurs propres $\mu_0 \pm i\gamma_0 t$ avec $\gamma_0 > 0$. Par analogie avec le cas initialement elliptique et par homogénéité du symbole principal, on considère le problème de Cauchy

$$\partial_t v(t\,,\varepsilon) = \varepsilon^{-1} i \underline{A}^{\mathrm{S}}(t) v(t\,;\varepsilon) \quad \text{avec} \quad v(0\,;\varepsilon) = v_0(\varepsilon) \tag{1.2.4}$$

avec encore

$$v_0(\varepsilon) = e^{-M(\varepsilon)} v_+ \quad \text{avec} \quad v_+ = \begin{pmatrix} 1 \\ -i\gamma_0 \end{pmatrix}.$$

Alors, les solutions sont $v(t\,;\varepsilon) = e^{-M(\varepsilon)} e^{it\mu_0/\varepsilon} e^{\gamma_0 t^2/(2\varepsilon)} v_+$. La croissance de ces solutions est alors

$$|v(t\,;\varepsilon)| \approx e^{-M(\varepsilon)} e^{\gamma_0 t^2/(2\varepsilon)}, \tag{1.2.5}$$

à comparer avec (1.1.12) dans le cas initialement elliptique. En particulier, le temps d'observation de l'instabilité, c'est-à-dire le temps t_ε pour lequel la norme L^2 de la solution $v(t\,;\varepsilon)$ soit d'ordre 1 pour $\varepsilon \to 0$, est ici de l'ordre de $\varepsilon^{1/2} M(\varepsilon)^{1/2}$, comparé à $\varepsilon M(\varepsilon)$ dans le cas initialement elliptique. Ainsi, l'échelle de temps long pour l'observation de l'instabilité est ici $t_\varepsilon/\varepsilon^{1/2}$, et non plus $t_\varepsilon/\varepsilon$ comme dans le cas initialement elliptique.

• Le cas d'une transition non semi-simple, non lisse avec

$$\underline{A}^{\mathrm{Ai}}(t,x) = \begin{pmatrix} 0 & 1 \\ -(t-t_\star(x)) & 0 \end{pmatrix}. \tag{1.2.6}$$

La fonction $t_\star(x) \geq 0$ représente un temps de transition. Pour $t < t_\star(x)$, les valeurs propres de $\underline{A}^{\mathrm{Ai}}$ sont $\pm\gamma_0\sqrt{t_\star(x)-t}$ et sont réelles. Pour $t > t_\star(x)$, les valeurs propres de $\underline{A}^{\mathrm{Ai}}$ sont $\pm i\gamma_0\sqrt{t-t_\star(x)}$ et sont imaginaires. La transition de l'hyperbolique vers l'elliptique n'est donc pas ici uniforme dans l'espace.

Dans l'optique d'adapter le cadre fonctionnel développé par Métivier dans [Mét05], nous devons traiter le terme $\begin{pmatrix} 0 & 0 \\ t_\star(x) & 0 \end{pmatrix}$ comme un terme de reste. Nous verrons par la suite que cela impose à $t_\star$ d'être relativement plat au voisinage de $x = 0$. Le cas non-dégénéré $t_\star(x) = x^2$ est notamment hors de portée de notre étude.

Comme précédemment, on étudie ici, avec $\varepsilon > 0$ un petit paramètre correspondant aux fortes oscillations, le problème de Cauchy

$$\partial_t v(t\,;\varepsilon) = \varepsilon^{-1} i\underline{A}^{\mathrm{Ai}}(t,0)v(t\,;\varepsilon) \quad \text{avec} \quad v(0\,;\varepsilon) = v_0(\varepsilon) \tag{1.2.7}$$

avec comme conditions initiales $v_0(\varepsilon) = e^{-M(\varepsilon)}v_+$, et le vecteur v_+ sera choisi plus tard. En posant $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, le système (1.2.7) est équivalent à

$$\partial_t^2 v_1 = \varepsilon^{-2} t v_1 \tag{1.2.8}$$

$$\partial_t v_1 = \varepsilon^{-1} i v_2 \tag{1.2.9}$$

où (1.2.8) est une équation différentielle scalaire, non autonome, du second ordre: c'est l'équation de Airy. Une base de solutions de cette équation est $(\mathrm{Ai}(\varepsilon^{-2/3}t), \mathrm{Ai}(\varepsilon^{-2/3}jt))$, où $j = e^{2i\pi/3}$ et Ai est la fonction de Airy (voir Lemma 3.6.1, Chapitre 3). La fonction $s \mapsto \mathrm{Ai}(js)$ vérifie en particulier la borne

$$|\mathrm{Ai}(js)| \approx e^{\frac{2}{3}s^{3/2}}$$

et c'est cette croissance exponentielle que nous utiliserons pour prouver l'instabilité. On note que cette croissance, indépendante de ε, est vérifiée dans l'échelle de temps $s = \varepsilon^{-2/3}t$. On obtient donc, similairement à (1.1.12) et (1.2.5), la croissance des solutions

$$|v(t\,;\varepsilon)| \approx e^{-M(\varepsilon)} e^{\frac{2}{3}t^{3/2}/\varepsilon}. \tag{1.2.10}$$

Le temps d'observation de l'instabilité sera donc, ici, $t_\varepsilon = \varepsilon^{2/3}M(\varepsilon)^{2/3}$.

Remarque 1.2.1. *Dans chacun des trois cas étudiés, nous avons mis en avant le temps d'observation de l'instabilité. Plus précisément, nous remarquons que ce temps d'instabilité n'appartient pas aux mêmes échelles de temps, suivant les cas. En effet, dans le cas initialement elliptique, le temps long d'instabilité est donné par $t_\varepsilon/\varepsilon = M(\varepsilon)$; dans le cas d'une transition lissement diagonalisable, par $t_\varepsilon/\varepsilon^{1/2} = M(\varepsilon)^{1/2}$; et finalement, dans le cas d'une transition non-semi simple et non lisse, par $t_\varepsilon/\varepsilon^{2/3} = M(\varepsilon)^{2/3}$.*

Remarque 1.2.2. *Au contraire d'une transition non semi-simple et non lisse, la forme normale d'une transition lissement diagonalisable vérifie* $t_\star \equiv 0$. *Cela s'explique par le fait que le cas lissement diagonalisable est stable par perturbation, au contraire de la matrice de Airy* $\begin{pmatrix} 0 & 1 \\ -t & 0 \end{pmatrix}$.

1.2.3 Solutions à oscillations rapides

Nous reprenons dans cette thèse la méthode développée par Métivier dans [Mét05] dans le cadre initialement elliptique en régularité Sobolev, et l'étendons au cas initialement elliptique en régularité Gevrey d'une part, et au cas des transitions en régularité Gevrey, d'autre part. Afin de ne pas alourdir les notations et cette introduction, on se concentre sur le cas où $u_0 = 0$, et où la fonction nulle est solution de (1.1.1) ; cela revient à écrire $f(u) = F(u)u$ pour un certain F. L'idée de la preuve de Métivier dans [Mét05] est d'étendre l'analyse décrite pour le cas initialement elliptique dans la Section 1.2.2 au cas général du système (1.1.1), en construisant une famille de solutions $(u_\varepsilon)_{\varepsilon>0}$ de (1.1.1) qui vérifient la croissance (1.1.12) dans le cas initialement elliptique traité dans [Mét05], puis la limite (1.1.13) qui prouve l'instabilité au sens d'Hadamard. Notons que la construction de telles solutions, comme dans la Section 1.1.5, repose sur un *théorème de Cauchy-Kovalevskaya en temps long* - c'est une des clés de l'article [Mét05].

Dans les Chapitres 2 et 3, nous étendons cette idée à la fois en régularité Gevrey et pour le cas de transitions. Il s'agit donc de mettre en avant les oscillations hautes fréquences, oscillations dont nous avons montré l'importance aussi dans le cas des transitions de l'hyperbolique vers l'elliptique dans la Section ci-avant. Ainsi, pour $\varepsilon > 0$ un petit paramètre, on considère l'ansatz suivant de solutions à oscillations hautes fréquences, en cherchant une famille de solutions $u(t, x\,;\varepsilon)$ du problème de Cauchy (1.1.1) sous la forme

$$u(t, x\,;\varepsilon) = \varepsilon^2 \mathbf{u}\left(\varepsilon^{-1/(1+\eta)} t, x, x \cdot \xi_0/\varepsilon\,;\varepsilon, \xi\right) \tag{1.2.11}$$

où la fonction $\mathbf{u}(s, x, \theta\,;\varepsilon)$ est une fonction 2π-périodique dans la variable θ. Le changement d'échelle "temps court/temps long" $s = \varepsilon^{-1/(1+\eta)} t$ est en lien direct avec la Remarque 1.2.1. Suivant les cas étudiés, l'instabilité n'est pas observée dans les mêmes échelles de temps, d'où l'introduction d'un paramètre $\eta \geq 0$. Précisément,

- $\eta = 0$ dans le cas initialement elliptique.
- $\eta = 1$ dans le cas d'une transition lissement diagonalisable.
- $\eta = 1/2$ dans le cas d'une transition non-semi simple, non lisse.

En injectant l'ansatz (1.2.11) dans le système (1.1.1), on obtient un nouveau système en $\mathbf{u}$:

$$\partial_s \mathbf{u} = \varepsilon^{-\eta/(1+\eta)} A(\xi_0) \partial_\theta \mathbf{u} + \varepsilon^{1/(1+\eta)} \left(\sum_j A_j \partial_{x_j} \mathbf{u} + F(\varepsilon^2 \mathbf{u}) \mathbf{u} \right) \tag{1.2.12}$$

où il est implicite que $A(\xi_0)$, A_j et F dépendent des variables $(\varepsilon^{1/(1+\eta)}s, x, \varepsilon^2\mathbf{u})$. L'ansatz (1.2.11) des solutions à oscillations rapides a donc permis de mettre en avant le symbole principal $A(\xi_0)$, et le facteur $\varepsilon^{1/(1+\eta)}$ devant les autres termes permettra de les considérer comme des restes.

Dans chacun des trois cas étudiés, le symbole principal $A(\xi_0)$, via un théorème de forme normale, peut s'écrire

$$A(t,x,u,\xi_0) = \underline{A}(t) + (A(t,x,u,\xi_0) - \underline{A}(t))$$

avec $\underline{A}(t)$ ne dépendant que du temps t, et égal à (1.2.1) dans le cas initialement elliptique, à (1.2.3) dans le cas d'une transition lissement diagonalisable, et à (1.2.6) dans le cas d'une transition non-semi simple et non-lisse. On peut donc alors réécrire (1.2.12) en

$$\partial_s \mathbf{u} - \varepsilon^{-\eta/(1+\eta)}\underline{A}(\varepsilon^{1/(1+\eta)}s)\partial_\theta \mathbf{u} = G \tag{1.2.13}$$

avec pour terme source

$$G = \varepsilon^{-\eta/(1+\eta)}\left(A(\xi_0) - \underline{A}(\varepsilon^{1/(1+\eta)}s)\right)\partial_\theta \mathbf{u} + \varepsilon^{1/(1+\eta)}\left(\sum_j A_j \partial_{x_j}\mathbf{u} + F(\varepsilon^2\mathbf{u})\mathbf{u}\right). \tag{1.2.14}$$

Afin de construire la famille de solutions de (1.1.1) qui prouveraient l'instabilité au sens d'Hadamard, il faut à présent procéder en deux temps. D'abord, expliciter une estimation a priori de croissance des solutions, du type (1.1.12) dans le cas initialement elliptique, (1.2.5) ou (1.2.10) dans les cas de transition: c'est l'objet de la Section 1.2.4. Puis montrer un théorème de Cauchy-Kovalevskaya en temps long d'existence de telles solutions: c'est l'objet de la Section 1.2.5.

1.2.4 Croissance des modes de Fourier et équation de point fixe

Comme $\mathbf{u}(\theta)$ est 2π-périodique, on peut décomposer $\mathbf{u}(\theta)$ selon ses modes de Fourier

$$\mathbf{u}(s,x,\theta\,;\varepsilon) = \sum_{n\in\mathbb{Z}} \mathbf{u}_n(s,x\,;\varepsilon)e^{in\theta}.$$

On considère ici le cas où le terme source G est identiquement nul, pour se concentrer sur l'équation

$$\partial_s \mathbf{u} - \varepsilon^{-\eta/(1+\eta)}\underline{A}(\varepsilon^{1/(1+\eta)}s)\partial_\theta \mathbf{u} = 0.$$

Cette équation devient, pour chaque composante de Fourier de $\mathbf{u}$, l'équation différentielle linéaire

$$\partial_s \mathbf{u}_n - in\varepsilon^{-\eta/(1+\eta)}\underline{A}(\varepsilon^{1/(1+\eta)}s)\mathbf{u}_n = 0$$

qui est, aux facteurs $n \in \mathbb{Z}$ et $\varepsilon^{-\eta/(1+\eta)}$ près, exactement une des équations (1.2.2), (1.2.4) ou (1.2.7), selon le cas étudié. On introduit alors le propagateur $U_n(s',s)$, pour chaque mode de Fourier $n \in \mathbb{Z}$, défini comme étant la solution du problème de Cauchy

$$\partial_s U_n(s',s) - in\varepsilon^{-\eta/(1+\eta)}\underline{A}(\varepsilon^{1/(1+\eta)}s)U_n(s',s) = 0 \quad \text{avec} \quad U_n(s',s') = \mathrm{Id}.$$

En se basant sur les résultats de la Section 1.2.2, on peut alors montrer que le propagateur vérifie la croissance

$$\left|U_n(s',s)\right| \lesssim C(\varepsilon)\,\exp\left(|n|\gamma_0\frac{1}{1+\eta}\left(s^{1+\eta}-s'^{1+\eta}\right)\right) \tag{1.2.15}$$

pour un certain $\gamma_0 > 0$, avec $C(\varepsilon) = 1$ dans les cas initialement elliptique et d'une transition lissement diagonalisable. En revanche, dans le cas d'une transition non-semi simple et non-lisse, la matrice

$$\varepsilon^{-1/3}\underline{A}^{\mathrm{Ai}}(\varepsilon^{2/3}s) = \begin{pmatrix} 0 & \varepsilon^{-1/3} \\ -\varepsilon^{1/3}s & 0 \end{pmatrix}$$

n'est pas homogène en temps. Le coefficient $\varepsilon^{-1/3}$ qui apparait donc en haut à droite de la matrice induit un coefficient $C(\varepsilon) = \varepsilon^{-1/3}$ dans l'estimation (1.2.15).

Nous voulons ici appuyer sur la souplesse et la généralité de la construction d'un cadre général à l'étude des différents cas (initialement elliptique, ou de transitions) par l'introduction d'une matrice générale $\underline{A}(t)$ (égale à (1.2.1), à (1.2.3) ou à (1.2.6) suivant les cas), et d'un paramètre η de changement d'échelle en temps, qui encode les différents temps d'observation de l'instabilité. En un sens, l'estimation (1.2.15) du propagateur encode en une seule inégalité la croissance exponentielle, en temps et en fréquence, attendue dans les cas non-hyperboliques - chaque cas étant différencié par le paramètre η de changement d'échelle temporelle.

Reprenons à présent le cas général où G est non identiquement nul. En utilisant la définition du propagateur $U_n(s',s)$, on peut transformer l'équation (1.2.13) en l'équation de point fixe

$$\mathbf{u}_n(s,x\,;\varepsilon) = U_n(0,s)\mathbf{u}_n(0,x\,;\varepsilon) + \int_0^s U_n(s',s)G_n(s',x,\mathbf{u}(s')\,;\varepsilon)ds' \tag{1.2.16}$$

décrite sur chaque mode de Fourier de $\mathbf{u}$. Nous allons à présent construire un espace de Banach adapté à la résolution en temps long de l'équation (1.2.16), dans lequel l'opérateur $\mathbf{u} \mapsto \int_0^s U_n(s',s)G_n(s',x,\mathbf{u}(s')\,;\varepsilon)ds'$ soit une contraction, et dans lequel les solutions conservent une croissance similaire à (1.2.15).

1.2.5 Méthode des séries majorantes, théorème de Cauchy-Kovalevskaya

Séries majorantes

Un outil puissant dans la construction de solutions analytiques pour un problème donné est *la méthode des séries majorantes.* Elle fut par exemple utilisée par Cauchy pour prouver que toute équation différentielle dont les coefficients sont analytiques, admet des solutions analytiques. Pour deux séries formelles d'une variable $\phi = \sum \phi_k x^k$ et $\psi = \sum \psi_k X^k$, avec de plus $\psi_k \geq 0$ pour tout $k \in \mathbb{N}$, on définit la relation

$$\phi \prec \psi \quad \Longleftrightarrow \quad \forall\, k \in \mathbb{N}\,, |\phi_k| \leq \psi_k \tag{1.2.17}$$

et on dit que ψ est une série majorante de ϕ. Cette relation entre séries formelles possède de nombreuses propriétés, parmi lesquelles

- la stabilité par rapport à la dérivation: si $\phi \prec \psi$, alors $\phi' \prec \psi'$.
- La stabilité par rapport au produit: si $\phi_j \prec \psi_j$, alors $\phi_1\phi_2 \prec \psi_1\psi_2$.
- Et surtout, si $\psi(X)$ converge pour $X > 0$, alors pour tout $|x| \leq X$, la série $\phi(x)$ converge.

Cette dernière propriété admet une réciproque. Il existe en effet des séries Φ, à coefficients positifs, telles que, pour toute série ϕ de rayon de convergence strictement inférieur à $R^{-1} > 0$, on ait $\phi \prec \Phi(RX)$. Un exemple de telles séries est $\frac{1}{1-X}$, en utilisant les formules de Cauchy pour les séries entières. Dans le cadre qui nous intéresse, nous avons en particulier besoin de contrôler des termes non-linéaires. On considère alors

$$\Phi(X) = \sum_k \Phi_k X^k \quad \text{avec} \quad \Phi_k = \frac{c_0}{k^2+1}$$

et le coefficient $c_0 > 0$ est choisi tel que $\Phi^2 \prec \Phi$. Notons aussi que Φ a pour rayon de convergence 1.

Comme nous voulons construire des solutions d'une équation d'évolution qui soient analytiques dans la variable x, et comme nous l'avons vu précédemment dans la Section 1.1.4, nous devons considérer un rayon de convergence décroissant linéairement en temps. On définit alors comme série majorante la série $X \mapsto \Phi(RX + \rho t)$, avec $R > 0$ et $\rho > 0$ deux paramètres, qui converge dans le domaine

$$\Omega_{R,\rho}(t) = \{x \,:\, R|x| + \rho t < 1\}\,.$$

Ce domaine d'analyticité, pour la variable d'espace, dépend du temps et est non vide dès que $t < \rho^{-1}$: c'est le *temps maximal d'analyticité*. Ce temps d'analyticité est en compétition avec le temps d'observation de l'instabilité t_ε évoqué dans la Section 1.2.2: on ne peut observer l'instabilité que si les solutions sont analytiques pour un temps assez grand. Notons aussi que R^{-1} est le rayon de convergence de la série majorante à $t = 0$.

Espaces de fonctions et temps d'instabilité

Nous sommes à présent en mesure de définir l'espace de Banach dans lequel nous allons travailler. On introduit un petit paramètres $\beta(\varepsilon) > 0$. Pour $s \geq 0$, on note $\mathbb{E}(s) = \mathbb{E}(s, \beta)$ l'espace des fonctions lisses $f(x, \theta) = \sum_n f_n(x) e^{in\theta}$ telles qu'il existe une constante $C(s)$ positive vérifiant

$$f_n(x) \prec C(s) \frac{1}{n^2+1} \exp\left(-\left(M - \gamma_0 \frac{s^{1+\eta}}{1+\eta} - \beta s\right)|n|\right) \Phi(Rx + \varepsilon^{1/(1+\eta)} \rho s). \tag{1.2.18}$$

On a pris soin ici de garder une marge de manoeuvre par rapport à la croissance du propagateur (1.2.15), et on introduit donc un nouveau temps d'observation de l'instabilité $\underline{s}_{insta}$, défini implicitement par

$$M - \gamma_0 \underline{s}_{insta}^{1+\eta}/(1+\eta) - \beta \underline{s}_{insta} = 0. \tag{1.2.19}$$

À la limite $\varepsilon \to 0$, on a encore $\underline{s}_{insta} \approx e^{-\sigma}$, en vérifiant que $\lim_{\varepsilon\to 0} \beta(\varepsilon) = 0$ dans la construction. On définit aussi le temps maximal d'analyticité $\underline{s}_{reg}$ par

$$\underline{s}_{reg} = \left(\varepsilon^{1/(1+\eta)}\rho\right)^{-1} \tag{1.2.20}$$

et le temps final $\underline{s}$ est défini par

$$\underline{s} = \min\left\{\underline{s}_{insta}, \underline{s}_{reg}\right\} \tag{1.2.21}$$

qui doit être au moins égal à $\underline{s}_{insta}$ pour pouvoir observer l'instabilité. Enfin, on définit l'espace $\mathbb{E} = \mathbb{E}(\beta)$ des fonctions lisses $f(s, x, \theta)$ telles que

$$f(s) \in \mathbb{E}(s) \quad , \quad \forall\, s \in [0, \underline{s}). \tag{1.2.22}$$

Cet espace de fonctions $\mathbb{E}$ possède de nombreuses propriétés. Notamment, toute fonction $f \in \mathbb{E}$ est analytique en θ, et $f(s)$ est analytique en x dans le domaine de convergence

$$\Omega_{R,\varepsilon^{1/(1+\eta)}\rho}(s) = \left\{x \,:\, R|x| + \varepsilon^{1/(1+\eta)}\rho s < 1\right\}$$

pour tout $s \in [0, \underline{s})$. Aussi, on peut munir $\mathbb{E}$ de la norme

$$|||f||| = \max\left\{C(s) > 0 \,:\, f(s) \text{ vérifie } (1.2.18)\right\}.$$

Une propriété importante, qui découle en parti du choix de Φ tel que $\Phi^2 \prec \Phi$, est que $|||\cdot|||$ est une norme d'algèbre, et que $(\mathbb{E}, |||\cdot|||)$ est une algèbre de Banach.

Régularisation par intégration en temps

Maintenant que nous avons défini le bon espace fonctionnel dans lequel travailler, il reste à montrer que l'opérateur $\mathbf{u} \mapsto \int_0^s U_n(s', s) G_n(s', x, \mathbf{u}(s')\,; \varepsilon) ds'$ est bien une contraction dans $\mathbb{E}$, avec G défini par (1.2.14). Le terme source G contient en particulier les opérateurs de dérivation ∂_θ et ∂_x. Même si ces opérateurs ne sont pas bornés sur $\mathbb{E}$, l'intégration en temps a un effet régularisant. Nous allons montrer ceci sur deux exemples simples mais reflétant au mieux l'essentiel du propos.

On définit ainsi

$$g(s, \theta) = \sum_{n\in\mathbb{Z}} \frac{1}{n^2+1} \exp\left(-\left(M - \gamma_0 \frac{s^{1+\eta}}{1+\eta} - \beta s\right)|n|\right) e^{in\theta} \tag{1.2.23}$$

est dans $\mathbb{E}$, mais pas $\partial_\theta g$ car $(\partial_\theta g)_n = ing_n$ et

$$|ing_n| \left(\frac{1}{n^2+1} \exp\left(-\left(M - \gamma_0 \frac{s^{1+\eta}}{1+\eta} - \beta s\right)|n|\right)\right)^{-1} = |n|$$

n'est pas borné pour $n \in \mathbb{Z}$, contredisant donc (1.2.18). De même, on considère

$$h(s, x) = \Phi(Rx + \varepsilon^{1/(1+\eta)}\rho s) \tag{1.2.24}$$

qui est bien dans $\mathbb{E}$, mais

$$\partial_x h = R\Phi'(Rx + \varepsilon^{1/(1+\eta)}\rho s)$$

n'est pas contrôlé par $\Phi(Rx + \varepsilon^{1/(1+\eta)}\rho s)$. En effet, les coefficients de Φ' sont égaux à $\frac{kc_0}{k^2+1}$, comparés aux coefficients $\frac{c_0}{k^2+1}$ de Φ.

En revanche, on observe dans les deux cas *une régularisation par intégration en temps.* Ainsi, dans le cas de g définie en (1.2.23), si on considère non pas les modes de Fourier de $\partial_\theta g$ mais

$$T(g)_n(s) = \int_0^s U_n(s', s)\left(\partial_\theta g\right)_n(s')ds',$$

la fonction $T(g) = \sum_{n\in\mathbb{Z}} = T(g)_n e^{in\theta}$ est bien dans $\mathbb{E}$. En effet, on vérifie que

$$\begin{aligned}
&|T(g)_n(s)| \\
&\leq \int_0^s \left|U_n(s', s)\right| \left|\partial_\theta g(s')\right|_n ds' \\
&\lesssim \int_0^s C(\varepsilon)\exp\left(|n|\gamma_0 \frac{1}{1+\eta}\left(s^{1+\eta} - s'^{1+\eta}\right)\right)\frac{|n|}{n^2+1}\exp\left(-\left(M - \gamma_0\frac{s'^{1+\eta}}{1+\eta} - \beta s'\right)|n|\right)ds'
\end{aligned}$$

grâce à l'estimation du propagateur (1.2.15). Puis on obtient

$$\begin{aligned}
&\int_0^s C(\varepsilon)\exp\left(|n|\gamma_0 \frac{1}{1+\eta}\left(s^{1+\eta} - s'^{1+\eta}\right)\right)\frac{|n|}{n^2+1}\exp\left(-\left(M - \gamma_0\frac{s'^{1+\eta}}{1+\eta} - \beta s'\right)|n|\right)ds' \\
&\leq C(\varepsilon)\frac{1}{n^2+1}\exp\left(-\left(M - \gamma_0\frac{s^{1+\eta}}{1+\eta} - \beta s\right)|n|\right)\int_0^s |n|e^{-\beta(s-s')|n|}ds' \\
&\leq C(\varepsilon)\beta^{-1}\frac{1}{n^2+1}\exp\left(-\left(M - \gamma_0\frac{s^{1+\eta}}{1+\eta} - \beta s\right)|n|\right)
\end{aligned}$$

par calcul explicite de l'intégrale. On vient donc de prouver que

$$|||T(g)||| \leq C(\varepsilon)\beta^{-1}.$$

Cette régularisation par intégration en temps a été utilisée par exemple dans [Uka01].

De la même façon, on peut régulariser $\partial_x h$. On définit

$$T(h)(s) = \int_0^s \partial_x h(s')ds'.$$

Alors, en intégrant la série coefficient par coefficient et par définition (1.2.24) de h, on a

$$\begin{aligned}
T(h)(s) &= \int_0^s R\Phi'(Rx + \varepsilon^{1/(1+\eta)}\rho s')ds' \\
&= \int_0^s R\left(\varepsilon^{1/(1+\eta)}\rho\right)^{-1}\partial_s\left(\Phi(Rx + \varepsilon^{1/(1+\eta)}\rho s')\right)ds' \\
&\prec R\left(\varepsilon^{1/(1+\eta)}\rho\right)^{-1}\Phi(Rx + \varepsilon^{1/(1+\eta)}\rho s)
\end{aligned}$$

et donc

$$|||T(h)||| \leq R\left(\varepsilon^{1/(1+\eta)}\rho\right)^{-1}.$$

Sur deux exemples simples, nous avons présenté le rôle important de l'intégration en temps dans le contrôle du terme source dans l'espace $\mathbb{E}$. Nous pouvons en fait généraliser ces calculs pour montrer que l'opérateur $\mathbf{u} \mapsto \int_0^s U_n(s', s) G_n(s', x, \mathbf{u}(s')\,; \varepsilon) ds'$ agit bien dans $\mathbb{E}$. Il nous reste alors à expliquer comment on peut estimer la norme de l'opérateur, et vérifier que celle-ci est petite, afin de pouvoir appliquer le théorème du point fixe de Banach dans $\mathbb{E}$.

1.2.6 Point fixe et borne sur les indices Gevrey

Dans le terme source G, défini par (1.2.14), l'opérateur ∂_θ a en facteur la fonction $\varepsilon^{-\eta/(1+\eta)}\,(A(\xi_0) - \underline{A})$, où $A(\xi_0)$ est le symbole principal, et $\underline{A}$ est égal à (1.2.1) dans le cas initialement elliptique, à (1.2.3) dans le cas d'une transition lissement diagonalisable, et à (1.2.6) dans le cas d'une transition non-semi simple et non-lisse. Selon chaque situation, on obtient un développement de $A(\xi_0) - \underline{A}$, par analyticité du symbole $A(\xi_0)$ dans les variables $(t, x, u) = (\varepsilon^{1/(1+\eta)} s, x, \varepsilon^2 \mathbf{u})$.

- Dans le cas initialement elliptique, $\eta = 0$ et on a

$$\varepsilon^{-\eta/(1+\eta)}\,(A(\xi_0) - \underline{A}) \approx \varepsilon s + x + \varepsilon^2 \mathbf{u}.$$

On peut montrer, ce que nous ne faisons pas ici pour simplifier le propos, que la fonction $\varepsilon^{-\eta/(1+\eta)}\,(A(\xi_0) - \underline{A})$ est bien dans $\mathbb{E}$, et que sa norme vérifie

$$|||\varepsilon^{-\eta/(1+\eta)}\,(A(\xi_0) - \underline{A})\,||| \lesssim \varepsilon \underline{s} + R^{-1} + \varepsilon^2 |||\mathbf{u}|||. \tag{1.2.25}$$

En effet, la norme $|||\cdot|||$ est une borne supérieure en temps, pour $s \in [0, \underline{s})$. De plus, on a $x \prec R^{-1}\Phi(Rx + \varepsilon^{1/(1+\eta)}\rho s)$.

- Dans le cas d'une transition lissement diagonalisable, $\eta = 1$ et on a

$$\varepsilon^{-\eta/(1+\eta)}\,(A(\xi_0) - \underline{A}) \approx s\left(\varepsilon^{1/2} s + x\right) + \varepsilon^{3/2}\mathbf{u}.$$

De même que dans le cas précédent, la fonction $\varepsilon^{-\eta/(1+\eta)}\,(A(\xi_0) - \underline{A})$ est dans $\mathbb{E}$ et sa norme vérifie

$$|||\varepsilon^{-\eta/(1+\eta)}\,(A(\xi_0) - \underline{A})\,||| \lesssim \underline{s}\left(\varepsilon^{1/2}\underline{s} + R^{-1}\right) + \varepsilon^{3/2}|||\mathbf{u}|||. \tag{1.2.26}$$

- Dans le cas d'une transition de type Airy, $\eta = 1/2$ et on a

$$\varepsilon^{-\eta/(1+\eta)}\,(A(\xi_0) - \underline{A}) \approx \varepsilon^{1/3} s\left(\varepsilon^{2/3} s + x\right) + \varepsilon^{-1/3} t_\star(x) + \varepsilon^{5/3}\mathbf{u}.$$

En supposant, sans perte de généralité, que le temps de transition vérifie $t_\star(s) = x^{2k}$ (on se souvient que $t_\star$ est positif, localement autour de $x = 0$), on obtient

$$|||\varepsilon^{-\eta/(1+\eta)}\,(A(\xi_0) - \underline{A})\,||| \lesssim \varepsilon^{1/3}\underline{s}\left(\varepsilon^{2/3}\underline{s} + R^{-1}\right) + \varepsilon^{-1/3}R^{-2k} + \varepsilon^{5/3}|||\mathbf{u}|||. \tag{1.2.27}$$

On obtient donc, dans chacun des cas, une estimation différente de la norme de $\varepsilon^{-\eta/(1+\eta)}\,(A(\xi_0) - \underline{A})$, ce qui donne une estimation différente de la norme de l'opérateur

$\mathbf{u} \mapsto \int_0^s U_n(s',s)G_n(s',x,\mathbf{u}(s')\,;\varepsilon)ds'$ dans $\mathbb{E}$. On rappelle que G est défini par (1.2.14). On peut en effet montrer, en suivant l'idée de régularisation par intégration en temps présentée à la Section ci-avant, que

$$\begin{aligned}&|||\int_0^s U_n(s',s)G_n(s',x,\mathbf{u}(s')\,;\varepsilon)ds'|||\\ &\lesssim C(\varepsilon)\left(\beta^{-1}\,|||\varepsilon^{-\eta/(1+\eta)}\left(A(\xi_0)-\underline{A}\right)|||+R\rho^{-1}+\varepsilon^{1/(1+\eta)}|||F(\varepsilon^2\mathbf{u})|||\right)\,|||\mathbf{u}|||\end{aligned}$$

et montrer que l'opérateur est une contraction dans $\mathbb{E}$ revient à montrer que

$$K = C(\varepsilon)\beta^{-1}\,|||\varepsilon^{-\eta/(1+\eta)}\left(A(\xi_0)-\underline{A}\right)|||+R\rho^{-1}+\varepsilon^{1/(1+\eta)}|||F(\varepsilon^2\mathbf{u})||| \tag{1.2.28}$$

est petit quand $\varepsilon \to 0$, dans chacun des trois cas étudiés, sous la contrainte de temps $\underline{s} = \underline{s}_{insta} < \underline{s}_{reg}$, sachant que $\underline{s}_{insta} \approx \varepsilon^{-\sigma}$.

Cependant, notre objectif n'est pas seulement de montrer l'existence de solutions via le théorème de point fixe de Banach. Il s'agit aussi de montrer que ces solutions vérifient elles aussi la croissance exponentielle en temps et fréquence décrite en (1.1.12) dans le cas initialement elliptique, (1.2.5) ou (1.2.10). Or, dans la définition (1.2.18) des espaces $\mathbb{E}(s)$, nous avons introduit une correction $e^{\beta s|n|}$, correction qui nous a permis de régulariser par intégration en temps l'opérateur non-borné ∂_θ. Cette correction introduit donc une erreur d'ordre $e^{\beta\underline{s}}$ pour la croissance des modes $\mathbf{u}_{\pm 1}(s)$ par rapport aux croissances (1.1.12), (1.2.5) ou (1.2.10) selon les cas. Comme $\underline{s} = \underline{s}_{insta} \approx \varepsilon^{-\sigma}$, on pose alors

$$\beta = \varepsilon^{\sigma}.$$

Nous pouvons alors conclure, au moins de manière heuristique, sur la construction de solutions analytiques menant à l'instabilité du système (1.1.1), dans chacun des trois cas décrits dans la Section 1.2.2:

• Dans le cas initialement elliptique, comme $C(\varepsilon) = 1$ et d'après l'estimation (1.2.25) et par définition (1.2.28) de K, on obtient

$$K \lesssim \varepsilon^{-\sigma}\,\left(\varepsilon\underline{s}+R^{-1}+\varepsilon^2|||\mathbf{u}|||\right)+R\rho^{-1}+\varepsilon|||F(\varepsilon^2\mathbf{u})|||.$$

Comme $\underline{s} \approx \varepsilon^{-\sigma}$, pour que K soit petit à la limite $\varepsilon \to 0$, il faut

$$\varepsilon^{1-2\sigma} < 1 \qquad \varepsilon^{-\sigma}R^{-1} < 1 \quad , \quad R\rho^{-1} < 1.$$

La première contrainte ci-dessus donne immédiatement la borne limite des indices Gevrey pour lesquels on peut montrer ici l'instabilité, à savoir $\sigma < 1/2$. Les deux autres contraintes doivent être combinées à la contrainte du temps maximal de régularité (1.2.20), qui doit être au moins égal à $\varepsilon^{-\sigma}$, d'où la triple contrainte

$$\varepsilon^{-\sigma}R^{-1} < 1 \quad , \quad R\rho^{-1} < 1 \quad , \quad \varepsilon^{-\sigma} < (\varepsilon\rho)^{-1}.$$

On en déduit ainsi $\varepsilon^{1-\sigma} < \rho^{-1} < R^{-1} < \varepsilon^{\sigma}$, contraintes solubles dès que $\sigma < 1/2$. On note donc que le rayon de convergence initial des solutions est soumis à une double contrainte:

ni trop grand pour pouvoir travailler dans un petit domaine, ni trop petit pour que le domaine de convergence contienne le temps d'observation de l'instabilité.

• Dans le cas d'une transition lissement diagonalisable, comme $C(\varepsilon) = 1$ et d'après l'estimation (1.2.26) et par définition (1.2.28) de K, on obtient

$$K \lesssim \varepsilon^{-\sigma} \left(\underline{s} \left(\varepsilon^{1/2} \underline{s} + R^{-1} \right) + \varepsilon^{3/2} |||\mathbf{u}||| \right) + R\rho^{-1} + \varepsilon^{1/2} |||F(\varepsilon^2 \mathbf{u})|||.$$

En suivant le même raisonnement que dans le cas initialement elliptique dont nous venons de parler, nous obtenons une première contrainte $\varepsilon^{1/2}\varepsilon^{-3\sigma} < 1$, qui donne comme limitation sur l'indice Gevrey $\sigma < 1/6$. De même que précédemment, nous obtenons ensuite comme contraintes sur R et ρ:

$$\varepsilon^{-2\sigma} R^{-1} < 1 \quad , \quad R\rho^{-1} < 1 \quad , \quad \varepsilon^{-\sigma} < \left(\varepsilon^{1/2} \rho \right)^{-1}$$

ce qui donne $\varepsilon^{1/2-\sigma} < \rho^{-1} < R^{-1} < \varepsilon^{2\sigma}$, contrainte soluble dès que $\sigma < 1/6$.

• Dans le cas d'une transition de type Airy, on a $C(\varepsilon) = \varepsilon^{-1/3}$. D'après l'estimation (1.2.26) et par définition (1.2.28) de K, on obtient

$$K \lesssim \varepsilon^{-1/3}\varepsilon^{-\sigma} \left(\varepsilon^{1/3} \underline{s} \left(\varepsilon^{2/3} \underline{s} + R^{-1} \right) + \varepsilon^{-1/3} R^{-2k} + \varepsilon^{5/3} |||\mathbf{u}||| \right) + \varepsilon^{-1/3} R\rho^{-1} + \varepsilon^{1/3} |||F(\varepsilon^2 \mathbf{u})|||.$$

Le facteur $\varepsilon^{-1/3}$ est dû, comme mentionné plus haut, au caractère non homogène de $\underline{A}^{\mathrm{Ai}}$. Comme précédemment, on obtient la contrainte $\varepsilon^{2/3}\varepsilon^{-3\sigma} < 1$, et donc la limitation $\sigma < 2/9$ sur l'indice Gevrey. La spécificité de la transition de type Airy, non uniforme dans l'espace à cause du temps de transition $t_\star(x) = x^{2k}$, intervient ici. En effet, les contraintes sur R et ρ sont maintenant

$$\varepsilon^{-2\sigma} R^{-1} < 1 \quad , \quad \varepsilon^{-2/3-\sigma} R^{-2k} < 1 \quad , \quad \varepsilon^{-1/3} R\rho^{-1} < 1 \quad , \quad \varepsilon^{-\sigma} < \left(\varepsilon^{2/3} \rho \right)^{-1} < 1.$$

La seconde contrainte $\varepsilon^{-2/3-\sigma} R^{-2k} < 1$ n'est pas anodine ici. Ainsi, si on considère seulement les trois dernières contraintes, on aboutit à

$$\varepsilon^{2/3-\sigma} < \rho^{-1} < R^{-1} < \varepsilon^{1/3+(2/3+\sigma)/(2k)}$$

qui est soluble si et seulement si $\varepsilon^{2/3-\sigma} < \varepsilon^{1/3+(2/3+\sigma)/(2k)}$, et donc si σ et k vérifient

$$\frac{1}{3} - \frac{1}{3k} > \sigma \left(1 + \frac{1}{2k} \right).$$

Comme le terme de droite de l'inégalité est strictement positif, on en déduit $k > 1$: le cas du temps de transition non-dégénéré $t_\star(x) = x^2$ est au-delà du cadre de ce travail, et nous verrons dans la Section 1.3 juste après comment traiter ce cas. Dans le cas dégénéré $t_\star(x) = x^4$, on montre l'instabilité si $\sigma < \min(2/9, 2/5) = 2/9$.

1.3 Caractère bien-posé d'une classe de systèmes faiblement hyperboliques

Afin de terminer l'étude développée dans les Chapitres 2 et 3 de cette thèse, et de montrer l'instabilité pour une transition non semi-simple et non lisse du type

$$\begin{pmatrix} 0 & 1 \\ -t+|x|^2 & 0 \end{pmatrix}$$

il nous faut comprendre le comportement du système dans le domaine d'hyperbolicité $\{(t,x) : 0 \leq t \leq |x|^2\}$. Nous avons vu dans la Section 1.2.5 que ce cas d'une transition générique était hors de portée de l'analyse développée précédemment, car le domaine hyperbolique est trop grand pour être négligé. La Figure (1.1) montre ainsi la différence entre les domaines d'hyperbolicité, dans le cas $t_\star(x) = x^2$ (à gauche) et dans le cas $t_\star(x) = x^4$ (à droite).

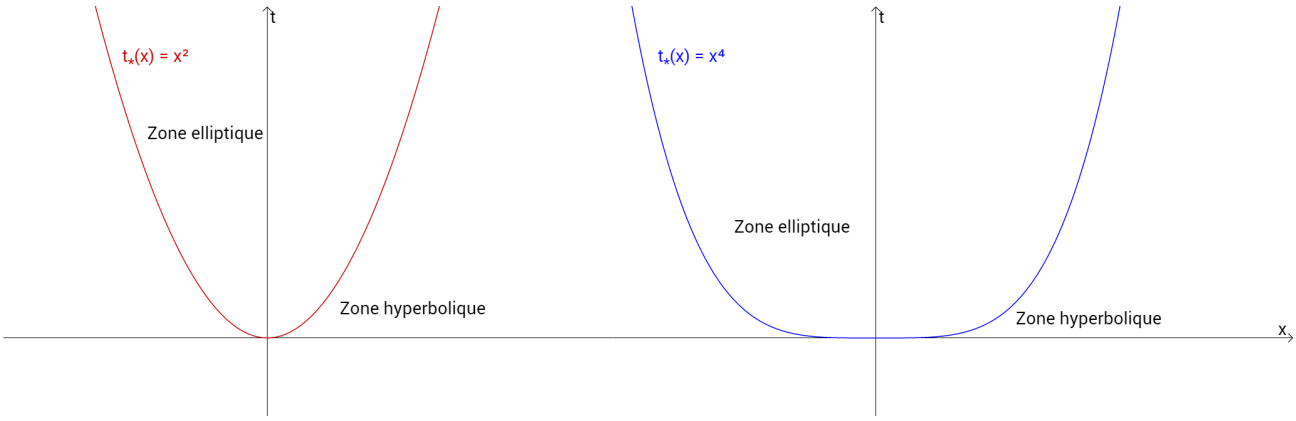

Figure 1.1: Domaines d'hyperbolicité

En utilisant les méthodes décrites dans les Chapitres 2 et 3, on peut montrer que le problème de Cauchy (1.1.1) en $t = t_\star(x)$ est mal-posé, en partant d'une famille de fonctions analytiques $(h_\varepsilon)_{\varepsilon>0}$ construite comme dans la Section précédente. Pour ramener cette construction au problème de Cauchy à $t = 0$, il faut donc pouvoir utiliser un résultat de caractère bien-posé du système (1.1.1), partant de $t = t_\star$ vers $t = 0$. Cela nous incite à considérer le système (1.1.1) pour les temps passés, c'est-à-dire de changer t en $-t$, et d'étudier le système faiblement hyperbolique

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ t+x^2 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{1.3.1}$$

pour les temps positifs et en régularité Gevrey. On note que ce système est directement lié à l'équation scalaire du second degré $\partial_t^2 u_1 = \partial_x \left((t+x^2)\partial_x u_1\right)$.

1.3.1 Le travail fondateur de Colombini, Janelli et Spagnolo

L'étude de telles équations type équation des ondes

$$\partial_t^2 v = \partial_x \left(a \partial_x v\right) \tag{1.3.2}$$

en régularité Gevrey remonte au travail fondateur [CJS83], dans le cas des coefficients $a = a(t)$ ne dépendant que du temps, positifs, et de régularité C^k pour $k \geq 1$. Le fait que a puisse s'annuler, par exemple en $t = 0$, indique que l'équation est *faiblement hyperbolique*. Pour étudier une telle équation, Colombini, Janelli et Spagnolo dans [CJS83] partent de l'énergie "classique" associée à (1.3.2), à savoir

$$E(t,\xi) = |\partial_t \hat{v}(t,\xi)|^2 + a(t)|\xi|^2 \, |\hat{v}(t,\xi)|^2$$

où $\hat{v}(t,\xi)$ est la transformée de Fourier de $v(t,x)$. En dérivant $E(t,\xi)$ en temps, on obtient alors, grâce à (1.3.2),

$$\partial_t E = a'(t)|\xi|^2 \, |\hat{v}(t,\xi)|^2 \, .$$

Avec en tête l'utilisation d'un lemme de type Gronwall et obtenir ainsi une borne de $E(t,\xi)$ en fonction de $E(0,\xi)$, il faut pouvoir borner $\partial_t E$ par E.

Pour ce faire, il faut pouvoir contrôler $|a'(t)|$ par $a(t)$. Un premier résultat de ce type est *l'inégalité de Glaeser*. Cette inégalité tire son nom de Georges Glaeser, qui l'utilisa dans [Gla63] pour étudier la dérivabilité de la racine carrée d'une fonction positive ou lisse[1]. Nous rappelons ici ce résultat

Lemme 1.3.1. *Soit f est une fonction positive et C^2 sur tout $\mathbb{R}$, et de dérivée seconde bornée sur $\mathbb{R}$. Alors*

$$|f'(t)|^2 \leq 2|f''|_{L^\infty} f(t) \quad , \quad \forall t \in \mathbb{R}. \tag{1.3.3}$$

Cette inégalité se montre aisément en utilisant la formule de Taylor avec reste intégral. En revenant au précédent calcul, cette inégalité donnerait

$$\partial_t E \leq |a'(t)||\xi|^2 \, |\hat{v}(t,\xi)|^2 \lesssim a(t)^{1/2}|\xi|^2 \, |\hat{v}(t,\xi)|^2 \, .$$

Mais cela ne serait pas suffisant pour montrer une inégalité du type $\partial_t E \lesssim E$, car $a(t)^{1/2}$ peut ne pas être contrôlé par $a(t)$ (par exemple: $a(t) = t^2$, pour $t \in [0,1]$).

Afin de palier ce problème, Colombini, Janelli et Spagnolo procèdent en deux temps: 1) introduire un petit paramètre ε, remplacer a par $a + \varepsilon$, ce qui rend ainsi l'équation (1.3.2) strictement hyperbolique, et poser

$$E_\varepsilon(t,\xi) = |\partial_t \hat{v}(t,\xi)|^2 + (a(t) + \varepsilon) \, |\xi|^2 \, |\hat{v}(t,\xi)|^2 \tag{1.3.4}$$

l'énergie approchée ; 2) démontrer une certaine généralisation de l'inégalité de Glaser, à savoir

Lemme 1.3.2. *Soit $f : [0,T] \to \mathbb{R}$ une fonction positive, C^k pour $k \geq 1$. Alors*

$$\left| \left(f^{1/k} \right)' \right|^k_{L^1([0,T])} \lesssim |f|_{C^k([0,T])} \, . \tag{1.3.5}$$

[1] Il est intéressant de noter que, dans cet article, Georges Glaeser indique que cette inégalité lui a été communiquée par le mathématicien français Malgrange. Cette inégalité devrait donc être l'inégalité de Malgrange, en somme.

Il s'agit du Lemme 1 de [CJS83], dont la preuve est donnée dans leur Section 2. Nous ferons des remarques sur ce Lemme plus bas, notamment en comparaison au Lemme 1.3.1.

En partant de (1.3.4), on calcule

$$\begin{aligned}\partial_t E_\varepsilon &= a'(t)|\xi|^2\,|\hat{v}(t,\xi)|^2 + 2\varepsilon|\xi|^2\,\mathrm{Re}\,\hat{v}(t,\xi)\cdot\partial_t\hat{v}(t,\xi)\\ &\leq \frac{|a'(t)|}{a(t)+\varepsilon}(a(t)+\varepsilon)|\xi|^2\,|\hat{v}(t,\xi)|^2 + \varepsilon^{1/2}|\xi|E_\varepsilon\\ &\leq \left(\frac{|a'(t)|}{a(t)+\varepsilon} + \varepsilon^{1/2}|\xi|\right)E_\varepsilon \qquad (1.3.6)\end{aligned}$$

grâce à l'inégalité de Cauchy-Schwartz. Comme

$$\Big((a(t)+\varepsilon)^{1/k}\Big)' = \frac{1}{k}\frac{a'(t)}{(a(t)+\varepsilon)^{1-1/k}}$$

on écrit

$$\begin{aligned}\frac{|a'(t)|}{a(t)+\varepsilon} &= \frac{|a'(t)|}{(a(t)+\varepsilon)^{1-1/k}}\times\frac{1}{(a(t)+\varepsilon)^{1/k}}\\ &= k\left|\Big((a(t)+\varepsilon)^{1/k}\Big)'\right|\frac{1}{(a(t)+\varepsilon)^{1/k}}\\ &\lesssim \left|\Big((a(t)+\varepsilon)^{1/k}\Big)'\right|\varepsilon^{-1/k}\end{aligned}$$

car $a(t)\geq 0$ pour tout $t\in[0,T]$. On obtient ensuite

$$\partial_t E_\varepsilon \lesssim \left(\left|\Big((a(t)+\varepsilon)^{1/k}\Big)'\right|\varepsilon^{-1/k} + \varepsilon^{1/2}|\xi|\right)E_\varepsilon.$$

En utilisant le lemme de Gronwall, il vient donc

$$\begin{aligned}E_\varepsilon(t,\xi) &\lesssim \exp\left(\varepsilon^{-1/k}\int_0^t\left|\Big((a(s)+\varepsilon)^{1/k}\Big)'\right|ds + t\varepsilon^{1/2}|\xi|\right)E_\varepsilon(0,\xi)\\ &\lesssim \exp\left(\varepsilon^{-1/k}\left|\Big((a+\varepsilon)^{1/k}\Big)'\right|_{L^1([0,T])} + T\varepsilon^{1/2}|\xi|\right)E_\varepsilon(0,\xi)\end{aligned}$$

pour tout $t\in[0,T]$. Pour conclure, on utilise le Lemme 1.3.2, en notant que la norme C^k de $a+\varepsilon$ est la norme C^k de a:

$$E_\varepsilon(t,\xi)\lesssim\exp\left(\varepsilon^{-1/k}|a|^{1/k}_{C^k([0,T])} + T\varepsilon^{1/2}|\xi|\right)E_\varepsilon(0,\xi).$$

Enfin, on détermine ε en fonction afin de minimiser l'exposant, ce qui donne $\varepsilon = |\xi|^{-2/(k+2)}$ et donc

$$E_\varepsilon(t,\xi)\lesssim e^{c|\xi|^{2/(k+2)}}E_\varepsilon(0,\xi)\quad,\quad\forall\,t\in[0,T]$$

pour un certain $c>0$.

De cette estimation d'énergie, on peut déduire le caractère bien-posé de (1.3.2), dans le cas où $a=a(t)$ est C^k, dans les espaces $\mathcal{G}^\sigma$ pour tout $2/(k+2)<\sigma\leq 1$. On note que la borne inférieure $2/(2+k)$ sur les indices de Gevrey pour lesquels il y a existence et unicité décroit avec k: plus a est régulier, meilleur est le résultat. En particulier, cela rejoint un résultat classique de caractère bien-posé en C^∞ dès lors que a est analytique.

1.3.2 Sur l'inégalité de Glaeser

Nous revenons à présent sur l'inégalité de Glaeser du Lemme 1.3.1 et sur l'estimation L^1 donnée dans [CJS83] et que nous rappelons dans le Lemme 1.3.2. Voici quelques remarques:

- On note que les hypothèses du Lemme 1.3.1 sont globales, c'est-à-dire vérifiées sur $\mathbb{R}$ tout entier. Au contraire, le Lemme 1 de [CJS83] repose sur des conditions vérifiées sur un segment et non sur $\mathbb{R}$.
- En revanche, le Lemme 1.3.2 donne un résultat moins précis que l'inégalité de Glaeser. Une certaine norme L^1 est contrôlée par une norme C^k, alors que Glaeser contrôle de manière ponctuelle $|a'(t)|$ en fonction de $a(t)^{1/2}$.
- Enfin, on note qu'une version locale de l'inégalité de Glaeser (i.e. sur un domaine borné de $\mathbb{R}$ ou $\mathbb{R}^d$) n'est pas forcément vérifiée. Le cas le plus simple est celui de $a(t) = t$ sur tout intervalle $[0, T]$, avec $T > 0$, qui vérifie $a'(0) = 1$ alors que $a(0) = 0$.

Reprenons à présent l'inégalité (1.3.6) sur la dérivée de E_ε. Supposons à présent que a vérifie l'inégalité de Glaeser sur $[0, T]$. Alors,

$$\partial_t E_\varepsilon \leq \exp\left(\frac{C}{(a(t)+\varepsilon)^{1/2}} + \varepsilon^{1/2}|\xi|\right) \leq \exp\left(C\varepsilon^{-1/2} + \varepsilon^{1/2}|\xi|\right)$$

où $C > 0$. On pose $\varepsilon = |\xi|^{-1}$, ce qui donne

$$E_\varepsilon(t,\xi) \lesssim e^{(C+1)t|\xi|^{1/2}} E_\varepsilon(0,\xi) \tag{1.3.7}$$

et conduit donc à un caractère bien posé pour $1/2 < \sigma \leq 1$. On note en particulier la croissance linéaire en temps de l'exposant, ce qui induit une décroissance linéaire en temps du rayon Gevrey.

Si ce résultat est moins bon que celui de [CJS83] que nous avons présenté au-dessus, les quelques inégalités que nous venons de montrer sont pourtant celles plus facilement adaptables au cas d'un coefficient $a = a(t, x)$ dépendant aussi de la variable d'espace.

1.3.3 Au delà de l'article de 1983 de Colombini, Janelli et Spagnolo

Dans la continuité du travail fondateur de [CJS83], Colombini et Nishitani ont étudié (1.3.2) dans le cas $a = a(t, x)$, avec $a \in C^2([0, T], \mathcal{G}^\sigma_\tau)$ pour $\sigma \in (0, 1]$, $\tau > 0$ et a positif sur $[0, T]$. Afin d'utiliser la méthode d'énergie que nous venons de décrire, la fonction $a(t) + \varepsilon(\xi)$ utilisée précédemment devient ici le symbole $a(t, x) + \langle\xi\rangle^{-c}$, avec $c \in (0, 2)$. Ce symbole approche le coefficient $a(t, x)$ à un poids $\langle\xi\rangle^{-c}$ petit à hautes fréquences. L'estimation donnée dans le Lemme 1.3.2 s'adaptant mal au calcul pseudo-différentiel car donnant une estimation L^1 en temps, Colombini et Nishitani lui préfèrent l'inégalité de Glaeser, par rapport à la variable temporelle et sur un segment $[0, T]$. Comme nous l'avons dit plus haut, l'inégalité de Glaeser sur un segment n'est pas toujours vérifiée, il faut donc ajouter la contrainte que $a(t, x)$ soit positif sur $[-\delta, T + \delta] \times B_r(x_0)$ pour un certain $\delta > 0$ et un certain $r > 0$. Sous cette hypothèse supplémentaire, l'équation (1.3.2) est localement

bien-posée dans les espaces de Gevrey $\mathcal{G}^\sigma$ pour $\sigma \in (1/2, 1]$. Il est à noter que cette borne ne dépend pas de la régularité du coefficient $a(t,x)$, et fait écho à l'estimation (1.3.7) plus haut.

Plus récemment, Colombini, Nishitani et Rauch ont étudié, dans [CNR], le système linéaire

$$\partial_t u = \sum_j A_j(t,x)\partial_{x_j} u + f \tag{1.3.8}$$

sous une hypothèse générale de faible hyperbolicité, c'est-à-dire uniquement de réalité du spectre du symbole principal. Sans hypothèse supplémentaire sur le détail du spectre , les auteurs prouvent qu'un tel système est localement bien-posé dans les espaces de Gevrey $\mathcal{G}^\sigma$, avec

$$1 \geq \sigma > \max\left\{\frac{1+6\theta}{2+6\theta}, \frac{2+4\theta}{3+4\theta}\right\}.$$

Le *baromètre* $\theta \in [0, N-1]$ permet d'évaluer à quel point le symbole $A(t,x,\xi)$ du système (1.3.8) est diagonalisable par blocs, avec des blocs de taille $\theta + 1$. Par exemple, si le symbole est diagonalisable, $\theta = 0$. Dans le cas de l'équation (1.3.1), où $N = 2$ et le symbole n'est pas diagonalisable, $\theta = N - 1 = 1$. On note d'ailleurs qu'on peut toujours prendre $\theta = N - 1$, où N est la taille de la matrice $A(t,x,\xi)$.

Afin d'étudier (1.3.8) sous la seule hypothèse de faible hyperbolicité du symbole, Colombini, Nishitani et Rauch basent leur preuve sur la construction d'un symétriseur inspiré des fonctions de Lyapounov pour les équations différentielles. En posant comme nouvelle inconnue $v = e^{(c_0 - ct)D^\sigma} u$, avec $D^\sigma = \mathrm{op}\left(\langle\cdot\rangle^\sigma\right)$, le symbole $A(t,x,\xi) = \sum_j A_j(t,x)\xi_j$ du système (1.3.8) devient

$$A(t,x,\xi) - c\langle\xi\rangle^\sigma =: M(t,x,\xi)$$

dont le spectre est compris dans $\{z \in \mathbb{C} : \mathrm{Re}\, z \leq -\langle\xi\rangle^\sigma\}$, pour c assez grand et pour σ bien choisi. En définissant alors la matrice définie positive

$$R(t,x,\xi) = \int_0^\infty \left(e^{sM(t,x,\xi)}\right)^* e^{sM(t,x,\xi)} ds$$

on obtient une fonction de Lyapunov de l'équation différentielle $X' = MX$, c'est-à-dire R vérifie $RM + M^*R$ est une matrice strictement négative. La preuve de Colombini, Nishitani et Rauch consiste à construire un opérateur pseudo-différentiel dont le symbole est analogue à $R(t,x,\xi)$, donner de bonnes estimations pour les semi-normes du symbole, puis de montrer une estimation d'énergie pour $\langle \mathrm{op}(R)v, v\rangle_{L^2}$.

Le cas des systèmes (1.1.1) faiblement hyperboliques de taille $N = 2$ peut se ramener, comme on l'a vu précédemment, à l'étude des équations scalaires (1.3.2): l'article [CNR] est donc bien une généralisation du résultat initial de [CJS83].

1.3.4 Caractère bien-posé en Gevrey pour des transitions faiblement hyperboliques

Limites de l'analyse de Colombini, Janelli et Spagnolo

Après une description de l'article fondateur [CJS83] et de ses deux principaux successeurs [CN07] et [CNR], nous revenons aux calculs présentés dans la Section 1.3.1, et plus partic-

ulièrement à l'inégalité (1.3.6). Avec en tête le système (1.3.1), qui correspond à l'équation (1.3.2) avec $a(t,x) = t + x^2$, on considère le cas $a(t) = t$, ce qui donne dans l'inégalité (1.3.6):

$$\begin{aligned} E_\varepsilon(t,\xi) &\lesssim \exp\left(\int_0^t \frac{a'(s)}{a(s)+\varepsilon} ds + t\varepsilon^{1/2}|\xi|\right) E_\varepsilon(0,\xi) \\ &\lesssim \frac{\varepsilon}{t+\varepsilon} e^{t\varepsilon^{1/2}|\xi|} E_\varepsilon(0,\xi). \end{aligned} \tag{1.3.9}$$

On peut à présent poser $\varepsilon = |\xi|^{-2}$ et obtenir une croissance polynomiale en $|\xi|$ de l'énergie, et non plus sous-exponentielle. L'énergie est donc contrôlée en norme Sobolev, et plus seulement en norme Gevrey. On note que ce calcul s'étend en fait à tout coefficient $a(t)$ dont la dérivée est positive au voisinage de $t = 0$.

Cette remarque est à mettre en parallèle à la preuve d'optimalité du résultat de [CJS83], donnée dans le même papier et reposant sur la construction d'un contre-exemple rapidement oscillant à $t = 0$. En particulier, le contre-exemple donné dans [CJS83] change de sens de variations une infinité de fois au voisinage de $t = 0$, ce qui nous empêche d'appliquer le raisonnement ci-dessus.

Aussi, comme nous l'avons mentionné plus haut dans la Section 1.3.2, le coefficient $a(t) = t$ ne vérifie pas l'inégalité de Glaeser sur les segments $[0,T]$ avec $T > 0$. L'inégalité (1.3.9) nous permet ainsi de contourner ce problème, et de prouver une estimation d'énergie meilleure que celle attendue d'après le travail de [CJS83].

Symbole adapté et métriques dans l'espace des phases

Dans notre travail présenté au Chapitre 4, nous étudions le système

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ t+x^2 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + F(u)u \tag{1.3.10}$$

avec $x \in \mathbb{R}$ et $F(u) = F(t,x,u)$ analytique. On notera pour la suite $a(t,x) = t + x^2$. La discussion précédente mettant en avant le poids $(a(t,0) + \varepsilon(\xi))^{-1}$ dans l'énergie, on considère l'opérateur pseudo-différentiel $\mathrm{op}(b)$ avec pour symbole

$$b(t,x,\xi) = \left(a(t,x) + \langle\xi\rangle^{-c}\right)^{-1/2} \tag{1.3.11}$$

avec $c \in (0,2)$. L'ordre du symbole b dépend du temps. En effet, pour $t = 0$, le symbole vérifie $b(0,x,\xi) \leq \langle\xi\rangle^{c/2}$ uniformément en x au voisinage de $x = 0$. En revanche, pour tout $t \geq \underline{t} > 0$, on a $b(t,x,\xi) \leq \underline{t}^{-1/2}$, uniformément en (t,x,ξ).

Pour réconcilier ces deux points de vue (ordre $c/2$ à $t = 0$, ordre 0 pour $t \geq \underline{t} > 0$) on utilise la notion de métrique dans l'espace des phases, en définissant la métrique

$$g^t_{(x,\xi)}(\mathrm{d}x, \mathrm{d}\xi) = b(t,x,\xi)^2 |\mathrm{d}x|^2 + \langle\xi\rangle^{-2} |\mathrm{d}\xi|^2$$

qui dépend du temps. Associée à cette métrique et pour tout poids $M = M(t,x,\xi) \geq 0$, on définit la classe de symboles $S(M, g^t)$ comme étant l'ensemble des fonctions f, C^∞ sur $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$, et qui vérifient les inégalités

$$\left|\partial_x^\alpha \partial_\xi^\beta f(t,x,\xi)\right| \lesssim M(t,x,\xi)\, b(t,x,\xi)^{|\alpha|} \langle\xi\rangle^{-|\beta|} \tag{1.3.12}$$

uniformément en (t, x, ξ), et pour tout $(\alpha, \beta) \in \mathbb{N}^d \times \mathbb{N}^d$ (sur les métriques dans l'espace des phases, les classes de symboles associées et le calcul pseudo-différentiel sur de telles métriques, voir l'Appendice 4.4.2 du Chapitre 4 et le livre de Nicolas Lerner [Ler11]). Comme $a(0, x) = x^2$ vérifie l'inégalité de Glaeser, on peut montrer en particulier que $b \in S(b, g^t)$, qui est inclus dans $S^{c/2}_{1,c/2}$.

Estimations d'énergie

En poursuivant la discussion de la Section 1.3.4, on définit le symétriseur

$$S = \mathrm{diag}(1, b) \tag{1.3.13}$$

et l'énergie

$$E = \frac{1}{2} \left| \mathrm{op}(S) e^{\tau(t) D^\sigma} u \right|_{L^2}. \tag{1.3.14}$$

avec $\tau(t) = \tau_0 - \boldsymbol{\tau} t$. En dérivant par rapport au temps, on obtient

$$\partial_t E = \mathrm{Re}\, \langle \mathrm{op}(S) \partial_t v, \mathrm{op}(S) v \rangle + \mathrm{Re}\, \langle \mathrm{op}(\partial_t S) v, \mathrm{op}(S) v_2 \rangle$$

en posant $v(t) = e^{\tau(t) D^\sigma} u(t)$. Comme u est solution du système (1.3.10), on calcule

$$\begin{aligned} \partial_t v &= -\boldsymbol{\tau} D^\sigma v + e^{\tau D^\sigma} \partial_t u \\ &= -\boldsymbol{\tau} D^\sigma v + e^{\tau D^\sigma} \begin{pmatrix} 0 & 1 \\ a(t,x) & 0 \end{pmatrix} \partial_x u + e^{\tau D^\sigma} F(u) u \\ &= -\boldsymbol{\tau} D^\sigma v + \begin{pmatrix} 0 & 1 \\ a^{(\tau)} & 0 \end{pmatrix} \partial_x v + F(u)^{(\tau)} v \end{aligned} \tag{1.3.15}$$

en notant l'opérateur de conjugaison de $a(t, x)$ avec l'opérateur Gevrey

$$a^{(\tau)} = e^{\tau D^\sigma} a\, e^{-\tau D^\sigma}$$

et de même

$$F(u)^{(\tau)} = e^{\tau D^\sigma} F(u)\, e^{-\tau D^\sigma}.$$

En reportant l'égalité (1.3.15) dans la dérivation de l'énergie, on obtient

$$\partial_t E = -\boldsymbol{\tau}\, \mathrm{Re}\, \langle \mathrm{op}(S) D^\sigma v, \mathrm{op}(S) v \rangle \tag{1.3.16}$$

$$+\mathrm{Re}\, \langle \mathrm{op}(S) \begin{pmatrix} 0 & 1 \\ a^{(\tau)} & 0 \end{pmatrix} \partial_x v, \mathrm{op}(S) v \rangle \tag{1.3.17}$$

$$+\mathrm{Re}\, \langle \mathrm{op}(\partial_t S) v, \mathrm{op}(S) v_2 \rangle \tag{1.3.18}$$

$$+\mathrm{Re}\, \langle \mathrm{op}(S) F(u)^{(\tau)} v, \mathrm{op}(S) v \rangle. \tag{1.3.19}$$

Le terme (1.3.16) est typique d'une estimation d'énergie Gevrey, et provient de la dérivation en temps de l'opérateur Gevrey $e^{\tau D^\sigma}$. On note en particulier que ce terme est donc d'ordre supérieur à E: le terme (1.3.16) contrôle $D^{\sigma/2} \mathrm{op}(S) v$ en norme L^2, alors que $E = \frac{1}{2} |\mathrm{op}(S) v|_{L^2}$. Le signe négatif de (1.3.16) permettra donc de contrôler les termes de restes dans l'estimation d'énergie.

Concentrons-nous à présent sur le terme (1.3.17), central dans notre analyse. En utilisant la définition (1.3.13) du symétriseur S, on obtient

$$\mathrm{op}(S)^2 \begin{pmatrix} 0 & 1 \\ a^{(\tau)} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \mathrm{op}(b)^2 a^{(\tau)} & 0 \end{pmatrix}.$$

Pour étudier l'opérateur $\mathrm{op}(b)^2 a^{(\tau)}$, nous allons utiliser à la fois les propriétés du calcul pseudo-différentiel sur les métriques non-plates, et les propriétés de l'opérateur $a^{(\tau)}$, en utilisant les résultats du Chapitre 5 sur la conjugaison par un opérateur Gevrey. En particulier,

$$a^{(\tau)} - a = \mathrm{op}(R) + \mathrm{op}\left(S_{1,0}^{-2(1-\sigma)}\right) \tag{1.3.20}$$

et R est dans $S(b^{-1}\langle\cdot\rangle^{\sigma-1}, g^t)$, en utilisant la définition (1.3.12). Puis, par définition (1.3.11), on obtient

$$a^{(\tau)} = \mathrm{op}(b^{-2}) - \mathrm{op}\left(\langle\cdot\rangle^{-c}\right) + \mathrm{op}(R) + \mathrm{op}\left(S_{1,0}^{-2(1-\sigma)}\right)$$

et finalement

$$\mathrm{op}(b)^2 a^{(\tau)} = \mathrm{op}(b)^2\mathrm{op}(b^{-2}) - \mathrm{op}(b)^2\mathrm{op}\left(\langle\cdot\rangle^{-c}\right) + \mathrm{op}(b)^2\mathrm{op}(R) + \mathrm{op}(b)^2\mathrm{op}\left(S_{1,0}^{-2(1-\sigma)}\right).$$

Dans le cadre des métriques non plates et du calcul pseudo-différentiel associé, un Lemme de composition est vérifié:

$$\mathrm{op}(b)^2\mathrm{op}(b^{-2}) = \mathrm{Id} + \mathrm{op}\left(S\left(b\langle\cdot\rangle^{-1}, g^t\right)\right).$$

En revenant au terme (1.3.17), on obtient donc

$$\begin{aligned}
&\mathrm{Re}\,\langle \mathrm{op}(S)\begin{pmatrix} 0 & 1 \\ a^{(\tau)} & 0 \end{pmatrix}\partial_x v, \mathrm{op}(S)v\rangle \\
&= \mathrm{Re}\,\langle\Big(\mathrm{Id} + \mathrm{op}\left(S\left(b\langle\cdot\rangle^{-1}, g^t\right)\right) - \mathrm{op}(b)^2\mathrm{op}\left(\langle\cdot\rangle^{-c}\right) + \mathrm{op}(b)^2\mathrm{op}(R) + \mathrm{op}(b)^2\mathrm{op}\left(S_{1,0}^{-2(1-\sigma)}\right)\Big)\partial_x v_1, v_2\rangle \\
&\quad + \mathrm{Re}\,\langle\partial_x v_2, v_1\rangle \\
&= \mathrm{Re}\,\langle\Big(\mathrm{op}\left(S\left(b\langle\cdot\rangle^{-1}, g^t\right)\right) - \mathrm{op}(b)^2\mathrm{op}\left(\langle\cdot\rangle^{-c}\right) + \mathrm{op}(b)^2\mathrm{op}(R) + \mathrm{op}(b)^2\mathrm{op}\left(S_{1,0}^{-2(1-\sigma)}\right)\Big)\partial_x v_1, v_2\rangle.
\end{aligned}$$

Grâce à la définition de b et de S, une annulation essentielle apparaît ainsi dans l'estimation d'énergie. Il ne reste donc plus qu'à contrôler les termes de reste, grâce au terme (1.3.16).

En utilisant le contrôle de $\mathrm{op}(S)D^{\sigma/2}v$ en norme L^2, on écrit, à termes de reste près venant de compositions d'opérateurs pseudo-différentiels,

$$\begin{aligned}
&\mathrm{Re}\,\langle\Big(\mathrm{op}\left(S\left(b\langle\cdot\rangle^{-1}, g^t\right)\right) - \mathrm{op}(b)^2\mathrm{op}\left(\langle\cdot\rangle^{-c}\right) + \mathrm{op}(b)^2\mathrm{op}(R) + \mathrm{op}(b)^2\mathrm{op}\left(S_{1,0}^{-2(1-\sigma)}\right)\Big)\partial_x v_1, v_2\rangle \\
&\approx \mathrm{Re}\,\langle\Big(\mathrm{op}\left(S\left(\langle\cdot\rangle^{-\sigma}, g^t\right)\right) + \mathrm{op}\left(S\left(bR\langle\cdot\rangle^{1-\sigma}, g^t\right)\right)\Big) D^{\sigma/2}v_1, \mathrm{op}(b)D^{\sigma/2}v_2\rangle \qquad (1.3.21) \\
&\quad + \mathrm{Re}\,\langle\Big(-\mathrm{op}\left(S\left(b\langle\cdot\rangle^{1-c-\sigma}, g^t\right)\right) + \mathrm{op}\left(S\left(b\langle\cdot\rangle^{-1+\sigma}, g^t\right)\right)\Big) D^{\sigma/2}v_1, \mathrm{op}(b)D^{\sigma/2}v_2\rangle. \qquad (1.3.22)
\end{aligned}$$

La première ligne (1.3.21) est constituée des termes principaux, et chaque opérateur agit bien dans L^2. En effet, le premier opérateur $\mathrm{op}\left(S\left(\langle\cdot\rangle^{-\sigma}, g^t\right)\right)$ agit bien de manière continue dans L^2, car $S\left(\langle\cdot\rangle^{-\sigma}, g^t\right) \subset S^{-\sigma}_{1,c/2}$. De plus, comme $R \in S(b^{-1}\langle\cdot\rangle^{\sigma-1}, g^t)$, on a $S\left(bR\langle\cdot\rangle^{1-\sigma}, g^t\right) \subset S(1, g^t)$, et les opérateurs de symbole dans ce dernier espace agissent bien continûment sur L^2.

La deuxième ligne (1.3.22) est constituée de termes de restes, mais c'est elle qui mène aux contraintes sur c et σ. En effet, on a d'une part

$$\mathrm{op}\left(S\left(b\langle\cdot\rangle^{1-c-\sigma}, g^t\right)\right) \in \mathrm{op}\left(S^{1-c/2-\sigma}_{1,c/2}\right)$$

car $b \in S(b, g^t) \subset S^{c/2}_{1,c/2}$. L'opérateur agit continuement sur L^2 à la condition que

$$1 - c/2 - \sigma \leq 0.$$

D'autre part, pour la même raison on a

$$\mathrm{op}\left(S\left(b\langle\cdot\rangle^{-1+\sigma}, g^t\right)\right) \in \mathrm{op}\left(S^{c/2-1+\sigma}_{1,c/2}\right)$$

qui agit continûment sur L^2 si

$$c/2 - 1 + \sigma \leq$$

et donc finalement

$$c = 2(1 - \sigma). \tag{1.3.23}$$

En résumé, si c vérifie (1.3.23), on a montré l'estimation

$$|(1.3.17)| \lesssim \left|\mathrm{op}(S)D^{\sigma/2}v\right|_{L^2}.$$

Il nous reste à étudier les termes (1.3.18) et (1.3.19). Le premier concerne la dérivée $\mathrm{op}(\partial_t S) = \begin{pmatrix} 0 & 0 \\ \mathrm{op}(\partial_t b) & 0 \end{pmatrix}$. Par définition (1.3.11) de b, on calcule

$$\partial_t b = -\frac{1}{2}\partial_t a\left(a(t,x) + \langle\xi\rangle^{-c}\right)^{-3/2} = -\frac{1}{2}b^3$$

car $\partial_t a = \partial_t(t + x^2) = 1$ dans cette introduction. Pour (t, x) suffisamment petits, on a $b \geq 1$ et donc $\partial_t b \leq -1/2$: on montre ainsi, dans la Section 4.3.3, que

$$\langle \mathrm{op}(\partial_t S)v, \mathrm{op}(b)v\rangle \leq 0.$$

Enfin, concernant les termes non-linéaires (1.3.19), nous avons besoin d'un lemme d'action d'opérateurs de la forme $F(u)^{(\tau)} = e^{\tau D^\sigma}F(u)e^{-\tau D^\sigma}$ dans $H^{\sigma/2}$. Un tel résultat fait l'objet de nos travaux du Chapitre 5, que nous décrirons juste après dans la Section 1.4. En supposant ici un tel résultat, il reste à contrôler les termes non-linéaires par $\left|\mathrm{op}(S)D^{\sigma/2}v\right|_{L^2}$. Dans (1.3.19), le terme d'ordre le plus grand est

$$\mathrm{Re}\,\langle \mathrm{op}(b)\left(F(u)^{(\tau)}\right)_{2,1} v_1, \mathrm{op}(b)v_2\rangle$$

car $\mathrm{op}(b)v_1$ n'est pas contrôlé en norme $H^{\sigma/2}$. On écrit alors

$$\mathrm{Re}\,\langle \mathrm{op}(b)\left(F(u)^{(\tau)}\right)_{2,1} v_1, \mathrm{op}(b)v_2\rangle \approx \mathrm{Re}\,\langle D^{-\sigma}\mathrm{op}(b)\left(F(u)^{(\tau)}\right)_{2,1} D^{\sigma/2}v_1, \mathrm{op}(b)D^{\sigma/2}v_2\rangle$$

et comme $F(u)^{(\tau)}$ agit continuement dans $H^{\sigma/2}$, il reste à prouver que l'opérateur $D^{-\sigma}\mathrm{op}(b)$ agit continuement dans L^2. Comme $b \in S(b, g^t) \subset S^{c/2}_{1,c/2}$, c'est chose faite dès lors que

$$0 \geq -\sigma + c/2 = 1 - 2\sigma$$

en utilisant (1.3.23), ce qui donne bien la borne inférieure sur les indices Gevrey $\sigma \geq 1/2$.

1.4 Opérateurs pseudo-différentiels sur les espaces de Gevrey

Dans le Chapitre 4 et comme nous venons de le voir, nous avons besoin de comprendre la conjugaison par l'opérateur Gevrey $e^{\tau D^\sigma}$ d'une fonction Gevrey $F \in \mathcal{G}^\sigma_{\tau_0}$, avec $\tau_0 \geq \tau$. On note

$$F^{(\tau)} = e^{\tau D^\sigma} F e^{-\tau D^\sigma}$$

l'opérateur conjugué. Deux questions se posent pour un tel opérateur: l'opérateur $F^{(\tau)}$ agit-il continuement dans L^2, même dans le cas limite $\tau = \tau_0$? Et cet opérateur est-il un opérateur pseudo-différentiel ? Si oui, à quel classe appartient le symbole ?

Dans un souci de généralité, nous avons étendu ces deux questions au cas de la conjugaison d'un opérateur pseudo-différentiel par un opérateur Gevrey, c'est-à-dire que nous avons considéré l'opérateur

$$\mathrm{op}(a)^{(\tau)} = e^{\tau D^\sigma}\mathrm{op}(a)e^{-\tau D^\sigma}.$$

Pour cela, nous avons besoin de considérer une classe de symboles $a = a(x, \xi)$ qui sont Gevrey dans la variable x, afin d'étendre le cas $F = F(x)$ des fonctions Gevrey. En nous basant sur les classes standard de symboles $S^0_{\rho,\delta}$, on introduit pour $1 \geq \rho > \delta \geq 0$, $s \in (1, +\infty)$ et $R > 0$ la classe $\mathrm{S}^0_{\rho,\delta}\mathrm{G}^s_R$ de symboles $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ qui vérifient

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha,\beta} R^{|\alpha+\beta|}\alpha!^s \beta! \langle\xi\rangle^{-\rho|\beta|+\delta|\alpha|} \tag{1.4.1}$$

pour tout $(\alpha, \beta) \in \mathbb{N}^d \times \mathbb{N}^d$, uniformément en $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$, et pour $C_{\alpha,\beta} > 0$ une suite bornée. Cette définition repose sur la définition des espaces de Gevrey "côté espace", comme décrit dans la Definition 5.2.1, afin de rester au plus près de la définition classique des espaces $S^0_{\rho,\delta}$.

On considère dans la suite les symboles $a \in \mathrm{S}^0_{\rho,\delta}\mathrm{G}^s_R$ à support compact B dans $\mathbb{R}^d_x$, uniformément en $\xi \in \mathbb{R}^d$. Ainsi, nous montrons que la transformée de Fourier par rapport à la première variable $\mathcal{F}_1 a(\cdot, \xi) = \hat{a}(\cdot, \xi)$ vérifie

$$\left| e^{\tau\langle\xi\rangle^{-\delta/s}\langle\eta\rangle^{1/s}} \hat{a}(\eta, \xi)\right|_{L^2_\eta} \lesssim 1$$

uniformément en $\xi \in \mathbb{R}^d$, avec $\tau < sR^{-1/s}$. On observe ainsi que, si $\delta > 0$, la fonction $\hat{a}(\cdot, \xi)$ a un rayon Gevrey qui décroit avec $|\xi|$: cette détérioration de la régularité Gevrey

pour de tels symboles est la clé d'un des résultats du Chapitre 5, à savoir que $\mathrm{op}(a)$ agit continuement dans les espaces de Gevrey pour des indices $\sigma \leq (1-\delta)/s$. La preuve de ce résultat repose essentiellement sur la preuve de l'action dans L^2 de l'opérateur $F^{(\tau)}$, que nous allons esquisser.

On considère donc une fonction F dans $\mathcal{G}^\sigma_\tau$, et v dans L^2. Alors, par définition de $F^{(\tau)}$, on a

$$\begin{aligned} \mathcal{F}\left(F^{(\tau)}v\right)(\xi) &= \int_\eta e^{\tau\langle\xi\rangle^\sigma-\tau\langle\eta\rangle^\sigma}\widehat{F}(\xi-\eta)\hat{v}(\eta)d\eta \\ &= \int_\eta e^{\tau\langle\xi\rangle^\sigma-\tau\langle\eta\rangle^\sigma-\tau\langle\xi-\eta\rangle^\sigma}f(\xi-\eta)\hat{v}(\eta)d\eta \end{aligned} \tag{1.4.2}$$

en posant $f(\zeta) = e^{\tau\langle\zeta\rangle^\sigma}\widehat{F}(\zeta)$, qui est dans L^2 car F est dans $\mathcal{G}^\sigma_\tau$. Une première idée est d'utiliser l'inégalité triangulaire sur la fonction $\langle\cdot\rangle^\sigma$, ce qui donne

$$e^{\tau\langle\xi\rangle^\sigma-\tau\langle\eta\rangle^\sigma-\tau\langle\xi-\eta\rangle^\sigma} \leq 1$$

uniformément en ξ et η, et donc

$$\left|\mathcal{F}\left(F^{(\tau)}v\right)(\xi)\right| \leq \int_\eta |f(\xi-\eta)|\,|\hat{v}(\eta)|\,d\eta$$

et comme f et $\hat{v}$ sont dans L^2, l'inégalité de Young donnerait ici $\mathcal{F}\left(F^{(\tau)}v\right) \in L^\infty$ et non dans L^2. Pour dépasser ce problème, qui vient d'une inégalité triangulaire trop imprécise, nous utilisons l'inégalité triangulaire améliorée suivante. Soit $K > 1$ et ξ, η vérifiant $|\xi-\eta| \leq |\eta|/K$. Alors,

$$|\langle\xi\rangle^\sigma - \langle\eta\rangle^\sigma| \leq \left(K^\sigma - (K-1)^\sigma\right)\langle\xi-\eta\rangle^\sigma$$

et il est important de noter que $K^\sigma-(K-1)^\sigma < 1$: on obtient *un gain en basse fréquence* par rapport à l'inégalité triangulaire, pour $\sigma \in (0,1)$ (on note en particulier qu'il n'y a pas de gain dans le cas analytique $\sigma = 1$). Cette inégalité a par exemple été utilisée dans [BMM16]. En revenant à l'égalité (1.4.2), on peut alors utiliser une décomposition en paraproduit, en découpant l'intégrale en trois régions, et utiliser l'inégalité triangulaire améliorée sur chacune des régions. Le gain Gevrey en basse fréquence assure alors que chaque partie est bornée dans L^2_ξ, ce qui assure le résultat annoncé.

Nous considérons à présent la question de savoir si l'opérateur conjugué $\mathrm{op}(a)^{(\tau)}$ est un opérateur pseudo-différentiel et, le cas échéant, de donner un développement asymptotique de son symbole. On sait déjà que c'est le cas (voir par exemple le Lemme 7.1 dans [CNR]) si τ est relativement petit:

$$\mathrm{op}(a)^{(\tau)} = \mathrm{op}(\widetilde{a})$$

si τ est relativement petit, avec $\widetilde{a}(x,\xi) = \int_{y,\eta} e^{-i\eta\cdot y}e^{\tau\langle\xi+\eta\rangle^\sigma-\tau\langle\xi\rangle^\sigma}a(x+y,\xi)dyd\eta$. On sait aussi que, pour τ relativement petit, ce symbole est dans $S^0_{1,0}$ si $a \in \mathrm{S}^0_{\rho,0}\mathrm{G}^s_R$. Nous montrons dans le Chapitre 5 que la limite pour les τ admissible est $sR^{-1/s}$, et nous donnons aussi une estimation des semi-normes pour tout $\underline{\tau} \in (\tau, sR^{-1/s})$, à savoir

$$\sup_{x\in B,\,\xi\in\mathbb{R}^d}\left|\langle\xi\rangle^{+|\beta|}\partial^\alpha_x\partial^\beta_\xi\widetilde{a}(x,\xi)\right| \lesssim_{B,\underline{\tau}} \sup_{\alpha\in\mathbb{N}^d}|a|_{\alpha,\beta}\left(\underline{\tau}-|\tau|\right)^{-(2|\beta|+|\alpha|)/\sigma}$$

où la constante implicite dans $\lesssim_{B,\underline{\tau}}$ est précisée dans le Lemme 5.5.1. Notre preuve repose encore une fois sur une décomposition en paraproduit et sur l'inégalité triangulaire améliorée pour $\langle\cdot\rangle^\sigma$. Ce résultat est notamment utilisé dans l'égalité (1.3.20) de la Section précédente.

Remarque 1.4.1. *Le fait que le rayon Gevrey soit limité à $\tau < sR^{-1/s}$ est une des principales raisons de la restriction, dans le Chapitre 4, au semi-linéaire. En effet, pour l'étude d'un système quasi-linéaire avec, par exemple, $a = a(t,x,u) = t + x^2 + u^2$, la régularité du coefficient a est limitée à celle de la solution u, qui est dans $\mathcal{G}^\sigma_\tau$: notre résultat ne permet pas de savoir si l'opérateur conjugué $a^{(\tau)}$ est un opérateur pseudo-différentiel, et on ne peut donc pas utiliser* (1.3.20) *comme dans la Section précédente.*

Chapter 2

On hyperbolicity and Gevrey well-posedness. Part one: the elliptic case.

In this paper we prove that the Cauchy problem for first-order quasi-linear systems of partial differential equations is ill-posed in Gevrey spaces, under the assumption of an initial ellipticity. The assumption bears on the principal symbol of the first-order operator. Ill-posedness means instability in the sense of Hadamard, specifically an instantaneous defect of Hölder continuity of the flow from G^σ to L^2, where $\sigma \in (0,1)$ depends on the initial spectrum. Building on the analysis carried out by G. Métivier [*Remarks on the well-posedness of the nonlinear Cauchy problem*, Contemp. Math. 2005], we show that ill-posedness follows from a long-time Cauchy-Kovalevskaya construction of a family of exact, highly oscillating, analytical solutions which are initially close to the null solution, and which grow exponentially fast in time. A specific difficulty resides in the observation time of instability. While in Sobolev spaces, this time is logarithmic in the frequency, in Gevrey spaces it is a power of the frequency. In particular, in Gevrey spaces the instability is recorded much later than in Sobolev spaces.

This Chapter is the article [Mor16a].

2.1 Introduction

We consider the Cauchy problem for first-order quasi-linear systems of partial differential equations

$$\partial_t u = \sum_{j=1}^{d} A_j(t,x,u)\partial_{x_j} u + f(t,x,u)\ , \qquad u(0,x) = h(x) \tag{2.1.1}$$

where $t \geq 0$, $x \in \mathbb{R}^d$, $u(t,x)$ and $f(t,x,u)$ are in $\mathbb{R}^N$ and $A_j(t,x,u) \in \mathbb{R}^{N\times N}$. We assume throughout the paper that the A_j and f are analytic in a neighborhood of $(0,0,0)$.

We prove that if the first-order operator is initially micro-locally elliptic, then the Cauchy problem (2.1.1) is ill-posed in Gevrey spaces. Our results extend Métivier's ill-posedness theorem [Mét05] for initially elliptic operators in Sobolev spaces.

While it may seem natural that Gevrey regularity, with associated sub-exponential Fourier rates of decay $O\left(e^{-|\xi|^\sigma}\right)$, with $\sigma < 1$, will not be sufficient to counteract the exponential growth of elliptic operators (think of $e^{t\xi}$, as is the case for the Cauchy-Riemann operator $\partial_t + i\partial_x$), the proof of ill-posedness requires a careful analysis of linear growth rates and linear and nonlinear errors. This ill-posedness result is Theorem 1, stated in Section 2.2.3. By ill-posedness, we mean the absence of a Hölder continuous dependence on the data, as measured from G^σ to L^2. The precise definition is given in Section 2.2.1. The larger σ, the stronger the result. Of course, well-posedness holds in the limiting case $\sigma = 1$, corresponding to analytic functions. Assuming only a property of micro-local ellipticity for the principal symbol of (2.1.1), we obtain, in Theorem 1, the bound $\sigma < 1/(m+1)$, where $m \geq 1$ is an algebraic multiplicity. Under an assumption of smooth partial diagonalization (see Assumption 2.2.6), we obtain, in Theorem 2, ill-posedness for any $\sigma < 1/2$ regardless of the algebraic multiplicity. Under stronger spectral assumptions (see Assumption 2.2.8), we obtain, in Theorem 3, ill-posedness for any $\sigma < 2/3$ and we outline the conditions which allow for an instability proof at an arbitrarily high Gevrey regularity.

We note that an equation may be simultaneously ill-posed in Sobolev spaces and well-posed in Gevrey spaces (for instance, the Prandtl equation [GVD10], [GVM15]). Besides well-posedness, the distinct but related phenomenon of Landau damping for Vlasov-Poisson occurs in Gevrey spaces [MV11], [BMM16], but not in Sobolev spaces [Bed16].

In the companion paper [Mor16b], we extend these results to systems transitioning from hyperbolicity to ellipticity, following [LMX10] and [LNT17].

2.1.1 Background: on Lax-Mizohata results

The question of the well-posedness of the Cauchy problem was first introduced and studied by Hadamard in [Had02]. Hadamard proved, in the case of linear second-order elliptic equations, that the associated solution flow is not regular in the vicinity of any solution of the system. The case of linear evolution systems of the form (2.1.1), that is with $A_j(t,x,u) \equiv A_j(t,x)$, $f(t,x,u) \equiv f(t,x)$ was first studied by Lax in [Lax05], where the proof was given that hyperbolicity of the system, i.e. reality of the spectrum of the principal symbol, was a necessary condition for (2.1.1) to be well-posed in the sense of Hadamard in C^k spaces. Lax's proof relied on separation of the spectrum. Mizohata extended Lax's result without this assumption in [Miz61]. Some cases of nonlinear systems were studied later by Wakabayashi in [Wak01] (here with stability also with respect to source term) and by Yagdjian in [Yag98] and [Yag02] (there in the special case of gauge invariant systems).

A first statement of a precise Lax-Mizohata result for first-order quasi-linear systems was given by Métivier in [Mét05], with a precise description of the lack of regularity of the flow. As we will adapt the methods used by Métivier, we want to take a close look at [Mét05].

2.1.2 On Métivier's result in Sobolev spaces

In Section 3 of [Mét05] Guy Métivier proves Hölder ill-posedness in Sobolev spaces for the Cauchy problem (2.1.1), as soon as hyperbolicity fails at $t = 0$. The initial defect of hyperbolicity means here that there are some $x_0 \in \mathbb{R}^d$, $\vec{u}_0 \in \mathbb{R}^N$ and $\xi_0 \in \mathbb{R}^d$ such that the principal symbol evaluated at $(0, x_0, \vec{u}_0, \xi_0)$:

$$A_0 := \sum_j A_j(0, x_0, \vec{u}_0)\xi_{0,j} \tag{2.1.2}$$

is supposed to have a couple of eigenvalues with non zero imaginary part, say $\pm i\gamma_0$, with eigenvectors $\vec{e}_\pm$. Hölder well-posedness, locally in time and space, would mean that initial data h_1 and h_2 in $H^\sigma(B_{r_0}(x_0))$, for some small $r_0 > 0$, would generate solutions u_1 and u_2 such that

$$||u_2 - u_1||_{L^2(\Omega)} \lesssim ||h_2 - h_1||^\alpha_{H^\sigma(B_{r_0}(x_0))} \tag{2.1.3}$$

for some space-time domain Ω, for some $\sigma \geq 0$, some $\alpha \in (0, 1]$. In order to disprove (2.1.3), Métivier chooses $h_1 \equiv \vec{u}_0$, and lets u_1 the Cauchy-Kovalevskaya solution issued from h_1, the existence of which is granted, locally in space and time, by the analyticity assumption on the coefficients A_j and f. Translating, Métivier is reduced to the case $\vec{u}_0 = 0$, $u_1 \equiv 0$, and the proof that (2.1.3) does not hold is reduced to the construction of a family $(u_\varepsilon)_{\varepsilon>0}$ of initially small, exact analytical solutions such that

$$\lim_{\varepsilon\to 0} \frac{||u_\varepsilon||_{L^2(\Omega_\varepsilon)}}{||u_\varepsilon(0)||^\alpha_{H^\sigma(B_0(x_0))}} = +\infty \tag{2.1.4}$$

for all Hölder exponent $\alpha \in (0, 1]$ and all Sobolev indices $\sigma > 0$, where Ω_ε is a small conical space-time domain centered at $(0, x_0)$.

To highlight the specific frequency ξ_0 at which the initial ellipticity occurs, Métivier looks for solutions of the form

$$u_\varepsilon(t, x) = \varepsilon \mathbf{u}(t/\varepsilon, x, (x - x_0) \cdot \xi_0/\varepsilon) \tag{2.1.5}$$

with ε a small parameter and $\mathbf{u}(s, x, \theta)$ is periodic in θ. Then $\mathbf{u}$ solves

$$\partial_s \mathbf{u} - A_0 \partial_\theta \mathbf{u} = G(\varepsilon \mathbf{u}) \tag{2.1.6}$$

where A_0 is defined by (2.1.2) and $G(\varepsilon\mathbf{u})$ comprises both linear and nonlinear "errors" terms. Factorizing the propagator, an equivalent fixed point equation is obtained

$$\mathbf{u} = e^{sA_0\partial_\theta}\mathbf{u}(0) + \int_0^s e^{(s-s')A_0\partial_\theta} G(\varepsilon\mathbf{u}(s'))ds'. \tag{2.1.7}$$

For equation (2.1.7), the goal is to prove:

- The existence of solutions over the space-time domain Ω_ε. This is a Cauchy-Kovalevskaya type of result, discussed in Section 2.1.2.
- The *wild growth estimate* (2.1.4). Since the instability develops in time, the existence domain Ω_ε must be large enough for (2.1.4) to be recorded. This point is discussed in Section 2.1.2.

Exponential growth of the solutions

As a consequence of the assumption of ellipticity on A_0 defined by (2.1.2), the propagator has an exponential growth in Fourier

$$\left|\left(e^{(s-s')A_0\partial_\theta}\mathbf{u}(s,x,\theta)\right)_n\right| \lesssim e^{|n|\gamma_0(s-s')}|\mathbf{u}_n(s,x)| \tag{2.1.8}$$

where we denote by $(\cdot)_n$ the n-th Fourier mode with respect to the periodic variable θ. We recall that ξ_0 is the distinguished frequency for which A_0, defined in (2.1.2), has a couple of non real eigenvalues associated with eigenvectors $\vec{e}_\pm$. We define well-chosen initial data

$$h_\varepsilon = \varepsilon^{M+1}\left(e^{\mp ix\cdot\xi_0/\varepsilon}\vec{e}_\pm\right) \quad , \quad \mathbf{h}_\varepsilon := \varepsilon^M\left(e^{\mp i\theta}\vec{e}_\pm\right) \tag{2.1.9}$$

for which the upper bound is attained:

$$\mathbf{f}_\varepsilon(s,\theta) := e^{sA_0\partial_\theta}\mathbf{h}_\varepsilon(\theta) \quad \text{satisfies} \quad |(\mathbf{f}_\varepsilon)_n| \approx \varepsilon^M e^{\gamma_0 s}\ , \quad \forall n\in\mathbb{Z}. \tag{2.1.10}$$

Above $\mathbf{f}_\varepsilon(s,\theta)$ is the free solution of (2.1.6), that is the solution of the equation when $G(\varepsilon\mathbf{u}) = 0$. One key observation in view of the Hadamard instability is that, for times of order $M|\ln(\varepsilon)|$, the free solution $\mathbf{f}_\varepsilon$ is of order 1 with respect to ε, whereas at time 0 it is of order ε^M. Roughly there are $f_\varepsilon(t,x) = \mathbf{f}_\varepsilon(t/\varepsilon, x, (x-x_0)\cdot\xi_0/\varepsilon)$, $h_\varepsilon(t,x) = \mathbf{h}_\varepsilon(t/\varepsilon, x, (x-x_0)\cdot\xi_0/\varepsilon)$ and Ω_ε a small conical space-time domain that contains the ball $B_\varepsilon((M|\ln(\varepsilon)|, x_0))$ of $\mathbb{R}_s\times\mathbb{R}^d_x$ for which there holds

$$\frac{||f_\varepsilon||_{L^2(\Omega_\varepsilon)}}{||h_\varepsilon||^\alpha_{H^\sigma}} \approx \varepsilon^{(d+1)/2}\varepsilon^{-\alpha(M-\sigma)} \tag{2.1.11}$$

and a suitable choice of M leads to (2.1.4) in the simplified case $u_\varepsilon = f_\varepsilon$, as $\varepsilon\to 0$.

Through a careful analysis of the quasilinear system, Métivier proved that the nonlinear solution $\mathbf{u}_\varepsilon$ is close enough to $\mathbf{f}_\varepsilon$ in such a way that the growth (2.1.10) of the free solution $\mathbf{f}_\varepsilon$ in long time $O(|\ln(\varepsilon)|)$ passes on to solutions $\mathbf{u}_\varepsilon$, such that

$$|\mathbf{u}_\varepsilon(s,x,\theta)| \gtrsim \varepsilon^M e^{\gamma_0 s} \tag{2.1.12}$$

in a whole neighborhood of $(s,x) = (M|\ln(\varepsilon)|, x_0)$. This estimate from below leads finally to (2.1.4).

In this sketch of analysis, we see in particular that the (projection over the temporal coordinate of) the existence domain Ω_ε introduced in Section 2.1.2 must be large enough to contain time intervals $[0, M|\ln(\varepsilon)|]$. In Gevrey spaces, this domain must be much larger, see Section 2.1.3.

Existence of solutions via a long-time Cauchy-Kovalevskaya result

In order to show that nonlinear solution $\mathbf{u}_\varepsilon$ of equation (2.1.7) actually exists for sufficiently long time $O(M|\ln(\varepsilon)|)$, Métivier proved a long-time Cauchy-Kovalevskaya theorem using techniques of majoring series developed by Wagschal in [Wag79] for the resolution of the nonlinear Goursat problem. A presentation of the method can also be found in [Car61], and is developed extensively in Section 2.4.1.

For formal series $\phi(x) = \sum_{k\in\mathbb{N}^d} \phi_k x^k$ and $\psi(x) = \sum_{k\in\mathbb{N}^d} \psi_k x^k$, with $\psi_k \geq 0$, we define the relation

$$\phi \prec \psi \quad \Longleftrightarrow \quad |\phi_k| \leq \psi_k \ , \ \forall k \in \mathbb{N}^d.$$

The method is based on the observation that, if ψ has convergence radius $R^{-1} > 0$ and $\phi \prec \psi$, then ϕ has a convergence radius at least equal to R^{-1}. Conversely, there are series of one variable $\Phi(z)$ with convergence radius equal to 1 that satisfy the following property: for any series ϕ with convergence radius less than R^{-1}, there is $C > 0$ such that $\phi \prec C\Phi(R\sum_j x_j)$. The norm of ϕ will be defined as the best constant C (see Definition 2.4.7). An example is $\Phi(z) = \frac{1}{1-z}$, which satisfies the previous property thanks to Cauchy's inequalities.

Based on those two observations, the method consists in shifting the focus from ϕ to Φ. The key is that Φ can be taken to be much simpler than the original, typically unknown, series. In this paper we choose Φ with convergence radius equal to one and also such that $\Phi^2 \prec \Phi$ (see point 4 in Lemma 2.4.3 in Section 2.4.1).

Now assume that we are given an initial datum $u(0,\cdot)$ in (2.1.7) such that $u(0,x) \prec \Phi(R\sum_j x_j)$. The Cauchy-Riemann operator $\partial_t + i\partial_x$ provides the simplest example of an elliptic Cauchy problem. On this example the radius of analyticity decays linearly in time: the datum u with $\hat{u}(0,\xi) = e^{-R^{-1}|\xi|}$ generates the solution $\hat{u}(t,\xi) = e^{-(R^{-1}-t)\xi}$, for $t > 0$ and $\xi > 0$. It makes sense to assume similarly a linearly decaying radius of convergence for the solutions to our elliptic problems. Thus after comparing $u(0)$ to $\Phi(R\sum_j x_j)$, we will compare $u(s)$ to $\Phi(R\sum_j x_j + \varepsilon\rho s)$, where R and ρ are parameters to be specified later. Note that the series $\Phi(R\sum_j x_j + \varepsilon\rho s)$ has converging radius $R^{-1}(1-\varepsilon\rho s)$, which is non zero for $s < (\varepsilon\rho)^{-1}$; this is hence the *maximal time of regularity* for the solutions.

For simplicity of exposition, consider equation (2.1.7) with source term satisfying $G(\varepsilon\mathbf{u}) \equiv \varepsilon\sum_j A_j(\varepsilon s, x, \vec{u}_0)\partial_{x_j}\mathbf{u}$, and with $A_0 \equiv 0$. The right-hand side of (2.1.7) reduces then to

$$\int_0^s \varepsilon \sum_j A_j(\varepsilon s', x, \vec{u}_0)\partial_{x_j}\mathbf{u}(s')\, ds'. \tag{2.1.13}$$

By assumption of analyticity of the A_j, we may control the series $A_j(\varepsilon s', x, \vec{u}_0)$ by the model $\Phi(R\sum_j x_j + \varepsilon\rho s)$, up to a multiplicative constant. Then (2.1.13) is controlled, in the sense of the binary relation $\prec$ and up to a multiplicative constant, by

$$\begin{aligned}
&\int_0^s \varepsilon\Phi(R\sum_j x_j + \varepsilon\rho s')\sum_j \partial_{x_j}\Phi\left(R\sum_j x_j + \varepsilon\rho s'\right) ds' \\
&\prec \int_0^s \varepsilon R\Phi(R\sum_j x_j + \varepsilon\rho s')\Phi'\left(R\sum_j x_j + \varepsilon\rho s'\right) ds' \\
&\prec \int_0^s \varepsilon R\Phi'\left(R\sum_j x_j + \varepsilon\rho s'\right) ds' \\
&\prec R\rho^{-1}\Phi\left(R\sum_j x_j + \varepsilon\rho s\right).
\end{aligned}$$

Above, we used $2\Phi\Phi' \prec \Phi'$, a consequence of $\Phi^2 \prec \Phi$ (the relation $\prec$ is compatible with derivation, see Lemma 2.4.3). We observed above the phenomenon of regularization (of ∂_{x_j}) by integration in time, as in [Uka01]. The "error" (2.1.13) is controlled at a cost of $R\rho^{-1}$.

To conclude to the existence of the family of analytic solutions $\mathbf{u}_\varepsilon$ exhibiting the growth (2.1.12) on sufficiently long time $O(M|\ln(\varepsilon)|)$, Métivier compared the maximal time of regularity $(\varepsilon\rho)^{-1}$, which then has to be greater than the instability time $M|\ln(\varepsilon)|$. This implies some constraints on R and ρ, and finally on the domain of existence Ω_ε. We will not go into more detail at this point, as those constraints will appear in the Gevrey analysis too.

2.1.3 Extension to Gevrey spaces

The aim of this article is to prove the same kind of Hölder ill-posedness as in [Mét05], under the assumption of analyticity of the coefficients of the A_j. But whereas [Mét05] holds in Sobolev spaces, we prove here instability in Gevrey spaces[1]. Following Métivier's method, we construct a family of solutions $(u_\varepsilon)_\varepsilon$ that satisfies

$$\lim_{\varepsilon\to 0} \frac{||u_\varepsilon||_{L^2(\Omega)}}{||u_\varepsilon(0)||^\alpha_{G^\sigma(B_0)}} = +\infty \tag{2.1.14}$$

where the Gevrey space $G^\sigma(B_0)$ is precisely defined in Section 2.2.1, with B_0 a ball of $\mathbb{R}^d$ containing the distinguished point x_0. Our goal in this Section is to informally describe the specific difficulties posed by the analysis in Gevrey spaces.

On the time of instability in Gevrey spaces

We first need to find a suitable replacement for the small coefficient ε^M of $\mathbf{h}_\varepsilon$ defined in (2.1.9) in the Sobolev framework. Indeed, the highly oscillating function $e^{ix\cdot\xi_0/\varepsilon}$ has Sobolev norm $||e^{ix\cdot\xi_0/\varepsilon}||_{H^\sigma(B_0)} \approx \varepsilon^{-\sigma}$ whereas the Gevrey norm satisfies (see Definition 2.2.1 and Lemma 2.3.3) $||e^{ix\cdot\xi_0/\varepsilon}||_{G^\sigma(B_0)} \approx e^{\varepsilon^{-\sigma}}$. Appropriate initial data are both small and highly oscillating. Thus we replace (2.1.9) by

$$h_\varepsilon = e^{-\varepsilon^{-\delta}}\left(e^{\mp ix\cdot\xi_0/\varepsilon}\vec{e}_\pm\right) \quad , \quad \mathbf{h}_\varepsilon := e^{-\varepsilon^{-\delta}}\left(e^{i\mp\theta}\vec{e}_\pm\right) \tag{2.1.15}$$

with $\sigma < \delta$. At the end of the analysis, we expect (2.1.12) to be replaced by

$$|\mathbf{u}_\varepsilon(s,x,\theta)| \gtrsim e^{-\varepsilon^{-\delta}} e^{\gamma_0 s}. \tag{2.1.16}$$

This leads to a typical observation time $\varepsilon^{-\delta}$. This is the time for which the time exponential growth associated with the ellipticity counterbalances the very small initial amplitude. This observation time is far bigger than the typical Sobolev time $O(|\ln(\varepsilon)|)$ described above in Section 2.1.2. Note that the limitation $\sigma < \delta$ ensures at least formally that the ratio (2.1.11) in Gevrey spaces G^σ diverges as $\varepsilon \to 0$ (see Remark 2.3.4).

[1] This has been suggested by Jeffrey Rauch, whom the author thanks warmly.

On the control of linear errors over long times

Typically the estimates for $G(\varepsilon\mathbf{u})$ (with notation introduced in (2.1.6)), which comprises both linear and nonlinear error terms, degrade over time. This is problematic in view of the resolution of the fixed point equation (2.1.7). By definition of A_0 in (2.1.2), the linear error comprises term

$$(\sum_j A_j(\varepsilon s, x, \varepsilon\mathbf{u})\xi_{0,j} - A_0)\partial_\theta \mathbf{u} \approx (\varepsilon s + |x - x_0| + \varepsilon\mathbf{u})\partial_\theta \mathbf{u}.$$

Suppose now, for simplicity of exposition, that $G(\varepsilon\mathbf{u}) = \varepsilon s\partial_\theta\mathbf{u}$, and recall that $s = O(\varepsilon^{-\delta})$ according to the sketch of analysis of Section 2.1.3. Suppose in addition that the linear bound (2.1.8) holds, and that we have an a priori control of the Fourier mode $n = -1$ of the solution $\mathbf{u}$ with a growth rate that is equal to the linear growth rate

$$|\mathbf{u}_{-1}(s)| \lesssim e^{-\varepsilon^{-\delta}} e^{\gamma_0 s}.$$

The amplitude $e^{-\varepsilon^{-\delta}}$ is the one previously discussed in Section 2.1.3. Then equation (2.1.7) for the Fourier mode $n = -1$ reduces to

$$\mathbf{u}_{-1}(s) - e^{-\varepsilon^{-\delta}} e^{\gamma_0 s}\vec{e}_+ = \int_0^s e^{-i(s-s')A_0}\left(\varepsilon s'(-i)\mathbf{u}_{-1}(s')\right) \mathrm{d}s'$$

where $\vec{e}_+$ is the eigenvector of A_0 associated to the eigenvalue with imaginary part $i\gamma_0$. For the right-hand side, we have the estimate:

$$\begin{aligned} \left|\int_0^s e^{-i(s-s')A_0}\left(\varepsilon s'(-i)\mathbf{u}_1(s')\right) \mathrm{d}s'\right| &\lesssim \int_0^s e^{\gamma_0(s-s')}\left(\varepsilon s' e^{-\varepsilon^{-\delta}} e^{\gamma_0 s'}\right) \mathrm{d}s' \\ &\lesssim \frac{1}{2}\varepsilon s^2 e^{-\varepsilon^{-\delta}} e^{\gamma_0 s} \end{aligned} \tag{2.1.17}$$

thanks to the upper bound (2.1.8). Hence $\mathbf{u}_{-1}(s)$ would satisfy (2.1.16) if $\varepsilon s^2 = o_{\varepsilon\to 0}(1)$ for any $s \in [0, \varepsilon^{-\delta})$, which would lead to the stringent constraint on the Gevrey index $\sigma < \delta < 1/2$.

Thus we need to consider the varying-coefficient operator $\sum_j A_j(\varepsilon s, x, \vec{u}_0)\xi_{0,j}\partial_\theta$, as opposed to [Mét05] where the constant-coefficient operator $A_0\partial_\theta$ was considered.

On linear growth bounds

As discussed in Section 2.1.3, we need to work with the varying-coefficient operator

$$\sum_j A_j(\varepsilon s, x, \vec{u}_0)\xi_{0,j}\partial_\theta.$$

We introduce first the propagator $U(s', s, x, \theta)$ which solves

$$\partial_s U(s', s, x, \theta) - \sum_j A_j(\varepsilon s, x, \vec{u}_0)\xi_{0,j}\,\partial_\theta\, U(s', s, x, \theta) \quad , \quad U(s', s', x, \theta) \equiv \mathrm{Id}.$$

As $\sum_j A_j(\varepsilon s, x, \vec{u}_0)\xi_{0,j}$ does not depend on θ, the Fourier coefficients $U_n(s', s, x)$ of the propagator satisfies the ODE

$$\partial_s U_n(s', s, x) - in \sum_j A_j(\varepsilon s, x, \vec{u}_0)\xi_{0,j}\, U_n(s', s, x) \quad , \quad U_n(s', s', x) \equiv \mathrm{Id}.$$

Then $U(\theta)$ acts diagonally on each Fourier component. Note that in the autonomous case $\sum_j A_j(\varepsilon s, x, \vec{u}_0)\xi_{0,j} \equiv \sum_j A_j(0, x, \vec{u}_0)\xi_{0,j}$, the propagator satisfies

$$U(s', s, x, \theta) = \exp\left((s - s') \sum_j A_j(0, x, \vec{u}_0)\xi_{0,j}\, \partial_\theta \right).$$

Using the propagator $U(s, s, x, \theta)$, fixed point equation (2.1.7) is replaced by

$$\mathbf{u}(s, x, \theta) = \mathbf{f}(s, x, \theta) + \int_0^s U(s', s, x, \theta) G(\varepsilon \mathbf{u}(s', x, \theta)) ds' \tag{2.1.18}$$

where $\mathbf{f}(s, x, \theta) = U(0, s, x, \theta)\mathbf{h}_\varepsilon(\theta)$ is the free solution, with $\mathbf{h}_\varepsilon$ defined in (2.1.15).

For the n-th Fourier coefficient $U_n(s', s, x)$ of the propagator, the derivation of bounds is described for instance in Section 4 of [LNT17]. Eigenvalues may cross at the distinguished point $(0, x_0)$. In particular, eigenvalues and eigenprojectors may not be smooth, although eigenvalues are continuous. Since we do not want to formulate any additional assumption on the symbol besides ellipticity (although see Section 2.1.3 below and Theorem 2), this forces us, in the derivation of upper bounds of $U_n(s', s, x)$, to resort to the procedure of approximate trigonalization described for instance in [LNT17].

In this procedure, a small error is produced in the rate of growth. On one side, an upper bound

$$\left|U_n(s', s, x)\right| \lesssim \omega^{-(m-1)} e^{|n|(s-s')(\mathrm{Im}\,\lambda_0 + R^{-1} + \varepsilon \underline{s} + \omega)} \tag{2.1.19}$$

is achieved, where λ_0 is an eigenvalue of A_0 with positive imaginary part which is maximal among the other eigenvalues, and m is the algebraic multiplicity of λ_0 in the spectrum. In (2.1.19) the parameter $\omega > 0$ is associated with the trigonalization error. The optimal choice of ω is described below in Section 2.1.3. The bound (2.1.19) holds for x in $B_{R^{-1}}(x_0)$ and s in $(0, \underline{s})$, where R^{-1} is the convergence radius and $\underline{s}$ the final time of observation. This is made precise in Lemma 2.3.1.

On the other side, the free solution satisfies a bound of the form

$$|\mathbf{f}_\varepsilon(s, x, \theta)| \gtrsim \omega^{-(m-1)}\, e^{-\varepsilon^{-\delta}} e^{s(\mathrm{Im}\,\lambda_0 - r - \varepsilon \underline{s} - \omega)} \tag{2.1.20}$$

for $(s, x) \in (0, \underline{s}) \times B_r(x_0)$ with r small. This is made precise in Lemma 2.3.2.

On the endgame

As we did in Section 2.1.3, suppose now that there holds $G(\varepsilon \mathbf{u}) = \varepsilon \sum_j A_j(\varepsilon s, x\vec{u}_0)\partial_{x_j}\mathbf{u}(s)$ and the linear bound (2.1.19). Suppose also that we have an a priori control of the Fourier mode $n = 1$ of the solution $\mathbf{u}$ with a growth rate that is equal to the linear growth rate

$$|\mathbf{u}_1(s)| \lesssim e^{-\varepsilon^{-\delta}} \omega^{-(m-1)} e^{(s-s')(\mathrm{Im}\,\lambda_0 + R^{-1} + \varepsilon \underline{s} + \omega)}. \tag{2.1.21}$$

In view of bound (2.1.19) and equation (2.1.18), there holds then for the Fourier mode $n = 1$ the bound

$$|\mathbf{u}_1(s) - \mathbf{f}_1(s)| \lesssim \int_0^s \omega^{-(m-1)} e^{(s-s')(\operatorname{Im}\lambda_0 + R^{-1} + \varepsilon\underline{s} + \omega)} \varepsilon \sum_j A_j(\varepsilon s', x\vec{u}_0)\partial_{x_j}\mathbf{u}(s')ds'.$$

Thanks to the majoring series method explained in Section 2.1.2 and based on (2.1.21) , we may expect to bound the above by

$$|\mathbf{u}_1(s) - \mathbf{f}_1(s)| \lesssim e^{-\varepsilon^{-\delta}} \omega^{-2(m-1)} e^{s(\operatorname{Im}\lambda_0 + R^{-1} + \varepsilon\underline{s} + \omega)} R\rho^{-1}. \tag{2.1.22}$$

To end the proof, it would suffice then to show that $\mathbf{u}_1$ has the same bound from below as $\mathbf{f}_1$ in (2.1.20). This is the case if the right-hand side of (2.1.22) satisfies

$$e^{-\varepsilon^{-\delta}} \omega^{-2(m-1)} e^{s(\operatorname{Im}\lambda_0 + R^{-1} + \varepsilon\underline{s} + \omega)} R\rho^{-1} \ll \omega^{-(m-1)}\, e^{-\varepsilon^{-\delta}} e^{s(\operatorname{Im}\lambda_0 - r - \varepsilon\underline{s} - \omega)} \tag{2.1.23}$$

for all $s \in (0, \underline{s})$, where $\ll$ is defined in (2.1.32). This is equivalent to

$$\omega^{-(m-1)}\, e^{\underline{s}(R^{-1} + r + \varepsilon\underline{s} + \omega)} R\rho^{-1} \ll 1. \tag{2.1.24}$$

As explained in Section 2.1.3, the final time $\underline{s}$ is of order $\varepsilon^{-\delta}$. In order for (2.1.24) to be satisfied, the argument of the exponential should be at most of order 1 as ε goes to 0. Hence R^{-1}, r and ω are chosen to be less than ε^{δ}. Note that we also get once again the constraint $\varepsilon\underline{s}^2 < 1$, which brings back the limitation $\sigma < \delta < 1/2$ on the Gevrey index.

Besides (2.1.24), another constraint shows up in the analysis. Recall that we work with the majoring series model $\Phi(R\sum_j x_j + \varepsilon\rho s)$. Its domain of analyticity is the conical space-time domain $\{(s,x) \,|\, R\sum_j |x_j| + \varepsilon\rho s < 1\}$. As the time of instability $\underline{s}$ is of order $\varepsilon^{-\delta}$, in order to see the instability the maximal regularity time $(\varepsilon\rho)^{-1}$ has to be greater than $\varepsilon^{-\delta}$. Hence another constraint

$$\varepsilon^{1-\delta} \ll \rho^{-1}. \tag{2.1.25}$$

Since ω and R^{-1} are of order ε^{δ}, we rewrite constraint (2.1.24) as $\rho^{-1} \ll \varepsilon^{(m-1)\delta} R^{-1}$ and then as

$$\rho^{-1} \ll \varepsilon^{m\delta}.$$

Finally we end up with a consistency inequality $\varepsilon^{1-\delta} \ll \varepsilon^{m\delta}$, equivalent to the limitation $\delta < 1/(m+1)$ of the Gevrey index. This is our principal result, detailed in Theorem 1.

On proving instability for higher Gevrey indices

We saw above in Section 2.1.3 that, in the general case, the consideration of the varying-coefficient operator $\sum_j A_j(\varepsilon\tau, x, \vec{u}_0)\xi_{0,j}\partial_\theta$ does not free us from the constraint $\sigma < 1/2$. Indeed, as discussed in Sec 2.1.3, we actually need to impose $\sigma < 1/(m+1)$, where $m \geq 1$ is the algebraic multiplicity of λ_0 in the spectrum.

We describe here a situation in which we improve the limiting Gevrey index.

Assume finally that (2.1.19) and (2.1.20) can be replaced by

$$\left|U_n(s', s)\right| \lesssim e^{|n|(s-s')(\operatorname{Im}\lambda_0 + \omega)} \tag{2.1.26}$$

and

$$|\mathbf{f}_\varepsilon(s,x,\theta)| \gtrsim e^{-\varepsilon^{-\delta}} e^{s(\operatorname{Im}\lambda_0 - \varepsilon^2 s^2 - r - \omega)} \tag{2.1.27}$$

respectively. Following the previous computations, we may then replace (2.1.23) by

$$e^{-\varepsilon^{-\delta}} e^{s(\operatorname{Im}\lambda_0+\omega)} R\rho^{-1} \ll e^{-\varepsilon^{-\delta}} e^{s(\operatorname{Im}\lambda_0 - \varepsilon^2 s^2 - r - \omega)}$$

and we finally get, instead of (2.1.24), the new constraint

$$e^{\underline{s}(\varepsilon^2 \underline{s}^2 + r + \omega)} R\rho^{-1} \ll 1.$$

It can be fulfilled for any δ in $(0, 2/3)$, which implies instability in Gevrey spaces G^σ with $\sigma < 2/3$. We show in Sections 2.2 and 2.3 that assumptions of maximality and semi-simplicity for the most unstable eigenvalue lead to (2.1.26) and (2.1.27). These correspond to the assumptions of Theorem 3.

Notations

- For all $z \in \mathbb{C}^m$ and $k \in \mathbb{N}^m$, we put

$$z^k = \prod_{i=1,\dots,m} z_i^{k_i} \tag{2.1.28}$$

- For all $k \in \mathbb{N}^m$

$$\binom{k_1 + \cdots + k_m}{k_1, \dots, k_m} = \frac{(k_1 + \cdots + k_m)!}{\prod_{i=1,\dots,m} k_i!} \tag{2.1.29}$$

- For all m and $i \in \{1, \dots, m\}$, we denote 1_i the m-uple with all coefficients null but the i-th:

$$1_i = (0, \dots, 0, 1, 0, \dots, 0) \tag{2.1.30}$$

- For all reals A and B we note

$$A \lesssim B \tag{2.1.31}$$

  if there is some constant independent of ε such that

$$A \leq CB.$$

- For any functions A and B of ε, we denote

$$A \ll B \quad \Longleftrightarrow \quad A = o_{\varepsilon \to 0}(B). \tag{2.1.32}$$

- For $r > 0$ and $x_0 \in \mathbb{R}^d$ we denote

$$B_r(x_0) = \left\{ x \in \mathbb{R}^d \;\middle|\; |x - x_0| < r \right\}. \tag{2.1.33}$$

2.2 Main assumptions and results

2.2.1 Definitions: Hölder well-posedness in Gevrey spaces

We recall the definition of Gevrey functions on an open set B of $\mathbb{R}^d$:

Definition 2.2.1 (Gevrey functions). *Let $\sigma \in (0,1)$. We define $G^\sigma(B)$ as the set of C^∞ functions f on B such that, for all compact $K \subset B$ there are constants $C_K > 0$ and $c_K > 0$ that satisfy*

$$|\partial^\alpha f|_{L^\infty(K)} \leq C_K c_K^{|\alpha|} |\alpha|!^{1/\sigma} \quad , \quad \forall \alpha \in \mathbb{N}^d. \tag{2.2.1}$$

We then define a family of norms on $G^\sigma(B)$, for all compact $K \subset B$ and $c > 0$ by

$$||f||_{\sigma,c,K} = \sup_\alpha |\partial^\alpha f|_{L^\infty(K)} c^{-|\alpha|} |\alpha|!^{-1/\sigma}. \tag{2.2.2}$$

For an introduction to Gevrey spaces and their properties, we refer to the book of Rodino [Rod93]. We introduce also space-time conical domains centered on $(0, x_0) \in \mathbb{R} \times \mathbb{R}^d$.

Definition 2.2.2 (Conical domains). *For $x_0 \in \mathbb{R}^d$, $R > 0$, $\rho > 0$ and $t \geq 0$ we define the set*

$$\Omega_{R,\rho,t}(x_0) = \left\{ x \in \mathbb{R}^d \;\middle|\; R|x - x_0|_1 + \rho t < 1 \right\} \tag{2.2.3}$$

with $|x|_1 = \sum_{j=1,\dots,d} |x_j|$ the L^1 norm on $\mathbb{R}^d$. Note that for all $t \geq \rho^{-1}$, $\Omega_{R,\rho,t}(x_0) = \emptyset$. We also denote

$$\Omega_{R,\rho}(x_0) = \bigcup_{t \geq 0} \{t\} \times \Omega_{R,\rho,t}(x_0) = \left\{ (t,x) \in \mathbb{R} \times \mathbb{R}^d \;\middle|\; 0 \leq t < \rho^{-1},\ R|x - x_0|_1 + \rho t < 1 \right\}. \tag{2.2.4}$$

Note that $\Omega_{R,\rho,t}$ is decreasing for the inclusion as a function of R, ρ and t. In particular, $\Omega_{R,0,0}(x_0)$ is $B_{R^{-1}}(x_0)$.

The question is whether the Cauchy problem (2.1.1) is well-posed in Gevrey spaces or not, in the following sense

Definition 2.2.3 (Hölder well-posedness). *We say that* (2.1.1) *is Hölder well-posed in G^σ locally around $x_0 \in \mathbb{R}^d$ if there are constants $r_0 > r_1 > 0$, $c > 0$, $C_{in} > 0$, C_{fin}, $\rho > 0$, $\alpha \in (0,1)$ such that for any h in $G^\sigma(B_{r_0}(x_0))$ with*

$$||h||_{\sigma,c,K} \leq C_{in} \qquad \forall K \text{ compact of } B_{r_0}(x_0)$$

and all $R > r_1^{-1}$ the Cauchy problem (2.1.1) *associated to h has a unique solution $u(t,x)$ in $C^1(\overline{\Omega}_{R,\rho}(x_0))$ with $|u|_{L^2(\Omega_{R,\rho}(x_0))} \leq C_{fin}$ and if moreover, given h_1 and h_2 in $G^\sigma(B_{r_0}(x_0))$ the corresponding solutions u_1 and u_2 satisfy the estimate for all $R > r_1^{-1}$ and K compact subset of $B_{r_0}(x_0)$*

$$|u_1 - u_2|_{L^2(\Omega_{R,\rho}(x_0))} \lesssim ||h_1 - h_2||^\alpha_{\sigma,c,K}.$$

2.2.2 Assumptions

We define the principal symbol evaluated at a distinguished frequency $\xi_0 \in \mathbb{R}^d$ by

$$A(t,x,u) = \sum_j A_j(t,x,u)\xi_{0,j} \quad , \forall (t,x,u) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^N. \tag{2.2.5}$$

Assumption 2.2.4. *We assume that for some $x_0 \in \mathbb{R}^d$ and $\vec{u}_0 \in \mathbb{R}^N$, the spectrum of $A(0,x_0,\vec{u}_0)$ is not real:*

$$\mathrm{Sp}A(0,x_0,\vec{u}_0) \not\subseteq \mathbb{R}. \tag{2.2.6}$$

That is, the principal symbol A is initially elliptic.

Notation 2.2.5. *We denote then*

$$A_0 = A(0,x_0,\vec{u}_0) \tag{2.2.7}$$

which is a constant matrix with non-real spectrum by (2.2.6)*. Among the nonreal eigenvalues of A_0, we denote λ_0 the one with maximal positive imaginary part, denoted γ_0. We denote $\vec{e}_+$ the associated eigenvector. We denote also*

$$\underline{A}(t,x) = A(t,x,\vec{u}_0). \tag{2.2.8}$$

Up to translations in x and u, which do not affect our assumptions, and by homogeneity in ξ, we may assume

$$x_0 = 0 \quad , \quad \vec{u}_0 = 0 \quad , \quad \xi_0 \in \mathbb{S}^{d-1}. \tag{2.2.9}$$

Under Assumption 2.2.4 alone, we prove instability for the Cauchy problem (2.1.1) in some Gevrey indices (Theorem 1 in Section 2.2.3 below). We now formulate additional assumptions which yield instability for higher Gevrey spaces (Theorems 2 and 3 below).

Assumption 2.2.6. *For some $x_0 \in \mathbb{R}^d$ and $\xi_0 \in \mathbb{S}^{d-1}$, the matrix A_0 has an eigenvalue λ_0 such that there holds $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$, and $\mathrm{Im}\,\lambda_0 > \mathrm{Im}\,\mu$, for any other eigenvalue μ of A_0. Besides, the eigenvalue λ_0 is semisimple (which means algebraic and geometric multiplicities coincide) and belongs to a branch of semisimple eigenvalues of A. Finally, $(0,x_0,\lambda_0)$ is not a coalescing point in the spectrum of $\underline{A}$.*

We denote P_0 the eigenprojector of A_0 associated with λ_0, and A_0^{-1} the partial inverse of A_0, defined by $P_0 A_0^{-1} = 0$, $A_0 A_0^{-1} = \mathrm{Id} - P_0$. We also denote $(t,x) = (x_0,\dots,x_d)$, so that $\partial_0 = \partial_t$, $\partial_j = \partial_{x_j}$.

Remark 2.2.7. *The non-coalescing assumption* 2.2.6 *implies (see* [Kat66]*, or Corollary* 2.2 *of* [Tex17]*) that there is a smooth (actually, analytical) branch λ of eigenvalues of $\underline{A}$ such that $\lambda(0,x_0) = \lambda_0$. The corresponding local eigenprojector $\underline{P}$ is smooth as well. The local semisimplicity assumption means that $\underline{A}\,\underline{P} = \lambda\underline{P}$, that is, in restriction to the eigenspace associated with λ, the symbol A is diagonal. A sufficient condition for semisimplicity is algebraic simplicity of the eigenvalue.*

Assumption 2.2.8. *With notation P_0 and A_0^{-1} introduced just above Remark* 2.2.7*,*

(i) there holds $P_0\partial_j\underline{A}(0,x_0)P_0 = 0$, *for all* $j \in \{0,\dots,d\}$.

Under condition (i), the matrix

$$P_0\partial_i\underline{A}A_0^{-1}\partial_j\underline{A}P_0 + P_0\partial_j\underline{A}A_0^{-1}\partial_i\underline{A}P_0 + P_0\partial_{ij}^2\underline{A}P_0 \tag{2.2.10}$$

(where derivatives of $\underline{A}$ are evaluated at $(0,x_0)$) has only non-zero eigenvalue (see [Kat66], or Proposition 2.6 of [Tex04]), which we denote μ_{ij}.

(ii) The matrix $(\operatorname{Im}\mu_{ij})_{0\le i,j\le d}$ *is negative definite.*

Remark 2.2.9. *Under Assumption* 2.2.6, *Assumption* 2.2.8 *implies (see* [Kat66], *or Proposition* 2.6 *of* [Tex04]*) that the Hessian of* $\operatorname{Im}\lambda$ *at* $(0,x_0)$ *is negative definite, hence* $(0,x_0)$ *is a local maximum, in space-time, for* $\operatorname{Im}\lambda$.

Assumption 2.2.10. *We assume that* $f(t,x,u)$ *is quadratic in* u *locally around* $u=\vec{u}_0$, *that is*

$$\partial_u f(t,x,u)\big|_{u=\vec{u}_0=0} \equiv 0 \tag{2.2.11}$$

2.2.3 Statement of the results

In the statement below we use notations introduced in Definitions 2.2.1 and 2.2.2.

Theorem 1. *Under Assumptions* 2.2.4 *and* 2.2.10, *the Cauchy problem* (2.1.1) *is not Hölder well-posed in Gevrey spaces* G^σ *for all* $\sigma \in (0,1/(m+1))$ *where* m *is the algebraic multiplicity of* λ_0. *That is for all* $c>0$, K *compact of* $\mathbb{R}^d$ *and* $\alpha \in (0,1]$, *there are sequences* $R_\varepsilon^{-1}\to 0$ *and* $\rho_\varepsilon^{-1}\to 0$, *a family of initial conditions* $h_\varepsilon \in G^\sigma$ *and corresponding solutions* u_ε *of the Cauchy problem on domains* $\Omega_{R_\varepsilon,\rho_\varepsilon}(x_0)$ *such that*

$$\lim_{\varepsilon\to 0} ||u_\varepsilon||_{L^2(\Omega_{R_\varepsilon,\rho_\varepsilon}(x_0))}/||h_\varepsilon||^\alpha_{\sigma,c,K} = +\infty. \tag{2.2.12}$$

The time of existence of the solutions u_ε *is at least of order* $\varepsilon^{1-\sigma}$.

We prove the instability for a larger band of Gevrey indices under stronger assumptions. First, the semisimplicity and non-coalescing Assumption 2.2.6 allows for a critical index equal to 1/2:

Theorem 2. *Under Assumptions* 2.2.6 *and* 2.2.10, *the result of Theorem* 1 *holds for any Gevrey index* σ *in* $(0,1/2)$.

Second, under Assumption 2.2.6, the null condition (i) and the sign condition (ii) in Assumption 2.2.8 allow for the critical index to go from 1/2 up to 2/3:

Theorem 3. *Under Assumptions* 2.2.6, 2.2.8 *and* 2.2.10, *the result of Theorem* 1 *holds for any Gevrey index* σ *in* $(0,2/3)$.

The rest of the paper is devoted to the proof of Theorems 1, 2 and 3.

Remark 2.2.11. *Higher-order null and sign conditions allow for a greater critical index. Precisely, under Assumption* 2.2.6, *if* $(0,x_0)$ *is a local maximum for* $\operatorname{Im}\lambda$, *and if there holds* $\lambda(\varepsilon s,x_0)-\lambda(0,x_0) = O(\varepsilon s)^{2k-1}$, *then our proof implies ill-posedness with a critical Gevrey index equal to* $2k/(2k+1)$. *These null and sign conditions can be expressed in terms of derivatives of* $\underline{A}$, *the partial inverse* A_0^{-1} *and the projector* P_0, *see* [Kat66], *or Remark* 2.7 *of* [Tex04]. *See also Remark* 2.6.5.

2.3 Highly oscillating solutions and reduction to a fixed point equation

2.3.1 Preparation of the equation

We want to compare two solutions of (2.1.1) with initial data h_1 and h_2 satisfying both

$$h_i(x=0)=0 \qquad \text{for } i=1,2$$

to fit with $\vec{u}_0=0$ in (2.2.9). We can choose h_1 analytic, which lead by Cauchy-Kovalevskaya theorem to an analytic solution u_1 in some small neighborhood of $(0,0)\in\mathbb{R}_t\times\mathbb{R}^d_x$. Then changing u into $u-u_1$ in (2.1.1) we get a new Cauchy problem

$$\partial_t u=\sum_j A_j(t,x,u)\partial_{x_j}u+F(t,x,u)u\ , \quad u(0,x)=h(x) \tag{2.3.1}$$

with $F(t,x,u)\in\mathbb{R}^{N\times N}$ is also analytic, by analyticity of f and u_1. We consider for h small analytical functions satisfying $h_{|x=0}=0$, as perturbations of the trivial datum $h\equiv 0$.

2.3.2 Highly oscillating solutions

As in [Mét05] we look for high oscillating solutions of (2.3.1) with the aim of seeing the expected growth. In this view we posit the following ansatz

$$u_\varepsilon(t,x)=\varepsilon\mathbf{u}(t/\varepsilon,x,x\cdot\xi/\varepsilon) \tag{2.3.2}$$

where the function $\mathbf{u}(s,x,\theta)$ is 2π-periodic in θ. We introduce for any analytical function $H(t,x,u)$ the compact notation

$$\mathbf{H}(s,x,\mathbf{u})=H\left(\varepsilon s,x,\varepsilon\mathbf{u}\right). \tag{2.3.3}$$

For $u_\varepsilon(t,x)$ to be solution of (2.3.1) it is then sufficient that $\mathbf{u}(s,x,\theta)$ solves the following equation

$$\partial_s\mathbf{u}=\mathbf{A}\,\partial_\theta\mathbf{u}+\varepsilon\left(\sum_j\mathbf{A}_j\partial_{x_j}\mathbf{u}+\mathbf{F}\,\mathbf{u}\right) \tag{2.3.4}$$

where we use the notation (2.3.3) for the $\mathbf{A}_j$ and $\mathbf{F}$, and A is defined by (2.2.5).

As we focus our study in a neighborhood of the distinguished point $(0,0)\in\mathbb{R}_t\times\mathbb{R}^d_x$ (recall that $x_0=0$), we rewrite now (2.3.4) as

$$\partial_s\mathbf{u}-\underline{\mathbf{A}}\partial_\theta\mathbf{u}=\mathbf{G}(s,x,\mathbf{u}) \tag{2.3.5}$$

where $\underline{\mathbf{A}}(s,x)=\underline{A}(\varepsilon s,x)$ in accordance with notation (2.3.3). We define the source term

$$\mathbf{G}=(\mathbf{A}-\underline{\mathbf{A}})\ \partial_\theta\mathbf{u}+\varepsilon\left(\sum_j\mathbf{A}_j\partial_{x_j}\mathbf{u}+\mathbf{F}\,\mathbf{u}\right) \tag{2.3.6}$$

using the notation (2.3.3).

2.3.3 Upper bounds for the propagator

To solve the Cauchy problem of the equation (2.3.5) with initial datum h_ε specified in Section 2.3.4, we first study the case $\mathbf{G} \equiv 0$, that is

$$\partial_s \mathbf{u}(s,x,\theta) - \underline{\mathbf{A}}(s,x)\partial_\theta \mathbf{u}(s,x,\theta) = 0. \tag{2.3.7}$$

Note that this equation is linear, non autonomous and non scalar. We define the matrix propagator $U(s',s,x,\theta)$ as the solution of

$$\partial_s U(s',s,x,\theta) - \underline{\mathbf{A}}(s,x)\partial_\theta U(s',s,x,\theta) = 0\ , \quad U(s',s',x,\theta) \equiv \mathrm{Id}. \tag{2.3.8}$$

and $U(s',s,x,\theta)$ is periodic in θ, following the ansatz (2.3.2).

Lemma 2.3.1 (Growth of the propagator). *The matrix propagator $U(s',s,x,\theta)$ satisfies the following growth of its Fourier modes in the θ variable:*

$$|U_n(s',s,x)| \lesssim \omega^{-(m-1)} \exp\left(\int_{s'}^{s} \gamma^\sharp(\tau\,;R,\omega)d\tau\,|n|\right) \quad , \quad \forall\, 0 \le s' \le s \text{ and } \forall\, n \in \mathbb{Z}. \tag{2.3.9}$$

- *Under Assumption* 2.2.4*, bound* (2.3.9) *holds with*

$$\gamma^\sharp(\tau\,;R,\omega) = \gamma_0 + \varepsilon\tau + R^{-1} + \omega \tag{2.3.10}$$

  where γ_0 is defined in Notation 2.2.5*, $m \ge 1$ is the algebraic multiplicity of λ_0. The bounds hold for $\omega > 0$ small enough, uniformly in x in the ball $B_{R^{-1}}(0)$.*

- *Under Assumption* 2.2.6*, bound* (2.3.9) *holds with $m = 1$ and*

$$\gamma^\sharp(\tau\,;R,\omega) = \gamma_0 + \varepsilon\tau + R^{-1} \tag{2.3.11}$$

  with $\omega = 0$, both uniformly in x in the ball $B_{R^{-1}}(0)$.

- *Under Assumptions* 2.2.6 *and* 2.2.8*, bound* (2.3.9) *holds with $\omega = 1$ and*

$$\gamma^\sharp(\tau\,;R,\omega) = \gamma_0 \tag{2.3.12}$$

  The bounds hold uniformly in x in the ball $B_{R^{-1}}(0)$.

In the framework of Assumption 2.2.4, the parameter ω is chosen in Proposition 2.6.2.

Proof. As $\underline{A}(t,x)$ does not depend on θ, equation (2.3.8) reads in Fourier transform in θ as

$$\partial_s U_n(s',s,x) - in\underline{A}(\varepsilon s,x)U_n(s',s,x)\ , \quad U_n(s',s,x) = \mathrm{Id}$$

where U_n is the n-th Fourier component of $U(\theta)$. That implies that operator $U(\theta)$ acts diagonally on each Fourier components.

The bounds (2.3.9) - (2.3.10) follow from elementary, and purely linear-algebraic, arguments detailed in Sections (4.2) and (4.3) of [LNT17].

The bounds (2.3.9) - (2.3.11) follow from a smooth partial diagonalization of symbol $\underline{A}$ over the eigenspace associated with λ. In particular, there is no diagonalization or trigonalization error, hence $m = 1$ in (2.3.9) and $\omega = 0$ in (2.3.11).

The bounds (2.3.9) - (2.3.12) follow from a smooth partial diagonalization as described above, and the fact that the imaginary part of λ is maximal at $(t, x) = (0, x_0)$, as described in Remark 2.2.9.

□

2.3.4 Free solutions

After getting the previous upper bounds for the propagator, we seek initial conditions h_ε that achieve the maximal growth. For this purpose, following again [Mét05] we introduce the following high-oscillating, small and well-polarized initial data

$$h_\varepsilon(x) = \varepsilon\, e^{-M(\varepsilon)} \mathrm{Re}\left(e^{-ix\cdot\xi_0/\varepsilon}\vec{e}_+ + e^{ix\cdot\xi_0/\varepsilon}\vec{e}_-\right) \tag{2.3.13}$$

which correspond in the ansatz (2.3.2) of high-oscillating solutions to

$$\mathbf{h}_\varepsilon(x,\theta) = e^{-M(\varepsilon)} \mathrm{Re}\left(e^{-i\theta}\vec{e}_+ + e^{i\theta}\vec{e}_-\right). \tag{2.3.14}$$

Here $\vec{e}_+$ is defined in Notation 2.2.5, and $\vec{e}_- = \overline{\vec{e}_+}$. The parameter $M(\varepsilon)$ is large in the limit $\varepsilon \to 0$, chosen such that the Gevrey norm of h_ε is small. We introduce also

$$\mathbf{f}_\varepsilon(s,x,\theta) = U(0,s,x,\theta)\mathbf{h}_\varepsilon(x,\theta) \tag{2.3.15}$$

which we call the free solution of equation (2.3.5) as it solves the equation for $\mathbf{G} \equiv 0$.

Growth of the free solution

Lemma 2.3.2 (Growth of the free solution)**.** *There holds*

$$|\mathbf{f}_\varepsilon(s,x,\theta)| \gtrsim \omega^{-(m-1)}\, e^{-M(\varepsilon)} \exp\left(\int_0^s \gamma^\flat(\tau\,; r,\omega)d\tau\right). \tag{2.3.16}$$

- *Under Assumption* 2.2.4*, bound* (2.3.16) *holds with*

$$\gamma^\flat(\tau\,; r,\omega) = \gamma_0 - \varepsilon\tau - r - \omega, \tag{2.3.17}$$

  pointwise in $(s,x,\theta) \in [0,\underline{s}) \times B_r(x_0) \times \mathbb{T}$.

- *Under Assumption* 2.2.6*, bound* (2.3.16) *holds with* $m = 1$ *and*

$$\gamma^\flat(\tau\,; r,\omega) = \gamma_0 - \varepsilon\tau - r, \tag{2.3.18}$$

  with $\omega = 0$*, pointwise in* $(s,x,\theta) \in [0,\underline{s}) \times B_r(0) \times \mathbb{T}$.

- *Under Assumptions* 2.2.6 *and* 2.2.8*, bound* (2.3.16) *holds with* $\omega = 1$ *and*

$$\gamma^\flat(\tau\,; r, \omega) = \operatorname{Im}\lambda(\varepsilon\tau, 0) - r. \tag{2.3.19}$$

pointwise in $(s, x, \theta) \in [0, \underline{s}) \times B_r(0) \times \mathbb{T}$.

Proof. Our choice of datum (2.3.13)-(2.3.15) allows an exact localization at the distinguished frequency ξ_0. Similarly to the proof of Lemma 2.3.1, the lower bounds follow from linear algebraic arguments detailed in [LNT17].

□

Smallness of the free solution and Gevrey index

The size of the Gevrey-σ norm of the initial data h_ε is linked to the exponent $M(\varepsilon)$ as shown by the following

Lemma 2.3.3. *For any* $\sigma \in (0, 1)$*,* $c > 0$ *and* K *a compact of* $\mathbb{R}^d$ *there holds*

$$||h_\varepsilon||_{\sigma,c,K} \lesssim \varepsilon \exp\left(-M(\varepsilon) + \frac{\varepsilon^{-\sigma}}{\sigma c^\sigma}\right). \tag{2.3.20}$$

We emphasize that the constant in the previous inequality does not depend on K*.*

Proof. First we have

$$\partial_x^k e^{\pm ix\cdot\xi_0/\varepsilon} = (\pm i\xi_0/\varepsilon)^k\, e^{\pm ix\cdot\xi_0/\varepsilon}\,, \quad \forall k \in \mathbb{N}^d\,, \quad \forall x \in \mathbb{R}^d$$

using notation (2.1.28) and then

$$|\partial_x^k e^{\pm ix\cdot\xi_0/\varepsilon}| \le C_d\, \varepsilon^{-|k|}\,, \quad \forall k \in \mathbb{N}^d\,, \quad \forall x \in \mathbb{R}^d$$

as $|\xi_0| = 1$, with $C_d > 0$ a constant depending only of the dimension d. So that for any compact K of $\mathbb{R}^d$ and by definition (2.3.13) of the initial data h_ε, there holds

$$c^{-|k|}|k|!^{-1/\sigma}|\partial_x^k h_\varepsilon|_{L^\infty(K)} \lesssim \varepsilon\, e^{-M(\varepsilon)}\, \varepsilon^{-|k|}\, c^{-|k|}|k|!^{-1/\sigma}\,, \quad \forall k \in \mathbb{N}^d.$$

By Definition 2.2.1 of the Gevrey norms, this implies

$$||h_\varepsilon||_{\sigma,c,K} \lesssim \varepsilon e^{-M(\varepsilon)} \sup_{k\in\mathbb{N}^d} \varepsilon^{-|k|}\, c^{-|k|}|k|!^{-1/\sigma}.$$

For any $t > 0$ we have

$$\frac{t^{|k|}}{|k|!} \le e^t \quad , \quad \forall t > 0\,, \quad \forall k \in \mathbb{N}^d$$

and note that the loss is smaller as $|k|$ is larger. This leads to

$$||h_\varepsilon||_{\sigma,c,K} \lesssim \varepsilon e^{-M(\varepsilon)} \sup_{k\in\mathbb{N}^d} \varepsilon^{-|k|}\, c^{-|k|} \left(t^{|k|}e^{-t}\right)^{-1/\sigma}$$

and then by putting $t = \varepsilon^{-\sigma}c^{-\sigma}$ into this last inequality, we finally obtain the inequality (2.3.20).

□

As we need h_ε to be small both in Gevrey-σ norm and in amplitude, we posit

$$M(\varepsilon) = \varepsilon^{-\delta}, \quad \delta \in (\sigma, 1). \tag{2.3.21}$$

Remark 2.3.4. *With the previous definition* (2.3.21)*, the initial data h_ε is exponentially small, both in Gevrey-σ norm and in absolute value. This last point is of importance, as we need h_ε to be small enough to see the exponential growth of the solution it generates in a sufficiently long time $T(\varepsilon)$ to be defined later. A constraint on this final time will lead to a constraint on the size $e^{-M(\varepsilon)}$ of h_ε, and then to the constraint $\sigma < \delta$ (see* (2.3.21)*) bearing on the admissible Gevrey regularity.*

2.3.5 Fixed point equation

Using the propagator $U(s', s, \theta)$, the free solution (2.3.13) and the Duhamel formula, we can express now (2.3.5) as the fixed point equation

$$\mathbf{u}(s, x, \theta) = \mathbf{f}_\varepsilon(s, x, \theta) + \int_0^s U(s', s, x, \theta)\mathbf{G}(s', \mathbf{u}(s', x, \theta))ds' \tag{2.3.22}$$

where $\mathbf{G}(\mathbf{u})$ is defined by (2.3.6). We denote the integral term

$$T(s, \mathbf{u}) = \int_0^s U(s', s)\mathbf{G}(s', \mathbf{u}(s'))ds' \tag{2.3.23}$$

which we split into three parts thanks to definition (2.3.6) like

$$\begin{aligned} T(s, \mathbf{u}) &= \int_0^s U(s', s) \left[(\mathbf{A} - \underline{\mathbf{A}})\, \partial_\theta \mathbf{u} + \varepsilon \left(\sum_j \mathbf{A}_j \partial_{x_j} \mathbf{u} + \mathbf{F}\,\mathbf{u} \right) \right] ds' \\ &= T^{[\theta]}(s, \mathbf{u}) + T^{[x]}(s, \mathbf{u}) + T^{[\mathbf{u}]}(s, \mathbf{u}) \end{aligned} \tag{2.3.24}$$

where we define

$$T^{[\theta]}(s, \mathbf{u}) = \int_0^s U(s', s)\ (\mathbf{A} - \underline{\mathbf{A}})\ \partial_\theta \mathbf{u}(s')ds' \tag{2.3.25}$$

$$T^{[x]}(s, \mathbf{u}) = \int_0^s U(s', s) \sum_j \left(\varepsilon \mathbf{A}_j(s', \mathbf{u}(s'))\right) \partial_{x_j} \mathbf{u}(s')ds' \tag{2.3.26}$$

$$T^{[\mathbf{u}]}(s, \mathbf{u}) = \int_0^s U(s', s)\ \left(\varepsilon \mathbf{F}(s', \mathbf{u}(s'))\right)\ \mathbf{u}(s')ds'. \tag{2.3.27}$$

2.3.6 Sketch of the proof

We have now reduced the initial question of finding a family of initial data h_ε generating a family of appropriately growing analytic solutions u_ε to the fixed point equation (2.3.22) for operator T. To find smooth solutions of this equation we have first to find a suitable functional space $\mathbb{E}$ with the following properties:

- The space $\mathbb{E}$ should be a Banach space to make use of the Banach fixed point theorem. Moreover functions of $\mathbb{E}$ should be smooth functions in variables (s, x, θ).

- The space $\mathbb{E}$ should be a Banach algebra equipped with norm $|||\cdot|||$ satisfying $|||\mathbf{uv}||| \leq |||\mathbf{u}|||\,|||\mathbf{v}|||$ as we deal with non linear terms $\mathbf{G}(\mathbf{u})$.

- We will need to precisely evaluate the action of derivation operators ∂_{x_j} and ∂_θ on $\mathbb{E}$. In an analytical framework, these are *a priori* not bounded operators, and as in [Uka01] and [Mét05] we should use time integration to get back boundedness in $\mathbb{E}$ with some loss in the bounds we should quantify.

- The space $\mathbb{E}$ should be invariant by the flow $U(s', s, x, \theta)$. In this view, we need estimates in $\mathbb{E}$ for the matrix flow $U_n(s', s, x)$.

- The operator T should be a contraction on $\mathbb{E}$ for well chosen parameters, and for small ε.

To this end, Section 2.4 will present the satisfying functional setting, and Section 2.5 will prove the contraction estimate for T.

In order to prove the Hadamard instability, the existence of solutions to the fixed point equations (2.3.22) is not sufficient. The key of the proof is to obtain for the solution $\mathbf{u}$ associated to $\mathbf{f}_\varepsilon$ the same kind of growth as $\mathbf{f}_\varepsilon$, as developed in Section 2.1.3, and this is the aim of Section 2.6. Finally, such a growth for $\mathbf{u}$ leads to the Hadamard instability of the Cauchy problem (2.3.1). This completes the proof of Theorems 1, 2 and 3 in Section 2.7.

2.4 Majoring series and functional spaces

2.4.1 Properties of majoring series

One aim of the paper is to construct a family of analytical solutions of the fixed point equation (2.3.22). We deal with functions of several variables: x, (s, x) or (s, x, u), and the question of analyticity of these functions with respect to all variables or only to some arises. In that purpose we consider formal series of μ variables, with complex coefficients that depend eventually on a parameter y in some open domain $\mathcal{O}$ of $\mathbb{C}^{\mu'}$. We denote such formal series

$$\phi(z, y) = \sum_{k \in \mathbb{N}^\mu} \phi_k(y) z^k \quad , \quad \phi_k(y) \in \mathbb{C}\,, \quad \forall\, k \in \mathbb{N}^\mu\,,\ \forall\, y \in \mathcal{O}$$

where we introduce formal unknowns $z = (z_1, \ldots, z_\mu)$. A formal series $\phi(z, y)$ is really a y-dependent sequence $(\phi_k(y))_k$ indexed by $k \in \mathbb{N}^\mu$. An important parameter is the dimension μ of the indices k. We define now the relation of majoring series between two formal series $\phi(z, y)$ and $\psi(Z, y)$, with z and Z denoting μ variables.

Definition 2.4.1 (Majoring series). *For $\phi(z,y)$ and $\psi(Z,y)$ formal series of respectively variable z and variable Z, and y a parameter in some open domain $\mathcal{O}$ of $\mathbb{C}^{\mu'}$, with furthermore*

$$\psi(Z,y) = \sum_{k\in\mathbb{N}^\mu} \psi_k(y) Z^k \quad \text{with} \quad \psi_k(y) \geq 0 \quad \forall k \in \mathbb{N}^\mu\,,\ \forall y \in \mathcal{O}$$

we define

$$\phi(z,y) \prec_y \psi(Z,y) \quad \Longleftrightarrow \quad \Big(\forall k \in \mathbb{N}^\mu\,,\ \forall y \in \mathcal{O}: \quad |\phi_k(y)| \leq \psi_k(y)\Big) \tag{2.4.1}$$

Remark 2.4.2. *In notation $\prec_y$ we emphasize that we consider y as a parameter in the formal series $\phi(z,y)$.*

In the following we sum up several classical properties of the relation (2.4.1) (see [Car61]).

Lemma 2.4.3. *Let ϕ and ψ be as in the previous definition, with $\phi \prec_y \psi$. Then*

1. *If ψ converges at a point (Z,y) with $Z_i \geq 0$ for all $i = 1,\dots,m$, then ϕ converges on all (z,y) such that $|z_i| \leq Z_i$, and*

$$|\phi(z_1,\dots,z_\mu,y)| \leq \psi(|z_1|,\dots,|z_\mu|,y) \tag{2.4.2}$$

2. *The relation $\prec_y$ is compatible with formal derivations: denoting ∂_i the formal derivation along the i-th variable, we have*

$$\phi \prec_y \psi \quad \Longrightarrow \quad \partial_i\phi(z,y) \prec_y \partial_i\psi(Z,y) \tag{2.4.3}$$

3. *The relation $\prec_y$ is compatible with multiplication:*

$$\phi_1 \prec_y \psi_1 \text{ and } \phi_2 \prec_y \psi_2 \quad \Longrightarrow \quad \phi_1\phi_2 \prec_y \psi_1\psi_2 \tag{2.4.4}$$

4. *There is a constant $c_0 > 0$ such that the series*

$$\Phi(z_1) = \sum_{k\geq 0} \frac{c_0}{k^2+1} z_1^k \tag{2.4.5}$$

satisfies

$$\Phi^2 \prec \Phi \tag{2.4.6}$$

The series Φ is analytic on $B_1(0)$, defined in (2.1.33).

Proof. We give here a short proof of this Lemma.

1. Assume that $\psi(Z,y)$ is converging at a point (Z,y), with all $Z_i \geq 0$. By definition of the majoring series, we have for all $k \in \mathbb{N}^\mu$ the inequality $|\phi_k(y)| \leq \psi_k(y)$. Since the

series $\sum_k \psi_k(y)Z^k$ is convergent, then for all $z \in \mathbb{C}^\mu$ such that $|z_i| \leq Z_i$ the series $\sum_k \phi_k(y)z^k$ converges and there holds by (2.1.28) and Definition 2.4.1

$$\begin{aligned}\left|\sum_{k\in\mathbb{N}^\mu} \phi_k(y)z^k\right| &\leq \sum_{k\in\mathbb{N}^\mu} |\phi_k(y)| \prod |z_j|^{k_j} \\ &\leq \sum_{k\in\mathbb{N}^\mu} \psi_k(y) \prod |z_j|^{k_j} \\ &= \psi(|z_1|, \ldots, |z_\mu|, y)\end{aligned}$$

Hence the importance of using two different notations for the μ variables, z and Z.

2. By definition of formal derivation ∂_i, there holds

$$\partial_i \phi(z, y) = \sum_{k\in\mathbb{N}^\mu} (k_i + 1)\phi_{k+1_i}(y)z^k$$

where 1_i is defined by (2.1.30) and for all $k \in \mathbb{N}^\mu$ there holds

$$|(k_i + 1)\phi_{k+1_i}(y)| \leq (k_i + 1)\psi_{k+1_i}(y)$$

by Definition 2.4.1, which is exactly the k-th coefficient of the formal series $\partial_i\psi(Z, y)$.

3. Let ϕ^1, ϕ^2, ψ^1 and ψ^2 be such that $\phi^1 \prec_y \psi^1$ and $\phi^2 \prec_y \psi^2$. By definition of the multiplication of two formal series, the coefficients of the formal series $\phi^1\phi^2(z, y)$ in z are

$$(\phi^1\phi^2)_k(y) = \sum_{p=0}^{k} \phi^1_p(y)\phi^2_{k-p}(y)$$

and then for all $y \in \mathcal{O}$ and $k \in \mathbb{N}^\mu$ there holds

$$\begin{aligned}\left|(\phi^1\phi^2)_k(y)\right| &\leq \sum_{p=0}^{k} \left|\phi^1_p(y)\right| \left|\phi^2_{k-p}(y)\right| \\ &\leq \sum_{p=0}^{k} \psi^1_p(y)\, \psi^2_{k-p}(y)\end{aligned}$$

because $\phi^1 \prec_y \psi^1$ and $\phi^2 \prec_y \psi^2$. As the right-hand side of the previous inequality is just $(\psi^1\psi^2)_k(y)$, this ends the proof.

4. For $\mu = 1$ and $\mu' = 0$, we consider the series

$$\Phi(z) = \sum_{k\in\mathbb{N}} \frac{c_0}{k^2 + 1} z^k$$

We compute

$$\Phi^2(z) = \sum_{k\in\mathbb{N}} \sum_{p=0}^{k} \frac{c_0}{p^2 + 1} \frac{c_0}{(k - p)^2 + 1} z^k.$$

To prove the existence of some $c_0 > 0$ such that (2.4.6) holds, it suffices to prove that

$$\sum_{p=0}^{k} \frac{k^2+1}{(p^2+1)((k-p)^2+1)}$$

is bounded for all $k \in \mathbb{N}$. Thanks to $k^2 \leq 2(p^2+(k-p)^2)$ there holds

$$\sum_{p=0}^{k} \frac{k^2+1}{(p^2+1)((k-p)^2+1)} \leq 4\sum_{p=0}^{k} \frac{1}{p^2+1} \leq 4\sum_{p\in\mathbb{N}} \frac{1}{p^2+1}$$

which suffices to end the proof.

□

After these abstract considerations we come back to series in the spatial variable x, where t a parameter. The principle behind the relation of majoring series is to replace unknown analytical functions by a fixed, well-known series. In this view we consider the series in d variables $(X_1, \ldots, X_d)$, with $t \in [0, \rho^{-1})$ a parameter and R and ρ some positive constants

$$\Phi(RX_1 + \cdots + RX_d + \rho t) = \sum_{k\in\mathbb{N}^d} \left(R^{|k|} \sum_{p\in\mathbb{N}} \frac{c_0}{(|k|+p)^2+1} \binom{|k|+p}{k,p} \rho^p t^p \right) X^k \tag{2.4.7}$$

using the notations (2.1.28) for X^k and (2.1.29) for $\binom{|k|+p}{k,p}$. We denote

$$\Phi_k(t) = R^{|k|} \sum_{p\in\mathbb{N}} \frac{c_0}{(|k|+p)^2+1} \binom{|k|+p}{k,p} \rho^p t^p \ , \quad \forall k \in \mathbb{N}^d \tag{2.4.8}$$

where it is implicit that $\Phi_k(t)$ depend also on R and ρ. Note that the series in the right hand side of (2.4.8) is convergent for $|t| < \rho^{-1}$. Since the series $\Phi(z)$ converges in $B_1(z=0)$, the series $\Phi(RX_1+\cdots+RX_d+\rho t)$ is convergent as a series in X and t variables on $\Omega_{R,\rho}(0)$ defined by (2.2.4).

From now on, we will note for convenience and with an abuse of notation

$$\Phi(RX + \rho t) = \Phi(RX_1 + \cdots + RX_d + \rho t) \tag{2.4.9}$$

as the reference series in the x variable, for some positive constants R and ρ. In the following Lemma we sum up properties for formal series ϕ in d variables with one parameter t that satisfy

$$\phi(x,t) \prec_t C\Phi(RX + \rho t)$$

for some $C > 0$. This is equivalent, thanks to (2.4.1), (2.4.7) and (2.4.8) to

$$|\phi_k(t)| \leq C\Phi_k(t) \quad , \quad \forall k \in \mathbb{N}^d \text{ and } 0 \leq t < \rho^{-1}. \tag{2.4.10}$$

Lemma 2.4.4. *For $\phi(x,t)$ a formal series in x with $\phi(x,t) \prec_t C\Phi(RX+\rho t)$ there holds*

1. *$\phi(x,t)$ is analytic as a series in x in the domain $\Omega_{R,\rho,t}(0)$ for all $0 \le t < \rho^{-1}$.*

2. *For all $0 \le t < \rho^{-1}$, there holds*

$$\partial_{x_j}\phi(x,t) \prec_t CR\Phi'(RX+\rho t) \tag{2.4.11}$$

with Φ' the derivative of Φ.

3. *For any $R \ge R_0$ and $\rho \ge \rho_0$, there holds*

$$\Phi(R_0X+\rho_0 t) \prec_{R_0,\rho_0,R,\rho,t} \Phi(RX+\rho t). \tag{2.4.12}$$

4. *For any $R > 0$, $\rho > 0$ and $0 \le t' < t < \rho^{-1}$, there holds*

$$\Phi(RX+\rho t') \prec_{t',t} \Phi(RX+\rho t). \tag{2.4.13}$$

Proof. 1. By the first property of Lemma 2.4.3, the formal series $\phi(x,t)$ is analytic in x on the domain of convergence of the series $\Phi(RX+\rho t)$ thought as a series in X variable. As it is just $\Omega_{R,\rho,t}(0)$, defined by (2.2.3), the function $\phi(x,t)$ is analytic on $\Omega_{R,\rho,t}(0)$ as a series in the x variable for all $0 \le t < \rho^{-1}$.

2. By the second property of Lemma 2.4.3 there holds $\partial_{x_j}\phi(x,t) \prec_t C\partial_{X_j}(\Phi(RX+\rho t))$ and as

$$\partial_{X_j}(\Phi(RX+\rho t)) = \partial_{X_j}(\Phi(RX_1+\cdots+RX_d+\rho t)) = R\Phi'(RX+\rho t)$$

for all $0 \le t < \rho^{-1}$, we finally get (2.4.11).

3. Thanks to notation (2.4.8) we have $\Phi(RX+\rho t) = \sum_{k\in\mathbb{N}^d}\Phi_k(t)X^k$ for all $0 \le t < \rho^{-1}$, where we recall it is implicit that the coefficients $\Phi_k(t) = \Phi_k(t,R,\rho)$ depend also on R and ρ. In the definition (2.4.8) we easily see that

$$\Phi_k(t,R_0,\rho_0) \le \Phi_k(t,R,\rho) \quad , \quad \forall R \ge R_0,\ \forall \rho \ge \rho_0,\ \forall 0 \le t < \rho^{-1}$$

which is exactly (2.4.12).

4. In the same way we see that, R and ρ being fixed, the coefficients $\Phi_k(t)$ are increasing functions of t:

$$\Phi_k(t') \le \Phi_k(t) \qquad \forall k \in \mathbb{N}^d,\ \forall 0 \le t' < t < \rho^{-1}$$

which is exactly (2.4.13).

□

The first property of the previous Lemma indicates that series controlled by Φ are analytic. Conversely the following Lemma proves that analytic functions are controlled by appropriate series:

Lemma 2.4.5. *Let $H(t,x,u)$ an analytic function in the neighborhood of $(0,0,0) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^N$. Then there are some positive constants C_H, R_H, ρ_H and a_H such that*

$$H(t,x,u) \prec C_H \Phi(R_H X + \rho_H t) \prod_{j=1}^{N} \frac{1}{1 - a_H u_j} \tag{2.4.14}$$

Proof. Formally we write

$$H(t,x,u) \quad = \quad \sum_{k_1,k_2,k_3} H_{k_1,k_2,k_3}\, t^{k_1}\, x^{k_2}\, u^{k_3}$$

with $k_1 \in \mathbb{N}$, $k_2 \in \mathbb{N}^d$ and $k_3 \in \mathbb{N}^N$. By the Cauchy relations for H, we know there are some positive constants C, r_1, r_2 and r_3 depending only on H such that

$$|H_{k_1,k_2,k_3}| \leq C \frac{1}{r_1^{k_1} r_2^{|k_2|} r_3^{|k_3|}} \quad , \quad \forall\, (k_1,k_2,k_3) \in \mathbb{N} \times \mathbb{N}^d \times \mathbb{N}^N.$$

We compare $|H_{k_1,k_2,k_3}|$ to the coefficients of the series $\Phi(R_H X + \rho_H t) \prod (1 - a_H u_j)^{-1}$:

$$\begin{aligned}
&\Phi(R_H X + \rho_H t) \prod_{j=1}^{N} \frac{1}{1 - a_H u_j} \\
&= \sum_{p \in \mathbb{N}} \frac{c_0}{p^2+1} (R_H X + \rho_H t)^p \sum_{q \in \mathbb{N}^N} a_H^{|q|} u^q \\
&= \sum_{p} \sum_{k_1 + |k_2| = p} \frac{c_0}{p^2+1} \binom{p}{k_1, k_2} (\rho_H t)^{k_1} (R_H X)^{k_2} \sum_{q} a_H^{|q|} u^q \\
&= \sum_{k_1,k_2,k_3} \frac{c_0}{(k_1+|k_2|)^2+1} \binom{k_1+|k_2|}{k_1, k_2} \rho_H^{k_1} R_H^{|k_2|} a_H^{|k_3|}\, t^{k_1} X^{k_2} u^{k_3}
\end{aligned}$$

Then we have for all $(k_1,k_2,k_3) \in \mathbb{N} \times \mathbb{N}^d \times \mathbb{N}^N$ we have

$$\begin{aligned}
|H_{k_1,k_2,k_3}| \quad &\leq \quad C \frac{1}{r_1^{k_1} r_2^{|k_2|} r_3^{|k_3|}} \\
&\leq \quad \frac{C}{c_0} \frac{(k_1+|k_2|)^2+1}{(\rho_H r_1)^{k_1} (R_H r_2)^{|k_2|} (a_H r_3)^{|k_3|}} \frac{c_0}{(k_1+|k_2|)^2+1} \binom{k_1+|k_2|}{k_1, k_2} \rho_H^{k_1} R_H^{|k_2|} a_H^{|k_3|}
\end{aligned}$$

thanks to $\binom{k_1+|k_2|}{k_1,k_2} \geq 1$ for all k_1, k_2. By choosing R_H, ρ_H and a_H such that $\rho_H r_1$, $R_H r_2$ and $a_H r_3$ are larger than 1, the term

$$\frac{(k_1+|k_2|)^2+1}{(\rho_H r_1)^{k_1} (R_H r_2)^{|k_2|} (a_H r_3)^{|k_3|}}$$

is bounded for all $(k_1,k_2,k_3) \in \mathbb{N} \times \mathbb{N}^d \times \mathbb{N}^N$. Then there is a constant $C_H > 0$ depending only on H, R_H, ρ_H and a_H such that for all $(k_1,k_2,k_3) \in \mathbb{N} \times \mathbb{N}^d \times \mathbb{N}^N$ there holds

$$|H_{k_1,k_2,k_3}| \leq C_H \frac{c_0}{(k_1+|k_2|)^2+1} \binom{k_1+|k_2|}{k_1, k_2} \rho_H^{k_1} R_H^{|k_2|} a_H^{|k_3|}$$

which implies

$$H(x,t,u) \prec C_H \Phi(R_H X + \rho_H t) \prod_{j=1}^{N} \frac{1}{1 - a_H u_j}.$$

□

Lemma 2.4.6. *There is $c_1 > 0$ such that*

$$\sum_{p\in\mathbb{Z}} \frac{c_1}{p^2+1} \frac{c_1}{(n-p)^2+1} \leq \frac{c_1}{n^2+1} \tag{2.4.15}$$

Proof. In the same way of the proof of the third point of Lemma 2.4.3, there holds

$$\sum_{p\in\mathbb{Z}} \frac{n^2+1}{(p^2+1)((n-p)^2+1)} \leq \sum_{p\in\mathbb{Z}} \frac{2(p^2+1+(n-p)^2+1)}{(p^2+1)((n-p)^2+1)} \leq 4 \sum_{p\in\mathbb{Z}} \frac{1}{p^2+1}$$

which suffices to end the proof. □

2.4.2 Definitions of functional spaces

Fixed time spaces $\mathbb{E}_s$

We consider trigonometric series in one variable θ with coefficients in the space of formal series in d variables x in the sense of Section 2.4.1, and we denote F_{d+1} the space of all such trigonometric series:

$$F_{d+1} = \left\{ \mathbf{v}(x,\theta) = \sum_{n\in\mathbb{Z}} \mathbf{v}_n(x) e^{in\theta} \;\middle|\; \mathbf{v}_n(x) = \sum_{k\in\mathbb{N}^d} \mathbf{v}_{n,k} x^k \right\}.$$

Definition 2.4.7 (Fixed time spaces $\mathbb{E}_s$)**.** *Given $s \in [0, (\varepsilon\rho)^{-1})$, $R > 0$, $\rho > 0$, $M' > 0$ and $\beta \in (0,1)$, we denote $\mathbb{E}_s = \mathbb{E}_s(R, \rho, M', \beta)$ the space of trigonometric series $\mathbf{v} \in F_{d+1}$ such that for some constant $C > 0$ there holds*

$$\mathbf{v}_n(x) \prec C \frac{c_1}{n^2+1} \exp\Big(- \big(M' - \int_0^s \gamma(\tau) d\tau\big) \langle n \rangle \Big) \Phi\left(RX + \varepsilon\rho s\right) \quad , \quad \forall n \in \mathbb{Z}. \tag{2.4.16}$$

where we denote

$$\gamma(\tau) = \gamma(\tau\,; R, \omega) := \gamma^\sharp(\tau\,; R, \omega) + \beta. \tag{2.4.17}$$

We define a norm on $\mathbb{E}_s$ with

$$\|\mathbf{v}\|_s = \inf\left\{ C > 0 \,|\quad (2.4.16) \text{ is satisfied } \right\}. \tag{2.4.18}$$

Note that in definition (2.4.17) of γ, the function $\gamma^\sharp$ corresponds to either one defined in Lemma 2.3.1. In previous Definition 2.4.7, it is implicit that space $\mathbb{E}_s$ depends on a positive function $\gamma^\sharp$.

Thanks to Lemma 2.4.4, for $s \in [0, (\varepsilon\rho)^{-1})$, all $\mathbf{v} \in \mathbb{E}_s$ are holomorphic in the x variable in the domain $\Omega_{R,\varepsilon\rho,s}$ defined by (2.2.3). We introduce also the growth time $\underline{s}_1$ defined implicitely as

$$M' = \int_0^{\underline{s}_1} \gamma(\tau)d\tau. \tag{2.4.19}$$

For $0 \leq s < \underline{s}_1$ we have $M' - \int_0^s \gamma(\tau)d\tau > 0$ and then analyticity of $\mathbf{v}$ in the θ variable. We will also see in Lemma 2.4.11 that if $0 \leq s < \underline{s}_1$, the space $(\mathbb{E}_s, ||\cdot||_s)$ is an algebra. After these considerations it is convenient to define the final time as

$$\underline{s} = \min\left\{\underline{s}_1, (\varepsilon\rho)^{-1}\right\}. \tag{2.4.20}$$

To simplify the notations, in all the following we will omit the parameters R, ρ, M' and β in $\mathbb{E}_s(R, \rho, M', \beta)$. All properties of spaces $\mathbb{E}_s$ do not depend on particular values of those parameters.

Spaces $\mathbb{E}$

We consider now trigonometric series

$$\mathbf{u}(s, x, \theta) = \sum_{n\in\mathbb{Z}} \mathbf{u}_n(s, x)e^{in\theta}$$

with coefficients $\mathbf{u}_n(s, x)$ being formal series in x whose coefficients depend smoothly on $s \in [0, \underline{s})$. We denote F_{d+2} the space of all such trigonometric series:

$$F_{d+2} = \left\{ \mathbf{u}(s, x, \theta) = \sum_{n\in\mathbb{Z}} \mathbf{u}_n(s, x)e^{in\theta} \;\middle|\; \mathbf{u}_n(s, x) = \sum_{k\in\mathbb{N}^d} \mathbf{u}_{n,k}(s)x^k \text{ with } \mathbf{u}_{n,k}(s)\ C^\infty \text{ in } s \right\}.$$

Definition 2.4.8 (Spaces $\mathbb{E}$). *We introduce*

$$\mathbb{E} = \{\mathbf{u} \in F_{d+2} \,|\, \forall\, 0 \leq s < \underline{s}, \quad \mathbf{u}(s) \in \mathbb{E}_s\} \tag{2.4.21}$$

and the corresponding norm

$$|||\mathbf{u}||| = \sup_{0\leq s<\underline{s}} \|\mathbf{u}(s)\|_s. \tag{2.4.22}$$

Recalling the definition of majoring series (2.4.1) and the definition of $\mathbb{E}_s$ (2.4.16), for all $\mathbf{u} \in \mathbb{E}$ there holds

$$\mathbf{u}_n(s, x) \prec_s |||\mathbf{u}||| \frac{c_1}{n^2+1} \exp\Big(- \big(M' - \int_0^s \gamma(\tau)d\tau\big) \langle n\rangle \Big) \Phi\left(RX + \varepsilon\rho s\right) \tag{2.4.23}$$

for all $n \in \mathbb{Z}$ and $s \in [0, \underline{s})$.

For $\mathbf{u}$ valued in $\mathbb{C}^N$, $\mathbf{u} \in \mathbb{E}$ means simply that each component of $\mathbf{u}$ is in $\mathbb{E}$, and $|||\mathbf{u}|||$ is then the maximum of the norms of the components.

We denote the ball of $\mathbb{E}$ of radius a, centered in $\mathbf{u} \in \mathbb{E}$ by

$$B_{\mathbb{E}}(\mathbf{u}, a) = \{\mathbf{v} \in \mathbb{E} \mid |||\mathbf{v} - \mathbf{u}||| < a\}. \tag{2.4.24}$$

2.4.3 Some properties of spaces $\mathbb{E}$

The spaces $\mathbb{E}_s$ are Banach spaces

Proposition 2.4.9. *For all $s \in [0, \underline{s})$, the space $\mathbb{E}_s$ equipped with the norm $||\cdot||_s$ is a Banach space.*

Proof. Any $\mathbf{v}$ in $\mathbb{E}_s$ is uniquely determined by the sequence of coefficients $(\mathbf{v}_{n,k})_{n\in\mathbb{Z},k\in\mathbb{N}^d}$, where

$$\mathbf{v}(x,\theta) = \sum_{n\in\mathbb{Z}} \mathbf{v}_n(x)e^{in\theta} \quad \text{with} \quad \mathbf{v}_n(x) = \sum_{k\in\mathbb{N}^d} \mathbf{v}_{n,k}x^k.$$

By the definition of majoring series (2.4.10) and notation (2.4.8), the definition (2.4.16) is equivalent to

$$|\mathbf{v}_{n,k}| \leq C\frac{c_1}{n^2+1}\exp\Big(-\big(M' - \int_0^s \gamma(\tau)d\tau\big)\langle n\rangle\Big)\Phi_k(\varepsilon s)\ ,\ \forall n\in\mathbb{Z},\ k\in\mathbb{N}^d,\ 0\leq s<(\varepsilon\rho)^{-1}$$

where γ is defined in (2.4.17). Thus the map

$$\mathcal{O}(s) : \mathbf{v}\in\mathbb{E}_s \mapsto (\mathbf{v}_{n,k}\mathcal{O}_{n,k}(s))_{n\in\mathbb{Z},k\in\mathbb{N}^d} \tag{2.4.25}$$

with

$$\mathcal{O}_{n,k}(s) = \left(\frac{c_1}{n^2+1}\exp\Big(-\big(M' - \int_0^s \gamma(\tau)d\tau\big)\langle n\rangle\Big)\Phi_k(\varepsilon s)\right)^{-1}$$

is onto $\ell^\infty(\mathbb{C}^{\mathbb{Z}\times\mathbb{N}^d})$. By definition of the norm in $\mathbb{E}_s$, the map $\mathcal{O}(s)$ is clearly an isometric isomorphism between $\mathbb{E}_s$ and $\ell^\infty(\mathbb{C}^{\mathbb{Z}\times\mathbb{N}^d})$. This implies that $(\mathbb{E}_s, ||\cdot||_s)$ is a Banach space. □

This implies immediately the following

Corollary 2.4.10. *The space $(\mathbb{E}, |||\cdot|||)$ is a Banach space.*

The spaces $\mathbb{E}_s$ are Banach algebra

Lemma 2.4.11. *For all $s\in[0,\underline{s})$, for all $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{E}_s$, the product $\mathbf{vw}$ is in $\mathbb{E}_s$ and we have*

$$||\mathbf{vw}||_s \leq ||\mathbf{v}||_s\,||\mathbf{w}||_s. \tag{2.4.26}$$

Proof. Starting with the definition of $\mathbb{E}_s$ (2.4.16), we obtain first for all $n\in\mathbb{Z}$ the

$$\begin{aligned}
&(\mathbf{vw})_n(x)\\
&= \sum_{p+q=n} \mathbf{v}_p(x)\mathbf{w}_q(x)\\
&\prec \sum_{p+q=n} \quad ||\mathbf{v}||_s\frac{c_1}{p^2+1}\exp\Big(-(M' - \int_0^s \gamma(\tau)d\tau)\langle p\rangle\Big)\Phi\left(RX+\varepsilon\rho s\right)\\
&\qquad\qquad \times||\mathbf{w}||_s\frac{c_1}{q^2+1}\exp\Big(-(M' - \int_0^s \gamma(\tau)d\tau)\langle q\rangle\Big)\Phi\left(RX+\varepsilon\rho s\right)\\
&\prec ||\mathbf{v}||_s\,||\mathbf{w}||_s\,\Phi^2\left(RX+\varepsilon\rho s\right)\sum_{p+q=n}\frac{c_1}{p^2+1}\frac{c_1}{q^2+1}\exp\Big(-(M' - \int_0^s \gamma(\tau)d\tau)(\langle p\rangle+\langle q\rangle)\Big).
\end{aligned}$$

Recalling that $\Phi^2 \prec \Phi$ by Lemma 2.4.3, we have

$$
\begin{aligned}
&(\mathbf{vw})_n(x) \\
&\prec ||\mathbf{v}||_s\, ||\mathbf{w}||_s\, \Phi\left(RX + \varepsilon\rho s\right) \sum_{p+q=n} \frac{c_1}{p^2+1}\frac{c_1}{q^2+1} \exp\Big(- (M' - \int_0^s \gamma(\tau)d\tau)(\langle p\rangle + \langle q\rangle)\Big) \\
&\prec ||\mathbf{v}||_s\, ||\mathbf{w}||_s \Phi\left(RX + \varepsilon\rho s\right) \exp\Big(- (M' - \int_0^s \gamma(\tau)d\tau)\, \langle n\rangle\Big) \sum_{p+q=n} \frac{c_1}{p^2+1}\frac{c_1}{q^2+1}
\end{aligned}
$$

because $\langle p\rangle + \langle q\rangle \geq \langle p+q\rangle = \langle n\rangle$ and $M' - \int_0^s \gamma(\tau)d\tau$ is positive for all $s < \underline{s}$, and γ is defined in (2.4.17). And by definition (2.4.15) of c_1 we have finally

$$
(\mathbf{vw})_n(x) \quad \prec \quad ||\mathbf{v}||_s\, ||\mathbf{w}||_s \frac{c_1}{n^2+1} \exp\Big(- (M' - \int_0^s \gamma(\tau)d\tau)\, \langle n\rangle\Big) \Phi\left(RX + \varepsilon\rho s\right)
$$

which implies the result.

□

This implies immediately the following

Corollary 2.4.12. *The space $\mathbb{E}$ is an algebra, and the norm $|||\cdot|||$ is an algebra norm.*

Action of holomorphic functions

Lemma 2.4.13. *Let $H(t,x,u)$ be a holomorphic function on a neighborhood of $(0,0,0) \in \mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^N_u$. Then for ε small enough there are constants C_H, R_H and ρ_H which depend only on H and c_0, such that for all $R \geq R_H$ and $\rho \geq \rho_H$,*

$$
\forall\, \mathbf{u} \in B_{\mathbb{E}(R,\rho)}(0,1): \quad |||\mathbf{H}(\mathbf{u})||| \leq C_H 2^N \tag{2.4.27}
$$

where $\mathbf{H}$ is defined by (2.3.3) and $|||\cdot|||$ is defined by (2.4.22).

Proof. Thanks to Lemma 2.4.5 we have

$$
H(t,x,u) \prec C_H \Phi(R_H X + \rho_H t) \prod_{j=1}^N \frac{1}{1 - a_H u_j}
$$

Let $\mathbf{u}$ be in $B_{\mathbb{E}}(0,1)$ with $\mathbb{E} = \mathbb{E}(R,\rho)$ for $R \geq R_H$ and $\rho \geq \rho_H$. For ε small enough we have $\varepsilon a_H < 1/2$ so that $|||\varepsilon a_H \mathbf{u}||| \leq 1/2$. We now prove that $\mathbf{H}(s,x,\mathbf{u})$ is indeed in $\mathbb{E}$. By Lemma 2.4.5 it suffices to prove that

$$
(s,x,\theta) \mapsto C_H \Phi(R_H X + \varepsilon\rho_H s) \prod_{j=1}^N \frac{1}{1 - \varepsilon a_H \mathbf{u}_j(s,x,\theta)}
$$

is in $\mathbb{E}$. Because $\mathbb{E}$ is a Banach algebra (Corollary 2.4.12)and $\varepsilon a_H < 1/2$, the operator

$$
\mathbf{u} \mapsto \prod_{j=1}^N (1 - \varepsilon a_H \mathbf{u}_j)^{-1}
$$

is a bounded operator and we have

$$\left|\left|\left|\prod_{j=1}^{N}\frac{1}{1-\varepsilon a_H \mathbf{u}_j(s,x,\theta)}\right|\right|\right| \leq \prod_{j=1}^{N}\frac{1}{1-\varepsilon a_H |||\mathbf{u}|||} \leq \left(\frac{1}{1-1/2}\right)^N = 2^N$$

By (2.4.12), we have $\Phi(R_H X + \varepsilon\rho_H s) \prec_s \Phi(RX+\varepsilon\rho s)$ for all $R \geq R_H$ and $\rho \geq \rho_H$, so that

$$\begin{aligned}\Phi(R_H X + \varepsilon\rho_H s)\Phi(RX+\varepsilon\rho s) &\prec_s \Phi(RX+\varepsilon\rho s)^2\\ &\prec_s \Phi(RX+\varepsilon\rho s)\end{aligned}$$

by (2.4.6). Hence $(s,x,\theta) \mapsto C_H\Phi(R_H X+\varepsilon\rho_H s)\prod_{j=1}^N(1-\varepsilon a_H \mathbf{u}_j(s,x,\theta))^{-1}$ is in $\mathbb{E}$, and then for all $\mathbf{u}\in\mathbb{E}$ in the ball $B_{\mathbb{E}}(0,1)$ the bound (2.4.27) holds.

□

In the operators $T^{[\theta]}$, $T^{[x]}$ and $T^{[\mathbf{u}]}$ defined by (2.3.25), (2.3.26) and (2.3.27), there appear A, $\underline{A}$, A_j and F. In Corollary 2.5.5, there will appear also A_{u_j}, all of which are analytic functions in variables $(t,x,u)\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^N$ in a neighborhood of $(0,0,0)\in\mathbb{R}_t\times\mathbb{R}^d_x\times\mathbb{R}^N_u$. The previous Lemma applies:

Corollary 2.4.14. *There are constants R_0 and ρ_0 such that for all $R\geq R_0$, $\rho\geq\rho_0$ and ε small enough:*

$$\forall\,\mathbf{u}\in B_{\mathbb{E}(R,\rho)}(0,1): \qquad |||\mathbf{H}(\mathbf{u})||| \lesssim 1 \tag{2.4.28}$$

with H equals to A, $\underline{A}$, A_j, F, or A_{u_j}.

2.4.4 Action of $U(s',s)$ on $\mathbb{E}$

Recall the growth of the Fourier modes of the propagator as showed in Lemma 2.3.1

$$|U_n(s',s,x)| \lesssim \omega^{-(m-1)}\exp\left(|n|\int_{s'}^{s}\gamma^\sharp(\tau)d\tau\right).$$

Here, as opposed to [Mét05], the propagator U_n does depend on x. As $U_n(s',s,x)$ is the solution of the differential equation (2.3.8) and as $\underline{A}(t,x)$ is analytic in x, so is $U_n(s',s,x)$. Using the Cauchy inequalities as in the proof of Lemma 2.4.5, we can prove in particular that

$$U_n(s',s,x) \prec_{s',s} \omega^{-(m-1)}\exp\left(|n|\int_{s'}^{s}\gamma^\sharp(\tau)d\tau\right)\Phi(R_0X) \tag{2.4.29}$$

for R_0 determined in Corollary 2.4.14. We use this result to determine precisely the action of the propagator on $\mathbb{E}$.

Lemma 2.4.15. *Given $\mathbf{u}$ in $\mathbb{E}=\mathbb{E}(R,\rho,M',\beta)$ then for all $n\in\mathbb{Z}$ ans $0\leq s'\leq s<\underline{s}$ there holds*

$$U_n(s',s)\mathbf{u}_n(s',x) \prec_{s',s} C_n(s',s)\,\omega^{-(m-1)}\,||\mathbf{u}(s')||_{s'}\frac{c_1}{n^2+1}\,e^{-(M'-\int_0^s\gamma(\tau)d\tau)\langle n\rangle}\Phi\left(RX+\varepsilon\rho s\right) \tag{2.4.30}$$

$$\textit{with} \quad C_n(s',s) = \exp\big(-\langle n\rangle\,\beta\,(s-s')\big) \le 1. \tag{2.4.31}$$

In particular we have

$$\|U(s',s)\mathbf{u}(s')\|_s \le \omega^{-(m-1)}\,\|\mathbf{u}(s')\|_{s'} \quad , \quad \forall\, 0 \le s' \le s < \underline{s}. \tag{2.4.32}$$

Proof. By the estimate (2.4.23) for $\mathbf{u} \in \mathbb{E}$ we have

$$\mathbf{u}_n(s',x) \prec_{s'} ||\mathbf{u}(s')||_{s'} \frac{c_1}{n^2+1} \exp\Big(-(M' - \int_0^{s'} \gamma(\tau)d\tau)\,\langle n\rangle\Big)\Phi\big(RX+\varepsilon\rho s'\big)$$

where γ is defined in (2.4.17). By estimate (2.4.29) and the multiplicative property of $\prec$ there holds

$$\begin{aligned}
&U_n(s',s)\mathbf{u}_n(s',x)\\
&\prec_{s',s} \omega^{-(m-1)} \exp\left(|n|\int_{s'}^{s} \gamma^\sharp(\tau)d\tau\right)\\
&\quad\times ||\mathbf{u}(s')||_{s'} \frac{c_1}{n^2+1} \exp\Big(-(M' - \int_0^{s'} \gamma(\tau)d\tau)\,\langle n\rangle\Big)\Phi\big(RX+\varepsilon\rho s'\big)\\
&\prec_{s',s} \omega^{-(m-1)}\,||\mathbf{u}(s')||_{s'} \frac{c_1}{n^2+1} \exp\Big(-(M' - \int_0^{s} \gamma(\tau)d\tau)\,\langle n\rangle\Big)\Phi\big(RX+\varepsilon\rho s\big)\\
&\quad\times \exp\left(-\langle n\rangle\int_{s'}^{s}\Big(\gamma(\tau)-\gamma^\sharp(\tau)\Big)\,d\tau\right)
\end{aligned}$$

because $\Phi(RX+\varepsilon\rho s') \prec_{s',s} \Phi(RX+\varepsilon\rho s)$ for $s' \le s < \underline{s}$ by (2.4.13). This gives us exactly (2.4.30) using (2.4.17), and then (2.4.32).

□

Remark 2.4.16. *The estimate* (2.4.32) *is not precise enough to show that T is a contraction in $\mathbb{E}$. The more precise estimate* (2.4.30) *is very important for the estimate* (2.5.6) *below.*

2.4.5 Norm of the free solution

Lemma 2.4.17 (Norm of the free solution)**.** *The free solution* $\mathbf{f}$ *defined by* (2.3.13) *satisfies*

$$|||\mathbf{f}||| \lesssim \omega^{-(m-1)}\,e^{M'-M(\varepsilon)}. \tag{2.4.33}$$

Proof. The Fourier decomposition of $\mathbf{f}_\varepsilon$ is given by $\mathbf{f}_\varepsilon = \mathbf{f}_{+1}e^{-i\theta} + \mathbf{f}_{-1}e^{i\theta}$ with $\mathbf{f}_\pm(s,x) = U_\mp(0,s,x)\vec{e}_\pm$. The Fourier coefficients $\mathbf{f}_\pm$ satisfy thanks to (2.4.29) the estimate

$$\mathbf{f}_{\pm 1}(s) \prec_s \omega^{-(m-1)}\,e^{-M(\varepsilon)}e^{\int_0^s \gamma^\sharp(\tau)d\tau}\Phi(R_0X). \tag{2.4.34}$$

Then by definition of $|||\cdot|||$ given by (2.4.22), and by definition (2.4.17) of γ, there holds

$$\begin{aligned}
|||\mathbf{f}_{\pm 1}||| &= \frac{2}{c_0c_1}\omega^{-(m-1)}\,e^{M'-M(\varepsilon)} \max_{[0,\underline{s})} e^{\int_0^s \gamma^\sharp(\tau)d\tau} e^{-\int_0^s \gamma(\tau)d\tau}\\
&= \frac{2}{c_0c_1}\omega^{-(m-1)}\,e^{M'-M(\varepsilon)} \max_{[0,\underline{s})} e^{-\int_0^s \beta d\tau}\\
&\lesssim \omega^{-(m-1)}\,e^{M'-M(\varepsilon)}
\end{aligned}$$

which ends the proof. □

2.5 Regularization by integration in time and contraction estimates

In this section we prove estimates in spaces $\mathbb{E}$ for the three operators $T^{[\theta]}$, $T^{[x]}$ and $T^{[\mathbf{u}]}$ defined respectively by (2.3.25), (2.3.26) and (2.3.27). Note that in the first two operators there appear derivation operators ∂_θ and ∂_{x_j}. As we will see in the next subsection, these are not bounded operators in $\mathbb{E}$. But thanks to some smoothing effect of the time-integration, as used in [Mét05], we will show that operators $T^{[\theta]}$, $T^{[x]}$ and $T^{[\mathbf{u}]}$ are in fact bounded in $\mathbb{E}$. We will follow in this section the work of [Uka01].

2.5.1 Lack of boundedness of derivation operators

In the following we make precise how the derivation operators ∂_{x_j} and ∂_θ act on $\mathbb{E}$.

Lemma 2.5.1 (Estimates for the derivation operators). *For any* $\mathbf{u}$ *in* $\mathbb{E}$*, we have the following estimates*

$$(\partial_\theta \mathbf{u})_n(s,x) \prec_s |n|\,|||\mathbf{u}|||\frac{c_1}{n^2+1}e^{-(M'-\int_0^s \gamma(\tau)d\tau)\langle n\rangle}\Phi\left(RX+\varepsilon\rho s\right) \tag{2.5.1}$$

$$(\partial_{x_j} \mathbf{u})_n(s,x) \prec_s R\,|||\mathbf{u}|||\frac{c_1}{n^2+1}e^{-(M'-\int_0^s \gamma(\tau)d\tau)\langle n\rangle}\Phi'\left(RX+\varepsilon\rho s\right) \tag{2.5.2}$$

for all $n \in \mathbb{Z}$ *and* $s \in [0, \underline{s})$.

Proof. The estimates (2.5.1) and (2.5.2) are straightforward. Indeed $(\partial_\theta \mathbf{u})_n = n\mathbf{u}_n$ for all $n \in \mathbb{Z}$ which implies (2.5.1). For (2.5.2) there holds $(\partial_{x_j}\mathbf{u})_n = \partial_{x_j}\mathbf{u}_n$ for all $n \in \mathbb{Z}$ and we get (2.5.2) thanks to the relation (2.4.11).

□

Remark 2.5.2 (Lack of boundedness of derivation operators). *Lemma* 2.5.1 *does not prove directly that the* ∂_{x_j} *and* ∂_θ *are not bounded operators on* $\mathbb{E}$*. But let us consider the function in* $\mathbb{E}$ *defined by its Fourier modes*

$$\mathbf{u}_n(s,x) = \frac{c_1}{n^2+1}e^{-(M'-\int_0^s \gamma(\tau)d\tau)\langle n\rangle}\Phi\left(RX+\varepsilon\rho s\right) \quad \forall n \in \mathbb{Z}$$

Then

$$\left(\partial_\theta \mathbf{u}\right)_n(s,x) = \frac{c_1 n}{n^2+1}e^{-(M'-\int_0^s \gamma(\tau)d\tau)\langle n\rangle}\Phi\left(RX+\varepsilon\rho s\right)$$

and $\partial_\theta \mathbf{u}$ *is not in* $\mathbb{E}$ *as we may not bound* $\frac{|n|}{n^2+1}$ *by* $\frac{1}{n^2+1}$*. Since* $\Phi' \prec \Phi$ *does not hold, the applications* $\partial_{x_j}\mathbf{u}$ *are not in* $\mathbb{E}$ *either. Hence the derivation operators* ∂_{x_j} *and* ∂_θ *are not bounded operators in* $\mathbb{E}$*.*

In the following, we will need exact estimates on terms like $\mathbf{v}\partial_\theta \mathbf{u}$, or $U(s',s)\partial_{x_j}\mathbf{u}(s')$.

Lemma 2.5.3 (Action of product and $U(s',s)$ on the lack of boundedness)**.** *For any* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{E}$*, for all* $n \in \mathbb{Z}$ *and* $0 \leq s' \leq< \underline{s}$*, there holds*

$$(\mathbf{v}\partial_\theta \mathbf{u})_n(s,x) \prec_s C|n|\,|||\mathbf{u}|||\,|||\mathbf{v}|||\frac{c_1}{n^2+1}e^{-(M'-\int_0^s \gamma(\tau)d\tau)\langle n\rangle}\Phi\left(RX+\varepsilon\rho s\right) \tag{2.5.3}$$

$$(\mathbf{v}\partial_{x_j} \mathbf{u})_n(s,x) \prec_s C'R\,|||\mathbf{u}|||\,|||\mathbf{v}|||\frac{c_1}{n^2+1}e^{-(M'-\int_0^s \gamma(\tau)d\tau)\langle n\rangle}\Phi'\left(RX+\varepsilon\rho s\right) \tag{2.5.4}$$

$$\begin{aligned}&(U(s',s,x,\theta)\partial_{x_j}\mathbf{u}(s',x,\theta))_n\\ &\quad\prec_{s',s} C_n(s',s)R\,\omega^{-(m-1)}\,||\mathbf{u}(s')||_{s'}\frac{c_1}{n^2+1}e^{-(M'-\int_0^s \gamma(\tau)d\tau)\langle n\rangle}\Phi'\left(RX+\varepsilon\rho s'\right)\end{aligned} \tag{2.5.5}$$

for some constants $C>0$ *and* $C'>0$ *independent of all parameters.*

Proof. To prove estimate (2.5.3) it suffices to get back to the proof of Lemma 2.4.11. Following the same computations we get

$$(\mathbf{v}\partial_\theta \mathbf{u})_n(s,x) \prec_s ||\mathbf{u}||_s\,||\mathbf{v}||_s\Phi\left(RX+\varepsilon\rho s\right)\exp\Big(-(M'-\int_0^s \gamma(\tau)d\tau)\,\langle n\rangle\Big)\sum_{p+q=n}\frac{c_1}{p^2+1}\frac{c_1|q|}{q^2+1}.$$

By adaptating the proof of the existence of some c_1 such that (2.4.15) in Lemma 2.4.6 there holds

$$\sum_{p+q=n}\frac{c_1}{p^2+1}\frac{c_1|q|}{q^2+1} \lesssim \frac{c_1|n|}{n^2+1} \quad , \quad \forall n\in\mathbb{Z}$$

and then (2.5.3) holds.

In the same way we have

$$\begin{aligned}&(\mathbf{v}\partial_{x_j} \mathbf{u})_n(s,x)\\ &\prec_s ||\mathbf{v}||_s\,||\mathbf{w}||_s\frac{c_1}{p^2+1}\exp\Big(-(M'-\int_0^s \gamma(\tau)d\tau)\,\langle n\rangle\Big)R\Phi'\left(RX+\varepsilon\rho s\right)\Phi\left(RX+\varepsilon\rho s\right)\end{aligned}$$

Thanks to Lemma 2.4.3, we differentiate the inequality $\Phi^2 \prec \Phi$ to get $2\Phi\Phi' \prec \Phi'$, hence estimate (2.5.4).

For estimate (2.5.5) it suffices to adapt the proof of Lemma 2.4.15, as $U(s',s)$ acts only on the size of the Fourier coefficients $\mathbf{u}_n(s,x)$ and not on the coefficients of the series $\mathbf{u}_{n,k}(s)$.

□

2.5.2 Integration in time and regularization of ∂_θ

Proposition 2.5.4. *For operator* $T^{[\theta]}$ *defined by* (2.3.25)*, for any* $\mathbf{u}\in B_{\mathbb{E}}(0,1)$ *there holds*

$$|||T^{[\theta]}(\mathbf{u})||| \lesssim \omega^{-(m-1)}\,\beta^{-1}|||(\mathbf{A}-\underline{\mathbf{A}})(\mathbf{u})|||\,|||\mathbf{u}|||. \tag{2.5.6}$$

Proof. By Lemma 2.4.13, the function $(\mathbf{A}-\underline{\mathbf{A}})(\cdot,\mathbf{u})$ is in $\mathbb{E}$. Applying first estimate (2.5.3) we get

$$\begin{aligned}&\big((\mathbf{A}-\underline{\mathbf{A}})(s',\mathbf{u}(s'))\partial_\theta\mathbf{u}(s')\big)_n\\ &\prec_{s'} |n|\,|||\mathbf{u}|||\,|||(\mathbf{A}-\underline{\mathbf{A}})(\mathbf{u})|||\frac{c_1}{n^2+1}e^{-(M'-\int_0^{s'} \gamma(\tau)d\tau)\langle n\rangle}\Phi\left(RX+\varepsilon\rho s'\right)\end{aligned}$$

where γ is defined in (2.4.17). Then by (2.5.5) there holds

$$\begin{aligned}&\left(U(s',s)(\mathbf{A}-\underline{\mathbf{A}})(s',\mathbf{u}(s'))\partial_\theta\mathbf{u}(s')\right)_n\\&\prec_{s',s} C_n(s',s)|n|\,\omega^{-(m-1)}\,|||\mathbf{u}|||\,|||(\mathbf{A}-\underline{\mathbf{A}})(\mathbf{u})|||\frac{c_1}{n^2+1}e^{-(M'-\int_0^s\gamma(\tau)d\tau)\langle n\rangle}\Phi\left(RX+\varepsilon\rho s\right).\end{aligned}$$

As integration in time and Fourier transform commute, we have

$$\left(T^{[\theta]}(\mathbf{u})\right)_n(s)=\int_0^s\left(U(s',s)(\mathbf{A}-\underline{\mathbf{A}})\partial_\theta\mathbf{u}(s')\right)_n ds'$$

and then

$$\begin{aligned}&\left(T^{[\theta]}(\mathbf{u})\right)_n(s)\\&\prec_s\int_0^s C_n(s',s)|n|\,\omega^{-(m-1)}\,|||\mathbf{u}|||\,|||(\mathbf{A}-\underline{\mathbf{A}})(\mathbf{u})|||\,\frac{c_1}{n^2+1}e^{-(M'-\int_0^s\gamma(\tau)d\tau)\langle n\rangle}\Phi\left(RX+\varepsilon\rho s\right)ds'\\&\prec_s\omega^{-(m-1)}\,|||\mathbf{u}|||\,|||(\mathbf{A}-\underline{\mathbf{A}})(\mathbf{u})|||\,\frac{c_1}{n^2+1}e^{-(M'-\int_0^s\gamma(\tau)d\tau)\langle n\rangle}\Phi\left(RX+\varepsilon\rho s\right)\int_0^s C_n(s',s)|n|\,ds'.\end{aligned}$$

To end the proof, we prove a uniform bound independent of n for the integral term $\int_0^s C_n(s',s)|n|\,ds'$. Recalling first the definition (2.4.31):

$$C_n(s',s)=\exp\left(-\beta\left(s-s'\right)\langle n\rangle\right)$$

there holds

$$\begin{aligned}\int_0^s C_n(s',s)|n|\,ds' &= \int_0^s\exp\left(-\beta\left(s-s'\right)\langle n\rangle\right)|n|ds'\\&= \exp\left(-\beta\,s\,\langle n\rangle\right)\int_0^s\exp\left(\beta\,s'\,\langle n\rangle\right)|n|ds'\\&\leq \beta^{-1}\end{aligned}$$

which ends the proof.

□

Thanks to the definition (2.2.8) of $\underline{A}$ and an expansion formula we make the previous result more precise:

Corollary 2.5.5. *For operator $T^{[\theta]}$ defined by* (2.3.25)*, for any $\mathbf{u}\in B_{\mathbb{E}}(0,1)$ there holds*

$$|||T^{[\theta]}(\mathbf{u})|||\lesssim\omega^{-(m-1)}\,\beta^{-1}\,\varepsilon\,|||\mathbf{u}|||^2. \tag{2.5.7}$$

Proof. By analyticity of $A(t,x,u)$ there are a family of matrices $A_{u_j}(t,x,u)$ depending analytically on (t,x,u) such that

$$A(t,x,u)-\underline{A}(t,x)=\sum_j A_{u_j}u_j.$$

This implies that

$$|||(\mathbf{A}-\underline{\mathbf{A}})(\mathbf{u})|||\leq\varepsilon|||\mathbf{u}|||$$

by definition of notation (2.3.3). □

2.5.3 Integration in time and regularization of ∂_{x_j}

After managing to deal with unbounded term $\partial_\theta \mathbf{u}$ we consider the other unbounded terms $\partial_{x_j}\mathbf{u}$. We consider then the operator $T^{[x]}$:

Proposition 2.5.6. *For operator $T^{[x]}$ defined by (2.3.26) and any $\mathbf{u} \in B_{\mathbb{E}}(0,1)$, there holds*

$$|||T^{[x]}(\mathbf{u})||| \lesssim \omega^{-(m-1)}\, R\rho^{-1}\, |||\mathbf{u}|||. \tag{2.5.8}$$

Proof. By Lemma 2.4.13, functions $\mathbf{A}_j(\cdot,\cdot,\mathbf{u}(\cdot))$ are in $\mathbb{E}$. Applying first estimate (2.5.4) we get

$$\begin{aligned}
&\left(\mathbf{A}_j(s',\mathbf{u}(s'))\partial_{x_j}\mathbf{u}(s')\right)_n \\
&\prec_{s'} R\,|||\mathbf{u}|||\,|||\mathbf{A}_j(\mathbf{u})|||\frac{c_1}{n^2+1}e^{-(M'-\int_0^{s'}\gamma(\tau)d\tau)\langle n\rangle}\Phi'\left(RX+\varepsilon\rho s'\right)
\end{aligned}$$

where we denote $|||\mathbf{A}_j(\mathbf{u})|||$ for $|||\mathbf{A}_j(\cdot,\cdot,\mathbf{u}(\cdot))|||$. Then by Lemma 2.4.15 there holds

$$\begin{aligned}
&\left(\sum_j U(s',s)\mathbf{A}_j(s',\mathbf{u}(s'))\partial_{x_j}\mathbf{u}(s')\right)_n \\
&\prec_{s',s} C_n(s',s)R\,\omega^{-(m-1)}\,|||\mathbf{u}|||\sum_j |||\mathbf{A}_j(\mathbf{u})|||\frac{c_1}{n^2+1}e^{-(M'-\int_0^{s}\gamma(\tau)d\tau)\langle n\rangle}\Phi'\left(RX+\varepsilon\rho s'\right) \\
&\prec_{s',s} R\,\omega^{-(m-1)}\,|||\mathbf{u}|||\sum_j |||\mathbf{A}_j(\mathbf{u})|||\frac{c_1}{n^2+1}e^{-(M'-\int_0^{s}\gamma(\tau)d\tau)\langle n\rangle}\Phi'\left(RX+\varepsilon\rho s'\right)
\end{aligned}$$

as $C_n(s',s)\le 1$. As integration in time and Fourier transform commute, we have

$$\left(T^{[x]}(\mathbf{u})\right)_n(s) = \int_0^s \left(U(s',s)\varepsilon\sum_j \mathbf{A}_j(s',\mathbf{u}(s'))\partial_{x_j}\mathbf{u}(s')\right)_n ds'$$

and then

$$\begin{aligned}
&\left(T^{[x]}(\mathbf{u})\right)_n \\
&\prec_s \int_0^s \varepsilon R\,\omega^{-(m-1)}\,|||\mathbf{u}|||\sum_j |||\mathbf{A}_j(\mathbf{u})|||\frac{c_1}{n^2+1}e^{-(M'-\int_0^{s}\gamma(\tau)d\tau)\langle n\rangle}\Phi'\left(RX+\varepsilon\rho s'\right)ds' \\
&\prec_s \varepsilon R\omega^{-(m-1)}\,|||\mathbf{u}|||\sum_j |||\mathbf{A}_j(\mathbf{u})|||\frac{c_1}{n^2+1}e^{-(M'-\int_0^{s}\gamma(\tau)d\tau)\langle n\rangle}\int_0^s \Phi'\left(RX+\varepsilon\rho s'\right)ds'.
\end{aligned}$$

By term-wise integration of the series, we have

$$\begin{aligned}
\int_0^s \Phi'\left(RX+\varepsilon\rho s'\right)ds' &= \int_0^s (\varepsilon\rho)^{-1}\partial_{s'}\left(\Phi\left(RX+\varepsilon\rho s'\right)\right)ds' \\
&\prec_s (\varepsilon\rho)^{-1}\Phi\left(RX+\varepsilon\rho s\right)
\end{aligned}$$

which suffices to end the proof.

□

2.5.4 Integration in time and product

As $\mathbb{E}$ is an algebra the operator $T^{[\mathbf{u}]}$ is directly bounded, with no need of a regularization by time result, on the contrary of operators $T^{[\theta]}$ and $T^{[x]}$. The following proposition gives us precisely

Proposition 2.5.7. *For the operator $T^{[u]}$ defined by* (2.3.27)*, for any $\mathbf{u} \in B_{\mathbb{E}}(0,1)$ there holds*

$$|||T^{[\mathbf{u}]}(\mathbf{u})||| \lesssim \omega^{-(m-1)}\, \beta^{-1}\, \varepsilon\, |||\mathbf{F}(\mathbf{u})|||\, |||\mathbf{u}|||. \tag{2.5.9}$$

Proof. As in the proof of Proposition 2.5.4 we have

$$\begin{aligned}
&\left(T^{[\mathbf{u}]}(\mathbf{u})\right)_n(s) \\
&\prec_s \int_0^s C_n^{\eta}(s',s)\, \omega^{-(m-1)}\, |||\mathbf{u}|||\, \varepsilon |||\mathbf{F}(\mathbf{u})||| \frac{c_1}{n^2+1} e^{-(M'-\int_0^s \gamma(\tau)d\tau)\langle n\rangle} \Phi\left(RX+\varepsilon\rho s\right) ds' \\
&\prec_s \varepsilon\omega^{-(m-1)}\, |||\mathbf{u}|||\, |||\mathbf{F}(\mathbf{u})||| \frac{c_1}{n^2+1} e^{-(M'-\int_0^s \gamma(\tau)d\tau)\langle n\rangle} \Phi\left(RX+\varepsilon\rho s\right) \int_0^s C_n(s',s)|n|\, ds'
\end{aligned}$$

and as

$$\int_0^s C_n(s',s)|n|\, ds' \lesssim \beta^{-1} \quad , \quad \forall n \in \mathbb{Z},\ \forall 0 \le s < \underline{s}$$

we get (2.5.9).

□

Using Assumption 2.2.10, we have in fact a more precise estimate:

Corollary 2.5.8. *Under Assumption* 2.2.10*, operator $T^{[\mathbf{u}]}$ defined by* (2.3.27) *satisfied for any $\mathbf{u} \in B_{\mathbb{E}}(0,1)$ the following bound*

$$|||T^{[\mathbf{u}]}(\mathbf{u})||| \lesssim \omega^{-(m-1)}\, \beta^{-1}\varepsilon\, |||\mathbf{u}|||^2. \tag{2.5.10}$$

2.5.5 Contraction estimates

The three previous subsections give us some precious estimates on operators $T^{[\theta]}$, $T^{[x]}$ and $T^{[\mathbf{u}]}$ in $\mathbb{E}$. In the perspective of using a fixed point theorem on the Banach space $\mathbb{E}$, we prove now estimates on the differences $T^{[\theta]}(\mathbf{u}) - T^{[\theta]}(\mathbf{v})$, $T^{[x]}(\mathbf{u}) - T^{[x]}(\mathbf{v})$ and $T^{[\mathbf{u}]}(\mathbf{u}) - T^{[\mathbf{u}]}(\mathbf{v})$ for $\mathbf{u}$ and $\mathbf{v}$ in the ball $B_{\mathbb{E}}(0,1)$.

Proposition 2.5.9 (Contraction estimates in $\mathbb{E}$)**.** *There are R_0, $\rho_0 > 0$ such that for all $R \ge R_0$, $\rho > \rho_0$ and $\varepsilon \in (0,1)$, we get the following estimates for all $\mathbf{u}$ and $\mathbf{v}$ in $B_{\mathbb{E}}(0,1)$:*

$$|||T(\mathbf{u})||| \lesssim \omega^{-(m-1)} \left(\beta^{-1}\left(\varepsilon|||\mathbf{F}(\mathbf{u})||| + |||\mathbf{A}(\mathbf{u}) - \underline{\mathbf{A}}(\mathbf{u})|||\right) + R\rho^{-1}\right) |||\mathbf{u}||| \tag{2.5.11}$$

$$|||T(\mathbf{u}) - T(\mathbf{v})||| \lesssim \omega^{-(m-1)} \left(\beta^{-1}\left(\varepsilon|||\mathbf{F}(\mathbf{u})||| + |||\mathbf{A}(\mathbf{u}) - \underline{\mathbf{A}}(\mathbf{u})|||\right) + R\rho^{-1}\right) |||\mathbf{u} - \mathbf{v}||| \tag{2.5.12}$$

Proof. Recalling that $T = T^{[\theta]} + T^{[x]} + T^{[\mathbf{u}]}$, we can apply directly Propositions 2.5.4, 2.5.6 and 2.5.7 to get (2.5.11).

To prove the contraction estimate (2.5.12), we write for all $\mathbf{u}$ and $\mathbf{v}$ in $B_{\mathbb{E}}(0,1)$ the following

$$T(\mathbf{u}) - T(\mathbf{v}) = \Big(T^{[\theta]}(\mathbf{u}) - T^{[\theta]}(\mathbf{v})\Big) + \Big(T^{[x]}(\mathbf{u}) - T^{[x]}(\mathbf{v})\Big) + \Big(T^{[\mathbf{u}]}(\mathbf{u}) - T^{[\mathbf{u}]}(\mathbf{v})\Big)$$

To get estimates on those three terms we first introduce some notations:

$$\begin{aligned}
T_H^{[\theta]}(s,\mathbf{u}) &= \int_0^s U(s',s)\,\mathbf{H}(\mathbf{u}(s'))\,\partial_\theta \mathbf{u}(s')ds' \\
T_H^{[x_j]}(s,\mathbf{u}) &= \int_0^s U(s',s)\,\mathbf{H}(\mathbf{u}(s'))\,\partial_{x_j} \mathbf{u}(s')ds' \\
T_H^{[\mathbf{u}]}(s,\mathbf{u}) &= \int_0^s U(s',s)\,\mathbf{H}(\mathbf{u}(s'))\,\mathbf{u}(s')ds'
\end{aligned}$$

with $H(t,x,u)$ holomorphic on the neighborhood of $(0,0,0) \in \mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^N_u$, and using notation (2.3.3). For example,

$$T^{[\theta]}(s,\mathbf{u}) = T_H^{[\theta]}(s,\mathbf{u}) \qquad \text{with } H = A - \underline{A} \tag{2.5.13}$$

Differences like $T^{[\theta]}(s,\mathbf{u}) - T^{[\theta]}(s,\mathbf{v})$ are now easier to write. For example

$$\begin{aligned}
T_H^{[\theta]}(s,\mathbf{u}) - T_H^{[\theta]}(s,\mathbf{v}) &= \int_0^s U(s',s)\,\big(\mathbf{H}(\mathbf{u}(s'))\,\partial_\theta \mathbf{u}(s') - \mathbf{H}(\mathbf{v}(s'))\,\partial_\theta \mathbf{v}(s')\big)\,ds' \\
&= \int_0^s U(s',s)\,\big(\mathbf{H}(\mathbf{u}(s')) - \mathbf{H}(\mathbf{v}(s'))\big)\,\partial_\theta \mathbf{v}(s')ds' \qquad (2.5.14) \\
&\quad + \int_0^s U(s',s)\,\mathbf{H}(\mathbf{u}(s'))\,\partial_\theta(\mathbf{u}-\mathbf{v})(s')ds' \qquad (2.5.15)
\end{aligned}$$

and these two terms are very similar to $T_H^{[\theta]}$. The same proof as Proposition 2.5.4 gives then directly

$$\left|\left|\left|\int_0^s U(s',s)\,\mathbf{H}(\mathbf{u}(s'))\,\partial_\theta(\mathbf{u}-\mathbf{v})(s')ds'\right|\right|\right| \lesssim \beta^{-1}\,|||\mathbf{H}(\mathbf{u})|||\,|||\mathbf{u}-\mathbf{v}|||$$

For the other term (2.5.14) we first note that for all (t,x,u) and (t,x,v) close to the distinguished point $(0,0,0) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^N$, with $u - v$ small enough, there holds

$$H(t,x,u) - H(t,x,v) = (u-v)\,\widetilde{H}(t,x,u,v)$$

with

$$\widetilde{H}(t,x,u,v) = \int_0^1 \partial_u H(t,x,v+y(u-v))dy.$$

Note that $\widetilde{H}$ is an analytic function of (t,x,u,v) near $(0,0,0,0)$. Hence an adaptation of the proof of Proposition 2.5.4 gives

$$\begin{aligned}&\left|\left|\left|\int_0^s U(s',s)\,\left(\mathbf{H}(\mathbf{u}(s'))-\mathbf{H}(\mathbf{v}(s'))\right)\,\partial_\theta\mathbf{v}ds'\right|\right|\right|\\&\lesssim \omega^{-(m-1)}\,\beta^{-1}\varepsilon\,|||\mathbf{u}-\mathbf{v}|||\,|||\widetilde{\mathbf{H}}(\mathbf{u},\mathbf{v})|||\,|||\mathbf{v}|||\\&\lesssim \omega^{-(m-1)}\,\beta^{-1}\varepsilon\,|||\mathbf{u}-\mathbf{v}|||\,|||\widetilde{\mathbf{H}}(\mathbf{u},\mathbf{v})|||\end{aligned}$$

as $\mathbf{v}\in B_{\mathbb{E}}(0,1)$, and recalling the prefactor ε in notation (2.3.3). In particular, for $H=A-\underline{A}$ we have just for all $\mathbf{u}$ and $\mathbf{v}$ in $B_{\mathbb{E}}(0,1)$ both

$$|||\mathbf{H}(\mathbf{u})|||\lesssim|||\mathbf{A}(\mathbf{u})-\underline{\mathbf{A}}(\mathbf{u})|||\quad\text{and}\quad|||\widetilde{\mathbf{H}}(\mathbf{u},\mathbf{v})|||\lesssim 1$$

thanks to Lemma 2.4.13. Finally there holds for all $\mathbf{u}$ and $\mathbf{v}$ in $B_{\mathbb{E}}(0,1)$:

$$|||T^{[\theta]}(\mathbf{u})-T^{[\theta]}(\mathbf{v})|||\quad\lesssim\quad\omega^{-(m-1)}\,\beta^{-1}\left(|||\mathbf{A}(\mathbf{u})-\underline{\mathbf{A}}(\mathbf{u})|||+\varepsilon\right)|||\mathbf{u}-\mathbf{v}|||.$$

For both $T^{[x]}(\mathbf{u})-T^{[x]}(\mathbf{v})$ and $T^{[\mathbf{u}]}(\mathbf{u})-T^{[\mathbf{u}]}(\mathbf{v})$ we do the same to finally get

$$\begin{aligned}|||T^{[x]}(\mathbf{u})-T^{[x]}(\mathbf{v})|||&\lesssim\omega^{-(m-1)}\,R\rho^{-1}|||\mathbf{u}-\mathbf{v}|||\\|||T^{[\mathbf{u}]}(\mathbf{u})-T^{[\mathbf{u}]}(\mathbf{v})|||&\lesssim\omega^{-(m-1)}\,\beta^{-1}\varepsilon|||\mathbf{u}-\mathbf{v}|||\end{aligned}$$

as ε is small.

□

Thanks to Corollary 2.5.5, we have a finer version of the contraction estimates:

Corollary 2.5.10 (Finer contraction estimates in $\mathbb{E}$)**.** *There are R_0, $\rho_0>0$ such that for all $\beta>0$, $R\geq R_0$, $\rho>\rho_0$ and $\varepsilon\in(0,1)$, we get the following estimates for all $\mathbf{u}$ and $\mathbf{v}$ in $B_{\mathbb{E}}(0,1)$:*

$$|||T(\mathbf{u})|||\lesssim\omega^{-(m-1)}\,\left(\beta^{-1}\,\varepsilon|||\mathbf{u}|||+R\rho^{-1}\right)|||\mathbf{u}|||\tag{2.5.16}$$

$$|||T(\mathbf{u})-T(\mathbf{v})|||\lesssim\omega^{-(m-1)}\,\left(\beta^{-1}\,\varepsilon|||\mathbf{u}|||+R\rho^{-1}\right)|||\mathbf{u}-\mathbf{v}|||.\tag{2.5.17}$$

2.6 Existence of solutions and estimates from below

2.6.1 Existence of solutions

Thanks to the Corollary 2.5.10, we can now solve the fixed point equation (2.3.22) in the ball $B_{\mathbb{E}}\left(0,|||\mathbf{f}_\varepsilon|||\right)$, provided that $|||\mathbf{f}_\varepsilon|||\leq 1/2$:

Corollary 2.6.1 (Existence of solutions)**.** *Let $R(\varepsilon)>R_0$, $\rho(\varepsilon)>\rho_0$, $\beta(\varepsilon)>0$ and $\underline{s}(\varepsilon)$ be such that*

$$\lim_{\varepsilon\to 0}\omega^{-(m-1)}\,\left(\beta^{-1}\varepsilon|||\mathbf{f}_\varepsilon|||+R\rho^{-1}\right)=0.\tag{2.6.1}$$

Then for any ε small enough, the fixed point equation (2.3.22), with $\mathbf{f}_\varepsilon$ defined by (2.3.15), has a unique solution $\mathbf{u}_\varepsilon$ in $B_{\mathbb{E}(R,\rho)}\left(0,2|||\mathbf{f}_\varepsilon|||\right)$. This solution satisfies

$$|||\mathbf{u}_\varepsilon-\mathbf{f}_\varepsilon|||\lesssim\omega^{-(m-1)}\,\left(\beta^{-1}\varepsilon|||\mathbf{f}_\varepsilon|||+R\rho^{-1}\right)|||\mathbf{f}_\varepsilon|||.\tag{2.6.2}$$

The proof of the Corollary is straigthforward using the estimates of Corollary 2.5.10, under the condition of smallness (2.6.1). For convenience we introduce

$$K(\varepsilon) = \omega^{-(m-1)} \left(\beta^{-1}\, \varepsilon |||\mathbf{f}_\varepsilon||| + R\rho^{-1}\right). \tag{2.6.3}$$

2.6.2 Bounds from below for the solutions

Recall that in Section 2.3.6, we explained that to prove Hadamard instability, we prove first that the solution $\mathbf{u}_\varepsilon$ of (2.3.22) has the same growth as $\mathbf{f}_\varepsilon$ given by Lemma 2.3.2. That is, the goal is to prove

$$|\mathbf{u}_\varepsilon(s,x,\theta)| \gtrsim \omega^{-(m-1)}\, e^{-M} \exp\left(\int_0^s \gamma^\flat(\tau\,;r)d\tau\right) \quad , \quad \forall\,(s,x,\theta) \in (\underline{s}-1,\underline{s}) \times B_r(0) \times \mathbb{T} \tag{2.6.4}$$

with $\gamma^\flat$ given by either (2.3.17) (under Assumption 2.2.4), (2.3.18) (under Assumption 2.2.6) or (2.3.19) (under Assumptions 2.2.6 and 2.2.8). It is indeed sufficient to prove this kind of estimate only on a small neighborhood of $(\underline{s},0) \times \mathbb{T}$, and not on all the domain $\Omega_{R,\varepsilon\rho}(0) \times \mathbb{T}$. To this effect in view of Lemma 2.3.2 it suffices to prove that

$$|(\mathbf{u}_\varepsilon - \mathbf{f}_\varepsilon)(s,x,\theta)| \lesssim C(\varepsilon)\omega^{-(m-1)}\, e^{-M(\varepsilon)} \exp\left(\int_0^s \gamma^\flat(\tau\,;r)d\tau\right) \tag{2.6.5}$$

for some constant $C(\varepsilon)$ such that $C(\varepsilon) \to 0$ as $\varepsilon \to 0$. The constant $C(\varepsilon)$ will depend on the parameters M', R, ρ, β and ω. Finding suitable parameters such that $C(\varepsilon) \to 0$ as $\varepsilon \to 0$ will depend on under which Assumption we work, as it is precised in Propositions 2.6.2, 2.6.3 and 2.6.4.

First, we decompose $\mathbf{u}_\varepsilon - \mathbf{f}_\varepsilon$ with its Fourier modes

$$(\mathbf{u}_\varepsilon - \mathbf{f}_\varepsilon)(s,x,\theta) = \sum_{n\in\mathbb{Z}} (\mathbf{u} - \mathbf{f}_\varepsilon)_n(s,x)e^{in\theta}.$$

Thanks to the first property of Lemma 2.4.3 and estimate (2.4.23), for all (s,x,θ) in $\Omega_{R,\varepsilon\rho}(0) \times \mathbb{T}$ there holds

$$\begin{aligned}
|(\mathbf{u}_\varepsilon - \mathbf{f}_\varepsilon)(s,x,\theta)| &\leq \sum_{n\in\mathbb{Z}} |(\mathbf{u}_\varepsilon - \mathbf{f}_\varepsilon)_n|(s,x) \\
&\leq |||\mathbf{u}_\varepsilon - \mathbf{f}_\varepsilon||| \sum_{n\in\mathbb{Z}} \frac{c_1}{n^2+1} \exp\left(-\left(M' - \int_0^s \gamma(\tau)d\tau\right)\langle n\rangle\right) \Phi\left(R|x|_1 + \varepsilon\rho s\right)
\end{aligned}$$

where γ is defined in (2.4.17). Then, as $M' - \int_0^s \gamma(\tau)d\tau > 0$ for any $s \in [0,\underline{s})$ (recall definition (2.4.19) of $\underline{s}_1$ and definition (2.4.20) of $\underline{s}$) and $\langle n\rangle \geq 1$ for all n, we have

$$\begin{aligned}
|(\mathbf{u}_\varepsilon - \mathbf{f}_\varepsilon)(s,x,\theta)| &\leq |||\mathbf{u}_\varepsilon - \mathbf{f}_\varepsilon||| \exp\left(-\left(M' - \int_0^s \gamma(\tau)d\tau\right)\langle n\rangle\right) \sum_{n\in\mathbb{Z}} \frac{c_1}{n^2+1} \Phi\left(R|x|_1 + \varepsilon\rho s\right) \\
&\leq |||\mathbf{u}_\varepsilon - \mathbf{f}_\varepsilon||| \exp\left(-\left(M' - \int_0^s \gamma(\tau)d\tau\right)\langle n\rangle\right) \sum_{n\in\mathbb{Z}} \frac{c_1}{n^2+1} \Phi(1)
\end{aligned}$$

and the last inequality holds because Φ is convergent in 1. As the series of the right-hand side of the previous inequality is convergent, there holds

$$|(\mathbf{u}_\varepsilon - \mathbf{f}_\varepsilon)(s,x,\theta)| \lesssim |||\mathbf{u}_\varepsilon - \mathbf{f}_\varepsilon||| \exp\left(-\left(M' - \int_0^s \gamma(\tau)d\tau\right)\langle n\rangle\right)$$

for all $(s,x,\theta) \in \Omega_{R,\varepsilon\rho}(0) \times \mathbb{T}$.

Next, by Lemma 2.4.17, estimate (2.6.1) of Corollary 2.6.1 and notation (2.6.3), we have successively

$$\begin{aligned} |(\mathbf{u}_\varepsilon - \mathbf{f}_\varepsilon)(s,x,\theta)| &\lesssim K(\varepsilon)\, |||\mathbf{f}_\varepsilon||| \exp\left(-\left(M' - \int_0^s \gamma(\tau)d\tau\right)\langle n\rangle\right) \\ &\lesssim K(\varepsilon)\, \omega^{-(m-1)} e^{M'-M(\varepsilon)} \exp\left(-\left(M' - \int_0^s \gamma(\tau)d\tau\right)\langle n\rangle\right) \\ &\lesssim K(\varepsilon)\, \omega^{-(m-1)} e^{-M(\varepsilon)} \exp\left(\int_0^s \gamma(\tau)d\tau\right) \end{aligned} \tag{2.6.6}$$

using $\langle n\rangle \geq 1$ for all $n \in \mathbb{Z}$. Note that estimate (2.6.6) holds pointwise for all (s,x,θ) in $\Omega_{R,\varepsilon\rho}(0) \times \mathbb{T}$. Now we focus our analysis to the smaller domain $(\underline{s}-1, \underline{s}) \times B_r(0) \times \mathbb{T}$. Having (2.6.5) in mind, we rewrite (2.6.6) to get

$$\begin{aligned} &|(\mathbf{u}_\varepsilon - \mathbf{f}_\varepsilon)(s,x,\theta)| \\ &\lesssim K(\varepsilon) \exp\left(\int_0^s (\gamma(\tau) - \gamma^\flat(\tau\,;r))d\tau\right) \omega^{-(m-1)}\, e^{-M} \exp\left(\int_0^s \gamma^\flat(\tau\,;r)d\tau\right) \\ &\lesssim K(\varepsilon) \exp\left(\int_0^{\underline{s}} (\gamma(\tau) - \gamma^\flat(\tau\,;r))d\tau\right) \omega^{-(m-1)}\, e^{-M} \exp\left(\int_0^s \gamma^\flat(\tau\,;r)d\tau\right) \\ &\lesssim K(\varepsilon) \exp\left(\underline{s}\,\beta + \int_0^{\underline{s}} (\gamma^\sharp(\tau\,;R,\omega) - \gamma^\flat(\tau\,;r,\omega))d\tau\right) \omega^{-(m-1)}\, e^{-M} \exp\left(\int_0^s \gamma^\flat(\tau\,;r,\omega)d\tau\right) \end{aligned} \tag{2.6.7}$$

by definition (2.4.20) of $\underline{s}$ and definition (2.4.17) of γ. So to get (2.6.5) we need

$$\lim_{\varepsilon\to 0} K(\varepsilon) \exp\left(\underline{s}\,\beta + \int_0^{\underline{s}} (\gamma^\sharp(\tau\,;R,\omega) - \gamma^\flat(\tau\,;r,\omega))d\tau\right) = 0.$$

If $K(\varepsilon) \to 0$ as in (2.6.1), and as $\omega(\varepsilon)$ is a small parameter, it suffices then to have

$$\lim_{\varepsilon\to 0} \exp\left(\underline{s}\,\beta + \int_0^{\underline{s}} (\gamma^\sharp(\tau\,;R,\omega) - \gamma^\flat(\tau\,;r,\omega))d\tau\right) = 0 \tag{2.6.8}$$

which brings another constraint on the parameters, after (2.6.1).

We recall also the constraint on the parameters M' and ρ coming from the competition between the growth time $\underline{s}_1$ defined in (2.4.19) and the regularity time $(\varepsilon\rho)^{-1}$. To see the growth of the solution, we need it to exist on a sufficiently large time compared to the growth time, that is we need $\underline{s} = \underline{s}_1$. This is equivalent to

$$\lim_{\varepsilon\to 0} \underline{s}_1 \varepsilon\rho = 0. \tag{2.6.9}$$

A last constraint on the parameters comes from the smallness of the norm of the free solution, that is

$$\lim_{\varepsilon\to 0} \omega^{-(m-1)}\, e^{M'-M} = 0 \tag{2.6.10}$$

following Lemma 2.4.17.

In constraint (2.6.8), recall that bound $\gamma^\sharp(\tau\,; R, \omega)$ is defined in Lemma 2.3.1. Under Assumption 2.2.4, the bound $\gamma^\sharp$ is given by (2.3.10) ; under Assumption 2.2.6, by (2.3.11) ; and under Assumptions 2.2.6 and 2.2.8, by (2.3.12). Similarly, recall that bound $\gamma^\flat(\tau\,; r, \omega)$ is defined in Lemma 2.3.2. Under Assumption 2.2.4, the bound $\gamma^\flat$ is given by (2.3.17) ; under Assumption 2.2.6, by (2.3.18) ; and under Assumptions 2.2.6 and 2.2.8, by (2.3.19). In each case, we combine altogether constraints (2.6.1), (2.6.8), (2.6.9) and (2.6.10), and we give in the following three Propositions a choice of parameters satisfying those constraints.

Proposition 2.6.2. *Under Assumption* 2.2.4*, with the following choice of parameters*

$$\omega = \varepsilon^\delta, \quad \beta = \varepsilon^\delta, \quad R^{-1} = \varepsilon^\delta, \quad \rho^{-1} = \varepsilon^{(1+(m-1)\delta)/2}, \quad M' = M(\varepsilon) - \min\{0, 1-(2m-1)\delta\}|\ln(\varepsilon)| \tag{2.6.11}$$

and the limitation on the Gevrey index

$$\sigma < \delta < 1/(m+1)$$

where m is the algebraic multiplicity of λ_0, the fixed point equation (2.3.22) *has a unique solution $\mathbf{u}_\varepsilon$ in $\mathbb{E}$ which satisfies*

$$|\mathbf{u}_\varepsilon(s,x,\theta)| \gtrsim \varepsilon^{-\delta(m-1)} e^{-M(\varepsilon)} \exp\left(\int_0^s \gamma^\flat(\tau\,; r, \omega) d\tau\right) \quad , \quad \forall\,(s,x,\theta) \in (\underline{s}-1, \underline{s}) \times B_r(0) \times \mathbb{T} \tag{2.6.12}$$

for any $r \lesssim \varepsilon^\delta$. Another consequence of (2.6.11) *is*

$$\underline{s} \approx \varepsilon^{-\delta}. \tag{2.6.13}$$

Proof. It is straightforward to verify that parameters given by (2.6.11) satisfy the four constraints (2.6.1), (2.6.8), (2.6.9) and (2.6.10). The aim of the proof is to show that those parameters are optimal, in some sens. For that, we assume that the constraints are satisfy and we get constraints directly on M', ρ, R, ω and β.

First, (2.6.9) being satisfied the final time is

$$\underline{s} = \underline{s}_1$$

defined by (2.4.19). In the asymptotic $\varepsilon \to 0$ there holds

$$\begin{aligned} \int_0^{\underline{s}_1} \gamma(\tau) d\tau &\sim \underline{s}_1 \gamma(\underline{s}_1) \\ &\approx \gamma_0 \underline{s}_1 \end{aligned}$$

which implies that

$$\underline{s}_1 \approx \frac{M'}{\gamma_0}.$$

Constraint (2.6.10) implies that $M' - M = -c(\varepsilon) + (m-1)\ln\omega$ with $\lim_{\varepsilon\to 0} c(\varepsilon) = +\infty$. We assume that $c(\varepsilon) = o\left(\varepsilon^{-\delta}\right)$ to get $M' \sim M$, hence

$$\underline{s} \approx M = \varepsilon^{-\delta}.$$

We also rewrite (2.6.9) as

$$\lim_{\varepsilon\to 0} \varepsilon^{1-\delta}\rho = 0. \tag{2.6.14}$$

Second, we focus on (2.6.8). By definitions (2.3.10) and (2.3.17) we have

$$\int_0^{\underline{s}} (\gamma^\sharp(\tau\,; R, \omega) - \gamma^\flat(\tau\,; r, \omega))d\tau \lesssim \underline{s}\left(\varepsilon\underline{s} + R^{-1} + r + \omega\right).$$

As $\underline{s} \approx \varepsilon^{-\delta}$, for (2.6.8) to be satisfied we need $\underline{s}\left(\beta + \varepsilon\underline{s} + R^{-1} + r + \omega\right)$ to be bounded, hence the choices

$$\beta = \varepsilon^\delta \quad , \quad r = \varepsilon^\delta \quad , \quad \omega = \varepsilon^\delta$$

and the constraints

$$\varepsilon\underline{s}^2 \lesssim 1 \quad , \quad R^{-1} \lesssim \varepsilon^\delta. \tag{2.6.15}$$

The first one implies in particular

$$\delta < 1/2.$$

The constraint (2.6.1) is now

$$\lim_{\varepsilon\to 0} \varepsilon^{-\delta(m-1)}\left(\varepsilon^{1-\delta}\varepsilon^{-\delta(m-1)}e^{M'-M} + R\rho^{-1}\right) = 0$$

using (2.4.33), and that is equivalent to both

$$\lim_{\varepsilon\to 0} e^{M'-M}\varepsilon^{1-\delta(2m-1)} = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} \varepsilon^{-\delta(m-1)}R\rho^{-1} = 0.$$

The first limit leads to the choice

$$M' = M - \min\{0, 1-(2m-1)\delta\}|\ln(\varepsilon)|$$

reminding that $\delta \in (0, 1/m)$. The second limit, combined with (2.6.14), gives us

$$\varepsilon^{1-\delta} \ll \rho^{-1} \ll \varepsilon^{\delta(m-1)}R^{-1} \tag{2.6.16}$$

using notation (2.1.32). We note then that in particular, R^{-1} has to be greater than $\varepsilon^{1-m\delta}$. As R^{-1} has to be also smaller than ε^δ, it implies the limitation

$$\varepsilon^{1-\delta} \ll \varepsilon^{\delta(m-1)}\varepsilon^\delta \tag{2.6.17}$$

which is equivalent to

$$\delta < 1/(m+1),$$

compatible with the previous limitation $\delta < 1/2$ as $m \geq 1$.

□

Proposition 2.6.3. *Under Assumption* 2.2.6*, with the following choice of parameters*

$$\omega = 0, \quad \beta = \varepsilon^{\delta}, \quad R^{-1} = \varepsilon, \quad \rho^{-1} = \varepsilon^{1-\delta/2}, \quad M' = M(\varepsilon) - (1-\delta)|\ln(\varepsilon)| \tag{2.6.18}$$

and the limitation on the Gevrey index

$$\sigma < \delta < 1/2$$

the fixed point equation (2.3.22) *has a unique solution* $\mathbf{u}_\varepsilon$ *in* $\mathbb{E}$ *which satisfies*

$$|\mathbf{u}_\varepsilon(s,x,\theta)| \gtrsim e^{-M(\varepsilon)} \exp\left(\int_0^s \gamma^\flat(\tau\,;r,\omega)d\tau\right) \quad , \quad \forall\,(s,x,\theta) \in (\underline{s}-1,\underline{s}) \times B_r(0) \times \mathbb{T} \tag{2.6.19}$$

for any $r \lesssim \varepsilon^{\delta}$*. Another consequence of* (2.6.11) *is*

$$\underline{s} \approx \varepsilon^{-\delta}. \tag{2.6.20}$$

Proof. The proof is the same the one of Proposition 2.6.2, with the difference that with Assumption 2.2.6, estimate (2.6.16) is replaced by $\varepsilon^{1-\delta} \ll \rho^{-1} \ll R^{-1}$ as $m = 1$. Hence constraint (2.6.17) is now $\varepsilon^{1-\delta} \ll \varepsilon^{\delta(m-1)}\varepsilon^{\delta}$ which is equivalent to $\delta < 1/2$.

□

Proposition 2.6.4. *Under Assumptions* 2.2.6 *and* 2.2.8*, with the following choice of parameters*

$$\omega = 1, \quad \beta = \varepsilon^{\delta}, \quad R^{-1} = \varepsilon, \quad \rho^{-1} = \varepsilon^{1-\delta/2}, \quad M' = M(\varepsilon) - (1-\delta)|\ln(\varepsilon)| \tag{2.6.21}$$

and the limitation on the Gevrey index

$$\sigma < \delta < 2/3$$

the fixed point equation (2.3.22) *has a unique solution* $\mathbf{u}_\varepsilon$ *in* $\mathbb{E}$ *which satisfies*

$$|\mathbf{u}_\varepsilon(s,x,\theta)| \gtrsim e^{-M(\varepsilon)} \exp\left(\int_0^s \gamma^\flat(\tau\,;r,\omega)d\tau\right) \quad , \quad \forall\,(s,x,\theta) \in (\underline{s}-1,\underline{s}) \times B_r(0) \times \mathbb{T} \tag{2.6.22}$$

for any $r \lesssim \varepsilon^{\delta}$*. Another consequence of* (2.6.11) *is*

$$\underline{s} \approx \varepsilon^{-\delta}. \tag{2.6.23}$$

Proof. The proof is the same the one of Proposition 2.6.2, with the difference that with Assumption 2.2.6, the bounds (2.3.10) and (2.3.17) are replaced by the sharper bounds (2.3.12) and (2.3.19), respectively. First, note that the parameter of trigonalization ω does not appear anymore, and is then taken equal to one. Second, thanks to Assumption 2.2.6, difference $\gamma^\sharp - \gamma^\flat$ is improved:

$$\gamma^\sharp(\tau\,;R,\omega) - \gamma^\flat(\tau\,;r,\omega) \lesssim \varepsilon^2\underline{s}^2 + r. \tag{2.6.24}$$

This implies in particular that

$$\int_0^{\underline{s}} (\gamma^\sharp(\tau\,;R,\omega) - \gamma^\flat(\tau\,;r,\omega))d\tau \lesssim \underline{s}\left(r + \varepsilon^2\underline{s}^2\right)$$

which no longer implies constraints (2.6.15). It suffices then to follow the rest of the proof of Proposition 2.6.2. □

Remark 2.6.5. *Estimate* (2.6.24) *in the previous proof shows that the limiting Gevrey index increases as* $\gamma^\sharp - \gamma^\flat$ *decreases (with* $\gamma^\sharp$ *and* $\gamma^\flat$ *the upper and lower rates of growth introduced in Lemmas* 2.3.1 *and* 2.3.2*). In particular, if the distinguished eigenvalue* λ *is very flat at the distinguished point* $(0, x_0)$*, then the limiting Gevrey index is close to* 1*, as claimed in Remark* 2.2.11*.*

2.7 Conclusion: Hadamard instability in Gevrey spaces

To close the proofs of Theorems 1, 2 and 3 we have now to get an estimate of the ratio

$$\frac{||u_\varepsilon||_{L^2(\Omega_{R,\rho}(0))}}{||h_\varepsilon||^\alpha_{\sigma,c,K}}$$

The previous Sections show the existence of a family of solutions $\mathbf{u}$ starting from $\mathbf{f}_\varepsilon$ of the fixed point equation (2.3.22). Thanks to the ansatz (2.3.2) which we recall here

$$u_\varepsilon(t,x) = \varepsilon \mathbf{u}(\varepsilon^{-1}\, t, x, x \cdot \xi_0/\varepsilon)$$

we have then a family of solutions u_ε existing in domains $\Omega_{R,\rho}(0)$, with R and ρ given by (2.6.11). As $\underline{s} < (\varepsilon\rho)^{-1}$ the domain of regularity $\Omega_{R,\rho}(0)$ for $\mathbf{u}$ contains the cube of size ε

$$C_\varepsilon = \{(t,x)\,|\,\varepsilon\underline{s} - \varepsilon < t < \varepsilon\underline{s}, \quad |x| < \varepsilon\}.$$

On one hand, thanks to estimate (2.6.4) with $r = \varepsilon$ there holds

$$\begin{aligned}
||u_\varepsilon||_{L^2(\Omega_{R,\rho})} &\geq ||u_\varepsilon||_{L^2(C_\varepsilon)} \\
&\gtrsim \inf_{\varepsilon\underline{s}-\varepsilon<t<\varepsilon\underline{s}} \left(\varepsilon^{-\delta(m-1)} e^{-M(\varepsilon)} \exp\left(\int_0^{t/\varepsilon} \gamma^\flat(\tau/\varepsilon) d\tau \right) \right) ||1||_{L^2(C_\varepsilon)} \\
&\gtrsim \varepsilon^{-\delta(m-1)} e^{-M(\varepsilon)} \exp\left((\underline{s}-1)\left(\gamma_0 - \varepsilon\underline{s} - r - \omega\right)\right) \varepsilon^{(d+1)/2} \\
&\gtrsim \varepsilon^{-\delta(m-1)} e^{-M(\varepsilon)} e^{\gamma_0 \underline{s}}\, \varepsilon^{(d+1)/2}
\end{aligned}$$

Next, by choice of $M' = M - (m\delta - 1)|\ln(\varepsilon)|$ we get

$$||u_\varepsilon||_{L^2(\Omega_{R,\rho})} \gtrsim \varepsilon^{-\delta(2m+1)+1} e^{-M'(\varepsilon)} e^{\gamma_0 \underline{s}}\, \varepsilon^{1+(d+1)/2}.$$

As $M' = \underline{s}\gamma = \underline{s}\gamma_0(1 + 2\varepsilon^\delta)$, this implies that

$$\begin{aligned}
||u_\varepsilon||_{L^2(\Omega_{R,\rho})} &\gtrsim e^{-\underline{s}\gamma_0(1+2\varepsilon^\delta)+\gamma_0\underline{s}}\, \varepsilon^{1+(d+1)/2-\delta(2m+1)} \\
&\gtrsim \varepsilon^{1+(d+1)/2-\delta(2m+1)}
\end{aligned}$$

thanks to $\underline{s}\varepsilon^\delta \approx 1$.

On the other hand, by Lemma 2.3.3 and definition (2.3.21) of M there holds

$$||h_\varepsilon||_{\sigma,c,K} \lesssim \varepsilon e^{-M} e^{c\varepsilon^{-\sigma}} = \varepsilon \exp(c\varepsilon^{-\sigma} - \varepsilon^{-\delta})$$

which is small as soon as $\sigma < \delta$. Combining those two estimates we have then

$$\frac{||u_\varepsilon||_{L^2(\Omega_{R,\rho})}}{||h_\varepsilon||^\alpha_{\sigma,c,K}} \gtrsim \varepsilon^{1+(d+1)/2-\delta(2m+1)-\alpha} \exp(-\alpha c \varepsilon^{-\sigma} + \alpha \varepsilon^{-\delta})$$

that tends to $+\infty$ as $\varepsilon \to 0$ because $\sigma < \delta$ no matter whether $1+(d+1)/2-\delta(2m+1)-\alpha$ is positive or negative , which ends the proof of Theorem 1.

The proofs of Theorems 2 and 3 rely on the exact same computations, using Proposition 2.6.3 and Proposition 2.6.4 respectively, instead of Proposition 2.6.2.

Chapter 3

On hyperbolicity and Gevrey well-posedness. Part two: Scalar or degenerate transitions

For first-order quasi-linear systems of partial differential equations, we formulate an assumption of a transition from initial hyperbolicity to ellipticity. This assumption bears on the principal symbol of the first-order operator. Under such an assumption, we prove a strong Hadamard instability for the associated Cauchy problem, namely an instantaneous defect of Hölder continuity of the flow from G^σ to L^2, with $0 < \sigma < \sigma_0$, the limiting Gevrey index σ_0 depending on the nature of the transition. We restrict here to scalar transitions, and non-scalar transitions in which the boundary of the hyperbolic zone satisfies a flatness condition. As in our previous work for initially elliptic Cauchy problems [B. Morisse, *On hyperbolicity and Gevrey well-posedness. Part one: the elliptic case*, arXiv:1611.07225], the instability follows from a long-time Cauchy-Kovalevskaya construction for highly oscillating solutions. This extends recent work of N. Lerner, T. Nguyen, and B. Texier [*The onset of instability in first-order systems*, to appear in J. Eur. Math. Soc.].

This Chapter is the article [Mor16b].

3.1 Introduction

We consider the following Cauchy problem, for first-order quasi-linear systems of partial differential equations:

$$\partial_t u = \sum_{j=1}^{d} A_j(t,x,u)\partial_{x_j} u + f(t,x,u)\ , \qquad u(0,x) = h(x). \tag{3.1.1}$$

The system is of size N, that is $u(t,x)$ and $f(t,x,u)$ are in $\mathbb{R}^N$ and the $A_j(t,x,u) \in \mathbb{R}^{N\times N}$. The time t is nonnegative, and x is in $\mathbb{R}^d$. We assume throughout the paper that the A_j and f are analytic in a neighborhood of some point $(0, x_0, u_0) \in \mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^N_u$.

Under assumptions of weak defects of hyperbolicity for the first-order operator, we prove ill-posedness of (3.1.1) in Gevrey spaces. Weak defect of hyperbolicity is here understood as a transition from hyperbolicity of the principal symbol at initial time, to ellipticity of the principal symbol for later times. Our results extend Métivier's ill-posedness theorem in Sobolev spaces for initially elliptic operators [Mét05], our own ill-posedness result in Gevrey spaces for initially elliptic operators [Mor16a], Lerner, Nguyen and Texier's theorem on systems transitioning from hyperbolicity to ellipticity [LNT17], and echo Lu's construction of WKB profiles [Lu16] which are destabilized by terms not present in the initial data.

Our proofs use Métivier's method developed in [Mét05] based on majoring series, hence the assumption of analyticity for the A_j and f. Our assumptions of weak defects of hyperbolicity mean that the operator in (3.1.1) experiences a transition in time from hyperbolicity to non hyperbolicity. The transition is possibly not uniform in space. Our assumptions bearing on the principal symbol, and the associated normal forms, are presented in Section 3.2. Our results are Theorems 4 and 5, stated in Section 3.2.4. The proofs comprise Sections 3.3 to 3.5.

In the companion paper [Mor17b], we consider the case of genuinely non-scalar transitions.

3.1.1 Background

A long-time Cauchy-Kovaleveskaya result for elliptic Cauchy problems

Previous Chapter 2 contains a long-time Cauchy-Kovalevskaya theorem, based on the paper [Mét05] by Métivier, which proves an Hadamard instability result for initially elliptic quasi-linear systems in Gevrey spaces. Precisely, the result of [Mor16a] asserts that the flow associated to the Cauchy problem (3.1.1) fails to be Hölder from a highly regular σ-Gevrey space to the very lowly regular L^2 space, locally in the x variable and for σ less than a critical exponent σ_0 depending on initial spectrum, under the assumption of initial ellipticity for the first-order differential operator.

Here initial ellipticity is understood as an initial defect of hyperbolicity. That is for some $(x_0, \vec{u}_0, \xi_0) \in \mathbb{R}^d_x \times \mathbb{R}^N_u \times \mathbb{R}^d_\xi$, the principal symbol at $(0, x_0, \vec{u}_0, \xi_0)$:

$$A_0 := \sum_{j=1}^{d} A_j(0, x_0, \vec{u}_0)\xi_{0,j}$$

has at least one couple of non-real eigenvalues, with imaginary part $\pm i\gamma_0$ associated to eigenvectors $\vec{e}_\pm$.

In Chapter 2 we posit in (2.3.2) the ansatz $u_\varepsilon(t,x) = \varepsilon \mathbf{u}(t/\varepsilon, x, (x - x_0)\cdot \xi_0/\varepsilon)$, where $\mathbf{u}(s, x, \theta)$ is periodic in the θ variable. We transform then the Cauchy problem (3.1.1) into the equation

$$\partial_s \mathbf{u} - \underline{A}(\varepsilon s, x)\partial_\theta \mathbf{u} = G(\mathbf{u}) \tag{3.1.2}$$

for some non-linear remainder term $G(\mathbf{u})$. The leading term $\underline{A}(t,x)$ is here the principal symbol

$$\underline{A}(t,x) = \sum_j A_j(t, x, \vec{u}_0)\xi_{0,j}. \tag{3.1.3}$$

The ellipticity condition is an open condition bearing on the principal symbol $\underline{A}$. In particular, ellipticity at $(0, x_0)$ implies ellipticity around $(0, x_0)$. The proof of Chapter 2 introduces the propagator U defined by

$$\partial_s U(s', s, x, \theta) - \underline{A}(\varepsilon s, x)\partial_\theta U(s', s, x, \theta) = 0 \quad , \quad U(s', s', x, \theta) = \mathrm{Id}. \tag{3.1.4}$$

By ellipticity, the propagator U has an exponential growth. We introduce an appropriate Banach space of functions of (s, x, θ) which are analytical in the x variable and whose Fourier coefficients in θ have an exponential growth which reflects the growth of the propagator. A fixed point argument shows existence and uniqueness, and exponential growth in this space, which implies the Hadamard instability.

The main issue in Chapter 2, compared to the previous analysis of Metivier [Mét05], is that in Gevrey spaces, the Hadamard instability is recorded at much longer times than in Sobolev spaces. The instability is observed thanks to highly oscillating, well-polarized initial data, which generate solutions growing exponentially both in time and frequency. Observing an instability means that at some time, the L^2 norm of the solution is far greater (with respect to the frequency) than the Sobolev or Gevrey norm of the initial datum. Considering the fundamental oscillation $e^{ix\cdot\xi}$ with frequency ξ, a simple computation leads to the Sobolev norm $||e^{ix\cdot\xi}||_{H^m} \simeq |\xi|^m$, whether the Gevrey norm is $||e^{ix\cdot\xi}||_{\sigma,c,K} \simeq e^{(\sigma c^\sigma)^{-1}|\xi|^\sigma}$ (see Definition 2.2.1 and Lemma 2.3.3 in Chapter 2). Hence the observation of the instability is recorded as a much longer time in Gevrey spaces than in Sobolev spaces.

Lerner, Morimoto and Xu's result on transition to ellipticity for scalar equations

In [LMX10], Lerner, Morimoto and Xu introduce the notion of transition to ellipticity for initially hyperbolic systems. A prototypical example is the Burgers equation with a complex forcing: $\partial_t u + u\partial_x u = i$. In the case of real data, the principal symbol is initially hyperbolic. Due to the complex forcing, the principal symbol is elliptic for ulterior times. For general equations (3.1.1) (with $N = 1$: scalar equations), under bracket conditions generalizing the situation for Burgers with complex forcing, and describing a transition from hyperbolicity to ellipticity, the authors in [LMX10] prove a strong form of instability, namely that if local C^2 solutions exist, then the complement of the analytic wave-front set of the datum is not empty. In particular, if the bracket conditions are formulated at $(x_0, u_0) \in \mathbb{R}^d \times \mathbb{R}$, it is shown in [LMX10] that for any analytical datum $\underline{h}$ such that $\underline{h}(x_0) = u_0$, there exists smooth initial data h close to $\underline{h}$ which do not generate local C^2 solutions, a result analogous to Lebeau's theorem for Kelvin-Helmholtz [Leb02]. The proof of [LMX10] relies strongly on a representation of solutions based on the method of characteristics, specific to scalar equations, which was developed earlier in [Mét85].

Lerner, Nguyen and Texier's result on transition to ellipticity for general systems

In [LNT17], Lerner, Nguyen and Texier extend the analysis of [LMX10] to systems (3.1.1). The result of [LNT17] shows an instantaneous lack of Hölder well-posedness of the flow,

with an arbitrarily large loss of derivatives, under appropriate assumptions of transition to ellipticity. The analysis of [LNT17] is based on the method of approximation of pseudo-differential flows introduced in [Tex16]. One key observation in [LNT17] is that for systems, many types of transitions may occur. The focus in [LNT17] is on genuinely non-scalar transitions (more about this specific point in Remark 3.1.1). For these, the propagator generically grows in time like the Airy function.

Defect of hyperbolicity in Maxwell systems

There is a strong analogy between the progression from [Mét05] to our present results and recent results [LT15], [Lu16] in geometric optics. In [LT15], Lu and Texier study large-amplitude solutions to Maxwell-based systems in the small wavelength limit. They show that in appropriate coordinates, resonances in frequency correspond to points of weak hyperbolicity. Thus at the resonances, the subprincipal symbol plays a role in the stability analysis. Under a Levi condition, hyperbolicity is violated around the resonances, and WKB solutions do not approximate exact solutions issued from appropriate nearby initial data, no matter how precise the order of the WKB approximation. This result is somehow analogous to Métivier's initial ellipticity result. Following [LT15], Lu studied in [Lu16] a situation in which WKB solutions are destabilized by terms which are not present in the initial data. That is, the Levi condition of [LT15] is satisfied initially, but higher-order harmonics of the WKB solutions, which are generated by the nonlinearities in the course of the propagation, are associated with higher-order resonances. For these resonances, the Levi conditions may not be satisfied, leading to instability. This framework is somehow similar to ours, with an instability which develops in time, starting from an initially hyperbolic situation.

AJouter [HKN16].

3.1.2 Overview of the paper

Our assumptions are based on the framework set out in [LNT17], the results of which we extend in two distinct ways: we prove existence of solutions up to the observation time at which the Hadamard instability is recorded, and we measure the deviation in Gevrey spaces.

We assume that for a specific frequency $\xi_0 \in \mathbb{R}^d$ the linear part of the principal symbol at $u = \vec{u}_0 \in \mathbb{R}^N$ defined by (3.1.3) has a real spectrum at time $t = 0$ while non real eigenvalues appear for $t > 0$. In this sense the operator experiences a transition from initial hyperbolicity ($t = 0$, real eigenvalues) to eventual ellipticity ($t > 0$, non-real eigenvalues).

A sharp difference with the initially elliptic case lies in the normal forms of the operators. Indeed, the elliptic case is reducible to the case where $\underline{A}$ is a triangular matrix with non real and conjugated diagonal entries.

By contrast, transitions in time appear in many ways. There is not one single normal form. Section 3.2 will be devoted to the descriptions of such transitions in time and the associated normal forms for systems of size $N = 2$. In particular, this paper focuses on two particular normal forms, described in the next paragraphs.

The smoothly diagonalizable case

Under Assumptions 3.2.2, 3.2.4 and 3.2.5 (see Proposition 3.2.6 below), there holds

$$\underline{A}(t,x) \approx \underline{A}^{\mathrm{S}}(t) := \begin{pmatrix} 0 & t \\ -\gamma_0^2 t & 0 \end{pmatrix}$$

with $\gamma_0 > 0$. Here $\approx$ means equality up to higher order terms in the Taylor expansion in time and space, and up to a change of basis. The matrix $\underline{A}^{\mathrm{S}}(t)$ is smoothly diagonalisable in $\mathbb{C}$, with smooth eigenvalues $\pm i\gamma_0 t$. This case is mostly scalar ; it is analogous to a degenerate Cauchy-Riemann problem.

Our analysis shows that our method in Chapter 2 is robust enough to allow for such a weak defect of hyperbolicity. We replace ansatz (2.3.2) therein by the new ansatz $u_\varepsilon(t,x) = \mathbf{u}(t/\varepsilon^{1/2}, x, (x-x_0)\cdot\xi_0/\varepsilon)$. For such $\underline{A}^{\mathrm{S}}(t)$, the growth for the associated propagator solving

$$\partial_s U^{\mathrm{S}}(s',s,\theta) - \underline{A}^{\mathrm{S}}(s)\partial_\theta U^{\mathrm{S}}(s',s,\theta) = 0$$

is like

$$|U_n^{\mathrm{S}}(s',s)| \lesssim \exp\left(\int_{s'}^{s} \gamma_{\mathrm{S}}^{\sharp}(\tau)d\tau\right) , \quad \forall\, 0 \le s' \le s \ , \ \forall\, n \in \mathbb{Z}^d \tag{3.1.5}$$

for the Fourier coefficients of $U^{\mathrm{S}}(s',s,\theta)$, with $\gamma_{\mathrm{S}}^{\sharp}(\tau) = \gamma_0\tau$.

The degenerate Airy case

Under Assumptions 3.2.2 and 3.2.7 (see Proposition 3.2.8 below), there holds

$$\underline{A}(t,x) \approx \underline{A}^{\mathrm{Ai}}(t,x) := \begin{pmatrix} 0 & 1 \\ -\gamma_0^2(t-t_\star(x)) & 0 \end{pmatrix}$$

where $\approx$ means equality up to higher order terms in the Taylor expansion in time and space and $t_\star(x) \ge 0$ in a whole neighborhood of $x = x_0$[1].

The time transition function $t_\star(x)$ defines the boundary between the elliptic and hyperbolic zones. Indeed, for $t < t_\star(x)$, the eigenvalues are $\pm\sqrt{t_\star(x)-t}$ while for $t > t_\star(x)$ the eigenvalues are $\pm i\sqrt{t-t_\star(x)}$.

The transition between hyperbolicity and ellipticity is thus not uniform in space, and depends on the space-dependent transition time $t_\star(x)$. In order to use and develop the method of Chapter 2, we have to treat the transition time as a remainder term and verify its smallness in the framework. From that view, the non degenerate case $t_\star(x) = O((x-x_0)^2)$ is out of reach of the method presented in this paper, and requires special attention - we devoted two companion papers [Mor17a] and [Mor17b] to the subject - more about this specific point in Remark 3.1.1 below. We will focus here on the degenerate case

$$t_\star(x) = O((x-x_0)^4).$$

Note that the cases of odd power of x are in contradiction with the assumption of non-negativity of $t_\star$ around $x = x_0$.

[1] If there were x_1 such that $t_\star(x_1) < 0$, we would be in the case of initial ellipticity and Métivier's result would apply.

We emphasize also the fact that the eigenvalues of $\underline{A}^{\mathrm{Ai}}$ are C^0 in time but not C^1, hence the stiffness of this case.

In this framework, we replace ansatz (2.3.2) in Chapter 2 by the ansatz $u_\varepsilon(t,x) = \mathbf{u}(t/\varepsilon^{2/3}, x, (x-x_0)\cdot\xi_0/\varepsilon)$. As such a transition is not semi-simple, the previous ansatz induces the following equation for the propagator

$$\partial_s U^{\mathrm{Ai}}(s',s,\theta) - \varepsilon^{-1/3}\underline{A}^{\mathrm{Ai}}(s,x_0)\partial_\theta U^{\mathrm{S}}(s',s,\theta) = 0$$

which as then a growth like

$$|U_n^{\mathrm{Ai}}(s',s)| \lesssim \varepsilon^{-1/3}\exp\left(\int_{s'}^{s}\gamma_{\mathrm{Ai}}^{\sharp}(\tau)d\tau\right)\ , \quad \forall\, 0\le s'\le s\ ,\ \forall\, n\in\mathbb{Z}^d \tag{3.1.6}$$

for the Fourier coefficients of $U^{\mathrm{Ai}}(s',s,\theta)$ with $\gamma_{\mathrm{Ai}}^{\sharp}(\tau) = \gamma_0\tau^{1/2}$ which is typical of the Airy growth.

Remark 3.1.1. *In* [LNT17] *, the authors allow for generic non-scalar transitions, for which $t_\star(x) = O((x-x_0)^2)$. In particular, the space-time domain $\{(t,x) : (x-x_0)^2 \le t\}$ is included in the domain of hyperbolicity. As we will see precisely in the course of the proof of Proposition* 3.5.4*, in our context this space-time domain is too large for the standard Cauchy-Kovalevskaya theorem to apply. Thus, in the case $t_\star(x) = O((x-x_0)^2)$, we need a specific Gevrey well-posedness result in that space-time domain before observing the instability develop in the elliptic domain. This Gevrey well-posedness result is the object of the article* [Mor17a]*, and the completion of the instability proof in the case $t_\star(x) = O((x-x_0)^2)$ is the object of the article* [Mor17b].

Example: compressible Euler with Van der Waals pressure law

Transitions of the principal symbol from hyperbolicity to ellipticity, as described in the above paragraphs, are observed in physical equations describing *phase transitions*. One such system (mentioned in both [Mét05] and [LNT17]) is the compressible Euler equations in one spatial dimension, with a Van der Waals pressure law:

$$\begin{cases} \partial_t u_1 + \partial_x u_2 = 0 \\ \partial_t u_2 + \partial_x(p(u_1)) = 0 \end{cases} \tag{3.1.7}$$

where p follows a Van der Waals equation of state, for which there holds $p'(u_1) \le 0$, for some $u_1 \in \mathbb{R}$. The system is hyperbolic (resp. elliptic) for $p'(u_1) > 0$ (resp. for $p'(u_1) < 0$). For solutions which leave the hyperbolic zone, a phase transition occurs. This corresponds for us to the catastrophic growth recorded in the elliptic zone. If for instance the elliptic zone is defined by $\{|u_1| \le \delta\}$, for some $\delta > 0$, then solutions may enter the elliptic zone only to leave it immediately, due to the exponential growth.

3.2 Main assumptions and results

3.2.1 Branching eigenvalues and defect of hyperbolicity

We look at the possible cases of a defect of hyperbolicity, that is transitions from initial hyperbolicity to ellipticity at time $t > 0$, following the work of Lerner, Nguyen and Texier

of [LNT17].

We introduce first

$$A(t,x,u) = \sum_j A_j(t,x,u)\xi_{0,j}. \tag{3.2.1}$$

We assume there are $(x_0, \vec{u}_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^N \times \mathbb{R}^d$ and $r_0 > 0$ such that the principal symbol defined by

$$\underline{A}(t,x) = A(t,x,\vec{u}_0) \tag{3.2.2}$$

satisfies

$$\mathrm{Sp}\,(\underline{A}(0,x)) \subseteq \mathbb{R}\,, \quad \forall x \in B_{r_0}(x_0) \tag{3.2.3}$$

which stands for initial and local hyperbolicity around $x_0 \in \mathbb{R}^d$. Note that, as soon as there is some $x_1 \in \mathbb{R}^d$ such that $\underline{A}(0,x_1)$ has non real spectrum, we are in the case of initial ellipticity treated in Chapter 2.

We assume also that, for small times $t > 0$, there are some x close to x_0 such that

$$\mathrm{Sp}\,(\underline{A}(t,x)) \not\subseteq \mathbb{R} \quad , \quad \forall\, t > 0. \tag{3.2.4}$$

Condition (3.2.3) stands for initial and local hyperbolicity around x_0 ; condition (3.2.4) expresses the ellipticity of $\underline{A}$ at time $t > 0$. Up to translations in x and u, which do not affect our forthcoming assumptions, and by homogeneity in ξ, we may assume

$$x_0 = 0 \quad , \quad \vec{u}_0 = 0 \quad , \quad |\xi_0| = 1. \tag{3.2.5}$$

Since the A_j have real coefficients, non-real eigenvalues of $\underline{A}(t,x)$ appear in conjugate pairs. For such a pair $\lambda_\pm(t,x)$, by reality of the eigenvalues at $t = 0$ we have a double eigenvalue $\lambda_-(0,x) = \lambda_+(0,x) \in \mathbb{R}$ of $\underline{A}(0,x)$. To avoid higher order transitions (which would involve eigenvalues of multiplicity 3 or greater), we assume the eigenvalues of $\underline{A}(0,0)$ to be distinct and simple, except for one double eigenvalue:

Assumption 3.2.1. *We assume the eigenvalues of $\underline{A}(0,0)$ to be distinct and simple, except for one double eigenvalue.*

We block diagonalize the principal symbol into $\underline{A}(t,x)^{(0)}$ and $\underline{A}(t,x)^{(1)}$. The block $\underline{A}(t,x)^{(0)}$ is a 2×2 matrix corresponding to the double eigenvalue, and the $(N-2)\times(N-2)$ block $\underline{A}^{(1)}$ has simple real eigenvalues at $t = 0$ in a whole neighborhood of $x = 0$. Thanks to Assumption 3.2.1 the block diagonalization is smooth. Therefore we focus our discussion on $\underline{A}^{(0)}$, and we may assume $N = 2$, that is $\underline{A} \equiv \underline{A}^{(0)}$.

The question is now to describe the possible matrices $\underline{A}(t,x)$ satisfying conditions (3.2.3) and (3.2.4). Following [LNT17], we reformulate the conditions (3.2.3) and (3.2.4) in terms of the characteristic polynomial of $\underline{A}$ defined as

$$P(\lambda,t,x) = \det\,(\lambda - \underline{A}(t,x)) \tag{3.2.6}$$

which is simply in the case $N = 2$

$$P(\lambda,t,x) = \left(\lambda - \frac{1}{2}\mathrm{Tr}\underline{A}(t,x)\right)^2 + \Delta(t,x) \tag{3.2.7}$$

where we define

$$\Delta(t,x) = \det \underline{A}(t,x) - \left(\frac{1}{2}\mathrm{Tr}\underline{A}(t,x)\right)^2. \tag{3.2.8}$$

Thus the real or complex nature of the spectrum depends on the sign of Δ. So condition (3.2.3) is equivalent in terms of Δ to

$$\Delta(0,x) \leq 0 \quad , \quad \forall\, x \in B_{r_0}(0). \tag{3.2.9}$$

As a double eigenvalue $\lambda_-(0,x) = \lambda_+(0,x) \in \mathbb{R}$ of $\underline{A}(0,x)$ corresponds to a double root of $P(\lambda,0,x)$, we formulate the following Assumption:

Assumption 3.2.2 (Branching eigenvalues)**.** *In addition to* (3.2.9)*, we assume that there exists some* $\lambda_0 \in \mathbb{R}$ *such that*

$$P(\lambda_0,0,0) = 0 \quad , \quad \partial_\lambda P(\lambda_0,0,0) = 0. \tag{3.2.10}$$

Remark 3.2.3. *Note that condition* (3.2.10) *is equivalent to*

$$\lambda_0 = \frac{1}{2}\mathrm{Tr}\underline{A}(0,0) \quad , \quad \Delta(0,0) = 0.$$

For condition (3.2.4) to be satisfied, that is for a conjugate pair of eigenvalues to appear as $t > 0$, $\Delta(t,x)$ has to be positive for $t > 0$. The eigenvalues of $\underline{A}$, which are the zeroes of P, are then expressed by the square roots of Δ. Even though the regularity of Δ, being an algebraic combination of the coefficients of $\underline{A}$, is analytic, the regularity of the square roots of Δ can be of course much weaker. How much rougher than Δ may $\sqrt{\Delta}$ be has been studied in particular by Glaeser [Gla63]. The question of the regularity of the eigenvalues and of the eigenvectors is here of importance as we work in the analytic framework: we may not use non-smooth (in time and space) changes of basis, since the methods we use, following [Mét05], strongly rely on analyticity. In particular, we may not diagonalize the principal symbol if the eigenvectors are not smooth.

3.2.2 The case of a smooth transition

For the square roots of Δ to be as smooth as Δ, the discriminant Δ has to be the square of a smooth function $\delta(t,x)$:

$$\Delta(t,x) = \delta(t,x)^2.$$

In this case, note that $\Delta(0,x) = \delta(0,x)^2 \geq 0$. Since we assume also that $\Delta(0,x) \leq 0$ by (3.2.9), we get

$$\delta(0,x) = 0 \quad , \quad \forall x \in B_{r_0}(0).$$

This is equivalent to the existence of some analytic function $\tilde{\delta}(t,x)$ such that

$$\delta(t,x) = t\tilde{\delta}(t,x).$$

We sum up all this in the following

Assumption 3.2.4 (Smooth transition). *There is a function $\delta(t,x)$ analytic in the t and x variables such that*

$$\Delta(t,x) = (t\delta(t,x))^2 \tag{3.2.11}$$

with

$$\delta(0,0) = \gamma_0 > 0. \tag{3.2.12}$$

Under Assumption 3.2.4, since $\Delta(0,x) \equiv 0$ the eigenvalues of $\underline{A}(0,x)$ are the double eigenvalue $\frac{1}{2}\mathrm{Tr}\underline{A}(0,x)$. There are two cases[2], as $\underline{A}(0,x)$ could be semi-simple or not. In what follows we add the assumption

Assumption 3.2.5 (Semi-simplicity). *The unique eigenvalue of $\underline{A}(0,0)$ is semi-simple, for all x near $x = 0$.*

This assumption is Hypothesis 1.5 in [LNT17]. We can now prove the following normal form result

Proposition 3.2.6 (Normal form for the smooth transition). *Under Assumptions* 3.2.2, 3.2.4 *and* 3.2.5, *there is an analytical change of basis $Q_0(t,x) \in \mathbb{R}^{2\times 2}$ such that*

$$Q_0^{-1}(t,x)\left(\underline{A}(t,x) - \frac{1}{2}\mathrm{Tr}\underline{A}(t,x)\,\mathrm{Id}\right)Q_0(t,x) = \begin{pmatrix} 0 & t \\ -t\delta^2 & 0 \end{pmatrix}. \tag{3.2.13}$$

Proof. We denote

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix} = \underline{A}(t,x) - \frac{1}{2}\mathrm{Tr}\underline{A}(t,x)\mathrm{Id}.$$

By definition (3.2.8) there holds $\Delta = -a_{11}^2 - a_{12}a_{21}$ and then, by (3.2.11) in Assumption 3.2.4

$$-a_{11}^2 - a_{12}a_{21} = t^2\delta(t,x)^2. \tag{3.2.14}$$

By Assumption 3.2.5 the matrix $(a_{ij})_{i,j}$ satisfies

$$a_{11}(0,x) = a_{12}(0,x) = a_{21}(0,x) = 0$$

so that there are smooth functions $\widetilde{a_{ij}}(t,x)$ such that $a_{ij}(t,x) = t\widetilde{a_{ij}}(t,x)$. Hence by (3.2.14) we get

$$\delta(t,x)^2 = -\widetilde{a_{11}}^2 - \widetilde{a_{12}}\widetilde{a_{21}}.$$

As $\delta(0,0)^2 > 0$ by Assumption 3.2.4 (2), the term $\widetilde{a_{12}}\widetilde{a_{21}}(0,0)$ is non zero. Hence either one of $\widetilde{a_{12}}(0,0)$ or $\widetilde{a_{21}}(0,0)$ is non zero. In the first case, the matrix

$$Q_0(t,x) = \begin{pmatrix} \widetilde{a_{11}} & 1 \\ \widetilde{a_{21}} & 0 \end{pmatrix}$$

is such that (3.2.13) holds. The second case is treated in the same way, which suffices to end the proof.

□

[2] As opposed to the case of a stiff transition, described in Section 3.2.3, where $\underline{A}(0,0)$ is not semi-simple.

3.2.3 The case of a stiff transition

If Δ is not the square of a function, its square roots are typically not as smooth as Δ. In fact, for any $k \in \mathbb{N}$ it is possible to find Δ such that it is analytic, but its square roots are C^k and not C^{k+1}. The first non degenerate case of this kind is when

$$\partial_t \Delta(0,0) > 0 \tag{3.2.15}$$

which implies that $\Delta(t,0)^{1/2} \sim t^{1/2}$ which is C^0 but not C^1 at $t = 0$. With $\Delta(0,0) = 0$ by Assumption 3.2.2, condition (3.2.15) and the implicit function theorem give the existence of an analytic function $t_\star(x)$ such that

$$\Delta(t,x) = 0 \Longleftrightarrow t = t_\star(x) \quad \text{locally around } (t,x) = (0,0). \tag{3.2.16}$$

Introducing

$$e(t,x) = \int_0^1 \partial_t \Delta((1-\tau)t_\star(x) + \tau t, x) d\tau$$

there holds

$$\Delta(t,x) = (t - t_\star(x))\, e(t,x). \tag{3.2.17}$$

As Δ is analytic, e is also analytic, and satisfies

$$e(0,0) = \partial_t \Delta(0,0) > 0$$

so that e is positive around $(0,0)$. Then the sign of $\Delta(t,x)$, hence the real or complex nature of the spectrum of $\underline{A}(t,x)$, is given by the sign of $t - t_\star(x)$, a situation comparable to the one described in Section 1.2.3 of [LNT17]:

- For (t,x) under the transition curve $\{(t_\star(x), x)\}$ the eigenvalues of $\underline{A}(t,x)$ are real.
- For (t,x) above the transition curve, the eigenvalues of $\underline{A}(t,x)$ have a non-zero imaginary part like $\pm i(t - t_\star(x))^{1/2}$.

The question is then to describe $t_\star$. First, as $\Delta(0,0) = 0$,

$$t_\star(0) = 0. \tag{3.2.18}$$

As $\Delta(0,x) \leq 0$ for $x \in B_{r_0}(0)$, we have

$$t_\star(x) \geq 0 \quad , \quad \forall x \in B_{r_0}(0)$$

which implies

$$\partial_x t_\star(0) = 0 \tag{3.2.19}$$

so that the Taylor expansion of $t_\star(\cdot)$ around $x = 0$ is as

$$t_\star(x) = \frac{1}{2} \sum_{j,k} \partial_{x_j} \partial_{x_k} t_\star(0)\, x_j x_k + O(x^3)$$

and the Hessian $\left(\partial_{x_j}\partial_{x_k}t_\star(0)\right)_{j,k}$ is a nonnegative matrix. But as we will see in the course of the proof of Proposition 3.5.4, the non degenerate case $\left(\partial_{x_j}\partial_{x_k}t_\star(0)\right)_{j,k} \neq 0$ cannot be dealt with our method. We then assume

$$\partial_{x_j}\partial_{x_k}t_\star(0) = 0 \quad , \quad \forall\, j,k = 1,\dots,d. \tag{3.2.20}$$

Just as before, inequality (3.2.9) implies that third order derivatives of $t_\star(\cdot)$ are null at $x = 0$, and there holds

$$t_\star(x) = O(x^4).$$

In order to sum up those assumptions in a more intrinsic way, we express derivatives of $t_\star$ by derivatives of Δ. By definition (3.2.16) of $t_\star$, there holds $\Delta(t_\star(x), x) = 0$ hence, differentiating with respect to x and taking $x = 0$:

$$\partial_x t_\star(0)\, \partial_t\Delta(0,0) + \partial_x\Delta(0,0) = 0.$$

As $\partial_t\Delta(0,0) > 0$, equality (3.2.19) is then equivalent to

$$\partial_x\Delta(0,0) = 0.$$

By Faà di Bruno formula on iterate derivatives applied to the equality $\Delta(t_\star(x), x) = 0$, we may prove by induction that $t_\star(x) = O(x^4)$ is equivalent to the following

Assumption 3.2.7 (Degenerate stiff transition)**.** *We assume*

$$\partial_x^\alpha\, \Delta(0,0) = 0 \quad , \quad \forall\, \alpha \in \mathbb{N}^d \text{ with } |\alpha| \leq 3.$$

We prove now a normal form expression for $\underline{A}(t,x)$:

Proposition 3.2.8 (Normal form for the stiff transition)**.** *Under Assumptions* 3.2.2 *and* 3.2.7*, there are an analytical change of basis $Q_0(t,x)$ and real analytical functions $t_\star(x)$ and $e(t,x)$ such that*

$$Q_0^{-1}\left(\underline{A}(t,x) - \frac{1}{2}\mathrm{Tr}\underline{A}(t,x)\,\mathrm{Id}\right)Q_0 = \begin{pmatrix} 0 & 1 \\ -(t-t_\star)e & 0 \end{pmatrix}. \tag{3.2.21}$$

Proof. By definition of Δ and denoting

$$\underline{A} - \frac{1}{2}\mathrm{Tr}\underline{A}\,\mathrm{Id} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix}$$

we get $\Delta = -a_{11}^2 - a_{12}a_{21}$. As $\Delta \sim t$ both a_{12} and a_{21} cannot both be zero at $(0,0)$. Assuming that $a_{21}(0,0) \neq 0$, the matrix

$$Q_0(t,x) = \begin{pmatrix} a_{11} & 1 \\ a_{21} & 0 \end{pmatrix}$$

is an analytical change of basis such that (3.2.21) holds.

□

Remark 3.2.9. *Note that, on the contrary of the normal form of the smooth transition given in Proposition* 3.2.6*, the normal form of the stiff transition is not semi-simple. This is of importance, as non-semisimplicity introduces an additional factor* $\varepsilon^{-1/3}$ *in the upper bound* (3.3.26) *of the Airy propagator, to be compared with the upper bound* (3.3.18) *in the smooth case.*

We add the following assumption:

Assumption 3.2.10 (Genuinely nonlinear zeroth-order perturbation)**.** *We assume that* $f(t,x,u)$ *is quadratic in* u *locally around* $u=\vec{u}_0$*, that is*

$$\partial_u f(t,x,u)\big|_{u=\vec{u}_0} \equiv 0$$

in a neighborhood of $(t,x)=(0,0)$*.*

3.2.4 Statement of the results

We recall first Definition 2.2.2 of conical domain of $\mathbb{R}_t\times\mathbb{R}^d_x$, centered here at $(t,x)=(0,0)$. We denote

$$\Omega_{R,\rho}=\bigcup_{t\geq 0}\{t\}\times\Omega_{R,\rho,t}=\left\{(t,x)\in\mathbb{R}\times\mathbb{R}^d \;\middle|\; 0\leq t<\rho^{-1},\ R|x|_1+\rho t<1\right\}. \tag{3.2.22}$$

Theorem 4 (Gevrey ill-posedness of the smooth case)**.** *Under Assumptions* 3.2.1, 3.2.2, 3.2.4, 3.2.5 *and* 3.2.10*, the Cauchy problem* (3.1.1) *is not Hölder well-posed in Gevrey spaces* G^σ *for all* $\sigma\in(0,\sigma_0)$ *with*

$$\sigma_0=1/3.$$

That is for all $c>0$*,* K *compact of* $\mathbb{R}^d$ *and* $\alpha\in(0,1]$*, there are sequences* $R_\varepsilon^{-1}\to 0$ *and* $\rho_\varepsilon^{-1}\to 0$*, a family of initial conditions* $h_\varepsilon\in G^\sigma$ *and corresponding solutions* u_ε *of the Cauchy problem on domains* $\Omega_{R_\varepsilon,\rho_\varepsilon}(0)$ *such that*

$$\lim_{\varepsilon\to 0}||u_\varepsilon||_{L^2(\Omega_{R_\varepsilon,\rho_\varepsilon})}/||h_\varepsilon||^\alpha_{\sigma,c,K}=+\infty. \tag{3.2.23}$$

The time of existence of the solutions u_ε *is at least of size* $\varepsilon^{1/2-\sigma/2}$*.*

Theorem 5 (Gevrey ill-posedness of the Airy case)**.** *Under Assumptions* 3.2.1, 3.2.2, 3.2.7 *and* 3.2.10*, the result of Theorem* 4 *holds for any Gevrey index* $\sigma\in(0,\sigma_0)$*, with*

$$\sigma_0=2/13.$$

Recall that a function f defined on an open set B of $\mathbb{R}^d$ is said to belong to the Gevrey space $G^\sigma(B)$ if for all compact $K\subset B$, there are constants $C_K>0$ and $c_K>0$ that satisfy

$$|\partial^\alpha f|_{L^\infty(K)}\leq C_K c_K^{|\alpha|}|\alpha|!^{1/\sigma}\quad,\quad\forall\alpha\in\mathbb{N}^d. \tag{3.2.24}$$

We then define a family of norms on $G^\sigma(B)$, for all compact $K\subset B$ and $c>0$ by

$$||f||_{\sigma,c,K}=\sup_\alpha|\partial^\alpha f|_{L^\infty(K)}c^{-|\alpha|}|\alpha|!^{-1/\sigma}. \tag{3.2.25}$$

Remark 3.2.11. *The limiting Gevrey index σ_0 is in both cases due in part to technical limitations. In the proof, in each case remainder terms are proved to be small in the spaces described later. The limiting index σ_0 is directly influenced by this smallness of the remainders. In the smooth case, a null remainder would imply $\sigma_0 = 1/2$, which is the expected limiting Gevrey index in this case. In the Airy case, a smaller remainder would imply a greater index σ_0, but it is not clear if the limit $1/2$ could be attained.*

Also, as pointed out in Remark 3.2.9, one main difference between both cases is the extra weight for the Airy propagator in the ansatz of highly oscillating solutions, as shown in Lemma 3.3.5. This implies a stronger constraint on the smallness of the remainder terms appearing in the Airy case, as explained in the proof of Proposition 3.5.4.

The proofs are given in Sections 3.3 to 3.5, with an appendix devoted to the Airy equation in Section 3.6. We introduce a functional framework that is flexible enough to simultaneously cover the smooth, semi-simple case (Theorem 4) and the stiff, non-semi-simple case (Theorem 5). We develop in Section 3.3 the ansatz of highly oscillating solutions which reduces the Cauchy problem (3.1.1) to a fixed point equation. In Sections 3.4 we recall properties of the spaces developed in [Mor16a], and use them to prove contraction estimates and existence of solutions. Finally, in Section 3.5.3 we prove that the constructed solutions satisfy a lower bound that leads to the Hadamard instability for Gevrey regularity $\sigma \in (0, \sigma_0)$.

3.3 Highly oscillating solutions and reduction to a fixed point equation

3.3.1 Highly oscillating solutions

As in Section 2.3.1 in Chapter 2, we first reduced (3.1.1) to the new Cauchy problem

$$\partial_t u = \sum_j A_j(t,x,u)\partial_{x_j} u + F(t,x,u)u \quad \text{with } u(0,x) = h(x) \tag{3.3.1}$$

where F is analytic in a neighborhood of $(0,0,0) \in \mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^N_u$ (see (3.2.5)), and h small analytic functions satisfying $h_{|x=0} = 0$, as perturbations of the trivial datum $h \equiv 0$.

Next, we adapt the ansatz of highly oscillating solutions of [Mét05] and [Mor16a] in order to take into account the different time scaling of the exponential growth. In this view we posit

$$u_\varepsilon(t,x) = \varepsilon^{2/(1+\eta)} \mathbf{u}\left(\varepsilon^{-1/(1+\eta)}\, t, x, x \cdot \xi_0/\varepsilon\right) \tag{3.3.2}$$

where

- The small parameter $\varepsilon > 0$ corresponds to high frequencies.
- The function $\mathbf{u}(s,x,\theta)$ is 2π-periodic in θ.
- The scaling term $\varepsilon^{2/(1+\eta)}$ insures the smallness of the nonlinear terms.

We introduce for any analytical function $H(t,x,u)$ the compact notation

$$\mathbf{H}(s,x,\mathbf{u}) = H\left(\varepsilon^{1/(1+\eta)}s, x, \varepsilon^{2/(1+\eta)}\mathbf{u}\right). \tag{3.3.3}$$

For $u_\varepsilon(t,x)$ to be solution of (3.3.1) it is then sufficient that $\mathbf{u}(s,x,\theta)$ solves the following equation

$$\partial_s \mathbf{u} = \varepsilon^{-\eta/(1+\eta)}\mathbf{A}\,\partial_\theta \mathbf{u} + \varepsilon^{1/(1+\eta)}\left(\sum_j \mathbf{A}_j \partial_{x_j}\mathbf{u} + \mathbf{F}\,\mathbf{u}\right) \tag{3.3.4}$$

where we use the notation (3.3.3) for $\mathbf{A}$ and $\mathbf{F}$, and A is defined by (3.2.1).

3.3.2 Remainder terms

We focus here on the term $\varepsilon^{-\eta/(1+\eta)}\mathbf{A}(\xi)\,\partial_\theta\mathbf{u}$ of the previous equation. To prove the expected growth of solutions of the initial problem, we decompose the symbol $A(t,x,u)$ in several pieces to highlight the leading term denoted by $\underline{A}^{\mathrm{S}}(t)$ for the smooth case, $\underline{A}^{\mathrm{Ai}}(t)$ for the Airy case, which will lead to the exponential growth.

First, by analyticity of the A_j and Taylor expansion formula, there is a family of analytical matrices $(A_{u_j})_{j=1,\dots,N}$ such that locally around $(0,0,0)\in\mathbb{R}_t\times\mathbb{R}^d_x\times\mathbb{R}^N_u$ there holds

$$A(t,x,u) = \underline{A}(t,x) + \sum_j A_{u_j}u_j. \tag{3.3.5}$$

In both smooth and Airy cases, we perform an analytical Taylor expansion on $\underline{A}(t,x)$ in order to highlight the principal term that lead to the exponential growth. This is made precise in the two following lemmas.

Lemma 3.3.1 (Expansion formula: smooth case)**.** *Following Proposition* 3.2.6*, we introduce the leading term* $\underline{A}^{\mathrm{S}}(t)$*, defined up to a change of basis and a trace terme as*

$$Q_0^{-1}(t,x)\left(\underline{A}^{\mathrm{S}}(t) - \frac{1}{2}\mathrm{Tr}\,\underline{A}(t,x)\right)Q_0(t,x) = \begin{pmatrix} 0 & t \\ -\gamma_0^2 t & 0\end{pmatrix} \tag{3.3.6}$$

and the analytical error term

$$Q_0^{-1}\,\mathrm{R}^{\mathrm{S}}\,Q_0 = \begin{pmatrix} 0 & 0 \\ -t(\delta^2-\delta(0,0)^2) & 0\end{pmatrix}. \tag{3.3.7}$$

Then there holds

$$\underline{A}(t,x) = \underline{A}^{\mathrm{S}}(t) + \mathrm{R}^{\mathrm{S}}(t,x) \tag{3.3.8}$$

and there are analytical matrices $\mathrm{R}^{\mathrm{S}}_t(t,x)$ *and* $\mathrm{R}^{\mathrm{S}}_x(t,x)$ *such that*

$$\mathrm{R}^{\mathrm{S}}(t,x) = t^2\mathrm{R}^{\mathrm{S}}_t(t,x) + tx\cdot\mathrm{R}^{\mathrm{S}}_x(t,x) \tag{3.3.9}$$

locally around $(0,0)\in\mathbb{R}_t\times\mathbb{R}^d_x$.

Proof. First the equality (3.2.13) of Lemma 3.2.6 implies that

$$Q_0^{-1}(t,x)\left(\underline{A}^{\mathrm{S}} - \frac{1}{2}\mathrm{Tr}\,\underline{A}(t,x)\right)Q_0(t,x) = \begin{pmatrix} 0 & t \\ -t\delta^2 & 0 \end{pmatrix}$$

hence (3.3.8). Second, by analyticity of δ and Taylor expansion formula, there are analytical functions $\mathrm{r}_t^{\mathrm{S}}$ and $\mathrm{r}_{x_j}^{\mathrm{S}}$ such that

$$\delta^2(t,x) - \delta(0,0)^2 = t\,\mathrm{r}_t^{\mathrm{S}}(t,x) + x\cdot \mathrm{r}_x^{\mathrm{S}}(t,x).$$

We finally introduce the matrices

$$Q_0^{-1}\,\mathrm{R}_t^{\mathrm{S}}\,Q_0 = \begin{pmatrix} 0 & 0 \\ -\mathrm{r}_t^{\mathrm{S}} & 0 \end{pmatrix} \text{ and } Q_0^{-1}\,\mathrm{R}_x^{\mathrm{S}}\,Q_0 = \begin{pmatrix} 0 & 0 \\ -\mathrm{r}_x^{\mathrm{S}} & 0 \end{pmatrix}$$

which leads to (3.3.9) and ends the proof. □

Lemma 3.3.2 (Expansion formula: Airy case)**.** *Following Proposition* 3.2.8*, we introduce the leading term* $\underline{A}^{\mathrm{S}}(t)$*, defined up to a change of basis and a trace term as*

$$Q_0^{-1}(t,x)\left(\underline{A}^{\mathrm{Ai}}(t) - \frac{1}{2}\mathrm{Tr}\,\underline{A}(t,x)\right)Q_0(t,x) = \begin{pmatrix} 0 & 1 \\ -\gamma_0^2 t & 0 \end{pmatrix} \tag{3.3.10}$$

and the analytical error term

$$Q_0^{-1}(t,x)\left(\mathrm{R}^{\mathrm{Ai}} - \frac{1}{2}\mathrm{Tr}\,\underline{A}(t,x)\right)Q_0(t,x) = \begin{pmatrix} 0 & 0 \\ -t(e-e(0,0)) + t_\star e & 0 \end{pmatrix}. \tag{3.3.11}$$

Then there holds

$$\underline{A}(t,x) = \underline{A}^{\mathrm{Ai}}(t) + \mathrm{R}^{\mathrm{Ai}}(t,x). \tag{3.3.12}$$

and there are analytical matrices $\mathrm{R}_t^{\mathrm{Ai}}$*,* $\mathrm{R}_x^{\mathrm{Ai}}$ *and* $\mathrm{R}_e^{\mathrm{Ai}}$ *such that*

$$\mathrm{R}^{\mathrm{Ai}}(t,x) = t^2\mathrm{R}_t^{\mathrm{Ai}}(t,x) + tx\cdot\mathrm{R}_x^{\mathrm{Ai}}(t,x) + t_\star\mathrm{R}_e^{\mathrm{Ai}}(t,x,\xi) \tag{3.3.13}$$

locally around $(0,0)$*.*

Proof. First equality (3.2.21) of Lemma 3.2.8 implies

$$Q_0^{-1}(t,x)\left(\underline{A}^{\mathrm{Ai}} - \frac{1}{2}\mathrm{Tr}\,\underline{A}(t,x)\right)Q_0(t,x) = \begin{pmatrix} 0 & 1 \\ -(t-t_\star(x))e(t,x) & 0 \end{pmatrix}$$

hence (3.3.12). Second, by analyticity of e and Taylor expansion formula, there are analytical functions e_t and e_{x_j} such that

$$e(t,x) - e(0,0) = te_t + x\cdot e_x$$

locally around $(0,0)$. Introducing the matrices

$$Q_0^{-1}\,\mathrm{R}_t^{\mathrm{Ai}}\,Q_0 = \begin{pmatrix} 0 & 0 \\ -e_t & 0 \end{pmatrix} \quad , \quad Q_0^{-1}\,\mathrm{R}_x^{\mathrm{Ai}}\,Q_0 = \begin{pmatrix} 0 & 0 \\ -e_x & 0 \end{pmatrix} \quad \text{and} \quad Q_0^{-1}\,\mathrm{R}_e^{\mathrm{Ai}}\,Q_0 = \begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix}$$

leads to (3.3.13) and ends the proof. □

In both Airy and smooth cases, we have then an expansion formula of the form

$$A(t,x,u) = \underline{A}^\eta(t) + \mathrm{R}^\eta(t,x) + A_u \cdot u$$

where η corresponds to the parameter introduced in the ansatz (3.3.2), adapted to each specific case. This parameter will be be precised in Lemma 3.3.4 in the smooth case, and in Lemma 3.3.5 in the Airy case. The remainder term R^η is R^{S} defined by (3.3.7) in the smooth case, and is R^{Ai} defined by (3.3.11) in the Airy case.

We rewrite now equation (3.3.4) as

$$\partial_s \mathbf{u} - \varepsilon^{-\eta/(1+\eta)} \underline{A}^\eta(\varepsilon^{1/(1+\eta)} s)\partial_\theta \mathbf{u} = \mathbf{G}^\eta(s,x,\mathbf{u}) \tag{3.3.14}$$

where we define the source term

$$\begin{aligned} \mathbf{G}^\eta(s,x,\mathbf{u}) &= \varepsilon^{-\eta/(1+\eta)} \left(\mathbf{R}^\eta + \varepsilon^{2/(1+\eta)} \mathbf{A}_u \cdot \mathbf{u}\right) \partial_\theta \mathbf{u} \\ &\quad + \varepsilon^{1/(1+\eta)} \left(\mathbf{A}(s,x,\mathbf{u}) \cdot \partial_x \mathbf{u} + \mathbf{F}(s,x,\mathbf{u})\mathbf{u}\right) \end{aligned} \tag{3.3.15}$$

using notation (3.3.3).

Remark 3.3.3. *Note that in Chapter* 2 *there are no remainder terms* $\mathbf{R}^\eta$*, as we consider the full varying-coefficient operator* $\underline{A}(\varepsilon s, x)\partial_\theta$ *in equation* (3.3.14).

3.3.3 Upper bounds for the propagators

To solve the Cauchy problem of equation (3.3.14) with initial datum h_ε specified in Section 3.3.4, we first study the case $\mathbf{G}^\eta \equiv 0$, that is

$$\partial_s \mathbf{u} - \varepsilon^{-\eta/(1+\eta)} \underline{A}^\eta(\varepsilon^{1/(1+\eta)} s)\partial_\theta \mathbf{u} = 0. \tag{3.3.16}$$

Note that this equation is linear, non autonomous and non scalar. For a general $\underline{A}^\eta(t)$ we define the matrix propagator $U^\eta(s',s,\theta)$ as the solution of

$$\partial_s U^\eta(s',s,\theta) - \varepsilon^{-\eta/(1+\eta)} \underline{A}^\eta(\varepsilon^{1/(1+\eta)} s)\partial_\theta U^\eta(s',s,\theta) = 0\ , \quad U^\eta(s',s',\theta) \equiv \mathrm{Id}$$

and $U^\eta(s',s,\theta)$ is periodic in θ, following the ansatz (3.3.2). The choice of the time scaling $s = \varepsilon^{-1/(1+\eta)} t$, that is the choice of η, is such that solutions of (3.3.16) have a typical exponential growth independent of ε. Both following Lemmas make the growth of the propagators explicit in both cases.

Lemma 3.3.4 (Growth of the propagator: the smooth case)**.** *Under Assumptions* 3.2.2, 3.2.4 *and* 3.2.5*, we put* $\eta = 1$*. The matrix propagator* $U^{\mathrm{S}}(s',s,\theta)$ *defined by*

$$\partial_s U^{\mathrm{S}}(s',s,\theta) - \underline{A}^{\mathrm{S}}(s)\partial_\theta U^{\mathrm{S}}(s',s,\theta) = 0 \quad , \quad U^{\mathrm{S}}(s',s',\theta) = \mathrm{Id} \tag{3.3.17}$$

satisfies the following growth of its Fourier modes in the θ *variable:*

$$|U^{\mathrm{S}}_n(s',s)| \lesssim \exp\left(\int_{s'}^{s} \gamma^\sharp_{\mathrm{S}}(\tau) d\tau\, |n|\right) \quad , \quad \forall\, 0 \le s' \le s\,, \forall\, n \in \mathbb{Z} \tag{3.3.18}$$

with

$$\gamma^\sharp_{\mathrm{S}}(\tau) = \gamma_0 \tau. \tag{3.3.19}$$

Proof. First, as $\underline{A}^{\mathrm{S}}$ is given by (3.3.6) up to a change of basis $Q_0(t,x)$ and a trace term $\frac{1}{2}\mathrm{Tr}\underline{A}$, we introduce

$$V_n(s',s,x) = \exp\left(in\int_{s'}^{s}\frac{1}{2}\mathrm{Tr}\underline{A}(\varepsilon^{1/(1+\eta)}\tau,x)d\tau\right)Q_0^{-1}\left(\varepsilon^{1/(1+\eta)}s,x\right)U_n(s',s) \tag{3.3.20}$$

which solves

$$\partial_s V_n(s',s) - in\begin{pmatrix}0 & \varepsilon^{(1-\eta)/(1+\eta)}s\\ -\gamma_0^2\varepsilon^{(1-\eta)/(1+\eta)}s & 0\end{pmatrix}V_n(s',s) = -\varepsilon^{1/(1+\eta)}Q_0^{-1}\partial_t Q_0\, V_n(s',s) \tag{3.3.21}$$

with initial condition $V_n(s',s',x) = Q_0^{-1}\left(\varepsilon^{1/(1+\eta)}s',x\right)$. We focus then on the autonomous differential system

$$\partial_s\widetilde{V}_n(s',s) - in\begin{pmatrix}0 & \varepsilon^{(1-\eta)/(1+\eta)}s\\ -\gamma_0^2\varepsilon^{(1-\eta)/(1+\eta)}s & 0\end{pmatrix}\widetilde{V}_n(s',s) = 0 \tag{3.3.22}$$

which becomes, with the choice $\eta = 1$, the ε-free matrix equation

$$\partial_s\widetilde{V}_n(s',s) - in\,s\begin{pmatrix}0 & 1\\ -\gamma_0^2 & 0\end{pmatrix}\widetilde{V}_n(s',s) = 0.$$

The complex constant change of basis

$$\underline{Q} = \begin{pmatrix}1 & 1\\ -i\gamma_0 & i\gamma_0\end{pmatrix}$$

leads us to the exact solution

$$\widetilde{V}_n(s',s) = \underline{Q}\begin{pmatrix}\exp(n\gamma_0(s^2-s'^2)/2) & 0\\ 0 & \exp(-n\gamma_0(s^2-s'^2)/2)\end{pmatrix}\underline{Q}^{-1}$$

which satisfies the upper bound

$$\left|\widetilde{V}_n(s',s)\right| \lesssim \exp\left(\int_{s'}^{s}\gamma_{\mathrm{S}}^{\sharp}(\tau)d\tau\,|n|\right) \tag{3.3.23}$$

with $\gamma_{\mathrm{S}}^{\sharp}$ defined in (3.3.19). Getting back to (3.3.21), we use Duhamel formula to write

$$V_n(s',s) = \widetilde{V}_n(s',s)Q_0^{-1}(\varepsilon^{1/2}s') - \int_{s'}^{s}\varepsilon^{1/2}\widetilde{V}_n(\tau,s)\left(Q_0^{-1}\partial_t Q_0\right)(\varepsilon^{1/2}\tau)\,V_n(s',\tau)d\tau. \tag{3.3.24}$$

Note that $V_n(s',s)$ depends also on x through $Q_0 = Q_0(t,x)$. Factorizing by the exponential growth $\exp(\int_{s'}^{s}\gamma_{\mathrm{S}}^{\sharp}(\tau)d\tau\,|n|)$, we get

$$\begin{aligned}
&\exp\left(-\int_{s'}^{s}\gamma_{\mathrm{S}}^{\sharp}\,|n|\right)\left|V_n(s',s)\right|\\
&\quad\lesssim \exp\left(-\int_{s'}^{s}\gamma_{\mathrm{S}}^{\sharp}\,|n|\right)\left|\widetilde{V}(s',s)\right|\\
&\qquad+\int_{s'}^{s}\varepsilon^{1/2}\exp\left(-\int_{\tau}^{s}\gamma_{\mathrm{S}}^{\sharp}\,|n|\right)\left|\widetilde{V}_n(\tau,s)\right|\left|Q_0^{-1}\partial_t Q_0\right|(\varepsilon^{1/2}\tau)\exp\left(-\int_{s'}^{\tau}\gamma_{\mathrm{S}}^{\sharp}\,|n|\right)|V_n(s',\tau)|d\tau
\end{aligned}$$

and the bound holds uniformly in x. We introduce then

$$y_\varepsilon(s',s) = \max_{x\in K_\varepsilon} \exp\left(-\int_{s'}^{s} \gamma_{\mathrm{S}}^{\sharp}\,|n|\right)\left|V_n(s',s,x)\right| \quad \text{with } K_\varepsilon = B_{R^{-1}}(0).$$

Thanks to the upper bound (3.3.23) there holds

$$y_\varepsilon(s',s) \lesssim 1 + \int_{s'}^{s} \varepsilon^{1/2}\,\left|Q_0^{-1}\partial_t Q_0\right|(\varepsilon^{1/2}\tau)\,y_\varepsilon(s',\tau)d\tau$$

and we make use of the Gronwall inequality to get

$$y_\varepsilon(s',s) \lesssim \exp\left(\varepsilon^{1/2}c\,(s-s')\right) \quad \text{with} \quad c = \max_{x\in K_\varepsilon}\left|Q_0^{-1}\partial_t Q_0\right|$$

As $\varepsilon^{1/2}s$ is small in our setting, we get the announced upper bound (3.3.18).

□

Lemma 3.3.5 (Growth of the propagator: the Airy case)**.** *Under Assumptions* 3.2.2 *and* 3.2.7 *we put* $\eta = 1/2$. *The matrix propagator* $U^{\mathrm{Ai}}(s',s,\theta)$ *defined by*

$$\partial_s U^{\mathrm{Ai}}(s',s,\theta) - \varepsilon^{-1/3}\underline{A}^{\mathrm{Ai}}(\varepsilon^{2/3}s)\partial_\theta U^{\mathrm{Ai}}(s',s,\theta) = 0 \quad , \quad U^{\mathrm{Ai}}(s',s',\theta) \equiv \mathrm{Id} \tag{3.3.25}$$

satisfies the following growth of its Fourier modes in the θ *variable:*

$$|U_n^{\mathrm{Ai}}(s',s)| \lesssim \varepsilon^{-1/3}\exp\left(\int_{s'}^{s}\gamma_{\mathrm{Ai}}^{\sharp}(\tau)d\tau\,|n|)\right) \quad , \quad \forall\, 0\le s'\le s<\varepsilon^{-2/3}\ ,\ \forall\, n\in\mathbb{Z} \tag{3.3.26}$$

with

$$\gamma_{\mathrm{Ai}}^{\sharp}(\tau) = \gamma_0\tau^{1/2}. \tag{3.3.27}$$

Proof. We proceed as in the previous proof of Lemma 3.3.4, by defining

$$V_n(s',s,x) = \exp\left(in\int_{s'}^{s}\frac{1}{2}\mathrm{Tr}\underline{A}(\varepsilon^{1/(1+\eta)}\tau,x)d\tau\right)Q_0^{-1}(\varepsilon^{1/(1+\eta)}s,x)U_n^{\mathrm{Ai}}(s',s,x)$$

and looking at the matrix equation

$$\partial_s\widetilde{V}_n(s',s) - in\begin{pmatrix}0 & \varepsilon^{-\eta/(1+\eta)}\\ -\gamma_0^2\varepsilon^{(1-\eta)/(1+\eta)}s & 0\end{pmatrix}\widetilde{V}_n(s',s) = 0. \tag{3.3.28}$$

As the eigenvalues of

$$\begin{pmatrix}0 & \varepsilon^{-\eta/(1+\eta)}\\ -\gamma_0^2\varepsilon^{(1-\eta)/(1+\eta)}s & 0\end{pmatrix}$$

are $\pm i\gamma_0\varepsilon^{(1-2\eta)/(1+\eta)}\sqrt{s}$, the choice $\eta = 1/2$ is natural in this case.

As in the proof of Lemma 3.3.4, to prove the upper bound (3.3.26) it suffices to prove the same bound for $\widetilde{V}_n(s',s)$. This is postponed to Section 3.6.2 of the Appendix. It uses classical bounds on the Airy function.

□

3.3.4 Free solutions

As in Section 2.3.4 in Chapter 2, we seek for high-oscillating, small and well-polarized initial data of the form

$$h_\varepsilon^\eta(x) = \varepsilon^{2/(1+\eta)} e^{-M(\varepsilon)} \mathrm{Re}\left(e^{ix\cdot\xi_0/\varepsilon}\vec{e}_+^\eta + e^{-ix\cdot\xi_0/\varepsilon}\vec{e}_-^\eta\right) \tag{3.3.29}$$

which correspond in the ansatz (3.3.2) of high-oscillating solutions to

$$\mathbf{h}_\varepsilon^\eta(\theta) = e^{-M(\varepsilon)} \mathrm{Re}\left(e^{i\theta}\vec{e}_+^\eta + e^{-i\theta}\vec{e}_-^\eta\right). \tag{3.3.30}$$

Here $\vec{e}_+^\eta$ and $\vec{e}_-^\eta$ are vectors chosen in each case such that $U^\eta(0,s)\mathbf{h}_\varepsilon^\eta$ satisfies the maximal growth of U^η, in either smooth or Airy case. The parameter $M(\varepsilon)$ is chosen such that both the Gevrey norm and the size of h_ε^η is small. Following Lemma 2.3.3, we posit

$$M(\varepsilon) = \varepsilon^{-\delta}, \quad \delta \in (\sigma, 1). \tag{3.3.31}$$

Remark 2.3.4 explains in particular the link between the long time of existence of solutions and the Gevrey weight $e^{-M(\varepsilon)}$, hence the constraint $\sigma < \delta$.

We introduce also

$$\mathbf{f}^\eta(s,\theta) = U^\eta(0,s)\mathbf{h}_\varepsilon^\eta(\theta) \tag{3.3.32}$$

which we call the free solution of equation (3.3.14), as it solves the equation with $\mathbf{G}^\eta \equiv 0$. In both smooth and Airy cases, we prove that for well-chosen $\vec{e}_+^\eta$ and $\vec{e}_-^\eta$, Lemma 2.3.2 still holds with

$$\gamma_\eta^\flat(\tau) = \gamma_0 \tau^\eta \tag{3.3.33}$$

for $\eta = 1$ (corresponding to the smooth case) and $\eta = 1/2$ (corresponding to the Airy case).

Lemma 3.3.6 (Growth of the free solution: the smooth case)**.** *We define*

$$\vec{e}_+^{\mathrm{S}} = Q_0(0,x)\begin{pmatrix}1\\-i\gamma_0\end{pmatrix} \quad \textit{and} \quad \vec{e}_-^{\mathrm{S}} = Q_0(0,x)\begin{pmatrix}1\\i\gamma_0\end{pmatrix}. \tag{3.3.34}$$

Then the free solution $\mathbf{f}^{\mathrm{S}}$ *of the smooth case satisfies*

$$|\mathbf{f}^{\mathrm{S}}(s)| \approx e^{-M(\varepsilon)} e^{\int_0^s \gamma_{\mathrm{S}}^\flat(\tau)d\tau} \quad , \quad \forall s \geq 0 \tag{3.3.35}$$

where $\approx$ *means equality up to a constant and with*

$$\gamma_{\mathrm{S}}^\flat(\tau) = \gamma_0 \tau. \tag{3.3.36}$$

Proof. We follow step by step the proof of Lemma 3.3.4. First, it is explicit that there holds

$$\widetilde{V}_{+1}(0,s)\begin{pmatrix}1\\-i\gamma_0\end{pmatrix} = e^{\gamma_0 s^2/2}\begin{pmatrix}1\\-i\gamma_0\end{pmatrix} \tag{3.3.37}$$

and also

$$\widetilde{V}_{-1}(0,s)\begin{pmatrix}1\\i\gamma_0\end{pmatrix} = e^{\gamma_0 s^2/2}\begin{pmatrix}1\\i\gamma_0\end{pmatrix}. \tag{3.3.38}$$

Then, thanks to (3.3.24) there holds

$$V_{+1}(0,s)\vec{e}_+^{\mathrm{S}} = \widetilde{V}_{+1}(0,s)\begin{pmatrix}1\\-i\gamma_0\end{pmatrix} - \int_0^s \varepsilon^{1/2}\widetilde{V}_{+1}(\tau,s)\,\left(Q_0^{-1}\partial_t Q_0\right)(\varepsilon^{1/2}\tau)\,V_n(0,\tau)\vec{e}_+^{\mathrm{S}}d\tau.$$

Using the upper bounds of $\widetilde{V}$ and V proved in Lemma 3.3.4, we get the following estimate for the integral term

$$\begin{aligned}&\left|\int_0^s \varepsilon^{1/2}\widetilde{V}_{+1}(\tau,s)\,\left(Q_0^{-1}\partial_t Q_0\right)(\varepsilon^{1/2}\tau)\,V_n(0,\tau)\vec{e}_+^{\mathrm{S}}d\tau\right|\\ &\lesssim \varepsilon^{1/2}\int_0^s \exp(\gamma_0(s^2-\tau^2)/2)\exp(\gamma_0\tau^2/2)d\tau\\ &\lesssim \varepsilon^{1/2}s\exp(\gamma_0 s^2/2).\end{aligned}$$

By (3.3.37) and as $\varepsilon^{1/2}s$ is small, we get

$$\left|V_{+1}(0,s)\vec{e}_+^{\mathrm{S}}\right| \approx e^{\gamma_0 s^2/2}$$

and the same holds for $V_{-1}(0,s)$. We end the proof by using formula (3.3.20).

□

In the Airy case, we make a careful analysis of the prefactor term coming from the crossing of eigenvalues.

Lemma 3.3.7 (Growth of the free solution: Airy case)**.** *We define*

$$\vec{e}_+^{\mathrm{Ai}} = Q_0(0,x)\begin{pmatrix}\mathrm{Ai}_1(0)\\-i\varepsilon^{1/3}j\mathrm{Ai}_1{}'(0)\end{pmatrix} \quad \textit{and} \quad \vec{e}_-^{\mathrm{Ai}} = Q_0(0,x)\begin{pmatrix}\mathrm{Ai}_1(0)\\i\varepsilon^{1/3}j\mathrm{Ai}_1{}'(0)\end{pmatrix} \tag{3.3.39}$$

where Ai_1 *is the Airy function defined in Lemma* 3.6.1 *of the Appendix, and* $j = e^{2i\pi/3}$*. Then the free solution* $\mathbf{f}^{\mathrm{Ai}}$ *of the Airy case satisfies*

$$|\mathbf{f}^{\mathrm{Ai}}(s)| \approx s^{-1/4}e^{-M(\varepsilon)}e^{\int_0^s \gamma_{\mathrm{Ai}}^{\flat}(\tau)d\tau} \quad , \quad \forall 0 \le s < \varepsilon^{-2/3} \tag{3.3.40}$$

with

$$\gamma_{\mathrm{Ai}}^{\flat}(\tau) = \gamma_0\tau^{1/2}. \tag{3.3.41}$$

We postpone the proof of this Lemma to Section 3.6.3 of the Appendix.

Remark 3.3.8. *Note that, on the contrary of Chapter* 2*, in each case there holds* $\gamma_\eta^\sharp = \gamma_\eta^\flat$*. This is due to the fact that we do not consider here the full varying-coefficient operator* $\underline{A}(\varepsilon^{1/(1+\eta)}s,x)\partial_\theta$ *but the reduced operator* $\underline{A}^\eta(s)\partial_\theta$*, which is a first-order approximation of operator* $\underline{A}(\varepsilon^{1/(1+\eta)}s,x)\partial_\theta$*.*

3.3.5 Fixed point equation

Using the propagator $U^\eta(s', s, \theta)$, the free solution (3.3.32) and the Duhamel formula, we can express now (3.3.14) as the fixed point equation

$$\mathbf{u}(s, x, \theta) = \mathbf{f}^\eta(s, \theta) + \int_0^s U^\eta(s', s, \theta)\mathbf{G}^\eta(s', \mathbf{u}(s'x, \theta))ds' \tag{3.3.42}$$

where $\mathbf{G}^\eta(\mathbf{u})$ is defined by (3.3.15). We denote the integral term

$$T^\eta(s, \mathbf{u}) = \int_0^s U^\eta(s', s)\mathbf{G}^\eta(s', \mathbf{u}(s'))ds' \tag{3.3.43}$$

which we split into three parts thanks to definition (3.3.15) like

$$T^\eta(s, \mathbf{u}) \tag{3.3.44}$$

$$= \int_0^s U^\eta(s', s)\left[\left(\varepsilon^{-\eta/(1+\eta)}\mathbf{R}^\eta + \varepsilon^{(2-\eta)/(1+\eta)}\mathbf{A}_u \cdot \mathbf{u}\right)\partial_\theta \mathbf{u} + \varepsilon^{1/(1+\eta)}\left(\mathbf{A} \cdot \partial_x \mathbf{u} + \mathbf{F}\right)\right] ds'$$

$$= T^{[\eta,\theta]}(s, \mathbf{u}) + T^{[\eta,x]}(s, \mathbf{u}) + T^{[\eta,\mathbf{u}]}(s, \mathbf{u}) \tag{3.3.45}$$

where we define

$$T^{[\eta,\theta]}(s, x, \mathbf{u}) = \int_0^s U^\eta(s', s)\left[\varepsilon^{-\eta/(1+\eta)}\mathbf{R}^\eta + \varepsilon^{(2-\eta)/(1+\eta)}\mathbf{A}_u \cdot \mathbf{u}\right]\partial_\theta \mathbf{u}(s')ds' \tag{3.3.46}$$

$$T^{[\eta,x]}(s, x, \mathbf{u}) = \int_0^s U^\eta(s', s)\left[\varepsilon^{1/(1+\eta)}\mathbf{A}(s', x, \mathbf{u}(s'))\right] \cdot \partial_x \mathbf{u}(s')ds' \tag{3.3.47}$$

$$T^{[\eta,\mathbf{u}]}(s, x, \mathbf{u}) = \int_0^s U^\eta(s', s)\left[\varepsilon^{1/(1+\eta)}\mathbf{F}(s', x, \mathbf{u}(s'))\right]\mathbf{u}(s')ds' \tag{3.3.48}$$

We have now reduced the initial question of finding a family of initial data h_ε generating a family of appropriately growing analytic solutions u_ε to the fixed point equation (3.3.42) for operator T^η. In order to prove Theorems 4 and 5 we refer to the proof of Gevrey instability in the case of initial ellipticity in Chapter 2. A sketch of the proof can be found in Section 2.3.6 therein.

3.4 Contraction estimates

We make use of spaces $\mathbb{E}$ constructed in Section 2.4 of Chapter 2 and their properties developed in Section 2.5 therein. The method is robust enough to be used in our context of transitions from hyperbolicity to ellipticity.

3.4.1 Functional spaces: definitions

We refer to Section 2.4.1 of Chapter 2 for definition and properties of the majoring series relation denoted by $\prec$. We recall the definition of Φ as the reference series in one variable

$$\Phi(z) = \sum_{k\geq 0} \frac{c_0}{k^2+1} z^k$$

with $c_0 > 0$ such that $\Phi^2 \prec \Phi$. We recall the notation

$$\Phi_k(\rho t) = R^{|k|} \sum_{p\in\mathbb{N}} \frac{c_0}{(|k|+p)^2+1} \binom{|k|+p}{k,p} \rho^p t^p \quad , \quad \forall k \in \mathbb{N}^d \tag{3.4.1}$$

of the positive coefficients of $\Phi(RX_1 + \cdots + RX_d + \rho t)$, with R and ρ both positive parameters. From now on, we will denote for convenience and with an abuse of notation $\Phi(RX + \rho t) = \Phi(RX_1 + \cdots + RX_d + \rho t)$.

Recall that, for any formal series $\phi(t,x) = \sum_{k\in\mathbb{N}^d} \phi_k(t)x^k$ in the x variable, with t a parameter, the notation $\phi(t,x) \prec_t \Phi(RX + \rho t)$ means

$$|\phi_k(t)| \leq \Phi_k(t) \quad , \quad \forall\, k \in \mathbb{N}^d \,,\ \forall t < \rho^{-1}.$$

We consider trigonometric series in one variable θ with coefficients in the space of formal series in d variables x in the sense of Section 2.4.1 of Chapter 2, and we denote F_{2d+1} the space of all such trigonometric series:

$$F_{2d+1} = \left\{ \mathbf{v}(x,\theta) = \sum_{n\in\mathbb{Z}} \mathbf{v}_n(x) e^{in\theta} \;\middle|\; \mathbf{v}_n(x) = \sum_{k\in\mathbb{N}^d} \mathbf{v}_{n,k} x^k \right\}.$$

We now define fixed time spaces $\mathbb{E}_s$ as in the previous Chapter, which we slightly modify as to take into account three differences:

- The renormalization in time is $t = \varepsilon^{1/(1+\eta)} s$ instead of $t = \varepsilon s$ in Chapter 2.
- The growth of the propagator as described in Lemmas 3.3.4 and 3.3.5 still depends on a nonnegative function $\gamma_\eta^\sharp(\tau)$, and spaces $\mathbb{E}$ are precisely developed on such functions. The difference lies in the final growth of time, as it will be more precise in (3.4.5).

Definition 3.4.1 (Fixed time spaces $\mathbb{E}_s$). *Given $\eta \geq 0$, $M' > 0$, $R > 0$, $\rho > 0$, $\beta \in (0,1)$ and $s \in [0, \left(\varepsilon^{1/(1+\eta)}\rho\right)^{-1})$, we denote $\mathbb{E}_s = \mathbb{E}_s(\eta, R, \rho, M', \beta)$ the space of trigonometric series $\mathbf{v} \in F_{2d+1}$ such that for some constant $C > 0$ there holds*

$$\mathbf{v}_n(x) \prec C \frac{c_1}{n^2+1} \exp\left(-\left(M' - \int_0^s \gamma_\eta(\tau) d\tau \right) \langle n \rangle \right) \Phi\left(RX + \varepsilon^{1/(1+\eta)} \rho s \right) \quad , \quad \forall\, n \in \mathbb{Z} \tag{3.4.2}$$

where we denote

$$\gamma_\eta(\tau) = \gamma_\eta^\sharp(\tau) + \beta. \tag{3.4.3}$$

We define a norm on $\mathbb{E}_s$ by

$$\|\mathbf{v}\|_s = \inf\{C > 0 \,|\quad (3.4.2) \text{ is satisfied } \}. \tag{3.4.4}$$

As in the discussion following the Definition 2.4.7 of Chapter 2, we introduce the growth time $\underline{s}_1(\eta)$ defined implicitely as

$$M' = \int_0^{\underline{s}_1(\eta)} \gamma_\eta(\tau) d\tau \tag{3.4.5}$$

and the final time as

$$\underline{s}(\eta) = \min\left\{\underline{s}_1(\eta), \left(\varepsilon^{1/(1+\eta)}\rho\right)^{-1}\right\} \tag{3.4.6}$$

where $\left(\varepsilon^{1/(1+\eta)}\rho\right)^{-1}$ is the regularity time. To simplify the notations, in all the following we will omit the parameters R, ρ, M' and β in $\mathbb{E}_s(\eta, R, \rho, M', \beta)$.

We consider now trigonometric series

$$\mathbf{u}(s, x, \theta) = \sum_{n\in\mathbb{Z}} \mathbf{u}_n(s, x)e^{in\theta}$$

with coefficients $\mathbf{u}_n(s, x)$ being formal series in x whose coefficients depend smoothly on $s \in [0, \underline{s}(\eta))$. We denote F_{2d+2} the space of all such trigonometric series:

$$F_{2d+2} = \left\{\mathbf{u}(s, x, \theta) = \sum_{n\in\mathbb{Z}} \mathbf{u}_n(s, x)e^{in\theta} \;\middle|\; \mathbf{u}_n(s, x) = \sum_{k\in\mathbb{N}^d} \mathbf{u}_{n,k}(s)x^k \text{ with } \mathbf{u}_{n,k}(s)\, C^\infty \text{ in } s\right\}$$

Definition 3.4.2 (Spaces $\mathbb{E}$). *We introduce*

$$\mathbb{E} = \{\mathbf{u} \in F_{2d+2} \,|\, \forall\, 0 \le s < \underline{s}(\eta)\,, \quad \mathbf{u}(s) \in \mathbb{E}_s\} \tag{3.4.7}$$

and the corresponding norm

$$|||\mathbf{u}||| = \sup_{0\le s<\underline{s}(\eta)} \|\mathbf{u}(s)\|_s \tag{3.4.8}$$

Note that $\mathbf{u}$ being in $\mathbb{E}$ is equivalent to

$$\mathbf{u}_n(s, x) \prec_s |||\mathbf{u}|||\frac{c_1}{n^2+1} \exp\left(\left(M' - \int_0^s \gamma_\eta(\tau)d\tau\right)\langle n\rangle\right)\Phi\left(RX + \varepsilon^{1/(1+\eta)}\rho s\right) \tag{3.4.9}$$

for all $n \in \mathbb{Z}$ and $s \in [0, \underline{s})$.

For $\mathbf{u}$ valued in $\mathbb{C}^N$, $\mathbf{u} \in \mathbb{E}$ means simply that each component of $\mathbf{u}$ is in $\mathbb{E}$, and $|||\mathbf{u}|||$ is then the maximum of the norms of the components. We denote the ball of $\mathbb{E}$ of radius a, centered in $\mathbf{u} \in \mathbb{E}$ by

$$B_{\mathbb{E}}(\mathbf{u}, a) = \{\mathbf{v} \in \mathbb{E} \mid |||\mathbf{v} - \mathbf{u}||| < a\} \tag{3.4.10}$$

3.4.2 Functional spaces: properties

Basic properties

We remind here basic properties of spaces $\mathbb{E}$. The proofs are the same as in Chapter 2, as those properties depend only on the nonnegativity of $\gamma_\eta^\sharp$.

Proposition 3.4.3 (Properties of $\mathbb{E}_s(\eta)$ and $\mathbb{E}(\eta)$.). *For any $\eta \in [0, 1]$ and $s \in [0, \underline{s}(\eta))$, there holds*

1. *The space $\mathbb{E}_s(\eta)$ is an algebra, and for any $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{E}_s(\eta)$ there holds*

$$||\mathbf{vw}||_s \le ||\mathbf{v}||_s\, ||\mathbf{w}||_s. \tag{3.4.11}$$

2. *The space* $\mathbb{E}_s(\eta)$ *is a Banach space.*

As an immediate corollary, there holds

1. *The space* $\mathbb{E}(\eta)$ *is an algebra, and for any* $\mathbf{v}$ *and* $\mathbf{w}$ *in* $\mathbb{E}(\eta)$ *there holds*

$$|||\mathbf{vw}||| \leq |||\mathbf{v}|||\,|||\mathbf{w}|||. \tag{3.4.12}$$

2. *The space* $\mathbb{E}(\eta)$ *is a Banach space.*

The action of analytic function $H(t,x,u)$ on the space $\mathbb{E}$, described in the Lemma 2.4.13 in Chapter 2, still holds as it relies on properties of the majoring series relation and on the definition of Φ described in Section 2.4.1 of Chapter 2.

Lemma 3.4.4. *Let* $H(t,x,u)$ *be an analytical function on a neighborhood of* $(0,0,0) \in \mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^N_u$. *There are constants* $C_H > 0$, $R_H > 0$ *and* $\rho_H > 0$ *which depend only on* H *and* c_0, *such that for all* $R \geq R_H$ *and* $\rho \geq \rho_H$ *and for* ε *small enough,*

$$\forall\, \mathbf{u} \in B_{\mathbb{E}(\eta,R,\rho)}(0,1): \quad |||\mathbf{H}(\cdot,\cdot,\mathbf{u})||| \lesssim 1 \tag{3.4.13}$$

where $\mathbf{H}$ *is defined by* (3.3.3) *and* $|||\cdot|||$ *is defined by* (3.4.8).

In the operators $T^{[\theta]}$, $T^{[x]}$ and $T^{[\mathbf{u}]}$ defined by (3.3.46), (3.3.47) and (3.3.48), there appears A, R^η, A_u and F, all of which are analytic functions in the variables $(t,x,u) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^N$. In the expansion formulas of both R^{S} and R^{Ai} there appear also analytical functions $\mathrm{R}^{\mathrm{S}}_t$, $\mathrm{R}^{\mathrm{S}}_t$, $\mathrm{R}^{\mathrm{Ai}}_t$, $\mathrm{R}^{\mathrm{Ai}}_x$ and $\mathrm{R}^{\mathrm{Ai}}_e$ as in Lemmas 3.3.1 and 3.3.2. The previous Lemma applies:

Corollary 3.4.5. *There are constants* R_0 *and* ρ_0 *such that for all* $\eta \geq 0$, $R \geq R_0$, $\rho \geq \rho_0$ *and* ε *small enough:*

$$\forall\, \mathbf{u} \in B_{\mathbb{E}(\eta,R,\rho)}(0,1): \qquad |||\mathbf{H}(\cdot,\cdot,\mathbf{u})||| \lesssim 1 \tag{3.4.14}$$

with H *either equals to* A, $\underline{A}$, $\mathrm{R}^{\mathrm{S}}_t$, $\mathrm{R}^{\mathrm{S}}_x$, $\mathrm{R}^{\mathrm{Ai}}_t$, $\mathrm{R}^{\mathrm{Ai}}_x$, $\mathrm{R}^{\mathrm{Ai}}_{t_\star}$, A_u *or* F.

Action of propagators on $\mathbb{E}$

To describe the action of both propagators U^{S} and U^{Ai}, we define here more general smooth matrix operators $U^\eta(s',s,\theta)$ for $\eta \geq 0$ that act diagonally on $\mathbf{u} \in \mathbb{E}$ as

$$\big(U^\eta(s',s,\theta)\mathbf{u}(s')\big)_n = U^\eta_n(s',s)\mathbf{u}_n(s') \quad , \quad \forall n \in \mathbb{Z},\ 0 \leq s' \leq s$$

and satisfied the upper bound for their Fourier modes

$$|U^\eta_n(s',s)| \lesssim C(U^\eta)\exp\left(\int_{s'}^{s} \gamma^\sharp_\eta(\tau)d\tau\right) \quad , \quad \forall n \in \mathbb{Z},\ 0 \leq s' \leq s. \tag{3.4.15}$$

for some $C(U^\eta) > 0$ depending eventually on ε. In the smooth case, thanks to Lemma 3.3.4 we have $\eta = 1$ and

$$C(U^{\mathrm{S}}) = 1. \tag{3.4.16}$$

whereas for the Airy case, thanks to Lemma 3.3.5 we have $\eta = 1/2$ and

$$C(U^{\mathrm{Ai}}) = \varepsilon^{-1/3} \tag{3.4.17}$$

For such matrix operators U^η, the same result as Lemma 2.4.15 in Chapter 2 still holds:

Lemma 3.4.6. *Given $\eta \geq 0$, $\beta > 0$ and $\mathbf{u}$ in $\mathbb{E}(\eta, \beta)$ then*

$$U_n^\eta(s', s)\mathbf{u}_n(s', x) \prec_{s',s} C(U^\eta)\, C_n^\eta(s', s)\, ||\mathbf{u}(s')||_{s'} \frac{c_1}{n^2+1} e^{-\left(M' - \int_0^s \gamma_\eta(\tau)d\tau\right)\langle n\rangle} \Phi\left(RX + \varepsilon^{1/(1+\eta)}\rho s\right) \tag{3.4.18}$$

where $C_n^\eta(s', s)$ is defined by

$$C_n^\eta(s', s) = \exp\left(-\beta(s - s')\langle n\rangle\right) \leq 1 \tag{3.4.19}$$

In particular we have

$$\|U^\eta(s', s)\mathbf{u}(s')\|_s \leq C(U^\eta)\, \|\mathbf{u}(s')\|_{s'} \qquad \forall\, 0 \leq s' \leq s < \underline{s} \tag{3.4.20}$$

The proof is exactly the same as in Chapter 2, as it relies only on the definition (3.4.3) of γ_η. The positive constant β acts as a perturbation of $\gamma_\eta^\sharp$ and introduces an error term like $e^{-\beta(s-s')|n|}$ in the growth of the n-th Fourier mode of the propagator. This explains why the prefactor term $C_n^\eta(s', s)$ is exactly the same as in Lemma 2.4.15 of Chapter 2.

Remark 3.4.7. *The estimate* (3.4.20) *is not precise enough to show that T is a contraction in $\mathbb{E}$. The more precise estimate* (3.4.18) *is very important for the estimate* (3.4.22) *below.*

Norm of the free solutions

In both smooth ($\eta = 1$) and Airy ($\eta = 1/2$) cases, we compute the norm in $\mathbb{E}$ of the free solution $\mathbf{f}^\eta$ defined in Lemmas 3.3.6 and 3.3.7. The proof of this result is the same as Lemma 2.4.17 of Chapter 2, using the precise estimates described in Lemmas 3.3.6 and 3.3.7.

Lemma 3.4.8 (Norm of the free solution)**.** *For $\eta = 1$ or $\eta = 1/2$ and $\beta > 0$, the free solution $\mathbf{f}^\eta$ satisfies*

$$|||\mathbf{f}^\eta||| \lesssim e^{M' - M(\varepsilon)} \tag{3.4.21}$$

Remark 3.4.9. *Note that, on the contrary of estimate* (4.33) *in Lemma* 2.4.17 *of Chapter* 2*, the previous estimate is not $|||\mathbf{f}^{\mathrm{Ai}}||| \lesssim \varepsilon^{-1/3} e^{M' - M(\varepsilon)}$ in the Airy case, thanks to the more precise estimate* (3.3.40)*.*

3.4.3 Estimates of remainder terms

As pointed out in Remark 3.3.3, our analysis differs from Chapter 2 with the presence of extra remainder terms $\mathbf{R}^\eta$. We compute carefully their norms.

Lemma 3.4.10 (Smooth case)**.** *In the framework of Lemma* 3.3.1*, the norm of the remainder term $\mathbf{R}^{\mathrm{S}}$ satisfies*

$$|||\varepsilon^{-1/2}\mathbf{R}^{\mathrm{S}}||| \lesssim \varepsilon^{1/2}\underline{s}^2 + \underline{s}R^{-1}.$$

Proof. By expansion formula (3.3.9) we have

$$\mathrm{R}^{\mathrm{S}}(t, x) = t^2 \mathrm{R}_t^{\mathrm{S}}(t, x) + tx \cdot \mathrm{R}_x^{\mathrm{S}}(t, x)$$

and then, as $t = \varepsilon^{1/2}s$ in the smooth case and by notation (3.3.3),

$$\varepsilon^{-1/2}\mathbf{R}^{\mathrm{S}}(s,x) = \varepsilon^{1/2}s^2\mathbf{R}^{\mathrm{S}}_t(s,x) + sx\cdot\mathbf{R}^{\mathrm{S}}_x(s,x).$$

As the norm $|||\cdot|||$ is defined by a supremum in time, first there holds

$$|||\varepsilon^{1/2}s^2\mathbf{R}^{\mathrm{S}}_t||| \lesssim \varepsilon^{1/2}\underline{s}^2|||\mathbf{R}^{\mathrm{S}}_t|||$$

To get a precise estimate of the term $x\cdot\mathbf{R}^{\mathrm{S}}_x(s,x)$, we first note that the coefficients $\Phi_k(\rho t)$ defined by (3.4.1) satisfy

$$1 \leq R^{-1}\Phi_k(\rho t) \quad , \quad \forall k\in\mathbb{N}^d \text{ with } |k|=1$$

so that for all $j = 1,\dots,d$ there holds

$$X_j \prec R^{-1}\Phi(RX + \varepsilon^{1/2}\rho s) \quad , \quad \forall 0\leq s<\underline{s}.$$

By inequality (3.4.12) in Proposition 3.4.3 we get then

$$|||x\cdot\mathbf{R}^{\mathrm{S}}_x(s,x)||| \lesssim R^{-1}|||\mathbf{R}^{\mathrm{S}}|||.$$

As $C(U^{\mathrm{S}}) = 1$ by (3.4.16), this ends the proof. □

Lemma 3.4.11 (Airy case)**.** *In the framework of Lemma* 3.3.2*, the norm of the remainder term* $\mathbf{R}^{\mathrm{Ai}}$ *satisfies*

$$|||\varepsilon^{-1/3}\mathbf{R}^{\mathrm{Ai}}||| \lesssim \varepsilon\underline{s}^2 + \varepsilon^{1/3}\underline{s}R^{-1} + \varepsilon^{-1/3}t_\star(R^{-1}).$$

Proof. The proof is the same as the previous one, with the differences that $\eta = 1/2$ and $C(U^{\mathrm{Ai}}) = \varepsilon^{-1/3}$. □

3.4.4 Contraction estimates

Regularization results

A crucial observation is that derivation operators ∂_θ and ∂_{x_j} are not bounded operators in spaces $\mathbb{E}$, as explained in Section 2.5.1 in Chapter 2. The main results in our previous paper are the description of the regularization effect of integration in time of derivation operators. These results are precised in Sections 2.5.2 through 2.5.4 in Chapter 2.

Those results still hold in our setting. We omit the proof of the following Lemmas, but give some indications on how to adapt the proofs of Chapter 2 of the similar results.

Proposition 3.4.12 (Regularization of ∂_θ)**.** *For operator* $T^{[\eta,\theta]}$ *defined by* (3.3.46)*, for any* $\mathbf{u}\in B_{\mathbb{E}(\eta,R,\rho)}(0,1)$ *and for* $\beta>0$*, there holds*

$$|||T^{[\eta,\theta]}(\mathbf{u})||| \lesssim C(U^\eta)\beta^{-1}\left(\varepsilon^{-\eta/(1+\eta)}|||\mathbf{R}^\eta||| + \varepsilon^{(2-\eta)/(1+\eta)}|||\mathbf{u}|||\right)|||\mathbf{u}||| \tag{3.4.22}$$

The proof is the same as Proposition 2.5.4 in Chapter 2, as it is based on Lemma 3.4.6 and the expression of prefactor C_n^η defined in (3.4.19) which is the same as prefactor (2.4.31) in Lemma 2.4.15 in Chapter 2.

As remainder terms $\mathbf{R}^\eta$ have different norms in spaces $\mathbb{E}$, given by Lemmas 3.4.10 and 3.4.11 , we give more precisely the following two results:

Corollary 3.4.13 (Smooth case). *In the smooth case, thanks to Lemma* 3.4.10*, there holds*

$$|||T^{[\mathrm{S},\theta]}(\mathbf{u})||| \lesssim \beta^{-1}\left(\varepsilon^{1/2}\underline{s}^2 + \underline{s}R^{-1} + \varepsilon^{1/2}|||\mathbf{u}|||\right)|||\mathbf{u}|||. \tag{3.4.23}$$

Corollary 3.4.14 (Airy case). *In the smooth case, thanks to Lemma* 3.4.11*, there holds*

$$|||T^{[\mathrm{Ai},\theta]}(\mathbf{u})||| \lesssim \varepsilon^{-1/3}\beta^{-1}\left(\varepsilon\underline{s}^2 + \varepsilon^{1/3}\underline{s}R^{-1} + \varepsilon^{-1/3}t_\star(R^{-1}) + \varepsilon|||\mathbf{u}|||\right)|||\mathbf{u}|||. \tag{3.4.24}$$

About the regularization of derivation operators ∂_{x_j}, the proof relies again on the simple computation given in Section 2.1.2 of Chapter 2. The difference in the time renormalization $\Phi(Rx + \varepsilon^{1/(1+\eta)}\rho s)$, instead of $\Phi(RX + \varepsilon\rho s)$ in Chapter 2, is a minor one for the proof.

Proposition 3.4.15 (Regularization of ∂_{x_j}). *For operator $T^{[\eta,x]}$ defined by* (3.3.47) *and any $\mathbf{u} \in B_{\mathbb{E}(\eta,R,\rho)}(0,1)$, there holds*

$$|||T^{[\eta,x]}(\mathbf{u})||| \lesssim C(U^\eta)\, R\rho^{-1}\, |||\mathbf{u}|||. \tag{3.4.25}$$

As $\mathbb{E}$ is an algebra the operator $T^{[\eta,\mathbf{u}]}$ is directly bounded, with no need of a regularization by time result, on the contrary of operators $T^{[\eta,\theta]}$ and $T^{[\eta,x]}$. The following proposition gives us precisely

Proposition 3.4.16 (Nonlinear term). *For the operator $T^{[\eta,\mathbf{u}]}$ defined by* (3.3.48)*, for any $\mathbf{u} \in B_{\mathbb{E}}(0,1)$ and $\beta > 0$ there holds*

$$|||T^{[\eta,\mathbf{u}]}(\mathbf{u})||| \lesssim C(U^\eta)\, \beta^{-1}\varepsilon^{1/(1+\eta)}|||\mathbf{u}|||^2. \tag{3.4.26}$$

Contraction estimates

Thanks to the results of the previous Section, we prove estimates for operator T^η defined in (3.3.43), as in Section 2.5.5 of Chapter 2. We omit once again the proof of this result, as it is the same as Proposition 2.5.9 in Chapter 2.

Proposition 3.4.17 (Contraction estimates in $\mathbb{E}$). *There are R_0, $\rho_0 > 0$ such that for all $\beta > 0$, $R \geq R_0$, $\rho > \rho_0$ and $\varepsilon \in (0,1)$, we get the following estimates for all $\mathbf{u}$ and $\mathbf{v}$ in $B_{\mathbb{E}}(0,1)$:*

$$|||T^\eta(\mathbf{u})||| \lesssim C(U^\eta)\left(\beta^{-1}\left(\varepsilon^{-\eta/(1+\eta)}|||\mathbf{R}^\eta||| + \varepsilon^{1/(1+\eta)}|||\mathbf{u}|||\right) + R\rho^{-1}\right)|||\mathbf{u}|||, \tag{3.4.27}$$

$$|||T^\eta(\mathbf{u}) - T^\eta(\mathbf{v})||| \lesssim C(U^\eta)\left(\beta^{-1}\left(\varepsilon^{-\eta/(1+\eta)}|||\mathbf{R}^\eta||| + \varepsilon^{1/(1+\eta)}|||\mathbf{u}|||\right) + R\rho^{-1}\right)|||\mathbf{u} - \mathbf{v}||| \tag{3.4.28}$$

For convenience we introduce

$$K^\eta(\varepsilon) = C(U^\eta)\left(\beta^{-1}\left(\varepsilon^{-\eta/(1+\eta)}|||\mathbf{R}^\eta||| + \varepsilon^{1/(1+\eta)}|||\mathbf{f}^\eta|||\right) + R\rho^{-1}\right). \tag{3.4.29}$$

3.5 Estimates from below and Hadamard instability

3.5.1 Existence of solutions

Thanks to the Proposition 3.4.17, we can now solve the fixed point equation (3.3.42) in the ball $B_{\mathbb{E}(\eta)}(0, |||\mathbf{f}^\eta|||)$:

Corollary 3.5.1 (Contraction and fixed point in $\mathbb{E}$). *Let $\eta > 0$ be fixed. Let $R(\varepsilon) > R_0$, $\rho(\varepsilon) > \rho_0$, $\beta(\varepsilon) > 0$ and $\underline{s}(\eta)$ be such that*

$$\lim_{\varepsilon \to 0} C(U^\eta) \left(\beta^{-1} \left(\varepsilon^{-\eta/(1+\eta)} |||\mathbf{R}^\eta||| + \varepsilon^{1/(1+\eta)} |||\mathbf{f}^\eta||| \right) + R\rho^{-1} \right) = 0 \tag{3.5.1}$$

Let assume that the propagator U^η satisfy the growth (3.4.15)*. Then for ε small enough, the fixed point equation* (3.3.42)*, with $\mathbf{f}^\eta$ defined by* (3.3.32)*, has a unique solution $\mathbf{u}$ in $B_{\mathbb{E}(\eta,R,\rho,\beta)}(0, 2|||\mathbf{f}^\eta|||)$. This solution satisfies*

$$|||\mathbf{u} - \mathbf{f}^\eta||| \lesssim K^\eta(\varepsilon) |||\mathbf{f}^\eta||| \tag{3.5.2}$$

with K^η defined in (3.4.29)*.*

The proof of the Corollary is straigthforward using the estimates of Proposition 3.4.17, under the condition of smallness for $K^\eta(\varepsilon)$ given by (3.5.1). This Corollary is in some sense an abstract result, as it deals with abstract propagator U^η with specific growth (3.4.15), described at the beginning of Section 3.4.2. We emphazise that U^η, except for both smooth and Airy cases, have not be proved to exist. Corollary 3.5.1 gives a result on fixed point equations (3.3.42), independently of the initial Cauchy equation.

3.5.2 Bounds from below

From now on we focus on both smooth and Airy cases, for which we have proved the existence and the growth of the propagators (see Lemmas 3.3.4 and 3.3.5) and the actual growth of the special free solution (see Lemmas 3.3.6 and 3.3.7).

We follow here Section 2.6.2 of Chapter 2. We aim to prove that, in the smooth case, the solutions have the same growth as $\mathbf{f}^{\mathrm{S}}$ given in Lemma 3.3.6, that is

$$|\mathbf{u}(s,x,\theta)| \gtrsim e^{-M(\varepsilon)} \exp\left(\int_0^s \gamma_{\mathrm{S}}^\flat(\tau) d\tau \right) \quad , \quad \forall\, (s,x,\theta) \in \Omega_{R,\varepsilon^{1/(1+\eta)}\rho} \times \mathbb{T} \tag{3.5.3}$$

with $\Omega_{R,\varepsilon^{1/(1+\eta)}\rho}$ defined by (3.2.22). In the Airy case, thanks to Lemma 3.3.7, we aim to prove

$$|\mathbf{u}(s,x,\theta)| \gtrsim s^{-1/4} e^{-M(\varepsilon)} \exp\left(\int_0^s \gamma_{\mathrm{Ai}}^\flat(\tau) d\tau \right) \quad , \quad \forall\, (s,x,\theta) \in \Omega_{R,\varepsilon^{2/3}\rho} \times \mathbb{T}. \tag{3.5.4}$$

As in [Mor16a], by some computations we prove the pointwise inequality

$$|(\mathbf{u} - \mathbf{f}^\eta)(s,x,\theta))| \lesssim K^\eta(\varepsilon)\, C(U^\eta) e^{-M(\varepsilon)} \exp\left(\int_0^s \gamma_\eta(\tau) d\tau \right) \tag{3.5.5}$$

which is inequality (2.6.5) in Chapter 2, holding for all $(s,x,\theta) \in \Omega_{R,\varepsilon^{1/(1+\eta)}\rho} \times \mathbb{T}$. Next, by definition (3.4.3) of γ_η, inequality (3.5.5) becomes

$$\begin{aligned}
&|(\mathbf{u}-\mathbf{f}^\eta)(s,x,\theta))| \\
&\lesssim K^\eta(\varepsilon)\, C(U^\eta) \exp\left(\underline{s}\beta + \int_0^s \left(\gamma_\eta^\sharp(\tau) - \gamma_\eta^\flat(\tau)\right) d\tau\right) e^{-M(\varepsilon)} \exp\left(\int_0^s \gamma_\eta^\flat(\tau) d\tau\right).
\end{aligned}$$

As $\gamma_\eta^\flat = \gamma_\eta^\sharp$ (see Remark 3.3.8), we finally get

$$|(\mathbf{u}-\mathbf{f}^\eta)(s,x,\theta))| \lesssim K(\varepsilon)\, C(U^\eta)\, e^{\underline{s}\beta}\, e^{-M(\varepsilon)} \exp\left(\int_0^s \gamma_\eta^\flat(\tau) d\tau\right). \tag{3.5.6}$$

Then in order to get (3.5.3) or (3.5.4) thanks to (3.5.6), limit (3.5.1) is not sufficient as the term $e^{\underline{s}\beta}$ could be large as ε tends to 0, as explained in Section 2.6.2 of Chapter 2. In both cases, we have then a stronger constraint on parameters R, ρ, β and M':

- In the smooth case, thanks to Corollary 3.4.13 and (3.5.3), we need

$$\lim_{\varepsilon\to 0} \left(\beta^{-1}\left(\varepsilon^{1/2}\underline{s}^2 + \underline{s}R^{-1} + \varepsilon^{1/2}e^{M'-M}\right) + R\rho^{-1}\right) e^{\underline{s}\beta} = 0. \tag{3.5.7}$$

- In the Airy case, thanks to Corollary 3.4.14 and (3.5.4), we need

$$\lim_{\varepsilon\to 0} \varepsilon^{-1/3}\underline{s}^{1/4} \left(\beta^{-1}\left(\varepsilon\underline{s}^2 + \varepsilon^{1/3}\underline{s}R^{-1} + \varepsilon^{-1/3}t_\star(R^{-1}) + \varepsilon^{2/3}e^{M'-M}\right) + R\rho^{-1}\right) e^{\underline{s}\beta} = 0. \tag{3.5.8}$$

The second constraint on the parameters comes from the competition between the characteristic growth time $\underline{s}_1(\eta)$ defined in (3.4.5) and the regularity time $(\varepsilon^{1/(1+\eta)}\rho)^{-1}$. To see the growth of the solution, hence the instability, we need it to exist on a sufficiently large time compared to the growth time, that is we need $\underline{s}$ to be $\underline{s}_1(\eta)$.

As M' is large in the limit $\varepsilon \to 0$, the implicit definition (3.4.5) of $\underline{s}_1(\eta)$ and definition (3.3.19) of $\gamma_{\mathrm{S}}^\sharp$ and definition (3.3.27) of $\gamma_{\mathrm{Ai}}^\sharp$ lead to the equivalent

$$\underline{s}_1(\eta) \approx M'^{1/(1+\eta)} \tag{3.5.9}$$

for $\eta = 1$ and $\eta = 1/2$. Hence the following constraints:

- In the smooth case,

$$\lim_{\varepsilon\to 0} M'^{1/2}\varepsilon^{1/2}\rho = 0. \tag{3.5.10}$$

- In the Airy case,

$$\lim_{\varepsilon\to 0} M'^{2/3}\varepsilon^{2/3}\rho = 0. \tag{3.5.11}$$

We focus now on both cases separately from now on, even if the way we find suitable R, ρ, β and M' which would satisfy both constraints is very similar in both cases. We sum up all of this in the two following Propositions.

Proposition 3.5.2 (Estimate from below: smooth case). *With the limitation of the Gevrey index*

$$\sigma < \delta < 1/3, \tag{3.5.12}$$

both constraints (3.5.7) *and* (3.5.10) *are satisfied. Then the fixed point equation* (3.3.42) *has a unique solution* $\mathbf{u}$ *in* $\mathbb{E}$ *and*

$$|\mathbf{u}(s,x,\theta)| \gtrsim e^{-M(\varepsilon)} e^{\gamma_0 \frac{1}{2} s^2} \quad , \quad \forall\, (s,x,\theta) \in \Omega_{R,\varepsilon^{1/2}\rho} \times \mathbb{T}. \tag{3.5.13}$$

There holds also

$$\underline{s} \approx \varepsilon^{-\delta/2}. \tag{3.5.14}$$

Proof. As in Chapter 2, we use notation $\ll$ defined in Notation (2.1.32) to rewrite all the constraints in a more useful way. Constraints (3.5.7) and (3.5.10) are equivalent to

$$\beta^{-1}\varepsilon^{1/2}\varepsilon^{-\delta}\, e^{\beta\varepsilon^{-\delta/2}} \ll 1 \tag{3.5.15}$$

$$\beta^{-1}\varepsilon^{-\delta/2}R^{-1}\, e^{\beta\varepsilon^{-\delta/2}} \ll 1 \tag{3.5.16}$$

$$\beta^{-1}\varepsilon^{1/2}e^{M'-M}\, e^{\beta\varepsilon^{-\delta/2}} \ll 1 \tag{3.5.17}$$

$$R\rho^{-1}\, e^{\beta\varepsilon^{-\delta/2}} \ll 1 \tag{3.5.18}$$

$$\varepsilon^{-\delta/2}\varepsilon^{1/2}\rho \ll 1. \tag{3.5.19}$$

This implies first, as in [Mor16a], that β as to be of size $\varepsilon^{\delta/2}$. We then posit

$$\beta = \varepsilon^{\delta/2}.$$

We have then

$$\varepsilon^{1/2}\varepsilon^{-3\delta/2} \ll 1 \tag{3.5.20}$$

$$\varepsilon^{-\delta}R^{-1} \ll 1 \tag{3.5.21}$$

$$\varepsilon^{1/2-\delta/2}e^{M'-M} \ll 1 \tag{3.5.22}$$

$$R\rho^{-1} \ll 1 \tag{3.5.23}$$

$$\varepsilon^{1/2-\delta/2}\rho \ll 1. \tag{3.5.24}$$

Asymptotic inequality (3.5.20) is equivalent to the limitation $\delta < 1/3$ on the Gevrey index. Next, as $\delta < 1$, asymptotic inequality (3.5.22) is satisfied as soon as $M' = M - |\ln(\varepsilon)|$.

Finally, inequalities (3.5.21), (3.5.23) and (3.5.24) are equivalent to

$$\varepsilon^{1/2-\delta/2} \ll \rho^{-1} \ll R^{-1} \ll \varepsilon^{\delta}. \tag{3.5.25}$$

This chain of asymptotic inequalities is satisfied as soon as $\varepsilon^{1/2-\delta/2} \ll \varepsilon^{\delta}$, which is equivalent again to the limitation $\delta < 1/3$ of the Gevrey index. Then the choice $R^{-1} = \varepsilon^{1/6+\delta/2}$ and $\rho^{-1} = \varepsilon^{1/3}$ satisfies the constraints.

□

Remark 3.5.3. *Note that in the case where* $\mathrm{R}^{\mathrm{S}} \equiv 0$*, both* (3.5.20) *and* (3.5.21) *disappear hence the limitation* $\delta < 1$ *in place of* $\delta < 1/3$*. This has to be put in parallel of Remark* 2.2.11 *in Chapter* 2 *, which describes a way to improve the result of Theorem* 3 *therein.*

Proposition 3.5.4 (Estimate from below: Airy case)**.** *With the limitation of the Gevrey index*

$$\sigma < \delta < 2/13 \tag{3.5.26}$$

both constraints (3.5.8) *and* (3.5.11) *are satisfied. Then the fixed point equation* (3.3.42) *has a unique solution* $\mathbf{u}$ *in* $\mathbb{E}$ *and*

$$|\mathbf{u}(s,x,\theta)| \gtrsim e^{-M(\varepsilon)} e^{\gamma_0 \frac{2}{3} s^{3/2}} \quad , \quad \forall\, (s,x,\theta) \in \Omega_{R,\varepsilon^{2/3}\rho} \times \mathbb{T}. \tag{3.5.27}$$

There holds also

$$\underline{s} \approx \varepsilon^{-2\delta/3}. \tag{3.5.28}$$

Proof. We follow here the same proof as the one of Proposition 3.5.2. Two differences appear: the extra weight $\varepsilon^{-1/3}$ coming from the specificity of the Airy propagator in Lemma 3.3.5 and the extra $t_\star(R^{-1})$ in the remainder term of Corollary 3.4.14.

Taking both those differences into account, the following constraints hold

$$\beta^{-1}\varepsilon^{2/3}\varepsilon^{-3\delta/2}\, e^{\beta\varepsilon^{-2\delta/3}} \ll 1 \tag{3.5.29}$$

$$\beta^{-1}\varepsilon^{-5\delta/6}R^{-1}\, e^{\beta\varepsilon^{-2\delta/3}} \ll 1 \tag{3.5.30}$$

$$\beta^{-1}\varepsilon^{-\delta/6}\varepsilon^{-2/3}t_\star(R^{-1})\, e^{\beta\varepsilon^{-2\delta/3}} \ll 1 \tag{3.5.31}$$

$$\beta^{-1}\varepsilon^{-\delta/6}\varepsilon^{1/3}e^{M'-M}\, e^{\beta\varepsilon^{-2\delta/3}} \ll 1 \tag{3.5.32}$$

$$\varepsilon^{-1/3}\varepsilon^{-\delta/6}R\rho^{-1}\, e^{\beta\varepsilon^{-2\delta/3}} \ll 1 \tag{3.5.33}$$

$$\varepsilon^{-2\delta/3}\varepsilon^{2/3}\rho \ll 1 \tag{3.5.34}$$

where there holds $\underline{s} \approx \varepsilon^{-2\delta/3}$. Again, those asymptotic inequalities imply that $\beta = \varepsilon^{2\delta/3}$ and inequality (3.5.29) is replaced by

$$\varepsilon^{2/3-13\delta/6} \ll 1$$

which gives the limitation $\delta < 4/13$ on the Gevrey index.

To find now R and ρ, we first use (3.5.30) and (3.5.33) to get

$$\varepsilon^{2/3-2\delta/3} \ll \rho^{-1} \ll \varepsilon^{1/3+\delta/6}R^{-1}. \tag{3.5.35}$$

For an asymptotic upper bound for R^{-1}, we have in this case two possibilities, thanks to (3.5.30) and (3.5.31). If we assume here that $t_\star$ is of order $k \geq 2$ in x, these two inequalities are equivalent to

$$R^{-1} \ll \varepsilon^{3\delta/2} \tag{3.5.36}$$

and

$$R^{-1} \ll \varepsilon^{\frac{1}{k}(2/3+5\delta/6)}. \tag{3.5.37}$$

The question is which one of (3.5.36) or (3.5.37) is a stronger constraint on R and ρ. By simple computations, we prove

$$3\delta/2 < \frac{1}{k}(2/3+5\delta/6) \quad \Longleftrightarrow \quad \delta < \frac{1}{9k/4-5/4}.$$

We are then reduced to study two cases:

- If $\delta < \frac{1}{9k/4-5/4}$, then $R^{-1} \ll \varepsilon^{\frac{1}{k}(2/3+5\delta/6)} \ll \varepsilon^{3\delta/2}$. With (3.5.35), we get the constraint $\varepsilon^{2/3-2\delta/3} \ll \varepsilon^{1/3+\frac{1}{k}(2/3+5\delta/6)}$, which is equivalent to $\delta < \frac{1/2-1/k}{1+1/k}$. We note in particular that the non degenerate Airy case $k=2$ is out of reach with our method. In the degenerate case $k=4$, we have the limitation $\delta < \frac{1/2-1/k}{1+1/k} = 4/21$, compatible with $\delta < \frac{1}{9k/4-5/4} = 4/31$.

- If $\delta > \frac{1}{9k/4-5/4}$, then $R^{-1} \ll \varepsilon^{3\delta/2} \ll \varepsilon^{\frac{1}{k}(2/3+5\delta/6)}$. With (3.5.35), we get the constraint $\varepsilon^{2/3-2\delta/3} \ll \varepsilon^{1/3+3\delta/2}$, which is equivalent to $\delta < 2/13$. It is incompatible with $\delta > \frac{1}{9k/4-5/4}$ when $k=2$.

In each of the previous cases, the case $k=2$ leads to a contradiction, hence proving that the non-degenerate Airy transition is out of reach of our method. In the degenerate case $k=4$, the previous analysis shows that the limiting Gevrey index is $2/13$.

□

Remark 3.5.5. *On the contrary of the smooth case and Remark* 3.5.3*, the limitation* $\delta < 2/13$ *still holds when* $\mathrm{R}^{\mathrm{Ai}} \equiv 0$*. This can be explained as* $t_\star$ *represents the transition time from hyperbolicity to ellipticity, and the domain of hyperbolicity is too large to be considered as an elliptic region.*

3.5.3 Conclusion: Hadamard instability in Gevrey spaces

To close the proofs of Theorem 4 and Theorem 5 we have now to get an estimate of the ratio

$$\frac{||u_\varepsilon||_{L^2(\Omega_{R,\rho})}}{||h_\varepsilon^\eta||^\alpha_{\sigma,c,K}}.$$

The previous Sections show the existence, in either the smooth or the Airy case, of a family of solutions $\mathbf{u}$ starting from $\mathbf{f}^\eta$ of the fixed point equation (3.3.42). Thanks to the ansatz (3.3.2) which we recall here

$$u_\varepsilon(t,x) = \varepsilon^{2/(1+\eta)}\mathbf{u}(\varepsilon^{-1/(1+\eta)}\, t, x, x\cdot\xi_0/\varepsilon)$$

we have then a family of solutions u_ε existing in domains $\Omega_{R,\rho}$, for some well-chosen parameters described in the proof of Proposition 3.5.2 or 3.5.4. In both cases we can verify that domains $\Omega_{R,\rho}$ contain the cube of size ε

$$C_\varepsilon = \{(t,x)\,|\,\underline{t}-\varepsilon < t < \underline{t}, \quad |x| < \varepsilon\}$$

where we denote simply $\underline{t} = \varepsilon^{1/(1+\eta)}\underline{s}$. The conclusion of the proof of Theorems 4 and 5 is the same as in Section 2.7 in [Mor16a].

3.6 Appendix: on the Airy equation

The purpose of this Appendix is to bring some crucial elements on the Airy equation, and to complete the proofs of Lemma 3.3.5 and 3.3.7. We recall here equation (3.3.28), with

$\eta = 1/2$, that appears on Lemma 3.3.5:

$$\partial_s \widetilde{V}_n(s',s) - in \begin{pmatrix} 0 & \varepsilon^{-1/3} \\ -\varepsilon^{1/3}\gamma_0^2 s & 0 \end{pmatrix} \widetilde{V}_n(s',s) = 0 \qquad \widetilde{V}_n(s',s') = \mathrm{Id}. \tag{3.6.1}$$

The aim here is to get upper bounds for the matrix flow $\widetilde{V}_n(s',s)$ for all $0 \le s' \le s$ and all $n \in \mathbb{Z}$, and hence to complete the proof of Lemma 3.3.5. For simplicity we denote

$$\widetilde{V}_n(s',s) = \begin{pmatrix} \widetilde{V}_{n,1,1} & \widetilde{V}_{n,1,2} \\ \widetilde{V}_{n,2,1} & \widetilde{V}_{n,2,2} \end{pmatrix} \qquad \widetilde{V}_{n,p,q}(s',s') = \delta(p,q). \tag{3.6.2}$$

3.6.1 Reduction to the scalar Airy equation and resolution

The vector equation (3.6.1) becomes the system of scalar equations

$$\left\{ \begin{array}{lcll} \partial_s \widetilde{V}_{n,1,1} & = & in\varepsilon^{-1/3}\, \widetilde{V}_{n,2,1} & \widetilde{V}_{n,1,1}(s',s') = 1 \\ \partial_s \widetilde{V}_{n,1,2} & = & in\varepsilon^{-1/3}\, \widetilde{V}_{n,2,2} & \widetilde{V}_{n,1,2}(s',s') = 0 \\ \partial_s \widetilde{V}_{n,2,1} & = & -\, in\varepsilon^{1/3}\gamma_0^2 s\, \widetilde{V}_{n,1,1} & \widetilde{V}_{n,2,1}(s',s') = 0 \\ \partial_s \widetilde{V}_{n,2,2} & = & -\, in\varepsilon^{1/3}\gamma_0^2 s\, \widetilde{V}_{n,1,2} & \widetilde{V}_{n,2,2}(s',s') = 1 \end{array} \right. \tag{3.6.3}$$

Differentiating the first equation and using next the third one, the entry $\widetilde{V}_{n,1,1}$ solves the second order scalar differential equation

$$\partial_s^2 \widetilde{V}_{n,1,1}(s',s) = (n\gamma_0)^2\, s\, \widetilde{V}_{n,1,1}(s',s) \tag{3.6.4}$$

with the initial condition for $\widetilde{V}_{n,1,1}$:

$$\widetilde{V}_{n,1,1}(s',s') = 1. \tag{3.6.5}$$

The initial condition for $\partial_s \widetilde{V}_{n,1,1}$ comes from the first equation of the system (3.6.3) and the initial condition for $\widetilde{V}_{n,2,1}$, as we have

$$\begin{aligned} \partial_s \widetilde{V}_{n,1,1}(s',s') &= in\varepsilon^{-1/3}\, \widetilde{V}_{n,2,1}(s',s') \\ &= 0. \end{aligned} \tag{3.6.6}$$

Note also that we can retrieve $\widetilde{V}_{n,2,1}$ thanks to the first line of (3.6.3), as

$$\widetilde{V}_{n,2,1}(s',s) = \frac{\varepsilon^{1/3}}{in}\, \partial_s \widetilde{V}_{n,1,1}(s',s) \tag{3.6.7}$$

Doing the same for the second and fourth equations, we obtain the same second order scalar differential equation for $\widetilde{V}_{n,1,2}$

$$\partial_s^2 \widetilde{V}_{n,1,2}(s',s) = (n\gamma_0)^2\, s\, \widetilde{V}_{n,1,2}(s',s) \tag{3.6.8}$$

with initial conditions

$$\widetilde{V}_{n,1,2}(s',s') = 0 \tag{3.6.9}$$

and

$$\partial_s \widetilde{V}_{n,1,2}(s',s') = in\varepsilon^{-1/3}. \tag{3.6.10}$$

We have also the relation

$$\widetilde{V}_{n,2,2}(s',s) = \frac{\varepsilon^{1/3}}{in}\,\partial_s \widetilde{V}_{n,1,2}(s',s). \tag{3.6.11}$$

Equations (3.6.4) and (3.6.8) are exactly the ε-independent scalar Airy equation

$$y_n''(s) = (|n|\gamma_0)^2\, s\, y_n(s) \tag{3.6.12}$$

which is a second-order scalar differential equation. The solutions of (3.6.12) are given by the following

Lemma 3.6.1 (Scalar Airy equation). *For $n \in \mathbb{Z}^*$, let $\mathrm{Ai_n}(z)$ be*

$$\mathrm{Ai_n}(z) = (2\pi)^{-1} \int_{\mathrm{Im}(\zeta)=a} \exp\big((|n|\gamma_0)(i\zeta^3/3 + i\zeta z)\big)\, d\zeta \tag{3.6.13}$$

for $a > 0$. Then for all $n \in \mathbb{Z}$, $\mathrm{Ai_n}$ is a holomorphic function in $\mathbb{C}$ independent of a, and the couple $(\mathrm{Ai_n}(\cdot), \mathrm{Ai_n}(j\cdot))$ is a basis of solutions of (3.6.12), with $j = e^{2i\pi/3}$.

To prove this Lemma, it suffices to adapt the proof following Definition 7.6.8 in [Hör83].

As $(\mathrm{Ai_n}(\cdot), \mathrm{Ai_n}(j\cdot))$ is a basis of solutions of equation (3.6.12), and as both entries $\widetilde{V}_{n,1,1}(s',s)$ and $\widetilde{V}_{n,1,2}(s',s)$ solve equations (3.6.4) and (3.6.8), there are $(\alpha_1(s'), \beta_1(s'))$ and $(\alpha_2(s'), \beta_2(s'))$ in $\mathbb{R}^2$ such that

$$\widetilde{V}_{n,1,1}(s',s) = \alpha_1(s')\mathrm{Ai_n}(s) + \beta_1(s')\mathrm{Ai_n}(js) \tag{3.6.14}$$

$$\widetilde{V}_{n,1,2}(s',s) = \alpha_2(s')\mathrm{Ai_n}(s) + \beta_2(s')\mathrm{Ai_n}(js) \tag{3.6.15}$$

By the relations (3.6.7) and (3.6.11), there holds also

$$\widetilde{V}_{n,2,1}(s',s) = \frac{\varepsilon^{1/3}}{in}\big(\alpha_1(s')\mathrm{Ai_n}'(s) + j\beta_1(s')\mathrm{Ai_n}'(js)\big) \tag{3.6.16}$$

$$\widetilde{V}_{n,2,2}(s',s) = \frac{\varepsilon^{1/3}}{in}\big(\alpha_2(s')\mathrm{Ai_n}'(s) + j\beta_2(s')\mathrm{Ai_n}'(js)\big). \tag{3.6.17}$$

This is equivalent to say that both vectors

$$\begin{pmatrix} \mathrm{Ai_n}(s) \\ -in^{-1}\varepsilon^{1/3}\mathrm{Ai_n}'(s) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathrm{Ai_n}(js) \\ -in^{-1}\varepsilon^{1/3} j\mathrm{Ai_n}'(js) \end{pmatrix} \tag{3.6.18}$$

forms a basis of solutions of the system (3.6.3). The functions $(\alpha_k(s'), \beta_k(s'))$ are determined by the initial conditions (3.6.5) and (3.6.6) for $k = 1$ and (3.6.9) and (3.6.10) for $k = 2$. We obtain the matrix representation of the $(\alpha_k(s'), \beta_k(s'))$:

$$\begin{pmatrix} \alpha_1(s') & \alpha_2(s') \\ \beta_1(s') & \beta_2(s') \end{pmatrix} = \frac{1}{D_n(s')} \begin{pmatrix} \frac{\varepsilon^{1/3}}{in} j\mathrm{Ai_n}'(js') & -\mathrm{Ai_n}(js') \\ -\frac{\varepsilon^{1/3}}{in}\mathrm{Ai_n}'(s') & \mathrm{Ai_n}(s') \end{pmatrix} \tag{3.6.19}$$

where $D_n(s')$ is the determinant of the basis (3.6.18), that is

$$D_n(s') := \frac{\varepsilon^{1/3}}{in}\mathrm{Ai_n}(s')j\mathrm{Ai_n}'(js') - \frac{\varepsilon^{1/3}}{in}\mathrm{Ai_n}(js')\mathrm{Ai_n}'(s')$$

which is in fact independent of s':

$$D_n(s') \equiv D_n(0) = \frac{\varepsilon^{1/3}}{in}(j-1)\mathrm{Ai_n}(0)\mathrm{Ai_n}'(0).$$

For simplicity we denote

$$C = \big((j-1)\mathrm{Ai_n}(0)\mathrm{Ai_n}'(0)\big)^{-1}. \tag{3.6.20}$$

Putting altogether equalities (3.6.14) to (3.6.19), we obtain

$$\begin{cases} \widetilde{V}_{n,1,1}(s',s) &= C\Big(j\mathrm{Ai_n}'(js')\mathrm{Ai_n}(s) - \mathrm{Ai_n}'(s')\mathrm{Ai_n}(js)\Big) \\ \widetilde{V}_{n,2,1}(s',s) &= C\frac{\varepsilon^{1/3}}{in}\Big(j\mathrm{Ai_n}'(js')\mathrm{Ai_n}'(s) - j\mathrm{Ai_n}'(s')\mathrm{Ai_n}'(js)\Big) \\ \widetilde{V}_{n,1,2}(s',s) &= C\frac{in}{\varepsilon^{1/3}}\Big(-\mathrm{Ai_n}(js')\mathrm{Ai_n}(s) + \mathrm{Ai_n}(s')\mathrm{Ai_n}(js)\Big) \\ \widetilde{V}_{n,2,2}(s',s) &= -C\Big(\mathrm{Ai_n}(js')\mathrm{Ai_n}'(s) - j\mathrm{Ai_n}(s')\mathrm{Ai_n}'(js)\Big). \end{cases} \tag{3.6.21}$$

3.6.2 Upper bounds for the propagator: proof of Lemma (3.4)

In order to prove Lemma 3.3.5, we derive asymptotic estimates of $\mathrm{Ai_n}(s)$ and $\mathrm{Ai_n}(js)$ when s real and $s \to +\infty$.

Lemma 3.6.2 (Asymptotic estimates for the Airy function)**.** *There holds for all $n \in \mathbb{Z}^*$ and $s \geq 1$, up to some complex constants:*

$$\mathrm{Ai_n}(s) \approx s^{-1/4}|n|^{-1/2}\exp(-|n|\gamma_0(2/3)s^{3/2}) \tag{3.6.22}$$

$$\mathrm{Ai_n}(js) \approx s^{-1/4}|n|^{-1/2}\exp(|n|\gamma_0(2/3)s^{3/2}) \tag{3.6.23}$$

$$\mathrm{Ai_n}'(s) \approx s^{1/4}|n|^{1/2}\exp(-|n|\gamma_0(2/3)s^{3/2}) \tag{3.6.24}$$

$$\mathrm{Ai_n}'(js) \approx s^{1/4}|n|^{1/2}\exp(|n|\gamma_0(2/3)s^{3/2}). \tag{3.6.25}$$

In particular, the Airy function $\mathrm{Ai_n}$ and its derivative satisfy the upper bounds

$$e^{|n|\gamma_0(2/3)s^{3/2}}\,|\mathrm{Ai_n}(s)| + e^{-|n|\gamma_0(2/3)s^{3/2}}\,|\mathrm{Ai_n}(js)| \lesssim |n|^{-1/2}(1+s)^{-1/4} \quad \forall 0 \leq s,\ \forall n \in \mathbb{Z}^* \tag{3.6.26}$$

and

$$e^{|n|\gamma_0(2/3)s^{3/2}}\,\big|\mathrm{Ai_n}'(s)\big| + e^{-|n|\gamma_0(2/3)s^{3/2}}\,\big|\mathrm{Ai_n}'(js)\big| \lesssim |n|^{1/2}s^{1/4} \quad \forall 0 \leq s,\ \forall n \in \mathbb{Z}^* \tag{3.6.27}$$

Proof. For $s \geq 1$, we put $a = is^{1/2}$ into the definition (3.6.13) to obtain

$$\begin{aligned} \mathrm{Ai_n}(s) &= (2\pi)^{-1}\int_{\mathrm{Im}\zeta = is^{1/2}} \exp\big((|n|\gamma_0)(i\zeta^3/3 + i\zeta z)\big)\,d\zeta \\ &= (2\pi)^{-1}\int_{\mathbb{R}} \exp\Big((|n|\gamma_0)(i(\xi + is^{1/2})^3/3 + i(\xi + is^{1/2})s)\Big)\,d\xi \\ &= (2\pi)^{-1}\,e^{-|n|\gamma_0(2/3)s^{3/2}}\int_{\mathbb{R}} \exp\Big((|n|\gamma_0)(i\xi^3/3 - \xi^2 s^{1/2})\Big)\,d\xi. \end{aligned}$$

By the change of variables $\xi \mapsto (|n|\gamma_0 s^{1/2})^{-1/2}\xi$ in the integral, there holds

$$\mathrm{Ai_n}(s) = \frac{|n|^{-1/2}s^{-1/4}}{2\pi\gamma_0^{1/2}}\, e^{-|n|\gamma_0(2/3)s^{3/2}} \int_{\mathbb{R}} \exp\left(i(|n|\gamma_0)^{-1/2}s^{-3/4}\xi^3/3 - \xi^2\right) d\xi.$$

As the last integral satisfies the asymptotic development, for $s \to +\infty$:

$$\int_{\mathbb{R}} \exp\left(i(|n|\gamma_0)^{-1/2}s^{-3/4}\xi^3/3 - \xi^2\right) d\xi = \sqrt{2\pi} + O\left(|n|^{-1/2}s^{-3/4}\right)$$

we obtain (3.6.22). By an analog computation we have (3.6.23), (3.6.24) and (3.6.25).

From those asymptotic estimates, we deduce immediately uniform bounds for $\mathrm{Ai_n}$ and the time derivative $\mathrm{Ai_n}'$.

□

Thanks to the previous Lemma, we end the proof of Lemma 3.3.5 by getting the upper bound of the propagator $\widetilde{V}_n(s', s)$. Combining the expression of $\widetilde{V}_n$ in function of $\mathrm{Ai_n}$ given by (3.6.21) with the estimates (3.6.26) and (3.6.27), we obtain the upper bounds for the coefficients of the matrix flow $\widetilde{V}_n(s', s)$, for $0 \le s' \le s \le \varepsilon^{-2/3}$:

$$\begin{cases} \left|\widetilde{V}_{n,1,1}(s',s)\right| & \approx & s'^{1/4}(1+s)^{-1/4}\exp\left(|n|\gamma_0\frac{2}{3}(s^{3/2}-s'^{3/2})\right) \\ \left|\widetilde{V}_{n,2,1}(s',s)\right| & \approx & \varepsilon^{1/3}s'^{1/4}s^{1/4}\exp\left(|n|\gamma_0\frac{2}{3}(s^{3/2}-s'^{3/2})\right) \\ \left|\widetilde{V}_{n,1,2}(s',s)\right| & \approx & \varepsilon^{-1/3}(1+s')^{-1/4}(1+s)^{-1/4}\exp\left(|n|\gamma_0\frac{2}{3}(s^{3/2}-s'^{3/2})\right) \\ \left|\widetilde{V}_{n,2,2}(s',s)\right| & \approx & (1+s')^{-1/4}s^{1/4}\exp\left(|n|\gamma_0\frac{2}{3}(s^{3/2}-s'^{3/2})\right) \end{cases} \tag{3.6.28}$$

As

$$\varepsilon^{-1/3}(1+s)^{-1/4} > s^{1/4} \qquad \forall 0 \le s < \varepsilon^{-2/3}$$

we obtain the upper bound for the propagator

$$|\widetilde{V}_n(s',s)| \lesssim \varepsilon^{-1/3}(1+s')^{-1/4}(1+s)^{-1/4}\exp(|n|\gamma_0(2/3)(s^{3/2}-s'^{3/2})) \quad \forall 0 \le s' \le s < \varepsilon^{-2/3}$$

which implies (3.3.26) and ends the proof of Lemma 3.3.5.

3.6.3 Growth of the free solution: proof of Lemma (3.6)

We prove here Lemma 3.3.7, following the proof of Lemma 3.3.6. We showed in it that it suffices to prove the lower bound for $\widetilde{V}_n$. Thanks to the equalities (3.6.21), a simple computation gives us

$$\widetilde{V}_{+1}(0,s)\begin{pmatrix}\mathrm{Ai}_1(0)\\ -i\varepsilon^{1/3}j\mathrm{Ai}_1'(0)\end{pmatrix} = \begin{pmatrix}\mathrm{Ai}_1(js)\\ -i\varepsilon^{1/3}j\mathrm{Ai}_1'(js)\end{pmatrix}$$

and also

$$\widetilde{V}_{-1}(0,s)\begin{pmatrix}\mathrm{Ai}_1(0)\\ i\varepsilon^{1/3}j\mathrm{Ai}_1'(0)\end{pmatrix} = \begin{pmatrix}\mathrm{Ai}_1(js)\\ i\varepsilon^{1/3}j\mathrm{Ai}_1'(js)\end{pmatrix}.$$

We denote

$$\widetilde{\mathbf{f}}(s,\theta) = \mathrm{Re}\left(\widetilde{V}_{+1}(0,s)\begin{pmatrix}\mathrm{Ai}_1(0)\\ -i\varepsilon^{1/3}j\mathrm{Ai}_1{}'(0)\end{pmatrix}e^{i\theta} + \widetilde{V}_{-1}(0,s)\begin{pmatrix}\mathrm{Ai}_1(0)\\ i\varepsilon^{1/3}j\mathrm{Ai}_1{}'(0)\end{pmatrix}e^{-i\theta}\right). \quad (3.6.29)$$

and we compute

$$\widetilde{\mathbf{f}}(s,\theta) = 2\mathrm{Re}\begin{pmatrix}\mathrm{Ai}_1(js)\cos(\theta)\\ -\varepsilon^{1/3}j\mathrm{Ai}_1{}'(js)\sin(\theta)\end{pmatrix}. \quad (3.6.30)$$

Next we denote $\widetilde{\mathbf{f}}_1(s)$ and $\widetilde{\mathbf{f}}_2(s)$ the two components of the vector $\widetilde{\mathbf{f}}$ defined by (3.6.29). Thanks to Lemma 3.6.2, we have

$$|\widetilde{\mathbf{f}}_1(s)| \approx \mathrm{Ai}_1(js) \approx s^{-1/4}\exp(\gamma_0 2/3 s^{3/2})$$

and

$$|\widetilde{\mathbf{f}}_2(s)| \approx \varepsilon^{1/3}\mathrm{Ai}_1{}'(js) \approx \varepsilon^{1/3}s^{1/4}\exp(\gamma_0 2/3 s^{3/2})$$

for $1 \leq s < \varepsilon^{-2/3}$. As $\varepsilon^{1/3}s^{1/4} < s^{-1/4}$ for $s < \varepsilon^{-2/3}$, the L^∞ norm of the vector $\widetilde{\mathbf{f}}(s,\theta)$ satisfies

$$|\widetilde{\mathbf{f}}(s,\theta)|_\infty = |\widetilde{\mathbf{f}}_1(s,\theta)| \approx s^{-1/4}\exp(\gamma_0 2/3 s^{3/2})$$

for all $1 \leq s < \varepsilon^{-2/3}$. Using the same steps as in the proof of Lemma 3.3.6, this suffices to end the proof of Lemma 3.3.7.

Chapter 4

On hyperbolicity and Gevrey well-posedness. Part three: a class of weakly hyperbolic systems.

4.1 Introduction

In this chapter we prove an energy estimate for systems of the form

$$\partial_t u = \begin{pmatrix} 0 & 1 \\ (t+x^2)e(t,x) & 0 \end{pmatrix} \partial_x u + F(u)u \tag{4.1.1}$$

where $x \in \mathbb{R}$, $F(u)$ is nonlinear in u, and e is a Gevrey function that is bounded away from zero and compactly supported around $(t,x) = (0,0)$. This result translates by classical arguments into a *local-in-time* well-posedness result in Gevrey spaces for the Cauchy problem for (4.1.1), and is easily extended into a general well-posedness for systems in several spatial dimensions:

$$\partial_t u = \sum_{1 \le j \le d} A_j(t,x)\partial_{x_j} u + f(t,x,u) \tag{4.1.2}$$

where x in $\mathbb{R}^d$, the A_j are in $\mathbb{R}^{2\times 2}$, f in $\mathbb{R}^2$, the A_j have some smoothness in time and are Gevrey regular in x, the nonlinearity f is analytic in all variables, and the principal symbol $A = \sum_j A_j(t,x)\xi_j$ experiences a transition from hyperbolicity to ellipticity. Precisely, in order to extend our result for (4.1.1) into a well-posedness result for (4.1.2), we assume

- hyperbolicity of the principal symbol A, that is the spectrum of $A(t,x,\xi)$ is real.
- At a distinguished point $(0,x_0,\xi_0) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^{d-1}$, the existence of a real and non semi-simple eigenvalue (semi-simplicity means simplicity as a zero of the minimal polynomial of $A(t,x,\xi)$).

- And finally we assume that A transitions from hyperbolicity to ellipticity at $(0, x_0, \xi_0)$, backwards in time. By transition from hyperbolicity to ellipticity we mean the phenomenon studied in Chapter 3. Here this transition is *not* degenerate, we will go back to this point in Section 4.1.3.

In a forthcoming version of this chapter, we expound on these Assumptions, and handle the general case of weakly hyperbolic systems of the form (4.1.2). In the present version of this chapter, we work exclusively with the prototypical example (4.1.1). The fact that (4.1.1) is one-dimensional ($x \in \mathbb{R}$) does not play any role in our analysis.

Further simplifying into $e \equiv 1$, $F(u) = \begin{pmatrix} 0 & 0 \\ 0 & u_1 \end{pmatrix}$, we find the system

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ t + x^2 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ u_1^2 \end{pmatrix},$$

which reduces to the wave-like equation in $u_1 \in \mathbb{R}$:

$$\partial_t^2 u_1 = \partial_x \left((t + x^2) \partial_x u_1 \right) + \partial_x (u_1^2) \tag{4.1.3}$$

The wave operator in (4.1.3) is singular at $(t, x) = (0, 0)$, and elliptic for negative times.

Our interest is in the Cauchy problem at $t = 0$, for forward times. Our present result has a double background: first in *well-posedness for weakly hyperbolic systems*, a line of research popularized in particular by Colombini and collaborators [CJS83], [CN07] and [CNR], and in *systems transitioning from hyperbolic to ellipticty*, a line of research initiated by Lerner, Morimoto and Xu in [LMX10].

4.1.1 Background: on weakly hyperbolic systems

The classical result of Colombini, Janelli and Spagnolo

We consider here the following second-order, linear scalar equation

$$\partial_t^2 v = \partial_x \left(a \partial_x v \right) \tag{4.1.4}$$

with $a = a(t)$ a nonnegative, $C^k([0, T])$ function for some $k \geq 1$. Such weakly hyperbolic, second-order scalar equations have long been studied by in Gevrey regularity.

A cornerstone of the domain is Colombini, Janelli and Spagnolo's paper [CJS83], which proved Gevrey well-posedness in the case of spatially-independent symbol $a(t)$. The work of Colombini, Janelli and Spagnolo is based on an energy estimate, which uses the particular structure of the wave equation (4.1.4) and a lemma of real analysis which extends the classical Glaeser's inequality[1], namely that if $a(t)$ is a C^k nonnegative function on $[0, T]$, then $a(t)^{1/k}$ is absolutely continuous on $[0, T]$ (see Lemma 1 in [CJS83], and [Gla63] for Glaeser's inequality).

[1] In fact, Lemma 1 in [CJS83] is a weaker version of Glaeser inequality: Lemma 1 states on the L^1 norm of $a^{1/k}$, where the Glaeser inequality is pointwise for $a(t)^{1/2}$.

In the case when $a = a(t)$, equation (4.1.4) transforms into the scalar ODE

$$\partial_t^2 w(t,\xi) = -a(t)|\xi|^2 w(t,\xi)$$

thanks to the Fourier transform, and where we denote $w(t,\xi) = \widehat{v}(t,\xi) \in \mathbb{C}$. As $a(t)$ is supposed to be only nonnegative (weak hyperbolicity), we introduce a small parameter $\varepsilon > 0$ (later on $\varepsilon = \varepsilon(\xi)$) and the approximat energy

$$E_\varepsilon(t,\xi) = |\partial_t w(t,\xi)|^2 + (a(t)+\varepsilon)\,|\xi|^2 |w(t,\xi)|^2$$

whose time derivative is

$$\partial_t E_\varepsilon = a'(t)|\xi|^2|w|^2 + 2\varepsilon|\xi|^2 \mathrm{Re}\, w\partial_t w.$$

Having in mind a Gårding-type inequality to fulfil an energy estimate, we bound the previous equality by

$$\partial_t E_\varepsilon \leq |a'(t)||\xi|^2|w|^2 + \varepsilon^{1/2}|\xi|\, E_\varepsilon$$

thanks to Cauchy-Schwarz's inequality. To bound the term $|a'(t)||\xi|^2|w|^2$, we need here to link $|a'|$ to $a+\varepsilon$ in order to bound $|a'(t)||\xi|^2|w|^2$ by the term $(a(t)+\varepsilon)\,|\xi|^2|w(t,\xi)|^2$ of the energy (up to a multiplicative constant). As $\left((a+\varepsilon)^{1/k}\right)' = \frac{1}{k}a'/(a+\varepsilon)^{1-1/k}$, we write

$$\begin{aligned} |a'(t)||\xi|^2|w|^2 &= \left|\frac{a'}{(a+\varepsilon)^{1-1/k}}\right| \frac{1}{(a+\varepsilon)^{1/k}}\,(a+\varepsilon)|\xi|^2|w|^2 \\ &= k\left|\left((a+\varepsilon)^{1/k}\right)'\right| \frac{1}{(a+\varepsilon)^{1/k}}\,(a+\varepsilon)|\xi|^2|w|^2. \end{aligned}$$

As a is nonnegative, there holds

$$\begin{aligned} \partial_t E_\varepsilon &\leq \left|\left((a+\varepsilon)^{1/k}\right)'\right| \frac{1}{(a+\varepsilon)^{1/k}}\, E_\varepsilon + \varepsilon^{1/2}|\xi|\, E_\varepsilon \\ &\leq \left|\left((a+\varepsilon)^{1/k}\right)'\right| \varepsilon^{-1/k}\, E_\varepsilon + \varepsilon^{1/2}|\xi|\, E_\varepsilon \end{aligned}$$

hence

$$\begin{aligned} E_\varepsilon(t,\xi) &\lesssim \exp\left(\varepsilon^{-1/k}\int_0^t \left|\left((a+\varepsilon)^{1/k}\right)'(s)\right| ds + t\varepsilon^{1/2}|\xi|\right) E_\varepsilon(0,\xi) \\ &\lesssim \exp\left(\varepsilon^{-1/k}|a|_{C^k}^{1/k} + T\varepsilon^{1/2}|\xi|\right) E_\varepsilon(0,\xi) \end{aligned}$$

for all $t \leq T$ thanks to Lemma 1 in [CJS83]. In order to optimize the exponential term, we put $\varepsilon = |\xi|^{-2/(k+2)}$ to get finally

$$E_\varepsilon(t,\xi) \lesssim e^{c|\xi|^{2/(k+2)}} E_\varepsilon(0,\xi)$$

for some constant $c > 0$.

Thanks to this (pointwise in frequency) energy estimate, the authors of [CJS83] proved that the Cauchy problem associated to (4.1.4) is well-posed in Gevrey spaces $\mathcal{G}_c^\sigma$ (see Definition 5.2.2) with $\sigma > 2/(k+2)$, where k is the regularity of the coefficient of equation (4.1.4). Note that, as the regularity of a grows, the range of Gevrey indices for which well-posedness holds grows as such.

Beyond the 1983 article of Colombini, Janelli and Spagnolo

The work of [CJS83] has been followed and extended notably by Colombini and Nishitani in [CN07] and by Colombini, Nishitani and Rauch in [CNR].

In [CN07], Colombini and Nishitani study the case when a depends also in x, that is, $a(t,x)$ is assumed to be nonnegative and in $C^2([0,T], G^s_R)$ (see Definition 5.2.1 for Gevrey spaces defined from the spatial viewpoint, and Proposition 5.2.3 for its link with $\mathcal{G}^\sigma_\tau$). Note that, as it is made explicit in Theorem 1.3 therein, it is assumed that $a(t,x)$ is in fact nonnegative in $[-\delta, T+\delta]$ for some $\delta > 0$. This additional assumption on a is crucial in the course of the proof of [CN07]. Indeed, in order to extend the energy-based study in [CJS83], the authors of [CN07] use a pseudo-differential calculus. In the context of symbols, Lemma 1 in [CJS83] is no longer helpful, as it leads to an L^1 estimate of the time derivative of a; instead, a pointwise inequality in (t,x) is needed, hence the use of Glaeser's inequality. For Glaeser's inequality to hold in a compact subspace of $\mathbb{R} \times \mathbb{R}^d$, the nonnegativity condition on a has to hold on a larger subspace containing the compact, see Appendix 4.4.1. Well-posedness is then proved for any $1 \leq s < 2$ - that is for any $1/2 < \sigma \leq 1$ thanks to Proposition 5.2.3 - extending the work of [CJS83].

The work of Colombini, Nishitani and Rauch in [CNR] explores a different way. Generic weakly hyperbolic systems (4.1.1) are considered, not only second-order scalar equations (4.1.4) as in [CJS83] or [CN07], i.e. the principal symbol $A(t,x,\xi)$ is there a $N \times N$ matrix with real spectrum but with potential eigenvalue crossings. To study such general symbols, the authors introduce a *block size barometer* $\theta = m-1$, which roughly measures the extent to which $A(t,x,\xi)$ can be smoothly block diagonalized by blocks of size m. For smoothly diagonalizable symbols, $\theta = 0$; on the other hand, $\theta = N-1$ if the symbol is not block diagonalizable at all - which is typically our framework, for $N = 2$. In order to get a general result on well-posedness in Gevrey spaces, regardless of the spectral details of the principal symbol of (4.1.1), a suitable Lyapunov symmetrizer is studied. In exchange for a general statement, the range of Gevrey indices for which well-posedness holds is quite reduced, and depends on θ. Precisely, well-posedness for (4.1.1) is proved for any

$$\sigma \geq \min\left\{\frac{1+6\theta}{2+6\theta}, \frac{2+4\theta}{3+4\theta}\right\}.$$

Note that in our framework there holds $\theta = 1$ which leads the lower bound $6/7$ for the Gevrey index.

4.1.2 Background: on systems transitioning away from hyperbolicity

The question of the instability of systems transitioning away from hyperbolicity has been first raised in [LMX10], extending the work [Mét05] on initially elliptic systems. In [LMX10] quasilinear *scalar* equations are considered, with analytic coefficients. It is assumed that these equations experience a transition from initial hyperbolicity to ellipticity for positive times. For such equations, it is proved in [LMX10] that the Cauchy problem with initial analytic data is strongly unstable with respect to C^∞ perturbation.

A similar instability result is established in [LNT17], in which quasilinear *systems* with smooth coefficients are considered. In various cases of transitions from initial hyperbolicity

to ellipticity, the Cauchy problem in Sobolev spaces is proved to be unstable, in the sense of Hadamard. That is, hypothetical flow of the system fails to be Hölder from Sobolev spaces to L^2. The article [Lu16] explores a similar theme in the context of high-frequency solutions of singularly perturbed symmetric hyperbolic systems.

In Chapter 3, we considered first order quasi-linear system (4.1.1) experiencing a transition from hyperbolicity to ellipticity. A typical example of symbols which falls into the class studied in Section 3.2.3 is

$$A(t,x,\xi) = \begin{pmatrix} 0 & 1 \\ -(t - t_\star(x,\xi)) & 0 \end{pmatrix} \tag{4.1.5}$$

in a neighborhood of $(0,0,\xi_0) \in \mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^d_\xi$, with

$$t_\star(x,\xi) = |x|^4 + |\xi - \xi_0|^2 \tag{4.1.6}$$

In such a case, we proved in Theorem 5 in Chapter 3 that (4.1.1) is not well-posed in Gevrey spaces for $\sigma \in (0, 2/13)$. As explained in Section 3.1.2, the term $|x|^4$ corresponds to a degenerate time transition. As we see in Figure 4.1, the hyperbolic domain $\{(t,x) \in [0,T] \times B_r(x_0) : t \leq |x|^4\}$ for $|x|^4$ is thinner than the hyperbolic domain $\{(t,x) \in [0,T] \times B_r(x_0) : t \leq |x|^2\}$ for $|x|^2$. This observation allowed us to treat the term $|x|^4$ as a remainder term. Having treated the case of *degenerate* transitions in Chapter 3, we now wish to handle *generic* transitions. These involve, as explained in [LNT17], time-transition functions of the form $t_\star(x) = x^2$, in one spatial dimension, and a Jordan block for the principal symbol, that is (4.1.1) with $t_\star(x) = x^2$.

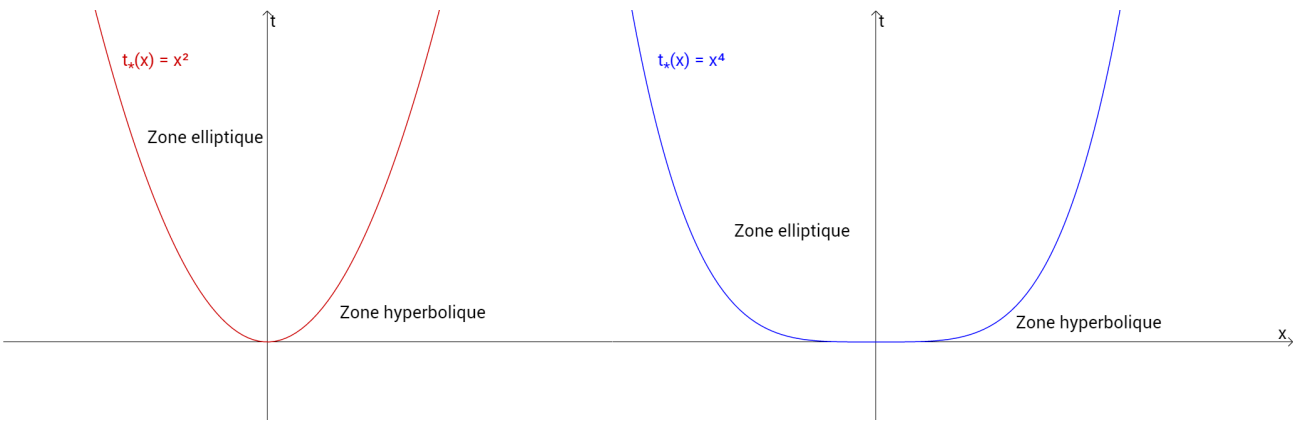

Figure 4.1: Comparison between degenerate x^4 and non-degenerate x^2

4.1.3 Generic time transitions

The proof of Chapter 3 in the case $t_\star(x) = |x|^2$ fails essentially due to the size of the hyperbolic domain $\{(t,x) \in [0,T] \times B_r(x_0) : t \leq |x|^2\}$ in the setting developed therein. The term $|x|^2$ may not be considered as a remainder term.

Thus in order to prove ill-posedness in the generic configuration, we have to handle the not so small hyperbolic region under the transition curve. This means proving a form

of *well-posedness* for $t < t_\star$. At $t = t_\star$ the unstable modes are turned on and the analysis of Chapter 3 should apply. For the analysis of Chapter 3 to go through, we must find suitable analytic data $(h_\varepsilon)_{\varepsilon>0}$ such that the Cauchy problem at $t = t_\star$ is ill-posed (with the difficulty that $t_\star$ is a function of x in 1d and of (x, ξ) in multi-d).

The outstanding question is then to find suitable *initial* (at $t = 0$, for all x) data which give rise to the suitable unstable data $h_\varepsilon(x)$ at $t = t_\star(x)$. In other words, we want to solve the *backward-in-time* Cauchy problem, in the hyperbolic zone, from $t = t_\star(x)$ to $t = 0$. This motivates the form of the principal symbol under consideration here, as we describe in the next Section.

4.1.4 Current result

As mentioned above, generic transitions from hyperbolic to ellipticity involve in one spatial dimension principal symbols of the form (4.1.5) with $t_\star(x) = x^2$. In order to study these transitions, we must understand the backward-in-time Cauchy problem for such operators. This motivates the form of our principal symbol in (4.1.1). The function e is assumed to be bounded away from zero and Gevrey (see Assumption 4.2.1). Under this assumption, we prove an energy estimate for solutions with compact support with regularity $\mathcal{G}^\sigma_\tau$ for any $\sigma \geq 1/2$ and $\tau > 0$ small. This is Theorem 6.

The proof relies on the construction of a suitable symmetrizer $\mathrm{op}(\mathrm{diag}(1, b))$ with symbol $b(t, x, \xi) = (t + x^2 + \langle\xi\rangle^{-c})^{-1/2}$ and a Gevrey energy estimate. An important observation is that the symbol b does not belong to a standard class of symbols. Indeed, $b(0, 0, \xi) = \langle\xi\rangle^{c/2}$ whereas $b(t, x, \xi) \in S^0_{1,0}$ when $t \geq \underline{t}$ and $|x| > r$. To reconcile both point of views, we make use of class of symbols defined with respect to a metric of the phase space, as described in [Ler11]. In Lemma 4.3.7, we prove that $b \in S(b, g^t)$ where the time-dependent metric g^t is defined in (4.3.8)[2]. Our paper relies also on Chapter 5 which contains our work on pseudo-differential operators with symbols which are Gevrey regular in the spatial variable.

Remark 4.1.1. *Our result is outside the range of the article* [CN07]. *The symbol $a(t, x)$, which is in our case similar to $t + x^2$, does not satisfy Glaeser's inequality for negative times. This result is also an improvement of the result given in* [CNR], *as we attain in our paper the lower bound $1/2$ for the Gevrey indices, compared to the lower bound $6/7$ as described above. The main difference is that, in our paper, we take care of the spectral details of the principal symbol, as we assume it is a 2 by 2 matrix, with a specific crossing of eigenvalues.*

[2] In the present work, we will in fact limit ourself mostly to the definition of the class of symbols $S(b, g^t)$ and the result of continuous action of $\mathrm{op}\left(S(1, g^t)\right)$ on L^2. We think that the use of generalized Sobolev spaces $H(b, g^t)$ as described in Definition 2.6.1 in [Ler11] would considerably simplify our computations, especially for the control of the remainder terms.

4.2 Main assumptions and results

We consider the Cauchy problem for the following first-order, 2 by 2 system

$$\partial_t u = \begin{pmatrix} 0 & 1 \\ a(t,x) & 0 \end{pmatrix} \partial_x u + F(t,x,u)u \tag{4.2.1}$$

where x is in $\mathbb{R}$, u in $\mathbb{R}^2$ and $F(t,x,u)$ is a 2×2 matrix. The coefficients of the matrix F are analytical in a neighborhood of $(0,x_0,u_0)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^2$, and $F(t,x,u_0)$ is a constant.

Assumption 4.2.1. *We assume that*

$$a(t,x) = \left(t + (x-x_0)^2\right) e(t,x)$$

where $e(t,x)$ has compact support $[0,T']\times B_{r'}(x_0)$ for some $T'>0$ and $r'>0$. Besides, e is in $C^2([0,T'],G^s_R)$, that is there is $C>0$ such that

$$|\partial_x^\alpha e(t,x)| \le CR^{|\alpha|}|\alpha|!^s \quad , \quad \forall\,\alpha\in\mathbb{N}\,,\ \forall\,(t,x)\in[0,T']\times B_{r'}(x_0).$$

There is also $0<T<T'$ and $0<r<r'$ such that

$$1/2 \le e(t,x) \le 2 \quad , \quad \forall\,(t,x)\in[0,T]\times B_r(x_0). \tag{4.2.2}$$

We denote

$$\underline{\tau} = sR^{-1/s} \tag{4.2.3}$$

the Gevrey regularity of a, in the Fourier viewpoint (see Definition 5.2.2 and Proposition 5.2.3 in Chapter 5).

The main result of our paper is an energy estimate in Gevrey space $\mathcal{G}^\sigma_\tau$ for any $\sigma\ge 1/2$ and $\tau>0$ small enough. The lower Gevrey index $1/2$ is the expected lower bound for the Gevrey regularity. To obtain such a result, we define a suitable symmetriser for A, introducing first the symbol

$$b(t,x,\xi) = \left(a(t,x) + \langle\xi\rangle^{-c}\right)^{-1/2} \tag{4.2.4}$$

for some $c\in(0,2)$ and denoting

$$\langle\xi\rangle = \left(\mu + |\xi|^2\right)^{1/2} \tag{4.2.5}$$

for some $\mu>1$ to be chosen later on. Section 4.3.1 will be devoted to prove that b is in the class of symbols $S(b,g^t)$, defined in (4.4.9) and the metric g^t defined in (4.3.8). This is done principally thanks to the non-negativity of a and Glaeser's inequality (see Lemma 4.3.1 and Section 4.4.1 below).

In all the following, we denote

$$D = \mathrm{op}\left(\langle\cdot\rangle\right) \quad \text{and} \quad D^\sigma = \mathrm{op}\left(\langle\cdot\rangle^\sigma\right). \tag{4.2.6}$$

Let $\sigma\in(0,1)$, $\tau>0$ and u in $\mathcal{G}^\sigma_\tau$. We introduce the Gevrey energy

$$E(\tau,u(t)) = \frac{1}{2}\left|e^{\tau D^\sigma}u_1(t)\right|^2_{L^2} + \frac{1}{2}\left|\mathrm{op}(b)e^{\tau D^\sigma}u_2(t)\right|^2_{L^2}. \tag{4.2.7}$$

Thanks to the assumption of "constancy outside a compact set" for (4.2.1), we use the result of sharp finite speed of propagation of [CR10]. We look for solutions with compact support in (t,x) included in $[0,T]\times B_r(x_0)$, which can be done if the initial datum u_0 has sufficiently small compact support (with respect to T and the finite speed propagation of (4.2.1)). The energy estimate yields local-in-time existence and uniqueness of solutions by classical arguments in Gevrey regularity for such systems.

Theorem 6. *For any $\tau_0 < \underline{\tau}$ with $\underline{\tau}$ defined in* (4.2.3)*, there is $\boldsymbol{\tau} > 0$ such that*

$$E\left(\tau_0 - \boldsymbol{\tau} t\right) \lesssim E(\tau_0) \quad , \quad \forall\, t \in [0, T_{\max}].$$

Section 4.3 is devoted to the proof of Theorem 6.

4.3 Proof of the energy estimate

In order to study (4.2.1) in Gevrey spaces, a classical approach is to introduce a Gevrey radius $\tau(t)$ which decreases linearly in time. Let $\tau_0 < \underline{\tau}$. We define

$$\tau(t) = \tau_0 - \boldsymbol{\tau} t. \tag{4.3.1}$$

with $\boldsymbol{\tau} > 0$ to be determined in the course of the proof. We define also the symbol

$$a_\sharp = a_\sharp(t,x,\xi) = a(t,x) + \langle \xi \rangle^{-c} \tag{4.3.2}$$

where the additional term $\langle \xi \rangle^{-c}$ makes the symbol $a_\sharp$ positive. This is a standard approach when dealing with weakly hyperbolic equations, see [CJS83]. Thanks to this notation, we may write the symbol b defined by (4.2.4) as $b = a_\sharp^{-1/2}$. Note that there holds

$$b(t,x,\xi) \leq \langle \xi \rangle^{c/2} \quad , \quad \forall\, (t,x,\xi) \in [0,T] \times B_r(x_0) \times \mathbb{R}. \tag{4.3.3}$$

4.3.1 Key preparatory Lemmas

In order to study the symbol b defined in (4.2.4), we first prove a Glaeser-type inequality for a in the following Section. In order to compute carefully some estimates on the derivatives of b, we prove first a local Glaeser inequality for a, as it is non-negative locally around $x = x_0$.

Lemma 4.3.1 (Glaeser inequality for a)**.** *Under Assumption* 4.2.1*, there is a neighborhood $[0,T]\times B_r(x_0)$ of $(0,x_0) \in \mathbb{R}_t \times \mathbb{R}_x$ and a constant $C_{T,r} > 0$ for which there holds*

$$\left(\partial_x a(t,x)\right)^2 \leq C_{T,r}\, a(t,x) \quad , \quad \forall\, (t,x) \in [0,T] \times B_r(x_0). \tag{4.3.4}$$

The proof is postponed to Appendix 4.4.1. It uses Lemma 4.4.3. Choosing T and r small enough, we may assume that a is smaller than 1 on $[0,T]\times B_r(x_0)$. We let then $\mu = \left(1 - |a|_{L^\infty([0,T]\times B_r(x_0))}\right)^{-2/c}$ so that

$$a_\sharp(t,x,\xi) \leq 1 \quad , \quad \forall\, (t,x,\xi) \in [0,T] \times B_r(x_0) \times \mathbb{R}. \tag{4.3.5}$$

The following Lemma gives precise estimates on the derivatives of b.

Lemma 4.3.2 (Derivatives of the symbol b). *There is a bounded sequence of constants $C_{\alpha,\beta} > 0$ for which there holds*

$$|\partial_x^\alpha \partial_\xi^\beta b(t,x,\xi)| \leq C_{\alpha,\beta} \widetilde{R}^{|\alpha+\beta|}\, \alpha!^s \beta!\, b(t,x,\xi)\, a_\sharp(t,x,\xi)^{-|\alpha|/2} \langle \xi \rangle^{-|\beta|} \quad , \quad \forall\, (\alpha,\beta) \in \mathbb{N} \times \mathbb{N} \tag{4.3.6}$$

for all (t,x) in $[0,T] \times B_r(x_0)$ and ξ in $\mathbb{R}$, and where $\widetilde{R}$ satisfies

$$\widetilde{R} = c(d) R. \tag{4.3.7}$$

The proof is postponed in Appendix 4.4.1. It relies on the Faà di Bruno formula (see Lemma 4.4.1) and the Glaeser inequality for a proved in Lemma 4.3.1. We follow through with some remarks on this result.

Remark 4.3.3. *The constant $c(d)$ in* (4.3.7) *depends uniquely on the dimension d, and is due to the combinatorics involved in the Faà di Bruno formula, and here $d = 1$.*

Remark 4.3.4. *Thanks to inequality* (4.3.3)*, Lemma* 4.3.2 *implies that $b \in S^{c/2}_{1,c/2} G^s_{\widetilde{R}}$, as defined in Definition* 5.2.7*. Without the Glaeser inequality described in Lemma* 4.3.1*, we would only prove that $b \in S^{c/2}_{1,c} G^s_{\widetilde{R}}$, knowing that c may be in* $(1,2)$*.*

The importance of the Glaeser inequality explains why we do not define b as $(\widetilde{a} + \langle \xi \rangle^{-c})^{-1/2}$ where $\widetilde{a}$ is defined in (4.3.18) *as the symbol of operator $a^{(\tau)}$, the Gevrey conjugation of a. Indeed the symbol $\widetilde{a}$ does not satisfy a priori the Glaeser inequality, as it is not real.*

Remark 4.3.5. *As a has compact support, $b(\cdot,\xi)$ is constant outside a compact set of $\mathbb{R}_t \times \mathbb{R}_x$ which does not depend on ξ.*

The bounds (4.3.6) show in particular that the symbol b has a variable order with respect to time and space. Indeed, for $(t,x) = (0,x_0)$, symbol $a_\sharp$ is equal to $\langle \xi \rangle^{-c}$, hence $b(t=0)$ is likely to be of order $c/2$. But as time goes, the order of b decreases. In fact, for $t \geq \underline{t} > 0$, there holds simply $a_\sharp \geq t \geq \underline{t}$, hence

$$|\partial_x^\alpha \partial_\xi^\beta b(t,x,\xi)| \leq C_{\alpha,\beta} \widetilde{R}^{|\alpha|}\, \alpha!^s \beta!\, b(t,x,\xi)\, \underline{t}^{-|\alpha|/2} \langle \xi \rangle^{-|\beta|}$$

for all $t \geq \underline{t}$. Then b is of order 0 for all $t \geq \underline{t}$.

A way to reconcile both points of view is to introduce the following time-dependent, non-flat metric in the phase space

$$g^t_{(x,\xi)}(\mathrm{d}x, \mathrm{d}\xi) = \frac{|\mathrm{d}x|^2}{a_\sharp(t,x,\xi)} + \frac{|\mathrm{d}\xi|^2}{\langle \xi \rangle^2}. \tag{4.3.8}$$

In order to use the properties described in Appendix 4.4.2, we verify that both g^t and b are admissible:

Lemma 4.3.6. *The metric g^t defined in* (4.3.8) *is an admissible metric, and the weight b defined in* (4.2.4) *is an admissible weight for the metric g^t, both in the sense of Definition* 2.2.15 *in* [Ler11].

This is Lemma 3.1 in [CN07], and we recall its proof in Appendix 4.4.2. In particular, Lemma 4.3.2 implies

Lemma 4.3.7. *For any k in $\mathbb{Z}$, the symbol b^k is in $S(b^k, g^t)$.*

Proof. The case $k = 1$ is just Lemma 4.3.2. Hence the result for any $k \geq 1$, thanks to Lemma 4.4.10. The case $k = -1$ is proved by the same proof as Lemma 4.3.2, as $b^{-1} = a_\sharp^{1/2}$. □

We denote also

$$\lambda(t, x, \xi) = b^{-1}\langle\xi\rangle. \tag{4.3.9}$$

In order to make the link between space $S(b, g^t)$ and the standard (i.e., with respect to flat metrics) classes of symbols, we state the following

Lemma 4.3.8. *The space $S(b, g^t)$ is embedded in $S^{c/2}_{1,c/2}$, and spaces $S^m_{1,0}$ are embedded in $S(\langle\cdot\rangle^m, g^t)$ for any m.*

Proof. The first proposition holds thanks to

$$g^t_{(x,\xi)}(\mathrm{d}x, \mathrm{d}\xi) \leq \frac{|\mathrm{d}x|^2}{\langle\xi\rangle^{-c}} + \frac{|\mathrm{d}\xi|^2}{\langle\xi\rangle^2}$$

which is implied by (4.3.3). The second holds thanks to

$$|\mathrm{d}x|^2 + \frac{|\mathrm{d}\xi|^2}{\langle\xi\rangle^2_\mu} \leq \frac{|\mathrm{d}x|^2}{a(t,x) + \langle\xi\rangle^{-c}} + \frac{|\mathrm{d}\xi|^2}{\langle\xi\rangle^2} = g^t_{(x,\xi)}(\mathrm{d}x, \mathrm{d}\xi)$$

by inequality (4.3.5). □

Also, in order to use the general spaces of symbols defined with respect to a metric, we use here the Weyl quantization, which we recall

$$\mathrm{op}(a)u(x) = \mathrm{op}_{1/2}(a)u(x) = \int e^{i(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi.$$

4.3.2 Time derivative of the energy

We compute here the time derivative of the energy E defined in (4.2.7). The energy E depends on time through the symbol b, the Gevrey weight and u.

We introduce

$$v(t) = e^{\tau(t)D^\sigma} u(t) \tag{4.3.10}$$

with $\tau(t)$ defined in (4.3.1) and D^σ in (4.2.6). There holds

$$\partial_t v(t) = -\boldsymbol{\tau} D^\sigma v(t) + e^{\tau(t)D^\sigma} \partial_t u(t).$$

As u solves system (4.2.1), v solves

$$\partial_t v = -\boldsymbol{\tau} D^\sigma v + e^{\tau D^\sigma} \left(A\partial_x u + F(u)u\right).$$

Thanks to the normal form for A described in Assumption 4.2.1, there holds

$$\partial_t v_1 = -\boldsymbol{\tau} D^\sigma v_1 + \partial_x v_2 + \Big(F(u)^{(\tau)} v\Big)_1 \tag{4.3.11}$$

$$\partial_t v_2 = -\boldsymbol{\tau} D^\sigma v_2 + a^{(\tau)} \partial_x v_1 + \Big(F(u)^{(\tau)} v\Big)_2 \tag{4.3.12}$$

We use here for $a^{(\tau)}$ and $F(u)^{(\tau)}$ notation (5.3.1) in Chapter 5 of the conjugation operator of a Gevrey function:

$$a^{(\tau)} = e^{\tau(t)D^\sigma} a\, e^{-\tau(t)D^\sigma} \quad \text{and} \quad F(u)^{(\tau)} = e^{\tau(t)D^\sigma} F(u)\, e^{-\tau(t)D^\sigma}$$

We compute the time derivative of the energy $E(\tau(t), u(t))$ defined in (4.2.7). Using notation v defined in (4.3.10), the energy satisfies

$$E(t, u(t)) = \frac{1}{2}|v_1|^2_{L^2} + \frac{1}{2}|\mathrm{op}(b) v_2|^2_{L^2}$$

Denoting here $\langle \cdot \rangle$ the $L^2(\mathbb{R}^d)$ scalar product, we compute

$$\partial_t E = \mathrm{Re}\,\langle \partial_t v_1, v_1 \rangle + \mathrm{Re}\,\langle \mathrm{op}(b)\partial_t v_2, \mathrm{op}(b) v_2 \rangle + \mathrm{Re}\,\langle \mathrm{op}(\partial_t b) v_2, \mathrm{op}(b) v_2 \rangle$$

Using (4.3.11) and (4.3.12), there holds

$$\partial_t E = -\boldsymbol{\tau} \mathrm{E}_1 + \mathrm{E}_2 + \mathrm{E}_3 + \mathrm{E}_4 \tag{4.3.13}$$

where

$$\mathrm{E}_1 = \mathrm{Re}\,\langle \mathrm{op}(b) D^\sigma v_2(t), \mathrm{op}(b) v_2 \rangle + \mathrm{Re}\,\langle D^\sigma v_1(t), v_1(t) \rangle \tag{4.3.14}$$

is the time-derivative of the Gevrey weight ;

$$\mathrm{E}_2 = \mathrm{Re}\,\langle \mathrm{op}(b) a^{(\tau)} \partial_x v_1(t), \mathrm{op}(b) v_2 \rangle + \mathrm{Re}\,\langle \partial_x v_2(t), v_1(t) \rangle \tag{4.3.15}$$

are linear terms in the equations ;

$$\mathrm{E}_3 = \mathrm{Re}\,\langle \mathrm{op}(\partial_t b) v_2(t), \mathrm{op}(b) v_2 \rangle \tag{4.3.16}$$

is the time-derivative of the symmetrizer ;

$$\mathrm{E}_4 = \mathrm{Re}\,\langle \mathrm{op}(b) \Big(F(u)^{(\tau)} v\Big)_2, \mathrm{op}(b) v_2 \rangle + \mathrm{Re}\,\langle \Big(F(u)^{(\tau)} v\Big)_1, v_1(t) \rangle \tag{4.3.17}$$

are the non-linear terms in the equation. The term E_1 is of higher order than the energy, thanks to the D^σ term coming from the time derivative of the Gevrey weight. The minus sign in front of E_1 is crucial in order to control the remainder terms E_2, E_3 and E_4. We focus now on each of those terms.

The term E_2

The crucial cancellations take place here. They rely on our choice of b defined in (4.2.4). As a is in $\mathcal{G}^{\sigma}_{\underline{\tau}}$ with $\underline{\tau}$ defined in (4.2.3) and by the results of Section 5.5 in Chapter 5 (see also [CNR]), there is a symbol $\tilde{a}$ in $S^0_{1,0}$ such that

$$a^{(\tau)} = \mathrm{op}(\tilde{a}) \tag{4.3.18}$$

for all $\tau = \tau(t)$, as $\tau(t) \leq \tau_0 < \underline{\tau}$ by definition (4.3.1). We may then write

$$a^{(\tau)} = \mathrm{op}(\widetilde{a}) = a + \mathrm{op}\left(\widetilde{a} - a\right) = \mathrm{op}\left(a_\sharp\right) + D^{-c} + \mathrm{op}\left(\widetilde{a} - a\right)$$

where D is defined in (4.2.6), so that the first term in E_2, namely

$$\mathrm{E}_{2,1} = \mathrm{Re}\,\langle \mathrm{op}(b) a^{(\tau)} \partial_x v_1, \mathrm{op}(b) v_2 \rangle$$

takes the form

$$\mathrm{E}_{2,1} = \mathrm{Re}\,\langle \mathrm{op}(b)\mathrm{op}(a_\sharp)\partial_x v_1, \mathrm{op}(b) v_2 \rangle + R_2$$

where R_2 comprises remainder terms:

$$R_2 = \mathrm{Re}\,\langle \mathrm{op}(b)\left(D^{-c} + \mathrm{op}\left(\widetilde{a} - a\right)\right)\partial_x v_1, \mathrm{op}(b) v_2 \rangle.$$

There holds

$$\mathrm{E}_{2,1} = \mathrm{Re}\,\langle \mathrm{op}(b)^2 \mathrm{op}(a_\sharp)\partial_x v_1, v_2 \rangle + R_2,$$

as $\mathrm{op}(b)^* = \mathrm{op}(b)$ in Weyl quantization and b is real, so that, applying twice Lemma 4.4.11,

$$\mathrm{op}(b)^2 = \mathrm{op}\left(b^2\right) + \mathrm{op}\left(S\left(b^2\lambda^{-1}, g^t\right)\right)$$

where λ is defined by (4.3.9). Thus

$$\mathrm{E}_{2,1} = \mathrm{Re}\,\langle \mathrm{op}(b^2)\mathrm{op}(a_\sharp)\partial_x v_1, v_2 \rangle + \widetilde{R}_2$$

where

$$\widetilde{R}_2 = R_2 + \mathrm{Re}\,\langle \mathrm{op}\left(S\left(b^2\lambda^{-1}, g^t\right)\right)\mathrm{op}(a_\sharp)\partial_x v_1, v_2 \rangle.$$

Now, by definition (4.2.4) of b, there holds

$$\mathrm{op}(b)^2\mathrm{op}(a_\sharp) = \mathrm{id} + \mathrm{op}\left(S(b^2 a_\sharp \lambda^{-1}, g^t)\right) = \mathrm{id} + \mathrm{op}\left(S(\lambda^{-1}, g^t)\right)$$

as $a_\sharp$ is in $S(a_\sharp, g^t)$ thanks to Lemma 4.3.7, and by use of Lemma 4.4.11. Thus the leading term in $\mathrm{E}_{2,1}$ cancels with the other term in E_2:

$$\mathrm{Re}\,\langle \mathrm{op}(b^2)\mathrm{op}(a_\sharp)\partial_x v_1, v_2 \rangle + \mathrm{Re}\,\langle \partial_x v_2, v_1 \rangle = \widetilde{R}_2 + \mathrm{Re}\,\langle \mathrm{op}\left(S(\lambda^{-1}, g^t)\right)\partial_x v_1, v_2 \rangle$$

and there holds

$$\begin{aligned}
\mathrm{E}_2 \;=\; & -\mathrm{Re}\,\langle \mathrm{op}(b) D^{-c}\partial_x v_1, \mathrm{op}(b) v_2 \rangle && (4.3.19)\\
& +\mathrm{Re}\,\langle \mathrm{op}\left(S(\lambda^{-1}, g^t)\right)\partial_x v_1, v_2 \rangle + \mathrm{Re}\,\langle \mathrm{op}\left(S\left(b^2\lambda^{-1}, g^t\right)\right)\mathrm{op}(a_\sharp)\partial_x v_1, v_2 \rangle && (4.3.20)\\
& +\mathrm{Re}\,\langle \mathrm{op}(b)\mathrm{op}(\widetilde{a} - a)\partial_x v_1, \mathrm{op}(b) v_2 \rangle. && (4.3.21)
\end{aligned}$$

The term E_3

We first note that $\partial_t b = -\frac{1}{2}\partial_t a\, a_\sharp^{-1} b$. Thanks to Assumption 4.2.1, function $\partial_t a(t,x)$ is positive. We may then write

$$\partial_t b = -\frac{1}{2}\left(\sqrt{\partial_t a}\, a_\sharp^{-1/2}\right)^2 b. \tag{4.3.22}$$

As $\sqrt{\partial_t a}$ depends only on (t,x) variables, it is in $S(1,g^t)$, hence $\sqrt{\partial_t a}\, a_\sharp^{-1/2}$ is in $S\left(a_\sharp^{-1/2}, g^t\right)$ by Lemma 4.4.10. Applying twice Lemma 4.4.11 in Appendix 4.4.2, there holds

$$\mathrm{op}\,(\partial_t b) = -\frac{1}{2}\left(\mathrm{op}\left(\sqrt{\partial_t a}\, a_\sharp^{-1/2}\right)\right)^2 \mathrm{op}(b) + \mathrm{op}\left(S\left(a_\sharp^{-3/2}\lambda^{-1}, g^t\right)\right)$$

where λ is defined in (4.3.9). This implies that

$$\begin{aligned}\mathrm{E}_3 \;=\; & -\mathrm{Re}\,\langle\left(\mathrm{op}\left(\sqrt{\partial_t a}\, a_\sharp^{-1/2}\right)\right)^2 \mathrm{op}(b)v_2, \mathrm{op}(b)v_2\rangle \\ & +\mathrm{Re}\,\langle \mathrm{op}\left(S\left(a_\sharp^{-3/2}\lambda^{-1}, g^t\right)\right) v_2, \mathrm{op}(b)v_2\rangle.\end{aligned}$$

The first term in the above right-hand side satisfies

$$\begin{aligned}\mathrm{Re}\,\langle\left(\mathrm{op}\left(\sqrt{\partial_t a}\, a_\sharp^{-1/2}\right)\right)^2 \mathrm{op}(b)v_2, \mathrm{op}(b)v_2\rangle \;&=\; \left|\mathrm{op}\left(\sqrt{\partial_t a}\, a_\sharp^{-1/2}\right)\mathrm{op}(b)v_2\right|^2 \\ &\geq\; 0.\end{aligned}$$

Thus

$$\mathrm{E}_3 \leq \mathrm{Re}\,\langle \mathrm{op}\left(S\left(a_\sharp^{-3/2}\lambda^{-1}, g^t\right)\right) v_2, \mathrm{op}(b)v_2\rangle. \tag{4.3.23}$$

4.3.3 Energy estimate

In Section 4.3.2, we observed cancellations in $\partial_t E$. The next step is to bound the remainder terms in E_2, E_3 and E_4 by a fraction of the negative term E_1 . This is done thanks to the properties of the pseudo-differential calculus described in Appendix 4.4.2 and Lemma 4.3.8.

Estimate of E_2

The term E_2, defined in (4.3.15) is equal, thanks to the previous computations, to the sum of (4.3.19), (4.3.20) and (4.3.21).

• First we focus on (4.3.19). As E_1 controls both $D^{\sigma/2}v_1$ and $\mathrm{op}(b)D^{\sigma/2}v_2$ in L^2 norm, we make both terms appear in (4.3.19) (possibly up to commutator terms). By lemma 4.3.7, $b \in S(b,g^t)$. By Lemma 4.3.8, $D^{\sigma/2} \in S(\langle\cdot\rangle^{\sigma/2}, g^t)$. Hence, by Lemma 4.4.11, the commutator $[\mathrm{op}(b)D^{\sigma/2}]$ belongs to $S\left(b\langle\cdot\rangle^{\sigma/2}\lambda^{-1}, g^t\right)$. Thus we may write

$$\begin{aligned}&\mathrm{Re}\,\langle \mathrm{op}(b)D^{-c}\partial_x v_1, \mathrm{op}(b)v_2\rangle \\ &= \mathrm{Re}\,\langle \mathrm{op}(b)D^{\sigma/2}\, D^{-c-\sigma}\partial_x D^{\sigma/2}v_1, \mathrm{op}(b)v_2\rangle \\ &= \mathrm{Re}\,\langle \mathrm{op}(b)\, D^{-c-\sigma}\partial_x D^{\sigma/2}v_1, D^{\sigma/2}\mathrm{op}(b)v_2\rangle \\ &\quad + \mathrm{Re}\,\langle \mathrm{op}\left(S\left(b\langle\cdot\rangle^{\sigma/2}\lambda^{-1}, g^t\right)\right) D^{-c-\sigma/2}\partial_x v_1, \mathrm{op}(b)v_2\rangle\end{aligned}$$

by commuting $\mathrm{op}(b)$ and $D^{\sigma/2}$ and using Lemma 4.4.11. Commuting both operators a second time we get

$$\mathrm{Re}\,\langle \mathrm{op}(b)D^{-c}\partial_x v_1, \mathrm{op}(b)v_2\rangle = \mathrm{Re}\,\langle \mathrm{op}(b)\, D^{-c-\sigma}\partial_x D^{\sigma/2}v_1, \mathrm{op}(b)D^{\sigma/2}v_2\rangle + R_{2,1} \quad (4.3.24)$$

denoting

$$\begin{aligned} R_{2,1} &= \mathrm{Re}\,\langle \mathrm{op}\left(S\left(b\langle\cdot\rangle^{\sigma/2}\lambda^{-1}, g^t\right)\right) D^{-c-\sigma/2}\partial_x v_1, \mathrm{op}(b)v_2\rangle \\ &\quad + \mathrm{Re}\,\langle \mathrm{op}(b)D^{-c-\sigma/2}\partial_x v_1, \mathrm{op}\left(S\left(b\langle\cdot\rangle^{\sigma/2}\lambda^{-1}, g^t\right)\right) v_2\rangle. \end{aligned} \quad (4.3.25)$$

Using both Lemma 4.3.8 and Lemma 4.4.11, there holds

$$R_{2,1} = \mathrm{Re}\,\langle \mathrm{op}\left(S\left(b\lambda^{-1}\langle\cdot\rangle^{1-c}, g^t\right)\right) v_1, \mathrm{op}(b)v_2\rangle.$$

Then, by definition (4.3.9) of λ and definition (4.2.4) of b,

$$R_{2,1} = \mathrm{Re}\,\langle \mathrm{op}\left(S\left(b^2\langle\cdot\rangle^{-c}, g^t\right)\right) v_1, \mathrm{op}(b)v_2\rangle$$

By (4.3.3) and definition of the symbol classes $S(M,g)$ in Section 4.4.2 of the Appendix, there holds

$$S\left(b^2\langle\cdot\rangle^{-c}, g^t\right) \subset S\left(1, g^t\right).$$

By Lemma 4.4.12, we get thus

$$|R_{2,1}| \lesssim |v_1|_{L^2}^2 + |\mathrm{op}(b)v_2|_{L^2}^2 \leq \mathrm{E}_1.$$

We focus now on the first term of the right-hand side of (4.3.24). Thanks to the bound (4.3.3) and Lemma 4.3.8, we note that $\mathrm{op}(b)D^{-c-\sigma}\partial_x \in \mathrm{op}\left(S(\langle\cdot\rangle^{1-c/2-\sigma}, g^t)\right)$. Thus

$$\left|\mathrm{Re}\,\langle \mathrm{op}(b)\, D^{-c-\sigma}\partial_x D^{\sigma/2}v_1, \mathrm{op}(b)D^{\sigma/2}v_2\rangle\right| \leq \left|\mathrm{op}\left(S(\langle\cdot\rangle^{1-c/2-\sigma}, g^t)\right)\, D^{\sigma/2}v_1\right|^2 + \left|\mathrm{op}(b)D^{\sigma/2}v_2\right|^2.$$

Then, as soon as σ and c satisfy the constraint

$$1 - c/2 - \sigma \leq 0 \quad (4.3.26)$$

there holds $\mathrm{op}\left(S(\langle\cdot\rangle^{1-c/2-\sigma}, g^t)\right) \subset \mathrm{op}\left(S\left(1, g^t\right)\right)$, hence the continuous action of the operator on L^2 by Lemma 4.4.12. Thus

$$|(4.3.19)| \leq \left|\mathrm{Re}\,\langle \mathrm{op}(b)\, D^{-c-\sigma}\partial_x D^{\sigma/2}v_1, \mathrm{op}(b)D^{\sigma/2}v_2\rangle\right| + |R_{2,1}| \lesssim \mathrm{E}_1. \quad (4.3.27)$$

• Second, we focus on (4.3.20). First, using Lemma 4.3.7 and Lemma 4.4.11, there holds

$$\begin{aligned} &\mathrm{Re}\,\langle \mathrm{op}\left(S(\lambda^{-1}, g^t)\right) \partial_x v_1, v_2\rangle + \mathrm{Re}\,\langle \mathrm{op}\left(S\left(b^2\lambda^{-1}, g^t\right)\right) \mathrm{op}(a_\sharp)\partial_x v_1, v_2\rangle \\ &= \mathrm{Re}\,\langle \mathrm{op}\left(S(\lambda^{-1}, g^t)\right) \partial_x v_1, v_2\rangle. \end{aligned}$$

As we did in the previous point for (4.3.19), we write

$$\begin{aligned}\mathrm{Re}\,\langle \mathrm{op}\left(S(\lambda^{-1},g^t)\right)\partial_x v_1, v_2\rangle &= \mathrm{Re}\,\langle D^{-\sigma/2}\mathrm{op}\left(S(\lambda^{-1},g^t)\right)D^{-\sigma/2}\partial_x D^{\sigma/2}v_1, D^{\sigma/2}v_2\rangle\\ &= \mathrm{Re}\,\langle \mathrm{op}\left(S\left(\langle\cdot\rangle^{1-\sigma}\lambda^{-1},g^t\right)\right)D^{\sigma/2}v_1, D^{\sigma/2}v_2\rangle\end{aligned}$$

using here again Lemma 4.4.11 and Lemma 4.3.8. By definition (4.3.9) of λ, there holds $\langle\cdot\rangle^{1-\sigma}\lambda^{-1}\leq\langle\cdot\rangle^{c/2-\sigma}$. This implies that

$$\mathrm{op}\left(S\left(\langle\cdot\rangle^{1-\sigma}\lambda^{-1},g^t\right)\right)\subset\mathrm{op}\left(S\left(\langle\cdot\rangle^{c/2-\sigma},g^t\right)\right)$$

and the continuous action of such operators on L^2, by Lemma 4.4.12, as soon as

$$c/2-\sigma\leq 0. \tag{4.3.28}$$

If (4.3.28) is satisfied, then

$$|(4.3.20)|\lesssim \mathrm{E}_1 \tag{4.3.29}$$

using the action of $\mathrm{op}\left(S(1,g^t)\right)$ described in Lemma 4.4.12.

• Third, we focus on (4.3.21). Thanks to Lemma 5.2 in [Mor17c] (in the case $k=1$, $m=0$, there holds indeed $\max\{m-(k+1)(1-\sigma),m-2+\sigma\}=-2(1-\sigma)$), the symbol $\widetilde{a}$ satisfies

$$\tilde{a}-a=i\partial_x a\,\partial_\xi\langle\xi\rangle^\sigma+S^{-2(1-\sigma)}_{1,0} \tag{4.3.30}$$

and we write thus

$$\mathrm{Re}\,\langle\mathrm{op}(b)\mathrm{op}(\widetilde{a}-a)\partial_x v_1,\mathrm{op}(b)v_2\rangle=\mathrm{Re}\,\langle\mathrm{op}(b)\mathrm{op}\left(i\partial_x a\,\partial_\xi\langle\cdot\rangle^\sigma\right)\partial_x v_1,\mathrm{op}(b)v_2\rangle+R_{2,2} \tag{4.3.31}$$

where

$$R_{2,2}=\mathrm{Re}\,\langle\mathrm{op}(b)\mathrm{op}\left(S^{-2(1-\sigma)}_{1,0}\right)\partial_x v_1,\mathrm{op}(b)v_2\rangle.$$

The sub-principal symbol $i\partial_x a\,\partial_\xi\langle\xi\rangle^\sigma$ is *a priori* in $S^{-(1-\sigma)}_{1,0}$, which would be insufficient to counterbalance both $\mathrm{op}(b)$ and ∂_x. Indeed, by Lemma 4.3.7 and Lemma 4.3.8, there holds $\mathrm{op}(b)\partial_x\in\mathrm{op}\left(S^{1+c/2}_{1,c/2}\right)$ *vs* the straigthforward estimate $\mathrm{op}\left(i\partial_x a\,\partial_\xi\langle\xi\rangle^\sigma\right)\in\mathrm{op}\left(S^{-(1-\sigma)}_{1,0}\right)$. But using the Glaeser inequality for a described in Lemma 4.3.1 and definition (4.3.8) of the metric g^t, we prove that in fact

$$i\partial_x a\,\partial_\xi\langle\xi\rangle^\sigma\in S(b^{-1}\langle\cdot\rangle^{\sigma-1},g^t). \tag{4.3.32}$$

Indeed for any α, β in $\mathbb{N}^d$, there holds

$$\begin{aligned}\left|\partial_x^\alpha\partial_\xi^\beta\left(i\partial_x a(t,x)\,\partial_\xi\langle\xi\rangle^\sigma\right)\right| &= \left|\partial_x^{\alpha+1}a(t,x)\,\partial_\xi^{\beta+1}\langle\xi\rangle^\sigma\right|\\ &\lesssim \left|\partial_x^{\alpha+1}a(t,x)\right|\langle\xi\rangle^{\sigma-1-|\beta|}\\ &\lesssim \left|\partial_x^{\alpha+1}a\right|_{L^\infty([0,T]\times B_r(x_0))}b^{-1+|\alpha|}\langle\xi\rangle^{\sigma-1-|\beta|}\end{aligned}$$

for $|\alpha|\geq 1$, as $b(t,x,\xi)\geq 1$ for all (t,x,ξ) thanks to inequality (4.3.5) on $a_\sharp$ and definition (4.2.4) of b. For $|\alpha|=0$, thanks to Lemma 4.3.1 and definition (4.2.4) of b, there holds

$$\left|\partial_x^\alpha\partial_\xi^\beta\left(i\partial_x a(t,x)\,\partial_\xi\langle\xi\rangle^\sigma\right)\right|\lesssim\left|\partial_x^2 a\right|_{L^\infty([0,T]\times B_r(x_0))}b^{-1+|\alpha|}\langle\xi\rangle^{\sigma-1-|\beta|}$$

thus, combining both cases, the proof of (4.3.32).

For the first term in the right-hand side of (4.3.31), we follow the same path as in the above treatment of (4.3.19) and (4.3.20), writing

$$\begin{aligned}&\mathrm{Re}\,\langle \mathrm{op}(b)\mathrm{op}\left(i\partial_x a\,\partial_\xi\langle\cdot\rangle^\sigma\right)\partial_x v_1, \mathrm{op}(b)v_2\rangle\\ &= \mathrm{Re}\,\langle \mathrm{op}(b)\mathrm{op}\left(i\partial_x a\,\partial_\xi\langle\cdot\rangle^\sigma\right)D^{-\sigma/2}\partial_x D^{\sigma/2}v_1, \mathrm{op}(b)D^{-\sigma/2}D^{\sigma/2}v_2\rangle\end{aligned}$$

and commuting twice operators $\mathrm{op}(b)$ and $D^{-\sigma/2}$, there holds

$$\begin{aligned}&\mathrm{Re}\,\langle \mathrm{op}(b)\mathrm{op}\left(i\partial_x a\,\partial_\xi\langle\cdot\rangle^\sigma\right)\partial_x v_1, \mathrm{op}(b)v_2\rangle\\ &= \mathrm{Re}\,\langle \mathrm{op}(b)\,D^{-\sigma/2}\mathrm{op}\left(i\partial_x a\,\partial_\xi\langle\cdot\rangle^\sigma\right)D^{-\sigma/2}\partial_x\,D^{\sigma/2}v_1, \mathrm{op}(b)D^{\sigma/2}v_2\rangle\\ &\quad+\mathrm{Re}\,\langle \mathrm{op}\left(S\left(b\langle\cdot\rangle^{-\sigma/2}\lambda^{-1}, g^t\right)\right)\mathrm{op}\left(i\partial_x a\,\partial_\xi\langle\cdot\rangle^\sigma\right)D^{-\sigma/2}\partial_x D^{\sigma/2}v_1, \mathrm{op}(b)D^{\sigma/2}v_2\rangle\\ &\quad+\mathrm{Re}\,\langle \mathrm{op}(b)\mathrm{op}\left(i\partial_x a\,\partial_\xi\langle\cdot\rangle^\sigma\right)D^{-\sigma/2}\partial_x D^{\sigma/2}v_1, \mathrm{op}\left(S\left(b\langle\cdot\rangle^{-\sigma/2}\lambda^{-1}, g^t\right)\right)D^{\sigma/2}v_2\rangle.\end{aligned}$$

We use then Lemma 4.4.11, Lemma 4.3.8 and (4.3.32), computing

$$\begin{aligned}&\mathrm{Re}\,\langle \mathrm{op}(b)\mathrm{op}\left(i\partial_x a\,\partial_\xi\langle\cdot\rangle^\sigma\right)\partial_x v_1, \mathrm{op}(b)v_2\rangle\\ &= \mathrm{Re}\,\langle \mathrm{op}\left(S\left(1, g^t\right)\right)\,D^{\sigma/2}v_1, \mathrm{op}(b)D^{\sigma/2}v_2\rangle\\ &\quad+\mathrm{Re}\,\langle \mathrm{op}\left(S\left(\lambda^{-1}, g^t\right)\right)\,D^{\sigma/2}v_1, \mathrm{op}(b)D^{\sigma/2}v_2\rangle\end{aligned}$$

hence, by Lemma 4.4.12,

$$|\mathrm{Re}\,\langle \mathrm{op}(b)\mathrm{op}\left(i\partial_x a\,\partial_\xi\langle\cdot\rangle^\sigma\right)\partial_x v_1, \mathrm{op}(b)v_2\rangle| \lesssim \mathrm{E}_1.$$

For the remainder term $R_{2,2}$, there holds

$$\begin{aligned}&\mathrm{Re}\,\langle \mathrm{op}(b)\mathrm{op}\left(S_{1,0}^{-2(1-\sigma)}\right)\partial_x v_1, \mathrm{op}(b)v_2\rangle\\ &= \mathrm{Re}\,\langle \mathrm{op}\left(S\left(b\langle\cdot\rangle^{-(1-\sigma)}, g^t\right)\right)D^{\sigma/2}v_1, \mathrm{op}(b)D^{\sigma/2}v_2\rangle\\ &\quad+\mathrm{Re}\,\langle \mathrm{op}\left(S\left(b\langle\cdot\rangle^{-(1-\sigma)}\lambda^{-1}, g^t\right)\right)D^{\sigma/2}v_1, \mathrm{op}(b)D^{\sigma/2}v_2\rangle\\ &\quad+\mathrm{Re}\,\langle \mathrm{op}(b)D^{\sigma/2}v_1, \mathrm{op}\left(S\left(b\langle\cdot\rangle^{-(1-\sigma)}\lambda^{-1}, g^t\right)\right)D^{\sigma/2}v_2\rangle\end{aligned}$$

using the same arguments as above concerning commutators. Thanks to inequality (4.3.3) on b, we prove $b\langle\cdot\rangle^{-(1-\sigma)} \le \langle\cdot\rangle^{c/2+\sigma-1}$ which implies

$$\mathrm{op}\left(S\left(b\langle\cdot\rangle^{-(1-\sigma)}, g^t\right)\right) \subset \mathrm{op}\left(S\left(\langle\cdot\rangle^{c/2+\sigma-1}, g^t\right)\right).$$

Hence, as soon as

$$c/2+\sigma-1 \le 0 \tag{4.3.33}$$

holds, operator $\mathrm{op}\left(S\left(b\langle\cdot\rangle^{-(1-\sigma)}, g^t\right)\right)$ acts on L^2 thanks to Lemma 4.4.12, thus

$$|(4.3.21)| \lesssim \mathrm{E}_1 \tag{4.3.34}$$

using again Lemma 4.4.12.

Combining all three constraints (4.3.26), (4.3.28) and (4.3.33), we get

$$c = 2(1-\sigma) \tag{4.3.35}$$

and

$$\sigma \geq 1/2 \tag{4.3.36}$$

which is the expected lower bound for the Gevrey index. Putting together estimates (4.3.27), (4.3.29) and (4.3.34), there is a constant $C_2 > 0$ such that

$$|\mathrm{E}_2| \leq C_2\, \mathrm{E}_1. \tag{4.3.37}$$

Estimate of E_3

Starting with the right-hand side of inequality (4.3.23), we compute

$$\begin{aligned}
&\mathrm{Re}\,\langle \mathrm{op}\left(S\left(a_\sharp^{-3/2}\lambda^{-1}, g^t\right)\right) v_2, \mathrm{op}(b) v_2\rangle \\
&= \mathrm{Re}\,\langle \mathrm{op}\left(S\left(a_\sharp^{-3/2}\lambda^{-1}, g^t\right)\right) D^{-\sigma/2}\mathrm{op}\left(b^{-1}\right)\mathrm{op}(b) D^{\sigma/2} v_2, \mathrm{op}(b) v_2\rangle \\
&\quad + \mathrm{Re}\,\langle \mathrm{op}\left(S\left(a_\sharp^{-3/2}\lambda^{-1}, g^t\right)\right) D^{-\sigma/2}\mathrm{op}\left(S\left(\lambda^{-1}, g^t\right)\right) D^{\sigma/2} v_2, \mathrm{op}(b) v_2\rangle
\end{aligned}$$

using Lemma 4.4.11. Using the Lemma once again, there holds

$$\begin{aligned}
&\mathrm{Re}\,\langle \mathrm{op}\left(S\left(a_\sharp^{-3/2}\lambda^{-1}, g^t\right)\right) D^{-\sigma/2}\mathrm{op}\left(b^{-1}\right)\mathrm{op}(b) D^{\sigma/2} v_2, \mathrm{op}(b) v_2\rangle \\
&\quad + \mathrm{Re}\,\langle \mathrm{op}\left(S\left(a_\sharp^{-3/2}\lambda^{-1}, g^t\right)\right) D^{-\sigma/2}\mathrm{op}\left(S\left(\lambda^{-1}, g^t\right)\right) D^{\sigma/2} v_2, \mathrm{op}(b) v_2\rangle \\
&= \mathrm{Re}\,\langle \mathrm{op}\left(S\left(b^2\lambda^{-1}\langle\cdot\rangle^{-\sigma/2}, g^t\right)\right)\mathrm{op}(b) D^{\sigma/2} v_2, \mathrm{op}(b) v_2\rangle \\
&\quad + \mathrm{Re}\,\langle \mathrm{op}\left(S\left(b^3\lambda^{-2}\langle\cdot\rangle^{-\sigma/2}, g^t\right)\right) D^{\sigma/2} v_2, \mathrm{op}(b) v_2\rangle
\end{aligned}$$

as $b = a_\sharp^{-1/2}$. As above, we make appear operator $D^{\sigma/2}$, defined in (4.2.6), writing

$$\begin{aligned}
&= \mathrm{Re}\,\langle \mathrm{op}\left(S\left(b^2\lambda^{-1}\langle\cdot\rangle^{-\sigma/2}, g^t\right)\right)\mathrm{op}(b) D^{\sigma/2} v_2, \mathrm{op}(b) v_2\rangle \\
&\quad + \mathrm{Re}\,\langle \mathrm{op}\left(S\left(b^3\lambda^{-2}\langle\cdot\rangle^{-\sigma/2}, g^t\right)\right) D^{\sigma/2} v_2, \mathrm{op}(b) v_2\rangle \\
&= \mathrm{Re}\,\langle \mathrm{op}\left(S\left(b^2\lambda^{-1}\langle\cdot\rangle^{-\sigma/2}, g^t\right)\right)\mathrm{op}(b) D^{\sigma/2} v_2, \mathrm{op}(b) D^{-\sigma/2} D^{\sigma/2} v_2\rangle \\
&\quad + \mathrm{Re}\,\langle \mathrm{op}\left(S\left(b^3\lambda^{-2}\langle\cdot\rangle^{-\sigma/2}, g^t\right)\right) D^{\sigma/2} v_2, \mathrm{op}(b) D^{-\sigma/2} D^{\sigma/2} v_2\rangle.
\end{aligned}$$

Next, commuting $\mathrm{op}(b)$ with $D^{-\sigma/2}$:

$$
\begin{aligned}
&= \mathrm{Re}\,\langle \mathrm{op}\left(S\left(b^2\lambda^{-1}\langle\cdot\rangle^{-\sigma/2}, g^t\right)\right)\mathrm{op}(b)D^{\sigma/2}v_2, \mathrm{op}(b)D^{-\sigma/2}D^{\sigma/2}v_2\rangle \\
&\quad + \mathrm{Re}\,\langle \mathrm{op}\left(S\left(b^3\lambda^{-2}\langle\cdot\rangle^{-\sigma/2}, g^t\right)\right)D^{\sigma/2}v_2, \mathrm{op}(b)D^{-\sigma/2}D^{\sigma/2}v_2\rangle \\
&= \mathrm{Re}\,\langle \mathrm{op}\left(S\left(b^2\lambda^{-1}\langle\cdot\rangle^{-\sigma}, g^t\right)\right)\mathrm{op}(b)D^{\sigma/2}v_2, \mathrm{op}(b)D^{\sigma/2}v_2\rangle \\
&\quad + \mathrm{Re}\,\langle \mathrm{op}\left(S\left(b^3\lambda^{-2}\langle\cdot\rangle^{-\sigma}, g^t\right)\right)D^{\sigma/2}v_2, \mathrm{op}(b)D^{\sigma/2}v_2\rangle \\
&\quad + \mathrm{Re}\,\langle \mathrm{op}\left(S\left(b^2\lambda^{-1}\langle\cdot\rangle^{-\sigma/2}, g^t\right)\right)\mathrm{op}(b)D^{\sigma/2}v_2, \mathrm{op}\left(S\left(b\langle\cdot\rangle^{-\sigma/2}\lambda^{-1}, g^t\right)\right)D^{\sigma/2}v_2\rangle \\
&\quad + \mathrm{Re}\,\langle \mathrm{op}\left(S\left(b^3\lambda^{-2}\langle\cdot\rangle^{-\sigma/2}, g^t\right)\right)D^{\sigma/2}v_2, \mathrm{op}\left(S\left(b\langle\cdot\rangle^{-\sigma/2}\lambda^{-1}, g^t\right)\right)D^{\sigma/2}v_2\rangle.
\end{aligned}
$$

As

$$
\begin{aligned}
b^2\lambda^{-1}\langle\xi\rangle^{-\sigma} &\leq \langle\xi\rangle^{3c/2-1-\sigma} \\
&\leq \langle\xi\rangle^{2-4\sigma}
\end{aligned}
$$

by equality (4.3.35) for c, there holds

$$
\mathrm{op}\left(S\left(b^2\lambda^{-1}\langle\cdot\rangle^{-\sigma}, g^t\right)\right) \subset \mathrm{op}\left(S\left(\langle\cdot\rangle^{2-4\sigma}, g^t\right)\right) \subset \mathrm{op}\left(S\left(1, g^t\right)\right)
$$

as $2-4\sigma \leq 0$ by (4.3.36). By Lemma 4.4.12 on the action of $\mathrm{op}\left(S(1,g^t)\right)$, there holds

$$
\left|\mathrm{Re}\,\langle \mathrm{op}\left(S\left(a_\sharp^{-3/2}\lambda^{-1}, g^t\right)\right)v_2, \mathrm{op}(b)v_2\rangle\right| \lesssim \mathrm{E}_1.
$$

Thus, by inequality (4.3.23), there is $C_3 > 0$ such that

$$
|\mathrm{E}_3| \leq C_3\,\mathrm{E}_1. \tag{4.3.38}
$$

Estimate of E_4

We write first, as before,

$$
\begin{aligned}
&\mathrm{Re}\,\langle \mathrm{op}(b)\left(F(u)^{(\tau)}v\right)_2, \mathrm{op}(b)v_2\rangle + \mathrm{Re}\,\langle\left(F(u)^{(\tau)}v\right)_1, v_1\rangle \\
&= \mathrm{Re}\,\langle D^{-\sigma/2}\mathrm{op}(b)\left(F(u)^{(\tau)}v\right)_2, \mathrm{op}(b)D^{\sigma/2}v_2\rangle + \mathrm{Re}\,\langle\left(F(u)^{(\tau)}v\right)_1, v_1\rangle \\
&\quad + R_{4,1}
\end{aligned} \tag{4.3.39}
$$

commuting $\mathrm{op}(b)$ with $D^{-\sigma/2}$, and where we denote

$$
R_{4,1} = \mathrm{Re}\,\langle \mathrm{op}(b)\left(F(u)^{(\tau)}v\right)_2, \mathrm{op}\left(b\langle\cdot\rangle^{-\sigma/2}\lambda^{-1}\right)v_2\rangle
$$

Next, by Lemma 4.3.8 and Lemma 4.4.11, there holds

$$
D^{-\sigma/2}\mathrm{op}(b) \subset \mathrm{op}\left(S\left(b\langle\cdot\rangle^{-\sigma}, g^t\right)\right)D^{\sigma/2}
$$

and inequalities (4.3.3) and (4.3.36) imply

$$\mathrm{op}\left(S\left(b\langle\cdot\rangle^{-\sigma}, g^t\right)\right) D^{\sigma/2} \subset \mathrm{op}\left(S\left(1, g^t\right)\right) D^{\sigma/2}$$

thus

$$\left|D^{-\sigma/2}\mathrm{op}(b)\left(F(u)^{(\tau)}v\right)_2\right|_{L^2} \lesssim \left|D^{\sigma/2}\left(F(u)^{(\tau)}v\right)_2\right|_{L^2}.$$

As u is in $\mathcal{G}^\sigma_\tau$, $u \mapsto F(u)$ analytic and $H^{\sigma/2}\mathcal{G}^\sigma_\tau$ an algebra thanks to Remark 5.3.2 in Chapter 5, Proposition 5.3.1 implies that $F(u)^{(\tau)}$ acts continuously in $H^{\sigma/2}$, hence

$$\left|D^{\sigma/2}\left(F(u)^{(\tau)}v\right)_2\right|_{L^2} \lesssim \left\|F(u)^{(\tau)}\right\|_{\mathcal{L}(H^{\sigma/2})}\left|D^{\sigma/2}v\right|_{L^2}.$$

Using Cauchy-Schwartz' inequality to get an estimate of (4.3.39), there holds

$$\begin{aligned}
&|(4.3.39)| \\
&\lesssim \left|D^{-\sigma/2}\mathrm{op}(b)\left(F(u)^{(\tau)}v\right)_2\right|_{L^2}\left|\mathrm{op}(b)D^{\sigma/2}v_2\right|_{L^2} + \left|\left(F(u)^{(\tau)}v\right)_1\right|_{L^2}|v_1|_{L^2} \\
&\lesssim \left\|F(u)^{(\tau)}\right\|_{\mathcal{L}(H^{\sigma/2})}\left|D^{\sigma/2}v\right|_{L^2}\left|\mathrm{op}(b)D^{\sigma/2}v_2\right|_{L^2} + \left\|F(u)^{(\tau)}\right\|_{\mathcal{L}(L^2)}|v_1|^2_{L^2} \\
&\lesssim \left(\left\|F(u)^{(\tau)}\right\|_{\mathcal{L}(H^{\sigma/2})} + \left\|F(u)^{(\tau)}\right\|_{\mathcal{L}(L^2)}\right)\mathrm{E}_1.
\end{aligned}$$

We apply the same scheme of proof for the remainder term $R_{4,1}$, as

$$\mathrm{op}\left(b\langle\cdot\rangle^{-\sigma/2}\lambda^{-1}\right) \subset \mathrm{op}\left(S\left(\langle\cdot\rangle^{c-1-\sigma/2}, g^t\right)\right) \subset \mathrm{op}\left(S\left(1, g^t\right)\right)$$

thanks to (4.3.35) and (4.3.36). We conclude by

$$|\mathrm{E}_4| \leq C_4\,\mathrm{E}_1 \tag{4.3.40}$$

for some $C_4 > 0$ depending essentially on $\left\|F(u)^{(\tau)}\right\|_{\mathcal{L}(H^{\sigma/2})} + \left\|F(u)^{(\tau)}\right\|_{\mathcal{L}(L^2)}$.

Conclusion

We prove Theorem 6 by taking $\boldsymbol{\tau} > C_2 + C_3 + C_4$, where the constants are defined respectively in (4.3.37), (4.3.38) and (4.3.40).

4.4 Appendices: two lemmas of real analysis and metrics in the phase space

4.4.1 Glaeser-type inequalities

We start by recalling the Faà di Bruno formula on iterated derivatives of composition of functions:

Lemma 4.4.1 (Faà di Bruno formula). *Let $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two C^∞ functions. Then for all α and β in $\mathbb{N}^d$ there holds*

$$\frac{\partial_x^\alpha \partial_\xi^\beta (g \circ f)}{\alpha!\beta!} = \sum_{1 \le k \le |\alpha+\beta|} \frac{g^{(k)} \circ f}{k!} \sum_{\substack{(\alpha_1,\beta_1)+\cdots+(\alpha_k,\beta_k)=(\alpha,\beta) \\ (\alpha_j,\beta_j)\neq(0,0)}} \prod_{1\le j\le k} \frac{\partial_x^{\alpha_j}\partial_\xi^{\beta_j} f}{\alpha_j!\beta_j!} \tag{4.4.1}$$

We recall that for a d-tuple $\alpha_j = (\alpha_j(1), \dots, \alpha_j(d))$, we denote $\alpha_j! = \prod_{1\le p\le d} \alpha_j(p)!$, and $\partial_x^{\alpha_j}$ means $\partial_{x_1}^{\alpha_j(1)} \circ \cdots \circ \partial_{x_d}^{\alpha_j(d)}$. For further use, we denote

$$N(\alpha, k) = \left| \left\{ (\alpha_1, \dots, \alpha_k) \,\middle|\, \alpha_1 + \cdots + \alpha_k = \alpha \,,\, \alpha_j \ge 1 \right\} \right|. \tag{4.4.2}$$

By combinatorial arguments, we may prove

$$N(\alpha, k) = \binom{\alpha - 1}{k - 1}.$$

By putting $f(y) = y^n$ and $g(x) = e^x$ in the Faà di Bruno formula, we obtain

$$n^\alpha = \sum_{1\le k\le|\alpha|} \binom{n}{k} \sum_{\substack{\alpha_1+\cdots+\alpha_k=\alpha \\ \alpha_j\ge 1}} \binom{\alpha}{\alpha_1, \dots, \alpha_k}.$$

Next we recall the classical Glaeser inequality (see [Gla63]):

Lemma 4.4.2 (Global Glaeser inequality). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a non negative C^2 function, such that $\partial_x^2 f$ is bounded. Then*

$$|\partial_x f(x)|^2 \le 2|\partial_x^2 f|_{L^\infty(\mathbb{R}^n)} f(x) \,, \quad \forall x \in \mathbb{R}^n. \tag{4.4.3}$$

The local result (inequality holds at any point) comes from a global assumption on f (non negativity of f, boundedness of $\partial_x^2 f$). The constant $2|\partial_x^2 f|_{L^\infty(\mathbb{R}^n)}$ is optimal. The proof of the Lemma is classical, and is based on the integral Taylor expansion formula.

Local versions of the previous statement, that is with assumptions valid only in an open set of $\mathbb{R}^n$, also exist. For any $x_0 \in \mathbb{R}^d$ and $r > 0$, we denote

$$B_r(x_0) = \left\{ x \in \mathbb{R}^d \,:\, |x - x_0| < r \right\}.$$

In all the following, we consider $f : B_r(x_0) \to \mathbb{R}$ a nonnegative, C^2 function. We give first a sharp version of a local Glaeser's inequality, used in the present paper. The proof is straightforward and omitted.

Lemma 4.4.3 (Sharp local Glaeser inequality).
Assuming that

$$\min_{x\in\overline{B}_r(x_0)} f(x) > 0$$

then, for any $p > 0$ and any $r' < r$, there holds

$$|\partial_x f(x)|^p \le \frac{\left(|\partial_x f|_{L^\infty(\overline{B}_{r'}(x_0))}\right)^p}{\min_{\overline{B}_{r'}(x_0)} f} f(x) \,, \quad \forall x \in \overline{B}_{r'}(x_0). \tag{4.4.4}$$

Remark 4.4.4. *Note that in this case, the Glaeser constant does not depend a priori of the L^∞ norm of the second order derivatives of f. We may indeed think of polynomials of degree 2 which are locally bounded from below by a positive constant and have a positive discriminant.*

Using Lemma 4.4.3, we prove here Lemmas 4.3.1 and 4.3.2.

Lemma 4.4.5 (Glaeser inequality for a)**.** *Under Assumption* 4.2.1*, there is a neighborhood* $[0,T]\times B_r(x_0)$ *of* $(0,x_0)\in\mathbb{R}_t\times\mathbb{R}_x$ *and a constant* $C_{T,r}>0$ *for which there holds*

$$(\partial_x a(t,x))^2 \le C_{T,r}\, a(t,x) \quad , \quad \forall\,(t,x)\in[0,T]\times B_r(x_0). \tag{4.4.5}$$

Proof of Lemma 4.3.1. Thanks to Assumption 4.2.1, there holds

$$(\partial_x a)^2 = a(t,x)\left(\frac{4x^2}{t+x^2}e + (t+x^2)\frac{(\partial_x e)^2}{e} + 4x\partial_x e\right)$$

and the term $\left(\frac{4x^2}{t+x^2}e + (t+x^2)\frac{(\partial_x e)^2}{e} + 4x\partial_x e\right)$ is locally bounded thanks to Lemma 4.4.3. □

Lemma 4.4.6 (Derivatives of the symbol b)**.** *We recall first definition* (4.2.4) *of b:*

$$b(t,x,\xi) = \left(a(t,x) + \langle\xi\rangle^{-c}\right)^{-1/2}.$$

There is a bounded sequence of constants $C_{\alpha,\beta}>0$ for which there holds

$$|\partial_x^\alpha\partial_\xi^\beta b(t,x,\xi)| \le C_{\alpha,\beta}\widetilde{R}^{|\alpha+\beta|}\,\alpha!^s\beta!\,b(t,x,\xi)\,a_\sharp(t,x,\xi)^{-|\alpha|/2}\,\langle\xi\rangle^{-|\beta|} \quad , \quad \forall\,(\alpha,\beta)\in\mathbb{N}\times\mathbb{N} \tag{4.4.6}$$

for all (t,x) in $[0,T]\times B_r(x_0)$ and ξ in $\mathbb{R}$, and where $\widetilde{R}$ satisfies

$$\widetilde{R} = c(1)R. \tag{4.4.7}$$

Proof of Lemma 4.3.2. By the Faà di Bruno formula (Lemma 4.4.1) on iterated derivatives of composition of functions, using the fact that $\partial_x^\alpha\partial_\xi^\beta a_\sharp \equiv 0$ as soon as $|\alpha|>0$ and $|\beta|>0$, we deduce

$$\frac{1}{\alpha!\beta!}\,\partial_x^\alpha\partial_\xi^\beta\left(a_\sharp^{-1/2}\right)$$
$$= \sum_{\substack{1\le k\le|\alpha|\\ 1\le k'\le|\beta|}} \frac{c_k c_{k'}}{k!k'!}\,a_\sharp^{-1/2-k-k'}\left(\sum_{\substack{\alpha_1+\cdots+\alpha_k=\alpha\\ \alpha_j\ge1}}\prod_{j=1}^{k}\frac{1}{\alpha_j!}\partial_x^{\alpha_j}a\right)\times\left(\sum_{\substack{\beta_1+\cdots+\beta_{k'}=\beta\\ \beta_j\ge1}}\prod_{j=1}^{k'}\frac{1}{\beta_j!}\partial_\xi^{\beta_j}\langle\cdot\rangle^{-c}\right)$$

where coefficients $c_{k+k'}$ are defined by $\left(y^{-1/2}\right)^{(k)} = c_k y^{-1/2-k}$.

Next, there holds

$$\left|\frac{1}{\beta_j!}\partial_\xi^{\beta_j}\langle\xi\rangle^{-c}\right| \le \langle\xi\rangle^{-c-|\beta_j|}$$

as $c < 2$, hence

$$\left| \sum_{\substack{\beta_1+\cdots+\beta_{k'}=\beta \\ \beta_j \geq 1}} \prod_{j=1}^{k'} \frac{1}{\beta_j!} \partial_\xi^{\beta_j} \langle \cdot \rangle^{-c} \right| \leq N(\beta, k') \langle \cdot \rangle^{-k'c-|\beta|}$$

where we denote

$$N(\beta, k') = \left| \left\{ (\beta_1, \ldots, \beta_{k'}) \,|\, \beta_1 + \cdots + \beta_{k'} = \beta \,,\, \beta_j \geq 1 \right\} \right| .$$

Thanks to the the bound (4.3.3), there holds $a_\sharp^{-1} \leq \langle \cdot \rangle^c$, hence

$$a_\sharp^{-k'} \left| \sum_{\substack{\beta_1+\cdots+\beta_{k'}=\beta \\ \beta_j \geq 1}} \prod_{j=1}^{k'} \frac{1}{\beta_j!} \partial_\xi^{\beta_j} \langle \cdot \rangle^{-c} \right| \leq N(\beta, k') \langle \cdot \rangle^{-|\beta|}.$$

We focus now on the sum

$$\sum_{\substack{\alpha_1+\cdots+\alpha_k=\alpha \\ \alpha_j \geq 1}} \prod_{j=1}^{k} \frac{1}{\alpha_j!} \partial_x^{\alpha_j} a.$$

If $|\alpha_j| = 1$, we may use Lemma 4.3.1 to bound $\partial_x^{\alpha_j}$. We introduce then

$$I_1(\alpha_1, \ldots, \alpha_k) = \{ j : |\alpha_j| = 1 \}$$

and there holds

$$|\partial_x^{\alpha_j} a| \leq \left(C_{T,r}^{1/2} \, a_\sharp^{1/2} \right)^{\alpha_j} \quad , \quad \forall \, j \in I_1$$

thus

$$\begin{aligned} \left| \sum_{\substack{\alpha_1+\cdots+\alpha_k=\alpha \\ \alpha_j \geq 1}} \prod_{j=1}^{k} \frac{1}{\alpha_j!} \partial_x^{\alpha_j} a \right| &\leq \sum_{\substack{\alpha_1+\cdots+\alpha_k=\alpha \\ \alpha_j \geq 1}} \prod_{j \in I_1} \frac{1}{\alpha_j!} |\partial_x^{\alpha_j} a| \prod_{j \notin I_1} \frac{1}{\alpha_j!} |\partial_x^{\alpha_j} a| \\ &\leq \sum_{\substack{\alpha_1+\cdots+\alpha_k=\alpha \\ \alpha_j \geq 1}} \prod_{j \in I_1} \frac{1}{\alpha_j!} \left(C_{T,r}^{1/2} \, a_\sharp^{1/2} \right)^{\alpha_j} \prod_{j \notin I_1} \frac{1}{\alpha_j!} |\partial_x^{\alpha_j} a|. \end{aligned}$$

For indices not in I_1, that is for $|\alpha_j| \geq 2$, we use the fact that a is in G_R^s, hence

$$\left| \sum_{\substack{\alpha_1+\cdots+\alpha_k=\alpha \\ \alpha_j \geq 1}} \prod_{j=1}^{k} \frac{1}{\alpha_j!} \partial_x^{\alpha_j} a \right| \leq \sum_{\substack{\alpha_1+\cdots+\alpha_k=\alpha \\ \alpha_j \geq 1}} \prod_{j \in I_1} \frac{1}{\alpha_j!} \left(C_{T,r}^{1/2} \, a_\sharp^{1/2} \right)^{\alpha_j} \prod_{j \notin I_1} \frac{1}{\alpha_j!} |a|_{s,R} \, R^{|\alpha_j|} \alpha_j!^s.$$

As the k-tuple $(\alpha_1, \ldots, \alpha_k)$ satisfies $\alpha_1 + \cdots + \alpha_k = \alpha$, there holds $|\alpha_1| + \cdots + |\alpha_k| = |\alpha|$ hence

$$\begin{aligned} |I_1(\alpha_1, \ldots, \alpha_k)| &= |\alpha| - \sum_{j \notin I_1} |\alpha_j| \\ &\leq |\alpha| - 2(k - |I_1|) \end{aligned}$$

which leads to $|I_1| \geq 2k - |\alpha|$. As $a_\sharp \leq 1$, we get

$$\left| \sum_{\substack{\alpha_1+\cdots+\alpha_k=\alpha \\ \alpha_j \geq 1}} \prod_{j=1}^{k} \frac{1}{\alpha_j!} \partial_x^{\alpha_j} a \right| \leq a_\sharp^{k-|\alpha|/2} \sum_{\substack{\alpha_1+\cdots+\alpha_k=\alpha \\ \alpha_j \geq 1}} \prod_{j \in I_1} \frac{1}{\alpha_j!} C_{T,r}^{1/2} \prod_{j \notin I_1} \frac{1}{\alpha_j!} \alpha_j!^s |a|_{s,R} R^{|\alpha_j|}.$$

We need then to compare $C_{T,r}^{1/2}$ with $|a|_{s,R}R$. Assume that

$$C_{T,r}^{1/2} \leq |a|_{s,R} R. \tag{4.4.8}$$

There holds

$$\begin{aligned} \left| \sum_{\substack{\alpha_1+\cdots+\alpha_k=\alpha \\ \alpha_j \geq 1}} \prod_{j=1}^{k} \frac{1}{\alpha_j!} \partial_x^{\alpha_j} a \right| &\leq a_\sharp^{k-|\alpha|/2} |a|_{s,R}^k R^{|\alpha|} \sum_{\substack{\alpha_1+\cdots+\alpha_k=\alpha \\ \alpha_j \geq 1}} \prod_{j=1}^{k} \frac{1}{\alpha_j!} \alpha_j!^s \\ &\leq a_\sharp^{k-|\alpha|/2} \, \alpha!^{s-1} |a|_{s,R}^k R^{|\alpha|} \sum_{\substack{\alpha_1+\cdots+\alpha_k=\alpha \\ \alpha_j \geq 1}} \prod_{j=1}^{k} \binom{\alpha}{\alpha_1, \ldots, \alpha_k}^{1-s}. \end{aligned}$$

Denote

$$C_s(\alpha, k) = \sum_{\substack{\alpha_1+\cdots+\alpha_k=\alpha \\ \alpha_j \geq 1}} \prod_{j=1}^{k} \binom{\alpha}{\alpha_1, \ldots, \alpha_k}^{1-s}.$$

As $\binom{\alpha}{\alpha_1,\ldots,\alpha_k} \geq 1$ and $s \geq 1$, there holds

$$\begin{aligned} C_s(\alpha, k) &\leq N(\alpha, k) \\ &\leq \binom{\alpha - 1}{k - 1} \\ &\leq 2^{k|\alpha|}. \end{aligned}$$

We put altogether all the inequalities:

$$\begin{aligned} &\left| \frac{1}{\alpha!\beta!} \partial_x^\alpha \partial_\xi^\beta \left(a_\sharp^{-1/2} \right) \right| \\ &\leq \sum_{\substack{1 \leq k \leq |\alpha| \\ 1 \leq k' \leq |\beta|}} \frac{|c_k| \, |c_{k'}|}{k! k'!} \, a_\sharp^{-1/2-k} \, a_\sharp^{k-|\alpha|/2} \, \alpha!^{s-1} |a|_{s,R}^k R^{|\alpha|} \, C_s(\alpha, k) \, N(\beta, k') \, \langle \cdot \rangle^{-|\beta|} \\ &\leq a_\sharp^{-1/2-|\alpha|/2} \alpha!^{s-1} R^{|\alpha|} \, \langle \cdot \rangle^{-|\beta|} \sum_{\substack{1 \leq k \leq |\alpha| \\ 1 \leq k' \leq |\beta|}} \frac{|c_k| \, |c_{k'}|}{k! k'!} \, N(\beta, k') |a|_{s,R}^k \, C_s(\alpha, k) \\ &\leq b(t, x, \xi) \, a_\sharp^{-|\alpha|/2} \alpha!^{s-1} R^{|\alpha|} \, \langle \cdot \rangle^{-|\beta|} \left(\sum_{1 \leq k \leq |\alpha|} \frac{|c_k|}{k!} |a|_{s,R}^k \, C_s(\alpha, k) \right) \left(\sum_{1 \leq k' \leq |\beta|} \frac{|c_{k'}|}{k'!} \, N(\beta, k') \right). \end{aligned}$$

By definition of $c_{k'}$ there holds

$$\begin{aligned} c_{k'} &= \prod_{j=0}^{k-1}(-1/2-j) \\ &= (-1/2)^k \prod_{j=0}^{k-1}(2j+1) \\ &= \left(\frac{-1}{2}\right)^k \frac{(2k)!}{\prod_{j=0}^{k-1}(2(j+1))} \\ &= \left(\frac{-1}{4}\right)^k \frac{(2k)!}{k!} \end{aligned}$$

□

We note that a local statement can be deduced from Lemma 4.4.2, using a C^∞ nonnegative function φ with compact support $\overline{B}_r(x_0)$, and equals to 1 in $\overline{B}_{r'}(x_0)$ for some $r' < r$. We may then extend any locally defined, nonnegative function into a globally defined, nonnegative one.

We first introduce some notations. For any domain $\mathcal{D} \subset B_r(x_0)$ and $j \in \mathbb{N}$, we denote

$$M_j(f;\mathcal{D}) = \sup\left\{|\partial_x^\alpha f| \,:\, x \in \mathcal{D}\,,\, |\alpha| = j\right\}.$$

For any $0 < r' < r$ we define

$$\mathcal{C}_{r',r}(x_0) = \left\{x \in \mathbb{R}^d \,:\, r' < |x - x_0| < r\right\}.$$

Lemma 4.4.7 (Local Glaeser inequality)**.** *Let $f : B_r(x_0) \to \mathbb{R}$ be a nonnegative C^2 function. Then*

$$|\partial_x f(x)|^2 \le G(f;x_0,r',r)f(x)\,, \quad \forall x \in \overline{B}_{r'}(x_0) \tag{4.4.9}$$

for any $r' < r$. The local Glaeser's constant $G(f;x_0,r',r)$ is defined by

$$G(f;x_0,r',r) = 2M_2(f;B_r(x_0)) + \frac{4}{r-r'}M_1\left(f;\mathcal{C}_{r',r}(x_0)\right) + \frac{4}{(r-r')^2}M_0\left(f;\mathcal{C}_{r',r}(x_0)\right). \tag{4.4.10}$$

We do not use Lemma 4.4.7 here, but include it since it may prove useful in further work on weakly hyperbolic systems.

Proof. Let φ be a C^∞ function with compact support $\overline{B}_r(x_0)$, satisfying also $0 \le \varphi \le 1$ and $\varphi(x) = 1$ for all $x \in \overline{B}_{r'}(x_0)$. Then the function $f\varphi$ satisfies the conditions for applying Lemma 4.4.2. Hence (4.4.3) leads to

$$|(f\varphi)'(x)|^2 \le 2M_2(f\varphi;\mathbb{R}^n)f(x)\varphi(x)$$

for all x in $\mathbb{R}^n$. As φ is identically one in $\overline{B}_{r'}(x_0)$, there holds

$$|f'(x)|^2 \le 2M_2(f\varphi;\mathbb{R}^n)f(x)$$

for all $x \in \overline{B}_{r'}(x_0)$.

To end the proof we have to give an upper bound of $M_2(f\varphi;\mathbb{R}^n)$, with respect to the distance $r-r'$. First there holds

$$M_2(f\varphi;\mathbb{R}^n) \leq M_2(f;B_r(x_0)) + 2M_1\left(f;\mathcal{C}_{r',r}(x_0)\right) M_1(\varphi;\mathbb{R}^n) + M_0\left(f;\mathcal{C}_{r',r}(x_0)\right) M_2(\varphi;\mathbb{R}^n).$$

Second, for any $x_{r'}$ such that $|x_{r'}| = r'$ we denote

$$x_r = x_0 + \frac{r}{r'}(x_{r'} - x_0)$$

the only point of $\overline{B}_r(x_0)$ such that $|x_r - x_0| = r$ and $x_{r'}$ is in the interval $[x_0, x_r]$. By the mean value theorem there is $s \in [0,1]$ such that

$$\varphi(x_r) - \varphi(x_{r'}) = (x_r - x_{r'}) \cdot \partial_x\varphi\left(x_0 + s\frac{r}{r'}(x_{r'} - x_0)\right)$$

thus

$$\left|\partial_x\varphi\left(x_0 + s\frac{r}{r'}(x_{r'} - x_0)\right)\right| = \frac{1}{r-r'}$$

as $\varphi(x_r) = 0$, $\varphi(x_{r'}) = 1$ and $|x_r - x_{r'}| = r - r'$ and then

$$M_1(\varphi) \geq \frac{1}{r-r'}.$$

By the same way we can prove also that

$$M_2(\varphi) \geq \frac{2}{(r-r')^2}.$$

To end the proof, it suffices to construct φ such that the previous lower bound are equalities. □

Remark 4.4.8. *In the estimate* (4.4.10) *appears the distance $r - r'$. In the worst case, it is the distance between the neighborhood of x_0 such that the Glaeser inequality holds, and the possible point $\tilde{x}$ such that $f(\tilde{x}) = 0$ and $\partial_x f(\tilde{x}) \neq 0$, at which Glaser inequality fails.*

For example, let take $f(x) = x$ in $[0, +\infty[$. Then $f'(x) = 1$ and there holds, for any $x_0 > 0$:

$$(f'(x))^2 \leq C(x_0)f(x)\ , \quad \forall\, x \in [x_0, x_0+1]$$

with $C(x_0) = 1/x_0$. By comparison, the constant $G(f;r',r)$ of the previous Lemma verifies

$$\begin{aligned} G(f;r',r) &\leq M_2(f;[x_0,x_0+1]) + \frac{2}{x_0}M_1(f;[0,x_0]) + \frac{2}{x_0^2}M_0(f;[0,x_0]) \\ &\lesssim \frac{1}{x_0} \end{aligned}$$

as $M_0(f;[0,x_0]) \leq x_0$.

4.4.2 Metrics in the phase space and pseudodifferential calculus

We follow here Chapter 2 of [Ler11]. Let g be a general metric on the phase space $\mathbb{R}^d \times \mathbb{R}^d$, that is

$$g_{x,\xi} = \frac{|\mathrm{d}x|^2}{\varphi(x,\xi)^2} + \frac{|\mathrm{d}\xi|^2}{\Phi(x,\xi)^2}.$$

For a positive function M on $\mathbb{R}^d \times \mathbb{R}^d$, we introduce the classes of symbols associated to the metric g:

Definition 4.4.9 (Definition of classes of symbols)**.** *The space of symbols $S(M,g)$ is defined as the set of C^∞ functions on $\mathbb{R}^d \times \mathbb{R}^d$ such that, for all (α,β) in $\mathbb{N}^d \times \mathbb{N}^d$, there is $C_{\alpha,\beta} > 0$ such that*

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq C_{\alpha,\beta} M(x,\xi)\varphi(x,\xi)^{-|\alpha|}\Phi(x,\xi)^{-|\beta|}.$$

By example, the standard classes of symbols can be expressed as $S^m_{\rho,\delta} = S\left(\langle\xi\rangle^m, \frac{|\mathrm{d}x|^2}{\langle\xi\rangle^{-2\delta}} + \frac{|\mathrm{d}\xi|^2}{\langle\xi\rangle^{2\rho}}\right)$.

We recall next the algebra property of general classes of symbols $S(M,g)$. Let M_1 and M_2 be both admissible weights for the metric g.

Lemma 4.4.10. *For any $f_j \in S(M_j,g)$ with $j = 1,2$, there holds*

$$f_1 f_2 \in S(M_1 M_2, g).$$

The proof is straightforward, using Leibniz formula and Definition 4.4.9.

We now state Theorem 2.3.7 in [Ler11], concerning the composition of operators with symbols in $S(M,g)$. For two symbols a_1 and a_2, we denote $a_1 \sharp a_2$ the symbol satisfying

$$\mathrm{op}(a_1)\mathrm{op}(a_2) = \mathrm{op}(a_1 \sharp a_2).$$

We introduce also

$$\lambda_g = \varphi\Phi.$$

Lemma 4.4.11 (Composition)**.** *Let g be an admissible metric on $\mathbb{R}^d \times \mathbb{R}^d$ (see Definition 2.2.15), M_1 and M_2 two admissible weights for g, and $a_j \in S(M_j,g)$. Then for all ν in $\mathbb{N}$ there holds*

$$a_1 \sharp a_2 - \left(\sum_{0\leq k<\nu} 2^{-k} \sum_{|\alpha|+|\beta|=k} \frac{(-i)^{|\beta|}}{\alpha!\beta!} \partial_\xi^\beta \partial_x^\alpha a \partial_\xi^\alpha \partial_x^\beta b\right) \in S(M_1 M_2 \lambda_g^{-\nu}, g).$$

We recall Theorem 2.5.1 of [Ler11].

Lemma 4.4.12 (Action)**.** *Let g be an admissible metric on $\mathbb{R}^d \times \mathbb{R}^d$, and $f \in S(1,g)$. Then $\mathrm{op}(f)$ acts continuously on L^2.*

Chapter 5

On the action of pseudo-differential operators on Gevrey spaces

5.1 Introduction

This paper is devoted to the study of a class of pseudo-differential operators acting in Gevrey spaces. We assume that the operators have symbols which are Gevrey regular in the spatial variable $x \in \mathbb{R}^d$ and satisfy estimates in (x, ξ) derivatives which are analogous to the ones enjoyed by symbols of the classical classes $S^m_{\rho,\delta}$. These symbols are precisely defined in Section 5.2. We give three results:

- The first, Proposition 5.3.1, states that if a function F belongs to $H^m\mathcal{G}^\sigma_\tau$ (a Gevrey space with Sobolev correction, defined in Section 5.2), then the operator $e^{\tau D^\sigma} F e^{-\tau D^\sigma}$ acts continuously in $H^m(\mathbb{R}^d)$. This result appeared in slightly different form in the article [BMM16] by Bedrossian, Masmoudi and Mouhot (see Lemma 3.3 therein). Its proof relies on a para-product decomposition and precise triangle-like inequalities in the spirit of [BMM16].

- Our second and main result, Theorem 8, describes the action of operators with symbols in $\mathrm{S}^0_{\rho,\delta}\mathrm{G}^s_R$ (classical symbols with Gevrey regularity, defined in Section 5.2) on Gevrey spaces. The proof relies again on a para-product decomposition.

- The third result is Lemma 5.5.1. Here we give precise bounds for the symbol of $e^{\tau D^\sigma}\mathrm{op}(p)e^{-\tau D^\sigma}$, where p belongs to $\mathrm{S}^0_{\rho,0}\mathrm{G}^s_R$. This completes Lemma 7.1 of the article [CNR] by Colombini, Nishitani and Rauch.

A classical reference on Gevrey spaces is Rodino's book [Rod93]. See also the paper [HR01] by Hua and Rodino, where slightly less general classes of symbols are studied. Questions about the action of pseudo-differential operators in Gevrey spaces naturally arise from the study of the Gevrey well-posedness of the Cauchy problem for first-order systems. The aforementioned article [CNR] focuses on Gevrey well-posedness, and so does our own line of research [Mor16a], [Mor16b], [Mor17a] and [Mor17b].

5.2 Classes of Gevrey regular symbols

5.2.1 Gevrey spaces

We start by two definitions of Gevrey spaces, one on the spatial side, the other on the Fourier side.

Definition 5.2.1 (Gevrey spaces: the spatial viewpoint). *For any $s \in [1, \infty)$ and $R > 0$, we define G^s_R to be the space of smooth functions f such that, for any compact set K of $\mathbb{R}^d$, there is a constant C_K for which there holds*

$$|\partial^\alpha_x f|_{L^\infty(K)} \leq C_K R^{|\alpha|} |\alpha!|^s \quad , \quad \forall \alpha \in \mathbb{N}^d. \tag{5.2.1}$$

We call R^{-1} the Gevrey radius and s the Gevrey (regularity) index.

For B a compact set of $\mathbb{R}^d$, we define $G^s_R(B)$ the space of functions f compactly supported on B and being in G^s_R. The associated norm is defined by

$$|f|_{s,R} = \sup_{\alpha \in \mathbb{N}^d} |\partial^\alpha_x f|_{L^\infty(B)} \left(R^{|\alpha|} |\alpha!|^s \right)^{-1}. \tag{5.2.2}$$

Definition 5.2.2 (Gevrey spaces: the Fourier viewpoint). *For any $\sigma \in (0, 1]$ and $\tau > 0$, we define $\mathcal{G}^\sigma_\tau$ to be the space of functions $f \in L^2$ such that $\exp(\tau \langle \cdot \rangle^\sigma) \hat{f}$ is in L^2. The associated norm is defined by*

$$|f|_{\sigma,\tau} = \left| e^{\tau \langle \cdot \rangle^\sigma} \widehat{f} \right|_{L^2}. \tag{5.2.3}$$

We call τ the Gevrey radius and σ the Gevrey (regularity) index.

Both previous definitions of Gevrey functions are linked, as shown by the following classical result (see [Rod93]):

Proposition 5.2.3 ($G^s_R(B)$ is included in $\mathcal{G}^{1/s}_\tau$). *For any compact set B of $\mathbb{R}^d$, the space $G^s_R(B)$ is included in the space $\mathcal{G}^\sigma_\tau$ for $\sigma = 1/s$ and $\tau < sR^{-1/s}$. Moreover there holds*

$$|u|_{\sigma,\tau} \leq |B|^{1/2} C \left(\tau s^{-1} R^{1/s} \right) |u|_{s,R}$$

with

$$C(y) \lesssim \frac{1}{1-y} P\left(\frac{y}{1-y} \right) \tag{5.2.4}$$

where P is a polynomial with degree at most $\lceil (3s-1)/2 \rceil$, and the implicit constant depends only on the Gevrey index s.

Proof. First we write $|u|_{\sigma,\tau} = e^\tau \left| e^{\tau(\langle \xi \rangle^\sigma - 1)} \widehat{u}(\xi) \right|_{L^2}$ and there holds

$$|u|_{\sigma,\tau} \leq e^\tau \sum_{n \geq 0} \frac{\tau^n}{n!} \left| (\langle \xi \rangle^\sigma - 1)^n \widehat{u}(\xi) \right|_{L^2}.$$

Next we compute $\langle\xi\rangle^\sigma - 1 = \sigma\int_0^1 \langle t\xi\rangle^{\sigma-2} t|\xi|^2 dt$ which implies, as $\sigma - 2 < 0$, that

$$\begin{aligned}\langle\xi\rangle^\sigma - 1 &\leq \sigma\int_0^1 (t|\xi|)^{\sigma-2}\, t|\xi|^2 dt\\ &\leq |\xi|^\sigma \int_0^1 \sigma t^{\sigma-1} dt\\ &\leq |\xi|^\sigma.\end{aligned}$$

Thus there holds

$$|u|_{\sigma,\tau} \leq e^\tau \sum_{n\geq 0} \frac{\tau^n}{n!}\, \left||\xi|^{n\sigma}\widehat{u}(\xi)\right|_{L^2}.$$

Let n be given in the following. By the support of u and inequalities (5.2.1), there holds

$$\left||\xi|^{n\sigma}\widehat{u}(\xi)\right|_{L^2} \leq |B|^{1/2}\, |u|_{s,R} R^m m!^s$$

for any $m \geq n\sigma$. Thus

$$\frac{\tau^n}{n!}\, \left||\xi|^{n\sigma}\widehat{u}(\xi)\right|_{L^2} \leq \frac{m!^s}{n!} |B|^{1/2}\, |u|_{s,R} \tau^n R^m.$$

Using Stirling's formula with $m < n\sigma + 1$, there is $\delta > 0$ such that

$$\begin{aligned}\frac{m!^s}{n!} &\leq (1+\delta)^s \frac{1}{n!}\left(\frac{n\sigma+1}{e}\right)^{(n\sigma+1)s} (2\pi(n\sigma+1))^{s/2}\\ &\leq (1+\delta)^{2s}\left(\frac{n\sigma+1}{e}\right)^{(n\sigma+1)s}\left(\frac{n}{e}\right)^{-n} (2\pi(n\sigma+1))^{s/2}\,(2\pi n)^{-1/2}.\end{aligned}$$

As $s\sigma = 1$, there holds

$$\left(\frac{n\sigma+1}{e}\right)^{(n\sigma+1)s}\left(\frac{n}{e}\right)^{-n} \leq \sigma^n (n\sigma+1)^s e^{1/\sigma - s}.$$

This implies finally

$$\frac{m!^s}{n!} \lesssim \sigma^n n^{(3s-1)/2}$$

hence

$$\frac{\tau^n}{n!}\, \left||\xi|^{n\sigma}\widehat{u}(\xi)\right|_{L^2} \lesssim |B|^{1/2}\, |u|_{s,R}\, n^{(3s-1)/2}\, (\sigma\tau R^\sigma)^n.$$

It now suffices to sum in $n \in \mathbb{N}$. □

We recall here some useful inequalities when dealing with Gevrey spaces. We define

$$\langle\xi\rangle = (1+|\xi|^2)^{1/2} \quad , \quad \forall\, \xi \in \mathbb{R}^d. \tag{5.2.5}$$

Lemma 5.2.4.

1. *Let $\sigma \in (0,1)$, ξ and η in $\mathbb{R}^d$ such that $|\xi - \eta| \leq \frac{1}{K}|\eta|$ for some $K > 1$. Then*

$$|\langle\xi\rangle^\sigma - \langle\eta\rangle^\sigma| \leq (K^\sigma - (K-1)^\sigma)\,\langle\xi - \eta\rangle^\sigma. \tag{5.2.6}$$

   Note that $K^\sigma - (K-1)^\sigma < 1$ for any $K > 1$.

2. *Let $\sigma \in (0,1)$, ξ and η in $\mathbb{R}^d$ such that $\frac{1}{K}|\xi - \eta| \leq |\eta| \leq K|\xi - \eta|$ for some $K > 1$. Then*

$$\langle\xi\rangle^\sigma \leq \langle\eta\rangle^\sigma + c'\langle\xi - \eta\rangle^\sigma \tag{5.2.7}$$

   for some $c' \in (0,1)$ depending on K.

3. *For any $\xi \in \mathbb{R}^d$, $\sigma \in (0,1)$, $\tau > 0$ and $m \geq 0$, there holds*

$$\langle\xi\rangle^m \lesssim \tau^{-m/\sigma} e^{\tau\langle\xi\rangle^\sigma} \tag{5.2.8}$$

Remark 5.2.5. *Note that the first point in the previous Lemma does not hold when $\sigma = 1$, i.e. in the analytic regularity.*

Proof. Denote $f(t) = \langle\eta + t(\xi - \eta)\rangle^\sigma$. The function f is differentiable on $[0,1]$, and there holds

$$f(1) - f(0) = \sigma(\xi - \eta) \cdot \int_0^1 (\eta + t(\xi - \eta))\langle\eta + t(\xi - \eta)\rangle^{\sigma-2} dt.$$

As $|\xi - \eta| \leq \frac{1}{K}|\eta|$, there holds $|\eta + t(\xi - \eta)| \geq (K - t)|\xi - \eta|$ hence

$$\begin{aligned} |f(1) - f(0)| &\leq \sigma|\xi - \eta| \int_0^1 \langle\eta + t(\xi - \eta)\rangle^{\sigma-1} dt \\ &\leq \sigma\langle\xi - \eta\rangle^\sigma \int_0^1 (K - t)^{\sigma-1} dt \end{aligned}$$

which is (5.2.6). We now turn to the proof of (5.2.7), from $|\eta| \geq K^{-1}|\xi - \eta|$ we deduce

$$1 + |\eta|^2 \geq 1 + K^{-2}|\xi - \eta|^2 = K^{-2}\left(K^2 + |\xi - \eta|^2\right) \geq K^{-2}\langle\xi - \eta\rangle^2$$

where we used $K > 1$. Thus, since $0 < \sigma$,

$$\langle\eta\rangle^\sigma \geq K^{-\sigma}\langle\xi - \eta\rangle^\sigma.$$

This implies

$$\langle\eta\rangle^\sigma + \langle\xi - \eta\rangle^\sigma \geq \left(1 + K^{-\sigma}\right)\langle\xi - \eta\rangle^\sigma.$$

Now assume in addition

$$|\xi| \leq c|\xi - \eta|, \quad \text{for some } c > 0. \tag{5.2.9}$$

If (5.2.9) holds with some $c \geq 1$, then it holds a fortiori with $c > 1$. Thus we may assume (5.2.9) for some $c > 1$, and then

$$\langle\xi\rangle \leq c\langle\xi - \eta\rangle,$$

so that

$$\langle\xi\rangle^\sigma \le c^\sigma \langle\xi-\eta\rangle^\sigma \le c^\sigma \left(1+K^{-\sigma}\right)^{-1}\left(\langle\eta\rangle^\sigma + \langle\xi-\eta\rangle^\sigma\right).$$

Thus we are done if (5.2.9) holds with

$$1 < c < \left(1+K^{-\sigma}\right)^{1/\sigma}.$$

Otherwise, there holds

$$|\xi| \ge \left(1+K^{-\sigma}\right)^{1/\sigma}|\xi-\eta| =: \widetilde{c}\,|\xi-\eta|,$$

and since $\widetilde{c} > 1$, we may then apply (5.2.6). This yields

$$\langle\xi\rangle^\sigma \le \langle\eta\rangle^\sigma + \left(\widetilde{c}^\sigma - (\widetilde{c}-1)^\sigma\right)\langle\xi-\eta\rangle^\sigma$$

and the result follows. The proof of (5.2.8) is trivial, hence omitted.

□

Remark 5.2.6. *Inequality* (5.2.6) *is somehow similar to inequality (3.11) in* [BMM16], *which we reproduce here:*

$$|\langle\xi\rangle^\sigma - \langle\eta\rangle^\sigma| \le \frac{\sigma}{(K-1)^{1-\sigma}}\langle\xi-\eta\rangle^\sigma$$

Note that the coefficient $\frac{\sigma}{(K-1)^{1-\sigma}}$ *may be strictly greater than* 1. *Inequality* (5.2.7) *is similar to inequality (3.12) in* [BMM16], *which we reproduce here:*

$$\langle\xi\rangle^\sigma \le \left(\frac{\langle\eta\rangle^\sigma}{\langle\xi\rangle^\sigma}\right)^{1-\sigma}\left(\langle\eta\rangle^\sigma + \langle\xi-\eta\rangle^\sigma\right)$$

for $|\eta| \ge |\xi-\eta|$. *Again, the coefficient* $\left(\frac{\langle\eta\rangle^\sigma}{\langle\xi\rangle^\sigma}\right)^{1-\sigma}$ *may be strictly greater than 1.*

5.2.2 Classes of symbols

We define a class of symbols $a(x,\xi)$ with Gevrey regularity in the spatial variable x.

Definition 5.2.7 (Class of symbols with Gevrey regularity)**.** *For* $s \in (1,\infty)$ *and* $R > 0$, *for* $m \in \mathbb{R}$, ρ *and* δ *such that* $0 < \delta < \rho \le 1$, *we define* $\mathrm{S}^m_{\rho,\delta}\mathrm{G}^s_R$ *to be the class of symbols* $a(x,\xi)$ *for which there is a bounded sequence of positive numbers* $C_{\alpha,\beta}$ *such that*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta} R^{|\alpha+\beta|}|\alpha!|^s|\beta!|\langle\xi\rangle^{m-\rho|\beta|+\delta|\alpha|} \tag{5.2.10}$$

uniformly in $x \in \mathbb{R}^d$ *and* $\xi \in \mathbb{R}^d$. *We denote*

$$|a|_{\alpha,\beta} = \sup_{(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d}\left|R^{-|\alpha+\beta|}|\alpha!|^{-s}|\beta!|^{-1}\langle\xi\rangle^{-m+\rho|\beta|-\delta|\alpha|}\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right|. \tag{5.2.11}$$

Remark 5.2.8. *Note that the space* G^s_R *is naturally in* $\mathrm{S}^0_{0,0}\mathrm{G}^s_R$, *with*

$$|a|_{\alpha,\beta} \le |a|_{s,R} \quad , \quad \forall\,(\alpha,\beta) \in \mathbb{Z}^d \times \mathbb{Z}^d.$$

Moreover spaces $\mathrm{S}^m_{\rho,\delta}\mathrm{G}^s_R$ *are naturally embedded in* $S^m_{\rho,\delta}$.

Remark 5.2.9. *A way to look at inequalities* (5.2.10) *is to put together the Gevrey term* $R^{|\alpha|}|\alpha!|^s$ *and the typical pseudo-differential term* $\langle\xi\rangle^{\delta|\alpha|}$, *which means that* $a(\cdot,\xi)$ *is in* $G^s_{R\langle\xi\rangle^\delta}$ *for all* $\xi \in \mathbb{R}^d$: *the Gevrey radius in* x *of the symbol decreases with* $|\xi|$ *if* $\delta > 0$.

5.3 Conjugation of a Gevrey function

We consider the Gevrey conjugation operator of a function F in $\mathcal{G}^\sigma_\tau$ with $\tau \geq 0$, and we denote

$$F^{(\tau)} = e^{\tau D^\sigma} F\, e^{-\tau D^\sigma} \tag{5.3.1}$$

where $D = \mathrm{op}(\langle\cdot\rangle)$.

Proposition 5.3.1. *Assume that $D^m F \in \mathcal{G}^\sigma_\tau$ for some $m \geq 0$. Then, for any $v \in H^m$, there holds*

$$\left|F^{(\tau)} v\right|_{H^m} \lesssim |D^m F|_{\sigma,\tau} |v|_{L^2} + |F|_{\sigma,\tau} |v|_{H^m}.$$

Remark 5.3.2. *This implies in particular that $H^m\mathcal{G}^\sigma_\tau$, the space of Gevrey functions with Sobolev correction of order m, is an algebra for any $m \geq 0$. With $m = 0$ in proposition 5.3.1, we see that if F is in $\mathcal{G}^\sigma_\tau$, then $F^{(\tau)}$ operates in L^2.*

Proof. In Fourier there holds

$$\mathcal{F}\left(D^m F^{(\tau)} v\right)(\xi) = \int_\eta e^{\tau\langle\xi\rangle^\sigma - \tau\langle\eta\rangle^\sigma} \langle\xi\rangle^m \widehat{F}(\xi-\eta)\widehat{v}(\eta) d\eta. \tag{5.3.2}$$

We use here a paraproduct decomposition and Lemma 5.2.4. Let $K > 1$. We divide the integral in $\eta \in \mathbb{R}^d$ in three frequency regions, defined as $\mathcal{R}_1 = \left\{\eta : |\xi-\eta| \leq \frac{1}{K}|\eta|\right\}$, $\mathcal{R}_2 = \left\{\eta : |\eta| \leq \frac{1}{K}|\xi-\eta|\right\}$ and $\mathcal{R}_3 = \left\{\eta : \frac{1}{K}|\xi-\eta| < |\eta| < K|\xi-\eta|\right\}$. We consider then each region successively:

• The case where $|\xi-\eta| \leq \frac{1}{K}|\eta|$: thanks to inequality (5.2.6) in Lemma 5.2.4, there is $c \in (0,1)$ such that $\langle\xi\rangle^\sigma - \langle\eta\rangle^\sigma \leq c\langle\xi-\eta\rangle^\sigma$, hence

$$e^{\tau\langle\xi\rangle^\sigma - \tau\langle\eta\rangle^\sigma} \leq e^{c\tau\langle\xi-\eta\rangle^\sigma}.$$

Besides, in the region under consideration, the Sobolev term satisfies $\langle\xi\rangle^m \lesssim \langle\eta\rangle^m$. This implies that

$$\begin{aligned}
&\int_{\eta\in\mathcal{R}_1} e^{\tau\langle\xi\rangle^\sigma - \tau\langle\eta\rangle^\sigma} \langle\xi\rangle^m \left|\widehat{F}(\xi-\eta)\right| |\widehat{v}(\eta)|\, d\eta \\
&\lesssim \int_{\eta\in\mathcal{R}_1} e^{c\tau\langle\xi-\eta\rangle^\sigma} \langle\eta\rangle^m \left|\widehat{F}(\xi-\eta)\right| |\widehat{v}(\eta)|\, d\eta \\
&\lesssim \int_{\eta\in\mathcal{R}_1} e^{-\tau(1-c)\langle\xi-\eta\rangle^\sigma} e^{\tau\langle\xi-\eta\rangle^\sigma} \left|\widehat{F}(\xi-\eta)\right| \langle\eta\rangle^m |\widehat{v}(\eta)|\, d\eta.
\end{aligned}$$

We use next Young's inequality to obtain

$$\begin{aligned}
&\left|\int_{\eta\in\mathcal{R}_1} e^{-\tau(1-c)\langle\xi-\eta\rangle^\sigma} e^{\tau\langle\xi-\eta\rangle^\sigma} \left|\widehat{F}(\xi-\eta)\right| \langle\eta\rangle^m |\widehat{v}(\eta)|\, d\eta\right|_{L^2_\xi} \\
&\lesssim \left|e^{-\tau(1-c)\langle\cdot\rangle^\sigma} e^{\tau\langle\cdot\rangle^\sigma} \left|\widehat{F}(\cdot)\right|\right|_{L^1} |v|_{H^m} \\
&\lesssim \left|e^{-\tau(1-c)\langle\cdot\rangle^\sigma}\right|_{L^2} |F|_{\sigma,\tau} |v|_{H^m}
\end{aligned}$$

using the extra Gevrey weight $e^{-\tau(1-c)\langle\xi-\eta\rangle^\sigma}$.

• The case where $|\eta| \leq \frac{1}{K}|\xi-\eta|$: thanks to inequality (5.2.6) in Lemma 5.2.4, there is $c \in (0,1)$ such that $\langle\xi\rangle^\sigma - \langle\xi-\eta\rangle^\sigma \leq c\langle\eta\rangle^\sigma$, hence

$$e^{\langle\xi\rangle^\sigma - \langle\xi-\eta\rangle^\sigma} \leq e^{c\langle\eta\rangle^\sigma}.$$

Besides, in the region under consideration, the Sobolev term satisfies $\langle\xi\rangle^m \lesssim \langle\xi-\eta\rangle^m$. This implies that

$$\begin{aligned}
&\int_{\eta\in\mathcal{R}_2} e^{\tau\langle\xi\rangle^\sigma - \tau\langle\eta\rangle^\sigma} \langle\xi\rangle^m \left|\widehat{F}(\xi-\eta)\right| |\widehat{v}(\eta)|\, d\eta \\
&\lesssim \int_{\eta\in\mathcal{R}_2} e^{\tau\langle\xi\rangle^\sigma - \tau\langle\eta\rangle^\sigma - \tau\langle\xi-\eta\rangle^\sigma} \langle\xi-\eta\rangle^m e^{\tau\langle\xi-\eta\rangle^\sigma} \left|\widehat{F}(\xi-\eta)\right| |\widehat{v}(\eta)|\, d\eta \\
&\lesssim \int_{\eta\in\mathcal{R}_2} e^{-\tau(1-c)\langle\eta\rangle^\sigma} \langle\xi-\eta\rangle^m\, e^{\tau\langle\xi-\eta\rangle^\sigma} \left|\widehat{F}(\xi-\eta)\right| \, |\widehat{v}(\eta)|\, d\eta.
\end{aligned}$$

We use next Young's inequality to obtain

$$\begin{aligned}
&\left|\int_{\eta\in\mathcal{R}_2} e^{-\tau(1-c)\langle\eta\rangle^\sigma} \langle\xi-\eta\rangle^m\, e^{\tau\langle\xi-\eta\rangle^\sigma} \left|\widehat{F}(\xi-\eta)\right| \, |\widehat{v}(\eta)|\, d\eta\right|_{L^2_\xi} \\
&\lesssim \left|e^{-\tau(1-c)\langle\cdot\rangle^\sigma} \, |\widehat{v}(\eta)|\right|_{L^1} |D^m F|_{\sigma,\tau} \\
&\lesssim \left|e^{-\tau(1-c)\langle\cdot\rangle^\sigma}\right|_{L^2} |D^m F|_{\sigma,\tau} \, |v|_{L^2}
\end{aligned}$$

using the extra Gevrey weight $e^{-\tau(1-c)\langle\xi-\eta\rangle^\sigma}$.

• The case where $\frac{1}{K}|\xi-\eta| < |\eta| < K|\xi-\eta|$: thanks to inequality (5.2.7) in Lemma 5.2.4, there is $c' \in (0,1)$ such that $\langle\xi\rangle^\sigma \leq c'\langle\xi-\eta\rangle^\sigma + \langle\eta\rangle^\sigma$, hence

$$e^{\tau\langle\xi\rangle^\sigma - \tau\langle\eta\rangle^\sigma} \leq e^{c'\langle\xi-\eta\rangle^\sigma}.$$

Besides, in the region under consideration, the Sobolev term satisfies $\langle\xi\rangle^m \lesssim \langle\eta\rangle^m + \langle\xi-\eta\rangle^m$ where the implicit constant depends on m, thus

$$\begin{aligned}
&\int_{\eta\in\mathcal{R}_3} e^{\tau\langle\xi\rangle^\sigma - \tau\langle\eta\rangle^\sigma} \langle\xi\rangle^m \left|\widehat{F}(\xi-\eta)\right| |\widehat{v}(\eta)|\, d\eta \\
&\lesssim \int_{\eta\in\mathcal{R}_3} e^{-(1-c')\tau\langle\xi-\eta\rangle^\sigma} \left(\langle\eta\rangle^m + \langle\xi-\eta\rangle^m\right) e^{\tau\langle\xi-\eta\rangle^\sigma} \left|\widehat{F}(\xi-\eta)\right| |\widehat{v}(\eta)|\, d\eta \\
&\lesssim \int_{\eta\in\mathcal{R}_3} e^{-(1-c')\tau\langle\xi-\eta\rangle^\sigma} e^{\tau\langle\xi-\eta\rangle^\sigma} \left|\widehat{F}(\xi-\eta)\right| \, \langle\eta\rangle^m \, |\widehat{v}(\eta)|\, d\eta \\
&\quad + \int_{\eta\in\mathcal{R}_3} e^{-(1-c')\tau\langle\xi-\eta\rangle^\sigma} \langle\xi-\eta\rangle^m e^{\tau\langle\xi-\eta\rangle^\sigma} \left|\widehat{F}(\xi-\eta)\right| |\widehat{v}(\eta)|\, d\eta
\end{aligned}$$

We use next Young's inequality to obtain

$$\begin{aligned}
&\left|\int_{\eta\in\mathcal{R}_3} e^{\tau\langle\xi\rangle^\sigma - \tau\langle\eta\rangle^\sigma} \langle\xi\rangle^m \left|\widehat{F}(\xi-\eta)\right| |\widehat{v}(\eta)|\, d\eta\right|_{L^2_\xi} \\
&\lesssim \left|e^{-(1-c')\tau\langle\cdot\rangle^\sigma}\right|_{L^2} \left(|D^m F|_{\sigma,\tau} \, |v|_{L^2} + |F|_{\sigma,\tau} \, |v|_{H^m}\right)
\end{aligned}$$

The result follows from (5.3.2), viewed as an integral over $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$.

□

5.4 Action of pseudo-differential operators on Gevrey spaces

In this Section, we consider symbols in $\mathrm{S}^0_{\rho,\delta}\mathrm{G}^s_R$ with compact support B of $\mathbb{R}^d_x$, uniformly in $\xi \in \mathbb{R}^d$. Note that for such a symbol a and any sufficiently smooth u, there holds

$$\mathrm{op}(a)u(x) = C_p u(x) \quad , \quad \forall\, x \in \mathbb{R}^d \setminus B \tag{5.4.1}$$

where we denote

$$\mathrm{op}(a)u(x) = \int e^{ix\cdot\xi} a(x,\xi)\widehat{u}(\xi)d\xi$$

the standard quantization, which we use in all the following. Equality (5.4.1) implies in particular that if u has compact support, containing B, then $\mathrm{op}(a)u$ has also compact support, contained in the compact support of u.

This additional assumption on the support of the symbol allows to use Proposition 5.2.3, parlaying the spatial Gevrey regularity into a Fourier Gevrey regularity for $a(\cdot,\xi)$. We may then use an adapted paraproduct decomposition to prove the continuous action of operators with symbols in $\mathrm{S}^0_{\rho,\delta}\mathrm{G}^s_R$. First we prove this result in the particular case $\rho = 1$, $\delta = 0$.

Theorem 7 (Action of $S^0_{1,0}G^s_R$ on $\mathcal{G}^\sigma_\tau$)**.** *Let $s \in (1,\infty)$ and $R > 0$. Let a be in $\mathrm{S}^0_{1,0}\mathrm{G}^s_R$, constant outside a compact set B of $\mathbb{R}^d_x$, uniformly in $\xi \in \mathbb{R}^d$. Then for any $\tau < sR^{-1/s}$ and $\sigma = \frac{1}{s}$ the operator* $\mathrm{op}(a)$ *acts continuously on $\mathcal{G}^\sigma_\tau$ with norm*

$$\|\mathrm{op}(a)\|_{\mathcal{L}(\mathcal{G}^\sigma_\tau)} \leq |B|^{1/2} C\left(\tau s^{-1} R^{1/s}\right) \sup_{\alpha\in\mathbb{N}^d} |a|_{\alpha,0}$$

where $C\left(\tau s^{-1}R^{1/s}\right)$ is defined in (5.2.4).

Proof. First, for fixed $\xi \in \mathbb{R}^d$, as $a(\cdot,\xi)$ is in G^s_R with compact support, Proposition 5.2.3 implies that $\widehat{a}(\cdot,\xi)$, the Fourier transform with respect to x of $a(\cdot,\xi)$, is in $\mathcal{G}^\sigma_\tau$ uniformly in $\xi \in \mathbb{R}^d$, with $\sigma = 1/s$ and $\tau < sR^{-1/s}$. That is, we may write

$$\widehat{a}(\zeta,\xi) = F_\xi(\zeta),$$

where for fixed $\xi \in \mathbb{R}^d$, $F_\xi(\cdot)$ belongs to $\mathcal{G}^\sigma_\tau$ with the uniform (in ξ) bound

$$|F_\xi(\cdot)|_{\sigma,\tau} \leq |B|^{1/2} C\left(\tau s^{-1} R^{1/s}\right) |a(\cdot,\xi)|_{s,R}.$$

thanks to Proposition 5.2.3. By definitions (5.2.11) of the semi-norms in Definition 5.2.7, there holds

$$|F_\xi(\cdot)|_{\sigma,\tau} \leq |B|^{1/2} C\left(\tau s^{-1} R^{1/s}\right) \sup_{\alpha\in\mathbb{N}^d} |a|_{\alpha,0}. \tag{5.4.2}$$

Let u be in G^σ_τ. The Fourier transform of $\mathrm{op}(p)u$ is

$$\int_\eta \widehat{a}(\xi-\eta,\xi)\hat{u}(\eta)d\eta.$$

Denoting $v(\eta)=e^{\tau\langle\eta\rangle^\sigma}\hat{u}(\eta)$, there holds

$$e^{\tau\langle\xi\rangle^\sigma}\mathcal{F}\left(\mathrm{op}(a)u\right)(\xi)=\int_\eta e^{\tau\langle\xi\rangle^\sigma-\tau\langle\eta\rangle^\sigma}F_\xi(\xi-\eta)v(\eta)d\eta$$

Proposition 5.3.1 now yields the result, since the bound (5.4.2) is uniform in ξ. □

In the general case $0<\delta<\rho\leq 1$, Remark 5.2.9 indicates a potential obstruction for the Gevrey index. This is made precise in the following

Theorem 8 (Action of $S^0_{\rho,\delta}G^s_R$ on $\mathcal{G}^\sigma_\tau$)**.** *Let $s\in(1,\infty)$, $R>0$ and $0<\delta<\rho\leq 1$. Let a be in $\mathrm{S}^0_{\rho,\delta}\mathrm{G}^s_R$, constant outside a compact set B of $\mathbb{R}^d_x$, uniformly in $\xi\in\mathbb{R}^d$. Then for any*

$$\sigma\leq(1-\delta)/s\quad\textit{and}\quad\tau'<\tau<sR^{-1/s}$$

the operator $\mathrm{op}(a)$ *acts continuously from $\mathcal{G}^\sigma_\tau$ into $\mathcal{G}^\sigma_{\tau'}$ with norm*

$$||\mathrm{op}(a)||_{\mathcal{L}(\mathcal{G}^\sigma_\tau,\mathcal{G}^\sigma_{\tau'})}\lesssim|B|^{1/2}C\left(\tau s^{-1}R^{1/s}\right)\sup_{\alpha\in\mathbb{N}^d}|a|_{\alpha,0}$$

where $C\left(\tau s^{-1}R^{1/s}\right)$ is defined in (5.2.4).

Proof. First, for fixed $\xi\in\mathbb{R}^d$, as $a(\cdot,\xi)$ is in G^s_R with compact support, Proposition 5.2.3 and Remark 5.2.9 implies that $\widehat{a}(\cdot,\xi)$, the Fourier transform with respect to x of $a(\cdot,\xi)$, is in $\mathcal{G}^\sigma_{\tau\langle\xi\rangle^{-\delta/s}}$ uniformly in $\xi\in\mathbb{R}^d$, with $\sigma=1/s$ and $\tau<sR^{-1/s}$. That is, we may write

$$e^{\tau\langle\xi\rangle^{-\delta/s}\langle\zeta\rangle^{1/s}}\hat{a}(\zeta,\xi)=F_\xi(\zeta),$$

where for fixed $\xi\in\mathbb{R}^d$, $F_\xi(\cdot)$ belongs to L^2 with the uniform (in ξ) bound

$$|F_\xi(\cdot)|_{L^2}\leq|B|^{1/2}C\left(\tau s^{-1}R^{1/s}\right)|a(\cdot,\xi)|_{s,R}.$$

thanks to Proposition 5.2.3. By definitions (5.2.11) of the semi-norms in Definition 5.2.7, there holds

$$|F_\xi(\cdot)|_{\sigma,\tau}\leq|B|^{1/2}C\left(\tau s^{-1}R^{1/s}\right)\sup_{\alpha\in\mathbb{N}^d}|a|_{\alpha,0}. \tag{5.4.3}$$

Let u be in $\mathcal{G}^\sigma_\tau$. Denoting $v(\eta)=e^{\tau\langle\eta\rangle^\sigma}\hat{u}(\eta)$, there holds

$$e^{\tau'\langle\xi\rangle^\sigma}\mathcal{F}\left(\mathrm{op}(a)u\right)(\xi)=\int_\eta e^{\tau'\langle\xi\rangle^\sigma-\tau\langle\eta\rangle^\sigma-\tau\langle\xi\rangle^{-\delta/s}\langle\xi-\eta\rangle^{1/s}}F_\xi(\xi-\eta)v(\eta)d\eta$$

We now decompose the integral into three regions, as in the proof of Proposition 5.3.1. Once we derive appropriate bounds on the exponential factor

$$W(\tau';\tau):=\exp\left(\tau'\langle\xi\rangle^\sigma-\tau\langle\eta\rangle^\sigma-\tau\langle\xi\rangle^{-\delta/s}\langle\xi-\eta\rangle^{1/s}\right)$$

the result follows from (5.4.3) by application of Young's inequality, as in the proof of Proposition 5.3.1. Thus we focus only on the above exponential factor. Here the multiplicative coefficient $K > 1$ is chosen in terms of τ and τ' .

• The case where $|\xi - \eta| \leq \frac{1}{K}|\eta|$: here the weight $\langle\xi\rangle^{-\delta/s}$ in the Gevrey radius of $\hat{a}(\cdot, \xi)$ is small, and

$$W(\tau'; \tau) \leq \exp\left(\tau'\langle\xi\rangle^{\sigma} - \tau\langle\eta\rangle^{\sigma}\right).$$

With (5.2.6) there holds

$$W(\tau'; \tau) \leq \exp\left(\tau'\left(K^{\sigma} - (K-1)^{\sigma}\right)\langle\xi - \eta\rangle^{\sigma} - (\tau - \tau')\langle\eta\rangle^{\sigma}\right)$$

Using now $|\xi - \eta| \leq \frac{1}{K}|\eta|$, there holds $\langle\xi - \eta\rangle^{\sigma} \leq \langle\eta\rangle^{\sigma}$, so that

$$W(\tau'; \tau) \leq \exp\left(-\left(\tau - (1 + (K^{\sigma} - (K-1)^{\sigma}))\tau'\right)\langle\eta\rangle^{\sigma}\right).$$

For K large enough, depending only on τ' and τ, with $\tau' < \tau$, there holds

$$\tau - (1 + (K^{\sigma} - (K-1)^{\sigma}))\tau',$$

thus $|W|_{L^2_\eta} < \infty$.

• The case where $|\eta| \leq \frac{1}{K}|\xi - \eta|$: since $|\eta| \leq K^{-1}|\xi - \eta|$, there holds inequality $|\xi| \leq (1 + K^{-1})|\xi - \eta|$, hence $\langle\xi\rangle \leq (1 + K^{-1})\langle\xi - \eta\rangle$. Thus, with $\sigma \leq (1 - \delta)/s$, we find the bound

$$W(\tau'; \tau) \leq \exp\left(\tau'\langle\xi\rangle^{\sigma} - \tau\langle\eta\rangle^{\sigma} - \tau\left(1 + 1/K\right)^{-\delta/s}\langle\xi - \eta\rangle^{(1-\delta)/s}\right).$$

Using inequality (5.2.6), this implies

$$\begin{aligned} W(\tau'; \tau) \quad \leq \quad & \exp\left(-(\tau - (K^{\sigma} - (K-1)^{\sigma})\,\tau')\langle\eta\rangle^{\sigma}\right) \\ & \times \exp\left(-\left(\tau\left(1 + 1/K\right)^{-\delta/s} - \tau'\right)\langle\xi - \eta\rangle^{(1-\delta)/s}\right) \end{aligned}$$

Since $K < 1$ and $\tau > \tau'$, there holds $\tau - (K^{\sigma} - (K-1)^{\sigma})\,\tau' > 0$. Thus

$$W(\tau'; \tau) \leq \exp\left(-\left(\tau\left(1 + 1/K\right)^{-\delta/s} - \tau'\right)\langle\xi - \eta\rangle^{(1-\delta)/s}\right)$$

and if K is large enough, depending only on τ and τ', there holds $\tau\left(1 + 1/K\right)^{-\delta/s} - \tau' > 0$. Thus $|W|_{L^2_\eta} < \infty$.

• The case where $\frac{1}{K}|\xi - \eta| < |\eta| < K|\xi - \eta|$: here we use inequality (5.2.7), which implies, since $\sigma \leq (1 - \delta)/s$,

$$W(\tau'; \tau) \leq \exp\left(-(\tau - \tau')\langle\eta\rangle^{\sigma}\right)\exp\left(-\left(\tau\langle\xi\rangle^{-\delta/s} - \tau'\langle\xi - \eta\rangle^{-\delta/s}\right)\langle\xi - \eta\rangle^{1/s}\right),$$

where $c' = c'(K) \in (0, 1)$. Since $|\xi| \leq (1 + K)|\xi - \eta|$ in the region under consideration, hence $\langle\xi\rangle \leq (1 = K)\langle\xi - \eta\rangle$, this implies

$$W \leq \exp\left(-\left(\tau(1 + K)^{-\delta/s} - \tau'\right)\langle\xi - \eta\rangle^{(1-\delta)/s}\right)$$

thus if K is large enough, depending only on τ and τ', there holds $|W|_{L^2_\eta} < \infty$.

□

5.5 A conjugation Lemma for operators

We consider here a symbol a in $\mathrm{S}^m_{\rho,0}\mathrm{G}^s_R$ for $\rho \in [0,1]$, $R > 0$ and $s \in (1,\infty)$, with compact support B of $\mathbb{R}^d_x$, uniformly in $\xi \in \mathbb{R}^d$. It is known (see Lemma 7.1 in [CNR]) that there is a symbol $\widetilde{a}$ such that

$$\mathrm{op}\,(\widetilde{a}) = \mathrm{op}(a)^{(\tau)} = e^{\tau D^\sigma}\mathrm{op}(a)e^{-\tau D^\sigma} \tag{5.5.1}$$

and which satisfies

$$\widetilde{a}(x,\xi) = \int_{y,\eta} e^{-i\eta\cdot y} e^{\tau\langle\xi+\eta\rangle^\sigma - \tau\langle\xi\rangle^\sigma} a(x+y,\xi)dyd\eta. \tag{5.5.2}$$

In Proposition 2.1 in [CNR], the symbol $\widetilde{a}$ is proved to be in $S^m_{1,0}$ for small τ. We extend here the result for all $|\tau| < sR^{-1/s}$, with in addition an estimate of the semi-norms of the symbol.

Lemma 5.5.1. *Given a in $\mathrm{S}^m_{\rho,0}\mathrm{G}^s_R$, for any $|\tau| < sR^{-1/s}$, the symbol defined by* (5.5.2) *is in $\mathrm{S}^m_{1,0}$. Moreover, for any $\underline{\tau} \in (|\tau|, sR^{-1/s})$, for any α, β in $\mathbb{N}^d$ there holds*

$$\sup_{x\in B,\,\xi\in\mathbb{R}^d}\left|\langle\xi\rangle^{-m+|\beta|}\,\partial^\alpha_x\partial^\beta_\xi\widetilde{a}(x,\xi)\right| \lesssim |B|^{1/2}C\left(\underline{\tau}s^{-1}R^{1/s}\right)\sup_{\alpha\in\mathbb{N}^d}|a|_{\alpha,\beta}\left(\underline{\tau}-|\tau|\right)^{-(2|\beta|+|\alpha|)/\sigma}$$

where constant C is defined in (5.2.4).

Proof. We compute the derivatives of the symbol $\widetilde{a}$. There holds

$$\begin{aligned}
&\partial^\alpha_x\partial^\beta_\xi\widetilde{a}(x,\xi)\\
&= \sum_{\beta_1+\beta_2=\beta}\binom{\beta}{\beta_1,\beta_2}\int_{y,\eta} e^{-i\eta\cdot y}\,\partial^{\beta_1}_\xi\left(e^{\tau\langle\xi+\eta\rangle^\sigma-\tau\langle\xi\rangle^\sigma}\right)\partial^\alpha_x\partial^{\beta_2}_\xi a(x+y,\xi)\,dyd\eta\\
&= \sum_{\beta_1+\beta_2=\beta}\binom{\beta}{\beta_1,\beta_2}\int_{\eta} e^{i\eta\cdot x}\,\partial^{\beta_1}_\xi\left(e^{\tau\langle\xi+\eta\rangle^\sigma-\tau\langle\xi\rangle^\sigma}\right)(i\eta)^\alpha\partial^{\beta_2}_\xi\hat{a}(\eta,\xi)\,d\eta.
\end{aligned} \tag{5.5.3}$$

We use now the fact that a is in $\mathrm{S}^m_{\rho,0}\mathrm{G}^s_R$ with compact support B in $\mathbb{R}^d_x$, uniformly in ξ. Thanks to Proposition 5.2.3, we may write

$$e^{\underline{\tau}\langle\eta\rangle^\sigma}\,\langle\xi\rangle^{-m+\rho|\beta_2|}\,\partial^{\beta_2}_\xi\hat{a}(\eta,\xi) = F_{\xi,\beta_2}(\eta),$$

where for fixed $\xi \in \mathbb{R}^d$ and $\beta_2 \in \mathbb{N}^d$, F_{ξ,β_2} is in L^2_η with bound

$$|F_{\xi,\beta_2}|_{L^2_\eta} \lesssim |B|^{1/2}C\left(\underline{\tau}s^{-1}R^{1/s}\right)\sup_{\alpha\in\mathbb{N}^d}|a|_{\alpha,\beta_2}$$

uniformly in $\xi \in \mathbb{R}^d$ and $\beta_2 \in \mathbb{N}^d$, and for all $\underline{\tau} < sR^{-1/s}$. The semi-norms of a are defined in (5.2.11). Next, as proved in the course of Proposition 2.1 in [CNR], there holds

$$\left|\partial^{\beta_1}_\xi\left(e^{\tau\langle\xi+\eta\rangle^\sigma-\tau\langle\xi\rangle^\sigma}\right)\right| \lesssim \langle\xi\rangle^{-|\beta_1|}\,\langle\eta\rangle^{2|\beta_1|}\,e^{\tau\langle\xi+\eta\rangle^\sigma-\tau\langle\xi\rangle^\sigma}.$$

This is proved using Faà di Bruno formula (see Lemma 5.1 in [Mor17a]) and inequality $\partial_\xi^{\beta_1}\left(\langle\xi+\eta\rangle^\sigma-\langle\xi\rangle^\sigma\right)\lesssim\langle\xi\rangle^{-|\beta_1|}\langle\eta\rangle^{2|\beta_1|}$. The integral in (5.5.3) satisfies thus

$$\begin{aligned}
&\left|\int_\eta e^{i\eta\cdot x}\,\partial_\xi^{\beta_1}\left(e^{\tau\langle\xi+\eta\rangle^\sigma-\tau\langle\xi\rangle^\sigma}\right)\,(i\eta)^\alpha\partial_\xi^{\beta_2}\hat{a}(\eta,\xi)\,d\eta\right|\\
&\lesssim\int_\eta\left|\partial_\xi^{\beta_1}\left(e^{\tau\langle\xi+\eta\rangle^\sigma-\tau\langle\xi\rangle^\sigma}\right)\right|\,|\eta|^\alpha e^{-\underline{\tau}\langle\eta\rangle^\sigma}\,\langle\xi\rangle^{m-\rho|\beta_2|}\,|F_{\xi,\beta_2}(\eta)|\,d\eta\\
&\lesssim\langle\xi\rangle^{m-|\beta_1|-\rho|\beta_2|}\int_\eta e^{\tau\langle\xi+\eta\rangle^\sigma-\tau\langle\xi\rangle^\sigma-\underline{\tau}\langle\eta\rangle^\sigma}\,\langle\eta\rangle^{2|\beta_1|+|\alpha|}\,|F_{\xi,\beta_2}(\eta)|\,d\eta.
\end{aligned}$$

Next, we use inequality (5.2.8) in Lemma 5.2.4 to get

$$\langle\eta\rangle^{2|\beta_1|+|\alpha|}\lesssim(\underline{\tau}-\tau)^{-(2|\beta_1|+|\alpha|)/\sigma}\,e^{(\underline{\tau}-\tau)\langle\eta\rangle^\sigma}$$

hence

$$\begin{aligned}
&\int_\eta e^{\tau\langle\xi+\eta\rangle^\sigma-\tau\langle\xi\rangle^\sigma-\underline{\tau}\langle\eta\rangle^\sigma}\,\langle\eta\rangle^{2|\beta_1|+|\alpha|}\,|F_{\xi,\beta_2}(\eta)|\,d\eta\\
&\lesssim(\underline{\tau}-\tau)^{-(2|\beta_1|+|\alpha|)/\sigma}\int_\eta e^{\tau\langle\xi+\eta\rangle^\sigma-\tau\langle\xi\rangle^\sigma-\tau\langle\eta\rangle^\sigma}\,|F_{\xi,\beta_2}(\eta)|\,d\eta
\end{aligned}$$

and we conclude using the proof of Proposition 5.3.1.

□

We recall also the asymptotic expansion of $\widetilde{a}$, as given in Proposition 2.1 in [CNR].

Lemma 5.5.2 (Asymptotic expansion of $\widetilde{a}$)**.** *For any $k\in\mathbb{N}$ there holds*

$$\widetilde{a}(x,\xi)=\sum_{|\alpha|\le k}\frac{i^\alpha}{\alpha!}\partial_x^\alpha a(x,\xi)\,(\tau\partial_\xi\langle\xi\rangle^\sigma)^\alpha+R \tag{5.5.4}$$

with R in $\mathrm{S}_{1,0}^{\max\{m-(k+1)(1-\sigma),m-2+\sigma\}}$.

This result is used in particular in our forthcoming papers [Mor17a] and [Mor17b].

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