Cauchy problem for the incompressible Navier-Stokes equation with an external force and Gevrey smoothing effect for the Prandtl equation

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Abstract

This thesis deals with equations of fluid dynamics. We consider the following two models: one is the Navier-Stokes equation in $\mathbb{R}^3$ with an external force, the other one is the Prandtl equation on the half plane $\mathbb{R}^2_+$. For the Navier-Stokes system, we focus on the local in time existence, uniqueness, long-time behavior and blow-up criterion. For the Prandtl equation on the half-plane, we consider the Gevrey regularity.

This thesis consists in four chapters. In the first chapter, we introduce some background on equations of fluid dynamics and recall the physical meaning of the above two models as well as some well-known mathematical results. Next, we state our main results and motivations briefly. At last we mention some open problems.

The second chapter is devoted to the Cauchy problem for the Navier-Stokes equation equipped with a small rough external force in $\mathbb{R}^3$. We show the local in time existence for this system for any initial data belonging to a critical Besov space with negative regularity. Moreover we obtain three kinds of uniqueness results for the above solutions. Finally, we study the long-time behavior and stability of priori global solutions.

The third chapter deals with a blow-up criterion for the Navier-Stokes equation with a time independent external force. We develop a profile decomposition for the forced Navier-Stokes equation. The decomposition enables us to connect the forced and the unforced equations, which provides the blow-up information from the unforced solution to the forced solution.

In Chapter 4, we study the Gevrey smoothing effect of the local in time solution to the Prandtl equation in the half plane. It is well-known that the Prandtl boundary layer equation is unstable for general initial data, and is well-posed in Sobolev spaces for monotonic initial data. Under a monotonicity assumption on the tangential velocity of the outflow, we prove Gevrey regularity for the solution to Prandtl equation in the half plane with initial data belonging to some Sobolev space.

Key words: Navier-Stokes equations, Prandtl equations, Gevrey space, Blow-up criterion

Abstract

Cette thèse est consacrée à l'etude des équations de la dynamique des fluides. La théorie des fluides est basée sur une hypothèse de continuum qui indique que le comportement macroscopique d'un fluide est le même que si il était complètement continu : la densité, la pression, la température et la vitesse sont supposées être infiniment petites et ils varient continuellement d'un point à un autre.

Nous considérons les deux modèles suivants:

1. Le premier est l'équation de Navier-Stokes homogène et incompressible $\mathbb{R}^3$ avec une force externe, qui décrit un fluide Newtonien, isotrope, homogène et incompressible dans tout l'espace $\mathbb{R}^3$,

$$(NSf)\begin{cases} \partial_t u_f - \Delta u_f + u_f \cdot \nabla u_f = -\nabla p_f + f, \\ \nabla \cdot u_f = 0, \\ u_f|_{t=0} = u_0. \end{cases} \tag{1}$$

Ici u_f est un champ de vecteurs à trois composants $u_f = (u_{f,1}, u_{f,2}, u_{f,3})$ représentant la vitesse du fluide, p_f est une fonction scalaire dénotant la pression, et toutes sont des fonctions inconnues de la variable d'espace $x \in \mathbb{R}^3$ et de la variable de temps $t > 0$.

2. L'autre est l'équation de Prandtl sur le demi-plan $\mathbb{R}^2_+$, qui est un modèle classique pour résoudre le problème de la couche limite. Pour simplifier, nous considérons l'équation de Prandtl sur le demi-plan avec un écoulement uniforme:

$$\begin{cases} u_t + uu_x + vu_y - u_{yy} = 0, \quad (t,x,y) \in]0,T[\times\mathbb{R}^2_+, \\ u_x + v_y = 0, \\ u|_{y=0} = v|_{y=0} = 0, \ \lim\limits_{y\to+\infty} u = 1, \\ u|_{t=0} = u_0(x,y)\,, \end{cases} \tag{2}$$

où u et v représentent la vitesse tangente, qui est inconnue, et la vitesse normale respectivement et v peut être représenté par u comme suit

$$v = -\int_0^y (\partial_x u)\, d\tilde{y}.$$

Pour le système Navier-Stokes, nous nous concentrons sur l'existence locale, l'unicité, le comportement à long terme et le critère de blow-up. Pour l'équation de Prandtl sur le demi-plan, nous considérons la régularité de Gevrey.

Cette thèse consiste en quatre chapitres. Dans le premier chapitre, nous introduisons quelques informations sur les équations de la dynamique des fluides et rappelons la signification physique des deux modèles ci-dessus ainsi que quelques résultats mathématiques bien connus. Ensuite, nous exposons brièvement nos principaux résultats et motivations. Enfin, nous mentionnons quelques problèmes ouverts.

Le deuxième chapitre est consacré au problème de Cauchy pour l'équation de Navier-Stokes avec une petite force externe rugueuse dans $\mathbb{R}^3$. La rugosité signifie que la force externe satisfait

$$\int_0^t e^{(t-s)\Delta}\mathbb{P}f(s)ds \in L^\infty(\mathbb{R}_+, L^{3,\infty})$$

qui contient beaucoup de cas singuliers intéressants bien connus. Nous montrons l'existence locale en temps pour ce système pour toutes les données initiales appartenant à un espace de Besov critique avec une régularité négative. Pour obtenir ce résultat, nous introduisons une équation de perturbation par rapport à (NSf) et sous une hypothèse de petitesse sur f nous prouvons le résultat d'existence ci-dessus en appliquant un argument de point fixe sur l'équation de perturbation. De plus, la solution peut être décomposée en une partie petite et une partie lisse. En utilisant un résultat de régularité via une itération, nous montrons que $u_f \in C_w([0,T^*), L^{3,\infty})$ et la partie lisse a une énergie finie quand $u_0 \in \dot{B}^{s_p}_{p,p} \cap L^{3,\infty}$ et $u_0 \in \dot{B}^{s_p}_{p,p} \cap L^2(\mathbb{R}^3)$ respectivement.

Nous obtenons également trois types de résultats d'unicité pour les solutions ci-dessus. Nous soulignons que sans l'hypothèse de petitesse sur la solution nous ne pouvons pas prouver que la solution ci-dessus est unique dans $L^\infty_t(L^{3,\infty})$. En fait, même pour (NS) l'unicité dans $L^\infty_t(L^{3,\infty})$ est toujours une problème ouvert (l'unicité ne tient que lorsque la solution est petite dans $L^\infty_t(L^{3,\infty})$). La raison pour laquelle nous nous concentrons sur l'unicité des solutions à (NSf) dans $L^\infty_t(L^{3,\infty})$ est que la singularité de la force externe limite la régularité des solutions. Dans notre cas, peu importe la régularité des données initiales, la solution correspondante à (NSf) appartient seulement à $L^\infty_t(L^{3,\infty})$. Cependant, nous montrons que la solution construite au chapitre 2 est unique dans le sens suivant: soit $u_f \in C_w([0,T^*), L^{3,\infty})$ une solution à (NSf) avec des données initiales $u_0 \in L^{3,\infty} \cap \dot{B}^{s_p}_{p,p}$ et $\bar{u}_f$ une autre solution à (NSf) avec les mêmes données initiales. On a

- si $\bar{u}_f - u_f$ se compose d'une petite partie et d'une partie lisse, alors $u_f \equiv \bar{u}_f$,
- si $\bar{u}_f - u_f \in C([0,T], L^{3,\infty})$, alors $u_f \equiv \bar{u}_f$,
- si $\bar{u}_f - u_f$ a une énergie finie et $3 < p < 5$, alors $u_f \equiv \bar{u}_f$.

Enfin, nous étudions le comportement à long terme et la stabilité d'une solution globale a priori. Nous montrons que si la solution u_f construite ci-dessus est globale, ce qui signifie que u_f appartient à $C_w(\mathbb{R}_+, L^{3,\infty})$, alors u_f appartient à $L^\infty(\mathbb{R}_+, L^{3,\infty})$. Ce résultat est valable pour des solutions globales à priori (sans hypothèse de petitesse sur les solutions) et pour les petites forces rugueuses (en particulier, on traite le cas $\Delta^{-1} f \sim \frac{1}{|x|}$). Ce résultat est un argument faible-fort de C. Calderón. Cependant, contrairement au cas non forcé, il est difficile d'obtenir que la perturbation ait une énergie locale dans le temps. Pour surmonter cette difficulté, nous introduisons un résultat de régularité via itération, qui satisfait une équation de perturbation plus générale. Nous prouvons aussi que u_f est stable.

Le troisième chapitre traite d'un critère de blow-up pour l'équation de Navier-Stokes avec une force externe indépendante du temps. Plus précisément, si $\Delta^{-1} f$ est petit dans L^3, alors pour tous $u_0 \in L^3$,

$$(BC) \qquad \sup_{0<t<T^*(u_0,f)} \|u_f(t,\cdot)\|_{L^3} < \infty \Rightarrow T^*(u_0,f) = \infty.$$

Ici $T^*(u_0, f)$ est la durée de vie de u_f. Nous remarquons que le problème principal est que l'unicité rétrograde de la chaleur n'est pas valide pour les équations de Navier-Stokes forcées. Par conséquent, pour obtenir le critère d'explosion, nous ne pouvons pas suivre le même argument. Nous développons une décomposition en profils pour l'équation de Navier-Stokes.

Puisque le critère de blow-up pour (NS) est connu, nous nous concentrons sur la facon de prouver (NSf) à partir de (NS). Nous utilisons une décomposition en profils pour les solutions à (NSf) pour prouver le résultat ci-dessus. Précisément, la décomposition permet de construire un lien entre l'équation forcée et l'équation non forcée, qui fournit l'information de la solution non forcée à la solution forcée. Plus précisément, nous pouvons décomposer u_f sous une forme constituée de la somme des profils de solutions à (NS), une solution à (NSf) et un reste. Nous montrons que l'information de blow-up de u_f est déterminée par l'information de blow-up des profils de solutions à (NS) par un argument utilisant la propriété scaling de ces solutions.

Nous soulignons également que l'on peut obtenir une décomposition en profils des solutions à l'équation de Navier-Stokes forcée avec une force externe $f \in \mathcal{L}^r(\mathbb{R}_+, \dot{B}^{s_p+\frac{2}{r}-2}_{p,p})$ avec $s_p + \frac{2}{r} > 0$ et des données initiales qui sont bornées dans $\dot{B}^{s_p}_{p,p}$ pour chaque $3 < p < \infty$. Et par le même argument que la preuve de (BC), on peut montrer le critère de blow-up comme (BC) en remplaçant L^3 par $\dot{B}^{s_p}_{p,p}$.

Au chapitre 4, nous étudions l'effet de lissage de Gevrey de la solution locale de l'équation de Prandtl dans le demi-plan. En raison de la dégénérescence de la variable tangentielle, les théories du caractère bien posé et la justification de la théorie de la couche limite de Prandtl demeurent des problèmes complexes dans la théorie mathématique de la mécanique des fluides.

Il est bien connu que l'équation de la couche limite de Prandtl est instable pour les données initiales générales et est bien posée dans les espaces de Sobolev pour les données initiales monotones. Sous une hypothèse de monotonicité sur la vitesse tangentielle du flux sortant, nous montrons la régularité de Gevrey pour la solution de l'équation de Prandtl dans le demi-plan avec des données initiales appartenant à un certain espace de Sobolev.

Il est bien connu que la difficulté principale pour l'équation de Prandtl est la dégénérescence en variable x, en raison de la présence de v:

$$v = -\int_0^y (\partial_x u)\, d\tilde{y}.$$

Pour surmonter cette dégénérescence, nous utilisons l'idée d'annulation pour effectuer des estimations sur une nouvelle fonction et plus sur la fonction u. En effet, on observe que

$$u_t + uu_x + vu_y - u_{yy} = 0,$$

et, avec $\omega = \partial_y u$,

$$\omega_t + u\omega_x + v\omega_y - \omega_{yy} = 0.$$

Pour éliminer le terme v dans le membre de gauche des deux équations ci-dessus, nous utilisons la condition de monotonicité $\partial_y u = \omega > 0$ et donc multiplions la deuxième équation par $-\frac{\partial_y \omega}{\omega}$, puis ajoutons l'équation résultante à la première; cela donne, dénotant $f = \omega - \frac{\partial_y \omega}{\omega} u$,

$$f_t + u\partial_x f - \partial_{yy} f = \text{termes de l'ordre inférieur.}$$

Notre observation principale pour la nouvelle équation est la structure subelliptique intrinsèque due à la condition de monotonie. En effet, dénotant $X_0 = \partial_t + u\partial_x$ et $X_1 = \partial_y$, nous pouvons réécrire l'équation ci-dessus à partir du type de Hörmander:

$$\Big(X_0 + X_1^* X_1\Big) f = \text{termes de l'ordre inférieur.}$$

Et en outre, un calcul direct nous montre que

$$\big[X_1,\ X_0\big] = (\partial_y u)\partial_x. \tag{3}$$

Ainsi la condition de crochet de Hörmander sera remplie formellement, fournie par $\partial_y u > 0$.

D'autre part par $\partial_y u > 0$, on a l'estimation sub-elliptique suivante:

$$\forall\, w \in C_0^\infty(K), \quad \|\Lambda^{1/3} w\|_{L^2} \lesssim \|\Big(X_0 + X_1^* X_1\Big) w\|_{L^2} + \|w\|_{L^2},$$

avec K un sous-ensemble compact de $\mathbb{R}^3_{t,x,y}$ et $\Lambda^d = \Lambda^d_x$ est le multiplicateur de Fourier du symbole $(|\xi|^2+1)^{\frac{d}{2}}$ par rapport à $x \in \mathbb{R}$.

Mots clefs: système de Navier-Stokes, critères d'explosion, l'équation de Prandtl, régularité Gevrey

Contents

Chapter 1

Introduction

In this thesis we focus on two topics: one concerns the solutions to the incompressible homogeneous Navier-Stokes equation with an external force in $\mathbb{R}^3$; the other one concerns the Gevery class smoothing effect for the Prandtl equation on the half-plane.

1.1 The Navier-Stokes equations

1.1.1 Physical meaning

Fluid theory is based on a continuum hypothesis which states that the macroscopic behavior of a fluid is the same as if the fluid were perfectly continuous: the density, pressure, temperature and velocity are taken to be well-defined at infinitely small points and are assumed to vary continuously from one point to another.

There are two representations of the fluid motion and of the associated physical quantities. In the Eulerian reference frame, the reference frame is fixed while the fluid moves. Thus the quantities are measured at a position x attached to the fixed frame. The velocity $u(t,x)$ is the velocity at time t of the fluid parcel that occupies the position x at that very instant t. In the Lagrangian reference frame, the reference frame is the initial state of the fluid. The quantities are attached to the parcels as they move.

More precisely, if $X_{x_0}(t)$ is the position of the parcel at time t whose position at time 0 was x_0, and if Q is some quantity attached to the parcels, we have two descriptions of the distribution of the values taken by Q at time t: the value $Q(t,x)$ taken at time t for the parcel which is located at that time at position x, and $Q_{x_0}(t)$ the value taken at time t for the parcel which was located at time 0 at position x_0. In particular, the velocity field $u(t,x)$ describes the velocities of the parcels as they move: $\frac{d}{dt}X_{x_0}(t) = u(t, X_{x_0}(t))$. This gives us the link between the variations of $Q_{x_0}(t)$ and those of $Q(t,x)$:

$$\frac{d}{dt}Q_{x_0}(t) = \partial_t Q(t,x)|_{x=X_{x_0}(t)} + \sum_{i=1}^{3} \partial_i Q(t,x)|_{x=X_{x_0}(t)} \frac{d}{dt}X_{x_0,i}(t).$$

The quantity $\frac{d}{dt}Q_{x_0}(t)$ is called the material derivative of Q and is denoted as $\frac{D}{Dt}Q$. Thus we have the following formula: **the material derivative**

$$\frac{D}{Dt}Q = \partial_t Q(t,x) + \sum_{i=1}^{3} u_i(t,x)\partial_i Q(t,x).$$

The convection theorem

If we consider a volume V_0 at time 0 filled of fluid parcels, and define V_t the volume filled by the parcels as they move, we have

$$V_t = \{y \in \mathbb{R}^3 : y = X_x(t),\ x \in V_0\}.$$

The volume element dy of V_t is given by $J(t,x)dx$, where J is the Jacobian of the transform $x \mapsto X_x(t)$. Let $\mathcal{J} := \det\big(\frac{\partial}{\partial x_i} J_j\big)_{1\le i,j\le 3}$; we have

$$\partial_t \frac{\partial}{\partial x_i} y_j = \frac{\partial}{\partial x_i}\partial_t y_j = \frac{\partial}{\partial x_i} u_j(t,x) = \sum_{k=1}^{3} \frac{\partial}{\partial y_k} u_j(t,t) \frac{\partial}{\partial x_i} y_k$$

and thus

$$\begin{aligned}
\partial_t \mathcal{J} &= \det(\partial_t \frac{\partial}{\partial x_i} y_1, \frac{\partial}{\partial x_i} y_2, \frac{\partial}{\partial x_i} y_3) + \det(\frac{\partial}{\partial x_i} y_1, \partial_t \frac{\partial}{\partial x_i} y_2, \frac{\partial}{\partial x_i} y_3) \\
&\quad + \det(\frac{\partial}{\partial x_i} y_1, \frac{\partial}{\partial x_i} y_2, \partial_t \frac{\partial}{\partial x_i} y_3) \\
&= \sum_{k=1}^{3} \frac{\partial}{\partial y_k} u_1(t,y) \det(\frac{\partial}{\partial x_i} y_k, \frac{\partial}{\partial x_i} y_2, \frac{\partial}{\partial x_i} y_3) \\
&\quad + \sum_{k=1}^{3} \frac{\partial}{\partial y_k} u_2(t,y) \det(\frac{\partial}{\partial x_i} y_1, \frac{\partial}{\partial x_i} y_k, \frac{\partial}{\partial x_i} y_3) \\
&\quad + \sum_{k=1}^{3} \frac{\partial}{\partial y_k} u_3(t,y) \det(\frac{\partial}{\partial x_i} y_1, \frac{\partial}{\partial x_i} y_2, \frac{\partial}{\partial x_i} y_k) \\
&= (\mathrm{div} u(t,x)) \mathcal{J},
\end{aligned}$$

so that, since $J(0,x) = 1$,

$$J(t,x) = \exp\{\int_0^t \mathrm{div} u(s, X_x(s)) ds\}.$$

Thus, we have that the divergence of u is the quantity that governs the deflation or the inflation of the volume of V_t.

Now, if $f(t,x)$ is a time-dependent field over $\mathbb{R}^3$, we may define

$$G(t) = \int_{V_t} f(t,y) dy.$$

We have

$$G(t) = \int_{V_0} f(t, X_x(t)) J(t,x) dx.$$

We use the fact that $\partial_t\big(f(t, X_x(t))\big) = \frac{D}{Dt} f(t,y)$, $\partial_t J(t,x) = \mathrm{div} u(t,y) J(t,x)$ and $J(t,x)dx = dy$ to get: **the convection theorem**

$$\frac{d}{dt}\int_{V_t} f(t,y)dy = \int_{V_t} \frac{D}{Dt} f(t,y) + f(t,y)\mathrm{div} u(t,y) dy.$$

Conservation of mass

We apply the convection theorem to the mass m of the parcels included in the volume V_t. If $\rho(t,y)$ is the density at time t and at position y, we have $m = \int_{V_t} \rho(t,y) dy$.

When the parcels move, their mass is conserved, so we find that $\frac{d}{dt}m = 0$. This gives: **the conservation of mass**

$$\frac{D}{Dt}\rho + \rho \mathrm{div} u = 0.$$

When the fluid is incompressible, the density of a given parcel cannot change, so $\frac{D}{Dt}\rho = 0$, hence we get: **incompressibility**

$$\mathrm{div} u = 0.$$

For an incompressible fluid, we find that $\partial_t \rho = -u \cdot \nabla \rho$. If the fluid is homogeneous, the density does not depend on the position, thus we get: **incompressibility and homogeneity**

$$\rho = Constant.$$

Newton's second law

We apply Newton's second law to a moving parcel of fluid. The momentum of the parcel at time t is given by $M := \int_{V_t} \rho(t,y)u(t,y)dy$. If $f(t,y)$ is the force density at time t and position y, the force applied to the parcel is $F = \int_{V_t} f(t,y)dy$. Newton's second law of mechanics then gives that

$$\frac{d}{dt}M = F.$$

The convection theorem gives then

$$\int_{V_t} \frac{D}{Dt}(\rho u) + \rho u \mathrm{div} u - f dy = 0,$$

combining with the conservation of mass, we have

$$\rho \frac{D}{Dt}u = f,$$

which can be written as

$$\rho\Big(\partial_t u + u \cdot \nabla u\Big) = f. \tag{1.1}$$

It remains to describe the force density f. This is a resultant of several forces: external forces (such as gravity) and internal forces. There are two important types of internal forces: the force induced by pressure and the force induced by friction.

Pressure

When a fluid is in contact with a body, it exerts on the surface of the body a force that is normal to the surface and called the pressure. The pressure is a scalar quantity, which does not depend of the direction of the normal.

Internal pressure is defined in an analogous way. The fluid parcel occupies a volume δV; the force exerted on the parcel induced by the pressure is then

$$F_P = -\int_V \nabla p dx.$$

This gives us the density for the pressure force: **Force density for the pressure**

$$f_P = -\nabla p.$$

Strain

Fluids are not rigid bodies. Thus, their motion implies deformations. Those deformations may be illustrated through the strain tensor. If the velocities and their derivatives are small enough, we may estimate for initial points x_0 and y_0 how the distance of the parcels will evolve. If $x(t) = X_{x_0}(t)$ and $y(t) = X_{y_0}(t)$, we have

$$\|x - y\|^2 \approx \|x_0 - y_0\|^2 + 2\int_0^t (x(s) - y(s)) \cdot Du(s, x(s))(x(s) - y(s))ds$$

where the matrix Du is

$$Du = (\partial_j u_i(s, x))_{1 \le i,j \le 3}.$$

Cauchy's strain tensor ϵ is defined as the symmetric part of Du:

$$\epsilon := \frac{1}{2}(Du + (Du)^T).$$

The antisymmetric part has a null contribution to the integral, and we find:

$$\|x - y\|^2 \approx \|x_0 - y_0\|^2 + 2\int_0^t (x(s) - y(s)) \cdot \epsilon(s, x(s))(x(s) - y(s))ds.$$

Cauchy's strain tensor the strain tensor at time t and position x is the matrix ϵ given by

$$\epsilon_{i,j} = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \text{ for } 1 \le i, j \le 3.$$

If we look at the infinitesimal displacement of y, we have

$$\frac{D}{Dt}y = u(t, y) = u(t, x) + \epsilon(y - x) + \frac{1}{2}(Du - (Du)^T)(y - x) + O(|y - x|^2).$$

$u(t, x)$ does not depend on y: it corresponds to an translation; $\frac{1}{2}(Du - (Du)^T)$ does not contribute to the distortion of distances, it corresponds to a rotation, ϵ_j corresponds to the deformation.

Stress

When a fluid is viscous, it reacts like an elastic body that resists deformations. Applying the theory of elasticity to the fluid motion, one can see that the deformations induce forces. If δV is a small parcel, the deformation of the parcel induces a force exerted on the border of δV; this force F_{visc} is given by a tensor $\mathbb{T}$ and we have

$$F_{visc} = \int_{\partial \delta V} \mathbb{T}\nu d\sigma,$$

which gives us the force density f_{visc} associated to the stress:

$$f_{visc,i} = \sum_{j=1}^{3} \partial_j T_{i,j} = \text{div} T_i.$$

When the fluid velocity and its derivatives are small enough, Stokes has shown that the relation between the stress tensor and the strain tensor is linear. Thus we find that f_{visc} is a sum of second derivatives of u. But due to the isotropy of the fluid, a change of referential through a rotation should not alter the relation between the force and the velocity. This gives that f_{visc} is determined only by two viscosity coefficients: **force density associated to the stress**

$$f_{visc} = \mu \Delta u + \lambda \nabla (\mathrm{div} u)$$

The above equation corresponds to a relationship between the tensor ϵ and the tensor $\mathbb{T}$:

$$\mathbb{T} = 2\mu\epsilon + \eta \mathrm{tr}(\epsilon)\mathrm{Id},$$

where μ is called the dynamical viscosity of the fluid and η the volume viscosity of the fluid. Fluids for which the above relation holds are called Newtonian fluids.

The equations of hydrodynamics

Let us consider a Newtonian isotropic fluid. We already have

$$\frac{D}{Dt}\rho + \rho \mathrm{div} u = 0$$

and

$$\rho \frac{D}{Dt} u = f.$$

The force density f is a superposition of external forces f_{ext} and internal forces f_{int}. In the external forces, one may have the gravity or the Coriolis force. In the internal forces, one has seen the force due to the pressure:

$$F_P = -\nabla p$$

and the force due to the viscosity:

$$f_{visc} = \mu \Delta u + \lambda \nabla (\mathrm{div} u).$$

In the absence of other internal forces, we obtain **the equations of hydrodynamics**

$$\frac{D}{Dt}\rho + \rho \mathrm{div} u = 0$$

and

$$\rho \frac{D}{Dt} u = -\nabla p + \mu \Delta u + \lambda \nabla (\mathrm{div} u) + f_{ext}.$$

Those equations are in number of four scalar equations with five unknown scalar quantities (u_1,u_2,u_3, ρ and p). The fifth equation depends on the nature of the fluid: it is a thermodynamical equation of state that links the pressure, the density and the temperature.

The Navier-Stokes equations

Let us consider the case of a Newtonian, isotropic, homogeneous and incompressible fluid. The above equations of hydrodynamics are transformed into the Navier-Stokes equations. Since ρ is constant, it is customary to divide the equations by ρ, and to replace the force density f_{ext} with a reduced density $f_r := \frac{1}{\rho} f_{ext}$, the pressure p with a reduced pressure $p_r = \frac{1}{\rho}p$ (which is called kinematic pressure), and the dynamical viscosity μ by the kinematic viscosity $\nu = \frac{1}{\rho}\mu$. We then have **The Navier-Stokes equations**:

$$\begin{cases} \partial_t u - \nu\Delta u + u\cdot\nabla u = -\nabla p_r + f_r, \\ \nabla\cdot u = 0. \end{cases} \tag{1.2}$$

ν is positive for a viscous fluid. In case of an ideal fluid($\nu = 0$), we obtain **The Euler equations**:

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\nabla p_r + f_r, \\ \nabla\cdot u = 0. \end{cases} \tag{1.3}$$

1.1.2 Mathematical aspects

In this paragraph, we consider mathematical problems for the Navier-Stokes equation in $\mathbb{R}^3$. Since we will compare the Navier-Stokes equation without an external force with the forced case, we use the following notations:

The Navier-Stokes equation without an external force

$$(NS)\begin{cases} \partial_t u - \Delta u + u\cdot\nabla u = -\nabla p, \\ \nabla\cdot u = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{1.4}$$

The Navier-Stokes equation with an external force

$$(NSf)\begin{cases} \partial_t u_f - \Delta u_f + u_f\cdot\nabla u_f = -\nabla p_f + f, \\ \nabla\cdot u_f = 0, \\ u_f|_{t=0} = u_0. \end{cases} \tag{1.5}$$

Here u and u_f are three-component divergence free vector fields $u = (u_1, u_2, u_3)$ and $u_f = (u_{f,1}, u_{f,2}, u_{f,3})$ representing the velocity of the fluids respectively, p and p_f are two scalar functions denoting the pressure respectively, and all are unknown functions of the space variable $x \in \mathbb{R}^3$ and of the time variable $t > 0$.

We introduce the Navier-Stokes scaling : $\forall\lambda > 0$, the vector field u_f is a solution to (NSf) with initial data u_0, if u_{λ,f_λ} is a solution to (NSf_λ) with initial data $u_{0,\lambda}$, where

$$\begin{gathered} u_{\lambda,f_\lambda}(t,x) := \lambda u_f(\lambda^2 t, \lambda x),\ f_\lambda(t,x) := \lambda^3 f(\lambda^2 t, \lambda x), \\ p_\lambda(t,x) := \lambda^2 p(\lambda^2 t, \lambda x) \ \text{ and } \ u_{0,\lambda} := \lambda u_0(\lambda x). \end{gathered}$$

Spaces which are invariant under the Navier-Stokes scaling are called critical spaces for Navier-Stokes equation. Examples of critical spaces of initial data for the Navier-Stokes in 3D are:

$$L^3(\mathbb{R}^3) \hookrightarrow \dot{B}_{p,q}^{-1+\frac{3}{p}}(\mathbb{R}^3)(p<\infty, q\le\infty) \hookrightarrow \mathrm{BMO}^{-1} \hookrightarrow \dot{B}_{\infty,\infty}^{-1}. \tag{1.6}$$

We will recall the definitions of function spaces in the last part of this section.

Existence

We begin by introducing existence results for (NS).

Weak solutions

We first introduce the weak formulation of (NS). From Leibniz's formula it is clear that when the vector field u is smooth and divergence-free, we have

$$u \cdot \nabla u = \mathrm{div}(u \otimes u), \quad \text{where} \;\; \mathrm{div}(u \otimes u)^j := \sum_{k=1}^{3} \partial_k (u_j u_k) = \mathrm{div}(u_j u),$$

so that (NS) may be written as

$$(NS) \begin{cases} \partial_t u - \Delta u + \mathrm{div}(u \otimes u) = -\nabla p, \\ \nabla \cdot u = 0, \\ u|_{t=0} = u_0. \end{cases}$$

The advantage of this formulation is that it makes sense for more singular vector fields than the previous formulation.

We now formally derive the well-known energy estimate. First, taking the $L^2(\mathbb{R}^3)$ scalar product of the system with the solution u gives

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + (u \cdot \nabla u|u)_{L^2} - (\Delta u|u)_{L^2} = -(\nabla p|u)_{L^2}.$$

Using formal integration by parts, we have

$$\begin{aligned} (u \cdot u|u)_{L^2} &= \sum_{1 \le j,k \le 3} \int_{\mathbb{R}^3} u_j (\partial_j u_k) u_k dx = \frac{1}{2} \sum_{1 \le j \le 3} \int_{\mathbb{R}^3} u_j \partial_j (|u|^2) dx \\ &= -\frac{1}{2} \int_{\mathbb{R}^3} (\mathrm{div} u)|u|^2 dx = 0, \end{aligned}$$

and

$$-(\Delta u|u)_{L^2} = \|\nabla u\|_{L^2}^2.$$

Again, integration by parts yields

$$-(\nabla p|u)_{L^2} = -\sum_{j=1}^{3} \int_{\mathbb{R}^r} u_j \partial_j p dx = \int_{\mathbb{R}^3} p \mathrm{div} u dx = 0.$$

It therefore turns out that, by time integration,

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2 ds = \|u_0\|_{L^2}^2.$$

It follows that the natural assumption for the initial data u_0 is that it is square integrable and divergence-free. This lead to Leray's weak solutions ([21]).

Theorem 1.1.1. *Let u_0 be a divergence-free vector field in $L^2(\mathbb{R}^d)$. Then (NS) has a weak solution u in the energy space*

$$L^\infty(\mathbb{R}_+, L^2) \cap L^2(\mathbb{R}_+, \dot{H}^1)$$

such that the energy inequality holds, namely,

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2 ds \le \|u_0\|_{L^2}^2.$$

Proving Leray's theorem relies on a compactness method:

- First, approximate solutions with compactly supported Fourier transforms satisfying the energy inequality are built.
- Next, a time and space compactness result is derived.
- Finally, the solution is obtained by passing to the limit.

In 2D, the Leray weak solutions are unique, however in 3D, Leray weak solutions are not known to be unique.

Strong solutions

Another important feature of the Navier-Stokes equation in the whole space $\mathbb{R}^3$ is that there is an explicit formula giving the pressure in terms of the velocity field. Indeed, in Fourier variables, the Leray projector $\mathbb{P}$ on the divergence-free vector fields is as follows:

$$\mathcal{F}(\mathbb{P}f)_j(\xi) = \hat{f}_j(\xi) - \frac{1}{|\xi|^2}\sum_{k=1}^{3} \xi_j\xi_k \hat{f}_k(\xi).$$

Also $\mathbb{P}$ can be written as,

$$(\mathbb{P}f)_j = f_j - \sum_{i=1}^{3} R_i R_j f,$$

where R_i is the Riesz transform $R_i = \partial_i(-\Delta)^{-\frac{1}{2}}$, for $1 \leq i \leq 3$. It is clear that $\mathbb{P}$ is a zero-order differential operator.

Therefore, applying the Leray projector to (NS) and (NSf), yields that u and u_f satisfy (formally) the following system respectively,

$$(NS)\begin{cases} \partial_t u - \Delta u + \mathbb{P}\nabla\cdot(u\otimes u) = 0, \\ \nabla\cdot u = 0, \\ u|_{t=0} = u_0, \end{cases}$$

and

$$(NSf)\begin{cases} \partial_t u_f - \Delta u_f + \mathbb{P}\nabla\cdot(u_f\otimes u_f) = \mathbb{P}f, \\ \nabla\cdot u_f = 0, \\ u_f|_{t=0} = u_0. \end{cases} \tag{1.7}$$

Moreover we can transform (NS) and (NSf) to the following integral forms:

$$u = e^{t\Delta}u_0 + B(u,u) \tag{1.8}$$

and

$$u_f = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}f(s)ds + B(u_f,u_f), \tag{1.9}$$

where

$$B(u,v) := -\frac{1}{2}\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes v + v\otimes u)ds.$$

Unforced case

Given $u_0 \in \mathcal{S}'(\mathbb{R}^3)$, in order to find a solution to (1.8), a natural approach is the iterate the transform

$$v \mapsto e^{t\Delta}u_0 + B(v,v)$$

and to find a fixed point u for this transform. This is the so-called Picard contraction method. A simple way to solve this problem is trying to find a Banach space $\mathcal{E}_T$ such that the bilinear transform $B(u,v)$ is bounded from $\mathcal{E}_T \times \mathcal{E}_T$ to $\mathcal{E}_T$. Then we need to consider the space E of initial data such that $e^{t\Delta}u_0 \in \mathcal{E}_T$ for any $u_0 \in E$. The Banach space $\mathcal{E}_T$ is called an admissible path space and E is called an adapted value space.

Remark 1.1.2. *To make sure that a solution to (1.8) is also a solution to (1.4), we need the solution is at least uniformly locally square integrable. Therefore an admissible space $\mathcal{E}_T$ should be a subspace of $L^2_{uloc,x}L^2_t((0,T)\times\mathbb{R}^3)$.*

Now we recall some well-known results about the existence of strong solutions (For the definitions of the following function spaces, see the last paragraph of this section).

- In 1984, T. Kato [18], $E = L^3$, $\mathcal{E}_T = K_p(T)(p > 3)$ (Kato's space). He proved that for any initial data $u_0 \in L^3$, there exists a unique maximal time T^* and a solution u to (NS) such that $u \in K_p(T)$ for any $T < T^*$.
  And using the fact that B is bounded from $L^p \times L^3$ to L^3, he proved that
  $$\forall\, u_0 \in L^3,\ \ \exists!\, T^* > 0 \ \text{ and } \ u \in C([0,T^*), L^3) \cap K_p \ \text{ solves } \ (NS).$$
  Moreover $T^* = \infty$ provided that u_0 is small enough in L^3.
- In 1998, F. Planchon [24], $E = \dot{B}^{s_p}_{p,\infty}(p > 3)$, $\mathcal{E}_T = K_p(\infty)$. For any initial data u_0 small enough in $\dot{B}^{s_p}_{p,\infty}$, there exists a unique small solution belonging to $K_p(\infty)$.
- H. Koch and D. Tataru [20] obtain a unique global in time solution for initial data small enough in a more general space, consisting of vector fields whose somponents are derivatives of BMO functions.

Remark 1.1.3. *Roughly speaking, the more singular the adapted value space is, the more decay in time required on the corresponding admissible space. For example, $E = \dot{B}^{s_p}_{p,q}$, $1 \le p,q < \infty$, its corresponding admissible space is $\mathcal{L}^r([0,T], \dot{B}^{s_p+\frac{2}{r}}_{p,q})$(for details, see [13]), where $s_p + \frac{2}{r} > 0$. It is clear that we need $r \to 2$ to make sure that the above relation holds, as $p \to \infty$.*

Forced case

A simple way to solve (1.9) is similar to solving (1.8). The difference is that we treat

$$e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}f(s)ds$$

as the first step of the iteration. Then we need to consider the space E of initial data and space F of external force such that $e^{t\Delta}u_0 \in \mathcal{E}_T$ for any $u_0 \in E$ and $\int_0^t e^{(t-s)\Delta}\mathbb{P}f(s)ds \in \mathcal{E}_T$ for any $f \in F$. This fact brings some troubles. For example,

- For any initial data $u_0 \in L^3(\mathbb{R}^3)$, $e^{t\Delta}u_0 \in K_p$ for any $p \geq 3$. Combining with the fact B is only continuous from $K_p \times K_p$ to K_p for $p > 3$, then we need F satisfying that for any $f \in F$

$$\int_0^t e^{(t-s)\Delta}\mathbb{P}f(s)ds \in K_p,$$

for some $p > 3$. In this case, M. Cannone and F. Planchon [7], proved the local in time well-posedness for any initial data $u_0 \in L^3(\mathbb{R}^3)$, if the external force f can be written as $f = \nabla \cdot V$ and $\sup_{0<t<T} t^{1-\frac{3}{p}}\|V\|_{L^{\frac{p}{2}}}$ is small enough for some $3 < p \leq 6$ and $T > 0$. Also they showed there exists a unique global solution to (NSf), provided $T = \infty$ and u_0 is small enough in $\dot{B}^{-1+\frac{3}{q}}_{q,\infty}$ with $3 < q < \frac{3p}{6-p}$. However, the above case misses the time-independent external force or more generally, misses the case when

$$\int_0^t e^{(t-s)\Delta}\mathbb{P}f(s)ds \in L^\infty(\mathbb{R}_+, L^3).$$

- In the case of $\mathcal{E}_\infty = L^\infty(\mathbb{R}_+, L^{3,\infty})$, B is continuous from $L^\infty(\mathbb{R}_+, L^{3.\infty}) \times L^\infty(\mathbb{R}_+, L^{3.\infty})$ to $L^\infty(\mathbb{R}_+, L^{3.\infty})$. M. Cannone and G. Karch [6] proved that there exist a solution $u_f \in C_w(\mathbb{R}_+, L^{3,\infty}(\mathbb{R}^3))$ to (NSf), if its initial data $u_0 \in L^{3,\infty}$ is small enough and the external force f satisfies that

$$\sup_{t>0} \| \int_0^t e^{(t-s)\Delta}\mathbb{P}f ds\|_{L^{3,\infty}}$$

is small enough in $L^\infty(\mathbb{R}_+, L^{3,\infty})$. But their result is only valid for small initial data u_0 in $L^{3,\infty}$.

In Chapter 2, we consider (NSf) with an external force given as [6] and initial data belonging to some critical Besov spaces with negative regularity. More precisely, we consider the force f satisfying that: $f \in C(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^3))$ such that for any $t > 0$

$$\int_0^t e^{(t-s)\Delta}\mathbb{P}f ds \in L^\infty(\mathbb{R}_+, L^{3,\infty}), \tag{1.10}$$

which belong to $C_w(\mathbb{R}_+, L^{3,\infty}(\mathbb{R}^3))$, see [6]. For any given $p > 3$, under the smallness assumption on f depending on p, we show the local and global existence to (NSf) for initial data u_0 belonging to $\dot{B}^{s_p}_{p,p}$. Moreover the solution can be decomposed as a small part and a smooth part. By using a regularity result via an iteration introduced in [13, 15], we show that $u_f \in C_w([0, T^*), L^{3,\infty})$ and the smooth part has finite energy when $u_0 \in \dot{B}^{s_p}_{p,p} \cap L^{3,\infty}$ and $u_0 \in \dot{B}^{s_p}_{p,p} \cap L^2(\mathbb{R}^3)$ respectively.

We use a Picard iteration on a perturbation equation instead of using it on (NSf) directly. More precisely, by using the existence of (NSf) for small initial data in $L^{3,\infty}$, there exists a unique small global solution $NSf(0)$ belonging to $L^\infty(\mathbb{R}_+, L^{3,\infty})$. Next, we using Picard iteration on the perturbation equation with initial data u_0 to obtain local in time existence. Hence the above solution to (NSf) can be written a small rough global in time part and a large smooth local in time part.

The reason why we focus on the forces satisfying (1.10) is that

1. there are many time independent external forces satisfying (1.10).

2. there are many rough external forces satisfying (1.10).

More precisely, the one-point stationary singular solutions to (NSf) of the following form (constructed by G. Tian and Z. Xin, [25]):

$$\begin{aligned} u_1(x) &= 2\frac{c|x|^2 - 2x_1|x| + cx_1^2}{|x|(c|x| - x_1)^2}, \quad u_2(x) = 2\frac{x_2(cx_1 - |x|)}{|x|(c|x| - x_1)^2} \\ u_3(x) &= 2\frac{x_3(cx_1 - |x|)}{|x|(x|x| - x_1)^2}, \quad p(x) = 4\frac{cx_1 - |x|}{|x|(c|x| - x_1)^2}, \end{aligned} \tag{1.11}$$

where $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and c is an arbitrary constant such that $|c| > 1$. By straightforward calculations, one can check that, indeed, the functions u_1, u_2, u_3 and p given by (1.11) satisfy (NSf) with $f \equiv 0$ in the point-wise sense for every $x \in \mathbb{R}^3 \backslash \{0\}$. On the other hand, if one treats (u, p) as a distributional or generalized solution to (NSf) in $\mathbb{R}^3$, they correspond to the very singular external force $f = (b\delta_0, 0, 0)$, where the parameter $b \neq 0$ depends on c and δ_0 stands for the Dirac mass. Actually, any external force f have the form of $f = (c_1\delta_0, c_2\delta_0, c_3\delta_0)$, satisfy (1.10) (Lemma 3.4 in [6], by M. Cannone and G. Karch).

Long-time behavior of global solutions

As mentioned in the above paragraph, for (NS), if we choose the adapted value space E as L^3 or $\dot{B}^{s_p}_{p,q}$ for $1 \leq p, q < \infty$, we obtain the local in time well-posedness for any initial data belonging to E. Then for any initial data $u_0 \in E$, there exists a maximal time $T^*(u_0)$ depending on u_0 such that for any $T < T^*$, the associated solution $NS(u_0)$ to (NS) belongs to $\mathcal{E}_T$.

In [13], I. Gallagher, D. Iftimie and F. Planchon proved that (a particular case): Let $u \in C(\mathbb{R}_+, L^3)$ be a priori global solution to (NS). Then

- this solution tends to zero at infinity in L^3,
- this solution is stable.

Forced case

The situation is more subtle when it comes to forced Navier-Stokes equations. We focus on the forces satisfying (1.10). From now on, we always assume that $\int_0^t e^{(t-s)\Delta}\mathbb{P}f ds \in L^\infty(\mathbb{R}_+, L^{3,\infty})$ is small enough.

After we obtain the local in time existence of (NSf) for any initial data $u_0 \in L^{3,\infty} \cap \dot{B}^{s_p}_{p,p}$ for some $p > 3$, we are interested in the long-time behavior of these (priori) global solutions. We mention that a solution $u_f \in C_w([0, T^*), L^{3,\infty})$ to (NSf) is global, which just means its corresponding life span $T^* = \infty$, one can't obtain that $u_f(t)$ has a uniform bound in $L^{3,\infty}$ as t goes to infinity in general.

We first recall some known results.

- The long-time behavior of small global solution to (NSf): in [6], M. Cannone and G. Karch proved the following result: let u_f and $\tilde{u}_f$ belonging to $C_w(\mathbb{R}_+, L^{3,\infty})$ be two small global solution to (NSf) with initial datas u_0 and $\tilde{u}_0$ respectively. Then

$$\lim_{t\to\infty} \|u(t) - \tilde{u}(t)\|_{L^{3,\infty}} = 0,$$

provided that

$$\lim_{t\to\infty} \|e^{t\Delta}(u_0 - \tilde{u}_0)\|_{L^{3,\infty}} = 0.$$

If f is time independent, C. Bjorland, L. Brandolese, D. Iftimie & M. E. Schonbek, in [3], proved that for small global solutions $u_f \in C_w(\mathbb{R}_+, L^{3,\infty})$ to (NSf) with initial data $u_0 \in L^{3,\infty}$, $\|u_f(t) - U\|_{L^{3,\infty}} \to 0$ as $t \to \infty$ if and only if $\|e^{t\Delta}(u_0 - U)\|_{L^{3,\infty}} \to 0$ as $t \to \infty$, where U is the corresponding steady state solution to (NSf). We point out that they use the iteration and semi-group properties to obtain the above results. However, this kind of method does not work any more for an arbitrary global solution in $C_w(\mathbb{R}_+, L^{3,\infty})$ to (NSf).

- The long-time behavior of priori large global solutions: In [3], C. Bjorland, L. Brandolese, D. Iftimie & M. E. Schonbek showed a long-time behavior of priori global solution with time independent external force. More precisely, suppose that $\Delta^{-1}f \in L^{3,\infty} \cap L^4$ and $\Delta^{-1}f$ is small enough in $L^{3,\infty}$. If $u_f \in L^\infty_{loc}(\mathbb{R}_+, L^{3,\infty}) \cap L^4_{loc}([0,\infty), L^4)$, then $u_f \in L^\infty(\mathbb{R}_+, L^{3,\infty})$.

However, we point out that there are some too strong conditions on the solutions in the above two results to cover some interesting cases: In [6], the long-time behavior result just holds for small global solutions. In [3], the condition $\Delta^{-1}f \in L^{3,\infty} \cap L^4$ excludes some important singular forces: $\Delta^{-1}f \sim \frac{1}{|x|}$, which satisfy (1.10).

In Chapter 2, we show that if the solution u_f is global, which means that u_f belongs to $C_w(\mathbb{R}_+, L^{3,\infty})$, then u_f belongs to $L^\infty(\mathbb{R}_+, L^{3,\infty})$. This result holds for a priori global solutions (without smallness assumption on solutions) and for small rough forces (in particular, they contain the case $\Delta^{-1}f \sim \frac{1}{|x|}$). Also we prove that the u_f is stable.

The reason why we can get rid of the restrictions given as above is that, as mentioned before, the solution can be decomposed as a small global solution to (NSf) and a local smooth large perturbative part, and the smooth part has a local in time finite energy. Then we have a global energy estimate, which implies that the smooth large part is bounded uniformly in time. Especially, in [3], the restriction on the external force is aimed to prove the perturbative part has a local in time finite energy.

Uniqueness

In this paragraph, we still assume that the external force satisfies (1.10) and $p > 3$. In Chapter 2, for any initial data in $L^{3,\infty} \cap \dot{B}^{s_p}_{p,p}$ we have constructed a local in time solution belonging to $C_w([0,T^*), L^{3,\infty})$, which can be decomposed as a large smooth part and a small rough part. Moreover if the solution is priori global, then it has a uniform bound in $L^{3,\infty}$ as t goes to infinity. It is natural to wonder whether this kind of solution is unique.

We point out that we cannot prove whether the above solution is unique in $L^\infty_t(L^{3,\infty})$ without the smallness assumption on the solution. Actually even for (NS) the uniqueness in $L^\infty_t(L^{3,\infty})$ is still open (the uniqueness just holds when the solution is small in $L^\infty_t(L^{3,\infty})$).

The reason we focus on the uniqueness of solutions to (NSf) in $L^\infty_t(L^{3,\infty})$ is that the singularity of external force limits the regularity of solutions. In our case, no matter how smooth the initial data is, its corresponding solution to (NSf) only belongs to $L^\infty_t(L^{3,\infty})$.

However, we show the solution constructed in Chapter 2 is unique in the following sense: Let $u_f \in C_w([0,T^*), L^{3,\infty})$ be a solution to (NSf) with initial data $u_0 \in L^{3,\infty} \cap \dot{B}^{s_p}_{p,p}$ and $\bar{u}_f$ be another solution to (NSf) with the same initial data. Then we have

- if $\bar{u}_f - u_f$ consists of a small part and a smooth part, then $u_f \equiv \bar{u}_f$,

- if $\bar{u}_f - u_f \in C([0,T], L^{3,\infty})$, then $u_f \equiv \bar{u}_f$,
- if $\bar{u}_f - u_f$ has finite energy and $3 < p < 5$, then $u_f \equiv \bar{u}_f$.

Blow up

In this thesis we are also interested in a blow up criterion for (NSf). Before stating our result, we recall some well-known results for the unforced case.

We suppose that X_T is such that any $u \in X_T$ satisfying (NS) belongs to $C([0,T], X)$. Setting

$$T^*_{X_T}(u_0) := \sup\{T > 0 | \exists! u := NS(u_0) \in X_T \text{ solving } (NS)\}$$

the Navier-Stokes blow-up problem is:
Question:

$$\text{Does} \quad \sup_{0<t<T^*_{X_T}(u_0)} \|u(t,\cdot)\|_X < \infty \ \text{ imply that } \ T^*_{X_T}(u_0) = \infty?$$

1. In the important work [12] of Escauriaza-Seregin-Šverák, it was established that for $X = L^3(\mathbb{R}^3)$, the answer is yes, by changing $\sup_{0<t<T^*(u_0)} \|u(t,\cdot)\|_{L^3}$ to $\limsup_{0<t<T^*(u_0)} \|u(t,\cdot)\|_{L^3}$. This extended a result in the foundational work of Leray [21] regarding the blow-up of $L^p(\mathbb{R}^3)$ norms at a singularity with $p > 3$, and of the "Ladyzhenskaya-Prodi-Serrin" type mixed norms $L^s_t(L^p_x)$, $\frac{2}{s} + \frac{3}{p} = 1$, $p > 3$, establishing a difficult "end-point" case of those results.

2. In [15], based on the work [19], I. Gallagher, G. S. Koch, F. Planchon gave an alternative proof this result in the setting of strong solutions using the method of "critical elements" of C. Kenig and F. Merle. In [14], I. Gallagher, G. S. Koch, F. Planchon extended the method in [15] to give a positive answer to the above question for $X = \dot{B}^{-1+\frac{3}{p}}_{p,q}(\mathbb{R}^3)$ for all $3 < p, q < \infty$.

3. Also in [1], D. Albritton proved a stronger blow-up criterion in $\dot{B}^{s_p}_{p,q}$ for $3 < p, q < \infty$ and his proof is based on elementary splitting arguments and energy estimates.

We mention that the above results depend on the backward uniqueness of heat equation strongly.

In Chapter 3, we focus on the Navier-Stokes equation with a time independent external force. The main trouble is that backward uniqueness of the heat equation is invalid for the forced Navier-Stokes equations. Therefore to obtain the blow up criterion we can't just follow the known road map.

However, since the blow-up criterion for (NS) is known, we focus on how to bring the blow-up information from (NS) to (NSf). In fact, the profile decomposition of the solutions to (NSf) plays a crucial role in establishing a connection between the solutions of (NS) and (NSf).

Roughly speaking, suppose that the external force f is time independent and satisfies $\Delta^{-1} f$ is small in L^3. Let $\{u_{0,n}\}_{n\in\mathbb{N}}$ be a bounded sequence in L^3. Then we have an orthogonal decomposition of the type

$$NSf(u_{0,n}) = NS(\Lambda_n(\varphi)) + NSf(\phi) + \text{Remainders},$$

where ϕ is a weak limit of $\{u_{0,n}\}_{n\in\mathbb{N}}$, φ is a profile of $\{u_{0,n}\}_{n\in\mathbb{N}}$ and Λ_n are Naiver-Stokes scaling operators. Moreover, we obtain that the life span $T_{0,n}$ of $NS(\Lambda_n(\varphi))$

is smaller than the life spans of $NSf(u_{0,n})$ and $NSf(\phi)$, and the remainders have uniform bounds on $[0, T_{0,n}]$ and the orthogonality of cores/scales which imply that for any $t \in [0, T_{0,n}]$

$$\|NSf(u_{0,n})(t)\|_{L^3} \geq \|NS(\Lambda_n(\varphi))(t)\|_{L^3}, \text{ as } n \to \infty.$$

Using the above idea, we obtain the following blow-up criterion for (NSf): Let $\Delta^{-1}f$ small in L^3, then

$$(BC) \qquad \limsup_{0<t<T^*(u_0,f)} \|u_f(t,\cdot)\|_{L^3} < \infty \Rightarrow T^*(u_0,f) = \infty.$$

Our profile decomposition method is not only valid for time-independent force, but also can be extended to more general time-dependent external forces. For example,

1. our method is valid for the strong solutions belonging to $C([0,T^*), L^3(\mathbb{R}^3))$ constructed in [7] with initial $u_0 \in L^3$, where the external force f can be written as $f = \nabla \cdot V$ and $\sup_{0<t<\infty} t^{1-\frac{3}{p}}\|V\|_{L^{\frac{p}{2}}}$ is small enough for some $3 < p \leq 6$. Actually our method only depends on smallness of U_f and the continuity in time of solution in space L^3, which are similar (U_f can by replace by some small solution with small initial data in L^3 constructed in [7]) with the solutions in [11], whose associated force is time-dependent. Therefore we focus on the case of f is time-independent in this thesis.

2. We also point out that one might get the profile decomposition of solutions to the forced Navier-Stokes with an external force $f \in \mathcal{L}^r(\mathbb{R}_+, \dot{B}^{s_p+\frac{2}{r}-2}_{p,p})$ with $s_p + \frac{2}{r} > 0$ and initial datas bounded in $\dot{B}^{s_p}_{p,p}$ for any $3 < p < \infty$ with a similar proof in [15]. And by the same argument in the proof of Theorem 3.1.4, one can show a blow-up criterion as (BC) by replacing L^3 by $\dot{B}^{s_p}_{p,p}$.

Open Problems

About the solutions to (NSf), we still have some unsolved interesting problems. As mentioned, we cannot prove whether the above solution is unique in $L^\infty_t(L^{3,\infty})$ without the smallness assumption on the solution. We are interested in the following questions:

$$\text{Does } \bar{u}_f - u_f = C_t(L^{3,\infty}) + \text{small rough part imply } u_f \equiv \bar{u}_f?$$

And

$$\text{Does } \bar{u}_f - u_f = \text{finite energy part} + \text{small rough part imply } u_f \equiv \bar{u}_f?$$

However, we can't give a positive answer to the above questions right now. Because the small rough part still limits the regularity.

About the Blow-up criterion, there are two weakness of our result, which we want to improve.

- To obtain profile decomposition of solutions to (NSf) we need to use the scales/cores orthogonality to deal with the source terms in corresponding perturbation equations, which is only valid for the space who can be approximated by C_0^∞. Hence we can not obtain a profile decomposition for a rough external force f, for example $\Delta^{-1} f \sim \frac{c}{|x|}$, as the singularity of f limits the regularity of the source terms.

- By the our profile decomposition method, we can not obtain that

$$T^* < \infty, \lim_{t \to T^*} \|u_f\|_{L^3} = \infty,$$

which is true for the case $f = 0$ proved by D. Albritton, in [1].

Function spaces

Let us first recall the definition of Besov spaces, in dimension $d \geq 1$.

Definition 1.1.4. *Let ϕ be a function in $\mathcal{S}(\mathbb{R}^d)$ such that $\hat{\phi} = 1$ for $|\xi| \leq 1$ and $\hat{\phi} = 0$ for $|\xi| > 2$, and define $\phi_j := 2^{dj}(2^j x)$. Then the frequency localization operators are defined by*

$$S_j := \phi_j * \cdot, \ \ \Delta_j := S_{j+1} - S_j.$$

Let f be in $\mathcal{S}'(\mathbb{R}^d)$. We say f belongs to $\dot{B}^s_{p,q}$ if

1. *the partial sum $\sum_{j=-m}^{m} \Delta_j f$ converges to f as a tempered distribution if $s < \frac{d}{p}$ and after taking the quotient with polynomials if not, and*

2. $$\|f\|_{\dot{B}^s_{p,q}} := \|2^{js} \|\Delta_j f\|_{L^p_x}\|_{\ell^q_j} < \infty.$$

We refer to [10] for the introduction of the following type of space in the context of the Navier-Stokes equations.

Definition 1.1.5. *Let $u(\cdot, t) \in \dot{B}^s_{p,q}$ for a.e. $t \in (t_1, t_2)$ and let Δ_j be a frequency localization with respect to the x variable. We shall say that u belongs to $\mathcal{L}^\rho([t_1, t_2], \dot{B}^s_{p,q})$ if*

$$\|u\|_{\mathcal{L}^\rho([t_1,t_2],\dot{B}^s_{p,q})} := \|2^{js}\|\Delta_j u\|_{L^\rho([t_1,t_2]L^p_x)}\|_{\ell^q_j} < \infty.$$

Note that for $1 \leq \rho_1 \leq q \leq \rho_2 \leq \infty$, we have

$$L^{\rho_1}([t_1, t_2], \dot{B}^s_{p,q}) \hookrightarrow \mathcal{L}^{\rho_1}([t_1, t_2], \dot{B}^s_{p,q}) \hookrightarrow \mathcal{L}^{\rho_2}([t_1, t_2], \dot{B}^s_{p,q}) \hookrightarrow L^{\rho_2}([t_1, t_2], \dot{B}^s_{p,q}).$$

Definition 1.1.6. *Let $p \geq 3$. Kato's space is defined as follow,*

$$K_p := \{u \in C(\mathbb{R}_+, L^p(\mathbb{R}^3)) : \|u\|_{K_p} := \sup_{t>0} t^{\frac{1}{2} - \frac{3}{2p}} \|u(t)\|_{L^p(\mathbb{R}^3)} < \infty\}.$$

We also recall the definition of the weak-L^p (or Marcinkiewicz space):

$$L^{p^*}(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{C} : \|f\|_{L^{p^*}} < \infty\},$$

which is equipped the following quasi-norm

$$\|f\|_{L^{p^*}} := \sup_{t>0} t[\lambda_f(t)]^{\frac{1}{p}},$$

where

$$\lambda_f(s) := m\{x : f(x) > s\}.$$

1.2 The Prandtl equations

One of the fundamental problems of fluid mechanics is to resolve the differences between inviscid flows and viscous flows with small viscosity. The issues include drag, vorticity production and boundary conditions:

- inviscid flow does not correctly describe drag on an object. In an irrotational flow ($\nabla \times u = 0$), there is no drag resisting the motion of an object in the flow. For rotational flow, but that does not account for the total drag.
- An inviscid flow does not produce vorticity.
- Along a boundary, an inviscid flow allows only the vanishing of the normal velocity (i.e. the flow cannot cross the boundary); whereas a viscous flow requires the vanishing of the velocity on the surface of a stationary object (i.e. the fluid sticks to boundary).

Consider the initial value problem for an incompressible flow over a half plane $\mathbb{R}^2_+$. The Euler equations for an inviscid flow are (without external force)

$$\begin{cases} \partial_t u^E + u^E \cdot \nabla u^E = -\nabla p^E, \\ \nabla \cdot u^E = 0, \\ u_2^E|_{y=0} = 0, \\ u^E|_{t=0} = u^{E,0}, \end{cases} \tag{1.12}$$

where $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$, and $u^E := (u_1^E, u_2^E)$ is the velocity and p^E is pressure.

The Navier-Stokes equations for a viscous flow are (without external force)

$$\begin{cases} \partial_t u^{NS} - \nu \Delta u^{NS} + u^{NS} \cdot \nabla u^{NS} = -\nabla p^{NS}, \\ \nabla \cdot u^{NS} = 0, \\ u^{NS}|_{y=0} = 0, \\ u^{NS}|_{t=0} = u^{NS,0}, \end{cases} \tag{1.13}$$

where $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$, and $u^{NS} := (u_1^{NS}, u_2^{NS})$ is the velocity and p^{NS} is pressure. In these equations, the Reynolds number $\mathrm{Re} = UL/\nu$ is the relevant non-dimensional parameter, in which U and L are characteristic values for the velocity and length scale. For typical flows, the viscosity ν is small, so that Re is large and the flow should be nearly inviscid.

L. Prandtl resolved the difference between viscous and inviscid flow, starting in 1904. This work contained the first development of boundary layer theory, which is now a standard part of singular perturbation theory. Prandtl found that the Euler equations are valid outside a thin "boundary layer" region. The boundary layer thickness is $\varepsilon = \sqrt{\nu}$. Viscous drag, vorticity production and relaxation of the no-slip boundary condition all occur inside the boundary layer.

The Prandtl equations for flow inside the boundary layer are

$$\begin{cases} \partial_t u^P - u^P \partial_x u^P + v^P \partial_Y u^P = (\partial_t + u_1^E \partial_x) u_1^E (y = 0) + \partial_{YY} u^P, \\ \partial_Y p^p = 0, \\ \partial_x u^P + \partial_Y v^P = 0, \\ u^P(Y = 0) = 0, \\ u^P(Y \to \infty) \to u_1^E(y = 0), \\ u^P|_{t=0} = u^{P,0}, \end{cases} \tag{1.14}$$

in which Y is a scaled variable normal to the boundary, as discussed below.

1.2.1 Derivation and basic properties of Prandtl's equations

Within the flow, the only parameter is the Reynolds number $\mathrm{Re} = UL/\nu$. Near a boundary, however, the relative distance to the boundary is a second parameter. This suggests that away from the boundary, yielding the Euler equations by the Reynolds number Re very small, but near a boundary, a different scaling may apply. Prandtl's boundary layer scaling is the following

$$Y = \frac{y}{\varepsilon}, \quad \mathbf{u} = (u, \varepsilon V)$$

so that $\partial_y = \varepsilon^{-1}\partial_Y$. This allows rapid variations normal to boundary and requires the normal velocity to be small near the boundary.

Under this scaling, the Navier-Stokes equations become

$$\begin{cases} \partial_t u + u\partial_x u + V\partial_Y u + \partial_x p = \nu\partial_x^2 u + (\frac{\nu}{\varepsilon})\partial_Y^2 u, \\ \partial_t V + u\partial_x V + V\partial_Y V + \varepsilon^{-2}\partial_Y p = \nu\partial_X^2 V + (\frac{\nu}{\varepsilon})\partial_Y^2 V, \\ \partial_x u + \partial_Y V = 0, \\ u = v = 0 \text{ on } Y = 0. \end{cases} \tag{1.15}$$

Set $\varepsilon = \sqrt{\nu}$ and take $\varepsilon \to 0$ to obtain Prandtl's Equations

$$\begin{cases} \partial_t u + u\partial_x u + V\partial_Y u + \partial_x p = \partial_Y^2 u, \\ \partial_Y p = 0, \\ \partial_x u + \partial_Y V = 0, \\ u = v = 0 \text{ on } Y = 0. \end{cases} \tag{1.16}$$

Since $p = p^P$ is independent of Y, set it to the limiting Euler value $p^P(t,x) = P^E(t,x,0)$ so that

$$\partial_x p^P(t,x) = \partial_x P^E(t,x,0) = -(\partial_t u_1^E + u_1^E\partial_x u_1^E)(t,x,0),$$

which implies that

$$\lim_{Y\to\infty} u^P(t,x,Y) = u_1^E(t,x,0).$$

Here is a summary of the properties of a solution to the Prandtl equation, showing that it accounts for the differences between inviscid and viscous flow that were mentioned before. The vorticity for the Navier-Stokes equations, written in the Prandtl scaling, is

$$\omega = \varepsilon\partial_x V - \varepsilon^{-1}\partial_Y u.$$

It follows that the vorticity in the Prandtl equations is

$$\omega^P = -\partial_Y u.$$

Since the flow is incompressible, the normal velocity is

$$v^P(t,x,Y) = -\int_0^Y \partial_x u^P(t,x,Y')dY'.$$

The boundary conditions for the Prandtl equations are

- no-slip at $Y = 0$, in the Navier-Stokes
- zero nomal velocity at $Y = \infty$, corresponding to $y = 0$, as required in Euler.

1.2.2 Gevrey regularity

In Chapter 4, we consider the Prandlt equation in the half plane. The results in Chapter 4 is a collection of a published paper (*SIAM J. Math. Anal.* 48 (2016), pages 1672–1726).

We first study the intrinsic subelliptic structure due to the monotonicity condition, and then deduce, basing on the subelliptic estimate, the Gevrey smoothing effect; that is, given a monotonic initial data belonging to some Sobolev space, the solution will lie in some Gevrey class at positive time, just like heat equation. It is different from the Gevrey propagation property obtained in the aforementioned works, where the initial data is supposed to be of some Gevrey class, for instance $G^{7/4}$ proved by D. Gérard-Varet and N. Masmoudi in [17], and the well-posedness is obtained in the same Gevrey space.

Because of the degeneracy in the tangential variable, the well-posedness theories and the justification of Prandtl's boundary layer theory remain as the challenging problems in the mathematical theory of fluid mechanics.

Under a monotonicity assumption on the tangential velocity of the outflow, Oleinik was the first to obtain the local existence of classical solutions for the initial-boundary value problems, and this result together with some of her works with collaborators are well presented in the monograph [23]. In addition to Oleinik's monotonicity assumption on the velocity field, by imposing a so called favorable condition on the pressure, Xin-Zhang [26] obtained the existence of global weak solutions to the Prandtl equation. All these well-posedness results were based on the Crocco transformation to overcome the main difficulty caused by the degeneracy and mixed type of the equation.

Just recently the well-posedness in Sobolev spaces was explored by an energy method instead of the Crocco transformation; see Alexandre et. all [2] and Masmoudi-Wong [22]. There are very few works concerned with the Prandtl equation without the monotonicity assumption. We mention that due to the degeneracy in x, it is natural to expect Gevrey regularity rather than analyticity for a subelliptic equation.

We recall that the Gevrey class, denoted by $G^s, s \geq 1$, is an intermediate space between analytic functions and C^∞ space. For a given domain Ω, the (global) Gevrey space $G^s(\Omega)$ is consist of such functions that $f \in C^\infty(\Omega)$ and that

$$\left\|\partial^\alpha f\right\|_{L^2(\Omega)} \leq L^{|\alpha|+1}(\alpha!)^s$$

for some constant L independent of α. The significant difference between Gevery ($s > 1$) and analytic ($s = 1$) classes is that there exist nontrivial Gevrey functions admitting compact support.

For simplicity, we consider the Prandlt equation on the half plane with a uniform out flow:

$$\begin{cases} u_t + uu_x + vu_y - u_{yy} = 0, \quad (t,x,y) \in]0,T[\times\mathbb{R}^2_+, \\ u_x + v_y = 0, \\ u|_{y=0} = v|_{y=0} = 0, \ \lim\limits_{y\to+\infty} u = 1, \\ u|_{t=0} = u_0(x,y)\,, \end{cases} \tag{1.17}$$

The approach

We introduce the main idea used in the proof. It is well-known that the main difficulty for Prandtl equation is the degeneracy in x variable, due to the presence of v:

$$v = -\int_0^y (\partial_x u)\, d\tilde{y}.$$

To overcome the degeneracy, we use the cancellation idea, introduced by Masmoudi-Wong [22], to perform the estimates on the new function and moreover on the original velocity function u. Precisely, observe that

$$u_t + uu_x + vu_y - u_{yy} = 0,$$

and, with $\omega = \partial_y u$,

$$\omega_t + u\omega_x + v\omega_y - \omega_{yy} = 0.$$

In order to eliminate the v term on the left-hand side of the above two equations, we use the monotonicity condition $\partial_y u = \omega > 0$ and thus multiply the second equation by $-\frac{\partial_y \omega}{\omega}$, and then add the resulting equation to the first one; this gives, denoting $f = \omega - \frac{\partial_y \omega}{\omega} u$,

$$f_t + u\partial_x f - \partial_{yy} f = \text{terms of lower order}.$$

Our main observation for the new equation is the intrinsic subelliptic structure due to the monotonicity condition. Indeed, denoting $X_0 = \partial_t + u\partial_x$ and $X_1 = \partial_y$, we can rewrite the above equation as of Hörmander's type:

$$\Big(X_0 + X_1^* X_1\Big) f = \text{terms of lower order}.$$

and moreover,direct computation show

$$\big[X_1,\ X_0\big] = (\partial_y u)\, \partial_x. \tag{1.18}$$

Thus Hörmander's bracket condition will be fulfilled formally, provided by $\partial_y u > 0$.

Now we introduce our main result in Chapter 4.

Theorem 1.2.1. *Let $u(t,x,y)$ be a classical local in time solution to Prandtl equation (1.17) on $[0,T]$ with the properties listed below:*

(i) There exist two constants $C_* > 1, \sigma > 1/2$ such that for any $(t,x,y) \in [0,T] \times \mathbb{R}^2_+$,

$$\begin{aligned} &C_*^{-1} \langle y \rangle^{-\sigma} \leq \partial_y u(t,x,y) \leq C_* \langle y \rangle^{-\sigma}, \\ &\big|\partial_y^2 u(t,x,y)\big| + \big|\partial_y^3 u(t,x,y)\big| \leq C_* \langle y \rangle^{-\sigma-1}, \end{aligned} \tag{1.19}$$

where $\langle y \rangle = (1+|y|^2)^{1/2}$.

(ii) There exists $c > 0, C_0 > 0$ and integer $N_0 \geq 7$ such that

$$\|e^{2cy}\partial_x u\|_{L^\infty\left([0,T];\ H^{N_0}(\mathbb{R}^2_+)\right)} + \|e^{2cy}\partial_x \partial_y u\|_{L^2\left([0,T];\ H^{N_0}(\mathbb{R}^2_+)\right)} \leq C_0\,. \tag{1.20}$$

Then for any $0 < T_1 < T$, there exists a constant L, such that for any $0 < t \leq T_1$,

$$\forall\, m > 1 + N_0, \quad \left\| e^{\tilde{c}y} \partial_x^m u(t) \right\|_{L^2(\mathbb{R}^2_+)} \leq t^{-3(m-N_0-1)}\, L^m\, (m!)^{3(1+\sigma)}\,, \tag{1.21}$$

Therefore, the solution u belongs to the Gevery class of index $3(1+\sigma)$ with respect to $x \in \mathbb{R}$ for any $0 < t \leq T_1$.

The solution described in the above theorem exists, for instance, if that the initial data u_0 can be written as

$$u_0(x, y) = u_0^s(y) + \tilde{u}_0(x, y),$$

where u_0^s is a function of y but independent of x such that $C^{-1} \langle y \rangle^{-\sigma} \leq \partial_y u_0^s(y) \leq C \langle y \rangle^{-\sigma}$ for some constant $C \geq 1$, and $\tilde{u}_0$ is a small perturbation such that its weighted Sobolev norm $\left\| e^{2cy} \tilde{u}_0 \right\|_{H^{2N_0+7}(\mathbb{R}^2_+)}$ is suitably small. Then using the arguments in [2], we can obtain the desired solution with the properties listed in Theorem 1.2.1 fulfilled.

The well-posedness problem of Prandtl's equation depends crucially on the choice of the underlying function spaces, especially on the regularity in the tangential variable x. If the initial datum is analytic in x, then the local in time solution exists(c.f. [5]), but the Cauchy problem is ill-posed in Sobolev space for linear and non linear Prandtl equation proved by D. Gérard-Varet and E. Dormy in 2010 (see [16]). Indeed, the main mathematical difficulty is the lack of control on the x derivatives. For example, v in (1.17) could be written as $-\int_0^y u_x(y')dy'$ by the divergence-free condition, and here we lose one derivatives in x-regularity.

The degeneracy cannot be balanced directly by any horizontal diffusion term, so that the standard energy estimates do not apply to establish the existence of local solution. But the results in our main Theorem 1.2.1 show that *the loss of derivative in tangential variable x can be partially compensated via the monotonicity condition*.

Under the hypothesis (4.1.2), the equation (1.17) is a non linear hypoelliptic equation of Hörmander type with a gain of regularity of order $\frac{1}{3}$ in x variable , so that any C^2 solution is locally C^∞, see [27, 28, 29]; for the corresponding linear operator, [8] obtained the regularity in the local Gevrey space G^3.

However, in this thesis we study the equation (1.17) as a boundary layer equation, so that the local property of solution is not of interest to the physics application, and our goal is then to study the global estimates in Gevrey class. In view of (1.19) we see u_y decays polynomially at infinite, so we only have a weighted subelliptic estimate . This explains why the Gevrey index, which is $3(1+\sigma)$, depends also on the decay index σ in (1.19).

Bibliography

[1] D. Albritton, Blow-up criteria for the Navier-Stokes equations in non-endpoint critical Besov spaces. arXiv: 1612.04439

[2] R. Alexandre, Y. Wang, C.-J.Xu and T.Yang, Well-posedness of The Prandtl Equation in Sobolev Spaces. *J. Amer. Math. Soc.*, DOI: http://dx.doi.org/10.1090/S0894-0347-2014-00813-4 Article electronically published on June 6, 2014. 40 pages, http://arxiv.org/abs/1203.5991.

[3] C. Bjorland, L. Brandolese, D. Iftimie & M. E. Schonbek, L^p-solutions of the steady-state Navier-Stokes equations with rough external forces. *Communications in Partial Differential Equations*, 2010, 36(2), 216-246

[4] C. P. Calderón, Existence of weak solutions for the Navier-Stokes equations with initial data in L^p. *Trans. Amer. Math. Soc.*318 (1990), no. 1, 179–200.

[5] M. Cannone, M. C. Lombardo and M. Sammartino : Well-posedness of the Prandtl equation with non compatible data. *Nonlinearity.*, 26(2013), 3077-3100.

[6] M. Cannone & G. Karch, About the regularized Navier-Stokes equations. *J. Math. Fluid Mech.* 7 (2005), no. 1, 1–28.

[7] M. Cannone & F. Planchon, On the non-stationary Navier-Stokes equations with an external force. *Adv. Differential Equations* 4 (1999), no. 5, 697–730.

[8] J.-Y Chemin, Théorèmes d'unicité pour le système de Navier-Stokes tridimensionnel. (French) [Uniqueness theorems for the three-dimensional Navier-Stokes system]. *J. Anal. Math.*77 (1999), 27–50.

[9] J.-Y Chemin, Remarques sur l'existence globale pour le système de Navier-Stokes incompressible. *SIAM J. Math. Anal.* 23 (1992), no. 1, 20–28.

[10] J.-Y Chemin & N. Lerner, Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes. *J. Differential Equations* 121 (1995), no. 2, 314–328.

[11] W. E and B. Enquist : Blow up of solutions of the unsteady Prandtl's equation, *Comm. Pure Appl. Math.*, 50(1997), 1287-1293.

[12] L. Escauriaza, G. Seregin & V. Šverák, $L_{3,\infty}$–solutions of Navier-Stokes equations and backward uniqueness. *Uspekhi Mat. Nauk.* 58 (2003), no. 2(350), 3–44.

[13] I. Gallagher, D. Iftimie & F. Planchon, Asymptotics and stability for global solutions to the Navier-Stokes equations. *Ann. Inst. Fourier (Grenoble)* 53 (2003), no. 5, 1387—1424.

[14] I. Gallagher, G. S. Koch & F. Planchon, Blow-up of critical Besov norms at a potential Navier-Stokes singularity. *Comm. Math. Phys.* 343 (2016), no. 1, 39–82.

[15] I. Gallagher, G. S. Koch & F. Planchon, A profile decomposition approach to the $L_t^\infty(L_x^3)$ Navier-Stokes regularity criterion. *Math. Ann.*355 (2013), no. 4, 1527–1559.

[16] D. Gérard-Varet and E. Dormy: On the ill-posedness of the Prandtl equation, *J. Amer. Math. Soc.*, 23(2010), 591-609.

[17] D. Gérard-Varet and N. Masmoudi : Well-posedness for the Prandtl system without analyticity or monotonicity, arXiv:1305.0221.

[18] T. Kato, Strong L^p solutions of the Navier-Stokes equation in $\mathbb{R}^m$, with applications to weak solutions. *Mathematische Zeitschrift*, 187, 1984, pages 471–480.

[19] C. E. Kenig & G. S. Koch, An alternative approach to regularity for the Navier-Stokes equations in critical spaces. *Ann. Inst. H. Poincaré Anal. Non Linéaire*28 (2011), no. 2, 159–187.

[20] H. Koch & D. Tataru, Well-posedness for the Navier-Stokes equations. *Adv. Math.* 157 (2001), no. 1, 22–35.

[21] J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.* 63 (1934), no. 1, 193–248.

[22] N. Masmoudi and T. K. Wong : Local-in-time existence and uniqueness of solution to the Prandtl equation by energy method. arXiv:1206.3629

[23] O. A. Oleinik and V. N. Samokhin, *Mathematical Models in Boundary Layers Theory*. Chapman & Hall/CRC, 1999.

[24] F. Planchon, Asymptotic behavior of global solutions to the Navier-Stokes equations in $\mathbb{R}^3$. *Rev. Mat. Iberoamericana* 14 (1998), no. 1, 71–93.

[25] G. Tian & Z. Xin, One-point singular solutions to the Navier-Stokes equations, *Topol. Meth. Nonlinear Anal.* 11(1998), 135-145.

[26] Z. Xin and L. Zhang : On the global existence of solutions to the Prandtl's system, *Adv. Math.*, 181(2004), 88-133.

[27] C.-J. Xu : Hypoellipticity of nonlinear second order partial differential equations. *J. Partial Differential Equations Ser. A* 1 (1988), 85–95.

[28] C.-J. Xu : Régularité des solutions pour les équations aux dérivées partielles quasi linéaires non elliptiques du second ordre. *C. R. Acad. Sci. Paris Sér. I Math.* 300 (1985), no. 9, 267–270.

[29] C.-J. Xu : Hypoellipticité d'équations aux dérivées partielles non linéaires. *Proceedings of the conference on partial differential equations, Vol. 1, 2 (Saint Jean de Monts, 1985)*, Exp. No. 7, 16 pp., Soc. Math. France, Paris, 1985.

Chapter 2

The incompressible Navier-Stokes equation with an external force

2.1 Introduction

We study the incompressible Navier-Stokes equations in $\mathbb{R}^3$,

$$(NSf)\begin{cases} \partial_t u_f - \Delta u_f + u_f \cdot \nabla u_f = -\nabla p + f, \\ \nabla \cdot u_f = 0, \\ u_f|_{t=0} = u_0. \end{cases}$$

Here u_f is a three-component vector field $u_f = (u_{1,f}, u_{2,f}, u_{3,f})$ representing the velocity of the fluid, p is a scalar denoting the pressure, and both are unknown functions of the space variable $x \in \mathbb{R}^3$ and of the time variable $t > 0$. Finally $f = (f_1, f_2, f_3)$ denotes a given external force defined on $[0,T] \times \mathbb{R}^3$ for some $T \in \mathbb{R}_+ \cup \{\infty\}$. We recall the Navier-Stokes scaling : $\forall \lambda > 0$, the vector field u_f is a solution to (NSf) with initial data u_0 if u_{λ, f_λ} is a solution to (NSf_λ) with initial data $u_{0,\lambda}$, where

$$u_{\lambda,f_\lambda}(t,x) := \lambda u_f(\lambda^2 t, \lambda x),\ f_\lambda(t,x) := \lambda^3 f(\lambda^2 t, \lambda x),$$
$$p_\lambda(t,x) := \lambda^2 p(\lambda^2 t, \lambda x) \text{ and } u_{0,\lambda} := \lambda u_0(\lambda x).$$

Spaces which are invariant under the Navier-Stokes scaling are called critical spaces for the Navier-Stokes equation. Examples of critical spaces of initial data for the Navier-Stokes equation in 3D are:

$$L^3(\mathbb{R}^3) \hookrightarrow \dot{B}^{-1+\frac{3}{p}}_{p,q}(\mathbb{R}^3)(p < \infty, q \le \infty) \hookrightarrow \mathrm{BMO}^{-1} \hookrightarrow \dot{B}^{-1}_{\infty,\infty}$$

(See below for definitions).

To put our results in perspective, let us first recall related results concerning the Cauchy problem for the classical (the case $f \equiv 0$) Navier-Stokes equation with possibly irregular initial data:

$$(NS)\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u = -\nabla p, \\ \nabla \cdot u = 0, \\ u|_{t=0} = u_0. \end{cases}$$

In the pioneering work [22], J. Leray introduced the concept of weak solutions to (NS) and proved global existence for datum $u_0 \in L^2$. However, their uniqueness has remained an open problem. In 1984, T. Kato [20] initiated the study of (NS) with initial data belonging to the space $L^3(\mathbb{R}^3)$ and obtained global existence in a subspace of $C([0,\infty), L^3(\mathbb{R}^3))$ provided the norm $\|u_0\|_{L^3(\mathbb{R}^3)}$ is small enough. The existence result for initial data small in the Besov space $\dot{B}_{p,q}^{-1+\frac{3}{p}}$ for $p, \in [1,\infty)$ and $q \in [1,\infty]$ can be found in [10, 16]. The function spaces $L^3(\mathbb{R}^3)$ and $\dot{B}_{p,q}^{-1+\frac{3}{p}}$ for $(p,q) \in [1,\infty)^2$ both guarantee the existence of local-in-time solution for any initial data. In 2001, H. Koch and D. Tataru [21] showed that global well-posedness holds as well for small initial data in the space BMO^{-1}. This space consists of vector fields whose components are derivatives of BMO functions. On the other hand, it has been shown by J. Bourgain and N. Pavlović [6] that the Cauchy problem with initial data in $\dot{B}^{-1}_{\infty,\infty}$ is ill-posed no matter how small the initial are. Also P. Germain showed the ill-posedness for initial data in $\dot{B}^{-1}_{\infty,q}$ for any $q > 2$, see [19].

However, the situation is more subtle when it comes to forced Navier-Stokes equations. In 1999, M. Cannone and F. Planchon [11] worked on constructing global mild solutions in $C([0,T), L^3(\mathbb{R}^3))$ to the Cauchy problem for the Navier-Stokes equations with an external force. They showed the local-in-time wellposedness for any initial data $u_0 \in L^3(\mathbb{R}^3)$, if the external force f can be written as $f = \nabla \cdot V$ and $\sup_{0<t<T} t^{1-\frac{3}{p}} \|V\|_{L^{\frac{p}{2}}}$ is small enough for some $3 < p \le 6$ and $T > 0$. Also they showed there exists a unique global solution to (NSf), provided $T = \infty$ and u_0 is small enough in $\dot{B}_{q,\infty}^{-1+\frac{3}{q}}$ with $3 < q < \frac{3p}{6-p}$. Later in 2005, M. Cannone and G. Karch [9] proved that there exists a solution $u_f \in C_w(\mathbb{R}_+, L^{3,\infty}(\mathbb{R}^3))$ to (NSf), if the initial data $u_0 \in L^{3,\infty}$ is small enough and the external force f satisfies that

$$\sup_{t>0} \left\| \int_0^t e^{(t-s)\Delta} \mathbb{P} f ds \right\|_{L^{3,\infty}} \tag{2.1}$$

is small enough depending on the norm of the bilinear operator B (defined in (2.3)) in $L^\infty(\mathbb{R}_+, L^{3,\infty})$.

The basic approach to obtain the above results is, in principle, always the same. One first transforms the Navier-Stokes equations (NSf) into an integral equation,

$$u_f(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \mathbb{P} f(s) ds + B(u_f, u_f)(t) \tag{2.2}$$

where

$$B(u,v) := -\frac{1}{2} \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes v + v \otimes u) ds, \tag{2.3}$$

$\mathbb{P}$ being the projection onto divergence free vector fields. It is customary to obtain the existence of a strong global ($T = \infty$) or local ($T < \infty$) solution $u_f \in X_T$ of (2.1), with X_T being an abstract critical Banach space, by means of the standard contraction lemma. For example, in [10, 9] the terms $e^{t\Delta} u_0$ and $\int_0^t e^{(t-s)\Delta} \mathbb{P} f(s) ds$ are treated as the first point of the iteration and they require that $e^{t\Delta} u_0$, $\int_0^t e^{(t-s)\Delta} \mathbb{P} f(s) ds$ both belong to the corresponding Banach space X_T. That is why in [10] V needs to have a suitable decay in time and in [9] the smallness is measured in $L^{3,\infty}(\mathbb{R}^3)$ and the initial data u_0 is restricted to $L^{3,\infty}$. The big difference between [11] and [9] is the following: in [11] the external force has good regularity and $e^{t\Delta} u_0$ belongs to Kato's space for initial data belonging to $\dot{B}^{s_p}_{p,\infty}$ for some $p > 3$ (see Definition 2.2.4), which allows the fixed point lemma to work in Kato's space. Therefore the solutions in [11] belong to $C([0,T^*), L^3)$; however in [9], the external force is rough, which limits the regularity of solution. Therefore in [9] the solutions to [9] only belong to $L^\infty_t(L^{3,\infty})$, even for small smooth initial data. That is the reason why these solutions lack uniqueness, unless the solution is small in $L^\infty_t(L^{3,\infty})$.

In this paper we consider (NSf) with an external force given in [9], however the class of initial data is different. More precisely, we consider the force f satisfying : $f \in C(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^3))$ with for any $t > 0$

$$\int_0^t e^{(t-s)\Delta}\mathbb{P} f ds \in L^\infty(\mathbb{R}_+, L^{3,\infty}),$$

which belongs to $C_w(\mathbb{R}_+, L^{3,\infty}(\mathbb{R}^3))$, see [9]. Under a smallness assumption on f (controlled by a universal small positive constant depending on the singularity of initial data), we first show the local and global existence to (NSf) for initial data u_0 belonging to $\dot{B}^{s_p}_{p,p}$. Moreover we obtain that the above solution belongs to $L^\infty_t(L^{3,\infty})$ when its initial data is in $L^{3,\infty} \cap \dot{B}^{s_p}_{p,p}$ for $p > 3$. Then we show the long-time behavior and stability of the above priori global solutions with initial data in $L^{3,\infty} \cap \dot{B}^{s_p}_{p,p}$. We need to mention that the uniqueness of solutions in $L^\infty_t(L^{3,\infty})$, even for smooth initial data, is a still open problem. However we show that if the difference between the above solution and another solution to (NSf) with the same initial data belongs to $C([0,T], L^{3,\infty})$ or has finite energy on some interval $[0,T]$, then they are equal on $[0,T]$.

The rest of the paper is organized as follows. In Section 2 we give some notations and the main results of this paper. Section 3 addresses the proof of the existence and uniqueness of solutions to (NSf) with initial data u_0 belonging to $\dot{B}^{s_p}_{p,p}$. Section 4 is devoted to the long-time behavior and stability of a priori global solution to (NSf) described in Section 2. The last section is devoted to a regularity result via an iteration. In the appendix, we recall several known results and properties of solutions in Besov spaces.

2.2 Notations and Main Results

Let us first recall the definition of Besov spaces, in dimension $d \geq 1$.

Definition 2.2.1. *Let ϕ be a function in $\mathcal{S}(\mathbb{R}^d)$ such that $\hat{\phi} = 1$ for $|\xi| \leq 1$ and $\hat{\phi} = 0$ for $|\xi| > 2$, and define $\phi_j := 2^{dj}(2^j x)$. Then the frequency localization operators are defined by*

$$S_j := \phi_j * \cdot, \quad \Delta_j := S_{j+1} - S_j.$$

Let f be in $\mathcal{S}'(\mathbb{R}^d)$. We say f belongs to $\dot{B}^s_{p,q}$ if

1. *the partial sum $\sum_{j=-m}^m \Delta_j f$ converges to f as a tempered distribution if $s < \frac{d}{p}$ and after taking the quotient with polynomials if not, and*

2. $$\|f\|_{\dot{B}^s_{p,q}} := \|2^{js}\|\Delta_j f\|_{L^p_x}\|_{\ell^q_j} < \infty.$$

We refer to [14] for the introduction of the following type of space in the context of the Navier-Stokes equations.

Definition 2.2.2. *Let $u(\cdot,t) \in \dot{B}^s_{p,q}$ for a.e. $t \in (t_1,t_2)$ and let Δ_j be a frequency localization with respect to the x variable (see Definition 3.1.1). We shall say that u belongs to $\mathcal{L}^\rho([t_1,t_2], \dot{B}^s_{p,q})$ if*

$$\|u\|_{\mathcal{L}^\rho([t_1,t_2],\dot{B}^s_{p,q})} := \|2^{js}\|\Delta_j u\|_{L^\rho([t_1,t_2]L^p_x)}\|_{\ell^q_j} < \infty.$$

Note that for $1 \le \rho_1 \le q \le \rho_2 \le \infty$, we have

$$L^{\rho_1}([t_1,t_2],\dot{B}^s_{p,q}) \hookrightarrow \mathcal{L}^{\rho_1}([t_1,t_2],\dot{B}^s_{p,q}) \hookrightarrow \mathcal{L}^{\rho_2}([t_1,t_2],\dot{B}^s_{p,q}) \hookrightarrow L^{\rho_2}([t_1,t_2],\dot{B}^s_{p,q}).$$

Let us introduce the following notation (also used in [17]): we define $s_p := -1 + \frac{3}{p}$ and

$$\mathbb{L}^{a:b}_{p,\bar{p}}(t_1,t_2) := \mathcal{L}^a([t_1,t_2];\dot{B}^{s_p+\frac{2}{a}}_{p,\bar{p}}) \cap \mathcal{L}^b([t_1,t_2];\dot{B}^{s_p+\frac{2}{b}}_{p,\bar{p}}), \mathbb{L}^{a:b}_{p}(t_1,t_2) := \mathbb{L}^{a:b}_{p,\bar{p}}(t_1,t_2)$$

$$\mathbb{L}^{a}_{p}(t_1,t_2) := \mathbb{L}^{a:a}_{p}(t_1,t_2),\ \mathbb{L}^{a:b}_{p}(T) := \mathbb{L}^{a:b}_{p}(0,T) \text{and}\ \mathbb{L}^{a:b}_{p}[T<T^*] := \cap_{T<T^*}\mathbb{L}^{a:b}_{p}(T).$$

Remark 2.2.3. *We point out that according to our notations, $u \in \mathbb{L}^{a:b}_p[T<T^*]$ merely means that $u \in \mathbb{L}^{a:b}_p(T)$ for any $T<T^*$ and does not imply that $u \in \mathbb{L}^{a:b}_p(T^*)$ (the notation does not imply any uniform control as $T \nearrow T^*$).*

Definition 2.2.4. *Let $p \ge 3$. Kato's space is defined as follow,*

$$K_p := \{u \in C(\mathbb{R}_+, L^p(\mathbb{R}^3)) : \|u\|_{K_p} := \sup_{t>0} t^{\frac{1}{2}-\frac{3}{2p}}\|u(t)\|_{L^p(\mathbb{R}^3)} < \infty\}.$$

In this paper we are also interested in the weak-strong uniqueness of solutions to (NSf). We introduce the following notations. We define that for any $T \in \mathbb{R}_+ \cup \{+\infty\}$

$$E(T) = L^\infty([0,T^*),L^2) \cap L^2([0,T^*),\dot{H}^1)$$

and

$$E_{loc}(T) = L^\infty_{loc}([0,T),L^2) \cap L^2_{loc}([0,T),\dot{H}^1).$$

We also recall the definition of Lorentz spaces $L^{p,q}$ with $1<p<\infty$ and $1 \le q \le \infty$.

Definition 2.2.5. *Let (X,λ) be a measure space. Let f be a scalar-valued λ-measurable function and*

$$\lambda_f(s) := \lambda\{x : f(x) > s\}.$$

Then the re-arrangement function f^* of f is defined by:

$$f^*(t) := \inf\{s : \lambda_f(s) \le t\}.$$

And for any $1<p<\infty$, the Lorentz spaces $L^{p,q}$ is defined by:

$$L^{p,q}(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{C},\ \ \|f\|_{L^{p,q}} < \infty\},$$

where

$$\|f\|_{L^{p,q}} = \begin{cases} \frac{q}{p}\Big[\int_0^\infty \big(t^{\frac{1}{p}}f^*(t)\big)dt\Big]^{\frac{1}{q}}, & q<\infty, \\ \sup_{t>0}\{t^{\frac{1}{p}}f^*(t)\}, & q=\infty. \end{cases}$$

We note that it is standard to use the above as a norm even if it does not satisfy the triangle inequality since one can find an equivalent norm that makes the space into a Banach space. In particular, $L^{p,\infty}$ agrees with the weak-L^p (or Marcinkiewicz space):

$$L^{p^*}(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{C} : \|f\|_{L^{p^*}} < \infty\},$$

which is equipped the following quasi-norm

$$\|f\|_{L^{p^*}} := \sup_{t>0} t[\lambda_f(t)]^{\frac{1}{p}}.$$

To deal with external forces and for simplicity of notation we introduce the following space (introduced in [9]),

$$\mathcal{Y} = \{f \in C(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^3)) : \int_0^t e^{(t-s)\Delta}\mathbb{P}f(s)ds \in C_w(\mathbb{R}_+, L^{3,\infty})\}$$

equipped with the norm

$$\|f\|_{\mathcal{Y}} := \sup_{t>0} \left\| \int_0^t e^{(t-s)\Delta}\mathbb{P}f(s)ds \right\|_{L^{3,\infty}}.$$

Remark 2.2.6. *We mention that $\mathcal{Y}$ contains many rough external forces.*
For example,

1. *For every $g \in C_w(\mathbb{R}_+, L^{\frac{3}{2},\infty})$, $f := \nabla g \in \mathcal{Y}$ and $\|f\|_Y \lesssim \|g\|_{L^\infty(\mathbb{R}_+, L^{\frac{3}{2}\infty})}$ (see Lemma 3.2 in [9]).*
2. *Every time-independent f satisfying $\Delta^{-1}f \in L^{3,\infty}$ belongs to $\mathcal{Y}$ (see Theorem 4.3 in [5]).*
3. *By Lemma 3.4 in [9], $\mathcal{Y}$ contains some really rough external force: $f := (c_1\delta_0, c_2\delta_0, c_3\delta_0)$, where δ_0 stands for the dirac mass.*

According to Theorem 2.1 in [9], there exists a constant $\varepsilon_0 > 0$ such that if $f \in \mathcal{Y}$ and $u_0 \in L^{3,\infty}$ satisfy that $\|u_0\|_{L^{3,\infty}} + \|f\|_{\mathcal{Y}} < \varepsilon_0$, there exists a unique solution to (NSf) with initial data u_0 and external force f, denoted by $NSf(u_0)$, which belongs to $C_w(\mathbb{R}_+, L^{3,\infty})$ such that

$$\|NSf(u_0)\|_{L^\infty(\mathbb{R}_+, L^{3,\infty})} \le 2(\|u_0\|_{L^{3,\infty}} + \|f\|_{\mathcal{Y}}).$$

In particular, we have $NSf(0) \in C_w(\mathbb{R}_+, L^{3,\infty})$ satisfying

$$\|NSf(0)\|_{L^\infty(\mathbb{R}_+, L^{3,\infty})} \le 2\|f\|_{\mathcal{Y}} < 2\varepsilon_0.$$

From now on, we denote $U_f := NSf(0)$.

Now let us state our main results. We first state a local in time existence result for (NSf) for initial data belonging to $\dot{B}^{s_p}_{p,p}$ for any $p > 3$ under a smallness assumption on f depending on p (it is no loss of generality to set p, p rather than p, q, which deduces some technical difficulties). Moreover we obtain a local in time existence result for (NSf) in $L^\infty_t(L^{3,\infty})$ for initial data belonging to $\dot{B}^{s_p}_{p,p} \cap L^{3,\infty}$.

Theorem 2.2.7 (Existence). *Let $p > 3$. There exists a small universal constant $c(p) > 0$ with the following properties:*

Suppose that $f \in \mathcal{Y}$ is a given external force satisfying that $\|f\|_{\mathcal{Y}} < c(p)$. Then

1. *for any initial data $u_0 \in \dot{B}^{s_p}_{p,p}$, a unique maximal time $T^*(u_0, f) > 0$ and a unique solution $u_f \in \mathbb{L}^{r_0:\infty}_p[T < T^*] + C_w([0, T^*), L^{3,\infty})$ to (NSf) with initial data u_0*

exist such that

$$u_f - U_f \in \mathbb{L}_p^{r_0:\infty}[T < T^*].$$

with $r_0 = \frac{2p}{p-1}$. *And if* $T^* < \infty$,

$$\limsup_{T \to T^*} \|u - U_f\|_{\mathbb{L}_p^{r_0:\infty}(T)} = \infty.$$

Moreover there exists a small constant $\eta > 0$ *depending on* f *and* p *such that*

$$\|u_0\|_{\dot{B}^{s_p}_{p,p}} < \eta \Rightarrow T^* = \infty \text{ and } \|u_f - U_f\|_{\mathbb{L}_p^{r_0:\infty}(\infty)} \le C(f)\|u_0\|_{\dot{B}^{s_p}_{p,p}}.$$

2. *if* $u_0 \in \dot{B}^{s_p}_{p,p} \cap L^{3,\infty}$, *the above solution* u_f *to* (NSf) *with initial data* u_0 *belongs to* $C_w([0,T^*), L^{3,\infty})$.
3. *if* $u_0 \in \dot{B}^{s_p}_{p,p} \cap L^2(\mathbb{R}^3)$, *the above solution* u_f *to* (NSf) *with initial data* u_0 *satisfies that*

$$u_f - U_f \in E_{loc}(T^*).$$

Our method is to transform (NSf) into the perturbation equation,

$$(PNS_{U_f}) \begin{cases} \partial_t v - \Delta v + v \cdot \nabla v + U_f \cdot \nabla v + v \cdot \nabla U_f = -\nabla \pi, \\ \nabla \cdot v = 0, \\ v|_{t=0} = v_0 := u_0, \end{cases}$$

The corresponding integral form of (PNS_{U_f}) is

$$v = e^{t\Delta} v_0 + B(v,v) + 2B(U_f, v), \tag{2.4}$$

where B is defined as (2.3). The reason why we focus on (PNS_{U_f}) is that (2.4) allows us to use the classical contraction lemma in the Besov space $\mathcal{L}^r([0,T], \dot{B}_{p,p}^{-1+\frac{3}{p}+\frac{2}{r}})$ with any $p > 3$ and some $r > 2$.

Also in order to control the energy estimate, we adopt the argument about the trilinear form $\int_0^T \int_{\mathbb{R}^3} (a \cdot \nabla b) \cdot c(t) dx dt$ in [18].

From Theorem 2.2.7, for any $u_0 \in \dot{B}^{s_p}_{p,p} \cap L^{3,\infty}$, there exists a solution $u_f \in C_w([0,T^*), L^{3,\infty})$. Actually $C_w([0,T^*), L^{3,\infty})$ is the highest regularity of solutions to (NSf), as the singularity of f limits it. Therefore the uniqueness of solutions to (NSf) in $C_w([0,T^*), L^{3,\infty})$ is crucial. We point out that we cannot prove that the above solution is unique in $L^\infty_t(L^{3,\infty})$ without the smallness assumption on the solution. Actually even if for (NS) the uniqueness in $L^\infty_t(L^{3,\infty})$ is still open (the uniqueness just holds when solution is small in $L^\infty_t(L^{3,\infty})$). However, we obtain that the above solution is unique in the following sense:

Theorem 2.2.8 (Uniqueness). *Let* $p > 3$. *There exists a universal small constant* $0 < c_1(p) \le c(p)$ *with the following properties:*

Suppose that $f \in \mathcal{Y}$ *is a given external force satisfying that* $\|f\|_{\mathcal{Y}} < c_1$ *and* $u_f \in C_w([0,T^*), L^{3,\infty})$ *is a solution to* (NSf) *constructed in Theorem 2.2.7 with initial data* $u_0 \in L^{3,\infty} \cap \dot{B}^{s_p}_{p,p}$. *Then* u_f *is unique in the following sense: Assume that* $\bar{u}_f \in C_w([0,T], L^{3,\infty})$ *for some* $T < T^*$ *is another solution to* (NSf) *with same initial data* u_0.

- *If* $u_f - \bar{u}_f \in \mathbb{L}_p^{r:\infty}(T) + \{U(t,x) \in C_w(\mathbb{R}_+, L^{3,\infty}) : \|U\|_{L^\infty(\mathbb{R}_+), L^{3,\infty}} < 2c_1\}$ *for some* $2 < r < \frac{2p}{p-3}$, *then* $u_f \equiv \bar{u}_f$ *on* $[0,T]$.

- *if* $u_f - \bar{u}_f \in C([0,T], L^{3,\infty})$, *then* $u_f \equiv \bar{u}_f$ *on* $[0,T]$
- *if* $3 < p < 5$ *and* $u_f - \bar{u}_f \in L^\infty([0,T], L^2) \cap L^2([0,T], \dot{H}^1)$, *then* $u_f \equiv \bar{u}_f$ *on* $[0,T]$.

We prove Theorem 2.2.7 and 2.2.8 in Section 3. Our method depends on an iteration regularity result developed in Section 5.

The global existence of the solutions described in Theorem 2.2.7 for large initial data $u_0 \in \dot{B}^{s_p}_{p,p}$ is still open, even for $f = 0$. We mention that even if a solution $u_f \in C_w([0,T^*), L^{3,\infty})$ to (NSf) is global, which just means its corresponding life span $T^* = \infty$, one cannot obtain that $u_f(t)$ has a uniform bound in $L^{3,\infty}$ as t goes to infinity in general. However, if u_f is a global solution to (NSf) with initial data $u_0 \in \dot{B}^{s_p}_{p,p} \cap L^{3,\infty}$ described in Theorem 2.2.7, the next theorem shows the solution belongs to $L^\infty(\mathbb{R}_+, L^{3,\infty})$.

Comparing with previous results of long-time behavior, our assumptions on u_f and f are all in critical spaces, but the class of initial data is smaller. For example, in [5] C. Bjorland, L. Brandolese, D. Iftimie & M. E. Schonbek proved that if the external force f is time-independent satisfying that $\Delta^{-1} f \in L^{3,\infty} \cap L^4$ and $\|\Delta^{-1} f\|_{L^{3,\infty}}$ is small, then for any priori global solution $u_f \in C_w(\mathbb{R}_+, L^{3,\infty}) \cap L^4_{loc}(\mathbb{R}_+, L^4)$ with initial data $u_0 = v_0 + w_0$ satisfying that $v_0 \in L^2$ and $\|w_0\|_{L^{3,\infty}}$ is smaller than a fixed small constant ϵ, then $u_f \in L^\infty(\mathbb{R}_+, L^{3,\infty})$. It clear that the space of initial data they are working on is larger than $L^{3,\infty} \cap \dot{B}^{s_p}_{p,p}$ and $\Delta^{-1} f \in L^{3,\infty} \cap L^4$ implies that $f \in \mathcal{Y}$. However the condition $\Delta^{-1} f \in L^{3,\infty} \cap L^4$ excludes some important singular force: $\Delta^{-1} f \sim \frac{1}{|x|}$, which belongs to $\mathcal{Y}$.

Theorem 2.2.9 (Long-time behavior of global solutions)**.** *Let* $p > 3$. *Suppose that* $f \in \mathcal{Y}$ *is a given external force such that* $\|f\|_{\mathcal{Y}} < c_1(p)$, *where* $c_1(p)$ *is the small constant in Theorem 2.2.8.*

Suppose that $u_f \in C_w([0,\infty), L^{3,\infty})$ *is an priori global solution to* (NSf) *described in Theorem 2.2.7, whose initial data* $u_0 \in L^{3,\infty} \cap \dot{B}^{s_p}_{p,p}$. *Then there exists a constant* M *independent of* u_f *such that*

$$\limsup_{t\to\infty} \|u_f(t)\|_{L^{3,\infty}} \le M.$$

The idea of the proof of long-time behavior, as in [16, 5], consists in decomposing the initial velocity in a small part plus a square integrable part. The small part remains small by the small data theory and the square-integrable part will become small at some point by using some energy estimates. More precisely, we split the initial data $u_0 = \bar{u}_0 + v_0$, where $\bar{u}_0$ is small enough in $L^{3,\infty}$ and $v_0 \in L^2(\mathbb{R}^3) \cap L^{3,\infty}$. By the global existence of (NSf) for small initial data (see [9]) we have $NSf(\bar{u}_0) \in L^\infty(\mathbb{R}_+, L^{3,\infty})$ and $v := u_f - NSf(\bar{u}_0)$ satisfies the perturbation equation $PNS_{NSf(\bar{u}_0)}$. Compared to the unforced case, it is hard to obtain that v has finite energy on $[0,T]$ for any $0 < T < \infty$ in general, which is the reason why the restriction on external force: $\Delta^{-1} f \in L^{3,\infty} \cap L^4$ is crucial in Theorem 4.7 in [5]. In our case, we have obtained that v has finite energy on $[0,T]$ for any $0 < T < \infty$ by Theorem 2.2.7.

We show the stability of priori global solutions constructed in Theorem 2.2.7 in the following theorem.

Theorem 2.2.10 (Stability of global solutions)**.** *Let* $p > 3$. *Suppose that* $f \in C(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^3))$ *satisfies the same conditions as Theorem 2.2.9 and that* u_f *is an priori global solution to* (NSf) *described in Theorem 2.2.7 with initial data* $u_0 \in \dot{B}^{s_p}_{p,p}$.

Then there exists an δ *(depending on* u_f*) with the following property.*
For any initial data $\bar{u}_0 \in \dot{B}^{s_p}_{p,p}$ *satisfying* $\|u_0 - \bar{u}_0\|_{\dot{B}^{s_p}_{p,p}} < \delta$, *there exist a global solution* $\bar{u}_f$

to (NSf) *with initial data* $\bar{u}_0$*, and*

$$\|u_f(t) - \bar{u}_f(t)\|_{\mathbb{L}_p^{r_0:\infty}(\infty)} \lesssim \|u_0 - \bar{u}\|_{\dot{B}^{s_p}_{p,p}}.$$

The stability result for the solution introduced as above is an extension of Theorem 3.1 in [16]. We prove it with a similar proof to Theorem 3.1 in [16], the difference between these two cases is that there is a small bounded in time and no-decay in time drift part in our case. The proofs of Theorem 2.2.9 and 2.2.10 are presented in Section 4.

2.3 Existence and uniqueness of (NSf)

The aim of this section is to prove Theorem 2.2.7 and Theorem 2.2.8. Let us recall the situation: Let $p > 3$ be fixed and the external force $f \in \mathcal{Y}$ and $\|f\|_{\mathcal{Y}} < c(p)$, where $c(p)$ is a small universal constant smaller than the constant ε in Theorem 2.1 of [5]. The class of initial data is $\dot{B}^{s_p}_{p,p}$.

2.3.1 Existence of (NSf)

By Theorem 2.1 in [5], there exists a unique solution $U_f := NSf(0) \in L^\infty(\mathbb{R}_+, L^{3,\infty})$ such that

$$\|U_f\|_{L^\infty(\mathbb{R}_+, L^{3,\infty})} \leq 2\|f\|_{\mathcal{Y}} < 2c(p). \tag{2.5}$$

Then we can transform the Cauchy problem of (NSf) into the Cauchy problem of (PNS_{U_f}):

$$(PNS_{U_f}) \begin{cases} \partial_t v - \Delta v + v \cdot \nabla v + U_f \cdot \nabla v + v \cdot \nabla U_f = -\nabla \pi, \\ \nabla \cdot v = 0, \\ v|_{t=0} = v_0, \end{cases}$$

whose integral form is

$$v(t,x) = e^{t\Delta} u_0 + B(v,v) + 2B(2U_f, v),$$

where B is defined in (2.3). We use a standard fixed point lemma to solve the above system: We first recall without proofs the following lemma.

Lemma 2.3.1. *Let* X *be a Banach space,* L *a linear operator from* $X \to X$ *such that a constant* $\lambda < 1$ *exists such that*

$$\forall x \in X, \;\; \|L(x)\|_X \leq \lambda \|x\|_X,$$

and B *a bilinear operator such that for some* γ*,*

$$\forall (x,y) \in X^2, \;\; \|B(x,y)\|_X \leq \gamma \|x\|_X \|y\|_X.$$

Then for all $x_1 \in X$ *such that*

$$\|x_1\|_X < \frac{(1-\lambda)^2}{4\gamma},$$

the sequence defined by

$$x^{(n+1)} = x_1 + L(x^{(n)}) + B(x^{(n)}, x^{(n)})$$

with $x^{(0)} = 0$ *converges in* X *and towards the unique solution of*

$$x = x_1 + L(x) + B(x, x)$$

such that

$$2\gamma\|x\|_X \leq (1 - \lambda).$$

In the proof of Theorem 2.2.7, we first show the local in time existence of (NSf) with initial data in $\dot{B}^{s_p}_{p,p}$. Next, we show the propagation of the regularity of the solution constructed above with initial data, in addition, belonging to $L^{3,\infty}$ or L^2.

Proof of Theorem 2.2.7. Let $u_0 \in \dot{B}^{s_p}_{p,p}$ be a divergence-free vector field. We note that v is the solution satisfying system (PNS_{U_f}) with initial data u_0.

Existence: It is clear that if there exists a solution v to (PNS_{U_f}) with initial data u_0 on $[0, T]$, then $v + U_f$ is a solution to (NSf) with initial data u_0. Hence to prove the first statement in Theorem 2.2.7, it is enough to prove that for any initial data $u_0 \in \dot{B}^{s_p}_{p,p}$, there exists a unique $T^* > 0$ and a unique solution $v \in \mathbb{L}^{r_0:\infty}_p$ to (PNS_{U_f}) with initial data u_0.

Now we start to prove the above statement by applying Lemma 2.3.1.

We choose $\mathcal{L}^{r_0}([0, T], \dot{B}^{s+\frac{2}{r_0}}_{p,p})$ as the Banach space in Lemma 2.3.1, where $r_0 = \frac{2p}{p-1}$. It is easy to check that $s_p + \frac{2}{r_0} > 0$. To apply Lemma 2.3.1, we need to obtain that $B(u, v)$ defined in (2.3) is a continuous bilinear operator from $\mathcal{L}^{r_0}([0, T], \dot{B}^{s+\frac{2}{r_0}}_{p,p}) \times \mathcal{L}^{r_0}([0, T], \dot{B}^{s+\frac{2}{r_0}}_{p,p})$ to $\mathcal{L}^{r_0}([0, T], \dot{B}^{s+\frac{2}{r_0}}_{p,p})$ and the linear operator $L(v) := 2B(2U_f, v)$ is continuous on $\mathcal{L}^{r_0}([0, T], \dot{B}^{s+\frac{2}{r_0}}_{p,p})$ with its norm strictly smaller than 1.

In fact, according to the first statement in Lemma 2.6.2 and the first statement of Proposition 2.6.3, we have that B is a continuous operator from $\mathcal{L}^{r_0}([0, T]; \dot{B}^{s_p+\frac{2}{r_0}}_{p,p}) \times \mathcal{L}^{r_0}([0, T]; \dot{B}^{s_p+\frac{2}{r_0}}_{p,p})$ to itself and hence, for some $\gamma > 0$

$$\|B(v_1, v_2)\|_{\mathcal{L}^{r_0}([0,T];\dot{B}^{s_p+\frac{2}{r_0}}_{p,p})} \leq \gamma\|v_1\|_{\mathcal{L}^{r_0}([0,T];\dot{B}^{s_p+\frac{2}{r_0}}_{p,p})}\|v_2\|_{\mathcal{L}^{r_0}([0,T];\dot{B}^{s_p+\frac{2}{r_0}}_{p,p})}.$$

According to the third statement in Proposition 2.6.3, replacing w by U_f, we have for any $v \in \mathcal{L}^{r_0}([0, T]; \dot{B}^{s_p+\frac{2}{r_0}}_{p,p})$,

$$\begin{aligned}\|B(2U_f, v)\|_{\mathcal{L}^{r_0}([0,T];\dot{B}^{s_p+\frac{2}{r_0}}_{p,p})} &\leq \|B(2U_f, v)\|_{\mathcal{L}^{r_0}([0,T];\dot{B}^{s_{\bar{p}}+\frac{2}{r_0}}_{\bar{p},p})} \\ &\leq 2C(p)\|U_f\|_{L^\infty(\mathbb{R}_+,L^{3,\infty})}\|v\|_{\mathcal{L}^{r_0}([0,T];\dot{B}^{s_p+\frac{2}{r_0}}_{p,p})},\end{aligned}$$

where $C(p) \to \infty$ as $p \to \infty$ and $\frac{1}{\bar{p}} = \frac{1}{3} + \frac{1}{6p}$. By taking $c(p) \leq (4C(p))^{-1}$, then by the above estimate and (2.5) , we have

$$\lambda := 2\gamma_1\bar{C}(p)\|U_f\|_{L^\infty(\mathbb{R}_+,L^{3,\infty})} < 1,$$

and

$$\|B(2U_f, v)\|_{\mathcal{L}^{r_0}([0,T];\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})} \leq \lambda \|v\|_{\mathcal{L}^{r_0}([0,T];\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})}.$$

Therefore according to Lemma 2.3.1 and the fact that

$$\|e^{t\Delta}u_0\|_{\mathcal{L}^{r_0}(\mathbb{R}_+,\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})} \lesssim \|u_0\|_{\dot{B}_{p,p}^{s_p}},$$

one can find a small enough number $\eta(p,f)$ such that, for any $u_0 \in \dot{B}_{p,p}^{s_p}$ with $\|u_0\|_{\dot{B}_{p,p}^{s_p}} < \eta$, there exists a unique global solution $v \in \mathcal{L}^{r_0}(\mathbb{R}_+, \dot{B}_{p,p}^{s_p+\frac{2}{r_0}})$ with initial data u_0 satisfying that

$$\|v\|_{\mathcal{L}^{r_0}(\mathbb{R}_+,\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})} \leq \frac{1-\lambda}{2\gamma}.$$

Moreover we notice that for any given $u_0 \in \dot{B}_{p,p}^{s_p}$ and any $T > 0$,

$$\begin{aligned}\|e^{t\Delta}u_0\|_{\mathcal{L}^{r_0}([0,T];\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})} &= \Big(\sum_{j\in\mathbb{Z}} \big(2^{j(s_p+\frac{2}{r_0})}\|\Delta_j e^{t\Delta}u_0\|_{L^{r_0}([0,T];L^p_x)}\big)^p\Big)^{\frac{1}{p}} \\ = \quad &\Big\|\big(1-e^{-r_0Tc_p2^{2j}}\big)^{\frac{1}{r_0}} 2^{js_p}\|\Delta_j u_0\|_{L^p}\Big\|_{\ell^p}.\end{aligned}$$

Next, an application of Lebesgue's dominated convergence theorem shows that

$$\lim_{t\to 0}\Big\|\big(1-e^{-r_0Tc_p2^{2j}}\big)^{\frac{1}{r_0}} 2^{js_p}\|\Delta_j u_0\|_{L^p}\Big\|_{\ell^p} = 0.$$

It follows that for any given $u_0 \in \dot{B}_{p,p}^{s_p}$, there exists T_0 such that

$$\|e^{t\Delta}u_0\|_{\mathcal{L}^{r_0}([0,T_0];\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})} < \frac{(1-\lambda)^2}{4\gamma}.$$

Therefore we have $v \in \mathcal{L}^{r_0}([0,T_0];\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})$.

Hence for any $u_0 \in \dot{B}_{p,p}^{s_p}$, there exists a $T^*(u_0,f) > 0$ such that $v \in \mathcal{L}^{r_0}([0,T^*),\dot{B}_{p,p}^{s_p})$. And according to Lemma 2.6.2, we obtain that $v \in \mathcal{L}^{r}([0,T^*);\dot{B}_{p,p}^{s_p+\frac{2}{r}})$ for any $r \in [r_0,\infty]$, which implies that $v \in \mathbb{L}_p^{r_0:\infty}[T<T^*]$.

When $T^* < \infty$, we claim that

$$\lim_{T\to T^*}\|v\|_{\mathbb{L}_p^{r_0:\infty}(T)} = \infty,$$

by a similar argument in [13]. In fact, if

$$\lim_{T\to T^*}\|v\|_{\mathbb{L}_p^{r_0:\infty}(T)} < \infty,$$

in particular,

$$v \in \mathcal{L}^{\infty}([0,T^*),\dot{B}_{p,p}^{s_p})$$

which implies that for any $\varepsilon > 0$, there exists $N(\varepsilon)$ such that for any $t' \in [0, T^*)$

$$\Big(\sum_{|j|>N(\varepsilon)} 2^{pjs_p} \|\Delta_j v(t')\|_{L^p}^p \Big)^{\frac{1}{p}} < \varepsilon.$$

Therefore for any fixed $t' \in [0, T^*)$,

$$\begin{aligned}
\|e^{t\Delta} v(t')\|_{\mathcal{L}^{r_0}([0,T];\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})} &= \Big\| \big(1 - e^{-r_0 T c_p 2^{2j}}\big)^{\frac{1}{r_0}} 2^{js_p} \|\Delta_j v(t')\|_{L^p} \Big\|_{\ell^p} \\
\lesssim \ & \Big(\sum_{|j|>N(\varepsilon)} 2^{pjs_p} \|\Delta_j v(t')\|_{L^p}^p \Big)^{\frac{1}{p}} + 2^{N(\varepsilon)s_p} (1 - e^{r_0 T}) \|v\|_{\mathcal{L}^\infty([0,T^*),\dot{B}_{p,p}^{s_p})} \\
\lesssim \ & \varepsilon + 2^{N(\varepsilon)s_p} (1 - e^{r_0 T}) \|v\|_{\mathcal{L}^\infty([0,T^*),\dot{B}_{p,p}^{s_p})},
\end{aligned}$$

which implies for any $t' \in [0, T^*)$, there exists a τ independent of $t' \in [0, T^*)$ such that

$$\|e^{t\Delta} v(t')\|_{\mathcal{L}^{r_0}([0,\tau];\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})} < \frac{(1-\lambda)^2}{4\gamma}.$$

Hence we obtain that $v \in \mathcal{L}^{r_0}([0, T^* + \tau/2], \dot{B}_{p,p}^{s_p+\frac{2}{r_0}})$, which contradicts the maximality of T^*.

To finish the proof of the first statement in Theorem 2.2.7, we need to prove v is the unique solution to (PNS_{U_f}) with initial data $u_0 \in \dot{B}_{p,p}^{s_p}$ in $\mathbb{L}_p^{r_0:\infty}[T < T^*]$. We suppose that $\bar{v} \in \mathbb{L}_p^{r_0:\infty}(T)$ for some $T < T^*$ is another solution to (PNS_{U_f}) with the same initial data u_0 and set $w := \bar{v} - v$. It is easy to check that w satisfies that

$$w = B(w, w) + B(2(U_f + v), w).$$

A similar argument as above implies that

$$\begin{aligned}
\|w\|_{\mathcal{L}_t^{r_0}(\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})} \leq & K_0 \|w\|^2_{\mathcal{L}_t^{r_0}(\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})} + K_0 \|v\|_{\mathcal{L}_t^{r_0}(\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})} \|w\|_{\mathcal{L}_t^{r_0}(\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})} \\
& + \lambda \|w\|_{\mathcal{L}_t^{r_0}(\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})},
\end{aligned}$$

for some $K_0 > 0$. This fact implies that one can find a $K_1 > K_0 > 0$ such that

$$\|w\|_{\mathcal{L}_t^{r_0}(\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})} \leq K_1 \|w\|^2_{\mathcal{L}_t^{r_0}(\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})} + K_1 \|v\|_{\mathcal{L}_t^{r_0}(\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})} \|w\|_{\mathcal{L}_t^{r_0}(\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})}.$$

We infer that

$$\|w\|_{\mathcal{L}_t^{r_0}(\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})} (K_1 \|w\|_{\mathcal{L}_t^{r_0}(\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})} + K_1 \|v\|_{\mathcal{L}_t^{r_0}(\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})} - 1) \geq 0. \tag{2.6}$$

By continuity of the norm of $\mathcal{L}_t^{r_0}(\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})$ with respect to the time, there exists $\tilde{T}$ such that for all $t \in [0, \tilde{T}]$

$$K_1 \|w\|_{\mathcal{L}_t^{r_0}(\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})} + K_1 \|v\|_{\mathcal{L}_t^{r_0}(\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})} - 1 < 0.$$

Therefore, for $t \in [0,\tilde{T}]$ relation (2.6) can hold only if $\|w\|_{\mathcal{L}^{r_0}_{\tilde{T}}(\dot{B}^{s_p+\frac{2}{r_0}}_{p,p})} = 0$, that is $w \equiv 0$ on $[0,\tilde{T}]$, by continuity again, $w \equiv 0$ on $[0,T]$ for any $T < T^*$.

The first statement of the theorem is proved.

Propagation of perturbations:

Next we turn to show the propagation of v. According to Theorem 2.5.1 by choosing $w = U_f$ and $\bar{w} = 0$, we have that v can be written as, for any $T \in [0,T^*)$

$$v = v^H + v^S,$$

where $v^H = H_{N_0} \in \mathbb{L}^{1:\infty}_p(\infty)$ and $v^S = W_{N_0} + Z_{N_0} \in L^\infty([0,T],L^{3,\infty})$ with N_0 being the largest integer such that $3(N_0-1) < p$. We first notice that in the case when $\bar{w} = 0$, H_N is a sum of a finite number of multilinear operators of order at most $N-1$, acting on $e^{t\Delta}u_0$ only. Hence according to Lemma 2.6.6 and an inductive argument, we obtain for any $N \geq 2$,

$$H_N \in L^\infty(\mathbb{R}_+,L^{3,\infty}),$$

which implies that $v^H \in L^\infty(\mathbb{R}_+,L^{3,\infty})$.

To prove the second statement of the theorem, we are left with the proof of $v \in C_w([0,T^*),L^{3,\infty})$. We notice that by Lemma 2 & 3 in [2], $e^{t\Delta}u_0 \in C_w([0,\infty),L^{3,\infty})$. This fact combined with Lemma 2.6.6 implies that for any $T \in [0,T^*)$

$$v = v^H + v^S = H_{N_0} + W_{N_0} + Z_{N_0} \in C_w([0,T^*),L^{3,\infty}).$$

The second statement of Theorem 2.2.7 is proved.

Finite energy of perturbations:

In the last part of the proof, we show that v has finite energy on $[0,T]$ for any $T < T^*$, if $u_0 \in \dot{B}^{s_p}_{p,p} \cap L^2$.

Now we suppose that $u_0 \in \dot{B}^{s_p}_{p,p} \cap L^2$ and $T \in [0,T^*)$ is fixed. We recall that

$$v = e^{t\Delta}u_0 + B(v,v) + B(2U_f,v).$$

It is clear that $e^{t\Delta}u_0 \in E(\infty)$. Hence we only need to prove $B(v,v)+B(2U_f,v) \in E(T)$.

By replacing v of $B(v,v)+B(2U_f,v)$ by $v^H + v^S$, we have

$$\begin{aligned} B(v,v)+B(2U_f,v) =& B(v^H, v^H + 2v^S + 2U_f) \\ &+ B(v^S, v^S + 2U_f). \end{aligned}$$

By applying Lemma 2.6.8 and the fact that $e^{t\Delta}u_0 \in E(\infty)$, we first obtain $v^H = H_{N_0} \in E(\infty)$. Again by Lemma 2.6.8, we obtain that

$$B(v^H, v^H + 2v^S + 2U_f) \in E(T),$$

provided that $v^H \in \mathbb{L}^{1:\infty}_p(\infty)$ and $v^S + U_f \in \mathcal{L}^\infty([0,T],\dot{B}^{s_q}_{q,\infty})$ where $q = \frac{3p}{p-2}$.

Now we turn to the proof of $B(v^S, v^S + 2U_f) \in E(T)$. We recall that

$$v^S + 2U_f \in L^\infty([0,T],L^{3,\infty}). \tag{2.7}$$

On the other hand, by $v^S \in \mathbb{L}^{r_0:\infty}_{\bar{p},p}(T)$ with some $r_0 = \frac{2p}{p-1}$ and some $\bar{p} < 3$, we have

$$v^S \in \mathbb{L}^{3:\infty}_{\bar{p},p}(T) \in \mathbb{L}^{3:\infty}_{6,\infty}(T),$$

provided that $\frac{2p}{p-1} < 3$ for any $p > 3$ and standard embedding $\mathbb{L}^{3:\infty}_{\bar{p},p}(T) \hookrightarrow \mathbb{L}^{3:\infty}_{6,\infty}(T)$. Hence by Lemma 2.6.9, we obtain

$$v^S \in L^2([0,T], L^{6,2}). \tag{2.8}$$

Thanks to (2.7) and (2.8), applying Lemma 2.6.8, we obtain

$$B(v^S, v^S + 2U_f) \in E(T).$$

Therefore we obtain $v \in E(T)$ Theorem 2.2.7 is proved. □

2.3.2 Uniqueness of (NSf)

Although the solutions in Theorem 2.2.7 need not be unique in $L^\infty_t(L^{3,\infty})$, the following arguement shows that the gap between two different solutions has infinite energy.

Proof of Theorem 2.2.8. Let $u_f \in C_w([0,T^*), L^{3,\infty})$ be a solution to (NSf) constructed in Theorem 2.2.7 with initial data $u_0 \in L^{3,\infty} \cap \dot{B}^{s_p}_{p,p}$.

We now prove the first statement in Theorem 2.2.8:

Assume that $\tilde{u}_f \in C_w([0,T], L^{3,\infty})$ for some $T < T^*$ is another solution to (NSf) with initial data u_0 and satisfies $w := \tilde{u}_f - u_f = w_1 + w_2$, where

$$w_1 \in \mathbb{L}^{r:\infty}_p(T) \;\text{ and }\; \|w_2\|_{L^\infty(\mathbb{R}_+, L^{3,\infty})} < 4c_1$$

for some $p > 3$, $2 < r < \frac{2p}{p-3}$ According to Theorem 2.2.7, u_f can be decomposed as

$$u_f = v + U_f,$$

where $v \in \mathbb{L}^{r:\infty}_p[T < T^*]$ and $U_f \in C_w(\mathbb{R}_+, L^{3,\infty})$ with $\|U_f\|_{L^\infty(\mathbb{R}_+, L^{3,\infty})} < 2c_1$.

We notice that w satisfies:

$$\begin{aligned} w &= B(w,w) + 2B(u_f, w) \\ &= B(w_1 + w_2, w) + B(2u_f, w) \\ &= B(w_1 + 2v, w) + B(w_2 + U_f, w). \end{aligned}$$

On the other hand, we notice that for any $q < 3$,

$$\mathcal{L}^\infty([0,T], \dot{B}^{s_q}_{q,\infty}) \hookrightarrow L^\infty([0,T], L^{3,\infty})$$

combining with $w_1, v \in \mathbb{L}^{r:\infty}_p(T)$ and $w \in L^\infty([0,T], L^{3,\infty})$, using Proposition 2.6.3, we obtain that, for any $\tau \in [0,T]$

$$\begin{aligned} \|B(w_1 + 2v, w)\|_{L^\infty([0,\tau], L^{3,\infty})} &\lesssim \|B(w_1 + 2v, w)\|_{\mathcal{L}^\infty([0,\tau], \dot{B}^{s_{\bar{p}}}_{\bar{p},p})} \\ &\le K\|w_1 + 2v\|_{\mathcal{L}^r([0,\tau]; \dot{B}^{s_p + \frac{2}{r}}_{p,p})} \|w\|_{L^\infty([0,\tau], L^{3,\infty})}. \end{aligned} \tag{2.9}$$

And according to Lemma 2.6.6, we obtain that

$$\|B(w_2 + U_f, w)\|_{L^\infty([0,\tau], L^{3,\infty})} \lesssim \|w_2 + U_f\|_{L^\infty(\mathbb{R}_+, L^{3,\infty})} \|w\|_{L^\infty([0,\tau], L^{3,\infty})}.$$

From the smallness of w_2 and U_f, which is

$$\|w_2\|_{L^\infty(\mathbb{R}_+, L^{3,\infty})} + \|U_f\|_{L^\infty(\mathbb{R}_+, L^{3,\infty})} < 6c_1,$$

we obtain that

$$\|B(w_2+U_f,w)\|_{L^\infty([0,\tau],L^{3,\infty})} \le \|w\|_{L^\infty([0,\tau],L^{3,\infty})}, \tag{2.10}$$

provided that c_1 is small enough.

By (2.9) and (2.10), we obtain that for any $\tau \in [0,T]$,

$$\|w\|_{L^\infty([0,\tau],L^{3,\infty})} \le K\|w_1+2v\|_{\mathcal{L}^r([0,\tau];\dot{B}^{s_p+\frac{2}{r}}_{p,p})}\|w\|_{L^\infty([0,\tau],L^{3,\infty})}.$$

By continuity of the norm of $\mathcal{L}^3_t(\dot{B}^{s_p+\frac{2}{3}}_{p,p})$ with respect to time, there exists N real numbers $(T_i)_{1\le i\le N}$ such that $T_1=0$ and $T_N=T$, satisfying that

$$[0,T]=\bigcup_{i=1}^{N-1}[T_i,T_{i+1}] \text{ and } \|w_1+2v\|_{\mathcal{L}^r([T_i,T_{i+1}];\dot{B}^{s_p+\frac{2}{r}}_{p,p})} \le \frac{1}{2K},$$

for all $i\in\{1,\dots,N-1\}$.

Now we prove that $w\equiv 0$ on $[T_i,T_{i+1}]$ for all $i\in\{1,\dots,N-1\}$ by induction. We first notice that

$$\begin{aligned}\|w\|_{L^\infty([0,T_2],L^{3,\infty})} &\le K\|w_1+2v\|_{\mathcal{L}^r([0,T_2];\dot{B}^{s_p+\frac{2}{r}}_{p,p})}\|w\|_{L^\infty([0,T_2],L^{3,\infty})}\\ &\le \frac{1}{2}\|w\|_{L^\infty([0,T_2],L^{3,\infty})},\end{aligned}$$

which implies that

$$w\equiv 0 \quad \text{on } [0,T_2].$$

Now we assume that $w\equiv 0$ on $[0,T_k]$ for some $k\ge 2$. Hence

$$\mathbf{1}_{[T_k,T]}(t)w = w = B(w_1+2v,\mathbf{1}_{[T_k,T]}(t)w)+B(w_2+U_f,\mathbf{1}_{[T_k,T]}(t)w).$$

Therefore we have the following bounds for w,

$$\begin{aligned}&\|w\|_{L^\infty([T_k,T_{k+1}],L^{3,\infty})} = \|w\|_{L^\infty([0,T_{k+1}],L^{3,\infty})}\\ \le\ &\frac{1}{2}\|B(w_1+2v,w)\|_{L^\infty([0,T_{k+1}],L^{3,\infty})}+\frac{1}{2}\|B(w_2+U_f,w)\|_{L^\infty([0,T_{k+1}],L^{3,\infty})}.\end{aligned}$$

Combining with (2.10), we have

$$\|w\|_{L^\infty([T_k,T_{k+1}],L^{3,\infty})} \le \|B(\mathbf{1}_{[T_k,T]}(w_1+2v),w)\|_{L^\infty([0,T_{k+1}],L^{3,\infty})}. \tag{2.11}$$

On the other hand, we notice that

$$B(w_1+2v,w)=B(w_1+2v,\mathbf{1}_{[T_k,T]}w)=B(\mathbf{1}_{[T_k,T]}(w_1+2v),w),$$

again by Lemma 2.6.3, we obtain that

$$\begin{aligned}&\|B(w_1+2v,w)\|_{L^\infty([0,T_{k+1}],L^{3,\infty})} = \|B(\mathbf{1}_{[T_k,T]}(w_1+2v),w)\|_{L^\infty([0,T_{k+1}],L^{3,\infty})}\\ \le\ &K\|\mathbf{1}_{[T_k,T]}(w_1+2v)\|_{\mathcal{L}^r([0,T_{k+1}];\dot{B}^{s_p+\frac{2}{r}}_{p,p})}\|w\|_{L^\infty([0,T_{k+1}],L^{3,\infty})}\\ =\ &K\|w_1+2v\|_{\mathcal{L}^r([T_k,T_{k+1}];\dot{B}^{s_p+\frac{2}{r}}_{p,p})}\|w\|_{L^\infty([T_k,T_{k+1}],L^{3,\infty})}\\ \le\ &\frac{1}{2}\|w\|_{L^\infty([T_k,T_{k+1}],L^{3,\infty})}.\end{aligned}$$

Hence, by the above estimate and (2.11), we have

$$\|w\|_{L^\infty([T_k,T_{k+1}],L^{3,\infty})} \leq \frac{1}{2}\|w\|_{L^\infty([T_k,T_{k+1}],L^{3,\infty})},$$

which implies that

$$w \equiv 0 \;\text{ on }\; [T_k, T_{k+1}].$$

Then we have $w \equiv 0$ on $[0,T]$. The first statement in Theorem 2.2.8 is proved.

Now we turn to prove the second statement in Theorem 2.2.8:

Assume that $\bar{u}_f \in C_w([0,T], L^{3,\infty})$ for some $T < T^*$ is another solution to (NSf) with same initial data u_0. We denote $w := \bar{u}_f - u_f$. By the assumption of the theorem, $w := \bar{u}_f - u_f \in C([0,T], L^{3,\infty})$ with $w(0) = 0$. We notice that w satisfies the following equation on $[0,T]$

$$w(t) = B(w + 2U_f, w) + B(2v, w),$$

where $v := u_f - U_f \in \mathbb{L}_p^{r_0:\infty}[T < T^*]$. According to Lemma 2.6.6, we have

$$\begin{aligned}\|B(w+&2U_f, w)\|_{L^\infty([0,t],L^{3,\infty})}\\ &\leq C\|w\|^2_{L^\infty([0,t],L^{3,\infty})} + C\|w\|_{L^\infty([0,t],L^{3,\infty})}\|U_f\|_{L^\infty([0,t],L^{3,\infty})}\\ &\leq C\|w\|_{L^\infty([0,t],L^{3,\infty})}(\|U_f\|_{L^\infty(\mathbb{R}_+,L^{3,\infty})} + \|w\|_{L^\infty([0,t],L^{3,\infty})}).\end{aligned} \tag{2.12}$$

According to the continuity of w in $L^{3,\infty}$ and the fact that $w(0) = 0$, one can choose a T_1 such that, combined with the smallness of U_f,

$$\|U_f\|_{L^\infty([0,t],L^{3,\infty})} + \|w\|_{L^\infty([0,t],L^{3,\infty})} \leq \frac{1}{3C},$$

which implies that

$$\|B(w + 2U_f, w)\|_{L^\infty([0,T_1],L^{3,\infty})} \leq \frac{1}{3}\|w\|_{L^\infty([0,T_1],L^{3,\infty})}. \tag{2.13}$$

By Lemma 2.6.3, by a similar argument as the above paragraph, we have that for any $t \in [0,T]$

$$\|B(2v, w)\|_{L^\infty([0,t],L^{3,\infty})} \leq C\|v\|_{\mathcal{L}^{r_0}([0,t],\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})}\|w\|_{L^\infty([0,t],L^{3,\infty})}.$$

By continuity of the norm of $\mathcal{L}^{r_0}([0,t], \dot{B}_{p,p}^{s_p+\frac{2}{r_0}})$ with respect to the time, there exists $T_2 > 0$ such that

$$C\|v\|_{\mathcal{L}^{r_0}([0,T_2],\dot{B}_{p,p}^{s_p+\frac{2}{r_0}})} < \frac{1}{3},$$

which implies that

$$\|B(2v, w)\|_{L^\infty([0,T_2],L^{3,\infty})} \leq \frac{1}{3}\|w\|_{L^\infty([0,T_2],L^{3,\infty})}. \tag{2.14}$$

According to (2.13) and (2.14), taking $T_0 = \min\{T_1, T_2\}$, we have

$$\begin{aligned}
&\|w\|_{L^\infty([0,T_0],L^{3,\infty})} \\
&\quad \leq \|B(w+2U_f,w)\|_{L^\infty([0,T_0],L^{3,\infty})} + \|B(2v,w)\|_{L^\infty([0,T_0],L^{3,\infty})} \\
&\quad \leq \frac{2}{3}\|w\|_{L^\infty([0,T_0],L^{3,\infty})},
\end{aligned}$$

which implies $w \equiv 0$ on $[0, T_1]$ and, by continuity, $w \equiv 0$ on $[0, T]$ too. Therefore we proved the second result in the theorem.

Now we are left with the proof of the last statement of the theorem. Since we need to apply Lemma 2.6.7 to obtain a uniform energy bound, we set $3 < p < 5$ to make sure that $v \in \mathcal{L}^p([0,T], \dot{B}^{s_p+\frac{2}{p}}_{p,p})$ with $s_p + \frac{2}{p} > 0$.

Assume that $\bar{u}_f \in C_w([0,T], L^{3,\infty})$ for some $T < T^*$ is another solution to (NSf) with same initial data u_0. We denote $w := \bar{u}_f - u_f$. By the assumption of the theorem, $\omega \in L^\infty([0,T], L^2) \cap L^2([0,T], \dot{H}^1)$ satisfies the following system:

$$\begin{cases}
\partial_t\omega - \Delta\omega + \omega\cdot\nabla\omega + u_f\cdot\nabla\omega + \omega\cdot\nabla u_f = -\nabla\pi, \\
\nabla\cdot\omega = 0, \\
\omega|_{t=0} = 0.
\end{cases}$$

Therefore we have the following energy equation, for any $t \in (0, T)$,

$$\|\omega(t)\|_{L^2}^2 + 2\int_0^t \|\nabla\omega(s)\|_{L^2}^2 ds = -2\int_0^t\int_{\mathbb{R}^3} \omega\cdot\nabla u_f\cdot\omega dxds.$$

According to Theorem 2.2.7, u_f can be written as $u_f = U_f + v$, where $U_f := NSf(0)$ is the solution to (NSf) with initial data 0 and $v \in \mathbb{L}_p^{r_0:\infty}[T < T^*]$ is the solution to (PNS_{U_f}) with initial data u_0. Therefore we have that

$$\begin{aligned}
&|\int_0^t\int_{\mathbb{R}^3} \omega\cdot\nabla u_f\cdot\omega dxds| \\
&\qquad\qquad \leq |\int_0^t\int_{\mathbb{R}^3} \omega\cdot\nabla(U_f)\cdot\omega dxds| + |\int_0^t\int_{\mathbb{R}^3} \omega\cdot\nabla v\cdot\omega dxds|.
\end{aligned}$$

By Young's inequality in Lorentz spaces, the first term on the right can be controlled by:

$$\begin{aligned}
|\int_0^t\int_{\mathbb{R}^3} \omega\cdot\nabla(U_f)\cdot\omega dxds| &= |\int_0^t\int_{\mathbb{R}^3} \omega\cdot\nabla\omega\cdot U_f dxds| \\
&\leq \int_0^t \|\omega(s)\|_{L^{6,2}}\|\nabla\omega(s)\|_{L^2}\|U_f(s)\|_{L^{3,\infty}} ds.
\end{aligned}$$

We observe now that $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^{6,2}(\mathbb{R}^3)$. This embedding follows from the Young inequality for Lorentz spaces after noticing that $(-\Delta)^{-\frac{1}{2}}$ is a convolution operator with a function bounded by $\frac{c}{|x|^2}$ whcih therefore belongs to $L^{\frac{3}{2},\infty}$. Hence

$$\int_0^t \|\omega(s)\|_{L^{6,2}}\|\nabla\omega(s)\|_{L^2}\|U_f(s)\|_{L^{3,\infty}} ds \leq \|U_f\|_{L^\infty(\mathbb{R}_+,L^{3,\infty})}\int_0^t \|\nabla\omega(s)\|_{L^2}^2 ds.$$

Since U_f is small enough in $L^\infty(\mathbb{R}_+, L^{3,\infty})$, we obtain

$$|\int_0^t \int_{\mathbb{R}^3} \omega \cdot \nabla(U_f) \cdot \omega dx ds| \le \frac{1}{3} \int_0^t \|\nabla\omega(s)\|_{L^2}^2 ds.$$

We recall that $v \in \mathbb{L}^{r_0:\infty}_{p,p}(T)$ with $3 < p < 5$ and one can take $r_0 = \frac{2p}{p-1}$. This implies $v \in \mathcal{L}^p([0,T], \dot{B}^{s_p+\frac{2}{p}}_{p,p})$ with $\frac{3}{p} + \frac{2}{p} > 1$. Applying Lemma 2.6.7, we obtain

$$|\int_0^t \int_{\mathbb{R}^3} \omega \cdot \nabla v \cdot \omega dx ds| \le C \int_0^t \|\omega(s)\|_{L^2}^2 \|v(s)\|^p_{\dot{B}^{s_4+\frac{2}{p}}_{p,p}} ds + \int_0^t \|\nabla\omega(s)\|_{L^2}^2 ds.$$

Then w satisfies the following energy inequality,

$$\|\omega(t)\|_{L^2}^2 + \int_0^t \|\nabla\omega(s)\|_{L^2}^2 ds \le C \int_0^t \|\omega(s)\|_{L^2}^2 \|v(s)\|^p_{\dot{B}^{s_p+\frac{2}{p}}_{p,p}} ds.$$

By Gronwall's inequality and the fact that $w|_{t=0} = 0$, we get

$$\|\omega(t)\|_{L^2}^2 + \int_0^t \|\nabla\omega(s)\|_{L^2}^2 ds \le 0.$$

Then $\omega \equiv 0$ on [0,T], which implies that $u_f \equiv \bar{u}_f$ on $[0,T]$. Hence we have proved the second statement of Theorem 2.2.8.

Theorem 2.2.8 is proved.

□

2.4 Long-time Behavior and Stability of Global Solutions

Let f be a given external force satisfying the assumption of Theorem 2.2.7. We consider a global in time solution u_f to (NSf) constructed in Theorem 2.2.7 with initial data $u_0 \in L^{3,\infty} \cap \dot{B}^{s_p}_{p,p}$. Also we are interested in the stability of this kind of global solutions.

2.4.1 Long-time behavior of global solutions

Now let us start to prove Theorem 2.2.9. In order to apply a weak-strong argument, we need to use the regularity result in Theorem 2.5.1 to obtain the local in time part has a local in time finite energy by a similar argument to the proof of the third statement of Theorem 2.2.7. However, we need to deal with a more complicated drift term than before.

Proof. Let $u_0 \in L^{3,\infty} \cap \dot{B}^{s_p}_{p,p}$. Suppose that $u_f \in C_w(\mathbb{R}_+, L^{3,\infty})$ is a solution to (NSf) with initial data u_0 such that

$$v := u_f - U_f \in \mathbb{L}^{r_0:\infty}_p[T < \infty],$$

where $U_f := NSf(0)$ and $r_0 = \frac{2p}{p-1}$. By the smallness assumption on f, we have $U_f \in L^\infty(\mathbb{R}_+, L^{3,\infty})$. Therefore to prove the theorem, we need to prove $v \in L^\infty(\mathbb{R}_+, L^{3,\infty})$. To achieve this goal, we only need to prove $v \in \mathbb{L}^{r_0:\infty}_p(\infty)$. More precisely, if $v \in \mathbb{L}^{r_0:\infty}_p(\infty)$, by choosing $T = \infty$, $w = U_f$ and $\bar{w} = 0$, Theorem 2.5.1 implies v can be written as

$$v = v^H + v^S,$$

where $v^H = H_{N_0} \in \mathbb{L}_p^{1:\infty}(\infty)$ and $v^S = W_{N_0} + Z_{N_0} \in L^\infty(\mathbb{R}_+, L^{3,\infty})$ with N_0 being the largest integer such that $3(N_0 - 1) < p$. We recall that in the case when $\bar{w} = 0$, $v^H = H_{N_0}$ is a sum of a finite number of multilinear operators of order at most $N_0 - 1$, acting on $e^{t\Delta}u_0$ only.

Hence according to $u_0 \in L^{3,\infty}$, Lemma 2.6.6 implies $H_{N_0} \in L^\infty(\mathbb{R}_+, L^{3,\infty})$. Thus $v \in L^\infty(\mathbb{R}_+, L^{3,\infty})$.

Now we start to prove that $v \in \mathbb{L}_p^{r_0:\infty}(\infty)$:

We use the method introduced by C.Calderón in [7] to prove results on weak solutions in L^p spaces, and used in [18] in the context of 2D Navier-Stokes equations: we split the initial data into two parts, $u_0 = \omega_0 + \bar{v}_0$, where $\omega_0 \in L^{3,\infty} \cap \dot{B}^{s_p}_{p,p} \cap L^2$ and $\bar{v}_0 \in L^{3,\infty} \cap \dot{B}^{s_p}_{p,p}$ such that

$$\|\bar{v}_0\|_{L^{3,\infty}} < \varepsilon(p) < c(p),$$

and its associated solution $\bar{v}$ to (PNS_{U_f}) satisfies that

$$\|\bar{v}\|_{\mathbb{L}_p^{r_0:\infty}(\infty)} \leq C(f)\|\bar{v}_0\|_{L^{3,\infty}}.$$

We define $\omega := v - \bar{v}$. It is easy to find that ω satisfies the following system,

$$\begin{cases} \partial_t \omega - \Delta\omega + \omega \cdot \nabla\omega + (U_f + \bar{v}) \cdot \nabla\omega + \omega \cdot \nabla(U_f + \bar{v}) = -\nabla\pi, \\ \nabla \cdot \omega = 0, \\ \omega|_{t=0} = \omega_0. \end{cases}$$

Also ω can be written as the following integral form

$$\omega = e^{t\Delta}\omega_0 + B(\omega, v + \bar{v} + 2U_f).$$

Step 1: We first show that for any $T \in (0, \infty)$, $\omega \in E(T)$. Suppose that $T > 0$ is fixed. We notice that $e^{t\Delta}\omega_0 \in E(T)$ provided $\omega_0 \in L^2$. Applying Theorem 2.5.1, by taking $w = U_f$ and $\bar{w} = v$, we obtain that ω can be written as

$$\omega = \omega^H + \omega^S,$$

where $\omega^H \in \mathbb{L}_p^{1:\infty}(\infty)$ and $\omega^S \in \mathbb{L}_{\tilde{p},p}^{r_0:\infty}$ for some $2 < \tilde{p} < 3$. Therefore we obtain

$$\omega^S \in \mathbb{L}^{3:\infty}_{6,\infty}(T),$$

provided that $r_0 = \frac{2p}{p-1} < 3$ for any $p > 3$. Hence by Lemma 2.6.8, we have

$$B(\omega^S, v + \bar{v} + 2U_f) \in E(T),$$

as $v + \bar{v} + 2U_f \in L^\infty([0,T], L^{3,\infty})$.

We recall that $\omega^H = H^E_{N_0}$, where $H^E_{N_0}$ can be written as

$$H^E_{N_0} = H^E_{N_0-1} + \sum_{M=0}^{N_0-2} B^M_{N_0-1,N_0-1}(\bar{v}^{\otimes M}, v_L^{\otimes(N_0-1-M)}),$$

where $B^M_{N_0-1,N_0-1}$ are (N_0-1)-linear operators and $v_L = e^{t\Delta}\omega_0$. We recall that

$$H_2^E = e^{t\Delta}\omega_0 \text{ and } H_3^E = H_2^E + B(e^{t\Delta}\omega_0, e^{t\Delta}\omega_0) + B(\bar{v}, e^{t\Delta}\omega_0).$$

Therefore by Lemma 2.6.8 and an inductive argument, we obtain that

$$H_{N_0}^E \in E(T),$$

provided that $\omega_0 \in L^2$ and $\bar{v} \in \mathbb{L}_p^{r_0:\infty}(T)$. Applying Lemma 2.6.8 again, we have

$$B(\omega^H, v + \bar{v} + 2U_f) \in E(T),$$

as $v + \bar{v} + 2U_f \in L^\infty([0,T], \dot{B}^{s_p}_{p,p})$ deduced by Lemma 2.6.4. Therefore we obtain that for any $T \in (0,\infty)$, $\omega \in E(T)$.

Step 2: In this we show a global energy estimate for ω. Let us write an energy estimate in L^2, starting at some time $t_0 \in (0,\infty)$. We get

$$\|\omega(t)\|_{L^2}^2 + 2\int_{t_0}^t \|\omega(s)\|_{L^2}^2 ds = \|\omega(t_0)\|_{L^2}^2 - 2\int_{t_0}^t \int_{\mathbb{R}^3} (\omega \cdot \nabla(\bar{v} + U_f) \cdot \omega dx ds.$$

We notice that

$$|\int_{t_0}^t \int_{\mathbb{R}^3} (\omega \cdot \nabla U_f) \cdot \omega dx ds| \leq \|U_f\|_{L^\infty(\mathbb{R}_+, L^{3,\infty})} \int_{t_0}^t \|\omega\|_{L^{6,2}} \|\nabla \omega\|_{L^2}.$$

We recall that $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^{6,2}(\mathbb{R}^3)$, which combined with the above relation implies that

$$|\int_{t_0}^t \int_{\mathbb{R}^3} (\omega \cdot \nabla U_f) \cdot \omega dx ds| \leq \|U_f\|_{L^\infty(\mathbb{R}_+, L^{3,\infty})} \int_{t_0}^t \|\nabla \omega(s)\|_{L^2}^2 ds$$

Since $\|U_f\|_{L^\infty(\mathbb{R}_+, L^{3,\infty})} \leq 2c_1(p)$ with $c_1(p)$ is small enough, hence we obtain

$$|\int_{t_0}^t \int_{\mathbb{R}^3} (\omega \cdot \nabla U_f) \cdot \omega dx ds| \leq \frac{1}{4} \int_{t_0}^t \|\nabla \omega(s)\|_{L^2}^2 ds. \tag{2.15}$$

On the other hand, by a similar argument as above, we have that $\bar{v}$ can be written as,

$$\bar{v} = \bar{v}^H + \bar{v}^S,$$

where $\bar{v}^H \in \mathbb{L}_p^{1:\infty}(\infty)$ and $\bar{v}^S \in \mathbb{L}_{\tilde{p},p}^{r_0;\infty}$ for some $2 < \tilde{p} < 3$. Hence

$$\begin{aligned} &|\int_{t_0}^t \int_{\mathbb{R}^3} (\omega \cdot \nabla \bar{v}) \cdot \omega dx ds| \\ &\quad \leq |\int_{t_0}^t \int_{\mathbb{R}^3} (\omega \cdot \nabla \bar{v}^H) \cdot \omega dx ds| + |\int_{t_0}^t \int_{\mathbb{R}^3} (\omega \cdot \nabla(\bar{v}^S) \cdot \omega dx ds|. \end{aligned}$$

We recall that $\bar{v}^H$ is a sum of a finite number of multilinear operators of order at most $N_0 - 1$, acting on $e^{t\Delta}u_0$ only, as $\bar{v} \in \mathbb{L}_p^{r_0:\infty}(\infty)$ is the small global solution to (PNS_{U_f}), which is the case of $\bar{w} = 0$. Then by Lemma 2.6.5 (for details see [16]), we obtain that there exists K only depending on p,

$$\sup_{t>0} t^{\frac{1}{2}} \|\bar{v}^H\|_{L^\infty} \lesssim \|\bar{v}_0\|_{\dot{B}^{s_p}_{p,p}} \leq K\varepsilon(p)$$

Therefore

$$
\begin{aligned}
|\int_{t_0}^t \int_{\mathbb{R}^3} (\omega \cdot \nabla \bar{v}^H) \cdot \omega dx ds| &\le \int_{t_0}^t \|\omega(s)\|_{L^2} \|\nabla \omega\|_{L^2} \sqrt{s} \|\bar{v}^H\|_{L^\infty} \frac{ds}{\sqrt{s}} \\
&\le \frac{1}{4} \int_{t_0}^t \|\nabla \omega(s)\|_{L^2}^2 ds + K\varepsilon^2 \int_{t_0}^t \|\omega(s)\|_{L^2}^2 \frac{ds}{\sqrt{s}}.
\end{aligned}
\tag{2.16}
$$

Again by Theorem 2.5.1, we also notice that there exists K_1 only depending on p

$$
\|\bar{v}^S\|_{L^\infty(\mathbb{R}_+, L^{3,\infty})} \lesssim \|\bar{W}_{N_0}\|_{\mathbb{L}^{r_0:\infty}_{\bar{p},p}(\infty)} + \|\bar{Z}_{N_0}\|_{\mathbb{L}^{r_N;\infty}_{p_{N_2}}(\infty)} \lesssim \|\bar{v}_0\|_{\dot{B}^{s_p}_{p,p}} \le K_1(\varepsilon(p)).
$$

Hence we obtain

$$
\begin{aligned}
&|\int_{t_0}^t \int_{\mathbb{R}^3} (\omega \cdot \nabla (\bar{v}^S) \cdot \omega dx ds| \\
&\quad \le \|\bar{v}^S\|_{L^\infty(\mathbb{R}_+, L^{3,\infty})} \int_{t_0}^t \|\omega(s)\|_{L^{6,2}} \|\nabla \omega(s)\|_{L^2} ds \\
&\quad \le \|\bar{v}^S\|_{L^\infty(\mathbb{R}_+, L^{3,\infty})} \int_{t_0}^t \|\nabla \omega(s)\|_{L^2}^2 ds \\
&\quad \le K_1 \varepsilon(p) \int_{t_0}^t \|\nabla \omega(s)\|_{L^2}^2 ds.
\end{aligned}
$$

Since $\varepsilon(p)$ is small enough, we have

$$
|\int_{t_0}^t \int_{\mathbb{R}^3} (\omega \cdot \nabla (\bar{v}^S) \cdot \omega dx ds| \le \frac{1}{4} \int_{t_0}^t \|\nabla \omega(s)\|_{L^2}^2 ds.
\tag{2.17}
$$

According to (2.15), (2.16) and (2.17), we have the following energy estimate for w,

$$
\|\omega(t)\|_{L^2}^2 + \frac{1}{2} \int_{t_0}^t \|\omega(s)\|_{L^2}^2 ds \le \|\omega(t_0)\|_{L^2}^2 + K^2 \varepsilon^2 \int_{t_0}^t \|\omega(s)\|_{L^2}^2 \frac{ds}{\sqrt{s}}.
$$

We now use Gronwall's Lemma, which yields

$$
\|\omega(t)\|_{L^2}^2 + \frac{1}{2} \int_{t_0}^t \|\omega(s)\|_{L^2}^2 ds \le \|\omega(t_0)\|_{L^2}^2 \left(\frac{t}{t_0}\right)^{K^2 \varepsilon^2}.
$$

Now by Sobolev embedding and interpolation we have

$$
\int_{t_0}^t \|\omega(s)\|_{\dot{B}^{s_p}_{p,p}}^4 ds \lesssim \int_{t_0}^t \|\omega(s)\|_{\dot{H}^{\frac{1}{2}}}^4 ds \le \int_{t_0}^t \|\omega(s)\|_{L^2}^2 \|\nabla \omega(s)\|_{L^2}^2 ds,
$$

which by the above estimate yields

$$
(t - t_0) \inf_{s \in [0,t]} \|\omega(s)\|_{\dot{B}^{s_p}_{p,p}}^4 \lesssim \|\omega(t_0)\|_{L^2}^4 \left(\frac{t}{t_0}\right)^{2K^2 \varepsilon^2}.
$$

Hence we obtain

$$
\inf_{s \in [0,t]} \|\omega(s)\|_{\dot{B}^{s_p}_{p,p}} \lesssim \|\omega(t_0)\|_{L^2} \left(\frac{t}{t_0}\right)^{K^2 \varepsilon^2 / 2} (t - t_0)^{-\frac{1}{4}}.
$$

In particular we can write, for all $t \geq t_0 + 1$,

$$\inf_{s\in[0,t]} \|\omega(s)\|_{\dot{B}^{s_p}_{p,p}} \lesssim \|\omega(t_0)\|_{L^2} t^{\frac{K^2\varepsilon^2}{2}-\frac{1}{4}}$$

which can be made arbitrarily small for $\varepsilon(p)\frac{1}{2K}$ and t large enough. It follows that one can find a time τ_0 such that

$$\|v(\tau_0)\|_{\dot{B}^{s_p}_{p,p}} \leq \eta(p).$$

By Theorem 2.2.7, we have $v \in \mathbb{L}^{r_0:\infty}_p(\infty)$.

Theorem 2.2.9 is proved. □

2.4.2 Stability of global solutions

We are now in a position to show the stability of an a priori global solution constructed in Theorem 2.2.7: let us prove Theorem 2.2.10.

Proof. Suppose that a divergence free vector field $u_0 \in \dot{B}^{s_p}_{p,p}$ generating a global solution $u_f \in \mathbb{L}^{r_0:\infty}_p[T < \infty] + C_w(\mathbb{R}_+, L^{3,\infty})$ with $r_0 = \frac{2p}{p-1}$ such that $v := u_f - U_f \in \mathbb{L}^{r_0:\infty}_p[T < \infty]$, where $U_f := NSf(0)$. According to Theorem 2.2.9, we obtain that actually

$$v \in \mathbb{L}^{r_0:\infty}_p(\infty).$$

Now let $\bar{u}_0 \in \dot{B}^{s_p}_{p,p}$ be another divergence free vector field. By Theorem 2.2.7, there exist a $T^*(\bar{u}_0)$ and a solution $\bar{u}_f \in \mathbb{L}^{r_0:\infty}_p[T < T^*(\bar{u}_0)] + C_w(\mathbb{R}_+, L^{3,\infty})$ such that $\bar{u}_f - U_f \in \mathbb{L}^{r_0:\infty}_p[T < T^*(\bar{u}_0)]$. We mention that the life span $T^*(\bar{u}_0)$ is priori finite.

We denote $w := \bar{u}_f - u_f$, then it is enough to prove that for $\|w|_{t=0}\|_{\dot{B}^{s_p}_{p,p}}$ small enough $w \in \mathbb{L}^{r_0:\infty}_p(\infty)$.

The function w satisfies the following system:

$$\begin{cases} \partial_t w - \Delta w + w \cdot \nabla w + (v + U_f) \cdot \nabla w + w \cdot \nabla(v + U_f) = -\nabla\pi, \\ \nabla \cdot w = 0, \\ w|_{t=0} = w_0. \end{cases}$$

We deduce from Proposition 4.1 in [16] and Lemma 2.6.2 & 2.6.3 that w satisfies the following estimate:

$$\begin{aligned} &\sup_{t\in[\alpha,\beta]} \|w(t)\|_{\dot{B}^{s_p}_{p,p}} + \|w\|_{\mathcal{L}^{r_0}([\alpha,\beta],\dot{B}^{s_p+\frac{2}{r_0}}_{p,p})} \\ \leq &K\|w(\alpha)\|_{\dot{B}^{s_p}_{p,p}} + K\|w\|^2_{\mathcal{L}^{r_0}([\alpha,\beta],\dot{B}^{s_p+\frac{2}{r_0}}_{p,p})} + K\|v\|_{\mathcal{L}^{r_0}([\alpha,\beta],\dot{B}^{s_p+\frac{2}{r_0}}_{p,p})} \|w\|_{\mathcal{L}^{r_0}([\alpha,\beta],\dot{B}^{s_p+\frac{2}{r_0}}_{p,p})} \end{aligned} \tag{2.18}$$

for some constant $K > 1$ and all times $\alpha, \beta \in [0, T]$. Then there exists N real numbers $(T_i)_{1\leq i\leq N}$ such that $T_1 = 0$ and $T_N = \infty$, satisfying

$$\mathbb{R}_+ = \cup_{i=1}^{N_1}[T_i, T_{i+1}] \text{ and } \|v\|_{\mathcal{L}^{r_0}([T_i,T_{i+1}],\dot{B}^{s_p+\frac{2}{r_0}}_{p,p})} \leq \frac{1}{4K}, \ \forall i \in \{1, ..N-1\}. \tag{2.19}$$

Suppose that

$$\|w_0\|_{\dot{B}^{s_p}_{p,p}} \leq \frac{1}{8KN(2K)^N}. \tag{2.20}$$

Then there exists a maximal time $T_0 \in \mathbb{R}_+ \cup \{\infty\}$ such that

$$\|w\|_{\mathcal{L}^{r_0}([0,T_0],\dot{B}^{s_p}_{p,p})} \leq \frac{1}{4K}. \tag{2.21}$$

If $T = \infty$ then the theorem is proved. Suppose now that $T_0 < \infty$. Then we can find an integer $k \in \{1, ..N_1\}$ such that

$$T_k \leq T_0 < T_{k+1}.$$

Then we have

$$\|w\|_{\mathcal{L}^{r_0}([T_i,T_{i+1}],\dot{B}^{s_p+\frac{2}{r_0}}_{p,p})} \leq 2K\|w(T_i)\|_{\dot{B}^{s_p}_{p,p}}$$

which implies that

$$\sup_{t\in[T_i,T_{i+1}]} \|w(t)\|_{\dot{B}^{s_p}_{p,p}} \leq 2K\|w(T_i)\|_{\dot{B}^{s_p}_{p,p}}.$$

By induction, we have for all $i \in \{1, ..., k-1\}$,

$$\|w(T_i)\|_{\dot{B}^{s_p}_{p,p}} \leq (2K)^{i-1}\|w_0\|_{\dot{B}^{s_p}_{p,p}}.$$

We conclude from the above two results that

$$\|w\|_{\mathcal{L}^{r_0}([T_i,T_{i+1}],\dot{B}^{s_p+\frac{2}{r_0}}_{p,p})} \leq (2K)^i\|w_0\|_{\dot{B}^{s_p}_{p,p}}$$

and

$$\sup_{t\in[T_i,T_{i+1}]} \|w(t)\|_{\dot{B}^{s_p}_{p,p}} \leq (2K)^i\|w_0\|_{\dot{B}^{s_p}_{p,p}}.$$

for all $i \leq k-1$. The same arguments as above also apply on the interval $[T_k, T_0]$ and yield

$$\|w\|_{\mathcal{L}^{r_0}([T_k,T_0],\dot{B}^{s_p+\frac{2}{r_0}}_{p,p})} \leq (2K)^k\|w_0\|_{\dot{B}^{s_p}_{p,p}}$$

and

$$\sup_{t\in[T_k,T_0]} \|w(t)\|_{\dot{B}^{s_p}_{p,p}} \leq (2K)^k\|w_0\|_{\dot{B}^{s_p}_{p,p}}.$$

On the other hand,

$$\begin{aligned}
\|w\|_{\mathcal{L}^{r_0}([0,T_0],\dot{B}^{s_p+\frac{2}{r_0}}_{p,p})} &\leq \sum_{i=1}^{k-1} \|w\|_{\mathcal{L}^{r_0}([T_i,T_{i+1}],\dot{B}^{s_p+\frac{2}{r_0}}_{p,p})} + \|w\|_{\mathcal{L}^{r_0}([T_k,T_0],\dot{B}^{s_p+\frac{2}{r_0}}_{p,p})} \\
&\leq N(2K)^N\|w_0\|_{\dot{B}^{s_p}_{p,p}} < \frac{1}{4K}.
\end{aligned}$$

Under assumption (2.21) this contracdicts the maximality of T_0. Then the theorem is proved.

$\square$

2.5 Regularity via iteration

Consider the following equation,

$$v(t,x) = e^{t\Delta}v_0 + B(v,v) + B(w,v) + B(\bar{w},v), \tag{2.22}$$

where B is defined in (2.3) . This section is devoted to showing the regularity of the solution to (2.22) by using an iteration method introduced in [16, 17] and we adopt a similar notation in [17].

Theorem 2.5.1 (Regularity). *Let $p > 3$ and $2 < r_0 < \frac{2p}{p-3}$. And let $w \in L^\infty(\mathbb{R}_+, L^{3,\infty})$ and $\bar{w} \in \mathbb{L}_p^{r_0:\infty}(\infty)$. Suppose that $v \in \mathbb{L}_p^{r_0:\infty}(T)$ for some $T > 0$ satisfies (2.22)with initial data $v_0 \in \dot{B}^{s_p}_{p,p}$.*

Then for any integer $N \geq 2$ such that $3(N-1) < p$, there are $H_N \in \mathbb{L}_p^{1;\infty}(\infty)$, $W_N \in \mathbb{L}_{\bar{p},p}^{r_0:\infty}$ for some $2 < \bar{p} < 3$ and $Z_N \in \mathbb{L}_{p_N}^{r_N;\infty}(T)$ with $p_N := \frac{p}{N}$ and $r_N = \max\{1, \frac{r_0}{N}\}$ such that,

$$v = H_N + W_N + Z_N. \tag{2.23}$$

In particular, by taking $N_0 := \max\{N \in \mathbb{N} : N \geq 2, 3(N-1) < p\}$, we obtain that v can be written as

$$v = v^H + v^S,$$

where $v^H := H_{N_0} \in \mathbb{L}_p^{1:\infty}(\infty)$ and $v^S := W_{N_0} + Z_{N_0} \in \mathbb{L}_{\tilde{p},p}^{r_0:\infty}(T) \hookrightarrow L^\infty([0,T], L^{3,\infty})$ with $\tilde{p} := \max\{\bar{p}, p_{N_0}\}$.

The argument leading to a similar result to the above theorem in the case $w = \bar{w} = 0$ can be found in [16] and [17] (in turn inspired by [24]). The idea of proving Theorem 2.5.1 is nearly the same as the idea in [16] and [17]. However, since in our case we need to handle two kinds of drift terms, the decomposition via iteration becomes much more complicated than those results. More precisely, there are two main difference with previous reuslts:

- the fact that one of the drift terms w does not have decay in time and cannot be approximated by smooth functions limits the decay in time and the regularity of W_N. That is no matter how many times we iterate, there is at least one term of W_N only belonging to $\mathbb{L}_{\bar{p},p}^{r_0:\infty}(T)$.
- Compared with the previous results in the case when $\bar{w} = 0$ (for details, see [16]), we cannot obtain that H_N belongs to Kato's spaces in general.

In the following, we adapt most of the notations in the proof of Lemma 3.3 in [17].

Proof. Let $v \in \mathbb{L}_p^{r_0:\infty}(T)$ for some $T > 0$ satisfies (2.22). We can write v as

$$v = v_L + B(v,v) + B(w,v) + B(\bar{w},v), \tag{2.24}$$

where

$$v_L := e^{t\Delta}v_0.$$

This gives the desired expansion when $N = 2$: We note that

$$v = H_2 + W_2 + Z_2,$$

where

$$H_2 = v_L, \;\; W_2 = B(w,v) \;\text{ and }\; Z_2 = B(v,v) + B(\bar{w},v).$$

Lemma 2.6.1 implies that $H_2 \in \mathbb{L}_p^{1:\infty}(\infty)$. According to the second and last statement in Proposition 2.6.3, we have

$$\|B(v,v)\|_{\mathbb{L}_{\frac{p}{2}}^{\frac{r_0}{2}:\infty}(T)} \lesssim \|v\|^2_{\mathbb{L}_p^{r_0:\infty}(T)}$$

and

$$\|B(\bar{v},v)\|_{\mathbb{L}_{\frac{p}{2}}^{\frac{r_0}{2}:\infty}(T)} \lesssim \|v\|_{\mathbb{L}_p^{r_0:\infty}(T)} \|\bar{w}\|_{\mathbb{L}_p^{r_0:\infty}(\infty)},$$

which implies $Z_2 \in \mathbb{L}_{\frac{p}{2}}^{\frac{r_0}{2}:\infty}(T)$.
Note that the fact that the bilinear term $B(v,v)$ and linear $B(\bar{w},v)$ allow to pass from an L^p to an $L^{\frac{p}{2}}$ integrability is a key feature in this proof.

We recall the embedding property $L^{3,\infty} \hookrightarrow \dot{B}^{s_q}_{q,\infty}$ for any $q > 3$. Combining with the above property with the last statement of Proposition 2.6.3 by taking $q = \frac{3p}{p-2}$, we obtain that

$$\|B(w,v)\|_{\mathbb{L}^{r_0:\infty}_{\frac{6p}{2p+1},p}(T)} \lesssim \|w\|_{L^\infty(\mathbb{R}_+,L^{3,\infty})} \|v\|_{\mathbb{L}_p^{r_0:\infty}(T)}.$$

Hence $W_2 \in \mathbb{L}^{r_0:\infty}_{\bar{p},p}(T)$ with $\bar{p} = \frac{6p}{2p+1} < 3$. Therefore we prove Theorem 2.5.1 in the case $N = 2$.

Next we plug the expansion (2.24) in to the term $Z_2(v) := B(v,v) + B(\bar{w},v)$, to find

$$\begin{aligned}
u =& v_L + B(w,v) + B(v,v) + B(\bar{w},v)\\
=& v_L + B(w,v) + B(\bar{w}, v_L + B(w,v) + B(v,v) + B(\bar{w},v))\\
&+ B(v_L + B(w,v) + B(v,v) + B(\bar{w},v), v_L + B(w,v) + B(v,v) + B(\bar{w},v))\\
=& v_L + B(v_L.v_L) + B(\bar{w},v_L) + B(w,v) + B(\bar{w},B(w,v)) + 2B(v_L,B(w,v))\\
&+ 2B(B(w,v),B(v,v)) + 2B(B(w,v),B(\bar{w},v)) + B(B(w,v),B(w,v))\\
&+ 2B(v_L,B(v,v)) + B(\bar{w},B(v,v)) + B(\bar{w},B(\bar{w},v)) + 2B(v_L,B(v,\bar{w}))\\
&+ 2B(B(v,v),B(\bar{w},v)) + B(B(v,v),B(v,v)) + B(B(\bar{w},v),B(\bar{w},v)).
\end{aligned}$$

This gives the expansion for $N = 3$:

$$\begin{aligned}
v =& H_3 + W_3 + Z_3 \;\text{ with }\; H_3 = H_2 + B(v_L,v_L) + B(\bar{w},v_L),\\
W_3 =& B(w,v) + B(\bar{w},B(w,v)) + 2B(v_L,B(w,v))\\
&+ 2B(B(w,v),B(v,v)) + 2B(B(w,v),B(\bar{w},v)) + B(B(w,v),B(w,v))
\end{aligned}$$

and

$$\begin{aligned}
Z_3 =& 2B(v_L,B(v,v)) + B(\bar{w},B(v,v)) + B(\bar{w},B(\bar{w},v)) + 2B(v_L,B(v,\bar{w}))\\
&+ 2B(B(v,v),B(\bar{w},v)) + B(B(v,v),B(v,v)) + B(B(\bar{w},v),B(\bar{w},v)).
\end{aligned}$$

The first statement of Proposition 2.6.3 implies that $H_3 \in \mathbb{L}_p^{1:\infty}(\infty)$ and the expected bounds of Z_3 follow again from product laws as soon as $\frac{p}{2} > 3$. Now we need to check that $W_3 \in \mathbb{L}^{r_0:\infty}_{\bar{p},p}(T)$. According to the previous arguments, we have $B(w,v) \in$

$\mathbb{L}^{r_0:\infty}_{\bar{p},p}(T)$. Hence we obtain that

$$B(w,v) \in L^\infty([0,T], \dot{B}^{s_q}_{q,\infty}), \forall q > 3,$$

provided that $\bar{p} < 3$. Again by the last statement of Proposition 2.6.3 and taking $q = \frac{3p}{p-2}$, we have the rest of terms in Z_3 belong to $\mathbb{L}^{r_0:\infty}_{\bar{p},p}(T)$, which implies that $W_3 \in \mathbb{L}^{r_0:\infty}_{\bar{p},p}(T)$.

Iterating further, the formulas immediately get very long and complicated, so let us argue by induction:

Assume that for any $2 \le N \le N_0$, there is an integer $K_N \ge 0$, and for any $0 \le k \le K_N$ some $(N+k)$-linear operators $B^M_{N+k,N}$(the parameter $M \in \{1,\dots,N+k\}$ measures the number of entries in which v and $\bar{w}$, rather than v_L, appears and the second parameter in the subscript denotes that the operators are generated in Nth step) such that

$$v = H_N + W_N + Z_N$$

with for any $N \ge 3$

$$H_N = H_{N-1} + \sum_{M=0}^{N-2} B^M_{N-1,N-1}(\bar{w}^{\otimes M}, v_L^{\otimes(N-1-M)}), \tag{2.25}$$

Z_N may be written as the form

$$\begin{aligned} Z_N = &\sum_{M=1}^{N} \sum_{\substack{J+L=M,\\ J\ge 1}} B^M_{N,N}(v^{\otimes J}, \bar{w}^{\otimes L}, v_L^{\otimes(N-M)}) \\ &+ \sum_{k=1}^{K_N}\sum_{M=0}^{N+k}\sum_{J+L=M} B^M_{N+k,N}(v^{\otimes J}, \bar{w}^{\otimes L}, v_L^{\otimes(N+k-M)}), \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} W_N = \sum_{M=1}^{N-1}\sum_{J+L=M}\sum_{l=1}^{J-1}\sum_{i+j=l}\sum_{i+j+m=J} & B^M_{N-1,N-1}(B(v,v)^{\otimes i}, B(w,v)^{\otimes m}, \\ & B(\bar{w},v)^{\otimes j}, \bar{w}^{\otimes L}, v_L^{\otimes(N-1-L-l-m)}) \\ & + W_{N-1}. \end{aligned} \tag{2.27}$$

we have used the following convention: for any $J+L=M$

$$B^M_{N+k,N}(\underbrace{u,\cdots,u}_{J\text{ terms}},\underbrace{v,\cdots,v}_{L\text{ terms}},\underbrace{w,\cdots,w}_{N-M\text{ terms}}) := B^M_{N+k,N}(u^{\otimes J}, v^{\otimes L}, w^{\otimes(N-M)})$$

Now let us prove that for any $2 \leq N \leq N_0$

$$\begin{aligned} Z_N = &\sum_{M=1}^{N} \sum_{J+L=M} \sum_{l=0}^{J-1} \sum_{i+j=l} \sum_{i+j+m=J} B^M_{N,N}(B(v,v)^{\otimes i}, B(w,v)^{\otimes m}, \\ &B(\bar{w},v)^{\otimes j}, \bar{w}^{\otimes L}, v_L^{\otimes(N-L-l-m)}) \\ &+ \sum_{M=0}^{N-1} B^M_{N,N}(\bar{w}^{\otimes M}, v_L^{\otimes(N-M)}) + Z_{N+1} \end{aligned} \tag{2.28}$$

where Z_{N+1} can be written in the following way: there exists an integer $K_{N+1} \geq 0$ for for all $0 \leq k \leq K_{N+1}$ and $0 \leq M \leq N+1+k$, some $N+1+k$-linear operators $B^M_{N+1+k,N+1}$, such that

$$\begin{aligned} Z_{N+1} = &\sum_{M=1}^{N+1} \sum_{\substack{J+L=M, \\ J \geq 1}} B^M_{N+1,N+1}(v^{\otimes J}, \bar{w}^{\otimes L}, v_L^{\otimes(N+1-M)}) \\ &+ \sum_{k=1}^{K_{N+1}} \sum_{M=0}^{N+k+1} \sum_{J+L=M} B^M_{N+k+1,N+1}(v^{\otimes J}, \bar{w}^{\otimes L}, v_L^{N+1+k-M}). \end{aligned} \tag{2.29}$$

In order to prove (2.28) and (2.29) we just need to use (2.24) again: replacing v by $v_L + B(v,v) + B(w,v) + B(\bar{w},v)$ in the argument of $B^M_{N,N}$ gives

$$\begin{aligned} &B^M_{N,N}(v^{\otimes J}, \bar{w}^{\otimes L}, v_L^{\otimes(N-M)}) \\ &\quad = B^M_{N,N}((v_L + B(w,v) + B(v,v) + B(\bar{w},v))^{\otimes J}, \bar{w}^{\otimes L}, v_L^{\otimes(N-M)}) \\ &\quad = B^M_{N,N}(v_L^{\otimes J}, \bar{w}^{\otimes L}, v_L^{\otimes(N-M)}) \\ &\qquad + \sum_{l=0}^{J-1} \sum_{i+j=l} \sum_{i+j+m=J} B^M_{N,N}(B(v,v)^{\otimes i}, B(w,v)^{\otimes m}, B(\bar{w},v)^{\otimes j}, \bar{w}^{\otimes L}, v_L^{\otimes(N-L-l-m)}) \\ &\qquad + \sum_{l=1}^{J} \sum_{i+j=l} \tilde{B}^M_{N+l,N}(v^{\otimes(2i+j)}, \bar{w}^{\otimes(L+j)}, v_L^{\otimes(N-L-l)}), \end{aligned}$$

where $\tilde{B}^M_{N+l,N}$ are some $N+l$-linear operators. Therefore we have

$$\begin{aligned} Z_{N+1} = &\sum_{k=1}^{K_N} \sum_{M=0}^{N+k} \sum_{J+L=M} B^M_{N+k,N}(v^{\otimes J}, \bar{w}^{\otimes L}, v_L^{\otimes(N+k-M)}) \\ &+ \sum_{M=1}^{N} \sum_{J+L=M} \sum_{l=1}^{J} \sum_{i+j=l} \tilde{B}^M_{N+l,N}(v^{\otimes(2i+j)}, \bar{w}^{\otimes(L+j)}, v_L^{\otimes(N-L-l)}) \end{aligned}$$

after reordering, this proves (2.28) and (2.29). Moreover (2.28) and (2.29) imply that (2.25) and (2.27) hold for the case that $N = N_0 + 1$.

To conclude the proof the theorem it remains to prove that $H_N \in \mathbb{L}^{1;\infty}_p(\infty)$, $W_N \in \mathbb{L}^{r_0;\infty}_{\bar{p},p}$ for some $2 < \bar{p} < 3$ and $Z_N \in \mathbb{L}^{r_N;\infty}_{p_N}(T)$ with $p_N := \frac{p}{N}$ and $r_N = \max\{1, \frac{r_0}{N}\}$. In fact, the above results again follow from estimates about the heat flow (see Lemma 2.6.1) and product laws in Proposition 2.6.3, which are based on a similar argument of the cases that $N = 2, N = 3$.

Now we take $N_0 := \max\{N \in \mathbb{N} : N \geq 2, 3(N-1) < p\}$. It is obvious that $p_{N_0} = \frac{p}{N_0} < 3$, which implies that

$$v^S = W_{N_0} + Z_{N_0} \in \mathbb{L}^{r_0:\infty}_{\tilde{p},p}(T) \hookrightarrow L^\infty([0,T], L^{3,\infty})$$

provided Lemma 2.6.4.

Theorem 2.5.1 is proved.

□

2.6 Appendix

2.6.1 Estimates on the heat equation

For the completeness of our proof, we give standard estimates for the heat kernel in Besov space. A similar result can be found in [12]. We first recall the long-time behavior of heat flow. We mention that the following lemmas only focus on critical Besov spaces.

Lemma 2.6.1. *Let $p, q \in [1, \infty)$ and $g \in \dot{B}^{s_p}_{p,q}$. Then we have that*

$$e^{t\Delta} g \in \mathbb{L}^{1:\infty}_{p,q}(\infty)$$

and

$$\lim_{t\to\infty} \|e^{t\Delta} g\|_{\dot{B}^{s_p}_{p,q}} = 0.$$

Proof. Let $g \in \dot{B}^{s_p}_{p,p}$. We notice that for any $j \in \mathbb{Z}$,

$$\|e^{t\Delta}\Delta_j g\|_{L^p} \lesssim e^{-t2^{2j}} \|\Delta_j g\|_{L^p} \lesssim 2^{-js_p} e^{-t2^{2j}} c_{j,q} \|g\|_{\dot{B}^{s_p}_{p,q}},$$

where $\|(c_{j,q})_{j\in\mathbb{Z}}\|_{\ell^q} = 1$. Then for any $r \in [1, \infty]$, we have

$$\|e^{t\Delta}\Delta_j g\|_{L^r(\mathbb{R}_+, L^p_x)} \lesssim 2^{-js_p} 2^{-\frac{2j}{r}} c_{j,q} \|g\|_{\dot{B}^{s_p}_{p,q}},$$

which implies that

$$\left\| \left(2^{j(s_p+\frac{2}{r})} \|e^{t\Delta}\Delta_j g\|_{L^r(\mathbb{R}_+, L^p_x)}\right)_{j\in\mathbb{Z}} \right\|_{\ell^q} \lesssim \|g\|_{\dot{B}^{s_p}_{p,q}}$$

Hence we have $e^{t\Delta} g \in \mathbb{L}^{1:\infty}_{p,q}(\infty)$.

Moreover for any $\varepsilon > 0$, one can choose an integer N such that for any $t \geq 0$

$$\Big(\sum_{|j|>N} 2^{qjs_p} \|e^{t\Delta}\Delta_j g\|^q_{L^p} \Big)^{\frac{1}{q}} < \frac{\varepsilon}{2}.$$

Also we have

$$\sum_{|j|\leq N} 2^{qjs_p} \|e^{t\Delta}\Delta_j g\|^q_{L^p} \lesssim 2^{-jqs_p} e^{-qt2^{-2N}} c^q_{j,q} \|g\|^q_{\dot{B}^{s_p}_{p,q}},$$

hence for the fixed N, there exists a $T(\varepsilon) > 0$ such that for any $t > T$,

$$\Big(\sum_{|j|\le N} 2^{qjs_p}\|e^{t\Delta}\Delta_j g\|_{L^p}^q\Big)^{\frac{1}{q}} < \frac{\varepsilon}{2}.$$

Therefore we have that for any $\varepsilon > 0$, there exists a $T(\varepsilon) > 0$, such that for any $t > T$

$$\|e^{t\Delta}g\|_{\dot{B}^{s_p}_{p,q}} < \varepsilon.$$

The lemma is proved. □

Lemma 2.6.2. *Let $p \in [1,\infty]$ and $r \in [1,\infty]$. Suppose that f is a function belonging to $\mathcal{L}^r_T(\dot{B}^{s_p+\frac{2}{r}-2}_{p,p})$. We denote that, for any $t \in [0,T]$*

$$H(f) := \int_0^t e^{(t-s)\Delta} f(s,\cdot)ds. \tag{2.30}$$

Then we have $H(f) \in \mathcal{L}^{\bar{r}}_T(\dot{B}^{s_p+\frac{2}{\bar{r}}}_{p,p})$ for any $\bar{r} \ge r$, and

$$\|H(f)\|_{\mathcal{L}^{\bar{r}}_T(\dot{B}^{s_p+\frac{2}{\bar{r}}}_{p,p})} \lesssim \|f\|_{\mathcal{L}^r_T(\dot{B}^{s_p+\frac{2}{r}-2}_{p,p})}.$$

Moreover, if $r < \infty$,

$$\lim_{t\to\infty} \|H(f)\|_{\dot{B}^{s_p}_{p,p}} = 0.$$

Proof. We first notice that

$$\begin{aligned}
\|\Delta_j H(f)\|_{L^{\bar{r}}_t L^p_x} &\le \Big\| \int_0^t \|e^{(t-s)\Delta}\Delta_j f(s,\cdot)\|_{L^p_x} ds\Big\|_{L^{\bar{r}}_t} \\
&\lesssim \Big\| \int_0^t e^{-c(t-s)2^{2j}}\|\Delta_j f(s,\cdot)\|_{L^p_x} ds\Big\|_{L^{\bar{r}}_t} \\
&\lesssim \|e^{-ct2^{2j}}\|_{L^{r'}_t}\|\Delta_j f\|_{L^r_t L^p_x},
\end{aligned}$$

where $\frac{1}{\bar{r}} + 1 = \frac{1}{r} + \frac{1}{r'}$. Since $f \in \mathcal{L}^r_T(\dot{B}^{s_p+\frac{2}{r}-2}_{p,p})$ we have

$$\|\Delta_j f\|_{L^r_t L^p_x} \lesssim 2^{-j(s_p+\frac{2}{r}-2)} d_{j,p}\|f\|_{\mathcal{L}^r_T(\dot{B}^{s_p+\frac{2}{r}-2}_{p,p})},$$

where $(d_{j,p}) \in \ell^p$ and $\|(d_{j,p})\|_{\ell^p} = 1$. We also notice that

$$\|e^{-ct2^{2j}}\|_{L^{r'}_t} \lesssim 2^{-\frac{2j}{r'}}.$$

Then we have

$$\begin{aligned}
\|\Delta_j H(f)\|_{L^{\bar{r}}_t L^p_x} &\lesssim 2^{-j(s_p+\frac{2}{r}+\frac{2}{r'}-2)} d_{j,p}\|f\|_{\mathcal{L}^r_T(\dot{B}^{s_p+\frac{2}{r}-2}_{p,p})} \\
&= 2^{-j(s_p+\frac{2}{\bar{r}})} d_{j,p}\|f\|_{\mathcal{L}^r_T(\dot{B}^{s_p+\frac{2}{r}-2}_{p,p})},
\end{aligned}$$

which implies that

$$\Big\|\big(2^{j(s_p+\frac{2}{\bar{r}})}\|\Delta_j H(f)\|_{L^{\bar{r}}_t L^p_x}\big)_{j\in\mathbb{Z}}\Big\|_{\ell^p} \lesssim \|f\|_{\mathcal{L}^r_T(\dot{B}^{s_p+\frac{2}{r}-2}_{p,p})}.$$

Thus we proved that $H(f) \in \mathcal{L}^{\bar{r}}_T(\dot{B}^{s_p+\frac{2}{\bar{r}}}_{p,p})$ for any $\bar{r} \geq r$, and

$$\|H(f)\|_{\mathcal{L}^{\bar{r}}_T(\dot{B}^{s_p+\frac{2}{\bar{r}}}_{p,p})} \lesssim \|f\|_{\mathcal{L}^{r}_T(\dot{B}^{s_p+\frac{2}{r}-2}_{p,p})}.$$

Now we suppose that $r < \infty$.
First we decompose $H(f)$ into two parts:

$$H_1(f) := \int_0^{\frac{t}{2}} e^{(t-s)\Delta} f(s,\cdot)ds,$$

and

$$H_2(f) := \int_{\frac{t}{2}}^{t} e^{(t-s)\Delta} f(s,\cdot)ds.$$

We notice that $H_1(f)$ can be written as

$$H_1(f) = e^{\frac{t\Delta}{2}} \int_0^{\frac{t}{2}} e^{(\frac{t}{2}-s)\Delta} f(s,\cdot)ds = e^{\frac{t\Delta}{2}} H(f)(\frac{t}{2}).$$

According the above argument, we have, for any $t > 0$, $H(f)(\frac{t}{2}) \in \dot{B}^{s_p}_{p,p}$. Applying Lemma 2.6.1, we have

$$\lim_{t\to\infty} \|e^{\frac{t\Delta}{2}} H(f)(\frac{t}{2})\|_{\dot{B}^{s_p}_{p,p}} = 0.$$

Now we turn to $H_2(f)$, we have

$$\begin{aligned} \|\Delta_j H_2(f)\|_{L^p} &\lesssim \int_{\frac{t}{2}}^{t} e^{-2^{2j}(t-s)} \|\Delta_j f(s)\|_{L^p} ds \\ &\lesssim 2^{2j(\frac{1}{r}-1)} \|\Delta_j f\|_{L^r([t/2,\infty);L^p)}, \end{aligned}$$

which implies that

$$\|H_2(f)(t)\|_{\dot{B}^{s_p}_{p,p}} \lesssim \|f\|_{\mathcal{L}^r([t/2,\infty);\dot{B}^{s_p+\frac{r}{2}-2}_{p,p})} \to 0, \quad \text{as} \quad t \to \infty.$$

Lemma 2.6.2 is proved. □

2.6.2 Product laws in Besov spaces

In this paragraph we recall the following product laws in Besov spaces, which use the theory of paraproducts. We only elected to state the results we needed previously, but it should be clear that we have not stated all possible estimates in their greatest generality.

Proposition 2.6.3. *1. Let $p > 3$ and $2 < r < \frac{2p}{p-3}$. Then there exists a constant $\gamma > 0$ such that for any $v, w \in \mathcal{L}^r([0,T], \dot{B}^{s_p+\frac{2}{r}}_{p,p})$, we have*

$$\|vw\|_{\mathcal{L}^r([0,T],\dot{B}^{s_p+\frac{2}{r}-1}_{p,p})} \leq \gamma \|v\|_{\mathcal{L}^r([0,T],\dot{B}^{s_p+\frac{2}{r}}_{p,p})} \|w\|_{\mathcal{L}^r([0,T],\dot{B}^{s_p+\frac{2}{r}}_{p,p})}. \tag{2.31}$$

2. *Let $p_1, p_2 \in (3,\infty)$, $2 < r < \frac{2p}{p-3}$ and $T \in \mathbb{R}_+ \cup \{\infty\}$. Suppose that $v \in \mathbb{L}^{r;\infty}_{p_1}(T)$ and $w \in \mathbb{L}^{r;\infty}_{p_2}(T)$. Then we have*

$$\|vw\|_{\mathcal{L}^{\frac{r}{2}}([0,T],\dot{B}^{s_p+\frac{4}{r}-1}_{p,p})} \lesssim \|v\|_{\mathbb{L}^{r;\infty}_{p_1}(T)} \|w\|_{\mathbb{L}^{r;\infty}_{p_2}(T)},$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

3. *Let $p > 3$. Suppose that $w \in L^\infty([0,T], L^{3,\infty})$ and $v \in \mathcal{L}^{r_0}([0,T]; \dot{B}^{s_p+\frac{2}{r_0}}_{p,p})$ for some $T \in \mathbb{R}_+ \cup \{+\infty\}$ with $r_0 = \frac{2p}{p-1}$, then we have*

$$\|vw\|_{\mathcal{L}^{r_0}([0,T],\dot{B}^{s_{\bar{p}}+\frac{2}{r_0}-1}_{\bar{p},p})} \leq C(p) \|w\|_{L^\infty([0,T],L^{3,\infty})} \|v\|_{\mathcal{L}^{r_0}([0,T];\dot{B}^{s_p+\frac{2}{r_0}}_{p,p})},$$

where $\frac{1}{\bar{p}} = \frac{1}{3} + \frac{1}{6p}$ and $C(p) \to \infty$ as $p \to \infty$.

Since the first two results in the proposition are standard and well-known, which can be found in [12, 16], we only give the proof of the last of the proposition.

Proof. For simplicity, we treat w and v as functions. We have

$$\Delta_j wv = \Delta_j T_w v + \Delta_j T_v w + \Delta_j R(u,v).$$

We first take q_1 such that $\frac{1}{\bar{p}} = \frac{1}{p} + \frac{1}{q_1} = \frac{1}{3} + \frac{1}{6p}$ implying that $q_1 = \frac{6p}{2p-5} > 3$.

About $\Delta_j T_w v$, we have

$$\|\Delta_j T_w v\|_{L^{r_0}(L^{\bar{p}})} \lesssim \|(S_j w)(\Delta_j v)\|_{L^{r_0}(L^{\bar{p}})} \lesssim \|S_j w\|_{L^\infty(L^{q_1})} \|v\|_{L^{r_0}(L^p)}.$$

And we notice that

$$\|S_j w\|_{L^\infty(L^{q_1})} \lesssim \sum_{j' \leq j} \|\Delta_j w\|_{L^\infty(L^{q_1})} \lesssim \sum_{j' \leq j} 2^{-j' s_{q_1}} c_{j,\infty} \|w\|_{\mathcal{L}^\infty([0,T],\dot{B}^{s_{q_1}}_{q_1,\infty})},$$

and

$$\|v\|_{L^{r_0}(L^p)} \lesssim 2^{-j(s_p+\frac{2}{r_0})} c_{j,p'} \|v\|_{\mathcal{L}^{r_0}([0,T];\dot{B}^{s_p+\frac{2}{r_0}}_{p,p})}.$$

Since $s_{q_1} < 0$, we have

$$\|2^{j(s_{\bar{p}}+\frac{2}{r_0}-1)} \|\Delta_j T_w v\|_{L^{r_0}(L^{p'})}\|_{\ell^{p'}} \lesssim \|w\|_{\mathcal{L}^\infty([0,T],\dot{B}^{s_{q_1}}_{q_1,\infty})} \|v\|_{\mathcal{L}^{r_0}([0,T];\dot{B}^{s_p+\frac{2}{r_0}}_{p,p})}.$$

This combined with Lemma 2.6.4, implies that

$$\|2^{j(s_{\bar{p}}+\frac{2}{r_0}-1)} \|\Delta_j T_w v\|_{L^{r_0}(L^{p'})}\|_{\ell^{p'}} \lesssim \|w\|_{L^\infty([0,T],L^{3,\infty})} \|v\|_{\mathcal{L}^{r_0}([0,T];\dot{B}^{s_p+\frac{2}{r_0}}_{p,p})}. \tag{2.32}$$

Now we choose $q := \frac{12p}{4p-1}$ and $p_1 := 4p$. It is easy to check such that $\frac{1}{\bar{p}} = \frac{1}{p_1} + \frac{1}{q} = \frac{1}{3} + \frac{1}{6p}$. We notice that

$$\|\Delta_j T_v w\|_{L^{r_0}(L^{\bar{p}})} \lesssim \|S_j v\|_{L^{r_0}(L^{p_1})} \|\Delta_j w\|_{L^\infty(L^q)},$$

and

$$\begin{aligned}\|S_j v\|_{L^{r_0}(L^{p_1})} &\lesssim \sum_{j'\le j}\|\Delta_j v\|_{L^{r_0}(L^{p_1})} \lesssim 2^{-j'(s_{p_1}+\frac{2}{r_0})}c_{j',p}\|v\|_{\mathcal{L}^{r_0}([0,T];\dot B^{s_{p_1}+\frac{2}{r_0}}_{p_1,p})}\\ &\lesssim 2^{-j'(s_{p_1}+\frac{2}{r_0})}c_{j',p}\|v\|_{\mathcal{L}^{r_0}([0,T];\dot B^{s_p+\frac{2}{r_0}}_{p,p})},\end{aligned}$$

and

$$\|\Delta_j w\|_{L^\infty(L^q)} \lesssim 2^{-js_q}c_{j,\infty}\|w\|_{\mathcal{L}^\infty([0,T],\dot B^{s_q}_{q,\infty})}.$$

Since $s_{p_1}+\frac{2}{r_0} = -1+\frac{3}{4p}+1-\frac{1}{p} = -\frac{1}{4p} < 0$, we have

$$\|2^{j(s_{\bar p}+\frac{2}{r_0}-1)}\|\Delta_j T_w v\|_{L^{r_0}(L^{\bar p})}\|_{\ell^p} \lesssim p\|w\|_{\mathcal{L}^\infty([0,T],\dot B^{s_q}_{q,\infty})}\|v\|_{\mathcal{L}^{r_0}([0,T];\dot B^{s_p+\frac{2}{r_0}}_{p,p})}. \tag{2.33}$$

Again by Lemma 2.6.4, we have

$$\|2^{j(s_{\bar p}+\frac{2}{r_0}-1)}\|\Delta_j T_w v\|_{L^{r_0}(L^{\bar p})}\|_{\ell^p} \lesssim p\|w\|_{L^\infty([0,T],L^{3,\infty})}\|v\|_{\mathcal{L}^{r_0}([0,T];\dot B^{s_p+\frac{2}{r_0}}_{p,p})}. \tag{2.34}$$

Now we turn to the remainder $\Delta_j R(w,v)$. We denote that $\frac{1}{\tilde p} := \frac{1}{p}+\frac{1}{q} = \frac{1}{3}+\frac{11}{12p}$. Since

$$\begin{aligned}\|\Delta_j R(w,v)\|_{L^{r_0}(L^{\tilde p})} &\lesssim \sum_{k\ge j-1}\|\Delta_k w\|_{L^\infty(L^q)}\|\Delta_k v\|_{L^{r_0}(L^p)}\\ &\lesssim \sum_{k\ge j}2^{-k(s_q+s_p+\frac{2}{r_0})}c_{k,\infty}c_{k,p}\|w\|_{\mathcal{L}^\infty([0,T],\dot B^{s_q}_{q,\infty})}\|v\|_{\mathcal{L}^{r_0}([0,T];\dot B^{s_p+\frac{2}{r_0}}_{p,p})},\end{aligned}$$

and

$$s_p+s_q+\frac{2}{r_0} = \frac{7}{4p} > 0$$

we have that, by applying Lemma 2.6.4,

$$\begin{aligned}\|2^{j(s_{\tilde p}+\frac{2}{r_0}-1)}\|\Delta_j R(w,v)\|_{L^{r_0}(L^{\tilde p})}\|_{\ell^p} &\lesssim p\|w\|_{\mathcal{L}^\infty([0,T],\dot B^{s_q}_{q,\infty})}\|v\|_{\mathcal{L}^{r_0}([0,T];\dot B^{s_p+\frac{2}{r_0}}_{p,p})}\\ &\lesssim p\|w\|_{L^\infty([0,T],L^{3,\infty})}\|v\|_{\mathcal{L}^{r_0}([0,T];\dot B^{s_p+\frac{2}{r_0}}_{p,p})}\end{aligned}$$

which is $R(w,v) \in \mathcal{L}^{r_0}([0,T];\dot B^{s_{\tilde p}+\frac{2}{r_0}-1}_{\tilde p,p'})$.

And we have $R(w,v) \in \mathcal{L}^{r_0}([0,T];\dot B^{s_{\bar p}+\frac{2}{r_0}-1}_{\bar p,p})$, as $\tilde p < \bar p$. Combining with (2.32) and (2.34) we get

$$\|wv\|_{\mathcal{L}^{r_0}([0,T];\dot B^{s_{\bar p}+\frac{2}{r_0}-1}_{\bar p,p})} \le C(p)\|w\|_{L^\infty([0,T],L^{3,\infty})}\|v\|_{\mathcal{L}^{r_0}([0,T];\dot B^{s_p+\frac{2}{r_0}}_{p,p})},$$

where $C(p)\to\infty$ as $p\to\infty$. The proposition is proved. □

We also recall the following standard embedding without proof. For details of the proof, one can check [4, 24].

Lemma 2.6.4. *Let $q_1 < 3 < q_2$. Then the following embeddings hold:*

$$\dot{B}^{s_{q_1}}_{q_1,\infty} \hookrightarrow L^{3,\infty} \hookrightarrow \dot{B}^{s_{q_2}}_{q_1,\infty}.$$

2.6.3 Properties of the bilinear operator B

We show a well-known results on the continuity of $B(u,v)$ in Kato's space by using the spatial decay of the convolution kernel appearing in B (see [24])

Lemma 2.6.5. *Let $p > 3$.Suppose that $u, v \in K_p(\mathbb{R}^3)$, then*

$$\|B(u,v)\|_{K_p} \lesssim \|u\|_{K_p}\|v\|_{K_p}. \tag{2.35}$$

Moreover if $p > 6$, then

$$\|B(u,v)\|_{K_\infty} \leq \|u\|_{K_p}\|v\|_{K_p}. \tag{2.36}$$

And we recall that B is a bounded operator from $L^\infty([0,T],L^{3,\infty})\times L^\infty([0,T],L^{3,\infty})$ to $L^\infty([0,T],L^{3,\infty})$ for any $T \in \mathbb{R}_+ \cup \{+\infty\}$ (see [5])

Lemma 2.6.6. *Suppose that $u, v \in L^\infty([0,T],L^{3,\infty})$ for some $T \in \mathbb{R}_+ \cup \{+\infty\}$. Then*

$$\|B(u,v)\|_{L^\infty([0,T],L^{3,\infty})} \lesssim \|u\|_{L^\infty([0,T],L^{3,\infty})}\|u\|_{L^\infty([0,T],L^{3,\infty})}.$$

Moreover, $B(u,v) \in C_w([0,T],L^{3,\infty})$.

The following lemma is a particular case of the result about the continuity of the trilinear form $\int_0^T\int_{\mathbb{R}^3}(a\cdot\nabla b)\cdot c dx dt$ proved by I.Gallagher & F. Planchon in [18].

Lemma 2.6.7. *Let $d \geq 2$ be fixed, and let r and q be two real numbers such that $2 \leq 2 < \infty, 2 < q < +\infty$. Suppose $a \in L^\infty(\mathbb{R}_+,L^2)\cap L^2(\mathbb{R}_+,\dot{H}^1)$ and $c \in \mathcal{L}^q([0,T],\dot{B}^{s_r+\frac{2}{q}}_{r,q})$. Then for every $0 \leq t \leq T$,*

$$|\int_0^t\int_{\mathbb{R}^3}(a\cdot\nabla a)\cdot c dx ds| \leq \|\nabla a\|^2_{L^2(\mathbb{R}_+,L^2)} + C\int_0^t \|a(s)\|^2_{L^2}\|c(s)\|^q_{\dot{B}^{s_r+\frac{2}{q}}_{r,q}} ds.$$

Now we recall that for any $T \in \mathbb{R}_+ \cup \{+\infty\}$

$$E(T) = L^\infty([0,T^*),L^2)\cap L^2([0,T^*),\dot{H}^1).$$

Lemma 2.6.8.

1. *Let $p > 3$ and $T > 0$. Suppose that $v \in E(T)$ and $\bar{v} \in \mathcal{L}^\infty([0,T],\dot{B}^{s_p}_{p,\infty})$. Then $B(v,\bar{v}) \in E(T)$.*
2. *Let $T \in (0,\infty)$. Suppose that $v \in L^\infty([0,T],L^{3,\infty})$ and $\bar{v} \in L^2([0,T],L^{6,2})$. Then $B(v,\bar{v}) \in E(T)$*

Proof. We denote $w := B(v,\bar{v})$, which satisfies the system

$$\begin{cases} \partial_t w - \Delta w + \bar{v}\cdot\nabla v + v\cdot\nabla\bar{v} = -\nabla\pi, \\ \nabla\cdot w = \nabla\cdot v = \nabla\cdot\bar{v} = 0, \\ w|_{t=0} = w_0 \end{cases}$$

For $v \in E(T)$ and $\bar{v} \in \mathcal{L}^\infty([0,T],\dot{B}^{s_p}_{p,\infty})$, by Proposition 4.2 in [16], we obtain that $B(v,\bar{v}) \in E(T)$.

Hence we are left with the proof of the second statement of the lemma. We now suppose that $v \in L^\infty([0,T], L^{3,\infty})$ and $\bar{v} \in L^2([0,T], L^{6,2})$.

First let J_ε be a smoothing operator that multiplies in the frequency space by a cut-off function bounded by 1 which is a smoothed out version of the characteristic function of the annulus $\{\varepsilon < |\xi| < \frac{1}{\varepsilon}\}$. Then we have for any $t \in [0,T]$

$$\begin{aligned}\|J_\varepsilon w(t)\|_{L^2}^2 + 2\int_0^t \|\nabla J_\varepsilon w(s)\|_{L^2}^2 ds = \|w_0\|_{L^2}^2 + 2\int_0^t \int_{\mathbb{R}^3} (v \cdot \nabla J_\varepsilon^2 w) \cdot \bar{v} dxds \\ + 2\int_0^t \int_{\mathbb{R}^3} (\bar{v} \cdot \nabla J_\varepsilon^2 w) \cdot v dxds.\end{aligned}$$

Then for any $t \in [0,T]$,

$$\begin{aligned} & |\int_0^t \int_{\mathbb{R}^3} (v \cdot \nabla J_\varepsilon^2 w) \cdot \bar{v} + \int_0^t \int_{\mathbb{R}^3} (\bar{v} \cdot \nabla J_\varepsilon^2 w) \cdot v dxds| \\ \leq \;& C\int_0^t \|\nabla J_\varepsilon^2 w\|_{L^2} \|\bar{v}\|_{L^{6,2}} \|v\|_{L^{3,\infty}} \\ \leq \;& \frac{1}{2}\int_0^t \|\nabla J_\varepsilon w\|_{L^2}^2 + \frac{C^2}{2} \|v\|_{L^\infty(\mathbb{R}_+, L^{3,\infty})}^2 \|\bar{v}\|_{L^2([0,T],L^{6,2})}^2,\end{aligned}$$

which implies that for any $t \in [0,T]$

$$\|J_\varepsilon w(t)\|_{L^2}^2 + \int_0^t \|\nabla J_\varepsilon w(s)\|_{L^2}^2 ds \lesssim \|w_0\|_{L^2}^2 + \|v\|_{L^\infty(\mathbb{R}_+, L^{3,\infty})}^2 \|\bar{v}\|_{L^2([0,T_1],L^{6,2})}^2.$$

By taking $\varepsilon \to 0$, we have $w \in E(T)$. □

Lemma 2.6.9. *Let $p > 3$. Suppose that $g \in \mathbb{L}^{3:\infty}_{6,\infty}[T < T^*]$ for some $T^* > 0$. then we have $g \in L^2([0,T], L^{6,2}(\mathbb{R}^3))$ for any $T < T^*$.*

Proof. Suppose that g is a function belonging to $\mathbb{L}^{3:\infty}_6[T < T^*]$ for some $T^* > 0$. Then for any fixed $T < T^*$, we have that

$$\|g\|_{\mathcal{L}^3([0,T],\dot{B}^{s_6}_{6,\infty})} \leq T^{\frac{1}{3}} \|g\|_{\mathcal{L}^\infty([0,T],\dot{B}^{s_6}_{6,\infty})}.$$

Hence we obtain $g \in \mathcal{L}^3([0,T], \dot{B}^{s_6}_{6,\infty}) \cap \mathcal{L}^3([0,T], \dot{B}^{s_6+\frac{2}{3}}_{6,\infty})$. Since that $s_6 < 0$ and $s_6 + \frac{2}{3} > 0$, Then by using Proposition 2.22 in [1], we have

$$\begin{aligned}\|g\|_{\mathcal{L}^3([0,T],\dot{B}^0_{6,1})} &\leq \|g\|^{\frac{1}{4}}_{\mathcal{L}^3([0,T];\dot{B}^{s_6}_{6,\infty})} \|g\|^{\frac{3}{4}}_{\mathcal{L}^3([0,T],\dot{B}^{s_6+\frac{2}{3}}_{6,\infty})} \\ &\leq T^{\frac{1}{3}} \|g\|_{\mathbb{L}^{3:\infty}_{6,\infty}(T)}.\end{aligned}$$

Now we are left with proving that

$$\mathcal{L}^3([0,T], \dot{B}^0_{6,1}) \hookrightarrow L^3([0,T], L^{6,2}).$$

By Littlewood-Paley decomposition,

$$\|g\|_{L^3([0,T],L^{6,2})} \leq \sum_{j\in\mathbb{Z}} \|\Delta_j g\|_{L^3([0,T],L^{6,2})}.$$

And $\Delta_j g$ can be written as the following convolution form:

$$\Delta_j g = \Delta_j(\Delta_j g) = 2^{3j} \int_{\mathbb{R}^3} h(2^j(x-y))\Delta_j g(y)dy.$$

By using Young's inequality,

$$\begin{aligned} \|\Delta_j g\|_{L^3([0,T],L^{6,2})} &\lesssim 2^{3j}\|h(2^j\cdot)\|_{L^1_x}\|\Delta_j g\|_{L^3([0,T],L^6)} \\ &\lesssim \|\Delta_j g\|_{L^3([0,T],L^6)} \lesssim c_j\|g\|_{\mathcal{L}^3([0,T],\dot{B}^0_{6,1})}, \end{aligned}$$

where $\sum_{j\in\mathbb{Z}} |c_j| = 1$. Then we have

$$\|g\|_{L^3([0,T],L^{6,2})} \lesssim \|g\|_{\mathcal{L}^3([0,T],\dot{B}^0_{6,1})},$$

which combined with the fact that

$$\|g\|_{L^2([0,T],L^{6,2})} \leq T^{\frac{1}{6}}\|g\|_{L^3([0,T],L^{6,2})}.$$

The lemma is proved. □

Bibliography

[1] H. Bahouri, J.-Y Chemin & R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations. *Springer-Verlag Berlin Heidelberg*, 2011.

[2] O. A. Barraza, Self-similar solutions in weak L^p-spaces of the Navier-Stokes equations. *Rev. Mat. Iberoamericana.* 12 (1996), no. 2, 411–439.

[3] O. A. Barraza, Regularity and stability for the solutions of the Navier-Stokes equations in Lorentz spaces. *Nonlinear Anal.* no. 6, Ser. A: Theory Methods, 747–764.

[4] J. Bergh & J. Löfström, Interpolation spaces: an introduction. *Springer*, 1976.

[5] C. Bjorland, L. Brandolese, D. Iftimie & M. E. Schonbek, L^p-solutions of the steady-state Navier-Stokes equations with rough external forces. *Communications in Partial Differential Equations*, 2010, 36(2), 216-246

[6] J. Bourgain & N. Pavlović, Ill-posedness of the Navier-Stokes equations in a critical space in 3D *J. Funct. Anal.* 255 (2008), no. 9, 2233–2247.

[7] C. P. Calderón, Existence of weak solutions for the Navier-Stokes equations with initial data in L^p. *Trans. Amer. Math. Soc.*318 (1990), no. 1, 179–200.

[8] M. Cannone, A generalization of a theorem by Kato on Navier-Stokes equations. *Rev. Mat. Iberoamericana* 13 (1997), no. 3, 515—541.

[9] M. Cannone & G. Karch, About the regularized Navier-Stokes equations. *J. Math. Fluid Mech.* 7 (2005), no. 1, 1–28.

[10] M. Cannone, Y. Meyer & F. Planchon, Solutions auto-similaires des équations de Navier-Stokes. *Séminaire sur les Équations aux Dérivées Partielles, 1993—1994,* No. VIII, 12 pp., École Polytech., Palaiseau, 1994.

[11] M. Cannone & F. Planchon, On the non-stationary Navier-Stokes equations with an external force. *Adv. Differential Equations* 4 (1999), no. 5, 697–730.

Comm. Partial Differential Equations 36 (2011), no. 2, 216—246.

[12] J.-Y Chemin, Théorèmes d'unicité pour le système de Navier-Stokes tridimensionnel. (French) [Uniqueness theorems for the three-dimensional Navier-Stokes system]. *J. Anal. Math.*77 (1999), 27–50.

[13] J.-Y Chemin, Remarques sur l'existence globale pour le système de Navier-Stokes incompressible. *SIAM J. Math. Anal.* 23 (1992), no. 1, 20–28.

[14] J.-Y Chemin & N. Lerner, Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes. *J. Differential Equations* 121 (1995), no. 2, 314–328.

[15] H. Fujita & T. Kato, On the Navier-Stokes initial value problem. I. *Arch. Rational Mech. Anal.* 16 1964 269—315.

[16] I. Gallagher, D. Iftimie & F. Planchon, Asymptotics and stability for global solutions to the Navier-Stokes equations. *Ann. Inst. Fourier (Grenoble)* 53 (2003), no. 5, 1387—1424.

[17] I. Gallagher, G. S. Koch & F. Planchon, Blow-up of critical Besov norms at a potential Navier-Stokes singularity. *Comm. Math. Phys.* 343 (2016), no. 1, 39–82.

[18] I. Gallagher & F. Planchon, On global infinite energy solutions to the Navier-Stokes equations in two dimensions. *Arch. Ration. Mech. Anal.* 161 (2002), no. 4, 307–337.

[19] P. Germain, The second iterate for the Navier-Stokes equation. *J. Funct. Anal.* 255 (2008), no. 9, 2248–2264.

[20] T. Kato, Strong L^p solutions of the Navier-Stokes equation in $\mathbb{R}^m$, with applications to weak solutions. *Mathematische Zeitschrift*, 187, 1984, pages 471–480.

[21] H. Koch & D. Tataru, Well-posedness for the Navier-Stokes equations. *Adv. Math.* 157 (2001), no. 1, 22–35.

[22] J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.* 63 (1934), no. 1, 193–248.

[23] J. Nečas, M. Růžička & V. Šverák, On Leray's self-similar solutions of the Navier-Stokes equations.

[24] F. Planchon, Asymptotic behavior of global solutions to the Navier-Stokes equations in $\mathbb{R}^3$. *Rev. Mat. Iberoamericana* 14 (1998), no. 1, 71–93.

[25] T. Yoneda, Ill-posedness of the 3D-Navier-Stokes equations in a generalized Besov space near BMO^{-1}. *J. Funct. Anal.* 258 (2010), no. 10, 3376–3387.

Chapter 3

Regularity Criterion for the Forced Navier-Stokes Equations in L^3

3.1 Introduction

We consider the incompressible Navier-Stokes equations with a time independent external force in $\mathbb{R}^3$,

$$(NSf) \begin{cases} \partial_t u_f - \Delta u_f + u_f \cdot \nabla u_f = f - \nabla p, \\ \nabla \cdot u = 0, \\ u_f|_{t=0} = u_0 \end{cases}$$

for $(t,x) \in (0,T) \times \mathbb{R}^3$, where u_f is the velocity vector field, $f(x)$ is the given external force defined in $\mathbb{R}^3$ and $p(t,x)$ is the associated pressure function. In this paper, we study the blow-up criterion for (NSf).

3.1.1 Blow-up problem in critical spaces

To put our results in perspective, we first recall the Navier-Stokes equations (without external force) blow-up problem in critical spaces. Consider the Navier-Stokes system:

$$(NS) \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u = -\nabla \pi, \\ \nabla \cdot u = 0, \\ u_f|_{t=0} = u_0 \end{cases}$$

where $u(t,\cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ is the unknown velocity field.
The spaces X appearing in the chain of continuous embeddings

$$\dot{H}^{\frac{1}{2}} \hookrightarrow L^3 \hookrightarrow \dot{B}^{-1+\frac{3}{p}}_{p,q} \hookrightarrow \dot{B}^{-1+\frac{3}{p'}}_{p',q'}, \ (3 < p \le p' < \infty, 3 < q \le q' < \infty)$$

are all critical with respect to the Navier-Stokes scaling in that $\|u_{0,\lambda}\|_X \equiv \|u\|_X$ for all $\lambda > 0$, where $u_{0,\lambda} := \lambda u(\lambda x)$ is the initial data which evolves as $u_\lambda := \lambda u(\lambda^2 t, \lambda x)$, as long as u_0 is the initial data for the solution $u(t,x)$. While the larger spaces $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$, BMO^{-1} and $\dot{B}^{-1}_{\infty,\infty}$ are also critical spaces and global well-posedness is known for the first two for small enough initial data in those spaces thanks to [4, 20, 23] (but only for finite p in the Besov case, see [3]), the ones in the chain above guarantee the existence and uniqueness of local-in-time solutions for any initial data. Specifically, there exist corresponding spaces $X_T = X_T((0,T) \times \mathbb{R}^3)$ such that for any $u_0 \in X$, there exists $T > 0$ and a unique strong solution $u \in X_T$ to the corresponding Duhamel-type integral equation,

$$\begin{aligned} u(t) &= e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u(s)\otimes u(s))ds \\ &= e^{t\Delta}u_0 + B(u,u), \end{aligned} \tag{3.1}$$

where

$$(v\otimes w)_{j,k} := v_j w_k, \quad [\nabla\cdot(v\otimes w)] := \textstyle\sum_{k=1}^3 \partial_k(v_j w_k)$$
$$\text{and } \mathbb{P}v := v + \nabla(-\Delta)^{-1}(\nabla\cdot v),$$

which results from applying the projection onto divergence-free vector fields operator $\mathbb{P}$ on (NS) and solving the resulting nonlinear heat equation. Moreover, X_T is such that any $u\in X_T$ satisfying (NS) belongs to $C([0,T],X)$. Setting

$$T^*_{X_T}(u_0) := \sup\{T>0|\exists! u := NS(u_0)\in X_T \text{ solving } (NS)\}$$

the Navier-Stokes blow-up problem is:
Question:

$$\text{Does} \sup_{0<t<T^*_{X_T}(u_0)} \|u(t,\cdot)\|_X < \infty \text{ imply that } T^*_{X_T}(u_0) = \infty?$$

In the important work [9] of Escauriaza-Seregin-Šverák, it was established that for $X = L^3(\mathbb{R}^3)$, the answer is yes. This extended a result in the foundational work of Leray [21] regarding the blow-up of $L^p(\mathbb{R}^3)$ norms at a singularity with $p>3$, and of the "Ladyzhenskaya-Prodi-Serrin" type mixed norms $L^s_t(L^p_x)$, $\frac{2}{s}+\frac{3}{p}=1,\ p>3$, establishing a difficult "end-point" case of those results. In [15], based on the work [18], I. Gallagher, G. S. Koch, F. Planchon gave an alternative proof this result in the setting of strong solutions using the method of "critical elements" of C. Kenig and F. Merle. In [14], I. Gallagher, G. S. Koch, F. Planchon extended the method in [15] to give a positive answer to the above question for $X = \dot{B}^{-1+\frac{3}{p}}_{p,q}(\mathbb{R}^3)$ for all $3<p,q<\infty$ (see Definition 3.1.1). Also in [1], D. Albritton proved a stronger blow-up criterion in $\dot{B}^{s_p}_{p,q}$ for $3<p,q<\infty$ and his proof is based on elementary splitting arguments and energy estimates.

We recall the main steps of the method of "critical elements": assume the above question's answer is no for some X and define

$$\infty > A_c := \inf\{\sup_{t\in[0,T^*_{X_T}(u_0))} \|NS(u_0)(t)\|_X \Big| u_0\in X \text{ with } T^*_{X_T}(u_0)<\infty\},$$

where $NS(u_0)$ is a solution to (NS) belonging to $C([0,T^*_{X_T}(u_0),X)$ with initial data $u_0\in X$. And define the set of initial data generating "critical elements"(possibly empty) as follows:

$$\mathcal{D}_c := \{u_0\in X | T^*(u_0)<\infty, \sup_{t\in[0,T^*(u_0))} \|NS(u_0)\|_X = A_c\}.$$

The main steps are:

1. If $A_c<\infty$, then $\mathcal{D}_c$ is non empty.

2. If $A_c<\infty$, then any $u_0\in\mathcal{D}_c$ satisfies $NS(u_0)(t)\to 0$ in $\mathcal{S}'$ as $t\nearrow T^*(u_0)$.

3. If $A_c < \infty$, by backward uniqueness of the heat equation (see [10]), for any $u_0 \in \mathcal{D}_c$, there exists a $t_0 \in (0, T^*(u_0))$ such that $NS(u_0)(t_0) = 0$, which contradicts to the fact that $A_c < \infty$.

In this paper, we consider the blow-up problem for the Navier-Stokes equation with a time-independent external force f, where $\Delta^{-1} f$ is small in L^3 and the initial data belongs to $L^3(\mathbb{R}^3)$.

According to Theorem 3.5.2, we know that there exists a universal constant $c > 0$ such that, if the given external force satisfies $\|\Delta^{-1} f\|_{L^3} < c$, then for any initial data $u_0 \in L^3$, there exists a unique maximal time $T^*(u_0, f) > 0$ and a unique solution to (NSf) u_f belonging to $\mathcal{C}([0,T]; L^3((\mathbb{R}^3))$ for any $T < T^*$ with initial data u_0. Again by Theorem 3.5.2, we have that if $T^*(u_0, f) = \infty$, then $u_f \in C([0,\infty), L^3(\mathbb{R}^3)) \cap L^\infty(\mathbb{R}_+, L^3(\mathbb{R}^3))$, and if $T^*(u_0) < \infty$, we have for any $p > 3$ and $2 < r < \frac{2p}{p-3}$,

$$\lim_{t \to T^*(u_0,f)} \|u_f - U_f\|_{\mathcal{L}^r([0,t], \dot{B}^{s_p + \frac{2}{r}}_{p,p})} = \infty,$$

where $U_f \in L^3$ is the unique small steady-state solution to (NSf) (for existence and uniqueness of small steady-state solution, see [2]) and the function space $\mathcal{L}^r_t(\dot{B}^{s_p+\frac{2}{r}}_{p,p})$ is defined in Definition 3.1.2. However, the above criterion is on the corresponding perturbation solution instead of solution u_f.

In this paper, we give the following blow-up criterion for (NSf): Let $\Delta^{-1} f$ be small in L^3, then

$$(BC) \qquad \limsup_{0<t<T^*(u_0,f)} \|u_f(t,\cdot)\|_{L^3} < \infty \Rightarrow T^*(u_0, f) = \infty.$$

We use a profile decomposition for the solutions to (NSf) to prove the above result. Precisely, the decomposition enables us to construct a connection between the forced and the unforced equation, which provides the blow-up information from the unforced solution to the forced solution. More precisely, we can decompose u_f in a form consisting of the sum of profiles of solutions to (NS), a solution to (NSf) and a remainder. We show that the blow-up information of u_f is determined by the blow-up information of the profiles of solutions to (NS) by an argument using the scaling property of those solutions. Compared with the “critical element” roadmap, we avoid using backward uniqueness of the heat equation (which is only true for the unforced case). We also mention that the method used in [1] can not be applied to our forced case, because the proof of [1] relies on the following scaling property: if u is solution to (NS) with initial data u_0, then $\lambda u(\lambda^2 t, \lambda x)$ is also a solution to (NS) with initial data $\lambda u_0(\lambda \cdot)$. However the above scaling property is not true for the Navier-Stokes equation with a time-independent force f satisfying $\Delta^{-1} f \in L^3$. In fact, for any solution u_f to (NSf) with initial data u_0, $\lambda u_f(\lambda^2 t, \lambda x)$ is no longer a solution to (NSf), unless f is self-similar (which means $f(t,x) \equiv \lambda^3 f(\lambda^2 t, \lambda x)$), hence does not satisfy $\Delta^{-1} f \in L^3$. (And his proof still relies on the backwards uniqueness of heat equation.)

We also point out that one can obtain a profile decomposition of solutions to the forced Navier-Stokes equation with an external force $f \in \mathcal{L}^r(\mathbb{R}_+, \dot{B}^{s_p+\frac{2}{r}-2}_{p,p})$ (Definition 3.1.2) with $s_p + \frac{2}{r} > 0$ and initial data bounded in $\dot{B}^{s_p}_{p,p}$ for any $3 < p < \infty$ with a similar proof as in [15]. And by the same argument as the proof of Theorem 3.1.4, one can show the blow-up criterion as (BC) by replacing L^3 by $\dot{B}^{s_p}_{p,p}$.

3.1.2 Notation and Statement of the Result

Let us first recall the definition of Besov spaces, in dimension $d \geq 1$.

Definition 3.1.1. *Let ϕ be a function in $\mathcal{S}(\mathbb{R}^d)$ such that $\hat{\phi} = 1$ for $|\xi| \leq 1$ and $\hat{\phi} = 0$ for $|\xi| > 2$, and define $\phi_j := 2^{dj}(2^j x)$. Then the frequency localization operators are defined by*

$$S_j := \phi_j * \cdot, \quad \Delta_j := S_{j+1} - S_j.$$

Let f be in $\mathcal{S}'(\mathbb{R}^d)$. We say f belongs to $\dot{B}^s_{p,q}$ if

1. *the partial sum $\sum_{j=-m}^{m} \Delta_j f$ converges to f as a tempered distribution if $s < \frac{d}{p}$ and after taking the quotient with polynomials if not, and*

2. $$\|f\|_{\dot{B}^s_{p,q}} := \|2^{js}\|\Delta_j f\|_{L^p_x}\|_{\ell^q_j} < \infty.$$

We refer to [8] for the introduction of the following type of space in the context of the Navier-Stokes equations.

Definition 3.1.2. *Let $u(\cdot,t) \in \dot{B}^s_{p,q}$ for a.e. $t \in (t_1,t_2)$ and let Δ_j be a frequency localization with respect to the x variable (see Definition 3.1.1). We shall say that u belongs to $\mathcal{L}^\rho([t_1,t_2],\dot{B}^s_{p,q})$ if*

$$\|u\|_{\mathcal{L}^\rho([t_1,t_2],\dot{B}^s_{p,q})} := \|2^{js}\|\Delta_j u\|_{L^\rho([t_1,t_2]L^p_x)}\|_{\ell^q_j} < \infty.$$

Note that for $1 \leq \rho_1 \leq q \leq \rho_2 \leq \infty$, we have

$$L^{\rho_1}([t_1,t_2],\dot{B}^s_{p,q}) \hookrightarrow \mathcal{L}^{\rho_1}([t_1,t_2],\dot{B}^s_{p,q}) \hookrightarrow \mathcal{L}^{\rho_2}([t_1,t_2],\dot{B}^s_{p,q}) \hookrightarrow L^{\rho_2}([t_1,t_2],\dot{B}^s_{p,q}).$$

Let us introduce the following notations (introduced in [14]): we define $s_p := -1 + \frac{3}{p}$ and

$$\mathbb{L}^{a:b}_p(t_1,t_2) := \mathcal{L}^a([t_1,t_2];\dot{B}^{s_p+\frac{2}{a}}_{p,p}) \cap \mathcal{L}^b([t_1,t_2];\dot{B}^{s_p+\frac{2}{b}}_{p,p}),$$

$$\mathbb{L}^a_p := \mathbb{L}^{a:a}_p, \ \mathbb{L}^{a:b}_p(T) := \mathbb{L}^{a:b}_p(0,T) \ \text{ and } \ \mathbb{L}^{a:b}_p[T < T^*] := \cap_{T<T^*}\mathbb{L}^{a:b}_p(T). \tag{3.2}$$

Remark 3.1.3. *We point out that according to our notations, $u \in \mathbb{L}^{a:b}_{p,q}[T < T^*]$ merely means that $u \in \mathbb{L}^{a:b}_{p,q}(T)$ for any $T < T^*$ and does not imply that $u \in \mathbb{L}^{a:b}_{p,q}(T^*)$(the notation does not imply any uniform control as $T \nearrow T^*$).*

Now let us state our main result.

Theorem 3.1.4. *Suppose that $\|\Delta^{-1} f\|_{L^3} < c$, where c is the small universal constant in Theorem 3.5.2. Let $u_0 \in L^3(\mathbb{R}^3)$ be a divergence free vector field and $u_f = NSf(u_0) \in C([0,T^*(u_0,f)),L^3(\mathbb{R}^3))$, where $T^*(u_0,f)$ is the maximal life span of u_f, be the unique strong solution of (NSf) with initial data u_0. If $T^*(u_0,f) < \infty$, then*

$$\limsup_{t \to T^*(u_0,f)} \|u(t)\|_{L^3(\mathbb{R}^3)} = \infty. \tag{3.3}$$

Remark 3.1.5. *Our profile decomposition method is not only valid for a time-independent force, but also can be extended to more general time-dependent external force. For example: our method is valid for solutions belonging to $C([0,T^*),L^3(\mathbb{R}^3))$ constructed in*

[6] with initial $u_0 \in L^3$, where the external force f can be written as $f = \nabla \cdot V$ and $\sup_{0<t<\infty} t^{1-\frac{3}{p}} \|V\|_{L^{\frac{p}{2}}}$ is small enough for some $3 < p \leq 6$. Actually our method only depends on the smallness of U_f and the continuity in time of solutions in space L^3, which are similar (U_f can by replaced by some small solution with small initial data in L^3 constructed in [6]) with the solutions in [6], whose associated force is time-dependent. After that we can obtain (BC) for any fixed small external force as above by a similar argument of the case that f is time independent.

The rest of this article is structured as follows. In Section 2, we give the proof Theorem 3.1.4, which relies on a profile decomposition of solutions to (NSf). Section 3 is devoted to showing the profile decomposition of solutions to (NSf). In Section 4, a perturbation result for (NS) is stated in an appropriate functional setting which provides the key estimate of Section 3. Finally in the Appendixs, we recall some well-posedness results for (NSf) and the corresponding steady-state equation. Also we collect standard Besov space estimates used throughout the paper in it.

3.2 Proof of the main theorem

Suppose that $\|\Delta^{-1} f\|_{L^3} < c$ is a fixed external force.

Let us define

$$\begin{aligned} A_c := \sup\{A > 0| \sup_{t\in[0,T^*(u_0,f))} \|NSf(u_0)(t)\|_{L^3} \leq A \\ \Longrightarrow T^*(u_0, f) = \infty, \forall u_0 \in L^3(\mathbb{R}^3)\}. \end{aligned} \tag{3.4}$$

Note that A_c is well-defined by small-data results. If A_c is finite, then A_c can be rewritten as

$$A_c = \inf\{ \sup_{t\in[0,T^*(u_0,f))} \|NSf(u_0)(t)\|_{L^3} | u_0 \in L^3 \text{ with } T^*(u_0, f) < \infty\}.$$

In the case when $A_c < \infty$, we introduce the (possibly empty) set of initial data generating a critical element as follows:

$$\mathcal{D}_c := \{u_0 \in L^3(\mathbb{R}^3) | T^*(u_0, f) < \infty, \sup_{t\in[0,T^*(u_0,f))} \|NSf(u_0)(t)\|_{L^3} = A_c\}.$$

Before proving Theorem 3.1.4, we prove the above set is empty.

Proposition 3.2.1 ($\mathcal{D}_c$ is empty)**.** *Suppose that $A_c < \infty$, then $\mathcal{D}_c = \emptyset$.*

Proof. We prove the proposition by contradiction. Assume $\mathcal{D}_c \neq \emptyset$, we take a $u_{0,c} \in \mathcal{D}_c$ and denote $u_c = NSf(u_{0,c})$. By the definition of $\mathcal{D}_c$, we have $T^*(u_{0,c}, f) < \infty$ and

$$\sup_{t\in[0,T^*(u_{0,c},f))} \|NSf(u_{0,c})(t)\|_{L^3} = A_c.$$

We choose a sequence $(s_n)_{n\in\mathbb{N}} \subset [0, T^*(u_{0,c}, f))$ such that $s_n \nearrow T^*(u_{0,c}, f)$. Let $u_{0,n} := u_c(s_n)$ and $u_n := NSf(u_{0,n})$. Since $A_c < \infty$, we know that $(u_{0,n})_{n\in\mathbb{N}}$ is a bounded sequence in $L^3(\mathbb{R}^3)$ and

$$\sup_{t\in[0,T^*(u_{0,n},f))} \|u_n(t)\|_{L^3} = A_c.$$

By Theorem 3.3.3 with the same notation, for any $t \leq \tau_n$, u_n has the following profile decomposition , for any $J \geq J_0$ and $n \geq n(J_0)$,

$$u_n = U^1 + \sum_{j=2}^{J} \Lambda_{j,n} U^j + w_n^J + r_n^J,$$

where $\tau_n = \min_{j \in I}\{\lambda_{j,n}^2 T^j\}$. After reordering, we can write

$$u_n = \sum_{j=1}^{J} \Lambda_{j,n} U^j + w_n^J + r_n^J$$

with $\Lambda_{j_0,n} \equiv \mathrm{Id}$ for some $1 \leq j_0 \leq J_0$ and for $j \leq J$ and n large enough,

$$\forall j \leq k \leq J_0, \;\; \lambda_{j,n}^2 T_j^* \leq \lambda_{k,n}^2 T_k^*.$$

First we claim that $j_0 > 1$. In fact, by Theorem 3.3.3,

$$\lambda_{1,n} T_1^* \leq T^*(u_{0,n}, f) = T^*(u_{0,c}, f) - s_n \to 0, \;\; \text{as} \;\; n \to \infty,$$

which implies that

$$\lim_{n \to \infty} \lambda_{1,n} = 0.$$

Hence $j_0 > 1$, which implies that with the new ordering $U^1 = NS(\phi_1)$, and $T_1^* < \infty$.

Now we take $s \in (0, T_1^*)$ and let $t_n = \lambda_{1,n}^2 s$. According to Proposition 3.3.4, we have

$$A_c^3 \geq \|u_n(t_n)\|_{L^3}^3 \geq \|U^1(s)\|_{L^3}^3 + \varepsilon(n, s),$$

where $\lim_{n \to \infty} \varepsilon(n, s) = 0$ for any fixed s. By the blow-up criterion for the Navier-Stokes equation (see [15])

$$\limsup_{t \to T_1^*} \|U^1(t)\|_{L^3(\mathbb{R}^3)} = \infty,$$

then we choose a $s_0 \in (0, T_1^*)$ such that

$$\|U^1(s_0)\|_{L^3(\mathbb{R}^3)} > 2A_c.$$

And we can take a corresponding $n_0 := n(s_0)$ such that $|\varepsilon(n_0, s_0)| \leq A_c^3$. Then we get

$$A_c^3 > 8A_c^3 - A_c^3 = 7A_c^3$$

which contradicts the fact that $A_c < \infty$. Then $\mathcal{D}_c = \emptyset$. □

Now we prove Theorem 3.1.4 by contradiction.

Proof of Theorem 3.1.4. We suppose that $A_c < \infty$ which means (3.3) fails.
Let us consider a sequence $u_{0,n}$ bounded in the space L^3 such that the life span of

$NSf(u_{0,n})$ satisfies $T^*(u_{0,n}, f) < \infty$ for each $n \in \mathbb{N}$ and such that

$$A_n := \sup_{t\in[0,T^*(u_{0,n},f))} \|NSf(u_{0,n})\|_{L^3(\mathbb{R}^3)}$$

satisfies

$$A_c \leq A_n \ \text{ and } \ A_n \to A_c, \ \ n \to \infty.$$

Then by Theorem 3.3.3 and after reordering as above, we have for any $J \geq J_0$ and $n \geq n(J_0)$

$$u_n := NSf(u_{0,n}, f) = \sum_{j=1}^{J} \Lambda_{j,n} U^j + w_n^J + r_n^J, \forall t \in [0, \tau_n]$$

and for any $n \geq n_0(J_0)$, recalling that T_j^* is the life span of U^j

$$\forall j \leq k \leq J_0, \ \ \lambda_{j,n}^2 T_j^* \leq \lambda_{k,n}^2 T_k^*,$$

where $U^{j_0} = NSf(\phi_{j_0})$ (j_0 is such that $\Lambda_{j_0,n} \equiv 1$) and $U^j = NS(\phi_j)$ for any $1 \leq j \leq J_0$ with $j \neq j_0$. Theorem 3.3.3 also ensures that there J_0 such that $T_{J_0}^* < \infty$ (if not we would have $\tau_n \equiv \infty$ and hence $T^*(u_{0,n}, f) \equiv \infty$, contrary to our assumption). On the other hand, we recall that $U^{j_0} := NSf(\phi_{j_0})$ with $1 \leq j_0 \leq J_0$, where ϕ_{j_0} is a weak limit of $(u_{0,n})_{n\geq 1}$. Therefore by the above re-ordering, two different cases need to be considered:

- $j_0 = 1$: the lower-bound of the life span of u_n is controlled by the life span of $U^{j_0} = U^1 = NSf(\phi_{j_0})$, which generates a critical element.
- $j_0 > 1$: the lower-bound of the life span of u_n is controlled by the life span of $\Lambda_{1,n} NS(\phi_1)$.

Case 1: $j_0 = 1$. In this case, by definition of A_c, we have $U^1 = NSf(\phi_1)$, $\Lambda_{1,n} \equiv \mathrm{Id}$ and

$$\sup_{s\in[0,T_1^*)} \|NSf(\phi_1)\|_{L^3} \geq A_c. \tag{3.5}$$

For any $s \in (0, T_1^*)$, setting $t_n := \lambda_{1,n}^2 s$, by Proposition 3.3.4

$$\begin{aligned} A_n^3 &\geq \sup_{t\in[0,T^*(u_{0,n},f))} \|NSf(u_{0,n})\|_{L^3}^3 \geq \|NSf(u_{0,n})(t_n)\|_{L^3}^3 \\ &\geq \|U^1(s)\|_{L^3}^3 + \varepsilon(n, s), \end{aligned}$$

where for any fixed $s \in [0, T_1^*)$

$$\lim_{n\to\infty} \varepsilon(n, s) = 0.$$

According to (3.5) and the fact that $A_n \to A_c$ as $n \to \infty$, we infer that

$$\sup_{s\in[0,T_1^*)} \|NSf(\phi_1)\|_{L^3} = A_c,$$

which means $\phi_1 \in \mathcal{D}_c$. This fact contradicts Proposition 3.2.1.
Case 2 $j_0 > 1$**:**. In this case, $U^1 = NS(\phi_1)$ and U^1 satisfies that

$$\limsup_{t\to T_1^*} \|U^1(t)\|_{L^3} = \infty, \tag{3.6}$$

and $\Lambda_{1,n} \neq \mathrm{Id}$.

On the other hand for any $s \in (0, T_1^*)$, setting $t_n := \lambda_{1,n}^2 s$,

$$\begin{aligned} A_n^3 &\geq \sup_{t\in[0,T^*(u_{0,n},f))} \|NSf(u_{0,n})\|_{L^3}^3 \geq \|NSf(u_{0,n})(t_n)\|_{\dot{H}^{1/2}}^2 \\ &\geq \|U^1(s)\|_{L^3}^3 + \varepsilon(n,s), \end{aligned}$$

where

$$\lim_{n\to\infty} \varepsilon(n,s) = 0, \quad \forall s \in [0, T_1^*).$$

Thanks to (3.6), one can take s_0 such that

$$\|U^1(s_0)\|_{L^3} > 2A_c$$

and choose $n_0 := n(s_0)$ such that $\varepsilon(n_0, s_0) \leq A_c^3$ and $A_{n_0}^3 \leq 2A_c^3$, then we have

$$\begin{aligned} 2A_c^3 &\geq \|U^1(s_0)\|_{L^3}^3 + \varepsilon(n_0, s_0) \\ &> 7A_c^3, \end{aligned}$$

which contradicts the fact that $A_c < \infty$. Then we prove that for any u_0, if $T^*(u_0, f) < \infty$

$$\limsup_{t\to T^*(u_0,f)} \|NSf(u_0)\|_{L^3(\mathbb{R}^3)} = \infty.$$

Theorem 3.1.4 is proved.

□

3.3 Profile decomposition

In [15] a profile decomposition of solutions to the Navier-Stokes equations associated with data in $\dot{B}_{p,p}^{s_p}$ is proved for $d < p < 2d + 3$, thus extending the result of [18]. In this section we use the idea of [15] to give a decomposition of solutions to the Navier-Stokes equations with a small external force and associated with initial data in L^3.

3.3.1 Profile decomposition of bounded sequence in L^3

Before stating the main result of this section, let us recall the following definition.

Definition 3.3.1. *We say that two sequences $(\lambda_{j,n}, x_{j,n})_{n\in\mathbb{N}} \in ((0,\infty) \times \mathbb{R}^3)^{\mathbb{N}}$ for $j \in \{1, 2\}$ are orthogonal, and we write $(\lambda_{1,n}, x_{1,n})_{n\in\mathbb{N}} \perp (\lambda_{2,n}, x_{2,n})_{n\in\mathbb{N}}$, if*

$$\lim_{n\to+\infty} \frac{\lambda_{1,n}}{\lambda_{2,n}} + \frac{\lambda_{2,n}}{\lambda_{1,n}} + \frac{|x_{1,n} - x_{2,n}|}{\lambda_{1,n}} = +\infty. \tag{3.7}$$

Similarly we say that a set of $(\lambda_{j,n}, x_{j,n})_{n\in\mathbb{N}}$*, for* $j \in \mathbb{N},\ j \geq 1$*, is orthogonal if for all* $j \neq j'$*,* $(\lambda_{j,n}, x_{j,n})_{n\in\mathbb{N}} \perp (\lambda_{j',n}, x_{j',n})_{n\in\mathbb{N}}$*.*

Next let us define, for any set of sequences $(\lambda_{j,n}, x_{j,n})_{n\in\mathbb{N}}$ (for $j \geq 1$), the scaling operator

$$\Lambda_{j,n}U_j(t,x) := \frac{1}{\lambda_{j,n}}U_j(\frac{t}{\lambda_{j,n}^2}, \frac{x - x_{j,n}}{\lambda_{j,n}}). \tag{3.8}$$

It is proved in [19] that any bounded (time-independent) sequence in $\dot{B}^{s_p}_{p,p}(\mathbb{R}^3)$ may be decomposed into a sum of rescaled functions $\Lambda_{j,n}\phi_j$, where the set of sequences $(\lambda_{j,n}, x_{j,n})_{n\in\mathbb{N}}$ is orthogonal, up to a small remainder term in $\dot{B}^{s_q}_{q,q}$, for any $q > p$. Since in this paper we only consider the initial data in L^3, we only state the profile decomposition result of bounded sequences in L^3 in [19]. The precise statement is in the spirit of the pioneering work [16].

Theorem 3.3.2. *Let* $(\varphi_n)_{n\geq 1}$ *be a bounded sequence of functions in* $L^3(\mathbb{R}^3)$ *and let* ϕ_1 *be any weak limit point of* $(\varphi_n)_{n\in\mathbb{N}}$*. Then, after possibly replacing* $(\varphi_n)_{n\in\mathbb{N}}$ *by a subsequence which we relabel* $(\varphi_n)_{n\geq 1}$*, there exists a subsequence of profiles* $(\phi_j)_{j\geq 2}$ *in* $L^3(\mathbb{R}^3)$*, and a set of sequences* $(\lambda_{j,n}, x_{j,n})_{n\in\mathbb{N}}$ *for* $j \in \mathbb{N}$ *with* $(\lambda_{1,n}, x_{1,n}) \equiv (1, 0)$ *which are orthogonal in the sense of Definition 3.3.1 such that, for all* $n, J \in \mathbb{N}$*, if we define* ψ_n^J *by*

$$\varphi_n = \sum_{j=1}^{J} \Lambda_{j,n}\phi_j + \psi_n^J,$$

the following properties hold:

- *the function* ψ_n^J *is a remainder in the sense that for any* $p > 3$*,*

$$\lim_{J\to\infty} \big(\limsup_{n\to\infty} \|\psi_n^J\|_{\dot{B}^{s_p}_{p,p}}\big) = 0; \tag{3.9}$$

- *There is a norm* $\|\cdot\tilde{\|}_{L^3}$ *which is equivalent to* $\|\cdot\|_{L^3}$ *such for each* $n \in \mathbb{N}$*,*

$$\sum_{j=1}^{\infty} \|\phi_j\tilde{\|}^3_{L^3(\mathbb{R}^3)} \leq \liminf_{n\to\infty} \|\varphi_n\tilde{\|}^3_{L^3(\mathbb{R}^3)}$$

  and, for any interger J*,*

$$\|\psi_n^J\tilde{\|}_{L^3} \leq \|\varphi_n\tilde{\|}_{L^3} + o(1)$$

  as n *goes to infinity.*

We mention that, in particular, for any $j \geq 2$, either $\lim_{n\to\infty}|x_{j,n}| = \infty$ or $\lim_{n\to\infty}\lambda_{j,n} \in \{0, \infty\}$ due to the orthogonality of scales/cores with $(\lambda_{1,n}, x_{1,n}) \equiv (1, 0)$, and also that

$$\sum_{j=1}^{\infty} \|\phi_j\|^3_{L^3} \lesssim \liminf_{n\to\infty} \|\varphi_n\|^3_{L^3}. \tag{3.10}$$

3.3.2 Profile decomposition of solutions to (NSf)

Theorem 3.3.3. *Suppose that* $\|\Delta^{-1}f\|_{L^3} < c$*, where* c *is the small universal constant in Theorem 3.5.2.*

Let $(u_{0,n})_{n\in\mathbb{N}}$ be a bounded sequence of divergence-free vector fields in $L^3(\mathbb{R}^3)$, and ϕ_1 be any weak limit point of $\{u_{0,n}\}$. Then, after possibly relabeling the sequence due to the extraction of a subsequence following an application of Theorem 3.3.2 with $\varphi_n := u_{0,n}$, defining $u_n := NSf(u_{0,n})$, $U^1 := NSf(\phi_1) \in C([0,T_1], L^3)$ and $U^j := NS(\phi_j) \in C([0,T_j], L^3)$ for any $j \geq 2$ (where T_j is any real number smaller than T_j^*, where T_j^* is the life span of U^j for $j \geq 1$, and $T^j = \infty$ if $T_j^* = \infty$), the following properties hold:

- *there is a finite (possibly empty) subset I of $\mathbb{N}$ such that*

$$\forall j \in I,\ \ T^j < \infty \ \text{ and } \ \forall j' \in \mathbb{N}\backslash I,\ \ U^{j'} \in C(\mathbb{R}_+, L^3(\mathbb{R}^3)).$$

  Moreover setting $\tau_n := \min_{j\in I} \lambda_{j,n}^2 T^j$ if I is nonempty and $\tau_n = \infty$ otherwise, we have

$$\sup_n \|u_n\|_{\mathcal{L}^\infty_{\tau_n}(L^3(\mathbb{R}^3))} < \infty; \tag{3.11}$$

- *there exists some large $J_0 \in \mathbb{N}$ such that for each $J > J_0$, there exists $N(J) \in \mathbb{N}$ such that for all $n > N(J)$, all $t \leq \tau_n$ and all $x \in \mathbb{R}^3$, setting $w_n^J := e^{t\Delta}(\psi_n^J)$ and defining r_n^J by*

$$u_n(t,x) = U^1 + \sum_{j=2}^{J} \Lambda_{j,n} U^j + w_n^J + r_n^J, \tag{3.12}$$

  then w_n^J and r_n^J are small remainders in the sense that, for any $3 < p < 5$,

$$\lim_{J\to\infty} \big(\limsup_{n\to\infty} \|w_n^J\|_{\mathbb{L}_p^{1:\infty}(\infty)}\big) = \lim_{J\to\infty} \big(\limsup_{n\to\infty} \|r_n^J\|_{\mathbb{L}_p^{p:\infty}(\tau_n)}\big) = 0. \tag{3.13}$$

We recall the following important orthogonality result without proof. Its proof is the same as the proof of Claim 3.3 of [15], as it just depends on orthogonality property on scales/core. To state the result, note first that an application of Theorem 3.3.3 yields a non- empty blow-up set $I \subset \{1, \ldots, J_0\}$. Then we can re-order those first J_0 profiles, thanks to the orthogonality (3.7) of the scales $\lambda_{j,n}$ so that for $n_0 = n_0(J_0)$ sufficiently large, we have

$$\forall n \geq n_0, \quad 1 \leq j \leq j' \leq J_0 \Rightarrow \lambda_{j,n}^2 T_j^* \leq \lambda_{j',n}^2 T_{j'}^* \tag{3.14}$$

(some of these terms may equal infinity).

Proposition 3.3.4. *Let $(u_{0,n})_{n\geq 1}$ be a bounded sequence in L^3 and for which the set I of blow-up profile indices resulting from an an application of Theorem 3.3.3 is non-empty. After re-ordering the profiles in the profile decomposition of $u_n := NSf(u_{0,n})$ such that (3.14) holds for some J_0, setting $t_n := \lambda_{1,n}^2 s$ for $s \in [0, T_1^*)$ one has (after possibly passing to a subsequence in n):*

$$\|u_n(t)\|_{L^3}^3 = \|(\Lambda_{1,n}U^1)(t_n)\|_{L^3}^3 + \|u(t_n) - (\Lambda_{1,n}U^1)(t_n)\|_{L^3}^3 + \varepsilon(n,s), \tag{3.15}$$

where $\varepsilon(n,s) \to 0$ as $n \to \infty$ for each fixed $s \in [0, T_1^*)$.

Proof of Theorem 3.3.3 . Let $(u_{0,n})_{n\geq 1}$ be a bounded sequence in L^3. We first use Theorem 3.3.2 to decompose the above sequence.

Then with the notation of Theorem 3.3.3

$$u_{0,n} = \sum_{j=1}^{J} \Lambda_{j,n}\phi_j + \psi_n^J.$$

We define

$$\begin{aligned} &u_n := NSf(u_{0,n}), \ \ U^1 := NSf(\phi_1) \in C([0,T_1^*), L^3(\mathbb{R}^3)), \\ &U^j := NS(\phi_j) \in C([0,T_j^*), L^3(\mathbb{R}^3)) \ \text{ and } \ w_n^J := e^{t\Delta}(\psi_n^J). \end{aligned}$$

By (3.9) and standard linear heat estimates we have

$$\lim_{J\to\infty}\Big(\limsup_{n\to\infty} \|w_n^J\|_{\mathbb{L}_p^{1:\infty}(\infty)}\Big) = 0.$$

According to (3.10), we have for any $p > 3$

$$\Big\|\big(\|\phi_j\|_{\dot{B}^{s_p}_{p,p}}\big)_{j=1}^{\infty}\Big\|_{\ell^p} \lesssim \Big\|\big(\|\phi_j\|_{L^3}\big)_{j=1}^{\infty}\Big\|_{\ell^3} \lesssim \liminf_{n\to\infty} \|u_{0,n}\|_{L^3}, \tag{3.16}$$

which implies that, for any $j \geq 2$,

$$U^j \in \mathbb{L}_p^{1:\infty}(T < T_j^*)$$

and there exists $J_0 > 0$ such that for any $j \geq J_0$, $T_j^* = \infty$. Moreover, for any $j \geq J_0$

$$U^j \in \mathbb{L}_p^{1:\infty}(\infty) \ \text{ and } \ \|U^j\|_{\mathbb{L}_p^{1:\infty}(\infty)} \lesssim \|\phi_j\|_{\dot{B}^{s_p}_{p,p}}.$$

Hence, I will be a subset of $\{1,\ldots,J_0\}$ which proves the first part of the first statement in Theorem 3.3.3.

From now on, we restrict $p \in (3,5)$. By the local Cauchy theory we can solve (NSf) with initial data $u_{0,n}$ for each integer n, and produce a unique solution $u_n \in C([0,T_n^*), L^3(\mathbb{R}^3))$, where T_n^* is the life span of u_n. Now we define, for any $J \geq 1$,

$$r_n^J := u_n - \sum_{j=1}^{J} \Lambda_{j,n}U^j - w_n^J,$$

where we recall that $\Lambda_{1,n}U^1 = U^1$. We mention that the life span of $\Lambda_{j,n}U^j$ is $\lambda_{j,n}^2 T_j^*$. Therefore, the function $r_n^J(t,x)$ is defined a priori for $t \in [0,t_n)$, where

$$t_n := \min(T_n^*; \tau_n)$$

with the notation of Theorem 3.3.3. Our main goal is to prove that r_n^J is actually defined on $[0,\tau_n]$ (at least if J and n are large enough), which will be a consequence of perturbation theory for the Navier-Stokes equations, see Proposition 3.4.1. In the process, we shall obtain the uniform limiting property, namely,

$$\lim_{J\to\infty}\Big(\limsup_{n\to\infty} \|r_n^J\|_{\mathbb{L}_p^{p:\infty}(\tau_n)}\Big) = 0. \tag{3.17}$$

Let us write the equation satisfied by r_n^J. We adapt the same method as [14] and [15]. It turns out to be easier to write that equation after a re-scaling in space-time. For convenience, let use re-order the functions $\Lambda_{j,n}U^j$, for $1 \leq j \leq J_0$, in such a way that, for some $n_0 = n_0(J_0)$ sufficiently large, we have as in [15],

$$\forall n \geq n_0, \;\; j \leq j' \leq J_0 \Rightarrow \lambda_{j,n} T_j^* \leq \lambda_{j',n} T_{j'}^*.$$

And we define $1 \leq j_0 \leq J_0$ as the integer such that $\Lambda_{j_0,n} \equiv \mathrm{Id}$. And $\Lambda_{j_0,n} U^{j_0} = NSf(\phi_{j_0}) = U_f + V^{j_0}$ (see Theorem 3.5.2). We note that $\lambda_{j,n}^2 T_j^*$ is the life span of $\Lambda_{j,n} U^j$.

The inverse of our dilation/translation operator $\Lambda_{j,n}$ is

$$\Lambda_{j,n}^{-1} f(s,y) := \lambda_{j,n} f(\lambda_{j,n}^2 s, \lambda_{j,n} y + x_{j,n}). \tag{3.18}$$

Then we define, for any integer J,

$$j \leq J, \;\; U_n^{j,1} := \Lambda_{1,n}^{-1} \Lambda_{j,n} U^j, \;\; R_n^{J,1} := \Lambda_{1,n}^{-1} r_n^J, \;\; V_n^{j_0,1} = \Lambda_{1,n}^{-1} V^{j_0}$$
$$U_f^1 := \Lambda_{1,n}^{-1} U_f, \;\; W_n^{J,1} := \Lambda_{1,n}^{-1} w_n^J \;\text{ and }\; U_n^1 := \Lambda_{1,n}^{-1} u_n.$$

Clearly we have

$$R^{J,1} = U_n^1 - \left(\sum_{j=1}^{J} U_n^{j,1} + W_n^{J,1}\right)$$

and $R_n^{J,1}$ is a divergence free vector field, solving the following system:

$$\begin{cases} \partial_t R_n^{J,1} - \Delta R_n^{J,1} + \mathbb{P}(R_n^{J,1} \cdot \nabla R_n^{J,1}) + Q(R_n^{J,1}, U_f^1 + G_n^{J,1}) = F_n^{J,1}, \\ R_n^{J,1}|_{t=0} = 0, \end{cases} \tag{3.19}$$

where we recall that $\mathbb{P} := \mathrm{Id} - \nabla\Delta^{-1}(\nabla\cdot)$ is the projection onto divergence free vector fields, and where

$$Q(a,b) := \mathbb{P}((a\cdot\nabla)b + (b\cdot\nabla)a)$$

for two vector fields a, b. Finally we have defined

$$G_n^{J,1} := \sum_{j \neq j_0}^{J} U_n^{j,1} + W_n^{J,1} + V_n^{j_0,1},$$

and

$$F_n^{J,1}, = -\frac{1}{2} Q(W_n^{J,1}, W_n^{J,1}) - \frac{1}{2} \sum_{j \neq j'}^{J} Q(U_n^{j,1}, U_n^{j',1}) - \sum_{j=1}^{J} Q(U_n^{j,1}, W_n^{J,1}).$$

In order to use perturbative bounds on this system, we need a uniform control on the drift term $G_n^{J,1}$ and smallness of the forcing term $F_n^{J,1}$. The results are the following.

Lemma 3.3.5. *Fix $T_1 < T_1^*$. The sequence $(G_n^{J,1})_{n \geq 1}$ is bounded in $\mathcal{L}^p([0,T_1], \dot{B}_{p,p}^{s_p + \frac{2}{p}})$, uniformly in J, which means that*

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|G_n^{J,1}\|_{\mathcal{L}^p([0,T_1], \dot{B}_{p,p}^{s_p+\frac{2}{p}})} = 0.$$

The proof of the above lemma is the same as the proof of Lemma 2.5 in [15], as it just depends on orthogonality property on scales/core.

Lemma 3.3.6. *Fix $T_1 < T_1^*$. The source term $F_n^{J,1}$ goes to zero for each $J \in \mathbb{N}$, as n goes to infinity, in the space $\mathcal{F} := \mathcal{L}^p([0,T_1], \dot{B}_{p,p}^{s_p+\frac{2}{p}-2}) + \mathcal{L}^{\frac{p}{2}}([0,T_1], \dot{B}_{p,p}^{s_p+\frac{4}{p}-2})$. In precisely,*

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|F_n^{J,1}\|_{\mathcal{F}} = 0.$$

Assuming these lemmas to be true, the end of the proof of the theorem is a direct consequence of Proposition 3.4.1.

□

Now let us prove Lemma 3.3.6.

Proof of Lemma 3.3.6. We first notice that

$$\begin{aligned}F_n^{J,1}, = &-\frac{1}{2}Q(W_n^{J,1}, W_n^{J,1}) - \frac{1}{2}\sum_{\substack{j\neq j', j\neq j_0\\ j'\neq j_0}}^{J} Q(U_n^{j,1}, U_n^{j',1}) - \sum_{j=1, j\neq j_0}^{J} Q(U_n^{j,1}, W_n^{J,1})\\ &-\sum_{j=1, j\neq j_0}^{J} Q(U_n^{j_0,1}, U_n^{j,1}) - Q(U_n^{j_0,1}, W_n^{J,1}).\end{aligned}$$

And we note that the structure of

$$A_n^J := -\frac{1}{2}Q(W_n^{J,1}, W_n^{J,1}) - \frac{1}{2}\sum_{\substack{j\neq j', j\neq j_0\\ j'\neq j_0}}^{J} Q(U_n^{j,1}, U_n^{j',1}) - \sum_{j=1, j\neq j_0}^{J} Q(U_n^{j,1}, W_n^{J,1})$$

is the same as the $G_n^{J,0}$ of Lemma 2.7 in [15]. As a consequence of Lemma 2.7 in [15], we obtain

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|A_n^J\|_{\mathcal{F}} = 0.$$

Hence to finish the proof of Lemma 3.3.6, we need to show

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|B_n^J\|_{\mathcal{F}} = 0,$$

where

$$B_n^J := -\sum_{j=1, j\neq j_0}^{J} Q(U_n^{j_0,1}, U_n^{j,1}) - Q(U_n^{j_0,1}, W_n^{J,1}).$$

By product laws and scaling invariance, we first have

$$\begin{aligned}\|Q(U_n^{j_0,1}, W_n^{J,1})\|_{\mathcal{L}^p([0,T_1], \dot{B}_{p,p}^{s_p+\frac{2}{p}-2})} &\lesssim \|W_n^{J,1}\|_{\mathcal{L}^p([0,T_1], \dot{B}_{p,p}^{s_p+\frac{2}{p}})} \|U_n^{j_0,1}\|_{\mathcal{L}^\infty([0,T_0], \dot{B}_{p,p}^{s_p})}\\ &\lesssim \|\psi_n^J\|_{\dot{B}_{p,p}^{s_p}} \|\phi_{j_0}\|_{\dot{B}_{p,p}^{s_p}},\end{aligned}$$

implies that

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|Q(U_n^{j_0,1}, W_n^{J,1})\|_{\mathcal{L}^p([0,T_1], \dot{B}_{p,p}^{s_p+\frac{2}{p}-2})} = 0.$$

Now we are left with proving that

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|\sum_{j\neq j'}^{J} Q(U_n^{j,1},U_n^{j',1})\|_{\mathcal{F}}=0.$$

We can write $\sum_{j=1,j\neq j_0}^{J} Q(U_n^{j_0,1},U_n^{j,1})$ as the following way:

$$\sum_{j=1,j\neq j_0}^{J} Q(U_n^{j_0,1},U_n^{j,1})=\sum_{j\neq j_0} Q(V_n^{j_0,1},U_n^{j,1})+\sum_{j\neq j_0} Q(\Lambda_{1,n}^{-1}U_f,U_n^{j,0}).$$

Since for any $j\neq j_0$ $U_n^{j,1}\in\mathbb{L}_p^{1:\infty}(T_0)$, $V_n^{j_0,1}\in\mathbb{L}_p^{p:\infty}(T_0)$ and $3<p<5$, by (3.20) in Proposition 3.3.7, we have for all $j',j\neq j_0$

$$\lim_{n\to\infty}\|Q(U_n^{j,1},V_n^{j_0,1})\|_{\mathcal{L}^{\frac{p}{2}}([0,T_1],\dot{B}_{p,p}^{s_p+\frac{4}{p}-2})}=0.$$

And according to $U_f\in L^3(\mathbb{R}^3)$, we have

$$\lim_{n\to\infty}\|Q(U_n^{j,1},\Lambda_{1,n}^{-1}U_f)\|_{\mathcal{L}^{p}([0,T_1],\dot{B}_{p,p}^{s_p+\frac{2}{p}-2})}=0$$

by Proposition 3.3.7. By the above two relations, we have

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|\sum_{j=1,j\neq j_0}^{J} Q(U_n^{j_0,1},U_n^{j,1})\|_{\mathcal{L}^{p}([0,T_1],\dot{B}_{p,p}^{s_p+\frac{2}{p}-2})}=0.$$

Lemma 3.3.6 is proved. □

3.3.3 Orthogonality Property

In this paragraph, we show the orthogonality properties used in the proof of Lemma 3.3.6. The first statement of Proposition 3.3.7 is just a particular case of orthogonality property given in [14] (see the proof Lemma 3.3 in [13]). By the same idea in [14], we give a orthogonality property in the case that one of the element in the product is time-independent.

Proposition 3.3.7. *We assume that $(\lambda_{1,n},x_{1,n})_{n\in\mathbb{N}}$ and $(\lambda_{2,n},x_{2,n})_{n\in\mathbb{N}}$ are orthogonal. Let $T\in\mathbb{R}_+\cup\{+\infty\}$. Then the following properties hold:*

1. *Let $p>3$ and $1-\frac{3}{p}<\frac{1}{a}<1$. Suppose that $v,w\in\mathcal{L}^{2a}([0,T],\dot{B}_{p,p}^{s_p+\frac{1}{a}})$. Then we have*

$$\lim_{n\to\infty}\|(\Lambda_{1,n}v)(\Lambda_{2,n}w)\|_{\mathcal{L}^{a}([0,T_n],\dot{B}_{p,p}^{s_p+\frac{2}{a}-1})}=0,\tag{3.20}$$

where $T_n:=\min\{\lambda_{1,n}^2T,\lambda_{2,n}^2T\}$.

2. *Let $p>3$ and $2<r<\frac{2p}{p-3}$. Suppose that $U\in L^3(\mathbb{R}^3)$ and $v\in\mathcal{L}^{r}([0,T],\dot{B}_{p,p}^{s_p+\frac{2}{r}})$.*

$$\lim_{n\to\infty}\|(\Lambda_{1,n}U)(\Lambda_{2,n}v)\|_{\mathcal{L}^{r}([0,T_n'],\dot{B}_{p,p}^{s_p+\frac{2}{r}-1})}=0,\tag{3.21}$$

where $T_n':=\lambda_{2.n}T$.

Proof. As we mentioned above, (3.20) is a particular case of orthogonality property given in [17], we only need to prove the second statement of the proposition.

For any given $\varepsilon > 0$ one can find two compactly supported (in space and time) functions v_ε and U_ε such that

$$\|v - v_\varepsilon\|_{\mathcal{L}^{2a}([0,T],\dot{B}^{s_p+\frac{1}{a}}_{p,p})} + \|U - U_\varepsilon\|_{L^3} \leq \varepsilon.$$

Product rules (along with the scale invariance of the scaling operators) gives that

$$\begin{aligned}
&\|(\Lambda_{1,n}v)(\Lambda_{2,n}(U - U_\varepsilon))\|_{\mathcal{L}^a([0,T_n],\dot{B}^{s_p+\frac{2}{a}-1}_{p,p})} \\
&+\|(\Lambda_{1,n}(v - v_\varepsilon))(\Lambda_{2,n}(U)\|_{\mathcal{L}^a([0,T_n],\dot{B}^{s_p+\frac{2}{a}-1}_{p,p})} \\
&+\|(\Lambda_{1,n}(v - v_\varepsilon))(\Lambda_{2,n}(U - U_\varepsilon))\|_{\mathcal{L}^a([0,T_n],\dot{B}^{s_p+\frac{2}{a}-1}_{p,p})} \lesssim \varepsilon.
\end{aligned}$$

Then it is enough to prove that for fixed $\varepsilon > 0$

$$\lim_{n\to\infty} \|(\Lambda_{1,n}U_\varepsilon)(\Lambda_{2,n}(v_\varepsilon)\|_{\mathcal{L}^r([0,T_n'],\dot{B}^{s_p+\frac{2}{r}-1}_{p,p})} = 0.$$

Again by Proposition 3.5.3, we have for some $3 < q < \frac{3q}{q-3}$ and small enough $\delta > 0$,

$$\|(\Lambda_{1,n}U_\varepsilon)(\Lambda_{2,n}(v_\varepsilon)\|_{\mathcal{L}^r([0,T_n'],\dot{B}^{s_p+\frac{2}{r}-1}_{p,p})} \lesssim \|\Lambda_{1,n}U_\varepsilon\|_{\dot{B}^{s_q+\delta}_{q,q}}\|\Lambda_{2,n}w_\varepsilon\|_{\mathcal{L}^r([0,T_n'],\dot{B}^{s_p+\frac{1}{a}-\delta}_{p,p})}.$$

According to the fact that

$$\|\Lambda_{1,n}U_\varepsilon\|_{\dot{B}^{s_q+\delta}_{q,q}} \lesssim \lambda_{1,n}^{-\delta}\|U\|_{L^3},$$

and

$$\|\Lambda_{2,n}v_\varepsilon\|_{\mathcal{L}^r([0,T_n],\dot{B}^{s_p+\frac{2}{r}-\delta}_{p,p})} \lesssim \lambda_{2,n}^{\delta}\|v_\varepsilon\|_{\mathcal{L}^{2a}([0,T],\dot{B}^{s_p+\frac{2}{r}}_{p,p})},$$

we have

$$\|(\Lambda_{1,n}U_\varepsilon)(\Lambda_{2,n}(v_\varepsilon)\|_{\mathcal{L}^r([0,T_n'],\dot{B}^{s_p+\frac{2}{r}-1}_{p,p})} \lesssim \left(\frac{\lambda_{2,n}}{\lambda_{1,n}}\right)^\delta \to 0, \quad n \to \infty,$$

if $\frac{\lambda_{2,n}}{\lambda_{1,n}} \to 0$. Hence we prove (3.21).

□

3.4 Estimates on perturbation equations

Now we consider the following perturbation equation,

$$\begin{cases} \partial_t w - \Delta w + \frac{1}{2}Q(w,w) + Q(w,g) + Q(w,U) = f, \\ w|_{t=0} = w_0, \end{cases} \tag{3.22}$$

Let us state the following perturbation result.

Proposition 3.4.1. *Let $T \in \mathbb{R}_+ \cup \{+\infty\}$ and $3 < p < 5$. Suppose that $U \in L^3(\mathbb{R}^3)$, $g \in \mathcal{L}^p([0,T],\dot{B}^{s_p+\frac{2}{p}}_{p,p}(\mathbb{R}^3))$ and $f \in \mathcal{F}([0,T]) := \mathcal{L}^p([0,T],\dot{B}^{s_p+\frac{2}{p}-2}_{p,p}) + \mathcal{L}^{\frac{p}{2}}([0,T],\dot{B}^{s_p+\frac{4}{p}-2}_{p,p})$. Assume that $\|U\|_{L^3(\mathbb{R}^3)} < c_1 < c$, where $c > 0$ is a universal small constant Theorem 3.5.2. Then there exists a constant C independent of T and ε_0 such that the following is true. If*

$$\|w_0\|_{\dot{B}^{s_p}_{p,p}} + \|f\|_{\mathcal{F}([0,T])} \leq \varepsilon_0 \exp\big(-C\|g\|_{\mathcal{L}^p([0,T],\dot{B}^{s_p+\frac{2}{p}}_{p,p})}\big),$$

then $w \in \mathbb{L}^p_p(T)$ *and*

$$\|w\|_{\mathbb{L}^p_p(T)} \leq C(\|w_0\|_{\dot{B}^{s_p}_{p,p}} + \|f\|_{\mathcal{F}([0,T])})\exp(C\|g\|_{\mathcal{L}^p([0,T],\dot{B}^{s_p+\frac{2}{p}}_{p,p})}).$$

The proof the proposition follows the estimates of [13] (see in particular Proposition 4.1 and Theorem 3.1 of [13]). The main difference is the absence of an exterior force and a small time-independent drift term in [13], but those terms are added with no difficulty to the estimates.

Proof. By Proposition 4.1 of [13], for any $\alpha, \beta \in [0,T]$, we have the following estimates

$$\begin{aligned} \|w\|_{\mathbb{L}^{p;\infty}_p([\alpha,\beta])} \leq K\|w(\alpha)\|_{\dot{B}^{s_p}_{p,p}} + K\|f\|_{\mathcal{F}([\alpha,\beta])} + K\|w\|^2_{\mathcal{L}^p([\alpha,\beta],\dot{B}^{s_p+2/p}_{p,p})} \\ + K(\|U\|_{L^3} + \|g\|_{\mathcal{L}^p([\alpha,\beta],\dot{B}^{s_p+2/p}_{p,p})})\|w\|_{\mathcal{L}^p([\alpha,\beta],\dot{B}^{s_p+2/p}_{p,p})}. \end{aligned} \tag{3.23}$$

We recall that c is a small enough number such that

$$K\|U\|_{L^3} < \frac{1}{4}.$$

And we claim that there exist N real numbers $(T_i)_{1\leq i\leq N}$ such that $T_1 = 0$ and $T_N = T$, satisfying $[0,T] = \cup_{i=1}^{N-1}[T_i, T_{i+1}]$ and

$$\|g\|_{\mathcal{L}^p([T_i,T_{i+1}],\dot{B}^{s_p+\frac{2}{p}}_{p,p})} \leq \frac{1}{4K}, \quad \forall i \in \{i, 1\ldots, N-1\}$$

Suppose that

$$\|w_0\|_{\dot{B}^{s_p}_{p,p}} + \|f\|_{\mathcal{F}([0,T])} \leq \frac{1}{8KN(4K)^N}. \tag{3.24}$$

By time continuity we can define a maximal time $\tilde{T} \in \mathbb{R}_+ \cup \{\infty\}$ such that

$$\|w\|_{\mathcal{L}^p([0,\tilde{T}],\dot{B}^{s_p+\frac{2}{p}}_{p,p})} \leq \frac{1}{4K}.$$

If $\tilde{T} \geq T$ then the proposition is proved. Indeed, by (3.23), we have ,

$$\|w\|_{\mathcal{L}^p([T_i,T_{i+1}],\dot{B}^{s_p+\frac{2}{p}}_{p,p})} \leq K\|w(T_i)\|_{\dot{B}^{s_p}_{p,p}} + K\|f\|_{\mathcal{F}([T_i,T_{i+1}])} + \frac{3}{4}\|w\|_{\mathcal{L}^p([T_i,T_{i+1}],\dot{B}^{s_p+\frac{2}{p}}_{p,p})},$$

which deduces that

$$\|w\|_{\mathcal{L}^p([T_i,T_{i+1}],\dot{B}^{s_p+\frac{2}{p}}_{p,p})} \leq 4K(\|w(T_i)\|_{\dot{B}^{s_p}_{p,p}} + \|f\|_{\mathcal{F}([T_i,T_{i+1}])}).$$

Hence according to (3.23) ,

$$\begin{aligned} \|w\|_{\mathcal{L}^\infty([T_i,T_{i+1}],\dot{B}^{s_p}_{p,p})} &\leq K\|w(T_i)\|_{\dot{B}^{s_p}_{p,p}} + K\|f\|_{\mathcal{F}([T_i,T_{i+1}])} + \frac{3}{4}\|w\|_{\mathcal{L}^p([T_i,T_{i+1}],\dot{B}^{s_p+\frac{2}{p}}_{p,p})} \\ &\leq 4K(\|w(T_i)\|_{\dot{B}^{s_p}_{p,p}} + \|f\|_{\mathcal{F}([T_i,T_{i+1}])}). \end{aligned}$$

Therefore,

$$\|w(T_i)\|_{\dot{B}^{s_p}_{p,p}} \leq (4K)^{i-1}(\|w(0)\|_{\dot{B}^{s_p}_{p,p}} + \|f\|_{\mathcal{F}([0,T])}),$$

which implies that

$$\|w\|_{\mathcal{L}^p([T_i,T_{i+1}],\dot{B}^{s_p+\frac{2}{p}}_{p,p})} \leq (4K)^i(\|w(0)\|_{\dot{B}^{s_p}_{p,p}} + \|f\|_{\mathcal{F}([0,T])}).$$

Hence,

$$\|w\|_{\mathcal{L}^p([0,T],\dot{B}^{s_p+\frac{2}{p}}_{p,p})} \leq N(4K)^N(\|w(0)\|_{\dot{B}^{s_p}_{p,p}} + \|f\|_{\mathcal{F}([0,T])}).$$

Take $N \sim \|g\|_{\mathcal{L}^p([0,T],\dot{B}^{s_p+\frac{2}{p}}_{p,p})}$, we have

$$\|w\|_{\mathcal{L}^p([0,T],\dot{B}^{s_p+\frac{2}{p}}_{p,p})} \lesssim (\|w(0)\|_{\dot{B}^{s_p}_{p,p}} + \|f\|_{\mathcal{F}([0,T])})\exp(C\|g\|_{\mathcal{L}^p([0,T],\dot{B}^{s_p+\frac{2}{p}}_{p,p})}).$$

And by (3.23), we have

$$\|w\|_{\mathcal{L}^\infty([0,T],\dot{B}^{s_p}_{p,p})} \lesssim (\|w_0\|_{\dot{H}^{1/2}} + \|f\|_{\mathcal{L}^4([\alpha,\beta],\dot{H}^{-1})})\exp(C\|g\|_{\mathcal{L}^p([0,T],\dot{B}^{s_p+\frac{2}{p}}_{p,p})}).$$

Thus the proposition is proved in the case $\tilde{T} \geq T$.

Now we turn to the proof in the case of $\tilde{T} < T$. We define an integer $K \in \{1, \dots, N-1\}$ such that

$$T_k \leq \tilde{T} < T_{k+1}.$$

Then for any $i \leq k-1$, we have

$$\|w\|_{\mathcal{L}^p([T_i,T_{i+1}],\dot{B}^{s_p+\frac{2}{p}}_{p,p})} \leq (4K)^i(\|w(0)\|_{\dot{B}^{s_p}_{p,p}} + \|f\|_{\mathcal{F}([0,T])}),$$

and

$$\|w(T_i)\|_{\dot{B}^{s_p}_{p,p}} \leq (4K)^{i-1}(\|w(0)\|_{\dot{B}^{s_p}_{p,p}} + \|f\|_{\mathcal{F}([0,T])}).$$

The same arguments as above also apply on the interval $[T_k, T]$ and yield,

$$\|w\|_{\mathcal{L}^4([T_k,T],\dot{H}^1)} \leq (4K)^N\|w_0\|_{\dot{H}^{1/2}} + CNK^2\|f\|_{\mathcal{L}^4([0,T],\dot{H}^{-1})},$$

and

$$\|w\|_{\mathcal{L}^\infty([T_k,T],\dot{H}^{1/2})} \leq (4K)^N\|w_0\|_{\dot{H}^{1/2}} + CNK^2\|f\|_{\mathcal{L}^4([0,T],\dot{H}^{-1})}.$$

Therefore we have

$$\|w\|_{\mathcal{L}^4([0,T],\dot{H}^1)} \leq N(4K)^N(\|w_0\|_{\dot{H}^{1/2}} + \|f\|_{\mathcal{L}^4([\alpha,\beta],\dot{H}^{-1})}) < \frac{1}{4k},$$

which contradicts to the maximality of $\tilde{T}$.

□

3.5 Appendix

3.5.1 Some results on the steady-state Navier-Stokes equations

In this part, we recall some existence results on the steady state Navier-Stokes equations, and the Navier-Stokes equations equipped with the same time-independent external force. The steady state Navier-Stokes system is defined as follows,

$$(SNS)\begin{cases} -\Delta U + U\cdot\nabla = f - \nabla\Pi, \\ \nabla\cdot U = 0, \end{cases}$$

where $f(x)$ is the external force defined on $\mathbb{R}^3$. Since we only care about the case of $U \in L^3$, we state the following result for $\Delta^{-1}f \in L^3$ without proof, which is a consequence of Theorem 2.2 in [2].

Proposition 3.5.1. *There exists an absolute constant $\delta > 0$ with the following property. If $f \in \mathcal{S}'$ satisfies $\Delta^{-1}f \in L^3(\mathbb{R}^3)$ and*

$$\|\Delta^{-1}f\|_{L^3(\mathbb{R}^3)} < \delta,$$

then there exists a unique solution to (SNS) such that

$$\|U\|_{L^3} \le 2\|\Delta^{-1}f\|_{L^3} < 2\delta.$$

Now we state a well-posedness result of (NSf), which is a particular case of results of Theorem 2.2.7, 2.2.8 and 2.2.9.

Theorem 3.5.2. *Suppose that f is a time-independent external force such that $\|\Delta^{-1}f\|_{L^3} < c$, where $c < \delta$ is a universal small constant. Let $U_f \in L^3(\mathbb{R}^3)$ be the unique solution to (SNS) with $\|U_f\|_{L^3} < 2\|\Delta^{-1}f\|_{L^3}$ (the existence of U_f is provided by Proposition 3.5.1) .Then we have*

1. *For any initial data $u_0 \in L^3(\mathbb{R}^3)$, there exists a unique maximal time $T^*(u_0,f) > 0$ and a unique solution u_f to (NSf) with initial data u_0 such that for any $T < T^*(u_0,f)$,*

$$u_f \in C([0,T], L^3(\mathbb{R}^3)).$$

Moreover there exists a constant $\delta_2(f)$ such that if $\|u_0 - U\|_{L^3} < \delta_2$, then $u_0 \in C_0(\mathbb{R}_+, L^3(\mathbb{R}^3))$. The solution u_f satisfies that for $3 < p < 5$

$$\lim_{T\to T^*(u_0,f)} \|u_f - U_f\|_{\mathbb{L}_p^{p:\infty}(T)} = \infty. \tag{3.25}$$

2. *Let $u_f \in C(\mathbb{R}_+, L^3(\mathbb{R}^3))$ with initial data $u_0 \in L^3(\mathbb{R}^3)$. Then $u_f \in L^\infty(\mathbb{R}_+, L^3(\mathbb{R}^3))$ and $u_f - U_f \in \mathbb{L}_p^{r_0:\infty}(\infty)$ for some $r_0 > 2$ and $p > 3$, and*

$$\lim_{t\to\infty} \|u_f - U\|_{\dot{B}^{s_p}_{p,p}} = 0.$$

3.5.2 Product laws and heat estimates

We first recall the following standard product laws in Besov space, which use the theory of para-products (for details, see [7, 13]).

Proposition 3.5.3. *1. Let $p > 3$, $q > 3$ and $r > 2$. Moreover assume that $s_q + s_p + \frac{2}{r} > 0$. We have, for any $|\varepsilon| < 1$ such that $1 - \frac{2}{r} + \varepsilon > 0$,*

$$\|vw\|_{\mathcal{L}^r([0,T],\dot{B}^{s_p+\frac{2}{r}}_{p,p})} \leq C(\varepsilon)\|v\|_{\mathcal{L}^\infty([0,T],\dot{B}^{s_q+\varepsilon}_{q,q})}\|w\|_{\mathcal{L}^r([0,T],\dot{B}^{s_p+\frac{2}{r}-\varepsilon}_{p,p})}.$$

2. let $p > 3$ and $2 < r < \frac{2p}{p-3}$. Then for any $\varepsilon \in \mathbb{R}$ such that $1 - \frac{2}{r} - |\varepsilon| > 0$, we have

$$\|vw\|_{\mathcal{L}^{\frac{2}{r}}([0,T],\dot{B}^{s_p+\frac{4}{r}-1}_{p,p})} \leq C(\varepsilon)\|v\|_{\mathcal{L}^r([0,T],\dot{B}^{s_p+\frac{2}{r}+\varepsilon}_{p,p})}\|w\|_{\mathcal{L}^r([0,T],\dot{B}^{s_p+\frac{2}{r}-\varepsilon}_{p,p})}.$$

Now let us recall the following standard heat estimate. For any $p \in [1, \infty]$, there exists some $c_0, c > 0$ such that for any $f \in \mathcal{S}'$ and $j \in \mathbb{Z}$,

$$\|\Delta_j(e^{t\Delta}f)\|_{L^p} \leq c_0 e^{-ct2^{2j}}\|\Delta_j f\|_{L^p}.$$

Hence for $0 < t \leq \infty$, recalling

$$B(u,v) := \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u(s)\otimes v(s))ds,$$

Young's inequality for convolutions implies that for any $\tilde{r} \in [r, \infty]$

$$\|B(u,v)\|_{\mathcal{L}^{\tilde{r}}([0,T],\dot{B}^{s+2+2(\frac{1}{\tilde{r}}-\frac{1}{r})}_{p,p})} \lesssim \|u\otimes v\|_{\mathcal{L}^r([0,T],\dot{B}^{s+1}_{p,p})}. \tag{3.26}$$

And we recall that B is a bounded operator from $L^\infty([0,T], L^{3,\infty}) \times L^\infty([0,T], L^{3,\infty})$ to $L^\infty([0,T], L^{3,\infty})$ for any $T \in \mathbb{R}_+ \cup \{+\infty\}$ (see [2])

Lemma 3.5.4. *Suppose that $u, v \in L^\infty([0,T], L^{3,\infty})$ for some $T \in \mathbb{R}_+ \cup \{+\infty\}$. Then*

$$\|B(u,v)\|_{L^\infty([0,T],L^{3,\infty})} \lesssim \|u\|_{L^\infty([0,T],L^{3,\infty})}\|u\|_{L^\infty([0,T],L^{3,\infty})}.$$

Bibliography

[1] D. Albritton, Blow-up criteria for the Navier-Stokes equations in non-endpoint critical Besov spaces. arXiv: 1612.04439

[2] C. Bjorland, L. Brandolese, D. Iftimie & M. E. Schonbek, L^p-solutions of the steady-state Navier-Stokes equations with rough external forces. *Communications in Partial Differential Equations*, 2010, 36(2), 216-246

[3] J. Bourgain & N. Pavlović, Ill-posedness of the Navier-Stokes equations in a critical space in 3D *J. Funct. Anal.* 255 (2008), no. 9, 2233–2247.

[4] M. Cannone, A generalization of a theorem by Kato on Navier-Stokes equations. *Rev. Mat. Iberoamericana* 13 (1997), no. 3, 515—541.

[5] M. Cannone, Y. Meyer & F. Planchon, Solutions auto-similaires des équations de Navier-Stokes. *Séminaire sur les Équations aux Dérivées Partielles, 1993—1994,* No. VIII, 12 pp., École Polytech., Palaiseau, 1994.

[6] M. Cannone & F. Planchon, On the non-stationary Navier-Stokes equations with an external force. *Adv. Differential Equations* 4 (1999), no. 5, 697–730.

[7] J.-Y Chemin, Théorèmes d'unicité pour le système de Navier-Stokes tridimensionnel. (French) [Uniqueness theorems for the three-dimensional Navier-Stokes system]. *J. Anal. Math.*77 (1999), 27–50.

[8] J.-Y Chemin & N. Lerner, Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes. *J. Differential Equations* 121 (1995), no. 2, 314–328.

[9] L. Escauriaza, G. Seregin & V. Šverák, $L_{3,\infty}$–solutions of Navier-Stokes equations and backward uniqueness. *Uspekhi Mat. Nauk.* 58 (2003), no. 2(350), 3–44.

[10] L. Escauriaza, G. Seregin & V. Šverák, Backward uniqueness for the heat operator in half-space. *Algebra i Analiz.* 15 (2003), no. 1, 201–214.

[11] H. Fujita & T. Kato, On the Navier-Stokes initial value problem. I. *Arch. Rational Mech. Anal.* 16 1964 269—315.

[12] I. Gallagher, Profile decomposition for solutions of the Navier-Stokes equations. *Bull. Soc. Math. France* 129 (2001), no. 2, 285–316.

[13] I. Gallagher, D. Iftimie & F. Planchon, Asymptotics and stability for global solutions to the Navier-Stokes equations. *Ann. Inst. Fourier (Grenoble)* 53 (2003), no. 5, 1387—1424.

[14] I. Gallagher, G. S. Koch & F. Planchon, Blow-up of critical Besov norms at a potential Navier-Stokes singularity. *Comm. Math. Phys.* 343 (2016), no. 1, 39–82.

[15] I. Gallagher, G. S. Koch & F. Planchon, A profile decomposition approach to the $L^\infty_t(L^3_x)$ Navier-Stokes regularity criterion. *Math. Ann.*355 (2013), no. 4, 1527–1559.

[16] P. Gérard, Description du défaut de compacité de l'injection de Sobolev. *ESAIM Control Optim. Calc. Var.*3 (1998), 213–233.

[17] T. Kato, Strong L^p solutions of the Navier-Stokes equation in $\mathbb{R}^m$, with applications to weak solutions. *Mathematische Zeitschrift*, 187, 1984, pages 471–480.

[18] C. E. Kenig & G. S. Koch, An alternative approach to regularity for the Navier-Stokes equations in critical spaces. *Ann. Inst. H. Poincaré Anal. Non Linéaire*28 (2011), no. 2, 159–187.

[19] G. S. Koch, Profile decompositions for critical Lebesgue and Besov space embeddings. *Indiana Univ. Math. J.* 59 (2010), no. 5, 1801–1830.

[20] H. Koch & D. Tataru, Well-posedness for the Navier-Stokes equations. *Adv. Math.* 157 (2001), no. 1, 22–35.

[21] J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.* 63 (1934), no. 1, 193–248.

[22] J. Nečas, M. Růžička & V. Šverák, On Leray's self-similar solutions of the Navier-Stokes equations. *Acta Math.* 176 (1996), no. 2, 283–294.

[23] F. Planchon, Asymptotic behavior of global solutions to the Navier-Stokes equations in $\mathbb{R}^3$. *Rev. Mat. Iberoamericana* 14 (1998), no. 1, 71–93.

[24] G. Seregin, A certain necessary condition of potential blow up for Navier-Stokes equations. *Comm. Math. Phys.* 312 (2012), no. 3, 833–845.

Chapter 4

Gevrey class smoothing effect for the Prandtl equation

4.1 Introduction

In this work, we study the regularity of solutions to the Prandtl equation which is the foundation of the boundary layer theory introduced by Prandtl in 1904, [24]. The results in this chapter is a collection of a published paper (*SIAM J. Math. Anal.* 48 (2016), pages 1672–1726).

The inviscid limit of an incompressible viscous flow with the non-slip boundary condition is still a challenging problem of mathematical analysis due to the appearance of a boundary layer, where the tangential velocity adjusts rapidly from nonzero at the area far away from the boundary to zero on the boundary. Prandtl equation describes the behavior of the flow near the boundary in the case of small viscosity limit, and it reads

$$\begin{cases} u_t + uu_x + vu_y + p_x = u_{yy}, \quad t > 0, \quad x \in \mathbb{R}, \quad y > 0, \\ u_x + v_y = 0, \\ u|_{y=0} = v|_{y=0} = 0, \ \lim\limits_{y \to +\infty} u = U(t,x), \\ u|_{t=0} = u_0(x,y)\,, \end{cases}$$

where $u(t,x,y)$ and $v(t,x,y)$ represent the tangential and normal velocities of the boundary layer, with y being the scaled normal variable to the boundary, while $U(t,x)$ and $p(t,x)$ are the valuesof the tangential velocity as $y \to \infty$and pressure of the outflow satisfying the Bernoulli law

$$\partial_t U + U\partial_x U + \partial_x q = 0.$$

Because of the degeneracy in tangential variable, the well-posedness theories and the justification of the Prandtl's boundary layer theory remain as the challenging problems in the mathematical theory of fluid mechanics. Up to now, there are only a few rigorous mathematical results (see [4, 13, 14, 15, 22] and referencesin). Under a monotonic assumption on the tangential velocity of the outflow, Oleinik was the first to obtain the local existence of classical solutions for the initial-boundary value problems, and this result together with some of her works with collaborators were well presented in the monograph [23]. In addition to Oleinik's monotonicity assumption on the velocity field, by imposing a so called favorable condition on the pressure, Xin-Zhang [26] obtained the existence of global weak solutions to the Prandtl equation. All these well-posedness results were based on the Crocco transformation to overcome the main difficulty caused by degeneracy and mixed type of the equation. Just recently the well-posedness in the Sobolev space was explored by virtue of energy method instead of the Crocco transformation; see Alexandre et. all [1] and Masmoudi-Wong [21]. There is very few work concerned with the Prandtl equation without the monotonicity assumption; we refer [2, 3, 20, 9, 25, 30] for the works in the analytic frame, and [12, 17] for the recent works concerned with the existence in Gevrey class. Recall Gevrey class, denoted by $G^s, s \geq 1$, is an intermediate space between analytic functions and C^∞ space. For a given domain, the Gevrey space $G^s(\Omega)$ is consist of such functions that $f \in C^\infty(\Omega)$ and that

$$\left\|\partial^\alpha f\right\|_{L^2(\Omega)} \leq L^{|\alpha|+1}(\alpha!)^s$$

for some constant L independent of α. The significant difference between Gevery ($s > 1$) and analytic ($s = 1$) classe is that there exist nontrivial Gevrey functions admitting compact support.

We mention that due to the degeneracy in x, it is natural to expect Gevrey regularity rather than analyticity for a subelliptic equation. We refer [5, 6, 7, 8] for the link between subellipticity and Gevrey regualiy. In this paper we first study the intrinsic subelliptic structure due to the monotonicity condition, and then deduce, basing on the subelliptic estimate, the Gevrey smoothing effect; that is, given a monotonic initial data belonging to some Sobolev space, the solution will lie in some Gevrey class at positive time, just like heat equation. It is different from the Gevrey propagation property obtained in the aforementioned works, where the initial data is supposed to be of some Gevrey class, for instance $G^{7/4}$ in [12], and the well-posedness is obtained in the same Gevrey space.

Now we state our main result. Without loss of generality, we only consider here the case of an uniform outflow $U = 1$, and the conclusion will still hold for Gevrey class outflow U. We mention that the Gevrey regularity for outflow U is well developed (see [18] for instance). For the uniform outflow, we get the constant pressure p due to the Bernoulli law. Then the Prandtl equation can be rewritten as

$$\begin{cases} u_t + uu_x + vu_y - u_{yy} = 0, \quad (t,x,y) \in]0,T[\times\mathbb{R}^2_+, \\ u_x + v_y = 0, \\ u|_{y=0} = v|_{y=0} = 0, \ \lim\limits_{y\to+\infty} u = 1, \\ u|_{t=0} = u_0(x,y). \end{cases} \tag{4.1.1}$$

The main result concerned with the Gevrey class regularity can be stated as follows.

Theorem 4.1.1. *Let $u(t,x,y)$ be a classical local in time solution to Prandtl equation (4.1.1) on $[0,T]$ with the properties subsequently listed below:*

(i) There exist two constants $C_* > 1, \sigma > 1/2$ such that for any $(t,x,y) \in [0,T]\times\mathbb{R}^2_+$,

$$\begin{aligned} &C_*^{-1}\left\langle y\right\rangle^{-\sigma} \leq \partial_y u(t,x,y) \leq C_*\left\langle y\right\rangle^{-\sigma}, \\ &\left|\partial_y^2 u(t,x,y)\right| + \left|\partial_y^3 u(t,x,y)\right| \leq C_*\left\langle y\right\rangle^{-\sigma-1}, \end{aligned} \tag{4.1.2}$$

where $\langle y\rangle = (1+|y|^2)^{1/2}$.

(ii) There exists $c > 0, C_0 > 0$ and integer $N_0 \geq 7$ such that

$$\|e^{2cy}\partial_x u\|_{L^\infty\left([0,T];\ H^{N_0}(\mathbb{R}^2_+)\right)} + \|e^{2cy}\partial_x\partial_y u\|_{L^2\left([0,T];\ H^{N_0}(\mathbb{R}^2_+)\right)} \leq C_0\,. \tag{4.1.3}$$

Then for any $0 < T_1 < T$, there exists a constant L, such that for any $0 < t \le T_1$,

$$\forall\, m > 1 + N_0, \quad \left\| e^{\tilde{c}y} \partial_x^m u(t) \right\|_{L^2(\mathbb{R}_+^2)} \le t^{-3(m-N_0-1)}\, L^m \,(m!)^{3(1+\sigma)}\,, \tag{4.1.4}$$

where $0 < \tilde{c} < c$. The constants L depends only on $C_0, T_1, C_*, c, \tilde{c}$ and σ. Therefore, the solution u belongs to the Gevery class of index $3(1+\sigma)$ with respect to $x \in \mathbb{R}$ for any $0 < t \le T_1$.

Remark 4.1.2.

1). The solution described in the above theorem exists, for instance, suppose that the initial data u_0 can be written as

$$u_0(x, y) = u_0^s(y) + \tilde{u}_0(x, y),$$

where u_0^s is a function of y but independent of x such that $C^{-1} \langle y \rangle^{-\sigma} \le \partial_y u_0^s(y) \le C \langle y \rangle^{-\sigma}$ for some constant $C \ge 1$, and $\tilde{u}_0$ is a small perturbation such that its weighted Sobolev norm $\left\| e^{2cy} \tilde{u}_0 \right\|_{H^{2N_0+7}(\mathbb{R}_+^2)}$ is suitably small. Then using the arguments in [1], we can obtain the desired solution with the properties listed in Theorem 4.1.1 fulfilled. Precisely, the solution $u(t, x, y)$ is a perturbation of a shear flow $u^s(t, y)$ such that property (i) in the above theorem holds for u, and moreover $e^{2cy}\,(u - u^s) \in L^\infty\left([0, T];\ H^{N_0+1}(\mathbb{R}_+^2)\right)$. Moreover following the analysis in [21] with some modifications, we can also obtain more general solutions with exponential decay rather than perturbative solutions around monotonic shear flows.

2). The well-posedness problem of Prandtl's equation depends crucially on the choice of the underlying function spaces, especially on the regularity in the tangential variable x. If the initial datum is analytic in x, then the local in time solution exists(c.f. [20, 25, 30]), but the Cauchy problem is ill-posedness in Sobolev space for linear and non linear Prandtl equation (cf. see [10, 11]). Indeed, the main mathematical difficulty is the lack of control on the x derivatives. For example, v in (4.1.1) could be written as $-\int_0^y u_x(y')dy'$ by the divergence-free condition, and here we lose one derivatives in x-regularity. The degeneracy can't be balanced directly by any horizontal diffusion term, so that the standard energy estimates do not apply to establish the existence of local solution. But the results in our main Theorem 4.1.1 shows that *the loss of derivative in tangential variable x can be partially compensated via the monotonicity condition*.

3). Under the hypothesis (4.1.2), the equation (4.1.1) is a non linear hypoelliptical equation of Hörmander type with a gain of regularity of order $\frac{1}{3}$ in x variable (see Proposition 4.2.4), so that any C^2 solution is locally C^∞, see [27, 28, 29]; for the corresponding linear operator, [8] obtained the regularity in the local Gevrey space G^3. However, in this paper we study the equation (4.1.1) as a boundary layer equation, so that the local property of solution is not of interest to the physics application, and our goal is then to study the global estimates in Gevrey class. In view of (4.1.2) we see u_y decays polynomially at infinite, so we only have a weighted subelliptic estimate (see Proposition 4.2.4). This explains why the Gevrey index, which is $3(1+\sigma)$, depends also on the decay index σ in (4.1.2).

4). Finally, the estimate (4.1.4) gives an explicit Gevrey norm of solutions for the Cauchy problem with respect to $t > 0$ when the initial datum is only in some finite order Sobolev space. Since the Gevrey class is an intermediate space between analytic space and Sobolev space, the qualitative study of solutions in Gevery class can help us to understand the Prandtl boundary layer theory which has been justified in analytic frame.

The approach

We end up the introduction with explaining the main idea used in the proof. It's well-known that the main difficulty for Prandtl equation is the degeneracy in x variable, due to the presence of v:

$$v = -\int_0^y (\partial_x u)\, d\tilde{y}.$$

To overcome the degeneracy, we use the cancellation idea, introduced by Masmoudi-Wong [21], to perform the estimates on the new function and moreover on the original velocity function u. Precisely, observe

$$u_t + uu_x + vu_y - u_{yy} = 0,$$

and, with $\omega = \partial_y u$,

$$\omega_t + u\omega_x + v\omega_y - \omega_{yy} = 0.$$

In order to eliminate the v term on the left sides of the above two equations, we use the monotonicity condition $\partial_y u = \omega > 0$ and thus multiply the second equation by $-\frac{\partial_y \omega}{\omega}$, and then add the resulting equation to the first one; this gives, denoting $f = \omega - \frac{\partial_y \omega}{\omega} u$,

$$f_t + u\partial_x f - \partial_{yy} f = \text{terms of lower order}.$$

Our main observation for the new equation is the intrinsic subelliptic structure due to the monotonicity condition. Indeed, denoting $X_0 = \partial_t + u\partial_x$ and $X_1 = \partial_y$, we can rewrite the above equation as of Hörmander's type:

$$\Big(X_0 + X_1^* X_1\Big) f = \text{terms of lower order}.$$

and moreover,direct computation shows

$$\big[X_1,\ X_0\big] = (\partial_y u)\, \partial_x. \tag{4.1.5}$$

Thus Hörmander's bracket condition will be fulfilled, provided by $\partial_y u > 0$, and consequently the following subelliptic estimate holds :

$$\forall\, w \in C_0^\infty(K), \quad \big\|\Lambda^{2/3} w\big\|_{L^2} \lesssim \Big\|\Big(X_0 + X_1^* X_1\Big) w\Big\|_{L^2} + \big\|w\big\|_{L^2},$$

with K a compact subset of $\mathbb{R}^3_{t,x,y}$ and $\Lambda^d = \Lambda^d_x$ is the Fourier multiplier of symbol $\langle \xi \rangle^d$ with respect to $x \in \mathbb{R}$. We refer to [16] for detail on general subelliptic operator. We remark the above subelliptic estimate is local, and as far as Prandtl equation is concerned, the situation is more complicated: on one side only global estimate is interesting, that is, we have to consider $y \geq 0$ rather than in a compacted subset of $\mathbb{R}_+$, on the other there are boundary and initial problems. When y varies in the half line $y \geq 0$ the Hörmander's bracket condition (4.1.5) is no longer true, since $\partial_y u \to 0$ as $y \to +\infty$. To over this difficulty we perform, following the arguments used in the classical (local) subelliptic estimate with some modification, a weighted subelliptic estimate of the following form: for any $w \in L^2\left([0,T], H^2(\mathbb{R}^2_+)\right)$,

$$\left\| \, |\partial_y u|^{1/2} \Lambda^{1/3} w \right\|_{L^2} \lesssim \left\| \Big(X_0 + X_1^* X_1 \Big) w \right\|_{L^2} + \|w\|_{L^2} + \text{terms from boundary conditions},$$

which indicates the loss-gain phenomenon, that is in order to gain $\Lambda^{1/3}$ regularity we have to loss $|\partial_y u|^{1/2}$ weight. Similarly as for as higher derivatives $\partial_x^m u$ are concerned, we can perform a equation for

$$f_m = \partial_x^m \omega - \frac{\partial_y \omega}{\omega} \partial_x^m u = \omega \partial_y \left(\frac{\partial_x^m u}{\omega} \right), \quad m \geq 1,$$

to cancel the bad term involving $\partial_x^m v$, and moreover the above weighted sublliptic estimate still holds for this equation. Moreover by Hardy inequality, in order to obtain the control of $\partial_x^m \omega$ and $\partial_x^m u$, it is sufficient to perform estimates on f_m (see Section 4.4 for detail).

Our choice of the weight function W_m^ℓ (see (4.2.2) below) is motivated by the loss-gain estimate. Recall

$$W_m^\ell = e^{2cy} \left(1 + \frac{2cy}{(3m+\ell)\sigma} \right)^{-\frac{(3m+\ell)\sigma}{2}} (1+cy)^{-1} \Lambda^{\frac{\ell}{3}}, \quad 0 \leq \ell \leq 3, \ \ m \in \mathbb{N}, \ y > 0$$

where the essential part is the factor

$$\left(1 + \frac{2cy}{(3m+\ell)\sigma} \right)^{-\frac{(3m+\ell)\sigma}{2}} \Lambda^{\frac{\ell}{3}}.$$

Thus as ℓ is increased by one, we gain $\Lambda^{\frac{1}{3}}$ regularity and meanwhile loss the weight $\langle y \rangle^{-\frac{\sigma}{2}} \sim |\partial_y u|^{\frac{1}{2}}$. Moreover

$$\left(1 + \frac{2cy}{(3m+\ell)\sigma} \right)^{-\frac{(3m+\ell)\sigma}{2}}$$

is bounded from below by e^{-cy} and goes to 0 as $y \to +\infty$, so we add the factor e^{2cy} in the expression of W_m^ℓ to guarantee the strictly positive lower bound. Another factor $(1+cy)^{-1}$ is introduced for the purpose that

$$\partial_y \left(e^{2cy} \left(1 + \frac{2cy}{(3m+\ell)\sigma} \right)^{-\frac{(3m+\ell)\sigma}{2}} (1+cy)^{-1} \right) \Bigg|_{y=0} = 0.$$

Observe the Prandtl equation is initial-boundary problem, and we will study the smoothing effect. Thus it is natural to introduce a cut-off function in time:

$$\phi_m^\ell = \phi^{3(m-(N_0+1))+\ell} = (t(T-t))^{3(m-(N_0+1))+\ell}, \quad m \geq N_0+1, \ 0 \leq \ell \leq 3,$$

which ensures that $\phi_m^\ell f_m$ vanishes at the endpoints.

Now we perform the equation for $G_m^\ell = \phi_m^\ell W_m^\ell f_m$:

$$\begin{cases} \left(\partial_t + u\partial + v\partial_y - \partial_y^2 \right) G_m^\ell = (\partial_t \phi_m^\ell) W_m^\ell f_m + \cdots, \\ \partial_y G_m^\ell \big|_{y=0} = 0, \\ G_m^\ell \big|_{t=0} = 0. \end{cases}$$

and have the energy estimate:

$$\|G_m^\ell\|_{L^\infty([0,T];\, L^2(\mathbb{R}^2_+))} + \|\partial_y G_m^\ell\|_{L^2([0,T]\times\mathbb{R}^2_+)} \lesssim m^{1/2}\|\phi^{-1/2}G_m^\ell\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \cdots .$$

and we have to control the first term on the right hand side, which arises from the commutator between ∂_t and the cut-off function ϕ_m^ℓ and is a crucial part to study the smoothing effect. Here we will make use of the weighted subelliptic estimate (see Section 4.3) to treat this term. More details can be found in Section 4.2.

The paper is organized as follows. In Section 4.2 we prove Theorem 4.1.1, and state some preliminaries lemmas used in the proof. The other sections are occupied by the proof of the preliminaries lemmas. Precisely, we prove in Section 4.3 a subelliptic estimate for the linearized Prandtl operator. Section 4.4 and Section 4.5 are devoted to presenting a crucial estimate for an auxilliary function and non linear terms. The last section is an appendix, where the equation fulfilled by the auxilliary function is deduced.

4.2 Proof for the main Theorem

We will prove in this section the Gevery estimate (4.1.4) by induction on m. As in [21], we consider the following auxilliary function

$$f_m = \partial_x^m \omega - \frac{\partial_y \omega}{\omega}\partial_x^m u = \omega \partial_y \left(\frac{\partial_x^m u}{\omega}\right), \quad m \geq 1, \tag{4.2.1}$$

where $\omega = \partial_y u > 0$ and u is a solution of equation (4.1.1) which satisfy the hypotheis (4.1.2). We also introduce the following inductive weight ,

$$W_m^\ell = e^{2cy}\left(1 + \frac{2cy}{(3m+\ell)\sigma}\right)^{-\frac{(3m+\ell)\sigma}{2}} (1+cy)^{-1}\Lambda^{\frac{\ell}{3}}, \quad 0 \leq \ell \leq 3, \;\; m \in \mathbb{N}, \;\; y > 0, \tag{4.2.2}$$

where $\Lambda^d = \Lambda_x^d$ is the Fourier multiplier of symbol $\langle \xi \rangle^d$ with respect to $x \in \mathbb{R}$. Notting

$$W_m^0 \geq e^{cy}(1+cy)^{-1} \geq c_0 e^{\tilde{c}y}, \tag{4.2.3}$$

for $0 < \tilde{c} < c$.

Since

$$\left|\frac{\partial_y \omega}{\omega}\right| \leq C_*^2 \langle y \rangle^{-1},$$

we have that , if u is smooth,

$$\|W_m^0 f_m\|_{L^2(\mathbb{R}^2_+)} \leq \|W_m^0 \partial_x^m \omega\|_{L^2(\mathbb{R}^2_+)} + C_*^2 \|W_m^0 \langle y \rangle^{-1} \partial_x^m u\|_{L^2(\mathbb{R}^2_+)}.$$

On the other hand, we have the following Poincaré type inequality.

Lemma 4.2.1. *There exist $C_1, \tilde{C}_1 > 0$ independents of $m \geq 1, 0 \leq \ell \leq 3$, such that*

$$\big\| \langle y \rangle^{-1} W_m^\ell \partial^m u\big\|_{L^2(\mathbb{R}^2_+)} + \big\| \langle y \rangle^{-1} W_m^\ell \partial_x^m \omega\big\|_{L^2(\mathbb{R}^2_+)} \leq C_1 \big\|W_m^\ell f_m\big\|_{L^2(\mathbb{R}^2_+)}. \tag{4.2.4}$$

As a result,

$$\big\|\Lambda^{-1} W_m^0 f_{m+1}\big\|_{L^2(\mathbb{R}^2_+)} \leq \tilde{C}_1 \big\|W_m^0 f_m\big\|_{L^2(\mathbb{R}^2_+)}, \tag{4.2.5}$$

and

$$\|\Lambda^{-1}\partial_y W_m^0 f_{m+1}\|_{L^2(\mathbb{R}^2_+)} \le \tilde{C}_1 \left(\|\partial_y W_m^0 f_m\|_{L^2(\mathbb{R}^2_+)} + \|W_m^0 f_m\|_{L^2(\mathbb{R}^2_+)}\right).$$

We will prove the above lemma in the section 4.4 as Lemma 4.4.2.

Since the initial datum of the equation (4.1.1) is only in Sobolev space H^{N_0+1}, we have to introduce the following cut-off function, with respect to $0 \le t \le T \le 1$, to study the Gevrey smoothing effect by using the hyopellicity,

$$\phi_m^\ell = \phi^{3(m-(N_0+1))+\ell} = (t(T-t))^{3(m-(N_0+1))+\ell}, \quad m \ge N_0+1,\ 0 \le \ell \le 3. \tag{4.2.6}$$

We will prove by induction an energy estimate for the function $\phi_m^0 W_m^0 f_m$. For this purpose we need the following lemma concerned with the link between $\phi_m^3 W_m^3 f_m$ and $\phi_{m+1}^0 W_{m+1}^0 f_{m+1}$, whose proof is postponed to the section 4.4 as Lemma 4.4.3 and Lemma 4.4.4.

Lemma 4.2.2. *There exists a constant C_2, depending only on the numbers σ, c and the constant C_* in Theorem 4.1.1, in particular, independents on m , such that for any $m \ge N_0+1$,*

$$\|\phi_{m+1}^0 W_{m+1}^0 f_{m+1}\|_{L^\infty([0,T];\, L^2(\mathbb{R}^2_+))} + \sum_{j=1}^2 \|\partial_y^j \Lambda^{-\frac{2(j-1)}{3}} \phi_{m+1}^0 W_{m+1}^0 f_{m+1}\|_{L^2([0,T]\times\mathbb{R}^2_+)}$$
$$\le C_2 \|\phi_m^3 W_m^3 f_m\|_{L^\infty([0,T];\, L^2(\mathbb{R}^2_+))} + C_2 \sum_{j=1}^2 \|\partial_y^j \Lambda^{-\frac{2(j-1)}{3}} \phi_m^3 W_m^3 f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)},$$

and

$$\begin{aligned}&\|\partial_y^3 \Lambda^{-1} \phi_{m+1}^0 W_{m+1}^0 f_{m+1}\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\le C_2 \|\phi_m^3 W_m^3 f_m\|_{L^\infty([0,T];\, L^2(\mathbb{R}^2_+))} + C_2 \sum_{j=1}^2 \|\partial_y^j \Lambda^{-\frac{2(j-1)}{3}} \phi_m^3 W_m^3 f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\quad + C_2 \|\partial_y^3 \Lambda^{-1} \phi_m^3 W_m^3 f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)},\end{aligned}$$

and

$$\|\langle y\rangle^{-\sigma/2} \partial_y \Lambda^{1/3} \Lambda_\delta^{-2} W_m^{\ell-1} f_m\|_{L^2(\mathbb{R}^2_+)} \le C_2 \|\partial_y \Lambda_\delta^{-2} W_m^\ell f_m\|_{L^2(\mathbb{R}^2_+)} + C_2 \|\Lambda_\delta^{-2} W_m^\ell f_m\|_{L^2(\mathbb{R}^2_+)}.$$

Now we prove Theorem 4.1.1 by induction on the estimate of $\phi_m^0 W_m^0 f_m$. The procedure of induction is as follows.

Initial hypothesis of the induction. From the hypothesis (4.1.2) and (4.1.3) of Theorem 4.1.1, we have firstly, in view of (4.2.1),

$$0 \le m \le N_0+1, \quad \|e^{2cy} f_m\|_{L^\infty\left([0,T];\, L^2(\mathbb{R}^2_+)\right)} + \sum_{i=1}^3 \|e^{2cy} \partial_y^i f_m\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} < C_0. \tag{4.2.7}$$

Hypothesis of the induction. Suppose that there exists $A > C_0 + 1$ such that, for some $m \ge N_0+1$ and for any $N_0+1 \le k \le m$, we have

$$\partial_y^3 \Lambda^{-1} \phi_k^0 W_k^0 f_k \in L^2([0,T]\times\mathbb{R}^2_+), \tag{4.2.8}$$

$$\begin{aligned}&\|\phi_k^0 W_k^0 f_k\|_{L^\infty([0,T];\, L^2(\mathbb{R}^2_+))} + \sum_{j=1}^{2} \|\partial_y^j \Lambda^{-\frac{2(j-1)}{3}} \phi_k^0 W_k^0 f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\ &\quad\le A^{k-5}\left((k-5)!\right)^{3(1+\sigma)}.\end{aligned} \tag{4.2.9}$$

Claim I_{m+1}: we claim that (4.2.8) and (4.2.9) are also true for $m+1$. As a result, (4.2.8) and (4.2.9) hold for all $k \ge N_0 + 1$ by induction.

Completeness of the proof for Theorem 4.1.1. .

Before proving the above **Claim I_{m+1}**, we remark that Theorem 4.1.1 is just its immediate consequence. Indeed, induction processus imply that for any $m > 1+N_0$, we have for any $0 < t < T$,

$$\left\|\phi_m^0 W_m^0 f_m(t)\right\|_{L^2(\mathbb{R}^2_+)} \le A^{m-5}\left((m-5)!\right)^{3(1+\sigma)} \le A^m \left(m!\right)^{3(1+\sigma)},$$

then with (4.2.2), (4.2.3), (4.2.4) and (4.2.6), we get that, for any $0 < t \le T_1 < T \le 1$,

$$t^{3(m-N_0-1)}\left\|e^{\tilde{c}y}\partial^m u\right\|_{L^2(\mathbb{R}^2_+)} \le (T-T_1)^{-3(m-N_0-1)}\left\|\phi_m^0 W_m^0 f_m\right\|_{L^2(\mathbb{R}^2_+)},$$

yields, for any $m > N_0+1$ and $0 < t \le T_1 < T \le 1$,

$$\begin{aligned} t^{3(m-N_0-1)}\left\|e^{\tilde{c}y}\partial^m u\right\|_{L^2(\mathbb{R}^2_+)} &\le (T-T_1)^{-3(m-N_0-1)} A^m \left(m!\right)^{3(1+\sigma)} \\ &\le (T-T_1)^{-3m} A^m \left(m!\right)^{3(1+\sigma)}. \end{aligned}$$

As a result, Theorem 4.1.1 follows if we take $L = (T-T_1)^{-3}A$. □

Now we begin to prove **Claim I_{m+1}**, and to do so it is sufficient to prove that the following:

Claim $E_{m,\ell}$, $0 \le \ell \le 3$: The following property hold for $0 \le \ell \le 3$,

$$\begin{aligned} &\partial_y^3 \Lambda^{-1} \phi_m^\ell W_m^\ell f_m \in L^2([0,T]\times\mathbb{R}^2_+), \\ &\|\phi_m^\ell W_m^\ell f_m\|_{L^\infty([0,T];\, L^2(\mathbb{R}^2_+))} + \textstyle\sum_{j=1}^2 \|\partial_y^j \Lambda^{-\frac{2(j-1)}{3}} \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\ &\quad\le A^{m-5+\frac{\ell}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{\ell(1+\sigma)}. \end{aligned} \tag{4.2.10}$$

In fact, **Claim $E_{m,3}$** yields $\partial_y^3\Lambda^{-1}\phi_m^3 W_m^3 f_m \in L^2([0,T]\times\mathbb{R}^2_+)$ and

$$\begin{aligned} &\|\phi_m^3 W_m^3 f_m\|_{L^\infty([0,T];\, L^2(\mathbb{R}^2_+))} + \sum_{j=1}^{2} \|\partial_y^j \Lambda^{-\frac{2(j-1)}{3}} \phi_m^3 W_m^3 f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\ &\le A^{m-5+\frac{1}{2}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{3(1+\sigma)} \\ &= A^{m-5+\frac{1}{2}}\left[\left((m+1)-5\right)!\right]^{3(1+\sigma)}, \end{aligned}$$

which, along with Lemma 4.2.2, yields $\partial_y^3\Lambda^{-1}\phi_{m+1}^0 W_{m+1}^0 f_{m+1} \in L^2([0,T]\times\mathbb{R}^2_+)$ and

$$\begin{aligned} &\|\phi_{m+1}^0 W_{m+1}^0 f_{m+1}\|_{L^\infty([0,T];\, L^2(\mathbb{R}^2_+))} + \sum_{j=1}^{2} \|\partial_y^j \Lambda^{-\frac{2(j-1)}{3}} \phi_{m+1}^0 W_{m+1}^0 f_{m+1}\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\ &\le C_2 A^{m-5+\frac{1}{2}}\left[\left((m+1)-5\right)!\right]^{3(1+\sigma)}, \end{aligned}$$

recalling C_2 is a constant depending only on the numbers σ, c and the constants C_0, C_* in Theorem 4.1.1. As a result, if we choose A in such a way that

$$A^{1/2} \geq C_2,$$

then we see (4.2.9) is also valid for $k = m+1$. Thus the desired **Claim $\boldsymbol{I_{m+1}}$** follows.

Proof of the Claim $\boldsymbol{E_{m,\ell}}$.

The rest of this section is devoted to proving Claim $\boldsymbol{E_{m,\ell}}$ holds for all $0 \leq \ell \leq 3$, supposing the inductive hypothesis (4.2.8) and (4.2.9) hold.

We will prove **Claim $\boldsymbol{E_{m,\ell}}$** by iteration on $0 \leq \ell \leq 3$. Obviously **Claim $\boldsymbol{E_{m,0}}$** holds, due to the hypothesis of induction (4.2.8) and (4.2.9) with $k = m$. Now supposing **Claim $\boldsymbol{E_{m,i}}$** holds for all $0 \leq i \leq \ell - 1$, i.e., for all $0 \leq i \leq \ell - 1$ we have

$$\begin{gathered} \partial_y^3 \Lambda^{-1} \phi_m^i W_m^i f_m \in L^2([0,T] \times \mathbb{R}_+^2), \\ \|\phi_m^i W_m^i f_m\|_{L^\infty([0,T];\, L^2(\mathbb{R}_+^2))} + \sum_{j=1}^2 \|\partial_y^j \Lambda^{-\frac{2(j-1)}{3}} \phi_m^i W_m^i f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ \leq A^{m-5+\frac{i}{6}} \left((m-5)!\right)^{3(1+\sigma)} (m-4)^{i(1+\sigma)}, \end{gathered} \tag{4.2.11}$$

we will prove in the remaining part **Claim $\boldsymbol{E_{m,\ell}}$** also holds. To do so, we first introduce the mollifier $\Lambda_\delta^{-2} = \Lambda_{\delta,x}^{-2}$ which is the Fourier multiplier with the symbol $\langle \delta\xi \rangle^{-2}$, $0 < \delta < 1$, and then consider the function $F = \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m$. Under the inductive assumption (4.2.11), we see F is a classical solution to the following problem (See the detail computation in Section 4.6 and the equation (4.6.1) fulfilled by f_m):

$$\begin{cases} \left(\partial_t + u\partial + v\partial_y - \partial_y^2\right) F = \mathcal{Z}_{m,\ell,\delta}, \\ \partial_y F\big|_{y=0} = 0, \\ F\big|_{t=0} = 0, \end{cases} \tag{4.2.12}$$

where

$$\begin{aligned} \mathcal{Z}_{m,\ell,\delta} =& \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell \mathcal{Z}_m + \Lambda_\delta^{-2} \left(\partial_t \phi_m^\ell\right) W_m^\ell f_m \\ &+ \left[u\partial_x + v\partial_y - \partial_y^2,\, \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell\right] f_m, \end{aligned} \tag{4.2.13}$$

with $\mathcal{Z}_m$ given in the appendix (seeing Section 4.6), that is,

$$\begin{aligned} \mathcal{Z}_m \quad = \quad & -\sum_{j=1}^{m} \binom{m}{j} (\partial^j u) f_{m+1-j} - \sum_{j=1}^{m-1} \binom{m}{j} (\partial^j v)(\partial_y f_{m-1}) \\ & - \left[\partial_y \left(\frac{\partial_y \omega}{\omega}\right)\right] \sum_{j=1}^{m-1} \binom{m}{j} (\partial^j v)(\partial^{m-j} u) - 2\left[\partial_y \left(\frac{\partial_y \omega}{\omega}\right)\right] f_m. \end{aligned}$$

The initial value and boundary value in (4.2.12) is take in the sense of trace in Sobolev space, due to the induction hypothesis (4.2.9) and the facts that $\partial_y \Lambda_\delta^{-2} \phi_m^\ell f_m|_{y=0} = 0$ (seeing (4.6.5) in the appendix) and

$$\partial_y \left(e^{2cy} \left(1 + \frac{2cy}{(3m+i)\sigma}\right)^{-(3m+i)\sigma/2} (1+cy)^{-1} \right)\Bigg|_{y=0} = 0.$$

We will prove an energy estimate for the equation (4.2.12). For this purpose, let $t \in [0,T]$, and take $L^2\left([0,t] \times \mathbb{R}_+^2\right)$ inner product with F on both sides of the first equation in (4.2.12); this gives

$$\mathrm{Re}\ \left(\left(\partial_t + u\partial + v\partial_y - \partial_y^2\right) F,\ F\right)_{L^2\left([0,t]\times\mathbb{R}_+^2\right)} = \mathrm{Re}\ \left(\mathcal{Z}_{m,\ell,\delta},\ F\right)_{L^2\left([0,t]\times\mathbb{R}_+^2\right)}.$$

Moreover observing the initial-boundary conditions in (4.2.12) and the facts that $u|_{y=0} = v|_{y=0} = 0$ and $\partial_x u + \partial_y v = 0$, we integrate by parts to obtain,

$$\mathrm{Re}\ \left(\left(\partial_t + u\partial + v\partial_y - \partial_y^2\right) F,\ F\right)_{L^2\left([0,t]\times\mathbb{R}_+^2\right)} = \frac{1}{2}\|F(t)\|_{L^2(\mathbb{R}_+^2)}^2 + \int_0^t \|\partial_y F(t)\|_{L^2(\mathbb{R}_+^2)}^2 dt.$$

Thus we infer

$$\|F\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}_+^2)\right)}^2 + \|\partial_y F\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}^2 \le 2\left|\left(\mathcal{Z}_{m,\ell,\delta},\ F\right)_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\right|,$$

and thus

$$\begin{aligned}
&\|F\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}_+^2)\right)}^2 + \|\partial_y F\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}^2 + \|\partial_y^2\Lambda^{-2/3}F\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}^2 \\
&\le 2\left|\left(\mathcal{Z}_{m,\ell,\delta},\ F\right)_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\right| + \|\partial_y^2\Lambda^{-2/3}F\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}^2 \\
&\le 2\left\|\phi^{1/2}\mathcal{Z}_{m,\ell,\delta}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\left\|\phi^{-1/2}F\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} + \|\partial_y^2\Lambda^{-2/3}F\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}^2.
\end{aligned} \tag{4.2.14}$$

In order to treat the first term on the right hand side, we need the following proposition, whose proof is postponed to Section 4.5.

Proposition 4.2.3. *Under the induction hypothesis (4.2.7) -(4.2.9) and (4.2.11), there exists a constant C_3, such that, using the notation $F = \Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m$ and*

$$\tilde{f} = \phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m$$

with ϕ defined in (4.2.6),

$$\begin{aligned}
&\left\|\phi^{1/2}\mathcal{Z}_{m,\ell,\delta}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \\
\le& mC_3\left\|\phi^{-1/2}F\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} + C_3\left\|\partial_y F\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \\
&+ C_3A^{m-6}\left((m-5)!\right)^{3(1+\sigma)},
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\Lambda^{-1/3}\phi^{1/2}\mathcal{Z}_{m,\ell-1,\delta}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \\
\le\ & mC_3\left\|\Lambda^{-1/3}\phi^{-1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \\
&+C_3\left\|\partial_y\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} + C_3A^{m-6}\left((m-5)!\right)^{3(1+\sigma)},
\end{aligned} \tag{4.2.15}$$

and

$$\begin{aligned}
&\left\|\Lambda^{-\frac{2}{3}}\partial_y\phi^{1/2}\mathcal{Z}_{m,\ell-1,\delta}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \\
&\le C_3\|\left\langle y\right\rangle^{-\sigma}\Lambda^{1/3}\tilde{f}\|_{L^2([0,T]\times\mathbb{R}_+^2)} + C_3\left\|\partial_y^2\Lambda^{-2/3}\tilde{f}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \\
&\quad +m\,C_3\Big(\left\|\Lambda^{-2/3}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \\
&+\left\|\Lambda^{-2/3}\phi^{-1/2}\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\Big) + C_3A^{m-6}\left((m-5)!\right)^{3(1+\sigma)}.
\end{aligned} \tag{4.2.16}$$

The constant C_3 depends only on σ, c, and the constant C_*, but is independent of m and δ.

Now combining (4.2.15) in the above proposition and (4.2.14), we have

$$\begin{aligned}
&\|F\|^2_{L^\infty\left([0,T];\, L^2(\mathbb{R}^2_+)\right)} + \sum_{j=1}^{2} \|\partial_y^j \Lambda^{-\frac{2(j-1)}{3}} F\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
\leq & 2mC_3\|\phi^{-1/2}F\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + (2C_3)^2\|\phi^{-1/2}F\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \frac{1}{2}\|\partial_y F\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&+ \left(A^{m-6}\left((m-5)!\right)^{3(1+\sigma)}\right)^2 + \|\partial_y^2\Lambda^{-2/3}F\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)},
\end{aligned}$$

which yields, denoting by $C_4 = 4C_3 + 10C_3^2 + 2$,

$$\begin{aligned}
&\|F\|^2_{L^\infty\left([0,T];\, L^2(\mathbb{R}^2_+)\right)} + \sum_{j=1}^{2} \|\partial_y^j \Lambda^{-\frac{2(j-1)}{3}} F\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\qquad \leq mC_4\|\phi^{-1/2}F\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + 2\left(A^{m-6}\left((m-5)!\right)^{3(1+\sigma)}\right)^2 \\
&\qquad\quad + 2\|\partial_y^2\Lambda^{-2/3}F\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)},
\end{aligned}$$

or equivalently,

$$\begin{aligned}
&\|F\|_{L^\infty\left([0,T];\, L^2(\mathbb{R}^2_+)\right)} + \sum_{j=1}^{2} \|\partial_y^j \Lambda^{-\frac{2(j-1)}{3}} F\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\leq C_4\left(m^{1/2}\|\phi^{-1/2}F\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \|\partial_y^2\Lambda^{-2/3}F\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\right) \\
&+2A^{m-6}\left((m-5)!\right)^{3(1+\sigma)}.
\end{aligned} \tag{4.2.17}$$

It remains to treat the right terms on the right hand side. To do so we need to study the subellipticity of the linearized Prandtl equation :

$$\mathcal{P}f = \partial_t f + u\partial_x f + v\partial_y f - \partial_y^2 f = h, \qquad (t,x,y) \in]0,T[\times\mathbb{R}^2_+, \tag{4.2.18}$$

where u, v is solution of Prandtl's equation (4.1.1) satisfying the condition (4.1.2) and (4.1.3). Then we have

Proposition 4.2.4. *Let $h, g \in L^2([0,T]\times\mathbb{R}^2_+)$ be given such that $\partial_y h, \partial_y g \in L^2([0,T]\times\mathbb{R}^2_+)$. Suppose that $f \in L^2\left([0,T];\ H^2(\mathbb{R}^2_+)\right)$ with $\partial_y^3 f \in L^2\left([0,T]\times\mathbb{R}^2_+\right)$, is a classical solution to the equation (4.2.18) with the following initial and boundary conditions :*

$$f(0,x,y) = f(T,x,y) = 0, \quad (x,y) \in \mathbb{R}^2_+, \tag{4.2.19}$$

and

$$\partial_y f(t,x,0) = 0, \quad \partial_t f(t,x,0) = \left(\partial_y^2 f\right)(t,x,0) + g(t,x,0), \quad (t,x) \in]0,T[\times\mathbb{R} \tag{4.2.20}$$

Then for any $\varepsilon > 0$ there exists a constant C_ε, depending only on ε, σ and the constants C_* , such that

$$
\begin{aligned}
&\left\| \langle y\rangle^{-\sigma/2} \Lambda^{1/3} f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y^2\Lambda^{-1/3} f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ & C_\varepsilon \left(\left\|\Lambda^{-1/3}h\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right) \quad (4.2.21)\\
&+C_\varepsilon \left(\left\| \langle y\rangle^{-\frac{\sigma}{2}} \partial_y\Lambda^{1/6} f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| \langle y\rangle^{\sigma} \Lambda^{-1/3}\partial_y g\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\
&+\varepsilon\left\|\Lambda^{-2/3}\partial_y h\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
&\|\partial_y^3\Lambda^{-2/3} f\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\quad\leq\tilde{C}\Big(\left\| \langle y\rangle^{-\sigma/2} \Lambda^{1/3} f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\Lambda^{-2/3}\partial_y h\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\qquad+\left\|\partial_y f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\Big),
\end{aligned}
$$

where $\tilde{C}$ is a constant depending only on σ, c, and C_*, C_0 in Theorem 4.1.1.

We will prove this proposition in next section 4.3. This subellipitic estimate gives a gain of regularity of order $\frac{1}{3}$ with respect to x variable, so it is sufficient to repeat the same procedure for 3 times to get 1 order of regularity.

Continuation of the proof of the Claim $\boldsymbol{E_{m,\ell}}$.
We now use the above subellipticity for the function $f = \tilde{f}$, with $\tilde{f}$ defined in Proposition 4.2.3, i.e.,

$$
f = \phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1} f_m = \Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac{1}{2}}W_m^{\ell-1} f_m.
$$

Similar to (4.2.12), we see f is a classical solution to the following problem:

$$
\begin{cases}
\left(\partial_t + u\partial + v\partial_y - \partial_y^2\right) f = \phi^{1/2}\mathcal{Z}_{m,\ell-1,\delta} + \left(\partial_t\phi^{1/2}\right)\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1} f_m,\\
\partial_y f\big|_{y=0} = 0,\\
f\big|_{t=0} = f\big|_{t=T} = 0,
\end{cases}
$$

where $\mathcal{Z}_{m,\ell-1,\delta}$ is defined in (4.2.13). The initial value and boundary value in (4.2.12) is take in the sense of trace in Sobolev space. The validity of **Claim $\boldsymbol{E_{m,\ell-1}}$** due to the inductive assumption (4.2.11) yields that $\partial_y^3 f \in L^2\left([0,T]\times\mathbb{R}^2_+\right)$. Next we calculate $\left(\partial_t f - \partial_y^2 f\right)\big|_{y=0}$. Firstly we have, seeing (4.6.6) in the appendix,

$$
\left(\partial_t f_m - \partial_y^2 f_m\right)|_{y=0} = -2\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right] f_m\Big|_{y=0}.
$$

Then

$$\begin{aligned}
\partial_t f\big|_{y=0} =&\Lambda_\delta^{-2}\left(\partial_t\phi^{3(m-N_0-1)+\ell-\frac12}\right)W_m^{\ell-1}f_m\big|_{y=0}+\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}W_m^{\ell-1}\partial_t f_m\big|_{y=0}\\
=&\Lambda_\delta^{-2}\left(\partial_t\phi^{3(m-N_0-1)+\ell-\frac12}\right)W_m^{\ell-1}f_m\big|_{y=0}+\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}W_m^{\ell-1}\partial_y^2 f_m\big|_{y=0}\\
&-2\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}W_m^{\ell-1}\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_m\big|_{y=0}\\
=&\Lambda_\delta^{-2}\left(\partial_t\phi^{3(m-N_0-1)+\ell-\frac12}\right)\Lambda^{(\ell-1)/3}f_m\big|_{y=0}+\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}W_m^{\ell-1}\partial_y^2 f_m\big|_{y=0}\\
&-2\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}\Lambda^{(\ell-1)/3}\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_m\big|_{y=0}.
\end{aligned}$$

This, along with the fact that

$$\begin{aligned}
&\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}W_m^{\ell-1}\partial_y^2 f_m\big|_{y=0}\\
&=\partial_y^2\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}W_m^{\ell-1}f_m\big|_{y=0}-\left[\partial_y^2,\,W_m^{\ell-1}\right]\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}f_m\big|_{y=0}\\
&=\partial_y^2 f\big|_{y=0}-\left(\frac{2c^2}{(3m+\ell-1)\sigma}+3c^2\right)\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}\Lambda^{(\ell-1)/3}f_m\big|_{y=0}
\end{aligned}$$

due to the fact that $\partial_y\Lambda_\delta^{-2}f_m|_{y=0}=0$ (seeing (4.6.5) in the appendix), gives

$$\begin{aligned}
&\left(\partial_t f-\partial_y^2 f\right)\big|_{y=0}\\
&=\Lambda_\delta^{-2}\left(\partial_t\phi^{3(m-N_0-1)+\ell-\frac12}\right)\Lambda^{(\ell-1)/3}f_m\big|_{y=0}\\
&\quad-\left(\frac{2c^2}{(3m+\ell-1)\sigma}+3c^2\right)\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}\Lambda^{(\ell-1)/3}f_m\big|_{y=0}\\
&\quad-2\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}\Lambda^{(\ell-1)/3}\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_m\big|_{y=0}\\
&\stackrel{\text{def}}{=}g\big|_{y=0}
\end{aligned}$$

with

$$\begin{aligned}
g\;=\;&\Lambda_\delta^{-2}\left(\partial_t\phi^{3(m-N_0-1)+\ell-\frac12}\right)\Lambda^{(\ell-1)/3}f_m\\
&-\left(\frac{2c^2}{(3m+\ell-1)\sigma}+3c^2\right)\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}\Lambda^{(\ell-1)/3}f_m\\
&-2\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}\Lambda^{(\ell-1)/3}\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_m.
\end{aligned}\tag{4.2.22}$$

Then using Proposition 4.2.4 for $h=\phi^{1/2}\mathcal{Z}_{m,\ell-1,\delta}+\left(\partial_t\phi^{1/2}\right)\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m$ and the above g, we have

$$\begin{aligned}
&\left\|\langle y\rangle^{-\sigma/2}\Lambda^{1/3}f\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}+\left\|\partial_y^2\Lambda^{-1/3}f\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\le C_\varepsilon\left(\left\|\Lambda^{-1/3}h\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}+\left\|\partial_y f\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}+\left\|f\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}\right)\\
&\quad+C_\varepsilon\left(\left\|\langle y\rangle^{-\frac{\sigma}{2}}\partial_y\Lambda^{1/6}f\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}+\left\|\langle y\rangle^{\sigma}\Lambda^{-1/3}\partial_y g\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}\right)\\
&\quad+\varepsilon\left\|\Lambda^{-2/3}\partial_y h\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}.
\end{aligned}$$

We claim, for any $\tilde\varepsilon > 0$,

$$
\begin{aligned}
& C_\varepsilon \left(\|\Lambda^{-1/3} h\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y f\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \|f\|_{L^2([0,T]\times\mathbb{R}^2_+)} \right) \\
& \quad + C_\varepsilon \left(\| \langle y\rangle^{-\frac{\sigma}{2}} \partial_y \Lambda^{1/6} f\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \| \langle y\rangle^{\sigma} \Lambda^{-1/3} \partial_y g\|_{L^2([0,T]\times\mathbb{R}^2_+)} \right) \\
& \quad + \varepsilon \|\Lambda^{-2/3} \partial_y h\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
\leq \; & \varepsilon\, C_5 \left(\| \langle y\rangle^{-\sigma/2} \Lambda^{1/3} f\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y^2 \Lambda^{-1/3} f\|_{L^2([0,T]\times\mathbb{R}^2_+)} \right) \\
& + \tilde\varepsilon\, m^{-(1+\sigma)/2} \left(\|F\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y F\|_{L^2([0,T]\times\mathbb{R}^2_+)} \right) \\
& + C_{\varepsilon,\tilde\varepsilon} m^{(1+\sigma)/2} A^{m-5+\frac{\ell-1}{6}} \left((m-5)!\right)^s m^{(\ell-1)(1+\sigma)},
\end{aligned}
\tag{4.2.23}
$$

where C_5 is a constant depending only on σ, c, and the constant C_*, but independent of m and δ, and $C_{\varepsilon,\tilde\varepsilon}$ is a constant depending only on $\varepsilon, \tilde\varepsilon, \sigma$, c, and the constant C_*, but independent of m and δ. Recall $F = \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m$. The proof of (4.2.23) is postponed to the end of this section. Now combining the above inequalities and letting ε be small enough, we infer for any $\tilde\varepsilon > 0$,

$$
\begin{aligned}
& \| \langle y\rangle^{-\sigma/2} \Lambda^{1/3} f\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y^2 \Lambda^{-1/3} f\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
& \quad \leq \tilde\varepsilon\, m^{-(1+\sigma)/2} \left(\|F\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y F\|_{L^2([0,T]\times\mathbb{R}^2_+)} \right) \\
& \qquad + C_{\tilde\varepsilon} m^{(1+\sigma)/2} A^{m-5+\frac{\ell-1}{6}} \left((m-5)!\right)^s m^{(\ell-1)(1+\sigma)}.
\end{aligned}
\tag{4.2.24}
$$

Now we come back to estimate the terms on the right side of (4.2.17). To do so we need the following technic Lemma, whose proof is presented at the end of Section 4.4.

Lemma 4.2.5. *Recall $F = \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m$ and $f = \phi^{1/2} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m$. There exists a constant C_6, depending only on σ, c, and the constant C_*, but independent of m and δ, such that*

$$
\begin{aligned}
& \|\phi^{-1/2} F\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y^2 \Lambda^{-2/3} F\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
& \quad \leq C_6 \left(m^{\sigma/2} \| \langle y\rangle^{-\frac{\sigma}{2}} \Lambda^{1/3} f\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y^2 \Lambda^{-1/3} f\|_{L^2([0,T]\times\mathbb{R}^2_+)} \right) \\
& \qquad + C_6 \left(\|\phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)} \right),
\end{aligned}
$$

and

$$
\begin{aligned}
& \|\partial_y^3 \Lambda^{-1} F\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
\leq & C_6 \|\partial_y^2 \Lambda^{-1/3} f\|_{L^2([0,T]\times\mathbb{R}^2_+)} + C_6 \|\partial_y^3 \Lambda^{-2/3} f\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
& + C_6 \left(\|\phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)} \right).
\end{aligned}
\tag{4.2.25}
$$

End of the proof of the Claim $E_{m,\ell}$.

We combine (4.2.24) and the first estimate in Lemma 4.2.5, to conclude

$$
\begin{aligned}
&\|\phi^{-1/2}F\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y^2\Lambda^{-2/3}F\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\quad\leq \tilde{\varepsilon}C_6 m^{-1/2}\left(\left\|F\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y F\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\
&\qquad + C_6 C_{\tilde{\varepsilon}} m^{\frac{1}{2}+\sigma} A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^s m^{(\ell-1)(1+\sigma)}\\
&\qquad + C_6\left(\|\phi_m^{\ell-1}W_m^{\ell-1}f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\
&\quad\leq \tilde{\varepsilon}C_6 m^{-1/2}\left(\left\|F\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y F\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\
&\qquad + \left(C_6 C_{\tilde{\varepsilon}} + C_6\right) m^{\frac{1}{2}+\sigma} A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}
\end{aligned}
$$

the last inequality using (4.2.11). This along with (4.2.17) yields

$$
\begin{aligned}
&\|F\|_{L^\infty\left([0,T];\, L^2(\mathbb{R}^2_+)\right)} + \sum_{j=1}^{2}\left\|\partial_y^j\Lambda^{-\frac{2(j-1)}{3}}F\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\leq \tilde{\varepsilon}C_4C_6\left(\left\|F\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \left\|\partial_y F\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\right)\\
&\quad + C_4\left(C_6C_{\tilde{\varepsilon}} + C_6\right) m^{1+\sigma} A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}\\
&+2A^{m-6}\left((m-5)!\right)^{3(1+\sigma)}.
\end{aligned}
$$

Consequently, letting $\tilde{\varepsilon} > 0$ be small sufficiently,

$$
\begin{aligned}
&\|F\|_{L^\infty\left([0,T];\, L^2(\mathbb{R}^2_+)\right)} + \sum_{j=1}^{2}\left\|\partial_y^j\Lambda^{-\frac{2(j-1)}{3}}F\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\leq C_7 m^{1+\sigma}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)} + C_7A^{m-6}\left((m-5)!\right)^{3(1+\sigma)}\\
&\leq C_8(m-4)^{1+\sigma}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}\\
&\quad + C_7A^{m-6}\left((m-5)!\right)^{3(1+\sigma)},
\end{aligned}
$$

where C_7, C_8 are two constants depending only on σ, c, and the constants C_0, C_* in Theorem 4.1.1, but is independent of m and δ. Now we choose A such that

$$
A \geq (2C_8 + 2C_7 + 1)^6.
$$

It then follows that, observing $\ell \geq 1$,

$$
\begin{aligned}
&\|F\|_{L^\infty\left([0,T];\, L^2(\mathbb{R}^2_+)\right)} + \sum_{j=1}^{2}\left\|\partial_y^j\Lambda^{-\frac{2(j-1)}{3}}F\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\quad\leq A^{m-5+\frac{\ell}{6}}\left((m-5)!\right)^s(m-4)^{\ell(1+\sigma)}.
\end{aligned}
$$

Observe the above constant A is independent of δ, and thus letting $\delta \to 0$, we see (4.2.11) holds for $i = \ell$. It remains to prove that $\partial_y^3\Lambda^{-1}\phi_m^\ell W_m^\ell f_m$. The above estimate together with (4.2.24) gives

$$
\left\|\langle y\rangle^{-\sigma/2}\Lambda^{1/3}f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y^2\Lambda^{-1/3}f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} < C_{m,1}
$$

with $C_{m,1}$ a constant depending on m but independent of δ, and thus, using the last estimate in Proposition 4.2.4 and (4.2.23),

$$\|\partial_y^3 \Lambda^{-2/3} f\|_{L^2([0,T]\times\mathbb{R}^2_+)} \le C_{m,2},$$

with $C_{m,2}$ a constant depending on m but independent of δ. As a result, combining (4.2.25), we conclude

$$\|\partial_y^3 \Lambda^{-1} F\|_{L^2([0,T]\times\mathbb{R}^2_+)} < C_{m,3}$$

with $C_{m,3}$ a constant depending on m but independent of δ. Thus letting $\delta \to 0$, we see $\partial_y^3 \Lambda^{-1} \phi_m^\ell W_m^\ell f_m \in L^2([0,T]\times\mathbb{R}^2_+)$. Thus **Claim $\boldsymbol{E_{m,\ell}}$** holds. This completes the proof of **Claim $\boldsymbol{I_{m+1}}$**, and thus the proof of Theorem 4.1.1.

We end up this section by the following

Proof of the estimate (4.2.23). In the proof we use C to denote different constants depending only on σ, c, and the constants C_0, C_* in Theorem 4.1.1, but is independent of m and δ.

(a) We first estimate $\|\Lambda^{-1/3} h\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}$, recalling

$$h = \phi^{1/2} \mathcal{Z}_{m,\ell-1,\delta} + \left(\partial_t \phi^{1/2}\right) \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m.$$

Using interpolation inequality gives, observing $\left|\partial_t \phi^{1/2}\right| \le \phi^{-1/2}$,

$$\begin{aligned}
&\|\Lambda^{-1/3}\left(\partial_t \phi^{1/2}\right) \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2(\mathbb{R}_x)} \\
&\quad\le m^{-1/2} \phi^{1/2} \| \left(\partial_t \phi^{1/2}\right) \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2(\mathbb{R}_x)} \\
&\qquad + m^{(\ell+1)/2} \phi^{-(\ell+1)/2} \|\Lambda^{-1-\frac{\ell-1}{3}} \left(\partial_t \phi^{1/2}\right) \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2(\mathbb{R}_x)} \\
&\quad\le m^{-1/2} \|\Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2(\mathbb{R}_x)} \\
&\qquad + m^{(\ell+1)/2} \|\Lambda^{-1-\frac{\ell-1}{3}} \phi^{-(\ell+2)/2} \Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2(\mathbb{R}_x)} \\
&\quad\le m^{-1/2} \|\Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2(\mathbb{R}_x)} + m^{(\ell+1)/2} \|\Lambda^{-1} \Lambda_\delta^{-2} \phi_{m-1}^0 W_{m-1}^0 f_m\|_{L^2(\mathbb{R}_x)},
\end{aligned}$$

the last inequality following from (4.2.2) which shows $W_i^0, i \ge 1$, is a decreasing sequence of functions as i varies in $\mathbb{N}$, and the fact that

$$\phi^{-(\ell+2)/2} \phi_m^{\ell-1} \le \phi_{m-1}^0.$$

Moreover, using (4.2.5) and the inductive assumptions (4.2.11) and (4.2.9), we compute, observing $\ell/2+1 \le 3(1+\sigma)$,

$$\begin{aligned}
&m^{-1/2} \|\Lambda_\delta^{-2} \phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2(\mathbb{R}_x)} + m^{(\ell+1)/2} \|\Lambda^{-1} \Lambda_\delta^{-2} \phi_{m-1}^0 W_{m-1}^0 f_m\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\quad\le m^{-1/2} \|\phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2(\mathbb{R}_x)} + \tilde{C}_1 m^{(\ell+1)/2} \|\phi_{m-1}^0 W_{m-1}^0 f_{m-1}\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\quad\le C m^{-1/2} A^{m-5+\frac{\ell-1}{6}} \left((m-5)!\right)^{3(1+\sigma)} (m-4)^{(\ell-1)(1+\sigma)}.
\end{aligned}$$

Thus we have, combining the above inequalities,

$$\begin{aligned}&\|\Lambda^{-1/3}\left(\partial_t\phi^{1/2}\right)\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\&\qquad\le Cm^{-1/2}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}.\end{aligned}\tag{4.2.26}$$

Similarly, we can show that

$$\begin{aligned}&\left\|\partial_y\Lambda^{-1/3}\left(\partial_t\phi^{1/2}\right)\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\\le\ \ &Cm^{-1/2}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}.\end{aligned}\tag{4.2.27}$$

Using (4.2.15) in Proposition 4.2.3, we have

$$\begin{aligned}&\left\|\Lambda^{-1/3}\phi^{1/2}\mathcal{Z}_{m,\ell-1,\delta}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\\le\ \ &mC_3\left\|\Lambda^{-1/3}\phi^{-1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\&+C_3\left\|\partial_y\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\&+C_3A^{m-6}\left((m-5)!\right)^s,\end{aligned}$$

and moreover repeating the arguments as in (4.2.26) and (4.2.27), with $\partial_t\phi^{1/2}$ there replaced by $\phi^{-1/2}$,

$$\begin{aligned}&mC_3\left\|\Lambda^{-1/3}\phi^{-1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\&+C_3\left\|\partial_y\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\\le\ \ &Cm^{1/2}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)},\end{aligned}$$

and thus

$$\left\|\Lambda^{-1/3}\phi^{1/2}\mathcal{Z}_{m,\ell-1,\delta}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\le Cm^{1/2}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}.$$

This along with (4.2.26) yields

$$\left\|\Lambda^{-1/3}h\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\le Cm^{1/2}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}.$$

(b) In this step we treat $\left\|\Lambda^{-2/3}\partial_yh\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}$. It follows from (4.2.27) that

$$\begin{aligned}&\left\|\Lambda^{-2/3}\partial_y\left(\partial_t\phi^{1/2}\right)\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\\le\ \ &Cm^{-1/2}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}.\end{aligned}$$

On the other hand, by (4.2.16) we have, recalling $\tilde f=f=\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m$,

$$\begin{aligned}&\|\Lambda^{-\frac{2}{3}}\partial_y\phi^{1/2}\mathcal{Z}_{m,\ell-1,\delta}\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\le C_3\|\left\langle y\right\rangle^{-\sigma}\Lambda^{1/3}f\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\&\quad+C_3\left\|\partial_y^2\Lambda^{-2/3}f\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+m\,C_3\Big(\left\|\Lambda^{-2/3}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\&\quad+\left\|\Lambda^{-2/3}\phi^{-1/2}\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\Big)+C_3A^{m-6}\left((m-5)!\right)^s\end{aligned}$$

and moreover similar to (4.2.26) and (4.2.27), we have

$$
\begin{aligned}
&m\,C_3\Big(\big\|\Lambda^{-2/3}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+\big\|\Lambda^{-2/3}\phi^{-1/2}\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\Big)\\
&\qquad\le Cm^{1/2}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)},
\end{aligned}
$$

since $\left|\partial_t\phi^{1/2}\right|\geq 1$. Combining the above three inequalities gives

$$
\begin{aligned}
\big\|\Lambda^{-2/3}\partial_y h\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \;\le\;& C\left(\|\left\langle y\right\rangle^{-\sigma}\Lambda^{1/3}f\|_{L^2([0,T]\times\mathbb{R}_+^2)}+\big\|\partial_y^2\Lambda^{-2/3}f\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\right)\\
&+Cm^{1/2}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}.
\end{aligned}
$$

(c) It follows from the inductive assumption (4.2.11) that, observing $\phi^{1/2}\le 1$,

$$
\begin{aligned}
\sum_{j=0}^{1}\big\|\partial_y^j f\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \;\le\;& \big\|\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+\big\|\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
\le\;& A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}.
\end{aligned}
$$

Now we estimate $\big\|\left\langle y\right\rangle^{\sigma}\Lambda^{-1/3}\partial_y g\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}$, with g is defined in (4.2.22). It is quite similar as in step (a). For instance,

$$
\begin{aligned}
&\|\left\langle y\right\rangle^{\sigma}\Lambda^{-1/3}\partial_y\Lambda_\delta^{-2}\left(\partial_t\phi^{3(m-N_0-1)+\ell-\frac12}\right)\Lambda^{(\ell-1)/3}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\le\big\|\Lambda^{-1/3}\phi^{-1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\partial_y f_m\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\le C\big\|\Lambda^{-1/3}\phi^{-1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\quad+C\big\|\partial_y\Lambda^{-1/3}\phi^{-1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}.
\end{aligned}
$$

Then similar to (4.2.26) and (4.2.27), we conclude

$$
\begin{aligned}
&\|\left\langle y\right\rangle^{\sigma}\Lambda^{-1/3}\partial_y\Lambda_\delta^{-2}\left(\partial_t\phi^{3(m-N_0-1)+\ell-\frac12}\right)\Lambda^{(\ell-1)/3}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\le Cm^{-1/2}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}.
\end{aligned}
$$

The other terms in (4.2.22) can be estimated similarly, and a classical commutator estimate (see Lemma 4.3.1 in the following section) will be used for treatment of the third term in (4.2.22). Thus we conclude

$$
\big\|\left\langle y\right\rangle^{\sigma}\Lambda^{-1/3}\partial_y g\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\le CA^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}.
$$

(d) It remains to estimate $\big\|\left\langle y\right\rangle^{-\sigma/2}\partial_y\Lambda^{1/6}f\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}$, and we have

$$
\begin{aligned}
&\big\|\left\langle y\right\rangle^{-\sigma/2}\partial_y\Lambda^{1/6}f\big\|^2_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&=\big\|\left\langle y\right\rangle^{-\sigma/2}\partial_y\Lambda^{1/6}\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}W_m^{\ell-1}f_m\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\le\big\|\left\langle y\right\rangle^{-\sigma}\partial_y\Lambda^{1/3}\Lambda_\delta^{-2}\phi_m^{\ell}W_m^{\ell-1}f_m\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\big\|\partial_y\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\le C\left(\big\|\partial_y\Lambda_\delta^{-2}\phi_m^{\ell}W_m^{\ell}f_m\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+\big\|\Lambda_\delta^{-2}\phi_m^{\ell}W_m^{\ell}f_m\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\right)\\
&\quad\times\big\|\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)},
\end{aligned}
$$

the last inequality following from the third estimate in Lemma 4.2.2. This, along with the inductive assumption (4.2.11) implies, for any $\tilde{\varepsilon} > 0$,

$$
\begin{aligned}
& \| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/3} f \|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
\leq\; & \tilde{\varepsilon} m^{-(1+\sigma)/2} \Big(\|F\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \|\partial_y F\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \Big) \\
& + C_{\tilde{\varepsilon}} m^{(1+\sigma)/2} A^{m-5+\frac{\ell-1}{6}} \left((m-5)!\right)^s m^{(\ell-1)(1+\sigma)},
\end{aligned}
$$

recalling $F = \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m$.

Now combining the estimates in the above steps (a)-(d), we obtain the desired (4.2.23). □

4.3 Subelliptic estimate

In this section we prove the Proposition 4.2.4. Since the following commutators estimates would be used in our proof, we state some results of them in below lemma. Throughout the paper we use $\big[Q_1,\ Q_2\big]$ to denote the commutator between two operators Q_1 and Q_2, which is defined by

$$
\big[Q_1,\ Q_2\big] = Q_1 Q_2 - Q_2 Q_1 = -\big[Q_2,\ Q_1\big].
$$

We have

$$
\big[Q_1,\ Q_2 Q_3\big] = Q_2\big[Q_1,\ Q_3\big] + \big[Q_1,\ Q_2\big] Q_3. \tag{4.3.1}
$$

Lemma 4.3.1. *Denote by $\big[\alpha\big]$ the largest integer less than or equal to $\alpha \geq 0$. For any $\tau \in \mathbb{R}$ and $a \in C_b^{[|\tau|]+1}(\mathbb{R}^2_+)$, the space of functions such that all their derivatives up to the order of $[|\tau|]+1$ are continuous and bounded, there exists $C > 0$ such that for suitable function f and any $0 < \delta < 1$,*

$$
\|[a, \Lambda^\tau \Lambda_\delta^{-2}] f\|_{L^2(\mathbb{R}^2_+)} \leq C \|\Lambda^{\tau-1} \Lambda_\delta^{-2} f\|_{L^2(\mathbb{R}^2_+)},
$$

and

$$
\|[a\partial_x, \Lambda^\tau \Lambda_\delta^{-2}] f\|_{L^2(\mathbb{R}^2_+)} \leq C \|\Lambda^{\tau} \Lambda_\delta^{-2} f\|_{L^2(\mathbb{R}^2_+)}.
$$

The constant C depends on only on τ and $\|a\|_{C_b^{[|\tau|]+1}(\mathbb{R}^2_+)}$.

Since $\Lambda^\tau \Lambda_\delta^{-2}$ is only a Fourier multiplier of x variable, so we can prove the above Lemma by direct calculus or pseudo-differential computation, cf. [16, 19]. In this section, we use above Lemma with $a = u$ or $a = v$ and $\tau = -1/3, -2/3$. So that with hypothesis (4.1.3), the constant in Lemma 4.3.1 depends only on the constant C_0 in Theorem 4.1.1.

Proof of the Proposition 4.2.4. Taking the operator $\Lambda^{-2/3}$ on both sides of (4.2.18), we see the function $\Lambda^{-2/3} f$ satisfies the following equation in $]0, T[\, \times \mathbb{R}^2_+$:

$$
\begin{aligned}
& \partial_t \Lambda^{-2/3} f + u\partial_x \Lambda^{-2/3} f + v\partial_y \Lambda^{-2/3} f - \partial_y^2 \Lambda^{-2/3} f \\
& = \Lambda^{-2/3} h + \big[u\partial_x + v\partial_y,\ \Lambda^{-2/3}\big] f,
\end{aligned} \tag{4.3.2}
$$

and that

$$
\Lambda^{-2/3} f\big|_{t=0} = \Lambda^{-2/3} f\big|_{t=T} = 0, \quad \partial_y \Lambda^{-2/3} f\big|_{y=0} = 0 \tag{4.3.3}
$$

due to (4.2.19) and (4.2.20), since $\Lambda^{-2/3}$ is an operator acing only on x variable. Recall $\big[u\partial_x + v\partial_y,\ \Lambda^{-2/3}\big]$ stands for the commutator between $u\partial_x + v\partial_y$ and $\Lambda^{-2/3}$.

Step 1). We will show in this step that

$$
\begin{aligned}
&\|\,(\partial_y u)^{1/2}\,\partial_x\Lambda^{-2/3} f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\leq 2\left|\mathrm{Re}\,\left(\partial_t\Lambda^{-2/3}f,\ \partial_y\partial_x\Lambda^{-2/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\right| + \|\partial_y^2\Lambda^{-1/3}f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\quad + C\left(\|\Lambda^{-1/3}h\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \|f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right).
\end{aligned}
\tag{4.3.4}
$$

To do so, we take $L^2([0,T]\times\mathbb{R}^2_+)$ inner product with the function $\partial_y\partial_x\Lambda^{-2/3}f \in L^2([0,T]\times\mathbb{R}^2_+)$ on the both sides of equation (4.3.2), and then consider the real parts; this gives

$$
\begin{aligned}
&-\mathrm{Re}\,\left(u\partial_x\Lambda^{-2/3}f,\ \partial_y\partial_x\Lambda^{-2/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&= \mathrm{Re}\,\left(\partial_t\Lambda^{-2/3}f,\ \partial_y\partial_x\Lambda^{-2/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)} - \mathrm{Re}\,\left(\partial_y^2\Lambda^{-2/3}f,\ \partial_y\partial_x\Lambda^{-2/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\quad + \mathrm{Re}\,\left(v\partial_y\Lambda^{-2/3}f,\ \partial_y\partial_x\Lambda^{-2/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\quad - \mathrm{Re}\,\left(\Lambda^{-2/3}h,\ \partial_y\partial_x\Lambda^{-2/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\quad - \mathrm{Re}\,\left(\big[u\partial_x + v\partial_y,\ \Lambda^{-2/3}\big]f,\ \partial_y\partial_x\Lambda^{-2/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}
\tag{4.3.5}
$$

We will treat the terms on both sides. For the term on left hand side we integrate by parts to obtain, here we use $u\big|_{y=0} = 0$,

$$
\begin{aligned}
&-\mathrm{Re}\,\left(u\partial_x\Lambda^{-2/3}f,\ \partial_y\partial_x\Lambda^{-2/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&= -\frac{1}{2}\Big\{\left(u\partial_x\Lambda^{-2/3}f,\ \partial_y\partial_x\Lambda^{-2/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\qquad + \left(\partial_y\partial_x\Lambda^{-2/3}f,\ u\partial_x\Lambda^{-2/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\Big\}\\
&= \frac{1}{2}\|\,(\partial_y u)^{1/2}\,\partial_x\Lambda^{-2/3}f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}
$$

Next we estimate the terms on the right hand side and have, by Cauchy-Schwarz's inequality ,

$$
\begin{aligned}
&\left|-\mathrm{Re}\,\left(\partial_y^2\Lambda^{-2/3}f,\ \partial_y\partial_x\Lambda^{-2/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\right|\\
&\quad\leq \frac{1}{2}\|\partial_y^2\Lambda^{-1/3}f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \frac{1}{2}\|\partial_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)},
\end{aligned}
$$

$$
\left|\mathrm{Re}\,\left(\Lambda^{-2/3}h,\ \partial_y\partial_x\Lambda^{-2/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\right| \leq \|\Lambda^{-1/3}h\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}
$$

and

$$
\begin{aligned}
&\left| -\mathrm{Re}\, \left(v\partial_y \Lambda^{-2/3} f,\ \partial_y\partial_x \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)} \right| \\
\leq\ & \left| \left(\partial_y f,\ \left[\Lambda^{-2/3},\ v\right] \partial_y\partial_x \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)} \right| \\
&+ \left| \left(v\partial_y f,\ \Lambda^{-2/3} \partial_y\partial_x \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)} \right| \\
\leq\ & C \|\partial_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)},
\end{aligned}
$$

the last inequality using Lemma 4.3.1. Finally

$$
\begin{aligned}
&\left| -\mathrm{Re}\, \left(\left[u\partial_x + v\partial_y,\ \Lambda^{-2/3}\right] f,\ \partial_y\partial_x \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)} \right| \\
\leq\ & \|\Lambda^{1/3} \left[u\partial_x + v\partial_y,\ \Lambda^{-2/3}\right] f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
\leq\ & 2\Big(\|\left[u\partial_x + v\partial_y,\ \Lambda^{1/3}\Lambda^{-2/3}\right] f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&+ \|\left[u\partial_x + v\partial_y,\ \Lambda^{1/3}\right] \Lambda^{-2/3} f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \Big) + \|\partial_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
\leq\ & C \left(\|f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \right),
\end{aligned}
$$

These inequalities, together with (4.3.5), yields the desired (4.3.4).

Step 2). In this step we will estimate the second term on the right hand side of (4.3.4) and show that for any $\varepsilon > 0$,

$$
\begin{aligned}
&\|\partial_y^2 \Lambda^{-1/3} f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\leq \varepsilon \,\|\, (\partial_y u)^{1/2}\, \partial_x \Lambda^{-2/3} f\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\quad + C_\varepsilon \left(\|\partial_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \|f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\Lambda^{-1/3} h\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \right),
\end{aligned}
\tag{4.3.6}
$$

with C_ε a constant depending on ε. We see that the function $\Lambda^{-1/3} f$ satisfies the equation in $]0,T[\ \times\mathbb{R}^2_+$,

$$
\begin{aligned}
&\partial_t \Lambda^{-1/3} f + (u\partial_x + v\partial_y)\, \Lambda^{-1/3} f - \partial_y^2 \Lambda^{-1/3} f \\
&= \Lambda^{-1/3} h + \left[u\partial_x + v\partial_y,\ \Lambda^{-1/3}\right] f,
\end{aligned}
\tag{4.3.7}
$$

with the boundary condition

$$
\Lambda^{-1/3} f\big|_{t=0} = \Lambda^{-1/3} f\big|_{t=T} = 0, \quad \partial_y \Lambda^{-1/3} f\big|_{y=0} = 0. \tag{4.3.8}
$$

Now we take $L^2([0,T]\times\mathbb{R}^2_+)$ inner product with the function $-\partial_y^2 \Lambda^{-1/3} f \in L^2([0,T]\times\mathbb{R}^2_+)$ on both sides of (4.3.7), and then consider the real parts; this gives

$$
\|\partial_y^2 \Lambda^{-1/3} f\|^2_{L^2(\mathbb{R}^3_+)} \leq \sum_{p=1}^{4} J_p, \tag{4.3.9}
$$

where

$$\begin{aligned}
J_1 &= \left| \mathrm{Re}\, \left(\partial_t \Lambda^{-1/3} f,\ \partial_y^2 \Lambda^{-1/3} f \right)_{L^2([0,T]\times\mathbb{R}_+^2)} \right|, \\
J_2 &= \left| \mathrm{Re}\, \left((u\partial_x + v\partial_y)\, \Lambda^{-1/3} f,\ \partial_y^2 \Lambda^{-1/3} f \right)_{[0,T]\times L^2([0,T]\times\mathbb{R}_+^2)} \right|, \\
J_3 &= \left| \mathrm{Re}\, \left(\Lambda^{-1/3} h,\ \partial_y^2 \Lambda^{-1/3} f \right)_{L^2([0,T]\times\mathbb{R}_+^2)} \right|, \\
J_4 &= \left| \mathrm{Re}\, \left(\left[u\partial_x + v\partial_y,\ \Lambda^{-1/3} \right] f,\ \partial_y^2 \Lambda^{-1/3} f \right)_{L^2([0,T]\times\mathbb{R}_+^2)} \right|.
\end{aligned}$$

Integrating by parts, we see

$$\left(\partial_t \Lambda^{-1/3} f,\ \partial_y^2 \Lambda^{-1/3} f \right)_{L^2([0,T]\times\mathbb{R}_+^2)} = -\left(\partial_t \partial_y \Lambda^{-1/3} f,\ \partial_y \Lambda^{-1/3} f \right)_{L^2([0,T]\times\mathbb{R}_+^2)},$$

which along with the fact

$$\mathrm{Re}\, \left(\partial_t \partial_y \Lambda^{-1/3} f,\ \partial_y \Lambda^{-1/3} f \right)_{L^2([0,T]\times\mathbb{R}_+^2)} = 0$$

due to (4.3.8), implies

$$J_1 = \left| \mathrm{Re}\, \left(\partial_t \Lambda^{-1/3} f,\ \partial_y^2 \Lambda^{-1/3} f \right)_{L^2([0,T]\times\mathbb{R}_+^2)} \right| = 0.$$

About J_2 we integrate by parts again and observe the boundary conditition (4.3.8), to compute

$$\begin{aligned}
& \mathrm{Re}\, \left(u\partial \Lambda^{-1/3} f,\ \partial_y^2 \Lambda^{-1/3} f \right)_{L^2([0,T]\times\mathbb{R}_+^2)} \\
= & -\mathrm{Re}\, \left(u\partial \Lambda^{-1/3} \partial_y f,\ \Lambda^{-1/3} \partial_y f \right)_{L^2([0,T]\times\mathbb{R}_+^2)} \\
& -\mathrm{Re}\, \left((\partial_y u) \partial_x \Lambda^{-1/3} f,\ \Lambda^{-1/3} \partial_y f \right)_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \\
= & \frac{1}{2} \left((\partial u)\, \Lambda^{-1/3} \partial_y f,\ \Lambda^{-1/3} \partial_y f \right)_{L^2([0,T]\times\mathbb{R}_+^2)} \\
& -\mathrm{Re}\, \left((\partial_y u) \partial_x \Lambda^{-1/3} f,\ \Lambda^{-1/3} \partial_y f \right)_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}.
\end{aligned}$$

This gives

$$\begin{aligned}
& \left| \mathrm{Re} \left(u\partial \Lambda^{-1/3} f,\ \partial_y^2 \Lambda^{-1/3} f \right)_{L^2([0,T]\times\mathbb{R}_+^2)} \right| \\
\leq\ & \| \Lambda^{-1/3} \left(\partial_y u \right) \partial_x \Lambda^{-1/3} f \|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \| \partial_y f \|_{L^2([0,T]\times\mathbb{R}_+^2)} + C \| \partial_y f \|_{L^2([0,T]\times\mathbb{R}_+^2)}^2 \\
\leq\ & C \Big(\| \left(\partial_y u \right)^{1/2} \partial_x \Lambda^{-2/3} f \|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} + \| f \|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \Big) \| \partial_y f \|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
& + C \big\| \partial_y f \big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}^2 \\
\leq\ & \varepsilon \, \| \left(\partial_y u \right)^{1/2} \partial_x \Lambda^{-2/3} f \|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}^2 + C_\varepsilon \left(\big\| \partial_y f \big\|_{L^2([0,T]\times\mathbb{R}_+^2)}^2 + \big\| f \big\|_{L^2([0,T]\times\mathbb{R}_+^2)}^2 \right).
\end{aligned}$$

Moreover integrating by part, we obtain

$$\begin{aligned}\left|\mathrm{Re}\left(v\partial_y\Lambda^{-1/3}f,\ \partial_y^2\Lambda^{-1/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\right| &= \frac{1}{2}\left|\left((\partial_y v)\,\partial_y\Lambda^{-1/3}f,\ \partial_y\Lambda^{-1/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\right|\\ &\leq C\|\partial_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}.\end{aligned}$$

Thus

$$\begin{aligned}J_2 \leq{}& \varepsilon\|\,(\partial_y u)^{1/2}\,\partial_x\Lambda^{-2/3}f\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\ &+C_\varepsilon\left(\|\partial_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\|f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right).\end{aligned}\tag{4.3.10}$$

It remains to estimate J_3 and J_4. Let $\tilde\varepsilon>0$ be an arbitrarily small number. Cauchy-Schwarz's inequality gives

$$\begin{aligned}J_3 &= \left|\mathrm{Re}\ \left(\Lambda^{-1/3}h,\ \partial_y^2\Lambda^{-1/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\right|\\ &\leq \tilde\varepsilon\,\|\partial_y^2\Lambda^{-1/3}f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+C_{\tilde\epsilon}\|\Lambda^{-1/3}h\|^2_{L^2(\mathbb{R}^3_+)},\end{aligned}$$

and for J_4, Lemma 4.3.1 implies

$$\begin{aligned}J_4 &= \left|\mathrm{Re}\ \left(\left[u\partial_x+v\partial_y,\ \Lambda^{-1/3}\right]f,\ \partial_y^2\Lambda^{-1/3}f\right)_{L^2(\mathbb{R}^2_+)}\right|\\ &\leq \tilde\varepsilon\,\|\partial_y^2\Lambda^{-1/3}f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+C_{\tilde\epsilon}\left(\|f\|^2_{L^2(\mathbb{R}^3_+)}+\|\partial_y f\|^2_{L^2(\mathbb{R}^3_+)}\right),\end{aligned}$$

where $\tilde C_\varepsilon$ is constant depending on $\tilde\varepsilon$. Now the above two estimates for J_3 and J_4, along with (4.3.9) - (4.3.10), gives

$$\begin{aligned}&\|\partial_y^2\Lambda^{-1/3}f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\leq\tilde\varepsilon\,\|\partial_y^2\Lambda^{-1/3}f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\varepsilon\,\|\,(\partial_y u)^{1/2}\,\partial_x\Lambda^{-2/3}f\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\ &\quad+C_\varepsilon\left(\|\partial_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\|f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\ &\quad+C_{\tilde\epsilon}\left(\|\Lambda^{-1/3}h\|^2_{L^2(\mathbb{R}^3_+)}+\|f\|^2_{L^2(\mathbb{R}^3_+)}+\|\partial_y f\|^2_{L^2(\mathbb{R}^3_+)}\right),\end{aligned}$$

and thus, letting $\tilde\varepsilon$ small sufficiently,

$$\begin{aligned}&\|\partial_y^2\Lambda^{-1/3}f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\leq C_\varepsilon\left(\|\partial_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\|f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\|\Lambda^{-1/3}h\|^2_{L^2(\mathbb{R}^3_+)}\right)\\ &\quad+\varepsilon\,\|\,(\partial_y u)^{1/2}\,\partial_x\Lambda^{-2/3}f\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.\end{aligned}$$

This is just the desired estimate (4.3.6).

Combining the estimates (4.3.4) and (4.3.6), we obtain, choosing ε sufficiently small,

$$\begin{aligned}
&\|\,(\partial_y u)^{1/2}\,\partial_x \Lambda^{-2/3} f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y^2 \Lambda^{-1/3} f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\leq C\left|\mathrm{Re}\,\left(\partial_t\Lambda^{-2/3} f,\ \partial_y\partial_x\Lambda^{-2/3} f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\right|\\
&\quad + C\left(\|\Lambda^{-1/3} h\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \|f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right).
\end{aligned}\tag{4.3.11}$$

Step 3) It remains to treat the first term on the right hand side of (4.3.11). In this step we will prove that, for any $\varepsilon_1 > 0$,

$$\begin{aligned}
&\left|\mathrm{Re}\,\left(\partial_t\Lambda^{-2/3} f,\ \partial_y\partial_x\Lambda^{-2/3} f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\right|\\
&\leq \varepsilon_1 \int_0^T\int_{\mathbb{R}}\left|\left(\partial_y^2\Lambda^{-1/2} f\right)(t,x,0)\right|^2 dxdt + C_{\varepsilon_1}\|\,\langle y\rangle\,\Lambda^{-1/3}\partial_y g\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\quad + \varepsilon_1^{-1} C\left(\|\,\langle y\rangle^{-\sigma/2}\,\Lambda^{1/6} f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\,\langle y\rangle^{-\sigma/2}\,\partial_y\Lambda^{1/6} f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right).
\end{aligned}\tag{4.3.12}$$

For this purpose we integrate by parts again and observe the boundary condition (4.3.3) , to compute

$$\begin{aligned}
&\left(\partial_t\Lambda^{-2/3} f,\ \partial_y\partial_x\Lambda^{-2/3} f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\\
=\ & -\left(\Lambda^{-2/3} f,\ \partial_t\partial_y\partial_x\Lambda^{-2/3} f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\\
=\ & \left(\partial_x\Lambda^{-2/3} f,\ \partial_t\partial_y\Lambda^{-2/3} f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\\
=\ & -\left(\partial_y\partial_x\Lambda^{-2/3} f,\ \partial_t\Lambda^{-2/3} f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\\
& + \int_0^T\int_{\mathbb{R}}\left(\partial_t\Lambda^{-2/3} f(t,x,0)\right)\left(\partial_x\Lambda^{-2/3} f(t,x,0)\right) dxdt,
\end{aligned}$$

which, along with the fact that

$$\begin{aligned}
&2\,\mathrm{Re}\,\left(\partial_t\Lambda^{-2/3} f,\ \partial_y\partial_x\Lambda^{-2/3} f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\\
=\ &\left(\partial_t\Lambda^{-2/3} f,\ \partial_y\partial_x\Lambda^{-2/3} f\right)_{L^2([0,T]\times\mathbb{R}^2_+)} + \left(\partial_y\partial_x\Lambda^{-2/3} f,\ \partial_t\Lambda^{-2/3} f\right)_{L^2([0,T]\times\mathbb{R}^2_+)},
\end{aligned}$$

yields, for any $\varepsilon_1 > 0$,

$$\begin{aligned}
&\left|\mathrm{Re}\,\left(\partial_t\Lambda^{-2/3} f,\ \partial_y\partial_x\Lambda^{-2/3} f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\right|\\
&= \frac{1}{2}\left|\int_0^T\int_{\mathbb{R}}\left(\partial_t\Lambda^{-2/3} f(t,x,0)\right)\left(\partial_x\Lambda^{-2/3} f(t,x,0)\right) dxdt\right|\\
&= \frac{1}{2}\left|\int_0^T\int_{\mathbb{R}}\left(\Lambda^{1/6}\partial_t\Lambda^{-2/3} f(t,x,0)\right)\left(\Lambda^{-1/6}\partial_x\Lambda^{-2/3} f(t,x,0)\right) dxdt\right|\\
&\leq \varepsilon_1\int_0^T\int_{\mathbb{R}}\left(\partial_t\Lambda^{-1/2} f(t,x,0)\right)^2 dxdt + \varepsilon_1^{-1}\int_0^T\int_{\mathbb{R}}\left(\Lambda^{1/6} f(t,x,0)\right)^2 dxdt.
\end{aligned}\tag{4.3.13}$$

Moreover observing

$$\Lambda^{1/6} f(t,x,0) = \left(\langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \right)(t,x,0),$$

it then follows from Sobolev inequality that

$$\begin{aligned} \left| \Lambda^{1/6} f(t,x,0) \right|^2 &\leq C \left(\| \langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \|^2_{L^2(\mathbb{R}_+)} + \| \partial_y \langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \|^2_{L^2(\mathbb{R}_+)} \right) \\ &\leq C \left(\| \langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \|^2_{L^2(\mathbb{R}_+)} + \| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f \|^2_{L^2(\mathbb{R}_+)} \right) \end{aligned}$$

with C a constant independent of t, x. And thus

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} \left(\Lambda^{1/6} f(t,x,0) \right)^2 dxdt \\ &\leq C \left(\| \langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \| \partial_y \langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \right) \\ &\leq C \left(\| \langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \| \langle y \rangle^{-\sigma/2} \Lambda^{1/6} \partial_y f \|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \right). \end{aligned} \tag{4.3.14}$$

Using the fact that

$$\partial_t \Lambda^{-1/2} f(t,x,0) = \left(\partial_y^2 \Lambda^{-1/2} f \right)(t,x,0) + \Lambda^{-1/2} g(t,x,0)$$

due to assumption (4.2.20), we conclude

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} \left(\partial_t \Lambda^{-1/2} f(t,x,0) \right)^2 dxdt \\ \leq\ & \int_0^T \int_{\mathbb{R}} \left| \left(\partial_y^2 \Lambda^{-1/2} f \right)(t,x,0) \right|^2 dxdt + \int_0^T \int_{\mathbb{R}} \left| \Lambda^{-1/2} g(t,x,0) \right|^2 dxdt. \end{aligned}$$

Moreover observe

$$\begin{aligned} \left| \Lambda^{-1/2} g(t,x,0) \right| &= \left| - \int_0^{+\infty} \partial_{\tilde{y}} \Lambda^{-1/2} g(t,x,\tilde{y}) d\tilde{y} \right| \\ &\leq \left(\int_0^{+\infty} \langle \tilde{y} \rangle^{-2\sigma} d\tilde{y} \right)^{1/2} \left(\int_0^{+\infty} \langle \tilde{y} \rangle^{2\sigma} \left| \Lambda^{-1/2} \partial_{\tilde{y}} g(t,x,\tilde{y}) \right|^2 d\tilde{y} \right)^{1/2}, \end{aligned}$$

which implies

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} \left| \Lambda^{-1/2} g(t,x,0) \right|^2 dxdt &\leq C \| \langle y \rangle^{\sigma} \Lambda^{-1/2} \partial_y g \|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\ &\leq C \| \langle y \rangle^{\sigma} \Lambda^{-1/3} \partial_y g \|^2_{L^2([0,T]\times\mathbb{R}^2_+)}, \end{aligned}$$

and thus

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} \left(\partial_t \Lambda^{-1/2} f(t,x,0) \right)^2 dxdt \\ \leq\ & \int_0^T \int_{\mathbb{R}} \left| \left(\partial_y^2 \Lambda^{-1/2} f \right)(t,x,0) \right|^2 dxdt + C \| \langle y \rangle^{\sigma} \Lambda^{-1/3} \partial_y g \|^2_{L^2([0,T]\times\mathbb{R}^2_+)}. \end{aligned}$$

This along with (4.3.13) and (4.3.14) yields the desired (4.3.12).

Step 4) Combining (4.3.11) and (4.3.12), we have, for any $\varepsilon_1 > 0$,

$$\begin{aligned}
&\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y^2 \Lambda^{-1/3} f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
\leq\ & \varepsilon_1 \int_0^T \int_{\mathbb{R}} \left| \left(\partial_y^2 \Lambda^{-1/2} f\right)(t,x,0)\right|^2 dx dt + C_{\varepsilon_1} \| \langle y\rangle^{\sigma} \Lambda^{-1/3} \partial_y g\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
& + \varepsilon_1^{-1} C \left(\| \langle y\rangle^{-\sigma/2} \Lambda^{1/6} f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \| \langle y\rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \right) \\
& + C \left(\|\partial_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \|f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\Lambda^{-1/3} h\|^2_{L^2(\mathbb{R}^3_+)} \right).
\end{aligned}$$

Moreover we use the monotonicity condition and interpolation inequality to get, for any $\varepsilon_2 > 0$

$$\begin{aligned}
&\| \langle y\rangle^{-\sigma/2} \Lambda^{1/6} f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\qquad \leq \varepsilon_2 \| \langle y\rangle^{-\sigma/2} \Lambda^{1/3} f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \varepsilon_2^{-1} \| \langle y\rangle^{-\sigma/2} f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\qquad \leq \varepsilon_2 \| \langle y\rangle^{-\sigma/2} \partial_x \Lambda^{-2/3} f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + C_{\varepsilon_2} \| \langle y\rangle^{-\sigma/2} f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\qquad \leq \varepsilon_2 \| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + C_{\varepsilon_2} \|f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}$$

From the above inequalities, we infer that, choosing ε_2 small enough,

$$\begin{aligned}
&\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y^2 \Lambda^{-1/3} f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\leq \varepsilon_1 \int_0^T \int_{\mathbb{R}} \left| \left(\partial_y^2 \Lambda^{-1/2} f\right)(t,x,0)\right|^2 dx dt \\
&\quad + C_{\varepsilon_1} \left(\| \langle y\rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \| \langle y\rangle^{\sigma} \Lambda^{-1/3} \partial_y g\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \right) \\
&\quad + C_{\varepsilon_1} \left(\|\partial_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \|f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\Lambda^{-1/3} h\|^2_{L^2(\mathbb{R}^3_+)} \right).
\end{aligned} \tag{4.3.15}$$

Step 5) In this step we treat the first term on the right side of (4.3.15), and show that, for any $0 < \varepsilon < 1$,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \left| \left(\partial_y^2 \Lambda^{-1/2} f\right)(t,x,0)\right|^2 dx dt \\
&\leq C \|(\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \varepsilon C \|\Lambda^{-2/3} \partial_y h\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\quad + C_\varepsilon \left(\|f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \right).
\end{aligned} \tag{4.3.16}$$

To do so, we integrate by parts to get

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}} \left| \left(\partial_y^2 \Lambda^{-1/2} f\right)(t,x,0)\right|^2 dx dt &= 2\mathrm{Re} \left(\partial_y^3 \Lambda^{-1/2} f,\ \partial_y^2 \Lambda^{-1/2} f\right)_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&= 2\mathrm{Re} \left(\partial_y^3 \Lambda^{-2/3} f,\ \partial_y^2 \Lambda^{-1/3} f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}$$

This yields

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \left| \left(\partial_y^2 \Lambda^{-1/2} f\right)(t,x,0)\right|^2 dxdt \\
&\leq \frac{\varepsilon}{2} \big\|\partial_y^3 \Lambda^{-2/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + 2\varepsilon^{-1} \big\|\partial_y^2 \Lambda^{-1/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\leq \varepsilon \big\|\partial_y^3 \Lambda^{-2/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C_\varepsilon \big\|\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)},
\end{aligned} \tag{4.3.17}$$

the last inequality holding because we can use (4.2.20) to integrate by parts and then obtain

$$\begin{aligned}
&\big\|\partial_y^2 \Lambda^{-1/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} = \left(\partial_y^2 \Lambda^{-2/3} f,\ \partial_y^2 f\right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\leq \left| \left(\partial_y^3 \Lambda^{-2/3} f,\ \partial_y f\right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \right| \\
&\leq \big\|\partial_y^3 \Lambda^{-2/3} f\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \big\|\partial_y f\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned} \tag{4.3.18}$$

Thus in order to prove (4.3.16) it suffices to estimate $\big\|\partial_y^3 \Lambda^{-2/3} f\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}$. We study the equation

$$\begin{aligned}
& \partial_t \Lambda^{-2/3} \partial_y f + u\partial_x \Lambda^{-2/3} \partial_y f + v\partial_y \Lambda^{-2/3} \partial_y f - \partial_y^3 \Lambda^{-2/3} f \\
= \quad & \Lambda^{-2/3} \partial_y h + \left[u\partial_x + v\partial_y,\ \Lambda^{-2/3}\right] \partial_y f - \Lambda^{-2/3} (\partial_y u)\partial_x f - \Lambda^{-2/3} (\partial_y v)\partial_y f,
\end{aligned}$$

which implies, by taking L^2 inner product with $-\partial_y^3 \Lambda^{-2/3} f$,

$$\begin{aligned}
&\|\partial_y^3 \Lambda^{-2/3} f\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} = -\mathrm{Re}\left(\partial_t \Lambda^{-2/3} \partial_y f,\ \partial_y^3 \Lambda^{-2/3} f\right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\quad - \mathrm{Re}\left(u\partial_x \Lambda^{-2/3} \partial_y f + v\partial_y \Lambda^{-2/3} \partial_y f,\ \partial_y^3 \Lambda^{-2/3} f\right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\quad + \mathrm{Re}\left(\Lambda^{-2/3} \partial_y h,\ \partial_y^3 \Lambda^{-2/3} f\right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} - \mathrm{Re}\left(\Lambda^{-2/3} (\partial_y v)\partial_y f,\ \partial_y^3 \Lambda^{-2/3} f\right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\quad + \mathrm{Re}\left(\left[u\partial_x + v\partial_y,\ \Lambda^{-2/3}\right] \partial_y f,\ \partial_y^3 \Lambda^{-2/3} f\right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\quad - \mathrm{Re}\left(\Lambda^{-2/3} (\partial_y u)\partial_x f,\ \partial_y^3 \Lambda^{-2/3} f\right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned}$$

Next we will treat the terms on the right hand side. Observing

$$\partial_t \Lambda^{-2/3} \partial_y f\big|_{y=0} = 0$$

due to (4.2.20), we integrate by part to compute

$$\begin{aligned}
& - \mathrm{Re}\ \left(\partial_t \Lambda^{-2/3} \partial_y f,\ -\partial_y^3 \Lambda^{-2/3} f\right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&= - \mathrm{Re}\ \left(\partial_t \partial_y^2 \Lambda^{-2/3} f,\ \partial_y^2 \Lambda^{-2/3} f\right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&= 0,
\end{aligned}$$

the last equality holding because

$$\partial_y^2 \Lambda^{-2/3} f\big|_{t=0} = \partial_y^2 \Lambda^{-2/3} f\big|_{t=T} = 0$$

due to (4.2.19). Since $u\big|_{y=0}$ then integrating by parts gives

$$
\begin{aligned}
& -\operatorname{Re}\left(u\partial_x\Lambda^{-2/3}\partial_y f,\ -\partial_y^3\Lambda^{-2/3}f\right)_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
= & -\operatorname{Re}\left(u\partial_x\Lambda^{-2/3}\partial_y^2 f,\ \partial_y^2\Lambda^{-2/3}f\right)_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
& -\operatorname{Re}\left((\partial_y u)\partial_x\Lambda^{-2/3}\partial_y f,\ \partial_y^2\Lambda^{-2/3}f\right)_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
= & \ \frac{1}{2}\left((\partial_x u)\Lambda^{-2/3}\partial_y^2 f,\ \Lambda^{-2/3}\partial_y^2 f\right)_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
& -\operatorname{Re}\left((\partial_y u)\partial_x\Lambda^{-2/3}\partial_y f,\ \partial_y^2\Lambda^{-2/3}f\right)_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
\le & \ \frac{1}{2}\|\partial_x u\|_{L^\infty}\|\Lambda^{-2/3}\partial_y^2 f\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
& +\|\Lambda^{-1/3}(\partial_y u)\partial_x\Lambda^{-2/3}\partial_y f\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+\|\partial_y^2\Lambda^{-1/3}f\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}.
\end{aligned}
$$

On the other hand, using Lemma 4.3.1 gives

$$
\begin{aligned}
& \|\Lambda^{-1/3}(\partial_y u)\partial_x\Lambda^{-2/3}\partial_y f\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
\le & \ 2\|\Lambda^{-1/3}\partial_x\Lambda^{-2/3}(\partial_y u)\partial_y f\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
& +2\|\Lambda^{-1/3}\left[\partial_y u,\ \partial_x\Lambda^{-2/3}\right]\partial_y f\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
\le & \ C\|\partial_y f\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
& -\operatorname{Re}\left(u\partial_x\Lambda^{-2/3}\partial_y f,\ -\partial_y^3\Lambda^{-2/3}f\right)_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
& \le C\left(\|\partial_y f\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+\|\partial_y^2\Lambda^{-1/3}f\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\right)\\
& \le \tilde{\varepsilon}\,\|\partial_y^3\Lambda^{-2/3}f\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+C_{\tilde{\varepsilon}}\|\partial_y f\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)},
\end{aligned}
$$

where the last inequality using (4.3.18). Using (4.3.18) we conclude

$$
\begin{aligned}
& -\operatorname{Re}\left(v\partial_y\Lambda^{-2/3}\partial_y f,\ -\partial_y^3\Lambda^{-2/3}f\right)_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
\le & \ \frac{\tilde{\varepsilon}}{2}\|\partial_y^3\Lambda^{-2/3}f\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+C_{\tilde{\varepsilon}}\|\partial_y^2\Lambda^{-1/3}f\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
\le & \ \tilde{\varepsilon}\,\|\partial_y^3\Lambda^{-2/3}f\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+C_{\tilde{\varepsilon}}\|\partial_y f\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}.
\end{aligned}
$$

Cauchy-Schwarz inequality gives, for any $\tilde{\varepsilon}>0$,

$$
\begin{aligned}
& \operatorname{Re}\left(\Lambda^{-2/3}\partial_y h,\ -\partial_y^3\Lambda^{-2/3}f\right)_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
& \le \tilde{\varepsilon}\|\partial_y^3\Lambda^{-2/3}f\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+\tilde{\varepsilon}^{-1}\|\Lambda^{-2/3}\partial_y h\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)},
\end{aligned}
$$

and

$$
\begin{aligned}
& -\ \mathrm{Re}\ \left(\Lambda^{-2/3}(\partial_y v)\partial_y f,\ -\partial_y^3\Lambda^{-2/3} f\right)_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \\
\leq\ & \tilde{\varepsilon}\big\|\partial_y^3\Lambda^{-2/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} + \tilde{\varepsilon}^{-1}\big\|\partial_y v\big\|^2_{L^\infty}\big\|\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathrm{Re}\ \left(\left[u\partial_x + v\partial_y,\ \Lambda^{-2/3}\right]\partial_y f,\ -\partial_y^3\Lambda^{-2/3} f\right)_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \\
\leq\ & \frac{\tilde{\varepsilon}}{2}\big\|\partial_y^3\Lambda^{-2/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} + 2\tilde{\varepsilon}^{-1}\big\|\left[u\partial_x + v\partial_y,\ \Lambda^{-2/3}\right]\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \\
\leq\ & \frac{\tilde{\varepsilon}}{2}\big\|\partial_y^3\Lambda^{-2/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} + C_{\tilde{\varepsilon}}\big\|\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \\
& + C_{\tilde{\varepsilon}}\big\|\partial_y^2\Lambda^{-1/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \\
\leq\ & \tilde{\varepsilon}\ \big\|\partial_y^3\Lambda^{-2/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} + C_{\tilde{\varepsilon}}\big\|\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)},
\end{aligned}
\tag{4.3.19}
$$

the second inequality using Lemma 4.3.1, while the last inequality following from (4.3.18). Finally,

$$
\begin{aligned}
& -\ \mathrm{Re}\ \left(\Lambda^{-2/3}(\partial_y u)\partial_x f,\ -\partial_y^3\Lambda^{-2/3} f\right)_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \\
\leq\ & \tilde{\varepsilon}\big\|\partial_y^3\Lambda^{-2/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} + \tilde{\varepsilon}^{-1}\big\|\Lambda^{-2/3}(\partial_y u)\partial_x f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \\
\leq\ & \tilde{\varepsilon}\big\|\partial_y^3\Lambda^{-2/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} + \tilde{\varepsilon}^{-1}\big\|(\partial_y u)\partial_x\Lambda^{-2/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \\
& + \tilde{\varepsilon}^{-1}\big\|\left[\partial_y u,\ \Lambda^{-2/3}\right]\partial_x f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \\
\leq\ & \tilde{\varepsilon}\big\|\partial_y^3\Lambda^{-2/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} + C_{\tilde{\varepsilon}}\big\|(\partial_y u)^{1/2}\partial_x\Lambda^{-2/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \\
& + C_{\tilde{\varepsilon}}\big\|f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}.
\end{aligned}
$$

This, along with (4.3.19) -(4.3.19), yields, for any $\tilde{\varepsilon} > 0$,

$$
\begin{aligned}
& \big\|\partial_y^3\Lambda^{-2/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \\
\leq\ & \tilde{\varepsilon}\big\|\partial_y^3\Lambda^{-2/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} + C_{\tilde{\varepsilon}}\big\|(\partial_y u)^{1/2}\partial_x\Lambda^{-2/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \\
& + C_{\tilde{\varepsilon}}\left(\big\|\Lambda^{-2/3}\partial_y h\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} + \big\|\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} + \big\|f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\right).
\end{aligned}
$$

Thus letting $\tilde{\varepsilon}$ be small enough, we have

$$
\begin{aligned}
& \big\|\partial_y^3\Lambda^{-2/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \\
\leq\ & C\big\|(\partial_y u)^{1/2}\partial_x\Lambda^{-2/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \\
& + C\left(\big\|\Lambda^{-2/3}\partial_y h\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} + \big\|f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} + \big\|\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\right).
\end{aligned}
\tag{4.3.20}
$$

This along with (4.3.17) yields the desired estimate (4.3.16).

Step 6) Now we combine (4.3.15) and (4.3.16) to conclude for any $0<\varepsilon,\varepsilon_1<1$,

$$\begin{aligned}
&\|\,(\partial_y u)^{1/2}\,\partial_x\Lambda^{-2/3}f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\|\partial_y^2\Lambda^{-1/3}f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ &\varepsilon_1 C\|\,(\partial_y u)^{1/2}\,\partial_x\Lambda^{-2/3}f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\varepsilon_1\varepsilon C\|\Lambda^{-2/3}\partial_y h\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+C_{\varepsilon_1,\varepsilon}\left(\|\,\langle y\rangle^{-\sigma/2}\,\partial_y\Lambda^{1/6}f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\|\,\langle y\rangle^{\sigma}\,\Lambda^{-1/3}\partial_y g\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\
&+C_{\varepsilon_1,\varepsilon}\left(\|\partial_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\|f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\|\Lambda^{-1/3}h\|^2_{L^2(\mathbb{R}^3_+)}\right),
\end{aligned}$$

which implies, choosing $\varepsilon_1>0$ sufficiently small,

$$\begin{aligned}
&\|\,(\partial_y u)^{1/2}\,\partial_x\Lambda^{-2/3}f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\|\partial_y^2\Lambda^{-1/3}f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ &\varepsilon\,\|\Lambda^{-2/3}\partial_y h\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+C_{\varepsilon}\left(\|\,\langle y\rangle^{-\sigma/2}\,\partial_y\Lambda^{1/6}f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\|\,\langle y\rangle^{\sigma}\,\Lambda^{-1/3}\partial_y g\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\
&+C_{\varepsilon}\left(\|\partial_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\|f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\|\Lambda^{-1/3}h\|^2_{L^2(\mathbb{R}^3_+)}\right),
\end{aligned}$$

with $\varepsilon>0$ arbitrarily small. This, along with

$$\begin{aligned}
\|\,\langle y\rangle^{-\sigma/2}\,\Lambda^{1/3}f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}&\leq C\|\,(\partial_y u)^{1/2}\,\Lambda^{1/3}f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\leq C\|\,\langle y\rangle^{-\sigma/2}\,\partial_x\Lambda^{-2/3}f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+C\|f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}
\end{aligned}$$

due to (4.1.2), implies, for any $\varepsilon>0$,

$$\begin{aligned}
&\|\,\langle y\rangle^{-\sigma/2}\,\Lambda^{1/3}f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\|\partial_y^2\Lambda^{-1/3}f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ &\varepsilon\|\Lambda^{-2/3}\partial_y h\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+C_{\varepsilon}\left(\|\,\langle y\rangle^{-\sigma/2}\,\partial_y\Lambda^{1/6}f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\|\,\langle y\rangle^{\sigma}\,\Lambda^{-1/3}\partial_y g\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\
&+C_{\varepsilon}\left(\|\partial_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\|f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\|\Lambda^{-1/3}h\|^2_{L^2(\mathbb{R}^3_+)}\right).
\end{aligned}$$

This is just the first estimate in Proposition 4.2.4. And the second estimate follows from (4.3.20) since $|\partial_y u|$ is bounded from above by $\langle y\rangle^{-\sigma}$. Thus the proof of Proposition 4.2.4 is complete. □

4.4 Property of inducative weight functions

This section is devoted to proving the Lemma 4.2.1, Lemma 4.2.2 and Lemma 4.2.5, used in Section 4.2.

Recall, for $m\geq N_0+1$ and $0\leq\ell\leq 3, y>0, 0\leq t\leq T<1$,

$$W_m^\ell=e^{2cy}\left(1+\frac{2cy}{(3m+\ell)\sigma}\right)^{-\frac{(3m+\ell)\sigma}{2}}(1+cy)^{-1}\Lambda^{\frac{\ell}{3}},\qquad \phi_m^\ell=\phi^{3(m-N_0-1)+\ell}.$$

thus

$$\phi_{m_1}^{\ell_1}\leq\phi_{m_2}^{\ell_2}\tag{4.4.1}$$

provided $N_0 + 1 \le m_2 \le m_1$ and $0 \le \ell_2 \le \ell_1 \le 3$.

Next we list some inequalities for the weight W_m^ℓ. Observe the function

$$\gamma \longrightarrow \left(1 + \frac{cy}{\gamma}\right)^{-\gamma}$$

is a monotonically decreasing function as γ varies in the interval $[1, +\infty[$ for $y \ge 0$. Thus

$$0 \le \ell \le 3, \quad \|W_{m_1}^\ell f\|_{L^2(\mathbb{R}_x)} \le \|W_{m_2}^\ell f\|_{L^2(\mathbb{R}_x)} \tag{4.4.2}$$

and

$$\forall\, 0 \le \ell \le i \le 3, \quad \|W_{m_1}^i f\|_{L^2(\mathbb{R}_x)} \le \|W_{m_2}^{i-\ell} \Lambda^{\ell/3} f\|_{L^2(\mathbb{R}_x)} \le \|W_{m_3}^i f\|_{L^2(\mathbb{R}_x)}, \tag{4.4.3}$$

provided that $m_1 \ge m_2 \ge 1$, and that $3m_2 + i - \ell \ge 3m_3 + i$. Moreover, since

$$\forall\, 0 \le \alpha \le 3,\ \forall\, \gamma \ge 1, \quad \left|\partial_y^\alpha e^{2cy} \left(1 + \frac{cy}{\gamma}\right)^{-\gamma} (1+cy)^{-1}\right| \le C_\alpha e^{2cy} \left(1 + \frac{cy}{\gamma}\right)^{-\gamma} (1+cy)^{-1},$$

with C_α a constant independent of γ, then the following estimates:

$$\|[\partial_y,\ W_m^i] f\|_{L^2(\mathbb{R}_+^2)} \le C \|W_m^i f\|_{L^2(\mathbb{R}_+^2)}, \tag{4.4.4}$$

$$\begin{aligned} \|[\partial_y^2,\ W_m^i] f\|_{L^2(\mathbb{R}_+^2)} &\le C \left(\|W_m^i f\|_{L^2(\mathbb{R}_+^2)} + \|W_m^i \partial_y f\|_{L^2(\mathbb{R}_+^2)}\right) \\ &\le \tilde{C} \left(\|W_m^i f\|_{L^2(\mathbb{R}_+^2)} + \|\partial_y W_m^i f\|_{L^2(\mathbb{R}_+^2)}\right) \end{aligned} \tag{4.4.5}$$

$$\begin{aligned} \|[\partial_y^3,\ W_m^i] f\|_{L^2(\mathbb{R}_+^2)} &\le C \left(\|W_m^i f\|_{L^2(\mathbb{R}_+^2)} + \|W_m^i \partial_y f\|_{L^2(\mathbb{R}_+^2)} + \|W_m^i \partial_y^2 f\|_{L^2(\mathbb{R}_+^2)}\right) \\ &\le \tilde{C} \left(\|W_m^i f\|_{L^2(\mathbb{R}_+^2)} + \|\partial_y W_m^i f\|_{L^2(\mathbb{R}_+^2)} + \|\partial_y^2 W_m^i f\|_{L^2(\mathbb{R}_+^2)}\right) \end{aligned} \tag{4.4.6}$$

hold for all integers m, i with $m \ge 1$ and $0 \le i \le 3$, where $C, \tilde{C}$ are two constants independent of m.

Lemma 4.4.1. *Under the assumption (4.1.2) and (4.1.3). Let c be the constant given in (4.2.2), and $\Lambda^{\tau_1}, \Lambda_\delta^{\tau_2}$ be the Fourier multiplier associate with the symbols $\langle \xi \rangle^{\tau_1}$ and $\langle \delta\xi \rangle^{\tau_2}$, respectively. Then there exists a constant C, such that for any $m, n \ge 1, 0 \le \ell \le 3$, and for any $0 < \tilde{c} < c$, we have*

$$\|e^{\tilde{c}y} \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \partial^m u\|_{L^2(\mathbb{R}_+^2)} \le C \|\Lambda^{\tau_1} \Lambda_\delta^{\tau_2} W_n^\ell f_m\|_{L^2(\mathbb{R}_+^2)}, \tag{4.4.7}$$

and

$$\|\Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \partial^m v\|_{L^\infty(\mathbb{R}_+;\ L^2(\mathbb{R}_x))} \le C \|\Lambda^{\tau_1} \Lambda_\delta^{\tau_2} W_n^\ell f_{m+1}\|_{L^2(\mathbb{R}_+^2)}. \tag{4.4.8}$$

Proof. In the proof we use C to denote different constants which are independent of m. Observe $\omega \in L^\infty$ and $\omega > 0$ then

$$\|e^{\tilde{c}y} \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \partial^m u\|_{L^2(\mathbb{R}_+^2)} \le C \|e^{\tilde{c}y} \Lambda^{\tau_1} \Lambda_\delta^{\tau_2} \frac{\partial^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)}$$

On the other hand, integrating by parts we have

$$
\begin{aligned}
\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial^m u}{\omega}\|^2_{L^2(\mathbb{R}^2_+)} &= \int_{\mathbb{R}}\int_0^\infty e^{2\tilde{c}y}\left(\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial^m u}{\omega}\right)\overline{\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial^m u}{\omega}}dydx\\
&= \frac{1}{2\tilde{c}}\int_{\mathbb{R}}\int_0^\infty (\partial_y e^{2\tilde{c}y})\left(\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial^m u}{\omega}\right)\overline{\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial^m u}{\omega}}dydx\\
&= -\frac{1}{2\tilde{c}}\int_{\mathbb{R}}\int_0^\infty e^{2\tilde{c}y}\left[\partial_y\left(\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial^m u}{\omega}\right)\right]\overline{\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial^m u}{\omega}}dydx\\
&\quad -\frac{1}{2\tilde{c}}\int_{\mathbb{R}}\int_0^\infty e^{2\tilde{c}y}\left(\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial^m u}{\omega}\right)\overline{\partial_y\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial^m u}{\omega}}dydx\\
&\le \frac{1}{\tilde{c}}\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial^m u}{\omega}\|_{L^2(\mathbb{R}^2_+)}\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_y\left(\frac{\partial^m u}{\omega}\right)\|_{L^2(\mathbb{R}^2_+)},
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial^m u}{\omega}\|_{L^2(\mathbb{R}^2_+)} &\le \|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_y\left(\frac{\partial^m u}{\omega}\right)\|_{L^2(\mathbb{R}^2_+)}\\
&= \|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}e^{\tilde{c}y}\omega^{-1}\omega\partial_y\left(\frac{\partial^m u}{\omega}\right)\|_{L^2(\mathbb{R}^2_+)}\\
&\le \|e^{\tilde{c}y}\omega^{-1}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\omega\partial_y\left(\frac{\partial^m u}{\omega}\right)\|_{L^2(\mathbb{R}^2_+)}\\
&\quad +\|\left[e^{\tilde{c}y}\omega^{-1},\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\right]\omega\partial_y\left(\frac{\partial^m u}{\omega}\right)\|_{L^2(\mathbb{R}^2_+)}.
\end{aligned}
$$

Thus we have, by the above inequalities,

$$
\begin{aligned}
\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial^m u\|_{L^2(\mathbb{R}^2_+)} &\le C\|e^{\tilde{c}y}\omega^{-1}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\omega\partial_y\left(\frac{\partial^m u}{\omega}\right)\|_{L^2(\mathbb{R}^2_+)}\\
&\quad +C\|\left[e^{\tilde{c}y}\omega^{-1},\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\right]\omega\partial_y\left(\frac{\partial^m u}{\omega}\right)\|_{L^2(\mathbb{R}^2_+)}.
\end{aligned}
$$

On the other hand, (4.1.2) and (4.1.3) enables us to use Lemma 4.3.1 to obtain

$$
\begin{aligned}
\|\left[e^{\tilde{c}y}\omega^{-1},\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\right]\omega\partial_y\left(\frac{\partial^m u}{\omega}\right)\|_{L^2(\mathbb{R}^2_+)} &\le C\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\omega\partial_y\left(\frac{\partial^m u}{\omega}\right)\|_{L^2(\mathbb{R}^2_+)}\\
&\le C\|e^{\tilde{c}y}\omega^{-1}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\omega\partial_y\left(\frac{\partial^m u}{\omega}\right)\|_{L^2(\mathbb{R}^2_+)}.
\end{aligned}
$$

As a result,

$$
\begin{aligned}
\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial^m u\|_{L^2(\mathbb{R}^2_+)} &\le C\|e^{\tilde{c}y}\omega^{-1}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\omega\partial_y\left(\frac{\partial^m u}{\omega}\right)\|_{L^2(\mathbb{R}^2_+)}\\
&\le C\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}W_n^\ell f_m\|_{L^2(\mathbb{R}^2_+)},
\end{aligned}
$$

the last inequality using the fact that $f_m = \omega\partial_y\left(\frac{\partial^m u}{\omega}\right)$, and that

$$
e^{\tilde{c}y}\omega^{-1} \le Ce^{\tilde{c}y}(1+y)^\sigma \le Ce^{2cy}\left(1+\frac{2cy}{\gamma}\right)^{-\gamma/2}
$$

for any $\gamma \geq 1$. This is just the desired (4.4.7). Now we prove (4.4.8). Recall $v(t,x,y) = -\int_0^y \partial u(t,x,y')dy'$. Then we have

$$\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial^m v = -\int_0^y \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial^{m+1}u(x,y')dy'$$

Therefore

$$\begin{aligned}\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial^m v\|_{L^\infty(\mathbb{R}_+;\, L^2(\mathbb{R}_x))} &\leq \big\|e^{-\tilde{c}y}\big\|_{L^2(\mathbb{R}_+)}\big\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_x^{m+1}u\big\|_{L^2(\mathbb{R}_+^2)}\\ &\leq C\big\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}W_n^\ell f_{m+1}\big\|_{L^2(\mathbb{R}_+^2)},\end{aligned}$$

the last inequality using (4.4.7). Thus the desired (4.4.8) follows and the proof of Lemma 4.4.1 is complete. □

We prove now Lemma 4.2.1, recall

$$f_m = \partial_x^m\omega - \frac{\partial_y\omega}{\omega}\partial_x^m u = \omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right).$$

Lemma 4.4.2. *There exists a constant C, such that*

$$\|\langle y\rangle^{-1}W_m^\ell\partial^m u\|_{L^2(\mathbb{R}_+^2)} + \|\langle y\rangle^{-1}W_m^\ell\partial^m\omega\|_{L^2(\mathbb{R}_+^2)} \leq C\|W_m^\ell f_m\|_{L^2(\mathbb{R}_+^2)}. \tag{4.4.9}$$

As a result, for some constant $\tilde{C}$,

$$\|\Lambda^{-1}W_m^0 f_{m+1}\|_{L^2(\mathbb{R}_+^2)} \leq \tilde{C}\|W_m^0 f_m\|_{L^2(\mathbb{R}_+^2)},$$

and

$$\|\Lambda^{-1}\partial_y W_m^0 f_{m+1}\|_{L^2(\mathbb{R}_+^2)} \leq \tilde{C}\left(\|\partial_y W_m^0 f_m\|_{L^2(\mathbb{R}_+^2)} + \|W_m^0 f_m\|_{L^2(\mathbb{R}_+^2)}\right).$$

Proof. In the proof we use C to denote different constants which depend only on σ, c, and C_* and are independent of m. We first prove (4.4.9). Observe

$$\begin{aligned}&\omega\langle y\rangle^{-1}\left(1+\frac{2cy}{(3m+\ell)\sigma}\right)^{-(3m+\ell)\sigma/2}(1+cy)^{-1}\\ &\leq C(1+y)^{-\sigma-1}\left(1+\frac{2cy}{(3m+\ell)\sigma}\right)^{-(3m+\ell)\sigma/2}\\ &\leq CR^{\sigma+1}(R+y)^{-\sigma-1}\left(1+\frac{2cy}{(3m+\ell)\sigma}\right)^{-(3m+\ell)\sigma/2},\end{aligned}$$

where $R \geq 1$ is a large number to be determined later. Thus using the notation

$$b_{m,\ell}^R(y) = \left(1+\frac{2cy}{(3m+\ell)\sigma}\right)^{-(3m+\ell)\sigma/2}(R+y)^{-\sigma-1},$$

we have

$$\begin{aligned}
&\|\langle y\rangle^{-1}W_m^\ell\partial^m u\|_{L^2(\mathbb{R}_+^2)}=\|\langle y\rangle^{-1}W_m^\ell(\omega\frac{\partial^m u}{\omega})\|_{L^2(\mathbb{R}_+^2)}\\
&\quad\le\|\omega\langle y\rangle^{-1}W_m^\ell\frac{\partial^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)}+\|\langle y\rangle^{-1}[W_m^\ell,\,\omega]\frac{\partial^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)}\\
&\quad\le CR^{\sigma+1}\|e^{2cy}b_{m,\ell}^R\frac{\Lambda^{\ell/3}\partial^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)}+\|\langle y\rangle^{-1}\left[W_m^\ell,\,\omega\right]\frac{\partial^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)}.
\end{aligned}$$

On the other hand, using Lemma 4.3.1

$$\begin{aligned}
\|\langle y\rangle^{-1}\left[W_m^\ell,\,\omega\right]\frac{\partial^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)}&\le R\|[\Lambda^{\frac{l}{3}},\,\omega]e^{2cy}b_{m,\ell}^R\frac{\partial^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)}\\
&\le CR\|e^{2cy}b_{m,\ell}^R\frac{\partial^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)}
\end{aligned}$$

Combining these inequalities we conclude

$$\|\langle y\rangle^{-1}W_m^\ell\partial^m u\|_{L^2(\mathbb{R}_+^2)}\le CR^{\sigma+1}\|e^{2cy}b_{m,\ell}^R\Lambda^{\ell/3}\frac{\partial^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)}. \tag{4.4.10}$$

Moreover, observe $u\big|_{y=0}=0$ and thus we have, by integrating by parts,

$$\begin{aligned}
&\|e^{2cy}b_{m,\ell}^R\Lambda^{\ell/3}\frac{\partial^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)}^2\\
&\quad=\int_{\mathbb{R}}\int_0^\infty e^{4cy}\left(b_{m,\ell}^R(y)\right)^2\left(\Lambda^{\ell/3}\frac{\partial^m u}{\omega}\right)\overline{\Lambda^{\ell/3}\frac{\partial^m u}{\omega}}dydx\\
&\quad=\frac{1}{4c}\int_{\mathbb{R}}\int_0^\infty\left(\partial_y e^{4cy}\right)\left(b_{m,\ell}^R(y)\right)^2\left(\Lambda^{\ell/3}\frac{\partial^m u}{\omega}\right)\overline{\Lambda^{\ell/3}\frac{\partial^m u}{\omega}}dydx\\
&\quad=-\frac{1}{2c}\int_{\mathbb{R}}\int_0^\infty e^{4cy}b_{m,\ell}^R(y)\left(\partial_y b_{m,\ell}^R(y)\right)\left(\Lambda^{\ell/3}\frac{\partial^m u}{\omega}\right)\overline{\Lambda^{\ell/3}\frac{\partial^m u}{\omega}}dydx\\
&\qquad-\frac{1}{4c}\int_{\mathbb{R}}\int_0^\infty e^{4cy}\left(b_{m,\ell}^R(y)\right)^2\left[\partial_y\left(\Lambda^{\ell/3}\frac{\partial^m u}{\omega}\right)\right]\overline{\Lambda^{\ell/3}\frac{\partial^m u}{\omega}}dydx\\
&\qquad-\frac{1}{4c}\int_{\mathbb{R}}\int_0^\infty e^{4cy}\left(b_{m,\ell}^R(y)\right)^2\left(\Lambda^{\ell/3}\frac{\partial^m u}{\omega}\right)\overline{\partial_y\Lambda^{\ell/3}\left(\frac{\partial^m u}{\omega}\right)}dydx,
\end{aligned}$$

which, along with the estimate

$$\left|\partial_y b_{m,\ell}^R\right|\le\left(c+(\sigma+1)R^{-1}\right)b_{m,\ell}^R,$$

gives

$$\begin{aligned}
&\|e^{2cy}b_{m,\ell}^R\Lambda^{\ell/3}\frac{\partial^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)}^2\\
&\qquad\le\frac{c+(\sigma+1)R^{-1}}{2c}\|e^{2cy}b_{m,\ell}^R\Lambda^{\ell/3}\frac{\partial^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)}^2\\
&\qquad+\frac{1}{2c}\|e^{2cy}b_{m,\ell}^R\Lambda^{\ell/3}\frac{\partial^m u}{\omega}\|_{L^2(\mathbb{R}_+^2)}\|e^{2cy}b_{m,\ell}^R\partial_y\left(\Lambda^{\ell/3}\frac{\partial^m u}{\omega}\right)\|_{L^2(\mathbb{R}_+^2)}.
\end{aligned}$$

Now we choose $R=1+2(\sigma+1)c^{-1}$, which gives $R\ge 1$ and

$$(\sigma+1)R^{-1}\le\frac{c}{2}.$$

Then we deduce, from the above inequalities,

$$\|e^{2cy}b^R_{m,\ell}\Lambda^{\ell/3}\frac{\partial^m u}{\omega}\|_{L^2(\mathbb{R}^2_+)} \leq \frac{2}{c}\|e^{2cy}b^R_{m,\ell}\partial_y\Lambda^{\ell/3}\left(\frac{\partial^m u}{\omega}\right)\|_{L^2(\mathbb{R}^2_+)}.$$

Moreover, observe $R \geq c^{-1}+1$ and the monotonicity assumption $\omega \geq C_*^{-1}(1+y)^{-\sigma}$, and thus

$$\begin{aligned} b^R_{m,\ell} &\leq c(1+y)^{-\sigma}(1+cy)^{-1}\left(1+\frac{2cy}{(3m+\ell)\sigma}\right)^{-(3m+\ell)\sigma/2} \\ &\leq cC_*\omega(1+cy)^{-1}\left(1+\frac{2cy}{(3m+\ell)\sigma}\right)^{-(3m+\ell)\sigma/2}. \end{aligned}$$

As a result, we obtain

$$\|e^{2cy}b^R_{m,\ell}\Lambda^{\ell/3}\frac{\partial^m u}{\omega}\|_{L^2(\mathbb{R}^2_+)} \leq C_*\|\omega W^\ell_m\partial_y\left(\frac{\partial^m u}{\omega}\right)\|_{L^2(\mathbb{R}^2_+)},$$

which along with (4.4.10) gives

$$\begin{aligned} &\|\langle y\rangle^{-1}W^\ell_m\partial^m u\|_{L^2(\mathbb{R}^2_+)} \\ \leq\ & C\|\omega W^\ell_m\partial_y\left(\frac{\partial^m u}{\omega}\right)\|_{L^2(\mathbb{R}^2_+)} \\ \leq\ & C\|W^\ell_m\omega\partial_y\left(\frac{\partial^m u}{\omega}\right)\|_{L^2(\mathbb{R}^2_+)} + C\|[\omega,\ W^\ell_m]\partial_y\left(\frac{\partial^m u}{\omega}\right)\|_{L^2(\mathbb{R}^2_+)}. \end{aligned}$$

Using the notation $\rho_{m,\ell}(y) = e^{2cy}\left(1+\frac{2cy}{(3m+\ell)\sigma}\right)^{-(3m+\ell)\sigma/2}(1+cy)^{-1}$,

$$\begin{aligned} \|[\omega,\ W^\ell_m]\partial_y\left(\frac{\partial^m u}{\omega}\right)\|_{L^2(\mathbb{R}^2_+)} &= \|[\omega,\ \Lambda^{\ell/3}]\rho_{m,\ell}(y)\partial_y\left(\frac{\partial^m u}{\omega}\right)\|_{L^2(\mathbb{R}^2_+)} \\ &= \|[\omega\langle y\rangle^{\sigma},\ \Lambda^{\ell/3}]\langle y\rangle^{-\sigma}\rho_{m,\ell}(y)\partial_y\left(\frac{\partial^m u}{\omega}\right)\|_{L^2(\mathbb{R}^2_+)} \\ &\leq C\|\langle y\rangle^{-\sigma}\rho_{m,\ell}(y)\partial_y\left(\frac{\partial^m u}{\omega}\right)\|_{L^2(\mathbb{R}^2_+)} \\ &\leq C\|\rho_{m,\ell}(y)\omega\partial_y\left(\frac{\partial^m u}{\omega}\right)\|_{L^2(\mathbb{R}^2_+)} \leq C\|W^\ell_m\omega\partial_y\left(\frac{\partial^m u}{\omega}\right)\|_{L^2(\mathbb{R}^2_+)}. \end{aligned}$$

Then, combining these inequalities we conclude,

$$\|\langle y\rangle^{-1}W^\ell_m\partial^m u\|_{L^2(\mathbb{R}^2_+)} \leq C\|W^\ell_m\omega\partial_y\left(\frac{\partial^m u}{\omega}\right)\|_{L^2(\mathbb{R}^2_+)} = C\|W^\ell_m f_m\|_{L^2(\mathbb{R}^2_+)}.$$

For the other terms in (4.4.9), we have

$$\begin{aligned} &\|\langle y\rangle^{-1}W^\ell_m\partial^m\omega\|_{L^2(\mathbb{R}^2_+)} \\ &\leq \|\langle y\rangle^{-1}W^\ell_m f_m\|_{L^2(\mathbb{R}^2_+)} + \|((\partial_y\omega)/\omega)\langle y\rangle^{-1}W^\ell_m\partial^m u\|_{L^2(\mathbb{R}^2_+)} \\ &\quad + \|[(\partial_y\omega)/\omega,\ W^\ell_m]\langle y\rangle^{-1}\partial^m u\|_{L^2(\mathbb{R}^2_+)} \\ &\leq \|\langle y\rangle^{-1}W^\ell_m f_m\|_{L^2(\mathbb{R}^2_+)} + C\|\langle y\rangle^{-1}W^\ell_m\partial^m u\|_{L^2(\mathbb{R}^2_+)} \\ &\leq C\|W^\ell_m f_m\|_{L^2(\mathbb{R}^2_+)}, \end{aligned}$$

Thus the desired estimate (4.4.9) follows. As a result, we have

$$\begin{aligned}\|\Lambda^{-1}W_m^0 f_{m+1}\|_{L^2(\mathbb{R}^2_+)} \leq & \|\Lambda^{-1}W_m^0\partial_x f_m\|_{L^2(\mathbb{R}^2_+)} + \|\Lambda^{-1}W_m^0\Big[\partial_x\big((\partial_y\omega)/\omega\big)\Big]\partial_x^m u\|_{L^2(\mathbb{R}^2_+)} \\ \leq & \|W_m^0 f_m\|_{L^2(\mathbb{R}^2_+)} + \|\left\langle y\right\rangle^{-1} W_m^0\partial_x^m u\|_{L^2(\mathbb{R}^2_+)} \leq C\|W_m^0 f_m\|_{L^2(\mathbb{R}^2_+)}.\end{aligned}$$

Similarly, we can deduce that, using (4.4.4),

$$\|\Lambda^{-1}\partial_y W_m^0 f_{m+1}\|_{L^2(\mathbb{R}^2_+)} \leq C\left(\|\partial_y W_m^0 f_m\|_{L^2(\mathbb{R}^2_+)} + \|W_m^0 f_m\|_{L^2(\mathbb{R}^2_+)}\right).$$

Thus the proof of Lemma 4.4.2 is complete. □

We prove now the Lemma 4.2.2 by the following 2 lemmas .

Lemma 4.4.3. *There exists a constant C such that, for any $m \geq 1$ and $1 \leq \ell \leq 3$,*

$$\|\left\langle y\right\rangle^{-\sigma/2}\partial_y\Lambda^{1/3}\Lambda_\delta^{-2}W_m^{\ell-1}f_m\|_{L^2(\mathbb{R}^2_+)} \leq C\|\partial_y\Lambda_\delta^{-2}W_m^\ell f_m\|_{L^2(\mathbb{R}^2_+)} + C\|\Lambda_\delta^{-2}W_m^\ell f_m\|_{L^2(\mathbb{R}^2_+)}.$$

Proof. We can write

$$\begin{aligned}\Lambda^{1/3}\Lambda_\delta^{-2}W_m^{\ell-1} &= e^{2cy}\left(1+\frac{2cy}{(3m+\ell-1)\sigma}\right)^{-\frac{(3m+\ell-1)\sigma}{2}}(1+cy)^{-1}\Lambda^{\frac{\ell}{3}}\Lambda_\delta^{-2} \\ &= a_{m,\ell}(y)\Lambda_\delta^{-2}W_m^\ell,\end{aligned}$$

where

$$a_{m,\ell}(y) = \left(1+\frac{2cy}{(3m+\ell-1)\sigma}\right)^{-\frac{(3m+\ell-1)\sigma}{2}}\left(1+\frac{2cy}{(3m+\ell)\sigma}\right)^{\frac{(3m+\ell)\sigma}{2}}.$$

Direct computation gives

$$\begin{aligned}&|a_{m,\ell}(y)| \\ &= \left(1+\frac{2cy}{(3m+\ell-1)\sigma}\right)^{\sigma/2}\left(1+\frac{2cy}{(3m+\ell-1)\sigma}\right)^{-\frac{(3m+\ell)\sigma}{2}}\left(1+\frac{2cy}{(3m+\ell)\sigma}\right)^{\frac{(3m+\ell)\sigma}{2}} \\ &\leq \left(1+\frac{2cy}{(3m+\ell-1)\sigma}\right)^{\sigma/2} \leq C\left\langle y\right\rangle^{\sigma/2}.\end{aligned}$$

Moreover observe $|\partial_y a_{m,\ell}(y)| \leq 2c\,|a_{m,\ell}(y)|$, and thus

$$|\partial_y a_{m,\ell}(y)| \leq C\left\langle y\right\rangle^{\sigma/2}.$$

As a result,

$$\begin{aligned}&\|\left\langle y\right\rangle^{-\sigma/2}\partial_y\Lambda^{1/3}\Lambda_\delta^{-2}W_m^{\ell-1}f_m\|_{L^2(\mathbb{R}^2_+)} = \|\left\langle y\right\rangle^{-\sigma/2}\partial_y\left(a_{m,\ell}\Lambda_\delta^{-2}W_m^\ell f_m\right)\|_{L^2(\mathbb{R}^2_+)} \\ \leq\ & \|\left\langle y\right\rangle^{-\sigma/2}a_{m,\ell}\partial_y\Lambda_\delta^{-2}W_m^\ell f_m\|_{L^2(\mathbb{R}^2_+)} + \|\left\langle y\right\rangle^{-\sigma/2}(\partial_y a_{m,\ell})\Lambda_\delta^{-2}W_m^\ell f_m\|_{L^2(\mathbb{R}^2_+)} \\ \leq\ & C\Big(\|\partial_y\Lambda_\delta^{-2}W_m^\ell f_m\|_{L^2(\mathbb{R}^2_+)} + \|\Lambda_\delta^{-2}W_m^\ell f_m\|_{L^2(\mathbb{R}^2_+)}\Big).\end{aligned}$$

The proof of Lemma 4.4.3 is thus complete. □

Lemma 4.4.4. *There exists a constant C, depending only on σ, c, and C_* , such that for any integers $m \geq N_0 + 1$, we have*

$$\|\phi^0_{m+1} W^0_{m+1} f_{m+1}\|_{L^\infty([0,T];\, L^2(\mathbb{R}^2_+))} + \sum_{j=1}^{2} \big\|\partial^j_y \Lambda^{-\frac{2(j-1)}{3}} \phi^0_{m+1} W^0_{m+1} f_{m+1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}$$
$$\leq C\big\|\phi^3_m W^3_m f_m\big\|_{L^\infty([0,T];\, L^2(\mathbb{R}^2_+))} + C\sum_{j=1}^{3} \big\|\partial^j_y \Lambda^{-\frac{2(j-1)}{3}} \phi^3_m W^3_m f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)},$$

and

$$\begin{aligned} &\big\|\partial^3_y \Lambda^{-1} \phi^0_{m+1} W^0_{m+1} f_{m+1}\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\ \leq\ & C\big\|\phi^3_m W^3_m f_m\big\|_{L^\infty([0,T];\, L^2(\mathbb{R}^2_+))} + C\sum_{j=1}^{2} \big\|\partial^j_y \Lambda^{-\frac{2(j-1)}{3}} \phi^3_m W^3_m f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\ & + C\big\|\partial^3_y \Lambda^{-1} \phi^3_m W^3_m f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}. \end{aligned}$$

Proof. In the proof we use C to denote different constants which are independent of m. In view of the definition (4.2.1) of f_m, we have, observing (4.4.1),

$$\begin{aligned} &\big\|\phi^0_{m+1} W^0_{m+1} f_{m+1}\big\|_{L^\infty([0,T];\, L^2(\mathbb{R}^2_+))} \\ \leq\ & \big\|\phi^3_m W^0_{m+1} \Lambda^1 f_m\big\|_{L^\infty([0,T];\, L^2(\mathbb{R}^2_+))} + C\big\| \langle y\rangle^{-1} \phi^3_m W^3_m \partial^m_x u\big\|_{L^\infty([0,T];\, L^2(\mathbb{R}^2_+))} \\ \leq\ & C\big\|\phi^3_m W^3_m f_m\big\|_{L^\infty([0,T];\, L^2(\mathbb{R}^2_+))}, \end{aligned}$$

the last inequality using (4.4.9) and (4.4.3). Similarly, using (4.4.4), we can deduce that

$$\begin{aligned} \big\|\partial_y \phi^0_{m+1} W^0_{m+1} f_{m+1}\big\|_{L^\infty([0,T];\, L^2(\mathbb{R}^2_+))} \leq & C\big\|\partial_y \phi^3_m W^3_m f_m\big\|_{L^\infty([0,T];\, L^2(\mathbb{R}^2_+))} \\ & + C\big\|\phi^3_m W^3_m f_m\big\|_{L^\infty([0,T];\, L^2(\mathbb{R}^2_+))}. \end{aligned}$$

The other terms

$$\big\|\partial^2_y \Lambda^{-2/3} \phi^0_{m+1} W^0_{m+1} f_{m+1}\big\|_{L^\infty([0,T];\, L^2(\mathbb{R}^2_+))}, \quad \big\|\partial^3_y \Lambda^{-1} \phi^0_{m+1} W^0_{m+1} f_{m+1}\big\|_{L^\infty([0,T];\, L^2(\mathbb{R}^2_+))}$$

can treated in the same way, thanks to (4.4.5) and (4.4.6). So we omit it here. Thus the proof of Lemma 4.4.4 is complete. □

Proof of Lemma 4.2.5. Observe

$$\begin{aligned} (1+y)^{-\frac{\sigma}{2}} &= \left(\frac{(3m+\ell-1)\sigma}{2c}\right)^{-\frac{\sigma}{2}} \left(\frac{2c}{(3m+\ell-1)\sigma} + \frac{2cy}{(3m+\ell-1)\sigma}\right)^{-\frac{\sigma}{2}} \\ &\geq C\left(\frac{(3m+\ell-1)\sigma}{2c}\right)^{-\frac{\sigma}{2}} \left(1 + \frac{2cy}{(3m+\ell-1)\sigma}\right)^{-\frac{\sigma}{2}} \\ &\geq Cm^{-\frac{\sigma}{2}} \left(1 + \frac{2cy}{(3m+\ell-1)\sigma}\right)^{-\frac{\sigma}{2}}. \end{aligned}$$

Then

$$(1+y)^{-\frac{\sigma}{2}}\left(1+\frac{2cy}{(3m+\ell-1)\sigma}\right)^{-\frac{(3m+\ell-1)\sigma}{2}} \geq Cm^{-\frac{\sigma}{2}}\left(1+\frac{2cy}{(3m+\ell-1)\sigma}\right)^{-\frac{(3m+\ell)\sigma}{2}} \quad (4.4.11)$$

Moreover we find

$$\begin{aligned}
&\left(1+\frac{2cy}{(3m+\ell-1)\sigma}\right)^{-\frac{(3m+\ell)\sigma}{2}}\\
&=\left(\frac{(3m+\ell)\sigma}{(3m+\ell-1)\sigma}\right)^{-\frac{(3m+\ell)\sigma}{2}}\left(\frac{(3m+\ell-1)}{(3m+\ell)}+\frac{2cy}{(3m+\ell)\sigma}\right)^{-\frac{(3m+\ell)\sigma}{2}}\\
&\geq\left(\frac{3m+\ell}{3m+\ell-1}\right)^{-\frac{(3m+\ell)\sigma}{2}}\left(1+\frac{2cy}{(3m+\ell)\sigma}\right)^{-\frac{(3m+\ell)\sigma}{2}}\\
&\geq C\left(1+\frac{2cy}{(3m+\ell)\sigma}\right)^{-\frac{(3m+\ell)\sigma}{2}},
\end{aligned}$$

which along with (4.4.11) gives

$$\left(1+\frac{2cy}{(3m+\ell)\sigma}\right)^{-\frac{(3m+\ell)\sigma}{2}} \leq Cm^{\frac{\sigma}{2}}(1+y)^{-\frac{\sigma}{2}}\left(1+\frac{2cy}{(3m+\ell-1)\sigma}\right)^{-\frac{(3m+\ell-1)\sigma}{2}}.$$

As a result, recalling

$$(1+y)^{-\frac{\sigma}{2}}\Lambda^{1/3}W_m^{\ell-1}=(1+y)^{-\frac{\sigma}{2}}e^{2cy}\left(1+\frac{2cy}{(3m+\ell-1)\sigma}\right)^{-\frac{(3m+\ell-1)\sigma}{2}}(1+cy)^{-1}\Lambda^{\frac{\ell}{3}},$$

we have, observing $\phi^{-\frac{1}{2}}\phi_m^\ell=\phi^{\frac{1}{2}}\phi_m^{\ell-1}$

$$\begin{aligned}
&\|\phi^{-\frac{1}{2}}\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\qquad\leq Cm^{\sigma/2}\left\|(1+y)^{-\frac{\sigma}{2}}\Lambda^{1/3}\phi^{\frac{1}{2}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2([0,T]\times\mathbb{R}_+^2)},
\end{aligned}$$

that is, recalling $F=\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m$ and $f=\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m$,

$$\|\phi^{-1/2}F\|_{L^2([0,T]\times\mathbb{R}_+^2)}\leq Cm^{\sigma/2}\left\|\langle y\rangle^{-\frac{\sigma}{2}}\Lambda^{1/3}f\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}.$$

Moreover, using (4.4.3) and (4.4.5) we have, observing $\phi_m^\ell\leq\phi^{1/2}\phi_m^{\ell-1}$,

$$\begin{aligned}
&\|\partial_y^2\Lambda^{-2/3}F\|_{L^2([0,T]\times\mathbb{R}_+^2)}=\left\|\partial_y^2\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\leq C\Big(\left\|\partial_y^2\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_m^\ell W_m^{\ell-1}f_m\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\quad+\|\partial_y\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_m^\ell W_m^{\ell-1}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\quad+\left\|\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_m^\ell W_m^{\ell-1}f_m\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}\Big)\\
&\leq C\left(\|\phi_m^{\ell-1}W_m^{\ell-1}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}+\left\|\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}\right)\\
&\quad+C\left\|\partial_y^2\Lambda^{-1/3}f\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}.
\end{aligned}$$

Then combining the above inequalities, the first estimate in Lemma 4.2.5 follows. The second one can be deduced similarly. In fact using (4.4.3) and (4.4.6) gives

$$\begin{aligned}
&\|\partial_y^3\Lambda^{-1}F\|_{L^2([0,T]\times\mathbb{R}^2_+)} = \|\partial_y^3\Lambda^{-1}\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\quad\leq C\big\|\partial_y^3\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_m^\ell W_m^{\ell-1} f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} + C\big\|\partial_y^2\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_m^\ell W_m^{\ell-1} f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\qquad + C\big\|\partial_y\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_m^{\ell-1} W_m^{\ell-1} f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} + C\big\|\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_m^{\ell-1} W_m^{\ell-1} f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\quad\leq C\big\|\partial_y^3\Lambda^{-2/3} f\big\|_{L^2([0,T]\times\mathbb{R}^2_+)} + C\big\|\partial_y^2\Lambda^{-1/3} f\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\qquad + C\left(\|\phi_m^{\ell-1} W_m^{\ell-1} f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \big\|\partial_y\phi_m^{\ell-1} W_m^{\ell-1} f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right).
\end{aligned}$$

This is just the second estimate in Lemma 4.2.5. The proof is thus complete. □

4.5 Estimates of the nonlinear terms

In this section we estimate the nonlinear terms $\mathcal{Z}_{m,\ell,\delta}$ defined in (4.2.13), and prove the Proposition 4.2.3. Recall

$$\begin{aligned}
\mathcal{Z}_{m,\ell,\delta} = &- \sum_{j=1}^{m}\binom{m}{j}\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial^j u) f_{m+1-j} - \sum_{j=1}^{m-1}\binom{m}{j}\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial^j v)\partial_y f_{m-j}\\
&- \Lambda_\delta^{-2}\phi_m^\ell W_m^\ell\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\sum_{j=1}^{m-1}\binom{m}{j}(\partial^j v)(\partial^{m-j}u)\\
&- 2\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right] f_m\\
&+ \Lambda_\delta^{-2}\left(\partial_t\phi_m^\ell\right) W_m^\ell f_m + \left[u\partial_x + v\partial_y - \partial_y^2, \Lambda_\delta^{-2}\phi_m^\ell W_m^\ell\right] f_m\\
=&\mathcal{J}_{m,\ell,\delta} + \Lambda_\delta^{-2}\left(\partial_t\phi_m^\ell\right) W_m^\ell f_m + \left[u\partial_x + v\partial_y - \partial_y^2, \Lambda_\delta^{-2}\phi_m^\ell W_m^\ell\right] f_m,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{J}_{m,\ell,\delta} \;=\; &-\sum_{j=1}^{m}\binom{m}{j}\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial^j u) f_{m+1-j} - \sum_{j=1}^{m-1}\binom{m}{j}\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial^j v)\partial_y f_{m-j}\\
&-\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\sum_{j=1}^{m-1}\binom{m}{j}(\partial^j v)(\partial^{m-j}u)\\
&-2\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right] f_m.
\end{aligned}$$

We remark it is suffices to prove the estimates (4.2.15) and (4.2.16) in Proposition 4.2.3, since the esimate (4.2.15) can be treated exactly similar as (4.2.15). Next we will proceed to prove (4.2.15) and (4.2.16) through the following Proposition 4.5.1 and Proposition 4.5.2. Proposition 4.5.2 is devoted to treating the term $\mathcal{J}_{m,\ell,\delta}$ in the definition of $\mathcal{Z}_{m,\ell,\delta}$, while the the other two terms are estimated in Proposition 4.5.1.

To simplify the notations, we will use C to denote different constants depending only on σ, c, and the constants C_0, C_* in Theorem 4.1.1, but independent of m and δ.

Proposition 4.5.1. *We have, denoting* $F = \Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m$ *and* $\tilde f = \phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1} f_m$,

$$
\begin{aligned}
&\|\phi^{1/2}\Lambda_\delta^{-2}\left(\partial_t\phi_m^\ell\right) W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\quad+\big\|\phi^{1/2}[u\partial_x+v\partial_y-\partial_y^2,\ \Lambda_\delta^{-2}\phi_m^\ell W_m^\ell] f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\qquad\le m\,C\big\|\phi^{-1/2}F\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\partial_y F\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\big\|\Lambda^{-\frac23}\partial_y\phi^{1/2}\Lambda_\delta^{-2}\left(\partial_t\phi_m^{\ell-1}\right)W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\quad+\big\|\Lambda^{-\frac23}\partial_y\phi^{1/2}[u\partial_x+v\partial_y-\partial_y^2,\ \Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}]f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ & C\big\|\left\langle y\right\rangle^{-\sigma}\Lambda^{\frac13}\tilde f\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\partial_y^2\Lambda^{-\frac23}\tilde f\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+mC\big\|\Lambda^{-\frac23}\phi^{-1/2}\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-\frac23}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}
$$

Proof. It is sufficient to prove the second estimate in Proposition 4.5.1, since the treatment of the first one is similar and easier and we omit it here for brevity. Observe

$$
\left|\partial_t\phi_m^{\ell-1}\right|\le 3m\phi_m^{\ell-2}\le 3m\phi_m^{\ell-1}\phi^{-1},
$$

and thus

$$
\begin{aligned}
&\|\Lambda^{-\frac23}\partial_y\phi^{1/2}\Lambda_\delta^{-2}\left(\partial_t\phi_m^{\ell-1}\right)W_m^{\ell-1}f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\qquad\le 3m\big\|\Lambda^{-\frac23}\phi^{-1/2}\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}
\tag{4.5.1}
$$

We write, using (4.3.1),

$$
\begin{aligned}
&\big\|\Lambda^{-\frac23}\partial_y\phi^{1/2}[u\partial_x+v\partial_y-\partial_y^2,\ \Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}]f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ &\big\|\left[u\partial+v\partial_y-\partial_y^2,\ \Lambda^{-\frac23}\partial_y\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\right]f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+\big\|\left[u\partial+v\partial_y-\partial_y^2,\ \Lambda^{-\frac23}\partial_y\phi^{1/2}\right]\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\stackrel{\text{def}}{=}\ &Q_{5,1}+Q_{5,2}.
\end{aligned}
$$

We first estimate $Q_{5,1}$. Observe

$$
\begin{aligned}
&\big\|\left[u\partial,\ \Lambda^{-\frac23}\partial_y\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\right]f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\ &\big\|\left[u\partial,\ \Lambda^{-\frac23}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\partial_y\right]\phi^{1/2}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+\big\|\left[u\partial,\ \Lambda^{-\frac23}\Lambda_\delta^{-2}\phi_m^{\ell-1}\left[\partial_y,\ W_m^{\ell-1}\right]\right]\phi^{1/2}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}
$$

On the other hand, we compute, using Lemma 4.3.1 and (4.4.4),

$$
\begin{aligned}
&\big\|\big[u\partial,\ \Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\partial_y\big]\phi^{1/2}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ &\big\|\big[u\partial,\ \Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\big]\partial_y\phi^{1/2}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\big[u\partial,\ \partial_y\big]\phi^{1/2}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ &C\big\|\partial_y\Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\phi^{1/2}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\phi^{1/2}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+\big\|(\partial_y u)\Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\partial_x\phi^{1/2}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+\big\|\big[\Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1},\ (\partial_y u)\big]\partial_x\phi^{1/2}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ &C\big\|\partial_y\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+C\big\|\left\langle y\right\rangle^{-\sigma}\Lambda^{\frac{1}{3}}\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}
$$

Similarly we also have, using again Lemma 4.3.1,

$$
\begin{aligned}
&\Big\|\Big[u\partial,\ \Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}\big[\partial_y,\ W_m^{\ell-1}\big]\Big]\phi^{1/2}f_m\Big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ &C\big\|\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}
$$

As a result, combining these inequalities, we have

$$
\begin{aligned}
&\big\|\big[u\partial,\ \Lambda^{-\frac{2}{3}}\partial_y\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\big]f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ &C\big\|\partial_y\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+C\big\|\left\langle y\right\rangle^{-\sigma}\Lambda^{\frac{1}{3}}\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}
$$

Similarly, repeating the above arguments with $u\partial_x$ replaced by $v\partial_y$ and ∂_y^2 respectively, one has

$$
\begin{aligned}
&\big\|\big[v\partial_y,\ \Lambda^{-\frac{2}{3}}\partial_y\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\big]f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ &C\big\|\partial_y^2\Lambda^{-\frac{2}{3}}\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-\frac{2}{3}}\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+C\big\|\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)},
\end{aligned}
$$

and

$$
\begin{aligned}
&\big\|\big[\partial_y^2,\ \Lambda^{-\frac{2}{3}}\partial_y\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\big]f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ &C\big\|\partial_y^2\Lambda^{-\frac{2}{3}}\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C\big\|\Lambda^{-\frac{2}{3}}\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+C\big\|\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}
$$

As a result, we conclude, combining these inequalities,

$$\begin{aligned}
Q_{5,1} &= \big\|\big[u\partial+v\partial_y-\partial_y^2,\ \Lambda^{-\frac{2}{3}}\partial_y\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\big]f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\le C\big\|\left\langle y\right\rangle^{-\sigma}\Lambda^{\frac{1}{3}}\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\quad+C\big\|\partial_y^2\Lambda^{-\frac{2}{3}}\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\quad+C\big\|\partial_y\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}+C\big\|\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}.
\end{aligned}$$

The term $Q_{5,2}$ can be treated similarly and easily, and we have

$$\begin{aligned}
Q_{5,2} &= \big\|\big[u\partial+v\partial_y-\partial_y^2,\ \Lambda^{-\frac{2}{3}}\partial_y\phi^{1/2}\big]\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\le C\big\|\left\langle y\right\rangle^{-\sigma}\Lambda^{\frac{1}{3}}\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\quad+C\big\|\partial_y^2\Lambda^{-\frac{2}{3}}\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\quad+C\big\|\partial_y\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}+C\big\|\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\big\|\Lambda^{-\frac{2}{3}}\partial_y\phi^{1/2}[u\partial_x+v\partial_y-\partial_y^2,\ \Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}]f_m\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
\le\ & C\big\|\left\langle y\right\rangle^{-\sigma}\Lambda^{\frac{1}{3}}\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&+C\big\|\partial_y^2\Lambda^{-\frac{2}{3}}\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&+C\big\|\partial_y\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}+C\big\|\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}.
\end{aligned}$$

This along with (4.5.1) gives the second estimate in Proposition 4.5.1. The proof is thus complete. □

Proposition 4.5.2. *Under the induction hypothesis (4.2.9), (4.2.10), we have, denoting $F=\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m$,*

$$\|\phi^{1/2}\mathcal{J}_{m,\ell,\delta}\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\le mC\|F\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+CA^{m-6}\left((m-5)!\right)^{3(1+\sigma)},$$

and

$$\begin{aligned}
&\|\Lambda^{-2/3}\partial_y\phi^{1/2}\mathcal{J}_{m,\ell-1,\delta}\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\le mC\Big(\|\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\quad+\|\Lambda^{-2/3}\partial_y\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\Big)\\
&\quad+CA^{m-6}\left((m-5)!\right)^{3(1+\sigma)},
\end{aligned}$$

where the constant $C>0$ is independent on m and $\delta>0$.

We first prove the first estimate in Proposition 4.5.2. In view of the definition given at the beginning of this section, we see,

$$
\begin{aligned}
&\|\phi^{1/2}\mathcal{J}_{m,\ell,\delta}\|_{L^2([0,T]\times\mathbb{R}^2_+)} \le \|\mathcal{J}_{m,\ell,\delta}\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le&\sum_{j=1}^{m}\binom{m}{j}\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial^j u)f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+\sum_{j=1}^{m-1}\binom{m}{j}\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial^j v)\partial_y f_{m-j}\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+\sum_{j=1}^{m-1}\binom{m}{j}\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell\left[\partial_y\left(\partial_y\omega/\omega\right)\right](\partial^j v)(\partial^{m-j}u)\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+2\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell\left[\partial_y\left(\partial_y\omega/\omega\right)\right]f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}
\tag{4.5.2}
$$

And we will proceed to estimate the each term on the right hand side of (4.5.2), and state as the following three Lemmas.

Lemma 4.5.3. *Under the same assumption as in Proposition 4.2.3, we have*

$$
\begin{aligned}
&\sum_{j=1}^{m-1}\binom{m}{j}\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial^j v)\partial_y f_{m-j}\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\quad & mC\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)}+CA^{m-6}\left((m-5)!\right)^{3(1+\sigma)}.
\end{aligned}
$$

Proof. We first split the summation as follows:

$$
\begin{aligned}
&\sum_{j=1}^{m-1}\binom{m}{j}\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial^j v)\partial_y f_{m-j}\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
=\quad&\sum_{j=m-2}^{m-1}\binom{m}{j}\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial^j v)\partial_y f_{m-j}\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+\sum_{j=1}^{m-3}\binom{m}{j}\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial^j v)\partial_y f_{m-j}\|_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}
$$

Moreover as for the last term on the right hand side, we use (4.4.3) to compute,

$$
\begin{aligned}
&\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial^j v)\partial_y f_{m-j}\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\quad&\|\phi_m^\ell W_m^0\Lambda^{\ell/3}(\partial^j v)\partial_y f_{m-j}\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\quad&\|\phi_m^\ell W_m^0(\partial^j v)\partial_y f_{m-j}\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\|\phi_m^\ell W_m^0\partial_x(\partial^j v)\partial_y f_{m-j}\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\le\quad&\|\phi_m^\ell W_m^0(\partial^j v)\partial_y f_{m-j}\|_{L^2([0,T]\times\mathbb{R}^2_+)}+\|\phi_m^\ell W_m^0(\partial^{j+1} v)\partial_y f_{m-j}\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+\|\phi_m^\ell W_m^0(\partial^j v)(\partial_y\partial_x f_{m-j})\|_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\sum_{j=1}^{m-1}\binom{m}{j}\big\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
\leq&\sum_{j=m-2}^{m-1}\binom{m}{j}\big\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&+\sum_{j=1}^{m-3}\binom{m}{j}\big\|\phi_m^\ell W_m^0(\partial^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&+\sum_{j=1}^{m-3}\binom{m}{j}\big\|\phi_m^\ell W_m^0(\partial^{j+1} v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&+\sum_{j=1}^{m-3}\binom{m}{j}\big\|\phi_m^\ell W_m^0(\partial^j v)(\partial_y\partial_x f_{m-j})\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}.
\end{aligned}\tag{4.5.3}
$$

Next we estimate step by step the terms on the right side of (4.5.3).

(a) We treat in this step the first term on the right hand side of (4.5.3), and prove that

$$
\begin{aligned}
&\sum_{j=m-2}^{m-1}\binom{m}{j}\big\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
\leq\;& m\,C\big\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}+CA^{m-6}\left((m-5)!\right)^{3(1+\sigma)}.
\end{aligned}\tag{4.5.4}
$$

To do so, direct computation gives

$$
\begin{aligned}
&\sum_{j=m-2}^{m-1}\binom{m}{j}\big\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
\leq&\sum_{j=m-2}^{m-1}\binom{m}{j}\big\|\Lambda_\delta^{-2}\Lambda^{\ell/3}e^{2cy}\left(1+cy\right)^{-1}\phi_m^\ell(\partial^j v)\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
\leq&\sum_{j=m-2}^{m-1}\binom{m}{j}\big\|e^{2cy}(\partial_y f_{m-j})\Lambda_\delta^{-2}\Lambda^{\ell/3}\left(1+cy\right)^{-1}\phi_m^\ell\partial^j v\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&+\sum_{j=m-2}^{m-1}\binom{m}{j}\big\|\left[e^{2cy}(\partial_y f_{m-j}),\ \Lambda_\delta^{-2}\Lambda^{\ell/3}\right]\left(1+cy\right)^{-1}\phi_m^\ell\partial^j v\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}.
\end{aligned}
$$

On the other hand, by (4.2.7),

$$
\begin{aligned}
&\sum_{j=m-2}^{m-1}\binom{m}{j}\big\|e^{2cy}(\partial_y f_{m-j})\Lambda_\delta^{-2}\Lambda^{\ell/3}\left(1+cy\right)^{-1}\phi_m^\ell\partial^j v\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\leq C\sum_{j=m-2}^{m-1}\binom{m}{j}\big\|(1+cy)^{-1}\big\|_{L^2(\mathbb{R}_+;\ L^\infty([0,T]\times\mathbb{R}_x))}\big\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial^j v\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&\leq C\sum_{j=m-2}^{m-1}\binom{m}{j}\big\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial^j v\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}.
\end{aligned}
$$

Similarly, we have, by virtue of Lemma 4.3.1,

$$\begin{aligned}
&\sum_{j=m-2}^{m-1}\binom{m}{j}\big\|\left[e^{2cy}(\partial_y f_{m-j}),\ \Lambda_\delta^{-2}\Lambda^{\ell/3}\right](1+cy)^{-1}\,\phi_m^\ell\partial^j v\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
\leq\ & C\sum_{j=m-2}^{m-1}\binom{m}{j}\big\|\,(1+cy)^{-1}\,\phi_m^\ell\partial^j v\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
\leq\ & C\sum_{j=m-2}^{m-1}\binom{m}{j}\big\|\phi_m^\ell\Lambda^{\ell/3}\partial^j v\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}.
\end{aligned}$$

Thus combining these inequalities, we obtain

$$\begin{aligned}
&\sum_{j=m-2}^{m-1}\binom{m}{j}\big\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial^j v)\partial_y f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
\leq\ & C\sum_{j=m-2}^{m-1}\binom{m}{j}\big\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial^j v\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
\leq\ & Cm\big\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial^{m-1}v\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
& +Cm^2\big\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial^{m-2}v\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}.
\end{aligned}$$

Moreover, observe

$$\begin{aligned}
&\big\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial^{m-2}v\big\|^2_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
\leq\ & \big\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial^{m-1}v\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\big\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial^{m-3}v\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))},
\end{aligned}$$

and thus

$$\begin{aligned}
&m^2\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial^{m-2}v\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&\quad\leq m\big\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial^{m-1}v\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}+m^3\big\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial^{m-3}v\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&\quad\leq m\big\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial^{m-1}v\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}+m^3\big\|\phi_{m-2}^0\partial^{m-3}v\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&\qquad+m^3\big\|\phi_{m-1}^0\partial^{m-2}v\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}.
\end{aligned}$$

Then we have, combining the above inequalities,

$$\begin{aligned}
&\sum_{j=m-2}^{m-1}\binom{m}{j}\big\|e^{2cy}(\partial_y f_{m-j})\Lambda_\delta^{-2}\Lambda^{\ell/3}\,(1+cy)^{-1}\,\phi_m^\ell\partial^j v\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
\leq\ & Cm\big\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial^{m-1}v\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}+Cm^3\big\|\phi_{m-2}^0\partial^{m-3}v\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
& +Cm^3\big\|\phi_{m-1}^0\partial^{m-2}v\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
\leq\ & Cm\big\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}+Cm^3\big\|\phi_{m-2}^0W_{m-2}^0f_{m-2}\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
& +Cm^3\big\|\phi_{m-1}^0W_{m-1}^0f_{m-1}\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))},
\end{aligned}$$

the last inequality following from (4.4.8). This, along with the estimate

$$\begin{aligned}
&m^3\big\|\phi_{m-2}^0W_{m-2}^0f_{m-2}\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}+m^3\big\|\phi_{m-1}^0W_{m-1}^0f_{m-1}\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
\leq\ & CA^{m-6}\left((m-5)!\right)^{3(1+\sigma)}
\end{aligned}$$

due to the inductive assumption (4.2.9), gives the desired estimate (4.5.4).

(b) We will estimate in this step the second and the third terms on the right hand side of (4.5.3), and prove that

$$
\begin{aligned}
&\sum_{j=1}^{m-3}\binom{m}{j}\left\|\phi_m^\ell W_m^0(\partial^j v)\partial_y f_{m-j}\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\quad+\sum_{j=1}^{m-3}\binom{m}{j}\left\|\phi_m^\ell W_m^0(\partial^{j+1} v)\partial_y f_{m-j}\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\le CA^{m-6}\left((m-5)!\right)^{3(1+\sigma)}.
\end{aligned}
\tag{4.5.5}
$$

For this purpose we write, denoting by $[m/2]$ the largest integer less than or equal to $m/2$,

$$
\begin{aligned}
&\sum_{j=1}^{m-3}\binom{m}{j}\left\|\phi_m^\ell W_m^0(\partial^{j+1} v)\partial_y f_{m-j}\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
\le&\sum_{j=1}^{[m/2]}\binom{m}{j}\left\|\phi_m^\ell W_m^0(\partial^{j+1} v)(\partial_y f_{m-j})\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&+\sum_{j=[m/2]+1}^{m-3}\binom{m}{j}\left\|\phi_m^\ell W_m^0(\partial^{j+1} v)\partial_y f_{m-j}\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
=&\,S_1+S_2.
\end{aligned}
\tag{4.5.6}
$$

We first treat S_1. Using the inequality

$$\phi_m^\ell\le\phi_m^0\le\phi_{j+3}^0\phi_{m-j}^0,\quad W_m^0\le W_{m-j}^0 \text{ for } j\ge 1,$$

gives

$$
\begin{aligned}
S_1=&\sum_{j=1}^{[m/2]}\binom{m}{j}\left\|\phi_m^\ell W_m^0(\partial^{j+1} v)\partial_y f_{m-j}\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
\le&\sum_{j=1}^{[m/2]}\binom{m}{j}\left\|\phi_{j+3}^0\partial^{j+1} v\right\|_{L^\infty([0,T]\times\mathbb{R}_+^2)}\left\|\phi_{m-j}^0 W_{m-j}^0\partial_y f_{m-j}\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}.
\end{aligned}
\tag{4.5.7}
$$

By Sobolev inequality, we have

$$
\begin{aligned}
&\left\|\phi_{j+3}^0\partial^{j+1} v\right\|_{L^\infty([0,T]\times\mathbb{R}_+^2)}\\
\le&\; C\left\|\phi_{j+3}^0\partial^{j+1} v\right\|_{L^\infty([0,T]\times\mathbb{R}_+;\,L^2(\mathbb{R}_x))}+C\left\|\phi_{j+3}^0\partial^{j+2} v\right\|_{L^\infty([0,T]\times\mathbb{R}_+;\,L^2(\mathbb{R}_x))}\\
\le&\; C\left\|\phi_{j+2}^0\partial^{j+1} v\right\|_{L^\infty([0,T]\times\mathbb{R}_+;\,L^2(\mathbb{R}_x))}+C\left\|\phi_{j+3}^0\partial^{j+2} v\right\|_{L^\infty([0,T]\times\mathbb{R}_+;\,L^2(\mathbb{R}_x))}\\
\le&\; C\left\|\phi_{j+2}^0 W_{j+2}^0 f_{j+2}\right\|_{L^\infty([0,T];\,L^2(\mathbb{R}_+^2))}+C\left\|\phi_{j+3}^0 W_{j+3}^0 f_{j+3}\right\|_{L^\infty([0,T];\,L^2(\mathbb{R}_+^2))},
\end{aligned}
$$

the secomd inequality using (4.4.1) and the last inequlaity following from (4.4.8). As a result, we use the hypothesis of induction (4.2.9) and the initial hypothesis of induction (4.2.7) to conclude that if $4 \leq j \leq [m/2]$ then

$$\begin{aligned}\|\phi^0_{j+3}\partial^{j+1}v\|_{L^\infty([0,T]\times\mathbb{R}^2_+)} &\leq C\left(A^{j-3}\left((j-3)!\right)^{3(1+\sigma)} + A^{j-2}\left((j-2)!\right)^{3(1+\sigma)}\right)\\ &\leq CA^{j-2}\left((j-2)!\right)^{3(1+\sigma)},\end{aligned}$$

and if $1 \leq j \leq 3$

$$\|\phi^0_{j+3}\partial^{j+1}v\|_{L^\infty([0,T]\times\mathbb{R}^2_+)} \leq C.$$

Moreover, using (4.4.4) and also the inductive assumption (4.2.9), we calculate, for any $1 \leq j \leq [m/2]$,

$$\begin{aligned}&\|\phi^0_{m-j}W^0_{m-j}\partial_y f_{m-j}\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ \leq\ &\|\partial_y\phi^0_{m-j}W^0_{m-j}f_{m-j}\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\phi^0_{m-j}\left[\partial_y,\ W^0_{m-j}\right]f_{m-j}\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ \leq\ &\|\partial_y\phi^0_{m-j}W^0_{m-j}f_{m-j}\|_{L^2([0,T]\times\mathbb{R}^2_+)} + C\|\phi^0_{m-j}W^0_{m-j}f_{m-j}\|_{L^\infty([0,T],\ L^2(\mathbb{R}^2_+))}\\ \leq\ &CA^{m-j-5}\left((m-j-5)!\right)^{3(1+\sigma)}.\end{aligned}$$

Putting these inequalities into (4.5.7) gives

$$\begin{aligned}S_1 &\leq C\sum_{j=4}^{[m/2]}\frac{m!}{j!(m-j)!}A^{j-2}\left((j-2)!\right)^{3(1+\sigma)}\left(A^{m-j-5}\left((m-j-5)!\right)^{3(1+\sigma)}\right)\\ &\quad + C\sum_{j=1}^{3}\frac{m!}{j!(m-j)!}\left(A^{m-j-5}\left((m-j-5)!\right)^{3(1+\sigma)}\right)\\ &\leq C\sum_{j=4}^{[m/2]}\frac{m!}{j^2(m/2)^5}A^{m-7}\left((m-7)!\right)^{3(1+\sigma)-1} + CA^{m-6}\left((m-5)!\right)^{3(1+\sigma)}\\ &\leq C(m-5)!A^{m-7}\left((m-7)!\right)^{3(1+\sigma)-1} + CA^{m-6}\left((m-5)!\right)^{3(1+\sigma)}\\ &\leq CA^{m-6}\left((m-5)!\right)^{3(1+\sigma)}.\end{aligned} \tag{4.5.8}$$

We now treat S_2. Using the inequality

$$\phi^\ell_m \leq \phi^0_m \leq \phi^0_{j+2}\phi^0_{m-j+1}, \quad W^0_m \leq W^0_{m-j+1} \text{ for } j \geq 1,$$

and thus

$$\begin{aligned}S_2 &= \sum_{j=[m/2]+1}^{m-3}\binom{m}{j}\|\phi^\ell_m W^0_m(\partial^{j+1}v)\partial_y f_{m-j}\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\leq \sum_{j=[m/2]+1}^{m-3}\binom{m}{j}\|\phi^0_{j+2}W^0_{j+2}f_{j+2}\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))}\\ &\qquad\qquad \times\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y f_{m-j}\|_{L^2([0,T]\times\mathbb{R}_+;\ L^\infty(\mathbb{R}_x))},\end{aligned} \tag{4.5.9}$$

the last inequality using (4.4.8). As for the last factor in the above inequality, we use Sobolev inequality, (4.4.1) and (4.4.2) to compute

$$
\begin{aligned}
&\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y f_{m-j}\|_{L^2([0,T]\times\mathbb{R}_+;\,L^\infty(\mathbb{R}_x))}\\
&\qquad\le C\big\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\qquad\quad + C\big\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y\partial_x f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\qquad\le C\big\|\phi^0_{m-j}W^0_{m-j}\partial_y f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C\big\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y\partial_x f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned}
$$

On the other hand, in view of the definition of f_m, we have

$$
\begin{aligned}
&\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y\partial_x f_{m-j}\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\le\big\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y f_{m-j+1}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\quad+\big\|\phi^0_{m-j+1}W^0_{m-j+1}(\partial_x^{m-j}\omega)\partial_x\left((\partial_y\omega)/\omega\right)\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\quad+\big\|\phi^0_{m-j+1}W^0_{m-j+1}(\partial_x^{m-j}u)\partial_x\partial_y\left((\partial_y\omega)/\omega\right)\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\le C\big\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y f_{m-j+1}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C\big\|\left\langle y\right\rangle^{-1}\phi^0_{m-j}W^0_{m-j}\partial_x^{m-j}\omega\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\quad+C\big\|\left\langle y\right\rangle^{-1}\phi^0_{m-j}W^0_{m-j}\partial_x^{m-j}u\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)},
\end{aligned}
$$

the last inequality using (4.4.1) and (4.4.2). Combining these inequalities, we conclude

$$
\begin{aligned}
&\big\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}_+;\,L^\infty(\mathbb{R}_x))}\\
\le\ & C\big\|\phi^0_{m-j}W^0_{m-j}\partial_y f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C\big\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y f_{m-j+1}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+C\big\|\left\langle y\right\rangle^{-1}\phi^0_{m-j}W^0_{m-j}\partial_x^{m-j}\omega\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+C\big\|\left\langle y\right\rangle^{-1}\phi^0_{m-j}W^0_{m-j}\partial_x^{m-j}u\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
\le\ & C\big\|\partial_y\phi^0_{m-j}W^0_{m-j}f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C\big\|\partial_y\phi^0_{m-j+1}W^0_{m-j+1}f_{m-j+1}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+C\big\|\phi^0_{m-j}W^0_{m-j}f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C\big\|\phi^0_{m-j+1}W^0_{m-j+1}f_{m-j+1}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)},
\end{aligned}
$$

where the last inequality follows from (4.4.9) and (4.4.4). This, along with the inductive assumptions (4.2.9), yields, if $\left[m/2\right]+1\le j\le m-4$ then

$$
\begin{aligned}
&\big\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}_+;\,L^\infty(\mathbb{R}_x))}\\
\le\ & CA^{m-j-5}\left((m-j-5)!\right)^{3(1+\sigma)} + CA^{m-j-4}\left((m-j-4)!\right)^{3(1+\sigma)}\\
\le\ & CA^{m-j-4}\left((m-j-4)!\right)^{3(1+\sigma)},
\end{aligned}
$$

and if $j=m-3$ then

$$
\big\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y f_{m-j}\big\|_{L^2([0,T]\times\mathbb{R}_+;\,L^\infty(\mathbb{R}_x))}\le C
$$

due to the initial hypothesis of induction (4.2.7). On the other hand, the inductive assumptions (4.2.9) yields, for any $\left[m/2\right]+1\le j\le m-3$,

$$
\big\|\phi^0_{j+2}W^0_{j+2}f_{j+2}\big\|_{L^\infty\left([0,T];\,L^2(\mathbb{R}^2_+)\right)}\le A^{j-3}\left((j-3)!\right)^{3(1+\sigma)}.
$$

Putting these estimates into (4.5.9), we have

$$
\begin{aligned}
S_2 &\leq C \sum_{j=\left[m/2\right]+1}^{m-4} \frac{m!}{j!(m-j)!} A^{j-3}\big((j-3)!\big)^{3(1+\sigma)} \Big(A^{m-j-4}\left((m-j-4)!\right)^{3(1+\sigma)}\Big) \\
&\quad + C \sum_{j=m-3}^{m-3} \frac{m!}{j!(m-j)!} A^{j-3}\big((j-3)!\big)^{3(1+\sigma)} \\
&\leq C \sum_{j=\left[m/2\right]+1}^{m-4} \frac{m!}{j^3(m-j)^4} A^{m-7}\big((j-3)!\big)^{3(1+\sigma)-1}\left((m-j-4)!\right)^{3(1+\sigma)-1} \\
&\quad + C A^{m-6}\big((m-5)!\big)^{3(1+\sigma)} \\
&\leq C(m-3)! A^{m-7}\left((m-7)!\right)^{3(1+\sigma)-1} + C A^{m-6}\big((m-5)!\big)^{3(1+\sigma)} \\
&\leq C A^{m-6}\left((m-5)!\right)^{3(1+\sigma)}.
\end{aligned}
$$

This along with (4.5.8) and (4.5.6) yields

$$
\sum_{j=1}^{m-3} \binom{m}{j} \left\|\phi_m^\ell W_m^0 (\partial^{j+1} v)\partial_y f_{m-j}\right\|_{L^2([0,T]\times\mathbb{R}_+^2)} \leq C A^{m-6}\left((m-5)!\right)^{3(1+\sigma)}.
$$

Similarly, we have

$$
\sum_{j=1}^{m-3} \binom{m}{j} \left\|\phi_m^\ell W_m^0 (\partial^{j} v)\partial_y f_{m-j}\right\|_{L^2([0,T]\times\mathbb{R}_+^2)} \leq C A^{m-6}\left((m-5)!\right)^{3(1+\sigma)}.
$$

Then the desired estimate (4.5.5) follows.

(c) It remains to prove that

$$
\sum_{j=1}^{m-3} \binom{m}{j} \left\|\phi_m^\ell W_m^0 (\partial^{j} v)(\partial_y\partial_x f_{m-j})\right\|_{L^2([0,T]\times\mathbb{R}_+^2)} \leq C A^{m-6}\left((m-5)!\right)^{3(1+\sigma)}. \tag{4.5.10}
$$

The proof is quite similar as in the previous step. To do so we first write

$$
\begin{aligned}
&\sum_{j=1}^{m-3} \binom{m}{j} \left\|\phi_m^\ell W_m^0 (\partial^{j} v)(\partial_y\partial_x f_{m-j})\right\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
&= \sum_{j=1}^{\left[m/2\right]} \binom{m}{j} \left\|\phi_m^\ell W_m^0 (\partial^{j} v)(\partial_y\partial_x f_{m-j})\right\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
&\quad + \sum_{j=\left[m/2\right]+1}^{m-3} \binom{m}{j} \left\|\phi_m^\ell W_m^0 (\partial^{j} v)(\partial_y\partial_x f_{m-j})\right\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\
&= \tilde{S}_1 + \tilde{S}_2.
\end{aligned}
$$

For the term $\tilde{S}_1$, we use

$$
\phi_m^\ell \leq \phi_m^0 \leq \phi_{j+2}^0 \phi_{m-j+1}^0, \quad W_m^0 \leq W_{m-j+1}^0 \text{ for } j \geq 2,
$$

to obtain

$$\tilde{S}_1 \leq \sum_{j=1}^{[m/2]} \binom{m}{j} \|\phi_{j+2}^0 \partial^j v\|_{L^\infty([0,T]\times\mathbb{R}_+^2)} \|\phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y \partial_x f_{m-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)}.$$

Then repeating the arguments used to estimate S_1 and S_2 in the previous step, we can deduce that

$$\tilde{S}_1 \leq C A^{m-6} \left((m-6)!\right)^{3(1+\sigma)}.$$

As for $\tilde{S}_2$, using the inequality

$$\phi_m^\ell \leq \phi_m^0 \leq \phi_{j+1}^0 \phi_{m-j+2}^0, \quad W_m^0 \leq W_{m-j+2}^0 \text{ for } j \geq 2,$$

gives

$$\begin{aligned} \tilde{S}_2 \leq & \sum_{j=[m/2]+1}^{m-3} \binom{m}{j} \|\phi_{j+1}^0 \partial^j v\|_{L^\infty([0,T]\times\mathbb{R}_+;\, L^2(\mathbb{R}_x))} \\ & \times \|\phi_{m-j+2}^0 W_{m-j+2}^0 \partial_y \partial_x f_{m-j}\|_{L^2([0,T]\times\mathbb{R}_+;\, L^\infty(\mathbb{R}_x))}. \end{aligned}$$

Then repeating the arguments used to estimate S_2 in the previous step, we have

$$\tilde{S}_2 \leq C A^{m-6} \left((m-5)!\right)^{3(1+\sigma)}.$$

This along with the estimate on $\tilde{S}_1$ yields (4.5.10). Finally, combining (4.5.3), (4.5.4), (4.5.5) and (4.5.10) gives the desired estimate in Lemma 4.5.3, and thus the proof is complete. □

Lemma 4.5.4. *Under the same assumption as in Proposition 4.2.3, we have*

$$\begin{aligned} & \sum_{j=1}^{m} \binom{m}{j} \|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell (\partial^j u) f_{m+1-j}\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ & \quad + \sum_{j=1}^{m-1} \binom{m}{j} \|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell \left[\partial_y \left(\partial_y \omega/\omega\right)\right] (\partial^j v)(\partial^{m-j} u)\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ \leq \; & m\, C \|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} + C A^{m-6} \left((m-5)!\right)^{3(1+\sigma)}. \end{aligned}$$

The proof of this Lemma is quite similar as in Lemma 4.5.3, so we omit it.

Lemma 4.5.5. *Under the same assumption as in Proposition 4.2.3, we have*

$$2\|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell \left[\partial_y \left((\partial_y \omega)/\omega\right)\right] f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} \leq C \|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}.$$

Proof. This is a just direct verification. Indeed, Lemma 4.3.1 gives

$$\begin{aligned} & \|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell \left[\partial_y \left((\partial_y \omega)/\omega\right)\right] f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ \leq \; & \|\left[\partial_y \left((\partial_y \omega)/\omega\right)\right] \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ & + \|\left[\partial_y \left((\partial_y \omega)/\omega\right),\, \Lambda_\delta^{-2} W_m^\ell\right] \phi_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ \leq \; & C \|\Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}. \end{aligned}$$

Then the desired estimate follows and thus the proof of Lemma 4.5.5 is complete. □

Proof of Proposition 4.5.2. In view of (4.5.2), we combine the estimates in Lemma 4.5.3-Lemma 4.5.5, to get the first estimate in Proposition 4.5.2. The second one can be treated quite similarly and the main difference is that we will use here additionally the inductive estimates on the terms of the following form

$$\|\partial_y^2 \Lambda^{-2/3} \phi_j^0 W_j^0 f_j\|_{L^2([0,T]\times\mathbb{R}_+^2)}, \quad 6 \le j \le m,$$

while in the proof of Lemma 4.5.3, we only used the estimates on the following two forms

$$\|\phi_j^0 W_j^0 f_j\|_{L^\infty([0,T];\, L^2(\mathbb{R}_+^2))}, \quad \|\partial_y \phi_j^0 W_j^0 f_j\|_{L^2([0,T]\times\mathbb{R}_+^2)}, \quad 6 \le j \le m.$$

So we omit the treatment of the second estimate for brevity, and thus the proof of Proposition 4.5.2 is complete. □

Completeness of the proof of Proposition 4.2.3. The estimates (4.2.15) follows from the combination of Propostion 4.5.1 and the first estimate in Proposition 4.5.2, while the estimate (4.2.16) in Proposition 4.2.3 follows from Propostion 4.5.1 and the second estimate in Proposition 4.5.2. The treatment of (4.2.15) is exactly the same as (4.2.15). The proof of Proposition 4.2.3 is thus complete. □

4.6 Appendix

Here we deduce the equation fulfilled by f_m (cf. [21]). Recall that

$$f_m = \partial_x^m \omega - \frac{\partial_y \omega}{\omega} \partial_x^m u, \quad m \ge 1,$$

where u is a smooth solution to Prandtl equation (4.1.1) and $\omega = \partial_y u$. We will verify that

$$\partial_t f_m + u\partial f_m + v\partial_y f_m - \partial_y^2 f_m = \mathcal{Z}_m, \tag{4.6.1}$$

where

$$\begin{aligned}
\mathcal{Z}_m \quad = \quad & -\sum_{j=1}^{m} \binom{m}{j} (\partial^j u) f_{m+1-j} - \sum_{j=1}^{m-1} \binom{m}{j} (\partial^j v)(\partial_y f_{m-j}) \\
& - \left[\partial_y \left(\frac{\partial_y \omega}{\omega}\right)\right] \sum_{j=1}^{m-1} \binom{m}{j} (\partial^j v)(\partial^{m-j} u) - 2\left[\partial_y \left(\frac{\partial_y \omega}{\omega}\right)\right] f_m.
\end{aligned}$$

To do so, we firstly notice that

$$u_t + uu_x + vu_y - u_{yy} = 0, \tag{4.6.2}$$

and

$$\omega_t + u\omega_x + v\omega_y - \omega_{yy} = 0.$$

Thus by Leibniz's formula, $\partial^m u, \partial_x^m \omega$ satisfy, respectively, the following equation

$$\begin{aligned}
&\partial_t \partial^m u + u\partial\partial^m u + v\partial_y \partial^m u - \partial_y^2 \partial^m u \\
&= -\sum_{j=1}^m \binom{m}{j} (\partial^j u)(\partial^{m-j+1} u) - \sum_{j=1}^m (\partial^j v)(\partial_y \partial^{m-j} u) \\
&= -\sum_{j=1}^m \binom{m}{j} (\partial^j u)(\partial^{m-j+1} u) - \sum_{j=1}^{m-1} (\partial^j v)(\partial_y \partial^{m-j} u) - (\partial^m v)(\partial_y u)
\end{aligned} \tag{4.6.3}$$

and

$$\begin{aligned}
&\partial_t \partial^m \omega + u\partial\partial^m \omega + v\partial_y \partial^m \omega - \partial_y^2 \partial^m \omega \\
&= -\sum_{j=1}^m \binom{m}{j} (\partial^j u)(\partial^{m-j+1} \omega) - \sum_{j=1}^m (\partial^j v)(\partial_y \partial^{m-j} \omega) \\
&= -\sum_{j=1}^m \binom{m}{j} (\partial^j u)(\partial^{m-j+1} \omega) - \sum_{j=1}^{m-1} (\partial^j v)(\partial_y \partial^{m-j} \omega) - (\partial^m v)(\partial_y \omega).
\end{aligned} \tag{4.6.4}$$

In order to eliminate the last terms on the right sides of the above two equations, we observe $\partial_y u = \omega > 0$ and thus multiply (4.6.3) by $-\frac{\partial_y \omega}{\omega}$, and then add the resulting equation to (4.6.4); this gives

$$\partial_t f_m + u\partial f_m + v\partial_y f_m - \partial_y^2 f_m = \mathcal{Z}_m$$

where

$$\begin{aligned}
\mathcal{Z}_m \quad = \quad & -\sum_{j=1}^m \binom{m}{j} (\partial^j u) f_{m+1-j} - \sum_{j=1}^{m-1} \binom{m}{j} (\partial^j v)(\partial_y f_{m-1}) \\
& -\left[\partial_y \left(\frac{\partial_y \omega}{\omega}\right)\right] \sum_{j=1}^{m-1} \binom{m}{j} (\partial^j v)(\partial^{m-j} u) + (\partial^m u) f_1 \\
& + \left(\partial_t \left(\frac{\partial_y \omega}{\omega}\right) + u\partial \left(\frac{\partial_y \omega}{\omega}\right) + v\partial_y \left(\frac{\partial_y \omega}{\omega}\right) - \partial_y^2 \left(\frac{\partial_y \omega}{\omega}\right)\right) \partial^m u \\
& -2\left[\partial_y \left(\frac{\partial_y \omega}{\omega}\right)\right] \partial_y \partial^m u.
\end{aligned}$$

On the other hand we notice that

$$\begin{aligned}
& \partial_t \left(\frac{\partial_y \omega}{\omega}\right) + u\partial \left(\frac{\partial_y \omega}{\omega}\right) + v\partial_y \left(\frac{\partial_y \omega}{\omega}\right) - \partial_y^2 \left(\frac{\partial_y \omega}{\omega}\right) \\
= \quad & \frac{1}{\omega} \left(\partial_t \partial_y \omega + u\partial\partial_y \omega + v\partial_y \partial_y \omega - \partial_y^2 \partial_y \omega\right) \\
& -\frac{\partial_y \omega}{\omega^2} \left(\partial_t \omega + u\partial\omega + v\partial_y \omega - \partial_y^2 \omega\right) + 2\frac{\partial_y \omega}{\omega} \partial_y \left(\frac{\partial_y \omega}{\omega}\right) \\
= \quad & -\partial\omega + \frac{(\partial u)(\partial_y \omega)}{\omega} + 2\frac{\partial_y \omega}{\omega} \partial_y \left(\frac{\partial_y \omega}{\omega}\right)
\end{aligned}$$

Therefore we have

$$
\begin{aligned}
\mathcal{Z}_m \;=\; & -\sum_{j=1}^{m}\binom{m}{j}(\partial^j u)f_{m+1-j}-\sum_{j=1}^{m-1}\binom{m}{j}(\partial^j v)(\partial_y f_{m-1}) \\
& -\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\sum_{j=1}^{m-1}\binom{m}{j}(\partial^j v)(\partial^{m-j}u)+(\partial^m u)f_1 \\
& +\left(\partial\omega-\frac{(\partial u)(\partial_y\omega)}{\omega}\right)\partial^m u+2\frac{\partial_y\omega}{\omega}\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\partial^m u-2\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\partial_y\partial^m u \\
= \; & -\sum_{j=1}^{m}\binom{m}{j}(\partial^j u)f_{m+1-j}-\sum_{j=1}^{m-1}\binom{m}{j}(\partial^j v)(\partial_y f_{m-1}) \\
& -\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\sum_{j=1}^{m-1}\binom{m}{j}(\partial^j v)(\partial^{m-j}u)+\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)^2\right]\partial^m u \\
& -2\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\partial^m\omega \\
= \; & -\sum_{j=1}^{m}\binom{m}{j}(\partial^j u)f_{m+1-j}-\sum_{j=1}^{m-1}\binom{m}{j}(\partial^j v)(\partial_y f_{m-1}) \\
& -\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\sum_{j=1}^{m-1}\binom{m}{j}(\partial^j v)(\partial^{m-j}u)-2\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_m.
\end{aligned}
$$

Next we will give the boundary value of $\partial_y f_m$ and $\partial_t f_m-\partial_y^2 f_m$. In view of (4.6.2), we infer, recalling $u|_{y=0}=v|_{y=0}=0$,

$$
\partial_y\omega\big|_{y=0}=\partial_y^2 u\big|_{y=0}=0.
$$

As a result, observing

$$
\partial_y f_m=\partial_y\partial^m\omega-\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\partial^m u-\left(\frac{\partial_y\omega}{\omega}\right)\partial_y\partial^m u,
$$

we have

$$
\partial_y f_m|_{y=0}=0. \tag{4.6.5}
$$

Direct verification shows

$$
\mathcal{Z}_m|_{y=0}=-2\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_m\Big|_{y=0},
$$

and thus

$$
\left(\partial_t f_m-\partial_y^2 f_m\right)|_{y=0}=\mathcal{Z}_m|_{y=0}=-2\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_m\Big|_{y=0}, \tag{4.6.6}
$$

due to the equation fulfilled by f_m.

Bibliography

[1] R. Alexandre, Y. Wang, C.-J.Xu and T.Yang, Well-posedness of The Prandtl Equation in Sobolev Spaces. *J. Amer. Math. Soc.*, DOI: http://dx.doi.org/10.1090/S0894-0347-2014-00813-4 Article electronically published on June 6, 2014. 40 pages, http://arxiv.org/abs/1203.5991.

[2] R. E. Caflisch and M. Sammartino: Existence and singularities for the Prandtl boundary layer equations, *Z. Angew. Math. Mech.*, 80(2000), 733-744.

[3] M. Cannone, M. C. Lombardo and M. Sammartino : Well-posedness of the Prandtl equation with non compatible data. *Nonlinearity.*, 26(2013), 3077-3100.

[4] W. E: Boundary layer theory and the zero-viscosity limit of the Navier-Stokes equation. *Acta Math. Sin. (Engl. Ser.)* 16(2000), 207-218.

[5] H. Chen, W.-X. Li, and C.-J. Xu. Gevrey hypoellipticity for linear and non- linear Fokker–Planck equations. *J. Differential Equations* 246(2009), 320-339

[6] H. Chen, W.-X. Li, and C.-J. Xu. Analytic smoothness effect of solutions for spatially homogeneous Landau equation. *J. Differential Equations*, 248(2010), 77-94.

[7] H. Chen, W.-X. Li, and C.-J. Xu. Gevrey hypoellipticity for a class of kinetic equations. *Comm. Partial Differential Equations*, 36(2011), 693-728.

[8] M. Derridj and C. Zuily : Sur la régularité Gevrey des opérateurs de Hörmander. *J.Math.Pures et Appl.* **52** (1973), 309-336.

[9] Y. Ding and N. Jiang :On Analytic Solutions of the Prandtl Equations with Robin Boundary Condition in Half Space, http://arxiv.org/pdf/1402.3158.pdf, preprint 2014.

[10] W. E and B. Enquist : Blow up of solutions of the unsteady Prandtl's equation, *Comm. Pure Appl. Math.*, 50(1997), 1287-1293.

[11] D. Gérard-Varet and E. Dormy: On the ill-posedness of the Prandtl equation, *J. Amer. Math. Soc.*, 23(2010), 591-609.

[12] D. Gérard-Varet and N. Masmoudi : Well-posedness for the Prandtl system without analyticity or monotonicity, arXiv:1305.0221,

[13] D. Gérard-Varet and T. Nguyen : Remarks on the ill-posedness of the Prandtl equation. *Asymptot. Anal.* 77 (2012), 71-88

[14] Y. Guo and T. Nguyen : A note on the Prandtl boundary layers, *Comm. Pure Appl. Math.* 64 (2011) 1416-1438, doi: 10.1002/cpa.20377

[15] L. Hong and J. K. Hunter : Singularity formation and instability in the unsteady inviscid and viscous Prandtl equations, *Commun. Math. Sci.* 1 (2003), 293-316.

[16] L. Hörmander. *The analysis of linear partial differential operators. III*, volume 275 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1985.

[17] I. Kukavica, N. Masmoudi, V. Vicol and T. Wong, On the Local Well-posedness of the Prandtl and Hydrostatic Euler Equations with Multiple Monotonicity Regions. *SIAM J. Math. Anal.* 46 (2014), 3865-3890.

[18] I. Kukavica and V. Vicol, On the analyticity and Gevrey-class regularity up to the boundary for the Euler equations. *Nonlinearity* 24 (2011) 765-796

[19] N. Lerner. *Metrics on the phase space and non-selfadjoint pseudo-differential operators*, volume 3 of *Pseudo-Differential Operators. Theory and Applications*. Birkhäuser Verlag, Basel, 2010.

[20] M. C. Lombardo, M. Cannone and M. Sammartino : Well-posedness of the boundary layer equations. *SIAM J. Math. Anal.*, 35(2003), 987-1004 (electronic).

[21] N. Masmoudi and T. K. Wong : Local-in-time existence and uniqueness of solution to the Prandtl equation by energy method. arXiv:1206.3629

[22] G. Métivier : *Small Viscosity and Boundary Layer Methods. Theory, Stability Analysis, and Applications*. Modeling and Simulation in Science, Engineering and Technology. Birkhauser Boston, Inc., Boston, MA, 2004. xxii+194 pp.

[23] O. A. Oleinik and V. N. Samokhin, *Mathematical Models in Boundary Layers Theory*. Chapman & Hall/CRC, 1999.

[24] L. Prandtl, Über Flüssigkeitsbewegungen bei sehr kleiner Reibung. In "*Verh. Int. Math. Kongr., Heidelberg 1904*", Teubner 1905, 484-494.

[25] M. Sammartino and R. E. Caflisch, Zero viscosity limit for analytic solutions of the Navier-Stokes equations on a half-space, I. Existence for Euler and Prandtl equations. *Comm. Math. Phys.*, 192(1998), 433-461; II. Construction of the Navier-Stokes solution. *Comm. Math. Phys.*, 192(1998), 463-491.

[26] Z. Xin and L. Zhang : On the global existence of solutions to the Prandtl's system, *Adv. Math.*, 181(2004), 88-133.

[27] C.-J. Xu : Hypoellipticity of nonlinear second order partial differential equations. *J. Partial Differential Equations Ser. A* 1 (1988), 85–95.

[28] C.-J. Xu : Régularité des solutions pour les équations aux dérivées partielles quasi linéaires non elliptiques du second ordre. *C. R. Acad. Sci. Paris Sér. I Math.* 300 (1985), no. 9, 267–270.

[29] C.-J. Xu : Hypoellipticité d'équations aux dérivées partielles non linéaires. *Proceedings of the conference on partial differential equations, Vol. 1, 2 (Saint Jean de Monts, 1985)*, Exp. No. 7, 16 pp., Soc. Math. France, Paris, 1985.

[30] P. Zhang and Z. Zhang, Long time well-posdness of Prandtl system with small and analytic initial data, arXiv:1409.1648, Preprint 2014