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# Observer Design for Different Classes of Nonlinear Delayed Systems.

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# Conception d'observateurs pour différentes classes de systèmes à retards non linéaires

Thèse de doctorat de l'Université Paris-Saclay  
préparée à l'Université Paris-Sud

École doctorale n°580 : Sciences et technologies  
de l'information et de la communication STIC  
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# List of Publications

## Journals

- Kahelras, M., Ahmed-Ali, T., Giri, F., Lamnabhi-Lagarrigue, F. (2018). Sampled-data chain-observer design for a class of delayed nonlinear systems. *International Journal of Control* 91(5), 1076-1090 [25].
- Ahmed-Ali, T., Giri, F., Krstic, M., Kahelras, M. (2018). PDE based observer design for nonlinear systems with large output delay. *Systems & Control Letters*, 113, 1-8 [8].
- Ahmed-Ali, T., Fridman, E., Giri, F., Kahelras, M., Lamnabhi-Lagarrigue, F., Burlion, L. (2019) Observer design for a class of parabolic systems with large delays and sampled measurements. *IEEE-TAC* 18-1958 (Accepted).

## Conferences

- Kahelras, M., Ahmed-Ali, T., Folin, T., Giri, F., Lamnabhi-Lagarrigue, F.. (2016) Observer design for triangular nonlinear systems using delayed sampled-output measurements. in: *IEEE Conference on Decision and Control*, Las Vegas, NV, USA, pp. 1447–1451 [24].
- Ahmed-Ali, T., Fridman, E., Giri, F., Kahelras, M., Lamnabhi-Lagarrigue, F., Burlion, L. (2018) Observer design for a class of parabolic systems with arbitrarily delayed measurements. *IEEE 57th Conference on Decision and Control (CDC)*, Miami Beach, FL, USA, 2018, pp. 2199-2204.

# Contents

<b>Résumé en français</b>	<b>10</b>
<b>General Introduction</b>	<b>15</b>
<b>1 Sampled-Data Observer Design For a Class of Triangular Delayed Nonlinear Systems</b>	<b>20</b>
1.1 Introduction . . . . .	20
1.2 Class of Systems . . . . .	21
1.3 Proposed Observer . . . . .	25
1.4 Observer Analysis . . . . .	27
1.5 Sampled-output chain-observer design . . . . .	38
1.5.1 System cascade representation . . . . .	38
1.5.2 Chain observer design . . . . .	39
1.5.3 Observer Analysis . . . . .	40
1.6 Simulation Results . . . . .	54
1.6.1 Observer application to a system satisfying all theoretical assumptions . . . . .	54
1.6.2 Observer application to a system not satisfying all theoretical assumptions . . . . .	57
1.7 Conclusion . . . . .	60
<b>2 Chain Observer Design For a Class of ODE-PDE Systems</b>	<b>61</b>
2.1 Introduction . . . . .	61
2.2 System description . . . . .	63
2.3 Cascade observer design . . . . .	65
2.3.1 System cascade representation . . . . .	65
2.3.2 System coordinate transformation . . . . .	67
2.4 Proposed Observer . . . . .	68
2.5 Observer Analysis . . . . .	69
2.6 Simulation Results . . . . .	77
2.7 Conclusion . . . . .	83

## CONTENTS

---

<b>3 Chain Observer Design For a Class of Nonlinear Parabolic Systems</b>	<b>84</b>
3.1 Introduction . . . . .	84
3.2 System description . . . . .	85
3.3 Observer design . . . . .	85
3.4 Well-posedness of the error system . . . . .	87
3.5 Observer analysis . . . . .	91
3.5.1 Extension to sampled-data case . . . . .	98
3.6 Example . . . . .	102
3.7 Conclusion . . . . .	105
<b>General Conclusion</b>	<b>106</b>
<b>A Control Theory Review</b>	<b>109</b>
<b>B Mathematical Review</b>	<b>111</b>
<b>C Basic Inequalities</b>	<b>113</b>
<b>D Finite Differences</b>	<b>115</b>
D.1 Finite differences approximations . . . . .	115
D.2 Numerical algorithms for parabolic equations . . . . .	116
<b>Bibliography</b>	<b>118</b>

# List of Figures

1.1	System to be observed . . . . .	22
1.2	$x_1(t)$ and it's estimate $\hat{x}_1(t)$ with $m = 4$ and a delay $d = 4s$ . .	56
1.3	$x_2(t)$ and it's estimate $\hat{x}_2(t)$ with $m = 4$ and a delay $d = 4s$ . .	56
1.4	$x_1(t)$ and it's estimate $\hat{x}_1(t)$ with $m = 4$ and a delay $d = 2s$ Lotka–Volterra . . . . .	58
1.5	$x_2(t)$ and it's estimate $\hat{x}_2(t)$ with $m = 4$ and a delay $d = 2s$ Lotka–Volterra . . . . .	59
2.1	ODE-PDE cascade system to be observed . . . . .	63
2.2	Observation error on $u(x, t)$ with $m = 1$ and a delay $D = 0.5s$	79
2.3	Observation error on $X(t)$ with $m = 1$ and a delay $D = 0.5s$ .	79
2.4	Observation error on $u(x, t)$ with $m = 2$ and a delay $D = 0.5s$	80
2.5	Observation error on $X(t)$ with $m = 2$ and a delay $D = 0.5s$ .	80
2.6	Observation error on $u(x, t)$ with $m = 2$ and a delay $D = 1s$ .	81
2.7	Observation error on $X(t)$ with $m = 2$ and a delay $D = 1s$ . .	81
2.8	Observation error on $u(x, t)$ with $m = 4$ and a delay $D = 1s$ .	82
2.9	Observation error on $X(t)$ with $m = 4$ and a delay $D = 1s$ . .	82
3.1	Delayed semi-linear diffusion equation to be observed . . . . .	85
3.2	The state $u(x, t)$ and its observations for $m = 1$ and $m = 2$ at $x = 0.1$ and $x = 0.6$ for a delay $D = 1s$ . . . . .	103
3.3	The state $u(x, t)$ and its observations for $m = 1$ and $m = 2$ at $x = 0.1$ and $x = 0.6$ for a delay $D = 2s$ . . . . .	104
A.1	Feedback connection . . . . .	110

# List of Tables

1.1	Dependence of number $m$ (characterising the simpler satisfactory observer) on system delay $d$ . Case $\theta = 1.1, K = [2 \ 1]^T$ . . .	55
1.2	Dependence of number $m$ (characterising the simpler satisfactory observer) on parameter $\theta$ . Case $d = 4s, K = [2 \ 1]^T$ . . . .	55
1.3	Dependence of number $m$ (characterising the simpler satisfactory observer) on system delay $d$ . Case $\theta = 1.1, K = [2 \ 1]^T$ . . .	59
1.4	Dependence of number $m$ (characterising the simpler satisfactory observer) on parameter $\theta$ . Case $d = 4s, K = [2 \ 1]^T$ . . . .	60
3.1	The maximum delay $D$ for a given number of sub-observers $m$ for the continuous case. . . . .	102
3.2	The maximum sampling period $h$ for a given number of sub-observers $m$ , with $D = 0.4s$ . . . . .	104
3.3	The maximum sampling period $h$ for a given number of sub-observers $m$ , with $D = 0.8s$ . . . . .	105
3.4	The maximum sampling period $h$ for a given number of sub-observers $m$ , with $D = 1.5s$ . . . . .	105

# Symbols and Abbreviations

## Symbols

$\mathbb{C}$	The set of complex numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^n$	The $n$ -dimensional Euclidean space
$\mathbb{R}^{n \times m}$	The space of real $n \times m$ matrices
$\dot{x}(t)$	The first derivative of $x$ with respect to time $t$
$\mathcal{C}(a, b)$	The space of continuous functions $\phi : [a, b] \rightarrow \mathbb{R}^n$ with the norm $\ \phi\ _C = \max_{\theta \in [a, b]}  \phi(\theta) $
$\mathcal{C}^m(a, b)$	The space of $m$ times continuous functions $\phi : [a, b] \rightarrow \mathbb{R}^n$ with the norm $\ \phi\ _{C^m} = \ \phi\ _C + \dots + \left\  \frac{d^m \phi}{d\theta^m} \right\ _C$
$L_p(a, b)$	The space of functions $\phi : [a, b] \rightarrow \mathbb{R}^n$ with the norm $\ \phi\ _{L_p} = \left[ \int_a^b  \phi(\theta) ^p d\theta \right]^{\frac{1}{p}}$ , $p \in \mathbb{N}$
$H^1(0, l)$	The Sobolev space of absolutely continuous functions $\phi : [0, l] \rightarrow \mathbb{R}^n$ with $\frac{d\phi}{d\theta} \in L_2(0, l)$
$P > 0 (P \geq 0)$	The symmetric matrix $P$ is positive (semi-positive) definite
$\lambda_{max}(P)$	The maximum eigenvalue of a symmetric matrix $P$
$\lambda_{min}(P)$	The minimum eigenvalue of a symmetric matrix $P$
*	The symmetric elements of the symmetric matrix
$A^T(x^T)$	The transpose of a matrix $A$ (a vector $x$ )
$I$	Identity matrix
$\text{diag}[a_1, \dots, a_n]$	A diagonal matrix with diagonal elements $a_1$ to $a_n$
$\min$	Minimum

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max	Maximum
sup	Supremum, the least upper bound
sgn(.)	The signum function

## Abbreviations

DPS	Distributed Parameter Systems
FDM	Finite Difference Method
FDS	Finite Dimensional System
IDS	Infinite Dimensional System
ISS	Input-to-State Stability
LMI	Linear Matrix Inequality
MAD	Maximum Admissible Delay
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
TDS	Time-Delay System
ZOH	Zero-Order-Hold

# Résumé en français

Les systèmes physiques en automatique et en théorie de commande sont souvent régis par des systèmes d'équations algébriques, équations différentielles ordinaires (ODE), et des équations aux dérivées partielles (EDP). Plusieurs modèles mathématiques de systèmes physiques présentent des paramètres à nature distribuée, c.à.d. des systèmes caractérisés par des variables d'état à coordonnées multiples, la combinaison temps-espace est celle qu'on retrouve le plus souvent. Les systèmes à paramètres distribués font référence à des systèmes contenant des retards temporels ou des équations aux dérivées partielles définies sur un espace à dimension infinie, d'où le terme systèmes à dimension infinie. Contrairement aux systèmes à dimension finie pour lesquels une théorie générale est établie pour l'étude de stabilité, observabilité et robustesse, le cas des systèmes à dimension infinie est beaucoup plus difficile à cause de la complexité de leurs modèles mathématiques. De nombreuses études ont été établies afin de résoudre quelques problèmes concernant ce type de systèmes [26, 34, 15]. Dans ce travail, on s'intéresse à quelques classes de systèmes à retard et à un type d'équations aux dérivées partielles.

Le retard temporel est un phénomène naturel qu'on retrouve dans la plupart des systèmes physiques et dans les applications d'ingénierie. Voici quelques exemples où on retrouve ce phénomène:

- Systèmes de commande en réseau,
- Procédés chimiques
- Systèmes de forage.

Plusieurs autres exemples peuvent être retrouvés dans les livres [17, 33, 38]. Rajoutons à cela que les composants de commandes tels que les capteurs et les actionneurs peuvent aussi introduire un retard en plus dans les systèmes de commande. Quand les retards sont relativement petits, ils peuvent être ignorés, cependant quand ce n'est pas le cas il a été bien établi que la négligence de ces retards pourra engendrer l'instabilité de tout le système.

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Quelques notions de base à propos du retard doivent être rappelées. Premièrement, les systèmes à retard sont des systèmes à dimension infinie, cela veut dire que l'état n'est pas un vecteur mais une fonction ou alors un vecteur de fonctions. Un autre défi majeur réside dans l'analyse de stabilité des systèmes à retard à l'aide des fonctionnels de Krasovskii. Et ce, contrairement au cas des systèmes ne contenant pas de retard où les fonctions de Lyapunov sont utilisées.

L'autre sujet qui sera étudié dans ce travail est la notion d'équations aux dérivées partielles. Une équation aux dérivées partielles est une équation différentielle qui contient des fonctions multi-variables inconnues et leurs dérivées à un certain ordre. La différence entre une équation différentielle ordinaire et une équation aux dérivées partielles c'est qu'une ODE comprend une fonction qui dépend d'une variable indépendante, contrairement à une PDE où la fonction et ses dérivées dépendent de plusieurs variables indépendantes. Cette différence fait que l'analyse des PDEs est beaucoup plus difficile que celle des ODEs. Une PDE de la fonction multi-variable  $u(x_1, x_2, \dots, x_n)$  est une équation de la forme:

$$f(x_1, x_2, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial^2 u}{\partial x_1 \partial x_1}, \dots, \frac{\partial^2 u}{\partial x_1 \partial x_n}, \dots) = 0$$

Des exemples d'équations aux dérivées partielles sont présentés : Le transfert de chaleur dans une barre métallique peut être modélisé par une PDE parabolique. Les vibrations le long d'une corde peuvent être modélisées par une PDE hyperbolique du second ordre. Le mouvement des fluides visqueux peut être modélisé par les Navier-Stokes équations.

Malgré leur complexité, les systèmes à retard peuvent être représentés comme une cascade de 2 sous-systèmes, où le premier sous-système est le système sans retard et le deuxième sous-système est une équation hyperbolique du premier ordre qui traduit le comportement du retard, cela veut dire qu'un système à retard constant  $y(t) = z(t - D)$  peut être remplacé par la PDE suivante:

$$\begin{aligned} \frac{\partial u(x, t)}{\partial t} &= \frac{\partial u(x, t)}{\partial x}, & x \in [0, D], \\ z(t) &= u(D, t), & y(t) = u(0, t). \end{aligned}$$

Ce système est un simple système de dimension infinie dans le monde complexe des PDEs. Le sujet principal de cette étude est la conception d'observateurs

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pour différentes classes de systèmes. Les observateurs d'état sont d'une grande importance en automatique à cause de leur utilité en matière d'estimation des variables. La présence de retard rend le problème d'observation plus complexe. Même si le retard est d'une nature distribuée, comme indiqué précédemment, la conception d'observateurs en présence du retard a souvent été traitée en utilisant des outils d'analyse de dimension finie, cela se traduit par l'étude en ignorant le retard et ensuite modifier l'observateur pour que la convergence exponentielle est préservée en présence de retard. L'approche majeure est l'utilisation d'un ou plusieurs prédicteurs de la sortie ou de l'état du système. Dans les cas complexes de systèmes non linéaires, le retard maximal admissible dépend du niveau de non linéarité qui est souvent de nature Lipchitzienne. Ce résultat a été illustré à l'aide des observateurs à grand gain où les prédicteurs utilisés ont été très utiles pour compenser l'effet du retard jusqu'à une certaine limite. Afin d'étendre cette limite le principe d'observateurs en chaîne a été introduit dans [19], et a aussi été utilisé pour quelques classes de systèmes non linéaires.

En parallèle avec la méthode de dimension finie citée dans le paragraphe précédent, la méthode de la transformation en backstepping de dimension infinie a été développée pour les systèmes linéaires dans [32, 33]. Dans cette approche, le retard est remplacé par une équation hyperbolique du premier ordre et connectée en série avec l'équation différentielle qui décrit les dynamiques de dimension finie du système, ce qui mène à une représentation en cascade ODE-PDE du système. Ainsi, le problème d'observation consiste à établir un observateur qui fournira les estimations en ligne de l'état à dimension finie de la partie ODE et l'état de dimension infinie de la partie PDE.

Le but de ce travail de recherche est la conception d'observateurs pour différentes classes de systèmes non linéaires. Cette thèse est composée de trois grands chapitres:

Chapitre 1: Conception d'observateurs échantillonnés pour une classe de systèmes non linéaires triangulaires. Chapitre 2: Conception d'observateur pour une classe de systèmes PDE-PDE. Chapitre 3: Conception d'observateur pour une classe de systèmes paraboliques non linéaires.

Dans le premier chapitre, on s'intéresse à la conception d'observateurs échantillonnés pour des systèmes non linéaires triangulaires où la sortie échantillonnée est affectée par un retard. Une autre difficulté majeure est considérée, il s'agit de la présence du signal de la sortie non retardée dans la matrice d'état, ce signal est inaccessible à la mesure, ce qui rend l'utilisation des observateurs existant dans la littérature inefficace. Dans la première partie du chapitre, un nouvel observateur est conçu où les effets du retard et de l'échantillonnage sont compensés à l'aide d'un prédicteur de la sortie.

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L'analyse de stabilité du système engendré par cet observateur est traitée à l'aide d'une fonctionnelle Lyapunov-Krasovskii. Des conditions suffisantes pour la convergence exponentielle sont établies en termes de valeurs maximales du retard et de l'intervalle d'échantillonnage. Dans la seconde partie du chapitre, le concept d'observateur en chaîne est invoqué et une nouvelle forme de l'observateur vu en première partie est présentée, où  $m$  sous observateurs élémentaires sont interconnectés en série. Le concept d'observateur en chaîne fonctionne de la manière suivante: le premier sous observateur reçoit la sortie échantillonnée, ensuite le  $j^{\text{ème}}$  sous observateur élémentaire reçoit une donnée virtuelle générée par le  $j - 1^{\text{ème}}$  sous observateur. Chaque sous observateur élémentaire compense l'effet d'une fraction du retard global du système, cela permet d'obtenir des observations plus exactes de l'état du système pour des valeurs plus grandes du retard. La nouvelle forme de l'observateur est analysée à l'aide d'outils similaires à ceux utilisés en première partie. En revanche, dans les conditions suffisantes on retrouve, en plus du terme de retard et d'intervalle d'échantillonnage, la présence du nombre de sous observateurs utilisés. Des résultats de simulation sont présentés, afin de prouver l'efficacité des observateurs proposés pour différentes valeurs du retard.

Dans le deuxième chapitre, le problème d'observation d'état pour des systèmes avec des sorties à retard est adressé, en considérant que le retard est une équation aux dérivées partielles hyperboliques du premier ordre connecté en série avec une équation différentielle ordinaire qui décrit les dynamiques du système à dimension finie, ce qui mène à une structure en cascade ODE-PDE. Le problème d'observation consiste à estimer l'état de dimension finie de la partie ODE et l'état de dimension infinie de la PDE, en utilisant la transformation en backstepping infinie. Afin d'obtenir un observateur dont l'état converge vers l'état du système pour de grandes valeurs du retard, le concept d'observateur en cascade est utilisé, dans cette approche on élabore un observateur composé de  $m$  sous observateurs à grand gain connectés en série. Des conditions suffisantes de la convergence exponentielle de l'état de l'observateur vers l'état du système sont obtenues à l'aide d'une fonctionnelle de Lyapunov bien définie. Des résultats de simulation sont présentés à la fin du chapitre pour confirmer les résultats théoriques.

On s'intéresse à un autre type de système dans le troisième chapitre, en effet une classe de systèmes paraboliques non linéaires est considérée où des mesures sont prises à des points distincts de l'intervalle. Ce problème a été traité dans le cas des petits retards dans [18], afin d'obtenir des résultats satisfaisants dans le cas de retards plus grands le concept de la représentation en cascade est invoqué. A l'aide d'une fonctionnelle Lyapunov-Krasovskii

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appropriée, des conditions suffisantes en termes de LMIs sont établies pour que l'observateur en chaîne fournisse des estimations qui convergent vers l'état infini du système réel. Dans ces conditions, on retrouve la présence d'un nombre minimal de sous observateurs élémentaires: plus le retard est grand, plus on augmente le nombre de sous observateurs. Une extension vers le cas échantillonné est présentée, la différence réside dans l'expression de la sortie réelle du système, ce qui nécessite la modification de la fonctionnelle Lyapunov-Krasovskii afin d'obtenir la stabilité du système d'erreur. Des résultats de simulation sont présentés à la fin du chapitre pour confirmer l'exactitude des résultats théoriques.

# General Introduction

The most widely used mathematical framework for control system studies consists of systems of algebraic equations, ordinary differential equations (ODE), and partial differential equations (PDE). Many systems from science and engineering are of a distributed parameter nature, i.e. systems characterized by state variables in two or more coordinates, time and space are the most frequent combination of independent variables. Distributed parameter systems (DPS) refer to systems whose dynamics involve time delay, partial differential equations, or functional differential equations defined on infinite dimensional spaces. Thus, they are called the infinite-dimensional systems (IDS) [37, 48]. Unlike the finite dimensional systems (FDS) for which there is a quite general control theoretical framework in the study of stability, observability and robustness [30], infinite dimensional systems are much more complicated to deal with because of the mathematical complexities of these models, studies have been conducted for this class of systems in [26, 34, 15]. In this work we focus on some classes of delayed systems and a type of partial differential equation.

Time-delay (also called dead-time or aftereffect) is a natural phenomenon in most physical systems and engineering applications. A short list of control applications in which delays arise includes:

- network controlled systems,
- traffic systems,
- chemical process control,
- drilling systems.

Several other examples can be found in the books [17, 33, 38]. Besides, control components, sensors and actuators, may also introduce additional delay in control systems. When these delays are relatively small, they can be safely ignored, however when this is not the case it is well established that the negligence of time-delays in control design may cause the instability of control

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systems. "This puts time-delay systems outside of the scope of standard methods that are commonly used for systems that have no delays, and it necessitates developing more reliable methods that can avoid the pitfalls of using traditional controllers on more complicated time-delay systems" [28]. For this reason, an intensive research activity has been devoted to control design for delayed systems, leading to thousands of papers and dozens of books especially over the last 15 years, the most recent significant books and surveys are [17, 33, 38, 39, 42, 50] see also the book [39] and references therein.

Some basic notions must be stated about time-delay. First, it should be known that time-delay systems (TDS) are not finite dimensional systems (FDS), but rather infinite dimensional systems (IDS), this means that the state is not a vector but a function or a vector of functions. Another major issue is that the stability analysis for time-delay systems is dealt with using Krasovskii functionals in comparison with non-delayed systems where Lyapunov functions are invoked.

Another important subject that will be dealt with during this work is the notion of partial differential equation (PDE). A partial differential equation is a differential equation that contains unknown multivariable functions and their partial derivatives up to a certain order. The main difference between the theory of PDEs and the theory of Ordinary Differential Equations (ODEs) is that an ODE involves a function (and its derivatives) of only one independent variable, while in PDE we deal with a function and its derivatives in several independent variables, this difference makes PDEs much harder to solve than ODEs [10, 36, 23]. A PDE for the multivariable function  $u(x_1, x_2, \dots, x_n)$  is an equation of the form

$$f(x_1, x_2, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial^2 u}{\partial x_1 \partial x_1}, \dots, \frac{\partial^2 u}{\partial x_1 \partial x_n}, \dots) = 0$$

Examples of partial differential equations are stated in the following:

- Heat flow in a rod is a system that can be represented by a parabolic PDE.
- Simply supported beam can be modeled by a second order hyperbolic PDE.
- The motion of viscous fluid substances is described by Navier–Stokes equations.

In spite of their complexity, delayed systems can be represented as a cascade of 2 sub-systems, where the first sub-system is the un-delayed part of the system and in the second part we let the delay be captured by a first order hyperbolic (PDE), this means that any time delayed system  $y(t) = z(t - D)$  can be replaced by

$$\begin{aligned} \frac{\partial u(x, t)}{\partial t} &= \frac{\partial u(x, t)}{\partial x}, \quad x \in [0, D], \\ z(t) &= u(D, t), \quad y(t) = u(0, t). \end{aligned}$$

This system is known to be a simple infinite dimensional model in the complex world of PDEs.

The main focus in this study will be the design of observers for different classes of delayed systems. State observer design is an important aspect of system control in view of the fact that observers are very useful, not only when sensing is not sufficient, but also when a better sensing reliability is needed [16], and the presence of time-delay makes the problem of observer design much more complex. Although time-delays are fundamentally of distributed parameter nature, observer design in presence of these elements has often been dealt with using finite-dimensional design tools. Accordingly, one starts with exponentially convergent state observers of ordinary differential equations (ODEs) (not involving time-delay) and modifies them so that exponential convergence is preserved in the presence of time-delay. The main approach is using one or several predictors of the output or the state. In the more challenging case of nonlinear systems, the maximum admissible delay (MAD) depends on the level of nonlinearity which, typically, is of globally Lipschitz nature. Roughly, the larger the Lipschitz constant, the smaller the MAD. This result has been illustrated using high-gain observers where the involved predictors proved to be useful in compensating the delay effect up to some upper limit. To get rid of this limitation, the concept of chain observer has been introduced in [19], and was extended to other cases of nonlinear systems [11, 6, 12, 25] and others.

In parallel with the above "finite-dimensional" research activity, the "infinite-dimensional" backstepping transformation design approach for linear systems has been developed in [32, 33]. It consists in letting the output sensor delay be captured through a first-order hyperbolic PDE connected in series with the ODE that describes the system finite-dimensional dynamics, leading to an ODE–PDE cascade representation of the system. Then, the observation problem consists in designing an observer that provides online

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estimates of both the (finite-dimensional) state of the ODE subsystem and the (infinite-dimensional) state of the PDE sensor.

## Outline of the thesis

The main interest of this thesis is the design of observers for different classes of nonlinear systems. The present work is organized in three chapters:

- Chapter 1: Sampled-data observer design for a class of triangular delayed nonlinear systems,
- Chapter 2: Chain observer design for a class of ODE-PDE systems,
- Chapter 3: Chain observer design for a class of nonlinear parabolic systems.

Chapter 1 focuses on a class of sampled nonlinear triangular systems where the sampled output is affected by a delay. Another major difficulty with the considered nonlinear systems is that the state matrix is dependent on the "undelayed output signal" which is not accessible to measurement, making existing observers inapplicable. In the first part of this chapter, a new observer is designed where the effects of time-delay and sampling are compensated for using an output predictor. The analysis of the proposed observer is treated using a suitable Lyapunov-Krasovskii functional, further the small gain technique is invoked. Sufficient conditions for the observer to be exponentially convergent are established in terms of the maximum time-delay and sampling interval. In the second part, the chain observer concept is presented, and a new extended form of the observer in part 1 is provided, this allows accurate observation of the state of the system for larger values of the delay. The new form of the observer is analysed using similar tools as in the first part.

Chapter 2 provides another concept for the study of the observation of triangular nonlinear systems, in this chapter, a new observer design method is developed for nonlinear systems with large transport delays. The new observer design is a generalization of the PDE-based backstepping-like observer design approach. First developed for delayed linear systems, this approach relies on a modelling of the output time-delay by a 1st order hyperbolic equation, leading to an ODE-PDE representation of the system, and on coordinate transformations of the innovative system. The major technical challenge, that is faced in the generalization of the approach to nonlinear systems, consists in making it applicable in the case of an arbitrarily large

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time-delay  $D$ . This issue is presently coped with by redesigning the cascade chain observer method to fit ODE–PDE cascade systems. A new class of observers is thus obtained involving a set of cascaded high-gain state observers and output predictors. The exponential stability of the observer is proved using a set of Lyapunov functionals.

Chapter 3 investigates the problem of observation for a class of nonlinear parabolic partial differential equations with delayed point measurements. The designed observer is of a chain form, this allows an accurate observation of the infinite dimensional state of the system for large delays. A Lyapunov-Krasovskii method is used for the stability analysis of the resulting error system, which is based on the application of Wirtinger’s and Halanay’s inequalities. Sufficient conditions for the exponential stabilization are derived in terms of Linear Matrix Inequalities (LMIs). By solving these LMIs, upper bounds on the delay and the number of sub-observers used in the chain that preserve the exponential stability are given. Later, sufficient conditions are derived for the case of sampled outputs in terms of new linear matrix inequalities. Solving these LMIs lead to the upper bounds on the delay, the upper bounds on the sampling intervals and the number of sub-observers used in the chain that preserve the exponential stability of the error dynamics.

Finally, this thesis is concluded with a summary of the work and directions for future work.

# Chapter 1

## Sampled-Data Observer Design For a Class of Triangular Delayed Nonlinear Systems

### 1.1 Introduction

The problem of designing sampled-output observers for continuous-time nonlinear systems is of prime importance in regard of the fact that nowadays control systems are implemented using digital means. This problem has been investigated since the early nineties [8], but it has witnessed a notable renewed interest on recent years, e.g. [2], [12], [18] because of the evolution of the digital field (sensors, actuators, computers,...). Just as for the time-delay case, the main underlying idea in all proposed sampled-data observer methods is to start the design with an exponentially convergent observer for ODEs (without output sampling) and modify these observers so that exponential convergence is preserved in presence of output measurement sampling. In [18] the sampling effect has been accounted for by using a zero-order-hold (ZOH) sampling of the output estimation error as innovation term in the observer state equation. This approach has proved to work well when applied to linear observable systems that are disturbed by a globally Lipschitz function of the state vector. To enlarge the admissible maximum sampling interval, the observer gain is let to be inter-sample exponentially decaying in [2] where exponential convergence conditions are expressed in terms of LMIs involving the sampling interval and other design parameters. The time-varying delay effect caused by output sampling can also be compensated for by inserting inter-sample output predictors. This idea has first been introduced and illustrated for triangular Lipschitz systems in [12].

## 1.2. CLASS OF SYSTEMS

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The problem of dealing simultaneously with both time-delay and output-sampling in observer design, has recently been investigated in [3], [4], [14]. The sampling and delay effects have been compensated for using inter-sample output predictors and state predictors. It was shown that the insertion of these predictors in any continuous-time observer, that is globally exponentially stable and robust with respect to output measurement errors, yields to an exponentially stable sampled-output observer.

In the first part of this chapter (from Section 1.2 to Section 1.4), sampling and delay effects are compensated for using only output predictors. Since no state predictors are involved, the new observer turns out to be much simpler, compared to those proposed in previously discussed works. Using the small gain method, sufficient conditions are established for the observer to be exponentially convergent. The sufficient conditions particularly involve the admissible maximum time-delay and sampling interval.

In the second part (Section 1.5), the previous system will be observed in the case of larger delays, this time the observer is composed of  $m$  elementary observers connected in series. The observer interconnection is such that the first elementary observer is directly driven by the output samples. Then, the  $j$ th elementary observer is driven by a virtual output generated by the  $(j - 1)$ th observer. Each elementary observer includes an output predictor which compensates for the effects of output sampling and a fraction of the system time-delay. In turn, using the small gain method, sufficient conditions are established for the observer to be exponentially convergent, however, in this case the sufficient conditions particularly involve the number of sub-observers, next to the admissible maximum time-delay and the sampling interval.

Simulation results will be presented in Section (1.6) in order to show the effectiveness of the proposed observers for different values of delay. A conclusion will end the chapter.

## 1.2 Class of Systems

The first class of systems under study is depicted by the following figure:

## 1.2. CLASS OF SYSTEMS

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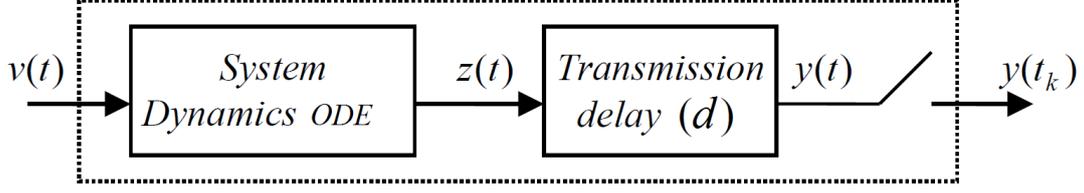


Figure 1.1 – System to be observed

This system is described by the following model:

$$\dot{x}(t) = A(v(t), z(t))x(t) + b(v(t), x(t)) \quad (1.1a)$$

$$z(t) = Cx(t) \quad (1.1b)$$

$$y(t) = z(t - d) = Cx(t - d) \quad (1.1c)$$

$$y(t_k) = z(t_k - d) = Cx(t_k - d) \quad (1.1d)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector;  $z(t) \in \mathbb{R}$  and  $y(t) \in \mathbb{R}$  are respectively the undelayed and delayed outputs;  $v(t) \in \mathbb{R}^m$  is a known external input; the integer  $n$  and the delay  $d$  are respectively known integer and real.

The  $t_k$ 's (for  $k = 0, 1, \dots$ ) denote the sampling instants. The set  $t_k$  is any partition of  $\mathbb{R}_+$  i.e. an increasing sequence such that  $t_0 = 0$ ,  $t_k \rightarrow \infty$  as  $k \rightarrow \infty$ , and  $0 < \tau < \infty$  with  $\tau = \sup_{0 \leq k \leq \infty} (t_k - t_{k-1})$ .

The row vector  $C \in \mathbb{R}^{1 \times n}$  and the matrices  $A(v, z) \in \mathbb{R}^{n \times n}$ ,  $b(v, x) \in \mathbb{R}^n$  are known and have the following triangular form:

$$A(v, z) = \begin{bmatrix} 0 & a_1(v, z) & 0 & \dots & 0 \\ 0 & 0 & a_2(v, z) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & \dots & a_{n-1}(v, z) \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix} \quad (1.2a)$$

$$b(v, x) = \begin{bmatrix} b_1(v, x_1) \\ b_2(v, x_1, x_2) \\ \vdots \\ b_n(v, x) \end{bmatrix} \quad (1.2b)$$

$$C = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \quad (1.2c)$$

where  $a(v, z) \in \mathbb{R}$  and  $b_i(v, x_1, \dots, x_i) \in \mathbb{R}$  are known functions.

## 1.2. CLASS OF SYSTEMS

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The observation problem will be dealt with in the next Section under the following assumptions:

- A1.** The functions  $a_i(v, z)$  are class  $\mathcal{C}^0$  with respect to  $v$  and  $\mathcal{C}^1$  with respect to  $z$ , while the functions  $b_i(v, x_1, \dots, x_i)$  are globally Lipschitz .
- A2.** All system signals  $(v, x, y, z)$  are bounded and an upper bound  $y_M$  on the output amplitude  $|y(t)|$  is known.
- A3.** The pair  $(A(v, z), C)$  is uniformly observable i.e.

$$|\det(O(v, z))| \geq \varepsilon_0 > 0, \quad (1.3)$$

$\forall v \in \mathbb{R}^m, \forall z \in \mathbb{R}$  for some (unknown) real constant  $\varepsilon_0$ , where  $O(v, z)$  denotes the  $n \times n$  matrix,

$$O(v, z) = \begin{bmatrix} C \\ CA(v, z) \\ \vdots \\ CA^{n-1}(v, z) \end{bmatrix} \quad (1.4)$$

- A4.** There is a real constant  $0 < l_\Gamma < \infty$ , such that:

$$\sup_{v \in \mathbb{R}^m, |z| \leq y_M} |\dot{\Gamma}(v, z)\Gamma^{-1}(v, z)| \leq l_\Gamma \quad (1.5)$$

where  $\Gamma(v, z) \in \mathbb{R}^{n \times n}$  is defined as follows:

$$\Gamma(v, z) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \prod_{i=1}^{n-1} a_i \end{bmatrix} \in \mathbb{R}^{n \times n} \quad (1.6)$$

The aim of this study is to design an observer providing online estimates  $\hat{x}(t)$  of the state vector  $x(t)$  such that the estimation error  $\hat{x}(t) - x(t)$  converges exponentially to the origin using only the external input  $v(t) \in \mathbb{R}^m$  and the delayed output samples  $y(t_k) = z(t_k - d) = Cx(t_k - d)$  for  $k = 0, 1, \dots$

## 1.2. CLASS OF SYSTEMS

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### Remark 1.

a) The complexity of this problem lies in:

- The presence of both delay and sampling effects
- The complexity of the model involving the signal  $z(t)$  (which is not accessible to measurements due to output delay) in the state matrix  $A(v, z)$ . This makes the first term on the right side of (1.1a) subject to a double uncertainty induced by the state vector  $x(t)$  and the undelayed output signal  $z(t)$ .

It turns out that the existing sampled-data observers are inappropriate for system (1.1a-1.1d).

b) In view of assumption A1, it follows from A2 that the functions  $a_i(v, z) (i = 1 \dots n)$  are also Lipschitz in  $z$  on the compact set  $|z(t)| \leq y_M$ . Since the input signal  $v(t)$  is bounded, there exists a real constant  $l_a$  such that, for  $i = 1 \dots n$  and all real numbers  $z_1, z_2 : |a_i(v, z_1) - a_i(v, z_2)| \leq l_a |z_1 - z_2|$ , where  $l_a$  is only dependent on the functions  $a_i(., .)$  and the supremum of  $|v(t)|$ .

c) Similarly, it readily follows from A2 that, there exists a Lipschitz constant  $l_b$  such that, for  $z_1, z_2 : |b(v, z_1) - b(v, z_2)| \leq l_b |z_1 - z_2|$ , for some constant  $l_b$  that is only dependent on  $b(., .)$  and on the supremum of  $|v(t)|$ . In this respect, let us note that instead of the globally Lipschitz assumption on  $b(., .)$ , one could alternately assume this function to be  $\mathcal{C}^1$  in  $x$ . Since the state vector trajectory  $x(t)$  lies in a known compact domain, say  $D_x$  (by assumption A1), one would conclude that  $b(., .)$  is Lipschitz on  $D_x$ . Then, the state estimation problem could be solved by replacing  $b(v(t), \hat{x}(t))$ , in the observer described in Section 1.3, by  $b(v(t), P(\hat{x}(t)))$  where  $P(., .)$  denotes the orthogonal projection on the domain  $D_x$ . This alternative has been illustrated (in the absence of output sampling and time-delay) in [46].

d) Assumption A3 is also usual in observer design literature. Presently, that assumption amounts to assuming that,  $\forall i, \inf_{|z| \leq y_M} |a_i(v, z)| \geq \varepsilon_a > 0$ ,  $v \in \mathbb{R}^m$  for some real constant  $\varepsilon_a > 0$ .

e) Assumption A4 is a technical condition induced by the fact that the state matrix  $A(v, z)$  is presently signal dependent. A similar assumption was required in the non-delayed non-sampled case dealt with in [43].

### 1.3. PROPOSED OBSERVER

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f) The class of systems defined by (1.1a)-(1.1d) is much wider than those in most existing works on sampled- and/or delayed-output based observer design, see e.g. [41], [27], [1], [2], [3], [12]. Indeed, in those works the state matrix  $A(v, z)$  is either constant or only dependent on the input signal  $v$ . The fact that  $z$  is presently inaccessible to measurements entails an extra difficulty.

## 1.3 Proposed Observer

Since no existing sampled-output observer is applicable to system (1.1a)-(1.1d), a new observer will now be constructed. To this end, some relevant expressions are immediately established from the problem formulation of Section 1.2. In this respect, the following saturation operator, suggested by assumption A1, will be used in the observer:

$$\sigma(\xi) = \text{sgn}(\xi)\min(y_M, |\xi|) \quad (1.7)$$

where  $\text{sgn}(\cdot)$  designates the sign function and  $y_M$  is as in assumption A1. On the other hand, it follows from equations (1.1a)-(1.1d) that the (delayed and non-delayed) outputs undergo the following ODEs, respectively:

$$\dot{y}(t) = CA(v(t-d), y(t))x(t-d) + Cb(v(t-d), x(t-d)) \quad (1.8)$$

$$\dot{z}(t) = CA(v(t), z(t))x(t) + Cb(v(t), x(t)) \quad (1.9)$$

In view of (1.1c), equation (1.9) also rewrites in the integral form:

$$z(t) = y(t) + \int_{t-d}^t C[A(v(s), z(s))x(s) + b(v(s), x(s))]ds \quad (1.10)$$

Considering the above observations and inspired from [27], the system model (1.1a)-(1.1d) suggests the following sampled-output observer:

For all  $t \in [t_k, t_{k+1})$ ,  $k = 0, 1, \dots$

$$\begin{aligned} \dot{\hat{x}}(t) &= A(v(t), \sigma(w_z(t)))\hat{x}(t) + b(v(t), \hat{x}(t)) \\ &\quad - \Gamma^{-1}(v(t), \sigma(w_z(t)))\Delta^{-1}K(C\hat{x}(t) - w_z(t)) \end{aligned} \quad (1.11a)$$

$$\dot{w}_y(t) = CA(v(t-d), \sigma(w_y(t)))\hat{x}(t-d) + Cb(v(t-d), \hat{x}(t-d)) \quad (1.11b)$$

$$w_y(t_k) = y(t_k) \quad (1.11c)$$

$$w_z(t) = w_y(t) + \int_{t-d}^t C[A(v(s), \sigma(w_z(s)))\hat{x}(s) + b(v(s), \hat{x}(s))]ds \quad (1.11d)$$

### 1.3. PROPOSED OBSERVER

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where:

$$\Delta = \text{diag} \left[ \frac{1}{\theta}, \dots, \frac{1}{\theta^n} \right] \in \mathbb{R}^{n \times n}, \text{ for any } \theta > 1, \quad (1.12)$$

and the gain  $K \in \mathbb{R}^n$  is chosen such that  $\bar{A} - KC$  is Hurwitz with:

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \quad (1.13)$$

A new feature of this observer is the presence of the saturation operator  $\sigma(\cdot)$ . This operator is used to ensure the boundedness of the observed signals  $w_z(t)$  and  $w_y(t)$ .

Note that  $K$  exists because the pair  $(A, C)$  is observable. In (1.11a), the initial state estimates  $\hat{x}(s)$  ( $-d < s < 0$ ) are arbitrarily chosen. In (1.11d), one uses the initial conditions  $w(s) = z(s) = y(s - d)$ , ( $-d \leq s < 0$ ). In view of (1.8)-(1.10), the variables  $w_y(t)$  and  $w_z(t)$  are nothing else than estimates of the delayed and undelayed outputs  $y(t)$  and  $z(t)$ , respectively. In fact,  $w_y(t)$  is a prediction of  $y(t)$  over the open intervals  $(t_k, t_{k+1})$ . It is generated by the predictor (1.11b)-(1.11c) from the output samples  $y(t_k), y(t_{k-1}), \dots$ . Finally, the filter (1.11d) is resorted to get the estimate  $w_z(t)$  of  $z(t)$ . Since none of  $w_y(t)$  and  $w_z(t)$  are a priori bounded, these signals enter through their saturated versions in the observer equations (1.11a)-(1.11d). Without using this saturation, it would not have been possible to ensure the boundedness of the various observer signals.

**Remark 2.**

a) The observer (1.11a)-(1.11d) is inspired by the high-gain observer proposed in [43] for continuous-time systems with no output delay and no output sampling. The main novelty of the new observer is the inter-sample predictor (1.11b)-(1.11d) providing the estimates  $w_z(t)$  (of the non-delayed output  $z(t)$ ) which is used in the innovation of the state equation (1.11a). Another new feature of the present observer, compared to [43], is the saturation operator  $\sigma(\cdot)$ . These novel features will prove to be useful in getting rid of the delay and sampling effects.

## 1.4. OBSERVER ANALYSIS

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b) Note that expression (1.11d) can be reformulated as follows:

$$w_z(t) = w_y(t) + \vartheta(t) \quad (1.14)$$

with  $\vartheta(t)$  defined by the following ODE:

$$\begin{aligned} \dot{\vartheta}(t) &= CA(v(t), \sigma(w_z(t)))\hat{x}(t) + Cb(v(t), \hat{x}(t)) \\ &\quad - CA(v(t-d), \sigma(w_z(t-d)))\hat{x}(t-d) \\ &\quad - Cb(v(t-d), \hat{x}(t-d)) \end{aligned} \quad (1.15)$$

$$\vartheta(0) = \int_{-d}^0 C[A(v(s), \sigma(w_z(s)))\hat{x}(s) + b(v(s), \hat{x}(s))]ds \quad (1.16)$$

## 1.4 Observer Analysis

The sampled-output observer defined by equations (1.11a)-(1.11d) is now going to be analyzed. For analysis purpose, the following errors are introduced:

$$\tilde{x} = \hat{x} - x, \quad e_y = w_y - y, \quad e_z = w_z - z \quad (1.17)$$

Subtracting system equations (1.1a), (1.8) and (1.10) from the corresponding observer equations, i.e. (1.11a), (1.11b) and (1.11d), one gets using (1.17) and rearranging terms the following equations describing the error dynamics:

For all  $t \in [t_k, t_{k+1})$ ,  $k = 0, 1, \dots$

$$\begin{aligned} \dot{\tilde{x}} &= [A(v, \sigma(w_z)) - \Gamma^{-1}(v, \sigma(w_z))\Delta^{-1}KC]\tilde{x} + [A(v, \sigma(w_z)) - A(v, z)]x \\ &\quad + (b(v, \hat{x}) - b(v, x)) + \Gamma^{-1}(v, \sigma(w_z))\Delta^{-1}Ke_z \end{aligned} \quad (1.18a)$$

$$\begin{aligned} \dot{e}_y(t) &= C[A(v(t-d), \sigma(w_y(t)))]\tilde{x}(t-d) + C[A(v(t-d), \sigma(w_y(t)))] \\ &\quad - A(v(t-d), y(t))x(t-d) + C[b(v(t-d), \hat{x}(t-d)) \\ &\quad - b(v(t-d), x(t-d))] \end{aligned} \quad (1.18b)$$

$$e_y(t_k) = 0 \quad (1.18c)$$

$$\begin{aligned} e_z(t) &= e_y(t) + \int_{t-d}^t CA(v(s), \sigma(w_z(s)))\tilde{x}(s)ds + \int_{t-d}^t C[A(v(s), \sigma(w_z(s)))] \\ &\quad - A(v(s), z(s))]x(s)ds \\ &\quad + \int_{t-d}^t C[b(v(s), \hat{x}(s)) - b(v(s), x(s))]ds \end{aligned} \quad (1.18d)$$

Note that the argument  $t$  has deliberately been omitted in (1.18a) to alleviate it. The main result is now stated in the following theorem.

#### 1.4. OBSERVER ANALYSIS

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**Theorem 1.1:** Let the proposed sampled-output observer (1.11a)-(1.11d) be applied to the system under study (1.1a)-(1.1d), subject to the stated assumptions A1-A4. Then, there exists a real constant  $1 < \theta^* < \infty$  such that if  $\theta > \theta^*$  then, there exist real constants  $0 < \tau^* < \infty$  and  $0 < d^* < \infty$  so that, if  $\tau < \tau^*$  and  $d < d^*$  then,  $\forall t \geq 0$  :

$$\|\tilde{x}(t)\| \leq \rho_x e^{-\alpha t/2}, \quad |e_y(t)| \leq \rho_y e^{-\alpha t/2}, \quad |e_z(t)| \leq \rho_z e^{-\alpha t/2}$$

for some real constants  $\alpha > 0$ ,  $\rho_x > 0$ ,  $\rho_y > 0$ ,  $\rho_z > 0$ .

**Proof:** The proof is partly inspired by [43], [27] and [25]. For convenience, the following coordinate change and notation are introduced:

$$\varepsilon = \Gamma(v, \sigma(w_z)) \Delta \tilde{x} \tag{1.19}$$

$$\tilde{A} = \bar{A} - KC \tag{1.20}$$

Then, equation (1.18a) rewrites in term of  $\varepsilon$  as follows:

$$\begin{aligned} \dot{\varepsilon} &= \Gamma(v, \sigma(w_z)) \Delta [A(v, \sigma(w_z)) - \Gamma^{-1}(v, \sigma(w_z)) \Delta^{-1} KC] \tilde{x} \\ &\quad + \Gamma(v, \sigma(w_z)) \Delta [A(v, \sigma(w_z)) - A(v, z)] x + \Gamma(v, \sigma(w_z)) \Delta (b(v, \hat{x}) - b(v, x)) \\ &\quad + \Gamma(v, \sigma(w_z)) \Delta \Gamma^{-1}(v, \sigma(w_z)) \Delta^{-1} K e_z + \dot{\Gamma}(v, \sigma(w_z)) \Delta \tilde{x} \\ &= \Gamma(v, \sigma(w_z)) \Delta [A(v, \sigma(w_z)) - \Gamma^{-1}(v, \sigma(w_z)) \Delta^{-1} KC] \Delta^{-1} \Gamma^{-1}(v, \sigma(w_z)) \Gamma(v, \sigma(w_z)) \Delta \tilde{x} \\ &\quad + \Gamma(v, \sigma(w_z)) \Delta [A(v, \sigma(w_z)) - A(v, z)] x + \Gamma(v, \sigma(w_z)) \Delta (b(v, \hat{x}) - b(v, x)) \\ &\quad + \Gamma(v, \sigma(w_z)) \Delta \Gamma^{-1}(v, \sigma(w_z)) \Delta^{-1} K e_z + \dot{\Gamma}(v, \sigma(w_z)) \Delta \tilde{x} \\ &= [\Gamma(v, \sigma(w_z)) \Delta A(v, \sigma(w_z)) \Delta^{-1} \Gamma^{-1}(v, \sigma(w_z)) - KC \Delta^{-1} \Gamma^{-1}(v, \sigma(w_z))] \Gamma(v, \sigma(w_z)) \Delta \tilde{x} \\ &\quad + \Gamma(v, \sigma(w_z)) \Delta [A(v, \sigma(w_z)) - A(v, z)] x + \Gamma(v, \sigma(w_z)) \Delta (b(v, \hat{x}) - b(v, x)) \\ &\quad + \Gamma(v, \sigma(w_z)) \Delta \Gamma^{-1}(v, \sigma(w_z)) \Delta^{-1} K e_z + \dot{\Gamma}(v, \sigma(w_z)) \Delta \tilde{x} \\ &= [\theta \tilde{A} - \theta KC] \varepsilon + \Gamma(v, \sigma(w_z)) \Delta [A(v, \sigma(w_z)) - A(v, z)] x \\ &\quad + \Gamma(v, \sigma(w_z)) \Delta (b(v, \hat{x}) - b(v, x)) + \Gamma(v, \sigma(w_z)) \Delta \Gamma^{-1}(v, \sigma(w_z)) \Delta^{-1} K e_z \\ &\quad + \dot{\Gamma}(v, \sigma(w_z)) \Gamma^{-1}(v, \sigma(w_z)) \varepsilon \\ &= \theta \tilde{A} \varepsilon + \Gamma(v, \sigma(w_z)) \Delta [A(v, \sigma(w_z)) - A(v, z)] x \\ &\quad + \Gamma(v, \sigma(w_z)) \Delta (b(v, \hat{x}) - b(v, x)) \\ &\quad + \Gamma(v, \sigma(w_z)) \Delta \Gamma^{-1}(v, \sigma(w_z)) \Delta^{-1} K e_z + \dot{\Gamma}(v, \sigma(w_z)) \Gamma^{-1}(v, \sigma(w_z)) \varepsilon \end{aligned} \tag{1.21}$$

## 1.4. OBSERVER ANALYSIS

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where the last equality is obtained using the following structural properties that one can easily check, using (1.6), (1.12) and (1.13):

$$\Gamma\Delta = \Delta\Gamma = \begin{bmatrix} \frac{1}{\theta} & 0 & \dots & \dots & 0 \\ 0 & \frac{a_1}{\theta^2} & 0 & \ddots & \vdots \\ \vdots & \ddots & \frac{a_1 a_2}{\theta^3} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & \frac{\prod_{i=1}^{n-1} a_i}{\theta^n} \end{bmatrix} \quad (1.22)$$

$$\Gamma A = \bar{A}\Gamma \quad (1.23a)$$

$$C\Delta^{-1}\Gamma^{-1} = \theta C \quad (1.23b)$$

$$\Gamma\Delta A\Delta^{-1}\Gamma^{-1} = \theta\bar{A} \quad (1.23c)$$

For all  $t \in [t_k, t_{k+1})$ ,  $k = 0, 1, \dots$

$$\begin{aligned} \dot{e}_y(t) &= a_1(v(t-d), \sigma(w_y(t)))\tilde{x}_2(t-d) \\ &\quad + [a_1(v(t-d), \sigma(w_y(t))) - a_1(v(t-d), y(t))]x_2(t-d) \\ &\quad + [b_1(v(t-d), \hat{x}(t-d)) - b_1(v(t-d), x(t-d))] \end{aligned} \quad (1.24a)$$

$$e_y(t_k) = 0 \quad (1.24b)$$

$$\begin{aligned} e_z(t) &= e_y(t) + \int_{t-d}^t a_1(v(s), \sigma(w_z(s)))\tilde{x}_2(s)ds \\ &\quad + \int_{t-d}^t [a_1(v(s), \sigma(w_z(s))) - a_1(v(s), z(s))]x_2(s)ds \\ &\quad + \int_{t-d}^t [b_1(v(s), \hat{x}(s)) - b_1(v(s), x(s))]ds \end{aligned} \quad (1.24c)$$

The proof is divided into two parts:

- Part1: Proof that the mapping  $e_z \rightarrow \varepsilon$  is input to state stable.
- Part2: Proof that the mapping  $e_y \rightarrow \varepsilon$  is also input to state stable.

For this we will show that  $\|\varepsilon\| \leq \eta(\sup_{0 \leq \tau \leq t} (|e_z(\tau)|))$  in the first part and that  $\|\varepsilon\| \leq \eta(\sup_{0 \leq \tau \leq t} (|e_y(\tau)|))$ . Finally, the small gain theorem will be invoked to establish the exponential convergence of all estimation errors ( $\tilde{x}, e_y, e_z$ ) (appendix A).

## 1.4. OBSERVER ANALYSIS

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### **Proof that the mapping $e_z \rightarrow \varepsilon$ is input to state stable**

In order to analyze the system (1.18a)-(1.18d), consider the candidate Lyapunov function :

$$V = \varepsilon^T P \varepsilon \quad (1.25)$$

where  $P = P^T$  is the unique positive definite matrix that satisfies :

$$P\tilde{A} + \tilde{A}^T P = -\mu I \quad (1.26)$$

where  $I$  is the identity matrix and  $\mu > 0$  is arbitrarily chosen. Note that  $P$  exists because we know that  $\tilde{A} = \bar{A} - KC$  is Hurwitz. Time-differentiation of  $V$  yields, using (1.25) and (1.21):

$$\begin{aligned} \dot{V} &= \dot{\varepsilon}^T P \varepsilon + \varepsilon^T P \dot{\varepsilon} \\ &= -\mu \theta \|\varepsilon\|^2 + 2\varepsilon^T P \Gamma(v, \sigma(w_z)) \Delta(A(v, \sigma(w_z)) - A(v, z))x \\ &\quad + 2\varepsilon^T P \Gamma(v, \sigma(w_z)) \Delta(b(v, \hat{x}) - b(v, x)) \\ &\quad + 2\varepsilon^T P \Gamma(v, \sigma(w_z)) \Delta \Gamma^{-1}(v, \sigma(w_z)) \Delta^{-1} K e_z \\ &\quad + 2\varepsilon^T P \dot{\Gamma}(v, \sigma(w_z)) \Gamma^{-1}(v, \sigma(w_z)) \varepsilon \end{aligned} \quad (1.27)$$

Next, we are going to show that the terms on the right side of (1.27) are bounded. First, recall that, by assumption A2, all terms  $a_i(v, z)$  ( $i = 1, \dots, n-1$ ) of the matrix  $A(v, z)$  are  $\mathcal{C}^1$  functions. Also, it follows from assumption A1 and equation (1.7) that:

$$|\sigma(w_z(t))| \leq y_M \quad \text{and} \quad \sigma(z(t)) = z(t), \quad \forall t \geq 0 \quad (1.28)$$

Then, one gets using Remark 1 (Part b):

$$|a_i(v, \sigma(w_z)) - a_i(v, z)| \leq l_a |e_z|, \quad i = 1, \dots, n-1 \quad (1.29)$$

Then, it follows from (1.6) and (1.12) that the second term:

$$\begin{aligned} \left\| 2\varepsilon^T P \Gamma(v, \sigma(w_z)) \Delta(A(v, \sigma(w_z)) - A(v, z))x \right\| &\leq \frac{2l_a \beta_1}{\theta} \|\varepsilon\| |e_z| \\ &\leq \frac{l_a \beta_1}{\theta} (\|\varepsilon\|^2 + |e_z|^2) \end{aligned} \quad (1.30)$$

using assumption A1 and the fact that  $\theta > 1$ , where  $\beta_1$  is a real constant depending on  $P$ ,  $y_M$  and the suprema of  $\|v(t)\|$  and  $\|x(t)\|$ , but is independent on  $\theta$ .

Again, using the triangular form of  $b(v, x)$ , the fact that the functions  $b_i(v, x)$  are globally Lipschitz and the diagonal form of  $\Gamma \Delta$ , it follows that:

$$\left\| \Delta(b(v(t), \hat{x}(t)) - b(v(t), x(t))) \right\| \leq \beta_2 \|\varepsilon\| \quad (1.31)$$

#### 1.4. OBSERVER ANALYSIS

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where  $\beta_2$  is a real constant independent on  $\theta$ . Then, making use of (1.6), assumptions A1-A2, and Remark 1 (Part c), one gets :

$$\left\| 2\varepsilon^T P \Gamma(v, \sigma(w_z)) \Delta(b(v, \hat{x}) - b(v, x)) \right\| \leq 2\beta_3 \beta_2 \|\varepsilon\|^2 \quad (1.32)$$

for some real constant  $\beta_3$ , independent on  $\theta$ , only dependent on  $y_M$ ,  $P$  and the functions  $a_i$ . The penultimate term on the right side of (1.26) is bounded as follows, using (1.22):

$$\begin{aligned} \left\| 2\varepsilon^T P \Gamma(v, \sigma(w_z)) \Delta \Gamma^{-1}(v, \sigma(w_z)) \Delta^{-1} K e_z \right\| &= \left\| 2\varepsilon^T P K e_z \right\| \\ &\leq \beta_4 (\|\varepsilon\|^2 + \|e_z\|^2) \end{aligned} \quad (1.33)$$

with  $\beta_4 = |PK|$ . Finally, the following bounding from above is readily got for last term on the right side of (1.26), using (1.5):

$$\left\| 2\varepsilon^T P \dot{\Gamma}(v, \sigma(w_z)) \Gamma^{-1}(v, \sigma(w_z)) \varepsilon \right\| \leq 2l_\Gamma \beta_5 \|\varepsilon\|^2 \quad (1.34)$$

with  $\|\beta_5\| = |P|$ . Using (1.34), (1.33), (1.32), and (1.30), it follows from (19) that:

$$\begin{aligned} \dot{V} &\leq -\gamma_\theta \|\varepsilon\|^2 + \beta_6 |e_z|^2 \\ \dot{V} &\leq -\alpha_\theta V + \beta_6 |e_z|^2 \end{aligned} \quad (1.35)$$

with

$$\alpha_\theta = \frac{\gamma_\theta}{\lambda_{\max}(P)} = \frac{1}{\lambda_{\max}(P)} \left( \mu\theta - \frac{l_a \beta_1}{\theta} - 2\beta_2 \beta_3 - \beta_4 - 2l_\Gamma \beta_5 \right) \quad (1.36)$$

$$\beta_6 = l_a \beta_1 + \beta_4 \quad (1.37)$$

using (1.24c) and the fact that  $\theta > 1$ . Clearly, from (1.36) one has:

$$\lim_{\theta \rightarrow +\infty} \alpha_\theta = \infty \quad (1.38)$$

In the sequel, we let  $\theta$  be sufficiently large so that:

$$\alpha_\theta > 1 \quad (1.39)$$

This is possible because none of the constants on the right side of (1.36) is dependent on  $\theta$ . Integrating inequality (1.35) gives, for all  $t \geq 0$ :

$$V(t) \leq e^{-\alpha_\theta t} V(0) + \beta_6 \int_0^t e^{-\alpha_\theta(t-s)} e_z^2(s) ds \quad (1.40)$$

#### 1.4. OBSERVER ANALYSIS

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Given any scalar  $\alpha$  such that  $0 < \alpha < \alpha_\theta/2$ , it follows, multiplying both sides of (1.40) by  $e^{\alpha t}$  :

$$e^{\alpha t}V(t) \leq M_0 + \beta_6 e^{\alpha t} \int_0^t e^{-\alpha_\theta(t-s)} e^{-\alpha s} e^{\alpha s} e_z^2(s) ds \quad (1.41)$$

with  $M_0 = V(0)$ , using the fact that  $e^{-\alpha_\theta t} < 1$ , when  $t > 0$ . Inequality (1.41) entails, successively:

$$\begin{aligned} e^{\alpha t}V(t) &\leq M_0 + \beta_6 e^{\alpha t} \int_0^t e^{-\alpha_\theta(t-s)} e_z^2(s) ds \\ &\leq M_0 + \beta_6 e^{(\alpha-\alpha_\theta)t} \int_0^t e^{(\alpha_\theta-\alpha)s} e^{\alpha s} e_z^2(s) ds \\ &\leq M_0 + \beta_6 e^{(\alpha-\alpha_\theta)t} \left( \int_0^t e^{(\alpha_\theta-\alpha)s} ds \right) \sup_{0 \leq s \leq t} (e^{\alpha s} e_z^2(s)) \\ &\leq M_0 + \frac{2\beta_6}{\alpha_\theta} \sup_{0 \leq s \leq t} (e^{\alpha s} e_z^2(s)) \end{aligned} \quad (1.42)$$

Taking square root of both sides of (1.42) gives:

$$e^{\frac{\alpha t}{2}} \sqrt{V(t)} \leq \sqrt{M_0} + \sqrt{\frac{2\beta_6}{\alpha_\theta}} \sup_{0 \leq s \leq t} (e^{\frac{\alpha s}{2}} |e_z(s)|) \quad (1.43)$$

From (1.24c) and (1.19) one has, using (1.22) and Remark 1 (Part c):

$$V \geq \lambda_{\min}(P) \|\varepsilon\|^2 \quad (1.44)$$

where  $\lambda_{\min}(P)$  denotes the minimal eigenvalue of  $P$  (which is clearly independent on  $\theta$ ). Then, it follows from (1.43):

$$e^{\frac{\alpha t}{2}} \|\varepsilon(t)\| \leq M_1 + \gamma_1 \sup_{0 \leq s \leq t} (e^{\frac{\alpha s}{2}} |e_z(s)|) \quad (1.45)$$

with

$$M_1 = \sqrt{\frac{M_0}{\lambda_{\min}(P)}}, \quad \gamma_1 = \sqrt{\frac{2\beta_6}{\alpha_\theta \lambda_{\min}(P)}} \quad (1.46)$$

Since the left side on (1.45) is non-decreasing with  $t$ , it follows that:

$$\sup_{0 \leq s \leq t} \left( e^{\frac{\alpha t}{2}} \|\varepsilon(s)\| \right) \leq M_1 + \gamma_1 \sup_{0 \leq s \leq t} (e^{\frac{\alpha s}{2}} |e_z(s)|) \quad (1.47)$$

This last inequality shows that the mapping  $e_z \rightarrow \varepsilon$  is input-to-state stable.

## 1.4. OBSERVER ANALYSIS

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**Proof that the mapping  $e_y \rightarrow \varepsilon$  is also input to state stable**  
Using (1.19), (1.29) and assumption A1, it follows from (1.24c) the following upper bounding of  $|e_z|$  by  $|e_y|$  and  $\|\varepsilon\|$  :

For all  $t \in [t_k, t_{k+1})$ ,  $k = 0, 1, \dots$

$$\begin{aligned}
|e_z(t)| &\leq |e_y(t)| + \int_{t-d}^t \|a_1(v(s), \sigma(w_z(s)))\tilde{x}_2(s)\| ds \\
&\quad + \int_{t-d}^t \|[a_1(v(s), \sigma(w_z(s))) - a_1(v(s), z(s))]x_2(s)\| ds \\
&\quad + \int_{t-d}^t \|b_1(v(s), \hat{x}(s)) - b_1(v(s), x(s))\| ds \tag{1.48}
\end{aligned}$$

for some real constant  $\beta_7$  (depending on the supremum of  $\|x\|$ , but independent on  $\theta$ ), where we have used the fact that

$$a_1(v(s), \sigma(w_z(s)))\tilde{x}_2(s) = \theta^2 \varepsilon_2(s) \tag{1.49}$$

(got from (1.19)):

$$\Gamma \Delta \tilde{x} = \begin{bmatrix} \frac{1}{\theta} \tilde{x}_1 \\ a_1 \tilde{x}_1 \\ \frac{\theta^2}{a_1 a_2} \tilde{x}_2 \\ \frac{a_1 a_2}{\theta^3} \tilde{x}_3 \\ \dots \\ \prod_{i=1}^{n-1} a_i \\ \frac{\prod_{i=1}^{n-1} a_i}{\theta^n} \tilde{x}_n \end{bmatrix} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \dots \\ \varepsilon_n \end{bmatrix} \tag{1.50}$$

The last quantity on the right side of (1.48) is handled using the Lipschitz and triangular nature of the  $b(v, x)$  (see (1.2b) and assumption A2). Accordingly, one has:

$$\|b_1(v(t), \hat{x}(t)) - b_1(v(t), x(t))\| \leq l_b \|\tilde{x}_1(t)\| \tag{1.51}$$

with  $l_b$  as in Remark 1 (Part c). From (1.50), it is immediately seen that  $\tilde{x}_1 = \theta \varepsilon_1$  which together with (1.51) gives:

$$\|b_1(v(t), \hat{x}(t)) - b_1(v(t), x(t))\| \leq \theta l_b \|\varepsilon_1(t)\| \tag{1.52}$$

## 1.4. OBSERVER ANALYSIS

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which yields, together with (1.48):

$$\begin{aligned}
|e_z(t)| &\leq |e_y(t)| + \theta(\theta + l_b) \int_{t-d}^t \|\varepsilon_1(s)\| ds + l_a \beta_7 \int_{t-d}^t |e_z(s)| ds \\
&\leq |e_y(t)| + \theta(\theta + l_b) \int_{t-d}^t e^{-\alpha s/2} e^{\alpha s/2} \|\varepsilon(s)\| ds \\
&\quad + l_a \beta_7 \int_{t-d}^t e^{-\alpha s/2} e^{\alpha s/2} |e_z(s)| ds \\
&\leq |e_y(t)| + \theta(\theta + l_b) \int_{t-d}^t e^{-\alpha s/2} ds \left( \sup_{t-d \leq s \leq t} e^{\alpha s/2} \|\varepsilon(s)\| \right) \\
&\quad + l_a \beta_7 \int_{t-d}^t e^{-\alpha s/2} ds \left( \sup_{t-d \leq s \leq t} e^{\alpha s/2} |e_z(s)| \right) \\
&\leq |e_y(t)| + \theta(\theta + l_b) d e^{-\alpha(t-d)/2} \left( \sup_{t-d \leq s \leq t} e^{\alpha s/2} \|\varepsilon(s)\| \right) \\
&\quad + l_a \beta_7 d e^{-\alpha(t-d)/2} \left( \sup_{t-d \leq s \leq t} e^{\alpha s/2} |e_z(s)| \right) \tag{1.53}
\end{aligned}$$

where the last inequality is obtained applying the mean value theorem to the integral quantity  $\int_{t-d}^t e^{-\alpha s/2} ds$ . Multiplying both sides of (1.53) by  $e^{\alpha t/2}$  yields:

$$\begin{aligned}
e^{\alpha t/2} |e_z(t)| &\leq e^{\alpha t/2} |e_y(t)| + \theta(\theta + l_b) d e^{\alpha d/2} \left( \sup_{t-d \leq s \leq t} e^{\alpha s/2} \|\varepsilon(s)\| \right) \\
&\quad + l_a \beta_7 d e^{\alpha d/2} \left( \sup_{t-d \leq s \leq t} e^{\alpha s/2} |e_z(s)| \right) \tag{1.54}
\end{aligned}$$

which implies:

$$\begin{aligned}
\sup_{0 \leq s \leq t} \left( e^{\alpha s/2} |e_z(s)| \right) &\leq \sup_{0 \leq s \leq t} \left( e^{\alpha s/2} |e_y(s)| \right) \\
&\quad + \theta(\theta + l_b) d e^{\alpha d/2} \left( \sup_{t-d \leq s \leq t} e^{\alpha s/2} \|\varepsilon(s)\| \right) \\
&\quad + l_a \beta_7 d e^{\alpha d/2} \left( \sup_{t-d \leq s \leq t} e^{\alpha s/2} |e_z(s)| \right) \tag{1.55}
\end{aligned}$$

Since  $d e^{\alpha d/2} \xrightarrow{d \rightarrow 0} 0$ , we let  $d$  be such that:

$$0 < l_a \beta_7 d e^{\alpha d/2} < 1 \tag{1.56}$$

Then, one gets from (1.56):

$$\begin{aligned}
\sup_{0 \leq s \leq t} \left( e^{\alpha s/2} |e_z(s)| \right) &\leq \frac{1}{1 - l_a \beta_7 d e^{\alpha d/2}} \sup_{0 \leq s \leq t} \left( e^{\alpha s/2} |e_y(s)| \right) \\
&\quad + \frac{\theta(\theta + l_b) d e^{\alpha d/2}}{1 - l_a \beta_7 d e^{\alpha d/2}} \sup_{t-d \leq s \leq t} \left( e^{\alpha s/2} \|\varepsilon(s)\| \right) \tag{1.57}
\end{aligned}$$

## 1.4. OBSERVER ANALYSIS

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In order to invoke the small gain theorem (appendix A) we start by integrating the expressions (1.24a)-(1.24c):

For all  $t \in [t_k, t_{k+1})$ ,  $k = 0, 1, \dots$

$$\begin{aligned} e_y(t) &= \int_{t_k}^t a_1(v(t-d), \sigma(w_y(t))) \tilde{x}_2(t-d) ds \\ &+ \int_{t_k}^t [a_1(v(t-d), \sigma(w_y(t))) - a_1(v(t-d), y(t))] x_2(t-d) ds \\ &+ \int_{t_k}^t [b_1(v(t-d), \hat{x}(t-d)) - b_1(v(t-d), x(t-d))] ds \end{aligned} \quad (1.58)$$

$$e_y(t_k) = 0 \quad (1.59)$$

Taking the norms of both sides of (1.24a) gives, using similar arguments as for (1.48) to (1.53):

For all  $t \in [t_k, t_{k+1})$ ,  $k = 0, 1, \dots$

$$\begin{aligned} |e_y(t)| &\leq \int_{t_k}^t \left\| \theta^2 \varepsilon_2(s-d) \right\| ds + l_a \beta_7 \int_{t_k}^t |e_y(s)| ds + \int_{t_k}^t \theta l_b \|\varepsilon_1(s-d)\| ds \\ &\leq \theta(\theta + l_b) \int_{t_k}^t \|\varepsilon(s-d)\| ds + l_a \beta_7 \int_{t_k}^t |e_y(s)| ds \end{aligned} \quad (1.60)$$

$$e_y(t_k) = 0 \quad (1.61)$$

where  $\beta_7$  is as in (1.53). Inequality (1.60) develops further as follows:

For all  $t \in [t_k, t_{k+1})$ ,  $k = 0, 1, \dots$

$$\begin{aligned} |e_y(t)| &\leq \theta(\theta + l_b) \int_{t_k}^t e^{-\alpha(s-d)/2} e^{\alpha(s-d)/2} \|\varepsilon(s-d)\| ds \\ &+ l_a \beta_7 \int_{t_k}^t e^{-\alpha s/2} e^{\alpha s/2} |e_y(s)| ds \\ &\leq \theta(\theta + l_b) \int_{t_k}^t e^{-\alpha s/2} ds \left( \sup_{t_k \leq s \leq t} e^{\alpha(s-d)/2} \|\varepsilon(s-d)\| \right) \\ &+ l_a \beta_7 \int_{t_k}^t e^{-\alpha s/2} ds \left( \sup_{t_k \leq s \leq t} e^{\alpha s/2} |e_y(s)| \right) \\ &\leq \theta(\theta + l_b) e^{\alpha d/2} \tau e^{-\alpha t_k/2} \left( \sup_{0 \leq s \leq t} e^{\alpha(s-d)/2} \|\varepsilon(s-d)\| \right) \\ &+ l_a \beta_7 \tau e^{-\alpha t_k/2} \left( \sup_{0 \leq s \leq t} e^{\alpha s/2} |e_y(s)| \right) \end{aligned} \quad (1.62)$$

where we have used the double inequality  $0 < \int_{t_k}^t e^{-\alpha s/2} ds \leq \tau e^{-\alpha t_k/2}$ . Multiplying both sides by  $e^{\alpha t/2}$  yields:

#### 1.4. OBSERVER ANALYSIS

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For all  $t \in [t_k, t_{k+1})$ ,  $k = 0, 1, \dots$

$$\begin{aligned} e^{\alpha t/2} |e_y(t)| &\leq \theta(\theta + l_b) e^{\alpha d/2} \tau e^{\alpha \tau/2} \left( \sup_{0 \leq s \leq t} e^{\alpha(s-d)/2} \|\varepsilon(s-d)\| \right) \\ &\quad + l_a \beta_7 \tau e^{\alpha \tau/2} \left( \sup_{0 \leq s \leq t} e^{\alpha s/2} |e_y(s)| ds \right) \end{aligned} \quad (1.63)$$

Since the right side terms are increasing functions of  $t$ , it follows that:

For all  $t \in [t_k, t_{k+1})$ ,  $k = 0, 1, \dots$

$$\begin{aligned} \sup_{0 \leq s \leq t} e^{\alpha s/2} |e_y(s)| &\leq \theta(\theta + l_b) e^{\alpha d/2} \tau e^{\alpha \tau/2} \left( \sup_{0 \leq s \leq t} e^{\alpha(s-d)/2} \|\varepsilon(s-d)\| \right) \\ &\quad + l_a \beta_7 \tau e^{\alpha \tau/2} \left( \sup_{0 \leq s \leq t} e^{\alpha s/2} |e_y(s)| ds \right) \end{aligned} \quad (1.64)$$

Since  $\tau e^{\alpha \tau/2} \xrightarrow{\tau \rightarrow 0} 0$ , we let  $\tau$  be such that:

$$0 < l_a \beta_7 \tau e^{\alpha \tau/2} < 1 \quad (1.65)$$

Then, one gets from (1.64):

$$\sup_{0 \leq s \leq t} (e^{\frac{\alpha s}{2}} |e_y(s)|) \leq \frac{\theta(\theta + l_b) e^{\alpha d/2} \tau e^{\alpha \tau/2}}{(1 - l_a \beta_7 \tau e^{\alpha \tau/2})} \left( \sup_{0 \leq s \leq t} e^{\alpha(s-d)/2} \|\varepsilon(s-d)\| \right) \quad (1.66)$$

This inequality shows that the mapping  $\varepsilon \rightarrow e_y$  is input-to-state stable. Using (1.66) it follows from (1.57) that:

$$\begin{aligned} \sup_{0 \leq s \leq t} (e^{\frac{\alpha s}{2}} |e_z(s)|) &\leq \frac{\theta(\theta + l_b) e^{\alpha d/2} \tau e^{\alpha \tau/2}}{(1 - l_a \beta_7 d e^{\alpha d/2})(1 - l_a \beta_7 \tau e^{\alpha \tau/2})} \left( \sup_{0 \leq s \leq t} e^{\alpha(s-d)/2} \|\varepsilon(s-d)\| \right) \\ &\quad + \frac{\theta(\theta + l_b) d e^{\alpha d/2}}{(1 - l_a \beta_7 d e^{\alpha d/2})} \left( \sup_{t-d \leq s \leq t} e^{\alpha s/2} \|\varepsilon(s)\| \right) \end{aligned} \quad (1.67)$$

Combining (1.67) and (1.45) one gets:

$$\begin{aligned} \sup_{0 \leq s \leq t} e^{\alpha s/2} \|\varepsilon(s)\| &\leq M_1 + \frac{\gamma_1 \theta(\theta + l_b) e^{\alpha d/2} \tau e^{\alpha \tau/2}}{(1 - l_a \beta_7 d e^{\alpha d/2})(1 - l_a \beta_7 \tau e^{\alpha \tau/2})} \left( \sup_{0 \leq s \leq t} e^{\alpha(s-d)/2} \|\varepsilon(s-d)\| \right) \\ &\quad + \frac{\gamma_1 \theta(\theta + l_b) d e^{\alpha d/2}}{(1 - l_a \beta_7 d e^{\alpha d/2})} \left( \sup_{t-d \leq s \leq t} e^{\alpha s/2} \|\varepsilon(s)\| \right) \end{aligned} \quad (1.68)$$

then,

$$\sup_{0 \leq s \leq t} (e^{\frac{\alpha s}{2}} \|\varepsilon(s)\|) \leq M_2 + \gamma_2 \sup_{0 \leq s \leq t} (e^{\frac{\alpha s}{2}} \|\varepsilon(s)\|) \quad (1.69)$$

## 1.4. OBSERVER ANALYSIS

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with

$$M_2 = M_1 + \frac{\gamma_1 \theta (\theta + l_b) e^{\alpha d/2} \tau e^{\alpha \tau/2}}{(1 - l_a \beta_7 d e^{\alpha d/2})(1 - l_a \beta_7 \tau e^{\alpha \tau/2})} \left( \sup_{-d \leq s \leq 0} e^{\alpha(s-d)/2} \|\varepsilon(s)\| \right) \quad (1.70)$$

where

$$\gamma_2 = \frac{\gamma_1 \theta (\theta + l_b) e^{\alpha d/2} [\tau e^{\alpha \tau/2} + d(1 - l_a \beta_7 \tau e^{\alpha \tau/2})]}{(1 - l_a \beta_7 d e^{\alpha d/2})(1 - l_a \beta_7 \tau e^{\alpha \tau/2})} \quad (1.71)$$

Let  $d$  and  $\tau$  be such that:

$$\gamma_2 < 1 \quad (1.72)$$

This is not an issue because  $\tau e^{\alpha \tau/2} \xrightarrow{\tau \rightarrow 0} 0$  and  $d e^{\alpha d/2} \xrightarrow{d \rightarrow 0} 0$ . Then, it follows from (1.69) that:

$$\sup_{0 \leq s \leq t} (e^{\frac{\alpha s}{2}} \|\varepsilon(s)\|) \leq \frac{M_2}{1 - \gamma_2} \quad (1.73)$$

which establishes that  $\varepsilon(t)$  is exponentially convergent to the origin. Then, using (1.22) and Remark 1 (Part c), it follows from (1.19) that in turn  $\tilde{x}(t)$  is also exponentially convergent to the origin and, by (1.66) and (1.67), the same result holds for  $e_y$  and  $e_z$ . This ends the proof of **Theorem 1.1**.

### Remark 3.

Admissible values of the delay  $d$  and the maximum sampling interval  $\tau$  are those satisfying conditions (1.56), (1.65) and (1.72). Accordingly, the maximum admissible values, say  $d_M$  and  $\tau_M$ , depend on the free parameter  $\alpha$  which has been introduced for analysis purpose. The smaller  $\alpha$  the larger the maximum admissible values. It follows that  $d_M$  and  $\tau_M$  are obtained by letting  $\alpha = 0$  in (1.56), (1.65) and (1.72). Doing so one gets:

$$d_M < \frac{1}{l_a \beta_7}, \quad \tau_M < \frac{1}{l_a \beta_7} \quad (1.74)$$

$$\frac{\tau_M + r_M(1 - l_a \beta_7 \tau_M)}{(1 - l_a \beta_7 d_M)(1 - l_a \beta_7 \tau_M)} < \frac{1}{\gamma_1 \theta (\theta + l_b)} \quad (1.75)$$

Then, the sets of admissible delay and sampling interval are respectively defined by:

$$0 < d < d_M \quad \text{and} \quad 0 < \tau < \tau_M \quad (1.76)$$

Inequalities (1.74)-(1.76) show that we are able to give explicit expressions for the admissible values of the delay  $d$  and the maximum sampling interval  $\tau$  for which the state of the proposed observer (1.11a)-(1.11d) converges to the state of the system (1.1a)-(1.1d).

## 1.5 Sampled-output chain-observer design

In this part, we deal with the same class of systems (1.1a)-(1.1d) but the delay in this case is not necessarily small. In order to enlarge the interval of admissible time-delay values, a set of predictors operating in cascade are implemented, this technique was introduced by [19]. We start by writing the system (1.1a)-(1.1d) as a cascade representation. Then we present the proposed chain observer for this class. After that, the observer will be analyzed and we will see that the observation errors will converge exponentially to the origin.

### 1.5.1 System cascade representation

Letting  $m \geq 1$  be any fixed integer, define the following signals, where  $j = 1, \dots, m$ :

For all  $t \geq 0$ ,

$$x_j(t) = x\left(t + j\frac{d}{m} - d\right), \quad (1.77a)$$

$$z_j(t) = z\left(t + j\frac{d}{m} - d\right) \quad (1.77b)$$

$$y_j(t) = z_j\left(t - \frac{d}{m}\right) = Cx_j\left(t - \frac{d}{m}\right), \quad (1.77c)$$

$$v_j(t) = v\left(t + j\frac{d}{m} - d\right) \quad (1.77d)$$

From, (1.77a)-(1.77d), one immediately gets the following useful relations:

$$x_{j-1}(t) = x_j\left(t - \frac{d}{m}\right), \quad (1.78a)$$

$$z_{j-1}(t) = z\left(t - \frac{d}{m}\right) \text{ for } j = 2, \dots, m \quad (1.78b)$$

$$y_j(t) = z_{j-1}(t), \quad \text{for } j \geq 2 \quad (1.78c)$$

Now, using (1.1a)-(1.1d), it readily follows from (1.77a)-(1.77d) that  $x_j(t)$  and  $z_j(t)$  undergo the following ODEs, where  $j = 1, \dots, m$ :

$$\dot{x}_j(t) = A(v_j(t), z_j(t))x_j(t) + b(v_j(t), x_j(t)) \quad (1.79a)$$

$$\dot{y}_j(t) = CA(v_j(t-r), y_j(t))x_j(t-r) + Cb(v_j(t-r), x_j(t-r)) \quad (1.79b)$$

$$\dot{z}_j(t) = CA(v_j(t), z_j(t))x_j(t) + Cb(v_j(t), x_j(t)) \quad (1.79c)$$

with

$$r = \frac{d}{m} \quad \text{and} \quad v_j(t) = v\left(t + j\frac{d}{m} - d\right) \quad (1.80)$$

## 1.5. SAMPLED-OUTPUT CHAIN-OBSERVER DESIGN

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In view of (1.78b) and (1.79b), one gets the following useful integral form:

$$\begin{aligned} \dot{z}_j(t) &= z_{j-1}(t) + C \int_{t-r}^t \left( A(v_j(s), z_j(s))x_j(s) + b(v_j(s), x_j(s)) \right) ds \\ &\text{for } j = 2 \dots m \end{aligned} \quad (1.81)$$

In view of (1.78c), (1.81) also writes:

$$\begin{aligned} \dot{z}_j(t) &= y_j(t) + C \int_{t-r}^t \left( A(v_j(s), z_j(s))x_j(s) + b(v_j(s), x_j(s)) \right) ds \\ &\text{for } j = 2 \dots m \end{aligned} \quad (1.82)$$

The link between the initial system representation (1.1a)-(1.1d) and the cascade representation (1.79a) is completed by the following relations:

$$y_1(t) = y(t), \quad x_m(t) = x(t) \quad (1.83)$$

these equations are direct consequences of the equations (1.77a)-(1.77d).

### 1.5.2 Chain observer design

For a given  $j = 1, \dots, m$ , the expressions in (1.77b)-(1.77c) show that the signals  $z_j(t)$  and  $y_j(t)$  represent respectively the un-delayed and delayed outputs of the system with state vector  $x_j(t)$  described by Equation (1.79a). Then, (1.83) shows that among all output signals, only  $y_1(t)$  is accessible to measurements (at sampling instants  $t_k$ ). Therefore, it is logical to start the state estimation process by estimating the state  $x_1(t)$  and the un-delayed output  $z_1(t)$ . Doing so, one gets using (1.78c) an estimate of  $y_2(t) = z_1(t)$  making possible the estimation of  $x_2(t)$  and  $z_2(t)$ . In turn, the latter entails the possibility of estimating  $x_3(t)$  and  $z_3(t)$ , making use of (1.78a)-(1.78c), and so on. This is the principle of the following sampled-output chain-observer:

The observer for  $x_1(t)$  takes the form,  
For all  $t \in [t_k, t_{k+1})$ ,  $k = 0, 1, \dots$

$$\begin{aligned} \dot{\hat{x}}_1(t) &= A(v_1(t), \sigma(\bar{z}_1(t)))\hat{x}_1(t) + b(v_1(t), \hat{x}_1(t)) \\ &\quad - \Gamma^{-1}(v_1(t), \sigma(\bar{z}_1(t)))\Delta^{-1}K(C\hat{x}_1(t) - \bar{z}_1(t)) \end{aligned} \quad (1.84a)$$

$$\dot{\bar{y}}_1(t) = CA(v_1(t-r), \sigma(\bar{y}_1))\hat{x}_1(t-r) + Cb(v_1(t-r), \hat{x}_1(t-r)) \quad (1.84b)$$

$$\bar{y}_1(t_k) = y(t_k) \quad (1.84c)$$

$$\begin{aligned} \bar{z}_1(t) &= \bar{y}_1(t) + \int_{t-r}^t [CA(v_1(s), \sigma(\bar{z}_1(s)))\hat{x}_1(s) + Cb(v_1(s), \hat{x}_1(s))] ds \\ &\quad (1.84d) \end{aligned}$$

## 1.5. SAMPLED-OUTPUT CHAIN-OBSERVER DESIGN

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where the initial conditions  $\hat{x}_j(0)$  and  $\bar{z}_j(0)$  ( $-r < s < 0$ ) for  $j = 1, \dots, m$  are arbitrarily chosen and the quantities  $\sigma$ ,  $K$ ,  $\theta$  and  $\Delta$  are as in (1.11a).

The observer for  $x_j(t)$  with  $j = 2, \dots, m$  takes the form:

$$\begin{aligned} \dot{\hat{x}}_j(t) &= A(v_j(t), \sigma(\bar{z}_j(t)))\hat{x}_j(t) + b(v_j(t), \hat{x}_j(t)) \\ &\quad - \Gamma^{-1}(v_j(t), \sigma(\bar{z}_j(t)))\Delta^{-1}K(C\hat{x}_j(t) - \bar{z}_j(t)) \end{aligned} \quad (1.85a)$$

$$\bar{z}_j(t) = \bar{z}_{j-1}(t) + \int_{t-r}^t \left[ CA(v_j(s), \sigma(\bar{z}_j(s)))\hat{x}_j(s) + Cb(v_j(s), \hat{x}_j(s)) \right] ds \quad (1.85b)$$

Comparing (1.84a)-(1.84d) and (1.85a)-(1.85b), it is seen that, for  $j = 1, \dots, m$ , the variable  $\bar{z}_j(t)$  is an estimate of the signal  $z_j(t)$ , while  $\bar{y}_1(t)$  is an estimate of  $y_1(t)$ . Also, the first observer variables ( $\bar{y}_1(t), \bar{z}_1(t)$ ) are obtained from the directly measured output samples  $y(t_k)$ ;  $k = 0, 1, 2, \dots$ . The variable  $\bar{z}_j(t)$ , for a given  $j = 2, \dots, n$ , is obtained using the previously estimated output  $\bar{z}_{j-1}(t)$ .

### 1.5.3 Observer Analysis

The proposed sampled-output chain observer defined by equations (1.84a) and (1.85a) will now be analysed. For analysis purpose, the following errors are introduced:

$$\tilde{x}_j = \hat{x}_j - x_j, \quad \tilde{y}_j = \bar{y}_j - y_j, \quad \tilde{z}_j = \bar{z}_j - z_j, \quad \text{for } j = 1, \dots, m \quad (1.86)$$

**Theorem 1.2:** Let the proposed sampled-output chain observer (1.84a)-(1.84d) and (1.85a)-(1.85b) be applied to the system (1.1a)-(1.1d), subject to the stated assumptions A1-A4. Then, there exists a real constant  $1 < \theta^* < \infty$  such that if  $\theta > \theta^*$  then, there exist real constants  $0 < \tau^* < \infty$  and  $0 < m^* < \infty$  so that, if  $\tau < \tau^*$  and  $m > m^*$  then,  $\forall t \geq 0$ :

$$\|\tilde{x}_j(t)\| \leq \rho_x e^{-\alpha t/2}, \quad |\tilde{y}_j(t)| \leq \rho_y e^{-\alpha t/2}, \quad |\tilde{z}_j(t)| \leq \rho_z e^{-\alpha t/2}$$

for  $j = 1..m$  and some real constants  $\alpha > 0$ ,  $\rho_x > 0$ ,  $\rho_y > 0$ ,  $\rho_z > 0$ .

**Proof:** The proof is divided into five parts. In part 1, the dynamics of the various estimation errors are established. In part 2, it is shown that the mapping  $\tilde{z}_j \rightarrow \varepsilon_j$  is ISS with  $\varepsilon_j = \Gamma(v, \sigma(\tilde{z}_j))\Delta\tilde{x}_j$ . In part 3, we show that the mapping  $(\varepsilon_1, \tilde{y}_1) \rightarrow \tilde{z}_1$  and  $(\varepsilon_j, \tilde{z}_{j-1}) \rightarrow \tilde{z}_j$  ( $j \geq 2$ ) are also ISS. In part 4, the result of the **Theorem 1.2** is established for  $j = 1$  and in part 5, it is established for  $(j = 2, \dots, m)$ [24, 25]. The Input to state stability analyses of Parts 2 and 3 are partly inspired by [27].

## 1.5. SAMPLED-OUTPUT CHAIN-OBSERVER DESIGN

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**Part 1.** Estimation errors dynamics.

The dynamics of the state estimation errors  $\tilde{x}_j(t)$ ,  $j = 1 \dots m$  are obtained, for  $j = 1$ , by subtracting (1.79a) from (1.84a) and, for  $j = 2, \dots, m$ , by subtracting (1.79a) from (1.85a). Doing so, one gets using (1.77c):

$$\begin{aligned} \dot{\tilde{x}}_j(t) &= [A(v_j(t), \sigma(\bar{z}_j)) - \Gamma^{-1}(v_j(t), \sigma(\bar{z}_j(t)))\Delta^{-1}KC]\tilde{x}_j(t) \\ &\quad + (A(v_j(t), \sigma(\bar{z}_j(t))) - A(v_j(t), z_j(t)))\tilde{x}_j(t) \\ &\quad + (b(v_j(t), \hat{x}_j(t)) - b(v(t), x_j(t))) \\ &\quad + \Gamma^{-1}(v_j(t), \sigma(\bar{z}_j(t)))\Delta^{-1}K\tilde{z}_j(t), \text{ for } j = 1 \dots m \end{aligned} \quad (1.87a)$$

The dynamics of  $\tilde{y}_1(t)$  is obtained by subtracting (1.79b) from (1.84b):

$$\begin{aligned} \dot{\tilde{y}}_1(t) &= CA(v_1(t-r), \sigma(\bar{y}_1(t)))\tilde{x}_1(t-r) + [CA(v_1(t-r), \sigma(\bar{y}_1(t))) \\ &\quad - CA(v_1(t-r), y_1(t))]x_1(t-r) + [Cb(v_1(t-r), \hat{x}_1(t-r)) \\ &\quad - Cb(v_1(t-r), x_1(t-r))], \text{ for } t_k < t < t_{k+1} \end{aligned} \quad (1.87b)$$

$$\tilde{y}_1(t_k) = 0 \quad (1.87c)$$

where the last equality is obtained using (1.84c). Similarly, the dynamics of the estimation error  $\tilde{z}_1(t)$  is obtained by subtracting (1.82) from (1.84d):

$$\begin{aligned} \tilde{z}_1(t) &= \tilde{y}_1(t) + \int_{t-r}^t CA(v_1(s), \sigma(\bar{z}_1(s)))\tilde{x}_1(s)ds + \int_{t-r}^t C[A(v_1(s), \sigma(\bar{z}_1(s))) \\ &\quad - A(v_1(s), z_1(s))]x_1(s)ds + \int_{t-r}^t C[b(v_1(s), \hat{x}_1(s)) - b(v_1(s), x_1(s))]ds \end{aligned} \quad (1.87d)$$

The dynamics of the estimation error  $\tilde{z}_j(t)$  for  $j = 2, \dots, m$  are obtained by subtracting (1.81) from (1.85b). This yields:

$$\begin{aligned} \tilde{z}_j(t) &= \tilde{z}_{j-1}(t) + \int_{t-r}^t CA(v_j(s), \sigma(\bar{z}_j(s)))\tilde{x}_j(s)ds + \int_{t-r}^t C[A(v_j(s), \sigma(\bar{z}_j(s))) \\ &\quad - A(v_j(s), z_j(s))]x_j(s)ds + \int_{t-r}^t C[b(v_j(s), \hat{x}_j(s)) - b(v_j(s), x_j(s))]ds \\ &\quad \text{for } j = 2, \dots, m \end{aligned} \quad (1.87e)$$

Just as in [43], introduce the following coordinate change:

$$\varepsilon_j(t) = \Gamma(v_j(t), \sigma(z_j(t)))\Delta\tilde{x}_j(t), \quad \text{for } j = 1, \dots, m \quad (1.88)$$

Then, equation (1.87a), for  $j = 1, \dots, m$ , rewrites in term of  $\varepsilon_j$  as follows,

## 1.5. SAMPLED-OUTPUT CHAIN-OBSERVER DESIGN

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where the argument  $t$  has been omitted to alleviate expressions:

$$\begin{aligned}
\dot{\varepsilon}_j &= \Gamma(v_j, \sigma(\bar{z}_j))\Delta[A(v_j, \sigma(\bar{z}_j)) - \Gamma^{-1}(v, \sigma(\bar{z}_j))\Delta^{-1}KC]\tilde{x}_j \\
&\quad + \Gamma(v_j, \sigma(\bar{z}_j))\Delta[A(v_j, \sigma(\bar{z}_j)) - A(v_j, z_j)]x_j + \Gamma(v_j, \sigma(\bar{z}_j))\Delta(b(v_j, \hat{x}_j) - b(v_j, x_j)) \\
&\quad + \Gamma(v_j, \sigma(\bar{z}_j))\Delta\Gamma^{-1}(v_j, \sigma(\bar{z}_j))\Delta^{-1}K\tilde{z}_j + \dot{\Gamma}(v_j, \sigma(\bar{z}_j))\Delta\tilde{x}_j \\
\varepsilon_j &= \theta\tilde{A}\varepsilon_j + \Gamma(v_j, \sigma(\bar{z}_j))\Delta[A(v_j, \sigma(\bar{z}_j)) - A(v_j, z_j)]x_j \\
&\quad + \Gamma(v_j, \sigma(\bar{z}_j))\Delta(b(v_j, \hat{x}_j) - b(v_j, x_j)) \\
&\quad + \Gamma(v_j, \sigma(\bar{z}_j))\Delta\Gamma^{-1}(v_j, \sigma(\bar{z}_j))\Delta^{-1}K\tilde{z}_j + \dot{\Gamma}(v_j, \sigma(\bar{z}_j))\Gamma^{-1}(v_j, \sigma(\bar{z}_j))\varepsilon_j \quad (1.89)
\end{aligned}$$

where the last equality is obtained as in (1.21) using the following structural properties that one can easily check, using (1.2a)-(1.2b), (1.6), (1.12), (1.13) and (1.88):

$$\tilde{x}_{j,1} = \theta\varepsilon_{j,1}, \quad \tilde{x}_{j,i} = \frac{\theta}{\prod_{k=1}^{i-1} a_k(v_j, \sigma(\bar{z}_j))} \varepsilon_{j,i}, \quad i = 2, \dots, n, \quad j = 1, \dots, m \quad (1.90)$$

where the function  $a_i$  are as in (1.2a). Similarly, in view of (1.2a)-(1.2b), equations (1.87b)-(1.87e) can be simplified as follows, where  $\tilde{x}_{j,i}(t)$  denotes the  $i$ -th components of the vector  $\tilde{x}_j(t)$ :

For all  $t \in [t_k, t_{k+1})$ ,  $k = 0, 1, \dots$

$$\begin{aligned}
\dot{\tilde{y}}_1(t) &= a_1(v_1(t-r), \sigma(\bar{y}_1(t)))\tilde{x}_{1,2}(t-r) + [a_1(v_1(t-r), \sigma(\bar{y}_1(t))) \\
&\quad - a_1(v_1(t-r), y_1(t))]x_{1,2}(t-r) + [b_1(v_1(t-r), \hat{x}_1(t-r)) \\
&\quad - b_1(v_1(t-r), x_1(t-r))], \quad (1.91a)
\end{aligned}$$

$$\tilde{y}_1(t_k) = 0 \quad (1.91b)$$

$$\begin{aligned}
\tilde{z}_1(t) &= \tilde{y}_1(t) + \int_{t-r}^t a_1(v_1(s), \sigma(\bar{z}_1(s)))\tilde{x}_{1,2}(s)ds + \int_{t-r}^t [a_1(v_1(s), \sigma(\bar{z}_1(s))) \\
&\quad - a_1(v_1(s), z_1(s))]x_{1,2}(s)ds + \int_{t-r}^t [b_1(v_1(s), \hat{x}_1(s)) - b_1(v_1(s), x_1(s))]ds \quad (1.91c)
\end{aligned}$$

$$\begin{aligned}
\tilde{z}_j(t) &= \tilde{z}_{j-1}(t) + \int_{t-r}^t a_1(v_j(s), \sigma(\bar{z}_j(s)))\tilde{x}_{j,2}(s)ds + \int_{t-r}^t [a_1(v_j(s), \sigma(\bar{z}_j(s))) \\
&\quad - a_1(v_j(s), z_j(s))]x_{j,2}(s)ds + \int_{t-r}^t [b_1(v_j(s), \hat{x}_j(s)) - b_1(v_j(s), x_j(s))]ds \\
j &= 2, \dots, m \quad (1.91d)
\end{aligned}$$

## 1.5. SAMPLED-OUTPUT CHAIN-OBSERVER DESIGN

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**Part 2.** Proof that the mapping  $\tilde{z}_j \rightarrow \varepsilon_j$  is ISS.

The proof is very similar to the proof of the input-to-state stability of the mapping  $e_z \rightarrow \varepsilon$  in Section 1.4. To analyse system (1.89), consider the candidate Lyapunov function :

$$V_j = \varepsilon_j^T P \varepsilon_j \quad (1.92)$$

with  $P = P^T$  the unique positive definite matrix that satisfies :

$$P\tilde{A} + \tilde{A}^T P = -\mu I \quad (1.93)$$

where  $I$  is the identity matrix and  $\mu > 0$  is arbitrarily chosen. Note that  $P$  exists because we know that  $\tilde{A} = \bar{A} - KC$  is Hurwitz. Time-differentiation of  $V_j$  yields, using (1.92) and (1.89):

$$\begin{aligned} \dot{V}_j &= \dot{\varepsilon}_j^T P \varepsilon_j + \varepsilon_j^T P \dot{\varepsilon}_j \\ &= -\mu\theta \|\varepsilon_j\|^2 + 2\varepsilon_j^T P \Gamma(v_j, \sigma(\bar{z}_j)) \Delta(A(v_j, \sigma(\bar{z}_j)) - A(v_j, z_j)) x_j \\ &\quad + 2\varepsilon_j^T P \Gamma(v_j, \sigma(\bar{z}_j)) \Delta(b(v_j, \hat{x}_j) - b(v_j, x_j)) \\ &\quad + 2\varepsilon_j^T P \Gamma(v_j, \sigma(\bar{z}_j)) \Delta \Gamma^{-1}(v_j, \sigma(\bar{z}_j)) \Delta^{-1} K \tilde{z}_j \\ &\quad + 2\varepsilon_j^T P \dot{\Gamma}(v_j, \sigma(\bar{z}_j)) \Gamma^{-1}(v_j, \sigma(\bar{z}_j)) \varepsilon_j \end{aligned} \quad (1.94)$$

The various terms on the right side of (1.94) will successively be bounded in the next lines. First, recall that, by assumption A2, all terms  $a_i(v, z)$  ( $i = 1 \dots n - 1$ ) of the matrix  $A(v, z)$  are  $C^1$  functions. Also, it follows from assumption A1 and (1.77b) that:

$$|\sigma(\tilde{z}_j(t))| \leq y_M \quad \text{and} \quad \sigma(z(t)) = z(t), \quad \forall t \geq 0 \quad (1.95)$$

Then, one gets using Remark 1 (Part b):

$$|a_i(v_j, \bar{z}_j) - a_i(v_j, z_j)| \leq l_a |\tilde{z}_j|, \quad i = 1, \dots, n - 1 \quad (1.96)$$

Then, it follows from (1.6) and (2.21) that the second term:

$$\begin{aligned} \left\| 2\varepsilon_j^T P \Gamma(v_j, \sigma(\bar{z}_j)) \Delta(A(v_j, \sigma(\bar{z}_j)) - A(v_j, z_j)) x \right\| &\leq \frac{2l_a \beta_1}{\theta} \|\varepsilon_j\| |\tilde{z}_j| \\ &\leq \frac{l_a \beta_1}{\theta} (\|\varepsilon_j\|^2 + |\tilde{z}_j|^2) \end{aligned} \quad (1.97)$$

using assumption A1 and the fact that  $\theta > 1$ , where  $\beta_1$  is a real constant depending on  $P$ ,  $y_M$  and the suprema of  $\|v(t)\|$  and  $\|x(t)\|$ , but is independent on  $\theta$ .

## 1.5. SAMPLED-OUTPUT CHAIN-OBSERVER DESIGN

---

Again, using the triangular form of  $b(v, x)$  and the diagonal form of  $\Gamma\Delta$ , it follows that:

$$\|\Delta(b(v_j(t), \hat{x}_j(t)) - b(v_j(t), x_j(t)))\| \leq \beta_2 \|\varepsilon_j\| \quad (1.98)$$

where  $\beta_2$  is a real constant independent on  $\theta$ . Then, one gets making use of (1.6), assumptions A1-A2, and Remark 1 (Part c):

$$\left\| 2\varepsilon_j^T P \Gamma(v_j, \sigma(\bar{z}_j)) \Delta(b(v_j, \hat{x}_j) - b(v_j, x_j)) \right\| \leq 2\beta_3 \beta_2 \|\varepsilon_j\|^2 \quad (1.99)$$

for some real constant  $\beta_3$ , independent on  $\theta$ , only dependent on  $y_M$ ,  $P$  and the functions  $a_i$ . The penultimate term on the right side of (1.94) is bounded as follows, using (1.22):

$$\begin{aligned} \left\| 2\varepsilon_j^T P \Gamma(v_j, \sigma(\bar{z}_j)) \Delta \Gamma^{-1}(v_j, \sigma(\bar{z}_j)) \Delta^{-1} K \tilde{z}_j \right\| &= \left\| 2\varepsilon_j^T P K \tilde{z}_j \right\| \\ &\leq \beta_4 (\|\varepsilon_j\|^2 + \|\tilde{z}_j\|^2) \end{aligned} \quad (1.100)$$

with  $\beta_4 = |PK|$ . Finally, the following bounding from above is readily got for last term on the right side of (1.26), using (1.5):

$$\left\| 2\varepsilon_j^T P \dot{\Gamma}(v_j, \sigma(\bar{z}_j)) \Gamma^{-1}(v_j, \sigma(\bar{z}_j)) \varepsilon_j \right\| \leq 2l_\Gamma \beta_5 \|\varepsilon_j\|^2 \quad (1.101)$$

with  $\|\beta_5\| = |P|$ . Using (1.101), (1.100), (1.99), and (1.97), it follows from (1.90) that:

$$\dot{V}_j \leq -\alpha_1(\theta) \|\varepsilon_j\|^2 + \beta_6 |\tilde{z}_j|^2 \leq -\alpha_1(\theta) V_j + \beta_6 |\tilde{z}_j|^2 \quad (1.102)$$

with

$$\alpha_1(\theta) = \frac{\gamma\theta}{\lambda_{\max}(P)} = \frac{1}{\lambda_{\max}(P)} \left( \mu\theta - \frac{l_a \beta_1}{\theta} - 2\beta_2 \beta_3 - \beta_4 - 2l_\Gamma \beta_5 \right) \quad (1.103)$$

$$\beta_6 = l_a \beta_1 + \beta_4 \quad (1.104)$$

using (1.91d) and the fact that  $\theta > 1$ . Clearly, from (1.103) one has:

$$\lim_{\theta \rightarrow +\infty} \alpha_1(\theta) = \infty \quad (1.105)$$

In the sequel, we let  $\theta$  be sufficiently large so that:

$$\alpha_1(\theta) > 1 \quad (1.106)$$

## 1.5. SAMPLED-OUTPUT CHAIN-OBSERVER DESIGN

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This is possible because none of the constants on the right side of (1.36) is dependent on  $\theta$ . Integrating inequality (1.102) gives, for all  $t \geq 0$  :

$$V_j(t) \leq e^{-\alpha_1(\theta)t} V_j(0) + \beta_6 \int_0^t e^{-\alpha_1(\theta)(t-s)} \tilde{z}_j^2(s) ds \quad (1.107)$$

Letting

$$\alpha(\theta) = \frac{\alpha_1(\theta)}{2} \quad (1.108)$$

it follows multiplying both sides of (1.107) by  $e^{\alpha(\theta)t}$  :

$$e^{\alpha(\theta)t} V_j(t) \leq M_0 + \beta_6 e^{\alpha(\theta)t} \int_0^t e^{-\alpha_1(\theta)(t-s)} e_z^2(s) ds \quad (1.109)$$

with  $M_0 = V_j(0)$ , using the fact that  $e^{-\alpha_1(\theta)t} < 1$ . Inequality (1.109) entails, successively:

$$\begin{aligned} e^{\alpha t} V_j(t) &\leq M_0 + \beta_6 e^{\alpha(\theta)t} \int_0^t e^{-\alpha_1(\theta)(t-s)} e^{-\alpha(\theta)s} e^{\alpha(\theta)s} \tilde{z}_j^2(s) ds \\ &\leq M_0 + \beta_6 e^{(\alpha(\theta) - \alpha_1(\theta))t} \int_0^t e^{(\alpha_1(\theta) - \alpha(\theta))s} e^{\alpha(\theta)s} \tilde{z}_j^2(s) ds \\ &\leq M_0 + \beta_6 e^{(\alpha(\theta) - \alpha_1(\theta))t} \left( \int_0^t e^{(\alpha_1(\theta) - \alpha(\theta))s} ds \right) \sup_{0 \leq s \leq t} (e^{\alpha(\theta)s} \tilde{z}_j^2(s)) \\ &\leq M_0 + \frac{2\beta_6}{\alpha_1(\theta)} \sup_{0 \leq s \leq t} (e^{\alpha(\theta)s} \tilde{z}_j^2(s)) \\ &\leq M_0 + \frac{\beta_6}{\alpha_1(\theta) - \alpha(\theta)} (1 - e^{(\alpha_1(\theta) - \alpha(\theta))t}) \sup_{0 \leq s \leq t} (e^{\alpha(\theta)s} \tilde{z}_j^2(s)) \\ &\leq M_0 + \frac{2\beta_6}{\alpha_1(\theta)} \sup_{0 \leq s \leq t} (e^{\alpha(\theta)s} \tilde{z}_j^2(s)) \\ &\leq M_0 + \frac{\beta_6}{\alpha(\theta)} \sup_{0 \leq s \leq t} (e^{\alpha(\theta)s} \tilde{z}_j^2(s)) \end{aligned} \quad (1.110)$$

Taking square root of both sides of (1.110) gives:

$$e^{\frac{\alpha(\theta)t}{2}} \sqrt{V_1(t)} \leq \sqrt{M_0} + \sqrt{\frac{\beta_6}{\alpha(\theta)}} \sup_{0 \leq s \leq t} (e^{\frac{\alpha(\theta)s}{2}} |\tilde{z}_1(s)|) \quad (1.111)$$

From (1.24c) and (1.19) one has, using (1.22) and Remark 1 (Part c):

$$V_j \geq \lambda_{\min}(P) \|\varepsilon_j\|^2 \quad (1.112)$$

## 1.5. SAMPLED-OUTPUT CHAIN-OBSERVER DESIGN

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where  $\lambda_{\min}(P)$  denotes the minimal eigenvalue of  $P$  (which is clearly independent on  $\theta$ ). Then, it follows from (1.111):

$$e^{\frac{\alpha(\theta)t}{2}} \|\varepsilon_j(t)\| \leq M_1 + \gamma_1 \sup_{0 \leq s \leq t} (e^{\frac{\alpha(\theta)s}{2}} |\tilde{z}_j(s)|) \quad (1.113)$$

with

$$M_1 = \sqrt{\frac{M_0}{\lambda_{\min}(P)}}, \quad \gamma_1 = \sqrt{\frac{2\beta_6}{\alpha(\theta)\lambda_{\min}(P)}} \quad (1.114)$$

Since the left side on (1.113) is increasing with  $t$ , it follows that:

$$\sup_{0 \leq s \leq t} e^{\frac{\alpha(\theta)s}{2}} \|\varepsilon_j(s)\| \leq M_1 + \gamma_1 \sup_{0 \leq s \leq t} (e^{\frac{\alpha(\theta)s}{2}} |\tilde{z}_j(s)|) \quad (1.115)$$

with the last inequality, we have shown that the mapping  $\tilde{z}_j(s) \rightarrow \varepsilon_j(s)$  is ISS.

**Part 3.** Proof that the mappings  $(\varepsilon_1, \tilde{y}_1) \rightarrow \tilde{z}_1$  and  $(\varepsilon_j, \tilde{z}_{j-1}) \rightarrow \tilde{z}_j$  ( $j \geq 2$ ) are ISS.

To keep compact the subsequent development, let us introduce the notation,

$$w_j = \begin{cases} |\tilde{y}_1| & \text{if } j = 1 \\ |\tilde{z}_{j-1}| & \text{if } j \geq 2 \end{cases} \quad (1.116)$$

Then, using (1.88), (1.100) and assumption A1, it follows from (1.91c)-(1.91d) that  $|\tilde{z}_j|$  is upper bounded by  $|w_j|$  and  $\|\varepsilon_j\|$  as follows:

$$\begin{aligned} |\tilde{z}_j(t)| &\leq |w_j(t)| + \int_{t-r}^t \|a_1(v_j(s), \sigma(\bar{z}_j(s))) \tilde{x}_{j,2}(s)\| ds \\ &\quad + \int_{t-r}^t \|a_1(v_j(s), \sigma(\bar{z}_j(s))) - a_1(v_j(s), z_j(s))\| x_{j,2}(s) ds \\ &\quad + \int_{t-r}^t \|b_1(v_j(s), \hat{x}_j(s)) - b_1(v_j(s), x_j(s))\| ds \\ &\quad j = 2, \dots, m \\ &\leq |w_j(t)| + \theta^2 \int_{t-r}^t \|\varepsilon_{j,2}(s)\| ds + l_a \beta_7 \int_{t-r}^t |\tilde{z}_j(s)| ds \\ &\quad + \int_{t-r}^t \|b_1(v_j(s), \hat{x}_j(s)) - b_1(v_j(s), x_j(s))\| ds \end{aligned} \quad (1.117)$$

for  $j = 1 \dots m$  and some real constant  $\beta_7$  depending on the supremum of  $\|x_j\|$  (which is the same as that of  $\|x\|$  due to (1.77a), but independent on

## 1.5. SAMPLED-OUTPUT CHAIN-OBSERVER DESIGN

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$\theta$ , where we have used (1.90). The last quantity on the right side of (1.117) is handled using the Lipschitz and the triangular nature of the  $b(v, x)$  (see (1.2b) and assumption A2). Accordingly, one has:

$$\|b_1(v_j(t), \hat{x}_j(t)) - b_1(v_j(t), x_j(t))\| \leq l_b \|\tilde{x}_{j,1}(t)\| \quad (1.118)$$

with  $l_b$  as in **Remark 1** (Part b). Using (1.90), (1.118) gives:

$$\|b_1(v_j(t), \hat{x}_j(t)) - b_1(v_j(t), x_j(t))\| \leq \theta l_b \|\varepsilon_{j,1}(t)\| \quad (1.119)$$

which yields, together with (1.117):

$$\begin{aligned} |\tilde{z}_j(t)| &\leq |w_j(t)| + \theta(\theta + l_b) \int_{t-r}^t \|\varepsilon_{j,2}(s)\| ds + l_a \beta_7 \int_{t-r}^t |\tilde{z}_j(s)| ds \\ &\leq |w_j(t)| + \theta(\theta + l_b) \int_{t-r}^t e^{-\alpha(\theta)s} e^{\alpha(\theta)s} \|\varepsilon_j(s)\| ds \\ &\quad + l_a \beta_7 \int_{t-r}^t e^{-\alpha(\theta)s} e^{\alpha(\theta)s} |\tilde{z}_j(s)| ds \\ &\leq |w_j(t)| + \theta(\theta + l_b) \int_{t-r}^t e^{-\alpha(\theta)s} ds \left( \sup_{t-r \leq s \leq t} (e^{\frac{\alpha(\theta)s}{2}} \|\varepsilon_j(s)\|) \right) \\ &\quad + l_a \beta_7 \int_{t-r}^t e^{-\frac{\alpha(\theta)s}{2}} |\tilde{z}_j(s)| ds \left( \sup_{t-r \leq s \leq t} e^{\frac{\alpha(\theta)s}{2}} |\tilde{z}_j(s)| \right) \\ &\leq |w_j(t)| + \theta(\theta + l_b) r e^{-\alpha(\theta)(t-r)/2} \left( \sup_{t-r \leq s \leq t} (e^{\frac{\alpha(\theta)s}{2}} \|\varepsilon_j(s)\|) \right) \\ &\quad + l_a \beta_7 r e^{-\alpha(\theta)(t-r)/2} \left( \sup_{t-r \leq s \leq t} e^{\frac{\alpha(\theta)s}{2}} |\tilde{z}_j(s)| \right) \end{aligned} \quad (1.120)$$

where the last inequality is obtained applying the mean value theorem to the integral quantity  $\int_{t-r}^t e^{-\alpha(\theta)s/2} ds$ . Multiplying both sides of (1.120) by  $e^{\alpha t/2}$  yields:

$$\begin{aligned} e^{\alpha(\theta)t/2} |\tilde{z}_j(t)| &\leq e^{\alpha(\theta)t/2} |w_j(t)| + \theta(\theta + l_b) r e^{\alpha(\theta)r/2} \left( \sup_{t-r \leq s \leq t} (e^{\frac{\alpha(\theta)s}{2}} \|\varepsilon_j(s)\|) \right) \\ &\quad + l_a \beta_7 r e^{\alpha(\theta)r/2} \left( \sup_{t-r \leq s \leq t} e^{\frac{\alpha(\theta)s}{2}} |\tilde{z}_j(s)| \right) \end{aligned} \quad (1.121)$$

which implies:

$$\begin{aligned} \sup_{0 \leq s \leq t} \left( e^{\alpha(\theta)s/2} |\tilde{z}_j(s)| \right) &\leq \sup_{0 \leq s \leq t} \left( e^{\alpha(\theta)s/2} |w_j(s)| \right) + \theta(\theta + l_b) r e^{\alpha(\theta)r/2} \left( \sup_{0 \leq s \leq t} e^{\alpha(\theta)s/2} \|\varepsilon_j(s)\| \right) \\ &\quad + l_a \beta_7 r e^{\alpha(\theta)r/2} \left( \sup_{0 \leq s \leq t} e^{\alpha(\theta)s/2} |\tilde{z}_j(s)| \right) \end{aligned} \quad (1.122)$$

$$\begin{aligned} &+ \theta(\theta + l_b) r e^{\alpha(\theta)r/2} \left( \sup_{-r \leq s \leq 0} e^{\alpha(\theta)s/2} \|\varepsilon_j(s)\| \right) \\ &+ l_a \beta_7 r e^{\alpha(\theta)r/2} \left( \sup_{-r \leq s \leq 0} e^{\alpha(\theta)s/2} |\tilde{z}_j(s)| \right) \end{aligned} \quad (1.123)$$

## 1.5. SAMPLED-OUTPUT CHAIN-OBSERVER DESIGN

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Since  $re^{\alpha r/2} \xrightarrow[r \rightarrow 0]{} 0$ , we let  $r$  be such that:

$$0 < l_a \beta_7 r e^{\alpha(\theta)r/2} < 1 \quad (1.124)$$

Then, one gets from (1.124):

$$\begin{aligned} \sup_{0 \leq s \leq t} \left( e^{\alpha(\theta)s/2} |\tilde{z}_j(s)| \right) &\leq \frac{1}{1 - l_a \beta_7 r e^{\alpha(\theta)r/2}} \sup_{0 \leq s \leq t} \left( e^{\alpha(\theta)s/2} |w_j(s)| \right) \\ &+ \frac{\theta(\theta + l_b) r e^{\alpha(\theta)r/2}}{1 - l_a \beta_7 r e^{\alpha(\theta)r/2}} \sup_{0 \leq s \leq t} \left( e^{\alpha(\theta)s/2} \|\varepsilon_j(s)\| \right) \end{aligned} \quad (1.125)$$

This proves the mappings  $(\varepsilon_1, \tilde{y}_1) \longrightarrow \tilde{z}_1$  and  $(\varepsilon_j, \tilde{z}_{j-1}) \longrightarrow \tilde{z}_j$  ( $j \geq 2$ ) are ISS.

**Part 4.** Proof of the theorem result for  $j = 1$ .  
Integrating (1.91a) gives:

For all  $t \in [t_k, t_{k+1})$ ,  $k = 0, 1, \dots$

$$\begin{aligned} \tilde{y}_1(t) &= \int_{t_k}^t a_1(v_1(s-r), \sigma(\tilde{y}_1(s))) \tilde{x}_{1,2}(s-r) ds + \int_{t_k}^t [a_1(v_1(s-r), \sigma(\tilde{y}_1(s))) \\ &\quad - a_1(v_1(s-r), y_1(t))] x_{1,2}(s-r) ds + \int_{t_k}^t [b_1(v_1(s-r), \hat{x}_1(s-r)) \\ &\quad - b_1(v_1(s-r), x_1(s-r))] ds, \end{aligned} \quad (1.126)$$

Taking the norms of both sides of (1.126) gives, using similar arguments as for (1.117) to (1.125):

For all  $t \in [t_k, t_{k+1})$ ,  $k = 0, 1, \dots$

$$\begin{aligned} |\tilde{y}_1(t)| &\leq \int_{t_k}^t \left\| \theta^2 \varepsilon_{1,2}(s-r) \right\| ds + l_a \beta_7 \int_{t_k}^t |\tilde{y}_1(s)| ds + \int_{t_k}^t \theta l_b \|\varepsilon_{1,1}(s-r)\| ds \\ &\quad + \|y_1(t_k)\| \\ &\leq \theta(\theta + l_b) \int_{t_k}^t \|\varepsilon_1(s-r)\| ds + l_a \beta_7 \int_{t_k}^t |\tilde{y}_1(s)| ds \end{aligned} \quad (1.127)$$

## 1.5. SAMPLED-OUTPUT CHAIN-OBSERVER DESIGN

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where  $\beta_7$  is as in (1.125). Inequality (1.126) develops further as follows:

$$\begin{aligned}
|\tilde{y}_1(t)| &\leq \theta(\theta + l_b) \int_{t_k}^t e^{-\alpha(s-r)/2} e^{\alpha(s-r)/2} \|\varepsilon_1(s-r)\| ds \\
&\quad + l_a \beta_7 \int_{t_k}^t e^{-\alpha s/2} e^{\alpha s/2} |\tilde{y}_1(s)| ds \\
&\leq \theta(\theta + l_b) \int_{t_k}^t e^{-\alpha(s-r)/2} ds \left( \sup_{t_k \leq s < t} e^{\alpha(\theta)(s-r)/2} \|\varepsilon_1(s-r)\| \right) \\
&\quad + l_a \beta_7 \int_{t_k}^t e^{-\alpha s/2} ds \left( \sup_{t_k \leq s < t} e^{\alpha s/2} |\tilde{y}_1(s)| \right) \\
&\leq \theta(\theta + l_b) e^{\alpha r/2} \tau e^{-\alpha t_k/2} \left( \sup_{t_k \leq s < t} e^{\alpha(\theta)(s-r)/2} \|\varepsilon_1(s-r)\| \right) \\
&\quad + l_a \beta_7 \tau e^{-\alpha t_k/2} \left( \sup_{0 \leq s \leq t} e^{\alpha s/2} |\tilde{y}_1(s)| ds \right) \text{ for } t_k < t < t_{k+1} \quad (1.128)
\end{aligned}$$

where we have used the double inequality  $0 < \int_{t_k}^t e^{-\alpha s/2} ds \leq \tau e^{-\alpha t_k/2}$ . Multiplying both sides by  $e^{\alpha t/2}$  yields:

For all  $t \in [t_k, t_{k+1})$ ,  $k = 0, 1, \dots$

$$\begin{aligned}
e^{\alpha t/2} |\tilde{y}_1(t)| &\leq \theta(\theta + l_b) e^{\alpha r/2} \tau e^{\alpha \tau/2} \left( \sup_{0 \leq s \leq t} e^{\alpha(s-r)/2} \|\varepsilon_1(s-r)\| \right) \\
&\quad + l_a \beta_7 \tau e^{\alpha \tau/2} \left( \sup_{0 \leq s \leq t} e^{\alpha s/2} |\tilde{y}_1(s)| ds \right) \quad (1.129)
\end{aligned}$$

Since the right side terms are non-decreasing functions of  $t$ , it follows that:

$$\begin{aligned}
\sup_{0 \leq s \leq t} e^{\alpha s/2} |\tilde{y}_1(s)| &\leq \theta(\theta + l_b) e^{\alpha r/2} \tau e^{\alpha \tau/2} \left( \sup_{0 \leq s \leq t} e^{\alpha(s-r)/2} \|\varepsilon_1(s-r)\| \right) \\
&\quad + l_a \beta_7 \tau e^{\alpha \tau/2} \left( \sup_{0 \leq s \leq t} e^{\alpha s/2} |\tilde{y}_1(s)| \right) \\
&\leq \theta(\theta + l_b) e^{\alpha r/2} \tau e^{\alpha \tau/2} \left( \sup_{0 \leq s \leq t} e^{\alpha(s)/2} \|\varepsilon_1(s)\| \right) \\
&\quad + \theta(\theta + l_b) e^{\alpha r/2} \tau e^{\alpha \tau/2} \left( \sup_{-r \leq s \leq 0} e^{\alpha(s)/2} \|\varepsilon_1(s)\| \right) \\
&\quad + l_a \beta_7 \tau e^{\alpha \tau/2} \left( \sup_{0 \leq s \leq t} e^{\alpha s/2} |\tilde{y}_1(s)| \right) \quad (1.130)
\end{aligned}$$

Since  $\tau e^{\alpha \tau/2} \xrightarrow{\tau \rightarrow 0} 0$ , we let  $\tau$  be such that:

$$0 < l_a \beta_7 \tau e^{\alpha \tau/2} < 1 \quad (1.131)$$

## 1.5. SAMPLED-OUTPUT CHAIN-OBSERVER DESIGN

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Then, one gets from (1.130):

$$\sup_{0 \leq s \leq t} (e^{\frac{\alpha s}{2}} |\tilde{y}_1(s)|) \leq \frac{\theta(\theta + l_b)e^{\alpha r/2}\tau e^{\alpha\tau/2}}{(1 - l_a\beta_7\tau e^{\alpha\tau/2})} \left( \sup_{0 \leq s \leq t} e^{\alpha s/2} \|\varepsilon_1(s)\| \right) + M_2 \quad (1.132)$$

with

$$M_2 = \frac{\theta(\theta + l_b)e^{\alpha r/2}\tau e^{\alpha\tau/2}}{(1 - l_a\beta_7\tau e^{\alpha\tau/2})} \left( \sup_{-r \leq s \leq 0} e^{\alpha s/2} \|\varepsilon_1(s)\| \right) \quad (1.133)$$

Using (1.132) it follows from (1.123) that:

$$\begin{aligned} \sup_{0 \leq s \leq t} (e^{\frac{\alpha s}{2}} |\tilde{z}_1(s)|) &\leq \frac{\theta(\theta + l_b)e^{\alpha r/2}\tau e^{\alpha\tau/2}}{(1 - l_a\beta_7re^{\alpha r/2})(1 - l_a\beta_7\tau e^{\alpha\tau/2})} \left( \sup_{0 \leq s \leq t} e^{\alpha s/2} \|\varepsilon_1(s)\| \right) \\ &\quad + \frac{\theta(\theta + l_b)re^{\alpha r/2}}{1 - l_a\beta_7re^{\alpha r/2}} \left( \sup_{0 \leq s \leq t} e^{\alpha s/2} \|\varepsilon_1(s)\| \right) \\ &\quad + \frac{M_2}{1 - l_a\beta_7re^{\alpha r/2}} \end{aligned} \quad (1.134)$$

Combining (1.134) and (1.115), one gets:

$$\begin{aligned} \sup_{0 \leq s \leq t} e^{\frac{\alpha s}{2}} \|\varepsilon_1(s)\| &\leq \frac{\gamma_1\theta(\theta + l_b)e^{\alpha r/2}\tau e^{\alpha\tau/2}}{(1 - l_a\beta_7re^{\alpha r/2})(1 - l_a\beta_7\tau e^{\alpha\tau/2})} \sup_{0 \leq s \leq t} e^{\frac{\alpha s}{2}} \|\varepsilon_1(s)\| \\ &\quad + \frac{\gamma_1\theta(\theta + l_b)re^{\alpha r/2}}{1 - l_a\beta_7re^{\alpha r/2}} \sup_{0 \leq s \leq t} e^{\frac{\alpha s}{2}} \|\varepsilon_1(s)\| + M_3 \\ &\leq \gamma_2 \left( \sup_{0 \leq s \leq t} e^{\frac{\alpha s}{2}} \|\varepsilon_1(s)\| \right) + M_3 \end{aligned} \quad (1.135)$$

with

$$M_3 = M_1 + \frac{M_2\gamma_1}{1 - l_a\beta_7re^{\alpha r/2}} \quad (1.136)$$

Let  $r$  and  $\tau$  be such that:

$$\gamma_2 < 1 \quad (1.137)$$

This is not an issue because  $\tau e^{\alpha\tau/2} \xrightarrow{\tau \rightarrow 0} 0$  and  $re^{\alpha r/2} \xrightarrow{r \rightarrow 0} 0$ . Then, it follows from (1.69) that:

$$\sup_{0 \leq s \leq t} (e^{\frac{\alpha s}{2}} \|\varepsilon_1(s)\|) \leq N_{\varepsilon,1} \quad \text{with} \quad N_{\varepsilon,1} = \frac{M_3}{1 - \gamma_2} \quad (1.138)$$

## 1.5. SAMPLED-OUTPUT CHAIN-OBSERVER DESIGN

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Then, (1.132) yields,

$$\sup_{0 \leq s \leq t} (e^{\frac{\alpha s}{2}} |\tilde{y}_1(s)|) \leq N_{y,1} \quad (1.139)$$

with

$$N_{y,1} = \frac{\theta(\theta + l_b)e^{\alpha r/2}\tau e^{\alpha \tau/2}}{1 - l_a\beta_7\tau e^{\alpha \tau/2}} \frac{M_3}{1 - \gamma_2} + M_2 \quad (1.140)$$

Using (1.138), (1.139) and (1.116), it follows from (1.123) that:

$$\sup_{0 \leq s \leq t} (e^{\frac{\alpha s}{2}} |\tilde{z}_1(s)|) \leq N_{z,1} \quad (1.141)$$

with

$$\begin{aligned} N_{z,1} = & \frac{1}{1 - l_a\beta_7\tau e^{\alpha \tau/2}} \left( \frac{\theta(\theta + l_b)e^{\alpha r/2}\tau e^{\alpha \tau/2}}{1 - l_a\beta_7\tau e^{\alpha \tau/2}} \frac{M_3}{1 - \gamma_2} + M_2 \right) \\ & + \frac{\theta(\theta + l_b)r e^{\alpha r/2}}{1 - l_a\beta_7r e^{\alpha r/2}} \frac{M_3}{1 - \gamma_2} \end{aligned} \quad (1.142)$$

Clearly, (1.138)–(1.142) imply that the three signals  $(\varepsilon_1(t), \tilde{y}_1(t), \tilde{z}_1(t))$  are exponentially convergent to the origin and the inequalities

$$|\tilde{z}_1(t)| \leq \rho_z(\theta)e^{-\alpha(\theta)t/2}, \quad |\tilde{y}_1(t)| \leq \rho_y(\theta)e^{-\alpha(\theta)t/2}, \quad \forall t \geq 0 \quad (1.143)$$

hold with any

$$\rho_z(\theta) \geq N_{z,1}, \quad \rho_y(\theta) \geq \frac{\theta(\theta + l_b)e^{\alpha r/2}\tau e^{\alpha \tau/2}}{1 - l_a\beta_7\tau e^{\alpha \tau/2}} \frac{M_3}{1 - \gamma_2} + M_2 \quad (1.144)$$

Then, using (1.90) and Remark 1 (Part d), it follows that  $\tilde{x}_1(t)$  is also exponentially convergent to the origin and the inequality  $\|\tilde{x}_1(t)\| \leq \rho_x(\theta)e^{-\alpha(\theta)t/2}$ ,  $\forall t \geq 0$ , holds with any  $\rho_x(\theta) \geq \frac{\theta^n}{\varepsilon^n} N_{\varepsilon,1}$ . A quick inspection of the expressions of  $N_{y,1}$ ,  $N_{z,1}$ ,  $N_{\varepsilon,1}$  shows that  $\rho_x(\theta)$ ,  $\rho_y(\theta)$ ,  $\rho_z(\theta)$  are increasing functions of  $\theta$ . On the other hand, conditions (1.124), (1.131) and (1.137) show that the sampling interval  $\tau$  and the fractional delay  $r$  are decreasing functions of  $\alpha(\theta)$  which, by (1.103) and (1.108), is an increasing function of  $\theta$ . These observations end Part 4, and so establishes **Theorem 1.2**, for  $j = 1$ .

**Part 5.** Proof of the theorem result for  $2 \leq j < m$ .

In this part, we will first show by induction that the following inequalities hold, for all  $\forall t \geq 0$  and  $1 \leq j \leq m$  :

$$\sup_{0 \leq s \leq t} (e^{\frac{\alpha s}{2}} |\tilde{z}_j(s)|) \leq N_{z,j} \quad (1.145)$$

$$\sup_{0 \leq s \leq t} (e^{\frac{\alpha s}{2}} \|\varepsilon_j(s)\|) \leq N_{\varepsilon,j} \quad (1.146)$$

## 1.5. SAMPLED-OUTPUT CHAIN-OBSERVER DESIGN

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for some constants  $N_{z,j}$ ,  $N_{\varepsilon,j}$ . We have already shown, in Part 4 of this proof, that inequalities (1.145) and (1.146) hold for  $j = 1$  (see (1.138)-(1.142)). So, assume that the inequalities hold for some  $2 \leq j < m$ . The rest of the proof will consist in showing that the inequalities also hold for  $j + 1$  i.e. with  $(\varepsilon_{j+1}(t), \tilde{z}_{j+1}(t))$ . To this end, (1.123) is rewritten for  $j + 1$  as follows, using (1.116):

$$\begin{aligned}
\sup_{0 \leq s \leq t} (e^{\frac{\alpha s}{2}} |\tilde{z}_{j+1}(s)|) &\leq \frac{1}{1 - l_a \beta_7 r e^{\alpha r/2}} \left( \sup_{0 \leq s \leq t} e^{\alpha s/2} |\tilde{z}_j(s)| \right) \\
&\quad + \frac{\theta(\theta + l_b) r e^{\alpha r/2}}{1 - l_a \beta_7 r e^{\alpha r/2}} \left( \sup_{0 \leq s \leq t} e^{\alpha s/2} \|\varepsilon_1(s)\| \right) \\
&\leq \frac{1}{1 - l_a \beta_7 r e^{\alpha r/2}} \left( \sup_{0 \leq s \leq t} e^{\alpha s/2} |\tilde{z}_j(s)| \right) \\
&\quad + \frac{\theta(\theta + l_b) r e^{\alpha r/2}}{1 - l_a \beta_7 r e^{\alpha r/2}} \left( M_1 + \gamma_1 \sup_{0 \leq s \leq t} e^{\alpha s/2} \|\tilde{z}_{j+1}(s)\| \right)
\end{aligned} \tag{1.147}$$

where the last inequality is obtained using (1.115). Again, because  $r e^{\alpha r/2} \xrightarrow{r \rightarrow 0} 0$ , we let  $r$  be such that:

$$\gamma_1 < 1, \quad \text{with} \quad \gamma_3 = \frac{\theta(\theta + l_b) r e^{\alpha r/2}}{1 - l_a \beta_7 r e^{\alpha r/2}} \gamma_1 \tag{1.148}$$

Then, (1.147) implies:

$$\sup_{0 \leq s \leq t} (e^{\frac{\alpha s}{2}} |\tilde{z}_{j+1}(s)|) \leq \frac{1}{(1 - \gamma_3)(1 - l_a \beta_7 r e^{\alpha r/2})} \left( \sup_{0 \leq s \leq t} e^{\alpha s/2} |\tilde{z}_j(s)| \right) + M_4 \tag{1.149}$$

with

$$M_4 = \frac{\theta(\theta + l_b) r e^{\alpha r/2} M_1}{(1 - \gamma_3)(1 - l_a \beta_7 r e^{\alpha r/2})} \tag{1.150}$$

Using (1.146), inequality (1.149) yields:

$$\sup_{0 \leq s \leq t} (e^{\frac{\alpha s}{2}} |\tilde{z}_{j+1}(s)|) \leq N_{z,j+1} \tag{1.151}$$

with

$$M_4 = \frac{N_{z,j}}{(1 - \gamma_3)(1 - l_a \beta_7 r e^{\alpha r/2})} \tag{1.152}$$

Then, using (1.151), one gets writing (1.113) for  $j + 1$  :

$$\sup_{0 \leq s \leq t} (e^{\frac{\alpha s}{2}} \|\varepsilon_{j+1}(s)\|) \leq N_{\varepsilon,j+1} \tag{1.153}$$

## 1.5. SAMPLED-OUTPUT CHAIN-OBSERVER DESIGN

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with  $N_{\varepsilon,j+1} = M_1 + \gamma_1 N_{z,j+1}$ . Inequalities (1.151)–(1.153) show that (1.145) and (1.146) also hold for  $j+1$ . We have thus proved that inequalities (1.145) and (1.146) hold for all  $1 \leq j \leq m$ . Then, it immediately follows that the signals  $(\varepsilon_j(t), \tilde{z}_j(t))$ , ( $j = 1 \dots m$ ) are exponentially convergent to the origin and the inequalities

$$|\tilde{z}_j(t)| \leq \rho_z(\theta) e^{-\alpha(\theta)t/2}, \quad (j = 2 \dots m), \quad \forall t \geq 0, \quad (1.154)$$

hold with

$$\rho_z(\theta) = \max(N_{z,1}, \dots, N_{z,m}). \quad (1.155)$$

Also, using (1.90) and Remark 1 (Part d), it follows from (1.146) that  $\tilde{x}_1(t)$  is also exponentially convergent to the origin and the inequality  $\|\tilde{x}_j(t)\| \leq \rho_x(\theta) e^{-\alpha(\theta)t/2}$ ,  $\forall t \geq 0$ , holds with

$$\rho_x(\theta) = \frac{\theta^n}{\varepsilon_a^n} \max(N_{\varepsilon,1}, \dots, N_{\varepsilon,m}) \quad (1.156)$$

Just as in Part 4, it is readily checked that  $\rho_x(\theta)$ ,  $\rho_y(\theta)$ ,  $\rho_z(\theta)$  are all increasing functions of  $\theta$ . This completes the proof of **Theorem 1.2**.

**Remark 4:** The maximal admissible values of the (fractional) delay  $r$  and the maximum sampling interval  $\tau$  are those satisfying conditions (1.124), (1.131) and (1.137). As pointed out in **Theorem 1.2**, the larger  $\theta$  the smaller the admissible values. Therefore, upper bounds  $r_M$  and  $\tau_M$  of these maximal admissible values are obtained by letting  $\alpha(\theta) = 0$  in (1.124), (1.131) and (1.137). Doing so, one gets:

$$r_M < \frac{1}{l_a \beta_\tau}, \quad \tau_M < \frac{1}{l_a \beta_\tau} \quad (1.157)$$

$$\frac{\tau_M + r_M(1 - l_a \beta_\tau \tau_M)}{(1 - l_a \beta_\tau r_M)(1 - l_a \beta_\tau \tau_M)} < \frac{1}{\gamma_1 \theta (\theta + l_b)} \quad (1.158)$$

Then, the sets of admissible delay and sampling interval are respectively defined by:

$$0 < r < r_M \quad \text{and} \quad 0 < \tau < \tau_M \quad (1.159)$$

Noting that  $r = d/m$  with  $m$  the number of observers in (1.84a)–(1.85b), it turns out that the condition  $0 < r < r_M$  can always be met (by increasing the number of observers).

## 1.6 Simulation Results

### 1.6.1 Observer application to a system satisfying all theoretical assumptions

To illustrate the performances of the chain observer (1.84a)-(1.84d) and (1.85a)-(1.85b), we consider the following second order system of the form (1.1a)-(1.1d) with the values,

$$\begin{aligned} A(v, z) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ b(v, x) &= \begin{bmatrix} \cos(x_1) \\ -\sin(x_2) - \sin(x_1) + 3\sin(v) \end{bmatrix} \\ C &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad d = 4s \end{aligned}$$

This system belongs to the class of systems defined in Section 1.2 to which the chain observer (1.84a)-(1.84d) and (1.85a)-(1.85b) applies whatever the initial conditions. The system is excited with the T-periodic input signal,

$$v(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 0.4T \\ 0 & \text{for } 0.4T \leq t \leq T \end{cases}$$

with  $T = 1s$ . The resulting output  $y(t)$  is periodically sampled with period  $\tau = 0.5s$  and its states are given the initial values  $x_1(0) = x_2(0) = 0$ . The chain observer is defined by Equations (1.84a)-(1.84d) and (1.85a)-(1.85b) with initial values  $\tilde{x}_1(0) = \tilde{x}_2(0) = 0.7$ . It is characterised, on one hand, by the gains  $\theta$  and  $K$  and, on the other hand, by the number  $m$  of underlying estimators. The latter is a measure of the observer complexity: the smaller  $m$  the simpler the observer. As a matter of fact, one seeks the simplest observer that converges rapidly. We consider this is the case when the convergence time is (nearly) three times the time-delay.

**Simplest observer selection for given gains.** Here, the observer gains are a priori set to  $\theta = 1.1$  and  $K = \begin{bmatrix} 2 & 1 \end{bmatrix}^T$ ; the latter is selected so that the matrix  $\bar{A} - KC$  has its eigenvalues both placed at  $-1$ . Then, the number  $m$  is increased until one gets satisfactory convergence properties. It turns out that the simpler chain observer is that composed of  $m = 4$  estimators. The resulting state estimates are shown, along with their true values,

## 1.6. SIMULATION RESULTS

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in Figure 1.2 and 1.3 which actually show satisfactory convergence.

**Simplest observer selection for different system delays.** To further illustrate the result of **Theorem 1.2**, the above simulation is repeated with different values of the system delay  $d$ . The observer parameters  $\theta$  and  $K$  are kept unchanged. As previously, for each delay value, the number  $m$  of estimators is increased until a satisfactory convergence quality is observed. Table 1.1 shows the couples  $(d, m)$  thus obtained.

$d$	0.5	1	2	5	10
$m$	1	1	2	4	8

Table 1.1 – Dependence of number  $m$  (characterising the simpler satisfactory observer) on system delay  $d$ . Case  $\theta = 1.1, K = [2 \ 1]^T$

**Observer complexity dependence on observer gain.** For a given system with delay, the complexity of the simpler satisfactory observer depends on the observer gains. To illustrate this dependence for the gain  $\theta$ , the above system with constant delay  $d = 4s$  is considered again. The chain observer is given a fixed gain  $K = [2 \ 1]^T$  but the parameter  $\theta$  is given different values. For each value, the number  $m$  for the simpler satisfactory observer is determined by simulation. The couples  $(\theta, m)$  thus obtained are shown in Table 1.2. Accordingly, the larger  $\theta$ , the larger  $m$ .

$\theta$	1.1	1.5	2	2.5
$m$	2	2	4	5

Table 1.2 – Dependence of number  $m$  (characterising the simpler satisfactory observer) on parameter  $\theta$ . Case  $d = 4s, K = [2 \ 1]^T$

## 1.6. SIMULATION RESULTS

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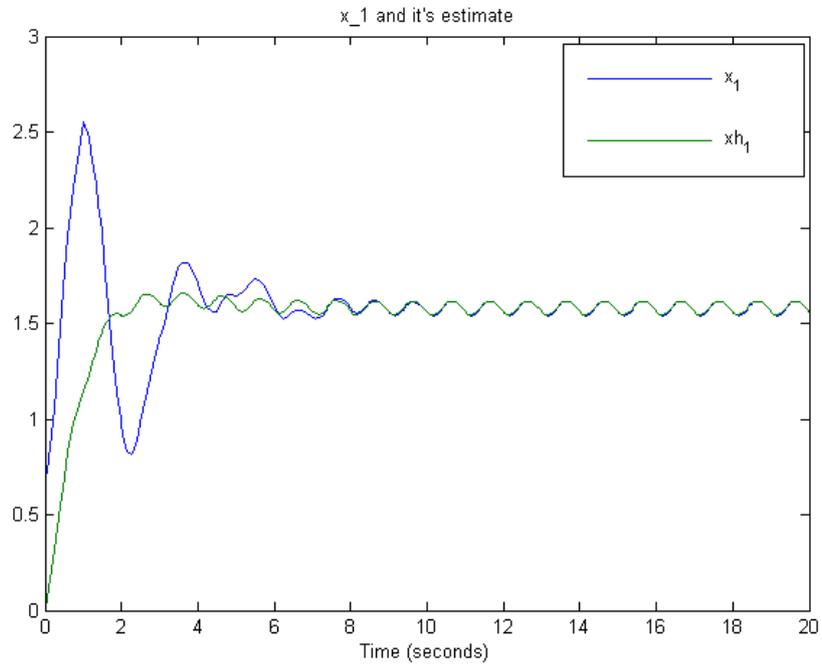


Figure 1.2 –  $x_1(t)$  and it's estimate  $\hat{x}_1(t)$  with  $m = 4$  and a delay  $d = 4s$

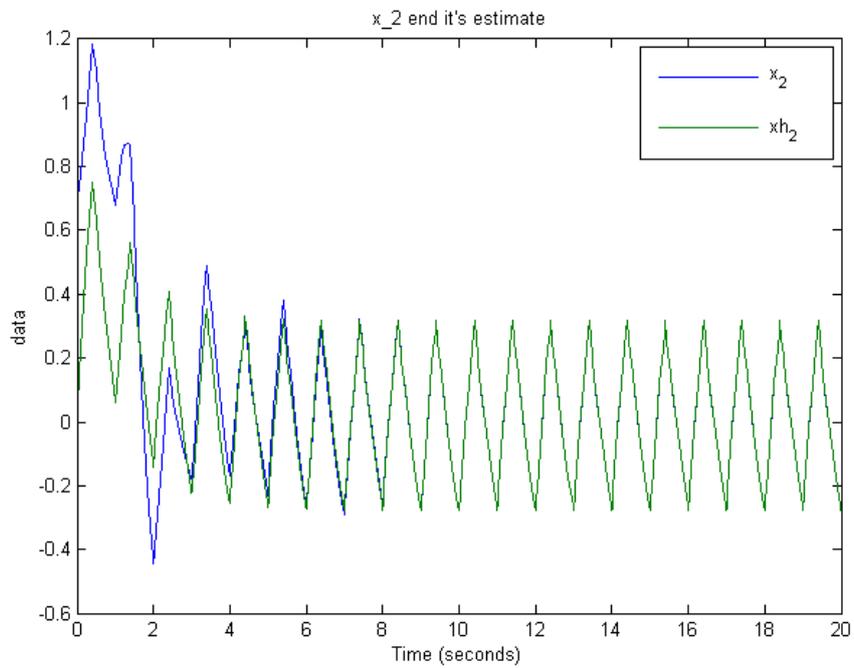


Figure 1.3 –  $x_2(t)$  and it's estimate  $\hat{x}_2(t)$  with  $m = 4$  and a delay  $d = 4s$

## 1.6. SIMULATION RESULTS

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### 1.6.2 Observer application to a system not satisfying all theoretical assumptions

**Theorem 1.2** states that the observer (1.84a)-(1.84d) and (1.85a)-(1.85b), when applied to the class of systems defined in Section 1.2, ensures global convergence of the estimates to their true values, provided the design parameters are appropriately selected. In this subsection, it checked that the observer still works well when applied to some systems not belonging to the class of Section 1.2, but then the globality feature might be lost. This is illustrated by considering the following Lotka–Volterra type system:

$$\begin{aligned}\dot{z}_1(t) &= z_1(t) - z_1(t)z_2(t) + v(t) \\ \dot{z}_2(t) &= -z_2(t) + z_1(t)z_2(t) \\ y(t) &= z_2(t - d)\end{aligned}$$

For the purpose of simulation, we let  $d = 4s$ . Using the variable change  $x_1 = z_2, x_2 = z_1$ , the above system is rewritten as follows:

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) + x_1(t)x_2(t) \\ \dot{x}_2(t) &= x_2(t) - x_1(t)x_2(t) + v(t) \\ y(t) &= x_1(t - d)\end{aligned}$$

which fits the vector form (1.1a-1.1d) with:

$$\begin{aligned}A(v, x) &= \begin{bmatrix} 0 & x_1 \\ 0 & 0 \end{bmatrix} \\ b(v, x) &= \begin{bmatrix} -x_1 \\ x_2 - x_1x_2 + v \end{bmatrix} \\ C &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad d = 2s\end{aligned}$$

Clearly, system observability (guaranteed by assumption A3) is lost whenever the variable  $x_1$  is zero (see also Remark 1 (Part d)). Bearing in mind this consideration, the system is excited with the input signal  $v(t) = 1 + 0.5\sin(t)$  (which stays always positive) and the initial conditions are let to be positive. In these operating conditions, it is readily checked that all assumptions (A1 to A4) hold because all system state variables turn out to be periodic and positive. The output  $y(t)$  is sampled with a fixed period  $\tau = 0.5s$  and its states are given the initial values  $x_1(0) = x_2(0) = 0.1$ . The chain observer, defined by Equations (1.84a)-(1.84d) and (1.85a)-(1.85b), is initialised with

## 1.6. SIMULATION RESULTS

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$\hat{x}_1(0) = \hat{x}_2(0) = 0.02$ . The observer design parameters  $\theta, K$  and  $m$  are selected following a similar procedure as in Subsection 1.6.1.

### Simplest observer selection for given gains

Here, the observer gains are a priori set to  $\theta = 1.1$  and  $K = \begin{bmatrix} 2 & 1 \end{bmatrix}^T$  so that the matrix  $\bar{A} - KC$  has its eigenvalues both placed at  $-1$ . Then, the number  $m$  is increased until one gets satisfactory convergence properties. The simpler chain observer thus obtained is that composed of  $m = 3$  estimators. The resulting state estimates are shown, along with their true values, in Figures 1.4-1.5 which actually show a satisfactory estimation quality.

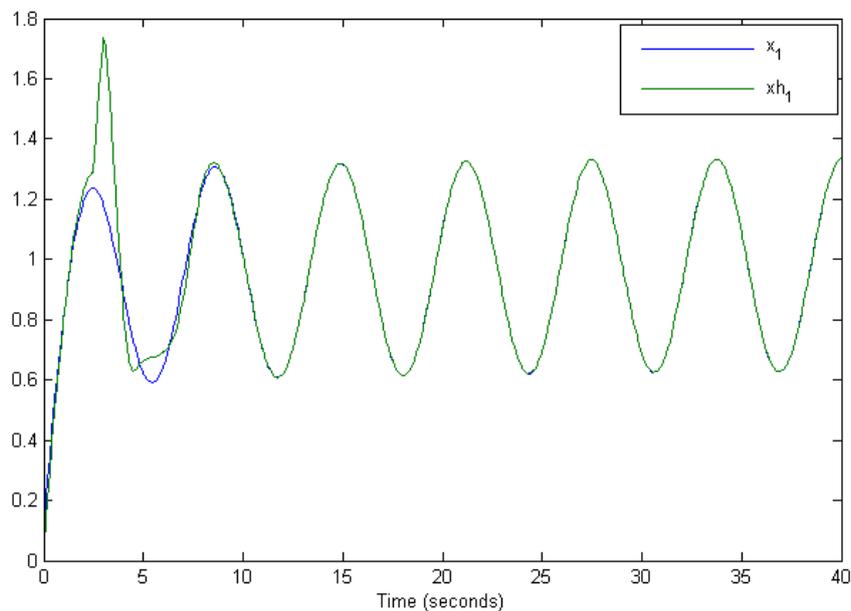


Figure 1.4 –  $x_1(t)$  and its estimate  $\hat{x}_1(t)$  with  $m = 4$  and a delay  $d = 2s$   
Lotka–Volterra

## 1.6. SIMULATION RESULTS

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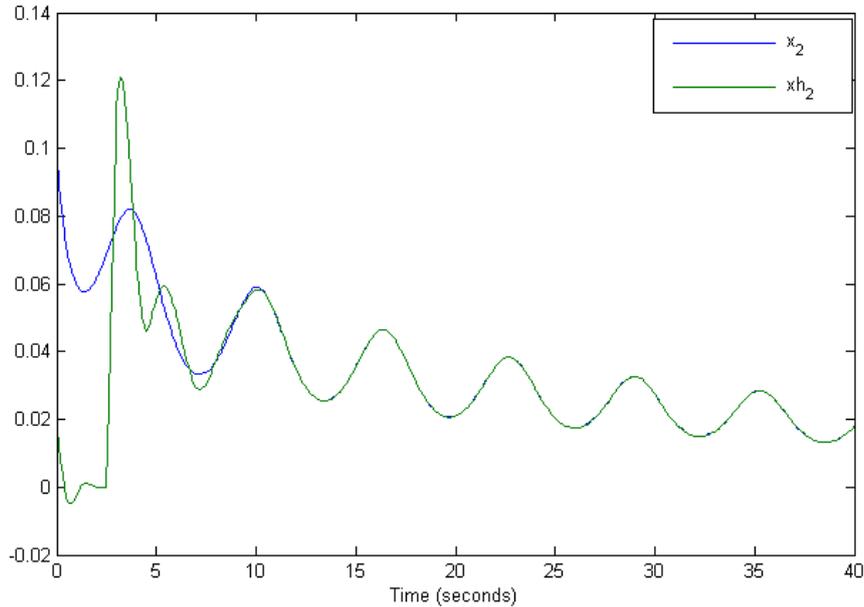


Figure 1.5 –  $x_2(t)$  and its estimate  $\hat{x}_2(t)$  with  $m = 4$  and a delay  $d = 2s$   
Lotka–Volterra

### Simplest observer selection for different system delays

The above simulation is repeated with different values of the delay  $d$ , while the observer parameters  $\theta$  and  $K$  are kept unchanged. As previously, for each delay value, the number  $m$  of estimators is increased until a satisfactory convergence quality is observed. Table 1.3 shows the couples  $(d, m)$  thus obtained.

$d$	0.5	1	2	5	8
$m$	1	1	2	6	8

Table 1.3 – Dependence of number  $m$  (characterising the simpler satisfactory observer) on system delay  $d$ . Case  $\theta = 1.1, K = [2 \ 1]^T$

### Observer complexity dependence on observer gain

Considering again the above system with constant delay  $d = 2s$  and the observer gain  $K = [2 \ 1]^T$ , we let the parameter  $\theta$  take different values and we seek the smallest number  $m$  that yields the best performances. The

## 1.7. CONCLUSION

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couples  $(\theta, m)$  thus obtained are shown in Table 1.4. Again, it is checked that  $m$  grows with  $\theta$ .

$\theta$	1.1	1.5	2	2.5
$m$	4	4	6	6

Table 1.4 – Dependence of number  $m$  (characterising the simpler satisfactory observer) on parameter  $\theta$ . Case  $d = 4s$ ,  $K = [2 \ 1]^T$

## 1.7 Conclusion

In this chapter we addressed the problem of state estimation for a class of triangular nonlinear systems (1.1a)-(1.1d) using sampled delayed output measurements. In addition to the fact that the delay is not necessarily small, the complexity of the problem lies in the interference of the delay and sampling and in the injection of the undelayed output (which is not accessible to measurements) in the state matrix  $A(v(t), z(t))$ .

The first proposed observer (1.1a)-(1.1d) features a simpler output predictor defined by two ODEs (while previous observers involved output and state predictors defined by several ODEs). The maximum sampling interval and time-delay for the observer to be exponentially convergent are well defined by inequalities (1.74) and (1.75).

The second part of the chapter was dedicated to present a new form of the first observer in order to be able to observe the states of the system (1.1a)-(1.1d) with larger delays, this new form is analytically described by equations (1.84a)-(1.84d) and (1.85a)-(1.85b), where this observer is composed of  $m$  sub-observers, we increase  $m$  until we get satisfactory results in other words the larger the delay  $d$  the bigger is the number of sub-observers  $m$  necessary to estimate the states of the system. Estimates of the maximum sampling interval and the minimal number of state estimators are provided in terms of inequalities (1.157)-(1.158). These results were published in [24, 25]. Possible future extensions of the present study include e.g. (1) removal of the requirement on the system state to be bounded and the prior knowledge of the output upper bound; and (2) enlargement of the class of systems to non-triangular form systems or to systems with nonlinear output function and non-constant delay.

# Chapter 2

## Chain Observer Design For a Class of ODE-PDE Systems

### 2.1 Introduction

In this chapter, the problem of state observation for systems with output delay (Fig. 2.1) is addressed by letting the time delay be what it is: a distributed parameter phenomenon. Accordingly, time delay is captured through a first-order hyperbolic PDE connected in series with the ODE that describes the system finite-dimensional dynamics, leading to an ODE–PDE cascade representation of the system. Then, the observation problem consists in designing an observer that provides online estimates of both the (finite-dimensional) state of the ODE subsystem and the (infinite-dimensional) state of the PDE sensor. This formulation of the observer design problem has been introduced in [31] and [32] where the ability of the backstepping design approach, to yield full-order observers with feedback-predictors, has been demonstrated for linear systems. Then, arbitrary time-delay sizes can be compensated for, due to the system linearity. This concept was developed for parabolic and second-order hyperbolic PDEs, as well as for several challenging physical problems such as turbulent flows and magneto-hydrodynamics [49]

This chapter aims at generalizing the PDE-based backstepping-like observer design approach of [32] and [33] to nonlinear systems. Specifically, the latter is described by an ODE of strict-feedback form with a globally Lipschitz nonlinearity. To cope with the system nonlinearity, we invoke the principle of high-gain observer design as in [4]. In the latter, we showed that high-gain observers, for cascade systems with parabolic PDEs, can be made exponentially convergent provided that the PDE domain length is sufficiently

## 2.1. INTRODUCTION

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small. A similar result can be obtained in the case of (first-order) hyperbolic PDEs.

A more challenging problem is one of designing exponentially convergent observers for ODE–PDE systems with nonlinear ODEs and PDEs of arbitrarily large domain length. This problem has yet to be solved both in the case of hyperbolic PDEs (of any order) and in the case of heat PDEs. In this chapter, we develop a solution in the case of first-order hyperbolic PDEs using the PDE-based backstepping-like observer design approach. One key idea is to redesign within the ODE–PDE framework the cascade observer concept, so far only developed in an ODE framework. Accordingly, we re-express the initial ODE–PDE system representation in the form of  $m \geq 2$  fictitious ODE–PDE subsystems involving first-order hyperbolic PDEs with domain length  $D/m$  ( $D$  being the arbitrarily-large time-delay) and appropriate boundary conditions defining the interaction between the subsystems. Then, we design an observer for each ODE–PDE subsystem using the high-gain observer principle backstepping-like design technique. It turns out that the global observer is composed of  $m$  (high-gain) observers connected in series. The interconnection is such that the first partial observer is directly driven by the physical system output. Then, the  $j$ th partial observer is driven by a virtual output generated by the  $(j - 1)$ th observer. Each partial observer includes an output predictor which compensates for the effects of the fractional time-delay  $D/m$ . The predictors are defined by simple first-order hyperbolic PDEs that are much simpler compared to some previous works which involved output and state predictors. We then use a backstepping-like transformation of the estimation error system and construct an appropriate Lyapunov stability functional to analyse the transformed system. Doing so, we obtain sufficient conditions for the cascade observer to be exponentially convergent. The sufficient conditions involve the minimal number  $m$  of partial observers: the larger the system delay, the larger the number  $m$ .

Compared with ODE-based delay-compensating observers (e.g. [19, 25]), the present observer is full-order in the sense that it estimates both the system (finite-dimensional) state and the sensor (infinite-dimensional) state. Also, the present output predictors feature a feedback structure, while those involved in ODE-based observers are open-loop.

In this chapter, we start by introducing the system under study and we put it under a cascade representation in Sections 2.2 and 2.3, we then present the proposed chain observer for this system in Section 2.4, this system is after that in Section 2.5 we analyse the error system using the backstepping-like

## 2.2. SYSTEM DESCRIPTION

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transformation in order to get sufficient conditions for the convergence of the observer to the state of the system in terms of time delay  $D$  and the number of sub-observers  $m$ . Simulation results are presented in Section 2.6 and a conclusion ends this chapter.

## 2.2 System description

The class of systems under study in this chapter is depicted by the following figure :

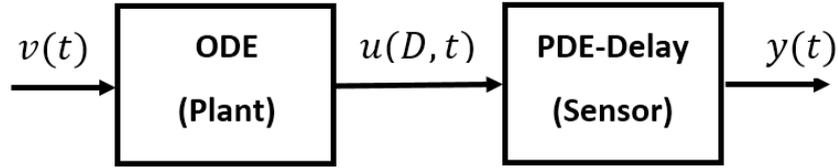


Figure 2.1 – ODE-PDE cascade system to be observed

This system consists of a finite-dimensional nonlinear subsystem connected in series with a time delay. Analytically, the considered output delayed system is described as follows:

$$\dot{X}(t) = AX(t) + f(X(t), v(t)) \quad (2.1a)$$

$$y(t) = CX(t - D) \quad (2.1b)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad (2.2a)$$

$$C = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{1 \times n} \quad (2.2b)$$

where  $X(t) \in \mathbb{R}^n$  is the system state vector,  $y(t) \in \mathbb{R}$  is the system output,  $v \in \mathcal{C}^0([0, \infty) : \Omega_v)$  is an external signal (control input) taking values in some known subset  $\Omega_v \subset \mathbb{R}$ ,  $f \in \mathcal{C}^2([0, \infty) : \mathbb{R}^{n \times n})$  is a known bounded vector field

## 2.2. SYSTEM DESCRIPTION

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with the triangular form:

$$f(X, v) = \begin{bmatrix} f_1(X_1, v) \\ f_2(X_1, X_2, v) \\ \vdots \\ f_n(X_1, \dots, X_n, v) \end{bmatrix} \quad (2.2c)$$

where  $f_i : \mathbb{R}^i \rightarrow \mathbb{R}$ . It is supposed that  $f(\cdot)$  is globally Lipschitz with respect to  $X$ , uniformly in  $v \in \Omega_v$ . That is, the following property holds:

$$\exists \beta_0 > 0, \forall X \in \mathbb{R}^n, \forall v \in \Omega_v : \|f_X(X)\| \leq \beta_0 \quad (2.3)$$

The positive real constant  $D$  denotes a time-delay that is arbitrarily large, but known. Both the input  $v$  and output  $y$  are accessible to measurements, but the state vector  $X(t)$  is not. Following the approach developed in [32] and [33], the output equation (2.1b) is given an equivalent representation in the form of a first-order hyperbolic equation. Accordingly, the system model (2.1a-2.2b) rewrites as follows:

$$\dot{X}(t) = AX(t) + f(X(t), v(t)) \quad (2.4a)$$

$$u(D, t) = CX(t) \quad (2.4b)$$

$$u_t(x, t) = u_x(x, t), \quad 0 \leq x < D, \quad t > 0 \quad (2.4c)$$

$$y(t) = u(0, t) \quad (2.4d)$$

The solution of (2.4b)–(2.4c) is well known to be:

$$u(x, t) = CX(t + x - D), \quad 0 \leq x < D, \quad t \geq 0 \quad (2.5)$$

That is, the boundary measurement (2.4d) gives the delayed output  $y(t) = CX(t - D)$ , which is identical to (2.1b).

We seek the design of an observer that provides accurate online estimates of the finite-dimensional state  $X(t)$  and the distributed state  $u(x, t)$  ( $0 \leq x < D$ ). The observer must only make use of the measurements of  $y(t), v(t)$ .

### Remark 1.

a) In the case of a zero function  $f(\cdot)$  (i.e. case of linear systems), exponentially convergent observers have been designed in [32] and [33] using the (infinite-dimensional) backstepping transformation approach. In such a linear context, there is no limitation on the delay size.

## 2.3. CASCADE OBSERVER DESIGN

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b) In the case of nonlinear systems (nonzero function  $f(\cdot)$ ) and a parabolic PDE (instead of (2.4c)), an exponentially convergent observer has been designed in [4] combining the backstepping transformation and the high-gain observer design technique (this motivated the triangular structure (2.2c) of the nonlinear function  $f(\cdot)$ ). The exponential convergence of the observer in [4] was established under the condition that the PDE domain length (presently, equivalent to the delay  $D$ ) is sufficiently small.

c) The present class of systems, described by (2.4a)–(2.4d), differs from that [4] in that the PDE is hyperbolic type and the domain length  $D$  is of arbitrary size which, together with the nonlinear function  $f(\cdot)$ , entails an extra difficulty. To get rid of this difficulty, a chain observer will be developed that is of quite different nature compared to those in [32]–[4].

## 2.3 Cascade observer design

The cascade observer design is performed in three steps. First, the initial system model (2.1a)–(2.2c) is given a cascade representation. Then, a set of backstepping transformations is introduced and a new cascade system representation is established in the transformed coordinates. Finally, the cascade observer is constructed on the basis of the new system representation.

### 2.3.1 System cascade representation

Letting  $m \geq 1$  be any integer, define the following signals where  $j = 1, \dots, m$  :

$$X_j(t) = X\left(t + j\frac{D}{m} - D\right), \quad t \geq 0 \quad (2.6a)$$

$$u_j(x, t) = u\left(x + (j-1)\frac{D}{m}, t\right), \quad \text{for } 0 \leq x \leq \frac{D}{m}; \quad t \geq 0 \quad (2.6b)$$

$$v_j(t) = v\left(t + j\frac{D}{m} - D\right) \quad (2.6c)$$

### 2.3. CASCADE OBSERVER DESIGN

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The following useful expressions are immediately obtained from (2.6a)–(2.6c) and (2.5):

$$u_j(x, t) = CX\left(t + x + (j-1)\frac{D}{m} - D\right),$$

$$\text{for } j = 1, \dots, m; 0 \leq x \leq \frac{D}{m} \quad (2.7a)$$

$$u_{j+1}(0, t) = u_j\left(\frac{D}{m}, t\right) = CX_j(t), \quad j = 2, \dots, m; \quad t \geq 0 \quad (2.7b)$$

$$X_m(t) = X(t) \quad (2.7c)$$

$$v_m(t) = v(t) \quad (2.7d)$$

$$u_m\left(\frac{D}{m}, t\right) = CX_m(t) = u(D, t) \quad (2.7e)$$

$$u_1(x, t) = u(x, t), \quad (2.7f)$$

$$u_1(0, t) = u(0, t) = CX(t - D) = y(t), \quad t \geq 0 \quad (2.7g)$$

In view of these relations, and equations (2.4a)–(2.4c), time-differentiation of (2.6a) and (2.6b) yields the following set of subsystems, with state functions  $(u_j(x, t), X_j(t)) (j = 1, \dots, m)$  :

Subsystem  $m$  :

$$\dot{X}_m(t) = AX_m(t) + f(X_m(t), v(t)), \quad t \geq 0 \quad (2.8a)$$

$$u_m\left(\frac{D}{m}, t\right) = CX_m(t), \quad t \geq 0 \quad (2.8b)$$

$$u_{m,t}(x, t) = u_{m,x}(x, t), \quad 0 \leq x \leq \frac{D}{m}, \quad t \geq 0 \quad (2.8c)$$

Subsystem  $j$ , ( $j = 2, \dots, m-1$ ) :

$$\dot{X}_j(t) = AX_j(t) + f(X_j(t), v_j(t)), \quad t \geq 0 \quad (2.9a)$$

$$u_j\left(\frac{D}{m}, t\right) = CX_j(t) = u_{j+1}(0, t), \quad t \geq 0 \quad (2.9b)$$

$$u_{j,t}(x, t) = u_{j,x}(x, t), \quad 0 \leq x \leq \frac{D}{m}, \quad t \geq 0 \quad (2.9c)$$

Subsystem 1 :

$$\dot{X}_1(t) = AX_1(t) + f(X_1(t), v_1(t)), \quad t \geq 0 \quad (2.10a)$$

$$u_1\left(\frac{D}{m}, t\right) = CX_1(t), \quad t \geq 0 \quad (2.10b)$$

$$u_{1,t}(x, t) = u_{1,x}(x, t), \quad 0 \leq x \leq \frac{D}{m}, \quad t \geq 0 \quad (2.10c)$$

## 2.3. CASCADE OBSERVER DESIGN

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Equations (2.8a) to (2.10c) constitute an equivalent representation of the system in the form of  $m$  cascaded ODE–PDE subsystems. A key feature of this representation is that all subsystems are of equal time-delay  $D/m$ . The larger  $m$ , the smaller this time-delay.

### 2.3.2 System coordinate transformation

Inspired by [32] and [4], we introduce the following  $m$  backstepping-like transformations :

$$\begin{aligned} p_j(x, t) &= u_j(x, t) - CM(x)M^{-1}\left(\frac{D}{m}\right)X_j(t); \\ j &= 1, \dots, m; \quad 0 \leq x \leq \frac{D}{m}, \quad t \geq 0 \end{aligned} \quad (2.11)$$

where  $M(x) \in \mathbb{R}^{n \times n}$  is defined by the ODE:

$$\frac{dM}{dx}(x) = AM(x), \quad M(0) = I \quad (2.12)$$

which simply solves by:

$$M(x) = e^{Ax} \quad (2.13)$$

Presently, this simplifies further as follows:

$$M(x) = I + \sum_{k=1}^{n-1} \frac{A^k x^k}{k!}, \quad M^{-1}(x) = I + \sum_{k=1}^{n-1} \frac{(-1)^k A^k x^k}{k!} \quad (2.14)$$

using the fact that  $A^n = 0$ , which is a property of (2.2a). Additional properties are:

$$AM(x) = M(x)A, \quad AM^{-1}(x) = M^{-1}(x)A \quad (2.15)$$

Using these properties and equations (2.8c), (2.9c) and (2.10c), it follows from (2.11) that the new state functions  $p_j(x, t) (j = 1, \dots, m)$  are governed by the following PDEs, for  $j = 1, \dots, m; \quad 0 \leq x \leq \frac{D}{m}; \quad t \geq 0$  :

$$p_{j,t}(x, t) = p_{j,x}(x, t) - CM(x)M^{-1}\left(\frac{D}{m}\right)f(X_j(t), v_j(t)) \quad (2.16a)$$

$$p_j\left(\frac{D}{m}, t\right) = 0 \quad (2.16b)$$

where the boundary conditions (2.16b) are immediately obtained from (2.11), using (2.12), (2.8b), (2.9b) and (2.10b). For convenience, the transformed

## 2.4. PROPOSED OBSERVER

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system cascade representation, is rewritten in terms of the state functions  $(X_j(t), p_j(x, t))$ , for  $j = 1, \dots, m$ ,  $0 \leq x < \frac{D}{m}$  and  $t \geq 0$  :

$$\dot{X}_j(t) = AX_j(t) + f(X_j(t), v_j(t)), \quad t \geq 0 \quad (2.17a)$$

$$p_{j,t}(x, t) = p_{j,x}(x, t) - CM(x)M^{-1}\left(\frac{D}{m}\right)f(X_j(t), v_j(t)) \quad (2.17b)$$

$$p_j\left(\frac{D}{m}, t\right) = 0 \quad (2.17c)$$

$$u_j(x, t) = p_j(x, t) + CM(x)M^{-1}\left(\frac{D}{m}\right)X_j(t); \quad (2.17d)$$

## 2.4 Proposed Observer

Inspired by [25] and [4], we propose the following high-gain type chain observer:

Observer for  $(X_1(t), p_1(x, t))$ , with  $0 \leq x \leq \frac{D}{m}$ ;  $t \geq 0$  :

$$\begin{aligned} \dot{\hat{X}}_1(t) &= A\hat{X}_1(t) + f(\hat{X}_1(t), v_1(t)) \\ &\quad - M\left(\frac{D}{m}\right)L(\hat{u}_1(0, t) - u_1(0, t)) \end{aligned} \quad (2.18a)$$

$$\hat{p}_{1,t}(x, t) = \hat{p}_{1,x}(x, t) - CM(x)M^{-1}\left(\frac{D}{m}\right)f(\hat{X}_1(t), v_1(t)) \quad (2.18b)$$

$$\hat{p}_1\left(\frac{D}{m}, t\right) = 0 \quad (2.18c)$$

$$\hat{u}_1(x, t) = \hat{p}_1(x, t) + CM(x)M^{-1}\left(\frac{D}{m}\right)\hat{X}_1(t), \quad (2.18d)$$

with  $L \in R^n$  is a gain to be chosen later, where  $u_1(0, t) = u(0, t)$  (by (2.7g)) is accessible to measurements, and  $\hat{X}_1(0)$  is arbitrary.

Observer for  $(X_j(t), p_j(x, t))$ , with  $j = 2, \dots, m$  and  $t \geq 0$  :

$$\begin{aligned} \dot{\hat{X}}_j(t) &= A\hat{X}_j(t) + f(\hat{X}_j(t), v_j(t)) \\ &\quad - M\left(\frac{D}{m}\right)L(\hat{u}_j(0, t) - \hat{u}_{j-1}\left(\frac{D}{m}, t\right)) \end{aligned} \quad (2.19a)$$

$$\hat{p}_{j,t}(x, t) = \hat{p}_{j,x}(x, t) - CM(x)M^{-1}\left(\frac{D}{m}\right)f(\hat{X}_j(t), v_j(t)) \quad (2.19b)$$

$$\hat{p}_j\left(\frac{D}{m}, t\right) = 0 \quad (2.19c)$$

$$\begin{aligned} \hat{u}_j(x, t) &= \hat{p}_j(x, t) + CM(x)M^{-1}\left(\frac{D}{m}\right)\hat{X}_j(t), \\ 0 \leq x \leq \frac{D}{m}, \quad t \geq 0, \end{aligned} \quad (2.19d)$$

where  $\hat{X}_j(0)$  is arbitrary and all other observer parameters are as in (2.18a)–(2.18d). Note that equations (2.19a)–(2.19d) define a set of sub-observers

## 2.5. OBSERVER ANALYSIS

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that must be run in order, starting with the sub-observer indexed by  $j = 2$ . Accordingly, when the  $j$ th sub-observer is being executed, the quantity  $\hat{u}_{j-1}\left(\frac{D}{m}, t\right)$ , has already been computed by the sub-observer of index  $j - 1$ . Then, in view of (2.7b), it makes a sense to use  $\hat{u}_{j-1}\left(\frac{D}{m}, t\right)$  as output measurement for sub-observer indexed by  $j$ .

**Remark 2.**

a) As already noticed, cascade observers can directly be designed on the basis of the initial system representation (2.1a)–(2.2c), using ODE-based tools, see e.g. [19]–[25].

b) The cascade observers obtained with the ODE-based observer design involve point-wise output predictors which are fully defined by ODEs aiming at estimating the system output only at times  $t - D + j\frac{D}{m}$ , for  $j = 1, \dots, m$  and all  $t$ , where  $D$  denotes the system time delay. The present ODE–PDE-based observer design approach also provides cascade observers that involve output predictors. But, the latter are presently distributed parameter, defined by the PDEs (2.18b)–(2.18d) and (2.19b)–(2.19d). Accordingly, the system output  $y(t + x - D) = u(x, t)$  is estimated, for all  $0 \leq x \leq D$  and  $t \geq 0$ , whereas the point-wise predictors only estimate the outputs  $y(t + x - D) = u(x, t)$  with  $x = j\frac{D}{m}$ ,  $j = 1, \dots, m$ . Furthermore, as pointed out in e.g. [33], the ODE-based observers essentially estimate the past state from  $\frac{D}{m}$  seconds back, and then advance it in an open-loop manner  $\frac{D}{m}$  seconds in the future. In contrast, the state estimation with the present PDE-based observer is performed all the time in closed-loop.

## 2.5 Observer Analysis

For analysis purpose, the following error notations are introduced:

$$\tilde{X}_j = \hat{X}_j - X_j, \quad \tilde{p}_j = \hat{p}_j - p_j, \quad \tilde{u}_j = \hat{u}_j - u_j, \quad \text{for } j = 1, \dots, m \quad (2.20)$$

$$\Delta = \text{diag} \left[ 1, \frac{1}{\theta}, \dots, \frac{1}{\theta^{n-1}} \right] \in \mathbb{R}^{n \times n}, \quad \text{for any } \theta > 1, \quad (2.21)$$

Also, the following properties are easily checked, using (2.2a), (2.2b) and (2.13):

$$\Delta A \Delta^{-1} = \theta A, \quad C \Delta = C, \quad C \Delta^{-1} = C. \quad (2.22a)$$

$$\Delta M(x) \Delta^{-1} = e^{\Delta A \Delta^{-1} x} = e^{\theta A x} \quad (2.22b)$$

$$\Delta M(x)^{-1} \Delta^{-1} = e^{-\Delta A \Delta^{-1} x} = e^{-\theta A x} \quad (2.22c)$$

## 2.5. OBSERVER ANALYSIS

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Now, differentiating the errors  $\tilde{X}_j = \hat{X}_j - X_j$  and  $\tilde{p}_j$  with respect to  $t$ , yields using (2.18b)–(2.18d) and (2.19b)–(2.19d):

$$\begin{aligned}\dot{\tilde{X}}_1(t) &= A\tilde{X}_1(t) + f(\hat{X}_1(t), v_1(t)) - f(X_1(t), v_1(t)) \\ &\quad - M\left(\frac{D}{m}\right)L\tilde{u}_1(0, t)\end{aligned}\quad (2.23a)$$

$$\begin{aligned}\tilde{p}_{1,t}(x, t) &= \tilde{p}_{1,x}(x, t) - CM(x)M^{-1}\left(\frac{D}{m}\right) \times \\ &\quad \left(f(\hat{X}_1(t), v_1(t)) - f(X_1(t), v_1(t))\right)\end{aligned}\quad (2.23b)$$

$$\tilde{p}_1\left(\frac{D}{m}, t\right) = 0 \quad (2.23c)$$

and, for  $j = 2, \dots, m$ :

$$\begin{aligned}\dot{\tilde{X}}_j(t) &= A\tilde{X}_j(t) + f(\hat{X}_j(t), v_j(t)) - f(X_j(t), v_j(t)) \\ &\quad - M\left(\frac{D}{m}\right)L(\hat{u}_j(0, t) - u_j(0, t) + \hat{u}_{j-1}\left(\frac{D}{m}, t\right) - u_{j-1}\left(\frac{D}{m}, t\right)) \\ &= A\tilde{X}_j(t) + f(\hat{X}_j(t), v_j(t)) - f(X_j(t), v_j(t)) \\ &\quad - M\left(\frac{D}{m}\right)L(\tilde{u}_j(0, t) + \tilde{u}_{j-1}\left(\frac{D}{m}, t\right))\end{aligned}\quad (2.24)$$

Using the equality  $0 = -u_j(0, t) + u_{j-1}\left(\frac{D}{m}, t\right)$  which is a consequence of (2.9b). Also, one gets from (2.19a)–(2.19d) that

$$\hat{u}_{j-1}\left(\frac{D}{m}, t\right) = C\hat{X}_{j-1}(t) \quad (2.25)$$

which together with (2.9b) gives

$$\tilde{u}_{j-1}\left(\frac{D}{m}, t\right) = C\tilde{X}_{j-1}(t) \quad (2.26)$$

Then, (2.24) writes for  $0 \leq x < \frac{D}{m}$ ;  $t \geq 0$ :

$$\begin{aligned}\dot{\tilde{X}}_j(t) &= A\tilde{X}_j(t) + f(\hat{X}_j(t), v_j(t)) - f(X_j(t), v_j(t)) \\ &\quad - M\left(\frac{D}{m}\right)L(\tilde{u}_j(0, t) + M\left(\frac{D}{m}\right)LC\tilde{X}_{j-1}(t))\end{aligned}\quad (2.27a)$$

$$\begin{aligned}\tilde{p}_{j,t}(x, t) &= \tilde{p}_{j,x}(x, t) - CM(x)M^{-1}\left(\frac{D}{m}\right) \left(f(\hat{X}_j(t), v_j(t)) \right. \\ &\quad \left. - f(X_j(t), v_j(t))\right)\end{aligned}\quad (2.27b)$$

$$\tilde{p}_j\left(\frac{D}{m}, t\right) = 0 \quad (2.27c)$$

Then, the main analysis result is stated in the following theorem, where the notation  $z[t]$  designates the function  $[0, \frac{D}{m}] \rightarrow \mathbb{R}$ ,  $x \rightarrow z[t](x) = z(x, t)$ :

## 2.5. OBSERVER ANALYSIS

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**Theorem 2.1:** Let the proposed chain observer (2.18a)–(2.18d) and (2.19a)–(2.19d) be applied to the ODE-PDE system (2.1a)–(2.2c), and denote  $L = \theta \Delta^{-1} L_0$  where  $L_0 \in \mathbb{R}^n$  is any vector gain such that the matrix  $A - L_0 C$  is Hurwitz. Then, there exist a scalar  $\theta^* > 0$  and an integer  $m^* > 0$  such that, for all  $\theta > \theta^*$  and all  $m > m^*$ , the chain observer is globally exponentially convergent in the sense that  $\|\hat{X}_j(t)\|$  and  $\int_0^D \hat{u}(x, t)^2 dx$  are exponentially converging to  $\|X_j(t)\|$  and  $\int_0^D u(x, t)^2 dx$  respectively as  $t \rightarrow \infty$  for all initial conditions  $X_j(0), \hat{X}_j(0) \in \mathbb{R}^n$ ,  $u_j[0], \hat{u}_j[0], p_j[0], \hat{p}_j[0] \in H^1(0, \frac{D}{m})$ , with  $(\hat{p}_j[0])(\frac{D}{m}) = 0$ ,  $(u_j[0])(\frac{D}{m}) = C X_j(0)$ .

**Proof:** First note that the selection of  $L \in \mathbb{R}^n$  so that  $A - LC$  is Hurwitz is not an issue because the pair  $(A, C)$  is observable. Also, introduce the following coordinate change:

$$Z_j(t) = \Delta M^{-1} \left( \frac{D}{m} \right) \tilde{X}_j(t), \quad j = 1, \dots, m \quad (2.28a)$$

or, equivalently,

$$\tilde{X}_j(t) = M \left( \frac{D}{m} \right) \Delta^{-1} Z_j(t), \quad j = 1, \dots, m \quad (2.28b)$$

Then it readily follows from (2.18d) and (2.19d) and (2.20) the following relation:

$$\begin{aligned} \tilde{u}_j(x, t) &= \tilde{p}_j(x, t) + CM(x)M^{-1} \left( \frac{D}{m} \right) \tilde{X}_j(t) \\ &= \tilde{p}_j(x, t) + CM(x)\Delta^{-1} Z_j(t), \quad \text{for } j = 1, \dots, m; \\ &0 \leq x \leq \frac{D}{m}; \quad t \geq 0 \end{aligned} \quad (2.29)$$

The rest of the proof is divided into two parts. First, we show that for  $j = 1$ ,  $\|Z_j(t)\|$  and  $\int_0^D e^x \tilde{p}_j(x, t)^2 dx$  are exponentially vanishing, in the second part of the proof we show that the same result holds for  $j = 2, \dots, m$ .

**Step 1.** Proof of the theorem results for  $j = 1$ . From (21a) one gets using

## 2.5. OBSERVER ANALYSIS

---

(22) and (24)–(25), for  $0 < x < \frac{D}{m}$ ;

$$\begin{aligned}
\dot{Z}_1(t) &= \Delta M^{-1} \left( \frac{D}{m} \right) (A \tilde{X}_1(t) + f(\hat{X}_1(t), v_1(t)) - f(X_1(t), v_1(t))) \\
&\quad - \Delta M^{-1} \left( \frac{D}{m} \right) \left( M \left( \frac{D}{m} \right) L (\tilde{p}_1(0, t) + C \Delta^{-1} Z_1(t)) \right) \\
&= \Delta M^{-1} \left( \frac{D}{m} \right) (A M \left( \frac{D}{m} \right) \Delta^{-1} Z_1(t) + f(\hat{X}_1(t), v_1(t)) - f(X_1(t), v_1(t))) \\
&\quad - \Delta M^{-1} \left( \frac{D}{m} \right) \left( M \left( \frac{D}{m} \right) L (\tilde{p}_1(0, t) + C \Delta^{-1} Z_1(t)) \right) \\
&= (\Delta A \Delta^{-1} - \Delta L C \Delta^{-1}) Z_1(t) \\
&\quad + \Delta M^{-1} \left( \frac{D}{m} \right) \left( \int_0^1 f_X(X(t) + s \tilde{X}(t), v_1(t)) ds \right) \\
&\quad \times M \left( \frac{D}{m} \right) \Delta^{-1} Z_1(t) - \Delta L \tilde{p}_1(0, t) \tag{2.30}
\end{aligned}$$

where the last equality is obtained using the mean value theorem and the property  $M(0) = I$ . Using (2.22a)–(2.22b), (2.23a)–(2.23c) and the equality  $L = \theta \Delta^{-1} L_0$ , equation (2.30) develops as follows, for  $0 \leq x \leq \frac{D}{m}$ ;  $t \geq 0$ :

$$\begin{aligned}
\dot{Z}_1(t) &= \theta(A - L_0 C) Z_1(t) + \Delta M^{-1} \left( \frac{D}{m} \right) \left( \int_0^1 f_X(X(t) + s \tilde{X}(t), v_1(t)) ds \right) \\
&\quad \times M \left( \frac{D}{m} \right) \Delta^{-1} Z_1(t) - \theta L_0 \tilde{p}_1(0, t) \tag{2.31a}
\end{aligned}$$

Similarly, (2.23b) rewrites using (2.28b):

$$\begin{aligned}
\tilde{p}_{1,t}(x, t) &= \tilde{p}_{1,x}(x, t) - C M(x) M^{-1} \left( \frac{D}{m} \right) \left( \int_0^1 f_X(X(t) + s \tilde{X}(t), v_1(t)) ds \right) \\
&\quad \times M \left( \frac{D}{m} \right) \Delta^{-1} Z_1(t) \tag{2.31b}
\end{aligned}$$

$$\tilde{p}_1 \left( \frac{D}{m}, t \right) = 0 \tag{2.31c}$$

where the last equality is identical to (2.23c). To analyse the system (2.31a)–(2.31c), consider the Lyapunov function candidate:

$$V_1 = Z_1(t)^T P Z_1(t) + \frac{a_1}{2} \int_0^{\frac{D}{m}} e^x \tilde{p}_1(x, t)^2 dx \tag{2.32}$$

where  $a_1 > 0$  is any real constant and  $P = P^T$  is the solution of this equation:

$$(A - L_0 C)^T P + P(A - L_0 C) = -\mu I \tag{2.33}$$

where  $\mu > 0$  is any real scalar and  $I \in \mathbb{R}^{n \times n}$  denotes the identity matrix. Time differentiation of (2.32) gives, using (2.31a)–(2.31c) and (2.33):

## 2.5. OBSERVER ANALYSIS

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$$\begin{aligned}
\dot{V}_1 &= \dot{Z}_1^T P Z_1 + Z_1^T P \dot{Z}_1 + a_1 \int_0^{\frac{D}{m}} e^x \tilde{p}_1(x, t) \tilde{p}_{1,t}(x, t) dx & (2.34) \\
&= -\mu\theta \|Z_1\|^2 + 2Z_1^T P \Delta M^{-1} \left(\frac{D}{m}\right) \left( \int_0^1 f_X(X(t) + s\tilde{X}(t), v_1(t)) ds \right) \\
&\quad \times M \left(\frac{D}{m}\right) \Delta^{-1} Z_1 - 2\theta Z_1^T(t) P L_0 \tilde{p}_1(0, t) \\
&\quad + a_1 \int_0^{\frac{D}{m}} e^x \tilde{p}_1(x, t) \tilde{p}_{1,x}(x, t) dx - a_1 \int_0^{\frac{D}{m}} \left[ e^x \tilde{p}_1(x, t) C M(x) M^{-1} \left(\frac{D}{m}\right) \right. \\
&\quad \left. \times \left( \int_0^1 f_X(X(t) + s\tilde{X}(t), v_1(t)) ds \right) M \left(\frac{D}{m}\right) \Delta^{-1} Z_1(t) \right] dx \\
&= -\mu\theta \|Z_1\|^2 + 2Z_1^T P \Delta M^{-1} \left(\frac{D}{m}\right) \left( \int_0^1 f_X(X(t) + s\tilde{X}(t), v_1(t)) ds \right) \\
&\quad \times M \left(\frac{D}{m}\right) \Delta^{-1} Z_1 - 2\theta Z_1^T(t) P L_0 \tilde{p}_1(0, t) - a_1 \frac{\tilde{p}_1^2(0, t)}{2} - \frac{a_1}{2} \int_0^{\frac{D}{m}} e^x \tilde{p}_1^2(x, t) dx \\
&\quad - a_1 \int_0^{\frac{D}{m}} e^x \tilde{p}_1(x, t) C M(x) M^{-1} \left(\frac{D}{m}\right) \left( \int_0^1 f_X(X(t) + s\tilde{X}(t), v_1(t)) ds \right) \\
&\quad \times M \left(\frac{D}{m}\right) \Delta^{-1} Z_1(t) dx & (2.35)
\end{aligned}$$

where we have used an integration by parts and boundary condition (2.23c). The second term on the right side of (2.35) is worked out as follows:

$$\begin{aligned}
&\Delta M^{-1} \left(\frac{D}{m}\right) \Delta^{-1} \Delta \left( \int_0^1 f_X(X(t) + s\tilde{X}(t), v_1(t)) ds \right) \Delta^{-1} \Delta M \left(\frac{D}{m}\right) \Delta^{-1} \\
&= \left( \Delta M^{-1} \left(\frac{D}{m}\right) \Delta^{-1} \right) \left[ \Delta \left( \int_0^1 f_X(X(t) + s\tilde{X}(t), v_1(t)) ds \right) \Delta^{-1} \right] \\
&\quad \Delta M \left(\frac{D}{m}\right) \Delta^{-1} & (2.36)
\end{aligned}$$

Using the triangular structure (2.2c) of the function  $f(\cdot)$  and the diagonal structure of the matrix  $\Delta$ , it is readily checked that, there exists a scalar  $\beta_1 > 0$  such that for all  $\theta > 1$  :

$$\left\| \Delta \left( \int_0^1 f_X(X(t) + s\tilde{X}(t), v_1(t)) ds \right) \Delta^{-1} \right\| \leq \beta_1 & (2.37)$$

Note that  $\beta_1$  depends on the Lipschitz constant of  $f(\cdot)$ , but is independent on  $\theta$ . Also, using the fact that  $\|A^k\| \leq 1$ , for  $k = 1, \dots, n-1$ , which is a consequence of (2.2a), it follows from (2.22c) that, for all  $\theta > 1$ , all  $D \in [0, \frac{m}{\theta}]$  and all  $x \in [0, \frac{D}{m}]$  :

$$\|\Delta M(x)^{-1} \Delta^{-1}\| \leq \beta_2 & (2.38)$$

## 2.5. OBSERVER ANALYSIS

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for some  $\beta_2 > 0$  independent on  $\theta$ . Using (2.36)-(2.38), one gets:

$$\begin{aligned} & \left\| 2Z_1^T(t)P\Delta M^{-1}\left(\frac{D}{m}\right)\left(\int_0^1 f_X(X(t) + s\tilde{X}(t), v_1(t))ds\right)M\left(\frac{D}{m}\right)\Delta^{-1}Z_1(t) \right\| \\ & \leq 2\beta_1\beta_2^2\|Z_1(t)\|^2 \end{aligned} \quad (2.39)$$

In turn, the third term on the right side of (2.35) is bounded from above as follows, using Young's inequality (Appendix C):

$$|2\theta Z_1^T(t)PL_0\tilde{p}_0(0,t)| \leq \frac{\theta}{\xi_1}\|Z_1(t)\|^2 + \theta\xi_1\|P\|^2\|L_0\|^2\tilde{p}_1^2(0,t) \quad (2.40)$$

For any  $\xi_1 > 0$ . Finally, the last term on the right side of (2.35) is bounded from above as follows, using Schwartz inequality (Appendix C):

$$\begin{aligned} & \left| a_1 \int_0^{\frac{D}{m}} e^x \tilde{p}_1(x,t) CM(x)M^{-1}\left(\frac{D}{m}\right)\left(\int_0^1 f_X(X(t) + s\tilde{X}(t), v_1(t))ds\right) \right. \\ & \quad \times M\left(\frac{D}{m}\right)\Delta^{-1}Z_1(t) dx \left. \right| \leq a_1 \left( \int_0^{\frac{D}{m}} e^x \tilde{p}_1(x,t) dx \right)^{1/2} \\ & \quad \times \left[ \int_0^{\frac{D}{m}} \left( e^{x/2} CM(x)M^{-1}\left(\frac{D}{m}\right)\left(\int_0^1 f_X(X(t) + s\tilde{X}(t), v_1(t))ds\right) \right. \right. \\ & \quad \left. \left. \times M\left(\frac{D}{m}\right)\Delta^{-1}Z_1(t) \right)^2 dx \right]^{1/2} \\ & \leq \frac{a_1\nu_1}{2} \int_0^{\frac{D}{m}} e^x \tilde{p}_1^2(x,t) dx + \frac{a_1}{2\nu_1} \int_0^{\frac{D}{m}} \left[ e^{x/2} CM(x)M^{-1}\left(\frac{D}{m}\right) \right. \\ & \quad \left. \times \left( \int_0^1 f_X(X(t) + s\tilde{X}(t), v_1(t))ds \right) M\left(\frac{D}{m}\right)\Delta^{-1}Z_1(t) \right]^2 dx \end{aligned} \quad (2.41)$$

whatever  $\nu_1 > 0$ , where the last inequality is obtained applying Young's inequality. In turn, the last term on the right side of (2.41) is worked out as follows, using (2.13):

$$\begin{aligned} & \frac{a_1}{2\nu_1} \int_0^{\frac{D}{m}} \left[ e^{x/2} CM(x)M^{-1}\left(\frac{D}{m}\right) \right. \\ & \quad \left. \times \left( \int_0^1 f_X(X(t) + s\tilde{X}(t), v_1(t))ds \right) M\left(\frac{D}{m}\right)\Delta^{-1}Z_1(t) \right]^2 dx \\ & = \frac{a_1}{2\nu_1} \int_0^{\frac{D}{m}} \left[ e^{x/2} C\Delta^{-1} \left[ \Delta M^{-1}\left(\frac{D}{m} - x\right)\Delta^{-1} \right] \left( \Delta \left( \int_0^1 f_X(X(t) \right. \right. \right. \right. \\ & \quad \left. \left. \left. \left. + s\tilde{X}(t), v_1(t))ds \right) \Delta^{-1} \right) \left( \Delta M^{-1}\left(\frac{D}{m}\right)\Delta^{-1} \right) Z_1(t) \right]^2 dx \end{aligned} \quad (2.42)$$

## 2.5. OBSERVER ANALYSIS

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Then, using (2.37) and (2.38) and the fact that  $\|C\| = 1$  and  $C\Delta^{-1} = C$ , it follows that, for  $\theta > 1$ ,  $D \in [0, \frac{m}{\theta}]$  :

$$\begin{aligned}
& \frac{a_1}{2\nu_1} \int_0^{\frac{D}{m}} \left[ e^{x/2} C M(x) M^{-1} \left( \frac{D}{m} \right) \right. \\
& \quad \times \left. \left( \int_0^1 f_X(X(t) + s\tilde{X}(t), v_1(t)) ds \right) M \left( \frac{D}{m} \right) \Delta^{-1} Z_1(t) \right]^2 dx \\
& \leq \frac{a_1 \beta_1 \beta_2^2}{2\nu_1} (e^{\frac{D}{m}} - 1) \|Z_1(t)\|^2 \\
& \leq \frac{a_1 \beta_1 \beta_2^2}{2\nu_1} (e^{\frac{1}{\theta}} - 1) \|Z_1(t)\|^2
\end{aligned} \tag{2.43}$$

Using (2.43), (2.41), (2.40) and (2.39), one gets from (2.35):

$$\begin{aligned}
\dot{V}_1 & \leq - \left( \mu\theta - 2\beta_1\beta_2^2 - \frac{\theta}{\xi_1} - \frac{a_1\beta_1\beta_2^2}{2\nu_1} (e^{\frac{1}{\theta}} - 1) \right) \|Z_1(t)\|^2 \\
& \quad - \frac{a_1}{2} (1 - \nu_1) \int_0^{\frac{D}{m}} e^x \tilde{p}_1^2(x, t) dx - \left( \frac{a_1}{2} - \theta\xi_1 \|P\|^2 \|L_0\|^2 \right) \tilde{p}_1(0, t)
\end{aligned} \tag{2.44}$$

Let the free parameters  $a_1$ ,  $\xi_1$ ,  $\nu_1$  be set as follows:

$$\frac{a_1}{2} = \theta\xi_1 \|P\|^2 \|L_0\|^2, \quad \frac{1}{\xi_1} = \frac{\mu}{2}, \quad \nu_1 = \frac{1}{2} \tag{2.45}$$

Then, inequality (2.44) simplifies as follows:

$$\begin{aligned}
\dot{V}_1 & \leq -\theta \left( \frac{\mu}{2} - \frac{2\beta_1\beta_2^2}{\theta} - \frac{4}{\mu} \|P\|^2 \|L_0\|^2 \beta_1\beta_2^2 (e^{\frac{1}{\theta}} - 1) \right) \|Z_1(t)\|^2 \\
& \quad - \frac{a_1}{4} \int_0^{\frac{D}{m}} e^x \tilde{p}_1^2(x, t) dx
\end{aligned} \tag{2.46}$$

Clearly,

$$\left( \frac{2\beta_1\beta_2^2}{\theta} - \frac{4}{\mu} \|P\|^2 \|L_0\|^2 \beta_1\beta_2^2 (e^{\frac{1}{\theta}} - 1) \right) \xrightarrow{\theta \rightarrow \infty} 0 \tag{2.47}$$

Then, there exists a  $\theta^* > 0$ , such that for all  $\theta > \theta^*$  and all  $D \in [0, \frac{m}{\theta}]$  :

$$\frac{\mu}{2} - \frac{2\beta_1\beta_2^2}{\theta} - \frac{4}{\mu} \|P\|^2 \|L_0\|^2 \beta_1\beta_2^2 (e^{\frac{1}{\theta}} - 1) \geq \frac{\mu}{4} \tag{2.48}$$

Then, (2.46) yields for all  $\theta > \theta^*$  and all  $D \in [0, \frac{m}{\theta}]$  :

$$\dot{V}_1 \leq -\theta \frac{\mu}{4} \|Z_1(t)\|^2 - \frac{a_1}{4} \int_0^{\frac{D}{m}} e^x \tilde{p}_1^2(x, t) dx \tag{2.49}$$

## 2.5. OBSERVER ANALYSIS

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which immediately gives, using (2.32):

$$\begin{aligned}\dot{V}_1 &\leq -\theta \frac{\mu}{4\lambda_{max}(P)} Z_1(t)^T P Z_1(t) - \frac{a_1}{4} \int_0^{\frac{D}{m}} e^x \tilde{p}_1^2(x, t) dx \\ &\leq -\min\left(\frac{\mu\theta}{4\lambda_{max}(P)}, \frac{1}{2}\right) V_1\end{aligned}\quad (2.50)$$

with  $\lambda_{max}(P)$  the largest eigenvalue value of  $P$ . Inequality (2.50), together with (2.32), immediately implies that  $\|Z_1(t)\|^2$  and  $\int_0^{\frac{D}{m}} e^x \tilde{p}_1^2(x, t) dx$  are exponentially vanishing as  $t \rightarrow \infty$ . By (2.28b) and (2.29), the same properties hold with  $\|X_1(t)\|^2$  and  $\int_0^{\frac{D}{m}} \tilde{u}_1^2(x, t) dx$ , respectively, which proves the stated **Theorem 2.1** for  $j = 1$  and ends **Step 1**.

**Step 2.** Proof of the theorem results for  $j = 2, \dots, m$ .

Just as for (2.23a), equation (2.27a) is first rewritten in terms of the state vector  $Z_j$ , defined by (2.28a), as follows:

$$\begin{aligned}\dot{Z}_j(t) &= \Delta M^{-1} \left(\frac{D}{m}\right) \left( A \tilde{X}_j(t) + f(\hat{X}_j(t), v_j(t)) - f(X_j(t), v_j(t)) \right) \\ &\quad - \Delta M^{-1} \left(\frac{D}{m}\right) \left( M \left(\frac{D}{m}\right) L (\tilde{u}_j(0, t) + \tilde{u}_{j-1} \left(\frac{D}{m}, t\right)) \right) \\ \dot{Z}_j(t) &= \Delta M^{-1} \left(\frac{D}{m}\right) \left( A \tilde{X}_j(t) + f(\hat{X}_j(t), v_j(t)) - f(X_j(t), v_j(t)) \right) \\ &\quad - \Delta L (\tilde{u}_j(0, t) + \tilde{u}_{j-1} \left(\frac{D}{m}, t\right))\end{aligned}\quad (2.51)$$

Using (2.22a)-(2.22b), (2.23a)-(2.23c) and the equality  $L = \theta \Delta^{-1} L_0$ , equation (2.51) develops as follows, for  $0 \leq x \leq \frac{D}{m}$ ;  $t \geq 0$ :

$$\begin{aligned}\dot{Z}_j(t) &= \theta(A - L_0 C) Z_j(t) + \Delta M^{-1} \left(\frac{D}{m}\right) \left( \int_0^1 f_X(X_j(t) + s \tilde{X}_j(t), v_j(t)) ds \right) \\ &\quad \times M \left(\frac{D}{m}\right) \Delta^{-1} Z_j(t) - \theta L_0 \tilde{p}_j(0, t) + \theta L_0 C M \left(\frac{D}{m}\right) \Delta^{-1} Z_{j-1}(t) \\ &\text{for } j = 2, \dots, m, \quad t > 0\end{aligned}\quad (2.52a)$$

This is completed by equations (2.27b)-(2.27c), which are rewritten as follows (using the mean value theorem and (2.28a):

$$\begin{aligned}\tilde{p}_{j,t}(x, t) &= \tilde{p}_{j,x}(x, t) - C M(x) M^{-1} \left(\frac{D}{m}\right) \left( \int_0^1 f_X(X_j(t) + s \tilde{X}_j(t), v_j(t)) ds \right) \\ &\quad \times M \left(\frac{D}{m}\right) \Delta^{-1} Z_j(t) \quad \text{for } j = 2, \dots, m, \quad t \geq 0\end{aligned}\quad (2.52b)$$

$$\tilde{p}_j \left(\frac{D}{m}, t\right) = 0 \quad \text{for } j = 2, \dots, m, \quad t \geq 0 \quad (2.52c)$$

## 2.6. SIMULATION RESULTS

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In fact, the system (2.52a)-(2.52c) is similar to the system (2.31a)-(2.31c), except for the last term on the right side of (2.52a) which stands as an external input in the system (2.52a)(2.52c). Consider the Lyapunov function candidate,

$$V_j = Z_j^T P Z_j + \frac{a_j}{2} \int_0^{\frac{D}{m}} e^x \tilde{p}_j(x, t)^2 dx \quad (2.53)$$

where  $a_j > 0$  is arbitrary and  $P$  is as in (2.33). Following mutatis-mutandis equations (2.35) through (2.49), one gets for all  $j = 2, \dots, m$  :

$$\begin{aligned} \dot{V}_j &\leq -\theta \frac{\mu}{4} \|Z_j(t)\|^2 - \frac{a_j}{4} \int_0^{\frac{D}{m}} e^x \tilde{p}_j^2(x, t) dx + 2\theta Z_j^T L_0 C M \left(\frac{D}{m}\right) \Delta^{-1} Z_{j-1}(t) \\ &\leq -\theta \frac{\mu}{4} \|Z_j(t)\|^2 - \frac{a_j}{4} \int_0^{\frac{D}{m}} e^x \tilde{p}_j^2(x, t) dx + \xi \|Z_j(t)\|^2 \\ &\quad + \frac{1}{\xi} \left\| \theta L_0 C M \left(\frac{D}{m}\right) \Delta^{-1} Z_{j-1}(t) \right\|^2 \end{aligned} \quad (2.54)$$

whatever  $\xi > 0$ , where the last inequality is obtained using Young's inequality. Letting  $\xi = \frac{\theta\mu}{8}$ , inequality (2.54) yields, for all  $j = 2, \dots, m$  :

$$\begin{aligned} \dot{V}_j &\leq -\theta \frac{\mu}{8\lambda_{\max}(P)} Z_j^T(t) P Z_j(t) - \frac{a_j}{4} \int_0^{\frac{D}{m}} e^x \tilde{p}_j^2(x, t) dx \\ &\quad + \frac{8}{\mu} \left\| \theta L_0 C M \left(\frac{D}{m}\right) \Delta^{-1} Z_{j-1}(t) \right\|^2 \\ &\leq -\min\left(\frac{\mu\theta}{8\lambda_{\max}(P)}, \frac{1}{2}\right) V_j + \frac{8}{\mu} \left\| L_0 C M \left(\frac{D}{m}\right) \Delta^{-1} \right\|^2 \|Z_{j-1}(t)\|^2 \end{aligned} \quad (2.55)$$

By **Step 1**,  $\|Z_1(t)\|^2$  is exponentially vanishing as  $t \rightarrow \infty$ . Then, using the comparison Lemma (e.g. [29], p. 102), it follows from (2.55) that  $V_2$  exponentially converges to zero as  $t \rightarrow \infty$  and, due to (2.53), so do  $\|Z_2(t)\|^2$  and  $\int_0^D e^x \tilde{p}_2(x, t)^2 dx$ . Then, proceeding with a mathematical induction, one shows that for  $j = 2, \dots, m$ ,  $V_j$  exponentially converges to zero as  $t \rightarrow \infty$  which, due to (2.53), implies that so do  $\|Z_j(t)\|^2$  and  $\int_0^D e^x \tilde{p}_j(x, t)^2 dx$ . By (2.28b) and (2.29), the same properties hold with  $\|X_j(t)\|^2$  and  $\int_0^D \tilde{u}_j(x, t)^2 dx$  respectively, for  $j = 2, \dots, m$ . This proves **Step 2** and completes the proof of the **Theorem 2.1**.

## 2.6 Simulation Results

To illustrate the observer design of the previous Sections, we consider the mass-spring system which is a classic example of nonlinear mechanical system. This is composed of body of mass  $M_0$  that is connected to a fixed

## 2.6. SIMULATION RESULTS

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point via a spring. One possible model of this system is the following Duffing equation:

$$M_0\ddot{z}(t) + c_v\dot{z}(t) + \kappa_r z(t) + \kappa_r a_r^2 z(t)^3 = F \quad (2.56)$$

where  $z$  denotes the body displacement,  $-c_v\dot{z}$  represents the viscous force,  $\kappa_r z(t) + \kappa_r a_r^2 z(t)^3$  the restoring force of the spring and  $F$  an external force applied to the body of mass  $M_0$ . Introducing the state vector  $X = [z(t) \ \dot{z}(t)]^T$  and the delayed output  $y(t) = z(t - D)$  and denoting the input signal  $v(t) = \frac{F(t)}{M_0}$ , equation (2.56) can be rewritten in the state space form (2.1a)(2.1b) with:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (2.57)$$

$$f(X, v) = \begin{bmatrix} 0 \\ -\frac{\kappa_r}{M_0}x_1 - \frac{\kappa_r}{M_0}x_1^3 - \frac{c_v}{M_0}x_2 + v \end{bmatrix} \quad (2.58)$$

It is shown (see e.g. [29], pp. 172–174) that if  $v(t)$  is bounded (which currently is the case) then so is the state vector  $X(t)$ . Let  $\Omega_X$  denotes any compact where  $X(t)$  takes its values. Then, the function  $f(X, v)$  is Lipschitz within the set  $\Omega_X$  (outside this set,  $f(X, v)$  can be extended by any  $\mathcal{C}^\infty$  globally Lipschitz function). Hence, the observer design of the previous sections can be applied to the system defined by (2.56). For the purpose of simulation, the system parameters are given the values  $a_r = c_v = \kappa_r = M_0 = 1$ . The delay  $D$  will be given different values and the external force  $F(t) = \cos(t)$ . The cascade observer defined by equations (2.18a)-(2.18d) and (2.19a)-(2.19d) is first considered with the following values of the delay and the design parameters:  $D = 0.5$ ,  $m = 1$ ,  $L_0 = [2 \ 1]^T$ ,  $\theta = 1.1$ . The estimation errors  $\tilde{X}(t)$  and  $\tilde{u}(x, t)$  are shown by the following figures.

## 2.6. SIMULATION RESULTS

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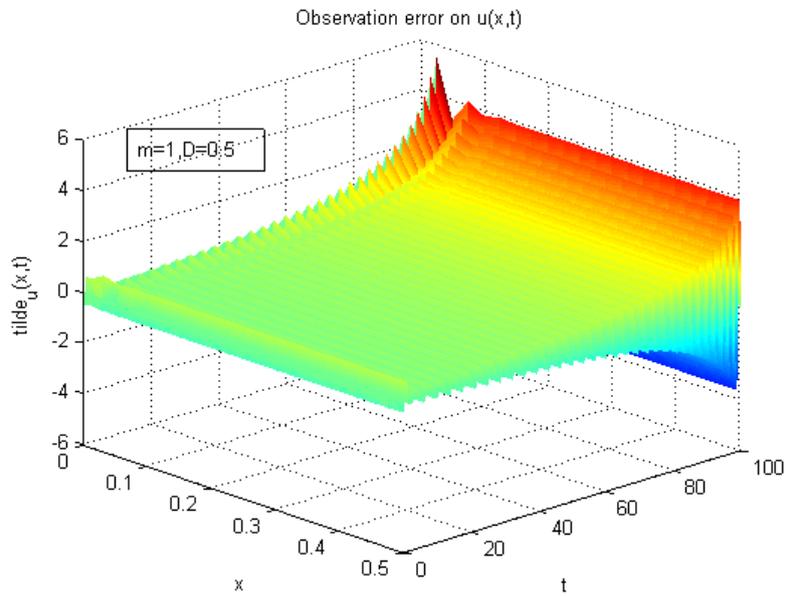


Figure 2.2 – Observation error on  $u(x,t)$  with  $m = 1$  and a delay  $D = 0.5s$

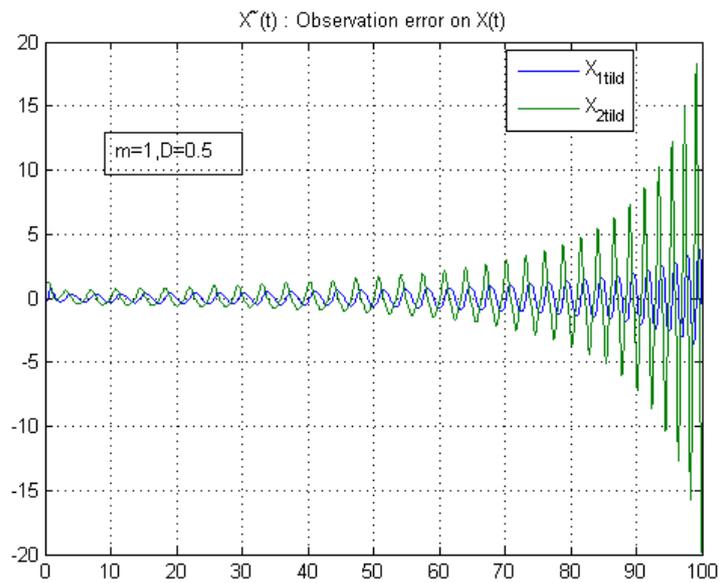


Figure 2.3 – Observation error on  $X(t)$  with  $m = 1$  and a delay  $D = 0.5s$

## 2.6. SIMULATION RESULTS

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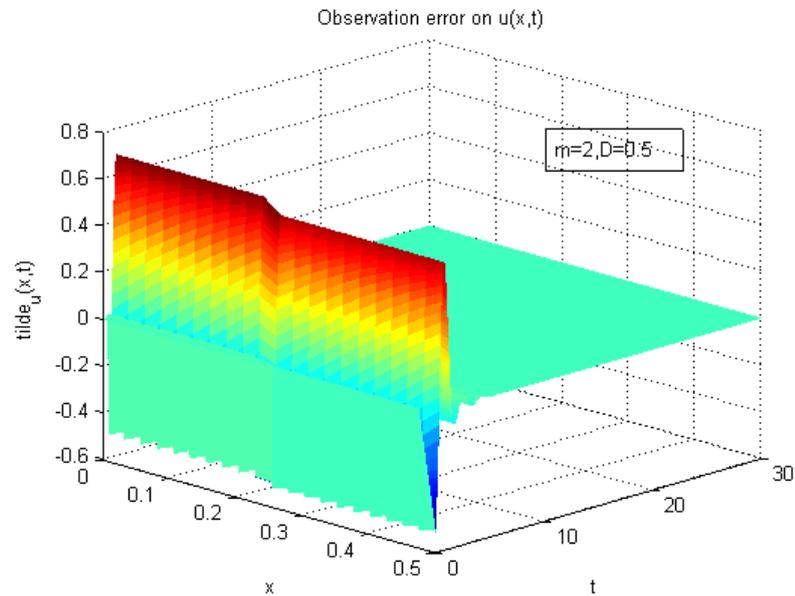


Figure 2.4 – Observation error on  $u(x,t)$  with  $m = 2$  and a delay  $D = 0.5s$

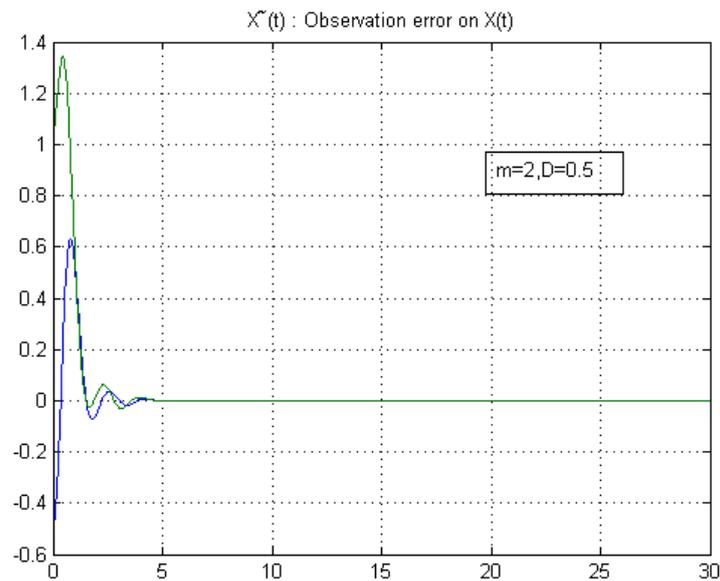


Figure 2.5 – Observation error on  $X(t)$  with  $m = 2$  and a delay  $D = 0.5s$

Clearly, the errors diverge, proving that a simple observer (involving a single predictor) is not enough to compensate for the delay  $D = 0.5$  (Figs. 2.2 and 2.3). Then, a cascade observer involving  $m = 2$  predictors is considered.

## 2.6. SIMULATION RESULTS

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The resulting estimation errors are plotted in Figs. 2.4 and 2.5 which show a rapid convergence of the errors to zero, confirming the result of the previous **Theorem 2.1**.

In the following, we test the cascade observer (2.18a)-(2.18d) and (2.19a)-(2.19d) to the mass-spring system (2.56) with a larger delay  $D = 1s$ . We start the simulations with two sub-observers  $m = 2$  :

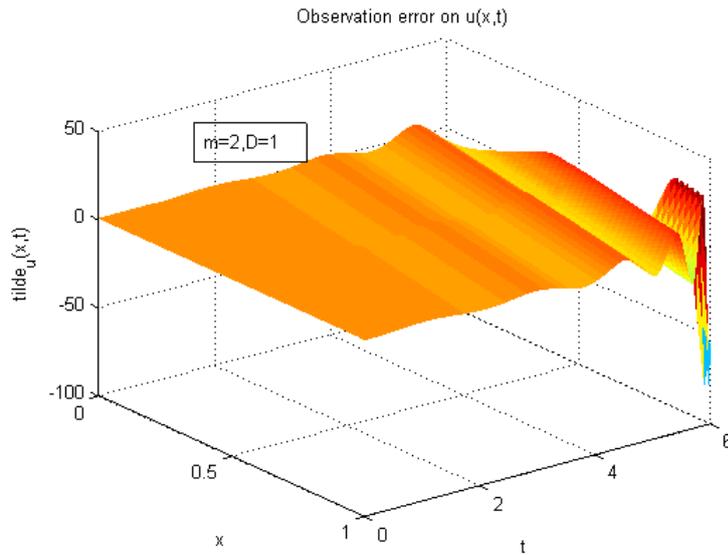


Figure 2.6 – Observation error on  $u(x, t)$  with  $m = 2$  and a delay  $D = 1s$

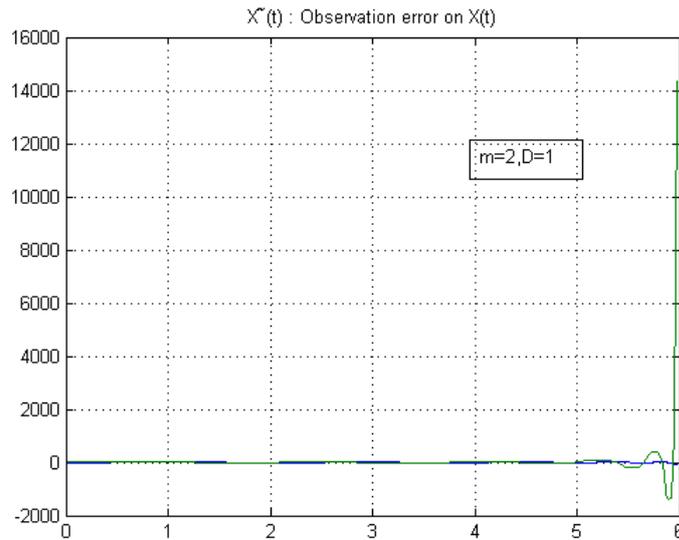


Figure 2.7 – Observation error on  $X(t)$  with  $m = 2$  and a delay  $D = 1s$

## 2.6. SIMULATION RESULTS

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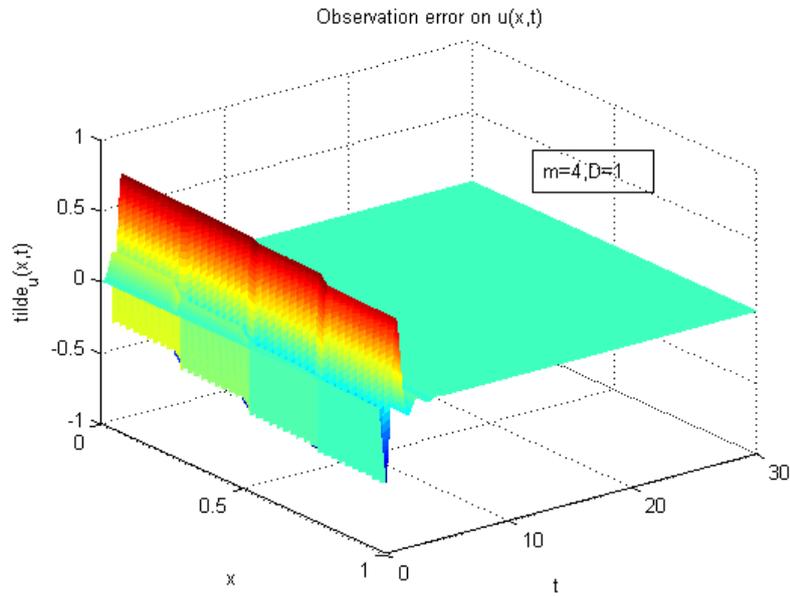


Figure 2.8 – Observation error on  $u(x,t)$  with  $m = 4$  and a delay  $D = 1s$

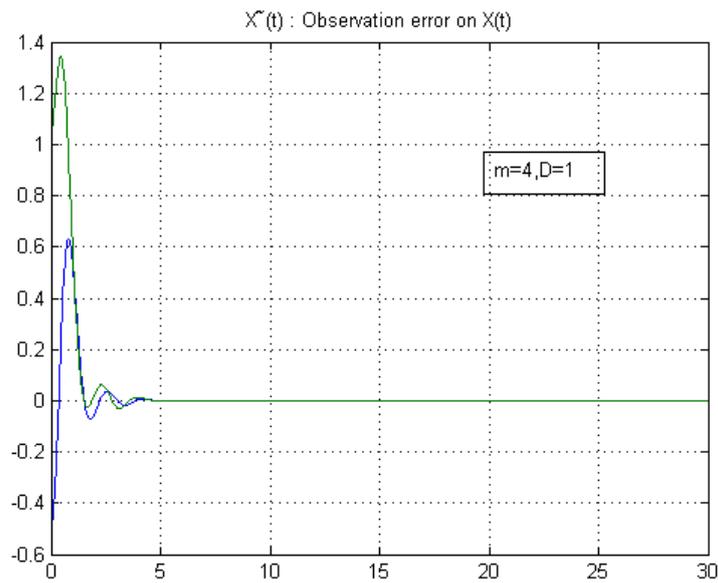


Figure 2.9 – Observation error on  $X(t)$  with  $m = 4$  and a delay  $D = 1s$

It is obvious that the cascade chain observer composed of two chain sub-observers  $m = 2$  is unable to achieve satisfactory results since both the finite dimensional estimation error  $\tilde{X}(t)$  and the infinite dimensional estimation

## 2.7. CONCLUSION

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$\tilde{u}(x, t)$  are diverging, this is shown in Figs. 2.6 and 2.7. On the other hand when a chain observer containing four sub-observers  $m = 4$  was invoked, the convergence of both  $\tilde{X}(t)$  and  $\tilde{u}(x, t)$  to the origin was obtained. These results could be summarized in the following statement: any undersized chain observer (with  $m = 1, 2$  or  $m = 3$ ) is unable to achieve satisfactory performances for a delay  $D = 1s$ .

## 2.7 Conclusion

In this part of our study, we presented a chain observer (2.18a)-(2.18d) and (2.19a)-(2.19d) for the class of systems depicted by Figure 2.1, analytically modeled by (2.1a)-(2.1b), that can be presented as an ODE-PDE cascade (2.4a)-(2.4d). The goal was to get online estimate of both the finite-dimensional state  $X(t)$  and the infinite-dimensional state  $u(x, t)$  over the  $x$ -domain  $(0, D)$ , where  $D > 0$  is allowed to be arbitrarily large. We have dealt with this observation problem by invoking the high-gain observer method, the backstepping-like design approach and the chain observer principle. The matrix function  $M(x)$ , which plays an instrumental role in the achievement of the observer exponential convergence, materializes the difference of the PDE-based observers and the standard ODE-based high-gain observers [25]. The current study can be pursued toward the extension of the proposed observer design approach to the case of sampled output measurement [5].

# Chapter 3

## Chain Observer Design For a Class of Nonlinear Parabolic Systems

### 3.1 Introduction

The problem of observer design for nonlinear PDEs with arbitrary delays measurements has yet to be solved. In this chapter, the problem is addressed for a class of parabolic PDEs (Fig. 3.1) under point measurements as in [18]. In the latter paper the results were confined to small delays. To compensate for the effect of the arbitrary-size delay, the concept of chain-observer is extended to fit this class of systems. Accordingly, the initial delay PDE system representation is re-expressed in the form of fictive delayed subsystems. The observer is composed of elementary observers connected in series. The interconnection is such that the first elementary observer is directly driven by the physical system output. Then, the elementary observer is driven by a virtual output generated by the previous observer. Each elementary observer can be viewed as a predictor which compensates for the effects of the fractional time-delay.

As in [18], using an appropriate Lyapunov-Krasovskii functional, sufficient conditions are established in terms of LMIs for the chain observer to be exponentially convergent. The sufficient conditions involve the minimal number of elementary observers: the larger the delay the larger the number of observers. Extension to sampled data delayed measurements is presented.

The chapter is organized as follows: first, the system under study is pre-

## 3.2. SYSTEM DESCRIPTION

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sented in Section 3.2, then the proposed observer for this class of systems is presented in Section 3.3, in Section 3.4 we study the well-posedness of the error system, the observer analysis and an extension to the sampled-data case are dealt with in Section 3.5; simulation results are presented in Section 3.6, a conclusion ends the chapter.

## 3.2 System description

In this chapter the system under study is a delayed semi-linear diffusion equation, depicted by the following:

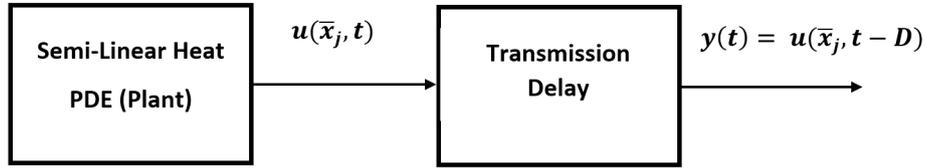


Figure 3.1 – Delayed semi-linear diffusion equation to be observed

It is represented mathematically by the following partial differential equation:

$$u_t(x, t) = u_{xx}(x, t) + f(u(x, t), x, t) \quad (3.1)$$

with Dirichlet conditions  $u(0, 0) = u(l, 0) = 0$ . The system output is,  $y_j(t) = u(\bar{x}_j, t - D)$  where  $\bar{x}_j = \frac{x_{j+1} + x_j}{2}$  ( $j = 0, \dots, N - 1$ ) and the points  $x_j$  divide the interval  $[0, l]$  such that  $0 = x_0 < \dots < x_N = l$ . It is supposed that  $x_{j+1} - x_j \leq \Delta$ . The constant  $D$  represents an arbitrary delay and  $N$  is the number of distributed sensors. It is also supposed that the function  $f$  is known, of class  $\mathcal{C}^1$ , and satisfying  $m_f \leq f_u \leq M_f$ , for some scalar constants  $m_f$  and  $M_f$ .

## 3.3 Observer design

We will present an observer, which ensures exponential convergence for an arbitrarily delay  $D$ . This chain is constituted by  $m$  sub-observers in cascade. Each sub-observer will estimate the state  $u(x, t + \frac{k}{m}D - D)$  by using the estimation provided by the previous one in the chain. The last sub-observer in the chain provides the estimation of the  $u(x, t)$ . As we will see below, by using a suitable Lyapunov functional, we will derive sufficient conditions

### 3.3. OBSERVER DESIGN

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involving both delay  $D$ , and the number of sub-observers in the chain  $m$ . As in [19] we introduce the following notations for the delayed states:

$$u^0(x, t) = u(x, t - D), \quad (3.2a)$$

$$u^k(x, t) = u(x, t + \frac{k}{m}D - D), \quad k = 1 \dots, \quad (3.2b)$$

Using these notations we easily check that:

$$u^{k+1}(x, t) = u^k(x, t - \frac{D}{m}) \quad (3.3a)$$

$$u^m(x, t) = u(x, t) \quad (3.3b)$$

where  $m$  is the number of sub-observers in the considered chain.

We propose the following observer structure:

for  $k = 1$  :

$$\begin{aligned} \hat{u}_t^1(x, t) &= \hat{u}_{xx}^1(x, t) + f(\hat{u}^1(x, t), x, t) - L(\hat{u}^1(\bar{x}_j, t - \frac{D}{m}) - y_j(t)), \\ &\forall x \in [x_j, x_{j+1}), \end{aligned} \quad (3.4a)$$

for  $k = 2, \dots, m$  :

$$\begin{aligned} \hat{u}_t^k(x, t) &= \hat{u}_{xx}^k(x, t) + f(\hat{u}^k(x, t), x, t) - L(\hat{u}^k(\bar{x}_j, t - \frac{D}{m}) - \hat{u}^{k-1}(\bar{x}_j, t)), \\ &\forall x \in [x_j, x_{j+1}), \end{aligned} \quad (3.4b)$$

It is readily checked that the observation error systems  $e^k(x, t) = \hat{u}^k(x, t) - u^k(x, t)$  undergoes the following equations:

for  $k = 1$  :

$$\begin{aligned} e_t^1(x, t) &= e_{xx}^1(x, t) + f(\hat{u}^1(x, t), x, t) - f(u^1(x, t), x, t) - L(e^1(\bar{x}_j, t - \frac{D}{m})), \\ &\forall x \in [x_j, x_{j+1}), \end{aligned} \quad (3.5a)$$

for  $k = 2, \dots, m$  :

$$\begin{aligned} e_t^k(x, t) &= e_{xx}^k(x, t) + f(\hat{u}^k(x, t), x, t) - f(u^k(x, t), x, t) \\ &\quad - L(\hat{u}^k(\bar{x}_j, t - \frac{D}{m}) - \hat{u}^{k-1}(\bar{x}_j, t)), \\ &\forall x \in [x_j, x_{j+1}), \end{aligned} \quad (3.5b)$$

### 3.4. WELL-POSEDNESS OF THE ERROR SYSTEM

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Using the fact that

$$u^k(x, t - \frac{D}{m}) = u^{k-1}(x, t)$$

then, for  $k = 1$  :

$$\begin{aligned} e_t^1(x, t) &= e_{xx}^1(x, t) + f(\hat{u}^1(x, t), x, t) - f(u^1(x, t), x, t) - Le^1(\bar{x}_j, t - \frac{D}{m}), \\ &\quad \forall x \in [x_j, x_{j+1}), \\ e^1(l, t) &= e^1(0, t) = 0, \end{aligned} \tag{3.6a}$$

for  $k = 2, \dots, m$  :

$$\begin{aligned} e_t^k(x, t) &= e_{xx}^k(x, t) + f(\hat{u}^k(x, t), x, t) - f(u^k(x, t), x, t) \\ &\quad - Le^k(\bar{x}_j, t - \frac{D}{m}) + Le^{k-1}(\bar{x}_j, t), \\ &\quad \forall x \in [x_j, x_{j+1}), \\ e^k(l, t) &= e^k(0, t) = 0, \end{aligned} \tag{3.6b}$$

which leads to:

for  $k = 1$  :

$$\begin{aligned} e_t^1(x, t) &= e_{xx}^1(x, t) + \Psi(x, t, e^1)(\hat{u}^1(x, t) - u^1(x, t)) - Le^1(\bar{x}_j, t - \frac{D}{m}), \\ &\quad x \in [x_j, x_{j+1}), \\ e^1(l, t) &= e^1(0, t) = 0, \end{aligned} \tag{3.7a}$$

for  $k = 2, \dots, m$  :

$$\begin{aligned} e_t^k(x, t) &= e_{xx}^k(x, t) + \Psi(x, t, e^k)(\hat{u}^k(x, t) - u^k(x, t)) \\ &\quad - Le^k(\bar{x}_j, t - \frac{D}{m}) + Le^{k-1}(\bar{x}_j, t), \\ &\quad \forall x \in [x_j, x_{j+1}), \\ e^k(l, t) &= e^k(0, t) = 0, \end{aligned} \tag{3.7b}$$

where

$$\Psi(x, t, e^k) = \int_0^1 f_u(\hat{u}^k + \theta e^k, x, t) d\theta. \tag{3.8}$$

## 3.4 Well-posedness of the error system

### Definition

A boundary value problem is well-posed in the sense of Hadamard if three conditions are met [9]:

### 3.4. WELL-POSEDNESS OF THE ERROR SYSTEM

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- existence: the solution must exist within a certain class of functions from which the solution is chosen,
- uniqueness: the solution must be unique within a certain class of functions from which the solution is chosen,
- continuous dependence of the solution on the data of the problem: if a small change occurs in the boundary conditions, initial conditions, coefficients of the equation, etc, then a small change will affect the solution as well.

A problem that violates any of the three properties of well-posedness is called an ill-posed problem.

#### Example of well-posed problem

Consider the second order hyperbolic (wave) partial differential equation:

$$u_{tt} = u_{xx}, \quad x \in \mathbb{R}, t > 0$$

with the conditions:

$$\begin{aligned} u(x, 0) &= 0 \\ u_t(x, 0) &= 0 \end{aligned}$$

The solution of this problem exists and it is unique:

$$u_1(x, t) = 0$$

We still have to check whether a small change in the initial data leads to a small change in the solution. For this, we replace one of the initial conditions to become:

$$\begin{aligned} u(x, 0) &= 0 \\ u_t(x, 0) &= \epsilon \sin\left(\frac{x}{\epsilon}\right) \end{aligned}$$

The solution becomes:

$$u_2(x, t) = \epsilon^2 \sin\left(\frac{x}{\epsilon}\right) \sin\left(\frac{t}{\epsilon}\right)$$

hence,

$$\|u_1(t) - u_2(t)\| \leq \epsilon^2 \left| \sin\left(\frac{t}{\epsilon}\right) \right| \leq \epsilon^2$$

Thus the small change in the initial data leads a small change in the solution at any positive time. This proves that this wave problem is well-posed.

### 3.4. WELL-POSEDNESS OF THE ERROR SYSTEM

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#### Example of ill-posed problem

Consider now the Laplace equation:

$$u_{tt} + u_{xx} = 0 \quad x \in \mathbb{R}, t > 0$$

with the conditions:

$$\begin{aligned} u(x, 0) &= 0 \\ u_t(x, 0) &= 0 \end{aligned}$$

The solution of this problem exists and it is unique:

$$u_1(x, t) = 0$$

Now consider the same problem but with a different initial condition (a small change in the second initial condition)

$$\begin{aligned} u(x, 0) &= 0 \\ u_t(x, 0) &= \epsilon \sin\left(\frac{x}{\epsilon}\right) \end{aligned}$$

this time the solution becomes:

$$u(x, t) = \epsilon^2 \sin\left(\frac{x}{\epsilon}\right) \sinh\left(\frac{t}{\epsilon}\right)$$

The small change in the initial condition is described by:

$$\|u_{1t}(0) - u_{2t}(0)\| = \epsilon$$

and the change in the solution is described by:

$$\|u_1(t) - u_2(t)\| \leq \epsilon^2 \left| \sinh\left(\frac{t}{\epsilon}\right) \right|$$

the term  $\epsilon^2 \left| \sinh\left(\frac{t}{\epsilon}\right) \right|$  is exponentially large since  $\epsilon$  is small. Thus a very small change in the initial conditions results in a large change in the solution for positive time. This proves that this problem is ill-posed due to the fact that the third condition of well-posedness is not satisfied.

#### Well-posedness problem of the error system under study

In this part we will establish the well-posedness of the error system (3.7a) for  $k = 1$  with Dirichlet boundary conditions:  $e^1(l, t) = e^1(0, t) = 0$ . First

### 3.4. WELL-POSEDNESS OF THE ERROR SYSTEM

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We write the system (3.7a) in the following form:

$$\begin{aligned}
 e_t^1(x, t) &= e_{xx}^1(x, t) + \Psi(x, t, e^1)e^1(x, t) - Le^1(x, t - \frac{D}{m}) \\
 &\quad + L \int_{\bar{x}_j}^x e_\xi^1(\xi, t - \frac{D}{m}) d\xi, \\
 &\quad \forall x \in [x_j, x_{j+1}), \\
 e^1(l, t) &= e^1(0, t) = 0,
 \end{aligned} \tag{3.9}$$

Introduce the Hilbert space  $H = L_2(0, l)$  with the norm  $\|\cdot\|_{L_2}$  and with the scalar product  $\langle \cdot, \cdot \rangle$ . The boundary-value problem (3.9) can be rewritten as a differential equation [18]:

$$\dot{w}(t) = Aw(t) + F(t, w(t)), \quad t \geq 0 \tag{3.10}$$

in  $H$  where the operator  $A$  is defined by:

$$A = \frac{\partial^2}{\partial x^2} \tag{3.11}$$

and has the dense domain:

$$\mathcal{D}(A) = \{w \in \mathcal{H}^2(0, l) : w(0) = w(l) = 0\}, \tag{3.12}$$

and the nonlinear term  $F : R \times \mathcal{H}^2(0, l) \rightarrow L_2(0, l)$  is defined as:

$$F(t, w(t)) = \Psi(x, t, e^1)e^1(x, t) - Le^1(x, t - \frac{D}{m}) + L \int_{\bar{x}_j}^x e_\xi^1(\xi, t - \frac{D}{m}) d\xi \tag{3.13}$$

So what we have done for our error system (3.7a) is to formulate it as an abstract differential equation (3.10) on the infinite-dimensional state space  $H = L_2(0, l)$ , where  $A$  is the unbounded operator on  $H$  defined by (3.11)-(3.12) and  $F$  is the nonlinear function on  $H$  defined by (3.13).

It is well-known that  $A$  generates a strongly continuous exponentially stable semigroup  $T$ , which satisfies the inequality  $\|T(t)\| \leq \kappa e^{-\delta t}$  with some constant  $\kappa \geq 1$  and decay rate  $\delta > 0$  [14]. The domain  $H_1 = \mathcal{D}(A) = A^{-1}H$  forms another Hilbert space with the graph inner product  $\langle x, y \rangle_1 = \langle Ax, Ay \rangle$ ,  $x, y \in H_1$ . The domain  $\mathcal{D}(A)$  is dense in  $H$  and the inequality  $\|Aw\|_{L_2} \geq \mu \|w\|_{L_2}$  holds for all  $w \in \mathcal{D}(A)$  and some constant  $\mu > 0$ . Operator  $-A$  is positive, so that its square root  $(-A)^{\frac{1}{2}}$  is well defined with:

$$H_{\frac{1}{2}} = \mathcal{D}((-A)^{\frac{1}{2}}) = \{w \in \mathcal{H}^1(0, l) : w(0) = w(l) = 0\}$$

### 3.5. OBSERVER ANALYSIS

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Moreover,  $H_{\frac{1}{2}}$  is a Hilbert space with the scalar product:

$$\langle u, v \rangle_{\frac{1}{2}} = \langle (-A)^{\frac{1}{2}}u, (-A)^{\frac{1}{2}}v \rangle$$

Denote by  $H(-A)_{\frac{1}{2}}$  the dual of  $H_{\frac{1}{2}}$  with respect to the pivot space  $H$ . Then  $A$  has an extension to a bounded operator  $A : H_{\frac{1}{2}} \rightarrow H_{-\frac{1}{2}}$ . We have  $H_1 \subset H_{\frac{1}{2}} \subset H$  with continuous embedding and the following inequality:

$$\left\| (-A)^{\frac{1}{2}}w \right\|_{L_2} \geq \mu \|w\|_{L_2} \text{ for all } w \in H_{\frac{1}{2}}$$

holds. All relevant material on fractional operator degrees can be found in [47].

A function  $w : [t_0, T) \rightarrow H_{\frac{1}{2}}$  is called a strong solution of (3.10) if

$$w(t) - w(t_0) = \int_{t_0}^t [Aw(s) + F(s, w(s))] ds \quad (3.14)$$

holds for all  $t \in [t_0, T)$ . Here, the integral is computed in  $H_{-\frac{1}{2}}$ .

Since the function  $\Psi$  is of class  $C^1$ , the following Lipschitz condition:

$$\left\| F(t_1, w_1) - F(t_2, w_2) \right\|_{L_2} \leq C[|t_1 - t_2| + \left\| (-A)^{\frac{1}{2}}(w_1 - w_2) \right\|_{L_2}]$$

with some constant  $C > 0$  holds locally in  $(t_i, w_i) \in R \times H_{\frac{1}{2}}$ ,  $i = 1, 2$ . Thus, Theorem 3.3.3 of [21] is applicable to (3.10) and by applying this theorem, a unique strong solution  $w(t) \in H_{\frac{1}{2}}$  of (3.10), initialized with  $z^{(0)} \in H_{\frac{1}{2}}$ , exists locally. Since  $\phi$  is bounded, there exists  $C_1 > 0$  such that

$$\left\| F(t, w) \right\|_{L_2} \leq C_1 \left\| (-A)^{\frac{1}{2}}w \right\|_{L_2}, \quad \forall w \in H_{\frac{1}{2}}, \quad \forall t \geq 0.$$

Hence, the strong solution initialized with  $z^{(0)} \in H_{\frac{1}{2}}$  exists for all  $t \geq t_0$  [21].

### 3.5 Observer analysis

**Theorem 3.1:** Given  $D$  and  $m$ , consider the system (3.1) and the observer (3.4a)-(3.4b). Given positive constants  $\Delta, \delta, L > M_f - \frac{\pi^2}{l^2}, R$  and  $\delta_1$  such that  $2\delta > \delta_1$ , if there exist positive scalars  $p_1, p_2, p_3, r$  and  $g$  such that :

$$\delta p_3 < p_2 ; \quad \frac{\Delta}{\pi} LR^{-1}(p_3 + p_2) < \delta_1 p_3 \quad (3.15)$$

and

$$\Phi_{m_f} < 0 ; \quad \Phi_{M_f} < 0 \quad (3.16)$$

### 3.5. OBSERVER ANALYSIS

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where

$$\Phi_\phi = \begin{pmatrix} \Phi_{11} - \lambda & \Phi_{12} & \Phi_{13} \\ \Phi_{12} & \Phi_{22} & \Phi_{23} \\ \Phi_{13} & \Phi_{23} & \Phi_{33} \end{pmatrix} \quad (3.17)$$

with

$$\begin{aligned} \Phi_{11} &= 2\delta p_1 + g - r e^{-2\delta \frac{D}{m}} + 2p_2 \left( \phi + \frac{\Delta}{2\pi} LR \right) \\ \Phi_{12} &= -p_2 + p_1 + p_3 \phi \\ \Phi_{13} &= r e^{-2\delta \frac{D}{m}} - p_2 L \\ \Phi_{22} &= \frac{\Delta LR p_3}{\pi} - 2p_3 + r \left( \frac{D}{m} \right)^2 \\ \Phi_{23} &= -L p_3 \\ \Phi_{33} &= -(r + g) e^{-2\delta \frac{D}{m}} \\ \lambda &= \frac{2\pi^2}{l^2} (p_2 - \delta p_3), \end{aligned} \quad (3.18)$$

then all the observation errors  $\int_0^1 \left( e^k(x, t) \right)^2 dx$  and  $\int_0^1 \left( e_x^k(x, t) \right)^2 dx$  ( $k = 1, \dots, m$ ) globally exponentially decay to zero as  $t \rightarrow +\infty$ . The above LMIs are always feasible for large enough  $m$ .

**Proof :**

The proof of the above theorem, will be performed by induction.

For  $k = 1$  consider the first observation error:

$$\begin{aligned} e_t^1(x, t) &= e_{xx}^1(x, t) + \Psi(x, t, e^1)(\hat{u}^1(x, t) - u^1(x, t)) - L e^1\left(\bar{x}_j, t - \frac{D}{m}\right), \\ &\quad \forall x \in [x_j, x_{j+1}), \\ e^1(l, t) &= e^1(0, t) = 0, \end{aligned} \quad (3.19)$$

and the following Lyapunov-Krasovskii functional as in [18]

$$\begin{aligned} V^1(t) &= p_1 \int_0^l \left( e^1(x, t) \right)^2 dx + p_3 \int_0^l \left( e_x^1(x, t) \right)^2 dx \\ &\quad + g \int_0^l \left[ \int_{t-\frac{D}{m}}^t e^{2\delta(s-t)} \left( e^1(x, s) \right)^2 ds \right] dx \\ &\quad + \frac{D}{m} r \int_0^l \left[ \int_{-\frac{D}{m}}^0 \int_{t+\theta}^t e^{2\delta(s-t)} \left( e_s^1(x, s) \right)^2 ds d\theta \right] dx \end{aligned} \quad (3.20)$$

### 3.5. OBSERVER ANALYSIS

---

As in [18], differentiating the above functional we find:

$$\begin{aligned}
\dot{V}^1(t) + 2\delta V^1(t) &= 2p_1 \int_0^l e^1(x, t) e_t^1(x, t) dx + 2p_3 \int_0^l e_x^1(x, t) e_{xt}^1(x, t) dx \\
&\quad - \frac{D}{m} r \int_0^l \int_{t-\frac{D}{m}}^t e^{2\delta(s-t)} e_s^1(x, s)^2 ds dx \\
&\quad + \int_0^l \left[ \left( \frac{D}{m} \right)^2 r (e_t^1(x, t))^2 + g(e^1(x, t))^2 \right. \\
&\quad \left. - g e^{-2\delta \frac{D}{m}} (e^1(x, t - \frac{D}{m}))^2 \right] dx + 2\delta p_1 \int_0^l (e^1(x, t))^2 dx \\
&\quad + 2\delta p_3 \int_0^l (e_x^1(x, t))^2 dx \tag{3.21}
\end{aligned}$$

using the descriptor method [17] for each  $j$  and by summing these expressions for all  $j = 0, \dots, N-1$ , this leads to:

$$\begin{aligned}
0 &= 2 \int_0^l \left[ p_2 e^1(x, t) + p_3 e_t^1(x, t) \right] \left[ -e_t^1(x, t) + e_{xx}^1(x, t) \right. \\
&\quad \left. + \Psi(x, t, e^1) e^1(x, t) - L e^1(x, t - \frac{D}{m}) \right] dx \\
&\quad + 2 \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} \left[ p_2 e^1(x, t) + p_3 e_t^1(x, t) \right] L \int_{\bar{x}_j}^x e_{\xi}^1(\xi, t - \frac{D}{m}) d\xi dx \tag{3.22}
\end{aligned}$$

Here  $p_2$  and  $p_3$  are free parameters. Using integration by parts:

$$2p_3 \int_0^l e_x^1(x, t) e_{xt}^1(x, t) dx = 2p_3 [e_x^1(x, t) e_t^1(x, t)]|_0^l - 2p_3 \int_0^l e_{xx}^1(x, t) e_t^1(x, t) dx \tag{3.23}$$

### 3.5. OBSERVER ANALYSIS

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Combining (3.21) with (3.22) we get:

$$\begin{aligned}
\dot{V}^1(t) + 2\delta V^1(t) &= \int_0^l (2\delta p_1 + g + 2p_2\phi)(e^1(x, t))^2 dx \\
&+ \int_0^l 2(p_1 - p_2 + p_3\phi)e^1(x, t)e_t^1(x, t) dx \\
&- \int_0^l 2p_2 L e^1(x, t)e^1(x, t - \frac{D}{m}) dx \\
&+ \int_0^l ((\frac{D}{m})^2 r - 2p_3)(e_t^1(x, t))^2 dx \\
&- \int_0^l 2p_3 L e_t^1(x, t)e^1(x, t - \frac{D}{m}) dx \\
&- \int_0^l g e^{-2\delta \frac{D}{m}} (e^1(x, t - \frac{D}{m}))^2 dx \\
&- \frac{D}{m} r \int_0^l \int_{t-\frac{D}{m}}^t e^{2\delta(s-t)} e_s^1(x, s)^2 ds dx \\
&+ 2\delta p_3 \int_0^l (e_x^1(x, t))^2 dx + 2p_2 \int_0^l e^1(x, t)e_{xx}^1(x, t) dx \\
&+ 2 \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} [p_2 e^1(x, t) + p_3 e_t^1(x, t)] L \int_{\bar{x}_j}^x e_\xi^1(\xi, t - \frac{D}{m}) d\xi dx
\end{aligned} \tag{3.24}$$

Using Jensen inequality:

$$\begin{aligned}
-\frac{D}{m} r \int_0^l \int_{t-\frac{D}{m}}^t e^{2\delta(s-t)} e_s^1(x, s) ds dx &\leq -r \int_0^l e^{-2\delta \frac{D}{m}} \left( \int_{t-\frac{D}{m}}^t e^{2\delta(s-t)} e_s^1(x, s) ds \right)^2 dx \\
&\leq -r e^{-2\delta \frac{D}{m}} \int_0^l \left( e^1(x, t) - e^1(x, t - \frac{D}{m}) \right)^2 dx
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
-r e^{-2\delta \frac{D}{m}} \int_0^l \left( e^1(x, t) - e^1(x, t - \frac{D}{m}) \right)^2 dx &= -r e^{-2\delta \frac{D}{m}} \int_0^l (e^1(x, t))^2 dx \\
&+ 2r e^{-2\delta \frac{D}{m}} \int_0^l e^1(x, t)e^1(x, t - \frac{D}{m}) dx \\
&- r e^{-2\delta \frac{D}{m}} \int_0^l \left( e^1(x, t - \frac{D}{m}) \right)^2 dx
\end{aligned} \tag{3.26}$$

### 3.5. OBSERVER ANALYSIS

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$$2p_2 \int_0^l e^1(x, t) e_{xx}^1(x, t) dx = 2p_2 (e^1(x, t) e_x^1(x, t))|_0^l - 2p_2 \int_0^l (e_x^1(x, t))^2 dx \quad (3.27)$$

$$2\delta p_3 \int_0^l (e_x^1(x, t))^2 dx + 2p_2 \int_0^l e^1(x, t) e_{xx}^1(x, t) dx = -2(p_2 - \delta p_3) \int_0^l (e_x^1(x, t))^2 dx \quad (3.28)$$

Using Wirtinger inequality [45]:

$$2\delta p_3 \int_0^l (e_x^1(x, t))^2 dx + 2p_2 \int_0^l e^1(x, t) e_{xx}^1(x, t) dx \leq -\frac{2\pi^2}{l^2} (p_2 - \delta p_3) \int_0^l (e_x^1(x, t))^2 dx \quad (3.29)$$

$$\begin{aligned} & 2 \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} \left[ p_2 e^1(x, t) + p_3 e_t^1(x, t) \right] L \int_{\bar{x}_j}^x e_\xi^1(\xi, t - \frac{D}{m}) d\xi dx \\ &= 2 \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} p_2 e^1(x, t) L \int_{\bar{x}_j}^x e_\xi^1(\xi, t - \frac{D}{m}) d\xi dx \\ & \quad + 2 \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} p_3 e_t^1(x, t) L \int_{\bar{x}_j}^x e_\xi^1(\xi, t - \frac{D}{m}) d\xi dx \end{aligned} \quad (3.30)$$

Using Young inequality:

$$\begin{aligned} & 2 \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} p_2 e^1(x, t) L \int_{\bar{x}_j}^x e_\xi^1(\xi, t - \frac{D}{m}) d\xi dx \leq L p_2 \bar{R} \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} (e^1(x, t))^2 dx \\ & + L p_2 \bar{R}^{-1} \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} \left( e_\xi^1(\xi, t - \frac{D}{m}) \right)^2 d\xi \end{aligned} \quad (3.31)$$

Using the fact that:

$$\sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} (e^1(x, t))^2 dx = \int_0^l (e^1(x, t))^2 dx \quad (3.32)$$

we get:

$$\begin{aligned} & 2 \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} p_2 e^1(x, t) L \int_{\bar{x}_j}^x e_\xi^1(\xi, t - \frac{D}{m}) d\xi dx \leq L p_2 \bar{R} \int_0^l (e^1(x, t))^2 dx \\ & + L p_2 \bar{R}^{-1} \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} \left( e_\xi^1(\xi, t - \frac{D}{m}) \right)^2 d\xi \end{aligned} \quad (3.33)$$

### 3.5. OBSERVER ANALYSIS

we apply Wirtinger inequality to the last term (3.33):

$$\begin{aligned} Lp_2\bar{R}^{-1} \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} \left( e_{\xi}^1(\xi, t - \frac{D}{m}) \right)^2 d\xi &\leq Lp_2\bar{R}^{-1} \sum_{j=0}^{N-1} \frac{\Delta^2}{\pi^2} \int_{x_j}^{x_{j+1}} \left( e_x^1(x, t - \frac{D}{m}) \right)^2 dx \\ &\leq Lp_2\bar{R}^{-1} \frac{\Delta^2}{\pi^2} \int_0^l \left( e_x^1(x, t - \frac{D}{m}) \right)^2 dx \end{aligned} \quad (3.34)$$

choosing  $\bar{R} = \frac{\Delta}{\pi}R$ , we get:

$$\begin{aligned} 2 \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} p_2 e^1(x, t) L \int_{\bar{x}_j}^x e_{\xi}^1(\xi, t - \frac{D}{m}) d\xi dx &\leq \frac{\Delta}{\pi} R L p_2 \int_0^l (e^1(x, t))^2 dx \\ + \frac{\Delta}{\pi} R^{-1} L p_2 \int_0^l \left( e_x^1(x, t - \frac{D}{m}) \right)^2 dx &\end{aligned} \quad (3.35)$$

$$\begin{aligned} 2 \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} p_3 e_t^1(x, t) L \int_{\bar{x}_j}^x e_{\xi}^1(\xi, t - \frac{D}{m}) d\xi dx &\leq \frac{\Delta}{\pi} R L p_3 \int_0^l (e_t^1(x, t))^2 dx \\ + \frac{\Delta}{\pi} R^{-1} L p_3 \int_0^l \left( e_x^1(x, t - \frac{D}{m}) \right)^2 dx &\end{aligned} \quad (3.36)$$

Combining (3.21)-(3.36) leads to:

$$\begin{aligned} \dot{V}^1(t) + 2\delta V^1(t) &\leq \left( 2\delta p_1 + g - r e^{-2\delta \frac{D}{m}} + 2p_2(\phi + \frac{\Delta}{2\pi}LR) - \frac{2\pi^2}{l^2}(p_2 - \delta p_3) \right) \\ &\quad \times \int_0^l (e^1(x, t))^2 dx + 2 \left( -p_2 + p_1 + p_3\phi \right) \int_0^l e^1(x, t) e_t^1(x, t) dx \\ &\quad + 2 \left( r e^{-2\delta \frac{D}{m}} - p_2 L \right) \int_0^l e^1(x, t) e^1(x, t - \frac{D}{m}) dx \\ &\quad + \left( \frac{\Delta L R p_3}{\pi} - 2p_3 + r \left( \frac{D}{m} \right)^2 \right) \int_0^l (e_t^1(x, t))^2 dx \\ &\quad + 2 \left( -L p_3 \right) \int_0^l e_t^1(x, t) e^1(x, t - \frac{D}{m}) dx \\ &\quad + \left( - (r + g) e^{-2\delta \frac{D}{m}} \right) \int_0^l \left( e^1(x, t - \frac{D}{m}) \right)^2 dx \end{aligned} \quad (3.37)$$

Set  $\eta = \text{col}\{e^1(x, t), e_t^1(x, t), e^1(x, t - \frac{D}{m})\}$ . Then (3.37) becomes:

$$\dot{V}^1(t) + 2\delta V^1(t) \leq \int_0^l \eta^T \Phi_{\phi} \eta dx + \frac{\Delta}{\pi} L R^{-1} (p_3 + p_2) \int_0^l \left( e_x^1(x, t - \frac{D}{m}) \right)^2 dx \quad (3.38)$$

### 3.5. OBSERVER ANALYSIS

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Since  $\Phi_\phi$  is affine in  $\phi$ , then under (3.16):

$$\int_0^l \eta^T \Phi_\phi \eta dx \leq 0. \quad (3.39)$$

From this we also deduce

$$\begin{aligned} \dot{V}^1(t) + 2\delta V^1(t) - \delta_1 V^1(t - \frac{D}{m}) &\leq \int_0^l \eta^T \Phi_\phi \eta dx + (\frac{\Delta}{\pi} LR^{-1}(p_3 + p_2) - \delta_1) \\ &\quad \times \int_0^l \left( e_x^1(x, t - \frac{D}{m}) \right)^2 dx. \end{aligned} \quad (3.40)$$

Then we conclude under conditions of **Theorem 3.1**, that

$$\dot{V}^1(t) + 2\delta V^1(t) - \delta_1 V^1(t - \frac{D}{m}) \leq 0. \quad (3.41)$$

By using Halanay's type Inequalities (Appendix C)  $V^1(t)$  is exponentially vanishing to zero.

Now by induction, let us suppose that the exponential convergence is guaranteed for  $k-1$ , and consider the observation for  $k$ . For this case the observation error is described by the following equation:

$$\begin{aligned} e_t^k(x, t) &= e_{xx}^k(x, t) + \Psi(x, t, e^k) e^k(x, t) - L e^k(\bar{x}_j, t - \frac{D}{m}) + L \int_0^{\bar{x}_j} e_x^{k-1}(x, t) dx, \\ &\quad \forall x \in [x_j, x_{j+1}), \\ e^k(l, t) &= e^k(0, t) = 0, \end{aligned} \quad (3.42)$$

The only difference between the above system and the one of the case  $k=1$  is in the disturbing term  $\int_0^{\bar{x}_j} e_x^{k-1}(x, t) dx$  which is supposed by induction to be exponentially vanishing to zero. By using the following Lyapunov-Krasovskii functional:

$$\begin{aligned} V^k(t) &= p_1 \int_0^l \left( e^k(x, t) \right)^2 dx + p_3 \int_0^l \left( e_x^k(x, t) \right)^2 dx \\ &\quad + g \int_0^l \left[ \int_{t-\frac{D}{m}}^t e^{2\delta(s-t)} \left( e^k(x, s) \right)^2 ds \right] dx \\ &\quad + \frac{D}{m} r \int_0^l \left[ \int_{-\frac{D}{m}}^0 \int_{t+\theta}^t e^{2\delta(s-t)} \left( e_s^k(x, s) \right)^2 ds d\theta \right] dx \end{aligned} \quad (3.43)$$

and similar arguments as for the case  $k=1$ , one can easily deduce that both  $\int_0^l \left( e^k(x, t) \right)^2 dx$  and  $\int_0^l \left( e_x^k(x, t) \right)^2 dx$  converge exponentially to zero.

### 3.5. OBSERVER ANALYSIS

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**Remark:** The LMIs in **Theorem 3.1** depend on the fraction  $D/m$ . If they are feasible for  $H_{max} = D/m$ , then choosing  $m \geq D/H_{max}$  we have always a feasible LMI. Then for each delay  $D$ , we can find a sufficiently large  $m$  such that the LMIs of the **Theorem 3.1** are verified.

#### 3.5.1 Extension to sampled-data case

In this part, we present briefly the extension of the above observer to sampled-measurements case. In this case the output is available only at sampling instants  $t_k$

$$0 = t_0 < t_1 < \dots < t_k < \dots, \quad \lim_{k \rightarrow \infty} t_k = \infty.$$

We assume that the sampling intervals may be variable, but upper-bounded by a known bound  $h$  :

$$t_{k+1} - t_k \leq h \quad \forall k = 0, 1, \dots$$

The proposed observer has the following form:

for  $k = 1$  :

$$\begin{aligned} \hat{u}_t^1(x, t) &= \hat{u}_{xx}^1(x, t) + f(\hat{u}^1(x, t), x, t) - L(\hat{u}^1(\bar{x}_j, t_k - \frac{D}{m}) - y_j(t_k)), \\ &\forall t \in [t_k, t_{k+1}), \quad \forall x \in [x_j, x_{j+1}), \end{aligned} \quad (3.44a)$$

for  $k = 2, \dots, m$  :

$$\begin{aligned} \hat{u}_t^k(x, t) &= \hat{u}_{xx}^k(x, t) + f(\hat{u}^k(x, t), x, t) - L(\hat{u}^k(\bar{x}_j, t - \frac{D}{m}) - \hat{u}^{k-1}(\bar{x}_j, t)), \\ &\forall t \in [t_k, t_{k+1}), \quad \forall x \in [x_j, x_{j+1}). \end{aligned} \quad (3.44b)$$

Then the observation error is described by the following equations:

for  $k = 1$  :

$$\begin{aligned} e_t^1(x, t) &= e_{xx}^1(x, t) + \Psi(x, t, e^1)(\hat{u}^1(x, t) - u^1(x, t)) - Le^1(\bar{x}_j, t_k - \frac{D}{m}), \\ &\forall t \in [t_k, t_{k+1}), \quad \forall x \in [x_j, x_{j+1}) \\ e^1(l, t) &= e^1(0, t) = 0, \end{aligned} \quad (3.45a)$$

for  $k = 2, \dots, m$  :

$$\begin{aligned} e_t^k(x, t) &= e_{xx}^k(x, t) + \Psi(x, t, e^k)(\hat{u}^k(x, t) - u^k(x, t)) \\ &\quad - Le^k(\bar{x}_j, t - \frac{D}{m}) + Le^{k-1}(\bar{x}_j, t), \\ &\forall x \in [x_j, x_{j+1}), \\ e^k(l, t) &= e^k(0, t) = 0, \end{aligned} \quad (3.45b)$$

### 3.5. OBSERVER ANALYSIS

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As we can easily see, the unique difference with the observer without sampling measurements is for the first sub-observer ( $k = 1$ ). In order to study the convergence of the case  $k = 1$ , we use the following modified Lyapunov-Krasvoskii functional inspired from [35, 44]:

$$\begin{aligned}
V^1(t) &= p_1 \int_0^l \left( e^1(x, t) \right)^2 dx + p_3 \int_0^l \left( e_x^1(x, t) \right)^2 dx \\
&+ g \int_0^l \left[ \int_{t-\frac{D}{m}}^t e^{2\delta(s-t)} \left( e^1(x, s) \right)^2 ds \right] dx \\
&+ \frac{D}{m} r \int_0^l \left[ \int_{-\frac{D}{m}}^0 \int_{t+\theta}^t e^{2\delta(s-t)} \left( e_s^1(x, s) \right)^2 ds d\theta \right] dx \\
&+ Wh^2 e^{2\delta h} \int_0^l \int_{t_k-\frac{D}{m}}^t e^{2\delta(s-t)} \left( e_s^1(x, s) \right)^2 ds dx \\
&- \frac{\pi^2}{4} W \int_0^l \int_{t_k-\frac{D}{m}}^{t-\frac{D}{m}} e^{2\delta(s-t)} \left[ e^1(x, s) - e^1\left(x, t_k - \frac{D}{m}\right) \right]^2 ds dx \quad (3.46)
\end{aligned}$$

By generalized Wirtinger's inequality [44], the W-terms expression in  $V^1$ ,

$$\begin{aligned}
&h^2 e^{2\delta h} \int_0^l \int_{t_k-\frac{D}{m}}^t e^{2\delta(s-t)} \left( e_s^1(x, s) \right)^2 ds dx \\
&- \frac{\pi^2}{4} \int_0^l \int_{t_k-\frac{D}{m}}^{t-\frac{D}{m}} e^{2\delta(s-t)} \left[ e^1(x, s) - e^1\left(x, t_k - \frac{D}{m}\right) \right]^2 ds dx \quad (3.47)
\end{aligned}$$

is nonnegative and does not grow in the jumps [44]. By differentiating it, we obtain:

$$\begin{aligned}
&h^2 e^{2\delta h} \int_0^l \left( e_t^1(x, s) \right)^2 dx \\
&- 2\delta h^2 e^{2\delta h} \int_0^l \int_{t_k-\frac{D}{m}}^t e^{2\delta(s-t)} \left( e_s^1(x, s) \right)^2 ds dx \\
&- \frac{\pi^2}{4} \int_0^l \left[ e^1\left(x, t - \frac{D}{m}\right) - e^1\left(x, t_k - \frac{D}{m}\right) \right]^2 dx \\
&+ \frac{\pi^2}{2} \delta \int_0^l \int_{t_k-\frac{D}{m}}^{t-\frac{D}{m}} e^{2\delta(s-t)} \left[ e^1(x, s) - e^1\left(x, t_k - \frac{D}{m}\right) \right]^2 ds dx. \quad (3.48)
\end{aligned}$$

### 3.5. OBSERVER ANALYSIS

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Using the following descriptor equation [17]:

$$\begin{aligned}
0 &= 2 \int_0^l \left[ p_2 e^1(x, t) + p_3 e_t^1(x, t) \right] \left[ -e_t^1(x, t) + e_{xx}^1(x, t) \right. \\
&\quad \left. + \Psi(x, t, e^1) e^1(x, t) - L e^1(x, t_k - \frac{D}{m}) \right] dx \\
&\quad + 2 \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} \left[ p_2 e^1(x, t) + p_3 e_t^1(x, t) \right] L \int_{\bar{x}_j}^x e_{\xi}^1(\xi, t_k - \frac{D}{m}) d\xi dx. \quad (3.49)
\end{aligned}$$

By adding and subtracting the term  $L e(x, t - \frac{D}{m})$  in (3.50), we get:

$$\begin{aligned}
0 &= 2 \int_0^l \left[ p_2 e^1(x, t) + p_3 e_t^1(x, t) \right] \left[ -e_t^1(x, t) + e_{xx}^1(x, t) \right. \\
&\quad \left. + \Psi(x, t, e^1) e^1(x, t) - L e^1(x, t - \frac{D}{m}) \right] dx \\
&\quad + L \int_0^l \left[ p_2 e^1(x, t) + p_3 e_t^1(x, t) \right] \left( e^1(x, t_k - \frac{D}{m}) - e^1(x, t - \frac{D}{m}) \right) dx \\
&\quad + 2 \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} \left[ p_2 e^1(x, t) + p_3 e_t^1(x, t) \right] L \int_{\bar{x}_j}^x e_{\xi}^1(\xi, t_k - \frac{D}{m}) d\xi dx. \quad (3.50)
\end{aligned}$$

Differentiating  $V(t)$  and using the descriptor equation (3.50), we get:

### 3.5. OBSERVER ANALYSIS

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$$\begin{aligned}
\dot{V}^1(t) + 2\delta V^1(t) &= \int_0^l (2\delta p_1 + g + 2p_2\phi)(e^1(x,t))^2 dx \\
&+ \int_0^l (p_1 - p_2 + p_3\phi)e^1(x,t)e_t^1(x,t) dx \\
&+ \int_0^l (-2p_2L)e^1(x,t)e^1(x,t - \frac{D}{m}) dx \\
&+ \left( (\frac{D}{m})^2 r + Wh^2 e^{2\delta h} \right) \int_0^l e_t^1(x,t) dx \\
&- (2p_3L) \int_0^l (e_t^1(x,t)e^1(x,t - \frac{D}{m})) dx \\
&- g e^{-2\delta \frac{D}{m}} \int_0^l (e^1(x,t - \frac{D}{m}))^2 dx \\
&- \frac{\pi^2}{4} W \int (e^1(x,t_k - \frac{D}{m}) - e^1(x,t - \frac{D}{m}))^2 dx \\
&+ 2p_2L \int_0^l e(x,t) ((e^1(x,t_k - \frac{D}{m}) - e^1(x,t - \frac{D}{m}))) dx \\
&- 2p_3L \int_0^l e_t(x,t) ((e^1(x,t_k - \frac{D}{m}) - e^1(x,t - \frac{D}{m}))) dx \\
&+ 2p_2 \int_0^l (e^1(x,t)e_{xx}^1(x,t)) dx + 2\delta p_3 \int_0^l (e_x^1(x,t))^2 dx \\
&- \frac{D}{m} r \int_0^l \int_0^l e^{2\delta(s-t)} e_s(x,s) ds dx \\
&+ 2 \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} \left[ p_2 e^1(x,t) + p_3 e_t^1(x,t) \right] L \int_{\bar{x}_j}^x e_\xi^1(\xi, t_k - \frac{D}{m}) d\xi dx
\end{aligned} \tag{3.51}$$

Using the same computation as in the continuous case and by considering  $\eta = \text{col}\{e^1(x,t), e_t^1(x,t), e^1(x,t - \frac{D}{m}), e^1(x,t_k - \frac{D}{m}) - e^1(x,t - \frac{D}{m})\}$ , we can easily derive the following theorem:

**Theorem 3.2:** Given  $D, h$  and  $m$ , consider the system (3.1) and the observer (3.44a)-(3.44b). Given positive constants scalars  $\Delta, \delta, L > M_f - \frac{\pi^2}{l^2}, R$  and  $\delta_1$  such that  $2\delta > \delta_1$ , if there exist positive scalars  $p_1, p_2, p_3, r, W$  and  $g$  such that:

$$\delta p_3 < p_2 ; \quad \frac{\Delta}{\pi} LR^{-1}(p_3 + p_2) < \delta_1 p_3 \tag{3.52}$$

and

$$\Phi_{m_f} < 0 \quad \Phi_{M_f} < 0 \tag{3.53}$$

### 3.6. EXAMPLE

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where

$$\Phi_\phi = \begin{pmatrix} \Phi_{11} - \lambda & \Phi_{12} & \Phi_{13} & p_2 L \\ \Phi_{12} & \Phi_{22} + Wh^2 e^{2\delta h} & \Phi_{23} & p_3 L \\ \Phi_{13} & \Phi_{23} & \Phi_{33} & 0 \\ p_2 L & p_3 L & 0 & -W \frac{\pi^2}{4} \end{pmatrix} \quad (3.54)$$

with  $\Phi_{11}, \Phi_{12}, \Phi_{13}, \Phi_{22}, \Phi_{23}$  and  $\Phi_{33}$  given by (3.18).

Then all the observation errors  $\int_0^1 \left( e^k(x, t) \right)^2 dx$  and  $\int_0^1 \left( e_x^k(x, t) \right)^2 dx$  ( $k = 1, \dots, m$ ) globally exponentially decay to zero as  $t \rightarrow +\infty$ .

## 3.6 Example

Let us consider the following example:

$$u_t = u_{xx}(x, t) + 1.02 \pi^2 \cos(u(x, t)) \quad (3.55)$$

with  $u(x, 0) = \sin(x)$  and let  $y_j = u(\bar{x}_j, t - D)$ ,  $j = 1, \dots, N - 1$ , where  $D$  is an arbitrarily delay and  $\hat{u}(x, 0) = 0$ .

### Continuous case

We choose  $L = 1$ ,  $\Delta = \frac{1}{50}$ ,  $\delta = 0.21$  and  $\delta_1 = 0.1$ . The following table shows the value of the delay  $D$  and the corresponding number of sub-observers  $m$  for which the LMIs of **Theorem 3.1** are feasible:

$D$	0.42	0.85	1.2	1.70	2.13
$m$	1	2	3	4	5

Table 3.1 – The maximum delay  $D$  for a given number of sub-observers  $m$  for the continuous case.

Numerical simulations of the above example are presented in the following. These figures show the state  $u(x, t)$  of the system (3.55) and its estimates at the points  $x = 0.1$  and  $x = 0.6$ . These results show that the state of the designed observer  $\hat{u}(x, t)$  converges to  $u(x, t)$  when  $D = 1s$  using only 1 sub-observer  $m = 1$ . However, when  $D = 2s$ , 2 sub-observers  $m = 2$  are required to get satisfactory results. Further simulations show that the observer (3.4a)-(3.4b) estimates the state of the system (3.55) for larger

### 3.6. EXAMPLE

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delays than those presented in the previous table. Future works will be dedicated to the improvement of the sufficient conditions presented here by the LMIs (3.15) and (3.16).

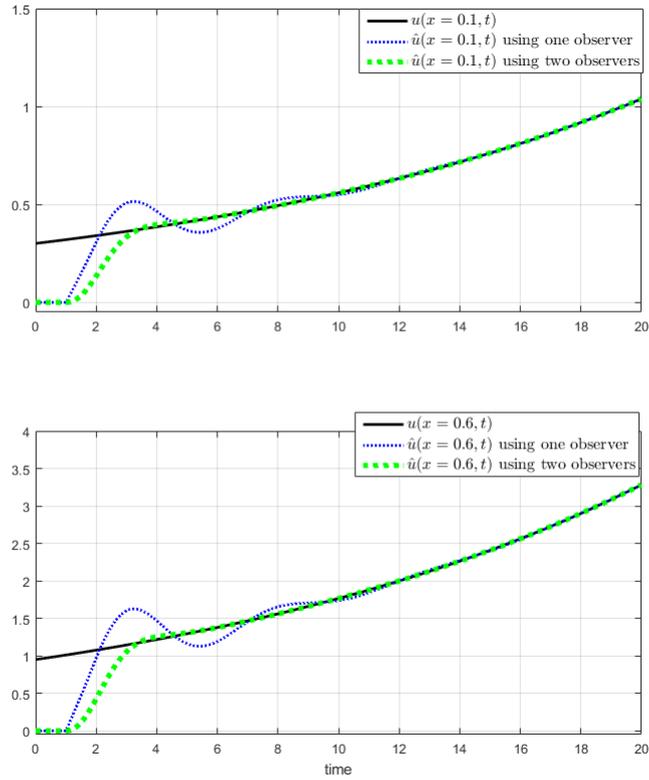


Figure 3.2 – The state  $u(x, t)$  and its observations for  $m = 1$  and  $m = 2$  at  $x = 0.1$  and  $x = 0.6$  for a delay  $D = 1s$

### 3.6. EXAMPLE

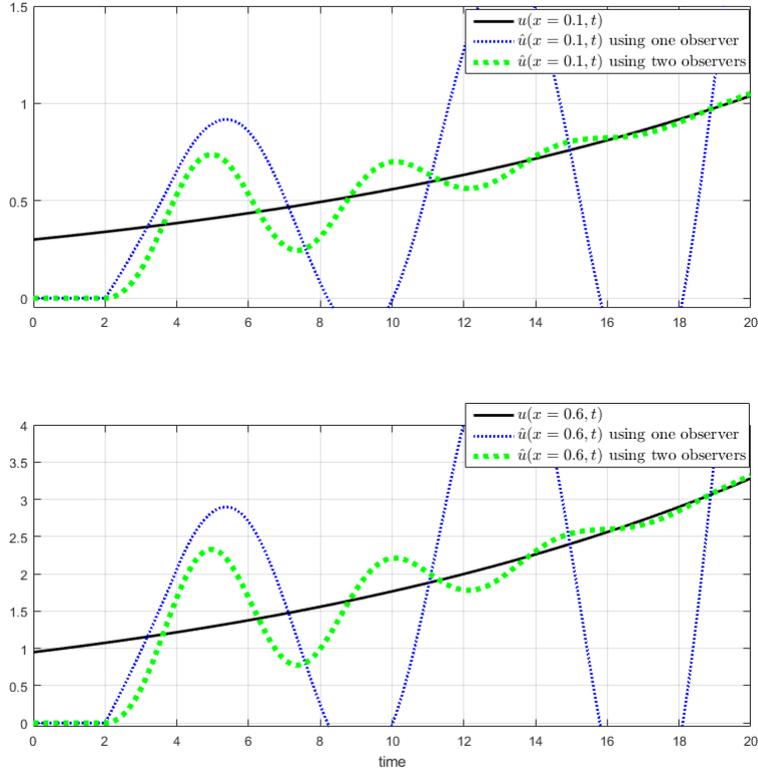


Figure 3.3 – The state  $u(x, t)$  and its observations for  $m = 1$  and  $m = 2$  at  $x = 0.1$  and  $x = 0.6$  for a delay  $D = 2s$

### Sampled Data Case

We consider the case where the outputs  $y_j$ ,  $j = 1, \dots, N - 1$ , are periodically sampled with period  $h$ . We use the same values of the parameters as for the continuous case. The following tables show, for different values of the delay  $D$ , the value of the sampling period  $h$  and the corresponding number of sub-observers  $m$  for which the LMIs of **Theorem 3.2** are feasible.

for  $D = 0.4$  :

$h$	0.05	0.37	0.47	0.51
$m$	1	2	3	4

Table 3.2 – The maximum sampling period  $h$  for a given number of sub-observers  $m$ , with  $D = 0.4s$

for  $D = 0.8$  :

### 3.7. CONCLUSION

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$h$	0.05	0.1	0.37	0.5
$m$	2	3	4	8

Table 3.3 – The maximum sampling period  $h$  for a given number of sub-observers  $m$ , with  $D = 0.8s$

for  $D = 1.5$  :

$h$	0.05	0.1	0.3	0.5
$m$	4	5	7	14

Table 3.4 – The maximum sampling period  $h$  for a given number of sub-observers  $m$ , with  $D = 1.5s$

## 3.7 Conclusion

In this chapter, a novel observer is proposed for a class of parabolic systems. The main advantage of the proposed algorithm is that it can handle arbitrary delay and sampled measurements. This result can be easily extended to classes of cascade ODE-parabolic PDE. The disadvantage of the above algorithm is that it needs a chain of PDEs, which can lead to some implementation issues. The simplification of this observer is under investigation.

# General Conclusion

This final chapter describes the concluding remarks drawn from this thesis and some of the future perspectives.

## Conclusions

The main interest of the present study is the design of various observers in order to estimate the state of different classes of nonlinear systems, where the outputs of these systems were delayed. In some of the cases, the problem of sampled output was treated as well.

Two major approaches were used in this thesis, the first one relies on "finite-dimensional" research tools, i.e. starting with exponentially convergent state observers of ordinary differential equations (ODEs) (not involving time-delay), then modifying them so that exponential convergence is preserved in the presence of time-delay. Mainly, one or several predictors of the output or the state are used in order to get satisfactory results. On the other hand, we have the "infinite -dimensional" design tools. This concept relies on the fact that the delay is modeled by a first order hyperbolic PDE, this leads to an ODE-PDE cascade representation of the original system. In order to observe this type of cascade systems, the PDE-based backstepping-like transformation was invoked.

Another important subject that was dealt with in this work was the chain observer structure. This design was used in order to give the different proposed observers the ability to give accurate observations in the case of arbitrary large pointwise delays.

The first chapter was dedicated to the study of state estimation for a class of nonlinear triangular systems using sampled delayed output measurement. The problem complexity lies, on one hand, in the interference of the output time-delay and sampling effects and, on the other hand, in the injection of

### 3.7. CONCLUSION

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the undelayed output (which is not accessible to measurements) in the state matrix. In the first part, a new observer was designed for this system that features a simple output predictor defined by two ODEs. The maximum sampling interval and time-delay for the observer to be exponentially convergent were obtained using an appropriate Lyapunov function in the analysis of the resulted error system, the input-to-state stability and the small gain theorem were invoked in the proof. A cascade chain representation of this observer was presented in the second part of this chapter. This new form will allow us to estimate the state of the system when delays are not necessarily small. This chain observer is composed of  $m$  sub-observers in series, when  $d$  is large we increase  $m$  until we get satisfactory results. In this part, we presented the estimates of the delay, the maximum sampling interval and the minimal number of state sub-observers to get accurate observations.

In the second chapter, the problem of state observation was addressed for another class of triangular nonlinear systems with delayed output, where the delay was captured by a first order hyperbolic partial differential equation, this led to the ODE-PDE cascade representation of the original system. The aim was to get online estimates of both the finite-dimensional state and the infinite-dimensional state where the delay is allowed to be large. This problem was dealt with using the high-gain chain observer defined by a set of ODE and PDE. Using a backstepping-like transformation on the estimation error system and a Lyapunov stability analysis on the transformed system, sufficient conditions were established for the chain observer to be exponentially convergent. The sufficient conditions involve the minimal number  $m$  of sub-observers: the larger the system delay the larger the number of sub-observers  $m$ .

The third chapter dealt with the observation of a class of nonlinear parabolic PDEs with output delay, where the measurements of the state were taken in a finite number of fixed sampling spatial points. The novelty of this new observer lies in the fact that the delay size is arbitrary, this was guaranteed using a cascade chain representation. Using an appropriate Lyapunov-Krasovskii functional, sufficient conditions were established in terms of LMIs for the chain observer to be exponentially convergent. The sufficient conditions involve the minimal number of elementary observers: the larger the delay the larger the number of observers. Extension to sampled data delayed measurements is presented as well.

Simulation results were presented in the three chapters in order to confirm the theoretical results.

### 3.7. CONCLUSION

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#### Perspectives

Even if the time-delay systems subject is extensively exploited in the literature, there are several points that have not been dealt with yet. Some of the problems that can be treated in future works are:

- Improvement and extension of the methods presented to various classes of nonlinear parabolic and hyperbolic systems.
- Observer design for an ODE-PDE cascade system where the PDE is of type nonlinear heat equation.
- Observer design for an ODE-PDE cascade system where the PDE is of type nonlinear wave equation.
- The study of the two previous cascade systems under delay measurements.

# Appendix A

## Control Theory Review

### Class $\mathcal{K}$ and class $\mathcal{KL}$ functions [29]

- A scalar continuous function  $\alpha(r)$ , defined for  $r \in [0, a)$ , belongs to class  $\mathcal{K}$  if it is strictly increasing and  $\alpha(0) = 0$ .
- scalar continuous function  $\beta(r, s)$ , defined for  $r \in [0, a)$  and  $s \in [0, \infty)$ , belongs to class  $\mathcal{KL}$  if, for each fixed  $s$ , the mapping  $\beta(r, s)$  belongs to class  $\mathcal{K}$  with respect to  $r$  and, for each fixed  $r$  the mapping  $\beta(r, s)$  is decreasing with respect to  $s$  and  $\beta(r, s) \rightarrow 0$  as  $s \rightarrow 0$

### Input to State Stability

Consider the system

$$\dot{x} = f(x, u) \tag{A.1}$$

where  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is locally Lipschitz in  $x$  and  $u$ . The input  $u(t)$  is a piecewise continuous, bounded function of  $t$  for all  $t \geq 0$ .

The system  $\dot{x} = f(x, u)$  is said to be input-to-state stable if there exist a class  $\mathcal{KL}$  function  $\beta$  and a class  $\mathcal{K}$  function  $\gamma$  such that for any  $t_0 \geq 0$ , any initial state  $x(t_0)$ , and any bounded input  $u(t)$ , the solution  $x(t)$  exists for all  $t \geq t_0$  and satisfies [30]

$$\|x(t)\| \leq \max\left\{\beta(\|x(t_0)\|, t - t_0), \gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right)\right\}, \quad \forall t \geq t_0 \tag{A.2}$$

Input-to-state stability of  $\dot{x} = f(x, u)$  implies the following properties:

- For any bounded input  $u(t)$ , the state  $x(t)$  is bounded;

- 
- $x(t)$  is ultimately bounded by  $\gamma\left(\sup_{t_0 \leq \tau \leq \infty} \|u(\tau)\|\right)$ ;
  - if  $u(t)$  converges to zero as  $t \rightarrow \infty$ , so does  $x(t)$ ;
  - The origin of the unforced system  $\dot{x} = f(x, 0)$  is globally asymptotically stable.

### Small Gain theorem

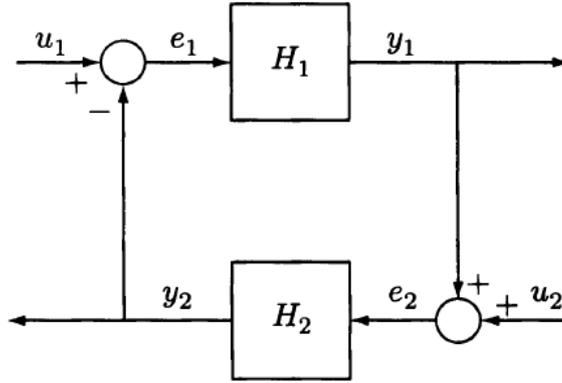


Figure A.1 – Feedback connection

Consider the feedback connection of figure A.1 where  $H_1 : \mathcal{L}_e^m \rightarrow \mathcal{L}_e^q$  and  $H_2 : \mathcal{L}_e^q \rightarrow \mathcal{L}_e^m$ . Suppose both systems are finite-gain  $\mathcal{L}$  stable; that is,

$$\|y_{1\tau}\|_{\mathcal{L}} \leq \gamma_1 \|e_{1\tau}\|_{\mathcal{L}} + \beta_1, \quad \forall e_1 \in \mathcal{L}_e^m, \quad \forall \tau \in [0, \infty) \quad (\text{A.3})$$

$$\|y_{2\tau}\|_{\mathcal{L}} \leq \gamma_2 \|e_{2\tau}\|_{\mathcal{L}} + \beta_2, \quad \forall e_2 \in \mathcal{L}_e^q, \quad \forall \tau \in [0, \infty) \quad (\text{A.4})$$

Suppose further that the feedback system is well defined in the sense that for every pair of inputs  $u_1 \in \mathcal{L}_e^m$  and  $u_2 \in \mathcal{L}_e^q$ , there exist unique outputs  $e_1, y_2 \in \mathcal{L}_e^m$  and  $e_2, y_1 \in \mathcal{L}_e^q$ . If  $H_1$  and  $H_2$  are represented by state models, we assume that the feedback connection has a well-defined state model. Define  $u = \text{col}(u_1, u_2)$ ,  $y = \text{col}(y_1, y_2)$ , and  $e = (e_1, e_2)$ . The next (small-gain) theorem gives a sufficient condition for the feedback connection to be finite-gain  $\mathcal{L}$  stable; that is the gain the mapping from  $u$  to  $e$ , or equivalently from  $u$  to  $y$ , is finite-gain  $\mathcal{L}$  stable [29].

**Theorem:** Under the preceding assumptions, the feedback connection is finite-gain  $\mathcal{L}$  stable if  $\gamma_1 \gamma_2 < 1$

# Appendix B

## Mathematical Review

### Mean value theorem

Assume that  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuously differentiable at each point  $x$  of an open set  $S \subset \mathbb{R}^n$ . Let  $x$  and  $y$  be two points of  $S$  such that the line segment  $L(x, y) \subset S$ . Then there is a point  $z$  of  $L(x, y)$  such that

$$f(y) - f(x) = \left. \frac{\partial f}{\partial x} \right|_{x=z} (y - x) \quad (\text{B.1})$$

When  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , a multidimensional version of the mean value theorem is given by

$$f(y) - f(x) = \int_0^1 \frac{\partial f}{\partial x}(x + \sigma(y - x)) d\sigma (y - x) \quad (\text{B.2})$$

which can be seen by setting  $J(x) = \left[ \frac{\partial f}{\partial x} \right](x)$ , and  $h(\sigma) = f(x + \sigma(y - x))$  for  $0 \leq \sigma \leq 1$ . By the chain rule,  $h'(\sigma) = J(x + \sigma(y - x))(y - x)$ . Using

$$f(y) - f(x) = h(1) - h(0) = \int_0^1 h'(\sigma) d\sigma \quad (\text{B.3})$$

### Comparison Lemma

Consider the scalar differential equation

$$\dot{u} = f(t, u), \quad u(t_0) = u_0 \quad (\text{B.4})$$

where  $f(t, u)$  is continuous in  $t$  and locally Lipschitz in  $u$ , for all  $u \in J \subset \mathbb{R}$ . Let  $[t_0, T)$  ( $T$  could be infinity) be the maximal interval of existence of the solution  $u(t)$ , and suppose  $u(t) \in J$  for all  $t \in [t_0, T)$ . Let  $v(t)$  be a continuous

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function whose upper right-hand derivative  $D^+v(t)$  satisfies the differential inequality

$$D^+v(t) \leq f(t, v(t)), \quad v(t_0) \leq u_0 \tag{B.5}$$

with  $v(t) \in J$  for all  $t \in [t_0, T)$ . Then

$$v(t) \leq u(t) \tag{B.6}$$

for all  $t \in [t_0, T)$ .

# Appendix C

## Basic Inequalities

### Young's inequality

$a$  and  $b$  are nonnegative real numbers

$$ab \leq \frac{\gamma}{2}a^2 + \frac{1}{2\gamma}b^2, \quad \forall \gamma > 0 \quad (\text{C.1})$$

### Cauchy–Schwartz inequality

$$\int_0^1 u(x)w(x)dx \leq \left( \int_0^1 u(x)^2 dx \right)^{1/2} \left( \int_0^1 w(x)^2 dx \right)^{1/2} \quad (\text{C.2})$$

### Poincaré inequality

For any  $w$ , continuously differentiable on  $[0, 1]$ ,

$$\int_0^1 w(x)^2 dx \leq 2w^2(1) + 4 \int_0^1 w_x(x)^2 dx \quad (\text{C.3})$$

$$\int_0^1 w(x)^2 dx \leq 2w^2(0) + 4 \int_0^1 w_x(x)^2 dx \quad (\text{C.4})$$

### Wirtinger's inequality [45]

Let  $w \in H^1(0, l)$ , be a scalar function with  $w(0) = 0$  or  $w(1) = 0$ .

$$\int_0^1 w(x)^2 dx \leq \frac{4}{\pi^2} \int_0^1 w_x(x)^2 dx \quad (\text{C.5})$$

Moreover,

$$\max_{x \in [0,1]} w(x)^2 \leq \int_0^1 w_x(x)^2 dx \quad (\text{C.6})$$

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**Halanay's type Inequalities** [20]

Let  $0 < \delta_1 < 2\delta$  and let  $V : [t_0 - h, \infty) \rightarrow [0, \infty)$  be an absolutely continuous function which satisfies

$$\dot{V}(t) \leq -2\delta V(t) + \delta_1 \sup_{-h \leq s \leq 0} V(t+s) \quad (\text{C.7})$$

Then

$$V(t) \leq e^{-2\alpha(t-t_0)} \sup_{-h \leq s \leq 0} V(t+s) \quad (\text{C.8})$$

where  $\alpha$  is the unique positive solution of the equation

$$\alpha = \delta - \frac{\delta_1 e^{2\alpha h}}{2} \quad (\text{C.9})$$

# Appendix D

## Finite Differences

### D.1 Finite differences approximations

In the third chapter, simulation results were given for parabolic equations, in the following the method used to achieve these simulations will be explained. The finite difference method (FDM) is one of the approaches for the numerical solution of PDEs. It proceeds by replacing those derivatives in the governing equations by finite differences. In general, a finite difference approximation to the value of some derivative of a scalar function  $u(x, t)$  at a point  $x_0$  in its domain, say  $u_x(x_0, t)$  or  $u_{xx}(x_0, t)$ , relies on a suitable combination of sampled function values at nearby points. Different types of approximation of the first derivative in  $x$  of  $u(x, t)$  are given [22]:

$$u_x(x, t) = \frac{u(x+h, t) - u(x, t)}{h} + O(h) \quad \text{forward} \quad (\text{D.1})$$

$$= \frac{u(x+h, t) - u(x-h, t)}{2h} + O(h^2) \quad \text{central} \quad (\text{D.2})$$

$$= \frac{u(x, t) - u(x-h, t)}{2h} + O(h) \quad \text{backward} \quad (\text{D.3})$$

The second derivative approximation of  $u(x, t)$  is:

$$u_{xx}(x, t) = \frac{u(x+h, t) - 2u(x, t) + u(x-h, t)}{h^2} + O(h^4) \quad (\text{D.4})$$

Similarly for  $u_t$  and  $u_{tt}$ .

## D.2. NUMERICAL ALGORITHMS FOR PARABOLIC EQUATIONS

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### D.2 Numerical algorithms for parabolic equations

Consider the heat equation

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < D, \quad t > 0 \quad (\text{D.5})$$

on an interval of length  $D$ , with constant  $\gamma > 0$ . We impose time-dependent Dirichlet boundary conditions

$$u(t, 0) = \alpha(t), \quad u(t, D) = \beta(t), \quad t > 0 \quad (\text{D.6})$$

and let the initial conditions be

$$u(0, x) = f(x), \quad 0 \leq x \leq D \quad (\text{D.7})$$

In order to effect a numerical approximation to the solution to this initial-boundary value problem, we begin by introducing a rectangular mesh consisting of nodes  $(t_j, x_m) \in \mathbb{R}^2$  with

$$0 = t_0 < t_1 < t_2 < \dots \quad \text{and} \quad 0 = x_0 < x_1 < \dots < x_n = D \quad (\text{D.8})$$

For simplicity, we maintain a uniform mesh spacing in both directions, with

$$\Delta t = t_{j+1} - t_j, \quad \text{and} \quad x_{m+1} - x_m = \frac{D}{n} \quad (\text{D.9})$$

representing, respectively, the time step size and the spatial mesh size. We will adopt the following notation

$$u_{j,m} = u(t_j, x_m), \quad \text{where} \quad t_j = j\Delta t, \quad x_m = m\Delta x \quad (\text{D.10})$$

to denote the numerical approximation to the solution value at the indicated node [40],[13].

Numerous finite difference schemes for the heat equation exist such as: Forward Euler, Lax-Friedrichs, Lax-Wendroff . . . , however not all of them are convergent and stable numerically. In our study we chose a scheme called Crank-Nicolson algorithm, this method is unconditionally stable (one that does not restrict the time step).

## D.2. NUMERICAL ALGORITHMS FOR PARABOLIC EQUATIONS

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### Crank-Nicolson Method

In this algorithm we approximate the first derivative of time  $u_t(x, t)$  by the backwards difference formula

$$\frac{\partial u(t_j, x_m)}{\partial t} = \frac{u(t_j, x_m) - u(t_{j-1}, x_m)}{\Delta t} + O(\Delta t) \quad (\text{D.11})$$

$$= \frac{u_{j,m} - u_{j-1,m}}{\Delta t} + O(\Delta t) \quad (\text{D.12})$$

The second-order space derivative is approximated by the centered difference formula, and hence

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(t_j, x_{m+1}) - 2u(t_j, x_m) + u(t_j, x_{m-1}))}{(\Delta x)^2} + O((\Delta x)^2) \quad (\text{D.13})$$

$$= \frac{u_{j,m+1} - 2u_{j,m} + u_{j,m-1}}{(\Delta x)^2} + O((\Delta x)^2) \quad (\text{D.14})$$

Combining these equations leads to the following formula that was implemented in MATLAB.

$$u_{j,m} = -\mu u_{j+1,m+1} + (1 + 2\mu)u_{j+1,m} - \mu u_{j+1,m} - \mu u_{j+1,m-1}, \quad (\text{D.15})$$

$$j = 0, 1, 2, \dots \quad (\text{D.16})$$

$$m = 1, \dots, n-1. \quad (\text{D.17})$$

in which the parameter

$$\mu = \frac{\gamma \Delta t}{(\Delta x)^2} \quad (\text{D.18})$$

# Bibliography

- [1] Ahmed-Ali T., Karafyllis, I., and Lamnabhi-Lagarrigue, F. (2013). Global exponential sampled-data observers for nonlinear systems with delayed measurements. *Systems and Control Letters* 62(7), 539-549.
- [2] Ahmed-Ali T., E. Fridman, F. Giri, L. Burlion, F. Lamnabhi-Lagarrigue. (2016). Using exponential time-varying gains for sampled-data stabilization and estimation. *Automatica* 67, 244–251.
- [3] Ahmed-Ali T., Karafyllis, I., Krstic, M. and Lamnabhi-Lagarrigue, F. (2016). “Robust Stabilization of Nonlinear Globally Lipschitz Delay Systems”, in ”Recent Results on Nonlinear Time Delayed Systems” (Eds. M. Malisoff, F. Mazenc, P. Pepe and I. Karafyllis) *Advances in Delays and Dynamics*, Springer.
- [4] Ahmed-Ali, T., Giri, F., Krstic, M., Lamnabhi-Lagarrigue, F. (2015). High-gain observer design for a class of nonlinear ODE-PDE cascade systems. *Systems Control Lett.* 83 19–27.
- [5] Ahmed-Ali, T., Karafyllis, I., Giri, F., Krstic, M. Lamnabhi-Lagarrigue, F. (2017). Exponential stability analysis of sampled-data ODE-PDE systems and application to observer design, *IEEE Trans. Automat. Control* 62 (6) 3091–3098.
- [6] Ahmed-Ali, T., Cherrier, E., and Lamnabhi-Lagarrigue, F. (2012). Cascade high predictors for a class of nonlinear systems. *IEEE Trans. on Aut. Control*, 57(1), 224-229.
- [7] Ahmed-Ali, T., Fridman, E., Giri, F., Burlion, L. and Lamnabhi-Lagarrigue, F. (2016). Using exponential time-varying gains for sampled-data stabilization and estimation, *Automatica*, 67, 244-251.
- [8] Ahmed-Ali, T., Giri, F., Krstic, M., Kahelras, M. (2018). PDE based observer design for nonlinear systems with large output delay. *Systems & Control Letters*, 113, 1-8.

## BIBLIOGRAPHY

---

- [9] Ashyralyev, A., Sobolevskii, P.E. (1994) .Well-posedness of parabolic differential and difference equations, Birkhäuser Basel.
- [10] Bellman, N.D., Adomian, G. (1985). Partial Differential Equations. New Methods for Their Treatment and Solution. Springer Netherlands.
- [11] Besançon, G., Georges, D., and Benayache, Z. (2007). Asymptotic state prediction for continuous-time systems with delayed input and application to control. The Proceedings of the European Control Conference, pp. 1786-1791.
- [12] Cacace F., Germani, A. and Manes, C. (2014). A Chain Observer for nonlinear systems with multiple time-varying measurement delays. SIAM J. Control Optimization 52(3), 1862-1885.
- [13] Coleman, M., P. (2013). An Introduction to Partial Differential Equations with MATLAB, Second Edition, Chapman and Hall/CRC, ISBN-13: 978-1439898468.
- [14] Curtain, R., and Zwart, H. (1995). An introduction to infinite-dimensional linear systems theory. New York: Springer-Verlag.
- [15] Desch, W., Kappel, F., and Kunisch, K. (2003). Control and Estimation of Distributed Parameter Systems, Birkhäuser Basel.
- [16] El Fadil, H., and Giri, F. (2011). Climatic sensorless maximum power point tracking in PV generation systems. Control Engineering Practice, 19(5), 513–521.
- [17] Fridman, E. (2014). Systems and Control: Foundations and Applications, Introduction to Time-Delay Systems: Analysis and Control. Birkhauser.
- [18] Fridman, E. and Blighovsky, A. (2012). Robust sampled-data control of a class of semilinear parabolic systems. Automatica, 48(5), 826-836.
- [19] Germani, A., Manes, C., & Pepe, P. (2002). A new approach to state observation of nonlinear systems with delayed output. IEEE Transactions on Automatic Control, 47(1), 96–101.
- [20] Halanay, A. (1966). Differential Equations: Stability, Oscillations, Time Lags. Academic Press, New York.
- [21] Henry, D. (1993). Geometric theory of semilinear parabolic equations. New York: Springer-Verlag.

## BIBLIOGRAPHY

---

- [22] Jichun, L., Yi-Tung, C. (2008). *Computational Partial Differential Equations Using MATLAB*, Chapman and Hall/CRC, ISBN 9781420089042.
- [23] Jost, J. (2007). *Partial Differential Equations*. Springer-Verlag New York.
- [24] Kahelras, M., Ahmed-Ali, T., Folin, T., Giri, F., Lamnabhi-Lagarrigue, F. (2016). Observer design for triangular nonlinear systems using delayed sampled-output measurements. in: *IEEE Conference on Decision and Control*, Las Vegas, NV, USA, 2016, pp. 1447–1451.
- [25] Kahelras, M., Ahmed-Ali, T., Giri, F., Lamnabhi-Lagarrigue, F. (2018). Sampled-data chain-observer design for a class of delayed nonlinear systems. *International Journal of Control* 91(5), 1076-1090.
- [26] Kappel, F., Kunisch, F. and Schappache, W. (1983). *Control Theory for Distributed Parameter Systems and Applications*, Springer-Verlag Berlin Heidelberg.
- [27] Karafyllis I., Kravaris, C. (2009). From continuous-time design to sampled-data design of observers. *IEEE Trans. on Aut. Control* 54 (9), 2169–2174.
- [28] Karafyllis, I., Malisoff, M., Mazenc, F., & Pepe, P. (2016). *Recent results on nonlinear delay control systems*. Switzerland: Springer.
- [29] Khalil, H. K. (2002). *Nonlinear Systems*, Prentice Hall, ISBN: 978-0130673893.
- [30] Khalil, H. K. (2014). *Nonlinear Control*, Pearson.
- [31] Kolmanovskii, V. B., & Myshkis, A. (1992). *Applied theory of functional differential equations*. In *Mathematics and Applications*, Vol. 85. Dordrecht: Kluwer Academy.
- [32] Krstic, M. , Smyshlyaev, A. (2008). Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays. *Systems Control Lett.* 57(9), 750–758.
- [33] Krstic, M. (2009). *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*, Birkhauser, ISBN: 978-0-8176-4698-1.
- [34] Li, X., and Yong, J. (1991). *Control Theory of Distributed Parameter Systems and Applications*, Springer-Verlag Berlin Heidelberg.

## BIBLIOGRAPHY

---

- [35] Liu, K. and Fridman, E. (2012). Wirtinger inequality and lyapunov-based sampled-data stabilization. *Automatica*, 48, 102-108.
- [36] Logan, J.D. (1998). *Applied Partial Differential Equations*. Springer-Verlag New York.
- [37] Lu, S-H. and Fong, I-K. (2000). Stability robustness of linear normal distributed parameter systems. *Systems & Control Letters* 41(5), 317–323.
- [38] Michiels, W., & Niculescu, S.-I. (2014). *Stability, control and computation for time-delay systems: An eigenvalue based approach*. Philadelphia, SIAM Series. *Advances in Design and Control*, DC 27.
- [39] Michiels, W. and Niculescu, S.-I. (2007). *Stability and Stabilization of Time-Delay Systems: An Eigenvalue-Based Approach*, SIAM.
- [40] Olver, P, J. (2014). *Introduction to Partial Differential Equations*. Springer International Publishing, ISBN: 978-3-319-02099-0.
- [41] Raff T., Kögel, M., Allgöwer, F. (2008) Observer with Sample-and-Hold Updating for Lipschitz Nonlinear Systems with Non-uniformly Sampled Measurements. *American Control Conference*, Washington, USA, pp. 5254-5257.
- [42] Richard, J.-P. (2003) Time-delay systems: An overview of some recent advances and open problems, *Automatica* 39(10), 1667–1694.
- [43] Schreier G., DeLeon, J., Glumineau, A., Boisliveau, R. (2001). Cascade nonlinear observers. Application to an experimental induction motor benchmark. *IEE Proc. Control Theory Appl.* 148(6), 509-515.
- [44] Selivanov, A., Fridman, E. (2016). Observer-based input-to-state stabilization of networked control systems with large uncertain delays. *Automatica* 74, 63-70.
- [45] Seuret, A., Gouaisbaut, F. (2013). Jensen’s and Wirtinger’s inequalities for time-delay systems *Proceedings of the 11th IFAC Workshop on Time-Delay Systems*, Grenoble: France, 46(3), 343-348.
- [46] Shim H., Y.I. Son, J.H. Seo. (2001). Semi-global observer for multi-output nonlinear systems. *Systems & Control Letters* 42(3), 233-244.
- [47] Tucsnak, M., and Weiss, G. (2009). *Observation and control for operator semigroups*. Birkhauser.

## BIBLIOGRAPHY

---

- [48] Vande Wouwer, A., Saucez, P. and Vilas Fernández, C. (2014). Simulation of ODE/PDE Models with MATLAB®, OCTAVE and SCILAB. Switzerland: Springer International Publishing.
- [49] Vazquez, R. , Krstic, M. (2007). Control of Turbulent and Magnetohydrodynamic Channel Flows, Birkhäuser.
- [50] Zhong, Q.-C. (2006). Robust Control of Time-delay Systems, Springer-Verlag London.

**Titre : Conception d'observateurs pour différentes classes de systèmes à retards non linéaires.**

**Mots clés :** Systèmes à retards, Observateurs échantillonnés et retardés, Equations aux dérivées partielles, Méthode de Lyapunov, Inégalités matricielles linéaires.

**Résumé :** Le retard est un phénomène naturel présent dans la majorité des systèmes physiques et dans les applications d'ingénierie, ainsi, les systèmes à retard ont été un domaine de recherche très actif en automatique durant les 60 dernières années. La conception d'observateur est un des sujets les plus importants qui a été étudié, ceci est dû à l'importance des observateurs en automatique et dans les systèmes de commande en absence de capteur pour mesurer une variable. Dans ce travail, l'objectif principal est de concevoir des observateurs pour différentes classes de systèmes à retard avec un retard arbitrairement grand, et ce en utilisant différentes approches. Dans la première partie de cette thèse, la conception d'un observateur a été réalisée pour une classe de systèmes non linéaires triangulaires avec une sortie échantillonnée et un retard arbitraire, l'autre difficulté majeure avec cette classe de systèmes est le fait que la matrice d'état dépend du signal de sortie non-retardé qui est immesurable. Un nouvel observateur en chaîne, composé de sous-observateurs en série est conçu pour compenser les retards arbitrairement grands.

Dans la seconde partie de ce travail, un nouvel observateur a été conçu pour un autre type de systèmes non linéaires triangulaires, où le retard a été considéré, cette fois-ci, comme une équation aux dérivées partielles de type hyperbolique du premier ordre. La transformation inverse en backstepping et le concept de l'observateur en chaîne ont été utilisés lors de la conception de cet observateur afin d'assurer son efficacité en cas de grands retards. Dans la dernière partie de cette thèse, la conception d'un nouvel observateur a été réalisée pour un type de système modélisé par des équations paraboliques non linéaires où les mesures sont issues d'un nombre fini de points du domaine spatial. Cet observateur est constitué d'une série de sous-observateurs en chaîne. Chaque sous-observateur compense une fraction du retard global. L'analyse de la stabilité des systèmes d'erreur a été fondée sur différentes fonctionnelles Lyapunov-Krasovskii. Par ailleurs, différents instruments mathématiques ont été employés au cours des différentes preuves présentées. Les résultats de simulation ont été présentés dans le but de confirmer l'exactitude des résultats théoriques

**Title :** Observer design for different classes of nonlinear delayed systems.

**Keywords :** Delayed systems, Sample data and delayed observers, Partial differential equations, Lyapunov method, Linear matrix inequalities.

**Abstract :** Time-delay is a natural phenomenon that is present in most physical systems and engineering applications, thus, delay systems have been an active area of research in control engineering for more than 60 years. Observer design is one of the most important subject that has been dealt with, this is due to the importance of observers in control engineering systems not only when sensing is not sufficient but also when a sensing reliability is needed. In this work, the main goal was to design observers for different classes of nonlinear delayed systems with an arbitrary large delay, using different approaches. In the first part, the problem of observer design is addressed for a class of triangular nonlinear systems with not necessarily small delay and sampled output measurements. Another major difficulty with this class of systems is the fact that the state matrix is dependent on the un-delayed output signal which is not accessible to measurement. A new chain observer, composed of sub-observers in series, is designed to compensate

for output sampling and arbitrary large delays. In the second part of this work, another kind of triangular nonlinear delayed systems was considered, where this time the delay was considered as a first order hyperbolic partial differential equation. The inverse backstepping transformation was invoked and a chain observer was developed to ensure its effectiveness in case of large delays. Finally, a new observer was designed for a class of nonlinear parabolic partial differential equations under point measurements, in the case of large delays. The observer was composed of several chained sub-observers. Each sub-observer compensates a fraction of the global delay. The stability analysis of the error systems were based on different Lyapunov-Krasovskii functionals. Also different mathematical tools have been used in order to prove the results. Simulation results were presented to confirm the accuracy of the theoretical results.

