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Adrien Dekkers

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Analyse mathématique de l'équation de Kuznetsov: problème de
Cauchy, questions d'approximations et problèmes aux bords
fractals

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Contents

Remerciements	ix
Introduction générale	xi
General introduction	xxi
I The Kuznetsov equation and other models of nonlinear acoustic	1
1 Cauchy Problem for the Kuznetsov Equation	3
1.1 Introduction française	3
1.2 Introduction	5
1.3 Preliminary remarks on L^2 -energies	8
1.4 Well-posedness for the inviscid case	11
1.4.1 Proof of Point 3 of Theorem 1.2.1	11
1.4.2 Proof of Point 5 of Theorem 1.2.1. Optimal estimations of the existence time	18
1.4.3 Proof of Point 4 of Theorem 1.2.1. Stability and uniqueness result	21
1.5 Well-posedness for the viscous case	22
1.5.1 Proof of Point 1 of Theorem 1.2.2	22
1.5.2 Proof of Point 2 of Theorem 1.2.2: Case $n \geq 3$	27
1.6 Illustration of the sharp behaviour of Point 1 in Theorem 1.2.2	36
2 Models of nonlinear acoustics viewed as an approximation of the Navier-Stokes and Euler compressible isentropic systems	39
2.1 Introduction française	39
2.2 Introduction	42
2.3 Isentropic Navier-Stokes system for a subsonic potential motion	44
2.4 Approximation of the Navier-Stokes system	46
2.4.1 Navier-Stokes system and the Kuznetsov equation	46
2.4.2 Navier-Stokes system and the KZK equation	56
2.4.3 Navier-Stokes system and the NPE equation	68
2.5 Approximation of the Euler system	73
2.6 The Kuznetsov equation and the KZK equation	79
2.6.1 Derivation of the KZK equation from the Kuznetsov equation	79
2.6.2 Well posedness of the models	80
2.6.3 Approximation of the solutions of the Kuznetsov equation by the solutions of the KZK equation	87

2.7	Approximation of the solutions of the Kuznetsov equation with the solutions of the NPE equation	93
2.8	The Kuznetsov equation and the Westervelt equation	95
2.8.1	Derivation of the Westervelt equation from the Kuznetsov equation	95
2.8.2	Approximation of the solutions of the Kuznetsov equation by the solutions of the Westervelt equation	95
2.9	Summary	97
II Propagation of linear and nonlinear waves in domains with fractal boundaries		101
3	Introduction to Part II	103
3.1	Introduction française	103
3.2	Introduction	104
4	Regularity of linear models on domains with fractal boundaries	107
4.1	First results and notations	107
4.1.1	Admissible domains	107
4.1.2	The Poisson equation with Dirichlet boundary conditions	109
4.1.3	The Poisson equation with homogeneous Robin boundary conditions	115
4.2	Well posedness of the linear wave equation with homogeneous Dirichlet boundary conditions	117
4.2.1	Existence of weak solution	117
4.2.2	Regularity results	119
4.3	Well posedness of the damped linear wave equation with homogeneous Dirichlet boundary conditions	123
4.3.1	Existence of weak solution	123
4.3.2	Regularity results	127
4.4	Well posedness of the damped linear wave equation with homogeneous Robin boundary conditions	131
5	Well-posedness and regularity of the non linear Westervelt equation on domains with fractal boundaries	133
5.1	Well-posedness of the Westervelt equation with homogeneous Dirichlet boundary conditions	133
5.2	Well posedness of the Westervelt equation with non homogeneous Dirichlet boundary conditions	137
5.3	Well posedness of the Westervelt equation with homogeneous Robin boundary conditions	140
5.3.1	Well posedness of the Westervelt equation with homogeneous Robin boundary condition on Lipschitz domains	141
5.3.2	Well posedness of the Westervelt equation with homogeneous Robin boundary conditions on admissible domains	144
5.4	Prefractal and fractal problem for the Westervelt equation on Koch Mixtures with Robin boundary conditions	146
5.4.1	Scale irregular Koch curves	146
5.4.2	Trace and extension theorems	148

5.4.3	Asymptotic analysis	149
5.4.4	Linear and Nonlinear problems with mixed boundary condition . . .	157
A	Expressions of the remainder terms	167
B	Résumé français	169
	Bibliography	179

List of Figures

1	Schéma de dérivation des modèles de l'acoustique non linéaire. Tous les modèles, les équations de Kuznetsov, KZK et NPE sont des approximations jusqu'aux termes d'ordre ϵ^3 du système isentropique de Navier-Stokes ou d'Euler.	xiii
2.1	Paraxial change of variables for the profiles $U(t - x_1/c, \epsilon x_1, \sqrt{\epsilon} \mathbf{x}')$	57
2.2	Periodic subsonic inflow-outflow boundary conditions for the Navier-Stokes system.	62
2.3	Paraxial change of variables for the profiles $U(\epsilon t, x_1 - ct, \sqrt{\epsilon} \mathbf{x}')$	69

List of Tables

- 2.1 Approximation results for models derived from Navier-Stokes and Euler systems 98
- 2.2 Approximation results for models derived from the Kuznetsov equation . . . 99

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Introduction générale

L'étude de la propagation d'ondes non-linéaires suscite un intérêt, en particulier à cause de récentes applications à l'imagerie ultrason (*i.e.* HIFU) ou des applications techniques et médicales comme la lithotripsie ou la thérapie. Ces nouvelles techniques reposent fortement sur la capacité à modéliser avec précision la propagation non-linéaire d'une pulsation sonore d'amplitude finie dans un milieu élastique thermo-visqueux. Les modèles les plus connus d'acoustique non linéaire, que nous considérerons dans cette thèse sont

1. l'équation de Kuznetsov qui se lit pour $\alpha = \frac{\gamma-1}{c^2}$, $\beta = 2$ comme

$$u_{tt} - c^2 \Delta u - \frac{\nu}{\rho_0} \varepsilon \Delta u_t = \alpha \varepsilon u_t u_{tt} + \beta \varepsilon \nabla u \cdot \nabla u_t, \quad x \in \mathbb{R}^n, \quad (1)$$

où c , ρ_0 , γ , ν sont la vitesse du son, la densité, le ratio des chaleurs spécifiques et la viscosité du milieu respectivement. Le coefficient ε représente un petit paramètre sans dimension apparaissant dans la dérivation de l'équation. Dans ce qui suit, nous pouvons juste supposer que α et β sont des constantes positives. C'est en fait une équation d'onde quasi-linéaire (amortie), initialement introduite par Kuznetsov [60] pour le potentiel de vitesse, voir aussi les Réfs. [37, 50, 55, 63] pour d'autres variations de sa dérivation;

2. l'équation de Khokhlov-Zabolotskaya-Kuznetsov (KZK)

$$c \partial_{\tau z}^2 I - \frac{(\gamma + 1)}{4\rho_0} \partial_\tau^2 I^2 - \frac{\nu}{2c^2 \rho_0} \partial_\tau^3 I - \frac{c^2}{2} \Delta_y I = 0, \quad (2)$$

qui peut être écrite pour les perturbations de la densité ou de la pression (voir les études physiques systématiques dans le livre [13] par Bakhvalov, Zhileikin, et Zabolotskaya);

3. l'Équation d'onde Non-linéaire Progressive (NPE)

$$\partial_{\tau z}^2 \xi + \frac{(\gamma + 1)c}{4\rho_0} \partial_z^2 [(\xi)^2] - \frac{\nu}{2\rho_0} \partial_z^3 \xi + \frac{c}{2} \Delta_y \xi = 0, \quad (3)$$

dérivée par McDonald et Kuperman dans la Réf. [70];

4. l'équation de Westervelt

$$\partial_t^2 \Pi - c^2 \Delta \Pi = \varepsilon \partial_t \left(\frac{\nu}{\rho_0} \Delta \Pi + \frac{\gamma + 1}{2c^2} (\partial_t \Pi)^2 \right), \quad (4)$$

qui est similaire à l'équation de Kuznetsov avec seulement un de ses deux termes non-linéaires, dérivée initialement par Westervelt [91] et plus tard par d'autres auteurs [1, 89].

L'équation de Kuznetsov (1) décrit l'évolution du potentiel de vitesse, c'est une équation d'onde quasi linéaire amortie, qui décrit la propagation d'une onde de grande amplitude dans un fluide. Elle est un des modèles dérivés du système de Navier-Stokes, et elle est appropriée pour les ondes planes, cylindriques et sphériques dans un fluide (voir [37] de Hamilton et Blackstock). La plupart des travaux sur l'équation de Kuznetsov (1) sont traités dans une dimension d'espace [50] ou dans un domaine borné de \mathbb{R}^n [55, 52, 53, 71]. Pour le cas visqueux, Kaltenbacher et Lasiecka [53] ont considéré le problème avec conditions de Dirichlet au bord et prouvé, pour des données initiales suffisamment petites, le caractère bien posé global pour $n \leq 3$. Meyer et Wilke [71] l'ont prouvé pour tout n . Dans [52], Kaltenbacher et Lasiecka ont prouvé le caractère bien posé local du problème avec conditions au bord de Neumann pour $n \leq 3$. Le travail des Réf. [52, 53] utilise des estimations d'énergie a priori, et la Réf. [71] la notion de régularité maximale.

L'équation de Westervelt (4) est aussi une approximation de l'équation de Kuznetsov, mais cette fois par une perturbation non-linéaire. De fait la seule différence entre ces deux modèles est que l'équation de Westervelt ne conserve qu'un des deux termes non-linéaires de l'équation de Kuznetsov, produisant des effets cumulatifs dans une propagation d'onde progressive selon Aanonsen, Barkve, Tjøtta et Tjøtta [1].

L'équation NPE (3) est habituellement utilisée pour décrire les vibrations en temps court et la propagation sur de longues distances, par exemple dans un guide d'onde océanique, où les phénomènes de réfractons sont importants, alors que l'équation de KZK (2) modélise typiquement la propagation d'ultrasons avec de forts phénomènes de diffraction, combinée avec des effets d'amplitude finie (voir Rozanova-Pierrat avec la Réf. [81] et les références utilisées). Bien que le contexte et l'utilisation physique des équations de KZK et NPE soient différents, il y a une bijection entre les variables de ces deux modèles et ils peuvent être représentés par le même type d'opérateur différentiel avec des coefficients constants positifs:

$$Lu = 0, \quad L = \partial_{tx}^2 - c_1 \partial_x (\partial_x \cdot)^2 - c_2 \partial_x^3 \pm c_3 \Delta_y, \quad \text{pour } t \in \mathbb{R}^+, x \in \mathbb{R}, y \in \mathbb{R}^{n-1}.$$

Ainsi, les résultats de la Réf. [80] sur les solutions de l'équation de KZK sont valides pour l'équation NPE. Voir aussi la Réf. [42] par Ito pour la décroissance exponentielle des solutions de ces modèles dans le cas visqueux.

Tous les modèles de Kuznetsov, KZK, NPE et Westervelt ont été dérivés jusqu'à de petits termes négligeables à partir de systèmes non-linéaires de Navier-Stokes (pour le milieu visqueux) et d'Euler (pour le cas non visqueux) compressibles et isentropiques. Mais toutes les dérivations physiques citées de ces modèles ne permettent pas de dire que leurs solutions approchent la solution du système de Navier-Stokes ou d'Euler. Les résultats sur le caractère bien posé des équations de KZK et NPE sont déjà connus, le premier travail l'expliquant pour l'équation de KZK est la Réf. [81] par Rozanova-Pierrat.

Nous nous sommes dès lors focalisés dans le Chapitre 1 sur le caractère bien posé du problème de Cauchy associé à l'équation de Kuznetsov dans \mathbb{R}^n pour les cas visqueux et non visqueux avec des données initiales suffisamment petites. Ces résultats correspondent à notre article [26] proposé avec Rozanova-Pierrat.

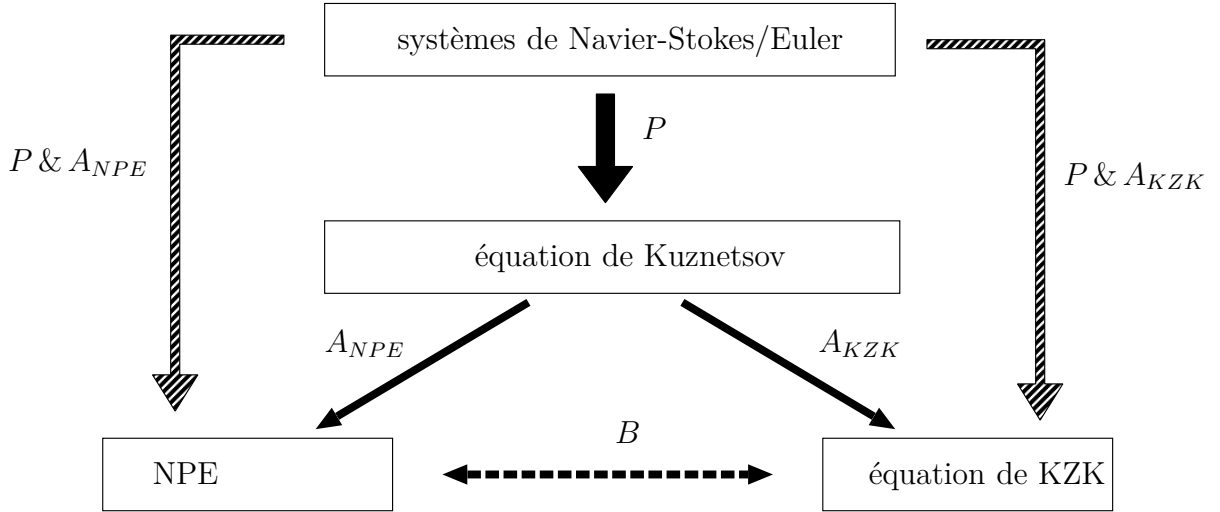
Dans le Chapitre 2, nous commençons à présenter le contexte initial du système de Navier-Stokes isentropique

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0, \tag{5}$$

$$\rho[\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}] = -\nabla p(\rho) + \varepsilon \nu \Delta \mathbf{v}, \tag{6}$$

(en fait, c'est aussi une approximation du système de Navier-Stokes compressible), qui décrit le mouvement d'une onde acoustique dans un milieu thermo-élastique homogène [13, 37, 65]. Nous systématisons dans le Chapitre 2 la dérivation de tous ces modèles en utilisant les idées de Rozanova-Pierrat dans la Réf. [81], consistant à utiliser des correcteurs dans les expansions de type Hilbert des *ansatzs* physiques correspondants.

Plus précisément, nous montrons que tous ces modèles sont des approximations du système de Navier-Stokes ou d'Euler jusqu'aux termes d'ordre trois en un petit paramètre sans dimension $\epsilon > 0$ mesurant la taille des perturbations de la pression, de la densité et de la vitesse par rapport à leur état constant $(p_0, \rho_0, 0)$ (voir Fig. 1).



P : petites perturbations (2.14)–(2.15)

A_{KZK} : approximation paraxiale de KZK (Fig. 2.1) B : bijection (2.95)

A_{NPE} : approximation paraxiale de NPE (Fig. 2.3)

Figure 1 – Schéma de dérivation des modèles de l'acoustique non linéaire. Tous les modèles, les équations de Kuznetsov, KZK et NPE sont des approximations jusqu'aux termes d'ordre ϵ^3 du système isentropique de Navier-Stokes ou d'Euler.

A l'aide des résultats connus sur le caractère bien posé des modèles, nous validons ensuite dans le Chapitre 2 ces approximations en obtenant des estimations en norme L^2 entre les solutions des modèles exacts et approchés considérés en étudiant d'abord l'approximation du système de Navier-Stokes puis l'approximation de l'équation de Kuznetsov. Il est à noter que pour le modèle exact nous pouvons considérer une solution faible peu régulière qui sera approchée par la solution régulière du modèle approché.

Ainsi nous avons été amenés dans la Partie II à étudier les solutions faibles d'équations d'ondes sur des domaines à bords fractals afin de considérer les domaines les plus généraux possibles sur lesquels de telles solutions faibles existent.

Pour en revenir au Chapitre 1 nous étudions le caractère bien posé du problème de Cauchy associé à l'équation (1). Dans le cas non visqueux pour $\nu = 0$, le problème de Cauchy pour l'équation de Kuznetsov est un cas particulier du système général quasi linéaire hyperbolique du second ordre considéré par Hughes, Kato et Marsden [40]. Le résultat de caractère bien posé local, prouvé dans [40], n'utilise pas des techniques d'estimations a priori

et est fondé sur la théorie des semi-groupes. Alors, grâce à [40], nous avons le caractère bien posé de (1) dans l'espace de Sobolev H^s avec un réel $s > \frac{n}{2} + 1$. De plus, pour étendre le caractère bien posé local au cas global (pour $n \geq 4$) et pour estimer l'intervalle de temps maximal sur lequel il existe une solution régulière, John [44] a développé des estimations a priori pour le problème de Cauchy associé à une équation d'onde quasi linéaire générale à l'aide d'une énergie de la forme

$$E_m[u](t) = \|\nabla u(t)\|_{H^m(\mathbb{R}^n)}^2 + \sum_{i=1}^{m+1} \|\partial_t^i u(t)\|_{H^{m+1-i}(\mathbb{R}^n)}^2.$$

Cette fois, à cause des non linéarités $u_t u_{tt}$ et $\nabla u \nabla u_t$ incluant les dérivées en temps, pour avoir une estimation a priori pour l'équation de Kuznetsov nous avons besoin de travailler avec les espaces de Sobolev H^s caractérisés par un entier s . Si nous appliquons directement les résultats généraux par John de la Réf. [44] à notre cas pour l'équation de Kuznetsov, nous obtenons le caractère bien posé pour des données initiales très régulières. Nous améliorons ce résultat et obtenons les résultats de John pour l'équation de Kuznetsov avec une régularité minimale des données initiales correspondant à la régularité obtenue par Hughes, Kato et Marsden [40]. Les estimations d'énergie nous permettent d'évaluer le temps d'existence maximal. Dans \mathbb{R}^2 et \mathbb{R}^3 l'optimalité des estimations obtenues pour le temps d'existence maximal est assurée par les résultats d'Alinhac [5]. Dans la Réf. [5] un blow-up géométrique pour les données petites est prouvé pour $\partial_t^2 u$ et Δu en temps fini et pour le même ordre que prédit par les estimations a priori.

Pour $n \geq 4$ et $\nu = 0$, nous améliorons aussi les résultats de John [44]. La petitesse des données initiales assure directement l'hyperbolicité de l'équation de Kuznetsov pour tout temps, *i.e.* elle assure que $1 - \alpha \epsilon u_t$ est strictement positif et borné pour tout temps. La preuve utilise les dérivées généralisées pour les équations d'ondes [44] et une estimation a priori de Klainerman [58, 59].

En présence du terme Δu_t pour le cas visqueux $\nu > 0$, la régularité des dérivées en temps d'ordre supérieur de u est différente (en comparaison au cas non visqueux), et la manière de contrôler les non linéarités change. Comme il a été montré dans [83] par Shibata, ce terme dissipatif change une vitesse finie de propagation pour l'équation d'onde en une vitesse infinie. En effet, la partie linéaire de l'équation (1) peut être vue comme deux compositions de l'opérateur de la chaleur $\partial_t - \Delta$ de la manière suivante:

$$u_{tt} - c^2 \Delta u - \nu \epsilon \Delta u_t = \partial_t (\partial_t u - \epsilon \nu \Delta u) - c^2 \Delta u.$$

Pour le cas visqueux nous prouvons les résultats sur le caractère bien posé global dans \mathbb{R}^n pour les données initiales suffisamment petites, dont nous spécifions la taille. Pour $n \geq 3$ nous établissons une estimation a priori qui nous donne aussi une condition suffisante pour l'existence de solutions globales avec une énergie initiale suffisamment petite. En considérant les espaces de Sobolev H^s caractérisés par un entier $s = m$ pair on contrôle l'énergie

$$E_{\frac{m}{2}}[u](t) = \|\nabla u(t)\|_{H^m(\mathbb{R}^n)}^2 + \sum_{i=1}^{\frac{m}{2}+1} \|\partial_t^i u(t)\|_{H^{m-2(i-1)}(\mathbb{R}^n)}^2.$$

Les mêmes résultats sont vrais dans $(\mathbb{R}/L\mathbb{Z}) \times \mathbb{R}^{n-1}$ pour $n \geq 2$ avec la périodicité et la valeur moyenne nulle selon une variable.

Intéressons nous dès lors au Chapitre 2. Comme il est montré dans la Fig. 1, l'équation de Kuznetsov vient du système de Navier-Stokes ou d'Euler seulement par de petites perturbations, mais pour obtenir les équations KZK et NPE nous avons besoin d'utiliser un

changement de variables paraxial en plus des petites perturbations. En outre, les équations de KZK et NPE peuvent aussi être obtenues à partir de l'équation de Kuznetsov juste en pratiquant le changement de variable paraxial correspondant. Nous pouvons noter que l'équation de Kuznetsov est une équation d'onde non-linéaire contenant des termes d'ordres différents en ϵ . Mais les approximations paraxiales pour KZK et NPE permettent d'avoir les équations approchées avec tous les termes de même ordre, *i.e.* les équations de KZK et NPE.

Portons notre attention sur le fait que l'*ansatz*, proposé initialement par Bakhvalov, Zhileikin, et Zabolotskaya dans la Réf. [13] et utilisé par Rozanova-Pierrat dans la Réf. [81] pour obtenir l'équation de KZK à partir des systèmes de Navier-Stokes ou d'Euler, est différent de l'*ansatz* que nous utilisons. De plus, cette nouvelle approximation des systèmes de Navier-Stokes et d'Euler est une amélioration en comparaison à la dérivation développée dans la Réf. [81], car dans cette référence le système de Navier-Stokes/Euler pouvait seulement être approchées jusqu'aux termes d'ordre $O(\epsilon^{\frac{5}{2}})$ (comparé à l'ordre $O(\epsilon^3)$ dans notre cas).

Les hypothèses principales pour la dérivation de tous ces modèles sont les suivantes:

- le mouvement est potentiel;
- l'état constant du milieu donné par $(p_0, \rho_0, 0)$ (0 pour la vitesse) est perturbé proportionnellement à un paramètre sans dimension $\epsilon > 0$ (par exemple, égal à 10^{-5} dans l'eau avec une puissance initiale de l'ordre de 0.3 W/cm^2);
- toutes les viscosités sont petites (d'ordre ϵ).

Pour garder le sens physique des problèmes d'approximation, nous considérerons particulièrement les cas bidimensionnel et tridimensionnel, *i.e.* \mathbb{R}^n avec $n = 2$ ou 3 , et dans la suite nous utiliserons la notation $x = (x_1, x') \in \mathbb{R}^n$ avec un axe $x_1 \in \mathbb{R}$ et la variable transversale $x' \in \mathbb{R}^{n-1}$.

Nous validons ainsi les approximations du système de Navier-Stokes compressible par les différents modèles : par l'équation de Kuznetsov, l'équation de KZK et l'équation NPE.

Puis nous faisons de même pour le système d'Euler dans le cas non visqueux. Les différences principales entre les cas visqueux et non visqueux sont le temps d'existence et la régularité des solutions. Typiquement dans le cas non visqueux, les solutions des modèles et aussi du système d'Euler lui-même (solutions fortes) peuvent entraîner la formation de fronts de choc en temps finis à cause de leurs non-linéarités [4, 26, 80, 84, 93]. Ainsi, elles sont seulement localement bien posées, alors que dans le cas visqueux tous les modèles d'approximations sont globalement bien posés pour des données initiales suffisamment petites [26, 67, 80].

Nous notons par \mathbf{U}_ϵ une solution du système de Navier-Stokes/Euler "exact" (voir l'Eq. (2.31))

$$\partial_t \mathbf{U}_\epsilon + \sum_{i=1}^n \partial_{x_i} \mathbf{G}_i(\mathbf{U}_\epsilon) - \epsilon \nu \begin{bmatrix} 0 \\ \Delta \mathbf{v}_\epsilon \end{bmatrix} = 0,$$

et par $\overline{\mathbf{U}}_\epsilon$ une solution approchée, construite par l'*ansatz* de dérivation à partir d'une solution régulière de l'un des modèles approchés (typiquement les équations de Kuznetsov, KZK et NPE), *i.e.* une fonction qui résout le système de Navier-Stokes/Euler jusqu'aux termes d'ordre ϵ^3 , désignés par $\epsilon^3 \mathbf{R}$ (voir l'Eq. (2.32)):

$$\partial_t \overline{\mathbf{U}}_\epsilon + \sum_{i=1}^n \partial_{x_i} \mathbf{G}_i(\overline{\mathbf{U}}_\epsilon) - \epsilon \nu \begin{bmatrix} 0 \\ \Delta \overline{\mathbf{v}}_\epsilon \end{bmatrix} = \epsilon^3 \mathbf{R}.$$

Pour avoir le terme de reste $\mathbf{R} \in C([0, T], L^2(\Omega))$ nous devons assurer que le terme de gauche de cette équation est dans $C([0, T], L^2(\Omega))$, *i.e.* nous avons besoin d'une solution $\bar{\mathbf{U}}_\varepsilon$ suffisamment régulière. La régularité minimale des données initiales pour avoir un tel $\bar{\mathbf{U}}_\varepsilon$ est donnée dans le Tableau 2.1 (voir aussi le Tableau 2.2 pour l'approximation de l'équation de Kuznetsov).

En choisissant pour le système exact les mêmes données initiales et au bord trouvées par l'*ansatz* pour $\bar{\mathbf{U}}_\varepsilon$ (le cas régulier) ou les données initiales prises dans un petit voisinage L^2 , *i.e.*

$$\|\mathbf{U}_\varepsilon(0) - \bar{\mathbf{U}}_\varepsilon(0)\|_{L^2(\Omega)} \leq \delta \leq \epsilon,$$

avec $\mathbf{U}_\varepsilon(0)$ non nécessairement régulier, mais assurant l'existence d'une solution faible admissible d'énergie bornée, nous prouvons l'existence de constantes $C > 0$ et $K > 0$ indépendantes de ε , δ et du temps t telles que

$$\text{pour tout } 0 \leq t \leq \frac{C}{\varepsilon} \quad \|(\mathbf{U}_\varepsilon - \bar{\mathbf{U}}_\varepsilon)(t)\|_{L^2(\Omega)}^2 \leq K(\epsilon^3 t + \delta^2)e^{Ket} \leq 9\epsilon^2$$

avec Ω un domaine où les deux solutions \mathbf{U}_ε et $\bar{\mathbf{U}}_\varepsilon$ existent. Il devient ainsi possible d'approcher une solution faible exacte peu régulière par une solution approchée régulière.

Comme les équations de KZK et NPE peuvent être vues comme des approximations de l'équation de Kuznetsov au vu de leur dérivation (voir la Figure 1), nous validons aussi l'approximation de l'équation de Kuznetsov par les équations de KZK et NPE, et aussi par l'équation de Westervelt (voir le Tableau 2.2).

Pour être capable de considérer l'approximation de l'équation de Kuznetsov par l'équation de KZK, nous établissons d'abord des résultats sur le caractère globalement bien posé de l'équation de Kuznetsov dans le demi espace, similaires au cadre précédent pour l'équation de KZK et le système de Navier-Stokes. Nous étudions deux cas : le problème périodique en temps purement aux bords dans les variables (z, τ, y) se déplaçant avec l'onde et le problème avec conditions initiales et au bord pour l'équation de Kuznetsov dans les variables initiales (t, x_1, x') avec des données venant de la solution de l'équation de KZK. Nous validons ces deux types d'approximations pour les cas visqueux et non visqueux.

Finalement nous validons l'approximation entre les équations de Kuznetsov et NPE et les équations de Kuznetsov et Westervelt respectivement (voir le Tableau 2.2). Nous pouvons les résumer de la manière suivante: si u est une solution de l'équation de Kuznetsov et \bar{u} est une solution de l'équation de NPE ou de KZK (pour le problème avec conditions initiales et aux bords) ou de Westervelt trouvée pour des données initiales assez proches

$$\|\nabla_{t,\mathbf{x}}(u(0) - \bar{u}(0))\|_{L^2(\Omega)} \leq \delta \leq \epsilon,$$

alors il existe $K > 0$, $C_1 > 0$, $C_2 > 0$ et $C > 0$ constantes indépendantes de ϵ , δ et du temps, telles que pour tout $t \leq \frac{C}{\epsilon}$ il est vérifié

$$\|\nabla_{t,\mathbf{x}}(u - \bar{u})\|_{L^2(\Omega)} \leq C_1(\epsilon^2 t + \delta)e^{C_2 \epsilon t} \leq K\epsilon.$$

Comme les estimations de la stabilité obtenues sont valables entre une solution régulière et une solution faible de Kuznetsov nous pouvons de nouveau approcher une solution moins régulière d'un modèle exact par la solution régulière d'un modèle approché.

Dans la Partie II, nous nous intéressons à la question des solutions faibles d'équation d'ondes. On se place dans le contexte des domaines bornés et on cherche la classe des bords la plus large pour que le problème soit bien posé faiblement. Ces équations incluent:

- l'équation des ondes avec des conditions de Dirichlet homogène en utilisant Evans [30],
- l'équation des ondes fortement amortie avec des conditions de Dirichlet homogènes et non homogènes ou des conditions de Robin homogènes,
- l'équation non-linéaire de Westervelt avec des conditions de Dirichlet homogènes et non homogènes ou des conditions de Robin homogènes.

La régularité des solutions de ces équations sur des domaines réguliers, typiquement avec un bord C^2 est bien connue, notamment le fait que, plus les données initiales sont régulières, plus la solution est régulière et ce jusqu'au bord. Nous pouvons citer Evans et la Réf. [30] pour l'équation des ondes ou les Réfs. [51, 52, 53, 54, 71] pour l'équation des ondes fortement amortie ou l'équation de Westervelt ainsi que leurs références utilisées. Nous pouvons nous demander si, sur des domaines moins réguliers, on peut avoir une solution faible continue ou C^1 jusqu'au bord. Les exemples de Arendt et Elst dans la Réf. [10] montrent l'apparition de problèmes pour la définition de la trace dès que le bord n'est plus C^1 . De plus si, pour un domaine au bord C^1 ou lipschitzien, on peut définir une normale intérieure presque partout, la question des conditions de Neumann ou Robin sur un bord moins régulier est plus délicate. Par ailleurs le fait de considérer un bord régulier C^2 comme dans [51, 52, 53, 54, 71] est une conséquence de ce que les dérivées spatiales sont au plus d'ordre 2 et peuvent ainsi être plus naturellement définies au bord. Dans le passé, les mathématiques se sont largement focalisées sur des domaines réguliers. Des ensembles comme celui de Von Koch ont principalement été considérés comme "pathologiques" et utilisés seulement pour produire des contre-exemples. Néanmoins, il y a eu un changement d'attitude lorsque les mathématiciens et les physiciens ont découvert que des structures semblable à celle de Von Koch apparaissaient dans la nature, comme par exemple la micro-structure des électrodes ou les côtes de l'Angleterre.

Un point clé pour résoudre les équations que nous étudierons sur des domaines à bords fractals est la compréhension du problème de Poisson sur ces domaines avec des conditions aux bords de Dirichlet

$$\begin{cases} -\Delta u = f \text{ sur } \Omega, \\ u|_{\Omega} = g \text{ sur } \partial\Omega \end{cases} \quad (7)$$

ou des conditions de Robin homogènes

$$\begin{cases} -\Delta u = f \text{ sur } \Omega, \\ \frac{\partial}{\partial n} u + au = 0 \text{ avec } a > 0 \text{ sur } \partial\Omega. \end{cases} \quad (8)$$

Pour le système (7) une approche générale passe par la formulation faible du problème de Dirichlet. Si u et $\partial\Omega$ sont suffisamment régulières on peut multiplier l'équation de Poisson dans le problème (7) par $v \in C_0^\infty(\Omega)$ et utiliser la formule de Green pour obtenir

$$\begin{cases} \int_{\Omega} \nabla u \nabla v \, dx = \int_{\Omega} f v \, dx \text{ for all } v \in C_0^\infty(\Omega), \\ u|_{\partial\Omega} = g, \end{cases}$$

qui est appelée une formulation faible du problème de Dirichlet. En introduisant les espaces de Sobolev $H^1(\Omega)$ et $H_0^1(\Omega)$ et en supposant qu'il existe $g^* \in H^1(\Omega)$ tel que la trace de g^* sur $\partial\Omega$ est g (une attention particulière doit être portée à la définition de la trace), on peut prouver, à l'aide du théorème de représentation de Riesz, qu'étant donné $f \in L^2(\Omega)$, $g^* \in H^1(\Omega)$, il existe un unique $u \in H^1(\Omega)$ tel que $-\Delta u = f$ au sens des distributions et $u - g^* \in H_0^1(\Omega)$.

Ceci soulève plusieurs questions:

- Comment définir la trace, habituellement définie pour des fonctions continues?
- Comment définir une extension g^* vérifiant $u = g$ au bord?

La réponse aux deux premières questions est connue si $\partial\Omega$ est assez régulier, on peut citer par exemple Raviart-Thomas [79], ou même lipschitzien avec le travail de Marschall [66].

- Peut-on utiliser la formule de Green? Dans le cas lipschitzien on a

$$\int_{\Omega} v \Delta u \, dx = \langle u, v \rangle_{(H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega))} - \int_{\Omega} \nabla u \nabla v \, dx.$$

- Est ce que u dépend uniquement ou continûment de f et g ?

Dans la Réf. [49] Jonsson et Wallin ont pu répondre à ces questions dans le cas où Ω est un (ϵ, δ) -domaine avec un bord $\partial\Omega$ qui est un d -ensemble pour la mesure de Hausdorff préservant l'inégalité de Markov. En se basant sur le travail de Lancia [62] on trouve un équivalent de la formule de Green faisant intervenir les espaces de Besov pour le terme de bord. Les résultats de Jonsson et Wallin sont à notre connaissance les premiers de ce type établis sur des domaines fractals. Les résultats de Jones [46] sur les d -ensembles et les domaines admettant des extensions W_p^k permettent de dire qu'en dimension 2 les (ϵ, δ) -domaines sont les domaines les plus généraux sur lesquels on peut définir des traces et des extensions des espaces de Sobolev et ainsi résoudre le problème de Poisson. Dans la Réf. [11], Arfi et Rozanova-Pierrat ont introduit un nouveau type de domaine à bords fractals dits les domaines admissibles. Ces domaines contiennent les (ϵ, δ) -domaines et sont plus généraux, ils forment la classe la plus large des domaines sur lesquels on peut définir des traces et des extensions aux espaces de Sobolev pour $\Omega \subset \mathbb{R}^n$ avec $n \geq 2$, et ainsi trouver une solution faible au problème de Poisson dépendant de manière unique et continue des données initiales.

En conséquence nous travaillerons principalement sur les domaines admissibles et résumons les résultats connus sur ces domaines. Il est à noter que le travail de la Réf. [30] par Evans nous fournit les propriétés spectrales ainsi que la régularité intérieure de la solution du problème de Poisson (7), *i.e.* le fait que pour un sous ensemble V inclus de manière compacte dans Ω , $V \subset\subset \Omega$, la solution sur Ω a sur V la même régularité que pour un domaine aux bords réguliers. La Réf. [11] par Arfi et Rozanova-Pierrat permet de donner des résultats similaires pour le problème de Poisson (8) et la Réf. [30] par Evans nous fournit encore les propriétés spectrales.

Une autre question importante est de savoir si les solutions des problèmes de Poisson (7) et (8) appartiennent à $C(\overline{\Omega})$ avec une estimation de la forme:

$$\|u\|_{L^\infty(\Omega)} \leq C \|f\|_{L^p(\Omega)}.$$

Pour le problème de Poisson (7) avec des conditions au bord de Dirichlet homogènes les travaux des Réfs. [77] par Nyström et [92] par Xie permettent de donner une réponse positive à cette questions en dimension $n = 2$ et 3 respectivement pour $p = 2$. Le travail de Daners dans la Réf. [25] nous donne aussi une réponse positive pour le problème de Poisson (8) si $p > n$. Ces estimations sont essentielles pour montrer que les solutions de nos modèles de type ondulatoires étudiés sont dans $C(\overline{\Omega})$ mais aussi pour traiter la non-linéarité de l'équation de Westervelt.

En utilisant une méthode de Galerkin comme dans la Réf. [30] par Evans nous obtenons la régularité de l'équation des ondes et de l'équation des ondes fortement amortie avec des conditions de Dirichlet homogènes avec l'aide d'une base de fonctions propres de $-\Delta$. Avec ces résultats de régularité nous traitons le caractère bien posé de l'équation de Westervelt avec des conditions de Dirichlet de la même façon que dans la preuve dans le Chapitre 1 du caractère bien posé global de l'équation de Kuznetsov sur \mathbb{R}^n . Les propriétés de la trace et de l'extension pour les domaines admissibles rappelées nous ont permis de traiter le cas des conditions de Dirichlet non homogènes. Ces résultats reposent sur des estimations dans des espaces où la solution et certaines de ses dérivées sont dans L^2 . Notons que nous avons utilisé une méthode similaire pour les problèmes avec conditions de Robin homogènes et obtenu le caractère bien posé et des estimations L^2 pour l'équation des ondes fortement amortie sur un domaine admissible, avec une méthode de Galerkin fondée sur une base de fonctions propres de $-\Delta$, ou pour l'équation de Westervelt sur un domaine lipschitzien de la même façon que dans la preuve dans le Chapitre 1 du caractère bien posé global de l'équation de Kuznetsov sur \mathbb{R}^n . Le cas de l'équation de Westervelt sur un domaine admissible avec des conditions de Robin homogènes a été traité à l'aide d'estimations L^p avec $p > n$ de la même manière.

En conclusion de cette Partie II, nous considérons un ensemble à bord fractal de type mixture de Koch, construit par récurrence à l'aide de familles de similitudes contractantes induisant ainsi une famille de domaines à bords pré-fractals et lipschitziens convergeant vers le domaine à bords fractals. En utilisant différents travaux par Capitanelli [19], Capitanelli et Vivaldi [20] ou Lancia [62] nous avons pu considérer la convergence asymptotique de type Mosco des solutions de l'équation de Westervelt avec conditions de Robin sur les domaines à bords pré-fractals qui approximent la solution sur le domaine à bords fractal de type mixture de Koch, une démarche souvent utilisée dans le cadre de l'optimisation de forme.

General introduction

There is a renewed interest in the study of nonlinear wave propagation, in particular because of recent applications to ultrasound imaging (*i.e.* HIFU) or technical and medical applications such as lithotripsy or thermotherapy. Such new techniques rely heavily on the ability to model accurately the nonlinear propagation of a finite-amplitude sound pulse in thermo-viscous elastic media. The most known nonlinear acoustic models, which we consider in this thesis, are :

1. the Kuznetsov equation (see Eq. (1)). It is actually a quasi-linear (damped) wave equation, initially introduced by Kuznetsov [60] for the velocity potential, see also Refs. [37, 50, 55, 63] for other different variations of its derivation;
2. the Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation (see Eq. (2)), which can be written for the perturbations of the density or of the pressure (see the systematic physical studies in the book[13] by Bakhvalov, Zhileïkin, et Zabolotskaya);
3. the Nonlinear Progressive wave Equation (NPE) (see Eq. (3)) derived by McDonald and Kuperman in Ref. [70];
4. the Westervelt equation (see Eq. (4)), which is similar to the Kuznetsov equation with only one of two nonlinear terms, derived initially by Westervelt[91] and later by other authors[1, 89].

The Kuznetsov equation 1 describes the evolution of the velocity potential, it is a weakly quasi-linear damped wave equation, that describes a propagation of a high amplitude wave in fluids. It is one of the models derived from the Navier-Stokes system, and it is well suited for the plane, cylindrical and spherical waves in a fluid(see [37] from Hamilton and Blackstock). Most of the works on the Kuznetsov equation (1) are treated in the one dimensional space [50] or in a bounded spatial domain of \mathbb{R}^n [52, 53, 55, 71]. For the viscous case Kaltenbacher and Lasiecka [53] have considered the Dirichlet boundary valued problem and proved for sufficiently small initial data the global well-posedness for $n \leq 3$. Meyer and Wilke [71] have proved it for all n . In [52] Kaltenbacher and Lasiecka have proved the local well-posedness of the Neumann boundary valued problem for $n \leq 3$. The work in Refs [52, 53] use a priori energy estimates and in Ref [71] the notion of maximal regularity.

The Westervelt equation (4) is also an approximation of the Kuznetsov equation, but this time by a nonlinear perturbation. Actually the only difference between these two models is that the Westervelt equation keeps only one of two non-linear terms of the Kuznetsov equation, producing cumulative effects in a progressive wave propagation according to Aanonsen, Barkve, Tjøtta et Tjøtta in [1].

The NPE equation is usually used to describe short-time pulses and a long-range propagation, for instance, in an ocean wave-guide, where the refraction phenomena are important,

while the KZK equation typically models the ultrasonic propagation with strong diffraction phenomena, combining with finite amplitude effects (see Rozanova-Pierrat with Ref. [81] and the references therein). Although the physical context and the physical use of the KZK and the NPE equations are different, there is a bijection between the variables of these two models and they can be presented by the same type of differential operator with constant positive coefficients:

$$Lu = 0, \quad L = \partial_{tx}^2 - c_1 \partial_x (\partial_x \cdot)^2 - c_2 \partial_x^3 \pm c_3 \Delta_y, \quad \text{for } t \in \mathbb{R}^+, \quad x \in \mathbb{R}, \quad y \in \mathbb{R}^{n-1}.$$

Therefore, the results on the solutions of the KZK equation from Ref. [80] are valid for the NPE equation. See also Ref. [42] by Ito for the exponential decay of the solutions of these models in the viscous case.

All the models of Kuznetsov, KZK, NPE, and Westervelt were derived from a compressible nonlinear isentropic Navier-Stokes (for viscous media) and Euler (for the inviscid case) systems up to some small negligible terms. But all cited physical derivations of these models don't allow to say that their solutions approximate the solution of the Navier-Stokes or Euler system. The results on the well-posedness of the KZK and NPE equations are already known, the first work explaining it for the KZK equation is Ref. [81] by Rozanova-Pierrat.

Therefore in Chapter 1 we have studied the well-posedness of the Cauchy problem associated to the Kuznetsov equation in \mathbb{R}^n in the viscous and inviscid cases for small enough initial data. This results correspond to our article [26] proposed with Rozanova-Pierrat.

In Chapter 2, we start to present the initial context of the isentropic Navier-Stokes system (5)–(6) (actually, it is also an approximation of the compressible Navier-Stokes system), which describes the acoustic wave motion in an homogeneous thermo-elastic medium [13, 37, 65]. We systematize in Chapter 2 the derivation of all these models using the ideas of Ref. [81], consisting to use correctors in the Hilbert type expansions of corresponding physical *ansatzs*.

More precisely, we show that all these models are approximations of the isentropic Navier-Stokes or Euler system up to third order terms of a small dimensionless parameter $\epsilon > 0$ measuring the size of the perturbations of the pressure, the density and the velocity to compare to their constant state $(p_0, \rho_0, 0)$ (see Fig 1).

With the known results on the well-posedness of these models, we validate in Chapter 2 these approximations obtaining L^2 -estimates between the solutions of the exact and approximated models considered by studying first the approximation of the Navier-Stokes system and then the approximation of the Kuznetsov equation. It is to be noted that we can consider for the exact model a weak solution with less regularity which will be approximated by the regular solution of the approximated model.

Therefore in Part II we have studied the weak solutions of waves equations on domains with fractal boundaries in order to consider the most general domains on which such weak solutions exist.

To come back to Chapter 1 we study the well-posedness of the Cauchy problem associated to Eq. (1). In the inviscid case for $\nu = 0$, the Cauchy problem for the Kuznetsov equation is a particular case of a general quasi-linear hyperbolic system of the second order considered by Hughes, Kato and Marsden [40]. The local well-posedness result, proved in [40], does not use a priori estimate techniques and is based on the semi-group theory. Hence, thanks to [40], we have the well-posedness of (1) in the Sobolev spaces H^s with a real $s > \frac{n}{2} + 1$. Actually, to extend the local well-posedness to a global one (for $n \geq 4$) and

to estimate the maximal time interval on which there exists a regular solution, John [44] has developed a priori estimates for the Cauchy problem for a general quasi-linear wave equation with an energy of the form

$$E_m[u](t) = \|\nabla u(t)\|_{H^m(\mathbb{R}^n)}^2 + \sum_{i=1}^{m+1} \|\partial_t^i u(t)\|_{H^{m+1-i}(\mathbb{R}^n)}^2.$$

This time, due to the non-linearities $u_t u_{tt}$ and $\nabla u \nabla u_t$ including the time derivatives, to have an a priori estimate for the Kuznetsov equation we need to work with Sobolev spaces H^s for a natural s . If we directly apply general results of John in Ref. [44] to our case of the Kuznetsov equation, we obtain a well-posedness result with a high regularity of the initial data. We improve this result and show John's results for the Kuznetsov equation with the minimal regularity on the initial data corresponding to the regularity obtained by Hughes, Kato and Marsden [40]. The energy estimates allow us to evaluate the maximal existence time interval. In \mathbb{R}^2 and \mathbb{R}^3 the optimality of the obtained estimations for the maximal existence time is ensured by the results of Alinhac [5]. In Ref. [5] a geometric blow-up for small data is proved for $\partial_t^2 u$ and Δu at a finite time of the same order as predicted by our a priori estimates.

For $n \geq 4$ and $\nu = 0$, we also improve the results of John [44]. The smallness of the initial data here directly ensures the hyperbolicity of the Kuznetsov equation for all time, *i.e.* it ensures that $1 - \alpha \varepsilon u_t$ is strictly positive and bounded for all time. The proof uses the generalized derivatives for the wave type equations [44] and a priori estimate of Klainerman [58, 59].

In the presence of the term Δu_t for the viscous case $\nu > 0$, the regularity of the higher order time derivatives of u is different (compared to the inviscid case), and the way to control the non-linearities in the a priori estimates becomes different. As it was shown in [83] by Shibata, this dissipative term changes a finite speed of propagation of the wave equation to the infinite one. Indeed, the linear part of Eq. (1) can be viewed as two compositions of the heat operator $\partial_t - \Delta$ in the following way:

$$u_{tt} - c^2 \Delta u - \nu \varepsilon \Delta u_t = \partial_t(\partial_t u - \varepsilon \nu \Delta u) - c^2 \Delta u.$$

For the viscous case we prove the global in time well-posedness results in \mathbb{R}^n for small enough initial data, the size of which we specify. For $n \geq 3$ we establish an a priori estimate which gives also a sufficient condition of the existence of a global solution for a sufficiently small initial energy. Considering the Sobolev spaces H^s given with an integer $s = m$ we control the energy

$$E_{\frac{m}{2}}[u](t) = \|\nabla u(t)\|_{H^m(\mathbb{R}^n)}^2 + \sum_{i=1}^{\frac{m}{2}+1} \|\partial_t^i u(t)\|_{H^{m-2(i-1)}(\mathbb{R}^n)}^2.$$

The same results hold in $(\mathbb{R}/L\mathbb{Z}) \times \mathbb{R}^{n-1}$ for $n \geq 2$ with a periodicity and mean value zero on one variable.

Therefore, let us pay attention to Chapter 2. As it is shown in Fig. 1, the Kuznetsov equation comes from the Navier-Stokes or Euler system only by small perturbations, but to obtain the KZK and the NPE equations we also need to perform in addition to the small perturbations a paraxial change of variables. Moreover, the KZK and the NPE equations can be also obtained from the Kuznetsov equation just performing the corresponding paraxial change of variables. We can notice that the Kuznetsov equation is a non-linear wave equation containing the terms of different order on ε . But the KZK- and NPE-paraxial

approximations allow to have the approximate equations with all terms of the same order, *i.e.* the KZK and NPE equations.

Let us pay attention that the *ansatz*, proposed initially by Bakhvalov, Zhileikin, and Zabolotskaya in Ref. [13] and used in Ref. [81] by Rozanova-Pierrat to obtain the KZK equation from the Navier-Stokes or Euler systems, is different from the *ansatz* that we use. Moreover, this new approximation of the Navier-Stokes and the Euler systems is an improvement compared to the derivation developed in Ref. [81], as in this reference the Navier-Stokes/Euler system could be only approximated up to $O(\varepsilon^{\frac{5}{2}})$ -terms (instead of $O(\varepsilon^3)$ in our case).

The main hypothesis for the derivation of all these models are the following

- the motion is potential;
- the constant state of the medium given by $(p_0, \rho_0, 0)$ (0 for the velocity) is perturbed proportionally to an dimensionless parameter $\epsilon > 0$ (for instance, equal to 10^{-5} in water with an initial power of the order of 0.3 W/cm^2);
- all viscosities are small (of order ϵ).

To keep a physical sense of the approximation problems, we consider especially the two or three dimensional cases, *i.e.* \mathbb{R}^n with $n = 2$ or 3 , and in the following we use the notation $x = (x_1, x') \in \mathbb{R}^n$ with one axis $x_1 \in \mathbb{R}$ and the transversal variable $x' \in \mathbb{R}^{n-1}$.

Hence, we validate the approximations of the compressible isentropic Navier-Stokes system by the different models: by the Kuznetsov, the KZK and the NPE equations.

Then we do the same for the Euler system in the inviscid case. The main difference between the viscous and the inviscid case is the time existence and regularity of the solutions. Typically in the inviscid case, the solutions of the models and also of the Euler system itself (actually strong solutions), due to their non-linearity, can provide shock front formations at a finite time [4, 26, 80, 84, 93]. Thus, they are only locally well-posed, while in the viscous media all approximative models are globally well-posed for small enough initial data [26, 67, 80].

We note by \mathbf{U}_ε a solution of the “exact” Navier-Stokes/Euler system (see Eq. (2.31))

$$\partial_t \mathbf{U}_\varepsilon + \sum_{i=1}^n \partial_{x_i} \mathbf{G}_i(\mathbf{U}_\varepsilon) - \varepsilon \nu \begin{bmatrix} 0 \\ \Delta \mathbf{v}_\varepsilon \end{bmatrix} = 0,$$

and by $\overline{\mathbf{U}}_\varepsilon$ an approximated solution, constructed by the derivation *ansatz* from a regular solution of one of the approximate models (typically of the Kuznetsov, the KZK or the NPE equations), *i.e.* a function which solves the Navier-Stokes/Euler system up to ε^3 terms, denoted by $\varepsilon^3 \mathbf{R}$ (see Eq. (2.32)):

$$\partial_t \overline{\mathbf{U}}_\varepsilon + \sum_{i=1}^n \partial_{x_i} \mathbf{G}_i(\overline{\mathbf{U}}_\varepsilon) - \varepsilon \nu \begin{bmatrix} 0 \\ \Delta \overline{\mathbf{v}}_\varepsilon \end{bmatrix} = \varepsilon^3 \mathbf{R}.$$

To have the remainder term $\mathbf{R} \in C([0, T], L^2(\Omega))$ we ensure that the left hand side in this equation is in $C([0, T], L^2(\Omega))$, *i.e.* we need a sufficiently regular solution $\overline{\mathbf{U}}_\varepsilon$. The minimal regularity of the initial data to have a such $\overline{\mathbf{U}}_\varepsilon$ is given in Table 2.1 (see also Table 2.2 for the approximations of the Kuznetsov equation).

Choosing for the exact system the same initial-boundary data found by the *ansatz* for $\overline{\mathbf{U}}_\varepsilon$ (the regular case) or the initial data taken in their small L^2 -neighbourhood, *i.e.*

$$\|\mathbf{U}_\varepsilon(0) - \overline{\mathbf{U}}_\varepsilon(0)\|_{L^2(\Omega)} \leq \delta \leq \epsilon,$$

with $\mathbf{U}_\varepsilon(0)$ not necessarily smooth, but ensuring the existence of an admissible weak solution of a bounded energy, we prove the existence of constants $C > 0$ and $K > 0$ independent of ε , δ and the time t such that

$$\text{for all } 0 \leq t \leq \frac{C}{\varepsilon} \quad \|(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon)(t)\|_{L^2(\Omega)}^2 \leq K(\varepsilon^3 t + \delta^2)e^{K\varepsilon t} \leq 9\varepsilon^2$$

with Ω a domain where both solutions \mathbf{U}_ε and $\overline{\mathbf{U}}_\varepsilon$ exist. Thus it is possible to approximate an exact weak solution with few regularities by a regular approximated solution.

As the KZK and NPE equations can be seen as approximations of the Kuznetsov equation due to their derivation (see Fig. 1), we also validate the approximation of the Kuznetsov equation by the KZK and NPE equations, and also by the Westervelt equation (see Table 2.2).

To be able to consider the approximation of the Kuznetsov equation by the KZK equation, we firstly establish global well-posedness results for the Kuznetsov equation in the half space similar to the previous framework for the KZK and the Navier-Stokes system. We study two cases: the purely time periodic boundary problem in the *ansatz* variables (z, τ, y) moving with the wave and the initial boundary-value problem for the Kuznetsov equation in the initial variables (t, x_1, x') with data coming from the solution of the KZK equation. We validate these two types of approximations for the viscous and inviscid cases.

Finally we validate the approximation between the Kuznetsov and NPE equation and the Kuznetsov and Westervelt equations respectively (see Table 2.2). We can summarize them in the following way: if u is a solution of the Kuznetsov equation and \overline{u} is a solution of the NPE or of the KZK (for the initial boundary value problem) or of the Westervelt equations found for rather closed initial data

$$\|\nabla_{t,\mathbf{x}}(u(0) - \overline{u}(0))\|_{L^2(\Omega)} \leq \delta \leq \epsilon,$$

then there exist constants $K > 0$, $C_1 > 0$, $C_2 > 0$ and $C > 0$ independent on ϵ , δ and on time, such that for all $t \leq \frac{C}{\epsilon}$ it holds

$$\|\nabla_{t,\mathbf{x}}(u - \overline{u})\|_{L^2(\Omega)} \leq C_1(\epsilon^2 t + \delta)e^{C_2 \epsilon t} \leq K\epsilon.$$

As the obtained stability estimates are true between a regular solution and a weak solution of Kuznetsov we can again approximate a solution with few regularities of an exact model by the regular solution of an approximated model.

In Part II we study the question of weak solutions of wave equations. We put ourselves in the context of bounded domains and look for the largest class of domain where the problem is well-posed in a weak sense. These equations include:

- the wave equation with homogeneous Dirichlet boundary conditions using Evans [30],
- the strongly damped wave equation with Dirichlet boundary conditions or homogeneous Robin boundary conditions,
- the non-linear Westervelt equation with Dirichlet boundary conditions or homogeneous Robin boundary conditions.

The regularity of the solutions of these equations on regular domains, typically with a C^2 boundary is well known, with the fact that more the initial data are regular, the more the solution is regular up to the boundary. We can cite Evans in Ref. [30] for the wave equation

and Refs. [51, 52, 53, 54, 71] and the references therein for the strongly damped wave equation and the Westervelt equation. The question is whether on less regular domains we can have a weak solution which is continuous or C^1 up to the boundary. The examples of Arendt and Elst in Ref.[10] show that problems appear for the definition of the trace as soon as the boundary is not C^1 . Moreover, if on a domain with a C^1 or Lipschitz boundary we can define an incoming normal vector almost everywhere, the question of Neumann or Robin boundary conditions is more complicated. We can add the fact that considering a C^2 boundary as in Refs. [51, 52, 53, 54, 71] is a consequence that the spatial derivatives are at most of order 2, in this case they can be defined naturally on the the boundary. In the past, mathematics has been concerned largely with regular domains. Domains like for example the Von Koch snowflake have mainly been considered as "pathological" and used only to produce counterexamples. Nevertheless, there has been a change of attitude as mathematicians and physicists have discovered that such Von Koch-like structures appear in nature with for example the English coasts or the microstructure of electrodes.

A key point to solve the equations that we will study on domains with fractal boundary is the understanding of the Poisson problem on domains with fractal boundary with Dirichlet boundary conditions (7) or homogeneous Robin boundary conditions (8). For system (7) a general approach is through the weak formulation of the Dirichlet problem. If u and $\partial\Omega$ are sufficiently smooth one may multiply the Poisson equation in (7) by $v \in C_0^\infty(\Omega)$ and use Green formula to end up with

$$\begin{cases} \int_{\Omega} \nabla u \nabla v \, dx = \int_{\Omega} f v \, dx \text{ for all } v \in C_0^\infty(\Omega) \\ u|_{\partial\Omega} = g, \end{cases}$$

which is called a weak formulation of the Dirichlet problem. Introducing the Sobolev spaces $H^1(\Omega)$ and $H_0^1(\Omega)$ and assuming that there exists $g^* \in H^1(\Omega)$ such that the trace of g^* to $\partial\Omega$ is g (attention must be paid to the definition of the trace), one may prove with the Riesz representation theorem that given $f \in L^2(\Omega)$, $g^* \in H^1(\Omega)$, there exists a unique $u \in H^1(\Omega)$ such that $-\Delta u = f$ in the sense of distributions and $u - g^* \in H_0^1(\Omega)$. This of course raises several questions:

- How is the trace defined as it is usually defined for continuous functions?
- When does there exist such an extension g^* satisfying $u = g$ on the boundary?

The answer to these two questions is already known if $\partial\Omega$ is regular enough, see for example Raviart-Thomas [79]) or even Lipschitz with the work of Marschall [66].

- Can we use the Green formula? In the Lipschitz case we have

$$\int_{\Omega} v \Delta u \, dx = \langle u, v \rangle_{(H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega))} - \int_{\Omega} \nabla u \nabla v \, dx.$$

- Does u depend uniquely and continuously on f and g ?

In Ref. [49] Jonsson and Wallin were able to answer this questions in the case where Ω is an (ϵ, δ) -domain with a boundary $\partial\Omega$ which is a so called d -set preserving Markov's inequality. With the work of Lancia [62] we find an equivalent of the Green formula using the Besov spaces for the boundary terms. To our knowledge, the results of Jonsson and Wallin are the first of this kind established on fractal domains. The results of Jones [46] on

d -sets and domains admitting $W^{k,p}$ extensions permit to say that, in dimension 2, (ϵ, δ) -domains are the most general domains on which we can define traces and extensions of the Sobolev spaces and then solve the Poisson problem. In Ref. [11], Arfi and Rozanova-Pierrat introduced a new type of domain with a fractal boundary called the admissible domains. These domains contained the (ϵ, δ) -domains and are more general, they are the largest class of domains on which we can define traces and extensions to the Sobolev spaces for $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, and then find a weak solution to the Poisson problem depending uniquely and continuously of the initial data.

As a consequence we will work mainly on admissible domains and resume the known results for these domains. It is to be noted that the work in Ref. [30] by Evans gives us the spectral properties as well as the interior regularity of the solution of the Poisson problem (7), *i.e.*, the fact that for a subset V compactly included in Ω , $V \subset\subset \Omega$, the solution on Ω has on V the same regularity than for a domain with regular boundaries. The work in Ref. [11] by Arfi and Rozanova-Pierrat permits to give similar results for the Poisson problem (8) and Ref. [30] by Evans gives us spectral properties again.

An other important question is whether the solutions of the Poisson problem (7) and (8) belong to $C(\overline{\Omega})$ with an estimate of the form:

$$\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^p(\Omega)}.$$

For the Poisson problem (7) with homogeneous Dirichlet boundary condition the works in Ref.[77] by Nyström and [92] by Xie permit to give a positive answer in dimension $n = 2$ and 3 respectively for $p = 2$. The work of Daners in Ref. [25] gives us also a positive answer for the Poisson problem (8) if $p > n$. These estimates are a key point to show that the solutions studied of our wave type models are in $C(\overline{\Omega})$ but also to treat the nonlinear term in the Westervelt equation.

Using a Galerkin method as in Ref.[30] by Evans we get the regularity of solutions of the wave equation and the strongly damped wave equation with homogeneous Dirichlet boundary conditions with the help of a basis of eigenfunctions of $-\Delta$. With these results on regularity we treat the well-posedness of the Westervelt equation with homogeneous Dirichlet boundary conditions in the same way than in the proof in Chapter 1 for the global well posedness for the Kuznetsov equation on \mathbb{R}^n . The recalled properties of the trace and extension in admissible domains permit us to treat the case of non homogeneous Dirichlet boundary conditions. These results rely on estimations in spaces where the solution and some of its derivatives are in L^2 . Note that we use a similar method for the problems with homogeneous Robin boundary conditions and obtain L^2 -estimate for the strongly damped wave equation on admissible domains, with again a Galerkin method with a basis of eigenfunctions of $-\Delta$, or the Westervelt equation on Lipschitz domain in the same way than the proof in Chapter 1 for the global well posedness for the Kuznetsov equation on \mathbb{R}^n . The case of the Westervelt equation on admissible domains with Robin boundary conditions has been shown using L^p estimates with $p > n$ in the same way.

We will conclude this Part II considering a domain with a fractal boundary of Koch mixture type constructed by induction with the help of families of contractive similitudes inducing a family of domains with prefractal and Lipschitz boundaries approximating the domain with fractal boundaries. Using different works by Capitanelli [19], Capitanelli and Vivaldi [20] or Lancia [62] we consider the asymptotic convergence of Mosco type of the solutions of the Westervelt equation with Robin boundary conditions on domains with a prefractal boundary, which approach the solution on the domain with a fractal boundary of Koch mixture type, an often used method in the case of shape optimization.

Part I

The Kuznetsov equation and other
models of nonlinear acoustic

Chapter 1

Cauchy Problem for the Kuznetsov Equation

1.1 Introduction française

L'équation de Kuznetsov [60] modélise la propagation d'ondes acoustiques non linéaires dans des milieux élastiques thermo-visqueux. Le problème de Cauchy pour l'équation de Kuznetsov se lit pour $\alpha = \frac{\gamma-1}{c^2}$, $\beta = 2$ et $\nu = \frac{\delta}{\rho_0}$ comme

$$u_{tt} - c^2 \Delta u - \nu \varepsilon \Delta u_t = \alpha \varepsilon u_t u_{tt} + \beta \varepsilon \nabla u \cdot \nabla u_t, \quad x \in \mathbb{R}^n, \quad (1.1)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \mathbb{R}^n, \quad (1.2)$$

où c , ρ_0 , γ , δ sont la vitesse du son, la densité, le ratio des chaleurs spécifiques et la viscosité du milieu respectivement. Nous pouvons nous référer à l'introduction générale.

Dans ce chapitre nous étudions le caractère bien posé du problème de Cauchy (1.1)–(1.2). Dans le cas non visqueux pour $\nu = 0$, le problème de Cauchy pour l'équation de Kuznetsov est un cas particulier du système général quasi linéaire hyperbolique du second ordre considéré par Hughes, Kato et Marsden [40] (voir Théorème 1.2.1 points 1 et 2 pour l'application de leurs résultats à l'équation de Kuznetsov). Le résultat de caractère bien posé local, prouvé dans [40], n'utilise pas des techniques d'estimations a priori et est basé sur la théorie des semi-groupes. Alors, grâce à [40], nous avons le caractère bien posé de (1.1)–(1.2) dans l'espace de Sobolev H^s avec un réel $s > \frac{n}{2} + 1$. De plus, pour étendre le caractère bien posé local au cas global (pour $n \geq 4$) et pour estimer l'intervalle de temps maximal sur lequel il existe une solution régulière, John [44] a développé des estimations a priori pour le problème de Cauchy associé à une équation d'onde quasi linéaire générale. Cette fois, à cause des non linéarités $u_t u_{tt}$ et $\nabla u \cdot \nabla u_t$ incluant les dérivées en temps, pour avoir une estimation a priori pour l'équation de Kuznetsov, nous avons besoin de travailler avec les espaces de Sobolev caractérisés par un entier s , dès lors dénoté dans ce qui suit par m . Si nous appliquons directement les résultats généraux de la référence [44] à notre cas pour l'équation de Kuznetsov, nous obtenons le caractère bien posé pour une grande régularité des données initiales. Nous améliorons ce résultat dans le Théorème 1.4.1 et montrons les résultats de John pour l'équation de Kuznetsov avec une régularité minimale des données initiales correspondant à la régularité obtenue par Hughes, Kato et Marsden [40]. Par exemple, nous prouvons une estimation d'énergie analogue dans H^m avec $m \geq \lfloor \frac{n}{2} \rfloor + 2$ au lieu de $m \geq \frac{3}{2}n + 4$ dans le cas de John (voir l'équation (1.24) dans la Proposition 1.4.1) et pour la version légèrement modifiée de l'estimation nous trouvons $m \geq \lfloor \frac{n}{2} \rfloor + 3$ au lieu de $m \geq \frac{3}{2}n + 7$ (voir l'équation (1.40) dans la Proposition 1.4.2). Les estimations d'énergie

nous permettent d'évaluer le temps d'existence maximal (voir le Théorème 1.2.1 Point 5 et le Théorème 1.4.2 pour plus de détails). Dans \mathbb{R}^2 et \mathbb{R}^3 l'optimalité des estimations obtenues pour le temps d'existence maximal est assurée par les résultats d'Alinhac [5]. Dans la référence [5] un blow-up géométrique pour les données petites est prouvé pour $\partial_t^2 u$ et Δu en temps fini et pour le même ordre que prédit par nos estimations a priori (voir le Théorème 1.2.1 Point 5, nos estimations du temps d'existence minimal correspondent aux résultats d'Alinhac sur les temps d'existence maximaux). D'autre part, le blow-up de $\partial_t^2 u$ et Δu est aussi confirmé par l'estimation de stabilité (1.12) dans le Théorème 1.2.1: si l'intervalle de temps d'existence maximal est fini et limité par T^* , l'équation (1.12) nous donne la divergence

$$\int_0^{T^*} \left(\|u_{tt}\|_{L^\infty(\mathbb{R}^n)} + \|\Delta u\|_{L^\infty(\mathbb{R}^n)} \right) dt = +\infty. \quad (1.3)$$

Pour $n \geq 4$ et $\nu = 0$, nous améliorons aussi les résultats de John [44] et montrons l'existence globale pour des données suffisamment petites $u_0 \in H^{m+1}(\mathbb{R}^n)$ et $u_1 \in H^m(\mathbb{R}^n)$ pour $m \geq n + 2$ au lieu de $m \geq \frac{3}{2}n + 7$ (voir la Proposition 1.4.4 et le Théorème 1.4.2). La petitesse des données initiales assure directement l'hyperbolicité de l'équation de Kuznetsov pour tout temps, *i.e.* elle assure que $1 - \alpha \varepsilon u_t$ est strictement positif et borné pour tout temps. La preuve utilise les dérivées généralisées pour les équations d'ondes [44] et une estimation a priori de Klainerman [58, 59] (voir Section 1.4.2).

Formulons à présent notre résultat principal sur le caractère bien posé dans le cas non visqueux avec le Théorème 1.2.1. Le Théorème 1.2.1 se fonde principalement sur les estimations a priori données dans les Sous-sections 1.4.1 (pour le Point 3) et 1.4.2 (pour le Point 5) et sur le résultat d'existence locale tiré de la référence [40] (Points 1 et 2). Le Point 4, prouvé dans la Sous-section 1.4.3, utilise les idées classiques de stabilité faible et forte, prouvées par exemple en détails pour l'équation de KZK dans [80] Théorème 1.1 Point 4 p. 785.

En analysant la structure de l'équation de Kuznetsov et les difficultés entraînées par ses termes non linéaires, nous commençons dans la Section 1.3 par des remarques préliminaires sur les propriétés d'énergie L^2 de l'équation de Kuznetsov à comparer avec ses versions simplifiées. Néanmoins en développant les estimations d'énergie dans les espaces de Sobolev, nous reconnaissons la structure de l'énergie L^2 de l'équation d'onde qui demeure inchangée.

En présence du terme Δu_t pour le cas visqueux $\nu > 0$, la régularité des dérivées en temps d'ordre supérieur de u est différente (en comparaison au cas non visqueux), et la manière de contrôler les non linéarités change.

Pour le cas visqueux, nous prouvons les résultats sur le caractère bien posé global dans \mathbb{R}^n (voir Section 1.5) pour les données initiales suffisamment petites, dont nous spécifions la taille (voir le Point 1 du Théorème 1.2.2 et la Sous-section 1.5.1 pour la preuve). Dans la Sous-section 1.5.2 pour $n \geq 3$ (voir le Point 2 du Théorème 1.2.2) nous établissons une estimation a priori qui nous donne aussi une condition suffisante pour l'existence de solutions globales avec une énergie initiale suffisamment petite du même ordre en ε que dans le Point 1 du Théorème 1.2.2. Les mêmes résultats sont vrais dans $(\mathbb{R}/L\mathbb{Z}) \times \mathbb{R}^{n-1}$ pour $n \geq 2$ avec la périodicité et la valeur moyenne nulle selon une variable (voir le Point 3 du Théorème 1.2.2).

Notons aussi que la condition d'hyperbolicité (1.9) est aussi satisfaite si nous requérons les conditions (1.13) et (1.15). Pour $\nu > 0$, le Point 4 du Théorème 1.2.1 est vérifié pour tout $n \in \mathbb{N}^*$. Le Point 1 du Théorème 1.2.2 est prouvé dans la Sous-section 1.5.1 en utilisant un théorème de l'analyse non linéaire [88] (voir le Théorème 1.5.2) et des résultats de régularité

pour l'équation d'onde fortement amortie suivant [32], qui peuvent aussi être utilisés pour $\Omega = (\mathbb{R}/L\mathbb{Z}) \times \mathbb{R}^{n-1}$ dans le Point 3. Le Point 2 du Théorème 1.2.2 est prouvé dans la Sous-section 1.5.2, en utilisant les estimations a priori données par la Proposition 1.5.1, voir aussi le Théorème 1.5.3. Le dernier point du Théorème 1.2.2 est un corollaire direct de l'inégalité de Poincaré

$$\|u\|_{L^2((\mathbb{R}/L\mathbb{Z}) \times \mathbb{R}^{n-1})} \leq C \|\partial_x u\|_{L^2((\mathbb{R}/L\mathbb{Z}) \times \mathbb{R}^{n-1})}, \quad (1.4)$$

vérifiée dans la classe des fonctions périodiques de moyenne nulle. L'estimation (1.4) permet d'avoir les mêmes estimations que dans le Lemme 1.5.1 (voir Section 1.5) pour $n = 2$, qui ne peuvent être vérifiées dans \mathbb{R}^2 . Ainsi, cela nous donne aussi l'existence globale pour les données initiales petites détaillée au Point 2.

1.2 Introduction

The Kuznetsov equation [60] models the propagation of non-linear acoustic waves in thermo-viscous elastic media. The Cauchy problem for the Kuznetsov equation reads for $\alpha = \frac{\gamma-1}{c^2}$, $\beta = 2$ and $\nu = \frac{\delta}{\rho_0}$ as

$$u_{tt} - c^2 \Delta u - \nu \varepsilon \Delta u_t = \alpha \varepsilon u_t u_{tt} + \beta \varepsilon \nabla u \cdot \nabla u_t, \quad x \in \mathbb{R}^n, \quad (1.5)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \mathbb{R}^n, \quad (1.6)$$

where c , ρ_0 , γ , δ are the velocity of the sound, the density, the ratio of the specific heats and the viscosity of the medium respectively. We can refer to the general introduction.

In this chapter we study the well-posedness properties of the Cauchy problem (1.5)–(1.6). In the inviscid case for $\nu = 0$, the Cauchy problem for the Kuznetsov equation is a particular case of a general quasi-linear hyperbolic system of the second order considered by Hughes, Kato and Marsden [40] (see Theorem 1.2.1 Points 1 and 2 for the application of their results to the Kuznetsov equation). The local well-posedness result, proved in [40], does not use a priori estimate techniques and is based on the semi-group theory. Hence, thanks to [40], we have the well-posedness of (1.5)–(1.6) in the Sobolev spaces H^s with a real $s > \frac{n}{2} + 1$. Therefore, actually, to extend the local well-posedness to a global one (for $n \geq 4$) and to estimate the maximal time interval on which there exists a regular solution, John [44] has developed a priori estimates for the Cauchy problem for a general quasi-linear wave equation. This time, due to the non-linearities $u_t u_{tt}$ and $\nabla u \cdot \nabla u_t$ including the time derivatives, to have an a priori estimate for the Kuznetsov equation we need to work with Sobolev spaces with a natural s , denoted in what follows by m . If we directly apply the general results of Ref. [44] to our case of the Kuznetsov equation, we obtain a well-posedness result with a high regularity of the initial data. We improve it in Theorem 1.4.1 and show John's results for the Kuznetsov equation with the minimal regularity on the initial data corresponding to the regularity obtained by Hughes, Kato and Marsden [40]. For instance, we prove the analogous energy estimates in H^m with $m \geq [\frac{n}{2} + 2]$ instead of John's $m \geq \frac{3}{2}n + 4$ (see Eq. (1.24) in Proposition 1.4.1) and its slight modified version in H^m with $m \geq [\frac{n}{2} + 3]$ instead of $m \geq \frac{3}{2}n + 7$ (see Eq. (1.40) in Proposition 1.4.2). The energy estimates allow us to evaluate the maximal existence time interval (see Theorem 1.2.1 Point 5 and Theorem 1.4.2 for more details). In \mathbb{R}^2 and \mathbb{R}^3 the optimality of obtained estimations for the maximal existence time is ensured by the results of Alinhac [5]. In Ref. [5] a geometric blow-up for small data is proved for $\partial_t^2 u$ and Δu at a finite time of

the same order as predicted by our a priori estimates (see Theorem 1.2.1 Point 5, our estimates of the minimum existence time correspond to Alinhac's maximum existence time results). On the other hand, the blow-up of $\partial_t^2 u$ and Δu is also confirmed by the stability estimate (1.12) in Theorem 1.2.1: if the maximal existence time interval is finite and limited by T^* , by Eq. (1.12), we have the divergence

$$\int_0^{T^*} \left(\|u_{tt}\|_{L^\infty(\mathbb{R}^n)} + \|\Delta u\|_{L^\infty(\mathbb{R}^n)} \right) dt = +\infty. \quad (1.7)$$

For $n \geq 4$ and $\nu = 0$, we also improve the results of John [44] and show the global existence for sufficiently small initial data $u_0 \in H^{m+1}(\mathbb{R}^n)$ and $u_1 \in H^m(\mathbb{R}^n)$ with $m \geq n+2$ instead of $m \geq \frac{3}{2}n+7$ (see Proposition 1.4.4 and Theorem 1.4.2). The smallness of the initial data here directly ensures the hyperbolicity of the Kuznetsov equation for all time, *i.e.* it ensures that $1 - \alpha\varepsilon u_t$ is strictly positive and bounded for all time. The proof uses the generalized derivatives for the wave type equations [44] and a priori estimate of Klainerman [58, 59] (see Section 1.4.2).

Let us now formulate our main well-posedness result for the inviscid case:

Theorem 1.2.1. (Inviscid case) *Let $\nu = 0$, $n \in \mathbb{N}^*$ and $s > \frac{n}{2} + 1$. For all $u_0 \in H^{s+1}(\mathbb{R}^n)$ and $u_1 \in H^s(\mathbb{R}^n)$ such that $\|u_1\|_{L^\infty(\mathbb{R}^n)} < \frac{1}{2\alpha\varepsilon}$, $\|u_0\|_{L^\infty(\mathbb{R}^n)} < M_1$, $\|\nabla u_0\|_{L^\infty(\mathbb{R}^n)} < M_2$, with M_1 and M_2 in \mathbb{R}_+^* the following results hold:*

1. *For all $T > 0$, there exists $T' > 0$, $T' \leq T$, such that there exists a unique solution u of (1.5)–(1.6) with the following regularity*

$$u \in C^r([0, T']; H^{s+1-r}(\mathbb{R}^n)) \text{ for } 0 \leq r \leq s, \quad (1.8)$$

$$\forall t \in [0, T'], \quad \|u_t(t)\|_{L^\infty(\mathbb{R}^n)} < \frac{1}{2\alpha\varepsilon}, \quad \|u\|_{L^\infty(\mathbb{R}^n)} < M_1, \quad \|\nabla u\|_{L^\infty(\mathbb{R}^n)} < M_2. \quad (1.9)$$

2. *The map $(u_0, u_1) \mapsto (u(t, \cdot), \partial_t u(t, \cdot))$ is continuous in the topology of $H^{s+1} \times H^s$ uniformly in $t \in [0, T']$.*
3. *Let T^* be the largest time on which such a solution is defined, and in addition $s \in \mathbb{N}$, *i.e.* $s = m \geq m_0 = \lfloor \frac{n}{2} + 2 \rfloor$. With the notation*

$$E_m[u](t) = \|\nabla u(t)\|_{H^m(\mathbb{R}^n)}^2 + \sum_{i=1}^{m+1} \|\partial_t^i u(t)\|_{H^{m+1-i}(\mathbb{R}^n)}^2, \quad (1.10)$$

there exist constants $C(n, c, \alpha) > 0$ and $\hat{C}(n, c, \alpha, \beta) > 0$ (see Theorem 1.4.1) such that if the initial data satisfies $\sqrt{E_{m_0}[u]}(0) \leq \frac{1}{C(n, c, \alpha)\varepsilon}$, then

$$T^* \geq \frac{1}{\varepsilon \hat{C}(n, c, \alpha, \beta) \sqrt{E_{m_0}[u]}(0)}, \text{ such that it holds (1.7)}. \quad (1.11)$$

4. *For two solutions u and v of the Kuznetsov equation for $\nu = 0$ defined on $[0, T^*[$ assume that u be regular as in (1.8)–(1.9), *i.e.* $u \in L^\infty([0, T^*]; H^{m+1}(\mathbb{R}^n))$, $u_t \in L^\infty([0, T^*]; H^m(\mathbb{R}^n))$ ($s = m$ as in Point 3), and*

$$v \in L^\infty([0, T^*]; H^1(\mathbb{R}^n)), \quad v_t \in L^\infty([0, T^*]; L^2(\mathbb{R}^n)) \text{ with } \|v\|_{L^\infty(\mathbb{R}^n)} < \frac{1}{2\alpha\varepsilon}$$

and with a bounded $\|\nabla v_t\|_{L^\infty(\mathbb{R}^n)}$ norm on $[0, T^*]$. Then it holds the following stability uniqueness result: there exist constants $C_1 > 0$ and $C_2 > 0$, independent on time, such that

$$\begin{aligned} (\|(u-v)_t\|_{L^2}^2 + \|\nabla(u-v)\|_{L^2}^2)(t) \leq C_1 \exp\left(C_2 \varepsilon \int_0^t \sup(\|u_{tt}\|_{L^\infty(\mathbb{R}^n)}, \|\Delta u\|_{L^\infty(\mathbb{R}^n)}) dl\right) \\ \cdot (\|u_1 - v_1\|_{L^2}^2 + \|\nabla(u_0 - v_0)\|_{L^2}^2). \end{aligned} \quad (1.12)$$

5. If $s = m \geq n + 2$, then for sufficiently small initial data (see Theorem 1.4.2 in Section 1.4.2)

- (a) $\liminf_{\varepsilon \rightarrow 0} \varepsilon^2 T^* > 0$ for $n = 2$,
- (b) $\liminf_{\varepsilon \rightarrow 0} \varepsilon \log(T^*) > 0$ for $n = 3$,
- (c) $T^* = +\infty$ for $n \geq 4$.

Theorem 1.2.1 is principally based on the a priori estimates given in Sections 1.4.1 (for Point 3) and 1.4.2 (for Point 5) and on the local existence result updated from Ref. [40] (Points 1 and 2). Point 4, proved in Section 1.4.3, uses the classical ideas of the weak-strong stability, for instance proved in details for the KZK equation in [80] Theorem 1.1 Point 4 p. 785.

Analysing the structure of the Kuznetsov equation and the difficulties involved by its non-linear terms, we start in Section 1.3 with preliminary remarks on the L^2 -energy properties for the Kuznetsov equation compared to its simplified versions. However when developing the energy estimates in the Sobolev spaces, we recognize the structure of the L^2 -energy of the wave equation which remains unchanged.

In the presence of the term Δu_t for the viscous case $\nu > 0$, the regularity of the higher order time derivatives of u is different (compared to the inviscid case), and the way to control the non-linearities in the a priori estimates becomes different.

For the viscous case we prove the global-in-time well-posedness results in \mathbb{R}^n (see Section 1.5) for small enough initial data, the size of which we specify (see Point 1 of Theorem 1.2.2 and Subsection 1.5.1 for its proof). In Subsection 1.5.2 for $n \geq 3$ (see Point 2 of Theorem 1.2.2) we establish an a priori estimate which gives also a sufficient condition of the existence of a global solution for a sufficiently small initial energy of the same order on ε as in Point 1 of Theorem 1.2.2. The same results (see Point 3 of Theorem 1.2.2) hold in $(\mathbb{R}/LZ) \times \mathbb{R}^{n-1}$ for $n \geq 2$ (with a periodicity and mean value zero on one variable).

Theorem 1.2.2. (*Viscous case*) Let $\nu > 0$, $n \in \mathbb{N}^*$, $s > \frac{n}{2}$ and $\mathbb{R}^+ = [0, +\infty[$. Considering the Cauchy problem for the Kuznetsov equation (1.5)–(1.6), the following results hold:

1. Let

$$X := H^2(\mathbb{R}^+; H^s(\mathbb{R}^n)) \cap H^1(\mathbb{R}^+; H^{s+2}(\mathbb{R}^n)),$$

the initial data

$$u_0 \in H^{s+2}(\mathbb{R}^n) \quad \text{and} \quad u_1 \in H^{s+1}(\mathbb{R}^n),$$

$r_* = O(1)$ be the positive constant defined in Eq. (1.59) and $C_1 = O(1)$ be the minimal constant such that the solution u^* of the corresponding linear Cauchy problem (1.56) satisfies

$$\|u^*\|_X \leq \frac{C_1}{\sqrt{\nu\varepsilon}} (\|u_0\|_{H^{s+2}(\mathbb{R}^n)} + \|u_1\|_{H^{s+1}(\mathbb{R}^n)}).$$

Then for all $r \in [0, r_*[$ and all initial data satisfying

$$\|u_0\|_{H^{s+2}(\mathbb{R}^n)} + \|u_1\|_{H^{s+1}(\mathbb{R}^n)} \leq \frac{\sqrt{\nu\epsilon}}{C_1} r, \quad (1.13)$$

there exists the unique solution $u \in X$ of the Cauchy problem for the Kuznetsov equation and $\|u\|_X \leq 2r$.

2. Let $n \geq 3$, $s = m \in \mathbb{N}$ be even and $m \geq [\frac{n}{2} + 3]$. With the notation

$$E_{\frac{m}{2}}[u](t) = \|\nabla u(t)\|_{H^m(\mathbb{R}^n)}^2 + \sum_{i=1}^{\frac{m}{2}+1} \|\partial_t^i u(t)\|_{H^{m-2(i-1)}(\mathbb{R}^n)}^2, \quad (1.14)$$

there exists a constant $\rho = O(1) > 0$ (see Theorem 1.5.3 Point 2), independent on time, such that for all initial data $u_0 \in H^{m+1}(\mathbb{R}^n)$ and $u_1 \in H^m(\mathbb{R}^m)$ satisfying

$$E_{\frac{m}{2}}[u](0) < \rho\epsilon, \quad (1.15)$$

there exists a unique $u \in C^0(\mathbb{R}^+; H^{m+1}(\mathbb{R}^n)) \cap C^i(\mathbb{R}^+; H^{m+2-2i}(\mathbb{R}^n))$, for $i = 1, \dots, \frac{m}{2} + 1$ with the bounded energy

$$\forall t \in \mathbb{R}^+, \quad E_{\frac{m}{2}}[u](t) \leq O\left(\frac{1}{\epsilon}\right) E_{\frac{m}{2}}[u](0) = O(1).$$

3. For $n \in \mathbb{N}^*$ in $\Omega = (\mathbb{R}/L\mathbb{Z}) \times \mathbb{R}^{n-1}$ with $s = m \in \mathbb{N}$ even and $m \geq [\frac{n}{2} + 3]$ there hold Points 1 and 2 in the class of periodic in one direction functions with the mean value zero

$$\int_{\mathbb{R}/L\mathbb{Z}} u(t, x, y) \, dx = 0. \quad (1.16)$$

Let us notice that the hyperbolicity condition (1.9) is also satisfied if we require conditions (1.13) and (1.15). For $\nu > 0$ Point 4 of Theorem 1.2.1 obviously holds for all $n \in \mathbb{N}^*$. Point 1 of Theorem 1.2.2 is proved in Subsection 1.5.1 using a theorem of a non-linear analysis [88] (see Theorem 1.5.2) and regularity results for the strongly damped wave equation following [32], which can also be used for $\Omega = (\mathbb{R}/L\mathbb{Z}) \times \mathbb{R}^{n-1}$ in point 3. Point 2 of Theorem 1.2.2 is proved in Subsection 1.5.2, using a priori estimates given in Proposition 1.5.1, see also Theorem 1.5.3. The last point of Theorem 1.2.2 is a direct corollary of the Poincaré inequality

$$\|u\|_{L^2((\mathbb{R}/L\mathbb{Z}) \times \mathbb{R}^{n-1})} \leq C \|\partial_x u\|_{L^2((\mathbb{R}/L\mathbb{Z}) \times \mathbb{R}^{n-1})}, \quad (1.17)$$

which holds in the class of periodic functions with the mean value zero. Estimate (1.17) allows to have the same estimate as in Lemma 1.5.1 (see Section 1.5) for $n = 2$, which fails in \mathbb{R}^2 . Thus, it also gives the global existence for rather small initial data detailed in Point 2.

1.3 Preliminary remarks on L^2 -energies

We can notice that Eq. (1.5) is a wave equation containing a dissipative term Δu_t and two non-linear terms: $\nabla u \nabla u_t$ describing local non-linear effects and $u_t u_{tt}$ describing global or

cumulative effects. Actually, the linear wave equation appears from Eq. (1.5) if we consider only the terms of the zero order on ε :

$$u_{tt} - c^2 \Delta u = 0. \quad (1.18)$$

The semi-group theory permits in the usual way to show that for $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$ there exists a unique solution of the Cauchy problem (1.18), (1.6)

$$u \in C^0(\mathbb{R}^+; H^1(\mathbb{R}^n)) \cap C^1(\mathbb{R}^+; L^2(\mathbb{R}^n)).$$

So the energy of the wave equation (1.18)

$$E(t) = \int_{\mathbb{R}^n} [(u_t)^2 + c^2(\nabla u)^2](t, x) dx, \quad (1.19)$$

is well defined and conserved

$$\frac{d}{dt} E(t) = 0.$$

For $\nu > 0$ and without non-linear terms, the Kuznetsov equation (1.5) becomes the well-known strongly damped wave equation:

$$u_{tt} - c^2 \Delta u - \nu \varepsilon \Delta u_t = 0, \quad (1.20)$$

which is well-posed [41]: for $m \in \mathbb{N}$, $u_0 \in H^{m+1}(\mathbb{R}^n)$ and $u_1 \in H^m(\mathbb{R}^n)$ there exists a unique solution of the Cauchy problem (1.20), (1.6)

$$u \in C^0(\mathbb{R}^+; H^{m+1}(\mathbb{R}^n)) \cap C^1(\mathbb{R}^+; H^m(\mathbb{R}^n)).$$

Multiplying Eq. (1.20) by u_t in $L^2(\mathbb{R}^n)$, we obtain for the energy of the wave equation (1.19)

$$\frac{d}{dt} E(t) = -2\nu \varepsilon \int_{\mathbb{R}^n} (\nabla u_t)^2(t, x) dx \leq 0,$$

which means that the energy $E(t)$ decreases in time, thanks to the viscosity term with $\nu > 0$. The decrease rate is found for more regular energies in [83] in accordance to the regularity of the initial conditions. Without the term $\nabla u \nabla u_t$ (local non-linear effects), the Kuznetsov equation becomes similar to the Westervelt equation, initially derived by Westervelt [91] before Kuznetsov. The Westervelt equation, historically derived [91] for the acoustic pressure fluctuation, has the following form

$$p_{tt} - c^2 \Delta p - \nu \varepsilon \Delta p_t = \frac{\gamma + 1}{c^2} \varepsilon p_t p_{tt}, \quad (1.21)$$

and can also be seen as an approximation of an isentropic Navier-Stokes system.

In the sequel we conveniently denote p by u . We multiply Eq. (1.21) by u_t and integrate over \mathbb{R}^n to obtain

$$\frac{1}{2} \frac{d}{dt} \left(\int_{\mathbb{R}^n} [(u_t)^2 + c^2(\nabla u)^2] dx \right) + \nu \varepsilon \int_{\mathbb{R}^n} (\nabla u_t)^2 dx = \frac{1}{3} \frac{\gamma + 1}{c^2} \varepsilon \frac{d}{dt} \left(\int_{\mathbb{R}^n} (u_t)^3 dx \right).$$

Then we have

$$\frac{1}{2} \frac{d}{dt} \left(\int_{\mathbb{R}^n} \left[\left(1 - \frac{2}{3} \frac{\gamma + 1}{c^2} \varepsilon u_t \right) (u_t)^2 + c^2(\nabla u)^2 \right] dx \right) + \nu \varepsilon \int_{\mathbb{R}^n} (\nabla u_t)^2 dx = 0.$$

For $\alpha = \frac{2}{3} \frac{\gamma+1}{c^2}$ we consider the energy

$$E_{nonl}(t) = \int_{\mathbb{R}^n} \left[(1 - \alpha \varepsilon u_t) (u_t)^2 + c^2 (\nabla u)^2 \right] dx, \quad (1.22)$$

which is monotonous decreasing for $\nu > 0$ and is conserved for $\nu = 0$. Let us also notice that, taking the same initial data for $\nu = 0$ and $\nu > 0$, we have:

$$\text{for all } \nu > 0 \text{ and } t > 0 \quad E_{nonl}(t, \nu = 0) > E_{nonl}(t, \nu) \geq 0,$$

in the assumption that $1 - \alpha \varepsilon u_t \geq 0$ almost everywhere.

While $\frac{1}{2} \leq 1 - \alpha \varepsilon u_t \leq \frac{3}{2}$, that is to say $\|u_t(t)\|_{L^\infty(\mathbb{R}^n)}$ remains small enough in time, then we can compare E_{nonl} to the energy of the wave equation

$$\frac{1}{2} E(t) \leq E_{nonl}(t) \leq \frac{3}{2} E(t).$$

Then a sufficiently regular solution of the Cauchy problem for the Westervelt equation has the energy E controlled by a decreasing in time function:

$$E(t) \leq 3E(0) - 4\nu\varepsilon \int_0^t \int_{\mathbb{R}^n} (\nabla u_t(\tau, x))^2 dx dl.$$

Now, let us consider the Kuznetsov equation (1.5). We multiply it by u_t and integrate on \mathbb{R}^n to obtain

$$\frac{1}{2} \frac{d}{dt} E_{nonl}(t) + \nu\varepsilon \int_{\mathbb{R}^n} (\nabla u_t)^2 dx = 2\varepsilon \int_{\mathbb{R}^n} \nabla u \nabla u_t u_t dx,$$

where $E_{nonl}(t)$ is given by Eq. (1.22) with $\alpha = \frac{2}{3} \frac{\gamma-1}{c^2}$. As

$$2\varepsilon \int_{\mathbb{R}^n} \nabla u \nabla u_t u_t dx = \varepsilon \frac{d}{dt} \int_{\mathbb{R}^n} u_t (\nabla u)^2 dx - \varepsilon \int_{\mathbb{R}^n} u_{tt} (\nabla u)^2 dx,$$

we find

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\int_{\mathbb{R}^n} \left[\left(1 - \frac{2}{3} \frac{\gamma-1}{c^2} \varepsilon u_t \right) (u_t)^2 + (c^2 - 2\varepsilon u_t) (\nabla u)^2 \right] dx \right. \\ \left. + 2\varepsilon \int_0^t \int_{\mathbb{R}^n} u_{tt} |\nabla u|^2 dx dl \right) + \nu\varepsilon \int_{\mathbb{R}^n} (\nabla u_t)^2 dx = 0. \end{aligned} \quad (1.23)$$

Thus, for $\alpha = \frac{2}{3} \frac{\gamma-1}{c^2}$, the function

$$F_\nu(t) = \int_{\mathbb{R}^n} \left[(1 - \alpha \varepsilon u_t) (u_t)^2 + (c^2 - 2\varepsilon u_t) (\nabla u)^2 \right] dx + 2\varepsilon \int_0^t \int_{\mathbb{R}^n} u_{tt} |\nabla u|^2 dx dl$$

is constant if $\nu = 0$ and decreases if $\nu > 0$. Let us notice that while $\frac{1}{2} \leq 1 - \alpha \varepsilon u_t \leq \frac{3}{2}$, the coefficient $c^2 - 2\varepsilon u_t$ is always positive (since c is the sound speed in the chosen medium, $c^2 \gg 1$), hence the first integral in $F_\nu(t)$ is positive, but a priori we don't know the sign of the second integral, *i.e.* the sign of u_{tt} . However, for $\nu = 0$, $F_{\nu=0}(t)$ is positive, as soon as $0 \leq 1 - \alpha \varepsilon u_1$:

$$F_{\nu=0}(t) = F_{\nu=0}(0) = \int_{\mathbb{R}^n} \left[(1 - \alpha \varepsilon u_1) (u_1)^2 + (c^2 - 2\varepsilon u_1) (\nabla u_0)^2 \right] dx \geq 0,$$

and, if we take the same initial data for the Cauchy problems with $\nu = 0$ and $\nu > 0$, for all $t > 0$ (for all time where $F_{\nu=0}$ exists) it holds $F_{\nu=0}(t) = F_{\nu=0}(0) > F_{\nu>0}(t)$.

For $n \geq 3$, we can control the term $2\varepsilon \int_{\mathbb{R}^n} \nabla u \nabla u_t u_t \, dx$ using the Hölder inequality and the Sobolev embeddings (which fails in \mathbb{R}^2):

$$\left| \int_{\mathbb{R}^n} \nabla u \nabla u_t u_t \, dx \right| \leq \|\nabla u\|_{L^n} \|\nabla u_t\|_{L^2} \|u_t\|_{L^{\frac{2n}{n-2}}} \leq C \|\nabla u\|_{L^n} \|\nabla u_t\|_{L^2}^2.$$

Indeed, in \mathbb{R}^2 we don't have any estimates of the form

$$\|u\|_{L^p(\mathbb{R}^2)} \leq \|\nabla u\|_{L^2(\mathbb{R}^2)},$$

with $p > 2$. But such an estimate is essential to control the nonlinear term. Then, instead of Eq. (1.23) for F_ν , we have the relation for E_{nonl} :

$$\frac{1}{2} \frac{d}{dt} E_{nonl}(t) + (\nu\varepsilon - 2\varepsilon C \|\nabla u\|_{L^n}) \int_{\mathbb{R}^n} (\nabla u_t)^2 \, dx \leq 0.$$

So, if a solution of the Kuznetsov equation u is such that $\|\nabla u(t)\|_{L^n}$ and $\|u_t(t)\|_{L^\infty}$ stay small enough for all time, then E_{nonl} decreases in time and, as previously for the Westervelt equation, thanks to $\frac{1}{2}E(t) \leq E_{nonl}(t) \leq \frac{3}{2}E(t)$, the energy E has a decreasing function for upper bound.

This fact leads us to look for global well-posedness results for the Cauchy problem for the Kuznetsov equation in the viscous case.

1.4 Well-posedness for the inviscid case

1.4.1 Proof of Point 3 of Theorem 1.2.1

Let us give an estimation of the maximum existence time for a solution of problem (1.5)–(1.6) with $\nu = 0$. For this we follow the work of John [44] with the use of a priori estimates. However we don't directly apply the general results of John, but we improve them for our specific problem as we can take less regular initial conditions in order to have suitable a priori estimates.

Proposition 1.4.1. *For a fixed $m \in \mathbb{N}$ with $m \geq m_0 = \left\lceil \frac{n}{2} + 2 \right\rceil$, let u be a local solution of problem (1.5)–(1.6) with $\nu = 0$ on $[0, T]$ satisfying (1.8) and (1.9) for $s = m$.*

For $t \in [0, T]$ we have for $E_m[u](t)$, defined in Eq. (1.10),

$$E_m[u](t) \leq B E_m[u](0) + C_m \max(\alpha, \beta) \varepsilon \int_0^t E_m[u](\tau)^{\frac{3}{2}} \, d\tau, \quad (1.24)$$

with constants $B = \frac{(3+2c^2)}{\min(1/2, c^2)} > 0$, depending only on c , and $C_m > 0$, depending only on m , on the dimension n and on c (only if $\min(1/2, c^2) = c^2$).

Proof. Following [44], let us consider

$$L_u v = v_{tt} - c^2 \Delta v - \alpha \varepsilon u_t v_{tt} - \beta \varepsilon \nabla u \nabla v_t, \quad (1.25)$$

where u is a local solution on $[0, T]$ of problem (1.5)–(1.6) with $\nu = 0$, satisfying (1.8) and (1.9) for $s = m$. We multiply Eq. (1.25) by v_t and integrate over \mathbb{R}^n

$$\begin{aligned} & \int_{\mathbb{R}^n} L_u v v_t dx \\ &= \frac{1}{2} \frac{d}{dt} \left(\int_{\mathbb{R}^n} v_t^2 + c^2 (\nabla v)^2 dx \right) - \alpha \varepsilon \int_{\mathbb{R}^n} u_t v_{tt} v_t dx - \beta \varepsilon \int_{\mathbb{R}^n} \nabla u \nabla v_t v_t dx \\ &= \frac{1}{2} \frac{d}{dt} \left(\int_{\mathbb{R}^n} v_t^2 + c^2 (\nabla v)^2 dx \right) - \frac{\alpha}{2} \varepsilon \left[\frac{d}{dt} \left(\int_{\mathbb{R}^n} u_t v_t^2 dx \right) - \int_{\mathbb{R}^n} u_{tt} v_t^2 dx \right] \\ & \quad + \frac{\beta}{2} \varepsilon \int_{\mathbb{R}^n} \Delta u (v_t)^2 dx. \end{aligned}$$

Hence, denoting by

$$I[v] = v_t^2 + c^2 (\nabla v)^2 - \alpha \varepsilon u_t v_t^2, \quad (1.26)$$

$$J[v] = 2L_u v v_t - [\alpha \varepsilon u_{tt} + \beta \varepsilon \Delta u] (v_t)^2, \quad (1.27)$$

we have the following equation

$$\frac{d}{dt} \int_{\mathbb{R}^n} I[v](t, x) dx = \int_{\mathbb{R}^n} J[v](t, x) dx. \quad (1.28)$$

Let $A = (A_0, A_1, \dots, A_n)$ be a multi-index, and $D^A = \partial_t^{A_0} \partial_{x_1}^{A_1} \dots \partial_{x_n}^{A_n}$. To prove estimate (1.24), we study $|\int_{\mathbb{R}^n} J[v](t, x) dx|$ for $v = D^A u$ with $|A| = A_0 + \dots + A_n \leq m$.

For $m \geq \left[\frac{n}{2} + 2\right]$ and a multi-index A with $|A| \leq m$ we estimate, thanks to the definition of $E_m[u]$ by Eq. (1.10),

$$\begin{aligned} \int_{\mathbb{R}^n} |u_{tt} (D^A u_t)^2| dx &\leq \|u_{tt}\|_{L^\infty(\mathbb{R}^n)} \|D^A u_t\|_{L^2(\mathbb{R}^n)}^2 \\ &\leq C \|u_{tt}\|_{H^{[\frac{n}{2}+1]}(\mathbb{R}^n)} E_m[u] \leq C E_m[u]^{\frac{3}{2}}, \end{aligned} \quad (1.29)$$

with a constant $C > 0$, depending only on n by the Sobolev embedding [2] Theorem 7.57 p. 228

$$H^s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n) \quad \text{for } s > \frac{n}{2}. \quad (1.30)$$

In the same way, using the Sobolev embedding (1.30), we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |\Delta u (D^A u_t)^2| dx &\leq \|\Delta u\|_{L^\infty(\mathbb{R}^n)} \|D^A u_t\|_{L^2(\mathbb{R}^n)}^2 \leq C \|\Delta u\|_{H^{[\frac{n}{2}+1]}(\mathbb{R}^n)} E_m[u] \\ &\leq C \|\nabla u\|_{H^m(\mathbb{R}^n)} E_m[u] \leq C E_m[u]^{\frac{3}{2}}. \end{aligned} \quad (1.31)$$

To calculate $L_u D^A u$ we apply the chain rule of differentiation to $D^A L_u u = 0$. As $L_u u = 0$ we suppose $|A| \geq 1$. By developing $D^A (\nabla u \nabla u_t) = \sum_{i=1}^n D^A (\partial_{x_i} u \partial_{x_i} u_t)$ with $D^A (u_t u_{tt})$, we have

$$L_u D^A u = \varepsilon \sum_j \left(C_j \alpha D^{A^{j1}} u_t D^{A^{j2}} u_t + \sum_{i=1}^n E_{ij} \beta D^{A^{j1}} \partial_{x_i} u D^{A^{j2}} \partial_{x_i} u \right), \quad (1.32)$$

where \sum_j is a finite sum, with C_j and E_{ij} depending only on $|A| \leq m$, and A^{j1} and A^{j2} are multi-index such that

$$\begin{cases} |A^{j1}| + |A^{j2}| = |A| + 1, \\ |A^{j1}| \geq 1, \quad |A^{j2}| \geq 1, \\ A_0^{j1} + A_0^{j2} = A_0 + 1, \quad A_i^{j1} + A_i^{j2} = A_i \quad \text{for } 1 \leq i \leq n. \end{cases} \quad (1.33)$$

Let us show for $m \geq \left[\frac{n}{2} + 2\right]$ the estimate

$$\left| \int_{\mathbb{R}^n} L_u D^A u D^A u_t dx \right| \leq C\varepsilon \max(\alpha, \beta) E_m[u]^{\frac{3}{2}}. \quad (1.34)$$

Without loss of generality, we consider two multi-indexes A^1 and A^2 satisfying (1.33) and divide the proof of (1.34) in two parts: we estimate $\int_{\mathbb{R}^n} |D^{A^1} u_t D^{A^2} u_t D^A u_t| dx$ first, and $\int_{\mathbb{R}^n} |D^{A^1} \partial_{x_i} u D^{A^2} \partial_{x_i} u D^A u_t| dx$ secondly. As for each part the proof is very similar, we give the details only for the first one.

To estimate $\int_{\mathbb{R}^n} |D^{A^1} u_t D^{A^2} u_t D^A u_t| dx$, we consider three cases:

Case 1 $1 < |A^1| < m$ and $1 < |A^2| < m$,

Case 2 $|A^1| \leq m$ and $|A^2| = 1$,

Case 3 $|A^2| \leq m$ and $|A^1| = 1$.

Let us detail **Case 1** (other cases can be treated in a similar way).

For $2 \leq |A^1| \leq m - 1$ and $2 \leq |A^2| \leq m - 1$, it holds

$$\int_{\mathbb{R}^n} |D^{A^1} u_t D^{A^2} u_t D^A u_t| dx \leq \|D^{A^1} u_t\|_{L^p(\mathbb{R}^n)} \|D^{A^2} u_t\|_{L^q(\mathbb{R}^n)} \|D^A u_t\|_{L^2(\mathbb{R}^n)},$$

with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ by the general Hölder inequality [17]. Hence, using the Sobolev embedding [2]

$$H^{m_1}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n) \quad \text{with } \frac{1}{p} = \frac{1}{2} - \frac{m_1}{n} \text{ and } 0 < m_1 < \frac{n}{2}, \quad (1.35)$$

we find

$$\int_{\mathbb{R}^n} |D^{A^1} u_t D^{A^2} u_t D^A u_t| dx \leq C \|D^{A^1} u_t\|_{H^{m_1}(\mathbb{R}^n)} \|D^{A^2} u_t\|_{H^{\frac{n}{2}-m_1}(\mathbb{R}^n)} \|D^A u_t\|_{L^2(\mathbb{R}^n)}.$$

In what follows by $C > 0$ is denoted as an arbitrary constant depending only on m and on n .

We have

$$\begin{aligned} \|D^{A^1} u_t\|_{H^{m_1}(\mathbb{R}^n)} &\leq \|\partial_t^{A_0^1} u_t\|_{H^{m_1+|A^1|-A_0^1}(\mathbb{R}^n)}, \\ \|D^{A^2} u_t\|_{H^{\frac{n}{2}-m_1}(\mathbb{R}^n)} &\leq \|\partial_t^{A_0^2} u_t\|_{H^{\frac{n}{2}-m_1+|A^2|-A_0^2}(\mathbb{R}^n)}. \end{aligned}$$

We need to find m for which there exists m_1 with $0 < m_1 < \frac{n}{2}$, such that

$$\begin{cases} m_1 + |A^1| - A_0^1 \leq m + 1 - (A_0^1 + 1), \\ \frac{n}{2} - m_1 + |A^2| - A_0^2 \leq m + 1 - (A_0^2 + 1), \end{cases} \quad (1.36)$$

or equivalently, by (1.33) $|A^2| = |A| + 1 - |A^1|$,

$$\begin{cases} m_1 + |A^1| \leq m, \\ \frac{n}{2} - m_1 + |A| + 1 - |A^1| \leq m. \end{cases}$$

As $m - |A| \geq 0$ it is sufficient to find m_1 , such that

$$\frac{n}{2} + 1 \leq m_1 + |A^1| \leq m$$

with $2 \leq |A^1| \leq m - 1$ and $0 < m_1 < \frac{n}{2}$. In particular, the last three inequalities imply that $m \geq \left[2 + \frac{n}{2}\right]$. For the existence of m_1 , we see that, for instance,

if $|A_1| = 2$ we can take $m_1 = \frac{n}{2} - \frac{1}{4}$,

if $2 < |A_1| < \frac{n}{2} + 1$ we can take $m_1 = \frac{n}{2} + 1 - |A_1|$,

if $\frac{n}{2} + 1 \leq |A_1| \leq m - 1$ we can take $m_1 = \frac{1}{4}$.

Moreover,

$$\|D^A u_t\|_{L^2(\mathbb{R}^n)} \leq \|\partial_t^{A_0} u_t\|_{H^{|A|-A_0}(\mathbb{R}^n)} \leq \|\partial_t^{A_0} u_t\|_{H^{m-A_0}(\mathbb{R}^n)}.$$

Then, thanks to relations (1.36), we conclude

$$\begin{aligned} \int_{\mathbb{R}^n} |D^{A^1} u_t D^{A^2} u_t D^A u_t| dx &\leq C \|\partial_t^{A_0^1} u_t\|_{H^{m-A_0^1}(\mathbb{R}^n)} \|\partial_t^{A_0^2} u_t\|_{H^{m-A_0^2}(\mathbb{R}^n)} \|\partial_t^{A_0} u_t\|_{H^{m-A_0}(\mathbb{R}^n)} \\ &\leq C E_m[u]^{\frac{3}{2}}. \end{aligned}$$

Consequently, for $m \geq \left[\frac{n}{2} + 2\right]$, and A^1 and A^2 , satisfying properties (1.33), it holds

$$\int_{\mathbb{R}^n} |D^{A^1} u_t D^{A^2} u_t D^A u_t| dx \leq C E_m[u]^{\frac{3}{2}}. \quad (1.37)$$

By the same argument, for $m \geq \left[\frac{n}{2} + 2\right]$ and A^1 and A^2 , satisfying properties (1.33), we control the terms of the form $\int_{\mathbb{R}^n} |D^{A^1} \partial_{x_i} u D^{A^2} \partial_{x_i} u D^A u_t| dx$:

$$\int_{\mathbb{R}^n} |D^{A^1} \partial_{x_i} u D^{A^2} \partial_{x_i} u D^A u_t| dx \leq C E_m[u]^{\frac{3}{2}}. \quad (1.38)$$

Thus, considering (1.32), (1.37) and (1.38) for $m \geq \left[\frac{n}{2} + 2\right]$ and for a multi-index A with $|A| \leq m$, we have estimate (1.34).

Thanks to estimates (1.29), (1.31) and (1.34), we are able to control each term of $J[D^A u]$ from Eq. (1.27):

$$\left| \int_{\mathbb{R}^n} J[D^A u](t, x) dx \right| \leq C \max(\alpha, \beta) \varepsilon E_m[u](t)^{\frac{3}{2}}. \quad (1.39)$$

With the hypothesis that u is a local solution of the inviscid Kuznetsov equation, u satisfies Eq. (1.9), *i.e.* $\|u_t(t)\|_{L^\infty} \leq \frac{1}{2\alpha\varepsilon}$ on $[0, T]$, which implies the equivalence of energies

$$\int_{\mathbb{R}^n} \frac{1}{2} (D^A u_t)^2 + c^2 (\nabla D^A u)^2 dx \leq \int_{\mathbb{R}^n} I[D^A u] dx \leq \int_{\mathbb{R}^n} \frac{3}{2} (D^A u_t)^2 + c^2 (\nabla D^A u)^2 dx.$$

We integrate relation (1.28) over $[0, t]$ with $t \leq T$ to obtain

$$\begin{aligned} &\|D^A u_t(t)\|_{L^2(\mathbb{R}^n)}^2 + \|\nabla D^A u(t)\|_{L^2(\mathbb{R}^n)}^2 \\ &\leq \frac{\left(\frac{3}{2} + c^2\right)}{\min(1/2, c^2)} (\|D^A u_t(0)\|_{L^2(\mathbb{R}^n)}^2 + \|\nabla D^A u(0)\|_{L^2(\mathbb{R}^n)}^2) \\ &\quad + \frac{1}{\min(1/2, c^2)} \int_0^t \int_{\mathbb{R}^n} J(\tau, x) dx d\tau. \end{aligned}$$

Then, using estimate (1.39), we find

$$\begin{aligned} &\|D^A u_t(t)\|_{L^2(\mathbb{R}^n)}^2 + \|\nabla D^A u(t)\|_{L^2(\mathbb{R}^n)}^2 \\ &\leq \frac{\left(\frac{3}{2} + c^2\right)}{\min(1/2, c^2)} (\|D^A u_t(0)\|_{L^2(\mathbb{R}^n)}^2 + \|\nabla D^A u(0)\|_{L^2(\mathbb{R}^n)}^2) \\ &\quad + \frac{1}{\min(1/2, c^2)} C \max(\alpha, \beta) \varepsilon \int_0^t E_m[u](\tau)^{\frac{3}{2}} d\tau. \end{aligned}$$

As we have this for all multi-index A with $|A| \leq m$, by summing, we obtain

$$E_m[u](t) \leq \frac{(3 + 2c^2)}{\min(1/2, c^2)} E_m[u](0) + \frac{C \max(\alpha, \beta)}{\min(1/2, c^2)} \varepsilon \int_0^t E_m[u](\tau)^{\frac{3}{2}} d\tau$$

with a constant $C > 0$, depending only on n and m . This gives the estimate (1.24). \square

Inequality (1.24), proved in Proposition 1.4.1, gives us an a priori estimate in order to have, with the help of the Gronwall Lemma, an estimation of the maximum existence time T^* . However, when m increases, C_m increases, and the maximum existence time, given by estimate (1.24), decreases whereas the initial conditions become more regular. Therefore, we prove the second a priori estimate (see Eq. (1.40)), playing a key role in order to avoid this problem:

Proposition 1.4.2. *Let conditions of Proposition 1.4.1 be satisfied. Then for $t \in [0, T]$ and $m \geq \lceil \frac{n}{2} + 3 \rceil$ we have*

$$E_m[u](t) \leq B E_m[u](0) + D_m \max(\alpha, \beta) \varepsilon \int_0^t E_{m-1}[u](\tau)^{\frac{1}{2}} E_m[u](\tau) d\tau, \quad (1.40)$$

with a constant $D_m > 0$, depending only on m , on n and on c and the same constant B as in Proposition 1.4.1.

Proof. Using the proof of Proposition 1.4.1 for A a multi-index with $|A| \leq m$ and $J[D^A u]$ from Eq. (1.27), we just have to show that for $m \geq \lceil \frac{n}{2} + 3 \rceil$

$$\left| \int_{\mathbb{R}^n} J[D^A u](s, x) dx \right| \leq C \varepsilon \sqrt{E_{m-1}[u] E_m[u]}.$$

Indeed, for $m \geq \lceil \frac{n}{2} + 3 \rceil$ and a multi-indexes A with $|A| \leq m$ we have

$$\int_{\mathbb{R}^n} |u_{tt}(D^A u_t)^2| dx \leq \|u_{tt}\|_{L^\infty} \|D^A u_t\|_{L^2}^2 \leq C \|u_{tt}\|_{H^{m-2}} E_m[u] \leq C \sqrt{E_{m-1}[u] E_m[u]},$$

and

$$\int_{\mathbb{R}^n} |\Delta u (D^A u_t)^2| dx \leq \|\Delta u\|_{L^\infty} \|D^A u_t\|_{L^2}^2 \leq C \|\nabla u\|_{H^{m-1}} E_m[u] \leq C \sqrt{E_{m-1}[u] E_m[u]}.$$

Now we consider two multi-indexes A^1 and A^2 with properties (1.33). As previously, we have to distinguish three cases:

Case 1: $1 < |A^1| < m$ and $1 < |A^2| < m$,

Case 2: $|A^1| \leq m$ and $|A^2| = 1$,

Case 3: $|A^2| \leq m$ and $|A^1| = 1$.

First, if $1 < |A^1| < m$ and $1 < |A^2| < m$, in Case 1 we have

$$\begin{aligned} \int_{\mathbb{R}^n} |D^{A^1} u_t D^{A^2} u_t D^A u_t| dx &\leq C \|\partial_t^{A_0^1} u_t\|_{H^{m_1+|A^1|-A_0^1}} \|\partial_t^{A_0^2} u_t\|_{H^{\frac{n}{2}-m_1+|A^2|-A_0^2}} \|D^A u_t\|_{L^2} \\ &\leq \|\partial_t^{A_0^1} u_t\|_{H^{m_1+|A^1|-A_0^1}} \|\partial_t^{A_0^2} u_t\|_{H^{\frac{n}{2}-m_1+|A^2|-A_0^2}} \sqrt{E_m[u]}. \end{aligned}$$

By symmetry of their roles we can take $|A_1| \leq \left\lceil \frac{m+1}{2} \right\rceil \leq |A_2|$. We look for m for which there exists m_1 , such that

$$\begin{cases} m_1 + |A^1| - A_0^1 \leq m - (A_0^1 + 1), \\ \frac{n}{2} - m_1 + |A^2| - A_0^2 \leq m + 1 - (A_0^2 + 1), \end{cases}$$

in order to have

$$\|\partial_t^{A_0^1} u_t\|_{H^{m_1+|A^1|-A_0^1}} \leq \sqrt{E_{m-1}[u]} \quad \text{and} \quad \|\partial_t^{A_0^2} u_t\|_{H^{\frac{n}{2}-m_1+|A^2|-A_0^2}} \leq \sqrt{E_m[u]}.$$

As $|A_2| = |A| + 1 - |A^1|$ and $m \geq |A|$, it is sufficient to find m_1 , such that

$$\frac{n}{2} + 1 \leq m_1 + |A^1| \leq m - 1,$$

with $2 \leq |A^1| \leq m - 1$, $|A_1| \leq \left\lceil \frac{m+1}{2} \right\rceil$ and $0 < m_1 < \frac{n}{2}$. This directly implies that $m \geq \left\lceil \frac{n}{2} + 3 \right\rceil$. In addition, this also implies, except if $m \leq 3$, that $|A^1| \leq m - 2$. Thus the existence of a required m_1 is justified in Case 1 of the proof of estimate (1.24). If $m \leq 3$, as $m \geq \left\lceil \frac{n}{2} + 3 \right\rceil$, we have only the case $m = 3$, $n = 1$ and $|A^1| = |A^2| = 2$, for which it holds

$$\begin{aligned} \int_{\mathbb{R}^n} |D^{A^1} u_t D^{A^2} u_t D^A u_t| dx &\leq \|D^{A^1} u_t\|_{L^2} \|D^{A^2} u_t\|_{L^\infty} \|D^A u_t\|_{L^2} \\ &\leq C \|D^{A^1} u_t\|_{L^2} \|D^{A^2} u_t\|_{H^1} \|D^A u_t\|_{L^2} \leq C \sqrt{E_{m-1}[u]} E_m[u], \end{aligned}$$

by the Sobolev embedding (1.30).

For $|A^1| \leq m$ and $|A^2| = 1$, in Case 2, we find

$$\int_{\mathbb{R}^n} |D^{A^1} u_t D^{A^2} u_t D^A u_t| dx \leq \|D^{A^1} u_t\|_{L^2} \|D^{A^2} u_t\|_{L^\infty} \|D^A u_t\|_{L^2},$$

where the two L^2 -norms are controlled by $\sqrt{E_m[u]}$:

$$\|D^{A^1} u_t\|_{L^2} \leq \|\partial_t^{A_0^1} u_t\|_{H^{m-A_0^1}} \leq \sqrt{E_m[u]}, \quad \|D^A u_t\|_{L^2} \leq \|\partial_t^{A_0} u_t\|_{H^{m-A_0}} \leq \sqrt{E_m[u]}.$$

As $A_0^2 \leq 1$, for the L^∞ -norm, for $m \geq \left\lceil \frac{n}{2} + 3 \right\rceil$, we have

$$\|D^{A^2} u_t\|_{L^\infty} \leq C \|D^{A^2} u_t\|_{H^{\left\lceil \frac{n}{2} + 1 \right\rceil}} \leq C \|\partial_t^{A_0^2} u_t\|_{H^{\left\lceil \frac{n}{2} + 2 \right\rceil - A_0^2}} \leq C \sqrt{E_{m-1}[u]}.$$

Case 3, *i.e.* for $|A^2| \leq m$ and $|A^1| = 1$, can be treated in the same way.

So, if $m \geq \left\lceil \frac{n}{2} + 3 \right\rceil$, and A^1 and A^2 satisfy properties (1.33), we obtain

$$\int_{\mathbb{R}^n} |D^{A^1} u_t D^{A^2} u_t D^A u_t| dx \leq C \sqrt{E_{m-1}[u]} E_m[u].$$

Using similar arguments, we can show with the same restrictions on m , A^1 and A^2 that

$$\int_{\mathbb{R}^n} |D^{A^1} \partial_{x_i} u D^{A^2} \partial_{x_i} u D^A u_t| dx \leq C \sqrt{E_{m-1}[u]} E_m[u].$$

Consequently,

$$\int_{\mathbb{R}^n} |L_u D^A u D^A u_t| dx \leq C \max(\alpha, \beta) \sqrt{E_{m-1}[u]} E_m[u],$$

from where follows the estimate (1.40). \square

Now let us give a first estimation of the lifespan T^* of a local solution of problem (1.5)–(1.6) with $\nu = 0$.

Theorem 1.4.1. *Let $m \geq m_0 = \left[\frac{n}{2} + 2\right]$ and let u be the unique solution on $[0, T^*[$ of problem (1.5)–(1.6) with $\nu = 0$ for*

$$u_0 \in H^{m+1}(\mathbb{R}^n), \quad u_1 \in H^m(\mathbb{R}^n) \quad \text{and} \quad \|u_1\|_{L^\infty(\mathbb{R}^n)} < \frac{1}{2\alpha\varepsilon}.$$

If $\sqrt{E_{m_0}[u](0)} \leq \frac{1}{4\sqrt{BC_\infty\alpha\varepsilon}}$, then

$$T^* > T_0 = \frac{1}{C_{m_0} \max(\alpha, \beta)\varepsilon\sqrt{B}E_{m_0}[u](0)} \quad (1.41)$$

and

$$u \in C^r([0, T_0]; H^{m+1-r}) \text{ for } 0 \leq r \leq m+1,$$

with

$$\forall t \in [0, T_0], \quad E_m[u](t) \leq C < +\infty.$$

Here B and C_{m_0} are the constants from estimate (1.24) and C_∞ is the embedding constant from the embedding of the Sobolev space $H^{\left[\frac{n}{2}+1\right]}(\mathbb{R}^n)$ in $L^\infty(\mathbb{R}^n)$.

Proof. Thanks to Point 1 of Theorem 1.2.1, for $u_0 \in H^{m+1}(\mathbb{R}^n)$, $u_1 \in H^m(\mathbb{R}^n)$ and $\|u_1\|_{L^\infty(\mathbb{R}^n)} < \frac{1}{2\alpha\varepsilon}$ there exists a unique solution u on a sufficiently small interval $[0, T]$ of problem (1.5)–(1.6) with $\nu = 0$, satisfying (1.8) and (1.9) for $s = m$. Moreover it implies that $E_m[u](0)$ is finite. Hence, we can add the hypothesis

$$\sqrt{E_{m_0}[u](0)} \leq \frac{1}{4\sqrt{BC_\infty\alpha\varepsilon}}$$

without adding further conditions of regularity on u_0 and u_1 as it can be reduced on a smallness condition on $\|u_0\|_{H^{m+1}(\mathbb{R}^n)} + \|u_1\|_{H^m(\mathbb{R}^n)}$.

Let us take T_0 , as defined in Eq. (1.41), and show by induction on $j \in \mathbb{N}$ with $m_0 \leq j \leq m$ that

$$\forall j \in \mathbb{N}, \text{ with } m_0 \leq j \leq m \quad \sup_{t \in [0, T_0]} E_j[u](t) < \infty.$$

For $j = m_0$, $u_0 \in H^{m+1}(\mathbb{R}^n) \subseteq H^{m_0+1}(\mathbb{R}^n)$ and $u_1 \in H^m(\mathbb{R}^n) \subseteq H^{m_0}(\mathbb{R}^n)$, and consequently

$$E_{m_0}[u](0) \leq E_m[u](0) < \infty.$$

For $t \geq 0$, while $\|u_t(t)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{2\alpha\varepsilon}$, it holds the estimate (1.24) with $m = m_0$. According to the Gronwall Lemma, applied to (1.24) with $m = m_0$, we have

$$E_{m_0}[u](t) \leq z(t),$$

where $z(t)$ is the solution of the Cauchy problem for an ordinary differential equation

$$z(t) = z_0 + C_{m_0} \max(\alpha, \beta)\varepsilon \int_0^t (z(\tau))^{3/2} d\tau \quad \text{with } z_0 = B E_{m_0}[u](0).$$

This problem can be solved explicitly:

$$z(t) = \frac{z_0}{\left(1 - \frac{1}{2}z_0^{1/2} C_{m_0} \max(\alpha, \beta)\varepsilon t\right)^2}.$$

We can see that, as long as $0 \leq t \leq T_0$, the function $z(t)$ has the finite upper bound $z(t) \leq 4z_0$. It implies the upper boundness of $E_{m_0}[u]$:

$$E_{m_0}[u](t) \leq 4B E_{m_0}[u](0). \quad (1.42)$$

Moreover, thanks to our notations,

$$\frac{\|u_t(t)\|_{L^\infty(\mathbb{R}^n)}}{C_\infty} \leq \|u_t(t)\|_{H^{\lfloor \frac{n}{2} + 1 \rfloor}} \leq \sqrt{E_{m_0}[u](t)},$$

from where, using inequality (1.42), we find

$$\|u_t(t)\|_{L^\infty(\mathbb{R}^n)} \leq 2C_\infty \sqrt{B E_{m_0}[u](0)} \leq \frac{1}{2\alpha\varepsilon},$$

since $\sqrt{E_{m_0}[u](0)} \leq \frac{1}{4\sqrt{B}C_\infty\alpha\varepsilon}$. Thus Eq. (1.9) holds on all interval $[0, T_0]$ and $\sup_{t \in [0, T_0]} E_{m_0}[u](t)$ is finite.

Let $j \in \mathbb{N}$, $m_0 \leq j \leq m - 1$ be such that $\sup_{t \in [0, T_0]} E_j[u](t) < \infty$.

Since Eq. (1.9) holds on all interval $[0, T_0]$, we can use the a priori estimate (1.40) and write that for all $t \in [0, T_0]$

$$E_{j+1}[u](t) \leq B E_{j+1}[u](0) + D_{j+1} \max(\alpha, \beta) \varepsilon \int_0^t \sqrt{E_j[u](\tau)} E_{j+1}[u](\tau) d\tau.$$

By the induction hypothesis $\sup_{t \in [0, T_0]} E_j[u](t)$ is bounded by a constant, denoted here by E^2 , and hence on $[0, T_0]$ it holds

$$E_{j+1}[u](t) \leq B E_{j+1}[u](0) + D_{j+1} \max(\alpha, \beta) E \varepsilon \int_0^t E_{j+1}[u](\tau) dl.$$

Applying the Gronwall Lemma, we obtain for $t \in [0, T_0]$

$$E_{j+1}[u](t) \leq B E_{j+1}[u](0) e^{D_{j+1} \max(\alpha, \beta) E \varepsilon t} \leq B E_{j+1}[u](0) e^{D_{j+1} \max(\alpha, \beta) E \varepsilon T_0}.$$

This means, as $E_{j+1}[u](0) \leq E_m[u](0) < +\infty$, that $\sup_{t \in [0, T_0]} E_{j+1}[u](t) < \infty$ and this finishes the proof. \square

Theorem 1.4.1 estimates the lifespan T^* as at least of the order $\frac{1}{\varepsilon}$, or more precisely, implies that

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon T^* > 0.$$

This result is independent on the dimension n . However, much better estimations for the lifespan can be obtained, if we use an inequality that takes into account the time decay of the solutions for $n > 1$, what we do in the next section.

1.4.2 Proof of Point 5 of Theorem 1.2.1. Optimal estimations of the existence time

In [44] John uses the group of linear transformations preserving the equation $u_{tt} - \Delta u = 0$. The generators of this group (the derivatives with respect to group parameters taken at the identity), here called generalized derivatives, include in addition to the derivatives $\partial_t, \partial_{x_1}, \dots, \partial_{x_n}$, first-order differential operators L_α with $\alpha = 0, \dots, n$ and Ω_{ik} with $1 \leq i < k \leq n$:

Definition 1.4.1. (Generalized derivatives [44]) The following operators

$$L_0 = t\partial_t + \sum_i x_i \partial_{x_i}, \quad L_i = x_i \partial_t + t \partial_{x_i} \text{ for } i = 1, \dots, n,$$

$$\Omega_{ik} = x_i \partial_{x_k} - x_k \partial_{x_i} \text{ for } 1 \leq i < k \leq n, \text{ and } \partial_t, \partial_{x_i} \text{ for } i = 1, \dots, n$$

are called the generalized derivatives. The operators

$$L_0, \dots, L_n, \Omega_{12}, \Omega_{13}, \dots, \Omega_{n-1n}, \partial_t, \partial_{x_1}, \dots, \partial_{x_n},$$

(taken in this order) are denoted respectively by $\Gamma_0, \dots, \Gamma_\mu$ with $\mu = \frac{1}{2}(n^2 + 3n + 2)$. For a multi-index $A = (A_0, \dots, A_\mu)$ we write in the usual way

$$|A| = A_0 + \dots + A_\mu, \quad \Gamma^A = (\Gamma_0)^{A_0} (\Gamma_1)^{A_1} \dots (\Gamma_\mu)^{A_\mu}.$$

Therefore, in the framework of the general derivatives, we define for $m \in \mathbb{N}$

$$E_{\infty, m}[u](t) = \sup_x \left| \sup_{|A| \leq m} [(\Gamma^A \partial_t u(t, x))^2 + (\Gamma^A \nabla u(x, t))^2] \right|, \quad (1.43)$$

$$E_{1, m}[u](t) = \sum_{|A| \leq m} (\|\Gamma^A \partial_t u\|_{L^2(\mathbb{R}^n)}^2 + \|\Gamma^A \nabla u\|_{L^2(\mathbb{R}^n)}^2)(t). \quad (1.44)$$

We give a remarkable estimate proved in Ref. [59] by Klainerman:

Proposition 1.4.3. (Klainerman 1987) For $n^* = [\frac{n}{2} + 1]$, $m \in \mathbb{N}$, and $t > 0$, as soon as u is such that $E_{1, m+n^*}[u](t)$ is finite, it holds

$$\sqrt{E_{\infty, m}[u](t)} \leq C_n (1+t)^{\frac{1-n}{2}} \sqrt{E_{1, m+n^*}[u](t)}. \quad (1.45)$$

Thanks to Proposition 1.4.3, we improve the results of John [44] for the case of the Kuznetsov equation and state:

Proposition 1.4.4. For n and m in \mathbb{N}^* , $m \geq n+2$, let u be a local solution on an interval $[0, T]$ of problem (1.5)–(1.6) with $\nu = 0$, satisfying (1.8) and (1.9) with $s = m$. Then for all $t \in [0, T]$, it holds

$$E_{1, m}[u](t) \leq B E_{1, m}[u](0) + C_m \max(\alpha, \beta) \varepsilon \int_0^t (1+\tau)^{(1-n)/2} E_{1, m}[u](\tau)^{\frac{3}{2}} d\tau, \quad (1.46)$$

with a positive constant $B > 0$, depending only on c , on α and on β , and with a positive constant $C_m > 0$, depending only on m , on n and on c .

Proof. The proof follows identically the proof of Proposition 1.4.1 up to Eq. (1.32) replacing everywhere D^A by Γ^A . This time Eq. (1.32) becomes

$$L_u \Gamma^A u = \varepsilon \sum_{j=0}^{\mu} \left(\alpha C_j \Gamma^{A^{j1}} u_t \Gamma^{A^{j2}} u_t + \sum_{i=1}^n \beta E_{ij} \Gamma^{A^{j1}} \partial_{x_i} u \Gamma^{A^{j2}} \partial_{x_i} u \right), \quad (1.47)$$

where μ is defined in Definition 1.4.1, C_j and E_{ij} depend only on $|A| \leq m$, and A^{j1} and A^{j2} are multi-indexes, such that

$$|A^{j1}| + |A^{j2}| \leq m + 1.$$

It follows that $|A^{j1}| \leq [\frac{m+1}{2}]$ or $|A^{j2}| \leq [\frac{m+1}{2}]$. Therefore, if we set $m' = [\frac{m+1}{2}]$, we obtain

$$\begin{aligned} |J[\Gamma^A u](\tau, x)| &\leq C_m \max(\alpha, \beta) \varepsilon \sqrt{\sup_{|B| \leq m'} \left((\Gamma^B \partial_t u(\tau, x))^2 + (\Gamma^B \nabla u(\tau, x))^2 \right)} \\ &\quad \cdot \sup_{|B| \leq m} \left((\Gamma^B \partial_t u(\tau, x))^2 + (\Gamma^B \nabla u(\tau, x))^2 \right) \\ &\leq C_m \max(\alpha, \beta) \varepsilon \sqrt{E_{\infty, m'}[u](\tau)} \sum_{|B| \leq m} \left((\Gamma^B \partial_t u(\tau, x))^2 + (\Gamma^B \nabla u(\tau, x))^2 \right), \end{aligned}$$

and thus

$$\left| \int_{\mathbb{R}^n} J[\Gamma^A u](\tau, x) dx \right| \leq C_m \max(\alpha, \beta) \varepsilon \sqrt{E_{\infty, m'}[u](\tau)} E_{1, m}[u](\tau).$$

By hypothesis on u ,

$$\|u_t(t)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{2\alpha\varepsilon} \text{ on } [0, T],$$

and then, by integrating of Eq. (1.28) on $[0, t]$ with $t \in [0, T]$, we have

$$\begin{aligned} \frac{1}{2} \|\partial_t \Gamma^A u(t)\|_{L^2(\mathbb{R}^n)}^2 + c^2 \|\nabla \Gamma^A u(t)\|_{L^2(\mathbb{R}^n)}^2 &\leq \frac{3}{2} \|\partial_t \Gamma^A u(0)\|_{L^2(\mathbb{R}^n)}^2 + c^2 \|\nabla \Gamma^A u(0)\|_{L^2(\mathbb{R}^n)}^2 \\ &\quad + C_m \max(\alpha, \beta) \varepsilon \int_0^t \sqrt{E_{\infty, m'}[u](\tau)} E_{1, m}[u](\tau) d\tau. \end{aligned}$$

By summing for $|A| \leq m$, we obtain

$$E_{1, m}[u](t) \leq B E_{1, m}[u](0) + C_m \max(\alpha, \beta) \varepsilon \int_0^t \sqrt{E_{\infty, m'}[u](\tau)} E_{1, m}[u](\tau) d\tau.$$

Now we use the Klainerman inequality (1.45), noticing that, if we take $m \geq n + 2$, we have

$$m' + n^* = \left[\frac{m+1}{2} \right] + \left[\frac{n}{2} + 1 \right] \leq m.$$

This finishes the proof. \square

We use the a priori estimate (1.46) to improve our estimation of the lifespan T^* as a function of n .

Theorem 1.4.2. *Let $m \geq n + 2$. For $u_0 \in H^{m+1}(\mathbb{R}^n)$ and $u_1 \in H^m(\mathbb{R}^n)$ with $\|u_1\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{2\alpha\varepsilon}$ we consider the local solution u of problem (1.5)–(1.6) with $\nu = 0$ on an interval $[0, T]$, satisfying (1.8) and (1.9) for $s = m$ as in Point 1 of Theorem 1.2.1. If $\sqrt{E_{1, m}[u](0)} \leq \frac{1}{4\sqrt{BC_\infty\alpha\varepsilon}}$, then*

$$E_{1, m}[u](t) \leq 4B E_{1, m}[u](0),$$

as long as

$$\begin{aligned} t &\leq \left(2C_m \max(\alpha, \beta) \varepsilon \sqrt{B E_{1, m}[u](0)} \right)^{-2} \text{ for } n = 2, \\ t &\leq 2 \exp \left(\frac{1}{C_m \max(\alpha, \beta) \varepsilon \sqrt{B E_{1, m}[u](0)}} \right) \text{ for } n = 3, \\ 1 &\leq \left(2C_m \max(\alpha, \beta) \varepsilon \sqrt{B E_{1, m}[u](0)} \right)^{-1} \text{ for } n \geq 4. \end{aligned}$$

Consequently,

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \varepsilon^2 T^* &> 0 \quad \text{for } n = 2, \\ \liminf_{\varepsilon \rightarrow 0} \varepsilon \log(T^*) &> 0 \quad \text{for } n = 3, \end{aligned}$$

and, for a small enough ε , $T^* = +\infty$ for $n \geq 4$, i.e. the solution u is global.

Proof. This is a direct consequence of the Gronwall lemma, used with the a priori estimate (1.46), as done by John in [44]. \square

Remark 1.4.1. The estimations, given for T^* in the case $n = 1, 2, 3$, are optimal, as soon as, thanks to Alinhac [5], they give the existence time of a smooth solution of the same order as Alinhac's blow-up time, i.e. up to the time of a geometrical blow-up formation.

1.4.3 Proof of Point 4 of Theorem 1.2.1. Stability and uniqueness result

By definition of u and v we have

$$(u - v)_{tt} - c^2 \Delta(u - v) = \alpha \varepsilon (u_t u_{tt} - v_t v_{tt}) + \beta \varepsilon (\nabla u \nabla u_t - \nabla v \nabla v_t). \quad (1.48)$$

We multiply this equation by $(u - v)_t$ and integrate on \mathbb{R}^n . By integration by parts we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\|(u - v)_t\|_{L^2}^2 + c^2 \|\nabla(u - v)\|_{L^2}^2 \right) &= \alpha \varepsilon \int_{\mathbb{R}^n} (u_t u_{tt} - v_t v_{tt})(u - v)_t \, dx \\ &\quad + \beta \varepsilon \int_{\mathbb{R}^n} (\nabla u \nabla u_t - \nabla v \nabla v_t)(u - v)_t \, dx. \end{aligned} \quad (1.49)$$

For the first right hand side term in Eq. (1.49) we find

$$\begin{aligned} \int_{\mathbb{R}^n} (u_t u_{tt} - v_t v_{tt})(u - v)_t \, dx &= \int_{\mathbb{R}^n} u_{tt}(u - v)_t^2 \, dx + \int_{\mathbb{R}^n} v_t(u - v)_{tt}(u - v)_t \, dx \\ &= \int_{\mathbb{R}^n} u_{tt}(u - v)_t^2 \, dx + \int_{\mathbb{R}^n} v_t \frac{1}{2} \frac{d}{dt} [(u - v)_t^2] \, dx \\ &= \int_{\mathbb{R}^n} u_{tt}(u - v)_t^2 \, dx + \frac{1}{2} \frac{d}{dt} \left[\int_{\mathbb{R}^n} v_t(u - v)_t^2 \, dx \right] - \frac{1}{2} \int_{\mathbb{R}^n} v_{tt}(u - v)_t^2 \, dx \\ &= \int_{\mathbb{R}^n} u_{tt}(u - v)_t^2 \, dx + \frac{1}{2} \frac{d}{dt} \left[\int_{\mathbb{R}^n} v_t(u - v)_t^2 \, dx \right] \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^n} (u_{tt} - v_{tt})(u - v)_t^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^n} u_{tt}(u - v)_t^2 \, dx \\ &= \frac{1}{2} \int_{\mathbb{R}^n} u_{tt}(u - v)_t^2 \, dx + \frac{1}{2} \frac{d}{dt} \left[\int_{\mathbb{R}^n} v_t(u - v)_t^2 \, dx \right] + \frac{1}{6} \frac{d}{dt} \left[\int_{\mathbb{R}^n} (u - v)_t^3 \, dx \right] \\ &= \frac{1}{2} \int_{\mathbb{R}^n} u_{tt}(u - v)_t^2 \, dx + \frac{d}{dt} \left[\int_{\mathbb{R}^n} \left(\frac{1}{6} u_t + \frac{1}{3} v_t \right) (u - v)_t^2 \, dx \right]. \end{aligned}$$

On one hand,

$$\left| \int_{\mathbb{R}^n} u_{tt}(u - v)_t^2 \, dx \right| \leq \|u_{tt}\|_{L^\infty} \|(u - v)_t\|_{L^2}^2,$$

and on the other hand, we can put the term $\alpha \varepsilon \frac{d}{dt} \left[\int_{\mathbb{R}^n} \left(\frac{1}{6} u_t + \frac{1}{3} v_t \right) (u - v)_t^2 \, dx \right]$ on the left hand side of Eq. (1.48) and combine it with $\frac{1}{2} \frac{d}{dt} (\|(u - v)_t\|_{L^2}^2)$, so that we obtain the term

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} A(u - v)_t^2 \, dx, \quad \text{with } A = 1 - \alpha \varepsilon \left(\frac{1}{3} u_t + \frac{2}{3} v_t \right).$$

We notice that with condition (1.9) on u_t and v_t (which keeps our model hyperbolic) we have $\frac{1}{2} \leq A(t) \leq \frac{3}{2}$ for $t < T^*$. Therefore, for the second term at the right hand side we find

$$\begin{aligned} \int_{\mathbb{R}^n} (\nabla u \nabla u_t - \nabla v \nabla v_t)(u - v)_t dx &= \int_{\mathbb{R}^n} \nabla v_t (\nabla u - \nabla v)(u - v)_t dx \\ &+ \int_{\mathbb{R}^n} \nabla u (\nabla u - \nabla v)_t (u - v)_t dx \\ &= \int_{\mathbb{R}^n} \nabla v_t (\nabla u - \nabla v)(u - v)_t dx - \frac{1}{2} \int_{\mathbb{R}^n} \Delta u (u - v)_t^2 dx. \end{aligned}$$

We estimate the obtained two terms:

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \nabla v_t (\nabla u - \nabla v)(u - v)_t dx \right| &\leq C \|\nabla v_t\|_{L^\infty} (\|\nabla(u - v)\|_{L^2}^2 + \|(u - v)_t\|_{L^2}^2), \\ \left| \int_{\mathbb{R}^n} \Delta u (u - v)_t^2 dx \right| &\leq \|\Delta u\|_{L^\infty} \|(u - v)_t\|_{L^2}^2. \end{aligned}$$

Thus, we find the following estimate

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\int_{\mathbb{R}^n} A(u - v)_t^2 + c^2 \|\nabla(u - v)\|_{L^2}^2 dx \right) &\leq C\varepsilon \sup(\|u_{tt}\|_{L^\infty}, \|\Delta u\|_{L^\infty}, \|\nabla v_t\|_{L^\infty}) \\ &\quad (\|(u - v)_t\|_{L^2}^2 + \|\nabla(u - v)\|_{L^2}^2). \end{aligned}$$

Applying the Gronwall Lemma, as $\frac{1}{2} \leq A(t) \leq \frac{3}{2}$ for $t < T^*$, from the last estimate we have

$$\begin{aligned} (\|(u - v)_t\|_{L^2}^2 + \|\nabla(u - v)\|_{L^2}^2)(t) &\leq C_1 \exp(C_2\varepsilon \int_0^t \sup(\|u_{tt}\|_{L^\infty}, \|\Delta u\|_{L^\infty}, \|\nabla v_t\|_{L^\infty}) ds) \cdot \\ &\quad \cdot (\|u_1 - v_1\|_{L^2}^2 + \|\nabla(u_0 - v_0)\|_{L^2}^2). \end{aligned}$$

With the hypothesis that $\|\nabla v_t\|_{L^\infty}$ is bounded on $[0, T^*[$, we obtain the result of Point 4 in Theorem 1.2.1 for all $t < T^*$.

Remark 1.4.2. *It is easy to verify that the same stability estimate also holds for $\nu > 0$: by the adding of the term $-\nu \Delta(u - v)_t$ to the equation, it gives a positive term $+\nu \int_0^t |\nabla(u - v)_t|^2 dx$ in the right hand side of the estimate and can be simply omitted.*

1.5 Well-posedness for the viscous case

1.5.1 Proof of Point 1 of Theorem 1.2.2

Let us show the global well-posedness, of the solution of the Cauchy problem (1.5)-(1.6). We start with the study of the linear problem, associated to the Kuznetsov equation.

Theorem 1.5.1. *Let $s \geq 0$ and X be the space defined in Point 1 of Theorem 1.2.2. Then the system*

$$\begin{cases} u_{tt} - c^2 \Delta u - \nu \varepsilon \Delta u_t = f, \\ u(0) = u_0, \quad u_t(0) = u_1 \end{cases} \quad (1.50)$$

has a unique solution $u \in X$, if and only if $f \in L^2(\mathbb{R}^+; H^s(\mathbb{R}^n))$, $u_0 \in H^{s+2}(\mathbb{R}^n)$ and $u_1 \in H^{s+1}(\mathbb{R}^n)$. Moreover it holds the following a priori estimate

$$\|u\|_X \leq C \left(\|f\|_{L^2(\mathbb{R}^+; H^s(\mathbb{R}^n))} + \|u_0\|_{H^{s+2}(\mathbb{R}^n)} + \|u_1\|_{H^{s+1}(\mathbb{R}^n)} \right) \quad (1.51)$$

with $\|u\|_X := \|u\|_{H^2(\mathbb{R}^+; H^s)} + \|u\|_{L^2(\mathbb{R}^+; H^{s+2})} + \|u_t\|_{L^2(\mathbb{R}^+; H^{s+2})}$.

Proof. First we take $f \in L^2(\mathbb{R}^+; H^s(\mathbb{R}^n))$, $u_0 \in H^{s+2}(\mathbb{R}^n)$ and $u_1 \in H^{s+1}(\mathbb{R}^n)$. We use the ideas of [32] (see Eq. (4.26)). For the sake of clarity, let us take $s = 0$. We take the inner product in $L^2(\mathbb{R}^n)$ of the equation with $-\Delta u_t$ and integrate by parts:

$$\frac{1}{2} \frac{d}{dt} \left(\|\nabla u_t\|_{L^2(\mathbb{R}^n)}^2 + c^2 \|\Delta u\|_{L^2(\mathbb{R}^n)}^2 \right) + \nu \epsilon \|\Delta u_t\|_{L^2(\mathbb{R}^n)}^2 = - \int_{\mathbb{R}^n} f \Delta u_t dx.$$

Using Young's inequality and integrating over $[0, t]$, we find

$$\begin{aligned} \frac{1}{2} \left(\|\nabla u_t\|_{L^2(\mathbb{R}^n)}^2 + c^2 \|\Delta u\|_{L^2(\mathbb{R}^n)}^2 \right) + \frac{\nu \epsilon}{2} \int_0^t \|\Delta u_\tau\|_{L^2(\mathbb{R}^n)}^2 dl \\ \leq \frac{1}{2} \|\nabla u_1\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{2} \|\Delta u_0\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{2\nu \epsilon} \int_0^t \int_{\mathbb{R}^n} |f|^2 dx dl. \end{aligned} \quad (1.52)$$

Since $f \in L^2(\mathbb{R}^+ \times \mathbb{R}^n)$ and $(u_0, u_1) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$, the last estimate implies that

$$\int_0^{+\infty} \int_{\mathbb{R}^n} |\Delta u_\tau|^2 dx dl < +\infty.$$

Since the domain of $-\Delta$ is H^2 , we obtain that

$$u, u_t \in L^2(\mathbb{R}^+; H^2(\mathbb{R}^n)), \quad \text{and} \quad u_{tt} \in L^2(\mathbb{R}^+ \times \mathbb{R}^n),$$

and hence, $u \in X$ for $s = 0$. For $s > 0$, as the equation is linear, we perform the same proof, using the fact that, the operator $\Lambda = (1 - \Delta)^{\frac{1}{2}}$, defined by its Fourier transform by the formula $(\widehat{\Lambda u})(\zeta) = (1 + |\zeta|^2)^{\frac{1}{2}} \hat{u}(\zeta)$, relies the norm of H^s with the L^2 -norm:

$$\Lambda^s = (1 - \Delta)^{\frac{s}{2}}, \quad \|u\|_{H^s(\mathbb{R}^n)} = \|\Lambda^s u\|_{L^2(\mathbb{R}^n)}. \quad (1.53)$$

The uniqueness of u follows from the linearity of the operator and the uniqueness of the solution of system (1.50) in the case $f = 0$ [41].

Conversely, if $u \in X$ is a solution of system (1.50), this implies that

$$u \in C(\mathbb{R}^+; H^{s+2}(\mathbb{R}^n)) \quad \text{and} \quad u_t \in H^1(\mathbb{R}^+; H^s(\mathbb{R}^n)) \cap L^2(\mathbb{R}^+; H^{s+2}(\mathbb{R}^n)).$$

Thanks to Theorem III.4.10.2 in [6], it follows that $u_t \in C(\mathbb{R}^+; H^{s+1}(\mathbb{R}^n))$. Then we have $u(0) \in H^{s+2}(\mathbb{R}^n)$ and $u_t(0) \in H^{s+1}(\mathbb{R}^n)$. Moreover, it reads directly from the definition of X , that $f \in L^2(\mathbb{R}^+; H^s(\mathbb{R}^n))$ for $u \in X$.

The a priori estimate follows from the closed graph theorem. \square

Let us notice that Theorem 1.5.1 states that problem (1.50) has L^2 -maximal regularity (see [22] Definition 2.1) on \mathbb{R}^+ .

To be able to give a sharp estimate of the smallness of the initial data and in the same time to estimate the bound of the corresponding solution of the Kuznetsov equation (see Point 1 of Theorem 1.2.2), we use the following theorem from [88], which allows us to establish our main result of the global well-posedness of the Cauchy problem for the Kuznetsov equation:

Theorem 1.5.2. (*Sukhinin*) *Let X be a Banach space, let Y be a separable topological vector space, let $L : X \rightarrow Y$ be a linear continuous operator, let U be the open unit ball in X , let $P_{LU} : LX \rightarrow [0, \infty[$ be the Minkowski functional of the set LU , and let $\Phi : X \rightarrow LX$ be a mapping satisfying the condition*

$$P_{LU}(\Phi(x) - \Phi(\bar{x})) \leq \Theta(r) \|x - \bar{x}\| \quad \text{for} \quad \|x - x_0\| \leq r, \quad \|\bar{x} - x_0\| \leq r$$

for some $x_0 \in X$, where $\Theta : [0, \infty[\rightarrow [0, \infty[$ is a monotone non-decreasing function. Set $b(r) = \max(1 - \Theta(r), 0)$ for $r \geq 0$.

Suppose that

$$w = \int_0^\infty b(r) dr \in]0, \infty], \quad r_* = \sup\{r \geq 0 \mid b(r) > 0\},$$

$$w(r) = \int_0^r b(t) dt \quad (r \geq 0) \quad \text{and} \quad f(x) = Lx + \Phi(x) \quad \text{for} \quad x \in X.$$

Then for any $r \in [0, r_*[$ and $y \in f(x_0) + w(r)LU$, there exists an $x \in x_0 + rU$ such that $f(x) = y$.

Remark 1.5.1. If either L is injective or $\text{Ker}(L)$ has a topological complement E in X such that $L(E \cap U) = LU$, then the assertion of Theorem 1.5.2 follows from the contraction mapping principle [88]. In particular, if L is injective, then the solution is unique.

Now, we have all elements to prove Point 1 of Theorem 1.2.2: for all $r \in [0, r_*[$ with $r_* = O(\epsilon^0) = O(1)$ (to be defined), as soon as the initial data are small as

$$\|u_0\|_{H^{s+2}(\mathbb{R}^n)} + \|u_1\|_{H^{s+1}(\mathbb{R}^n)} \leq C\sqrt{\epsilon}r \quad \text{with} \quad C = O(1), \quad (1.54)$$

then the unique solution $u \in X$ satisfies $\|u\|_X \leq 2r$ ($r = O(1)$).

Remark 1.5.2. It is very important to notice that here all physical coefficients of the Cauchy problem for the Kuznetsov equation are expressed in function of the powers of ϵ (ϵ is the dimensionless parameter characterising the medium perturbation as explained in [81] and [82]). In particular, if we take into account in Point 3 of Theorem 1.2.1 that $c^2 = O(\frac{1}{\epsilon})$, we obtain the same types of smallness of the initial energy for the inviscid case as in Point 2 of Theorem 1.2.2: $\sqrt{E_{m_0}[u](0)} \leq O(\sqrt{\epsilon})$. But, if we want to understand the smallness of the initial data by their norms without the calculus of the initial energy, the results of Point 1 of Theorem 1.2.2 can be useful. The sharp character of Point 1 of Theorem 1.2.2 can be illustrated by the following direct energy estimation approach, presented in Section 1.6.

Let suppose that Point 2 of Theorem 1.2.2 holds (see also Eq. (1.14)). Thus, for $n \geq 3$, $m \geq \lfloor \frac{n}{2} + 3 \rfloor$ if

$$\sqrt{E_{\frac{m}{2}}[u](0)} = \sqrt{\|\nabla u(0)\|_{H^m(\mathbb{R}^n)}^2 + \sum_{i=1}^{\frac{m}{2}+1} \|\partial_i^i u(0)\|_{H^{m-2(i-1)}(\mathbb{R}^n)}^2} \leq O(\sqrt{\epsilon}),$$

then it follows in a sufficient way (see Section 1.6 for more details) that for $u_0 \in H^{m+1}(\mathbb{R}^n)$ and for $u_1 \in H^m(\mathbb{R}^n)$ it holds

$$\|\nabla u_0\|_{H^m(\mathbb{R}^n)} + \|u_1\|_{H^m(\mathbb{R}^n)} \leq O(\sqrt{\epsilon^{m+1}}), \quad (1.55)$$

which implies the existence of a unique global solution $u \in C^0(\mathbb{R}^+; H^{m+1}(\mathbb{R}^n)) \cap C^1(\mathbb{R}^+; H^m(\mathbb{R}^n))$ of problem (1.5)–(1.6) such that for all $t \in \mathbb{R}^+$

$$E_{\frac{m}{2}}[u](t) \leq O\left(\frac{1}{\epsilon}\right) E_{\frac{m}{2}}[u](0) = O(1).$$

Thus we see that by this approach the sufficient condition to have for all $t \geq 0$, $E_{\frac{m}{2}}[u](t)$ bounded by a constant of order zero on ϵ is given by Eq. (1.55) and depends on the smooth properties of the initial data (the more they are regular, the smaller they should be). Hence, it is much more restrictive to compare to (1.54).

Proof. For $u_0 \in H^{s+2}(\mathbb{R}^n)$ and $u_1 \in H^{s+1}(\mathbb{R}^n)$ let us denote by $u^* \in X$ the unique solution of the linear problem

$$\begin{cases} u_{tt}^* - c^2 \Delta u^* - \nu \varepsilon \Delta u_t^* = 0, \\ u^*(0) = u_0 \in H^{s+2}(\mathbb{R}^n), \quad u_t^*(0) = u_1 \in H^{s+1}(\mathbb{R}^n). \end{cases} \quad (1.56)$$

In addition, according to Theorem 1.5.1, we take

$$X := H^2(\mathbb{R}^+; H^s(\mathbb{R}^n)) \cap H^1(\mathbb{R}^+; H^{s+2}(\mathbb{R}^n)),$$

this time for $s > \frac{n}{2}$ (we need it to control the non-linear terms), and introduce the Banach spaces

$$X_0 := \{u \in X \mid u(0) = u_t(0) = 0\} \quad (1.57)$$

and $Y = L^2(\mathbb{R}^+; H^s(\mathbb{R}^n))$. Then by Theorem 1.5.1, the linear operator

$$L : X_0 \rightarrow Y, \quad u \in X_0 \mapsto L(u) := u_{tt} - c^2 \Delta u - \nu \varepsilon \Delta u_t \in Y,$$

is a bi-continuous isomorphism.

Let us now notice that if v is the unique solution of the non-linear Cauchy problem

$$\begin{cases} v_{tt} - c^2 \Delta v - \nu \varepsilon \Delta v_t - \alpha \varepsilon (v + u^*)_t (v + u^*)_{tt} - \beta \varepsilon \nabla(v + u^*) \cdot \nabla(v + u^*)_t = 0, \\ v(0) = 0, \quad v_t(0) = 0, \end{cases} \quad (1.58)$$

then $u = v + u^*$ is the unique solution of the Cauchy problem for the Kuznetsov equation (1.5)–(1.6). Let us prove the existence of such a v , using Theorem 1.5.2.

We suppose that $\|u^*\|_X \leq r$ and define for $v \in X_0$

$$\Phi(v) := \alpha \varepsilon (v + u^*)_t (v + u^*)_{tt} + \beta \varepsilon \nabla(v + u^*) \cdot \nabla(v + u^*)_t.$$

For w and z in X_0 such that $\|w\|_X \leq r$ and $\|z\|_X \leq r$, we estimate

$$\begin{aligned} \|\Phi(w) - \Phi(z)\|_Y &= \|\alpha \varepsilon (u_t^*(w - z))_{tt} + (w - z)_t u_{tt}^* + w_t w_{tt} - z_t z_{tt} \\ &\quad + \beta \varepsilon (\nabla u^* \nabla(w - z))_t + \nabla(w - z) \nabla u_t^* + \nabla w \nabla w_t - \nabla z \nabla z_t\|_Y \\ &= \|\alpha \varepsilon (u_t^*(w - z))_{tt} + (w - z)_t u_{tt}^* + w_t (w - z)_{tt} + (w - z)_t z_{tt} \\ &\quad + \beta \varepsilon (\nabla u^* \nabla(w - z))_t + \nabla(w - z) \nabla u_t^* + \nabla w \nabla(w - z)_t + \nabla(w - z) \nabla z_t\|_Y \end{aligned}$$

by applying the triangular inequality

$$\begin{aligned} \|\Phi(w) - \Phi(z)\|_Y &\leq \alpha \varepsilon \left(\|u_t^*(w - z)_{tt}\|_Y + \|(w - z)_t u_{tt}^*\|_Y \right. \\ &\quad \left. + \|w_t (w - z)_{tt}\|_Y + \|(w - z)_t z_{tt}\|_Y \right) \\ &\quad + \beta \varepsilon \left(\|\nabla u^* \nabla(w - z)_t\|_Y + \|\nabla(w - z) \nabla u_t^*\|_Y \right. \\ &\quad \left. + \|\nabla w \nabla(w - z)_t\|_Y + \|\nabla(w - z) \nabla z_t\|_Y \right). \end{aligned}$$

Now, for all a and b in X with $s \geq s_0 > \frac{n}{2}$ it holds

$$\begin{aligned} \|a_t b_{tt}\|_Y &\leq \|a_t\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^n)} \|b_{tt}\|_Y \\ &\leq C_{H^1(\mathbb{R}^+; H^{s_0}) \rightarrow L^\infty(\mathbb{R}^+ \times \mathbb{R}^n)} \|a_t\|_{H^1(\mathbb{R}^+; H^s(\mathbb{R}^n))} \|b\|_X \\ &\leq C_{H^1(\mathbb{R}^+; H^{s_0}) \rightarrow L^\infty(\mathbb{R}^+ \times \mathbb{R}^n)} \|a\|_X \|b\|_X, \end{aligned}$$

where $C_{H^1(\mathbb{R}^+; H^{s_0}) \rightarrow L^\infty(\mathbb{R}^+ \times \mathbb{R}^n)}$ is the embedding constant of $H^1(\mathbb{R}^+; H^{s_0})$ into the space $L^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$, independent on s , but depending only on the dimension n . In the same way, for all a and b in X it holds

$$\|\nabla a \nabla b_t\|_Y \leq C_{H^1(\mathbb{R}^+; H^{s_0}) \rightarrow L^\infty(\mathbb{R}^+ \times \mathbb{R}^n)} \|a\|_X \|b\|_X.$$

Taking a and b equal to u^* , w , z or $w - z$, as $\|u^*\|_X \leq r$, $\|w\|_X \leq r$ and $\|z\|_X \leq r$, we obtain

$$\|\Phi(w) - \Phi(z)\|_Y \leq 4(\alpha + \beta) C_{H^1(\mathbb{R}^+; H^{s_0}) \rightarrow L^\infty(\mathbb{R}^+ \times \mathbb{R}^n)} \varepsilon r \|w - z\|_X.$$

By the fact that L is a bi-continuous isomorphism, there exists a minimal constant $C_\varepsilon = O\left(\frac{1}{\varepsilon\nu}\right) > 0$ (coming from the inequality $C_0 \varepsilon \nu \|u\|_X^2 \leq \|f\|_Y \|u\|_X$ for u , a solution of the linear problem (1.50) with homogeneous initial data [for a constant $C_0 = O(1) > 0$ maximal]) such that

$$\forall u \in X_0 \quad \|u\|_X \leq C_\varepsilon \|Lu\|_Y.$$

Hence, for all $f \in Y$

$$P_{LU_{X_0}}(f) \leq C_\varepsilon P_{U_Y}(f) = C_\varepsilon \|f\|_Y.$$

Then we find for w and z in X_0 , such that $\|w\|_X \leq r$, $\|z\|_X \leq r$, and also with $\|u^*\|_X \leq r$, that

$$P_{LU_{X_0}}(\Phi(w) - \Phi(z)) \leq \Theta(r) \|w - z\|_X,$$

where $\Theta(r) := 4C_\varepsilon(\alpha + \beta) C_{H^1(\mathbb{R}^+; H^{s_0}) \rightarrow L^\infty(\mathbb{R}^+ \times \mathbb{R}^n)} \varepsilon r$. Thus we apply Theorem 1.5.2 for $f(x) = L(x) - \Phi(x)$ and $x_0 = 0$. Therefore, knowing that $C_\varepsilon = \frac{C_0}{\varepsilon\nu}$, we have that for all $r \in [0, r_*[$ with

$$r_* = \frac{\nu}{4C_0(\alpha + \beta) C_{H^1(\mathbb{R}^+; H^{s_0}) \rightarrow L^\infty(\mathbb{R}^+ \times \mathbb{R}^n)}} = O(1), \quad (1.59)$$

for all $y \in \Phi(0) + w(r)LU_{X_0} \subset Y$ with

$$w(r) = r - 2\frac{C_0}{\nu} C_{H^1(\mathbb{R}^+; H^{s_0}) \rightarrow L^\infty(\mathbb{R}^+ \times \mathbb{R}^n)} (\alpha + \beta) r^2,$$

there exists a unique $v \in 0 + rU_{X_0}$ such that $L(v) - \Phi(v) = y$. But, if we want that v be the solution of the non-linear Cauchy problem (1.58), then we need to impose $y = 0$, and thus to ensure that $0 \in \Phi(0) + w(r)LU_{X_0}$. Since $-\frac{1}{w(r)}\Phi(0)$ is an element of Y and $LU_{X_0} = Y$, there exists a unique $z \in X_0$ such that

$$Lz = -\frac{1}{w(r)}\Phi(0). \quad (1.60)$$

Let us show that $\|z\|_X \leq 1$, what will implies that $0 \in \Phi(0) + w(r)LU_{X_0}$. Noticing that by definition of Φ

$$\begin{aligned} \|\Phi(0)\|_Y &\leq \alpha \varepsilon \|u_t^* u_{tt}^*\|_Y + \alpha \varepsilon \|u_t^* u_t^*\|_Y \\ &\leq \frac{B}{4} \alpha \varepsilon \|u^*\|_X^2 \\ &\leq \frac{B}{4} \alpha \varepsilon r^2 \end{aligned}$$

and using (1.60), we find

$$\begin{aligned} \|z\|_X &\leq C_\epsilon \|Lz\|_Y = C_\epsilon \frac{\|\Phi(0)\|_Y}{w(r)} \\ &\leq \frac{C_\epsilon C_{H^1(\mathbb{R}^+; H^{s_0}) \rightarrow L^\infty(\mathbb{R}^+ \times \mathbb{R}^n)}(\alpha + \beta)\epsilon r}{(1 - 2C_\epsilon C_{H^1(\mathbb{R}^+; H^{s_0}) \rightarrow L^\infty(\mathbb{R}^+ \times \mathbb{R}^n)}(\alpha + \beta)\epsilon r)} < \frac{1}{2}, \end{aligned}$$

as soon as $r < r^*$.

Consequently, $z \in U_{X_0}$ and $\Phi(0) + w(r)Lz = 0$.

Then we conclude that for all $r \in [0, r_*[$, if $\|u^*\|_X \leq r$, there exists a unique $v \in rU_{X_0}$ such that $L(v) - \Phi(v) = 0$, *i.e.* the solution of the non-linear Cauchy problem (1.58). Thanks to the maximal regularity and the a priori estimate following from inequality (1.52) with $f = 0$, there exists a constant $C_1 = O(\epsilon^0) > 0$, such that

$$\|u^*\|_X \leq \frac{C_1}{\sqrt{\nu\epsilon}} (\|u_0\|_{H^{s+2}(\mathbb{R}^n)} + \|u_1\|_{H^{s+1}(\mathbb{R}^n)}).$$

Thus, for all $r \in [0, r_*[$ and $\|u_0\|_{H^{s+2}(\mathbb{R}^n)} + \|u_1\|_{H^{s+1}(\mathbb{R}^n)} \leq \frac{\sqrt{\nu\epsilon}}{C_1}r$, the function $u = u^* + v \in X$ is the unique solution of the Cauchy problem for the Kuznetsov equation and $\|u\|_X \leq 2r$. \square

1.5.2 Proof of Point 2 of Theorem 1.2.2: Case $n \geq 3$

Knowing the existence of a solution u of the Kuznetsov equation in

$$X = H^2(\mathbb{R}^+; H^s(\mathbb{R}^n)) \cap H^1(\mathbb{R}^+; H^{s+2}(\mathbb{R}^n)),$$

we notice that this directly implies that

$$u \in C(\mathbb{R}^+; H^{s+2}(\mathbb{R}^n)) \quad \text{and} \quad u_t \in H^1(\mathbb{R}^+; H^s(\mathbb{R}^n)) \cap L^2(\mathbb{R}^+; H^{s+2}(\mathbb{R}^n)).$$

By Theorem III.4.10.2 in [6], it implies that $u_t \in C(\mathbb{R}^+; H^{s+1}(\mathbb{R}^n))$, which gives that

$$u \in C^1(\mathbb{R}^+; H^{s+1}(\mathbb{R}^n)) \cap C(\mathbb{R}^+; H^{s+2}(\mathbb{R}^n))$$

and, this time with the help of the Kuznetsov equation, $u_{tt} \in C(\mathbb{R}^+; H^{s-1}(\mathbb{R}^n))$. Consequently, in the viscous case, the regularity of the time derivatives of the order greater than two of the solutions differs from the regularity, obtained in Section 1.4 for the inviscid case. Thus we have to consider estimates with different energies: the energy $E_{\frac{m}{2}}[u](t)$, defined in Eq. (1.14), and the energy

$$S_{\frac{m}{2}}[u](t) = \sum_{i=1}^{\frac{m}{2}+1} \|\nabla \partial_t^i u(t)\|_{H^{m-2(i-1)}(\mathbb{R}^n)}^2, \quad (1.61)$$

defined, as $E_{\frac{m}{2}}[u](t)$, for $m \in \mathbb{N}$ and m even, which respect to the obtained regularity of u and its derivatives.

Lemma 1.5.1. *Let $n \in \mathbb{N}^*$, $n \geq 3$, $m \in \mathbb{N}$, and u be the solution of problem (1.5)-(1.6). Then for $m \geq \lceil \frac{n}{2} + 3 \rceil$, m even, and all multi-index $A = (A_0, A_1, \dots, A_n)$ with*

$|A| - A_0 \leq m - 2A_0$ it holds

$$\begin{aligned} & \frac{d}{dt} \left(\int_{\mathbb{R}^n} ((1 - \alpha \varepsilon u_t)(D^A u_t)^2 + c^2 (\nabla D^A u)^2)(\tau, x) \, dx \right) \\ & \quad + 2\nu \varepsilon \int_{\mathbb{R}^n} (\nabla D^A u_t)^2(\tau, x) \, dx \\ & \leq C_m \max(\alpha, \beta) \varepsilon \sqrt{E_{\frac{m}{2}}[u](\tau)} S_{\frac{m}{2}}[u](\tau) \end{aligned} \quad (1.62)$$

with a constant $C_m > 0$, depending only on m and on the dimension n .

Proof. Following notations of the proof of Proposition 1.4.1, we redefine

$$L_u v := v_{tt} - c^2 \Delta v - \nu \varepsilon \Delta v_t - \alpha \varepsilon u_t v_{tt} - \beta \varepsilon \nabla u \cdot \nabla v_t,$$

where u is the solution of problem (1.5). For this new $L_u v$ with the additional term $\nu \varepsilon \Delta v_t$, we have a modified version of relation (1.28)

$$\frac{d}{dt} \int_{\mathbb{R}^n} I[v](t, x) \, dx + 2\nu \varepsilon \int_{\mathbb{R}^n} (\nabla v_t)^2 \, dx = \int_{\mathbb{R}^n} J[v](t, x) \, dx, \quad (1.63)$$

where $I[v]$ and $J[v]$ are defined in Eqs. (1.26)–(1.27). We still take $v = D^A u$ with $A = (A_0, A_1, \dots, A_n)$, but this time $|A| - A_0 \leq m - 2A_0$ and m is even. Then we just need to show

$$\left| \int_{\mathbb{R}^n} J[D^A u](t, x) \, dx \right| \leq \varepsilon C_m \max(\alpha, \beta) \sqrt{E_{\frac{m}{2}}[u](t)} S_{\frac{m}{2}}[u](t). \quad (1.64)$$

For $n \geq 3$, $m \geq \left[\frac{n}{2} + 3\right]$ and m even, we have, thanks to the Hölder inequality,

$$\int_{\mathbb{R}^n} |u_{tt}(D^A u_t)^2| \, dx \leq \|u_{tt}\|_{L^{\frac{n}{2}}(\mathbb{R}^n)} \|D^A u_t\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}^2.$$

Noticing, that, thanks to Ref. [2] Theorem 7.57 p. 228, for $s > \frac{n}{2}$ there hold the continuous embeddings $H^s(\mathbb{R}^n) \subset C_B^0(\mathbb{R}^n) \subset L^{\frac{n}{2}}(\mathbb{R}^n)$ (where C_B^0 is the Banach space of bounded continuous functions equal to zero at the infinity), we can write for $m \geq \left[\frac{n}{2} + 3\right]$

$$\|u_{tt}\|_{L^{\frac{n}{2}}(\mathbb{R}^n)} \leq C \|u_{tt}\|_{H^{[\frac{n}{2}+1]}(\mathbb{R}^n)} \leq C \|u_{tt}\|_{H^{m-2}(\mathbb{R}^n)} \leq C \sqrt{E_{\frac{m}{2}}[u]}. \quad (1.65)$$

In addition, with the help of the Gagliardo-Nirenberg-Sobolev inequality

$$\|v\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C \|\nabla v\|_{L^2(\mathbb{R}^n)}, \quad (1.66)$$

we also have

$$\|D^A u_t\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C \|\nabla D^A u_t\|_{L^2(\mathbb{R}^n)} \leq C \|\nabla D_t^{A_0+1} u\|_{H^{|A|-A_0}(\mathbb{R}^n)}.$$

With the hypothesis that $|A| - A_0 \leq m - 2A_0$, there hold $2A_0 \leq m$ and

$$\|\nabla D_t^{A_0+1} u\|_{H^{|A|-A_0}(\mathbb{R}^n)} \leq \|\nabla D_t^{A_0+1} u\|_{H^{m-2A_0}(\mathbb{R}^n)}.$$

Therefore, all norms $\|\nabla D_t^{A_0+1} u\|_{H^{m-2A_0}(\mathbb{R}^n)}^2$, for the chosen n , m and A_0 , are present in $S_{\frac{m}{2}}$. Hence, we find

$$\int_{\mathbb{R}^n} |u_{tt}(D^A u_t)^2| \, dx \leq C \|u_{tt}\|_{H^{m-2}(\mathbb{R}^n)} \|\nabla D^A u_t\|_{L^2(\mathbb{R}^n)}^2 \leq C \sqrt{E_{\frac{m}{2}}[u]} S_{\frac{m}{2}}[u], \quad (1.67)$$

and in the same way,

$$\begin{aligned} \int_{\mathbb{R}^n} |\Delta u (D^A u_t)^2| dx &\leq \|\Delta u\|_{L^{\frac{n}{2}}(\mathbb{R}^n)} \|D^A u_t\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}^2 \leq C \|\Delta u\|_{H^{[\frac{n}{2}+1]}(\mathbb{R}^n)} \|\nabla D^A u_t\|_{L^2(\mathbb{R}^n)}^2 \\ &\leq C \sqrt{E_{\frac{m}{2}}[u]} S_{\frac{m}{2}}[u]. \end{aligned}$$

To calculate $L_u D^A u$ we use expression (1.32) with multi-indexes A^{j1} and A^{j2} satisfying (1.33). As in the proof of Proposition 1.4.1, without loss of generality, we consider two multi-indexes A^1 and A^2 with the same properties (1.33). We perform two steps:

Step 1 we prove

$$\int_{\mathbb{R}^n} |D^{A^1} u_t D^{A^2} u_t D^A u_t| dx \leq C \sqrt{E_{\frac{m}{2}}[u]} S_{\frac{m}{2}}[u], \quad (1.68)$$

Step 2 we prove

$$\int_{\mathbb{R}^n} |D^{A^1} \partial_{x_i} u D^{A^2} \partial_{x_i} u D^A u_t| dx \leq C \sqrt{E_{\frac{m}{2}}[u]} S_{\frac{m}{2}}[u]. \quad (1.69)$$

Step 1. Thanks to properties (1.33) of A_1 and A_2 and to the symmetry of the general case

$$\int_{\mathbb{R}^n} |(D_t^{A_0^1} D_x^{(A_1^1, \dots, A_n^1)} u_t)(D_t^{A_0^2} D_x^{(A_1^2, \dots, A_n^2)} u_t)(D^A u_t)| dx,$$

we divide our proof on three typical cases:

Case 1 $|A^1| - A_0^1 \geq 0$, $A_0^1 \geq 0$, $|A^2| - A_0^2 > 0$ and $A_0^2 > 0$, *i.e.* a non trivial presence of $D_t^{A_0^2}$ and $D_x^{(A_1^2, \dots, A_n^2)}$ is imposed,

Case 2 $|A^1| - A_0^1 = 0$, $A_0^1 > 0$, $|A^2| - A_0^2 > 0$ and $A_0^2 = 0$, *i.e.* we consider the integrals of the form $\int_{\mathbb{R}^n} |(D_t^{A_0^1} u_t)(D_x^{(A_1^2, \dots, A_n^2)} u_t)(D^A u_t)| dx$,

Case 3 $|A^1| - A_0^1 = 0$, $A_0^1 > 0$, $|A^2| - A_0^2 = 0$ and $A_0^2 > 0$, *i.e.* we consider only non-trivial time derivatives $\int_{\mathbb{R}^n} |(D_t^{A_0^1} u_t)(D_t^{A_0^2} u_t)(D^A u_t)| dx$.

Step 1, Case 1. By the generalized Hölder inequality with $\frac{1}{p} + \frac{1}{q} = \frac{n+2}{2n}$, we have

$$\int_{\mathbb{R}^n} |D^{A^1} u_t D^{A^2} u_t D^A u_t| dx \leq \|D^{A^1} u_t\|_{L^p(\mathbb{R}^n)} \|D^{A^2} u_t\|_{L^q(\mathbb{R}^n)} \|D^A u_t\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}.$$

By the Sobolev embeddings (1.35) of $H^{m_1} \subset L^p$ and $H^{m_2} \subset L^q$ with $m_1 + m_2 = \frac{n}{2} - 1$ and $0 < m_1 < \frac{n}{2} - 1$, we find

$$\int_{\mathbb{R}^n} |D^{A^1} u_t D^{A^2} u_t D^A u_t| dx \leq C \|D^{A^1} u_t\|_{H^{m_1}(\mathbb{R}^n)} \|D^{A^2} u_t\|_{H^{m_2}(\mathbb{R}^n)} \|\nabla D^A u_t\|_{L^2(\mathbb{R}^n)},$$

where we have also applied the Gagliardo-Nirenberg-Sobolev inequality (1.66). Hence,

$$\begin{aligned} \int_{\mathbb{R}^n} |D^{A^1} u_t D^{A^2} u_t D^A u_t| dx \\ \leq C \|\partial_t^{A_0^1} u_t\|_{H^{m_1+|A^1|-A_0^1}(\mathbb{R}^n)} \|\nabla \partial_t^{A_0^2} u_t\|_{H^{m_2+|A^2|-A_0^2-1}(\mathbb{R}^n)} S_{\frac{m}{2}}[u]^{\frac{1}{2}}. \end{aligned} \quad (1.70)$$

Now we are looking for $0 < m_1 < \frac{n}{2} - 1$, such that

$$\begin{cases} m_1 + |A^1| - A_0^1 \leq m - 2A_0^1, \\ m_2 + |A^2| - A_0^2 - 1 \leq m - 2A_0^2, \end{cases} \quad (1.71)$$

in order to have

$$\|\partial_t^{A_0^1} u_t\|_{H^{m_1+|A^1|-A_0^1}(\mathbb{R}^n)} \leq \sqrt{E_{\frac{m}{2}}[u]} \quad \text{and} \quad \|\nabla \partial_t^{A_0^2} u_t\|_{H^{m_2+|A^2|-A_0^2-1}(\mathbb{R}^n)} \leq \sqrt{S_{\frac{m}{2}}[u]}. \quad (1.72)$$

Since $m_2 = \frac{n}{2} - 1 - m_1$, and by (1.33), $|A^2| = |A| + 1 - |A^1|$ and $A_0^2 = A_0 + 1 - A_0^1$, system (1.71) is equivalent to

$$\begin{cases} m_1 + |A^1| + A_0^1 \leq m, \\ \frac{n}{2} - 1 - m_1 + |A| + 1 - |A^1| + A_0 + 1 - A_0^1 - 1 \leq m. \end{cases}$$

The last system, thanks to $|A| + A_0 \leq m$, corresponding to the assumptions of the Proposition, is satisfied if

$$\frac{n}{2} \leq m_1 + |A^1| + A_0^1 \leq m.$$

Using (1.33), we find that

$$|A^1| + A_0^1 = |A| + A_0 + 2 - (|A^2| + A_0^2).$$

Therefore, since for Case 1 $|A^2| \geq 2$ and $A_0^2 \geq 1$, recalling that (again by (1.33)) $|A| + A_0 \leq m$, we obtain

$$1 \leq |A^1| + A_0^1 \leq m - 1.$$

Thus, we distinguish three sub-cases:

For $n \geq 3$, $\frac{n}{2} \leq |A^1| + A_0^1 \leq m - 1$ taking $m_1 = \frac{1}{4}$, we obtain (1.72).

For $n \geq 5$, $2 \leq |A^1| + A_0^1 < \frac{n}{2}$ as $m \geq \left[\frac{n}{2} + 3\right]$, it is sufficient to take $m_1 = \frac{n}{2} - (|A^1| + A_0^1)$.

For $n \geq 3$, $|A^1| + A_0^1 = 1$ instead of finding m_1 , we notice, that we have only two possibility: either $D^{A^1} = \partial_t$ and $A^2 = A$, which gives estimate (1.67), or $D^{A^1} = \partial_{x_i}$ with $A_0^2 = A_0 + 1$ and $|A^2| - A_0^2 = |A| - A_0 - 1 > 0$. For the last case, by the generalized Hölder inequality, we have

$$\int_{\mathbb{R}^n} |\partial_{x_i} u_t D^{A^2} u_t D^A u_t| dx \leq \|\partial_{x_i} u_t\|_{L^n(\mathbb{R}^n)} \|D^{A^2} u_t\|_{L^2(\mathbb{R}^n)} \|D^A u_t\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}. \quad (1.73)$$

For $m \geq \left[\frac{n}{2} + 3\right]$ the first norm in Eq. (1.73) can be estimated using the continuous embedding $H^s(\mathbb{R}^n) \subset L^n(\mathbb{R}^n)$ holding for $s > \frac{n}{2}$:

$$\|\partial_{x_i} u_t\|_{L^n(\mathbb{R}^n)} \leq C \|\partial_{x_i} u_t\|_{H^{\left[\frac{n}{2}+1\right]}(\mathbb{R}^n)} \leq C \|u_t\|_{H^{m-1}(\mathbb{R}^n)} \leq C \sqrt{E_{\frac{m}{2}}[u]}.$$

With the help of the Gagliardo-Nirenberg-Sobolev inequality (1.66), we also estimate the second norm in (1.73)

$$\|D^A u_t\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C \|\nabla D^A u_t\|_{L^2(\mathbb{R}^n)} \leq C \sqrt{S_{\frac{m}{2}}[u]}, \quad (1.74)$$

and for the last one we directly have

$$\|D^{A^2} u_t\|_{L^2(\mathbb{R}^n)} \leq \|\nabla \partial_t^{A_0+2} u\|_{H^{|A|-A_0-2}(\mathbb{R}^n)} \leq \|\nabla \partial_t^{A_0+2} u\|_{H^{m-2A_0-2}(\mathbb{R}^n)} \leq \sqrt{S_{\frac{m}{2}}[u]}.$$

Thus we obtain as previously estimate (1.68) of Step 1.

This permits to conclude Case 1 of Step 1.

Step 1, Case 2. We have $|A^1| - A_0^1 = 0$, $A_0^1 > 0$, $|A^2| - A_0^2 > 0$ and $A_0^2 = 0$. Therefore, by (1.33), $A_0^1 = 1 + A_0$, and, updating (1.70), we directly have

$$\int_{\mathbb{R}^n} |D_t^{A_0^1} u_t D_x^{(A_1^2, \dots, A_n^2)} u_t D^A u_t| dx \leq C \|\partial_t^{A_0+1} u_t\|_{H^{m_1}(\mathbb{R}^n)} \|\nabla u_t\|_{H^{m_2+|A^2|-1}(\mathbb{R}^n)} S_{\frac{m}{2}}[u]^{\frac{1}{2}}$$

with $m_1 + m_2 = \frac{n}{2} - 1$, $0 < m_1 < \frac{n}{2} - 1$. Now we need to find m_1 , belonging to $]0, \frac{n}{2} - 1[$, such that

$$\begin{cases} m_1 \leq m - 2(A_0 + 1), \\ m_2 + |A^2| - 1 \leq m, \end{cases} \quad (1.75)$$

in order to have

$$\|\partial_t^{A_0+1} u_t\|_{H^{m_1}(\mathbb{R}^n)} \leq \sqrt{E_{\frac{m}{2}}[u]} \quad \text{and} \quad \|\nabla u_t\|_{H^{m_2+|A^2|-1}(\mathbb{R}^n)} \leq \sqrt{S_{\frac{m}{2}}[u]}.$$

From $1 + |A| = |A^1| + |A^2|$, by (1.33), with the relation $|A^1| = A_0^1 = 1 + A_0$ it follows that

$$|A^2| = |A| - A_0. \quad (1.76)$$

Therefore, as $m_2 = \frac{n}{2} - m_1 - 1$, system (1.75) is equivalent to

$$\begin{cases} m_1 + 2A_0 \leq m - 2, \\ \frac{n}{2} - 2 \leq m_1 + m - |A| + A_0. \end{cases}$$

By the assumption of the proposition

$$m - |A| + A_0 \geq 2A_0, \quad (1.77)$$

hence the last system is satisfied if we have m_1 such that

$$\frac{n}{2} - 2 \leq m_1 + 2A_0 \leq m - 2.$$

Knowing that $|A^2| > 0$ (by the assumption of Case 2), Eq. (1.76) implies that $|A| - A_0 > 0$. Thus, relation (1.77) gives $2A_0 \leq m - 1$, or more precisely

$$2A_0 \leq m - 2,$$

since m is even. So, a m_1 with $0 < m_1 < \frac{n}{2} - 1$ exists if $m - 2A_0 > 2$. Indeed, if $2A_0 < \frac{n}{2} - 2$ we can take $m_1 = \frac{n}{2} - 2 - 2A_0$, and if $m - 3 \geq 2A_0 \geq \frac{n}{2} - 2$ we can take $m_1 = \frac{1}{2}$.

Let us now consider the limit case $2A_0 = m - 2$. Then we have $|A^1| = A_0^1 = \frac{m}{2}$. Moreover, from (1.77) viewed, thanks to Eq. (1.76), as $|A^2| + 2A_0 \leq m$, follows that $1 \leq |A^2| \leq 2$. We apply the generalized Hölder inequality and estimate (1.66) to obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |\partial_t^{\frac{m}{2}} u_t D_x^{(A_1^2, \dots, A_n^2)} u_t D^A u_t| dx &\leq \|\partial_t^{\frac{m}{2}} u_t\|_{L^2(\mathbb{R}^n)} \|D_x^{(A_1^2, \dots, A_n^2)} u_t\|_{L^n(\mathbb{R}^n)} \|D^A u_t\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \\ &\leq C \|\partial_t^{\frac{m}{2}} u_t\|_{L^2(\mathbb{R}^n)} \|D_x^{(A_1^2, \dots, A_n^2)} u_t\|_{L^n(\mathbb{R}^n)} \sqrt{S_{\frac{m}{2}}[u]}. \end{aligned}$$

Moreover,

$$\|\partial_t^{\frac{m}{2}} u_t\|_{L^2(\mathbb{R}^n)} \leq \sqrt{E_{\frac{m}{2}}[u]}.$$

Using the continuity of the embedding $H^s(\mathbb{R}^n) \subset L^n(\mathbb{R}^n)$ for $s > \frac{n}{2}$, we also find for $m \geq \left[\frac{n}{2} + 3\right]$

$$\begin{aligned} \|D^{A^2} u_t\|_{L^n(\mathbb{R}^n)} &\leq C \|D^{A^2} u_t\|_{H^{[\frac{n}{2}+1]}(\mathbb{R}^n)} \leq C \|\nabla u_t\|_{H^{[\frac{n}{2}+2]}(\mathbb{R}^n)} \\ &\leq C \|\nabla u_t\|_{H^m(\mathbb{R}^n)} \leq C \sqrt{S_{\frac{m}{2}}[u]}. \end{aligned}$$

Hence, estimate (1.68) of Step 1 is also proved for Case 2.

Step 1, Case 3. Let us notice that thanks to relations (1.33), from $|A^1| = A_0^1$ and $|A^2| = A_0^2$ it follows $|A| = A_0$. We start as usual with the generalized Hölder inequality

$$\int_{\mathbb{R}^n} |D_t^{A_0^1} u_t D_t^{A_0^2} u_t D_t^{A_0} u_t| dx \leq \|D_t^{A_0^1} u_t\|_{L^p(\mathbb{R}^n)} \|D_t^{A_0^2} u_t\|_{L^q(\mathbb{R}^n)} \|D_t^{A_0} u_t\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}$$

with $\frac{1}{p} + \frac{1}{q} = \frac{n+2}{2n}$. Then we apply the Gagliardo-Nirenberg-Sobolev inequality (1.66) and its more general version, which can be viewed as the embedding of the Sobolev space $W_{q^*}^1(\mathbb{R}^n)$ in the Lebesgue space $L^q(\mathbb{R}^n)$ with $\frac{1}{q} = \frac{1}{q^*} - \frac{1}{n}$ and $1 \leq q^* < n$:

$$\int_{\mathbb{R}^n} |D_t^{A_0^1} u_t D_t^{A_0^2} u_t D_t^{A_0} u_t| dx \leq C \|D_t^{A_0^1} u_t\|_{L^p(\mathbb{R}^n)} \|\nabla D_t^{A_0^2} u_t\|_{L^{q^*}(\mathbb{R}^n)} \|\nabla D_t^{A_0} u_t\|_{L^2(\mathbb{R}^n)}$$

with $\frac{1}{p} + \frac{1}{q^*} = \frac{n+4}{2n}$. We notice that if we want to use the Sobolev embeddings (1.35) to L^p and to L^{q^*} , it is only possible if $\frac{1}{p}$ and $\frac{1}{q^*}$ are smaller than $\frac{1}{2}$, or equivalently, if $\frac{1}{p} + \frac{1}{q^*} = \frac{n+4}{2n} < 1$. Knowing that $\frac{n+4}{2n} < 1$ for $n \geq 5$, $\frac{n+4}{2n} > 1$ for $n = 3$ and $\frac{n+4}{2n} = 1$ for $n = 4$, we treat separately two cases: $n \geq 5$ and $n = 3$ or 4 .

For $n = 3$ or 4 , we choose $p = \frac{n}{2}$ and $q = \frac{2n}{n-2}$, implying $q^* = 2$. Thus, for $n = 3$ we use the continuous embedding $H^2(\mathbb{R}^3) \subset L^{\frac{3}{2}}(\mathbb{R}^3)$ [2] (since $2 > \frac{3}{2}$) and for $n = 4$ we use $H^2(\mathbb{R}^4) \subset L^2(\mathbb{R}^4)$ to obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |D_t^{A_0^1} u_t D_t^{A_0^2} u_t D_t^{A_0} u_t| dx &\leq \|D_t^{A_0^1} u_t\|_{L^{\frac{n}{2}}(\mathbb{R}^n)} \|\nabla D_t^{A_0^2} u_t\|_{L^2(\mathbb{R}^n)} \|\nabla D_t^{A_0} u_t\|_{L^2(\mathbb{R}^n)} \\ &\leq C \|D_t^{A_0^1} u_t\|_{H^2(\mathbb{R}^n)} S_{\frac{m}{2}}[u]. \end{aligned}$$

If $m - 2A_0^1 \geq 2$, then we directly have

$$\|D^{A^1} u_t\|_{H^2(\mathbb{R}^n)} \leq \|D^{A^1} u_t\|_{H^{m-2A_0^1}(\mathbb{R}^n)} \leq \sqrt{E_{\frac{m}{2}}[u]}.$$

Recalling that m is even, and, by our assumption $|A^1| + A_0^1 \leq m$, $2A_0^1 \leq m$, there is only one additional possibility: $m - 2A_0^1 = 0$, i.e. $A_0^1 = \frac{m}{2}$.

For $A_0^1 = \frac{m}{2}$, thanks to (1.33) and the assumption $2A_0 \leq m$, we necessarily have $|A_0^2| = 1$, and consequently, by (1.74),

$$\int_{\mathbb{R}^n} |\partial_t^{\frac{m}{2}} u_t u_{tt} \partial_t^{\frac{m}{2}} u_t| dx \leq C \|u_{tt}\|_{H^2(\mathbb{R}^n)} \|\partial_t^{\frac{m}{2}} u_t\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}^2 \leq \sqrt{E_{\frac{m}{2}}[u]} S_{\frac{m}{2}}[u].$$

Thus for $n = 3$ and $n = 4$ we find estimate (1.68).

Now, for $n \geq 5$, when $\frac{1}{p} + \frac{1}{q^*} = \frac{n+4}{2n} < 1$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} |D_t^{A_0^1} u_t D_t^{A_0^2} u_t D_t^{A_0} u_t| dx &\leq C \|D_t^{A_0^1} u_t\|_{L^p(\mathbb{R}^n)} \|\nabla D_t^{A_0^2} u_t\|_{L^{q^*}(\mathbb{R}^n)} \|\nabla D_t^{A_0} u_t\|_{L^2(\mathbb{R}^n)} \\ &\leq C \|D_t^{A_0^1} u_t\|_{H^{m_1}(\mathbb{R}^n)} \|\nabla D_t^{A_0^2} u_t\|_{H^{m_2}(\mathbb{R}^n)} \sqrt{S_{\frac{m}{2}}[u]} \end{aligned}$$

with $m_1 + m_2 = \frac{n}{2} - 2$ and $0 < m_1 < \frac{n}{2} - 2$ by the Sobolev embeddings (1.35) which give us $H^{m_1} \subset L^p$ and $H^{m_2} \subset L^{q^*}$. We look for m_1 such that

$$m_1 \leq m - 2A_0^1, \quad m_2 \leq m - 2A_0^2 \quad (1.78)$$

in order to have

$$\|D_t^{A_0^1} u_t\|_{H^{m_1}} \leq \sqrt{E_{\frac{m}{2}}[u]} \quad \text{and} \quad \|\nabla D_t^{A_0^2} u_t\|_{H^{m_2}} \leq \sqrt{S_{\frac{m}{2}}[u]}.$$

As $m_2 = \frac{n}{2} - 2 - m_1$ and $A_0^2 = A_0 + 1 - A_0^1$, system (1.78) is equivalent to

$$\begin{cases} m_1 + 2A_0^1 \leq m, \\ \frac{n}{2} - 2 \leq m - 2A_0 + m_1 + 2A_0^1 - 2. \end{cases}$$

As $m - 2A_0 \geq 0$, it is sufficient to have m_1 such that

$$\frac{n}{2} \leq m_1 + 2A_0^1 \leq m$$

with $0 < m_1 < \frac{n}{2} - 2$ and $1 \leq A_0^1 \leq \frac{m}{2}$. If $2 \leq A_0^1 < \frac{n}{4}$ we can take $m_1 = \frac{n}{2} - 2A_0^1$. And if $\frac{n}{4} \leq A_0^1 \leq \frac{m}{2} - 1$ we can take $m_1 = \frac{1}{4}$.

If $A_0^1 = 1$, then necessary $A_0^2 = A_0$, and using estimates (1.65) and (1.74) we directly find

$$\int_{\mathbb{R}^n} |u_{tt} (D_t^{A_0} u_t)^2| dx \leq C \|u_{tt}\|_{L^{\frac{n}{2}}(\mathbb{R}^n)} \|D_t^{A_0^2} u_t\|_{L^{\frac{2m}{n-2}}(\mathbb{R}^n)}^2 \leq C \sqrt{E_{\frac{m}{2}}[u]} S_{\frac{m}{2}}[u].$$

If $A_0^1 = \frac{m}{2}$ we are in a symmetric case as $A_0^2 = 1$. This conclude the proof of Case 3 and of Step 1, *i.e.* of estimate (1.68).

Step 2. Let us show estimate (1.69). Thanks to properties (1.33) of A_1 and A_2 and to the symmetry of the general case

$$\int_{\mathbb{R}^n} |(D_t^{A_0^1} D_x^{(A_1^1, \dots, A_n^1)} u_{x_i})(D_t^{A_0^2} D_x^{(A_1^2, \dots, A_n^2)} u_{x_i})(D^A u_t)| dx,$$

we divide our proof on two typical cases:

Case 1 $|A^1| - A_0^1 \geq 0$, $A_0^1 > 0$, $|A^2| - A_0^2 \geq 0$ and $A_0^2 > 0$, *i.e.* a non trivial presence of $D_t^{A_0^1}$ and $D_t^{A_0^2}$ is imposed,

Case 2 $|A^1| - A_0^1 > 0$, $A_0^1 = 0$, $|A^2| - A_0^2 \geq 0$ and $A_0^2 > 0$, *i.e.* we consider the integrals of the form $\int_{\mathbb{R}^n} |(D_x^{A_1^1 + \dots + A_n^1} u_{x_i})(D_t^{A_0^2} D_x^{A_1^2 + \dots + A_n^2} u_{x_i})(D^A u_t)| dx$ with a non-trivial $D_t^{A_0^2}$.

Case 1. Using estimate $\|D^A u_t\|_{L^2} \leq \sqrt{E_{\frac{m}{2}}[u]}$, we have

$$\begin{aligned} & \int_{\mathbb{R}^n} |(D_t^{A_0^1} D_x^{(A_1^1, \dots, A_n^1)} u_{x_i})(D_t^{A_0^2} D_x^{(A_1^2, \dots, A_n^2)} u_{x_i})(D^A u_t)| dx \\ & \leq C \|\nabla \partial_t^{A_0^1} u\|_{H^{m_1 + |A^1| - A_0^1}(\mathbb{R}^n)} \|\nabla \partial_t^{A_0^2} u\|_{H^{m_2 + |A^2| - A_0^2}(\mathbb{R}^n)} \sqrt{E_{\frac{m}{2}}[u]} \end{aligned}$$

with $m_1 + m_2 = \frac{n}{2}$ and $0 < m_1 < \frac{n}{2}$.

Let us find m_1 with $0 < m_1 < \frac{n}{2}$ such that

$$\begin{cases} m_1 + |A^1| - A_0^1 \leq m - 2(A_0^1 - 1), \\ m_2 + |A^2| - A_0^2 \leq N - 2(A_0^2 - 1) \end{cases} \quad (1.79)$$

in order to have

$$\|\nabla \partial_t^{A_0^1} u\|_{H^{m_1+|A^1|-A_0^1}(\mathbb{R}^n)} \leq \sqrt{S_{\frac{m}{2}}[u]} \quad \text{and} \quad \|\nabla \partial_t^{A_0^2} u\|_{H^{m_2+|A^2|-A_0^2}(\mathbb{R}^n)} \leq \sqrt{S_{\frac{m}{2}}[u]}.$$

As $m_2 = \frac{n}{2} - m_1$, $|A^1| + |A^2| = |A| + 1$, and $A_0^1 + A_0^2 = A_0 + 1$, system (1.79) is equivalent to

$$\begin{cases} m_1 + |A^1| + A_0^1 \leq m + 2, \\ \frac{n}{2} + |A| + A_0 + 2 \leq m + 2 + m_1 + |A^1| + A_0^1. \end{cases}$$

By our assumption $|A| + A_0 \leq m$, and hence the last system is satisfied if m_1 verifies

$$\frac{n}{2} \leq m_1 + |A^1| + A_0^1 \leq m + 2.$$

In our case $A_0^1 > 0$, thus $2 \leq |A^1| + A_0^1 \leq m$, which implies the existence of such a m_1 with $0 < m_1 < \frac{n}{2}$. Indeed, if $m \geq |A^1| + A_0^1 \geq \frac{n}{2}$ we can take $m_1 = 1$, else if $2 \leq |A^1| + A_0^1 < \frac{n}{2}$ it is possible to take $m_1 = \frac{n}{2} - (|A^1| + A_0^1)$. This concludes Case 1 of Step 2.

Case 2. Thanks to (1.33), the conditions $|A^1| > 0$ with $A_0^1 = 0$ imply that $|A| - A_0 > 0$. Consequently, with $m_1 + m_2 = \frac{n}{2}$ and $0 < m_1 < \frac{n}{2}$ as in the previous case, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^n} |D_x^{A^1} \partial_{x_i} u D^{A^2} \partial_{x_i} u D^A u_t| dx \\ & \leq C \|\nabla u\|_{H^{m_1+|A^1|}(\mathbb{R}^n)} \|\nabla \partial_t^{A_0^2} u\|_{H^{m_2+|A^2|-A_0^2}(\mathbb{R}^n)} \|\nabla \partial_t^{A_0} u_t\|_{H^{|A|-A_0-1}(\mathbb{R}^n)} \\ & \leq C \|\nabla u\|_{H^{m_1+|A^1|}(\mathbb{R}^n)} \|\nabla \partial_t^{A_0^2} u\|_{H^{m_2+|A^2|-A_0^2}(\mathbb{R}^n)} \sqrt{S_{\frac{m}{2}}[u]}. \end{aligned}$$

In the aim to have

$$\|\nabla u\|_{H^{m_1+|A^1|}(\mathbb{R}^n)} \leq \sqrt{E_{\frac{m}{2}}[u]} \quad \text{and} \quad \|\nabla \partial_t^{A_0^2} u\|_{H^{m_2+|A^2|-A_0^2}(\mathbb{R}^n)} \leq \sqrt{S_{\frac{m}{2}}[u]},$$

we need to find m_1 with $0 < m_1 < \frac{n}{2}$, such that

$$\begin{cases} m_1 + |A^1| \leq m, \\ m_2 + |A^2| - A_0^2 \leq m - 2(A_0^2 - 1). \end{cases}$$

As $m_2 = \frac{n}{2} - m_1$, $|A^2| = |A| + 1 - |A^1|$ and $A_0^2 = A_0 + 1$ it is equivalent to solve

$$\begin{cases} m_1 + |A^1| \leq m, \\ \frac{n}{2} - m_1 + |A| + 1 - |A^1| + A_0 + 1 - 2 \leq m. \end{cases}$$

As $m - |A| - A_0 \geq 0$, the last system is satisfied if m_1 verifies

$$\frac{n}{2} \leq m_1 + |A^1| \leq m.$$

By assumptions of this case it told $1 \leq |A^1| \leq m$, what guarantees the existence of such m_1 with $0 < m_1 < \frac{n}{2}$. Indeed, if $1 \leq |A^1| < \frac{n}{2}$, then we can take $m_1 = \frac{n}{2} - |A^1|$, and if $\frac{n}{2} \leq |A^1| \leq m - 1$, then we can take $m_1 = \frac{1}{2}$. In the case $|A^1| = m$, corresponding to $D^{A^2} = \partial_t$, we directly obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |D_x^{A^1} \partial_{x_i} u \partial_{x_i} u_t D^A u_t| dx &\leq C \|D_x^{A^1} \partial_{x_i} u\|_{L^2(\mathbb{R}^n)} \|\partial_{x_i} u_t\|_{L^n(\mathbb{R}^n)} \|D^A u_t\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \\ &\leq C \|\nabla u\|_{H^m} \|\nabla u_t\|_{H^m(\mathbb{R}^n)} \|\nabla D^A u_t\|_{L^2(\mathbb{R}^n)} \leq C \sqrt{E_{\frac{m}{2}}[u]} S_{\frac{m}{2}}[u]. \end{aligned}$$

This completes the proof of Step 2 and hence the proof of estimate (1.69).

Thus, estimates (1.68) and (1.69) imply

$$\left| \int_{\mathbb{R}^n} L_u D^A u D^A u_t dx \right| \leq C \max(\alpha, \beta) \varepsilon \sqrt{E_{\frac{m}{2}}[u]} S_{\frac{m}{2}}[u],$$

from where follows (1.64). \square

Thanks to Lemma 1.5.1, we have the following energy decay result:

Theorem 1.5.3. *Let $n \geq 3$, $m \in \mathbb{N}$ be even and $m \geq \lceil \frac{n}{2} + 3 \rceil$. For $u_0 \in H^{m+1}(\mathbb{R}^n)$ and $u_1 \in H^m(\mathbb{R}^n)$, satisfying the smallness condition according to Point 1 of Theorem 1.2.2, there exists a unique global solution*

$$u \in C^1(\mathbb{R}^+; H^{m-1}(\mathbb{R}^n)) \cap C(\mathbb{R}^+; H^m(\mathbb{R}^n))$$

of problem (1.5)–(1.6) and the energy $E_{\frac{m}{2}}[u](0) < \infty$ is well-defined. Then

1. it holds the a priori estimate

$$\frac{d}{dt} E(t) + \sqrt{2} \varepsilon S_{\frac{m}{2}}[u](t) \left(\sqrt{2} \nu - C_m \max(\alpha, \beta) \sqrt{E(t)} \right) \leq 0, \quad (1.80)$$

where, denoting by V the set of all multi-indices $A = (A_0, A_1, \dots, A_n)$ with $|A| - A_0 \leq m - 2A_0$,

$$E(t) = \sum_{A \in V} \int_{\mathbb{R}^n} (1 - \alpha \varepsilon u_t) (D^A u_t)^2 + c^2 (\nabla D^A u)^2(t, x) dx.$$

2. if in addition $\sqrt{E_{\frac{m}{2}}[u](0)} \leq \frac{\sqrt{2} \nu}{\sqrt{\frac{3}{2} + c^2} C_m \max(\alpha, \beta)} = O(\sqrt{\varepsilon})$, then

$$\forall t \in \mathbb{R}^+, \quad E_{\frac{m}{2}}[u](t) \leq (3 + 2c^2) E_{\frac{m}{2}}[u](0) = O(1). \quad (1.81)$$

Proof. We sum (1.62) on all $A \in V$ to obtain

$$\frac{d}{dt} E(t) + 2\nu \varepsilon S_{\frac{m}{2}}[u] \leq C_m \max(\alpha, \beta) \varepsilon \sqrt{E_{\frac{m}{2}}[u]} S_{\frac{m}{2}}[u].$$

While $\|u_t(t)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{2\alpha\varepsilon}$ it holds

$$\frac{1}{2} E_{\frac{m}{2}}[u](t) \leq E(t) \leq \left(\frac{3}{2} + c^2\right) E_{\frac{m}{2}}[u](t),$$

and consequently,

$$\frac{d}{dt}E(t) + 2\nu\varepsilon S_{\frac{m}{2}}[u](t) \leq \sqrt{2}C_m \max(\alpha, \beta)\varepsilon\sqrt{E(t)}S_{\frac{m}{2}}[u](t).$$

Thus, if for all time $\sqrt{E(t)} < \frac{\sqrt{2}\nu}{\max(\alpha, \beta)C_m}$, and in particular,

$$E(0) \leq \left(\frac{3}{2} + c^2\right) E_{\frac{m}{2}}[u](0) < 2 \left(\frac{\nu}{C_m \max(\alpha, \beta)}\right)^2, \quad (1.82)$$

then we have the decreasing of E in time:

$$\frac{d}{dt}E(t) < 0 \quad \text{and} \quad E(t) \leq E(0).$$

Moreover, for all time $t \geq 0$

$$\begin{aligned} \|u_t(t)\|_{L^\infty(\mathbb{R}^n)} &\leq C_\infty \sqrt{E_{\frac{m}{2}}[u](t)} \leq C_\infty \sqrt{2}\sqrt{E(t)} \leq C_\infty \sqrt{2}\sqrt{E(0)} \\ &< 2C_\infty \frac{\nu}{C_m \max(\alpha, \beta)} < \frac{1}{2\alpha\varepsilon}. \end{aligned}$$

To be able to write $2C_\infty \frac{\nu}{C_m \max(\alpha, \beta)} < \frac{1}{2\alpha\varepsilon}$, we recall that, using the physical values of coefficients, $\varepsilon \ll 1$, $c^2 = O(\frac{1}{\varepsilon})$, $\alpha = \frac{\gamma-1}{c^2} < \beta = 2$, and consequently, as $\nu = O(1)$, the last inequality becomes

$$\frac{C_\infty}{C_m} \nu < \frac{1}{2\alpha\varepsilon},$$

which is obviously true in the case of $\varepsilon \ll 1$ (and, for instance, taking $C_m = 2C_\infty$). Hence, if Eq. (1.82) holds, then for all time $\|u_t(t)\|_{L^\infty} < \frac{1}{2\alpha\varepsilon}$ and the well-posedness of the Cauchy problem is ensured with the following energy estimate

$$E_{\frac{m}{2}}[u](t) \leq 2E(0) \leq (3 + 2c^2)E_{\frac{m}{2}}[u](0).$$

□

1.6 Illustration of the sharp behaviour of Point 1 in Theorem 1.2.2

Theorem 1.6.1. *Let $n \geq 3$, $m \in \mathbb{N}$ be even, $m \geq [\frac{n}{2} + 3]$. For $u_0 \in H^{m+1}(\mathbb{R}^n)$ and $u_1 \in H^m(\mathbb{R}^n)$ if*

$$\begin{aligned} &\|\nabla u_0\|_{H^m(\mathbb{R}^n)} + \|u_1\|_{H^m(\mathbb{R}^n)} \\ &\leq \sqrt{\frac{1}{1 + \frac{(2c^2+2)^{m+2}-1}{(2c^2+2)^2-1}} \frac{2\nu^2}{(\frac{3}{2} + c^2)C_m^2 \max(\alpha^2, \beta^2)}} = O(\sqrt{\varepsilon^{m+1}}), \end{aligned} \quad (1.83)$$

then $\sqrt{E_{\frac{m}{2}}[u](0)} \leq \frac{\sqrt{2}\nu}{\sqrt{\frac{3}{2} + c^2}C_m \max(\alpha, \beta)} = O(\sqrt{\varepsilon})$, so that by Theorem 1.5.3 Point 2 there exists a unique global solution $u \in C^0(\mathbb{R}^+; H^{m+1}(\mathbb{R}^n)) \cap C^1(\mathbb{R}^+; H^m(\mathbb{R}^n))$ of the Cauchy problem associated to the Kuznetsov equation such that for all $t \in \mathbb{R}^+$

$$\sqrt{E_{\frac{m}{2}}[u](t)} \leq \sqrt{\left(\frac{3}{2} + c^2\right) \left(1 + \frac{(2c^2+2)^{m+2}-1}{(2c^2+2)^2-1}\right)} (\|\nabla u_0\|_{H^m(\mathbb{R}^n)} + \|u_1\|_{H^m(\mathbb{R}^n)}). \quad (1.84)$$

Proof. We want to show (1.83). To do it, we perform the induction on $i \in \{0; 1; \dots; \frac{m}{2}\}$ proving that the time derivatives of the solution of the Cauchy problem (1.5)–(1.6) u at $t = 0$ satisfy for all $i \in \{0; 1; \dots; \frac{m}{2}\}$ and for $k \in \mathbb{N}$, $0 \leq k \leq i$ the following estimate

$$\|\partial_t^k u_t(0)\|_{H^{m-2k}(\mathbb{R}^n)} \leq a_k (\|\nabla u_0\|_{H^m(\mathbb{R}^n)} + \|u_1\|_{H^m(\mathbb{R}^n)}), \quad (1.85)$$

with $a_0 = 1$, $a_1 = 2c^2 + 2$ and

$$a_{k+1} = a_k + 2c^2 a_{k-1} + 2 \sum_{i=0}^k a_i + 1 \quad \text{for } 1 \leq k \leq \frac{m}{2} - 1.$$

For $i = 0$ the proof is direct. For $i = 1$ from the Kuznetsov equation we have

$$u_{tt}(0) = \frac{1}{1 - \alpha \varepsilon u_1} (c^2 \Delta u_0 + \nu \varepsilon \Delta u_1 + \beta \varepsilon \nabla u_0 \nabla u_1).$$

As for a small enough ε it holds $\|\frac{1}{1 - \alpha \varepsilon u_1}\|_\infty \leq 2$, taking the $\|\cdot\|_{H^{m-2}(\mathbb{R}^n)}$ -norm of the last equality we obtain

$$\begin{aligned} \|u_{tt}(0)\|_{H^{m-2}(\mathbb{R}^n)} &\leq 2(c^2 \|\Delta u_0\|_{H^{m-2}(\mathbb{R}^n)} + \nu \varepsilon \|\Delta u_1\|_{H^{m-2}(\mathbb{R}^n)} \\ &\quad + \beta \varepsilon \|\nabla u_0 \nabla u_1\|_{H^{m-2}(\mathbb{R}^n)}). \end{aligned} \quad (1.86)$$

Thanks to [2] we have for all $l \in \mathbb{N}$ and for all $k \in \mathbb{N}$, $0 \leq l \leq m$ and $0 \leq k \leq m - l$ the continuous embedding of the product

$$H^{m-l}(\mathbb{R}^n) \times H^{k+l}(\mathbb{R}^n) \hookrightarrow H^k(\mathbb{R}^n). \quad (1.87)$$

Thus we can write for (1.86)

$$\begin{aligned} \|u_{tt}(0)\|_{H^{m-2}(\mathbb{R}^n)} &\leq 2(c^2 \|\nabla u_0\|_{H^m(\mathbb{R}^n)} + \nu \varepsilon \|u_1\|_{H^m(\mathbb{R}^n)} \\ &\quad + \beta \varepsilon K \|\nabla u_0\|_{H^{m-1}(\mathbb{R}^n)} \|\nabla u_1\|_{H^{m-1}(\mathbb{R}^n)}), \end{aligned}$$

and by Young's inequality we find

$$\begin{aligned} \|u_{tt}(0)\|_{H^{m-2}(\mathbb{R}^n)} &\leq 2 \left[c^2 \|\nabla u_0\|_{H^m(\mathbb{R}^n)} + \nu \varepsilon \|u_1\|_{H^m(\mathbb{R}^n)} \right. \\ &\quad \left. + \frac{1}{2} \beta \varepsilon K \left(\|\nabla u_0\|_{H^m(\mathbb{R}^n)}^2 + \|u_1\|_{H^m(\mathbb{R}^n)}^2 \right) \right]. \end{aligned} \quad (1.88)$$

Choosing ε small enough such that

$$\beta \varepsilon K \|\nabla u_0\|_{H^m(\mathbb{R}^n)} \leq 1, \quad \beta \varepsilon K \|u_1\|_{H^m(\mathbb{R}^n)} \leq 1, \quad \nu \varepsilon \leq \frac{1}{2},$$

from (1.88) it follows

$$\begin{aligned} \|u_{tt}(0)\|_{H^{m-2}(\mathbb{R}^n)} &\leq 2 \left[\left(c^2 + \frac{1}{2} \right) \|\nabla u_0\|_{H^m(\mathbb{R}^n)} + \left(\frac{1}{2} + \frac{1}{2} \right) \|u_1\|_{H^m(\mathbb{R}^n)} \right] \\ &\leq (2c^2 + 2) (\|\nabla u_0\|_{H^m(\mathbb{R}^n)} + \|u_1\|_{H^m(\mathbb{R}^n)}). \end{aligned}$$

Let define now the induction hypothesis: for $i \in \{0; 1; \dots; \frac{m}{2} - 1\}$ for $k \in \mathbb{N}$, $0 \leq k \leq i$ it holds estimate (1.85). Now we want to show it for $i + 1$, by the induction hypothesis we just need to show

$$\|\partial_t^{i+1} u_t(0)\|_{H^{m-2(i+1)}(\mathbb{R}^n)} \leq a_{i+1} (\|\nabla u_0\|_{H^m(\mathbb{R}^n)} + \|u_1\|_{H^m(\mathbb{R}^n)}).$$

Deriving i -times on time the Kuznetsov equation, for $i \geq 1$ we obtain

$$\begin{aligned} \partial_t^i u_{tt}(0) = & \frac{1}{1 - \alpha u_1} \left(c^2 \Delta \partial_t^i u(0) + \nu \varepsilon \Delta \partial_t^i u_t(0) + \alpha \varepsilon \sum_{k=0}^{i-1} C_i^k \partial_t^{i-k} u_t(0) \partial_t^k u_{tt}(0) \right. \\ & \left. + \beta \varepsilon \sum_{k=0}^i C_i^k \nabla \partial_t^{i-k} u(0) \nabla \partial_t^k u_t(0) \right). \end{aligned}$$

We take the $\|\cdot\|_{H^{m-2(i+1)}}$ -norm of this equation and in the same way as for $i = 1$ we show that

$$\begin{aligned} & \|\partial_t^{i+1} u_t(0)\|_{H^{m-2(i+1)}(\mathbb{R}^n)} \\ & \leq \left(2c^2 a_{i-1} + a_i + 2 \sum_{k=0}^i a_k + 1 \right) (\|\nabla u_0\|_{H^m(\mathbb{R}^n)} + \|u_1\|_{H^m(\mathbb{R}^n)}) \\ & \leq a_{i+1} (\|\nabla u_0\|_{H^m(\mathbb{R}^n)} + \|u_1\|_{H^m(\mathbb{R}^n)}). \end{aligned}$$

This concludes the induction.

With the induction result we have for $k \in \mathbb{N}$, $0 \leq k \leq \frac{m}{2}$

$$\|\partial_t^k u_t(0)\|_{H^{m-2k}(\mathbb{R}^n)} \leq a_k (\|\nabla u_0\|_{H^m(\mathbb{R}^n)} + \|u_1\|_{H^m(\mathbb{R}^n)}),$$

where

$$a_k \leq (2c^2 + 2)^k.$$

Therefore we can write

$$\begin{aligned} E_{\frac{m}{2}}[u](0) & \leq \left(1 + \sum_{i=0}^{\frac{m}{2}} a_i^2 \right) (\|\nabla u_0\|_{H^m(\mathbb{R}^n)} + \|u_1\|_{H^m(\mathbb{R}^n)})^2 \\ & \leq \left(1 + \frac{(2c^2 + 2)^{m+2} - 1}{(2c^2 + 2)^2 - 1} \right) (\|\nabla u_0\|_{H^m(\mathbb{R}^n)} + \|u_1\|_{H^m(\mathbb{R}^n)})^2. \end{aligned}$$

Hence, taking the initial data satisfying estimate (1.83) we have the following estimate for the initial energy

$$E_{\frac{m}{2}}[u](0) \leq \frac{2\nu^2}{\left(\frac{3}{2} + c^2\right) C_m^2 \max(\alpha^2, \beta^2)}.$$

Consequently, by Theorem 1.5.3 Point 2 for all $t \in \mathbb{R}^+$ we obtain estimate (1.84). \square

Chapter 2

Models of nonlinear acoustics viewed as an approximation of the Navier-Stokes and Euler compressible isentropic systems

2.1 Introduction française

L'étude de la propagation d'ondes non-linéaires suscite un renouveau d'intérêt, en particulier à cause de récentes applications à l'imagerie ultrason (par exemple HIFU) ou des applications techniques et médicales comme la lithotripsie ou la thérapie. Ces techniques nouvelles reposent fortement sur la capacité à modéliser précisément la propagation non-linéaire d'une pulsation sonore d'amplitude finie dans un milieu élastique thermo-visqueux. Les modèles les plus connus d'acoustique non linéaire, que nous considérons dans ce chapitre, sont

1. l'équation de Kuznetsov (voir Eq. (2.13) et Eq. (2.23));
2. l'équation de Khokhlov-Zabolotskaya-Kuznetsov (KZK) (voir Eq. (2.63));
3. l'Équation d'onde Non-linéaire Progressive (NPE);
4. l'équation de Westervelt (voir Eq. (2.146)).

Nous pouvons nous référer à l'introduction générale pour l'analyse physique de ces modèles. Ils ont tous été dérivés jusqu'à de petits termes négligeables à partir de systèmes non-linéaires de Naviers-Stokes (pour le milieu visqueux) et d'Euler (pour le cas non visqueux) compressibles et isentropiques. Mais toutes les dérivations physiques de ces modèles citées ne permettent pas de dire que leurs solutions approchent la solution du système de Navier-Stokes ou d'Euler. Nous commençons dans la Section 2.3 à présenter le contexte initial du système de Navier-Stokes isentropique (en fait, c'est aussi une approximation du système de Navier-Stokes compressible (2.5)–(2.8)), qui décrit le mouvement d'une onde acoustique dans un milieu thermo-élastique homogène [13, 37, 65]. Nous systématisons dans ce chapitre la dérivation de tous ces modèles en utilisant les idées de la Réf. [81], consistant à utiliser des correcteurs des *ansatzs* physiques correspondants dans les expansions de type Hilbert.

Plus précisément, nous montrons que tous ces modèles sont des approximations du système de Navier-Stokes ou d'Euler jusqu'aux termes d'ordre trois en un petit paramètre

sans dimension $\epsilon > 0$ mesurant la taille des perturbations de la pression, la densité et la vitesse par rapport à leur état constant $(p_0, \rho_0, 0)$ (voir Fig. 1).

Dans la Section 2.4, nous validons les approximations du système de Navier-Stokes compressible par les différents modèles: l'équation de Kuznetsov (Sous-section 2.4.1), l'équation de KZK (Sous-section 2.4.2) et l'équation NPE (Sous-section 2.4.3).

Dans la Section 2.5, nous faisons de même pour le système d'Euler dans le cas non visqueux. Les différences principales entre les cas visqueux et non visqueux sont le temps d'existence et la régularité des solutions. Typiquement, dans le cas non visqueux, les solutions des modèles et aussi du système d'Euler lui-même (solutions fortes) peuvent entraîner la formation de fronts de choc en temps finis à cause de leurs non-linéarités[4, 26, 80, 84, 93], ou comme dans le Théorème 1.2.1 du Chapitre 1 pour le cas de Kuznetsov non visqueux. Ainsi, elles sont seulement localement bien posées, alors que dans le cas visqueux tous les modèles d'approximations sont globalement bien posés pour des données initiales suffisamment petites[26, 80], ou encore comme dans les Théorèmes 1.2.1 et 1.2.2 du Chapitre 1. Ces propriétés d'existence des solutions pour les cas visqueux et non visqueux peuvent aussi impliquer des différences dans la définition du domaine sur lequel les approximations ont lieu: par exemple[81], pour l'approximation entre l'équation de KZK et le système de Navier-Stokes le domaine d'approximation est le demi-espace, mais pour le cas non visqueux analogue de l'équation de KZK et du système d'Euler c'est un cône (voir aussi en conclusion le Tableau 2.1).

Dans les Sections 2.4 et 2.5 nous notons par \mathbf{U}_ϵ une solution du système de Navier-Stokes/Euler "exact" (voir l'Eq. (2.31)):

$$\partial_t \mathbf{U}_\epsilon + \sum_{i=1}^n \partial_{x_i} \mathbf{G}_i(\mathbf{U}_\epsilon) - \epsilon \nu \begin{bmatrix} 0 \\ \Delta \mathbf{v}_\epsilon \end{bmatrix} = 0,$$

et par $\overline{\mathbf{U}}_\epsilon$ une solution approchée, construite par l'*ansatz* de dérivation à partir d'une solution régulière de l'un des modèles approchés (typiquement les équations de Kuznetsov, KZK et NPE), *i.e.* une fonction qui résout le système de Navier-Stokes/Euler jusqu'aux termes d'ordre ϵ^3 , dénotés par $\epsilon^3 \mathbf{R}$ (voir l'Eq. (2.32)):

$$\partial_t \overline{\mathbf{U}}_\epsilon + \sum_{i=1}^n \partial_{x_i} \mathbf{G}_i(\overline{\mathbf{U}}_\epsilon) - \epsilon \nu \begin{bmatrix} 0 \\ \Delta \overline{\mathbf{v}}_\epsilon \end{bmatrix} = \epsilon^3 \mathbf{R}.$$

Pour avoir le terme de reste $\mathbf{R} \in C([0, T], L^2(\Omega))$ nous devons assurer que le terme de gauche de cette équation est dans $C([0, T], L^2(\Omega))$, *i.e.* nous avons besoin d'une solution $\overline{\mathbf{U}}_\epsilon$ suffisamment régulière. La régularité minimale des données initiales pour avoir un tel $\overline{\mathbf{U}}_\epsilon$ est donnée dans le Tableau 2.1 (voir aussi le Tableau 2.2 pour l'approximation de l'équation de Kuznetsov).

En choisissant pour le système exact les mêmes données initiales et au bord trouvées par l'*ansatz* pour $\overline{\mathbf{U}}_\epsilon$ (le cas régulier) ou les données initiales prises dans un petit voisinage L^2 , *i.e.*

$$\|\mathbf{U}_\epsilon(0) - \overline{\mathbf{U}}_\epsilon(0)\|_{L^2(\Omega)} \leq \delta \leq \epsilon, \tag{2.1}$$

avec $\mathbf{U}_\epsilon(0)$ non nécessairement régulier, mais assurant l'existence d'une solution faible admissible d'énergie bornée (voir la Définition 2.4.1), nous prouvons l'existence de constantes $C > 0$ et $K > 0$ indépendantes de ϵ , δ et du temps t telles que

$$\text{pour tout } 0 \leq t \leq \frac{C}{\epsilon} \quad \|(\mathbf{U}_\epsilon - \overline{\mathbf{U}}_\epsilon)(t)\|_{L^2(\Omega)}^2 \leq K(\epsilon^3 t + \delta^2) e^{K\epsilon t} \leq 9\epsilon^2 \tag{2.2}$$

avec Ω un domaine où les deux solutions \mathbf{U}_ε et $\overline{\mathbf{U}}_\varepsilon$ existent (voir les Théorèmes 2.4.3, 2.4.6 et 2.4.10).

Dans le cas visqueux tous les modèles d'approximation vérifient l'existence globale de solutions classiques pour les données initiales suffisamment petites sur leur domaine d'approximation Ω , qui varie selon les différents modèles (voir Tableau 2.1): il correspond à \mathbb{R}^n , $\mathbb{T}_{x_1} \times \mathbb{R}^{n-1}$ et $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ pour l'équation de Kuznetsov, l'équation NPE et l'équation de KZK respectivement. Si nous prenons des données initiales régulières $\mathbf{U}_\varepsilon(0) = \overline{\mathbf{U}}_\varepsilon(0)$, la même chose est vraie pour le système de Navier-Stokes avec la même régularité pour les solutions [67], à l'exception du cas du demi espace pour l'approximation entre le système de Navier-Stokes et l'équation de KZK, d'abord considérée dans la Réf. [81], où, dû aux conditions de bords périodiques, venant des conditions initiales pour l'équation de KZK, nous prouvons seulement l'existence locale. Pour l'obtenir, nous utilisons la Réf. [81] Théorème 5.5. Nous l'adaptions au cadre du nouvel *ansatz* (2.64)–(2.65) et corrigeons plusieurs ambiguïtés dans sa preuve (voir la Sous-Section 2.4.2 Théorème 2.4.5), qui nous permet dans le Théorème 2.4.6 de la Sous-Section 2.4.2 d'établir le résultat d'approximation entre l'équation de KZK et le système de Navier-Stokes en suivant la Réf. [81] Théorème 5.7 en adaptant juste les estimations de stabilité de l'approximation.

Pour obtenir l'estimation (2.2) nous n'avons pas besoin de la régularité d'une solution classique du système de Navier-Stokes (ou d'Euler), ce peut être une solution faible (au sens de Hoff [38] pour le système de Navier-Stokes ou une des solutions au sens de Luo et al. [64] pour le système d'Euler) satisfaisant les conditions d'admissibilité données dans la Définition 2.4.1 (voir aussi la Réf. [23] p.52 et la Réf. [81] Définition 5.9).

Pour le cas non visqueux, donné en Section 2.5, nous vérifions que le temps d'existence des solutions (fortes) pour tous les modèles n'est pas plus petit que $O(\frac{1}{\varepsilon})$ et que l'estimation (2.2) est toujours vérifiée.

Comme les équations de KZK et NPE peuvent être vues comme des approximations de l'équation de Kuznetsov au vu de leur dérivation (voir la Figure 1), nous validons aussi l'approximation de l'équation de Kuznetsov par les équations de KZK et NPE, et aussi par l'équation de Westervelt, dans les Sections 2.6, 2.7 et 2.8 (voir le Tableau 2.2).

Pour pouvoir considérer l'approximation de l'équation de Kuznetsov par l'équation de KZK (voir la Section 2.6), nous établissons d'abord des résultats sur le caractère globalement bien posé de l'équation de Kuznetsov dans le demi espace, similaires au cadre précédent pour l'équation de KZK et le système de Navier-Stokes dans la Sous-Section 2.4.2. Nous étudions deux cas : le problème périodique en temps purement aux bords dans les variables (z, τ, y) se déplaçant avec l'onde et le problème avec conditions initiales et au bord pour l'équation de Kuznetsov dans les variables initiales (t, x_1, x') avec des données venant de la solution de l'équation de KZK. Nous validons ces deux types d'approximations en Sous-Section 2.6.3 pour les cas visqueux et non visqueux.

Finalement dans les Section 2.7 et 2.8 nous validons l'approximation entre les équations de Kuznetsov et NPE et les équations de Kuznetsov et Westervelt respectivement (voir le Tableau 2.2). Nous pouvons les résumer de la manière suivante: si u est une solution de l'équation de Kuznetsov et \overline{u} est une solution de l'équation de NPE ou de KZK (pour le problème avec conditions initiales et aux bords) ou de Westervelt trouvée pour des données initiales assez proches

$$\|\nabla_{t,\mathbf{x}}(u(0) - \overline{u}(0))\|_{L^2(\Omega)} \leq \delta \leq \epsilon,$$

alors il existe des constantes $K > 0$, $C_1 > 0$, $C_2 > 0$ et $C > 0$ indépendantes de ϵ , δ et du

temps, telles que pour tout $t \leq \frac{C}{\epsilon}$ il est vérifié

$$\|\nabla_{t,\mathbf{x}}(u - \bar{u})\|_{L^2(\Omega)} \leq C_1(\epsilon^2 t + \delta)e^{C_2 \epsilon t} \leq K\epsilon.$$

2.2 Introduction

There is a renewed interest in the study of nonlinear wave propagation, in particular because of recent applications to ultrasound imaging (*e.g.* HIFU) or technical and medical applications such as lithotripsy or thermotherapy. Such new techniques rely heavily on the ability to model accurately the nonlinear propagation of a finite-amplitude sound pulse in thermo-viscous elastic media. The most known nonlinear acoustic models, which we consider in this chapter, are

1. the Kuznetsov equation (see Eq. (2.13) and Eq. (2.23));
2. the Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation (see Eq. (2.63));
3. the Nonlinear Progressive wave Equation (NPE) (see Eq. (2.93) and Eq. (2.94));
4. the Westervelt equation (see Eq. (2.146)).

We can refer to the general introduction for the physical analysis of these models. All these models were derived from a compressible nonlinear isentropic Navier-Stokes (for viscous media) and Euler (for the inviscid case) systems up to some small negligible terms. But all cited physical derivations of these models don't allow to say that their solutions approximate the solution of the Navier-Stokes or Euler system. The first work explaining it for the KZK equation is Ref. [81]. We start in Section 2.3 to present the initial context of the isentropic Navier-Stokes system (actually, it is also an approximation of the compressible Navier-Stokes system (2.5)–(2.8)), which describes the acoustic wave motion in an homogeneous thermo-elastic medium [13, 37, 65]. In this chapter, we systematize the derivation of all these models using the ideas of Ref. [81], consisting to use correctors in the Hilbert type expansions of corresponding physical *ansatzs*.

More precisely, we show that all these models are approximations of the isentropic Navier-Stokes or Euler system up to third order terms of a small dimensionless parameter $\epsilon > 0$ measuring the size of the perturbations of the pressure, the density and the velocity to compare to their constant state $(p_0, \rho_0, 0)$ (see Fig. 1).

In Section 2.4, we validate the approximations of the compressible isentropic Navier-Stokes system by the different models: by the Kuznetsov (Subsection 2.4.1), the KZK (Subsection 2.4.2) and the NPE equations (Subsection 2.4.3).

In Section 2.5 we do the same for the Euler system in the inviscid case. The main difference between the viscous and the inviscid case is the time existence and regularity of the solutions. Typically in the inviscid case, the solutions of the models and also of the Euler system itself (actually strong solutions), due to their non-linearity, can provide shock front formations at a finite time [4, 26, 80, 84, 93], or see Theorem 1.2.1 in Chapter 1 for the inviscid Kuznetsov equation. Thus, they are only locally well-posed, while in the viscous media all approximative models are globally well-posed for small enough initial data [26, 80], for the Kuznetsov equation we have the results of Theorems 1.2.1 and 1.2.2 in Chapter 1. These existence properties of solutions for the viscous and the inviscid cases may also imply the differences in the definition of the domain where the approximations hold:

for example[81], for the approximation between the KZK equation and the Navier-Stokes system the approximation domain is a half-space, but for the analogous inviscid case of the KZK and the Euler system it is a cone (see also the concluding Table 2.1).

In Sections 2.4 and 2.5 we denote by \mathbf{U}_ε a solution of the “exact” Navier-Stokes/Euler system (see Eq. (2.31))

$$\partial_t \mathbf{U}_\varepsilon + \sum_{i=1}^n \partial_{x_i} \mathbf{G}_i(\mathbf{U}_\varepsilon) - \varepsilon \nu \begin{bmatrix} 0 \\ \Delta \mathbf{v}_\varepsilon \end{bmatrix} = 0,$$

and by $\overline{\mathbf{U}}_\varepsilon$ an approximate solution, constructed by the derivation *ansatz* from a regular solution of one of the approximate models (typically of the Kuznetsov, the KZK or the NPE equations), *i.e.* a function which solves the Navier-Stokes/Euler system up to ε^3 terms, denoted by $\varepsilon^3 \mathbf{R}$ (see Eq. (2.32)):

$$\partial_t \overline{\mathbf{U}}_\varepsilon + \sum_{i=1}^n \partial_{x_i} \mathbf{G}_i(\overline{\mathbf{U}}_\varepsilon) - \varepsilon \nu \begin{bmatrix} 0 \\ \Delta \overline{\mathbf{v}}_\varepsilon \end{bmatrix} = \varepsilon^3 \mathbf{R}.$$

To have the remainder term $\mathbf{R} \in C([0, T], L^2(\Omega))$ we ensure that the left hand side term in this is in $C([0, T], L^2(\Omega))$, *i.e.* we need a regular enough solution $\overline{\mathbf{U}}_\varepsilon$. The minimal regularity of the initial data to have such a $\overline{\mathbf{U}}_\varepsilon$ is given in Table 2.1 (see also Table 2.2 for the approximations of the Kuznetsov equation).

Choosing for the exact system the same initial-boundary data found by the *ansatz* for $\overline{\mathbf{U}}_\varepsilon$ (the regular case) or the initial data taken in their small L^2 -neighbourhood, *i.e.*

$$\|\mathbf{U}_\varepsilon(0) - \overline{\mathbf{U}}_\varepsilon(0)\|_{L^2(\Omega)} \leq \delta \leq \varepsilon, \quad (2.3)$$

with $\mathbf{U}_\varepsilon(0)$ not necessarily smooth, but ensuring the existence of an admissible weak solution of a bounded energy (see Definition 2.4.1), we prove the existence of constants $C > 0$ and $K > 0$ independent of ε , δ and the time t such that

$$\text{for all } 0 \leq t \leq \frac{C}{\varepsilon} \quad \|(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon)(t)\|_{L^2(\Omega)}^2 \leq K(\varepsilon^3 t + \delta^2) e^{K\varepsilon t} \leq 9\varepsilon^2 \quad (2.4)$$

with Ω a domain where both the solutions \mathbf{U}_ε and $\overline{\mathbf{U}}_\varepsilon$ exist (see Theorems 2.4.3, 2.4.6 and 2.4.10).

In the viscous case, all approximative models have a global existence of classical solutions for small enough initial data in its approximative domain Ω , which is various for different models (see Table 2.1): it is equal to \mathbb{R}^n , $\mathbb{T}_{x_1} \times \mathbb{R}^{n-1}$ and $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ for the Kuznetsov equation, the NPE equation and the KZK equation respectively. If we take regular initial data $\mathbf{U}_\varepsilon(0) = \overline{\mathbf{U}}_\varepsilon(0)$, the same thing is true for the Navier-Stokes system with the same regularity for the solutions [67], except for the case of the half-space for the approximation between the Navier-Stokes system and the KZK equation, firstly considered in Ref. [81], when, due to the time periodic boundary conditions, coming from the initial conditions for the KZK equation, we only prove a local existence. To obtain it we use Ref. [81] Theorem 5.5. We updated it in the framework of the new *ansatz* (2.64)–(2.65) and corrected several misleading in its proof (see Subsection 2.4.2 Theorem 2.4.5), what allows us in Theorem 2.4.6 of Subsection 2.4.2 to establish the approximation result between the KZK equation and the Navier-Stokes system following Ref. [81] Theorem 5.7 just updating the stability approximation estimate.

To obtain estimate (2.4) we don't need the regularity of the classical solution of the Navier-Stokes (or Euler) system, it can be a weak solution (in the sense of Hoff [38] for the Navier-Stokes system or one of solutions in the sense of Luo and al. [64] for the Euler system) satisfying the admissible conditions given in Definition 2.4.1 (see also Ref. [23] p.52 and Ref. [81] Definition 5.9).

For the inviscid case, given in Section 2.5, we verify that the existence time of (strong) solutions of all models is not less than $O(\frac{1}{\epsilon})$ and estimate (2.4) still holds.

As the KZK and NPE equations can be seen as approximations of the Kuznetsov equation due to their derivation (see Figure 1), we also validate the approximation of the Kuznetsov equation by the KZK and NPE equations, and also by the Westervelt equation, in Section 2.6, 2.7 and 2.8 (see Table 2.2).

To be able to consider the approximation of the Kuznetsov equation by the KZK equation (see Section 2.6), we firstly establish global well-posedness results for the Kuznetsov equation in the half space similar to the previous framework for the KZK and the Navier-Stokes system in Subsection 2.4.2. We study two cases: the purely time periodic boundary problem in the *ansatz* variables (z, τ, y) moving with the wave and the initial boundary-value problem for the Kuznetsov equation in the initial variables (t, x_1, x') with data coming from the solution of the KZK equation. We validate these two types approximations in Subsection 2.6.3 for the viscous and inviscid cases.

Finally in Sections 2.7 and 2.8 we validate the approximation between the Kuznetsov and NPE equation and the Kuznetsov and Westervelt equations respectively (see Table 2.2). We can summarize them in the following way: if u is a solution of the Kuznetsov equation and \bar{u} is a solution of the NPE or of the the KZK (for the initial boundary value problem) or of the Westervelt equations found for rather closed initial data

$$\|\nabla_{t,\mathbf{x}}(u(0) - \bar{u}(0))\|_{L^2(\Omega)} \leq \delta \leq \epsilon,$$

then there exist $K > 0$, $C_1 > 0$, $C_2 > 0$ and $C > 0$ independent on ϵ , δ and on time, such that for all $t \leq \frac{C}{\epsilon}$ it holds

$$\|\nabla_{t,\mathbf{x}}(u - \bar{u})\|_{L^2(\Omega)} \leq C_1(\epsilon^2 t + \delta)e^{C_2 \epsilon t} \leq K\epsilon.$$

2.3 Isentropic Navier-Stokes system for a subsonic potential motion

To describe the acoustic wave motion in an homogeneous thermo-elastic medium, we start from the Navier-Stokes system in \mathbb{R}^n

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0, \tag{2.5}$$

$$\rho[\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}] = -\nabla p + \eta \Delta \mathbf{v} + \left(\zeta + \frac{\eta}{3}\right) \nabla \cdot \operatorname{div}(\mathbf{v}), \tag{2.6}$$

$$\begin{aligned} \rho T[\partial_t S + (\mathbf{v} \cdot \nabla) S] &= \kappa \Delta T + \zeta (\operatorname{div} \mathbf{v})^2 \\ &+ \frac{\eta}{2} \left(\partial_{x_k} v_i + \partial_{x_i} v_k - \frac{2}{3} \delta_{ik} \partial_{x_i} v_i \right)^2, \end{aligned} \tag{2.7}$$

$$p = p(\rho, S), \tag{2.8}$$

where the pressure p is given by the state law $p = p(\rho, S)$. The density ρ , the velocity \mathbf{v} , the temperature T and the entropy S are unknown functions in system (2.5)–(2.8). The

coefficients β , κ and η are constant viscosity coefficients. The wave motion is supposed to be potential and the viscosity coefficients are supposed to be small in terms of a dimensionless small parameter $\epsilon > 0$:

$$\eta \Delta \mathbf{v} + \left(\zeta + \frac{\eta}{3} \right) \nabla \cdot \operatorname{div}(\mathbf{v}) = \left(\zeta + \frac{4}{3} \eta \right) \Delta \mathbf{v} := \beta \Delta \mathbf{v}, \quad \text{with } \beta = \epsilon \tilde{\beta}.$$

Any constant state $(\rho_0, \mathbf{v}_0, S_0, T_0)$ is a stationary solution of system (2.5)–(2.8). Further we always take $\mathbf{v}_0 = 0$ using a Galilean transformation. Perturbation near this constant state $(\rho_0, 0, S_0, T_0)$ introduces small increments in terms of the same dimensionless small parameter $\epsilon > 0$:

$$\begin{aligned} T(x, t) &= T_0 + \epsilon \tilde{T}(x, t) & \text{and} & & S(x, t) &= S_0 + \epsilon^2 \tilde{S}(x, t), \\ \rho_\epsilon(x, t) &= \rho_0 + \epsilon \tilde{\rho}_\epsilon(x, t) & \text{and} & & \mathbf{v}_\epsilon(x, t) &= \epsilon \tilde{\mathbf{v}}_\epsilon(x, t), \end{aligned}$$

where the perturbation of the entropy is of order $O(\epsilon^2)$, since it is the smallest size on ϵ of right hand terms in Eq (2.7), due to the smallness of the viscosities (see Eq. (2.9)).

Actually, ϵ is the Mach number, which is supposed to be small [13] ($\epsilon = 10^{-5}$ for the propagation in water with an initial power of the order of 0.3 W/cm^2):

$$\frac{\rho - \rho_0}{\rho_0} \sim \frac{T - T_0}{T_0} \sim \frac{|\mathbf{v}|}{c_0} \sim \epsilon,$$

where $c_0 = \sqrt{p'(\rho_0)}$ is the speed of sound in the unperturbed media.

Using the transport heat equation (2.7) up to the terms of the order of ϵ^3

$$\epsilon^2 \rho_0 T_0 \partial_t \tilde{S} = \epsilon^2 \tilde{\kappa} \Delta \tilde{T} + \mathcal{O}(\epsilon^3), \quad (2.9)$$

the approximate state equation

$$p = p_0 + c^2 \epsilon \tilde{\rho}_\epsilon + \frac{1}{2} (\partial_\rho^2 p)_S \epsilon^2 \tilde{\rho}_\epsilon^2 + (\partial_S p)_\rho \epsilon^2 \tilde{S} + \mathcal{O}(\epsilon^3)$$

(where the notation $(\cdot)_S$ means that the expression in brackets is constant in S), can be replaced [13, 65, 37] by

$$p = p_0 + c^2 \epsilon \tilde{\rho}_\epsilon + \frac{(\gamma - 1)c^2}{2\rho_0} \epsilon^2 \tilde{\rho}_\epsilon^2 - \epsilon \tilde{\kappa} \left(\frac{1}{C_V} - \frac{1}{C_p} \right) \nabla \cdot \mathbf{v}_\epsilon + \mathcal{O}(\epsilon^3),$$

using $T = \frac{p}{\rho R}$ from the theory of ideal gaze and taking

$$p(\rho, S) = R \rho^\gamma e^{\frac{S - S_0}{C_V}}.$$

Here $\gamma = C_p/C_V$ denotes the ratio of the heat capacities at constant pressure and at constant volume respectively.

Hence, system (2.5)–(2.8) becomes an isentropic Navier-Stokes system

$$\partial_t \rho_\epsilon + \operatorname{div}(\rho_\epsilon \mathbf{v}_\epsilon) = 0, \quad (2.10)$$

$$\rho_\epsilon [\partial_t \mathbf{v}_\epsilon + (\mathbf{v}_\epsilon \cdot \nabla) \mathbf{v}_\epsilon] = -\nabla p(\rho_\epsilon) + \epsilon \nu \Delta \mathbf{v}_\epsilon, \quad (2.11)$$

with the approximate state equation $p(\rho, S) = p(\rho_\epsilon) + O(\epsilon^3)$:

$$p(\rho_\epsilon) = p_0 + c^2 (\rho_\epsilon - \rho_0) + \frac{(\gamma - 1)c^2}{2\rho_0} (\rho_\epsilon - \rho_0)^2, \quad (2.12)$$

and with a small enough and positive viscosity coefficient:

$$\epsilon \nu = \beta + \kappa \left(\frac{1}{C_V} - \frac{1}{C_p} \right).$$

2.4 Approximation of the Navier-Stokes system

2.4.1 Navier-Stokes system and the Kuznetsov equation

We consider system (2.10)–(2.12) as the exact model. The state law (2.12) is a Taylor expansion of the pressure up to the terms of the third order on ϵ . Therefore an approximation of system (2.10)–(2.12) for \mathbf{v}_ϵ and ρ_ϵ up to terms $O(\epsilon^3)$ would be optimal. In the framework of the nonlinear acoustic between the known approximative models derived from system (2.10)–(2.12) are the Kuznetsov, the KZK and the NPE equations. In this section we focus on the first of these models, *i.e.* on the Kuznetsov equation.

Initially the Kuznetsov equation was derived by Kuznetsov [60] from the isentropic Navier-Stokes system (2.10)–(2.12) for the small velocity potential $\mathbf{v}_\epsilon(\mathbf{x}, t) = -\nabla\tilde{u}(\mathbf{x}, t)$, $\mathbf{x} \in \mathbb{R}^n$, $t \in \mathbb{R}^+$:

$$\partial_t^2 \tilde{u} - c^2 \Delta \tilde{u} = \partial_t \left((\nabla \tilde{u})^2 + \frac{\gamma - 1}{2c^2} (\partial_t \tilde{u})^2 + \frac{\epsilon \nu}{\rho_0} \Delta \tilde{u} \right). \quad (2.13)$$

The derivation was latter discussed by a lot of authors [37, 50, 63].

Unlike in these physical derivations we introduce a Hilbert expansion type construction with a corrector $\epsilon^2 \rho_2(\mathbf{x}, t)$ for the density perturbation, considering the following *ansatz*

$$\rho_\epsilon(\mathbf{x}, t) = \rho_0 + \epsilon \rho_1(\mathbf{x}, t) + \epsilon^2 \rho_2(\mathbf{x}, t), \quad (2.14)$$

$$\mathbf{v}_\epsilon(\mathbf{x}, t) = -\epsilon \nabla u(\mathbf{x}, t). \quad (2.15)$$

The use of the second order corrector in (2.14) allows to ensure the approximation of (2.11) up to terms of order ϵ^3 (see Subsection 2.4.1) and to open the question about the approximation between the exact solution of the isentropic Navier-Stokes system (2.10)–(2.12) and its approximation given by the solution of the Kuznetsov equation, as it was done for the KZK equation[81].

Derivation of the Kuznetsov equation from an isentropic Navier-Stokes system

Putting expressions for the density and velocity (2.14)–(2.15) into the isentropic Navier-Stokes system (2.10)–(2.12), we obtain for the momentum conservation (2.11)

$$\begin{aligned} \rho_\epsilon [\partial_t \mathbf{v}_\epsilon + (\mathbf{v}_\epsilon \cdot \nabla) \mathbf{v}_\epsilon] + \nabla p(\rho_\epsilon) - \epsilon \nu \Delta \mathbf{v}_\epsilon &= \epsilon \nabla (-\rho_0 \partial_t u + c^2 \rho_1) \\ + \epsilon^2 \left[-\rho_1 \nabla(\partial_t u) + \frac{\rho_0}{2} \nabla((\nabla u)^2) + c^2 \nabla \rho_2 + \frac{(\gamma - 1)c^2}{2\rho_0} \nabla(\rho_1^2) + \nu \nabla \Delta u \right] &+ O(\epsilon^3). \end{aligned} \quad (2.16)$$

In order to have an approximation up to the terms $O(\epsilon^3)$ we put the terms of order one and two in ϵ equal to 0, what allows us to find the expressions for the density correctors:

$$\rho_1(\mathbf{x}, t) = \frac{\rho_0}{c^2} \partial_t u(\mathbf{x}, t), \quad (2.17)$$

$$\rho_2(\mathbf{x}, t) = -\frac{\rho_0(\gamma - 2)}{2c^4} (\partial_t u)^2 - \frac{\rho_0}{2c^2} (\nabla u)^2 - \frac{\nu}{c^2} \Delta u. \quad (2.18)$$

Indeed, we start by making $\epsilon \nabla(-\rho_0 \partial_t u + c^2 \rho_1) = 0$ and find the first order perturbation of the density ρ_1 given by Eq. (2.17). Consequently, if ρ_1 satisfies (2.17), then Eq. (2.16)

becomes

$$\begin{aligned} \rho_\varepsilon[\partial_t \mathbf{v}_\varepsilon + (\mathbf{v}_\varepsilon \cdot \nabla) \mathbf{v}_\varepsilon] + \nabla p(\rho_\varepsilon) - \varepsilon \nu \Delta \mathbf{v}_\varepsilon &= \varepsilon \nabla(-\rho_0 \partial_t u + c^2 \rho_1) \\ \varepsilon^2 \nabla \left[-\frac{\rho_0}{2c^2} (\partial_t u)^2 + \frac{\rho_0}{2} (\nabla u)^2 + c^2 \rho_2 + \frac{(\gamma-1)\rho_0}{2c^2} (\partial_t u)^2 + \nu \Delta u \right] &+ O(\varepsilon^3). \end{aligned} \quad (2.19)$$

Thus, taking the corrector ρ_2 by formula (2.18), we ensure that

$$\rho_\varepsilon[\partial_t \mathbf{v}_\varepsilon + (\mathbf{v}_\varepsilon \cdot \nabla) \mathbf{v}_\varepsilon] + \nabla p(\rho_\varepsilon) - \varepsilon \nu \Delta \mathbf{v}_\varepsilon = O(\varepsilon^3). \quad (2.20)$$

Now we put these expressions of ρ_1 from (2.17) and ρ_2 from (2.18) with *ansatz* (2.14)–(2.15) in Eq. (2.10) of the mass conservation to obtain

$$\begin{aligned} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon \mathbf{v}_\varepsilon) &= \varepsilon \frac{\rho_0}{c^2} \left[\partial_t^2 u - c^2 \Delta u - \right. \\ \left. \varepsilon \partial_t \left((\nabla u)^2 + \frac{\gamma-2}{2c^2} (\partial_t u)^2 + \frac{\nu}{\rho_0} \Delta u \right) - \varepsilon u_t \Delta u \right] &+ O(\varepsilon^3). \end{aligned} \quad (2.21)$$

Then we notice that the right hand term of the order ε in Eq. (2.21) is actually the linear wave equation up to smaller on ε terms:

$$\partial_t^2 u - c^2 \Delta u = O(\varepsilon).$$

Hence, we express

$$\varepsilon u_t \Delta u = \varepsilon \frac{1}{c^2} u_t u_{tt} + O(\varepsilon^2) = \varepsilon \frac{1}{2c^2} \partial_t ((u_t)^2) + O(\varepsilon^2),$$

and putting it in Eq. (2.21), we finally have

$$\begin{aligned} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon \mathbf{v}_\varepsilon) &= \varepsilon \frac{\rho_0}{c^2} \left[\partial_t^2 u - c^2 \Delta u - \right. \\ \left. \varepsilon \partial_t \left((\nabla u)^2 + \frac{\gamma-1}{2c^2} (\partial_t u)^2 + \frac{\nu}{\rho_0} \Delta u \right) \right] &+ O(\varepsilon^3). \end{aligned} \quad (2.22)$$

The right hand side of Eq. (2.22) gives us the Kuznetsov equation

$$\partial_t^2 u - c^2 \Delta u = \varepsilon \partial_t \left((\nabla u)^2 + \frac{\gamma-1}{2c^2} (\partial_t u)^2 + \frac{\nu}{\rho_0} \Delta u \right), \quad (2.23)$$

which is the first order approximation of the isentropic Navier-Stokes system up to the terms $O(\varepsilon^3)$. Moreover, if u is a solution of the Kuznetsov equation, then with the relations for the density perturbations (2.17) and (2.18) and with *ansatz* (2.14)–(2.15) we have

$$\partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon \mathbf{v}_\varepsilon) = O(\varepsilon^3), \quad (2.24)$$

$$\rho_\varepsilon[\partial_t \mathbf{v}_\varepsilon + (\mathbf{v}_\varepsilon \cdot \nabla) \mathbf{v}_\varepsilon] + \nabla p(\rho_\varepsilon) - \varepsilon \nu \Delta \mathbf{v}_\varepsilon = O(\varepsilon^3). \quad (2.25)$$

Hence, it is clear that the standard physical perturbative approach without the corrector ρ_2 (it is sufficient to take $\rho_2 = 0$ in our calculus) can't ensure (2.24)–(2.25).

Let us also notice, as it was originally mentioned by Kuznetsov, that the Kuznetsov equation (2.23) contains terms of different orders, and hence, it is a wave equation with small size non-linear perturbations $\partial_t (\nabla u)^2$, $\partial_t (\partial_t u)^2$ and the viscosity term $\partial_t \Delta u$.

Approximation of the solutions of the isentropic Navier-Stokes system by the solutions of the Kuznetsov equation

Let us calculate the remainder terms in (2.24)–(2.25), which are denoted respectively by $\varepsilon^3 R_1^{NS-Kuz}$ and $\varepsilon^3 \mathbf{R}_2^{NS-Kuz}$:

$$\begin{aligned} \varepsilon^3 R_1^{NS-Kuz} &= \varepsilon^3 \left[\frac{1}{c^2} \partial_t u \left(\frac{\rho_0(\gamma-2)}{2c^4} \partial_t [(\partial_t u)^2] + \frac{\rho_0}{c^2} \partial_t [(\nabla u)^2] + \frac{\nu}{c^2} \partial_t \Delta u \right) \right. \\ &\quad \left. - \frac{\rho_0}{c^2} \partial_t u \Delta u - \nabla \rho_2 \cdot \nabla u - \rho_2 \Delta u \right] + \varepsilon^4 \frac{1}{c^2} \partial_t u (\nabla \rho_2 \cdot \nabla u + \rho_2 \Delta u), \end{aligned} \quad (2.26)$$

$$\varepsilon^3 \mathbf{R}_2^{NS-Kuz} = \varepsilon^3 \left[\frac{\rho_1}{2} \nabla [(\nabla u)^2] - \rho_2 \nabla \partial_t u \right] + \varepsilon^4 \frac{\rho_2}{2} \nabla [(\nabla u)^2]. \quad (2.27)$$

If u is a sufficiently regular solution of the Cauchy problem for the Kuznetsov equation in \mathbb{R}^n

$$\begin{cases} \partial_t^2 u - c^2 \Delta u = \varepsilon \partial_t \left((\nabla u)^2 + \frac{\gamma-1}{2c^2} (\partial_t u)^2 + \frac{\nu}{\rho_0} \Delta u \right), \\ u(0) = u_0, \quad u_t(0) = u_1, \end{cases} \quad (2.28)$$

then, taking ρ_1 and ρ_2 according to formulas (2.17)–(2.18), we define $\bar{\rho}_\varepsilon$ and $\bar{\mathbf{v}}_\varepsilon$ by formulas (2.14)–(2.15) and obtain a solution of the following approximate system

$$\partial_t \bar{\rho}_\varepsilon + \operatorname{div}(\bar{\rho}_\varepsilon \bar{\mathbf{v}}_\varepsilon) = \varepsilon^3 R_1^{NS-Kuz}, \quad (2.29)$$

$$\bar{\rho}_\varepsilon [\partial_t \bar{\mathbf{v}}_\varepsilon + (\bar{\mathbf{v}}_\varepsilon \cdot \nabla) \bar{\mathbf{v}}_\varepsilon] + \nabla p(\bar{\rho}_\varepsilon) - \varepsilon \nu \Delta \bar{\mathbf{v}}_\varepsilon = \varepsilon^3 \mathbf{R}_2^{NS-Kuz} \quad (2.30)$$

with $p(\bar{\rho}_\varepsilon)$ from the state law (2.12). With notations $\mathbf{U}_\varepsilon = (\rho_\varepsilon, \rho_\varepsilon \mathbf{v}_\varepsilon)^t$ and $\bar{\mathbf{U}}_\varepsilon = (\bar{\rho}_\varepsilon, \bar{\rho}_\varepsilon \bar{\mathbf{v}}_\varepsilon)^t$, the exact (2.10)–(2.11) and the approximated (2.29)–(2.30) Navier-Stokes systems can be respectively rewritten in the following forms[23, 81]:

$$\partial_t \mathbf{U}_\varepsilon + \sum_{i=1}^n \partial_{x_i} \mathbf{G}_i(\mathbf{U}_\varepsilon) - \varepsilon \nu \begin{bmatrix} 0 \\ \Delta \mathbf{v}_\varepsilon \end{bmatrix} = 0, \quad (2.31)$$

$$\partial_t \bar{\mathbf{U}}_\varepsilon + \sum_{i=1}^n \partial_{x_i} \mathbf{G}_i(\bar{\mathbf{U}}_\varepsilon) - \varepsilon \nu \begin{bmatrix} 0 \\ \Delta \bar{\mathbf{v}}_\varepsilon \end{bmatrix} = \varepsilon^3 \mathbf{R}^{NS-Kuz} \quad (2.32)$$

with $\mathbf{R}^{NS-Kuz} = \begin{bmatrix} R_1^{NS-Kuz} \\ \mathbf{R}_2^{NS-Kuz} \end{bmatrix}$ from (2.26)–(2.27) and

$$\mathbf{G}_i(\mathbf{U}_\varepsilon) = \begin{bmatrix} \rho_\varepsilon v_i \\ \rho_\varepsilon v_i \mathbf{v}_\varepsilon + p(\rho_\varepsilon) \mathbf{e}_i \end{bmatrix}, \quad \partial_{x_i} \mathbf{G}_i(\mathbf{U}_\varepsilon) = D\mathbf{G}_i(\mathbf{U}_\varepsilon) \partial_{x_i} \mathbf{U}_\varepsilon. \quad (2.33)$$

The well-posedness results for the Cauchy problems (2.10)–(2.12)[67] and (2.28)[26] or see Theorem 1.2.2 from Chapter 1, allow us to establish the global existence and the unicity of the classical solutions \mathbf{U}_ε and $\bar{\mathbf{U}}_\varepsilon$, considered in the Kuznetsov approximation framework:

Theorem 2.4.1. *There exists a constant $k > 0$ such that if the initial data $u_0 \in H^5(\mathbb{R}^3)$ and $u_1 \in H^4(\mathbb{R}^3)$ for the Cauchy problem for the Kuznetsov equation (2.28) are small enough*

$$\|u_0\|_{H^5(\mathbb{R}^3)} + \|u_1\|_{H^4(\mathbb{R}^3)} < k,$$

then there exist global in time solutions $\bar{\mathbf{U}}_\varepsilon = (\bar{\rho}_\varepsilon, \bar{\rho}_\varepsilon \bar{\mathbf{v}}_\varepsilon)^t$ of the approximate Navier-Stokes system (2.32) and $\mathbf{U}_\varepsilon = (\rho_\varepsilon, \rho_\varepsilon \mathbf{v}_\varepsilon)^t$ of the exact Navier-Stokes system (2.31) respectively, with the same regularity corresponding to

$$\bar{\rho}_\varepsilon - \rho_0, \rho_\varepsilon - \rho_0 \in C([0, +\infty[; H^3(\mathbb{R}^3))) \cap C^1([0, +\infty[; H^2(\mathbb{R}^3))) \quad (2.34)$$

and

$$\bar{\mathbf{v}}_\varepsilon, \mathbf{v}_\varepsilon \in C([0, +\infty[; H^3(\mathbb{R}^3)) \cap C^1([0, +\infty[; H^1(\mathbb{R}^3)), \quad (2.35)$$

both considered with the state law (2.12) and with the same initial data

$$(\bar{\rho}_\varepsilon - \rho_\varepsilon)|_{t=0} = 0, \quad (\bar{\mathbf{v}}_\varepsilon - \mathbf{v}_\varepsilon)|_{t=0} = 0, \quad (2.36)$$

where $\bar{\rho}_\varepsilon|_{t=0}$ and $\bar{\mathbf{v}}_\varepsilon|_{t=0}$ are constructed as the functions of the initial data for the Kuznetsov equation u_0 and u_1 according to formulas (2.14)–(2.15) and (2.17)–(2.18):

$$\bar{\rho}_\varepsilon|_{t=0} = \rho_0 + \varepsilon \frac{\rho_0}{c^2} u_1 - \varepsilon^2 \left[\frac{\rho_0(\gamma - 2)}{2c^4} u_1^2 + \frac{\rho_0}{2c^2} (\nabla u_0)^2 + \frac{\nu}{c^2} \Delta u_0 \right], \quad (2.37)$$

$$\bar{\mathbf{v}}_\varepsilon|_{t=0} = -\varepsilon \nabla u_0. \quad (2.38)$$

Proof. On one hand, Theorem 1.2 in Ref. [26] or Theorem 1.2.2 in Chapter 1 applied for $n = 3$ with $m = 4$ ensures that for $u_0 \in H^5(\mathbb{R}^3)$ and $u_1 \in H^4(\mathbb{R}^3)$ there exists a constant $k_2 > 0$ such that if

$$\|u_0\|_{H^5(\mathbb{R}^3)} + \|u_1\|_{H^4(\mathbb{R}^3)} < k_2, \quad (2.39)$$

then the Cauchy problem for the Kuznetsov equation (2.28) has a unique global in time solution

$$u \in C([0, +\infty[, H^5(\mathbb{R}^3)) \cap C^1([0, +\infty[, H^4(\mathbb{R}^3)) \cap C^2([0, +\infty[, H^2(\mathbb{R}^3)). \quad (2.40)$$

On the other hand, the Cauchy problem for the Navier-Stokes system is also globally well-posed in \mathbb{R}^3 for sufficiently small initial data (see Ref. [67] Theorem 7.1, p. 100): there exists a constant $k_1 > 0$ such that if the initial data

$$\rho_\varepsilon(0) - \rho_0 \in H^3(\mathbb{R}^3), \quad \mathbf{v}_\varepsilon(0) \in H^3(\mathbb{R}^3) \quad (2.41)$$

satisfy

$$\|\rho_\varepsilon(0) - \rho_0\|_{H^3(\mathbb{R}^3)} + \|\mathbf{v}_\varepsilon(0)\|_{H^3(\mathbb{R}^3)} < k_1,$$

then the Cauchy problem (2.10)–(2.12) with the initial data (2.41) has a unique solution $(\rho_\varepsilon, \mathbf{v}_\varepsilon)$ globally in time satisfying (2.34) and (2.35).

Thus, for the initial solutions of the Kuznetsov equation we need to impose $u_0 \in H^5(\mathbb{R}^3)$ to have $\Delta u_0 \in H^3(\mathbb{R}^3)$ to be able to ensure that $\rho_\varepsilon - \rho_0|_{t=0} \in H^3(\mathbb{R}^3)$. The regularity $u_1 \in H^4(\mathbb{R}^3)$ comes from the well-posedness of the Kuznetsov problem and obviously ensures $\mathbf{v}_\varepsilon|_{t=0} \in H^3(\mathbb{R}^3)$, what is necessary [67] to have a global solution of the exact Navier-Stokes system (2.31).

As $\bar{\rho}_\varepsilon$ and $\bar{\mathbf{v}}_\varepsilon$ are defined by *ansatz* (2.14)–(2.15) with ρ_1 and ρ_2 given in (2.17) and (2.18) respectively, the regularity of u ensures for $\bar{\rho}_\varepsilon - \rho_0$ and $\bar{\mathbf{v}}_\varepsilon$ at least the same regularity as given in (2.34) and (2.35). To find it we use the following Sobolev embedding for the multiplication (see for example Ref. [15] or [57]):

$$H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \hookrightarrow H^s(\mathbb{R}^n) \quad \text{for } s > \frac{n}{2}, \quad (2.42)$$

$$(u, v) \mapsto uv.$$

Moreover, considering formulas (2.26)–(2.27) with u as defined in (2.40), all terms in R_1^{NS-Kuz} and \mathbf{R}_2^{NS-Kuz} are in $H^2(\mathbb{R}^3)$. Therefore, as $2 > \frac{3}{2}$, we use embedding (2.42) to find that

$$R_1^{NS-Kuz} \in C([0, +\infty[, H^2(\mathbb{R}^3)) \quad \text{and} \quad \mathbf{R}_2^{NS-Kuz} \in C([0, +\infty[, H^2(\mathbb{R}^3)).$$

Hence, the $L^2(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3)$ norms of the remainder terms $R_1^{NS-Kuz}(t)$ and $\mathbf{R}_2^{NS-Kuz}(t)$ are bounded for $t \in [0, +\infty[$.

Finally, it is important to notice that, as $\mathbf{U}_\varepsilon(0) = \overline{\mathbf{U}}_\varepsilon(0)$,

$$\begin{aligned} \|\rho_\varepsilon(0) - \rho_0\|_{H^3(\mathbb{R}^3)} + \|\mathbf{v}_\varepsilon(0)\|_{H^3(\mathbb{R}^3)} &= \|\overline{\rho}_\varepsilon(0) - \rho_0\|_{H^3(\mathbb{R}^3)} + \|\overline{\mathbf{v}}_\varepsilon(0)\|_{H^3(\mathbb{R}^3)} \\ &\leq C(\|u_0\|_{H^5(\mathbb{R}^3)} + \|u_1\|_{H^4(\mathbb{R}^3)}). \end{aligned}$$

Thus, there exists $k > 0$ (necessarily $k \leq k_2$) such that $\|u_0\|_{H^5} + \|u_1\|_{H^4} < k$ implies the global existences of \mathbf{U}_ε and $\overline{\mathbf{U}}_\varepsilon$. \square

The stability estimate which we obtain between the exact solution of the Navier-Stokes system \mathbf{U}_ε and the solution of the Kuznetsov equation presented by $\overline{\mathbf{U}}_\varepsilon$ does not require for \mathbf{U}_ε to have the regularity of a classical solution and allows to approximate less regular solutions of the Navier-Stokes system with initial data in a small L^2 neighbourhood of $\overline{\mathbf{U}}_\varepsilon(0)$. To define the minimal regularity property of \mathbf{U}_ε for which stability estimate (2.4) holds, we introduce admissible weak solutions of a bounded energy using the entropy of the Euler system (system (2.31) with $\nu = 0$)

$$\eta(\mathbf{U}_\varepsilon) = \rho_\varepsilon h(\rho_\varepsilon) + \rho_\varepsilon \frac{\mathbf{v}_\varepsilon^2}{2} = H(\rho_\varepsilon) + \frac{1}{\rho_\varepsilon} \frac{\mathbf{m}^2}{2}, \quad (2.43)$$

which is convex [23] with $h'(\rho_\varepsilon) = \frac{p(\rho_\varepsilon)}{\rho_\varepsilon^2}$, $\mathbf{v}_\varepsilon = \frac{\mathbf{m}}{\rho_\varepsilon}$. Thus, the first and second derivatives of η are [81]

$$\eta'(\mathbf{U}_\varepsilon) = \begin{bmatrix} H'(\rho_\varepsilon) - \frac{1}{\rho_\varepsilon^2} \frac{\mathbf{m}^2}{2} \\ \frac{\mathbf{m}}{\rho_\varepsilon} \end{bmatrix}^t = \begin{bmatrix} H'(\rho_\varepsilon) - \frac{\mathbf{v}_\varepsilon^2}{2} \\ \mathbf{v}_\varepsilon \end{bmatrix}^t, \quad (2.44)$$

$$\eta''(\mathbf{U}_\varepsilon) = \begin{bmatrix} H''(\rho_\varepsilon) + \frac{\mathbf{m}^2}{\rho_\varepsilon^3} & -\frac{\mathbf{m}}{\rho_\varepsilon^2} \\ -\frac{\mathbf{m}}{\rho_\varepsilon^2} & \frac{1}{\rho_\varepsilon} \end{bmatrix} = \begin{bmatrix} H''(\rho_\varepsilon) + \frac{\mathbf{v}_\varepsilon^2}{\rho_\varepsilon} & -\frac{\mathbf{v}_\varepsilon}{\rho_\varepsilon} \\ -\frac{\mathbf{v}_\varepsilon}{\rho_\varepsilon} & \frac{1}{\rho_\varepsilon} \end{bmatrix}, \quad (2.45)$$

knowing that $\eta''(\mathbf{U}_\varepsilon)$ is strictly positive defined.

Definition 2.4.1. *The function $\mathbf{U}_\varepsilon = (\rho_\varepsilon, \rho_\varepsilon \mathbf{v}_\varepsilon)$ is called an admissible weak solution of a bounded energy of the Cauchy problem for the Navier-Stokes system (2.10)–(2.12) if it satisfies the following properties:*

1. *The pair $(\rho_\varepsilon, \mathbf{v}_\varepsilon)$ is a weak solution of the Cauchy problem for the Navier-Stokes system (2.10)–(2.12) (in the distributional sense).*
2. *The function \mathbf{U}_ε satisfies in the sense of distributions (see Ref. [23, p.52])*

$$\partial_t \eta(\mathbf{U}_\varepsilon) + \nabla \cdot \mathbf{q}(\mathbf{U}_\varepsilon) - \varepsilon \nu \mathbf{v}_\varepsilon \Delta \mathbf{v}_\varepsilon \leq 0, \quad \text{where } \mathbf{q}(\mathbf{U}_\varepsilon) = \mathbf{v}_\varepsilon (\eta(\mathbf{U}_\varepsilon) + p(\rho_\varepsilon)), \quad (2.46)$$

or equivalently, for any positive test function ψ in $\mathcal{D}(\mathbb{R}^n \times [0, \infty[)$ the function \mathbf{U}_ε satisfies

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^n} \left(\partial_t \psi \eta(\mathbf{U}_\varepsilon) + \nabla \psi \cdot \mathbf{q}(\mathbf{U}_\varepsilon) + \varepsilon \nu |\nabla \cdot \mathbf{v}_\varepsilon|^2 \psi + \varepsilon \nu \mathbf{v}_\varepsilon \cdot [\nabla \cdot \mathbf{v}_\varepsilon \nabla \psi] \right) dx dt \\ &+ \int_{\mathbb{R}^n} \psi(x, 0) \eta(\mathbf{U}_\varepsilon(0)) dx \geq 0. \end{aligned}$$

3. The function \mathbf{U}_ϵ satisfies the equality (with the notation $\mathbf{v}_\epsilon = (v_1, \dots, v_n)$)

$$\begin{aligned} & - \int_{\mathbb{R}^n} \frac{\mathbf{U}_\epsilon^2(t)}{2} dx + \int_0^t \int_{\mathbb{R}^n} \left(\sum_{i=1}^n \mathbf{G}_i(\mathbf{U}_\epsilon) \partial_{x_i} \mathbf{U}_\epsilon - \epsilon \nu \nabla(\rho_\epsilon v_i) \cdot \nabla v_i \right) dx ds \\ & + \int_{\mathbb{R}^n} \frac{\mathbf{U}_\epsilon^2(0)}{2} dx = 0. \end{aligned}$$

Let us notice that any classical solution of (2.31), for instance the solution defined in Theorem 2.4.1, satisfies the entropy condition (2.46) by the equality and obviously it is regular enough to perform the integration by parts resulting in the relation of point 3. For existence results of global weak solutions of the Cauchy problem for the Navier-Stokes system (2.31) with sufficiently small initial data around the constant state $(\rho_0, 0)$ (actually, $\rho_0 - \rho(0)$ is small in L^∞ , $\mathbf{v}(0)$ is small in L^2 and bounded in L^{2^n}) and with the pressure $p(\rho) = K\rho^\gamma$ with $\gamma \geq 1$, we refer to results of D. Hoff [38, 39]. For fixing the idea of the regularity of a global weak solution we summarize the results of Hoff in the following theorem:

Theorem 2.4.2. [38] *Let for $n = 3$ $\beta = 0$ and for $n = 2$ β be arbitrary small, N be a given arbitrary large constant. There exists a constant $C_0 > 0$ such that if the initial data of (2.31) with $p(\rho) = K\rho^\gamma$ ($\gamma \geq 1$) satisfy the following smallness condition*

$$\begin{aligned} & \|\rho_0 - \rho(0)\|_{L^\infty(\mathbb{R}^n)}^2 + \int_{\mathbb{R}^n} [(\rho_0 - \rho(0))^2 + |\mathbf{v}(0)|^2] (1 + |x|^2)^\beta dx \leq C_0, \\ & \|\mathbf{v}(0)\|_{L^{2^n}(\mathbb{R}^n)} \leq N, \end{aligned}$$

then there exists a global weak solution (ρ, \mathbf{v}) (in the distributional sense) such that

1. $\rho - \rho_0 \in L^\infty(\mathbb{R}^n \times [0, \infty[)$,
2. $\mathbf{v} \in H^1(\mathbb{R}^n)$ for all $t > 0$,
3. for all $t \geq \tau > 0$ $\mathbf{v}(\cdot, t) \in L^\infty(\mathbb{R}^n)$,
4. for all $\tau > 0$ $\mathbf{v} \in C^{\alpha, \frac{\alpha}{2\alpha+2}}(\mathbb{R}^n \times [\tau, \infty[)$ for all $\alpha \in]0, 1[$ when $n = 2$ and $\mathbf{v} \in C^{\frac{1}{2}, \frac{1}{8}}(\mathbb{R}^n \times [\tau, \infty[)$ when $n = 3$,
5. $\epsilon \nu \operatorname{div} \mathbf{v} + p(\rho) - p(\rho_0) \in H^1(\mathbb{R}^n) \cap C^\alpha(\mathbb{R}^n)$ for almost all $t > 0$ with $\alpha = \frac{1}{2}$ for $n = 2$ and $\alpha = \frac{1}{10}$ when $n = 3$.

In addition, $(\rho, \mathbf{v}) \rightarrow (\rho_0, 0)$ as $t \rightarrow +\infty$ in the sense that for all $q \in]2, +\infty[$

$$\lim_{T \rightarrow \infty} \left(\|\rho - \rho_0\|_{L^\infty(\mathbb{R}^n \times [T, \infty[)} + \|\mathbf{v}(\cdot, T)\|_{L^q(\mathbb{R}^n)} \right) = 0.$$

Therefore, from Theorem 2.4.2 it follows that a weak solution of the isentropic compressible Navier-Stokes system (2.10)–(2.12) is also an admissible weak solution of a bounded energy in the sense of Definition 2.4.1. But in the following we only consider the question of the validity of the stability estimate (2.4) for initial data closed to $\overline{\mathbf{U}}_\epsilon(0)$ in L^2 norm (thus for initial data not necessarily satisfying Theorem 2.4.2) and we don't consider the existence question of an admissible weak solution of a bounded energy of the Cauchy problem for the Navier-Stokes system. Thanks to Theorem 2.4.1 for classical solutions of two models and to Definition 2.4.1 containing the minimal conditions on \mathbf{U}_ϵ necessary for saying that it is in a small L^2 -neighbourhood of the regular solution of the Kuznetsov equation, we validate the approximation of \mathbf{U}_ϵ by $\overline{\mathbf{U}}_\epsilon$ following the ideas of Ref. [81].

Theorem 2.4.3. *Let $\nu > 0$ and $\epsilon > 0$ be fixed and all assumptions of Theorem 2.4.1 hold. Then there exist constants $C > 0$ and $K > 0$, independent on ϵ and the time t , such that*

1. for all $t \leq \frac{C}{\epsilon}$

$$\|(\mathbf{U}_\epsilon - \overline{\mathbf{U}}_\epsilon)(t)\|_{L^2(\mathbb{R}^3)}^2 \leq K\epsilon^3 t e^{Ket} \leq 4\epsilon^2;$$

2. for all $b \in]0, 1[$ during all time $t \leq \frac{C}{\epsilon} \ln(\frac{1}{\epsilon})$ it holds

$$\|(\mathbf{U}_\epsilon - \overline{\mathbf{U}}_\epsilon)(t)\|_{L^2(\mathbb{R}^3)} \leq 2\epsilon^b.$$

Moreover, if the initial conditions for the Kuznetsov equation are such that

$$u_0 \in H^{s+2}(\mathbb{R}^n), \quad u_1 \in H^{s+1}(\mathbb{R}^n) \text{ for } s > \frac{n}{2}, \quad n \geq 2$$

and sufficiently small (in the sense of Ref. [26] Theorem 1.2 or Theorem 1.2.2 in Chapter 1), then there exists the unique global in time solution of the Cauchy problem for the Kuznetsov equation

$$\overline{p}_\epsilon - \rho_0 \in C([0, +\infty[; H^s(\mathbb{R}^n)) \cap C^1([0, +\infty[; H^{s-1}(\mathbb{R}^n)), \quad (2.47)$$

$$\overline{\mathbf{v}}_\epsilon \in C([0, +\infty[; H^{s+1}(\mathbb{R}^n)) \cap C^1([0, +\infty[; H^s(\mathbb{R}^n)) \quad (2.48)$$

and the remainder terms $(\mathbf{R}_1^{NS-Kuz}, \mathbf{R}_2^{NS-Kuz})$, defined in Eqs. (2.26)–(2.27), belong to $C([0, +\infty[, H^{s-1}(\mathbb{R}^n))$.

If in addition there exists an admissible weak solution of a bounded energy of the Cauchy problem for the Navier-Stokes system (2.31) (for instance if $\mathbf{U}_\epsilon(0)$ satisfies conditions of Theorem 2.4.2 there is a global such weak solution) on a time interval $[0, T_{NS}[$ for the initial data

$$\|\mathbf{U}_\epsilon(0) - \overline{\mathbf{U}}_\epsilon(0)\|_{L^2(\mathbb{R}^n)} \leq \delta \leq \epsilon,$$

then it holds for all $t < \min\{\frac{C}{\epsilon}, T_{NS}\}$ the stability estimate (2.4):

$$\|(\mathbf{U}_\epsilon - \overline{\mathbf{U}}_\epsilon)(t)\|_{L^2(\mathbb{R}^n)}^2 \leq K(\epsilon^3 t + \delta^2) e^{Ket} \leq 9\epsilon^2.$$

Proof. In terms of entropy, system (2.32), having by the assumption the unique classical solution $\overline{\mathbf{U}}_\epsilon$, can be rewritten as follows

$$\partial_t \eta(\overline{\mathbf{U}}_\epsilon) + \nabla \cdot \mathbf{q}(\overline{\mathbf{U}}_\epsilon) - \epsilon \nu \overline{\mathbf{v}}_\epsilon \cdot \Delta \overline{\mathbf{v}}_\epsilon = \epsilon^3 \left(\frac{\eta(\overline{\mathbf{U}}_\epsilon) + p(\overline{p}_\epsilon)}{\overline{p}_\epsilon} R_1^{NS-Kuz} + \overline{\mathbf{v}}_\epsilon \cdot \mathbf{R}_2^{NS-Kuz} \right), \quad (2.49)$$

with $\mathbf{R}^{NS-Kuz} = (R_1^{NS-Kuz}, \mathbf{R}_2^{NS-Kuz})$ defined in Eq. (2.26)–(2.27). To abbreviate the notations, we denote the remainder term of the entropy equation in system (2.49) by

$$\overline{R}^{NS-Kuz} = \left(\frac{\eta(\overline{\mathbf{U}}_\epsilon) + p(\overline{p}_\epsilon)}{\overline{p}_\epsilon} R_1^{NS-Kuz} + \overline{\mathbf{v}}_\epsilon \cdot \mathbf{R}_2^{NS-Kuz} \right).$$

In the same time, it is assumed that for \mathbf{U}_ϵ it holds (2.46) in the sense of distributions.

Let us estimate in the sense of distributions

$$\frac{\partial}{\partial t} \left(\eta(\mathbf{U}_\epsilon) - \eta(\overline{\mathbf{U}}_\epsilon) - \eta'(\overline{\mathbf{U}}_\epsilon)(\mathbf{U}_\epsilon - \overline{\mathbf{U}}_\epsilon) \right). \quad (2.50)$$

First we find from systems (2.46) and (2.49) that in the sense of distributions

$$\begin{aligned} \frac{\partial}{\partial t}(\eta(\mathbf{U}_\varepsilon) - \eta(\overline{\mathbf{U}}_\varepsilon)) &\leq -\nabla \cdot (\mathbf{q}(\mathbf{U}_\varepsilon) - \mathbf{q}(\overline{\mathbf{U}}_\varepsilon)) + \varepsilon\nu(\mathbf{v}_\varepsilon \cdot \Delta \mathbf{v}_\varepsilon - \overline{\mathbf{v}}_\varepsilon \cdot \Delta \overline{\mathbf{v}}_\varepsilon) - \varepsilon^3 \overline{\mathbf{R}}^{NS-Kuz} \\ &= -\nabla \cdot (\mathbf{q}(\mathbf{U}_\varepsilon) - \mathbf{q}(\overline{\mathbf{U}}_\varepsilon)) + \varepsilon\nu \sum_{i=1}^n \partial_{x_i} (\mathbf{v}_\varepsilon \partial_{x_i} \mathbf{v}_\varepsilon - \overline{\mathbf{v}}_\varepsilon \partial_{x_i} \overline{\mathbf{v}}_\varepsilon) \\ &\quad - \varepsilon\nu \sum_{i=1}^n (\partial_{x_i} \mathbf{v}_\varepsilon \partial_{x_i} \mathbf{v}_\varepsilon - \partial_{x_i} \overline{\mathbf{v}}_\varepsilon \partial_{x_i} \overline{\mathbf{v}}_\varepsilon) - \varepsilon^3 \overline{\mathbf{R}}^{NS-Kuz}. \end{aligned}$$

Then we notice that

$$-\frac{\partial}{\partial t}(\eta'(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon)) = -\partial_t \overline{\mathbf{U}}_\varepsilon^t \eta''(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon) - \eta'(\overline{\mathbf{U}}_\varepsilon)(\partial_t \mathbf{U}_\varepsilon - \partial_t \overline{\mathbf{U}}_\varepsilon),$$

where in the sense of distributions

$$\begin{aligned} -\partial_t \overline{\mathbf{U}}_\varepsilon^t \eta''(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon) &= -\left[-\sum_{i=1}^n D\mathbf{G}_i(\overline{\mathbf{U}}_\varepsilon) \partial_{x_i} \overline{\mathbf{U}}_\varepsilon \right]^t \eta''(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon) \\ &\quad - \left(\begin{bmatrix} 0 \\ \varepsilon\nu \Delta \overline{\mathbf{v}}_\varepsilon \end{bmatrix} + \varepsilon^3 \mathbf{R}^{NS-Kuz} \right)^t \eta''(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon), \end{aligned}$$

and

$$\begin{aligned} -\eta'(\overline{\mathbf{U}}_\varepsilon)(\partial_t \mathbf{U}_\varepsilon - \partial_t \overline{\mathbf{U}}_\varepsilon) &= -\eta'(\overline{\mathbf{U}}_\varepsilon) \left(-\sum_{i=1}^n \partial_{x_i} (\mathbf{G}_i(\mathbf{U}_\varepsilon) - \mathbf{G}_i(\overline{\mathbf{U}}_\varepsilon)) \right) \\ &\quad - \eta'(\overline{\mathbf{U}}_\varepsilon) \varepsilon\nu \begin{bmatrix} 0 \\ \Delta \mathbf{v}_\varepsilon - \Delta \overline{\mathbf{v}}_\varepsilon \end{bmatrix} + \varepsilon^3 \eta'(\overline{\mathbf{U}}_\varepsilon) \mathbf{R}^{NS-Kuz} \\ &= \sum_{i=1}^n \partial_{x_i} (\eta'(\overline{\mathbf{U}}_\varepsilon) (\mathbf{G}_i(\mathbf{U}_\varepsilon) - \mathbf{G}_i(\overline{\mathbf{U}}_\varepsilon))) \\ &\quad - \sum_{i=1}^n \partial_{x_i} \overline{\mathbf{U}}_\varepsilon^t \eta''(\overline{\mathbf{U}}_\varepsilon) (\mathbf{G}_i(\mathbf{U}_\varepsilon) - \mathbf{G}_i(\overline{\mathbf{U}}_\varepsilon)) \\ &\quad - \eta'(\overline{\mathbf{U}}_\varepsilon) \varepsilon\nu \begin{bmatrix} 0 \\ \Delta \mathbf{v}_\varepsilon - \Delta \overline{\mathbf{v}}_\varepsilon \end{bmatrix} + \varepsilon^3 \eta'(\overline{\mathbf{U}}_\varepsilon) \mathbf{R}^{NS-Kuz}. \end{aligned}$$

Thanks to the convex property of the entropy we have

$$\eta''(\mathbf{U}) D\mathbf{G}_i(\mathbf{U}) = (D\mathbf{G}_i(\mathbf{U}))^t \eta''(\mathbf{U}),$$

and consequently

$$\begin{aligned} (D\mathbf{G}_i(\overline{\mathbf{U}}_\varepsilon) \partial_{x_i} \overline{\mathbf{U}}_\varepsilon)^t \eta''(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon) &= \partial_{x_i} \overline{\mathbf{U}}_\varepsilon^t (D\mathbf{G}_i(\overline{\mathbf{U}}_\varepsilon))^t \eta''(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon) \\ &= \partial_{x_i} \overline{\mathbf{U}}_\varepsilon^t \eta''(\overline{\mathbf{U}}_\varepsilon) D\mathbf{G}_i(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon). \end{aligned}$$

Moreover, we compute in the sense of distributions

$$\begin{aligned} -\left[\begin{bmatrix} 0 \\ \varepsilon\nu \Delta \overline{\mathbf{v}}_\varepsilon \end{bmatrix} \right]^t \eta''(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon) &= -\varepsilon\nu \Delta \overline{\mathbf{v}}_\varepsilon (\mathbf{v}_\varepsilon - \overline{\mathbf{v}}_\varepsilon) - \varepsilon\nu \Delta \overline{\mathbf{v}}_\varepsilon \frac{\rho_\varepsilon - \overline{\rho}_\varepsilon}{\overline{\rho}_\varepsilon} (\mathbf{v}_\varepsilon - \overline{\mathbf{v}}_\varepsilon) \\ &= -\varepsilon\nu \sum_{i=1}^n \partial_{x_i} (\partial_{x_i} \overline{\mathbf{v}}_\varepsilon (\mathbf{v}_\varepsilon - \overline{\mathbf{v}}_\varepsilon)) + \varepsilon\nu \sum_{i=1}^n \partial_{x_i} \overline{\mathbf{v}}_\varepsilon \partial_{x_i} (\mathbf{v}_\varepsilon - \overline{\mathbf{v}}_\varepsilon) - \varepsilon\nu \Delta \overline{\mathbf{v}}_\varepsilon \frac{\rho_\varepsilon - \overline{\rho}_\varepsilon}{\overline{\rho}_\varepsilon} (\mathbf{v}_\varepsilon - \overline{\mathbf{v}}_\varepsilon), \end{aligned}$$

and

$$\begin{aligned} -\eta'(\bar{\mathbf{U}}_\varepsilon)\varepsilon\nu \begin{bmatrix} 0 \\ \Delta \mathbf{v}_\varepsilon - \Delta \bar{\mathbf{v}}_\varepsilon \end{bmatrix} &= -\varepsilon\nu \bar{\nabla}_\varepsilon \cdot (\Delta \mathbf{v}_\varepsilon - \Delta \bar{\mathbf{v}}_\varepsilon) \\ &= -\varepsilon\nu \sum_{i=1}^n \partial_{x_i} (\bar{\nabla}_\varepsilon \partial_{x_i} (\mathbf{v}_\varepsilon - \bar{\mathbf{v}}_\varepsilon)) + \varepsilon\nu \sum_{i=1}^n \partial_{x_i} \bar{\nabla}_\varepsilon \partial_{x_i} (\mathbf{v}_\varepsilon - \bar{\mathbf{v}}_\varepsilon). \end{aligned}$$

We integrate expression (2.50) over \mathbb{R}^n and notice that the integrals of the terms in divergence form in the development of (2.50) are equal to zero. For the regular case in the framework of Theorem 2.4.1 it is due to the regularity given by (2.34) and (2.35) and the following Sobolev embedding[2]

$$H^s(\mathbb{R}^n) \hookrightarrow C_0(\mathbb{R}^n) := \{f \in C(\mathbb{R}^n) \mid |f(x)| \rightarrow 0 \text{ as } \|x\| \rightarrow +\infty\} \text{ for } s > \frac{n}{2}, \quad (2.51)$$

which allows us to use the fact that

$$\forall f \in C_0(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} \nabla \cdot f(x) \, dx = 0.$$

In the case of a weak admissible solution \mathbf{U}_ε it follows from its bounded energy property (see Definition 2.4.1 point 3) which implies that $\rho_\varepsilon - \rho_0$ and \mathbf{v}_ε tend to 0 for $|x| \rightarrow +\infty$ and also implies the existence of the integrals over \mathbb{R}^n . Therefore, we obtain the following estimate in which each term is well-defined in the sense of distributions on $[0, +\infty[\cap[0, T_{NS}]$

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^3} \eta(\mathbf{U}_\varepsilon) - \eta(\bar{\mathbf{U}}_\varepsilon) - \eta'(\bar{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon - \bar{\mathbf{U}}_\varepsilon) dx \leq \\ &- \sum_{i=1}^3 \int_{\mathbb{R}^3} \partial_{x_i} \bar{\mathbf{U}}^t \eta''(\bar{\mathbf{U}}_\varepsilon) (\mathbf{G}_i(\mathbf{U}_\varepsilon) - \mathbf{G}_i(\bar{\mathbf{U}}_\varepsilon) - D\mathbf{G}_i(\bar{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon - \bar{\mathbf{U}}_\varepsilon)) dx \\ &- \varepsilon\nu \int_{\mathbb{R}^3} \sum_{i=1}^3 (\partial_{x_i} \mathbf{v}_\varepsilon \partial_{x_i} \mathbf{v}_\varepsilon - \partial_{x_i} \bar{\mathbf{v}}_\varepsilon \partial_{x_i} \bar{\mathbf{v}}_\varepsilon) dx \\ &+ 2\varepsilon\nu \int_{\mathbb{R}^3} \sum_{i=1}^3 \partial_{x_i} \bar{\nabla}_\varepsilon \partial_{x_i} (\mathbf{v}_\varepsilon - \bar{\mathbf{v}}_\varepsilon) dx + \varepsilon\nu \int_{\mathbb{R}^3} \Delta \bar{\nabla}_\varepsilon \frac{\rho_\varepsilon - \bar{\rho}_\varepsilon}{\bar{\rho}_\varepsilon} (\mathbf{v}_\varepsilon - \bar{\mathbf{v}}_\varepsilon) dx \\ &- \varepsilon^3 \int_{\mathbb{R}^3} (\bar{\mathbf{R}}^{NS-Kuz} - \eta'(\bar{\mathbf{U}}_\varepsilon) \mathbf{R}^{NS-Kuz}) dx - \varepsilon^3 \int_{\mathbb{R}^3} [\mathbf{R}^{NS-Kuz}]^t \eta''(\bar{\mathbf{U}}_\varepsilon) (\mathbf{U}_\varepsilon - \bar{\mathbf{U}}_\varepsilon) dx. \end{aligned} \quad (2.52)$$

Now we study lower bounds of the left hand side and upper bounds of the right hand side of (2.52) in order to obtain a suitable estimate. For the right hand side of Eq. (2.52) we notice that

$$\begin{aligned} &-\varepsilon\nu \int_{\mathbb{R}^3} \sum_{i=1}^3 (\partial_{x_i} \mathbf{v}_\varepsilon \partial_{x_i} \mathbf{v}_\varepsilon - \partial_{x_i} \bar{\mathbf{v}}_\varepsilon \partial_{x_i} \bar{\mathbf{v}}_\varepsilon) dx + 2\varepsilon\nu \int_{\mathbb{R}^3} \sum_{i=1}^3 \partial_{x_i} \bar{\nabla}_\varepsilon \partial_{x_i} (\mathbf{v}_\varepsilon - \bar{\mathbf{v}}_\varepsilon) dx \\ &= -\varepsilon\nu \int_{\mathbb{R}^3} \sum_{i=1}^3 (\partial_{x_i} (\mathbf{v}_\varepsilon - \bar{\mathbf{v}}_\varepsilon))^2 dx \leq 0, \end{aligned}$$

hence this term can be passed in the left hand side of Eq.(2.52) and omitted in the estimation. As the entropy is convex it holds

$$\exists \delta_0 > 0 : \quad \eta(\mathbf{U}_\varepsilon) - \eta(\bar{\mathbf{U}}_\varepsilon) - \eta'(\bar{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon - \bar{\mathbf{U}}_\varepsilon) \geq \delta_0 |\mathbf{U}_\varepsilon - \bar{\mathbf{U}}_\varepsilon|^2.$$

Then using also its continuity, we find

$$\begin{aligned} \delta_0 \int_{\mathbb{R}^3} |\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon|^2(t) dx &\leq \int_0^t \frac{d}{ds} \left(\int_{\mathbb{R}^3} \eta(\mathbf{U}_\varepsilon) - \eta(\overline{\mathbf{U}}_\varepsilon) - \eta'(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon) dx \right) ds \\ &\quad + C_0 \int_{\mathbb{R}^3} |\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon|^2(0) dx. \end{aligned}$$

On the right hand side of (2.52), by the Taylor expansion we also have

$$\mathbf{G}_i(\mathbf{U}_\varepsilon) - \mathbf{G}_i(\overline{\mathbf{U}}_\varepsilon) - D\mathbf{G}_i(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon) \leq C|\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon|^2.$$

With the boundness on $[0; +\infty[$ of $R_1(t)$ and $\mathbf{R}_2(t)$ in the L^2 and L^∞ norms, and thanks to the regularity of $\overline{\mathbf{U}}_\varepsilon$ defined in (2.47) and (2.48) (see also (2.34) and (2.35) for the case $\mathbf{U}_\varepsilon(0) = \overline{\mathbf{U}}_\varepsilon(0)$) and the energy boundedness of \mathbf{U}_ε , we estimate the other terms in Eq. (2.52) in the following way

$$\begin{aligned} \varepsilon\nu \int_{\mathbb{R}^3} \Delta \overline{\mathbf{v}}_\varepsilon \frac{\rho_\varepsilon - \overline{\rho}_\varepsilon}{\overline{\rho}_\varepsilon} (\mathbf{v}_\varepsilon - \overline{\mathbf{v}}_\varepsilon) dx &\leq K\varepsilon \|\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon\|_{L^2(\mathbb{R}^3)}^2, \\ -\varepsilon^3 \int_{\mathbb{R}^3} (\overline{\mathbf{R}}^{NS-Kuz} - \eta'(\overline{\mathbf{U}}_\varepsilon)\mathbf{R}^{NS-Kuz}) dx &\leq K\varepsilon^3, \\ -\varepsilon^3 \int_{\mathbb{R}^3} [\mathbf{R}^{NS-Kuz}]^t \eta''(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon) dx \\ &\leq \varepsilon^3 \|\eta''(\overline{\mathbf{U}}_\varepsilon)\|_{L^\infty(\mathbb{R}^3)} \|\mathbf{R}^{NS-Kuz}\|_{L^2(\mathbb{R}^3)} \|\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon\|_{L^2(\mathbb{R}^3)} \\ &\leq K\varepsilon^3 \|\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon\|_{L^2(\mathbb{R}^3)}. \end{aligned}$$

Now, by integrating on $[0, t]$, we obtain from (2.52) the following inequality

$$\begin{aligned} \int_{\mathbb{R}^3} |\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon|^2(t) dx &\leq \int_0^t \left[(C\|\nabla \overline{\mathbf{U}}_\varepsilon\|_{L^\infty} + K\varepsilon) \|\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon\|_{L^2(\mathbb{R}^3)}^2 \right. \\ &\quad \left. + K\varepsilon^3 + K\varepsilon^3 \|\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon\|_{L^2(\mathbb{R}^3)} \right] ds + C_1 \int_{\mathbb{R}^3} |\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon|^2(0) dx. \end{aligned}$$

Here K , C and C_1 are generic constants of order $O(\varepsilon^0)$ which do not depend on time. Using once more the regularity properties (2.34) and (2.35), we have the boundness of $\|\nabla \overline{\mathbf{U}}_\varepsilon\|_{L^\infty}$. But knowing that $\overline{\rho}_\varepsilon$ and $\overline{\mathbf{v}}_\varepsilon$ are defined by *ansatz* (2.14)–(2.15), we deduce that $\|\nabla \overline{\mathbf{U}}_\varepsilon\|_{L^\infty} \leq C\varepsilon$. Therefore,

$$\begin{aligned} \|\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon\|_{L^2}^2 &\leq \int_0^t K \left(\varepsilon \|\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon\|_{L^2(\mathbb{R}^3)}^2 + \varepsilon^3 + \varepsilon^3 \|\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon\|_{L^2(\mathbb{R}^3)} \right) ds \\ &\quad + C_1 \int_{\mathbb{R}^3} |\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon|^2(0) dx. \end{aligned}$$

Then applying the Gronwall Lemma we have directly

$$\|(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon)(t)\|_{L^2(\mathbb{R}^3)}^2 \leq K(\varepsilon^3 t + \delta^2) e^{Ket},$$

since Ket is a non-decreasing in time function and $\varepsilon^3 \sqrt{v} < K\varepsilon v$ for all $v \in \mathbb{R}^+$. In addition, to find the estimate of Point 2 for the regular case $\mathbf{U}_\varepsilon(0) = \overline{\mathbf{U}}_\varepsilon(0)$, we notice that

$$\|\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon\|_{L^2(\mathbb{R}^3)} \leq v,$$

where v is the solution of the following Cauchy problem

$$\begin{cases} (v^2)' = K(\varepsilon^3 + \varepsilon^3 v + \varepsilon v^2), \\ v(0) = 0. \end{cases}$$

The study of this problem gives us

$$\begin{aligned} & \frac{1}{K\varepsilon} \ln \left(1 + v(t) + \frac{1}{\varepsilon^2} v(t)^2 \right) \\ & - \frac{1}{K} \frac{2}{\sqrt{4-\varepsilon^2}} \left[\arctan \left(\frac{2}{\sqrt{4\varepsilon^2-\varepsilon^4}} \left[v(t) + \frac{\varepsilon^2}{2} \right] \right) - \arctan \left(\frac{\varepsilon}{\sqrt{4-\varepsilon^2}} \right) \right] = t. \end{aligned}$$

The boundness of the function $\arctan x$ implies

$$\begin{aligned} 1 + v(t) + \frac{1}{\varepsilon^2} v(t)^2 & \leq e^{\frac{2\varepsilon}{\sqrt{4-\varepsilon^2}}} e^{\arctan \left[\frac{2}{\sqrt{4\varepsilon^2-\varepsilon^4}} \left(v(t) + \frac{\varepsilon^2}{2} \right) \right] - \arctan \left(\frac{\varepsilon}{\sqrt{4-\varepsilon^2}} \right)} e^{K\varepsilon t} \\ & \leq e^{\frac{2\varepsilon}{\sqrt{4-\varepsilon^2}}} e^{\frac{\pi}{2}} e^{K\varepsilon t} \leq c_0^2 e^{K\varepsilon t} \end{aligned}$$

with $c_0^2 = e^{\frac{2}{\sqrt{3}}} e^{\frac{\pi}{2}}$ which for instance is less than 3.5 . Therefore, the estimate

$$\|\mathbf{U}_\varepsilon - \bar{\mathbf{U}}_\varepsilon\|_{L^2(\mathbb{R}^3)} \leq c_0 \varepsilon e^{K\varepsilon t}$$

gives the result as soon as $c_0 \varepsilon e^{\varepsilon K t} \leq 2\varepsilon^b$, with $b \leq 1$, *i.e.* for $t \leq \frac{C}{\varepsilon}$ when $b = 1$, and for $t \leq \frac{C}{\varepsilon} \ln(\frac{1}{\varepsilon})$ in the case $b < 1$.

We finish the proof with the remark on the minimal regularity of the initial data for the Kuznetsov equation such that the approximation is possible, *i.e.* the remainder terms R_1^{NS-Kuz} and \mathbf{R}_2^{NS-Kuz} keep bounded for a finite time interval. Indeed, if $u_0 \in H^{s+2}(\mathbb{R}^n)$ and $u_1 \in H^{s+1}(\mathbb{R}^n)$ with $s > \frac{n}{2}$ then $u \in C([0, +\infty[; H^{s+2}(\mathbb{R}^n))$ and

$$u_t \in C([0, +\infty[; H^{s+1}(\mathbb{R}^n)), \quad u_{tt} \in C([0, +\infty[; H^{s-1}(\mathbb{R}^n)).$$

Since $\bar{\rho}_\varepsilon$ is defined by (2.14) with (2.17) and (2.18) and $\bar{\mathbf{v}}_\varepsilon$ by (2.15) respectively, we exactly find the regularity (2.47) and (2.48). Thus by the regularity of the left-hand side part for the approximated Navier-Stokes system (2.29)–(2.30) we obtain the desired regularity for the right-hand side. \square

2.4.2 Navier-Stokes system and the KZK equation

Derivation of the KZK equation from an isentropic Navier-Stokes system

In the present section we focus on the derivation from the isentropic Navier-Stokes system of the Khokhlov-Zabolotskaya-Kuznetsov equation (KZK) in non-linear media using the following acoustical properties of beam's propagation

1. The beams are concentrated near the x_1 -axis ;
2. The beams propagate along the x_1 -direction;
3. The beams are generated either by an initial condition or by a forcing term on the boundary $x_1 = 0$.

The different type of derivations of the KZK equation are discussed in Ref. [81].

This time we perform it in two steps:

1. We introduce small perturbations around a constant state of the compressible isentropic Navier-Stokes system according to the Kuznetsov *ansatz* (2.14)–(2.15):

$$\begin{aligned} \partial_t \rho_\varepsilon + \nabla \cdot (\rho_\varepsilon \mathbf{v}_\varepsilon) = & \varepsilon [\partial_t \rho_1 - \rho_0 \Delta u] \\ & + \varepsilon^2 [\partial_t \rho_2 - \nabla \rho_1 \nabla u - \rho_1 \Delta u] + O(\varepsilon^3), \end{aligned} \quad (2.53)$$

and we have again (2.16) for the conservation of momentum.

2. We perform the paraxial change of variable[81] (see Fig. 2.1):

$$\tau = t - \frac{x_1}{c}, \quad z = \varepsilon x_1, \quad y = \sqrt{\varepsilon} x'. \quad (2.54)$$

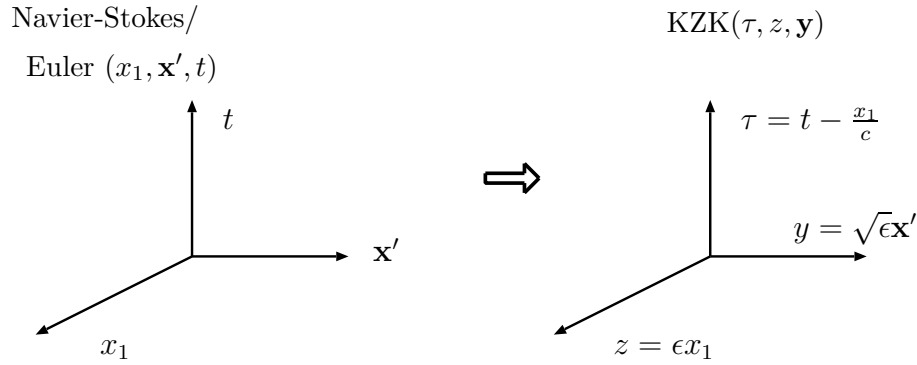


Figure 2.1 – Paraxial change of variables for the profiles $U(t - x_1/c, \varepsilon x_1, \sqrt{\varepsilon} x')$.

Since the gradient ∇ in the coordinates (τ, z, y) becomes depending on ε

$$\tilde{\nabla} = \left(\varepsilon \partial_z - \frac{1}{c} \partial_\tau, \sqrt{\varepsilon} \nabla_y \right)^t,$$

if we denote

$$u(x, t) = \Phi(t - x_1/c, \varepsilon x_1, \sqrt{\varepsilon} x') = \Phi(\tau, z, y), \quad (2.55)$$

we need to take attention to have the paraxial correctors of the order $O(1)$:

$$\rho_1(x, t) = I(\tau, z, y), \quad \rho_2(x, t) = H(\tau, z, y) = J(\tau, z, y) + O(\varepsilon),$$

where actually $H(\tau, z, y)$ is the profile function obtained from ρ_2 (see Appendix A Eq. (A.1)) containing not only the terms of the order $O(1)$ but also terms up to ε^2 . Hence, we denote by J all terms of H of order 0 on ε which are significant in order to have an approximation up to the terms $O(\varepsilon^3)$.

In new variables (τ, z, y) Eq. (2.16) becomes

$$\begin{aligned} \rho_\varepsilon [\partial_t \mathbf{v}_\varepsilon + (\mathbf{v}_\varepsilon \cdot \nabla) \mathbf{v}_\varepsilon] + \nabla p(\rho_\varepsilon) - \varepsilon \nu \Delta \mathbf{v}_\varepsilon = & \varepsilon \tilde{\nabla} [-\rho_0 \partial_\tau \Phi + c^2 I] \\ & + \varepsilon^2 \left[-I \tilde{\nabla} (\partial_\tau \Phi) + \frac{\rho_0}{2} \tilde{\nabla} \left(\frac{1}{c^2} (\partial_\tau \Phi)^2 \right) \right. \\ & \left. + c^2 \tilde{\nabla} J + \frac{\gamma - 1}{2\rho_0} c^2 \tilde{\nabla} (I^2) + \nu \tilde{\nabla} \left(\frac{1}{c^2} \partial_\tau^2 \Phi \right) \right] + O(\varepsilon^3). \end{aligned} \quad (2.56)$$

Consequently, we find the correctors of the density as functions of Φ :

$$I(\tau, z, y) = \frac{\rho_0}{c^2} \partial_\tau \Phi(\tau, z, \mathbf{y}), \quad (2.57)$$

$$J(\tau, z, y) = -\frac{\rho_0(\gamma-1)}{2c^4} (\partial_\tau \Phi)^2 - \frac{\nu}{c^4} \partial_\tau^2 \Phi. \quad (2.58)$$

Indeed, we start by making $\varepsilon \tilde{\nabla}[-\rho_0 \partial_\tau \Phi + c^2 I] = 0$ and find the first order perturbation of the density I given by Eq. (2.57). Moreover, if ρ_1 satisfies (2.57), then Eq. (2.56) becomes

$$\begin{aligned} \rho_\varepsilon [\partial_t \mathbf{v}_\varepsilon + (\mathbf{v}_\varepsilon \cdot \nabla) \mathbf{v}_\varepsilon] + \nabla p(\rho_\varepsilon) - \varepsilon \nu \Delta \mathbf{v}_\varepsilon &= \varepsilon \tilde{\nabla}[-\rho_0 \partial_\tau \Phi + c^2 I] \\ \varepsilon^2 \tilde{\nabla} \left[-\frac{\rho_0}{2c^2} (\partial_\tau \Phi)^2 + \frac{\rho_0}{2c^2} (\partial_\tau \Phi)^2 + c^2 J + \frac{(\gamma-1)\rho_0}{2c^2} (\partial_\tau \Phi)^2 + \frac{\nu}{c^2} \partial_\tau^2 \Phi \right] &+ O(\varepsilon^3). \end{aligned} \quad (2.59)$$

Thus, taking the corrector J in the expansion of ρ_ε

$$\rho_\varepsilon(\mathbf{x}, t) = \rho_0 + \varepsilon I(\tau, z, \mathbf{y}) + \varepsilon^2 J(\tau, z, \mathbf{y}), \quad (2.60)$$

by formula (2.58), we ensure that

$$\rho_\varepsilon [\partial_t \mathbf{v}_\varepsilon + (\mathbf{v}_\varepsilon \cdot \nabla) \mathbf{v}_\varepsilon] + \nabla p(\rho_\varepsilon) - \varepsilon \nu \Delta \mathbf{v}_\varepsilon = O(\varepsilon^3). \quad (2.61)$$

Now we put these expressions of I from (2.57) and J from (2.58) with the paraxial approximation in Eq. (2.53) of the mass conservation to obtain

$$\begin{aligned} \partial_t \rho_\varepsilon + \nabla \cdot (\rho_\varepsilon \mathbf{v}_\varepsilon) &= \varepsilon^2 \left[\frac{\rho_0}{c^2} (2c \partial_{z\tau}^2 \Phi - c^2 \Delta_y \Phi) - \frac{\rho_0}{2c^4} (\gamma+1) \partial_\tau [(\partial_\tau \Phi)^2] - \frac{\nu}{c^4} \partial_\tau^3 \Phi \right] \\ &+ O(\varepsilon^3). \end{aligned} \quad (2.62)$$

All terms of the second order on ε in relation (2.62) give us the equation on Φ , which is the KZK equation. If we use relation (2.57), we obtain the usual form of the KZK equation often written[12, 81] for the first perturbation I of the density ρ_ε :

$$c \partial_{\tau z}^2 I - \frac{(\gamma+1)}{4\rho_0} \partial_\tau^2 I^2 - \frac{\nu}{2c^2 \rho_0} \partial_\tau^3 I - \frac{c^2}{2} \Delta_y I = 0. \quad (2.63)$$

We notice that, as the Kuznetsov equation, this model still contains terms describing the wave propagation $\partial_{\tau z}^2 I$, the non-linearity $\partial_\tau^2 I^2$ and the viscosity effects $\partial_\tau^3 I$ of the medium but also adds a diffraction effects by the transversal Laplacian $\Delta_y I$. This corresponds to the description of the quasi-one-dimensional propagation of a signal in a homogeneous nonlinear isentropic medium. By our derivation (see also (2.80)–(2.81)) we obtain that the KZK equation is the second order approximation of the isentropic Navier-Stokes system up to term of $O(\varepsilon^3)$. In this sense, since the entropy and the pressure in Section 2.3 are approximated up to terms of the order of ε^3 , the *ansatz* (2.60)–(2.68) (for the KZK equations) is optimal, as the equations of the Navier-Stokes system are approximated up to $O(\varepsilon^3)$ -terms.

Let us compare our *ansatz*

$$u(x_1, \mathbf{x}', t) = \Phi(t - x_1/c, \varepsilon x_1, \sqrt{\varepsilon} x'), \quad (2.64)$$

$$\rho_\varepsilon(x_1, \mathbf{x}', t) = \rho_0 + \varepsilon I(t - x_1/c, \varepsilon x_1, \sqrt{\varepsilon} x') + \varepsilon^2 J(t - x_1/c, \varepsilon x_1, \sqrt{\varepsilon} x') \quad (2.65)$$

to the *ansatz* introduced in Ref. [81] by defining a corrector $\epsilon^2 v_2$ for the velocity perturbation along the propagation axis in the initial *ansatz*, proposed by Khokhlov and Zabolotskaya [12]:

$$\rho_\epsilon(x_1, \mathbf{x}', t) = \rho_0 + \epsilon I\left(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon} \mathbf{x}'\right), \quad (2.66)$$

$$\mathbf{v}_\epsilon(x_1, \mathbf{x}', t) = \epsilon(v_1 + \epsilon v_2; \sqrt{\epsilon} \mathbf{w})\left(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon} \mathbf{x}'\right). \quad (2.67)$$

This time, the assumption to work directly with the velocity potential (2.64) immediately implies the following velocity expansion

$$\mathbf{v}_\epsilon(\mathbf{x}, t) = -\epsilon \left(-\frac{1}{c} \partial_\tau \Phi + \epsilon \partial_z \Phi; \sqrt{\epsilon} \nabla_y \Phi \right) (\tau, z, \mathbf{y}), \quad (2.68)$$

where we recognize the velocity *ansatz* of Ref. [81] with

$$v_1 = \frac{1}{c} \partial_\tau \Phi = \frac{c}{\rho_0} I, \quad \mathbf{w} = \nabla_y \Phi = \frac{c^2}{\rho_0} \partial_\tau^{-1} \nabla_y I,$$

but for the corrector v_2 this time

$$v_2 = -\partial_z \Phi = -\frac{c^2}{\rho_0} \partial_\tau^{-1} \partial_z I$$

instead of (see Ref. [81] and formula (2.71) for definition of the operator ∂_τ^{-1})

$$v_2^{Rozanova} = -\frac{c^2}{\rho_0} \partial_\tau^{-1} \partial_z I + \frac{(\gamma - 1)}{2\rho_0^2} c I^2 + \frac{\nu}{c\rho_0^2} \partial_\tau I.$$

If we add the second order correctors v_2 for the velocity to J for the density, we obtain exactly all terms of the corrector $v_2^{Rozanova}$. But the *ansatz* (2.66)–(2.67) is not optimal since the equation of momentum in transverse direction keeps the non-zero terms [81] of the order of $\epsilon^{\frac{5}{2}}$.

Well posedness of the KZK equation

We use Ref. [80] to give results on the well posedness of the Cauchy problem:

$$\begin{cases} c \partial_{\tau z}^2 I - \frac{(\gamma+1)}{4\rho_0} \partial_\tau^2 I^2 - \frac{\nu}{2c^2\rho_0} \partial_\tau^3 I - \frac{c^2}{2} \Delta_y I = 0 \text{ on } \mathbb{T}_\tau \times \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\ I(\tau, 0, y) = I_0(\tau, y) \text{ on } \mathbb{T}_\tau \times \mathbb{R}^{n-1} \end{cases} \quad (2.69)$$

in the class of L -periodic functions with respect to the variable τ and with mean value zero

$$\int_0^L I(\ell, z, y) d\ell = 0. \quad (2.70)$$

The introduction of the operator ∂_τ^{-1} , defined by formula

$$\partial_\tau^{-1} I(\tau, z, y) := \int_0^\tau I(\ell, z, y) d\ell + \int_0^L \frac{\ell}{L} I(\ell, z, y) d\ell, \quad (2.71)$$

allows us to consider instead of Eq. (2.63) the following equivalent equation

$$c \partial_z I - \frac{(\gamma + 1)}{4\rho_0} \partial_\tau I^2 - \frac{\nu}{2c^2\rho_0} \partial_\tau^2 I - \frac{c^2}{2} \partial_\tau^{-1} \Delta_y I = 0 \text{ on } \mathbb{T}_\tau \times \mathbb{R}_+ \times \mathbb{R}^{n-1},$$

for which it holds the following theorem [42, 80]:

Theorem 2.4.4. [80] Consider the Cauchy problem for the KZK equation:

$$\begin{cases} c\partial_z I - \frac{(\gamma+1)}{4\rho_0}\partial_\tau I^2 - \frac{\nu}{2c^2\rho_0}\partial_\tau^2 I - \frac{c^2}{2}\partial_\tau^{-1}\Delta_y I = 0 & \text{on } \mathbb{T}_\tau \times \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\ I(\tau, 0, y) = I_0(\tau, y) & \text{on } \mathbb{T}_\tau \times \mathbb{R}^{n-1}, \end{cases} \quad (2.72)$$

with the operator ∂_τ^{-1} defined by formula (2.71), $\nu \geq 0$, and $\int_0^L I_0(\ell, y) d\ell = 0$, the following results hold true

1. (Local existence) For $s > \left\lfloor \frac{n}{2} \right\rfloor + 1$ there exists a constant $C(s, L)$ such that for any initial data $I_0 \in H^s(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$ on an interval $[0, T[$ with

$$T \geq \frac{1}{C(s, L)\|I_0\|_{H^s(\mathbb{T}_\tau \times \mathbb{R}^{n-1})}}$$

problem (2.72) has a unique solution I such that

$$I \in C([0, T[, H^s(\mathbb{T}_\tau \times \mathbb{R}^{n-1})) \cap C^1([0, T[, H^{s-2}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})),$$

which satisfies the zero mean value condition (2.70).

2. (Shock formation) Let T^* be the largest time on which such a solution is defined, then we have

$$\int_0^{T^*} \sup_{\tau, y} (|\partial_\tau I(\tau, t, y)| + |\nabla_y I(\tau, t, y)|) dt = +\infty.$$

3. (Global existence) If $\nu > 0$ we have the global existence for small enough data: there exists a constant $C_1 > 0$ such that

$$\|I_0\|_{H^s(\mathbb{T}_\tau \times \mathbb{R}^{n-1})} \leq C_1 \Rightarrow T^* = +\infty.$$

4. (Exponential decay) [42] If $\nu > 0$, $s \in \mathbb{N}$ and $s \geq \left\lfloor \frac{n+1}{2} \right\rfloor$, then there exists a constant $C_2 > 0$ such that $\|I_0\|_{H^s(\mathbb{T}_\tau \times \mathbb{R}^{n-1})} \leq C_2$ implies for all $z \geq 0$

$$\|I(z)\|_{H^s(\mathbb{T}_\tau \times \mathbb{R}^{n-1})} \leq C\|I_0\|_{H^s(\mathbb{T}_\tau \times \mathbb{R}^{n-1})} e^{-\ell z},$$

where $C > 0$ and $\ell \in]0, 1[$ are constants.

Remark 2.4.1. [80] We note that when $\nu = 0$, all the corresponding statements of Theorem 2.4.4 remain valid for $0 > t > -C$ with a suitable C .

Remark 2.4.2. In the study of the well-posedness of the KZK equation we reverse the usual role of the time with the main space variable along the propagation axis z : for $\nu > 0$ the solution $I(\tau, z, y) = I(t - \frac{z}{c}, \epsilon x_1, \sqrt{\epsilon} x')$ is defined for $x_1 > 0$, as it is global on $z \in \mathbb{R}^+$. Hence if we want to compare the KZK equation to other models such as the Kuznetsov equation or the Navier-Stokes system we need the well posedness results for these models on the half space

$$\{x_1 > 0, \quad t > 0, \quad x' \in \mathbb{R}^{n-1}\}, \quad (2.73)$$

taking into account the fact that the boundary conditions for the exact system come from the initial condition I_0 of the Cauchy problem (2.72) associated to the KZK equation.

Entropy estimate for the isentropic Navier-Stokes equation on the half space and the associated existence result

We follow now Section 5.2 in Ref. [81] updating it for the new *ansatz* and correct the proof of Theorem 5.5. See Ref. [81] for more details.

We consider the Cauchy problem for the KZK equation (2.72) for an initial data

$$I(t, 0, y) = I_0(t, y) \quad (\tau = t \text{ for } x_1 = 0)$$

L -periodic in t with mean value zero. Theorem 2.4.4 ensures that for any initial data I_0 , defined in $\mathbb{T}_t \times \mathbb{R}^{n-1}$ with small enough H^s ($s > [\frac{n}{2}] + 1$) norm (with respect to ν), there exists a unique solution of the KZK equation (2.63) I , which as a function of (τ, z, y) is global on $z \in \mathbb{R}^+$, periodic in τ of period L and mean value zero, and decays for $z \rightarrow \infty$ [80].

Therefore, see Remark 2.4.2, we consider our approximation problem between the isentropic Navier-Stokes system (2.10)–(2.11) and the KZK equation in the half space (2.73).

By I_0 we find I and thus also Φ and J , using Eq. (2.57)–(2.58). This allows us to construct the density and velocities $\bar{\rho}_\epsilon$ and $\bar{\mathbf{v}}_\epsilon$ in accordance with the *ansatz* (2.60) and (2.68). Thus, by I we construct the function $\bar{\mathbf{U}}_\epsilon = (\bar{\rho}_\epsilon, \bar{\rho}_\epsilon \bar{\mathbf{v}}_\epsilon)^t$.

In particular, for $t = 0$ we have functions defined for $x_1 > 0$ because I is well-defined for any $z > 0$

$$\begin{aligned} \bar{\rho}_\epsilon(0, x_1, x') &= \rho_0 + \epsilon I\left(-\frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon} x'\right) + \epsilon^2 J\left(-\frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon} x'\right), \\ \bar{\mathbf{v}}_\epsilon(0, x_1, x') &= (\bar{v}_1, \bar{\mathbf{v}}'_\epsilon)\left(-\frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon} x'\right), \end{aligned}$$

where

$$\bar{v}_1 = \epsilon \frac{c}{\rho_0} I + \epsilon^2 \frac{c^2}{\rho_0} \partial_z \partial_\tau^{-1} I, \quad \bar{\mathbf{v}}'_\epsilon = \sqrt{\epsilon} \frac{c^2}{\rho_0} \nabla_y \partial_\tau^{-1} I$$

and for $x_1 = 0$ we have L -periodic functions with mean value zero

$$\bar{\rho}_\epsilon(t, 0, x') = \rho_0 + \epsilon I(t, 0, \sqrt{\epsilon} x') + \epsilon^2 J(t, 0, \sqrt{\epsilon} x'), \quad (2.74)$$

$$\bar{\mathbf{v}}_\epsilon(t, 0, x') = (\bar{v}_1, \bar{\mathbf{v}}'_\epsilon)(t, 0, \sqrt{\epsilon} x'). \quad (2.75)$$

It is important to notice that the solution $\bar{\mathbf{v}}_\epsilon$ in system (2.10)–(2.11) is small on the boundary: $\bar{\mathbf{v}}_\epsilon|_{x_1=0} = \epsilon \tilde{\mathbf{v}}_\epsilon|_{x_1=0}$. Therefore, we have $|\bar{\mathbf{v}}_\epsilon|_{x_1=0}| < c$, which corresponds to the “subsonic” boundary case. More precisely, when the first velocity component is positive $\bar{v}_1|_{x_1=0} > 0$, we have a subsonic inflow boundary condition, and when it is negative $\bar{v}_1|_{x_1=0} < 0$, we have a subsonic outflow boundary condition, see Fig. 2.2. We also notice that, due to Eq. (2.68), the first component of the velocity $\bar{\mathbf{v}}_1$ on the boundary has the following form

$$\begin{aligned} \bar{v}_1|_{x_1=0} &= \left(\epsilon \frac{c}{\rho_0} I + \epsilon^2 G(I) \right) (t, 0, \sqrt{\epsilon} x') = \left(\epsilon \frac{c}{\rho_0} I + \epsilon^2 G(I) \right) \Big|_{z=0} \\ &= \epsilon \frac{c}{\rho_0} I_0(t, y) + \epsilon^2 G(I_0)(t, y), \end{aligned}$$

where

$$G(I) = \frac{c^2}{\rho_0} \partial_z \partial_\tau^{-1} I = \frac{c^2}{\rho_0} \partial_\tau^{-1} \left(\frac{(\gamma+1)}{4c\rho_0} \partial_\tau I^2 + \frac{\nu}{2c^3\rho_0} \partial_\tau^2 I + \frac{c}{2} \partial_\tau^{-1} \Delta_y I \right). \quad (2.76)$$

Therefore, the boundary conditions for $\bar{\mathbf{v}}_1$ are defined by the initial conditions for KZK equation and are L -periodic in t and have mean value zero. In addition, the sign of $\bar{v}_1|_{x_1=0}$ is the same as the sign of I_0 (because the term $G(I_0)$ is of a higher order of smallness on ϵ).

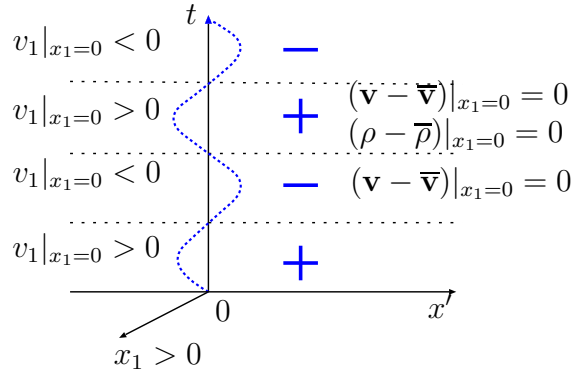


Figure 2.2 – Periodic subsonic inflow-outflow boundary conditions for the Navier-Stokes system.

Remark 2.4.3. As the viscosity term $\epsilon\nu\bar{\mathbf{v}}_\epsilon$, where ϵ is a fixed small enough parameter, ν is a constant, and in our case $\bar{\mathbf{v}}_\epsilon$ is of the order of ϵ , the boundary layer phenomenon can be excluded.

Theorem 2.4.5. Let $n \leq 3$. Suppose that the initial data of the KZK Cauchy problem $I_0(t, y) = I_0(t, \sqrt{\epsilon}x')$ is such that

1. I_0 is L -periodic in t and with mean value zero,
2. for fixed t , I_0 has the same sign for all $y \in \mathbb{R}^{n-1}$, and for $t \in]0, L[$ the sign changes, i.e. $I_0 = 0$, only for a finite number of times,
3. $I_0(t, y) \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1})$ for $s \geq 10$,
4. I_0 is sufficiently small in the sense of Theorem 2.4.4 such that [80, p.20]

$$\|I_0\|_{H^s} < \frac{\nu}{2c^2\rho_0} \frac{C_1(L)}{C_2(s)}.$$

Consequently, there exists a unique global solution in time $I(\tau, z, y)$ of (2.72) for $z = \epsilon x_1 > 0$, moreover, the functions $\bar{\rho}_\epsilon$, $\bar{\mathbf{v}}_\epsilon = (\bar{v}_1, \bar{\mathbf{v}}_\epsilon)$, defined by the ansatz (2.60)-(2.68) and Eq. (2.57)-(2.58) in the half space (2.73) are smooth with $\Omega = \mathbb{T}_t \times \mathbb{R}_y^{n-1}$:

$$\bar{\rho}_\epsilon \in C\left([0, \infty[, H^{s-4}(\Omega)\right) \cap C^1\left([0, \infty[; H^{s-6}(\Omega)\right), \quad (2.77)$$

$$\bar{\mathbf{v}}_\epsilon \in C\left([0, \infty[; H^{s-4}(\Omega)\right) \cap C^1\left([0, \infty[; H^{s-6}(\Omega)\right). \quad (2.78)$$

The Navier-Stokes system (2.10)-(2.11) in the half space with initial data (2.36) and following boundary conditions

$$(\bar{\mathbf{v}}_\epsilon - \mathbf{v}_\epsilon)|_{x_1=0} = 0,$$

with positive first component of the velocity $v_1|_{x_1=0} > 0$ (i.e. at points where the fluid enters the domain) has the additional boundary condition

$$(\bar{\rho}_\epsilon - \rho_\epsilon)|_{x_1=0} = 0.$$

When $v_1|_{x_1=0} \leq 0$ there is no any boundary condition for ρ_ϵ .

Then, for all finite times $T > 0$ there exists a unique solution $U_\epsilon = (\rho_\epsilon, \rho_\epsilon u_\epsilon)$ of the Navier-Stokes system (2.10)–(2.11) with the following smoothness on $[0, T]$

$$\rho_\epsilon \in C\left([0, T], H^3\left(\{x_1 > 0\} \times \mathbb{R}^{n-1}\right)\right) \cap C^1\left([0, T], H^2\left(\{x_1 > 0\} \times \mathbb{R}^{n-1}\right)\right)$$

and

$$u_\epsilon \in C\left([0, T], H^3\left(\{x_1 > 0\} \times \mathbb{R}^{n-1}\right)\right) \cap C^1\left([0, T], H^1\left(\{x_1 > 0\} \times \mathbb{R}^{n-1}\right)\right).$$

Remark 2.4.4. [81] The restriction to have the same sign for I_0 for all fixed time avoids a change in the type of the boundary condition applied to the tangential variables for the Navier-Stokes system. Moreover, Zabolotskaya [12] takes as the initial conditions for the KZK equation (which correspond to the boundary condition for v_1) the expression

$$I(\tau, 0, y) = -F(y) \sin \tau$$

with an amplitude distribution $F(y) \geq 0$. Especially, for a Gaussian beam [12]

$$F(y) = e^{-y^2},$$

while for a beam with a polynomial amplitude [12]

$$F(y) = \begin{cases} (1 - y^2)^2, & y \leq 1, \\ 0, & y > 1. \end{cases}$$

Proof. As previously, we use the fact that the entropy for the isentropic Euler system $\eta(\mathbf{U}_\epsilon)$, defined by Eq. (2.43) is a convex function[23].

Let us multiply the Navier-Stokes system (2.31), from the left, by $2\mathbf{U}_\epsilon^T \eta''(\mathbf{U}_\epsilon)$

$$2\mathbf{U}_\epsilon^T \eta''(\mathbf{U}_\epsilon) \partial_t \mathbf{U}_\epsilon + \sum_{i=1}^n 2\mathbf{U}_\epsilon^T \eta''(\mathbf{U}_\epsilon) DG_i(\mathbf{U}_\epsilon) \partial_{x_i} \mathbf{U}_\epsilon - \epsilon \nu 2\mathbf{U}_\epsilon^T \eta''(\mathbf{U}_\epsilon) \begin{bmatrix} 0 \\ \Delta \mathbf{v}_\epsilon \end{bmatrix} = 0.$$

We notice that

$$\mathbf{U}_\epsilon^T \eta''(\mathbf{U}_\epsilon) \begin{bmatrix} 0 \\ \Delta \mathbf{v}_\epsilon \end{bmatrix} = 0,$$

and, therefore, we have

$$2\mathbf{U}_\epsilon^T \eta''(\mathbf{U}_\epsilon) \partial_t \mathbf{U}_\epsilon = \partial_t [\mathbf{U}_\epsilon^T \eta''(\mathbf{U}_\epsilon) \mathbf{U}_\epsilon] - \mathbf{U}_\epsilon^T \partial_t \eta''(\mathbf{U}_\epsilon) \mathbf{U}_\epsilon.$$

Moreover, by virtue of $\eta''(U) DG_i(U) = (DG_i(U))^T \eta''(U)$ we find

$$2\mathbf{U}_\epsilon^T \eta''(\mathbf{U}_\epsilon) DG_i(\mathbf{U}_\epsilon) \partial_{x_i} \mathbf{U}_\epsilon = \partial_{x_i} [\mathbf{U}_\epsilon^T \eta''(\mathbf{U}_\epsilon) DG_i(\mathbf{U}_\epsilon) \mathbf{U}_\epsilon] - \mathbf{U}_\epsilon^T \partial_{x_i} [\eta''(\mathbf{U}_\epsilon) DG_i(\mathbf{U}_\epsilon)] \mathbf{U}_\epsilon.$$

Integrating over $[0, t] \times \{x_1 > 0\}$ ($x' \in \mathbb{R}^{n-1}$), we obtain

$$\begin{aligned} & \int_0^t \int_{x_1 > 0} \partial_t [\mathbf{U}_\epsilon^T \eta''(\mathbf{U}_\epsilon) \mathbf{U}_\epsilon] dx ds + \int_0^t \int_{x_1 > 0} \sum_{i=1}^n \partial_{x_i} [\mathbf{U}_\epsilon^T \eta''(\mathbf{U}_\epsilon) DG_i(\mathbf{U}_\epsilon) \mathbf{U}_\epsilon] dx ds \\ & - \int_0^t \int_{x_1 > 0} \mathbf{U}_\epsilon^T \partial_t \eta''(\mathbf{U}_\epsilon) \mathbf{U}_\epsilon dx ds - \int_0^t \int_{x_1 > 0} \sum_{i=1}^n \mathbf{U}_\epsilon^T \partial_{x_i} [\eta''(\mathbf{U}_\epsilon) DG_i(\mathbf{U}_\epsilon)] \mathbf{U}_\epsilon dx ds = 0. \end{aligned}$$

Integrating by parts we result in

$$\begin{aligned} & \int_{x_1>0} \mathbf{U}_\epsilon^T \eta''(\mathbf{U}_\epsilon) \mathbf{U}_\epsilon dx - \int_{x_1>0} \mathbf{U}_\epsilon^T \eta''(\mathbf{U}_\epsilon) \mathbf{U}_\epsilon |_{t=0} dx \\ & - \int_0^t \int_{x_1>0} \mathbf{U}_\epsilon^T \left[\partial_t \eta''(\mathbf{U}_\epsilon) + \sum_{i=1}^n \partial_{x_i} [\eta''(\mathbf{U}_\epsilon) DG_i(\mathbf{U}_\epsilon)] \right] \mathbf{U}_\epsilon dx ds \\ & - \int_0^t \int_{\mathbb{R}^{n-1}} \mathbf{U}_\epsilon^T \eta''(\mathbf{U}_\epsilon) DG_1(\mathbf{U}_\epsilon) \mathbf{U}_\epsilon |_{x_1=0} dx' ds = 0. \end{aligned}$$

We recall that $\eta''(\mathbf{U}_\epsilon)$ is positive definite, consequently for some $C > 0$ and $\delta_0 > 0$

$$C|\mathbf{U}_\epsilon|^2 \geq \mathbf{U}_\epsilon^T \eta''(\mathbf{U}_\epsilon) \mathbf{U}_\epsilon \geq \delta_0 |\mathbf{U}_\epsilon|^2.$$

Therefore, we obtain for the initial data

$$\mathbf{U}_0 = \left[\begin{array}{c} \rho_0 + \epsilon I + \epsilon^2 J \\ \epsilon (\rho_0 + \epsilon I + \epsilon^2 J) \left(\frac{\epsilon}{\rho_0} I + \epsilon G(I), \sqrt{\epsilon} \bar{w} \right) \end{array} \right] \left(-\frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon} x' \right) \quad (2.79)$$

and the relation

$$\begin{aligned} \delta_0 \int_{x_1>0} \mathbf{U}_\epsilon^2 dx - C \int_{x_1>0} \mathbf{U}_0^2 dx - \int_0^t \int_{\mathbb{R}^{n-1}} \mathbf{U}_\epsilon^T \eta''(\mathbf{U}_\epsilon) DG_1(\mathbf{U}_\epsilon) \mathbf{U}_\epsilon |_{x_1=0} dx' ds \\ \leq C_1 \int_0^t \int_{x_1>0} \mathbf{U}_\epsilon^2 dx ds. \end{aligned}$$

As in Ref. [35], C_1 is an upper bound for the eigenvalues of the symmetric matrix

$$\partial_t \eta''(\mathbf{U}_\epsilon) + \sum_{i=1}^n \partial_{x_i} [\eta''(\mathbf{U}_\epsilon) DG_i(\mathbf{U}_\epsilon)].$$

Let us now consider the integral on the boundary. With notation $\mathbf{v}_\epsilon = (v_1, \mathbf{v}'_\epsilon)^t$ for the velocity and $H''(\rho) = \frac{p'(\rho)}{\rho}$, we see with $DG_1(\mathbf{U}_\epsilon)$ coming from (2.33) that

$$\begin{aligned} & \mathbf{U}_\epsilon^T \eta''(\mathbf{U}_\epsilon) DG_1(\mathbf{U}_\epsilon) \mathbf{U}_\epsilon \\ & = (\rho_\epsilon, \rho_\epsilon \mathbf{v}_\epsilon)^T \begin{pmatrix} H''(\rho_\epsilon) + \frac{\mathbf{v}_\epsilon^2}{\rho_\epsilon} & -\frac{\mathbf{v}_\epsilon}{\rho_\epsilon} \\ -\frac{\mathbf{v}_\epsilon}{\rho_\epsilon} & \frac{1}{\rho_\epsilon} Id_n \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -v_1^2 + p'(\rho_\epsilon) & 2v_1 & 0 \\ -v_1 \mathbf{v}'_\epsilon & \mathbf{v}'_\epsilon & v_1 Id_{n-1} \end{pmatrix} \begin{pmatrix} \rho_\epsilon \\ \rho_\epsilon \mathbf{v}_\epsilon \end{pmatrix} \\ & = (\rho_\epsilon, \rho_\epsilon v_1, \rho_\epsilon \mathbf{v}'_\epsilon)^T \begin{pmatrix} v_1 \left(\frac{\mathbf{v}_\epsilon^2}{\rho_\epsilon} - \frac{p'(\rho_\epsilon)}{\rho_\epsilon} \right) & \frac{-v_1^2}{\rho_\epsilon} + \frac{p'(\rho_\epsilon)}{\rho_\epsilon} & -v_1 \frac{\mathbf{v}'_\epsilon}{\rho_\epsilon} \\ \frac{-v_1^2}{\rho_\epsilon} + \frac{p'(\rho_\epsilon)}{\rho_\epsilon} & \frac{v_1}{\rho_\epsilon} & 0 \\ -v_1 \frac{\mathbf{v}'_\epsilon}{\rho_\epsilon} & 0 & \frac{v_1}{\rho_\epsilon} Id_{n-1} \end{pmatrix} \begin{pmatrix} \rho_\epsilon \\ \rho_\epsilon v_1 \\ \rho_\epsilon \mathbf{v}'_\epsilon \end{pmatrix} \\ & = \rho_\epsilon p'(\rho_\epsilon) v_1. \end{aligned}$$

Let us consider the initial condition $I_0(t, y)$ for the KZK equation of the type described in Remark 2.4.4. We suppose (without loss of generality) that $I_0 = 0$ for $t \in]0, L[$ only once. More precisely, we suppose that the sign of v_1 is changing in the following way:

- $v_1 \leq 0$ for $t \in [0 + (k-1)L, \frac{L}{2} + (k-1)L]$ ($k = 1, 2, 3, \dots$),
- $v_1 > 0$ for $t \in]\frac{L}{2} + (k-1)L, kL[$ ($k = 1, 2, 3, \dots$).

If $t \in [0, \frac{L}{2}]$ (for $k = 1$), the first component of the velocity $v_1|_{x_1=0} < 0$ is negative, and thus we have

$$\rho_\varepsilon p'(\rho_\varepsilon) v_1 < 0.$$

If $t \in]\frac{L}{2}, L[$, the first component of velocity is positive $v_1|_{x_1=0} > 0$, then we also impose $\rho_\varepsilon|_{x_1=0} = \rho_0 + \varepsilon I_0(t, y) + \varepsilon^2 J$, where $I_0(t, y)$ is the initial condition for the KZK equation and J coming from Eq. (2.58). For the term

$$\rho_\varepsilon p'(\rho_\varepsilon) v_1 > 0$$

we see that on the boundary it has the form

$$\begin{aligned} \rho_\varepsilon p'(\rho_\varepsilon) v_1 &= \varepsilon \left(\frac{c}{\rho_0} I_0 + \frac{c^2}{\rho_0} \partial_z \partial_\tau^{-1} I_0 \right) (\rho_0 + \varepsilon I_0(t, y) + \varepsilon^2 J) p'(\rho_0 + \varepsilon I_0(t, y) + \varepsilon^2 J) \\ &\leq C_0 \varepsilon I_0 \end{aligned}$$

for some constant $C_0 > 0$ independent on ε . Consequently, for $k \geq 1$

$$\begin{aligned} \int_0^{kL} \int_{\mathbb{R}^{n-1}} \rho_\varepsilon p'(\rho_\varepsilon) v_1|_{x_1=0} dx' ds &\leq \sum_{j=1}^k \int_{] \frac{L}{2} + (j-1)L, jL[} \int_{\mathbb{R}^{n-1}} \rho_\varepsilon p'(\rho_\varepsilon) v_1|_{x_1=0} dx' ds \\ &\leq \sum_{j=1}^k \int_{] \frac{L}{2} + (j-1)L, jL[} \int_{\mathbb{R}^{n-1}} C_0 \varepsilon I_0 \leq K k \varepsilon \|I_0\|_{H^s}, \end{aligned}$$

where $K = O(1)$ is a positive constant independent of k .

However for $t > 0$ we have $k \geq 1$ such that $t \in [(k-1)L, kL[$ and it implies on one hand if $t \in [(k-1)L, (k-1)L + \frac{L}{2}[$

$$\int_0^t \int_{\mathbb{R}^{n-1}} \rho_\varepsilon p'(\rho_\varepsilon) v_1|_{x_1=0} dx' ds \leq \int_0^{(k-1)L} \int_{\mathbb{R}^{n-1}} \rho_\varepsilon p'(\rho_\varepsilon) v_1|_{x_1=0} dx' ds$$

and on the other hand if $t \in [(k-1)L + \frac{L}{2}, kL[$

$$\int_0^t \int_{\mathbb{R}^{n-1}} \rho_\varepsilon p'(\rho_\varepsilon) v_1|_{x_1=0} dx' ds \leq \int_0^{kL} \int_{\mathbb{R}^{n-1}} \rho_\varepsilon p'(\rho_\varepsilon) v_1|_{x_1=0} dx' ds.$$

As a consequence, we obtain for all $t > 0$

$$\int_0^t \int_{\mathbb{R}^{n-1}} \rho_\varepsilon p'(\rho_\varepsilon) v_1|_{x_1=0} dx' ds \leq K \left(\left[\frac{t}{L} \right] + 1 \right) \varepsilon \|I_0\|_{H^s}.$$

Therefore we deduce the estimate, as $\delta_0 > 0$

$$\int_{x_1>0} \mathbf{U}_\varepsilon^2 dx \leq \frac{C}{\delta_0} \int_{x_1>0} \mathbf{U}_0^2 dx + \varepsilon \frac{K}{\delta_0} \left(\left[\frac{t}{L} \right] + 1 \right) \|I_0\|_{H^s} + \frac{C_1}{\delta_0} \int_0^t \int_{x_1>0} \mathbf{U}_\varepsilon^2 dx ds.$$

By the Gronwall lemma we find

$$\|\mathbf{U}_\varepsilon\|_{L^2}^2(t) \leq \frac{C}{\delta_0} \left(\|\mathbf{U}_0\|_{L^2}^2 + \varepsilon \frac{K}{C} \left(\left[\frac{t}{L} \right] + 1 \right) \|I_0\|_{H^s} \right) e^{\frac{C_1}{\delta_0} t}$$

remaining bounded for all finite times.

Thus, for all $T < +\infty$ we obtain that

$$\mathbf{U}_\epsilon \in L^\infty \left([0, T], L^2 \left(\{x_1 > 0\} \times \mathbb{R}^{n-1} \right) \right).$$

If $I_0 = 0$ for $t \in]0, L[$ a finite number of times m , we obtain the same result for \mathbf{U}_ϵ .

Hence, by Ref. [35] we have proved that the chosen boundary conditions ensure the local well-posedness for the Navier-Stokes system in the half space, which can be viewed as a symmetrizable incompletely parabolic system. We apply now the theory of incompletely parabolic problems [35, p. 352] with the result of global well-posedness of the Navier-Stokes system in the half space with the Dirichlet boundary conditions [68] for the velocity and with the initial data $\rho_\epsilon(0) - \rho_0 \in H^3(\{x_1 > 0\} \times \mathbb{R}^{n-1})$ and $\mathbf{v}_\epsilon(0) \in H^3(\{x_1 > 0\} \times \mathbb{R}^{n-1})$ small enough. Hence, for sufficient regular initial data $\mathbf{U}_0 \in H^3(\{x_1 > 0\} \times \mathbb{R}^{n-1})$ ($n \leq 3$) for all finite time $T < \infty$, we obtain by the energy method that $\mathbf{U}_\epsilon \in L^\infty([0, T], H^3(\{x_1 > 0\} \times \mathbb{R}^{n-1}))$.

To ensure that \mathbf{U}_0 defined in Eq. (2.79) belongs $H^3(\{x_1 > 0\} \times \mathbb{R}^{n-1})$ we need to take $I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1})$ such that

$$\bar{\rho}_\epsilon \in C([0, +\infty[; H^3(\{x_1 > 0\} \times \mathbb{R}^{n-1})), \bar{\mathbf{v}}_\epsilon \in C([0, +\infty[; H^3(\{x_1 > 0\} \times \mathbb{R}^{n-1})).$$

By Theorem 2.4.4, $I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1})$ implies while $s - 2k \geq 0$ that

$$I(\tau, z, y) \in C^k(\{x_1 > 0\}; H^{s-2k}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})),$$

but we can also say[42], thanks to Point 4 of Theorem 2.4.4, that

$$\partial_z^k I(\tau, z, y) \in L^2(\{x_1 > 0\}; H^{s-2k}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})).$$

Considering the expressions of $\bar{\rho}_\epsilon$ and $\bar{\mathbf{v}}_\epsilon$

$$\bar{\rho}_\epsilon = \rho_0 + \epsilon I - \frac{\epsilon^2}{\rho_0} \left(\frac{\gamma - 1}{2} I^2 - \frac{\nu}{c^2} \partial_\tau I \right), \quad \bar{\mathbf{v}}_\epsilon = \frac{c^2}{\rho_0} \left(\frac{\epsilon}{c} I - \epsilon^2 \partial_\tau^{-1} \partial_z I; \epsilon^{\frac{3}{2}} \partial_\tau^{-1} \nabla_y I \right),$$

the least regular term is $\partial_\tau^{-1} \partial_z I$. Thus we need to ensure

$$\partial_z I \in C([0, +\infty[; H^3(\{x_1 > 0\} \times \mathbb{R}^{n-1})),$$

which leads us to take $s \geq 10$ in order to have

$$\partial_z^k I(\tau, z, y) \in L^2(\{x_1 > 0\}; H^{s-2k}(\mathbb{T}_\tau \times \mathbb{R}^{n-1}))$$

for $k \leq 4$ with $s - 2k \geq 2$ as we want to have the continuity on time. This choice of the regularity for I_0 allows us to control the boundary terms appearing from the integration by parts in the energy method. Indeed, we can perform analogous computations as in Ref. [23] p.103 to control the spatial derivative of \mathbf{U}_ϵ of the order less or equal to 3 and directly verify that all boundary terms are controlled by $t \|I_0\|_{H^s}$, what is actually is a consequence of the well-posedness[68] in H^3 .

Thus, we obtain the existence of the unique local solution of the Navier-Stokes system with

$$\rho_\epsilon \in C \left([0, T], H^3 \left(\{x_1 > 0\} \times \mathbb{R}^{n-1} \right) \right) \cap C^1 \left([0, T], H^2 \left(\{x_1 > 0\} \times \mathbb{R}^{n-1} \right) \right)$$

and

$$u_\epsilon \in C \left([0, T], H^3 \left(\{x_1 > 0\} \times \mathbb{R}^{n-1} \right) \right) \cap C^1 \left([0, T], H^1 \left(\{x_1 > 0\} \times \mathbb{R}^{n-1} \right) \right).$$

□

Approximation of the solutions of the isentropic Navier-Stokes system with the solutions of the KZK equation

Knowing from Subsection 2.4.2 that the KZK equation can be derived from the compressible isentropic Navier-Stokes system (2.10)–(2.11) using the *ansatz* (2.64)–(2.65) with I and J given by (2.57) and (2.58) respectively, we obtain the following expansion of the Navier-Stokes equations

$$\begin{aligned} \partial_t \rho_\varepsilon + \nabla \cdot (\rho_\varepsilon \mathbf{v}_\varepsilon) = & \varepsilon^2 \left[\frac{\rho_0}{c^2} (2c \partial_{z\tau}^2 \Phi - c^2 \Delta_y \Phi) - \frac{\rho_0}{2c^4} (\gamma + 1) \partial_\tau [(\partial_\tau \Phi)^2] - \frac{\nu}{c^4} \partial_\tau^3 \Phi \right] \\ & + \varepsilon^3 R_1^{NS-KZK} \end{aligned} \quad (2.80)$$

and

$$\begin{aligned} \rho_\varepsilon [\partial_t \mathbf{v}_\varepsilon + (\mathbf{v}_\varepsilon \cdot \nabla) \mathbf{v}_\varepsilon] + \nabla p(\rho_\varepsilon) - \varepsilon \nu \Delta \mathbf{v}_\varepsilon = & \varepsilon \tilde{\nabla} [-\rho_0 \partial_\tau \Phi + c^2 I] \\ & + \varepsilon^2 \tilde{\nabla} \left[c^2 J + \frac{(\gamma - 1) \rho_0}{2c^2} (\partial_\tau \Phi)^2 + \frac{\nu}{c^2} \partial_\tau^2 \Phi \right] + \varepsilon^3 \mathbf{R}_2^{NS-KZK}, \end{aligned} \quad (2.81)$$

where R_1^{NS-KZK} and \mathbf{R}_2^{NS-KZK} are the remainder terms given in Appendix A. So, as it was previously explained for the approximation of the Navier-Stokes by the Kuznetsov equation in Subsection 2.4.1, if we consider a solution of the KZK equation I and define by it the functions Φ and J , then we define according to *ansatz* (2.64)–(2.65) $\bar{\rho}_\varepsilon$ and $\bar{\mathbf{v}}_\varepsilon$ (see Eq. (2.68)), which solve the approximate system (2.29)–(2.30) with the remainder terms R_1^{NS-KZK} and \mathbf{R}_2^{NS-KZK} and, as previously, with $p(\bar{\rho}_\varepsilon)$ from the state law (2.12) :

$$\partial_t \bar{\rho}_\varepsilon + \operatorname{div}(\bar{\rho}_\varepsilon \bar{\mathbf{v}}_\varepsilon) = \varepsilon^3 R_1^{NS-KZK}, \quad (2.82)$$

$$\bar{\rho}_\varepsilon [\partial_t \bar{\mathbf{v}}_\varepsilon + (\bar{\mathbf{v}}_\varepsilon \cdot \nabla) \bar{\mathbf{v}}_\varepsilon] + \nabla p(\bar{\rho}_\varepsilon) - \varepsilon \nu \Delta \bar{\mathbf{v}}_\varepsilon = \varepsilon^3 \mathbf{R}_2^{NS-KZK}. \quad (2.83)$$

As usual, we denote by $\mathbf{U}_\varepsilon = (\rho_\varepsilon, \rho_\varepsilon \mathbf{v}_\varepsilon)^t$ the solution of the exact Navier-Stokes system and by $\bar{\mathbf{U}}_\varepsilon = (\bar{\rho}_\varepsilon, \bar{\rho}_\varepsilon \bar{\mathbf{v}}_\varepsilon)^t$ the solution of (2.82)–(2.83).

We work on $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ ($n = 2$ or 3) due to the domain of the well-posedness for the KZK equation. In this case the Navier-Stokes system is locally well-posed with non homogeneous boundary conditions of $\bar{\mathbf{U}}_\varepsilon$, as they are directly determined by the initial condition I_0 of the KZK equation (2.72) according to Theorem 2.4.5. Knowing the existence results for two problems, we validate the approximation of \mathbf{U}_ε by $\bar{\mathbf{U}}_\varepsilon$ following Ref. [81] and Subsection 2.4.1:

Theorem 2.4.6. [81] *Let $n = 2$ or 3 , $s \geq 10$ and Theorem 2.4.5 hold. Then there exist constants $C > 0$ and $K > 0$ such that we have the following stability estimate*

$$0 \leq t \leq \frac{C}{\varepsilon}, \quad \|\mathbf{U}_\varepsilon - \bar{\mathbf{U}}_\varepsilon\|_{L^2(\mathbb{R}_+ \times \mathbb{R}^{n-1})}^2(t) \leq K \varepsilon^3 t e^{K \varepsilon t} \leq 9 \varepsilon^2.$$

Remark 2.4.5. *The regularity of $I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1})$ with $s > 8$ (see Table 2.1) is minimal to ensure that R_1^{NS-KZK} and \mathbf{R}_2^{NS-KZK} , see Appendix A, belongs to $C([0, +\infty[; L^2(\mathbb{R}_+ \times \mathbb{R}^{n-1}))$.*

Indeed, if $I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1})$ with $s > \max\{8, \frac{n}{2}\}$, then for $0 \leq k \leq 4$

$$I(\tau, z, y) \in C^k(\{z > 0\}; H^{s-2k}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})).$$

Let us denote $\Omega = \mathbb{T}_\tau \times \mathbb{R}^{n-1}$. Given the equations for $\bar{\rho}_\varepsilon$ by (2.60) with (2.57) and (2.58) and for $\bar{\mathbf{v}}_\varepsilon$ by (2.68) respectively, we have for $0 \leq k \leq 2$

$$\partial_z^k \bar{\rho}_\varepsilon(\tau, z, y) \in C(\{z > 0\}; H^{s-1-2k}(\Omega)), \quad \partial_z^k \bar{\mathbf{v}}_\varepsilon(\tau, z, y) \in C(\{z > 0\}; H^{s-2-2k}(\Omega)),$$

but we can also say [42] thanks to Point 4 of Theorem 2.4.4 that

$$\partial_z^k \bar{\rho}_\varepsilon(\tau, z, y) \in L^2(\{z > 0\}; H^{s-1-2k}(\Omega)), \quad \partial_z^k \bar{\mathbf{v}}_\varepsilon(\tau, z, y) \in L^2(\{z > 0\}; H^{s-2-2k}(\Omega)).$$

This implies for $0 \leq k \leq 2$ (as $s > 8$) that $s - 2 - 2k > 2$ and

$$\begin{aligned} \partial_z^k \bar{\rho}_\varepsilon(\tau, z, y) &\in C(\mathbb{T}_\tau; L^2(\{z > 0\}; H^{s-1-2k}(\mathbb{R}^{n-1}))), \\ \partial_z^k \bar{\mathbf{v}}_\varepsilon(\tau, z, y) &\in C(\mathbb{T}_\tau; L^2(\{z > 0\}; H^{s-2-2k}(\mathbb{R}^{n-1}))). \end{aligned}$$

Hence we find

$$\bar{\rho}_\varepsilon(t, x_1, x'), \quad \bar{\mathbf{v}}_\varepsilon(t, x_1, x') \in C([0, +\infty[; H^2(\{x_1 > 0\} \times \mathbb{R}^{n-1})).$$

As in addition for $0 \leq k \leq 1$, considering $\bar{\rho}_\varepsilon$ and $\bar{\mathbf{v}}_\varepsilon$ as functions of (τ, z, y) ,

$$\partial_z^k \partial_\tau \bar{\rho}_\varepsilon \in C(\{z > 0\}; H^{s-2-2k}(\Omega)), \quad \partial_z^k \partial_\tau \bar{\mathbf{v}}_\varepsilon \in C(\{z > 0\}; H^{s-3-2k}(\Omega)),$$

we deduce in the same way that

$$\partial_t \bar{\rho}_\varepsilon(t, x_1, x'), \quad \partial_t \bar{\mathbf{v}}_\varepsilon(t, x_1, x') \in C([0, +\infty[; H^1(\{x_1 > 0\} \times \mathbb{R}^{n-1})).$$

These regularities of $\bar{\rho}_\varepsilon$ and $\bar{\mathbf{v}}_\varepsilon$ viewed as functions of (t, x_1, x') allow to have all left-hand terms in the approximated Navier-Stokes system (2.82)–(2.83) of the regularity $C([0, T]; L^2(\{x_1 > 0\} \times \mathbb{R}^{n-1}))$ and the remainder terms in the right-hand side inherit it.

2.4.3 Navier-Stokes system and the NPE equation

Derivation of the NPE equation

The NPE equation (Nonlinear Progressive wave Equation), initially derived by McDonald and Kuperman [70], is an example of a paraxial approximation aiming to describe short-time pulses and a long-range propagation, for instance, in an ocean wave-guide, where the refraction phenomena are important. To compare to the KZK equation we use the following paraxial change of variables

$$u(t, x_1, x') = \Psi(\varepsilon t, x_1 - ct, \sqrt{\varepsilon} x') = \Psi(\tau, z, y), \quad (2.84)$$

with

$$\tau = \varepsilon t, \quad z = x_1 - ct, \quad y = \sqrt{\varepsilon} x'. \quad (2.85)$$

For the velocity we have

$$\mathbf{v}_\varepsilon(t, x_1, x') = -\varepsilon \nabla u(t, x_1, x') = -\varepsilon (\partial_z \Psi, \sqrt{\varepsilon} \nabla_y \Psi)(\tau, z, y). \quad (2.86)$$

If we compare the NPE equation to the isentropic Navier-Stokes system this method of approximation does not allow to keep the Kuznetsov *ansatz* of perturbations (2.14)–(2.15) imposing (2.17)–(2.18) just by introducing the new paraxial profiles Ψ for u , ξ for ρ_1 and

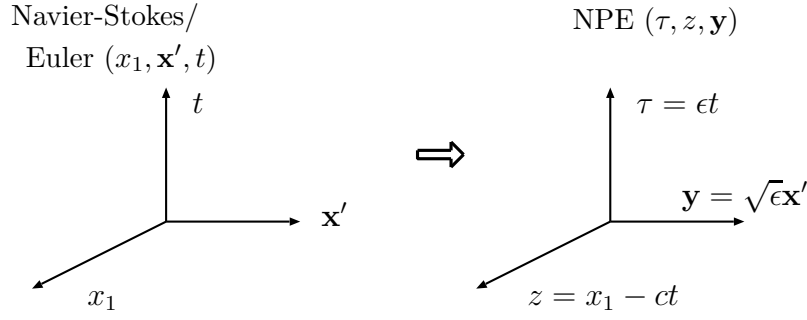


Figure 2.3 – Paraxial change of variables for the profiles $U(\epsilon t, x_1 - ct, \sqrt{\epsilon}x')$.

χ for ρ_2 and taking the term of order 0 in ϵ as it was done in the case of the KZK-approximation. This time the paraxial change of variables (2.85) for ρ_1 and ρ_2 defined in (2.17)–(2.18) gives

$$\begin{aligned}\rho_1 &= -\frac{\rho_0}{c}\partial_z\Psi + \epsilon\frac{\rho_0}{c^2}\partial_\tau\Psi, \\ \rho_2 &= -\frac{\rho_0(\gamma-2)}{2c^2}(\partial_z\Psi)^2 - \frac{\rho_0}{2c^2}(\partial_z\Psi)^2 - \frac{\nu}{\rho_0}\partial_z^2\Psi \\ &\quad + \epsilon\left[\frac{\rho_0(\gamma-2)}{2c^3}\partial_z\Psi\partial_\tau\Psi - \frac{\rho_0}{2c^2}(\nabla_y\Psi)^2 - \frac{\nu}{c^2}\Delta_y\Psi\right] \\ &\quad + \epsilon^2\left(-\frac{\rho_0(\gamma-2)}{2c^4}\right)(\partial_\tau\Psi)^2.\end{aligned}$$

Thus one of the terms in the ρ_1 -extension takes part of the second order corrector of ρ_ϵ :

$$\rho_\epsilon(t, x_1, x') = \rho_0 + \epsilon\xi(\tau, z, y) + \epsilon^2\chi(\tau, z, y), \quad (2.87)$$

with

$$\xi(\tau, z, y) = -\frac{\rho_0}{c}\partial_z\Psi, \quad (2.88)$$

$$\chi(\tau, z, y) = \frac{\rho_0}{c^2}\partial_\tau\Psi - \frac{\rho_0(\gamma-1)}{2c^2}(\partial_z\Psi)^2 - \frac{\nu}{c^2}\partial_z^2\Psi. \quad (2.89)$$

The obtained *ansatz* (2.86)–(2.87) applied to the Navier-Stokes system gives

$$\begin{aligned}\partial_t\rho_\epsilon + \operatorname{div}(\rho_\epsilon\mathbf{v}_\epsilon) &= \epsilon^2\left(-\frac{2\rho_0}{c}\right)\left(\partial_{\tau z}^2\Psi - \frac{(\gamma+1)}{4}\partial_z(\partial_z\Psi)^2 - \frac{\nu}{2\rho_0}\partial_z^3\Psi + \frac{c}{2}\Delta_y\Psi\right) \\ &\quad + \epsilon^3\mathbf{R}_1^{NS-NPE},\end{aligned}$$

$$\begin{aligned}\rho_\epsilon[\partial_t\mathbf{v}_\epsilon + (\mathbf{v}_\epsilon\cdot\nabla)\mathbf{v}_\epsilon] + \nabla p(\rho_\epsilon) - \epsilon\nu\Delta\mathbf{v}_\epsilon &= \epsilon\nabla\left(\xi + \frac{\rho_0}{c}\partial_z\Psi\right) \\ &\quad + c^2\epsilon^2\nabla\left[\chi - \frac{\rho_0}{c^2}\partial_\tau\Psi + \frac{\rho_0(\gamma-1)}{2c^2}(\partial_z\Psi)^2 + \frac{\nu}{c^2}\partial_z^2\Psi\right] + \epsilon^3\mathbf{R}_2^{NS-NPE}.\end{aligned}$$

The remainder term in the conservation of mass is given by

$$\begin{aligned}\epsilon^3\mathbf{R}_1^{NS-NPE} &= \epsilon^3\left(\partial_\tau\chi - \nabla_y\xi\nabla_y\Psi - \xi\Delta_y\Psi - \partial_z\chi\partial_z\Psi - \chi\partial_z^2\Psi\right) \\ &\quad + \epsilon^4(-\nabla_y\chi\nabla_y\Psi - \chi\Delta_y\Psi),\end{aligned} \quad (2.90)$$

while in the conservation of momentum along the x_1 axis it is given by

$$\begin{aligned} \varepsilon^3 \mathbf{R}_2^{NS-NPE} \cdot \vec{e}_1 = \varepsilon^3 \left[-\frac{\rho_0}{c} \partial_z \Psi \partial_{\tau z}^2 \Psi + \frac{\rho_0}{2} \partial_z (\nabla_y \Psi)^2 + \nu \partial_z \Delta_y \Psi + \frac{\xi}{2} \partial_z (\partial_z \Psi)^2 \right. \\ \left. + c \chi \partial_z^2 \Psi \right] + \varepsilon^4 \left(\frac{\xi}{2} \partial_z (\nabla_y \Psi)^2 - \chi \partial_{\tau z}^2 \Psi + \frac{\chi}{2} \partial_z (\partial_z \Psi)^2 \right) + \varepsilon^5 \frac{\chi}{2} \partial_z (\nabla_y \Psi)^2, \end{aligned} \quad (2.91)$$

and along all transversal direction x_j to the propagation x_1 -axis

$$\begin{aligned} \varepsilon^3 \mathbf{R}_2^{NS-NPE} \cdot \vec{e}_j = \varepsilon^{\frac{7}{2}} \left[-\frac{\rho_0}{c} \partial_z \Psi \partial_{\tau y_j}^2 \Psi + \frac{\rho_0}{2} \partial_{y_j} (\nabla_y \Psi)^2 + \nu \partial_{y_j} \Delta_y \Psi + \frac{\xi}{2} \partial_{y_j} (\partial_z \Psi)^2 \right. \\ \left. + c \chi \partial_{zy_j}^2 \Psi \right] + \varepsilon^{\frac{9}{2}} \left(\frac{\xi}{2} \partial_{y_j} (\nabla_y \Psi)^2 - \chi \partial_{\tau y_j}^2 \Psi + \frac{\chi}{2} \partial_{y_j} (\partial_z \Psi)^2 \right) + \varepsilon^{\frac{11}{2}} \frac{\chi}{2} \partial_{y_j} (\nabla_y \Psi)^2. \end{aligned} \quad (2.92)$$

As all previous models, for this *ansatz*, the NPE equation

$$\partial_{\tau z}^2 \Psi - \frac{(\gamma + 1)}{4} \partial_z (\partial_z \Psi)^2 - \frac{\nu}{2\rho_0} \partial_z^3 \Psi + \frac{c}{2} \Delta_y \Psi = 0 \quad (2.93)$$

appears as the second order approximation of the isentropic Navier-Stokes system up to the terms of the order of $\mathcal{O}(\varepsilon^3)$. In the sequel we work with the NPE equation satisfied by ξ (see Eq. (2.88) for the definition)

$$\partial_{\tau z}^2 \xi + \frac{(\gamma + 1)c}{4\rho_0} \partial_z^2 [(\xi)^2] - \frac{\nu}{2\rho_0} \partial_z^3 \xi + \frac{c}{2} \Delta_y \xi = 0. \quad (2.94)$$

Looking at Figs 2.1 and 2.3 together with (2.63) and (2.93) we see that we have a bijection between the variables of the KZK and NPE equations defined by the relations

$$z_{NPE} = -c\tau_{KZK} \text{ and } \tau_{NPE} = \varepsilon\tau_{KZK} + \frac{z_{KZK}}{c}, \quad (2.95)$$

which implies for the derivatives

$$\partial_{\tau_{NPE}} = c\partial_{z_{KZK}} \text{ and } \partial_{z_{NPE}} = -\frac{1}{c}\partial_{\tau_{KZK}}.$$

Thus, as it was mentioned in Introduction, the known mathematical results for the KZK equation can be directly applied for the NPE equation.

Well posedness of the NPE equation

We consider the Cauchy problem:

$$\begin{cases} \partial_{\tau z}^2 \xi + \frac{(\gamma+1)c}{4\rho_0} \partial_z^2 [(\xi)^2] - \frac{\nu}{2\rho_0} \partial_z^3 \xi + \frac{c}{2} \Delta_y \xi = 0 \text{ on } \mathbb{R}_+ \times \mathbb{T}_z \times \mathbb{R}^{n-1}, \\ \xi(0, z, y) = \xi_0(z, y) \text{ on } \mathbb{T}_z \times \mathbb{R}^{n-1}, \end{cases} \quad (2.96)$$

in the class of L -periodic functions with respect to the variable z and with mean value zero along z . The introduction of the operator ∂_z^{-1} defined similarly to ∂_τ^{-1} in Eq. (2.71) allows us to consider instead of Eq. (2.94) the following equivalent equation

$$\partial_\tau \xi + \frac{(\gamma + 1)c}{4\rho_0} \partial_z [(\xi)^2] - \frac{\nu}{2\rho_0} \partial_z^2 \xi + \frac{c}{2} \Delta_y \partial_z^{-1} \xi = 0 \text{ on } \mathbb{R}_+ \times \mathbb{T}_z \times \mathbb{R}^{n-1}.$$

As a consequence we can use the results of Subsection 2.4.2 if we replace τ by z . In the same time for the viscous case it holds the following theorem:

Theorem 2.4.7. *Let $n \geq 2$, $\nu > 0$, $s > \max\left(4, \left[\frac{n}{2}\right] + 1\right)$ and $\xi_0 \in H^s(\mathbb{T}_z \times \mathbb{R}^{n-1})$ with zero mean value along z . Then there exists a constant $k_2 > 0$ such that if*

$$\|\xi_0\|_{H^s(\mathbb{T}_z \times \mathbb{R}^2)} < k_2, \quad (2.97)$$

then the Cauchy problem for the NPE equation (2.96) has a unique global in time solution

$$\xi \in \bigcap_{i=0}^2 C^i([0, +\infty[, H^{s-2i}(\mathbb{T}_z \times \mathbb{R}^2)) \quad (2.98)$$

satisfying the zero mean value condition along z . Moreover for Ψ according with Eq. (2.88) we have

$$\Psi := -\frac{c}{\rho_0} \partial_z^{-1} \xi \in \bigcap_{i=0}^2 C^i([0, +\infty[, H^{s-2i}(\mathbb{T}_z \times \mathbb{R}^2))$$

and it also satisfies the zero mean value condition along z , i.e. $\int_0^L \Psi(\tau, l, y) dl = 0$.

Proof. For $\xi_0 \in H^s(\mathbb{T}_z \times \mathbb{R}^{n-1})$ small enough, the existence of a global in time solution

$$\xi \in \bigcap_{i=0}^1 C^i([0, +\infty[, H^{s-2i}(\mathbb{T}_z \times \mathbb{R}^{n-1}))$$

of the Cauchy problem for the NPE equation (2.96) comes from Theorem 2.4.4. We also have the desired regularity by a simple bootstrap argument. Moreover the formula for ∂_z^{-1} (see the equivalent definition of ∂_τ^{-1} in Eq. (2.71)) implies for $s \geq 1$ by the Poincaré inequality

$$\|\partial_z^{-1} \xi\|_{H^s(\mathbb{T}_z \times \mathbb{R}^{n-1})} \leq C \|\partial_z \partial_z^{-1} \xi\|_{H^s(\mathbb{T}_z \times \mathbb{R}^{n-1})} \leq C \|\xi\|_{H^s(\mathbb{T}_z \times \mathbb{R}^{n-1})},$$

which gives us the same regularity for Ψ . □

Approximation of the solutions of the isentropic Navier-Stokes system by the solutions of the NPE equation

By Subsection 2.4.2, this time the approximation domain is $\mathbb{T}_{x_1} \times \mathbb{R}^{n-1}$. Let ξ be a sufficiently regular solution of the Cauchy problem (2.96) for the NPE equation in $\mathbb{T}_z \times \mathbb{R}^{n-1}$. Then, taking ξ and χ according to formulas (2.88)-(2.89), with Ψ defined using the operator ∂_z^{-1} equivalent to ∂_τ^{-1} (see Eq. (2.71)), we define $\bar{\rho}_\varepsilon$ and $\bar{\mathbf{v}}_\varepsilon$ by formulas (2.86)-(2.87). For $\bar{\rho}_\varepsilon$ and $\bar{\mathbf{v}}_\varepsilon$ we obtain a solution of the approximate system (2.29)-(2.30) defined on $\mathbb{R}_+ \times \mathbb{T}_{x_1} \times \mathbb{R}^{n-1}$ with $p(\bar{\rho}_\varepsilon)$ from the state law (2.12), but with the remainder terms R_1^{NS-NPE} and R_2^{NS-NPE} defined respectively in Eqs. (2.90)-(2.92) instead of R_1^{NS-Kuz} and R_2^{NS-Kuz} .

In what follows we consider the three dimensional case, knowing, thanks to the energy method used in Ref. [67] on \mathbb{R}^3 , that the Cauchy problem for the Navier-Stokes system is globally well-posed in $\mathbb{T}_{x_1} \times \mathbb{R}^2$ for sufficiency small initial data (see Ref. [67] Theorem 7.1, p. 100 or Ref. [18]):

Theorem 2.4.8. *There exists a constant $k_1 > 0$ such that if the initial data*

$$\rho_\varepsilon(0) - \rho_0 \in H^3(\mathbb{T}_{x_1} \times \mathbb{R}^2), \quad \mathbf{v}_\varepsilon(0) \in H^3(\mathbb{T}_{x_1} \times \mathbb{R}^2) \quad (2.99)$$

satisfy

$$\|\rho_\varepsilon(0) - \rho_0\|_{H^3(\mathbb{T}_{x_1} \times \mathbb{R}^2)} + \|\mathbf{v}_\varepsilon(0)\|_{H^3(\mathbb{T}_{x_1} \times \mathbb{R}^2)} < k_1,$$

and $\rho_\varepsilon(0) - \rho_0$ and $\mathbf{v}_\varepsilon(0)$ have a zero mean value among x_1 then the Cauchy problem (2.10)-(2.12) on $\mathbb{T}_{x_1} \times \mathbb{R}^2$ with the initial data (2.99) has a unique global in time solution $(\rho_\varepsilon, \mathbf{v}_\varepsilon)$ such that

$$\rho_\varepsilon - \rho_0 \in C([0, +\infty[; H^3(\mathbb{T}_{x_1} \times \mathbb{R}^2)) \cap C^1([0, +\infty[; H^2(\mathbb{T}_{x_1} \times \mathbb{R}^2)), \quad (2.100)$$

which implies

$$\rho_\varepsilon - \rho_0 \in C([0, +\infty[; H^3(\mathbb{T}_{x_1} \times \mathbb{R}^2)) \cap C^1([0, +\infty[; H^1(\mathbb{T}_{x_1} \times \mathbb{R}^2)) \quad (2.101)$$

and

$$\mathbf{v}_\varepsilon \in C([0, +\infty[; H^3(\mathbb{T}_{x_1} \times \mathbb{R}^2)) \cap C^1([0, +\infty[; H^1(\mathbb{T}_{x_1} \times \mathbb{R}^2)). \quad (2.102)$$

Moreover for all time for $\rho_\varepsilon - \rho_0$ and \mathbf{v}_ε have a zero mean value along x_1 .

The existence results for the Cauchy problems of the Navier-Stokes system (2.10)-(2.12) and the NPE equation (2.96) allow us to establish the global existence of \mathbf{U}_ε and $\overline{\mathbf{U}}_\varepsilon$, considered in the NPE approximation framework:

Theorem 2.4.9. *Let $n = 3$. There exists a constant $k > 0$ such that if the initial datum $\xi_0 \in H^5(\mathbb{T}_z \times \mathbb{R}^2)$ for the Cauchy problem for the NPE equation (2.96) (necessarily $k \leq k_2$, see Theorem 2.4.7) is sufficiently small*

$$\|\xi_0\|_{H^5(\mathbb{T}_z \times \mathbb{R}^{n-1})} < k,$$

and has a zero mean value then there exist global in time solutions $\overline{\mathbf{U}}_\varepsilon = (\overline{\rho}_\varepsilon, \overline{\rho}_\varepsilon \overline{\mathbf{v}}_\varepsilon)^t$ of the approximate Navier-Stokes system (2.32) and $\mathbf{U}_\varepsilon = (\rho_\varepsilon, \rho_\varepsilon \mathbf{v}_\varepsilon)^t$ of the exact Navier-Stokes system (2.31) respectively, with the same regularity corresponding to (2.101) and (2.102) and a zero mean value in the x_1 -direction, both considered with the state law (2.12) and with the same initial data

$$(\overline{\rho}_\varepsilon - \rho_\varepsilon)|_{t=0} = 0, \quad (\overline{\mathbf{v}}_\varepsilon - \mathbf{v}_\varepsilon)|_{t=0} = 0, \quad (2.103)$$

where $\overline{\rho}_\varepsilon|_{t=0}$ and $\overline{\mathbf{v}}_\varepsilon|_{t=0}$ are constructed as the functions of the initial datum for NPE equation ξ_0 according to formulas (2.86)–(2.89).

Proof. The proof is essentially the same as for Theorem 2.4.1. According to Theorem 2.4.7 with $s = 5$, the datum ξ_0 is regular enough so that

$$\rho_\varepsilon - \rho_0|_{t=0} \in H^3(\mathbb{T}_{x_1} \times \mathbb{R}^2) \text{ and } \mathbf{v}_\varepsilon|_{t=0} \in [H^3(\mathbb{T}_{x_1} \times \mathbb{R}^2)]^3$$

constructed with the help of formulas (2.86)–(2.89) in order to apply Theorem 2.4.8. These formulas together with Theorem 2.4.7 imply that $\overline{\rho}_\varepsilon$ and $\overline{\mathbf{v}}_\varepsilon$ have the desired regularity. \square

Thanks to Theorem 2.4.9 we validate the approximation of \mathbf{U}_ε by $\overline{\mathbf{U}}_\varepsilon$ following Ref. [81]:

Theorem 2.4.10. *Let $\nu > 0$ and $\epsilon > 0$ be fixed and all assumptions of Theorem 2.4.9 hold. Then estimates of Theorem 2.4.3 hold in $L^2(\mathbb{T}_{x_1} \times \mathbb{R}^2)$.*

The proof being the same as in Theorem 2.4.3 is omitted. In fact it is due to the same Eqs. (2.31) and (2.32) with just different remainders terms of the same order on ε .

It is also easy to see using the previous arguments that the minimum regularity of the initial data (see Table 2.1) to have the remainder terms

$$R_1^{NS-NPE} \text{ and } \mathbf{R}_2^{NS-NPE} \in C([0, +\infty[; L^2(\mathbb{T}_{x_1} \times \mathbb{R}^2))$$

corresponds to $\xi_0 \in H^s(\mathbb{T}_{x_1} \times \mathbb{R}^2)$ with $s \geq 4$ since then for $0 \leq k \leq 2$

$$\xi(\tau, z, y) \in C^k([0, +\infty[; H^{s-2k}(\mathbb{T}_z \times \mathbb{R}^2)),$$

which finally implies with formulas (2.86)–(2.89) that

$$\begin{aligned} \bar{p}_\varepsilon(t, x_1, x') &\in C([0, +\infty[; H^2(\mathbb{T}_{x_1} \times \mathbb{R}^2)) \cap C^1([0, +\infty[; L^2(\mathbb{T}_{x_1} \times \mathbb{R}^2)), \\ \bar{\mathbf{v}}_\varepsilon(t, x_1, x') &\in C([0, +\infty[; H^3(\mathbb{T}_{x_1} \times \mathbb{R}^2)) \cap C^1([0, +\infty[; H^1(\mathbb{T}_{x_1} \times \mathbb{R}^2)). \end{aligned}$$

2.5 Approximation of the Euler system

Let us consider the following isentropic Euler system:

$$\partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon \mathbf{v}_\varepsilon) = 0, \tag{2.104}$$

$$\rho_\varepsilon [\partial_t \mathbf{v}_\varepsilon + (\mathbf{v}_\varepsilon \cdot \nabla) \mathbf{v}_\varepsilon] + \nabla p(\rho_\varepsilon) = 0 \tag{2.105}$$

with $p(\rho_\varepsilon)$ given in Eq. (2.12). We use all notations of Section 2.4 just taking $\nu = 0$.

Let us consider two and three dimensional cases. The entropy η of the isentropic Euler system, defined in Eq. (2.43), is of class C^3 and in addition $\eta''(U_\varepsilon)$ is positive definite for $\rho_\varepsilon > 0$. Moreover, from (2.31) we see that $G_i \in C^\infty$ with respect to U_ε for $\rho_\varepsilon > 0$. Then we can apply Theorem 5.1.1 p. 98 in Ref. [23] which gives us the local well-posedness of the Euler system:

Theorem 2.5.1. [23] *In \mathbb{R}^n for $n = 2$ or 3 , suppose the initial data $\mathbf{U}_\varepsilon(0)$ be continuously differentiable on \mathbb{R}^n , take value in some compact set with $\rho_\varepsilon(0) > 0$, and*

$$\text{for } i = 1, \dots, n, \partial_{x_i} \mathbf{U}_\varepsilon(0) \in [H^s(\mathbb{R}^n)]^{n+1} \text{ with } s > n/2.$$

Then there exists $0 < T_\infty \leq +\infty$, and a unique continuously differentiable function \mathbf{U}_ε on $\mathbb{R}^3 \times [0, T_\infty[$ taking value with $\rho_\varepsilon > 0$, which is a classical solution of the Cauchy problem associated to (2.31) with $\nu = 0$. Furthermore for $i = 1, \dots, n$

$$\partial_{x_i} \mathbf{U}_\varepsilon(t) \in \bigcap_{k=0}^s C^k([0, T_\infty[; [H^{s-k}(\mathbb{R}^n)]^{n+1}).$$

The interval $[0, T_\infty[$ is maximal in that if $T_\infty < +\infty$ then

$$\int_0^{T_\infty} \sup_{i=1, \dots, n} \|\partial_{x_i} \mathbf{U}_\varepsilon\|_{[L^\infty(\mathbb{R}^n)]^{n+1}} dt = +\infty,$$

and/or the range of $\mathbf{U}_\varepsilon(t)$ escapes from every compact subsets of $\mathbb{R}_+^ \times \mathbb{R}^n$ as $t \rightarrow T_\infty$.*

Remark 2.5.1. *A sufficient condition for the initial data to apply Theorem 2.5.1 is to have $\rho_\varepsilon(0) - \rho_0 \in H^3(\mathbb{R}^n)$ and $\mathbf{v}_\varepsilon(0) \in (H^3(\mathbb{R}^n))^n$ with $\rho_\varepsilon(0) > 0$.*

To approximate the solutions of the Euler system and the Kuznetsov, the NPE or the KZK equations, we need to know for which time (how long) they exist. In the difference to the viscous case, the inviscid models can provide blow-up phenomena as indicated in Theorem 2.5.1 for the Euler system, in Theorem 1.4.2 for the Kuznetsov equation and for the KZK and the NPE equations see Theorem 1.3 in Ref. [80]. Let us start by summarizing what is known on the blow-up time for the Euler system [4, 84, 85, 86, 87, 93].

Due to our framework of the non-linear acoustic, it is important for us to have a potential motion (the irrotational case) and to consider the compressible isentropic Euler system (2.104)–(2.105) with initial data defining a perturbation of order ε around the constant state $(\rho_0, 0)$:

Theorem 2.5.2. (Existence time for the Euler system)

1. In \mathbb{R}^n for $n = 2$ or 3 , suppose the initial data

$$\mathbf{U}_\varepsilon(0) = (\rho_{\varepsilon,0}, \rho_{\varepsilon,0} \mathbf{v}_{\varepsilon,0})^t$$

be a perturbation of order ε around the constant state $(\rho_0, 0)$ (see Eq. (2.106)) and take value such that for $i = 1, \dots, n$, $\partial_{x_i} \mathbf{U}_\varepsilon(0) \in [H^s(\mathbb{R}^n)]^{n+1}$ with $s > n/2$. Then according to Theorem 2.5.1 there exists a unique classical solution of the Cauchy problem associated to (2.31) with $\nu = 0$ with a regularity given in Theorem 2.5.1. Moreover considering a generic constant $C > 0$ independent on ε , the existence time T_ε is estimated by $T_\varepsilon \geq \frac{C}{\varepsilon}$.

2. [84, 85, 86, 87] If $\nabla \times \mathbf{v}_{\varepsilon,0} = 0$ and if

$$\left(\frac{\rho_{\varepsilon,0}}{\rho_0} \right)^{\frac{\gamma-1}{2}} - 1 \text{ and } \mathbf{v}_{\varepsilon,0} \text{ belong to the energy space } X^m$$

a dense subspaces of $H^m(\mathbb{R}^n)$ with $m \geq 4$ (for instance they can belong to $\mathcal{D}(\mathbb{R}^n)$, see p.7-8 in Ref. [86] for the exact definition of X^m) then

$$T_\varepsilon \geq \frac{C}{\varepsilon^2} \text{ for } n = 2, \text{ and } T_\varepsilon \geq \exp\left(\frac{C}{\varepsilon}\right) - 1 \text{ for } n = 3.$$

The regularity is given by energy estimates on X^m which implies at least the same regularity as in Theorem 2.5.1 if for $i = 1, \dots, n$, $\partial_{x_i} \mathbf{U}_\varepsilon(0) \in [H^{m-1}(\mathbb{R}^n)]^{n+1}$.

Proof. The first point is a direct consequence of the proof of Theorem 5.1.1 p. 98 in Ref. [23]. For the second point we refer to Refs. [84, 85, 86, 87] in order to have estimations of T_ε with the help of energy estimates in the considered energy spaces which are dense subspaces of the usual Sobolev spaces. □

Let us pay attention to the optimality of the lifespan in the previous results for two [4] and three dimensional cases [93]. The following theorem tells us that the lower bound for the lifespan of the compressible Euler system in the irrotational case found in Theorem 2.5.2 is optimal:

Theorem 2.5.3. (Blow-up for the Euler system)

1. [4] In \mathbb{R}^2 , we consider the initial data given by

$$\rho_\varepsilon(0) = \rho_0 + \varepsilon\rho_{\varepsilon,0} \text{ and } \mathbf{v}_\varepsilon(0) = \varepsilon\mathbf{v}_{\varepsilon,0}, \quad (2.106)$$

with $\rho_{\varepsilon,0}$ and $\mathbf{v}_{\varepsilon,0}$ of regularity C^∞ with a compact support. Moreover

$$\mathbf{v}_{\varepsilon,0}(x) = v_r|x|_2\vec{e}_r + v_\theta|x|_2\vec{e}_\theta,$$

with $\rho_{\varepsilon,0}$, v_r , $v_\theta \in \mathcal{D}(\mathbb{R}^2)$ depending only on $r = |x|_2 = \sqrt{x_1^2 + x_2^2}$ for $x = (x_1, x_2)^t$.

Then the Euler system (2.104)–(2.105) with initial data (2.106) admits a C^∞ solution for $t \in [0, T_\varepsilon[$ with

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 T_\varepsilon = C > 0.$$

2. [93] In \mathbb{R}^3 , we consider the initial data given by (2.106) with $\rho_{\varepsilon,0}$ and $\mathbf{v}_{\varepsilon,0}$ of regularity C^∞ with a compact support. Moreover

$$\mathbf{v}_{\varepsilon,0}(x) = v_r|x|_3\vec{e}_r,$$

with $\rho_{\varepsilon,0}$ and $v_r \in \mathcal{D}(\mathbb{R}^3)$ depending only on $r = |x|_3 = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for $x = (x_1, x_2, x_3)^t$. Then the Euler system (2.104)–(2.105) with initial data (2.106) admits a C^∞ solution for $t \in [0, T_\varepsilon[$ with

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \ln(T_\varepsilon) = C > 0.$$

Now let us consider the derivation of the Kuznetsov equation of Subsection 2.4.1 in the assumption $\nu = 0$. Taking *ansatz* (2.14)–(2.15) for ρ_ε and \mathbf{v}_ε and imposing (2.17)–(2.18) for ρ_1 and ρ_2 with $\nu = 0$, we derive as in Subsection 2.4.1 the inviscid Kuznetsov equation with the notation $\alpha = \frac{\gamma-1}{c^2}$

$$\begin{cases} \partial_t^2 u - c^2 \Delta u = \varepsilon \partial_t \left((\nabla u)^2 + \frac{\alpha}{2} (\partial_t u)^2 \right), \\ u(0) = u_0, \quad u_t(0) = u_1. \end{cases} \quad (2.107)$$

Thanks to Theorem 1.1 in Ref. [26], or Theorem 1.2.1 in Chapter 1 we have a local well posedness result for the inviscid Kuznetsov equation. Ref. [26] or Theorem 1.2.1 in Chapter 1 allow us to give a result on the lower bound of the lifespan T_ε of the Kuznetsov equation. The method is similar to the case of the Euler system (2.104)–(2.105). It is based on the use of a group of linear transformations preserving the equation $u_{tt} - \Delta u = 0$, initially proposed by John [44]. We formulate the lifespan and blow-up time results for the inviscid Kuznetsov equation in Theorem 1.4.2 in Chapter 1

Remark 2.5.2. In \mathbb{R}^2 and \mathbb{R}^3 we see that the lifespan of the inviscid Kuznetsov equation corresponds to the blow-up time estimation for the compressible isentropic Euler system in Theorems 2.5.2 and 2.5.3, a result in accordance with the fact that the inviscid Kuznetsov equation is an approximation of the Euler system. We also notice that in the two cases (for the Euler system and the Kuznetsov equation) having a longer existence time requires more regularity on the initial data.

Theorem 2.5.4. *Let $n = 2$ or 3 . If the initial data $u_0 \in H^4(\mathbb{R}^n)$ and $u_1 \in H^3(\mathbb{R}^n)$ for the Cauchy problem for the inviscid Kuznetsov equation (2.107) satisfy*

$$\|u_0\|_{H^4(\mathbb{R}^n)} + \|u_1\|_{H^3(\mathbb{R}^n)} \leq l$$

with l small enough, there exists $T_\varepsilon^ > 0$ and $C > 0$, independent on ε , satisfying*

$$T_\varepsilon^* \geq \frac{C}{\varepsilon}$$

such that there exist local in time solutions

$$\overline{\mathbf{U}}_\varepsilon = (\overline{\rho}_\varepsilon, \overline{\rho}_\varepsilon \overline{\mathbf{v}}_\varepsilon)^t \text{ and } \mathbf{U}_\varepsilon = (\rho_\varepsilon, \rho_\varepsilon \mathbf{v}_\varepsilon)^t \text{ on } [0, T_\varepsilon^*[$$

of the approximate Euler system given by (2.32) and of the exact Euler system given by (2.31) with $\nu = 0$, both considered with the state law (2.12) and with the same initial data (2.36). In addition, the solutions have the same regularity corresponding to

$$\mathbf{U}_\varepsilon - (\rho_0, 0)^t \in \bigcap_{k=0}^3 C^k([0, T_\varepsilon^*]; [H^{3-k}(\mathbb{R}^n)]^{n+1}). \quad (2.108)$$

Here $\overline{\rho}_\varepsilon|_{t=0}$ and $\overline{\mathbf{v}}_\varepsilon|_{t=0}$ are constructed as the functions of the initial data for the Kuznetsov equation u_0 and u_1 by formulas (2.37)–(2.38) according to (2.14)–(2.15) and (2.17)–(2.18) taken with $\nu = 0$.

Proof. Taking $u_0 \in H^4(\mathbb{R}^n)$ and $u_1 \in H^3(\mathbb{R}^n)$ with $\|u_0\|_{H^4(\mathbb{R}^n)} + \|u_1\|_{H^3(\mathbb{R}^n)} \leq l$ and l small enough, the Cauchy problem for the inviscid Kuznetsov equation (2.107) is locally well-posed according to Theorem 1.4.2. Moreover the solution u belongs to $\bigcap_{k=0}^4 C^k([0, T_{\varepsilon,1}]; H^{4-k}(\mathbb{R}^n))$ with $T_{\varepsilon,1} \geq \frac{C_1}{\varepsilon}$ and $C_1 > 0$ independent of ε .

As $u_0 \in H^4(\mathbb{R}^n)$ and $u_1 \in H^3(\mathbb{R}^n)$, it ensures that

$$\rho_\varepsilon - \rho_0|_{t=0} \in H^3(\mathbb{R}^n) \text{ and } \mathbf{v}_\varepsilon|_{t=0} \in [H^3(\mathbb{R}^n)]^3.$$

Therefore $\rho_\varepsilon|_{t=0} > 0$ if u_0 and u_1 small enough.

By Theorem 2.5.2 it is sufficient to have a local solution \mathbf{U}_ε on $[0, T_{\varepsilon,2}[$ of the exact Euler system (see (2.31) with $\nu = 0$) verifying (2.108) with T_ε^* corresponding to $T_{\varepsilon,2}$, $T_{\varepsilon,2} \geq \frac{C_2}{\varepsilon}$ with $C_2 > 0$ independent on ε .

Now we consider $T_\varepsilon^* = \min(T_{\varepsilon,1}, T_{\varepsilon,2})$, and we have $T_\varepsilon^* \geq \frac{C}{\varepsilon}$ with $C > 0$ independent on ε . As $\overline{\rho}_\varepsilon$ and $\overline{\mathbf{v}}_\varepsilon$ are defined by *ansatz* (2.14)–(2.15) with ρ_1 and ρ_2 given in Eqs. (2.17)–(2.18), the regularity of u implies for $\overline{\mathbf{U}}_\varepsilon$ at least the same regularity as given in (2.108). To find it we use the Sobolev embedding (2.42) for the multiplication. \square

Knowing the existence results for the two problems, we validate the approximation of \mathbf{U}_ε by the solution of the Kuznetsov equation, *i.e.* by $\overline{\mathbf{U}}_\varepsilon$, following Ref. [81].

Theorem 2.5.5. (Approximation of the Euler system by the Kuznetsov equation) *Let $n = 2$ or 3 and $u_0 \in H^4(\mathbb{R}^n)$, $u_1 \in H^3(\mathbb{R}^n)$ be the initial data for the Kuznetsov equation and $\mathbf{U}_\varepsilon(0) = \overline{\mathbf{U}}_\varepsilon(0)$ for the Euler energy respectively. For*

$$\|u_0\|_{H^4(\mathbb{R}^n)} + \|u_1\|_{H^3(\mathbb{R}^n)} \leq l$$

with l small enough, there is the local existence of \mathbf{U}_ε and $\overline{\mathbf{U}}_\varepsilon$ for $t \in [0, T_\varepsilon^[$ with T_ε^* given by Theorem 2.5.4 and the same regularity (2.108). Moreover there exist constants $C > 0$ and $K > 0$ independent on ε and on the time t , such that*

$$\forall t \leq \frac{C}{\varepsilon} \quad \|(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon)(t)\|_{L^2(\mathbb{R}^3)}^2 \leq Kt\varepsilon^3 e^{Ket} \leq 4\varepsilon^2. \quad (2.109)$$

Proof. The local existence of \mathbf{U}_ε and $\overline{\mathbf{U}}_\varepsilon$ comes from Theorem 2.5.4.

We make use of the convex entropy as in Ref. [23] for the isentropic Euler equation and the rest follows exactly as in the proof of Theorem 2.4.3 except that $\nu = 0$.

We finish the proof with the remark on the minimal regularity of the initial data for the Kuznetsov equation such that the approximation is possible, *i.e.* the remainder terms R_1^{NS-Kuz} and \mathbf{R}_2^{NS-Kuz} keep bounded for a finite time interval. Indeed, if $u_0 \in H^{s+2}(\mathbb{R}^n)$ and $u_1 \in H^{s+1}(\mathbb{R}^n)$ with $s > \frac{n}{2}$ then $u \in C([0, T_\varepsilon^*]; H^{s+2}(\mathbb{R}^n))$ and

$$u_t \in C([0, T_\varepsilon^*]; H^{s+1}(\mathbb{R}^n)), \quad u_{tt} \in C([0, T_\varepsilon^*]; H^s(\mathbb{R}^n)).$$

Since $\bar{\rho}_\varepsilon$ is defined by (2.14) with (2.17)–(2.18) and $\bar{\mathbf{v}}_\varepsilon$ by (2.15), with $\nu = 0$, respectively, we exactly find the regularity

$$\begin{aligned} \bar{\rho}_\varepsilon &\in C([0, T_\varepsilon^*]; H^{s+1}(\mathbb{R}^n)) \cap C^1([0, T_\varepsilon^*]; H^s(\mathbb{R}^n)), \\ \bar{\mathbf{v}}_\varepsilon &\in C([0, T_\varepsilon^*]; H^{s+1}(\mathbb{R}^n)) \cap C^1([0, T_\varepsilon^*]; H^s(\mathbb{R}^n)). \end{aligned}$$

Thus by the regularity of the left-hand side part for the approximated Navier-Stokes system (2.29)–(2.30) we obtain the desired regularity for the right-hand side. \square

Theorem 2.5.6. (Approximation of the Euler system by the NPE equation) *Let $n = 2$ or 3 . There exists a constant $k > 0$ such that if the initial datum $\xi_0 \in H^5(\mathbb{T}_z \times \mathbb{R}^2)$ for the Cauchy problem for the NPE equation (2.96) with $\nu = 0$ is sufficiently small*

$$\|\xi_0\|_{H^5(\mathbb{T}_z \times \mathbb{R}^{n-1})} < k\varepsilon,$$

and has a zero mean value then there exist local in time solutions $\overline{\mathbf{U}}_\varepsilon$ of the approximate Euler system (2.32) and \mathbf{U}_ε of the exact Euler system (2.31) with $\nu = 0$ respectively, with the same regularity corresponding to (2.101) and (2.102) on $[0, T_\varepsilon^]$ instead of $[0, +\infty[$ and a zero mean value in the x_1 -direction, both considered with the state law (2.12) and with the same initial data (2.103) where $\bar{\rho}_\varepsilon|_{t=0}$ and $\bar{\mathbf{v}}_\varepsilon|_{t=0}$ are constructed as the functions of the initial datum for NPE equation ξ_0 according to formulas (2.86)–(2.89) with $\nu = 0$. Moreover there exists $C > 0$ independent of ε such that $T_\varepsilon^* > \frac{C}{\varepsilon}$ and for $t \leq \frac{C}{\varepsilon}$ we have inequality (2.109) on $\mathbb{T}_{x_1} \times \mathbb{R}^{n-1}$.*

Proof. The work of Dafermos in Ref. [23] can always be applied on $\mathbb{T}_{x_1} \times \mathbb{R}^{n-1}$ for $n = 2$ or 3 instead of \mathbb{R}^n so we have an equivalent of Theorem 2.5.1 and we also have the same equivalent of Theorem 2.5.2. This is due to the fact that the energy estimate in the articles of Sideris [84, 85, 86, 87] are always true on $\mathbb{T}_{x_1} \times \mathbb{R}$ and $\mathbb{T}_{x_1} \times \mathbb{R}^2$. In all these cases we must also suppose that we have a mean value equal to zero in the direction x_1 . As by Theorem 2.4.4 the NPE equation is locally well posed on $[0, T_\varepsilon]$ with $T_\varepsilon \geq \frac{C}{\varepsilon}$ if $\|\xi_0\|_{H^5(\mathbb{T}_z \times \mathbb{R}^{n-1})} < k\varepsilon$, we have an equivalent of Theorems 2.5.4 and 2.5.5 for the exact compressible isentropic Euler system and its approximation by the NPE equation on $\mathbb{T}_{x_1} \times \mathbb{R}^{n-1}$ for $n = 2$ or 3 as $\xi_0 \in H^5(\mathbb{T}_z \times \mathbb{R}^{n-1})$ also implies $\bar{\rho}_\varepsilon|_{t=0}$ and $\bar{\mathbf{v}}_\varepsilon|_{t=0}$ in $H^3(\mathbb{T}_{x_1} \times \mathbb{R}^{n-1})$.

It is also easy to see using the previous arguments that the minimum regularity of the initial data (see Table 2.1) to have the remainder terms

$$R_1^{NS-NPE} \text{ and } \mathbf{R}_2^{NS-NPE} \in C([0, T_\varepsilon^*]; L^2(\mathbb{T}_{x_1} \times \mathbb{R}^{n-1}))$$

corresponds to $\xi_0 \in H^s(\mathbb{T}_{x_1} \times \mathbb{R}^{n-1})$ with $s \geq 4$ since then for $0 \leq k \leq 2$

$$\xi(\tau, z, y) \in C^k([0, T_\varepsilon^*]; H^{s-2k}(\mathbb{T}_z \times \mathbb{R}^{n-1})),$$

which finally implies with formulas (2.87), (2.86), (2.88) and (2.89) with $\nu = 0$ that

$$\begin{aligned}\bar{\rho}_\varepsilon(t, x_1, x') &\in C([0, T_\varepsilon^*]; H^2(\mathbb{T}_{x_1} \times \mathbb{R}^{n-1})) \cap C^1([0, T_\varepsilon^*]; L^2(\mathbb{T}_{x_1} \times \mathbb{R}^{n-1})), \\ \bar{\mathbf{v}}_\varepsilon(t, x_1, x') &\in C([0, T_\varepsilon^*]; H^3(\mathbb{T}_{x_1} \times \mathbb{R}^{n-1})) \cap C^1([0, T_\varepsilon^*]; H^1(\mathbb{T}_{x_1} \times \mathbb{R}^{n-1})).\end{aligned}$$

□

Remark 2.5.3. *If we allow the Euler system to have not the classical, but an admissible weak solution with the bounded energy (see Definition 2.4.1 and take $\nu = 0$) taking the initial data in a small on ε L^2 -neighbourhood of $\bar{\mathbf{U}}_\varepsilon(0)$, then we also formally have estimate (2.4). But, thanks to Ref. [64] it is known that the Euler system can provide infinitely many admissible weak solutions, and thus there are no sense to approximate them.*

For the approximation by the KZK equation the inviscid case has already been studied in Ref. [81]. The key point is that we must restrict our spatial domain to a cone in order to take into account the fact that the KZK equation is only locally well posed.

Theorem 2.5.7. [81] *Suppose that there exists the solution I of the KZK Cauchy problem (2.72) with $I_0 \in H^s(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$ for $s > \max\{10, \lfloor \frac{n}{2} \rfloor + 1\}$, and $\nu = 0$ such that $I(\tau, z, y)$ is L -periodic with respect to τ and defined for $|z| \leq R$ and $y \in \mathbb{R}_y^{n-1}$. Also we assume*

$$z \mapsto I(\tau, z, y) \in C([-R, R]; H^s(\mathbb{T}_\tau \times \mathbb{R}_y^{n-1})) \cap C^1([-R, R]; H^{s-2}(\mathbb{T}_\tau \times \mathbb{R}_y^{n-1}))$$

(the uniqueness and the existence of such a solution is proved by Theorem 2.4.4).

Let $\bar{\mathbf{U}}_\varepsilon = (\bar{\rho}_\varepsilon, \bar{\rho}_\varepsilon \bar{\mathbf{v}}_\varepsilon)^t$ be the approximate solution of the isentropic Euler system (2.82)–(2.83) with $\nu = 0$ deduced from a solution of the KZK equation. Then the function $\bar{\mathbf{U}}_\varepsilon(t, x_1, x')$ is defined in

$$\mathbb{T}_t \times (\Omega_\varepsilon = \{x_1 | x_1 < \frac{R}{\varepsilon} - ct\} \times \mathbb{R}_{x'}^{n-1})$$

and is smooth enough according to the regularity of I .

Let us now consider the solution \mathbf{U}_ε of the Euler System (2.31) with $\nu = 0$ in a cone

$$C(t) = \{0 < s < t\} \times Q_\varepsilon(s) = \{x = (x_1, x') : |x_1| \leq \frac{R}{\varepsilon} - Ms, M \geq c, x' \in \mathbb{R}^{n-1}\}$$

with the initial data

$$(\rho_\varepsilon - \bar{\rho}_\varepsilon)|_{t=0} = 0, \quad (\mathbf{v}_\varepsilon - \bar{\mathbf{v}}_\varepsilon)|_{t=0} = 0.$$

Consequently, (see Ref. [23] p. 62) there exists T_0 such that for the time interval $0 \leq t \leq \frac{T_0}{\varepsilon}$ there exists the classical solution $\mathbf{U}_\varepsilon = (\rho_\varepsilon, \rho_\varepsilon \mathbf{v}_\varepsilon)$ of the Euler system (2.31) with $\nu = 0$ in a cone

$$C(T) = \{0 < t < T | T < \frac{T_0}{\varepsilon}\} \times Q_\varepsilon(t)$$

with

$$\|\nabla \mathbf{U}_\varepsilon\|_{L^\infty([0, \frac{T_0}{\varepsilon}]; H^{s-1}(Q_\varepsilon))} < \varepsilon C \text{ for } s > \lfloor \frac{n}{2} \rfloor + 1.$$

Moreover, there exists $K > 0$ such that for any ε small enough, the solutions \mathbf{U}_ε and $\bar{\mathbf{U}}_\varepsilon$ which were determined as above in cone $C(T)$ with the same initial data, satisfy the estimate for $0 < t < \frac{T_0}{\varepsilon}$

$$\|(\mathbf{U}_\varepsilon - \bar{\mathbf{U}}_\varepsilon)(t)\|_{L^2(Q_\varepsilon(t))}^2 \leq c_0^2 \varepsilon^3 t e^{2Ket} \leq 4\varepsilon^2$$

with $c_0^2 > 0$.

Remark 2.5.4. The regularity of $I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1})$ with $s > 8$ (see Table 2.1) is minimal to ensure that \mathbf{R}_1^{NS-KZK} and \mathbf{R}_2^{NS-KZK} , see Appendix A, are in $C([0, \frac{T_0}{\varepsilon}]; L^2(Q_\varepsilon))$.

$$I(\tau, z, y) \in C^k([\cdot - R, R]; H^{s-2k}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})).$$

Let us denote $\Omega = \mathbb{T}_\tau \times \mathbb{R}^{n-1}$. Given the equations for $\bar{\rho}_\varepsilon$ by (2.60) with (2.57) and (2.58) and for $\bar{\mathbf{v}}_\varepsilon$ by (2.68) with $\nu = 0$ respectively, we have for $0 \leq k \leq 2$

$$\partial_z^k \bar{\rho}_\varepsilon(\tau, z, y) \in C([\cdot - R, R]; H^{s-2k}(\Omega)), \quad \partial_z^k \bar{\mathbf{v}}_\varepsilon(\tau, z, y) \in C([\cdot - R, R]; H^{s-2-2k}(\Omega)),$$

but we can also say that

$$\partial_z^k \bar{\rho}_\varepsilon(\tau, z, y) \in L^2([\cdot - R, R]; H^{s-2k}(\Omega)), \quad \partial_z^k \bar{\mathbf{v}}_\varepsilon(\tau, z, y) \in L^2([\cdot - R, R]; H^{s-2-2k}(\Omega)).$$

This implies for $0 \leq k \leq 2$ (as $s > 8$) that $s - 2 - 2k > 2$ and

$$\begin{aligned} \partial_z^k \bar{\rho}_\varepsilon(\tau, z, y) &\in C(\mathbb{T}_\tau; L^2(\{x_1 | x_1 < \frac{R}{\varepsilon} - ct\}; H^{s-2k}(\mathbb{R}^{n-1}))), \\ \partial_z^k \bar{\mathbf{v}}_\varepsilon(\tau, z, y) &\in C(\mathbb{T}_\tau; L^2(\{x_1 | x_1 < \frac{R}{\varepsilon} - ct\}; H^{s-2-2k}(\mathbb{R}^{n-1}))). \end{aligned}$$

Hence we find

$$\bar{\rho}_\varepsilon(t, x_1, x'), \quad \bar{\mathbf{v}}_\varepsilon(t, x_1, x') \in C([0, \frac{T_0}{\varepsilon}]; H^2(Q_\varepsilon)).$$

As in addition for $0 \leq k \leq 1$, considering $\bar{\rho}_\varepsilon$ and $\bar{\mathbf{v}}_\varepsilon$ as functions of (τ, z, y) ,

$$\partial_z^k \partial_\tau \bar{\rho}_\varepsilon \in C([\cdot - R, R]; H^{s-1-2k}(\Omega)), \quad \partial_z^k \partial_\tau \bar{\mathbf{v}}_\varepsilon \in C([\cdot - R, R]; H^{s-3-2k}(\Omega)),$$

we deduce in the same way that

$$\partial_t \bar{\rho}_\varepsilon(t, x_1, x'), \quad \partial_t \bar{\mathbf{v}}_\varepsilon(t, x_1, x') \in C([0, \frac{T_0}{\varepsilon}]; H^1(Q_\varepsilon)).$$

These regularities of $\bar{\rho}_\varepsilon$ and $\bar{\mathbf{v}}_\varepsilon$ viewed as functions of (t, x_1, x') allow to have all left-hand terms in the approximated Euler system (2.82)–(2.83) with $\nu = 0$ of the regularity $C([0, \frac{T_0}{\varepsilon}]; L^2(Q_\varepsilon))$ and the remainder terms in the right-hand side inherit it.

2.6 The Kuznetsov equation and the KZK equation

2.6.1 Derivation of the KZK equation from the Kuznetsov equation

If the velocity potential is given [60] by Eq. (2.55), we directly obtain from the Kuznetsov equation (2.23) with the paraxial change of variable (2.54) that

$$\begin{aligned} &\partial_t^2 u - c^2 \Delta u - \varepsilon \partial_t \left((\nabla u)^2 + \frac{\gamma - 1}{2c^2} (\partial_t u)^2 + \frac{\nu}{\rho_0} \Delta u \right) \\ &= \varepsilon \left[2c \partial_{\tau z}^2 \Phi - \frac{\gamma + 1}{2c^2} \partial_\tau (\partial_\tau \Phi)^2 - \frac{\nu}{\rho_0 c^2} \partial_\tau^3 \Phi - c^2 \Delta_y \Phi \right] + \varepsilon^2 R^{Kuz-KZK} \end{aligned} \quad (2.110)$$

with

$$\begin{aligned} \varepsilon^2 R^{Kuz-KZK} = & \varepsilon^2 \left(-c^2 \partial_z^2 \Phi + \frac{2}{c} \partial_\tau (\partial_\tau \Phi \partial_z \Phi) - \partial_\tau (\nabla_y \Phi)^2 + \frac{2\nu}{c\rho_0} \partial_\tau^2 \partial_z \Phi - \frac{\nu}{\rho_0} \partial_\tau \Delta_y \Phi \right) \\ & + \varepsilon^3 \left(-\partial_\tau (\partial_z \Phi)^2 - \frac{\nu}{\rho_0} \partial_\tau \partial_z^2 \Phi \right). \end{aligned} \quad (2.111)$$

Therefore, we find that the right-hand side ε -order term in Eq. (2.110) is exactly the KZK equation (2.63). Due to its well posedness domain, to validate the approximation between the solutions of the KZK and the Kuznetsov equations, we need to study the well posedness of the Kuznetsov equation on the half space with boundary conditions coming from the initial condition for the KZK equation.

2.6.2 Well posedness of the models

Well posedness of the Kuznetsov equation in the half space with periodic boundary conditions

Let us consider the following periodic in time problem for the Kuznetsov equation in the half space $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ with periodic in time Dirichlet boundary conditions:

$$\begin{cases} u_{tt} - c^2 \Delta u - \nu \varepsilon \Delta u_t = \alpha \varepsilon u_t u_{tt} + \beta \varepsilon \nabla u \cdot \nabla u_t & \text{on } \mathbb{T}_t \times \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\ u|_{x_1=0} = g & \text{on } \mathbb{T}_t \times \mathbb{R}^{n-1}, \end{cases} \quad (2.112)$$

where g is a L -periodic in time and of mean value zero function. For this we use Ref. [21] and thus we directly obtain the following result of maximal regularity:

Theorem 2.6.1. [21] *Let $n = 3$, $\Omega = \mathbb{R}_+ \times \mathbb{R}^{n-1}$ and $p \in]1, +\infty[$. Then there exists a unique solution $u \in W_p^2(\mathbb{T}_t; L^p(\Omega)) \cap W_p^1(\mathbb{T}_t; W_p^2(\Omega))$ with the mean value zero*

$$\forall x \in \Omega \quad \int_{\mathbb{T}_t} u(s, x) ds = 0 \quad (2.113)$$

of the following system

$$\begin{cases} u_{tt} - c^2 \Delta u - \nu \varepsilon \Delta u_t = f & \text{on } \mathbb{T}_t \times \Omega, \\ u = g & \text{on } \mathbb{T}_t \times \partial\Omega \end{cases} \quad (2.114)$$

if and only if the functions

$$f \in L^p(\mathbb{T}_t; L^p(\Omega)) \text{ with } g \in W_p^{2-\frac{1}{2p}}(\mathbb{T}_t; L^p(\partial\Omega)) \cap W_p^1(\mathbb{T}_t; W_p^{2-\frac{1}{p}}(\partial\Omega)) \quad (2.115)$$

and they are of mean value zero:

$$\forall x \in \Omega \quad \int_{\mathbb{T}_t} f(l, x) dl = 0 \text{ and } \forall x' \in \partial\Omega \quad \int_{\mathbb{T}_t} g(l, x') dl = 0. \quad (2.116)$$

Moreover, we have the following stability estimate

$$\begin{aligned} \|u\|_{W_p^2(\mathbb{T}_t; L^p(\Omega)) \cap W_p^1(\mathbb{T}_t; W_p^2(\Omega))} \leq & C \left(\|f\|_{L^p(\mathbb{T}_t; L^p(\Omega))} \right. \\ & \left. + \|g\|_{W_p^{2-\frac{1}{2p}}(\mathbb{T}_t; L^p(\partial\Omega)) \cap W_p^1(\mathbb{T}_t; W_p^{2-\frac{1}{p}}(\partial\Omega))} \right). \end{aligned}$$

Proof. On one hand, if f and g satisfy (2.115)–(2.116), the necessity of the conditions is shown in Ref. [21]. On the other hand, the conditions (2.115)–(2.116) are sufficient by a direct application of the trace theorems recalled in Ref. [21] and proved in Ref. [27] for example. \square

The results of Ref. [21] allow to see that Theorem 2.6.1 does not depend on n , moreover if we look at the case $p = 2$ the linearity of the operator $\partial_t^2 - c^2\Delta - \nu\Delta\partial_t$ from (2.114) implies that we can work with $H^s(\Omega)$ instead of $L^2(\Omega)$:

Lemma 2.6.1. *Let $n \in \mathbb{N}^*$, $\Omega = \mathbb{R}_+ \times \mathbb{R}^{n-1}$, $s \geq 0$ then there exists a unique solution*

$$u \in X = \left\{ u \in H^2(\mathbb{T}_t; H^s(\Omega)) \cap H^1(\mathbb{T}_t; H^{s+2}(\Omega)) \mid \forall x \in \Omega \int_{\mathbb{T}_t} u(s, x) ds = 0 \right\} \quad (2.117)$$

with the mean value zero (see Eq. (2.113)) of system (2.114) if and only if

$$f \in L^2(\mathbb{T}_t; H^s(\Omega)) \text{ and } g \in \mathbb{F}_{\mathbb{T}} = H^{\frac{7}{4}}(\mathbb{T}_t; H^s(\partial\Omega)) \cap H^1(\mathbb{T}_t; H^{s+\frac{3}{2}}(\partial\Omega)) \quad (2.118)$$

both satisfying (2.116).

Moreover we have the following stability estimate

$$\|u\|_X \leq C(\|f\|_{L^2(\mathbb{T}_t; H^s(\Omega))} + \|g\|_{\mathbb{F}_{\mathbb{T}}}).$$

Here $H^2(\mathbb{T}_t; H^s(\Omega)) \cap H^1(\mathbb{T}_t; H^{s+2}(\Omega))$ is endowed with its usual norm denoted here and in the sequel by $\|\cdot\|_X$.

Now we can use the maximal regularity result for system (2.114) with Theorem 1.5.2 in Chapter 1 and the same method as for the Cauchy problem associated with the Kuznetsov equation used in our previous work [26] or in Subsection 1.5.1 of Chapter 1. We will just have to use the boundary conditions of problem (2.112) as the initial condition of the corresponding Cauchy problem in \mathbb{R}^n .

Theorem 2.6.2. *Let $\nu > 0$, $n \in \mathbb{N}^*$, $\Omega = \mathbb{R}_+ \times \mathbb{R}^{n-1}$, $s > \frac{n}{2}$. Let X be defined by (2.117) and the boundary condition $g \in \mathbb{F}_{\mathbb{T}}$ be defined by (2.118) and in addition, let g be of the mean value zero (see Eq. (2.116)).*

Then there exist $r^ = O(1)$ and $C_1 = O(1)$ such that for all $r \in [0, r^*[$, if*

$$\|g\|_{\mathbb{F}_{\mathbb{T}}} \leq \frac{\sqrt{\nu\varepsilon}}{C_1}r,$$

there exists a unique solution $u \in X$ of the periodic problem (2.112) satisfying (2.113) and such that $\|u\|_X \leq 2r$.

Proof. For $g \in \mathbb{F}_{\mathbb{T}}$ defined in (2.118) and satisfying (2.116), let us denote by $u^* \in X$ the unique solution of the linear problem (2.114) with $f = 0$ and $g \in \mathbb{F}_{\mathbb{T}}$.

In addition, according to Theorem 2.6.1, we take X defined in (2.117), this time for $s > \frac{n}{2}$ (we need it to control the non-linear terms), and introduce the Banach spaces

$$X_0 := \{u \in X \mid u|_{\partial\Omega} = 0 \text{ on } \mathbb{T}_t \times \partial\Omega\} \quad (2.119)$$

and

$$Y = \left\{ f \in L^2(\mathbb{T}_t; H^s(\Omega)) \mid \forall x \in \Omega \int_{\mathbb{T}_t} f(s, x) ds = 0 \right\}.$$

Hence, for all $f \in Y$

$$P_{LU_{X_0}}(f) \leq C_\epsilon P_{U_Y}(f) = C_\epsilon \|f\|_Y.$$

Then we find for w and z in X_0 , such that $\|w\|_X \leq r$, $\|z\|_X \leq r$, and also with $\|u^*\|_X \leq r$, that with $\Theta(r) := 4C_\epsilon(\alpha + \beta)C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)}\varepsilon r$ it holds

$$P_{LU_{X_0}}(\Phi(w) - \Phi(z)) \leq \Theta(r)\|w - z\|_X.$$

Thus we apply Theorem 1.5.2 in Chapter 1 with $f(x) = L(x) - \Phi(x)$ and $x_0 = 0$. Therefore, knowing that $C_\epsilon = \frac{C_0}{\varepsilon\nu}$, we have, that for all $r \in [0, r_*[$ with

$$r_* = \frac{\nu}{4C_0(\alpha + \beta)C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)}} = O(1), \quad (2.121)$$

for all $y \in \Phi(0) + w(r)LU_{X_0} \subset Y$ with

$$w(r) = r - 2\frac{C_0}{\nu}C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)}(\alpha + \beta)r^2,$$

there exists a unique $v \in 0 + rU_{X_0}$ such that $L(v) - \Phi(v) = y$. But, if we want that v be the solution of the non-linear problem (2.120), then we need to impose $y = 0$ and thus, to ensure that $0 \in \Phi(0) + w(r)LU_{X_0}$. Since $-\frac{1}{w(r)}\Phi(0)$ is an element of Y and $LU_{X_0} = Y$, there exists a unique $z \in X_0$ such that

$$Lz = -\frac{1}{w(r)}\Phi(0). \quad (2.122)$$

Let us show that $\|z\|_X \leq 1$, what will implies that $0 \in \Phi(0) + w(r)LU_{X_0}$. Noticing that

$$\begin{aligned} \|\Phi(0)\|_Y &\leq \alpha\varepsilon\|v_t v_{tt}\|_Y + \beta\varepsilon\|\nabla v \nabla v_t\|_Y \\ &\leq (\alpha + \beta)\varepsilon C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)}\|v\|_X^2 \\ &\leq (\alpha + \beta)\varepsilon C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)}r^2 \end{aligned}$$

and using (2.122), we find

$$\begin{aligned} \|z\|_X &\leq C_\epsilon \|Lz\|_Y = C_\epsilon \frac{\|\Phi(0)\|_Y}{w(r)} \\ &\leq \frac{C_\epsilon C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)}(\alpha + \beta)\varepsilon r}{(1 - 2C_\epsilon C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)}(\alpha + \beta)\varepsilon r)} < \frac{1}{2}, \end{aligned}$$

as soon as $r < r^*$.

Consequently, $z \in U_{X_0}$ and $\Phi(0) + w(r)Lz = 0$. Then we conclude that for all $r \in [0, r_*[$, if $\|u^*\|_X \leq r$, there exists a unique $v \in rU_{X_0}$ such that $L(v) - \Phi(v) = 0$, *i.e.* the solution of the non-linear problem (2.120). Thanks to the maximal regularity and a priori estimate following from Theorem 2.6.1 with $f = 0$, there exists a constant $C_1 = O(\varepsilon^0) > 0$, such that

$$\|u^*\|_X \leq \frac{C_1}{\sqrt{\nu\varepsilon}}\|g\|_{\mathbb{F}_T}.$$

Thus, for all $r \in [0, r_*[$ and $\|g\|_{\mathbb{F}_T} \leq \frac{\sqrt{\nu\varepsilon}}{C_1}r$, the function $u = u^* + v \in X$ is the unique solution of the time periodic problem for the Kuznetsov equation and $\|u\|_X \leq 2r$. \square

Well posedness of the initial boundary value problem in the half space for the Kuznetsov equation

We work on $\Omega = \mathbb{R}_+ \times \mathbb{R}^{n-1}$ and we are going to study the initial boundary value problem for the Kuznetsov equation on this space, *i.e.* the perturbation of an imposed initial condition by a source on the boundary, which will later be determined by the solution of the KZK equation.

Lemma 2.6.2. *Let $s \geq 0$, $n \in \mathbb{N}$. There exists a unique solution*

$$u \in \mathbb{E} := H^2(\mathbb{R}_+; H^s(\Omega)) \cap H^1(\mathbb{R}_+; H^{s+2}(\Omega)) \quad (2.123)$$

of the linear problem

$$\begin{cases} u_{tt} - c^2 \Delta u - \nu \varepsilon \Delta u_t = f & \text{in } \mathbb{R}_+ \times \Omega, \\ u = g & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ u(0) = u_0, \quad u_t(0) = u_1 & \text{in } \Omega \end{cases} \quad (2.124)$$

if and only if the data satisfy the following conditions

- $f \in L^2(\mathbb{R}_+; H^s(\Omega))$,
- for the boundary condition

$$g \in \mathbb{F}_{\mathbb{R}_+} = H^{7/4}(\mathbb{R}_+; H^s(\partial\Omega)) \cap H^1(\mathbb{R}_+; H^{s+3/2}(\partial\Omega)); \quad (2.125)$$

- $u_0 \in H^{s+2}(\Omega)$ and $u_1 \in H^{s+1}(\Omega)$;
- $g(0) = u_0$ and $g_t(0) = u_1$ on $\partial\Omega$ in the trace sense.

In addition, the solution satisfies the stability estimate

$$\|u\|_{\mathbb{E}} \leq C(\|f\|_{L^2(\mathbb{R}_+; H^s(\Omega))} + \|g\|_{\mathbb{F}_{\mathbb{R}_+}} + \|u_0\|_{H^{s+2}} + \|u_1\|_{H^{s+1}}).$$

In order to prove this result we will use the subsequent lemma to remove the inhomogeneity g .

Lemma 2.6.3. *Let $s \geq 0$, $n \in \mathbb{N}$. There exists a unique solution $w \in \mathbb{E}$ defined by (2.123) of the following linear problem*

$$\begin{cases} w_{tt} - \nu \varepsilon \Delta w_t = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ w = g & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ w(0) = 0, \quad w_t(0) = 0 & \text{in } \Omega \end{cases} \quad (2.126)$$

if and only if the data satisfy the following conditions

- $g \in \mathbb{F}_{\mathbb{R}_+}$ defined in (2.125),
- for the compatibility: for all $x \in \partial\Omega$, $g(0) = 0$ and $g_t(0) = 0$.

Moreover, the solution w satisfies the stability estimate

$$\|w\|_{\mathbb{E}} \leq C\|g\|_{\mathbb{F}_{\mathbb{R}_+}}.$$

Proof. First we prove the sufficiency. By assumption (2.125), we have

$$\partial_t g \in H^{3/4}(\mathbb{R}_+; H^s(\partial\Omega)) \cap L^2(\mathbb{R}_+; H^{s+3/2}(\partial\Omega)).$$

Thanks to § 3 p. 288 in Ref. [61], we obtain a unique solution

$$v \in H^1(\mathbb{R}_+; H^s(\Omega)) \cap L^2(\mathbb{R}_+; H^{s+2}(\Omega))$$

of the parabolic problem

$$v_t - \nu\varepsilon\Delta v = 0 \text{ in } \mathbb{R}_+ \times \Omega, \quad v = \partial_t g \text{ on } \mathbb{R}_+ \times \partial\Omega, \quad v(0) = 0 \text{ in } \Omega.$$

Next we define for $t \in \mathbb{R}_+$ and $x \in \Omega$ the function

$$w(t, x) := \int_0^t v(l, x) dl.$$

We have $w(0) = 0$ and $w_t(0) = 0$. Moreover, it satisfies

$$w_{tt} - \nu\varepsilon\Delta w_t = 0, \quad w(t)|_{\partial\Omega} = \int_0^t g_t(l) dl = g(t),$$

as $g(0) = 0$. Therefore, w is a solution of problem (2.126). The necessity follows from the spatial trace theorem ensuring that the trace operator $Tr_{\partial\Omega} : u \mapsto u|_{\partial\Omega}$, considering as a map

$$H^1(\mathbb{R}_+; H^s(\Omega)) \cap L^2(\mathbb{R}_+; H^{s+2}(\Omega)) \rightarrow H^{3/4}(\mathbb{R}_+; H^s(\partial\Omega)) \cap L^2(\mathbb{R}_+; H^{s+3/2}(\partial\Omega)), \quad (2.127)$$

is bounded and surjective by Lemma 3.5 in Ref. [27]. For the compatibility condition, thanks to Lemma 11 in Ref. [29], we also know that the temporal trace $Tr_{t=0} : g \mapsto g|_{t=0}$, considered as a map

$$H^{3/4}(\mathbb{R}_+; H^s(\partial\Omega)) \cap L^2(\mathbb{R}_+; H^{s+3/2}(\partial\Omega)) \rightarrow H^{s+1/2}(\partial\Omega) \quad (2.128)$$

is well defined and bounded. Moreover, the spatial trace

$$H^{s+1/2}(\Omega) \rightarrow H^s(\partial\Omega) \quad (2.129)$$

is bounded by Theorem 1.5.1.1 from Ref. [34].

To obtain uniqueness, let w be a solution to (2.126) with $g = 0$. Since w_t solve a heat problem with homogeneous data, we obtain $w_t = 0$ and therefore also $w = 0$ by the initial condition $w(0) = 0$. The stability estimate follows from the closed graph theorem. \square

Let us prove Lemma 2.6.2:

Proof. We obtain the uniqueness for (2.124) from the fact that in the case $g = 0$ we can consider $-\Delta$ as a self-adjoint and non negative operator with homogeneous Dirichlet boundary conditions and we can use Ref. [32]. To verify the necessity of the conditions on the data, we suppose that $u \in \mathbb{E}$ defined in (2.123) is a solution of (2.124). Then

$$u, u_t \in H^1(\mathbb{R}_+; H^s(\Omega)) \cap L^2(\mathbb{R}_+; H^{s+2}(\Omega)) \text{ and thus } f \in L^2(\mathbb{R}_+; H^s(\Omega)).$$

Taking as in the previous proof the spatial trace $Tr_{\partial\Omega}$ as in (2.127) we have

$$g, g_t \in H^{3/4}(\mathbb{R}_+; H^s(\partial\Omega)) \cap L^2(\mathbb{R}_+; H^{s+3/2}(\partial\Omega)), \text{ which implies } g \in \mathbb{F}_{\mathbb{R}_+}.$$

By the Sobolev embedding $H^1(\mathbb{R}_+; H^{s+2}(\Omega)) \hookrightarrow C(\mathbb{R}_+; H^{s+2}(\Omega))$, it follows that $u_0 \in H^{s+2}(\Omega)$ and we also have the temporal trace

$$u \mapsto u|_{t=0} : H^1(\mathbb{R}_+; H^s(\Omega)) \cap L^2(\mathbb{R}_+; H^{s+2}(\Omega)) \rightarrow H^{s+1}(\Omega)$$

by Lemma 3.7 in Ref. [27]. For the compatibility condition we use (2.128) and (2.129) as in the proof of Lemma 2.6.3.

It remains to prove the sufficiency of the conditions. We extend u_0 , u_1 and f in odd functions among x_1 on \mathbb{R}^n so that we have $\tilde{u}_0 \in H^{s+2}(\mathbb{R}^n)$, $\tilde{u}_1 \in H^{s+1}(\mathbb{R}^n)$ and $\tilde{f} \in L^2(\mathbb{R}_+; H^s(\mathbb{R}^n))$. We consider the problem

$$\begin{cases} \tilde{u}_{tt} - c^2 \Delta \tilde{u} - \nu \varepsilon \Delta \tilde{u}_t = \tilde{f} & \text{in } \mathbb{R}_+ \times \mathbb{R}^n, \\ \tilde{u}(0) = \tilde{u}_0, \quad \tilde{u}_t(0) = \tilde{u}_1 & \text{in } \mathbb{R}^n. \end{cases}$$

By Theorem 4.1 in Ref.[26] or Theorem 1.5.1 in Chapter 1 we obtain the existence of its unique solution

$$\tilde{u} \in H^2(\mathbb{R}_+; H^s(\mathbb{R}^n)) \cap H^1(\mathbb{R}_+; H^{s+2}(\mathbb{R}^n)).$$

Let $\bar{u} \in \mathbb{E}$, defined in (2.123), denote the restriction of \tilde{u} to Ω and let $\bar{g} := g - \bar{u}|_{\partial\Omega}$. By the spatial trace theorem $\bar{u}|_{\partial\Omega} \in \mathbb{F}_{\mathbb{R}_+}$, and hence $\bar{g} \in \mathbb{F}_{\mathbb{R}_+}$. Then the solution u of the non homogeneous linear problem (2.124) is given by $u = v + \bar{u}$, where v solves problem (2.124) with $f = u_0 = u_1 = 0$ and $g = \bar{g}$. From Lemma 2.6.3 we have a unique solution $\bar{v} \in \mathbb{E}_u$ of the problem (2.126) with $g = \bar{g}$. Then the function $w := v - \bar{v}$ solves the following system

$$\begin{cases} w_{tt} - \Delta w - \nu \varepsilon \Delta w_t = c^2 \Delta \bar{v} & \text{in } \mathbb{R}_+ \times \Omega, \\ w = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ w(0) = 0, \quad w_t(0) = 0 & \text{in } \Omega, \end{cases}$$

which thanks to Theorem 2.6 in Ref. [32] has a unique solution $w \in \mathbb{E}$ defined in (2.123). The function $u := w + \bar{v} + \bar{u}$ is the desired solution of (2.124) and the stability estimate follows from the closed graph theorem. This concludes the proof of Lemma 2.6.2. \square

The next theorem follows from the maximal regularity result and Theorem 1.5.2 in Chapter 1. Its proof is similar to the proof of Theorem 2.6.2 and hence is omitted.

Theorem 2.6.3. *Let $\nu > 0$, $n \in \mathbb{N}^*$, $\Omega = \mathbb{R}_+ \times \mathbb{R}^{n-1}$ and $s > \frac{n}{2}$. Considering the initial boundary value problem for the Kuznetsov equation in the half space with the Dirichlet boundary condition*

$$\begin{cases} u_{tt} - c^2 \Delta u - \nu \varepsilon \Delta u_t = \alpha \varepsilon u_t u_{tt} + \beta \varepsilon \nabla u \nabla u_t & \text{in } [0, +\infty[\times \Omega, \\ u = g & \text{on } [0, \infty[\times \partial\Omega, \\ u(0) = u_0, \quad u_t(0) = u_1 & \text{in } \Omega, \end{cases} \quad (2.130)$$

the following results hold: there exists constants $r^ = O(1)$ and $C_1 = O(1)$, such that for all initial data satisfying*

- $g \in \mathbb{F}_{\mathbb{R}_+} := H^{7/4}([0, \infty[; H^s(\partial\Omega)) \cap H^1([0, \infty[; H^{s+3/2}(\partial\Omega))$,
- $u_0 \in H^{s+2}(\Omega)$, $u_1 \in H^{s+1}(\Omega)$,
- $g(0) = u_0|_{\partial\Omega}$ and $g_t(0) = u_1|_{\partial\Omega}$,

and such that for $r \in [0, r^*[$

$$\|u_0\|_{H^{s+2}(\Omega)} + \|u_1\|_{H^{s+1}(\Omega)} + \|g\|_{\mathbb{F}_{[0,T]}} \leq \frac{\nu\varepsilon}{C_1}r,$$

there exists a unique solution of problem (2.130)

$$u \in H^2([0, \infty[; H^s(\Omega)) \cap H^1([0, \infty[; H^{s+2}(\Omega)),$$

such that

$$\|u\|_{H^2([0, \infty[; H^s(\Omega)) \cap H^1([0, \infty[; H^{s+2}(\Omega))} \leq 2r.$$

2.6.3 Approximation of the solutions of the Kuznetsov equation by the solutions of the KZK equation

Given Theorem 2.6.2 for the viscous case, we consider the Cauchy problem associated to the KZK equation (2.72) for small enough initial data in order to have a time periodic solution I defined on $\mathbb{R}_+ \times \mathbb{R}^{n-1}$. If $\nu > 0$, to compare the solutions of the Kuznetsov and the KZK equations we consider two cases. The first case is considered in Sub-subsection 2.6.3, when the Kuznetsov equation can be considered as a time periodic boundary problem coming just from the initial condition I_0 of problem (2.72). In Sub-subsection 2.6.3 we study the second case, when the solution of the KZK equation taken for $\tau = 0$ gives $I(0, z, y)$ defined on $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ from which we deduce according to the derivation *ansatz* both an initial condition for the Kuznetsov equation at $t = 0$ and a corresponding boundary condition. In this second situation, it also makes sense to consider the inviscid case, briefly commented in the end of Sub-subsection 2.6.3.

Approximation problem for the Kuznetsov equation with periodic boundary conditions

Let $\Omega_1 = \mathbb{T}_\tau \times \mathbb{R}^{n-1}$ and $s \geq \left[\frac{n}{2}\right] + 1$. Suppose that a function $I_0(t, y) = I_0(t, \sqrt{\varepsilon}x')$ is such that $I_0 \in H^s(\Omega_1)$ small enough and $\int_{\mathbb{T}_\tau} I_0(s, y) ds = 0$. Then by Theorem 2.4.4 there is a unique solution $I(\tau, z, y)$ of the Cauchy problem for the KZK equation (2.72) such that

$$z \mapsto I(\tau, z, y) \in C([0, \infty[, H^s(\Omega_1)) \tag{2.131}$$

with $\int_{\mathbb{T}_\tau} I(l, z, y) dl = 0$. We use the operator ∂_τ^{-1} defined in (2.71). Formula (2.71), which implies that $\partial_\tau^{-1}I$ is L -periodic in τ and of mean value zero, gives us the estimate

$$\|\partial_\tau^{-1}I\|_{H^s(\Omega_1)} \leq C\|\partial_\tau \partial_\tau^{-1}I\|_{H^s(\Omega_1)} = C\|I\|_{H^s(\Omega_1)}.$$

So $\partial_\tau^{-1}I|_{z=0} \in H^s(\Omega_1)$, and hence by (2.131)

$$z \mapsto \partial_\tau^{-1}I(\tau, z, y) \in C([0, \infty[, H^s(\Omega_1)),$$

with $\int_{\mathbb{T}_\tau} \partial_\tau^{-1}I(s, z, y) ds = 0$.

We define on $\mathbb{T}_t \times \mathbb{R}_+ \times \mathbb{R}^{n-1}$

$$\bar{u}(t, x_1, x') := \frac{c^2}{\rho_0} \partial_\tau^{-1}I(\tau, z, y) = \frac{c^2}{\rho_0} \partial_\tau^{-1}I\left(t - \frac{x_1}{c}, \varepsilon x_1, \sqrt{\varepsilon}x'\right) \tag{2.132}$$

with the paraxial change of variable (2.54) associated to the KZK equation. Thus \bar{u} is L -periodic in time and of mean value zero. Now we consider the Kuznetsov problem (2.112) associated to the following boundary condition, imposed by the initial condition for the KZK equation:

$$g(t, x') := \bar{u}(t, 0, x') = \frac{c^2}{\rho_0} \partial_\tau^{-1} I_0(\tau, y). \quad (2.133)$$

Taking $\tilde{I} := \frac{\rho_0}{c^2} \partial_\tau \Phi$ (see Eq. (2.57)), let \tilde{I} be the solution of the Kuznetsov equation written in the following form with the remainder $R^{Kuz-KZK}$ defined in Eq. (2.111):

$$\begin{cases} c \partial_z \tilde{I} - \frac{(\gamma+1)}{4\rho_0} \partial_\tau \tilde{I}^2 - \frac{\nu}{2c^2 \rho_0} \partial_\tau^2 \tilde{I} - \frac{c^2}{2} \Delta_y \partial_\tau^{-1} \tilde{I} + \varepsilon \frac{\rho_0}{2c^2} R^{Kuz-KZK} = 0, \\ \tilde{I}|_{z=0} = I_0, \end{cases} \quad (2.134)$$

where we can recognize the system associated to the KZK equation (2.72).

Now we can formulate the following approximation result

Theorem 2.6.4. *Let $\nu > 0$. For $s > \frac{n}{2} + 2$ and $I_0 \in H^{s+\frac{3}{2}}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$ small enough in $H^{s+\frac{3}{2}}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$, there exists a unique global solution I of the Cauchy problem for the KZK equation (2.72) such that*

$$z \mapsto I(\tau, z, y) \in C([0, \infty[, H^{s+\frac{3}{2}}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})).$$

In addition, there exists a unique global solution \tilde{I} of the Kuznetsov problem (2.134), in the sense $\tilde{I} := \frac{\rho_0}{c^2} \partial_\tau \Phi$, with $\Phi(\tau, z, y) := u(t, x_1, x')$ with the paraxial change of variable (2.54) and

$$u \in H^2(\mathbb{T}_t; H^s(\mathbb{R}^+ \times \mathbb{R}^{n-1})) \cap H^1(\mathbb{T}_t; H^{s+2}(\mathbb{R}^+ \times \mathbb{R}^{n-1})),$$

is the global solution of the periodic problem (2.112) for the Kuznetsov equation with g defined by I_0 as in Eq. (2.133). Moreover there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\frac{1}{2} \frac{d}{dz} \|I - \tilde{I}\|_{L^2(\mathbb{T}_\tau \times \mathbb{R}^{n-1})}^2 \leq C_1 \|I - \tilde{I}\|_{L^2(\mathbb{T}_\tau \times \mathbb{R}^{n-1})}^2 + C_2 \varepsilon \|I - \tilde{I}\|_{L^2(\mathbb{T}_\tau \times \mathbb{R}^{n-1})},$$

which implies

$$\|I - \tilde{I}\|_{L^2(\mathbb{T}_\tau \times \mathbb{R}^{n-1})}(z) \leq \frac{C_2}{2} \varepsilon z e^{\frac{C_1}{2} z} \leq \frac{C_2}{C_1} \varepsilon (e^{\frac{C_1}{2} z} - 1)$$

and $\|I - \tilde{I}\|_{L^2(\mathbb{T}_\tau \times \mathbb{R}^{n-1})}(z) \leq K\varepsilon$ while $z \leq C$ with $K > 0$, and $C > 0$ independent of ε .

Proof. For $s > \frac{n}{2} + 2$, the global well-posedness of I comes from Theorem 2.4.4 if $I_0 \in H^{s+\frac{3}{2}}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$ is small enough. Moreover, since g is given by Eq. (2.133), thanks to the definition of ∂_τ^{-1} in (2.71) and the fact that $I_0 \in H^{s+\frac{3}{2}}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$, we have

$$g \in H^{s+\frac{3}{2}}(\mathbb{T}_t \times \mathbb{R}^{n-1}) \text{ and } \partial_t g \in H^{s+\frac{3}{2}}(\mathbb{T}_t \times \mathbb{R}^{n-1}).$$

And thus

$$g \in H^{\frac{7}{4}}(\mathbb{T}_t; H^s(\mathbb{R}^{n-1})) \cap H^1(\mathbb{T}_t; H^{s+2-\frac{1}{2}}(\mathbb{R}^{n-1})).$$

Therefore we can use Theorem 2.6.2 which implies the global existence of the periodic in time solution

$$u \in H^2(\mathbb{T}_t; H^s(\mathbb{R}^+ \times \mathbb{R}^{n-1})) \cap H^1(\mathbb{T}_t; H^{s+2}(\mathbb{R}^+ \times \mathbb{R}^{n-1})),$$

of the Kuznetsov periodic boundary value problem (2.112) as I_0 is small enough in $H^{s+\frac{3}{2}}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$. Therefore, it also implies the global existence of \tilde{I} defined in (2.57) which is the solution of the exact Kuznetsov system (2.134).

Now we subtract the equations in systems (2.72) and (2.134):

$$\begin{aligned} c\partial_z(I - \tilde{I}) - \frac{\gamma+1}{2\rho_0}(I - \tilde{I})\partial_\tau I - \frac{\gamma+1}{2\rho_0}\tilde{I}\partial_\tau(I - \tilde{I}) - \frac{\nu}{2c^2\rho_0}\partial_\tau^2(I - \tilde{I}) \\ - \frac{c^2}{2}\partial_\tau^{-1}\Delta_y(I - \tilde{I}) = \varepsilon\frac{\rho_0}{2c^2}R^{Kuz-KZK}. \end{aligned}$$

Denoting $\Omega_1 = \mathbb{T}_\tau \times \mathbb{R}^{n-1}$, we multiply this equation by $(I - \tilde{I})$, integrate over $\mathbb{T}_\tau \times \mathbb{R}^{n-1}$ and perform a standard integration by parts which gives

$$\begin{aligned} \frac{c}{2}\frac{d}{dz}\|I - \tilde{I}\|_{L^2(\Omega_1)}^2 - \frac{\gamma+1}{2\rho_0}\int_{\Omega_1}\partial_\tau I(I - \tilde{I})^2 d\tau dy \\ - \frac{\gamma+1}{2\rho_0}\int_{\Omega_1}\tilde{I}(I - \tilde{I})\partial_\tau(I - \tilde{I}) d\tau dy \\ + \frac{\nu}{2c^2\rho_0}\int_{\Omega_1}(\partial_\tau(I - \tilde{I}))^2 d\tau dy = \varepsilon\frac{\rho_0}{2c^2}\int_{\Omega_1}R^{Kuz-KZK}(I - \tilde{I}) d\tau dy. \end{aligned}$$

Let us notice that

$$\begin{aligned} \int_{\Omega_1}\tilde{I}(I - \tilde{I})\partial_\tau(I - \tilde{I}) d\tau dy &= \int_{\Omega_1}[(\tilde{I} - I) + I]\frac{1}{2}\partial_\tau(I - \tilde{I})^2 d\tau dy = \\ &= -\frac{1}{2}\int_{\Omega_1}\partial_\tau I(I - \tilde{I})^2 d\tau dy, \end{aligned}$$

and as for $s > \frac{n}{2} + 2$ and $u \in H^2(\mathbb{T}_t; H^s(\Omega)) \cap H^1(\mathbb{T}_t; H^{s+2}(\Omega))$ we also have

$$R^{Kuz-KZK} \in C(\mathbb{R}_+; L^2(\mathbb{T}_\tau \times \mathbb{R}^{n-1})). \quad (2.135)$$

This comes from the fact that in system (2.134) the worst term outside the remainder is $\partial_\tau^2 \tilde{I}$ with \tilde{I} given by Eq. (2.57). As $\partial_t^3 u \in L^2(\mathbb{T}_t; H^{s-2}(\Omega))$, we need to take $s > \frac{n}{2} + 2$ to have $\partial_\tau^2 \tilde{I}$ in $L^\infty(\mathbb{R}_+; L^2(\mathbb{T}_\tau \times \mathbb{R}^{n-1}))$. Therefore

$$\left| \int_{\Omega_1} R^{Kuz-KZK}(I - \tilde{I}) d\tau dy \right| \leq \|R^{Kuz-KZK}\|_{L^2(\Omega_1)} \|I - \tilde{I}\|_{L^2(\Omega_1)} \leq C \|I - \tilde{I}\|_{L^2(\Omega_1)}$$

with a constant $C > 0$ independent on z thanks to (2.135). It leads to the estimate

$$\frac{1}{2}\frac{d}{dz}\|I - \tilde{I}\|_{L^2(\Omega_1)}^2 \leq K \sup_{\Omega_1} |\partial_\tau I(\tau, z, y)| \|I - \tilde{I}\|_{L^2(\Omega_1)}^2 + C\varepsilon \|I - \tilde{I}\|_{L^2(\Omega_1)},$$

in which, due to the regularity of I for s and I_0 (see also Point 1 and 3 of Theorem 2.4.4) the term $\sup_{\Omega_1} |\partial_\tau I(\tau, z, y)|$ is bounded by a constant $C > 0$ independent on z . With this we have the desired estimate and the other results follow from Gronwall's Lemma. \square

Remark 2.6.1. Here the regularity $I_0 \in H^{s+\frac{3}{2}}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$ for $s > \frac{n}{2} + 2$ is the minimal regularity to ensure (2.135).

Approximation problem for the Kuznetsov equation with initial-boundary conditions

Let as previously $\Omega_1 = \mathbb{T}_\tau \times \mathbb{R}^{n-1}$, but $s \geq \left[\frac{n+1}{2}\right]$. Suppose that a function $I_0(t, y) = I_0(t, \sqrt{\varepsilon}x')$ is such that $I_0 \in H^s(\Omega_1)$ and $\int_{\mathbb{T}_\tau} I_0(s, y) ds = 0$. Then by Theorem 2.4.4 there is a unique solution $I(\tau, z, y)$ of the Cauchy problem (2.72) for the KZK equation such that

$$z \mapsto I(\tau, z, y) \in C([0, \infty[, H^s(\Omega_1)).$$

We define \bar{u} and g as in Eqs. (2.132) and (2.133) respectively. Thus, for $R^{Kuz-KZK}$ defined in Eq. (2.111), \bar{u} is the solution of the following system

$$\begin{cases} \partial_t^2 \bar{u} - c^2 \Delta \bar{u} - \varepsilon \partial_t \left((\nabla \bar{u})^2 + \frac{\gamma-1}{2c^2} (\partial_t \bar{u})^2 + \frac{\nu}{\rho_0} \Delta \bar{u} \right) = \varepsilon^2 R^{Kuz-KZK} & \text{in } \mathbb{T}_t \times \Omega, \\ \bar{u} = g & \text{on } \mathbb{T}_t \times \partial\Omega. \end{cases} \quad (2.136)$$

We study for $T > 0$ the solution u of the Dirichlet boundary-value problem (2.130) for the Kuznetsov equation on $[0, T] \times \mathbb{R}_+ \times \mathbb{R}^{n-1}$, taking $u_0 := \bar{u}(0)$ and $u_1 := \bar{u}_t(0)$ and considering the time periodic function g defined by Eq. (2.133) as a function on $[0, T]$. Now we have the following stability result.

Theorem 2.6.5. *Let $T > 0$, $\nu > 0$, $n \geq 2$, $\Omega = \mathbb{R}^+ \times \mathbb{R}^{n-1}$ and $I_0 \in H^s(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$, $s \in \mathbb{R}^+$. Let I be the solution of the KZK equation. By I the solution \bar{u} of the approximated Kuznetsov problem (2.136) is constructed using (2.132) and with g defined in (2.133).*

Then there hold

1. *If $s \geq 6$ for $n = 2, 3$, or else $\left[\frac{s}{2}\right] > \frac{n}{2} + 1$, there exists $k > 0$ such that $\|I_0\|_{H^s} < k$ implies the global well-posedness of the Cauchy problem for the KZK equation. Its solution is denoted for $0 \leq k \leq \left[\frac{s}{2}\right]$ by*

$$I \in C^k(\{z > 0\}; H^{s-2k}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})),$$

thus

$$\bar{u} \in C^k(\{z > 0\}; H^{s-2k}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})), \quad \partial_t \bar{u} \in C^k(\{z > 0\}; H^{s-2k}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})),$$

or again

$$\bar{u} \in H^2(\mathbb{T}_t, H^{\left[\frac{s}{2}\right]-1}(\Omega)) \cap H^1(\mathbb{T}_t, H^{\left[\frac{s}{2}\right]}(\Omega)). \quad (2.137)$$

The regularity of $I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1})$ (see Table 2.2) is minimal to ensure that $R^{Kuz-KZK}$, see Eq. (2.111), is in $C([0, +\infty[; L^2(\mathbb{R}_+ \times \mathbb{R}^{n-1}))$.

2. *If $\left[\frac{s}{2}\right] > \frac{n}{2} + 2$, taking the same initial data for the exact boundary-value problem for the Kuznetsov equation (2.130) as for \bar{u} , i.e.*

$$\begin{aligned} u(0) = \bar{u}(0) &= \frac{c^2}{\rho_0} \partial_\tau^{-1} I\left(-\frac{x_1}{c}, \varepsilon x_1, \sqrt{\varepsilon}x'\right) \in H^{\left[\frac{s}{2}\right]}(\Omega), \\ u_t(0) = \bar{u}_t(0) &= \frac{c^2}{\rho_0} \partial_\tau I\left(-\frac{x_1}{c}, \varepsilon x_1, \sqrt{\varepsilon}x'\right) \in H^{\left[\frac{s}{2}\right]-1}(\Omega), \end{aligned}$$

there exists $k > 0$ such that $\|I_0\|_{H^s} < k$ implies the well-posedness of the exact Kuznetsov equation (2.130) considered with Dirichlet boundary condition

$$g = \frac{c^2}{\rho_0} \partial_\tau^{-1} I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1}) \subset H^{7/4}([0, T]; H^{[\frac{s}{2}]-2}(\partial\Omega)) \\ \cap H^1([0, T]; H^{[\frac{s}{2}]-2+3/2}(\partial\Omega))$$

and the regularity

$$u \in H^2([0, T], H^{[\frac{s}{2}]-1}(\Omega)) \cap H^1([0, T], H^{[\frac{s}{2}]}(\Omega)). \quad (2.138)$$

Moreover, there exists constants $K > 0$, and $C > 0$ independent of ε such that for all $t \leq \frac{C}{\varepsilon}$ we have $C_1 > 0$ and $C_2 > 0$ with

$$\sqrt{\|(u - \bar{u})_t(t)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(t)\|_{L^2(\Omega)}^2} \leq C_1 \varepsilon^2 t e^{C_2 \varepsilon t} \leq K \varepsilon. \quad (2.139)$$

3. In addition, let u be a solution of the Dirichlet boundary-value problem (2.130) for the Kuznetsov equation, with g defined by Eq. (2.133) and $u_0 \in H^{m+2}(\Omega)$, $u_1 \in H^{m+1}(\Omega)$ with $m > \frac{n}{2}$ and

$$\|(u - \bar{u})_t(0)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(0)\|_{L^2(\Omega)}^2 \leq \delta^2 \leq \varepsilon^2. \quad (2.140)$$

There exists $K > 0$ and $C > 0$ independent of ε such that for all $t \leq \frac{C}{\varepsilon}$ we have $C_1 > 0$ and $C_2 > 0$ with

$$\sqrt{\|(u - \bar{u})_t(t)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(t)\|_{L^2(\Omega)}^2} \leq C_1(\varepsilon^2 t + \delta^2) e^{C_2 \varepsilon t} \leq K \varepsilon. \quad (2.141)$$

Proof. Let \bar{u} and g be defined by (2.132) and (2.133) by the solution I of the Cauchy problem (2.72) for the KZK equation with $I|_{z=0} = I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1})$ and $s \geq 6$ for $n = 2, 3$, or else $[\frac{s}{2}] > \frac{n}{2} + 1$. In this case, \bar{u} is the global solution of the approximated Kuznetsov system (2.136), what is a direct consequence of Theorem 2.4.4. If $I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1})$ with the chosen s , then for $0 \leq k \leq [\frac{s}{2}]$

$$I(\tau, z, y) \in C^k(\{z > 0\}; H^{s-2k}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})).$$

Let us denote $\Omega_1 = \mathbb{T}_\tau \times \mathbb{R}^{n-1}$. Given the equation for \bar{u} by (2.132), we have

$$\bar{u}(\tau, z, y) \text{ and } \partial_\tau \bar{u}(\tau, z, y) \in C^k(\{z > 0\}; H^{s-2k}(\Omega_1)), \text{ if } 0 \leq k \leq [\frac{s}{2}], \\ \partial_\tau^2 \bar{u}(\tau, z, y) \in C^k(\{z > 0\}; H^{s-1-2k}(\Omega_1)), \text{ if } 0 \leq k \leq [\frac{s}{2}] - 1,$$

but we can also say [42] thanks to Point 4 of Theorem 2.4.4 that

$$\bar{u}(\tau, z, y) \text{ and } \partial_\tau \bar{u}(\tau, z, y) \in H^k(\{z > 0\}; H^{s-2k}(\Omega_1)), \\ \partial_\tau^2 \bar{u}(\tau, z, y) \in H^k(\{z > 0\}; H^{s-1-2k}(\Omega_1)).$$

This implies as for the chosen s that

$$\bar{u}(t, x_1, x') \text{ and } \partial_t \bar{u}(t, x_1, x') \in L^2(\mathbb{T}_t; H^{[\frac{s}{2}]}(\Omega)) \cap H^2(\mathbb{T}_t; H^{[\frac{s}{2}]-1}(\Omega)), \\ \partial_t^2 \bar{u}(t, x_1, x') \in L^2(\mathbb{T}_t; H^{[\frac{s}{2}]-1}(\Omega)) \cap H^2(\mathbb{T}_t; H^{[\frac{s}{2}]-2}(\Omega)).$$

This implies

$$\begin{aligned}\bar{u}(t, x_1, x') &\in C^1([0, +\infty[; H^{[\frac{s}{2}]-1}(\Omega)), \\ \partial_t^2 \bar{u}(t, x_1, x') &\in C([0, +\infty[; H^{[\frac{s}{2}]-2}(\Omega)).\end{aligned}$$

With the chosen s , these regularities of $\bar{u}(t, x_1, x')$ give us the regularity (2.137) and allow to have all left-hand terms in the approximated Kuznetsov system (2.136) of the desired regularity, *i.e* in $C([0, +\infty[; L^2(\Omega))$. In addition for $[\frac{s}{2}] > \frac{n}{2} + 2$ with the chosen g , $u_0 = \bar{u}(0)$ and $u_1 = \bar{u}_t(0)$ in the conditions of the theorem we have

$$u_0 \in H^{[\frac{s}{2}]}(\Omega), \quad u_1 \in H^{[\frac{s}{2}]-1}(\Omega)$$

with

$$g \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1}) \text{ and } \partial_t g \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1}),$$

which implies

$$g \in H^{7/4}(\]0, T[; H^{[\frac{s}{2}]-2}(\partial\Omega)) \cap H^1(\]0, T[; H^{[\frac{s}{2}]-2+3/2}(\partial\Omega))$$

with $[\frac{s}{2}] - 2 > \frac{n}{2}$ as required by Theorem 2.6.3 to have the local well-posedness of u , the solution of the Kuznetsov equation associated to system (2.130). This completes the local well-posedness results and we deduce that u has the desired regularity (2.138) announced in the Theorem. Moreover, we have $R^{Kuz-KZK}$ in $C([0, +\infty[, L^2(\Omega))$.

To validate the approximation we will only demonstrate the estimate in point (3) as it directly implies the estimate in point (2). We take again $I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1})$ with $[\frac{s}{2}] > \frac{n}{2} + 2$ to define \bar{u} and g and consider u to be a solution of the Dirichlet boundary-value problem (2.130) for the Kuznetsov equation under the conditions $u_0 \in H^{m+2}(\Omega)$, $u_1 \in H^{m+1}(\Omega)$ with $m > \frac{n}{2}$ satisfying (2.140). Now we subtract the Kuznetsov equation from the approximated Kuznetsov equation (see system (2.136)), multiply by $(u - \bar{u})_t$ and integrate over Ω to obtain as in Ref. [26] the following stability estimation:

$$\begin{aligned}\frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} A(t, x) (u - \bar{u})_t^2 + c^2 (\nabla(u - \bar{u}))^2 dx \right) \\ \leq C\varepsilon \sup(\|u_{tt}\|_{L^\infty(\Omega)}; \|\Delta u\|_{L^\infty(\Omega)}; \|\nabla \bar{u}_t\|_{L^\infty(\Omega)}) \\ \cdot \left(\|(u - \bar{u})_t\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})\|_{L^2(\Omega)}^2 \right) \\ + \varepsilon^2 \int_{\Omega} R^{Kuz-KZK} (u - \bar{u})_t dx\end{aligned}$$

where $\frac{1}{2} \leq A(t, x) \leq \frac{3}{2}$ for $0 \leq t \leq T$ and $x \in \Omega$. By regularity of the solutions $\sup(\|u_{tt}\|_{L^\infty(\Omega)}; \|\Delta u\|_{L^\infty(\Omega)}; \|\nabla \bar{u}_t\|_{L^\infty(\Omega)})$ is bounded in time on $[0, T]$. Moreover, we have $\|R^{Kuz-KZK}(t)\|_{L^2(\Omega)}$ bounded for $t \in [0, T]$ by the regularity of \bar{u} where $R^{Kuz-KZK}$ is defined in (2.111). Then after integration on $[0, t]$, we can write

$$\begin{aligned}\|(u - \bar{u})_t(t)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(t)\|_{L^2(\Omega)}^2 \\ \leq 3(\|(u - \bar{u})_t(0)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(0)\|_{L^2(\Omega)}^2) \\ + C_1 \varepsilon \int_0^t \|(u - \bar{u})_t(s)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(s)\|_{L^2(\Omega)}^2 ds \\ + C_2 \varepsilon^2 \int_0^t \sqrt{\|(u - \bar{u})_t(s)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(s)\|_{L^2(\Omega)}^2} ds.\end{aligned}$$

As $\|(u - \bar{u})_t(0)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(0)\|_{L^2(\Omega)}^2 \leq \delta^2 \leq \varepsilon^2$, we finally find by the Gronwall Lemma

$$\sqrt{\|(u - \bar{u})_t(t)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(t)\|_{L^2(\Omega)}^2} \leq C_1(\varepsilon^2 t + \delta^2)e^{C_2 \varepsilon t} \leq K\varepsilon$$

for $t \leq \frac{C}{\varepsilon}$ what allows us to conclude. \square

For the inviscid media we use (2.4) on the cone $C(t)$ defined in Theorem 2.5.7 instead of \mathbb{R}^n when we compare the Euler system and the inviscid Kuznetsov equation. Therefore the triangular inequality permits us to validate the approximation between the Kuznetsov and KZK equations in the inviscid case as their respective approximations with the Euler system are validated by (2.4) in the cone.

2.7 Approximation of the solutions of the Kuznetsov equation with the solutions of the NPE equation

Now let us go back to the NPE equation introduced in Section 2.4.3 and consider its *ansatz* (2.86)–(2.89). As previously we start with the viscous case $\nu > 0$. Then we can rewrite the Kuznetsov equation

$$\begin{aligned} & \partial_t^2 u - c^2 \Delta u - \varepsilon \partial_t \left((\nabla u)^2 + \frac{\gamma - 1}{2c^2} (\partial_t u)^2 + \frac{\nu}{\rho_0} \Delta u \right) \\ &= \varepsilon \left(-2c \partial_{\tau z}^2 \Psi - c^2 \Delta_y \Psi + \frac{\nu}{\rho_0} c \partial_z^3 \Psi + \frac{\gamma + 1}{2} c \partial_z (\partial_z \Psi)^2 \right) + \varepsilon^2 R^{Kuz-NPE} \end{aligned}$$

with

$$\begin{aligned} \varepsilon^2 R^{Kuz-NPE} &= \varepsilon^2 \left(\partial_\tau^2 \Psi - \frac{\nu}{\rho_0} \partial_z^2 \partial_\tau \Psi + \frac{\nu}{\rho_0} c \Delta_y \partial_z \Psi - (\gamma - 1) \partial_\tau \Psi \partial_z^2 \Psi \right. \\ &\quad \left. - 2(\gamma - 1) \partial_z \Psi \partial_{\tau z}^2 \Psi - 2 \partial_z \Psi \partial_{\tau z}^2 \Psi + 2c \nabla_y \Psi \nabla_y \partial_z \Psi \right) \\ &\quad + \varepsilon^3 \left(-\frac{\nu}{\rho_0} \Delta_y \partial_\tau \Psi + 2 \frac{\gamma - 1}{c} \partial_\tau \Psi \partial_{\tau z}^2 \Psi + \frac{\gamma - 1}{c} \partial_z \Psi \partial_\tau^2 \Psi \right. \\ &\quad \left. - 2 \nabla_y \Psi \nabla_y \partial_\tau \Psi \right) + \varepsilon^4 \left(-\frac{\gamma - 1}{c^2} \partial_\tau \Psi \partial_\tau^2 \Psi \right). \end{aligned} \tag{2.142}$$

We obtain the NPE equation satisfied by $\partial_z \Psi$ modulo a multiplicative constant:

$$\partial_{\tau z}^2 \Psi - \frac{\gamma + 1}{4} \partial_z (\partial_z \Psi)^2 - \frac{\nu}{2\rho_0} \partial_z^3 \Psi + \frac{c}{2} \Delta_y \Psi = 0.$$

In the sequel we will work with ξ defined by (2.88) which satisfies the Cauchy problem (2.96) for the NPE equation. This time in relation with the KZK equation we used the bijection (2.95). We also update our notation for $\Omega_1 = \mathbb{T}_z \times \mathbb{R}_y^{n-1}$ and $s > \frac{n}{2} + 1$. Suppose that $\xi_0 \in H^{s+2}(\mathbb{T}_z \times \mathbb{R}_y^{n-1})$ and $\int_{\mathbb{T}_z} \xi_0(z, y) dz = 0$. Then there is a constant $r > 0$ such that if $\|\xi_0\|_{H^{s+2}(\mathbb{T}_z \times \mathbb{R}_y^{n-1})} < r$, then, by Theorem 2.4.4, there is a unique solution $\xi \in C([0, \infty[; H^{s+2}(\mathbb{T}_z \times \mathbb{R}_y^{n-1}))$ of the NPE Cauchy problem (2.96) satisfying

$$\int_{\mathbb{T}_z} \xi(\tau, z, y) dz = 0 \quad \text{for any } \tau \geq 0, y \in \mathbb{R}^{n-1}.$$

We define $\partial_{x_1}\bar{u}(t, x_1, x') := -\frac{c}{\rho_0}\xi(\tau, z, y)$ with the change of variable (2.85) and

$$\bar{u}(t, x_1, x') = -\frac{c}{\rho_0}\partial_z^{-1}\xi(\tau, z, y) = \left(-\frac{c}{\rho_0}\right) \left(\int_0^z \xi(\tau, s, y) ds + \int_0^L \frac{s}{L}\xi(\tau, s, y) ds \right).$$

We notice $u_1(x_1, x') := \partial_t\bar{u}(0, x_1, x')$ and $u_0(x_1, x') := -\frac{c}{\rho_0}\partial_z^{-1}\xi_0(z, y)$ and consequently we have $u_0 \in H^{s+2}(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1})$, $u_1 \in H^s(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1})$. Thus for these initial data there exists

$$\bar{u} \in C([0, \infty[; H^{s+1}(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1})) \cap C^1([0, \infty[; H^s(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1}))$$

the unique solution on $\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1}$ of the approximated Kuznetsov system

$$\begin{cases} \bar{u}_{tt} - c^2\Delta\bar{u} - \nu\varepsilon\Delta\bar{u}_t - \alpha\varepsilon\bar{u}_t\bar{u}_{tt} - \beta\varepsilon\nabla\bar{u}\nabla\bar{u}_t = \varepsilon^2 R^{Kuz-NPE}, \\ \bar{u}(0) = u_0 \in H^{s+2}(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1}), \quad \bar{u}_t(0) = u_1 \in H^{s+1}(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1}) \end{cases} \quad (2.143)$$

with $R^{Kuz-NPE}$ defined in (2.142). If we consider the Cauchy problem (2.28) for the Kuznetsov equation on $\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1}$ with u_0 and u_1 derived from ξ_0 we have

$$\|u_0\|_{H^{s+2}(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1})} + \|u_1\|_{H^s(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1})} \leq C\|\xi_0\|_{H^{s+2}(\mathbb{T}_z \times \mathbb{R}_y^{n-1})}.$$

Hence, if $\|\xi_0\|_{H^{s+2}(\mathbb{T}_z \times \mathbb{R}_y^{n-1})}$ is small enough [26], we have a unique solution

$$u \in C([0, \infty[; H^{s+1}(\Omega)) \cap C^1([0, \infty[; H^s(\Omega))$$

bounded in time of the Kuznetsov equation.

Theorem 2.7.1. *For the defined above solutions u of the exact Cauchy problem (2.28) and \bar{u} of the approximated Cauchy problem (2.143) for the Kuznetsov equation on $\Omega = \mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1}$. Then there exist constants $K > 0$, $C > 0$, $C_1 > 0$ and $C_2 > 0$ such that for all $t < \frac{C}{\varepsilon}$ we have estimate (2.139) and in addition Point 3 of Theorem 2.6.5.*

Proof. The global existence of u and \bar{u} has already been shown. The proof of the approximation estimate follows exactly as in Theorem 2.6.5 and is thus omitted. \square

Remark 2.7.1. *The case $\nu = 0$ implies the same approximation result except that u and \bar{u} are only locally well posed on an interval $[0, T]$.*

Remark 2.7.2. *We can see for $n = 2$ or 3 , using the previous arguments that the minimum regularity of the initial data (see Table 2.2) to have the remainder terms*

$$R^{Kuz-NPE} \in C([0, +\infty[; L^2(\mathbb{T}_{x_1} \times \mathbb{R}^{n-1}))$$

corresponds to $\xi_0 \in H^s(\mathbb{T}_{x_1} \times \mathbb{R}^{n-1})$ with $s \geq 4$ since then for $0 \leq k \leq 2$

$$\xi(\tau, z, y) \in C^k([0, +\infty[; H^{s-2k}(\mathbb{T}_z \times \mathbb{R}^{n-2})),$$

which finally implies with formula $\bar{u} = -\frac{c}{\rho_0}\partial_z^{-1}\xi$ that with $\Omega = \mathbb{T}_{x_1} \times \mathbb{R}^{n-1}$

$$\begin{aligned} \bar{u}(t, x_1, x') &\in C([0, +\infty[; H^4(\Omega)), \quad \partial_t\bar{u}(t, x_1, x') \in C([0, +\infty[; H^2(\Omega)), \\ \partial_t^2\bar{u}(t, x_1, x') &\in C([0, +\infty[; L^2(\Omega)). \end{aligned}$$

In the same way for $n \geq 4$ we can take $\xi_0 \in H^s(\Omega)$ with $s > \frac{n}{2} + 2$ for the minimal regularity as it implies

$$\begin{aligned} \bar{u}(t, x_1, x') &\in C([0, +\infty[; H^s(\Omega)), \quad \partial_t\bar{u}(t, x_1, x') \in C([0, +\infty[; H^{s-2}(\Omega)), \\ \partial_t^2\bar{u}(t, x_1, x') &\in C([0, +\infty[; H^{s-4}(\Omega)). \end{aligned}$$

2.8 The Kuznetsov equation and the Westervelt equation

2.8.1 Derivation of the Westervelt equation from the Kuznetsov equation

We consider the Kuznetsov equation (2.23). Similarly as in Ref. [1] we set

$$\Pi = u + \frac{1}{2c^2}\varepsilon\partial_t[u^2] \quad (2.144)$$

and obtain

$$\partial_t^2\Pi - c^2\Delta\Pi = \varepsilon\partial_t\left(\Delta u + \frac{\gamma+1}{2c^2}(\partial_t u)^2 + \frac{1}{c^2}u(\partial_t^2 - c^2\Delta)u\right).$$

By Definition (2.144) of Π we have

$$\partial_t^2\Pi - c^2\Delta\Pi = \varepsilon\partial_t\left(\Delta\Pi + \frac{\gamma+1}{2c^2}(\partial_t\Pi)^2\right) + \varepsilon^2 R^{Kuz-Wes},$$

where

$$\begin{aligned} \varepsilon^2 R^{Kuz-Wes} = & \varepsilon^2\partial_t\left[-\frac{1}{2c^2}\Delta(u\partial_t u) - \frac{\gamma+1}{2c^4}\partial_t u\partial_t^2(u^2) \right. \\ & \left. + \frac{1}{c^2}u\partial_t\left((\nabla u)^2 + \frac{\gamma-1}{2c^2}(\partial_t u)^2 + \frac{\nu}{\rho_0}\Delta u\right)\right] \\ & + \varepsilon^3\partial_t\left[-\frac{\gamma+1}{8c^6}[\partial_t^2(u^2)]^2\right]. \end{aligned} \quad (2.145)$$

We recognize the Westervelt equation

$$\partial_t^2\Pi - c^2\Delta\Pi = \varepsilon\partial_t\left(\Delta\Pi + \frac{\gamma+1}{2c^2}(\partial_t\Pi)^2\right). \quad (2.146)$$

2.8.2 Approximation of the solutions of the Kuznetsov equation by the solutions of the Westervelt equation

For the well-posedness of the Westervelt equation we refer to our work [26] on the Kuznetsov equation where our results can be directly applied. For u solution of the Cauchy problem (2.28) for the Kuznetsov equation we set

$$\bar{\Pi} = u + \frac{1}{2c^2}\varepsilon\partial_t[u^2],$$

and we have $\bar{\Pi}$ solution of the Cauchy problem

$$\begin{cases} \partial_t^2\bar{\Pi} - c^2\Delta\bar{\Pi} = \varepsilon\partial_t\left(\Delta\bar{\Pi} + \frac{\gamma+1}{2c^2}(\partial_t\bar{\Pi})^2\right) + \varepsilon^2 R^{Kuz-Wes}, \\ \bar{\Pi}(0) = \Pi_0, \partial_t\bar{\Pi}(0) = \Pi_1 \end{cases} \quad (2.147)$$

with $R^{Kuz-Wes}$ defined by (2.145) and in accordance with the definition of $\bar{\Pi}$

$$\Pi_0 = u_0 + \frac{1}{c^2}\varepsilon u_0 u_1, \quad (2.148)$$

$$\Pi_1 = u_1 + \frac{1}{c^2}\varepsilon u_1^2 + \frac{1}{c^2}\varepsilon u_0 \partial_t^2 u(0) \quad (2.149)$$

$$= u_1 + \frac{1}{c^2}\varepsilon u_1^2 + \frac{1}{c^2}\varepsilon u_0 \frac{1}{1 - \frac{\gamma-1}{c^2}\varepsilon u_1} \left(c^2\Delta u_0 + \frac{\nu}{\rho_0}\varepsilon\Delta u_1 + 2\varepsilon\nabla u_0\nabla u_1 \right)$$

with u_0 and u_1 initial data of the the Cauchy problem (2.28) for the Kuznetsov equation.

For $s > \frac{n}{2}$, if we take $u_0 \in H^{s+4}(\mathbb{R}^n)$ and $u_1 \in H^{s+3}(\mathbb{R}^3)$, we have $\Pi_0 \in H^{s+3}(\mathbb{R}^n) \subset H^{s+2}(\mathbb{R}^n)$ and $\Pi_1 \in H^{s+1}(\mathbb{R}^n)$ with

$$\|\Pi_0\|_{H^{s+2}(\mathbb{R}^n)} + \|\Pi_1\|_{H^{s+1}(\mathbb{R}^n)} \leq C(\|u_0\|_{H^{s+4}(\mathbb{R}^n)} + \|u_1\|_{H^{s+3}(\mathbb{R}^n)}),$$

so similarly to our previous work [26] we obtain

Theorem 2.8.1. *Let $n \geq 1$, $s > \frac{n}{2}$, $u_0 \in H^{s+4}(\mathbb{R}^n)$ and $u_1 \in H^{s+3}(\mathbb{R}^n)$. Then there exists a constant $k_2 > 0$ such that if*

$$\|u_0\|_{H^{s+4}(\mathbb{R}^n)} + \|u_1\|_{H^{s+3}(\mathbb{R}^n)} < k_3, \quad (2.150)$$

then the Cauchy problem for the Westervelt equation

$$\begin{cases} \partial_t^2 \Pi - c^2 \Delta \Pi = \varepsilon \partial_t \left(\Delta \Pi + \frac{\gamma+1}{2c^2} (\partial_t \Pi)^2 \right), \\ \bar{\Pi}(0) = \Pi_0, \partial_t \bar{\Pi}(0) = \Pi_1 \end{cases} \quad (2.151)$$

with Π_0 and Π_1 defined by Eqs. (2.148) and (2.149), has a unique global in time solution

$$\Pi \in H^2([0, +\infty[, H^s(\mathbb{R}^n)) \cap H^1([0, +\infty[, H^{s+2}(\mathbb{R}^n)) \quad (2.152)$$

and if $s \geq 1$

$$\Pi \in C([0, +\infty[, H^{s+2}(\mathbb{R}^n)) \cap C^1([0, +\infty[, H^{s+1}(\mathbb{R}^n)) \cap C^2([0, +\infty[, H^{s-1}(\mathbb{R}^n)) \quad (2.153)$$

Moreover we have $\bar{\Pi}$ global in time solution of the approximated Cauchy problem (2.147) with the same regularity.

For Π solution of the Cauchy problem (2.151) we set \bar{u} such that $\Pi = \bar{u} + \frac{\varepsilon}{c^2} \bar{u} \partial_t \bar{u}$ and we obtain

$$\begin{aligned} \partial_t^2 \bar{u} - c^2 \Delta \bar{u} - \varepsilon \frac{\nu}{\rho_0} \Delta \partial_t \bar{u} - \varepsilon \frac{\gamma-1}{c^2} \partial_t \bar{u} \partial_t^2 \bar{u} - 2\varepsilon \nabla \bar{u} \cdot \nabla \partial_t \bar{u} \\ + \varepsilon \left(\frac{1}{c^2} \partial_t \bar{u} \partial_t^2 \bar{u} - \partial_t \bar{u} \Delta \bar{u} + \frac{1}{c^2} \bar{u} \partial_t^3 \bar{u} - \bar{u} \Delta \partial_t \bar{u} \right) = \varepsilon^2 R_1^{Wes-Kuz} \end{aligned}$$

with

$$\begin{aligned} R_1^{Wes-Kuz} = & \left[\frac{\nu}{\rho_0 c^2} (2\partial_t \bar{u} \Delta \partial_t \bar{u} + 2(\nabla \partial_t \bar{u})^2 + \partial_t^2 \bar{u} \Delta \bar{u} + \bar{u} \Delta \partial_t^2 \bar{u} + 2\nabla \bar{u} \cdot \nabla \partial_t^2 \bar{u}) \right. \\ & \left. + \frac{\gamma+1}{c^4} ((\partial_t \bar{u})^2 + \bar{u} \partial_t^2 \bar{u}) \partial_t^2 \bar{u} + \frac{\gamma+1}{c^4} (3\partial_t \bar{u} \partial_t^2 \bar{u} + \bar{u} \partial_t^3 \bar{u}) \partial_t \bar{u} \right] \\ & + \varepsilon \frac{\gamma+1}{c^6} ((\partial_t \bar{u})^2 + \bar{u} \partial_t^2 \bar{u}) (3\partial_t \bar{u} \partial_t^2 \bar{u} + \bar{u} \partial_t^3 \bar{u}). \end{aligned}$$

And as

$$\partial_t^2 \bar{u} - c^2 \Delta \bar{u} = O(\varepsilon)$$

if we inject this in the term $\left(\frac{1}{c^2} \partial_t \bar{u} \partial_t^2 \bar{u} - \partial_t \bar{u} \Delta \bar{u} + \frac{1}{c^2} \bar{u} \partial_t^3 \bar{u} - \varepsilon \bar{u} \Delta \partial_t \bar{u} \right)$ we have

$$\partial_t^2 \bar{u} - c^2 \Delta \bar{u} - \varepsilon \frac{\nu}{\rho_0} \Delta \partial_t \bar{u} - \varepsilon \frac{\gamma-1}{c^2} \partial_t \bar{u} \partial_t^2 \bar{u} - 2\varepsilon \nabla \bar{u} \cdot \nabla \partial_t \bar{u} = \varepsilon^2 R^{Wes-Kuz}. \quad (2.154)$$

Now we can write the following approximation result for the Westervelt equation

Theorem 2.8.2. *Let $\nu > 0$, $n \geq 2$, $s > \frac{n}{2}$ with $s \geq 1$, $\bar{u}_0 \in H^{s+4}(\mathbb{R}^n)$ and $\bar{u}_1 \in H^{s+3}(\mathbb{R}^n)$, there exists $k > 0$ such that $\|\bar{u}_0\|_{H^{s+4}(\mathbb{R}^n)} + \|\bar{u}_1\|_{H^{s+3}(\mathbb{R}^n)} < k$ implies the global existence of Π with the regularities (2.152) and (2.153) which is the solution of the Cauchy problem (2.151) with Π_0 and Π_1 defined by Eqs. (2.148) and (2.149). Moreover for $u_0 \in H^{s+2}(\mathbb{R}^n)$ and $u_1 \in H^{s+1}(\mathbb{R}^n)$ we have u exact solution of the Cauchy problem (2.28) for the Kuznetsov equation. Let \bar{u} such that*

$$\Pi = \bar{u} + \frac{\varepsilon}{c^2} \bar{u} \partial_t \bar{u},$$

as a consequence \bar{u} is a solution of the approximated Kuznetsov equation (2.154) and if u and \bar{u} satisfy (2.140) with $u(0) = u_0$, $\partial_t u(0) = u_1$, $\bar{u}(0) = \bar{u}_0$, $\partial_t \bar{u}(0) = \bar{u}_1$, there exists constants $K > 0$ and $C > 0$ independent of ε such that for all $t \leq \frac{C}{\varepsilon}$ we have $C_1 > 0$ and $C_2 > 0$ with estimate (2.141).

Proof. The existence of u and \bar{u} has already been shown. The proof of the approximation estimate follows exactly the proof of Theorem 2.6.5 and hence it is omitted. \square

Remark 2.8.1. *For the minimal regularity (see Table 2.2) of u_0 and u_1 to ensure that $R^{Kuz-Wes}$, see Eq. (2.145), is in $C([0, +\infty[; L^2(\mathbb{R}_+ \times \mathbb{R}^{n-1}))$, if $u_0 \in H^{s+2}(\mathbb{R}^3)$ and $u_1 \in H^{s+1}(\mathbb{R}^3)$ for $s \geq 3$ then*

$$\begin{aligned} u &\in C([0, +\infty[; H^5(\mathbb{R}^3)), \quad \partial_t u \in C([0, +\infty[; H^4(\mathbb{R}^3)), \\ \partial_t^2 u &\in C([0, +\infty[; H^2(\mathbb{R}^3)), \quad \partial_t^3 u \in C([0, +\infty[; L^2(\mathbb{R}^3)). \end{aligned}$$

Taking $\bar{\Pi}$ as in (2.144) we obtain

$$\bar{\Pi} \in C([0, +\infty[; H^4(\mathbb{R}^3)), \quad \partial_t \bar{\Pi} \in C([0, +\infty[; H^2(\mathbb{R}^3)), \quad \partial_t^2 \bar{\Pi} \in C([0, +\infty[; L^2(\mathbb{R}^3)).$$

Injecting this result in the approximated Westervelt equation in system (2.147) we obtain $R^{Kuz-Wes} \in C([0, +\infty[; L^2(\mathbb{R}^3))$. In the same way if $n \geq 4$ we take $u_0 \in H^{s+2}(\mathbb{R}^n)$ and $u_1 \in H^{s+1}(\mathbb{R}^n)$ with $s > \frac{n}{2} + 1$.

2.9 Summary

We summarize all obtained approximation results in two comparative tables: Table 2.1 for the approximations of the Navier-Stokes and Euler systems and Table 2.2 for the approximations of the Kuznetsov equation.

Table 2.1 – Approximation results for models derived from Navier-Stokes and Euler systems

	Kuznetsov		KZK		NPE	
	Navier-Stokes	Euler	Navier-Stokes	Euler	Navier-Stokes	Euler
Theorem	Theorem 2.4.3	Theorem 2.5.5	Theorem 2.4.6	Theorem 2.5.7	Theorem 2.4.10	Theorem 2.5.5
Ansatz	$\begin{aligned} \rho_\varepsilon &= \rho_0 + \varepsilon\rho_1 + \varepsilon^2\rho_2, \\ \mathbf{v}_\varepsilon &= -\varepsilon\nabla u, \\ \rho_1 &= \frac{\rho_0}{c^2}\partial_t u, \\ \rho_2 &\text{ from (2.18)} \end{aligned}$		paraxial approximation $\begin{aligned} u &= \Phi(t - \frac{x_1}{c}, \varepsilon x_1, \sqrt{\varepsilon}\mathbf{x}') \\ \rho_\varepsilon &= \rho_0 + \varepsilon I + \varepsilon^2 J, \\ \mathbf{v}_\varepsilon &\text{ from (2.68), } I = \frac{\rho_0}{c^2}\partial_\tau \Phi, \\ &J \text{ from (2.58)} \end{aligned}$		paraxial approximation $\begin{aligned} u &= \Psi(\varepsilon t, x_1 - ct, \sqrt{\varepsilon}\mathbf{x}') \\ \rho_\varepsilon &= \rho_0 + \varepsilon\xi + \varepsilon^2\chi, \\ \mathbf{v}_\varepsilon &\text{ from (2.86), } \xi = -\frac{\rho_0}{c}\partial_z \Psi, \\ &\chi \text{ from (2.89)} \end{aligned}$	
Models	$\begin{aligned} \partial_t^2 u - c^2 \Delta u &= \\ \varepsilon \partial_t \left((\nabla u)^2 + \frac{\gamma-1}{2c^2} (\partial_t u)^2 \right. \\ &\left. + \frac{\nu}{\rho_0} \Delta u \right) \end{aligned}$		$\begin{aligned} c \partial_{\tau z}^2 I - \frac{(\gamma+1)}{4\rho_0} \partial_\tau^2 I^2 \\ - \frac{\nu}{2c^2\rho_0} \partial_\tau^3 I - \frac{c^2}{2} \Delta_y I = 0 \end{aligned}$		$\begin{aligned} \partial_{\tau z}^2 \xi + \frac{(\gamma+1)c}{4\rho_0} \partial_z^2 (\xi^2) \\ - \frac{\nu}{2\rho_0} \partial_z^3 \xi + \frac{c}{2} \Delta_y \xi = 0 \end{aligned}$	
Approximation Order	$O(\varepsilon^3)$					
Domain Ω	\mathbb{R}^3		the half space $\{x_1 > 0, x' \in \mathbb{R}^{n-1}\}$	the cone $\{ x_1 < \frac{R}{\varepsilon} - ct\}$ $\times \mathbb{R}_{x'}^{n-1}$	$\mathbb{T}_{x_1} \times \mathbb{R}^2$	
Approximation	$\ U_\varepsilon - \bar{U}_\varepsilon\ _{L^2} \leq \varepsilon$ for $t \leq \frac{T}{\varepsilon}$					
Initial data regularity	$u_0 \in H^5(\Omega)$ $u_1 \in H^4(\Omega)$	$u_0 \in H^4(\Omega)$ $u_1 \in H^3(\Omega)$	$I_0 \in H^{10}(\Omega)$	$I_0 \in H^{10}(\Omega)$	$\xi_0 \in H^5(\Omega)$	$\xi_0 \in H^5(\Omega)$
Data regularity for remainder boundness	$u_0 \in H^{s+2}(\Omega)$ $u_1 \in H^{s+1}(\Omega)$ $s > \frac{n}{2}$	$u_0 \in H^{s+2}(\Omega)$ $u_1 \in H^{s+1}(\Omega)$ $s > \frac{n}{2}$	$I_0 \in H^8(\Omega)$	$I_0 \in H^8(\Omega)$	$\xi_0 \in H^4(\Omega)$	$\xi_0 \in H^4(\Omega)$

Table 2.2 – Approximation results for models derived from the Kuznetsov equation

	KZK		NPE	Westervelt
	periodic boundary condition problem	initial boundary value problem		
Theorem	Theorem 2.6.4	Theorem 2.6.5	Theorem 2.7.1	Theorem 2.8.2
Derivation	paraxial approximation $u = \Phi(t - \frac{x_1}{c}, \varepsilon x_1, \sqrt{\varepsilon} \mathbf{x}')$		paraxial approximation $u = \Psi(\varepsilon t, x_1 - ct, \sqrt{\varepsilon} \mathbf{x}')$	$\Pi = u + \frac{1}{c^2} \varepsilon u \partial_t u$
Approximation domain	the half space $\{x_1 > 0, x' \in \mathbb{R}^{n-1}\}$		$\mathbb{T}_{x_1} \times \mathbb{R}^2$	\mathbb{R}^3
Approximation order	$O(\varepsilon)$		$O(\varepsilon)$	$O(\varepsilon^2)$
Estimation	$\ I - I_{approx}\ _{L^2(\mathbb{T}_t \times \mathbb{R}^{n-1})} \leq \varepsilon$ $z \leq K$	$\ (u - \bar{u})_t(t)\ _{L^2}$ $+\ \nabla(u - \bar{u})(t)\ _{L^2}$ $\leq K\varepsilon.$ $t < \frac{T}{\varepsilon}$	$\ (u - \bar{u})_t(t)\ _{L^2}$ $+\ \nabla(u - \bar{u})(t)\ _{L^2}$ $\leq K\varepsilon$ $t < \frac{T}{\varepsilon}$	$\ (u - \bar{u})_t(t)\ _{L^2}$ $+\ \nabla(u - \bar{u})(t)\ _{L^2}$ $\leq K\varepsilon$ $t < \frac{T}{\varepsilon}$
Initial data regularity	$I_0 \in H^{s+\frac{3}{2}}(\mathbb{T}_t \times \mathbb{R}_{x'}^{n-1})$ for $s \geq \frac{n}{2} + 2$	$I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}_{x'}^{n-1})$ for $[\frac{s}{2}] > \frac{n}{2} + 2$	$\xi_0 \in H^{s+2}(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1})$ for $s > \frac{n}{2} + 1$	$u_0 \in H^{s+4}(\mathbb{R}^n)$ $u_1 \in H^{s+3}(\mathbb{R}^3)$ for $s > \frac{n}{2}$
Data regularity for remainder boundness	$I_0 \in H^{s+\frac{3}{2}}(\mathbb{T}_t \times \mathbb{R}_{x'}^{n-1})$ for $s \geq \frac{n}{2} + 2$	$I_0 \in H^6(\mathbb{T}_t \times \mathbb{R}_{x'}^{n-1})$ for $n = 2, 3,$ $I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}_{x'}^{n-1})$ for $[\frac{s}{2}] > \frac{n}{2} + 1, n \geq 4$	$\xi_0 \in H^4(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1})$ for $n = 2, 3.$ $\xi_0 \in H^s(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1})$ for $s > \frac{n}{2} + 2, n \geq 4.$	$u_0 \in H^{s+2}(\mathbb{R}^n)$ $u_1 \in H^{s+1}(\mathbb{R}^n)$ for $s \geq 3, n = 2, 3.$ $u_0 \in H^{s+2}(\mathbb{R}^n)$ $u_1 \in H^{s+1}(\mathbb{R}^n)$ for $s \geq \frac{n}{2} + 1, n \geq 4.$

Part II

Propagation of linear and nonlinear waves in domains with fractal boundaries

Chapter 3

Introduction to Part II

3.1 Introduction française

Cette partie porte principalement sur l'étude des propriétés de régularité des solutions faibles d'équations sur Ω un domaine ouvert borné de \mathbb{R}^n , $n \geq 2$ avec un bord fractal. Nous pouvons nous référer à l'introduction générale. Nous commençons dans la Section 4.1 par donner les propriétés connues des domaines admissibles (voir la Définition 4.1.5) qui sont la classe la plus générale dès lors que l'on veut définir une trace ou une extension sur le domaine Ω (voir Théorème 4.1.1) ainsi qu'une injection de Sobolev (voir Théorème 4.1.2). Nous poursuivons dans les sous Sections 4.1.2 et 4.1.3 en donnant les propriétés connues du problème de Poisson avec des conditions de Dirichlet

$$\begin{cases} -\Delta u = f \text{ sur } \Omega, \\ u|_{\Omega} = g \text{ sur } \partial\Omega, \end{cases} \quad (3.1)$$

ou de Robin homogène

$$\begin{cases} -\Delta u = f \text{ sur } \Omega, \\ \frac{\partial}{\partial n} u + au = 0 \text{ avec } a > 0 \text{ sur } \partial\Omega, \end{cases} \quad (3.2)$$

notamment concernant les estimations de la forme

$$\|u\|_{L^\infty(\Omega)} \leq C \|\Delta u\|_{L^p(\Omega)}. \quad (3.3)$$

Les Sections 4.2 et 4.3 sont dédiées respectivement à l'équation des ondes et à l'équation des ondes fortement amorties et à la régularité des solutions au sens faible de ces équations pour des conditions de Dirichlet homogènes en se basant sur une méthode de Galerkin comme chez Evans [30]. Il est à noter que pour de telles conditions aux bords le domaine considéré a juste besoin d'être un ouvert borné quelconque, le point clé étant l'inégalité de Poincaré. La Section 4.4 est dédiée au caractère bien posé de l'équation des ondes fortement amortie avec des conditions de Robin homogènes.

Avec les résultats de la Section 4.3 sur l'équation des ondes fortement amorties nous avons traité le caractère bien posé au sens faible de l'équation de Westervelt avec des conditions de Dirichlet homogène dans la Section 5.1 de la même façon que dans la preuve

du Point 1 du Théorème 1.2.2 dans le Chapitre 1 Section 1.5.1 pour le caractère bien posé global de l'équation de Kuznetsov sur \mathbb{R}^n en utilisant le Théorème 1.5.2. Si nous pouvons encore considérer un domaine borné quelconque en dimension $n = 3$, la nécessité d'un contrôle de la norme L^∞ de la solution de l'équation de Poisson avec une estimation du type (3.3) nous a amenés à nous restreindre à des domaines dits admissibles (voir la Définition 4.1.5) en dimension $n = 2$ par Nyström [77]. En Section 5.2 à l'aide des résultats sur les traces et les extensions de la Section 4.1 en nous plaçant sur les domaines admissibles, nous avons pu traiter le cas de l'équation de Westervelt avec des conditions de Dirichlet non homogènes.

La Section 5.3 est dédiée à l'étude de l'équation de Westervelt avec des conditions de Robin homogènes. Les résultats de régularité concernant l'inégalité (3.3) connus par la Section 4.1 pour l'équation de Poisson avec conditions de Robin homogènes nous ont amené à nous placer dans les espaces H^s sur un domaine à bords lipschitziens et sur les espaces $W^{s,p}$ avec $p > n$ sur un domaine admissible pour traiter du caractère bien posé au sens faible de l'équation de Westervelt.

Dans la Section 5.4, nous finissons cette Partie en considérant un ensemble à bord fractal de type mixture de Koch construit par récurrence à l'aide de familles de similitudes contractantes induisant ainsi une famille de domaines à bords préfractals et lipschitziens convergeant vers le domaine à bords fractals. En utilisant différents travaux de Capitanelli [19], Capitanelli et Vivaldi [20] ou Lancia [62], nous avons pu considérer la convergence asymptotique de type Mosco des solutions de l'équation de Westervelt avec conditions de Robin sur les domaines à bords préfractals qui approximent la solution sur le domaine à bords fractal de type mixture de Koch, une démarche souvent utilisée dans le cadre de l'optimisation de forme. De fait dans la Sous-section 5.4.3, nous montrons la convergence asymptotique de type Mosco des formes variationnelles associées à l'équation de Westervelt en considérant un bord purement fractal et des conditions de Robin aux bords partout. Néanmoins la nécessité d'un contrôle uniforme des normes des solutions sur le préfractal, indépendant de la génération du préfractal, nous a amené à imposer une condition de Dirichlet homogène sur une partie du bord du domaine pour pouvoir utiliser l'inégalité de Poincaré. L'utilisation de l'inégalité de Poincaré est essentielle pour faire apparaître dans les estimations des constantes ne dépendant que des aires des domaines qui, dans notre cas, sont bornées uniformément.

3.2 Introduction

The main topic of this Part is the study of the regularity properties of the weak solutions of equations on Ω of a bounded open set of \mathbb{R}^n , $n \geq 2$ with a fractal boundary. We can refer to the general introduction.

We begin in Section 4.1 by giving the known properties of admissible domains (see Definition 4.1.5) which are the most general class as soon as we want to define a trace or an extension on the domain Ω (see Theorem 4.1.1) or Sobolev embeddings (see Theorem 4.1.2). In the sequel with Subsections 4.1.2 and 4.1.3 we give the known properties of the Poisson problem with the Dirichlet boundary conditions (3.1) or the homogeneous Robin boundary conditions (3.2), also giving estimates of form (3.3). Sections 4.2 and 4.3 are dedicated respectively to the wave equation and the strongly damped wave equation and to the well-posedness of these equations in a weak sense for the homogeneous Dirichlet boundary conditions basing ourselves on a Galerkin method as in Evans [30]. Let us note that for such

boundary conditions the considered domain only needs to be any bounded open domain, the key point being the Poincaré inequality.

With the results in Section 4.3 on the strongly damped wave equation we have treated in Section 5.1 the well-posedness in a weak sense of the Westervelt equation with the homogeneous Dirichlet boundary conditions in the same way that in the proof of Point 1 of Theorem 1.2.2 in Chapter 1 Section 1.5.1 for the global well-posedness of the Westervelt equation on \mathbb{R}^n using Theorem 1.5.2. If we can again consider any bounded open domain in dimension $n = 3$, the necessity to control the L^∞ -norm of the solution with an estimate of the form (3.3) leads us to restrict ourselves on the admissible domains in dimension $n = 2$ by [77]. In Section 5.2 with the help of results on traces and extensions in Section 4.1 using the admissible domains we can treat the case of the Westervelt equation with non homogeneous boundary conditions. Section 4.4 is used to show the well-posedness of the strongly damped wave equation with the homogeneous Robin boundary conditions.

Section 5.3 is dedicated to the study of the Westervelt equation with homogeneous Robin boundary conditions. The regularity results concerning estimate (3.3) known by Section 4.1 for the Poisson equation lead us to work with spaces of type H^s on bounded Lipschitz domains and on spaces $W^{s,p}$ with $p > n$ on an admissible domain to treat the well-posedness in a weak sense of the Westervelt equation. In Section 5.4, we will conclude this Part considering a domain with a fractal boundary of Koch mixture type constructed by induction with the help of families of contractive similitudes inducing a family of domains with prefractal and Lipschitz boundaries approximating the domain with fractal boundaries. Using different works by Capitanelli [19], Capitanelli and Vivaldi [20] or Lancia [62], we consider the asymptotic convergence of Mosco type of the solutions of the Westervelt equation with the Robin boundary conditions on domains with a prefractal boundary which approach the solution on the domain with a fractal boundary of Koch mixture type, a method often used in the case of the shape optimisation. In fact in Subsection 5.4.3 we show the convergence of Mosco type of the variational form associated to the Westervelt equation considering a fractal boundary everywhere with the Robin boundary condition. Nevertheless the necessity of a uniform control of the norms of the solutions on the prefractal, independent of the generation of the prefractal, leads us to impose an homogeneous Dirichlet boundary condition on a part of the domain's boundary in order to use the Poincaré inequality. In the estimates, the use of the Poincaré inequality is essential to make appear constants depending only of the areas of our domains which in our case are uniformly bounded.

Chapter 4

Regularity of linear models on domains with fractal boundaries

In this Chapter, we study linear equations such as the Poisson equation, the wave equations and the strongly damped wave equation.

4.1 First results and notations

4.1.1 Admissible domains

Thanks to article [11] and the references therein we introduce the following definitions :

Definition 4.1.1. (*d-set*) Let F be a Borel subset of \mathbb{R}^n and m_d be the d -dimensional Hausdorff measure, $0 < d \leq n$. The set F is called a d -set, if there exists positive constants $c_1, c_2 > 0$,

$$c_1 r^d \leq m_d(F \cap B_r(x)) \leq c_2 r^d, \quad \text{for } \forall x \in F, 0 < r \leq 1$$

where $B_r(x) \subset \mathbb{R}^n$ denotes the Euclidean ball of radius r and centered at x .

- In \mathbb{R}^n , Lipschitz domains and regular domains are n -sets with $(n - 1)$ -sets as boundaries.
- [48] In \mathbb{R}^n , the (ε, δ) domains are n -sets with a possibly fractal d -set boundary.

Definition 4.1.2. (*Markov's local inequality*) A closed subset V in \mathbb{R}^n preserves Markov's local inequality if for every fixed $k \in \mathbb{N}^*$, there exists a constant $c = c(V, n, k) > 0$, such that

$$\max_{V \cap \overline{B_r(x)}} |\nabla P| \leq \frac{c}{r} \max_{V \cap \overline{B_r(x)}} |P|$$

for all polynomials $P \in \mathcal{P}_k$ and all closed balls $\overline{B_r(x)}$, $x \in V$ and $0 < r \leq 1$.

The Markov's inequality means geometrically that the sets considered are not too "flat" everywhere [47] for example it is not contained in a plane of \mathbb{R}^4 .

Definition 4.1.3. (*Space $C_p^k(\Omega)$*) For a set $\Omega \subset \mathbb{R}^n$ of positive Lebesgue measure,

$$C_p^k(\Omega) = \left\{ f \in L_p(\Omega) \mid f_{k,\Omega}^\sharp(x) = \sup_{r>0} r^{-k} \inf_{P \in \mathcal{P}_{k-1}} \frac{1}{\mu(B_r(x))} \int_{B_r(x) \cap \Omega} |f - P| dy \in L_p(\Omega) \right\}$$

with the norm $\|f\|_{C_p^k} = \|f\|_{L^p} + \|f_{k,\Omega}^\sharp\|_{L^p}$.

This spaces have been introduced in [36] in order to treat the extendability of Sobolev functions on optimal spaces.

Definition 4.1.4. (Trace) For an arbitrary open set Ω of \mathbb{R}^n the trace operator Tr is defined for $u \in L^1_{loc}(\Omega)$ by

$$Tru(x) = \lim_{r \rightarrow 0} \frac{1}{\lambda(\Omega \cap B_r(x))} \int_{\Omega \cap B_r(x)} u(y) d\lambda,$$

where λ is the Lebesgue measure. The trace operator Tr is considered for all $x \in \overline{\Omega}$ for which the limit exists.

Introduced for example in [48], this local definition of the trace permits to extend the notion of trace employed for continuous functions.

Definition 4.1.5. (Admissible domain) A domain $\Omega \subset \mathbb{R}^n$ is called admissible if it is an n -set such that for $1 < p < \infty$ and $k \in \mathbb{N}^*$ $W^{k,p}(\Omega) = C^k_p(\Omega)$ as set with equivalent norms, with a closed d -set boundary $\partial\Omega$, $0 < d < n$, preserving local Markov's inequality.

Now, we can give the trace theorem as in [11].

Theorem 4.1.1. Let $1 < p < +\infty$, $k \in \mathbb{N}^*$ be fixed. Let Ω be an admissible domain in \mathbb{R}^n . Then for $\beta = k - \frac{n-d}{p} > 0$, the following trace operators

1. $Tr : W^{k,p}(\mathbb{R}^n) \rightarrow B^{p,p}_\beta(\partial\Omega)$,
2. $Tr_\Omega : W^{k,p}(\mathbb{R}^n) \rightarrow W^{k,p}(\Omega)$,
3. $Tr_{\partial\Omega} : W^{k,p}(\Omega) \rightarrow B^{p,p}_\beta(\partial\Omega)$

are linear continuous and surjective with linear bounded right inverse, i.e. extension, operators $E : B^{p,p}_\beta(\partial\Omega) \rightarrow W^{k,p}(\mathbb{R}^n)$, $E_\Omega : W^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^n)$, $E_{\partial\Omega} : B^{p,p}_\beta(\partial\Omega) \rightarrow W^{k,p}(\Omega)$.

The definition of the Besov space $B^{p,p}_\beta(\partial\Omega)$ on a close d -set $\partial\Omega$ can be found, for instance, in [48] p.135. The next proposition was shown in [11] with the help of [62].

Proposition 4.1.1. (Green formula) Let Ω be an admissible domain in \mathbb{R}^n ($n \geq 2$) with a d -set boundary $\partial\Omega$ such that $n - 2 < d < n$. Then for all $u, v \in H^1(\Omega)$ with $\Delta u \in L_2(\Omega)$ it holds the Green formula

$$\int_\Omega v \Delta u dx = \left\langle \frac{\partial u}{\partial n}, Trv \right\rangle_{(B^{2,2}_\beta(\partial\Omega))', B^{2,2}_\beta(\partial\Omega)} - \int_\Omega \nabla u \nabla v dx,$$

where $\beta = 1 - \frac{n-d}{2} > 0$ and the Besov space $B^{2,2}_\beta(\partial\Omega)$ and dual Besov space $(B^{2,2}_\beta(\partial\Omega))' = B^{-\beta}_{-2}(\partial\Omega)$.

Article [11] also tells us that the Sobolev's embeddings stay true for a bounded admissible domain.

Theorem 4.1.2. (Sobolev's embeddings) Let $\Omega \subset \mathbb{R}^n$ be a bounded n -set with $W^k_p(\Omega) = C^k_p(\Omega)$, $1 < p < +\infty$, $k, l \in \mathbb{N}^*$. Then there hold the following compact embeddings

1. $W^{k+l,p}(\Omega) \subset\subset W^{l,p}(\Omega)$,
2. $W^{k,p}(\Omega) \subset\subset L^q(\Omega)$,

with $q \in [1, +\infty[$ if $kp = n$, $q \in [1, +\infty]$ if $kp > n$, and with $q \in \left[1, \frac{pn}{n-kp}\right[$ if $kp < n$. Moreover if $kp < n$ we have the continuous embedding

$$W^{k,p}(\Omega) \hookrightarrow L^{\frac{pn}{n-kp}}(\Omega).$$

In addition, the Poincaré inequality stays true on bounded admissible domains:

Theorem 4.1.3. (the Poincaré inequality) *Let $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ be a bounded connected admissible domain. For all $u \in W_0^{1,p}(\Omega)$ with $1 \leq p < +\infty$, there exists $C > 0$ depending only on Ω , p and n such that*

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}.$$

Therefore the semi-norm $\|\cdot\|_{W_0^{1,p}(\Omega)}$, defined by $\|u\|_{W_0^{1,p}(\Omega)} := \|\nabla u\|_{L^p(\Omega)}$, is a norm which is equivalent to $\|\cdot\|_{W^{1,p}(\Omega)}$ on $W_0^{1,p}(\Omega)$.

Moreover for all $u \in W^{1,p}(\Omega)$ there exists $C > 0$ depending only on Ω , p and n such that

$$\left\| u - \frac{1}{\lambda(\Omega)} \int_{\Omega} u \, d\lambda \right\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}.$$

Proof. The result for $u \in W_0^{1,p}(\Omega)$ comes from the boundness of Ω . The result for $u \in W^{1,p}(\Omega)$ comes from the compactness of the embedding $W^{1,p}(\Omega) \subset\subset L^p(\Omega)$ from Theorem 4.1.2 and following for instance the proof in [30] (see section 5.8.1 Theorem 1). \square

We also have by [11] the following theorem for the Dirichlet-to-Neumann operator.

Theorem 4.1.4. *Let Ω be a bounded admissible domain in \mathbb{R}^n ($n \geq 2$) with a d -set boundary $\partial\Omega$ such that $n - 2 < d < n$. Then for $\beta = 1 - \frac{n-d}{2} > 0$ the Poincaré-Steklov operator*

$$A : B_{\beta}^{2,2}(\partial\Omega) \rightarrow B_{-\beta}^{2,2}(\partial\Omega)$$

mapping $u|_{\partial\Omega}$ to $\partial_{\nu}u|_{\partial\Omega}$ is a linear bounded self adjoint operator with $\ker A \neq 0$.

4.1.2 The Poisson equation with Dirichlet boundary conditions

In a way, the theorems coming from [11] show that the function spaces on admissible domain share a lot of properties with the same function spaces considered on regular domains or domains with Lipschitz boundary. Nevertheless differences in the regularity of solutions occur (see Theorems 4.1.8 and 4.1.9) when we consider partial differential equations even as simple as the Poisson equation on admissible domains. Using the results from [49] and [11] we have the following well-posedness result for the Laplace equation.

Theorem 4.1.5. *Let Ω be a bounded admissible domain in \mathbb{R}^n ($n \geq 2$) with a d -set boundary $\partial\Omega$ such that $n - 2 < d < n$. For $\beta = 1 - \frac{n-d}{2} > 0$, given $f \in L^2(\Omega)$ and $g \in B_{\beta}^{2,2}(\partial\Omega)$ the Poisson problem (3.1) has a unique weak solution $u \in H^1(\Omega)$ in the sense that*

$$\forall v \in H_0^1(\Omega), \quad \int_{\Omega} \nabla u \nabla v = \int_{\Omega} f v,$$

and

$$Tr_{\partial\Omega} u = g.$$

Furthermore, the mapping $\{f, g\} \mapsto u$ is a bounded linear operator from $L^2(\Omega) \times B_{\beta}^{2,2}(\partial\Omega)$ to $H^1(\Omega)$.

Proof. First we use the operator $E_{\partial\Omega}$ defined in Theorem 4.1.2 to obtain $\bar{g} \in H^1(\Omega)$ such that $Tr_{\partial\Omega}\bar{g} = g$ and $\|\bar{g}\|_{H^1(\Omega)} \leq C\|g\|_{B_{\beta}^{2,2}(\partial\Omega)}$ with $C > 0$. Then the researched solution u of the Poisson problem (3.1) is defined as $u = w + \bar{g}$ where $w \in H_0^1(\Omega)$ is given as a function such that

$$\forall v \in H_0^1(\Omega) \quad \int_{\Omega} \nabla w \nabla v = \int_{\Omega} f v - \int_{\Omega} \nabla \bar{g} \nabla v.$$

Applying the Lax-Milgram theorem with the Poincaré inequality such a function w exists and is unique and we can easily deduce the boundness of the mapping $\{f, g\} \mapsto u$ as the Lax-Milgram theorem and the Poincaré inequality give us the estimate

$$\|w\|_{H^1(\Omega)} \leq C_1 \|\nabla w\|_{L^2(\Omega)} \leq C_2 \|f\|_{L^2(\Omega)} + \|\nabla \bar{g}\|_{L^2(\Omega)},$$

with $C_1 > 0$ and $C_2 > 0$. □

The following theorem can be shown using [30] section 6.3.1 page 309.

Theorem 4.1.6. [30] *Let Ω be an arbitrary bounded domain in \mathbb{R}^n ($n \geq 2$). Assume $f \in L^2(\Omega)$. Suppose furthermore that $u \in H_0^1(\Omega)$ is the weak solution of the Poisson problem (3.1) with $g = 0$. Then*

1. $u \in H_{loc}^2(\Omega)$ and for each open subset $V \subset\subset \Omega$ we have the estimate

$$\|u\|_{H^2(V)} \leq C \|f\|_{L^2(\Omega)},$$

where the constant $C > 0$ depends only on the open subset V and on the domain Ω itself.

2. Assume $f \in H^m(\Omega)$. Then

$$u \in H_{loc}^{m+2}(\Omega)$$

and for each open subset $V \subset\subset \Omega$ we have the estimate

$$\|u\|_{H^{m+2}(V)} \leq C \|f\|_{H^m(\Omega)},$$

with a constant $C > 0$ depending only on V and Ω .

3. Assume $f \in C^\infty(\Omega)$. Then $u \in C^\infty(\Omega)$.

Remark 4.1.1. *We see that for u solution of the Poisson problem (3.1) with homogeneous Dirichlet boundary conditions the interior regularity on $V \subset\subset \Omega$ does not depend on the geometry of Ω . Nevertheless if we take non homogeneous Dirichlet boundary conditions the solution can only be defined if we have appropriate trace and extensions theorems according to the proof of Theorem 4.1.5, which implies that we need to consider at least a bounded admissible domain.*

Let us also notice that as $-\Delta$ with homogeneous Dirichlet data is a symmetric elliptic operator we have by [30] (Section 6.5.1 p. 334):

Theorem 4.1.7. [30] *Let Ω be a bounded arbitrary connected domain in \mathbb{R}^n ($n \geq 2$). We consider the symmetric elliptic operator $-\Delta$ with homogeneous Dirichlet data on Ω . Then*

1. *The eigenvalues of $-\Delta$ are at most countable, of finite multiplicity and real.*

2. If we repeat each eigenvalue according to its multiplicity, we have

$$\{\lambda_k\}_{k=1}^{\infty},$$

where

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$$

and

$$\lambda_k \rightarrow +\infty \text{ for } k \rightarrow +\infty.$$

3. There exists an orthonormal basis $\{w_k\}_{k=1}^{\infty}$ of $L^2(\Omega)$, where $w_k \in H_0^1(\Omega) \cap C^\infty(\Omega)$ is an eigenfunction corresponding to λ_k : for $k = 1, 2, \dots$ in a weak sense

$$\begin{cases} -\Delta w_k = \lambda_k w_k \text{ in } \Omega, \\ w_k = 0 \text{ on } \partial\Omega. \end{cases}$$

Remark 4.1.2. The key point to work on arbitrary domain is the compactness of the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ which is necessary with respect to the proof presented in Ref. [30]. By the weak formulation, $\{w_k\}_{k=1}^{\infty}$ is an orthogonal basis of $H_0^1(\Omega)$.

The results in Theorems 4.1.1–4.1.7 permit to see that the regularity of the datum f in the Poisson problem (3.1) with the homogeneous Dirichlet boundary conditions ensures the regularity of the solution in the interior of the domain even if Ω has a fractal boundary. As we only have local regularity results we have to consider in which case we can expand these results. For this we recall the result from [30] for regular domains.

Theorem 4.1.8. [30] Let Ω be a bounded open set. Assume $f \in H^m(\Omega)$ for $m \in \mathbb{N}$ and $u \in H_0^1(\Omega)$ is the weak solution of the Poisson problem (3.1) with $g = 0$. Assume finally

$$\partial\Omega \text{ is of class } C^{m+2}.$$

Then

$$u \in H^{m+2}(\Omega)$$

and we have the estimate

$$\|u\|_{H^{m+2}(\Omega)} \leq C \|f\|_{H^m(\Omega)},$$

with a constant $C > 0$ depending only on m, Ω .

Remark 4.1.3. The work of Grisvard [34] tells us that in dimension $n = 2$ this result is also true for convex polygonal domains.

As a consequence, in a bounded domain Ω with a smooth boundary if $f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ is a weak solution of the Poisson problem (3.1) with $g = 0$ then $u \in H^2(\Omega)$. This is no longer true for domains with a fractal boundary even if the source term f is very regular. For example we have:

Theorem 4.1.9. [76] Let $\Omega \subset \mathbb{R}^2$ be von Koch's snowflake. Let $f \in \mathcal{D}(\Omega)$ be non negative and non identically zero. Let $u \in H_0^1(\Omega)$ be the weak solution of the Poisson problem (3.1) with $g = 0$. Then $u \notin H^2(\Omega)$.

This theorem implies that for von Koch's snowflake as $u \notin H^2(\Omega)$, we have not proved that the solution $u \in C(\overline{\Omega})$, a fact observed in reality for example in the case of a drum membrane. In dimension $n = 3$ we can use the next theorem:

Theorem 4.1.10. [92] *Let Ω be an arbitrary open set in \mathbb{R}^3 . If $u \in H_0^1(\Omega)$ and $\Delta u \in L^2(\Omega)$, then*

$$\|u\|_{L^\infty(\Omega)} \leq \frac{1}{\sqrt{2\pi}} \|\nabla u\|_{L^2(\Omega)}^{1/2} \|\Delta u\|_{L^2(\Omega)}^{1/2}.$$

The constant $\frac{1}{\sqrt{2\pi}}$ is the best possible for all Ω .

Using Theorems 4.1.5, 4.1.6 and 4.1.10 we can deduce the result:

Corollary 4.1.1. *Let Ω be a bounded arbitrary domain in \mathbb{R}^3 . Assume $f \in L^2(\Omega)$. Suppose furthermore that $u \in H_0^1(\Omega)$ is the weak solution of the Poisson problem (3.1) with $g = 0$. Then*

$$u \in C(\overline{\Omega})$$

and we have the estimate

$$\|u\|_{L^\infty(\Omega)} \leq C \|f\|_{L^2(\Omega)}$$

with a constant $C > 0$ depending only on Ω .

Proof. On Ω a bounded arbitrary domain we have, by the Lax-Milgram theorem for $u \in H_0^1(\Omega)$ the weak solution of the Poisson problem (3.1) with $g = 0$, the estimate with $C > 0$

$$\|\nabla u\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)} = C \|\Delta u\|_{L^2(\Omega)}.$$

Then we use the estimate of Theorem 4.1.10 to conclude. □

For a similar result in dimension $n = 2$ we use again [76]. The domain used are the Non Tangentially Accessible (NTA) domains introduced in [43] with the help of the notion of Harnack chain :

Definition 4.1.6. [43] (Harnack chain) *An M non-tangential ball in a domain Ω is a ball $B(A, r)$ in Ω whose distance from $\partial\Omega$ is comparable to its radius:*

$$Mr > d(B(A, r), \partial\Omega) > M^{-1}r.$$

For P_1, P_2 in Ω , a Harnack chain from P_1 to P_2 in Ω is a sequence of M non-tangential balls such that the first ball contains P_1 , the last contains P_2 , and such that consecutive balls have non empty intersections.

Definition 4.1.7. [43] (NTA domain) *A bounded domain $\Omega \subset \mathbb{R}^n$ is called NTA when there exist constants M and r_0 such that:*

1. *Corkscrew condition: For any $Q \in \partial\Omega$, $r < r_0$, there exists $A = A_r(Q) \in \Omega$ such that $M^{-1}r < |A - Q| < r$ and $d(A, \partial\Omega) > M^{-1}r$.*
2. *$\mathbb{R}^n \setminus \overline{\Omega}$ satisfies the Corkscrew condition.*
3. *Harnack chain condition: If $\epsilon > 0$ and P_1 and P_2 belong to Ω , $d(P_j; \partial\Omega) > \epsilon$ and $|P_1 - P_2| < C\epsilon$, then there exists a Harnack chain from P_1 to P_2 whose length depends on C and not on ϵ .*

Let us describe the geometry of NTA domains in the plane. There is a close connection between NTA domains and the theory of quasi-conformal mappings. By a quasicircle is meant the image of a circle by a quasi conformal mapping. A domain bounded by a quasicircle is called a quasidisc. For the theory on quasi-conformal mappings we can refer to [31] and [90] for example.

Definition 4.1.8. *A simple closed curve in the plane is said to satisfy Ahlfors' three point condition if for any points z_1, z_2 of the curve and any z_3 on the arc between z_1 and z_2 of smaller diameter the distance between z_1 and z_3 is bounded by a constant times the distance between z_1 and z_2 .*

With this comes the next theorem:

Theorem 4.1.11. [3, 45] *Let Ω be a bounded simply connected subset of the plane. Then the following statements are equivalent:*

1. Ω is a quasidisc.
2. $\partial\Omega$ satisfies the Ahlfors' three point condition.
3. Ω is an NTA domain.

Proof. (1) \Leftrightarrow (2) is due to [3] and (1) \Leftrightarrow (3) is due to [45]. □

With the work of [46] on quasiconformal mappings Theorem 4.1.1 implies:

Corollary 4.1.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected set. Then Ω is an admissible domain if and only if it is an NTA domain.*

For $f \in L^2(\Omega)$ we note $Gf \in H_0^1(\Omega)$ the Green potential solution of the Poisson problem (3.1) with $g = 0$. The work in [77] gives us the following theorem

Theorem 4.1.12. *Let $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ be a bounded simply connected NTA-domain. Let $q_0 = 1 + \frac{1}{1-\beta(M)} > 2$, where $\beta(M) > 0$ is a constant describing the behavior of the Green function near the boundary (see Lemma 4.1.1 below) and M is a constant which appears in the Definition 4.1.7 of Ω . Then there exists constants $\epsilon = \epsilon(\Omega)$ and $C = C(\Omega, q)$ such that if $\frac{n}{n-1} < q < q_0 + \epsilon$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$, then the following inequality is valid for all $f \in L^p(\Omega)$,*

$$\|\nabla Gf\|_{L^q(\Omega)} \leq C\|f\|_{L^p(\Omega)} = C\|\Delta Gf\|_{L^p(\Omega)}.$$

Remark 4.1.4. *The same results hold true for Lipschitz's domain but the work of Dahlberg [24] permits to ensure $q_0 \geq 4$ in dimension $n = 2$ and $q_0 \geq 3$ in dimension $n \geq 3$.*

The $\beta = \beta(M) > 0$ in the statement of Theorem 4.1.12 is the β described in the following lemma.

Lemma 4.1.1. *Let $Q_0 \in \partial\Omega$, $\Delta(Q_0, r) := B(Q_0, r) \cap \partial\Omega$ for all $r > 0$, $w(x, \Delta(Q_0, r), \Omega)$ denotes the harmonic measure on Ω and $d(y, \partial\Omega)$ denotes the Euclidean distance from y to $\partial\Omega$.*

Then there exists a constant $C = C(n)$ such that if

$$Cr < r_0 \text{ and } x \in \Omega \setminus B(Q_0, Cr),$$

then the following estimate is valid with $\beta = \beta(M) > 0$ for all $y \in B(Q_0, r) \cap \Omega$,

$$G(x, y) \leq C(M, n) \frac{d(y, \partial\Omega)^\beta}{r^{n-2+\beta}} w(x, \Delta(Q_0, r), \Omega),$$

where G is the Green potential associated to Ω for the Poisson problem (3.1) with $g = 0$.

The work in [51] and [54] on the well-posedness of the Westervelt equation need estimates that are true for bounded domains with a regular C^2 -boundary. We present their analogous versions, which are necessary to obtain the similar results of well-posedness for admissible domains.

Proposition 4.1.2. *Let Ω be a bounded connected admissible domain in \mathbb{R}^n for $n = 2$ or 3 with a d -set boundary $\partial\Omega$ such that $n - 2 < d < n$. Let*

$$\beta_1 = 1 - \frac{n-d}{2} \text{ and } \beta_2 = 2 - \frac{n-d}{2}.$$

For $w \in H^1(\Omega)$ with $\Delta w \in L^2(\Omega)$ and $Tr_{\partial\Omega} w \in B_{\beta_2}^{2,2}(\partial\Omega)$ we have

$$H^1(\Omega) \subset L^2(\Omega) \text{ with } \|w\|_{L^2(\Omega)} \leq \tilde{C}_0(\|\nabla w\|_{L^2(\Omega)} + \|Tr_{\partial\Omega} w\|_{B_{\beta_1}^{2,2}(\partial\Omega)}), \quad (4.1)$$

$$\|\nabla w\|_{L^2(\Omega)} \leq \hat{C}_0(\|\Delta w\|_{L^2(\Omega)} + \|Tr_{\partial\Omega} w\|_{B_{\beta_2}^{2,2}(\partial\Omega)}), \quad (4.2)$$

$$H^1(\Omega) \subset L^6(\Omega) \text{ with } \|w\|_{L^6(\Omega)} \leq \tilde{C}_1(\|\nabla w\|_{L^2(\Omega)} + \|Tr_{\partial\Omega} w\|_{B_{\beta_1}^{2,2}(\partial\Omega)}), \quad (4.3)$$

$$\|w\|_{L^\infty(\Omega)} \leq \tilde{C}_2(\|\Delta w\|_{L^2(\Omega)} + \|Tr_{\partial\Omega} w\|_{B_{\beta_2}^{2,2}(\partial\Omega)}), \quad (4.4)$$

$$L^{\frac{6}{5}}(\Omega) \subset H^{-1}(\Omega), \text{ with } \|w\|_{H^{-1}(\Omega)} \leq \tilde{C}_3 \|w\|_{L^{\frac{6}{5}}(\Omega)}. \quad (4.5)$$

Moreover for $n = 2$ we fix $p_1 > 2$ and $p'_1 > 2$ such that $2 < p_1 < q_0 + \epsilon$ (see Theorem 4.1.12) and $\frac{1}{p_1} + \frac{1}{p'_1} = \frac{1}{2}$ and note $C_{p_1} > 0$, $C_{p'_1} > 0$ such that

$$\|\nabla w\|_{L^{p_1}(\Omega)} \leq C_{p_1}(\|\Delta w\|_{L^2(\Omega)} + \|Tr_{\partial\Omega} w\|_{B_{\beta_2}^{2,2}(\partial\Omega)}), \quad (4.6)$$

$$\|w\|_{L^{p'_1}(\Omega)} \leq C_{p'_1}(\|\nabla w\|_{L^2(\Omega)} + \|Tr_{\partial\Omega} w\|_{B_{\beta_1}^{2,2}(\partial\Omega)}). \quad (4.7)$$

Proof. The estimates (4.1) and (4.3) are a direct consequence of Proposition 3 in Ref. [11] as the norm $\sqrt{\|\nabla \cdot\|_{L^2(\Omega)}^2 + \|Tr_{\partial\Omega} \cdot\|_{L^2(\partial\Omega)}^2}$ is equivalent to the H^1 -norm and by Ref. [11] for instance $B_{\beta_1}^{2,2}(\partial\Omega) \subset\subset L^2(\partial\Omega)$.

Estimate (4.5) comes from Theorem 4.1.2 and duality. In dimension $n = 2$ we have by Theorem 4.1.1

$$E_{\partial\Omega}(Tr_{\partial\Omega} w) \in H^2(\Omega) \text{ with } \|E_{\partial\Omega}(Tr_{\partial\Omega} w)\|_{H^2(\Omega)} \leq C \|Tr_{\partial\Omega} w\|_{B_{\beta_2}^{2,2}(\partial\Omega)}$$

and $Tr_{\partial\Omega}[w - E_{\partial\Omega}(Tr_{\partial\Omega} w)] = 0$.

Then it implies $w - E_{\partial\Omega}(Tr_{\partial\Omega} w) \in H_0^1(\Omega)$ and $\Delta[w - E_{\partial\Omega}(Tr_{\partial\Omega} w)] \in L^2(\Omega)$. So by Theorem 4.1.12 we take $p_1 > 2$ such that $2 < p_1 < q_0 + \epsilon$ to obtain

$$\|\nabla[w - E_{\partial\Omega}(Tr_{\partial\Omega} w)]\|_{L^{p_1}(\Omega)} \leq C(p_1, \Omega) \|\Delta[w - E_{\partial\Omega}(Tr_{\partial\Omega} w)]\|_{L^{\frac{2p_1}{2+p_1}}(\Omega)}.$$

But $1 < \frac{2p_1}{2+p_1} < 2$ and Ω bounded so $L^2(\Omega) \hookrightarrow L^{\frac{2p_1}{2+p_1}}(\Omega)$ so we can obtain estimate (4.6) by the fact that

$$\begin{aligned} \|\nabla w\|_{L^{p_1}(\Omega)} &\leq \|\nabla[w - E_{\partial\Omega}(Tr_{\partial\Omega} w)]\|_{L^{p_1}(\Omega)} + \|\nabla E_{\partial\Omega}(Tr_{\partial\Omega} w)\|_{L^{p_1}(\Omega)} \\ &\leq C \|\Delta[w - E_{\partial\Omega}(Tr_{\partial\Omega} w)]\|_{L^{\frac{2p_1}{2+p_1}}(\Omega)} + C \|E_{\partial\Omega}(Tr_{\partial\Omega} w)\|_{H^2(\Omega)} \\ &\leq C \|\Delta w\|_{L^2(\Omega)} + C \|\Delta E_{\partial\Omega}(Tr_{\partial\Omega} w)\|_{L^2(\Omega)} + C \|E_{\partial\Omega}(Tr_{\partial\Omega} w)\|_{H^2(\Omega)} \\ &\leq C \|\Delta w\|_{L^2(\Omega)} + C \|E_{\partial\Omega}(Tr_{\partial\Omega} w)\|_{H^2(\Omega)} \\ &\leq C \|\Delta w\|_{L^2(\Omega)} + C \|Tr_{\partial\Omega} w\|_{B_{\beta_2}^{2,2}(\partial\Omega)}. \end{aligned}$$

We can deduce estimate (4.2) in the same way but also estimate (4.4) as $W_0^{1,p_1}(\Omega) \subset L^\infty(\Omega)$ by Theorem 4.1.2. In dimension $n = 3$ we use again $E_{\partial\Omega}(Tr_{\partial\Omega}w)$ with Corollary 4.1.1 and Theorem 4.1.1 to obtain estimate (4.4).

The proof of estimate (4.7) is not different from the proof of estimate (4.3) as $p'_1 > 2$ and we are in dimension $n = 2$. \square

Remark 4.1.5. *Estimates (4.1)–(4.5) are very similar to those used in [51] and [54] for a regular domain, with the Besov spaces replacing $H^{3/2}(\partial\Omega)$ and $H^{1/2}(\Omega)$*

$$\begin{aligned} \|w\|_{L^2(\Omega)} &\leq \tilde{C}_0(\|\nabla w\|_{L^2(\Omega)} + \|Tr_{\partial\Omega}w\|_{H^{1/2}(\partial\Omega)}), \\ \|\nabla w\|_{L^2(\Omega)} &\leq \hat{C}_0(\|\Delta w\|_{L^2(\Omega)} + \|Tr_{\partial\Omega}w\|_{H^{3/2}(\partial\Omega)}), \\ \|w\|_{L^6(\Omega)} &\leq \tilde{C}_1(\|\nabla w\|_{L^2(\Omega)} + \|Tr_{\partial\Omega}w\|_{H^{1/2}(\partial\Omega)}), \\ \|\nabla w\|_{L^6(\Omega)} &\leq \hat{C}_1(\|\Delta w\|_{L^2(\Omega)} + \|Tr_{\partial\Omega}w\|_{H^{3/2}(\partial\Omega)}), \\ \|w\|_{L^\infty(\Omega)} &\leq \tilde{C}_2(\|\Delta w\|_{L^2(\Omega)} + \|Tr_{\partial\Omega}w\|_{H^{3/2}(\partial\Omega)}), \\ \|w\|_{H^{-1}(\Omega)} &\leq \tilde{C}_3\|w\|_{L^{\frac{6}{5}}(\Omega)}. \end{aligned}$$

Nevertheless the Theorem 4.1.12 tells us that, in a general NTA-domain or Lipschitz domain, we do not have the estimate

$$\|\nabla w\|_{L^6(\Omega)} \leq C(\|\Delta w\|_{L^2(\Omega)} + \|Tr_{\partial\Omega}w\|_{B_{\beta_2}^{2,2}(\partial\Omega)}),$$

which implies to make a sly modification in the proof of [54]. In dimension $n = 2$ this estimate stays true for convex polygonal domains by the work of [34], which allows to extend the results of well-posedness in [51, 52, 53, 54] found initially for a regular C^2 boundary.

4.1.3 The Poisson equation with homogeneous Robin boundary conditions

Definition 4.1.9. For $a > 0$, we define the norm $\|\cdot\|_{\tilde{H}^1(\Omega)}$ in $H^1(\Omega)$ by

$$\|u\|_{\tilde{H}^1(\Omega)} = \int_{\Omega} |\nabla u|^2 dx + a \int_{\partial\Omega} |Tr_{\partial\Omega}u|^2 dm_d. \quad (4.8)$$

For $f \in L^2(\Omega)$, we say that

$$u \in H^1(\Omega)$$

is a weak solution of the Poisson problem (3.2) provided for all $v \in H^1(\Omega)$

$$(u, v)_{\tilde{H}^1(\Omega)} = \int_{\Omega} \nabla u \nabla v dx + a \int_{\partial\Omega} Tr_{\partial\Omega}u Tr_{\partial\Omega}v dm_d = \int_{\Omega} f v dx = (f, v)_{L^2(\Omega)}.$$

Proposition 3 in [11] tells us that $\|\cdot\|_{\tilde{H}^1(\Omega)}$ is equivalent to the usual norm on $H^1(\Omega)$ if Ω is a bounded admissible domain so the Lax-Milgram Theorem gives us:

Theorem 4.1.13. *Let Ω be a bounded admissible domain in \mathbb{R}^n . For all $f \in L^2(\Omega)$ and $a > 0$ there exists a unique weak solution $u \in H^1(\Omega)$ of the Poisson problem (3.2) and we have the estimate*

$$\|u\|_{\tilde{H}^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

Now we give a result on eigenvalues that will be very useful:

Theorem 4.1.14. *Let Ω be a bounded connected admissible domain in \mathbb{R}^n ($n \geq 2$) with a d -set boundary $\partial\Omega$ such that $n - 2 < d < n$. Set $\lambda \in \mathbb{C}$ an eigenvalue of the Poisson problem (3.2) such that there exists u weak solution of*

$$\begin{cases} -\Delta u = \lambda u \text{ in } \Omega, \\ \frac{\partial}{\partial n} u + au = 0 \text{ on } \partial\Omega. \end{cases}$$

We have the following assertions:

1. Each eigenvalue is real.
2. If we repeat each eigenvalue according to its finite multiplicity the set of all eigenvalues $\{\lambda_i\}$ is countable and we can write

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$$

with

$$\lambda_k \rightarrow +\infty \text{ when } k \rightarrow +\infty.$$

3. There exists an orthonormal basis $\{w_k\}_{k=1}^{+\infty}$ of $L^2(\Omega)$, where $w_k \in H^1(\Omega)$ is an eigenfunction corresponding to λ_k .

Proof. If we consider the unique solution u of the Poisson Problem (3.2) then the operator

$$S : f \mapsto u$$

is a bounded, linear operator mapping $f \in L^2(\Omega)$ into $H^1(\Omega)$ by Theorem 4.1.13.

But Theorem 4.1.2 implies the compactness of the embedding $\Lambda : H^1(\Omega) \rightarrow L^2(\Omega)$, so $\tilde{S} = \Lambda \circ S$ is a bounded, linear, compact operator mapping $L^2(\Omega)$ into itself.

We also have that \tilde{S} is symmetric positive in $L^2(\Omega)$. The theory of compact, symmetric operators that we can find in [17, 30] for example permits to obtain the theorem as $\tilde{S}w = \frac{1}{\lambda}w$ if and only if $-\Delta w = \lambda w$ and moreover by definition of S and \tilde{S} we have $w = \lambda\tilde{S}w \in H^1(\Omega)$. \square

Remark 4.1.6. *As the solutions space is $H^1(\Omega)$ we need to use admissible domains in order to have the compactness of the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ which is different to the case of homogeneous Dirichlet boundary conditions where we could use arbitrary domains.*

Definition 4.1.10. *For Ω an admissible bounded domain in \mathbb{R}^n ($n = 2$ or 3),*

$$f \in L^p(\Omega)$$

with $p \geq 2$ and $a > 0$, we take the norm $\|\cdot\|_{\tilde{H}^1(\Omega)}$ on $H^1(\Omega)$ defined in (4.8) and we define

$$u \in H^1(\Omega)$$

as a weak solution of the Poisson problem (3.2) such that

$$\forall v \in H^1(\Omega), \quad (u, v)_{\tilde{H}^1(\Omega)} = (f, v)_{L^2(\Omega)}.$$

Given this definition we have the equivalent of Theorem 4.1.13 for $f \in L^p(\Omega)$ as for Ω a bounded domain we have $L^p(\Omega) \hookrightarrow L^2(\Omega)$ if $p \geq 2$.

Definition 4.1.11. For Ω a bounded admissible domain and $p \geq 2$, we will denote by $\mathcal{D}(-\Delta) \subset H^1(\Omega)$ the domain in $L^p(\Omega)$ of operator $-\Delta$ on Ω in a weak sense with the homogeneous Robin boundary conditions in accordance with Theorem 4.1.13:

$$\begin{aligned} -\Delta : \mathcal{D}(-\Delta) \subset H^1(\Omega) &\rightarrow L^p(\Omega) \\ u &\mapsto -\Delta u. \end{aligned}$$

For the continuity of the solution of the Poisson problem (3.2) we have the following result coming directly from [25] with just a generalization to the class of admissible domains coming from the trace, extension and embedding results of Theorems 4.1.1 and 4.1.2.

Theorem 4.1.15. For $p > n$ and Ω an admissible domain in \mathbb{R}^n ($n = 2$ or 3) with a d -set boundary $\partial\Omega$ such that $n - 2 < d < n$. Let u be the unique solution of the Poisson problem (3.2) for $f \in L^p(\Omega)$ then

$$\|u\|_{L^\infty(\Omega)} \leq C \max\left(1, \frac{1}{a}\right) \|f\|_{L^p(\Omega)}.$$

4.2 Well posedness of the linear wave equation with homogeneous Dirichlet boundary conditions

4.2.1 Existence of weak solution

In this subsection we recall the results of Evans [30] for the weak well-posedness of the linear wave equation holding on all arbitrary bounded domains in \mathbb{R}^n ($n \geq 2$).

Definition 4.2.1. [30] For $f \in L^2((0, T); L^2(\Omega))$, $u_0 \in H_0^1(\Omega)$, and $u_1 \in L^2(\Omega)$, we say a function $u \in L^2([0, T]; H_0^1(\Omega))$ with $\partial_t u \in L^2([0, T]; L^2(\Omega))$ and $\partial_t^2 u \in L^2([0, T]; H^{-1}(\Omega))$ is a weak solution of the hyperbolic initial/boundary value problem

$$\begin{cases} u_{tt} - c^2 \Delta u = f & \text{on }]0, T] \times \Omega, \\ u|_{\partial\Omega} = 0 & \text{on } [0; T] \times \partial\Omega, \\ u(0) = u_0, \quad u_t(0) = u_1, \end{cases} \quad (4.9)$$

provided

$$\langle u_{tt}, v \rangle_{(H^{-1}(\Omega), H_0^1(\Omega))} + c^2 (\nabla u, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)},$$

for each $v \in H_0^1(\Omega)$ and a.e. time $0 \leq t \leq T$, with

$$u(0) = u_0, \quad u_t(0) = u_1.$$

Here $\langle \cdot, \cdot \rangle_{(H^{-1}(\Omega), H_0^1(\Omega))}$ means the duality product in $H_0^1(\Omega)$ and $(\cdot, \cdot)_{L^2(\Omega)}$ is the inner product.

To construct our weak solution of problem (4.9) for an arbitrary domain Ω , we use Galerkin's method. We select smooth functions $w_k = w_k(x)$, ($k \geq 1$) such that

$$\{w_k\}_{k=1}^{\infty} \text{ is an orthogonal basis of } H_0^1(\Omega) \quad (4.10)$$

and

$$\{w_k\}_{k=1}^{\infty} \text{ is an orthonormal basis of } L^2(\Omega). \quad (4.11)$$

We use the normalized eigenfunctions of the operator $-\Delta$ on Ω with homogeneous Dirichlet boundary conditions, defined in Theorem 4.1.7, that is to say

$$-\Delta w_k = \lambda_k w_k \text{ in the weak sense, i.e. } \forall v \in H_0^1(\Omega) \quad (\nabla w_k, \nabla v)_{L^2(\Omega)} = \lambda_k (w_k, v)_{L^2(\Omega)}. \quad (4.12)$$

We fix a positive integer m and write

$$u_m(t) := \sum_{i=1}^m d_m^k(t) w_k, \quad (4.13)$$

where we intend to select the coefficients $d_m^k(t)$ ($0 \leq t \leq T$, $k = 1, \dots, m$) to satisfy using the initial conditions

$$d_m^k(0) = (u_0, w_k)_{L^2(\Omega)} \quad (k = 1, \dots, m) \quad (4.14)$$

$$\partial_t d_m^k(0) = (u_1, w_k)_{L^2(\Omega)} \quad (k = 1, \dots, m) \quad (4.15)$$

and

$$(\partial_t^2 u_m, w_k)_{L^2(\Omega)} + c^2 (\nabla u_m, \nabla w_k)_{L^2(\Omega)} = (f, w_k)_{L^2(\Omega)} \quad (0 \leq t \leq T, k = 1, \dots, m). \quad (4.16)$$

Theorem 4.2.1. ([30] p. 380) *For each integer $m = 1, 2, \dots$, there exists a unique function u_m of form (4.13) satisfying (4.14)-(4.16).*

Our plan is hereafter to make $m \rightarrow \infty$, and so we need some estimates uniform on m .

Theorem 4.2.2. ([30] p. 381) *There exists a constant $C > 0$, depending only on the domain Ω and the final time T , such that*

$$\begin{aligned} \max_{0 \leq t \leq T} (\|u_m(t)\|_{H_0^1(\Omega)}^2 + \|\partial_t u_m(t)\|_{L^2(\Omega)}^2) + \|\partial_t^2 u_m\|_{L^2((0,T);H^{-1}(\Omega))}^2 \\ \leq C (\|f\|_{L^2((0,T);L^2(\Omega))}^2 + \|u_0\|_{H_0^1(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2) \end{aligned} \quad (4.17)$$

for $m = 1, 2, \dots$

Now we pass to limits in our Galerkin approximations.

Theorem 4.2.3. ([30] p. 384) *For Ω an arbitrary bounded domain, there exists a weak solution of the wave equation problem (4.9) in the sense of Definition 4.2.1.*

Theorem 4.2.4. ([30] p. 385) *For Ω an arbitrary bounded domain, a weak solution of the wave equation problem (4.9) is unique.*

4.2.2 Regularity results

Theorems 4.1.6 and 4.1.9 tell us that when we take higher regularity for the data of the wave equation problem (4.9) on an arbitrary bounded domain Ω in \mathbb{R}^n ($n \geq 2$) we can not have the same regularity results presented in [30] for regular domains. For example we can not have in general $u \in L^\infty([0, T]; H^2(\Omega))$ for a weak solution u of problem (4.9). Nevertheless we have the following result which improves [30] concerning the interior regularity:

Theorem 4.2.5. *Let Ω be an arbitrary bounded domain in \mathbb{R}^n ($n \geq 2$):*

(i) *Assume*

$$u_0 \in H_0^1(\Omega), \quad u_1 \in L^2(\Omega), \quad f \in L^2([0, T]; L^2(\Omega)),$$

and suppose also $u \in L^2([0, T]; H_0^1(\Omega))$ with $\partial_t u \in L^2([0, T]; L^2(\Omega))$ and $\partial_t^2 u \in L^2([0, T]; H^{-1}(\Omega))$ is the weak solution of the wave equation problem (4.9). Then in fact

$$u \in L^\infty([0, T]; H_0^1(\Omega)), \quad \partial_t u \in L^\infty([0, T]; L^2(\Omega)),$$

and we have the estimate

$$\begin{aligned} \operatorname{ess\,sup}_{0 \leq t \leq T} (\|u(t)\|_{H_0^1(\Omega)} + \|\partial_t u(t)\|_{L^2(\Omega)}) + \|\partial_t^2 u\|_{L^2([0, T]; H^{-1}(\Omega))} \\ \leq C(\|f\|_{L^2([0, T]; L^2(\Omega))} + \|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)}). \end{aligned}$$

(ii) *If, in addition*

$$u_0 \in H^2(\Omega), \quad u_1 \in H_0^1(\Omega), \quad \partial_t f \in L^2([0, T]; L^2(\Omega)),$$

then

$$\begin{aligned} u \in L^\infty([0, T]; H_{loc}^2(\Omega)) \cap L^\infty([0, T]; H_0^1(\Omega)), \quad \partial_t u \in L^\infty([0, T]; H_0^1(\Omega)), \\ \partial_t^2 u \in L^\infty([0, T]; L^2(\Omega)), \quad \partial_t^3 u \in L^2([0, T]; H^{-1}(\Omega)), \end{aligned}$$

with the following estimate for each open subset $V \subset\subset \Omega$

$$\begin{aligned} \operatorname{ess\,sup}_{0 \leq t \leq T} (\|u(t)\|_{H^2(V)} + \|u(t)\|_{H_0^1(\Omega)} + \|\partial_t u(t)\|_{H_0^1(\Omega)} + \|\partial_t^2 u(t)\|_{L^2(\Omega)}) + \|\partial_t^3 u\|_{L^2([0, T]; H^{-1}(\Omega))} \\ \leq C(\|f\|_{H^1([0, T]; L^2(\Omega))} + \|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^1(\Omega)}) \end{aligned} \quad (4.18)$$

with a constant $C > 0$ depending of V , Ω and T .

Proof. In the previous section we have proved for u_m of form (4.13) satisfying (4.14)–(4.16) the energy estimate (4.17). The unique weak solution u of the wave equation problem (4.9) is constructed in [30] (see p. 384) as the weak limit of a subsequence $\{u_{m_l}\}_{l=1}^\infty \subset \{u_m\}_{m=1}^\infty$ with

$$\begin{cases} u_{m_l} \rightharpoonup u \text{ weakly in } L^2([0, T]; H_0^1(\Omega)), \\ \partial_t u_{m_l} \rightharpoonup \partial_t u \text{ weakly in } L^2([0, T]; L^2(\Omega)), \\ \partial_t^2 u_{m_l} \rightharpoonup \partial_t^2 u \text{ weakly in } L^2([0, T]; H^{-1}(\Omega)). \end{cases} \quad (4.19)$$

Passing to limits in estimate (4.17) for $m = m_l \rightarrow \infty$, we deduce (i).

Assume now the hypothesis of assertion (ii). We consider again u_m of form (4.13) satisfying (4.14)-(4.16). We also suppose $m = m_l$, where m_l is defined for the convergence result (4.19). Fix a positive integer m and next differentiate identity (4.16) with respect to t . Writing $\tilde{u}_m := \partial_t u_m$ we obtain

$$(\partial_t \tilde{u}_m, w_k)_{L^2(\Omega)} + c^2(\nabla \tilde{u}_m, \nabla w_k)_{L^2(\Omega)} = (\partial_t f, w_k)_{L^2(\Omega)} \quad (0 \leq t \leq T, k = 1, \dots, m).$$

We have $\tilde{u}_m(0) = \partial_t u_m(0) = \sum_{i=1}^m (u_1, w_i)_{L^2(\Omega)} w_i$ and $\partial_t \tilde{u}_m(0) = \sum_{k=0}^m \partial_t^2 d_m^k(0) w_k$, then we use identities (4.16)

$$\begin{aligned} \partial_t^2 d_m^k(0) &= (f(0), w_k)_{L^2(\Omega)} - \sum_{l=1}^m c^2(\nabla w_k, \nabla w_l)_{L^2(\Omega)} d_m^l(0) \\ &= (f(0), w_k)_{L^2(\Omega)} - c^2 \|\nabla w_k\|_{L^2(\Omega)}^2 (u_0, w_k)_{L^2(\Omega)} \\ &= (f(0), w_k)_{L^2(\Omega)} - c^2 \lambda_k (u_0, w_k)_{L^2(\Omega)} \\ &= (f(0) + c^2 \Delta u_0, w_k) \end{aligned}$$

with (4.12). We have $\partial_t f \in L^2([0, T]; L^2(\Omega))$, $u_1 \in H_0^1(\Omega)$ and $f(0) + c^2 \Delta u_0 \in L^2(\Omega)$ as $u_0 \in H^2(\Omega)$ and $f \in H^1([0, T]; L^2(\Omega)) \subset C([0, T]; L^2(\Omega))$. It implies by [30] (see Section 7.2.2 p. 980) with the results of the last section that a subsequence of \tilde{u}_m weakly converges to \tilde{u} unique weak solution of the wave equation problem

$$\begin{cases} \tilde{u}_{tt} - c^2 \Delta \tilde{u} = \partial_t f & \text{on }]0, T] \times \Omega, \\ \tilde{u}|_{\partial\Omega} = 0 & \text{on } [0; T] \times \partial\Omega, \\ \tilde{u}(0) = u_0, \quad \tilde{u}_t(0) = f(0) - \Delta u_0. \end{cases}$$

As $\tilde{u}_m = \partial_t u_m$ and $\partial_t u_m$ weakly converge to $\partial_t u$ by (4.19) the unicity of the limit implies

$$\tilde{u} = \partial_t u.$$

Then we have u the weak solution of the Dirichlet initial-valued problem for the linear wave equation (4.9) with

$$\begin{aligned} u &\in L^\infty([0, T]; H_0^1(\Omega)), \quad \partial_t u \in L^\infty([0, T]; H_0^1(\Omega)), \\ \partial_t^2 u &\in L^\infty([0, T]; L^2(\Omega)), \quad \partial_t^3 u \in L^2([0, T]; H^{-1}(\Omega)) \end{aligned}$$

and, by point (i) of this theorem applied to u and u_t , we deduce

$$\begin{aligned} &\sup_{0 \leq t \leq T} (\|u(t)\|_{H_0^1(\Omega)}^2 + \|\partial_t u(t)\|_{H_0^1(\Omega)}^2 + \|\partial_t^2 u(t)\|_{L^2(\Omega)}^2) + \|\partial_t^3 u\|_{L^2([0, T]; H^{-1}(\Omega))}^2 \\ &\leq C(\|f\|_{H^1([0, T]; L^2(\Omega))}^2 + \|u_0\|_{H_0^1(\Omega)}^2 + \|u_1\|_{H_0^1(\Omega)}^2 + \|f(0)\|_{L^2(\Omega)}^2 + \|\Delta u_0\|_{L^2(\Omega)}^2) \quad (4.20) \\ &\leq C(\|f\|_{H^1([0, T]; L^2(\Omega))}^2 + \|u_0\|_{H_0^1(\Omega)}^2 + \|u_1\|_{H_0^1(\Omega)}^2 + \|\Delta u_0\|_{L^2(\Omega)}^2). \end{aligned}$$

Set an open subset $V \subset\subset \Omega$ by Theorem 4.1.6 we have

$$\|u\|_{H^2(V)}^2 \leq C \|\Delta u\|_{L^2(\Omega)}^2 \leq C \left(\frac{1}{c^2} \|u_{tt} - f\|_{L^2(\Omega)}^2 \right)$$

as u solution of the linear problem (4.9) and by the Poincaré inequality, so

$$\begin{aligned} \|u\|_{H^2(V)}^2 &\leq C(\|u_{tt}\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2) \\ &\leq C(\|u_{tt}\|_{L^2(\Omega)}^2 + \|f\|_{H^1([0,T];L^2(\Omega))}^2) \\ &\leq C(\|f\|_{H^1([0,T];L^2(\Omega))}^2 + \|u_0\|_{H_0^1(\Omega)}^2 + \|u_1\|_{H_0^1(\Omega)}^2 + \|\Delta u_0\|_{L^2(\Omega)}^2), \end{aligned}$$

by (4.20), and this allows to conclude point (ii) using estimate (4.20). \square

Remark 4.2.1. *The analysis of the proof shows that the condition $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ can be taken instead as $u_0 \in H_0^1(\Omega)$ and $\Delta u_0 \in L^2(\Omega)$ which is less regular.*

Theorem 4.2.6. *Let Ω be an arbitrary bounded domain in \mathbb{R}^n ($n \geq 2$). Set $m \in \mathbb{N}^*$. Assume*

$$u_0 \in H^{m+1}(\Omega), \quad u_1 \in H^m(\Omega), \quad \text{and} \quad \frac{d^k}{dt^k} f \in L^2([0, T]; H^{m-k}(\Omega)) \quad (k = 0, \dots, m).$$

Suppose also the following m^{th} -order compatibility conditions holds

$$\left\{ \begin{array}{l} g_0 := u_0 \in H_0^1(\Omega), \quad h_1 := u_1 \in H_0^1(\Omega), \quad \text{for } l \in \mathbb{N}^*, \\ g_{2l} := \left(\frac{d^{2l-2}}{dt^{2l-2}} f \right) (0) + c^2 \Delta g_{2l-2} \in H_0^1(\Omega) \quad \text{if } 2l \leq m, \\ h_{2l+1} := \left(\frac{d^{2l-1}}{dt^{2l-1}} f \right) (0) + c^2 \Delta g_{2l-1} \in H_0^1(\Omega) \quad \text{if } 2l + 1 \leq m. \end{array} \right. \quad (4.21)$$

Then the weak solution u of problem (4.9) satisfies

$$\frac{d^k}{dt^k} u \in L^\infty([0, T]; H_{loc}^{m-1-k}(\Omega)) \cap L^\infty([0, T]; H_0^1(\Omega)) \quad k = 0, \dots, m,$$

$$\frac{d^{m+1}}{dt^{m+1}} u \in L^\infty([0, T]; L^2(\Omega)),$$

and we have the following estimate for $V \subset\subset \Omega$

$$\begin{aligned} &\text{ess sup}_{0 \leq t \leq T} \left(\sum_{k=0}^m \left(\left\| \frac{d^k}{dt^k} u \right\|_{H^{m+1-k}(V)} + \left\| \frac{d^k}{dt^k} u \right\|_{H_0^1(\Omega)} \right) + \left\| \frac{d^{m+1}}{dt^{m+1}} u \right\|_{L^2(\Omega)} \right) \\ &\leq C \left(\sum_{k=0}^m \left\| \frac{d^k}{dt^k} f \right\|_{L^2([0,T]; H^{m-k}(\Omega))} + \|u_0\|_{H^{m+1}(\Omega)} + \|u_1\|_{H^m(\Omega)} \right). \end{aligned}$$

Remark 4.2.2. *This result of regularity on arbitrary bounded domains is new compared to [30] where regular domains were considered.*

Proof. We prove it by an induction, the case $m = 1$ following from Theorem 4.2.5 above.

Next assume the theorem is valid for some positive integer m and suppose $u_0 \in H^{m+2}(\Omega)$, $u_1 \in H^{m+1}(\Omega)$, and $\frac{d^k f}{dt^k} \in L^2([0, T]; H^{m+1-k}(\Omega))$ ($k = 0, \dots, m + 1$). Suppose also the

$(m + 1)^{th}$ -order compatibility conditions obtained. Now set $\tilde{u} := \partial_t u$. Differentiating the wave equation with respect to t , we check that \tilde{u} is the unique, weak solution of

$$\begin{cases} \tilde{u}_{tt} - c^2 \Delta \tilde{u} = \partial_t f & \text{in } [0, T] \times \Omega, \\ \tilde{u} = 0 & \text{on } \partial U \times [0, T], \\ \tilde{u}(0) = u_1, \tilde{u}_t(0) = f(0) + \Delta u_0. \end{cases}$$

In particular, for $m = 1$, we rely upon Theorem 4.2.5 to obtain the regularity of \tilde{u} . Since u_0, u_1 and f satisfy the $(m + 1)^{th}$ -order compatibility conditions, $u_1, f(0) + \Delta u_0$ and $\partial_t f$ satisfy the m^{th} -order compatibility conditions. Thus applying the induction assumption, we see

$$\begin{aligned} \frac{d^k}{dt^k} \tilde{u} &\in L^\infty([0, T]; H_{loc}^{m-1-k}(\Omega)) \cap L^\infty([0, T]; H_0^1(\Omega)) \quad k = 0, \dots, m, \\ \frac{d^{m+1}}{dt^{m+1}} \tilde{u} &\in L^\infty([0, T]; L^2(\Omega)), \end{aligned}$$

and we have the following estimate for $V \subset\subset \Omega$

$$\begin{aligned} &\text{ess sup}_{0 \leq t \leq T} \left(\sum_{k=0}^m \left(\left\| \frac{d^k}{dt^k} \tilde{u} \right\|_{H^{m+1-k}(V)} + \left\| \frac{d^k}{dt^k} \tilde{u} \right\|_{H_0^1(\Omega)} \right) + \left\| \frac{d^{m+1}}{dt^{m+1}} \tilde{u} \right\|_{L^2(\Omega)} \right) \\ &\leq C \left(\sum_{k=0}^m \left\| \frac{d^k}{dt^k} f_t \right\|_{L^2([0, T]; H^{m-k}(\Omega))} + \|u_1\|_{H^{m+1}(\Omega)} + \|f(0) + \Delta u_0\|_{H^m(\Omega)} \right). \end{aligned}$$

Since $\tilde{u} = \partial_t u$, we can rewrite

$$\begin{aligned} &\text{ess sup}_{0 \leq t \leq T} \left(\sum_{k=1}^{m+1} \left(\left\| \frac{d^k}{dt^k} u \right\|_{H^{m+2-k}(V)} + \left\| \frac{d^k}{dt^k} u \right\|_{H_0^1(\Omega)} \right) + \left\| \frac{d^{m+2}}{dt^{m+2}} u \right\|_{L^2(\Omega)} \right) \quad (4.22) \\ &\leq C \left(\sum_{k=1}^{m+1} \left\| \frac{d^k}{dt^k} f \right\|_{L^2([0, T]; H^{m+1-k}(\Omega))} + \|u_1\|_{H^{m+1}(\Omega)} + \|f(0)\|_{H^m(\Omega)} + \|\Delta u_0\|_{H^m(\Omega)} \right) \\ &\leq C \left(\sum_{k=0}^{m+1} \left\| \frac{d^k}{dt^k} f \right\|_{L^2([0, T]; H^{m+1-k}(\Omega))} + \|u_1\|_{H^{m+1}(\Omega)} + \|u_0\|_{H^{m+2}(\Omega)} \right), \end{aligned}$$

using the inequality

$$\|f\|_{C([0, T]; H^m(\Omega))} \leq C(\|f\|_{L^2([0, T]; H^m(\Omega))} + \|\partial_t f\|_{L^2([0, T]; H^m(\Omega))}).$$

Moreover, Theorem 4.2.5 ensures $u \in L^\infty([0, T]; H_0^1(\Omega))$ with an estimate that we can add to (4.22). We can also write for $V \subset\subset W \subset\subset \Omega$ that

$$-c^2 \Delta u = f - \partial_t^2 u.$$

So, by Theorem 4.1.6 we have

$$\begin{aligned} \|u\|_{H^{m+2}(V)} &\leq C \left(\frac{1}{c^2} \|f - \partial_t^2 u\|_{H^m(W)} \right) \\ &\leq C(\|f\|_{H^m(W)} + \|\partial_t^2 u\|_{H^m(W)}) \\ &\leq C(\|f\|_{L^2([0, T]; H^m(\Omega))} + \|\partial_t f\|_{L^2([0, T]; H^m(\Omega))} + \|\partial_t^2 u\|_{H^m(W)} + \|\nabla u\|_{H_0^1(\Omega)}). \end{aligned}$$

Taking the essential supremum with respect to t , adding this inequality to (4.22) and making estimates as in Theorem 4.2.5 the theorem follows. \square

4.3 Well posedness of the damped linear wave equation with homogeneous Dirichlet boundary conditions

4.3.1 Existence of weak solution

In this subsection for $0 < T \leq +\infty$, we consider Ω an arbitrary bounded domain in \mathbb{R}^n .

Definition 4.3.1. For $f \in L^2([0, T]; L^2(\Omega))$, $u_0 \in H_0^1(\Omega)$, and $u_1 \in L^2(\Omega)$, we say a function

$$u \in L^2([0, T]; H_0^1(\Omega)) \text{ with } \partial_t u \in L^2([0, T]; H_0^1(\Omega)) \text{ and } \partial_t^2 u \in L^2([0, T]; H^{-1}(\Omega))$$

is a weak solution of the hyperbolic initial/boundary value problem

$$\left\{ \begin{array}{l} u_{tt} - c^2 \Delta u - \nu \Delta u_t = f \text{ on }]0, T] \times \Omega, \\ u|_{\partial\Omega} = 0 \text{ on } [0; T] \times \partial\Omega, \\ u(0) = u_0, \quad u_t(0) = u_1, \end{array} \right. \quad (4.23)$$

provided

$$\langle u_{tt}, v \rangle_{(H^{-1}(\Omega), H_0^1(\Omega))} + c^2 (\nabla u, \nabla v)_{L^2(\Omega)} + \nu (\nabla u_t, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)}, \quad (4.24)$$

for each $v \in H_0^1(\Omega)$ and a.e. time $0 \leq t \leq T$, with

$$u(0) = u_0, \quad u_t(0) = u_1.$$

Galerkin approximations

As for the linear wave equation we employ Galerkin's method by selecting smooth functions w_k , $k \geq 1$, satisfying (4.10)–(4.12) as the normalized eigenfunctions of the operator $-\Delta$ on Ω with homogeneous Dirichlet boundary conditions.

We fix a positive integer m and define u_m as in (4.13) such that the coefficients $d_m^k(t)$ defined in (4.13) satisfy (4.14), (4.15) and for $(0 \leq t \leq T, k = 1, \dots, m)$

$$(\partial_t^2 u_m, w_k)_{L^2(\Omega)} + c^2 (\nabla u_m, \nabla w_k)_{L^2(\Omega)} + \nu (\nabla \partial_t u_m, \nabla w_k)_{L^2(\Omega)} = (f, w_k)_{L^2(\Omega)}. \quad (4.25)$$

Theorem 4.3.1. For each integer $m = 1, 2, \dots$, there exists a unique function u_m of the form (4.13) satisfying (4.14), (4.15) and (4.25).

Proof. Let u_m be given by (4.13) the finite decomposition over the eigenfunctions of the Dirichlet-laplacian. Furthermore, we have $e^{kl} := (\nabla w_l, \nabla w_k)_{L^2(\Omega)}$. We also write $f^k = (f, w_k)_{L^2(\Omega)}$. Consequently system (4.25) becomes the linear system of ordinary differential equations

$$\partial_t^2 d_m^k(t) + \sum_{l=1}^m c^2 e^{kl} d_m^l(t) + \sum_{l=1}^m \nu e^{kl} \partial_t d_m^l(t) = f^k(t) \quad (0 \leq t \leq T, k = 1, \dots, m), \quad (4.26)$$

subject to the initial conditions (4.14) and (4.15). According to Cauchy-Lipschitz theory for ordinary differential equations, there exists a unique function $d_m(t) = (d_m^1(t), \dots, d_m^m(t))$, satisfying (4.14), (4.15) and solving (4.26) for $0 \leq t \leq T$. \square

Energy estimates

Our plan is hereafter to pass to the limit for $m \rightarrow \infty$, and so we need some uniform estimates in m .

Theorem 4.3.2. *For $T = +\infty$, let u_m be of form (4.13) the finite decomposition over the eigenfunctions of the Dirichlet-laplacian satisfying (4.14), (4.15) and (4.25). Then there exists a constant $C > 0$, depending only on Ω , such that for $m = 1, 2, \dots$*

$$\begin{aligned} \max_{0 \leq t} (\|u_m(t)\|_{H_0^1(\Omega)}^2 + \|\partial_t u_m(t)\|_{L^2(\Omega)}^2) + \|\nabla \partial_t u_m\|_{L^2([0, +\infty[; L^2(\Omega))}^2 \\ + \|\nabla u_m\|_{L^2([0, +\infty[; L^2(\Omega))}^2 + \|\partial_t^2 u_m\|_{L^2((0, +\infty[; H^{-1}(\Omega))}^2 \\ \leq C(\|f\|_{L^2([0, +\infty[; L^2(\Omega))}^2 + \|u_0\|_{H_0^1(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2). \end{aligned} \quad (4.27)$$

Proof. Multiply equality (4.25) by $\partial_t d_m^k(t)$, sum $k = 1, \dots, m$, and recall (4.13) to obtain

$$(\partial_t^2 u_m, \partial_t u_m)_{L^2(\Omega)} + c^2(\nabla u_m, \nabla \partial_t u_m)_{L^2(\Omega)} + \nu(\nabla \partial_t u_m, \nabla \partial_t u_m)_{L^2(\Omega)} = (f, \partial_t u_m)_{L^2(\Omega)}$$

for a.e. $t \geq 0$. Using successively Cauchy-Schwarz's, Poincaré's and Young's inequalities we observe

$$\begin{aligned} (f, \partial_t u_m)_{L^2(\Omega)} &\leq \|f\|_{L^2(\Omega)} \|\partial_t u_m\|_{L^2(\Omega)} \\ &\leq K \|f\|_{L^2(\Omega)} \|\nabla \partial_t u_m\|_{L^2(\Omega)} \\ &\leq \frac{K^2}{2\nu} \|f\|_{L^2(\Omega)}^2 + \frac{\nu}{2} \|\nabla \partial_t u_m\|_{L^2(\Omega)}^2. \end{aligned}$$

Thus we obtain that for $t \geq 0$ after integration in time

$$\begin{aligned} \|\partial_t u_m(t)\|_{L^2(\Omega)}^2 + c^2 \|u_m(t)\|_{H_0^1(\Omega)}^2 + \nu \int_0^t \|\nabla \partial_t u_m(s)\|_{L^2(\Omega)}^2 ds \\ \leq C(\|f\|_{L^2([0, +\infty[; L^2(\Omega))}^2 + c^2 \|u_0\|_{H_0^1(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2) \end{aligned}$$

with $C > 0$ independent on t . Since $t \geq 0$ was arbitrary, we deduce from this estimate that

$$\begin{aligned} \max_{0 \leq t} (\|\partial_t u_m(t)\|_{L^2(\Omega)}^2 + \|u_m(t)\|_{H_0^1(\Omega)}^2) + \int_0^{+\infty} \|\nabla \partial_t u_m(s)\|_{L^2(\Omega)}^2 ds \\ \leq C(\|f\|_{L^2([0, +\infty[; L^2(\Omega))}^2 + \|u_0\|_{H_0^1(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2). \end{aligned} \quad (4.28)$$

We multiply equations (4.25) by $d_m^k(t)$ and sum over $k = 1, \dots, m$. By definition (4.13) of u_m we have for a.e. $t \geq 0$

$$(\partial_t^2 u_m, u_m)_{L^2(\Omega)} + c^2(\nabla u_m, \nabla u_m)_{L^2(\Omega)} + \nu(\nabla \partial_t u_m, \nabla u_m)_{L^2(\Omega)} = (f, u_m)_{L^2(\Omega)}.$$

We can write using successively Cauchy-Schwarz's, Poincaré's and Young's inequalities

$$\begin{aligned} (f, u_m)_{L^2(\Omega)} &\leq \|f\|_{L^2(\Omega)} \|u_m\|_{L^2(\Omega)} \\ &\leq K \|f\|_{L^2(\Omega)} \|\nabla u_m\|_{L^2(\Omega)} \\ &\leq \frac{K^2}{2c^2} \|f\|_{L^2(\Omega)}^2 + \frac{c^2}{2} \|\nabla u_m\|_{L^2(\Omega)}^2 \end{aligned}$$

with a constant $K > 0$ independent on time. Then we have

$$\frac{d}{dt} \left(\frac{\nu}{2} \|\nabla u_m(t)\|_{L^2(\Omega)}^2 \right) + \frac{c^2}{2} \|\nabla u_m(s)\|_{L^2(\Omega)}^2 ds \leq \frac{K^2}{2c^2} \|f\|_{L^2(\Omega)}^2 - \int_{\Omega} \partial_t^2 u_m u_m dx.$$

Integrating over $[0, t]$ we obtain

$$\begin{aligned} \frac{\nu}{2} \|\nabla u_m(t)\|_{L^2(\Omega)}^2 + \frac{c^2}{2} \int_0^t \|\Delta u_m(s)\|_{L^2(\Omega)}^2 ds \\ \leq \frac{\nu}{2} \|\nabla u_0\|_{L^2(\Omega)}^2 + \frac{K^2}{2c^2} \|f\|_{L^2([0,+\infty[;L^2(\Omega))}^2 - \int_0^t \int_{\Omega} \partial_t^2 u_m u_m dx ds. \end{aligned}$$

We have

$$\begin{aligned} - \int_0^t \int_{\Omega} \partial_t^2 u_m u_m dx &= \int_0^t \left[-\frac{d}{dt} \left(\int_{\Omega} \partial_t u_m u_m dx \right) + \int_{\Omega} (\partial_t u_m)^2 dx \right] ds \\ &= - \int_{\Omega} \partial_t u_m u_m dx + \int_{\Omega} \partial_t u_m(0) u_m(0) dx \\ &\quad + \|\partial_t u_m\|_{L^2([0,t];L^2(\Omega))}^2, \end{aligned}$$

so by Young's inequality

$$\begin{aligned} - \int_0^t \int_{\Omega} \partial_t^2 u_m u_m dx &\leq \frac{1}{2} \|\partial_t u_m\|_{L^\infty([0,t];L^2(\Omega))}^2 + \frac{1}{2} \|u_m\|_{L^\infty([0,t];L^2(\Omega))}^2 \\ &\quad + \frac{1}{2} \|\partial_t u_m(0)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u_m(0)\|_{L^2(\Omega)}^2 \\ &\quad + \|\partial_t u_m\|_{L^2([0,t];L^2(\Omega))}^2, \end{aligned}$$

then by the Poincaré inequality and using the initial conditions involved in (4.14)–(4.15)

$$\begin{aligned} - \int_0^t \int_{\Omega} \partial_t^2 u_m u_m dx &\leq \frac{1}{2} \|\partial_t u_m\|_{L^\infty([0,+\infty[;L^2(\Omega))}^2 + \frac{K^2}{2} \|u_m\|_{L^\infty([0,+\infty[;H_0^1(\Omega))}^2 \\ &\quad + \frac{1}{2} \|u_1\|_{L^2(\Omega)}^2 + \frac{K^2}{2} \|\nabla u_0\|_{L^2(\Omega)}^2 \\ &\quad + K^2 \|\partial_t u_m\|_{L^2([0,+\infty[;H_0^1(\Omega))}^2 \end{aligned}$$

with a constant $K > 0$ depending only on Ω and by estimate (4.28) we obtain

$$- \int_0^t \int_{\Omega} \partial_t^2 u_m u_m dx \leq C (\|f\|_{L^2([0,+\infty[;L^2(\Omega))}^2 + \|\nabla u_0\|_{L^2(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2)$$

with $C > 0$ independent on t . We can deduce the existence of $C > 0$ such that

$$\int_0^{+\infty} \|\nabla u_m(s)\|_{L^2(\Omega)}^2 ds \leq C (\|f\|_{L^2([0,+\infty[;L^2(\Omega))}^2 + \|\nabla u_0\|_{L^2(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2) \quad (4.29)$$

Fix any $v \in H_0^1(\Omega)$, $\|v\|_{H_0^1(\Omega)} \leq 1$, we consider the decomposition $v = v^1 + v^2$, where

$$v^1 \in \text{span}\{w_k\}_{k=1}^m \text{ and } (v^2, w_k)_{L^2(\Omega)} = 0 \text{ (} k = 1, \dots, m \text{)}.$$

Note $\|v^1\|_{H_0^1(\Omega)} \leq 1$. Then the decomposition (4.13) and identities (4.25) imply

$$\begin{aligned} \langle \partial_t^2 u_m, v \rangle_{(H^{-1}(\Omega), H_0^1(\Omega))} &= (\partial_t^2 u_m, v)_{L^2(\Omega)} = (\partial_t^2 u_m, v^1)_{L^2(\Omega)} \\ &= (f, v^1)_{L^2(\Omega)} - c^2 (\nabla u_m, \nabla v^1)_{L^2(\Omega)} - \nu (\nabla \partial_t u_m, \nabla v^1)_{L^2(\Omega)}. \end{aligned}$$

Thus by the Cauchy-Schwartz inequality and the Poincare inequality

$$\begin{aligned} |\langle \partial_t^2 u_m, v \rangle_{(H^{-1}(\Omega), H_0^1(\Omega))}| &\leq \|f\|_{L^2(\Omega)} \|v_1\|_{L^2(\Omega)} + c^2 \int_{\Omega} |\nabla u_m \nabla v_1| dx + \nu \int_{\Omega} |\nabla \partial_t u_m \nabla v_1| dx \\ &\leq C \|f\|_{L^2(\Omega)} \|v^1\|_{H_0^1(\Omega)} + \|u_m\|_{H_0^1(\Omega)} \|v^1\|_{H_0^1(\Omega)} + \|\partial_t u_m\|_{H_0^1(\Omega)} \|v^1\|_{H_0^1(\Omega)}, \end{aligned}$$

then

$$|\langle \partial_t^2 u_m, v \rangle_{(H^{-1}(\Omega), H_0^1(\Omega))}| \leq C (\|f\|_{L^2(\Omega)} + \|u_m\|_{H_0^1(\Omega)} + \|\partial_t u_m\|_{H_0^1(\Omega)}),$$

since $\|v^1\|_{H_0^1(\Omega)} \leq 1$. Consequently, by the estimates (4.28) and (4.29),

$$\begin{aligned} \int_0^{+\infty} \|\partial_t^2 u_m\|_{H^{-1}(\Omega)}^2 dt &\leq C \int_0^{+\infty} (\|f\|_{L^2(\Omega)}^2 + \|u_m\|_{H_0^1(\Omega)}^2 + \|\partial_t u_m\|_{H_0^1(\Omega)}^2) dt \\ &\leq C (\|f\|_{L^2([0, +\infty[; L^2(\Omega))}^2 + \|u_0\|_{H_0^1(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2) \end{aligned}$$

and combining with the estimates (4.28) and (4.29) we obtain estimate (4.27). \square

Existence and uniqueness

Now we pass to limits in our Galerkin approximations.

Theorem 4.3.3. *There exists a weak solution of the damped wave equation problem (4.23) in the sense of Definition 4.3.1.*

Proof. According to the energy estimates (4.27), we see that

- the sequence $\{u_m\}_{m=1}^{\infty}$ is bounded in $L^2([0, T]; H_0^1(\Omega))$,
- $\{\partial_t u_m\}_{m=1}^{\infty}$ is bounded in $L^2([0, T]; H_0^1(\Omega))$,
- $\{\partial_t^2 u_m\}_{m=1}^{\infty}$ is bounded in $L^2([0, T]; H^{-1}(\Omega))$.

Then there exists a subsequence $\{u_{m_l}\}_{l=1}^{\infty} \subset \{u_m\}_{m=1}^{\infty}$ and $u \in L^2([0, T]; H_0^1(\Omega))$, with $\partial_t u \in L^2([0, T]; H_0^1(\Omega))$, $\partial_t^2 u \in L^2([0, T]; H^{-1}(\Omega))$ such that

$$\left\{ \begin{array}{l} u_{m_l} \rightharpoonup u \text{ weakly in } L^2([0, T]; H_0^1(\Omega)), \\ \partial_t u_{m_l} \rightharpoonup \partial_t u \text{ weakly in } L^2([0, T]; H_0^1(\Omega)), \\ \partial_t^2 u_{m_l} \rightharpoonup \partial_t^2 u \text{ weakly in } L^2([0, T]; H^{-1}(\Omega)). \end{array} \right. \quad (4.30)$$

In the same way that for the wave equation problem (see [30] p. 384) we show that u is a weak solution of the damped wave equation problem (4.23). \square

Theorem 4.3.4. *A weak solution of the damped wave equation problem (4.23) is unique.*

Proof. It suffices to show that the only weak solution of (4.23) with $f \equiv 0$ and $u_0 \equiv u_1 \equiv 0$ is

$$u \equiv 0.$$

For this we take $\partial_t u \in L^2([0, T]; H_0^1(\Omega))$ and so by definition of u as a weak solution of the damped wave equation problem (4.23)

$$\int_0^T \langle \partial_t^2 u, \partial_t u \rangle_{(H^{-1}(\Omega), H_0^1(\Omega))} + c^2 (\nabla u, \nabla \partial_t u)_{L^2(\Omega)} + \nu (\nabla \partial_t u, \nabla \partial_t u)_{L^2(\Omega)} dt = 0.$$

Then

$$\frac{1}{2} \|\partial_t u(T)\|_{L^2(\Omega)}^2 + \frac{c^2}{2} \|\nabla u(T)\|_{L^2(\Omega)}^2 + \int_0^T \|\nabla u_t(s)\|_{L^2(\Omega)} ds = 0,$$

which implies $\nabla u_t \equiv 0$ and so $\nabla u \equiv 0$ as $\nabla u_0 = 0$ and so $u \equiv 0$ as $u \in H_0^1(\Omega)$ as a weak solution. \square

4.3.2 Regularity results

Theorem 4.3.5. *Let Ω be an arbitrary bounded domain in \mathbb{R}^n ($n \geq 2$).*

Then,

(i) *Assume*

$$u_0 \in H_0^1(\Omega), \quad u_1 \in L^2(\Omega), \quad f \in L^2([0, T]; L^2(\Omega))$$

and suppose also $u \in L^2([0, T]; H_0^1(\Omega))$ with $\partial_t u \in L^2([0, T]; H_0^1(\Omega))$ and $\partial_t^2 u \in L^2([0, T]; H^{-1}(\Omega))$ is a weak solution of the damped wave equation problem (4.23) in the sense of (4.24). Then we also have

$$u \in L^\infty([0, T]; H_0^1(\Omega)), \quad \partial_t u \in L^\infty([0, T]; L^2(\Omega))$$

and we have the estimate

$$\begin{aligned} & \operatorname{ess\,sup}_{0 \leq t \leq T} (\|u(t)\|_{H_0^1(\Omega)} + \|\partial_t u(t)\|_{L^2(\Omega)}) + \int_0^T \|\nabla \partial_t u(s)\|_{L^2(\Omega)} ds + \|\partial_t^2 u\|_{L^2([0, T]; H^{-1}(\Omega))} \\ & \leq C(\|f\|_{L^2([0, T]; L^2(\Omega))} + \|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)}). \end{aligned}$$

Moreover we can take $T = +\infty$

(ii) *If, in addition*

$$\Delta u_0 \in L^2(\Omega), \quad u_1 \in H_0^1(\Omega), \quad f \in L^2([0, +\infty[; L^2(\Omega)),$$

then we have

$$\begin{aligned} & u \in L^\infty([0, +\infty[; H_0^1(\Omega)) \cap L^2([0, +\infty[; H_0^1(\Omega)) \cap L^2([0, +\infty[; H_{loc}^2(\Omega)), \\ & \partial_t u \in L^\infty([0, +\infty[; H_0^1(\Omega)) \cap L^2([0, +\infty[; H_0^1(\Omega)) \cap L^2([0, +\infty[; H_{loc}^2(\Omega)), \\ & \partial_t^2 u \in L^2([0, +\infty[; L^2(\Omega)), \\ & \Delta u \in L^\infty([0, +\infty[; L^2(\Omega)) \cap L^2([0, +\infty[; L^2(\Omega)), \\ & \Delta \partial_t u \in L^2([0, +\infty[; L^2(\Omega)) \end{aligned}$$

with the estimates

$$\begin{aligned} & \operatorname{ess\,sup}_{0 \leq t} (\|\Delta u(t)\|_{L^2(\Omega)}^2 + \|\nabla \partial_t u(t)\|_{L^2(\Omega)}^2) + \int_0^\infty \|\Delta \partial_t u(s)\|_{L^2(\Omega)}^2 ds \\ & \leq C(\|f\|_{L^2([0, +\infty[; L^2(\Omega))}^2 + \|\Delta u_0\|_{L^2(\Omega)}^2 + \|\nabla u_1\|_{L^2(\Omega)}^2) \end{aligned} \quad (4.31)$$

and

$$\int_0^{+\infty} \|\Delta u(s)\|_{L^2(\Omega)}^2 ds \leq C(\|f\|_{L^2([0, +\infty[; L^2(\Omega))}^2 + \|\Delta u_0\|_{L^2(\Omega)}^2 + \|\nabla u_1\|_{L^2(\Omega)}^2). \quad (4.32)$$

Proof. In the previous section we have proved, for

$$u_m(t) = \sum_{i=1}^m d_m^k(t) w_k,$$

of the form (4.13) satisfying (4.14)–(4.16), estimate (4.27) and also the convergence result (4.30) with u unique weak solution of problem (4.23). Passing to limits in (4.27) as $m = m_l \rightarrow \infty$, we deduce (i).

Assume now the hypothesis of assertion (ii). We consider again u_m of the form (4.13) satisfying (4.14), (4.15) and (4.25). We multiply by $-\lambda_k \partial_t d_m^k(t)$ and sum over $k = 1, \dots, m$. By definition (4.13) of u_m and (4.12) of λ_k we have

$$\int_{\Omega} -\partial_t^2 u_m \Delta \partial_t u_m - c^2 \nabla u_m \nabla \Delta \partial_t u_m - \nu \nabla \partial_t u_m \nabla \Delta \partial_t u_m \, dx = -(f, \Delta \partial_t u_m)_{L^2(\Omega)}.$$

But by Definition (4.13) of u_m as a linear combination of Laplacian's eigenfunctions w_k , which satisfy (4.10)–(4.12), we have $\partial_t^2 u_m \in H_0^1(\Omega)$ and $\Delta \partial_t u_m \in H_0^1(\Omega)$ so by the Green formula we have

$$\int_{\Omega} \nabla \partial_t^2 u_m \nabla \partial_t u_m + c^2 \Delta u_m \Delta \partial_t u_m + \nu \Delta \partial_t u_m \Delta \partial_t u_m \, dx = -(f, \Delta \partial_t u_m)_{L^2(\Omega)}.$$

We can write by Young's inequality

$$|(f, \Delta \partial_t u_m)_{L^2(\Omega)}| \leq \frac{2}{\nu} \|f\|_{L^2(\Omega)}^2 + \frac{\nu}{2} \|\Delta \partial_t u_m\|_{L^2(\Omega)}^2.$$

Then we have

$$\frac{d}{dt} \left(\frac{1}{2} \|\nabla \partial_t u_m(t)\|_{L^2(\Omega)}^2 + \frac{c^2}{2} \|\Delta u_m(t)\|_{L^2(\Omega)}^2 + \frac{\nu}{2} \int_0^t \|\Delta \partial_t u_m(s)\|_{L^2(\Omega)}^2 \, ds \right) \leq \frac{2}{\nu} \|f\|_{L^2(\Omega)}^2.$$

Integrating over $[0, t]$ we obtain

$$\begin{aligned} & \frac{1}{2} \|\nabla \partial_t u_m(t)\|_{L^2(\Omega)}^2 + \frac{c^2}{2} \|\Delta u_m(t)\|_{L^2(\Omega)}^2 + \frac{\nu}{2} \int_0^t \|\Delta \partial_t u_m(s)\|_{L^2(\Omega)}^2 \, ds \\ & \leq \frac{2}{\nu} \int_0^t \|f(s)\|_{L^2(\Omega)}^2 \, ds + \frac{1}{2} \|\nabla \partial_t u_m(0)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\Delta u_m(0)\|_{L^2(\Omega)}^2 \\ & \leq \frac{2}{\nu} \|f\|_{L^2([0, +\infty[; L^2(\Omega))}^2 + \frac{1}{2} \|\nabla u_1\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\Delta u_0\|_{L^2(\Omega)}^2, \end{aligned} \quad (4.33)$$

so we have (4.31) taking the weak limit of a subsequence. Then

$$\begin{aligned} \partial_t u & \in L^\infty([0, +\infty[; H_0^1(\Omega)), \\ \Delta u & \in L^\infty([0, +\infty[; L^2(\Omega)), \\ \Delta \partial_t u & \in L^2([0, +\infty[; L^2(\Omega)). \end{aligned}$$

The linearity of the equation implies for all $T \geq 0$

$$\partial_t^2 u \in L^2([0, T]; L^2(\Omega)),$$

as $\Delta u \in L^\infty([0, +\infty[; L^2(\Omega)) \subset L^2([0, T]; L^2(\Omega))$ for all $T \geq 0$.

Moreover as $\partial_t u_m \in H_0^1(\Omega)$ and $\Delta \partial_t u_m \in L^2(\Omega)$ by Proposition 4.1.2 we have

$$\|\nabla \partial_t u_m\|_{L^2(\Omega)} \leq C \|\Delta \partial_t u_m\|_{L^2(\Omega)},$$

which implies by estimate (4.33)

$$\begin{aligned} \|\nabla \partial_t u_m\|_{L^2([0,+\infty[;L^2(\Omega))} &\leq C \|\Delta \partial_t u_m\|_{L^2([0,+\infty[;L^2(\Omega))} \\ &\leq C(\|f\|_{L^2([0,+\infty[;L^2(\Omega))}^2 + \|\Delta u_0\|_{L^2(\Omega)}^2 + \|\nabla u_1\|_{L^2(\Omega)}^2), \end{aligned}$$

and taking the weak limit of a subsequence

$$\|\partial_t u\|_{L^2([0,+\infty[;H_0^1(\Omega))} \leq C(\|f\|_{L^2([0,+\infty[;L^2(\Omega))}^2 + \|\Delta u_0\|_{L^2(\Omega)}^2 + \|\nabla u_1\|_{L^2(\Omega)}^2). \quad (4.34)$$

By Proposition 4.1.2 we also have

$$\|\nabla u_m\|_{L^2(\Omega)} \leq C \|\Delta u_m\|_{L^2(\Omega)},$$

which implies in the same way as estimate (4.34)

$$\|u\|_{L^\infty([0,+\infty[;H_0^1(\Omega))} \leq C(\|f\|_{L^2([0,+\infty[;L^2(\Omega))}^2 + \|\Delta u_0\|_{L^2(\Omega)}^2 + \|\nabla u_1\|_{L^2(\Omega)}^2). \quad (4.35)$$

We multiply equations (4.25) by $-\lambda_k d_m^k(t)$ and sum over $k = 1, \dots, m$. By definition (4.13) of u_m and (4.12) of λ_k we have

$$\int_{\Omega} -\partial_t^2 u_m \Delta u_m - c^2 \nabla u_m \nabla \Delta u_m - \nu \nabla \partial_t u_m \nabla \Delta u_m \, dx = -(f, \Delta u_m)_{L^2(\Omega)}.$$

But by definition (4.13) of u_m as the w_k satisfy (4.10)–(4.12), we have $\partial_t^2 u_m \in H_0^1(\Omega)$ and $\Delta \partial_t u_m \in H_0^1(\Omega)$ so by the Green formula we have

$$\int_{\Omega} \nabla \partial_t^2 u_m \nabla u_m + c^2 \Delta u_m \Delta u_m + \nu \Delta \partial_t u_m \Delta u_m \, dx = -(f, \Delta u_m)_{L^2(\Omega)}.$$

We can write by Young's inequality

$$|(f, \Delta \partial_t u_m)_{L^2(\Omega)}| \leq \frac{2}{c^2} \|f\|_{L^2(\Omega)}^2 + \frac{c^2}{2} \|\Delta \partial_t u_m\|_{L^2(\Omega)}^2.$$

Then we have

$$\frac{d}{dt} \left(\frac{\nu}{2} \|\Delta u_m(t)\|_{L^2(\Omega)}^2 + \frac{c^2}{2} \int_0^t \|\Delta u_m(s)\|_{L^2(\Omega)}^2 \, ds \right) \leq \frac{2}{c^2} \|f\|_{L^2(\Omega)}^2 - \int_{\Omega} \nabla \partial_t^2 u_m \nabla u_m \, dx.$$

Integrating over $[0, t]$ we obtain

$$\begin{aligned} \frac{\nu}{2} \|\Delta u_m(t)\|_{L^2(\Omega)}^2 + \frac{c^2}{2} \int_0^t \|\Delta u_m(s)\|_{L^2(\Omega)}^2 \, ds \\ \leq \frac{\nu}{2} \|\Delta u_0\|_{L^2(\Omega)}^2 + \frac{2}{c^2} \|f\|_{L^2([0,+\infty[;L^2(\Omega))}^2 - \int_0^t \int_{\Omega} \nabla \partial_t^2 u_m \nabla u_m \, dx \, ds. \end{aligned}$$

We have

$$\begin{aligned} - \int_0^t \int_{\Omega} \nabla \partial_t^2 u_m \nabla u_m \, dx &= \int_0^t \left[-\frac{d}{dt} \left(\int_{\Omega} \nabla \partial_t u_m \nabla u_m \, dx \right) + \int_{\Omega} (\nabla \partial_t u_m)^2 \, dx \right] \, ds \\ &= - \int_{\Omega} \nabla \partial_t u_m \nabla u_m \, dx + \int_{\Omega} \nabla \partial_t u_m(0) \nabla u_m(0) \, dx \\ &\quad + \|\partial_t u_m\|_{L^2([0,t];H_0^1(\Omega))}^2, \end{aligned}$$

so by Young's inequality

$$\begin{aligned}
-\int_0^t \int_{\Omega} \nabla \partial_t^2 u_m \nabla u_m \, dx &\leq \frac{1}{2} \|\partial_t u_m\|_{L^\infty([0,t]; H_0^1(\Omega))}^2 + \frac{1}{2} \|u_m\|_{L^\infty([0,t]; H_0^1(\Omega))}^2 \\
&\quad + \frac{1}{2} \|\nabla \partial_t u_m(0)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla u_m(0)\|_{L^2(\Omega)}^2 \\
&\quad + \|\partial_t u_m\|_{L^2([0,t]; H_0^1(\Omega))}^2 \\
&\leq \frac{1}{2} \|\partial_t u_m\|_{L^\infty([0,+\infty]; H_0^1(\Omega))}^2 + \frac{1}{2} \|u_m\|_{L^\infty([0,+\infty]; H_0^1(\Omega))}^2 \\
&\quad + \frac{1}{2} \|\nabla u_1\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla u_0\|_{L^2(\Omega)}^2 \\
&\quad + \|\partial_t u_m\|_{L^2([0,+\infty]; H_0^1(\Omega))}^2
\end{aligned}$$

and by estimates (4.33), (4.34) and (4.35) we obtain

$$-\int_0^t \int_{\Omega} \nabla \partial_t^2 u_m \nabla u_m \, dx \leq C(\|f\|_{L^2([0,+\infty]; L^2(\Omega))}^2 + \|\Delta u_0\|_{L^2(\Omega)}^2 + \|\nabla u_1\|_{L^2(\Omega)}^2)$$

with $C > 0$ independent on t . We can deduce

$$\int_0^{+\infty} \|\Delta u_m(s)\|_{L^2(\Omega)}^2 \, ds \leq C(\|f\|_{L^2([0,+\infty]; L^2(\Omega))}^2 + \|\Delta u_0\|_{L^2(\Omega)}^2 + \|\nabla u_1\|_{L^2(\Omega)}^2)$$

and taking a convergent subsequence

$$\int_0^{+\infty} \|\Delta u(s)\|_{L^2(\Omega)}^2 \, ds \leq C(\|f\|_{L^2([0,+\infty]; L^2(\Omega))}^2 + \|\Delta u_0\|_{L^2(\Omega)}^2 + \|\nabla u_1\|_{L^2(\Omega)}^2).$$

The linearity of the equation gives us $\partial_t^2 u \in L^2([0, +\infty[; L^2(\Omega))$ as

$$\|\partial_t^2 u\|_{L^2([0,+\infty[; L^2(\Omega))} \leq C(\|f\|_{L^2([0,+\infty[; L^2(\Omega))} + \|\Delta u\|_{L^2([0,+\infty[; L^2(\Omega))} + \|\Delta \partial_t u\|_{L^2([0,+\infty[; L^2(\Omega))}).$$

By Theorem 4.1.6 we can also deduce

$$\partial_t u \in L^2([0, +\infty[; H_{loc}^2(\Omega)),$$

with the estimate for $V \subset\subset \Omega$

$$\|\partial_t u\|_{L^2([0,+\infty[; H^2(V))} \leq C\|\Delta \partial_t u\|_{L^2([0,+\infty[; L^2(\Omega))},$$

which implies

$$\|\partial_t u\|_{L^2([0,+\infty[; H^2(V))}^2 \leq C(\|f\|_{L^2([0,+\infty[; L^2(\Omega))}^2 + \|\Delta u_0\|_{L^2(\Omega)}^2 + \|\nabla u_1\|_{L^2(\Omega)}^2).$$

The same type of arguments permits to deduce

$$u \in L^\infty([0, +\infty[; H_{loc}^2(\Omega)) \cap L^2([0, +\infty[; H_{loc}^2(\Omega)),$$

as $\Delta u \in L^\infty([0, +\infty[; L^2(\Omega)) \cap L^2([0, +\infty[; L^2(\Omega))$. □

4.4 Well posedness of the damped linear wave equation with homogeneous Robin boundary conditions

In this section Ω is an admissible domain in \mathbb{R}^n ($n = 2$ or 3) with a d -set boundary $\partial\Omega$ such that $n - 2 < d < n$. We denote by m_d the d -dimensional Hausdorff measure.

Definition 4.4.1. For $a > 0$ we take $H^1(\Omega)$ endowed with the norm $\|\cdot\|_{\tilde{H}^1(\Omega)}$ defined in (4.8). For $f \in L^2([0, T]; L^2(\Omega))$, $u_0 \in H^1(\Omega)$, and $u_1 \in L^2(\Omega)$, we say a function $u \in L^2([0, T]; H^1(\Omega))$ with $\partial_t u \in L^2([0, T]; H^1(\Omega))$ and $\partial_t^2 u \in L^2([0, T]; H^{-1}(\Omega))$ is a weak solution of the hyperbolic initial/boundary value problem

$$\left\{ \begin{array}{l} u_{tt} - c^2 \Delta u - \nu \Delta u_t = f \quad \text{on }]0, T] \times \Omega, \\ \frac{\partial}{\partial n} u + au = 0 \quad \text{on } [0, T] \times \partial\Omega, \\ u(0) = u_0, \quad u_t(0) = u_1, \end{array} \right. \quad (4.36)$$

provided

$$\langle u_{tt}, v \rangle_{(H^{-1}(\Omega), H^1(\Omega))} + c^2(u, v)_{\tilde{H}^1(\Omega)} + \nu(u_t, v)_{\tilde{H}^1(\Omega)} = (f, v)_{L^2(\Omega)}, \quad (4.37)$$

for each $v \in H^1(\Omega)$ and a.e. time $0 \leq t \leq T$, with

$$\frac{\partial}{\partial n} u + au = 0 \quad \text{on } [0; T] \times \partial\Omega$$

and

$$u(0) = u_0, \quad u_t(0) = u_1.$$

Now we can give the following theorem on the existence of weak solution.

Theorem 4.4.1. Set $a > 0$. For every $f \in L^2([0, T]; L^2(\Omega))$, $u_0 \in H^1(\Omega)$, and $u_1 \in L^2(\Omega)$ there exists a unique weak solution of the hyperbolic initial/boundary value problem (4.36) in the sense of formulation (4.37).

Proof. The proof is essentially the same as for homogeneous boundary conditions in Subsection 4.3.1 using the Galerkin method replacing $H_0^1(\Omega)$ by $H^1(\Omega)$ with the scalar products associated to $\|\cdot\|_{\tilde{H}^1(\Omega)}$. The basis of $L^2(\Omega)$ chosen for the Galerkin approximation is the basis of eigenvalues for the Poisson equation with homogeneous Robin boundary conditions defined in Theorem 4.1.14. By construction it is also an orthogonal basis of $H^1(\Omega)$ with the scalar product associated to $\|\cdot\|_{\tilde{H}^1(\Omega)}$ as for u and v in $\tilde{H}^1(\Omega)$ with $-\Delta u \in L^2(\Omega)$ and $\frac{\partial}{\partial n} u + au = 0$ in the weak sense of Definition 4.1.9

$$(-\Delta u, v)_{L^2(\Omega)} = (u, v)_{\tilde{H}^1(\Omega)}.$$

□

Then we have the higher regularity result.

Theorem 4.4.2. *Let Ω be an admissible domain in \mathbb{R}^n ($n \geq 2$) with a d -set boundary $\partial\Omega$ such that $n - 2 < d < n$. Then,*

(i) *Assume*

$$u_0 \in H^1(\Omega), \quad u_1 \in L^2(\Omega), \quad f \in L^2([0, T]; L^2(\Omega)),$$

and suppose also $u \in L^2([0, T]; H^1(\Omega))$ with $\partial_t u \in L^2([0, T]; H^1(\Omega))$ and $\partial_t^2 u \in L^2([0, T]; H^{-1}(\Omega))$ is a weak solution of the damped wave equation problem (4.36) in the sense of formulation (4.37). Then we also have

$$u \in L^\infty([0, T]; H^1(\Omega)), \quad \partial_t u \in L^\infty([0, T]; L^2(\Omega)),$$

and we have the estimate

$$\begin{aligned} & \operatorname{ess\,sup}_{0 \leq t \leq T} (\|u(t)\|_{\tilde{H}^1(\Omega)} + \|\partial_t u(t)\|_{L^2(\Omega)}) + \int_0^T \|\partial_t u(s)\|_{\tilde{H}^1(\Omega)} ds + \|\partial_t^2 u\|_{L^2([0, T]; H^{-1}(\Omega))} \\ & \leq C(\|f\|_{L^2([0, T]; L^2(\Omega))} + \|u_0\|_{\tilde{H}^1(\Omega)} + \|u_1\|_{L^2(\Omega)}). \end{aligned}$$

(ii) *If in addition*

$$\Delta u_0 \in L^2(\Omega), \quad u_1 \in H^1(\Omega), \quad f \in L^2([0, +\infty[; L^2(\Omega)),$$

where in the sense of Definition 4.1.9

$$\forall v \in H^1(\Omega) \quad (u_0, v)_{\tilde{H}^1(\Omega)} = (g, v)_{L^2(\Omega)} \text{ for an arbitrary } g \in L^2(\Omega),$$

then we have

$$\begin{aligned} & u \in L^\infty([0, +\infty[; H^1(\Omega)) \cap L^2([0, +\infty[; H^1(\Omega)), \\ & \partial_t u \in L^\infty([0, +\infty[; H^1(\Omega)) \cap L^2([0, +\infty[; H^1(\Omega)), \\ & \partial_t^2 u \in L^2([0, +\infty[; L^2(\Omega)), \\ & \Delta u \in L^\infty([0, +\infty[; L^2(\Omega)) \cap L^2([0, +\infty[; L^2(\Omega)), \\ & \Delta \partial_t u \in L^2([0, +\infty[; L^2(\Omega)) \end{aligned}$$

with the estimates

$$\begin{aligned} & \operatorname{ess\,sup}_{0 \leq t} (\|\Delta u(t)\|_{L^2(\Omega)}^2 + \|\partial_t u(t)\|_{\tilde{H}^1(\Omega)}^2) + \int_0^\infty \|\Delta \partial_t u(s)\|_{L^2(\Omega)}^2 ds \\ & \leq C(\|f\|_{L^2([0, +\infty[; L^2(\Omega))}^2 + \|\Delta u_0\|_{L^2(\Omega)}^2 + \|u_1\|_{\tilde{H}^1(\Omega)}^2) \end{aligned} \quad (4.38)$$

and

$$\int_0^{+\infty} \|\Delta u(s)\|_{L^2(\Omega)}^2 ds \leq C(\|f\|_{L^2([0, +\infty[; L^2(\Omega))}^2 + \|\Delta u_0\|_{L^2(\Omega)}^2 + \|u_1\|_{\tilde{H}^1(\Omega)}^2). \quad (4.39)$$

Proof. The proof is similar to Theorem 4.3.5 and thus is omitted. \square

Chapter 5

Well-posedness and regularity of the non linear Westervelt equation on domains with fractal boundaries

Now we will treat the well-posedness of the non linear Westervelt equation.

5.1 Well-posedness of the Westervelt equation with homogeneous Dirichlet boundary conditions

In this section Ω is an arbitrary bounded domain in \mathbb{R}^3 or a bounded admissible domain in \mathbb{R}^2 and \mathcal{A} designates the operator $-\Delta$ on Ω in a weak sense with homogeneous Dirichlet boundary conditions

$$\begin{aligned}\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset L^2(\Omega) &\rightarrow L^2(\Omega) \\ u &\mapsto -\Delta u.\end{aligned}$$

Here $\mathcal{D}(\mathcal{A})$ is the domain of \mathcal{A} , $u \in \mathcal{D}(\mathcal{A})$ if and only if $u \in H_0^1(\Omega)$ and $-\Delta u \in L^2(\Omega)$. The operator \mathcal{A} is linear self-adjoint and coercive in the sense where for $u \in \mathcal{D}(\mathcal{A})$

$$(\mathcal{A}u, u)_{L^2(\Omega)} = (\nabla u, \nabla u)_{L^2(\Omega)}$$

and we will note for $u \in \mathcal{D}(\mathcal{A})$

$$\|u\|_{\mathcal{D}(\mathcal{A})} = \|\Delta u\|_{L^2(\Omega)}.$$

Theorem 5.1.1. *For*

$$X := H^1([0, +\infty[; \mathcal{D}(\mathcal{A})) \cap H^2([0, +\infty[; L^2(\Omega)) \quad (5.1)$$

there exists a unique weak solution, in the sense of formulation (4.24),

$$u \in X$$

of the boundary-valued problem

$$\begin{cases} \partial_t^2 u + c^2 \mathcal{A}u + \nu \varepsilon \partial_t \mathcal{A}u = f & \text{on } [0, +\infty[\times \Omega, \\ u = 0 & \text{on } [0, +\infty[\times \partial\Omega, \\ u(0) = u_0, \partial_t u(0) = u_1, \end{cases} \quad (5.2)$$

if and only if $f \in L^2([0, +\infty[; L^2(\Omega))$, $u_0 \in \mathcal{D}(\mathcal{A})$ and $u_1 \in H_0^1(\Omega)$.

Moreover we have the estimate

$$\|u\|_X \leq C(\|f\|_{[0, +\infty[; L^2(\Omega)} + \|\Delta u_0\|_{L^2(\Omega)} + \|u_1\|_{H_0^1(\Omega)}).$$

Proof. Theorem 4.3.5 gives us directly one side of the equivalence. If $f \in L^2([0, +\infty[; L^2(\Omega))$, $u_0 \in \mathcal{D}(\mathcal{A})$ and $u_1 \in H_0^1(\Omega)$ there exists a unique weak solution u of problem (5.2) in the sense of (4.24) with $\partial_t^2 u$, Δu and $\Delta \partial_t u$ in $L^2([0, +\infty[; L^2(\Omega))$. Estimates (4.31) and (4.32) imply $u \in X$ with the desired estimate.

Now let us consider a weak solution

$$u \in H^1([0, +\infty[; \mathcal{D}(\mathcal{A})) \cap H^2([0, +\infty[; L^2(\Omega))$$

of the Cauchy problem (5.2) for the strongly damped wave equation. By linearity u is unique and by definition of u we have

$$f \in L^2([0, +\infty[; L^2(\Omega))$$

and

$$u_0 \in \mathcal{D}(\mathcal{A}).$$

Now if we look at u_t and consider the semi group generated by $-\mathcal{A}$ as u_t satisfies a parabolic type problem

$$\partial_t(\partial_t u) + \mathcal{A}(\partial_t u) = f - \mathcal{A}u,$$

we can write

$$\partial_t u(t) = \exp^{-t\mathcal{A}} u_1 + \int_0^t \exp^{-(t-s)\mathcal{A}} (f - \mathcal{A}u) ds,$$

with $f - \mathcal{A}u \in L^2([0, +\infty[; L^2(\Omega))$. Then the theory of L^p maximal regularity in [8] for parabolic problems tells us for all $T > 0$

$$t \mapsto \int_0^t \exp^{-(t-s)\mathcal{A}} (f - \mathcal{A}u) ds \in L^2([0, T]; \mathcal{D}(\mathcal{A})) \cap L^2([0, T]; L^2(\Omega)).$$

By definition for all $T > 0$

$$u_t \in L^2([0, T]; \mathcal{D}(\mathcal{A})) \cap L^2([0, T]; L^2(\Omega)),$$

so for all $T > 0$

$$t \mapsto \exp^{-t\mathcal{A}} u_1 \in L^2([0, T]; \mathcal{D}(\mathcal{A})) \cap L^2([0, T]; L^2(\Omega)).$$

As we can write for \mathcal{A} with $v \in \mathcal{D}(\mathcal{A})$ and $w \in H_0^1(\Omega)$

$$|(\mathcal{A}v, w)_{L^2(\Omega)}| = |(\nabla v, \nabla w)_{L^2(\Omega)}| \leq \|v\|_{H_0^1(\Omega)} \|w\|_{H_0^1(\Omega)}$$

and

$$(\mathcal{A}v, v)_{L^2(\Omega)} = (\nabla v, \nabla v)_{L^2(\Omega)} = \|v\|_{H_0^1(\Omega)}^2,$$

the analysis in [7] tells us $u_1 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H_0^1(\Omega)$.

□

We will use Theorem 1.5.2 to prove the following global well-posedness result on the nonlinear Westervelt equation whose form slightly differs from the Westevelt equation studied in Chapter 2 due to the fact that it has been derivated in time. We found again this form in the study of the Westervelt equation on regular domains [54].

Theorem 5.1.2. *Let Ω be an arbitrary bounded domain in \mathbb{R}^3 or a bounded admissible domain in \mathbb{R}^2 . In addition let $\nu > 0$, $\mathbb{R}^+ = [0, +\infty[$ and X be the Hilbert space defined in (5.1). Taking*

$$u_0 \in \mathcal{D}(\mathcal{A}), \quad u_1 \in H_0^1(\Omega) \quad \text{and} \quad f \in L^2(\mathbb{R}^+; L^2(\Omega)),$$

there exists $C_1 = O(1)$ the minimal constant such that the weak solution, in the sense of (4.24), $u^* \in X$ of the corresponding non homogeneous linear boundary-valued problem (5.2) satisfies

$$\|u^*\|_X \leq \frac{C_1}{\nu\epsilon} (\|f\|_{L^2(\mathbb{R}^+; L^2(\Omega))} + \|u_0\|_{\mathcal{D}(\mathcal{A})} + \|u_1\|_{H_0^1(\Omega)}).$$

Then there exists $r_* > 0$ with $r_* = O(1)$ such that for all $r \in [0, r_*[$ and all data satisfying

$$\|f\|_{L^2(\mathbb{R}^+; L^2(\Omega))} + \|u_0\|_{\mathcal{D}(\mathcal{A})} + \|u_1\|_{H_0^1(\Omega)} \leq \frac{\nu\epsilon}{C_1} r,$$

there exists the unique weak solution $u \in X$ of the boundary valued problem for the Westervelt equation

$$\begin{cases} \partial_t^2 u + c^2 \mathcal{A}u + \nu\epsilon \mathcal{A} \partial_t u = \alpha\epsilon u \partial_t^2 u + \alpha\epsilon (\partial_t u)^2 + f & \text{on } [0, +\infty[\times \Omega, \\ u = 0 & \text{on } [0, +\infty[\times \partial\Omega, \\ u(0) = u_0, \quad \partial_t u(0) = u_1. \end{cases} \quad (5.3)$$

It is a weak solution in the sense where $u = u^* + v$ where u^* is defined as a weak solution in the sense of (4.24) and $v \in X$ is the solution of an homogeneous non linear initial-boundary valued problem depending on u^* and determined with Theorem 1.5.2.

That is to say $\forall w \in L^2([0, +\infty[, H_0^1(\Omega))$

$$\begin{aligned} & \int_0^{+\infty} (\partial_t^2 v, w)_{L^2(\Omega)} + c^2 (\nabla v, \nabla w)_{L^2(\Omega)} + \nu\epsilon (\nabla \partial_t v, \nabla w)_{L^2(\Omega)} dt = \\ & = \int_0^{+\infty} (\alpha\epsilon (v + u^*) \partial_t^2 (v + u^*) + \alpha\epsilon [\partial_t (v + u^*)]^2, w)_{L^2(\Omega)} dt, \\ & \text{with } (v(0), \partial_t v(0)) = (0, 0). \end{aligned}$$

Moreover

$$\|u\|_X \leq 2r.$$

Proof. For $u_0 \in \mathcal{D}(\mathcal{A})$ and $u_1 \in H_0^1(\Omega)$ and $f \in L^2(\mathbb{R}^+; L^2(\Omega))$ let us denote, by Theorem 5.1.1, $u^* \in X$ is the unique weak solution of the linear problem

$$\begin{cases} u_{tt}^* + c^2 \mathcal{A}u^* + \nu\epsilon \mathcal{A}u_t^* = f & \text{on } [0, +\infty[\times \Omega, \\ u = 0 & \text{on } [0, +\infty[\times \partial\Omega, \\ u^*(0) = u_0 \in \mathcal{D}(\mathcal{A}), \quad u_t^*(0) = u_1 \in H_0^1(\Omega), \end{cases}$$

in the sense of the variational formulation (4.24). In addition, according to Theorem 5.1.1, we take X defined in (5.1) and introduce the Banach spaces

$$X_0 := \{u \in X \mid u(0) = u_t(0) = 0\}$$

and $Y = L^2(\mathbb{R}^+; L^2(\Omega))$. Then by Theorem 5.1.1, the linear operator

$$L : X_0 \rightarrow Y, \quad u \in X_0 \mapsto L(u) := u_{tt} + c^2 \mathcal{A}u + \nu \varepsilon \mathcal{A}u_t \in Y,$$

is a bi-continuous isomorphism.

Let us now notice that if v is the unique weak solution of the non-linear boundary valued problem

$$\begin{cases} v_{tt} + c^2 \mathcal{A}v + \nu \varepsilon \mathcal{A}v_t - \alpha \varepsilon (v + u^*)(v + u^*)_{tt} - \alpha \varepsilon [(v + u^*)_t]^2 = 0 & \text{on } [0, +\infty[\times \Omega, \\ v = 0 & \text{on } [0, +\infty[\times \partial\Omega, \\ v(0) = 0, \quad v_t(0) = 0, \end{cases} \tag{5.4}$$

then $u = v + u^*$ is the unique weak solution of the boundary valued problem for the Westervelt equation (5.3). Let us prove the existence of a such v , using Theorem 1.5.2.

We suppose that $\|u^*\|_X \leq r$ and define for $v \in X_0$

$$\Phi(v) := \alpha \varepsilon (v + u^*)(v + u^*)_{tt} + \alpha \varepsilon [(v + u^*)_t]^2.$$

For w and z in X_0 such that $\|w\|_X \leq r$ and $\|z\|_X \leq r$, we estimate

$$\|\Phi(w) - \Phi(z)\|_Y,$$

by applying the triangular inequality. The key point is that it appears terms of the form $\|ab_{tt}\|_Y$ and $\|a_t b_t\|_Y$ with a and b in X and we have the estimate

$$\|ab_{tt}\|_Y \leq \|a\|_{L^\infty(\mathbb{R}^+ \times \Omega)} \|b_{tt}\|_Y.$$

By Corollary 4.1.1 in \mathbb{R}^3 and Proposition 4.1.2 for admissible domains in \mathbb{R}^2 we have

$$\|ab_{tt}\|_Y \leq C \|a\|_{L^\infty(\mathbb{R}^+; \mathcal{D}(\mathcal{A}))} \|b\|_X$$

and the Sobolev embedding implies

$$\begin{aligned} \|ab_{tt}\|_Y &\leq C \|a\|_{H^1(\mathbb{R}^+; \mathcal{D}(\mathcal{A}))} \|b\|_X \\ &\leq B_1 \|a\|_X \|b\|_X, \end{aligned}$$

with a constant $B_1 > 0$ depending only on Ω . Moreover we have

$$\|a_t b_t\|_Y \leq \sqrt{\int_0^{+\infty} \|a_t\|_{L^\infty(\Omega)} \|b_t\|_{L^2(\Omega)} ds}.$$

Therefore, by Corollary 4.1.1 in \mathbb{R}^3 and Proposition 4.1.2 for admissible domains in \mathbb{R}^2 we have

$$\begin{aligned} \|a_t b_t\|_Y &\leq C \sqrt{\int_0^{+\infty} \|a_t\|_{\mathcal{D}(\mathcal{A})} \|b_t\|_{L^2(\Omega)} ds} \\ &\leq C \|a_t\|_{L^2(\mathbb{R}^+; \mathcal{D}(\mathcal{A}))} \|b_t\|_{L^\infty(\mathbb{R}^+; L^2(\Omega))} \\ &\leq C \|a\|_X \|b_t\|_{H^1(\mathbb{R}^+; L^2(\Omega))} \end{aligned}$$

also using Sobolev's embeddings. As a result we find

$$\|a_t b_t\|_Y \leq B_2 \|a\|_X \|b\|_X,$$

with a constant $B_2 > 0$ depending only on Ω . Taking a and b equal to u^* , w , z or $w - z$, as $\|u^*\|_X \leq r$, $\|w\|_X \leq r$ and $\|z\|_X \leq r$, we obtain

$$\|\Phi(w) - \Phi(z)\|_Y \leq \alpha B \varepsilon r \|w - z\|_X,$$

with a constant $B > 0$ depending only on Ω . The final result follows as in the proof of Point 1 of Theorem 1.2.2 in Chapter 1 Subsection 1.5.1 for the global well posedness for the Kuznetsov equation on \mathbb{R}^n using Theorem 1.5.2 at the difference that here we add a non homogeneous term, which implies

$$\|u^*\|_X \leq \frac{C_1}{\nu \varepsilon} (\|f\|_Y + \|u_0\|_{\mathcal{D}(\mathcal{A})} + \|u_1\|_{H_0^1(\Omega)}).$$

Whereas when $f = 0$ we have

$$\|u^*\|_X \leq \frac{C'_1}{\sqrt{\nu \varepsilon}} (\|u_0\|_{\mathcal{D}(\mathcal{A})} + \|u_1\|_{H_0^1(\Omega)}).$$

□

5.2 Well posedness of the Westervelt equation with non homogeneous Dirichlet boundary conditions

In this section Ω is an admissible domain in \mathbb{R}^n ($n = 2$ or 3) with a d -set boundary $\partial\Omega$ such that $n - 2 < d < n$ and \mathcal{A} designates again the Laplace operator $-\Delta$ on Ω in a weak sense with homogeneous Dirichlet boundary conditions.

The fact to have non homogeneous Dirichlet boundary conditions implies the use of traces and extensions operators which leads us to leave the field of arbitrary domains.

Theorem 5.2.1. *Let $\beta_2 = 2 - \frac{n-d}{2}$ and*

$$F := H^1([0, +\infty[; B_{\beta_2}^{2,2}(\partial\Omega)) \cap H^{\frac{7}{4}}([0, +\infty[; L^2(\partial\Omega)). \quad (5.5)$$

For $u_0 \in H^2(\Omega)$, $u_1 \in H^1(\Omega)$, $g \in F$ and $f \in L^2([0, +\infty[; L^2(\Omega))$ with the compatibility conditions

$$g(0) = Tr_{\partial\Omega} u_0, \quad \partial_t g(0) = Tr_{\partial\Omega} u_1$$

there exists a unique weak solution \tilde{u} of the problem

$$\begin{cases} \partial_t^2 \tilde{u} - c^2 \Delta \tilde{u} - \nu \varepsilon \Delta \partial_t \tilde{u} = f \text{ in } [0, +\infty[\times \Omega, \\ \tilde{u} = g \text{ on } \partial\Omega, \\ \tilde{u}(0) = u_0, \quad \partial_t \tilde{u}(0) = u_1. \end{cases} \quad (5.6)$$

It is a weak solution in the way where $\tilde{u} = u^ + \bar{g}$ with*

$$\bar{g} \in X_1 := H^2([0, +\infty[; L^2(\Omega)) \cap H^1([0, +\infty[; H^2(\Omega)), \quad (5.7)$$

such that

$$Tr_{\partial\Omega}\bar{g} = g$$

and for X defined in (5.1)

$$u^* \in X$$

is the unique weak solution of the system

$$\begin{cases} \partial_t^2 u + c^2 \mathcal{A}u + \nu \varepsilon \partial_t \mathcal{A}u = f - \partial_t^2 \bar{g} + c^2 \Delta \bar{g} + \nu \varepsilon \Delta \partial_t \bar{g} \text{ in } [0, +\infty[\times \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u(0) = u_0 - \bar{g}(0), \quad \partial_t u(0) = u_1 - \partial_t \bar{g}(0) \end{cases} \quad (5.8)$$

in the sense of formulation (4.24). Moreover we have the estimate

$$\|u^*\|_X \leq C(\|f\|_{[0, +\infty[; L^2(\Omega)} + \|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^1(\Omega)} + \|g\|_F).$$

Proof. As $g \in F$ defined in (5.5) the existence of

$$\bar{g} \in X_1 \text{ defined in (5.7),}$$

with $Tr_{\partial\Omega}\bar{g} = g$ comes from the extension operator $E_{\partial\Omega}$ introduced in Theorem 4.1.1 along with the fact that by Lemma 3.5 in [28]

$$\begin{aligned} & H^2([0, +\infty[; L^2(\mathbb{R}^{n-1} \times \mathbb{R}^+)) \cap H^1([0, +\infty[; H^2(\mathbb{R}^{n-1} \times \mathbb{R}^+)) \\ & \hookrightarrow H^{\frac{7}{4}}([0, +\infty[; L^2(\mathbb{R}^{n-1})) \cap H^1([0, +\infty[; H^{\frac{3}{2}}(\mathbb{R}^{n-1})) \end{aligned}$$

with a bounded right inverse. Moreover the boundness of $E_{\partial\Omega}$ implies

$$\|\bar{g}\|_{X_1} \leq C\|g\|_X. \quad (5.9)$$

Let us define u^* as a solution of system (5.8).

The regularity of \bar{g} implies

$$\begin{aligned} & -\partial_t^2 \bar{g} + c^2 \Delta \bar{g} + \nu \varepsilon \Delta \partial_t \bar{g} \in L^2([0, T]; L^2(\Omega)), \\ & u_0 - \bar{g}(0) \in H^2(\Omega), \quad u_1 - \partial_t \bar{g}(0) \in H^1(\Omega). \end{aligned}$$

The compatibility conditions also allow to have

$$\begin{aligned} & Tr_{\partial\Omega}(u_0 - \bar{g}(0)) = Tr_{\partial\Omega}(u_0) - g(0) = 0, \\ & Tr_{\partial\Omega}(u_1 - \partial_t \bar{g}(0)) = Tr_{\partial\Omega}(u_1) - \partial_t g(0) = 0. \end{aligned}$$

Then we can apply Theorem 5.1.1 to obtain the existence of a unique weak solution u^* of system (5.8) with the desired regularity. The regularity of u_0 , u_1 and \bar{g} with the estimate in Theorem 5.1.1 allows to give the desired estimate. \square

If we consider the problem for the Westervelt equation with non homogeneous Dirichlet boundary data:

$$\left\{ \begin{array}{l} \partial_t^2 u - c^2 \Delta u - \nu \varepsilon \Delta \partial_t u = \alpha \varepsilon u \partial_t^2 u + \alpha \varepsilon (\partial_t u)^2 + f \text{ in } [0, +\infty[\times \Omega, \\ u = g \text{ on } \partial\Omega, \\ (u(0), \partial_t u(0)) = (u_0, u_1) \text{ on } \Omega, \end{array} \right. \quad (5.10)$$

we are looking for solution of the form $u = v + \tilde{u}$, where \tilde{u} solves the strongly damped system (5.6) in a weak sense in the conditions of Theorem 5.2.1. It lead us to v solution of the following system

$$\begin{aligned} \partial_t^2 v + c^2 \mathcal{A}v + \nu \varepsilon \mathcal{A} \partial_t v &= \alpha \varepsilon (v + \tilde{u}) \partial_t^2 (v + \tilde{u}) + \alpha \varepsilon [\partial_t (v + \tilde{u})]^2 \text{ in } [0, +\infty[\times \Omega, \\ v &= 0 \text{ on } \partial\Omega, \\ (v(0), \partial_t v(0)) &= (0, 0) \text{ on } \Omega, \end{aligned} \quad (5.11)$$

the well posedness of which is determined in the following theorem using Theorem 1.5.2.

Theorem 5.2.2. *Let X and X_1 be defined by (5.1) and (5.7) respectively. For $u^* \in X$ and $\bar{g} \in X_1$, there exists $r^* > 0$ with $r^* = O(1)$ such that for $r \in [0, r^*[$ and all data satisfying*

$$\|u^*\| \leq \frac{r}{2} \text{ and } \|\bar{g}\|_{X_1} \leq \frac{r}{2}$$

there exists the unique weak solution $v \in X$ of problem (5.11) with $\tilde{u} = u^ + \bar{g}$ and $\|v\|_X \leq r$.*

Proof. As in the proof of Theorem 5.1.2 we introduce the Banach spaces

$$X_0 := \{u \in X \mid u(0) = u_t(0) = 0\}$$

and $Y = L^2(\mathbb{R}^+; L^2(\Omega))$. Then by Theorem 5.1.1, the linear operator

$$L : X_0 \rightarrow Y, \quad u \in X_0 \mapsto L(u) := u_{tt} + c^2 \mathcal{A}u + \nu \varepsilon \mathcal{A}u_t \in Y$$

is a bi-continuous isomorphism.

Now set $u^* \in X$ and $\bar{g} \in X_1$ with $\|u^*\|_X \leq \frac{r}{2}$ and $\|\bar{g}\|_{X_1} \leq \frac{r}{2}$ and consider

$$\Phi(v) = \alpha \varepsilon (v + u^* + \bar{g}) \partial_t^2 (v + u^* + \bar{g}) + \alpha \varepsilon [\partial_t (v + u^* + \bar{g})]^2.$$

The properties of X and X_1 allow to show for w and z in X with $\|w\|_X \leq r$ and $\|z\|_X \leq r$

$$\|\Phi(w) - \Phi(z)\|_Y \leq \alpha B \varepsilon r \|w - z\|_X$$

with a constant $B > 0$ depending only on Ω . The final result follows as in the proof of global well posedness for the Kuznetsov equation on \mathbb{R}^n and the use of Theorem 1.5.2. \square

Now we can give the result on the weak well-posedness of the Westervelt system (5.10):

Theorem 5.2.3. Let $u_0 \in H^2(\Omega)$, $u_1 \in H^1(\Omega)$, $g \in F$ defined in (5.5) and take $f \in L^2([0, +\infty[; L^2(\Omega))$ with the compatibility conditions

$$g(0) = Tr_{\partial\Omega}u_0, \quad \partial_t g(0) = Tr_{\partial\Omega}u_1.$$

There exist $r^* > 0$ and $C > 0$ such that for $r \in [0, r^*[$

$$\|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^1(\Omega)} + \|g\|_F + \|f\|_{L^2([0, +\infty[; L^2(\Omega))} \leq Cr$$

implies that there exists a unique weak solution u of the Westervelt system (5.10) with non homogeneous Dirichlet boundary conditions in the way where $u = v + u^* + \bar{g}$. We have $u^* \in X$, $\bar{g} \in X_1$ defined in (5.1) and (5.7) such that $\tilde{u} = u^* + \bar{g}$ unique weak solution of the strongly damped problem (5.6) and $v \in X$ unique solution of the system (5.11) in the sense where $\forall w \in L^2([0, +\infty[; H_0^1(\Omega))$

$$\begin{aligned} & \int_0^{+\infty} (\partial_t^2 v, w)_{L^2(\Omega)} + c^2(\nabla v, \nabla w)_{L^2(\Omega)} + \nu\varepsilon(\nabla \partial_t v, \nabla w)_{L^2(\Omega)} dt = \\ & = \int_0^{+\infty} (\alpha\varepsilon(v + \tilde{u})\partial_t^2(v + \tilde{u}) + \alpha\varepsilon[\partial_t(v + \tilde{u})]^2, w)_{L^2(\Omega)} dt, \\ & \text{with } (v(0), \partial_t v(0)) = (0, 0). \end{aligned}$$

Moreover, we have the estimates

$$\|u^*\|_X \leq r, \quad \|\bar{g}\|_{X_1} \leq \frac{r}{2}, \quad \|v\|_X \leq \frac{r}{2}.$$

Proof. According to Theorem 5.2.2 we have $v \in X$ with $\|v\|_X \leq r$ solution of system (5.11) as soon as $\|u^*\|_X \leq \frac{r}{2}$ and $\|\bar{g}\|_{X_1} \leq \frac{r}{2}$ for $r \in [0, r^*[$ with $r^* > 0$. But according to Theorem 5.2.1, if $\tilde{u} = u^* + \bar{g}$ is a unique weak solution of the strongly damped problem (5.6), then we have the estimates

$$\|u^*\|_X \leq C(\|f\|_{[0, +\infty[; L^2(\Omega)} + \|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^1(\Omega)} + \|g\|_F)$$

and

$$\|\bar{g}\|_{X_1} \leq C\|g\|_F.$$

So there exists a constant $C > 0$ such that

$$\|f\|_{L^2([0, +\infty[; L^2(\Omega))} + \|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^1(\Omega)} + \|g\|_F \leq Cr$$

implies

$$\|u^*\|_X \leq \frac{r}{2}, \quad \text{and} \quad \|\bar{g}\|_{X_1} \leq \frac{r}{2},$$

which permits to conclude. □

5.3 Well posedness of the Westervelt equation with homogeneous Robin boundary conditions

In this section Ω is an admissible domain in \mathbb{R}^n ($n = 2$ or 3) with a d -set boundary $\partial\Omega$ such that $n - 2 < d < n$. We note m_d the d -dimensional Hausdorff measure.

5.3.1 Well posedness of the Westervelt equation with homogeneous Robin boundary condition on Lipschitz domains

In this subsection only Ω is a Lipschitz domain in \mathbb{R}^n ($n = 2$ or 3) with a $(n - 1)$ -set boundary $\partial\Omega$. We denote by $\mathcal{D}(-\Delta) \subset H^1(\Omega)$ the domain in $L^2(\Omega)$ of the operator $-\Delta$ on Ω in a weak sense with homogeneous Robin boundary conditions in accordance with Definition 4.1.9 and Theorem 4.1.13:

$$\begin{aligned} -\Delta : \mathcal{D}(-\Delta) \subset H^1(\Omega) &\rightarrow L^2(\Omega) \\ u &\mapsto -\Delta u. \end{aligned}$$

We say that $u \in \mathcal{D}(-\Delta)$ if and only if $u \in H^1(\Omega)$, u satisfies homogeneous Robin boundary condition and $-\Delta u \in L^2(\Omega)$. The operator $-\Delta$ is linear self-adjoint and coercive in the sense where for all $u \in \mathcal{D}(-\Delta)$

$$(-\Delta u, u)_{L^2(\Omega)} = (u, u)_{\tilde{H}^1(\Omega)}$$

and we will use the notation

$$\|u\|_{\mathcal{D}(-\Delta)} = \|\Delta u\|_{L^2(\Omega)}.$$

To show the well-posedness of the Westervelt equation on Lipschitz domain we will use the following result coming from Theorem 3.14 in [75].

Theorem 5.3.1. *For Ω a bounded Lipschitz domain in \mathbb{R}^n , and $f \in L^{\frac{p}{2}}(\Omega)$ with $p > n$ let $u \in H^1(\Omega)$ be the unique weak solution of the Poisson problem (3.2) in the sense of Definition 4.1.9, then $u \in C(\bar{\Omega})$ and we have the following estimate*

$$\|u\|_{L^\infty(\Omega)} \leq C \|f\|_{L^{\frac{p}{2}}(\Omega)}.$$

Remark 5.3.1. *In dimension $n = 2$ or 3 , we can take $f \in L^2(\Omega)$ as $4 > n$ and $\frac{4}{2} = 2$.*

Theorem 5.3.2. *Let Ω be an arbitrary bounded Lipschitz domain in \mathbb{R}^n , $n = 2$ or 3 . Let $\nu > 0$, $\mathbb{R}^+ = [0, +\infty[$ and X be the Hilbert space defined in (5.1) by X defined by*

$$X := H^1([0, +\infty[; \mathcal{D}(-\Delta)) \cap H^2([0, +\infty[; L^2(\Omega)). \quad (5.12)$$

For

$$u_0 \in \mathcal{D}(-\Delta), \quad u_1 \in H^1(\Omega) \quad \text{and} \quad f \in L^2(\mathbb{R}^+; L^2(\Omega))$$

there exists $C_1 = O(1)$ the minimal constant such that the weak solution $u^* \in X$ of the corresponding boundary Cauchy problem (4.36), in the sense of formulation (4.37), satisfies

$$\|u^*\|_X \leq \frac{C_1}{\nu\epsilon} (\|f\|_{L^2(\mathbb{R}^+; L^2(\Omega))} + \|u_0\|_{\mathcal{D}(-\Delta)} + \|u_1\|_{\tilde{H}^1(\Omega)}).$$

Then there exists $r_* > 0$ with $r_* = O(1)$ such that for all $r \in [0, r_*[$ and all data satisfying

$$\|f\|_{L^2(\mathbb{R}^+; L^2(\Omega))} + \|u_0\|_{\mathcal{D}(-\Delta)} + \|u_1\|_{\tilde{H}^1(\Omega)} \leq \frac{\nu\epsilon}{C_1} r,$$

there exists the unique weak solution $u \in X$ of the boundary valued problem for the Westervelt equation

$$\begin{cases} \partial_t^2 u + c^2 \mathcal{A}u + \nu \varepsilon \mathcal{A} \partial_t u = \alpha \varepsilon u \partial_t^2 u + \alpha \varepsilon (\partial_t u)^2 + f & \text{on } [0, +\infty[\times \Omega, \\ \frac{\partial}{\partial n} u + au = 0 & \text{on } [0, +\infty[\times \partial \Omega, \\ u(0) = u_0, \quad \partial_t u(0) = u_1. \end{cases} \quad (5.13)$$

It is a weak solution in the sense where $u = u^* + v$ with u^* is defined as a weak solution in the sense of (4.37) and v is the solution of an homogeneous non linear initial-boundary valued problem depending on u^* and determined with Theorem 1.5.2.

That is to say $\forall w \in L^2([0, +\infty[; H^1(\Omega))$

$$\begin{aligned} & \int_0^{+\infty} (\partial_t^2 v, w)_{L^2(\Omega)} + c^2 (v, w)_{\tilde{H}^1(\Omega)} + \nu \varepsilon (\partial_t v, w)_{\tilde{H}^1(\Omega)} dt = \\ & = \int_0^{+\infty} (\alpha \varepsilon (v + u^*) \partial_t^2 (v + u^*) + \alpha \varepsilon [\partial_t (v + u^*)]^2, w)_{L^2(\Omega)} dt, \\ & \text{with } (v(0), \partial_t v(0)) = (0, 0). \end{aligned}$$

Moreover $\|u\|_X \leq 2r$.

Proof. For $u_0 \in \mathcal{D}(-\Delta)$, $u_1 \in H^1(\Omega)$ and $f \in L^2(\mathbb{R}^+; L^2(\Omega))$ let us denote, by Theorem 4.4.2, $u^* \in X$, where X defined in (5.12), the unique weak solution, in the sense of formulation (4.37) of the linear problem

$$\begin{cases} u_{tt}^* - c^2 \Delta u^* - \nu \varepsilon \Delta u_t^* = f, \\ \frac{\partial}{\partial n} u^* + au^* = 0, \\ u^*(0) = u_0 \in \mathcal{D}(-\Delta), \quad u_t^*(0) = u_1 \in H^1(\Omega). \end{cases}$$

By Theorem 4.4.2 $\partial_t^2 u^*$, Δu^* and $\Delta \partial_t u^*$ are in $L^2(\mathbb{R}^+; L^2(\Omega))$ so $u^* \in X$ and the estimates (4.38) and (4.39) give us the estimate on $\|u^*\|_X$. In addition we introduce the Banach spaces

$$X_0 := \{u \in X \mid u(0) = u_t(0) = 0\} \quad \text{and} \quad Y = L^2(\mathbb{R}^+; L^2(\Omega)).$$

Then by Theorem 4.4.2, the linear operator

$$L : X_0 \rightarrow Y, \quad u \in X_0 \mapsto L(u) := \partial_t^2 u - c^2 \Delta u - \nu \varepsilon \partial_t \Delta u \in Y$$

is a bi-continuous isomorphism.

Let us now notice that if $v \in X_0$ is the unique solution of the non-linear Cauchy problem

$$\begin{cases} v_{tt} - c^2 \Delta v - \nu \varepsilon \Delta v_t - \alpha \varepsilon (v + u^*) (v + u^*)_{tt} - \alpha \varepsilon [(v + u^*)_t]^2 = 0, \\ \frac{\partial}{\partial n} v + av = 0, \\ v(0) = 0, \quad v_t(0) = 0, \end{cases} \quad (5.14)$$

then $u = v + u^*$ is the unique solution of the Cauchy problem for the Westervelt equation (5.3). Let us prove the existence of a such v , using Theorem 1.5.2.

We suppose that $\|u^*\|_X \leq r$ and define for $v \in X_0$

$$\Phi(v) := \alpha\varepsilon(v + u^*)(v + u^*)_{tt} + \alpha\varepsilon[(v + u^*)_t]^2.$$

For w and z in X_0 such that $\|w\|_X \leq r$ and $\|z\|_X \leq r$, we estimate

$$\|\Phi(w) - \Phi(z)\|_Y,$$

by applying the triangular inequality. The key point is that it appears terms of the form $\|ab_{tt}\|_Y$ and $\|a_t b_t\|_Y$ with a and b in X . We have the estimate

$$\|ab_{tt}\|_Y \leq \|a\|_{L^\infty(\mathbb{R}^+ \times \Omega)} \|b_{tt}\|_Y.$$

By Theorem 5.3.1 we have

$$\|ab_{tt}\|_Y \leq C \|a\|_{L^\infty(\mathbb{R}^+; \mathcal{D}(-\Delta))} \|b\|_X,$$

and the Sobolev's embedding implies

$$\begin{aligned} \|ab_{tt}\|_Y &\leq C \|a\|_{H^1(\mathbb{R}^+; \mathcal{D}(-\Delta))} \|b\|_X \\ &\leq B_1 \|a\|_X \|b\|_X \end{aligned}$$

with a constant $B_1 > 0$ depending only on Ω . Moreover we have

$$\|a_t b_t\|_Y \leq \sqrt{\int_0^{+\infty} \|a_t\|_{L^\infty(\Omega)}^2 \|b_t\|_{L^2(\Omega)}^2 ds},$$

and hence, by Theorem 5.3.1 we find

$$\begin{aligned} \|a_t b_t\|_Y &\leq C \sqrt{\int_0^{+\infty} \|a_t\|_{\mathcal{D}(-\Delta)}^2 \|b_t\|_{L^2(\Omega)}^2 ds} \\ &\leq C \|a_t\|_{L^2(\mathbb{R}^+; \mathcal{D}(-\Delta))} \|b_t\|_{L^\infty(\mathbb{R}^+; L^2(\Omega))}. \end{aligned}$$

The Sobolev's embedding implies

$$\|a_t b_t\|_Y \leq C \|a_t\|_X \|b_t\|_{H^1(\mathbb{R}^+; L^2(\Omega))},$$

which gives us

$$\|a_t b_t\|_Y \leq B_2 \|a\|_X \|b\|_X$$

with a constant $B_2 > 0$ depending only on Ω . Taking a and b equal to u^* , w , z or $w - z$, as

$$\|u^*\|_X \leq r, \quad \|w\|_X \leq r \quad \text{and} \quad \|z\|_X \leq r,$$

we obtain

$$\|\Phi(w) - \Phi(z)\|_Y \leq \alpha B \varepsilon r \|w - z\|_X$$

with a constant $B > 0$ depending only on Ω . The final result follows as in the proof of Point 1 of Theorem 1.2.2 in Chapter 1 Subsection 1.5.1 for the global well posedness for the Kuznetsov equation on \mathbb{R}^n using Theorem 1.5.2 at the difference that here we add a non homogeneous term f , which implies

$$\|u^*\|_X \leq \frac{C_1}{\nu\varepsilon} (\|f\|_Y + \|u_0\|_{\mathcal{D}(-\Delta)} + \|u_1\|_{\tilde{H}^1(\Omega)}).$$

Whereas for $f = 0$ we have

$$\|u^*\|_X \leq \frac{C'_1}{\sqrt{\nu\varepsilon}} (\|u_0\|_{\mathcal{D}(-\Delta)} + \|u_1\|_{\tilde{H}^1(\Omega)}).$$

□

5.3.2 Well posedness of the Westervelt equation with homogeneous Robin boundary conditions on admissible domains

Here Ω is an admissible bounded domain in \mathbb{R}^n ($n = 2$ or 3) with a d -set boundary $\partial\Omega$ such that $n - 2 < d < n$. For $p \geq 2$, we denote by $\mathcal{D}(-\Delta) \subset H^1(\Omega)$ the domain in $L^p(\Omega)$ of the operator $-\Delta$ on Ω in a weak sense with homogeneous Robin boundary conditions in accordance with Theorem 4.1.13 and Definition 4.1.10:

$$-\Delta : \mathcal{D}(-\Delta) \subset H^1(\Omega) \rightarrow L^p(\Omega) \\ u \mapsto -\Delta u.$$

For $p \geq 2$ we have $L^p(\Omega) \subset L^2(\Omega)$ and by Theorem 4.1.14 the spectrum of $-\Delta$ in $L^p(\Omega)$ is contained in \mathbb{R}_+^* as it is contained in the spectrum of $-\Delta$ in $L^2(\Omega)$. We need a result on maximal L^p regularity which is a direct application of Theorem 4.1 in [22] to the linear system for the strongly damped wave equation with Robin boundary conditions and homogenous initial data:

$$\begin{cases} \partial_t^2 u - c^2 \Delta u - \nu \Delta \partial_t u = f \text{ on } [0, T] \times \Omega, \\ \frac{\partial}{\partial n} u + au = 0 \text{ on } [0, T] \times \partial\Omega, \\ u(0) = \partial_t u(0) = 0 \text{ in } \Omega. \end{cases} \quad (5.15)$$

Theorem 5.3.3. [22] For $p \geq 2$, let $-\Delta$ be defined on $L^p(\Omega)$ in accordance with Definition 4.1.9, which defines the weak solution of the Poisson problem with homogeneous Robin boundary conditions. If $-\Delta$ is a sectorial operator on $L^p(\Omega)$ which admits a bounded RH^∞ functional calculus of angle β with $0 < \beta < \frac{\pi}{2}$ then system (5.15) has L^p -maximal regularity.

Remark 5.3.2. It is an application of a general theorem using UMD spaces. UMD spaces have been introduced in Ref.[16]. By Ref. [56], if A is a sectorial operator on an UMD spaces X with property (α) and admits a bounded H^∞ calculus of angle β , then A already admits a RH^∞ calculus of angle β . For the definition of Banach spaces having property (α) see Ref. [78]. For $p > 1$, $L^p(\Omega)$ is an UMD space having property (α) according to Ref.[22] before Theorem 4.1.

As a consequence we have:

Theorem 5.3.4. For $p \geq 2$ and $T > 0$, there exists a unique weak solution $u \in X^p$ with

$$X^p := W^{1,p}([0, T]; \mathcal{D}(-\Delta)) \cap W^{2,p}([0, T]; L^p(\Omega)) \quad (5.16)$$

of the homogeneous boundary-valued problem (5.15)

if and only if $f \in L^p([0, T]; L^p(\Omega))$. Moreover we have the estimate

$$\|u\|_{X^p} \leq C \|f\|_{L^p([0, T]; L^p(\Omega))}.$$

It is a weak solution in the sense that the operator $-\Delta$ is defined in accordance with Definition 4.1.9, which defines the weak solution of the Poisson problem with homogeneous Robin boundary conditions.

Proof. The equivalence is a direct consequence of Theorem 5.3.3. If we want to prove Theorem 5.3.4, by Remark 5.3.2 we only need to ensure that $-\Delta$ is a sectorial operator on $L^p(\Omega)$ which admits a bounded H^∞ calculus of angle β with $\beta < \frac{\pi}{2}$. With the work in [9], particularly Theorem 5.6, this is true. The key point according to Theorem 4.1.14, which is true on admissible domain, is that we have for $z \in \mathbb{C}$ such that $|\arg(z)| < \frac{\pi}{2}$

$$\|e^{z\Delta}\|_{L^2 \rightarrow L^2} \leq e^{-\lambda_1|z|}$$

with $\lambda_1 > 0$. The estimate in Theorem 5.3.4 is a consequence of the closed graph theorem. \square

Now we consider the non-homogeneous damped wave problem (4.36):

Theorem 5.3.5. *For $p \geq 2$ and $T > 0$, we define X^p as in Definition (5.16) and moreover we define*

$$(L^p(\Omega), \mathcal{D}(-\Delta))_p = \{(u_0, u_1) \in L^p(\Omega) \times L^p(\Omega) \mid \exists u \in X^p \text{ with } u(0) = u_0, u_t(0) = u_1\}. \quad (5.17)$$

There exists a unique weak solution $u \in X^p$ of the damped wave equation problem (4.36) if and only if $f \in L^p([0, T]; L^p(\Omega))$ and $(u_0, u_1) \in (L^p(\Omega), \mathcal{D}(-\Delta))_p$. Moreover we have the estimate

$$\|u\|_{X^p} \leq C(\|f\|_{L^p([0, T]; L^p(\Omega))} + \|(u_0, u_1)\|_{(L^p(\Omega), \mathcal{D}(-\Delta))_p}).$$

Proof. For $(u_0, u_1) \in (L^p(\Omega), \mathcal{D}(-\Delta))_p$, we have by definition $w \in X^p$ such that

$$w(0) = u_0 \quad \text{and} \quad w_t(0) = u_1.$$

In particular,

$$\partial_t^2 w - c^2 \Delta w - \nu \Delta \partial_t w \in L^p([0, T]; L^p(\Omega)).$$

So in the sense of Theorem 5.3.4 if we take \tilde{w} the unique weak solution in X^p of

$$\begin{cases} \partial_t^2 \tilde{w} - c^2 \Delta \tilde{w} - \nu \Delta \partial_t \tilde{w} = f - (\partial_t^2 w - c^2 \Delta w - \nu \Delta \partial_t w) \text{ on } [0, T] \times \Omega, \\ \frac{\partial}{\partial n} \tilde{w} + a \tilde{w} = 0 \text{ on } [0, T] \times \partial \Omega, \\ \tilde{w}(0) = \partial_t \tilde{w}(0) = 0 \text{ in } \Omega, \end{cases}$$

we have by the linearity $u = w + \tilde{w}$ which is the weak solution of the damped wave equation problem (4.36). The unicity comes from the unicity of the solution when $u_0 = u_1 = 0$ by Theorem 5.3.4. The other side of the equivalence comes directly from the definition of X^p and $(L^p(\Omega), \mathcal{D}(-\Delta))_p$.

The estimate is a consequence of the close graph theorem. \square

Remark 5.3.3. *Since*

$$\mathcal{D}(-\Delta) \times \mathcal{D}(-\Delta) \hookrightarrow (L^p(\Omega), \mathcal{D}(-\Delta))_p$$

we have a similar estimate in Theorem 5.3.5 for the solutions of the damped wave equation problem (4.36), when $(u_0, u_1) \in \mathcal{D}(-\Delta) \times \mathcal{D}(-\Delta)$ replacing $\|(u_0, u_1)\|_{(L^p(\Omega), \mathcal{D}(-\Delta))_p}$ by $\|u_0\|_{\mathcal{D}(-\Delta)} + \|u_1\|_{\mathcal{D}(-\Delta)}$.

Therefore we can treat the Westervelt equation with the homogeneous Robin boundary conditions on admissible domains.

Theorem 5.3.6. *Let Ω be an admissible domain in \mathbb{R}^n ($n = 2$ or 3) with a d -set boundary $\partial\Omega$ such that $n - 2 < d < n$. We define X^p as in (5.16). Let $\nu > 0$, $p > n$ and $T > 0$. We consider the problem for the Westervelt equation with the homogeneous Robin boundary conditions (5.13). Let the initial data*

$$u_0 \in \mathcal{D}(-\Delta) \quad \text{and} \quad u_1 \in \mathcal{D}(-\Delta)$$

with the non homogeneous datum $f \in L^p([0, T]; L^p(\Omega))$, $C_1 > 0$ be the minimal constant such that the solution $u^ \in X^p$ of the corresponding non homogeneous linear Cauchy problem (4.36) satisfies*

$$\|u^*\|_{X^p} \leq C_1(\|f\|_{L^p([0, T]; L^p(\Omega))} + \|u_0\|_{\mathcal{D}(-\Delta)} + \|u_1\|_{\mathcal{D}(-\Delta)}).$$

Then there exists $r_ > 0$ such that for all $r \in [0, r_*[$ and all data satisfying*

$$\|f\|_{L^2([0, T]; L^2(\Omega))} + \|u_0\|_{\mathcal{D}(-\Delta)} + \|u_1\|_{\mathcal{D}(-\Delta)} \leq \frac{1}{C_1}r, \tag{5.18}$$

there exists a unique weak solution $u \in X^p$ of problem (5.13) for the Westervelt equation and $\|u\|_{X^p} \leq 2r$.

Proof. The proof is essentially the same as in Theorem 5.3.2 and thus is omitted. We just replace $L^2([0, +\infty[; L^2(\Omega))$ by $L^p([0, T]; L^p(\Omega))$ and the space X defined in (5.12) by the space X^p defined in (5.1). We also use the Theorems 5.3.5 and 4.1.15 to have the required estimates. □

5.4 Prefractal and fractal problem for the Westervelt equation on Koch Mixtures with Robin boundary conditions

Now let us fix a type of fractal boundaries of our domain, choosing Koch mixtures, as defined in [19]. The advantage of a fixed fractal geometry is to be able to consider the approximations questions of how the solutions on the prefractal domain converge to the solution on a fractal one. This kind of convergence is named Mosco-convergence.

5.4.1 Scale irregular Koch curves

We recall briefly some notations introduced in Section 2 page 1223 of [19] for scale irregular Koch curves built on two families of contractive similitudes. Let $\mathcal{B} = \{1, 2\}$: for $a \in \mathcal{B}$ let $2 < l_a < 4$, and for each $a \in \mathcal{B}$ let

$$\Psi^{(a)} = \{\psi_1^{(a)}, \dots, \psi_4^{(a)}\}$$

be the family of contractive similitudes $\psi_i^{(a)} : \mathbb{C} \rightarrow \mathbb{C}$, $i = 1, \dots, 4$, with contraction factor l_a^{-1} defined in [20].

Let $\Xi = \mathcal{B}^{\mathbb{N}}$; we call $\xi \in \Xi$ an environment. We define the usual left shift S on Ξ . For $\mathcal{O} \subset \mathbb{R}^2$, we set

$$\Phi^{(a)}(\mathcal{O}) = \bigcup_{i=1}^4 \psi_i^{(a)}(\mathcal{O})$$

and

$$\Phi_m^{(\xi)}(\mathcal{O}) = \Phi^{(\xi_1)} \circ \dots \circ \Phi^{(\xi_m)}(\mathcal{O}).$$

Let K be the line segment of unit length with $A = (0, 0)$ and $B = (1, 0)$ as endpoints. We set, for each m in \mathbb{N} ,

$$K^{(\xi),m} = \Phi_m^{(\xi)}(K).$$

$K^{(\xi),m}$ is the so-called m -th prefractal curve. The fractal $K^{(\xi)}$ associated with the environment sequence ξ is defined by

$$K^{(\xi)} = \overline{\bigcup_{m=1}^{+\infty} \Phi_m^{(\xi)}(\Gamma)},$$

where $\Gamma = \{A, B\}$. For $\xi \in \Xi$, we set $i|m = (i_1, \dots, i_m)$ and $\psi_{i|m} = \psi_{i_1}^{(\xi_1)} \circ \dots \circ \psi_{i_m}^{(\xi_m)}$. We define the volume measure $\mu^{(\xi)}$ as the unique Radon measure on $K^{(\xi)}$ such that

$$\mu^{(\xi)}(\psi_{i|m}(K^{(S^m \xi)})) = \frac{1}{4^n}$$

(see Section 2 in [14]) as, for each $a \in \mathcal{B}$, the family $\Phi^{(a)}$ has 4 contractive similitudes.

The fractal set $K^{(\xi)}$ and the volume measure $\mu^{(\xi)}$ depend on the oscillations in the environment sequence ξ . We denote by $h_a^{(\xi)}(m)$ the frequency of the occurrence of a in the finite sequence $\xi|m$, $m \geq 1$:

$$h_a^{(\xi)}(m) = \frac{1}{m} \sum_{i=1}^m 1_{\{\xi_i=a\}}, \quad a = 1, 2.$$

Let p_a be a probability distribution on \mathcal{B} , and suppose that ξ satisfies

$$h_a^{(\xi)}(m) \xrightarrow{m \rightarrow +\infty} p_a,$$

(where $0 \leq p_a \leq 1$, $p_1 + p_2 = 1$) and

$$|h_a^{(\xi)}(m) - p_a| \leq \frac{C_0}{m}, \quad a = 1, 2, \quad (n \geq 1),$$

with some constant $C_0 \geq 1$, that is, we consider the case of the fastest convergence of the occurrence factors.

Under these conditions, the measure $\mu^{(\xi)}$ has the property that there exist two positive constants C_1, C_2 , such that (see [73] and [74]),

$$C_1 r^{d^{(\xi)}} \leq \mu^{(\xi)}(K^{(\xi)} \cap B_r(x)) \leq C_2 r^{d^{(\xi)}} \quad \text{for all } x \in K^{(\xi)}, 0 < r \leq 1,$$

where $B_r(x) \subset \mathbb{R}^2$ denotes the Euclidean ball of radius r and centered at x with

$$d^{(\xi)} = \frac{\ln 4}{p_1 \ln p_1 + p_2 \ln p_2}.$$

According to Definition 4.1.1 we can say that $K^{(\xi)}$ is a $d^{(\xi)}$ set with respect to the $d^{(\xi)}$ -dimensional Hausdorff measure $m_{d^{(\xi)}}$. Thus, the measure $\mu^{(\xi)}$ is also $d^{(\xi)}$ -dimensional and equivalent to $m_{d^{(\xi)}}$.

5.4.2 Trace and extension theorems

As in [20], we collect some preliminary results: in particular, we state trace theorems for the specific geometry of the prefractal and fractal problem. We will use Definition 4.1.4 to define the trace operator.

Let Ω_0 be the square $\{(x, y) : 0 < x < 1, -1 < y < 0\}$ with vertices $A = (0, 0)$, $B = (1, 0)$, $C = (1, -1)$, and $D = (0, -1)$. On each of the 4 sides we construct either a scale irregular Koch curve or the corresponding approximating prefractal curve. More precisely, we consider the set $\Omega^{(\xi)}$ bounded by four scale irregular Koch curves $K_j^{(\xi)}$, $j = 1, 2, 3, 4$ with endpoints A and B , B and C , C and D , D and A respectively. Moreover, we consider the set $\Omega^{(\xi),m}$ bounded by 4 approximating prefractal curves $K_j^{(\xi),m}$, $j = 1, 2, 3, 4$ starting from the segments K_j with endpoints A and B , B and C , C and D , D and A respectively.

While the fractal boundary $\partial\Omega^{(\xi)}$ is irregular, the prefractal boundary $\partial\Omega^{(\xi),m}$ is polygonal, so we can easily give well posedness results for partial differential equations with domains having such boundary and use the classic Lebesgue measure on such a boundary. In this way we are going to give well-posedness results on the solution u of the Westervelt equation on a domain $\Omega^{(\xi)}$ by a convergence argument on the functions u_m solutions of the Westervelt equation on a domain $\Omega^{(\xi),m}$.

The following theorem characterizes the trace to the set $\partial\Omega^{(\xi),m}$ of Sobolev spaces $H^\sigma(\mathbb{R}^2)$. It is proved in [48]. Set

$$l^{(\xi)}(m) = \prod_{i=1}^m l_{\xi_i}. \tag{5.19}$$

Theorem 5.4.1. *Let $u \in H^\sigma(\mathbb{R}^2)$ and $l^{(\xi)}(m)$ as in (5.19). Then, for $\frac{1}{2} < \sigma \leq 1$*

$$\|Tr_{\partial\Omega^{(\xi),m}} u\|_{L^2(\partial\Omega^{(\xi),m})}^2 \leq C_\sigma \frac{4^m}{l^{(\xi)}(m)} \|u\|_{H^\sigma(\mathbb{R}^2)}^2, \tag{5.20}$$

where C_σ is independent of m .

The following theorem characterizes the trace to the set $\partial\Omega^{(\xi)}$ of Sobolev spaces $H^\sigma(\mathbb{R}^2)$ (for the proof, see [48]).

Theorem 5.4.2. *Let $u \in H^\sigma(\mathbb{R}^2)$. Then, for $\frac{1}{2} < \sigma \leq 1$*

$$\|Tr_{\partial\Omega^{(\xi)}} u\|_{L^2(\partial\Omega^{(\xi)})}^2 \leq C_\sigma \|u\|_{H^\sigma(\mathbb{R}^2)}^2. \tag{5.21}$$

The following Theorem extends functions of $H^1(\Omega^{(\xi),m})$ to the space $H^1(\mathbb{R}^2)$ by an operator whose norm is independent of the (increasing) number of sides. It is a particular case of the extension theorem due to Jones (Theorem 1 in [46]) as the domains $\Omega^{(\xi),m}$ are (ε, ∞) -domains with ε independent of m (for the proof, see [19]).

Theorem 5.4.3. *For any $m \in \mathbb{N}$, there exists a bounded linear extension operator $E_{\Omega^{(\xi),m}} : H^1(\Omega^{(\xi),m}) \rightarrow H^1(\mathbb{R}^2)$, whose norm is independent of m , that is*

$$\|E_{\Omega^{(\xi),m}} v\|_{H^1(\mathbb{R}^2)} \leq C_J \|v\|_{H^1(\Omega^{(\xi),m})} \tag{5.22}$$

with C_J independent of m .

We also use another extension result (for the proof, see [19]).

Theorem 5.4.4. *There exists a bounded linear extension operator $E_{\Omega^{(\xi)}} : H^\sigma(\Omega^{(\xi)}) \rightarrow H^\sigma(\mathbb{R}^2)$, $\frac{1}{2} < \sigma \leq 1$, such that*

$$\|E_{\Omega^{(\xi)}}v\|_{H^\sigma(\mathbb{R}^2)} \leq C_\Omega \|v\|_{H^\sigma(\Omega^{(\xi)})}. \quad (5.23)$$

We conclude this section with the following approximation results for the measure μ^ξ proved in [20] and [19].

Theorem 5.4.5. [20] *Let*

$$\sigma_m := \frac{l^{(\xi)}(m)}{4^m} \quad (5.24)$$

with $l^{(\xi)}(m)$ as in (5.19). For any function $g \in H^1(\Omega^{(\xi)})$

$$\sigma_m \int_{\partial\Omega^{(\xi),m}} Tr_{\partial\Omega^{(\xi),m}} g ds \rightarrow \int_{\partial\Omega^{(\xi)}} Tr_{\partial\Omega^{(\xi)}} g d\mu^{(\xi)} \text{ for } m \rightarrow +\infty. \quad (5.25)$$

Theorem 5.4.6. [19] *Let σ_m defined by equation (5.24). For any function $g \in C(\overline{\Omega^{(\xi)}})$*

$$\sigma_m \int_{\partial\Omega^{(\xi),m}} Tr_{\partial\Omega^{(\xi),m}} g ds \rightarrow \int_{\partial\Omega^{(\xi)}} Tr_{\partial\Omega^{(\xi)}} g d\mu^{(\xi)} \text{ for } m \rightarrow +\infty. \quad (5.26)$$

Remark 5.4.1. $\sigma_m = \frac{1}{Vol(\partial\Omega^m)} \xrightarrow{m \rightarrow +\infty} 0$ while $Vol(\partial\Omega^m) \xrightarrow{m \rightarrow +\infty} +\infty$.

5.4.3 Asymptotic analysis

From now on, when it does not give rise to misunderstanding, in the notation we suppress the super index (ξ) by writing simply Ω , Ω^m , $l(m)$ and similar expressions. We consider the problem with Robin boundary conditions on an open set Ω associated to the Westervelt equation

$$\begin{cases} \partial_t^2 u - c^2 \Delta u - \nu \varepsilon \Delta \partial_t u = \alpha \varepsilon \partial_t [u \partial_t u] & \text{on }]0, T] \times \Omega, \\ \partial_n u + au = b & \text{on }]0, T] \times \partial\Omega, \\ u(0) = u_0, \quad u_t(0) = u_1 & \text{on } \Omega \end{cases} \quad (5.27)$$

with the compatibility condition

$$b = (\partial_n u_0 + au_0)|_{\partial\Omega} \quad 0 = (\partial_n u_1 + au_1)|_{\partial\Omega}.$$

Let Ω^* be a bounded possibly regular open set such that $\overline{\Omega}^m \subset \Omega^*$ for all n and $\overline{\Omega} \subset \Omega^*$.

Definition 5.4.1. *For an open set U and $T > 0$ we define the Hilbert space*

$$H(U) = \{u \in H^1([0, T]; H^1(U)) \mid \partial_t u(0) \text{ and } \partial_t u(T) \in L^2(U)\}. \quad (5.28)$$

For $\phi \in C^1([0, T] \times \mathcal{D}(\Omega^*))$ and $u \in H(\Omega^*)$ we define

$$\begin{aligned}
F^m[u, \phi] := & \int_0^T \int_{\Omega^m} -\partial_t u \partial_t \phi + c^2 \nabla u \nabla \phi - \nu \varepsilon \nabla u \nabla \partial_t \phi + \alpha \varepsilon (u \partial_t u) \partial_t \phi \, d\lambda \, dt \\
& + \int_0^T \int_{\partial\Omega^m} c^2 a \operatorname{Tr}_{\partial\Omega^m} u \operatorname{Tr}_{\partial\Omega^m} \phi - \nu \varepsilon a_m \operatorname{Tr}_{\partial\Omega^m} u \operatorname{Tr}_{\partial\Omega^m} \partial_t \phi - c^2 b_m \operatorname{Tr}_{\partial\Omega^m} \phi \, ds \, dt \\
& + \int_{\Omega^m} \partial_t u(T) \phi(T) \, d\lambda - \int_{\Omega^m} \partial_t u(0) \phi(0) \, d\lambda \\
& + \nu \varepsilon \int_{\Omega^m} \nabla u(T) \nabla \phi(T) \, d\lambda - \nu \varepsilon \int_{\Omega^m} \nabla u(0) \nabla \phi(0) \, d\lambda \\
& - \alpha \varepsilon \int_{\Omega^m} (u \partial_t u)(T) \phi(T) \, d\lambda + \alpha \varepsilon \int_{\Omega^m} (u \partial_t u)(0) \phi(0) \, d\lambda \\
& + \nu \varepsilon \int_{\partial\Omega^m} a_m \operatorname{Tr}_{\partial\Omega^m} u(T) \operatorname{Tr}_{\partial\Omega^m} \phi(T) \, ds - \nu \varepsilon \int_{\partial\Omega^m} a_m \operatorname{Tr}_{\partial\Omega^m} u(0) \operatorname{Tr}_{\partial\Omega^m} \phi(0) \, ds
\end{aligned} \tag{5.29}$$

with $a_m \geq 0$ and $b_m \in \mathbb{R}$, and for $a \geq 0$ and $b \in \mathbb{R}$

$$\begin{aligned}
F[u, \phi] := & \int_0^T \int_{\Omega} -\partial_t u \partial_t \phi + c^2 \nabla u \nabla \phi - \nu \varepsilon \nabla u \nabla \partial_t \phi + \alpha \varepsilon (u \partial_t u) \partial_t \phi \, d\lambda \, dt \\
& + \int_0^T \int_{\partial\Omega} c^2 a \operatorname{Tr}_{\partial\Omega} u \operatorname{Tr}_{\partial\Omega} \phi - \nu \varepsilon a \operatorname{Tr}_{\partial\Omega} u \operatorname{Tr}_{\partial\Omega} \partial_t \phi - c^2 b \operatorname{Tr}_{\partial\Omega} \phi \, d\mu \, dt \\
& + \int_{\Omega} \partial_t u(T) \phi(T) \, d\lambda - \int_{\Omega} \partial_t u(0) \phi(0) \, d\lambda \\
& + \nu \varepsilon \int_{\Omega} \nabla u(T) \nabla \phi(T) \, d\lambda - \nu \varepsilon \int_{\Omega} \nabla u(0) \nabla \phi(0) \, d\lambda \\
& - \alpha \varepsilon \int_{\Omega} (u \partial_t u)(T) \phi(T) \, d\lambda + \alpha \varepsilon \int_{\Omega} (u \partial_t u)(0) \phi(0) \, d\lambda \\
& + \nu \varepsilon \int_{\partial\Omega} a \operatorname{Tr}_{\partial\Omega} u(T) \operatorname{Tr}_{\partial\Omega} \phi(T) \, d\mu - \nu \varepsilon \int_{\partial\Omega} a \operatorname{Tr}_{\partial\Omega} u(0) \operatorname{Tr}_{\partial\Omega} \phi(0) \, d\mu
\end{aligned} \tag{5.30}$$

where $\operatorname{Tr}_{\partial\Omega^m}$ and $\operatorname{Tr}_{\partial\Omega}$ denotes the trace of u on the boundaries of Ω^m and Ω respectively.

Definition 5.4.2. *We shall say that u is a weak solution of the Westervelt problem (5.27) on $[0, T] \times \Omega$ if*

- $u \in H(\Omega)$ defined in (5.28).
- For every $\phi \in C^1([0, T] \times \mathcal{D}(\overline{\Omega}))$

$$F[u, \phi] = 0,$$

where F defined in (5.30).

- $u(0) = u_0$ and $u_t(0) = u_1$ on Ω .

The expression $F[u, \phi] = 0$ can be obtained multiplying the Westervelt equation from system (5.27) by $\phi \in C^1([0, T] \times \mathcal{D}(\overline{\Omega}))$ integrating on $[0, T] \times \Omega$ and doing integration by parts. In the same way for $F^m[u, \phi]$, defined in equation (5.29),

$$\forall \phi \in C^1([0, T] \times \mathcal{D}(\overline{\Omega})) \quad F^m[u, \phi] = 0,$$

is the weak solution of the following problem

$$\begin{cases} \partial_t^2 u - c^2 \Delta u - \nu \varepsilon \Delta \partial_t u = \alpha \varepsilon \partial_t [u \partial_t u] & \text{on }]0, T] \times \Omega^m, \\ \partial_n u + a_m u = b_m & \text{on }]0, T] \times \partial \Omega^m, \\ u(0) = u_0|_{\Omega^m}, \quad u_t(0) = u_1|_{\Omega^m} & \text{on } \Omega^m \end{cases} \quad (5.31)$$

with the compatibility condition

$$b_m = (\partial_n u_0 + a_m u_0)|_{\partial \Omega^m} \quad 0 = (\partial_n u_1 + a_m u_1)|_{\partial \Omega^m}.$$

In order to state our main result, we also need to recall the notion of M -convergence of functionals introduced in [72].

Definition 5.4.3. A sequence of functionals $G^m : H \rightarrow (-\infty, +\infty]$ is said to M -converge to a functional $G : H \rightarrow (-\infty, +\infty]$ in a Hilbert space H , if

1. (*lim sup condition*) For every $u \in H$ there exists u_m converging strongly in H such that

$$\overline{\lim} G^m[u_m] \leq G[u], \quad \text{as } m \rightarrow +\infty. \quad (5.32)$$

2. (*lim inf condition*) For every v_m converging weakly to u in H

$$\underline{\lim} G^m[v_m] \geq G[u], \quad \text{as } m \rightarrow +\infty. \quad (5.33)$$

Our main result is the following theorem.

Theorem 5.4.7. For $\phi \in C^1([0, T]; \mathcal{D}(\Omega^*))$, $a_m = a\sigma_m$, $b_m = b\sigma_m$, with σ_m defined by equation (5.24), the sequence of functionals $u \mapsto F^m[u, \phi]$ defined in (5.29), M -converges in $H(\Omega^*)$ defined in (5.28) to the following functional $u \mapsto F[u, \phi]$ defined in (5.30) as $m \rightarrow +\infty$. More precisely in our case if v_m converges weakly to u in $H(\Omega^*)$

$$\forall \phi \in C^1([0, T]; \mathcal{D}(\Omega^*)) \quad F^m[v_m, \phi] \rightarrow F[u, \phi] \quad \text{as } m \rightarrow +\infty.$$

Proof. We consider $\phi \in C^1([0, T]; \mathcal{D}(\Omega^*))$, $a_m = a\sigma_m$, $b_m = b\sigma_m$, with σ_m defined by equation (5.24).

Proof of "lim sup" condition.

We take a given function $u \in H(\Omega^*)$ defined by (5.28) so that $F[u, \phi]$ is well defined and we are going to consider $F^m[u, \phi]$. As $\Omega^m \rightarrow \Omega$ in the sense that

$$\lambda(\Omega \setminus \Omega^m \cup \Omega^m \setminus \Omega) \rightarrow 0$$

and $u \in H(\Omega^*)$ defined in equation (5.28) the dominated convergence theorem gives directly for $m \rightarrow +\infty$

$$\begin{aligned} & \int_0^T \int_{\Omega^m} -\partial_t u \partial_t \phi + c^2 \nabla u \nabla \phi - \nu \varepsilon \nabla u \nabla \partial_t \phi + \alpha \varepsilon (u \partial_t u) \partial_t \phi \, d\lambda \, dt \\ \rightarrow & \int_0^T \int_{\Omega} -\partial_t u \partial_t \phi + c^2 \nabla u \nabla \phi - \nu \varepsilon \nabla u \nabla \partial_t \phi + \alpha \varepsilon (u \partial_t u) \partial_t \phi \, d\lambda \, dt, \end{aligned} \quad (5.34)$$

$$\begin{aligned} & \int_{\Omega^m} [\partial_t u(T)\phi(T) - \partial_t u(0)\phi(0) + \nu\varepsilon(\nabla u(T)\nabla\phi(T) - \nabla u(0)\nabla\phi(0))] d\lambda \\ & \rightarrow \int_{\Omega} [\partial_t u(T)\phi(T) - \partial_t u(0)\phi(0) + \nu\varepsilon(\nabla u(T)\nabla\phi(T) - \nabla u(0)\nabla\phi(0))] d\lambda \end{aligned} \quad (5.35)$$

and

$$\begin{aligned} & \int_{\Omega^m} -\alpha\varepsilon[(u\partial_t u)(T)\phi(T) - (u\partial_t u)(0)\phi(0)]d\lambda \\ & \rightarrow \int_{\Omega} -\alpha\varepsilon[(u\partial_t u)(T)\phi(T) - (u\partial_t u)(0)\phi(0)]d\lambda. \end{aligned} \quad (5.36)$$

Moreover by Theorem 5.4.5, as $\phi \in C^1([0, T]; \mathcal{D}(\Omega^*))$, for $m \rightarrow +\infty$

$$\int_{\partial\Omega^m} -c^2 b_m Tr_{\partial\Omega^m} \phi ds \rightarrow \int_{\partial\Omega} -c^2 b Tr_{\partial\Omega} \phi d\mu.$$

But

$$\left| \int_{\partial\Omega^m} -c^2 b_m Tr_{\partial\Omega^m} \phi ds \right| \leq C \|\phi\|_{H^1(\Omega)}$$

uniformly in n on $[0, T]$ by Theorems 5.4.1 and 5.4.4. So by the dominated convergence theorem, as the time interval $[0, T]$ is compact, for $m \rightarrow +\infty$

$$\int_0^T \int_{\partial\Omega^m} -c^2 b_m Tr_{\partial\Omega^m} \phi ds dt \rightarrow \int_0^T \int_{\partial\Omega} -c^2 b Tr_{\partial\Omega} \phi d\mu dt. \quad (5.37)$$

As u belongs to $H(\Omega^*)$, there exists a sequence $u_m \in C([0, T] \times \overline{\Omega^*}) \cap H(\Omega^*)$ such that $u_m \rightarrow u$ in $H(\Omega^*)$. Then in the same way as in the proof of Theorem 5.4.5 (see [20])

$$\begin{aligned} & \left| \sigma_m \int_{\partial\Omega^m} Tr_{\partial\Omega^m} u Tr_{\partial\Omega^m} \phi ds - \int_{\partial\Omega} Tr_{\partial\Omega} u Tr_{\partial\Omega} \phi d\mu \right| \\ & \leq \left| \sigma_m \int_{\partial\Omega^m} Tr_{\partial\Omega^m} u Tr_{\partial\Omega^m} \phi ds - \sigma_m \int_{\partial\Omega^m} Tr_{\partial\Omega^m} u_m Tr_{\partial\Omega^m} \phi ds \right| \\ & \quad + \left| \sigma_m \int_{\partial\Omega^m} Tr_{\partial\Omega^m} u_m Tr_{\partial\Omega^m} \phi ds - \int_{\partial\Omega} Tr_{\partial\Omega} u_m Tr_{\partial\Omega} \phi d\mu \right| \\ & \quad + \left| \int_{\partial\Omega} Tr_{\partial\Omega} u_m Tr_{\partial\Omega} \phi d\mu - \int_{\partial\Omega} Tr_{\partial\Omega} u Tr_{\partial\Omega} \phi d\mu \right|. \end{aligned} \quad (5.38)$$

The first term on the right-hand side in (5.38) can be estimated by using Theorems 5.4.1 and 5.4.4

$$\left| \sigma_m \int_{\partial\Omega^m} Tr_{\partial\Omega^m} u Tr_{\partial\Omega^m} \phi ds - \sigma_m \int_{\partial\Omega^m} Tr_{\partial\Omega^m} u_m Tr_{\partial\Omega^m} \phi ds \right| \leq C^* \|u - u_m\|_{H^1(\Omega^*)} \|\phi\|_{L^\infty}, \quad (5.39)$$

with $C^* > 0$ independent of n , which implies by definition of u_m , for all $\delta > 0$ if m big enough for all $n \in \mathbb{N}$ as $\|u - u_m\|_{H(\Omega^*)} \rightarrow 0$ when $m \rightarrow +\infty$

$$\int_0^T \left| \sigma_m \int_{\partial\Omega^m} Tr_{\partial\Omega^m} u Tr_{\partial\Omega^m} \phi ds - \sigma_m \int_{\partial\Omega^m} Tr_{\partial\Omega^m} u_m Tr_{\partial\Omega^m} \phi ds \right| dt \leq \frac{\delta}{2}. \quad (5.40)$$

The third term on the right-hand side in (5.38) can be estimated by using Theorems 5.4.2 and 5.4.4

$$\left| \int_{\partial\Omega} Tr_{\partial\Omega} u_m Tr_{\partial\Omega} \phi d\mu - \int_{\partial\Omega} Tr_{\partial\Omega} u Tr_{\partial\Omega} \phi d\mu \right| \leq C^{**} \|u - u_m\|_{H^1(\Omega^*)} \|\phi\|_{L^\infty}. \quad (5.41)$$

with a constant $C^{**} > 0$ independent of m . This implies by definition of u_m that, for all $\delta > 0$ if m big enough,

$$\int_0^T \left| \int_{\partial\Omega} Tr_{\partial\Omega} u_m Tr_{\partial\Omega} \phi \, d\mu - \int_{\partial\Omega} Tr_{\partial\Omega} u Tr_{\partial\Omega} \phi \, d\mu \right| dt \leq \frac{\delta}{2}. \quad (5.42)$$

Finally, for the second term on the right-hand side in (5.38) we use Theorem 5.4.6: so we obtain for $m \rightarrow +\infty$

$$\sigma_m \int_{\partial\Omega^m} Tr_{\partial\Omega^m} u_m Tr_{\partial\Omega^m} \phi \, ds \rightarrow \int_{\partial\Omega} Tr_{\partial\Omega} u_m Tr_{\partial\Omega} \phi \, d\mu. \quad (5.43)$$

Then, by using (5.38), (5.39), (5.41) and (5.43), we have

$$\begin{aligned} & \int_{\partial\Omega^m} \sigma_m Tr_{\partial\Omega^m} u(T) Tr_{\partial\Omega^m} \phi(T) - \sigma_m Tr_{\partial\Omega^m} u(0) Tr_{\partial\Omega^m} \phi(0) \, ds \\ & \rightarrow \int_{\partial\Omega} Tr_{\partial\Omega} u(T) Tr_{\partial\Omega} \phi(T) - Tr_{\partial\Omega} u(0) Tr_{\partial\Omega} \phi(0) \, ds. \end{aligned} \quad (5.44)$$

Moreover, in the same way as for (5.39) and (5.41) we have by Theorems 5.4.1, 5.4.2 and 5.4.4

$$\left| \sigma_m \int_{\partial\Omega^m} Tr_{\partial\Omega^m} u_m Tr_{\partial\Omega^m} \phi \, ds - \int_{\partial\Omega} Tr_{\partial\Omega} u_m Tr_{\partial\Omega} \phi \, d\mu \right| \leq C \|u_m\|_{H^1(\Omega^*)}$$

independently of n , and with (5.43) we obtain by the dominated convergence theorem for $m \rightarrow +\infty$ that

$$\int_0^T \left| \sigma_m \int_{\partial\Omega^m} Tr_{\partial\Omega^m} u_m Tr_{\partial\Omega^m} \phi \, ds - \int_{\partial\Omega} Tr_{\partial\Omega} u_m Tr_{\partial\Omega} \phi \, d\mu \right| dt \rightarrow 0. \quad (5.45)$$

So, by the convergence results (5.38), (5.40), (5.42), (5.45) for $m \rightarrow +\infty$

$$\int_0^T \sigma_m \int_{\partial\Omega^m} Tr_{\partial\Omega^m} u Tr_{\partial\Omega^m} \phi \, ds \, dt \rightarrow \int_0^T \int_{\partial\Omega} Tr_{\partial\Omega} u Tr_{\partial\Omega} \phi \, d\mu \, dt. \quad (5.46)$$

In the same way we have for $m \rightarrow +\infty$

$$\int_0^T \sigma_m \int_{\partial\Omega^m} Tr_{\partial\Omega^m} u Tr_{\partial\Omega^m} \partial_t \phi \, ds \, dt \rightarrow \int_0^T \int_{\partial\Omega} Tr_{\partial\Omega} u Tr_{\partial\Omega} \partial_t \phi \, d\mu \, dt. \quad (5.47)$$

By making together (5.34), (5.35), (5.36) (5.37), (5.44), (5.46) and (5.47), we conclude that for $m \rightarrow +\infty$ for all $\phi \in C^1([0, T]; \mathcal{D}(\Omega^*))$

$$F^m[u, \phi] \rightarrow F[u, \phi].$$

This proves the "lim sup" condition.

Proof of the "lim inf" condition. Now, let v_m be a sequence such that

$$v_m \rightharpoonup u \quad \text{in } H(\Omega^*),$$

with $H(\Omega^*)$ defined by (5.28). Then we have

$$\partial_t v_m \rightharpoonup \partial_t u \quad \text{in } L^2([0, T]; L^2(\Omega^*)), \quad (5.48)$$

and

$$v_m \rightharpoonup u, \nabla v_m \rightharpoonup \nabla u \quad \text{in } L^2([0, T]; L^2(\Omega^*)). \quad (5.49)$$

Moreover, as $L^2([0, T]; H^1(\Omega^*)) \subset\subset L^2([0, T]; L^2(\Omega^*))$,

$$v_m \rightarrow u, \partial_t v_m \rightarrow \partial_t u \quad \text{in } L^2([0, T]; L^2(\Omega^*)). \quad (5.50)$$

We also have as $H^1([0, T]; H^1(\Omega^*)) \subset\subset C([0, T]; H^1(\Omega^*))$ that

$$v_m(0) \rightarrow u(0), v_m(T) \rightarrow u(T) \quad \text{in } H^1(\Omega^*) \quad (5.51)$$

and

$$\partial_t v_m(0) \rightarrow \partial_t u(0), \partial_t v_m(T) \rightarrow \partial_t u(T) \quad \text{in } L^2(\Omega^*). \quad (5.52)$$

Let $\phi \in C^1([0, T]; \mathcal{D}(\Omega^*))$, we will show that our goal is to prove that

$$F^m[v_m, \phi] \xrightarrow{m \rightarrow +\infty} F[u, \phi].$$

First we study the convergence of the terms with $\int_0^T \int_{\Omega^m}$

$$\begin{aligned} \left| \int_0^T \int_{\Omega^m} \partial_t v_m \partial_t \phi \, d\lambda \, ds - \int_0^T \int_{\Omega} \partial_t u \partial_t \phi \, d\lambda \, ds \right| &\leq \left| \int_0^T \int_{\Omega^*} \partial_t v_m \mathbb{1}_{\Omega^m} \partial_t \phi \, d\lambda \, ds \right. \\ &\quad \left. - \int_0^T \int_{\Omega^*} \partial_t v_m \mathbb{1}_{\Omega} \partial_t \phi \, d\lambda \, ds \right| \\ &\quad + \left| \int_0^T \int_{\Omega^*} \partial_t v_m \mathbb{1}_{\Omega} \partial_t \phi \, d\lambda \, ds \right. \\ &\quad \left. - \int_0^T \int_{\Omega^*} \partial_t u \mathbb{1}_{\Omega} \partial_t \phi \, d\lambda \, ds \right|. \end{aligned}$$

By $\mathbb{1}_{\Omega}$ is denoted the characteristic function of the set Ω .

The second term on the right hand side tends to zero as $m \rightarrow +\infty$ by (5.48) as

$$\mathbb{1}_{\Omega} \partial_t \phi \in L^2([0, T]; L^2(\Omega^*)) = L^2([0, T]; L^2(\Omega^*))'.$$

For the first term

$$\left| \int_0^T \int_{\Omega^*} \partial_t v_m (\mathbb{1}_{\Omega^m} - \mathbb{1}_{\Omega}) \partial_t \phi \, d\lambda \, ds \right| \leq \|(\mathbb{1}_{\Omega^m} - \mathbb{1}_{\Omega})\phi\|_{L^2([0, T] \times \Omega^*)} \|\partial_t v_m\|_{L^2([0, T] \times \Omega^*)},$$

but $\|\partial_t v_m\|_{L^2([0, T] \times \Omega^*)}$ is bounded by (5.48) and by the dominated convergence theorem

$$\|(\mathbb{1}_{\Omega^m} - \mathbb{1}_{\Omega})\phi\|_{L^2([0, T] \times \Omega^*)} \xrightarrow{m \rightarrow +\infty} 0.$$

Then when $m \rightarrow +\infty$

$$\int_0^T \int_{\Omega^m} \partial_t v_m \partial_t \phi \, d\lambda \, ds \rightarrow \int_0^T \int_{\Omega} \partial_t u \partial_t \phi \, d\lambda \, ds.$$

Using (5.49) we can deduce in the same way

$$\begin{aligned} &\int_0^T \int_{\Omega^m} -\partial_t v_m \partial_t \phi + c^2 \nabla v_m \nabla \phi - \nu \varepsilon \nabla v_m \nabla \partial_t \phi \, d\lambda \, dt \\ &\xrightarrow{m \rightarrow +\infty} \int_0^T \int_{\Omega} -\partial_t u \partial_t \phi + c^2 \nabla u \nabla \phi - \nu \varepsilon \nabla u \nabla \partial_t \phi \, d\lambda \, dt. \end{aligned} \quad (5.53)$$

For the quadratic term we have

$$\begin{aligned} \left| \int_0^T \int_{\Omega^m} (v_m \partial_t v_m) \partial_t \phi d\lambda dt - \int_0^T \int_{\Omega} (u \partial_t u) \partial_t \phi d\lambda dt \right| &\leq \left| \int_0^T \int_{\Omega^m} (v_m \partial_t v_m) \partial_t \phi d\lambda dt \right. \\ &\quad \left. - \int_0^T \int_{\Omega^m} (u \partial_t u) \partial_t \phi d\lambda dt \right| \\ &\quad + \left| \int_0^T \int_{\Omega^m} (u \partial_t u) \partial_t \phi d\lambda dt \right. \\ &\quad \left. - \int_0^T \int_{\Omega} (u \partial_t u) \partial_t \phi d\lambda dt \right|. \end{aligned}$$

As by the prefractal construction of the sequence $(\Omega^m)_{m \in \mathbb{N}^*} \mathbb{1}_{\Omega^m} \rightarrow \mathbb{1}_{\Omega}$ and $(u \partial_t u) \partial_t \phi \in L^1([0, T] \times \Omega^*)$, the dominated convergence theorem implies that the second term in the right hand side tends to 0 when $m \rightarrow +\infty$. For the first term on the right hand side we use the fact that

$$(v_m \partial_t v_m) - (u \partial_t u) = v_m (\partial_t v_m - \partial_t u) + (v_m - u) \partial_t u,$$

so we obtain by the Cauchy-Schwartz inequality as $\partial_t \phi$ is bounded on $[0, T] \times \Omega^*$

$$\begin{aligned} \left| \int_0^T \int_{\Omega^m} [(v_m \partial_t v_m) - (u \partial_t u)] \partial_t \phi d\lambda dt \right| &\leq \int_0^T \int_{\Omega^m} |v_m (\partial_t v_m - \partial_t u) \partial_t \phi| d\lambda dt \\ &\quad + \int_0^T \int_{\Omega^m} |(v_m - u) \partial_t u \partial_t \phi| d\lambda dt \\ &\leq C \|\partial_t v_m - \partial_t u\|_{L^2([0, T] \times \Omega^*)} \|v_m\|_{L^2([0, T] \times \Omega^*)} \\ &\quad + C \|v_m - u\|_{L^2([0, T] \times \Omega^*)} \|u\|_{L^2([0, T] \times \Omega^*)}. \end{aligned}$$

But by (5.50) the right hand side term tends to zero when n goes to infinity. Then we deduce

$$\int_0^T \int_{\Omega^m} \alpha \varepsilon (v_m \partial_t v_m) \partial_t \phi d\lambda dt \rightarrow_{m \rightarrow +\infty} \int_0^T \int_{\Omega} \alpha \varepsilon (u \partial_t u) \partial_t \phi d\lambda dt. \quad (5.54)$$

Now we consider the boundary terms in $F^m[v_m, \phi]$, we see that we always have (5.37). Moreover for n and $m \in \mathbb{N}$

$$\begin{aligned} &\left| \int_0^T \int_{\partial \Omega^m} \sigma_m \text{Tr}_{\partial \Omega^m} v_m \text{Tr}_{\partial \Omega^m} \phi ds dt - \int_0^T \int_{\partial \Omega} \text{Tr}_{\partial \Omega} u \text{Tr}_{\partial \Omega} \phi d\mu dt \right| \\ &\leq \left| \int_0^T \int_{\partial \Omega^m} \sigma_m \text{Tr}_{\partial \Omega^m} v_m \text{Tr}_{\partial \Omega^m} \phi ds dt - \int_0^T \int_{\partial \Omega^m} \sigma_m \text{Tr}_{\partial \Omega^m} u \text{Tr}_{\partial \Omega^m} \phi ds dt \right| \\ &\quad + \left| \int_0^T \int_{\partial \Omega^m} \sigma_m \text{Tr}_{\partial \Omega^m} u \text{Tr}_{\partial \Omega^m} \phi ds dt - \int_0^T \int_{\partial \Omega} \text{Tr}_{\partial \Omega} u \text{Tr}_{\partial \Omega} \phi d\mu dt \right|. \quad (5.55) \end{aligned}$$

The second term on the right hand side in (5.55) tends to zero as $m \rightarrow +\infty$ by (5.46). For the first term on the right hand side in (5.55) we have

$$\begin{aligned} &\left| \int_0^T \int_{\partial \Omega^m} \sigma_m \text{Tr}_{\partial \Omega^m} v_m \text{Tr}_{\partial \Omega^m} \phi ds dt - \int_0^T \int_{\partial \Omega^m} \sigma_m \text{Tr}_{\partial \Omega^m} u \text{Tr}_{\partial \Omega^m} \phi ds dt \right| \\ &\leq \sigma_m \|\text{Tr}_{\partial \Omega^m} v_m - \text{Tr}_{\partial \Omega^m} u\|_{L^2([0, T]; L^2(\partial \Omega^m))} \|\text{Tr}_{\partial \Omega^m} \phi\|_{L^2([0, T]; L^2(\partial \Omega^m))}. \quad (5.56) \end{aligned}$$

From Theorems 5.4.1 and 5.4.4 for $\frac{1}{2} < p < 1$ denoting by E_{Ω^*} the extension of $H^p(\Omega^*)$ on $H^p(\mathbb{R}^2)$ we have

$$\begin{aligned} \sigma_m \|Tr_{\partial\Omega^m} v_m - Tr_{\partial\Omega^m} u\|_{L^2([0,T];L^2(\partial\Omega^m))}^2 &\leq C \|E_{\Omega^*}(v_m - u)\|_{L^2([0,T];H^p(\mathbb{R}^2))} \\ &\leq C \|(v_m - u)\|_{L^2([0,T];H^p(\Omega^*))}, \end{aligned} \quad (5.57)$$

and in the same way

$$\sigma_m \|Tr_{\partial\Omega^m} \phi\|_{L^2([0,T];L^2(\partial\Omega^m))}^2 \leq C \|\phi\|_{L^2([0,T];H^1(\Omega^*))} \quad (5.58)$$

with $C > 0$ independent of n . As

$$v_m \rightharpoonup u \text{ in } L^2([0, T]; H^1(\Omega^*))$$

and

$$H^1(\Omega^*) \subset\subset H^q(\Omega^*) \text{ for } \frac{1}{2} < q < 1,$$

we have

$$v_m \rightarrow u \text{ strongly in } L^2([0, T]; H^q(\Omega^*)). \quad (5.59)$$

By (5.56), (5.57), (5.58), (5.59)

$$\left| \int_0^T \int_{\partial\Omega^m} \sigma_m Tr_{\partial\Omega^m} v_m Tr_{\partial\Omega^m} \phi \, ds \, dt - \int_0^T \int_{\partial\Omega^m} \sigma_m Tr_{\partial\Omega^m} u Tr_{\partial\Omega^m} \phi \, ds \, dt \right| \xrightarrow{m \rightarrow +\infty} 0.$$

Then the analysis of (5.55) gives us when $m \rightarrow +\infty$

$$\int_0^T \int_{\partial\Omega^m} \sigma_m Tr_{\partial\Omega^m} v_m Tr_{\partial\Omega^m} \phi \, ds \, dt \rightarrow \int_0^T \int_{\partial\Omega} Tr_{\partial\Omega} u Tr_{\partial\Omega} \phi \, d\mu \, dt. \quad (5.60)$$

In the same way that for (5.60) we have when $m \rightarrow +\infty$

$$\int_0^T \int_{\partial\Omega^m} \sigma_m Tr_{\partial\Omega^m} v_m Tr_{\partial\Omega^m} \partial_t \phi \, ds \, dt \rightarrow \int_0^T \int_{\partial\Omega} Tr_{\partial\Omega} u Tr_{\partial\Omega} \partial_t \phi \, d\mu \, dt. \quad (5.61)$$

The same type of arguments with (5.51) permits to say when $m \rightarrow +\infty$

$$\begin{aligned} \nu \varepsilon \int_{\partial\Omega^m} a_m Tr_{\partial\Omega^m} v_m(T) Tr_{\partial\Omega^m} \phi(T) \, ds - \nu \varepsilon \int_{\partial\Omega^m} a_m Tr_{\partial\Omega^m} v_m(0) Tr_{\partial\Omega^m} \phi(0) \, ds \\ \rightarrow \nu \varepsilon \int_{\partial\Omega} a Tr_{\partial\Omega} u(T) Tr_{\partial\Omega} \phi(T) \, d\mu - \nu \varepsilon \int_{\partial\Omega} a Tr_{\partial\Omega} u(0) Tr_{\partial\Omega} \phi(0) \, d\mu. \end{aligned} \quad (5.62)$$

The key point is that as $H^1(\Omega^*) \subset\subset H^q(\Omega^*)$ for $\frac{1}{2} < q < 1$ the weak convergence in $H^1(\Omega^*)$ implies the strong convergence in $H^q(\Omega^*)$.

For the time boundary terms we have

$$\left| \int_{\Omega^m} \partial_t v_m(T) \phi(T) \, d\lambda - \int_{\Omega} \partial_t u(T) \phi(T) \, d\lambda \right| \xrightarrow{m \rightarrow +\infty} 0$$

as when $m \rightarrow +\infty$ $\partial_t v_m(T) \rightharpoonup \partial_t u(T)$ in $L^2(\Omega^*)$ and $\mathbb{1}_{\Omega^m} \phi(T) \rightarrow \mathbb{1}_{\Omega} \phi(T)$ in $L^2(\Omega^*)$. In the same way when $m \rightarrow +\infty$ we have

$$\begin{aligned} \int_{\Omega^m} \partial_t v_m(T) \phi(T) - \partial_t v_m(0) \phi(0) \, d + \nu \varepsilon \nabla v_m(T) \nabla \phi(T) \, d\lambda - \nu \varepsilon \nabla v_m(0) \nabla \phi(0) \, d\lambda \\ \rightarrow \int_{\Omega} \partial_t u(T) \phi(T) - \partial_t u(0) \phi(0) + \nu \varepsilon \nabla u(T) \nabla \phi(T) - \nu \varepsilon \nabla u(0) \nabla \phi(0) \, d\lambda. \end{aligned} \quad (5.63)$$

For $l \in \{0, T\}$ we use the fact that when $m \rightarrow +\infty$ $\partial_t v_m(l) \rightarrow \partial_t u(l)$ in $L^2(\Omega^*)$ and $\mathbb{1}_{\Omega^m} v_m(l) \phi(l) \rightarrow \mathbb{1}_{\Omega} u(l) \phi(l)$ in $L^2(\Omega^*)$ by (5.51) and by (5.52) to obtain when $m \rightarrow +\infty$

$$\begin{aligned} & \int_{\Omega^m} -\alpha\varepsilon(v_m \partial_t v_m)(T) \phi(T) + \alpha\varepsilon(v_m \partial_t v_m)(0) \phi(0) \, d\lambda \\ & \rightarrow \int_{\Omega} -\alpha\varepsilon(u \partial_t u)(T) \phi(T) + \alpha\varepsilon(u \partial_t u)(0) \phi(0) \, d\lambda. \end{aligned} \quad (5.64)$$

So by (5.53), (5.54), (5.60), (5.61), (5.62), (5.63), (5.64) and (5.37) we have

$$\forall \phi \in C^1([0, T]; \mathcal{D}(\Omega^*)) \quad F^m[v_m, \phi] \rightarrow F[u, \phi],$$

as $m \rightarrow +\infty$ and this concludes the proof. \square

5.4.4 Linear and Nonlinear problems with mixed boundary condition

Let Ω_0 be as previously in Subsection 5.4.2 the square $\{(x, y) : 0 < x < 1, -1 < y < 0\}$ with vertices $A = (0, 0)$, $B = (1, 0)$, $C = (1, -1)$, and $D = (0, -1)$. On the side between A and B we construct either a scale irregular Koch curve or the corresponding approximating prefractal curve. This time only others points are related by straight lines. More precisely, we consider the set $\Omega^{(\xi)}$ bounded by a scale irregular Koch curves $K^{(\xi)}$, with endpoints A and B , and segments $\Gamma_1 = [B, C]$, $\Gamma_0 = [C, D]$ and $\Gamma_3 = [D, A]$. Moreover, we consider the set $\Omega^{(\xi).m}$ bounded by an approximating prefractal curves $K^{(\xi).m}$, starting from the segments with endpoints A and B , and segments $\Gamma_1 = [B, C]$, $\Gamma_0 = [C, D]$ and $\Gamma_3 = [D, A]$. From now on, when it does not give rise to misunderstanding, in the notation we suppress the super index (ξ) by writing simply Ω , Ω^m .

We start by considering the Poisson mixed boundary problem with the Robin boundary conditions on a fractal boundary K

$$\left\{ \begin{array}{ll} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ \frac{\partial}{\partial n} u = 0 & \text{on } \Gamma_1, \\ \frac{\partial}{\partial n} u + au = 0 & \text{on } K, \\ \frac{\partial}{\partial n} u = 0 & \text{on } \Gamma_3, \end{array} \right. \quad (5.65)$$

and the analogous prefractal problem

$$\left\{ \begin{array}{ll} -\Delta u = f & \text{in } \Omega^m, \\ u = 0 & \text{on } \Gamma_0, \\ \frac{\partial}{\partial n} u = 0 & \text{on } \Gamma_1, \\ \frac{\partial}{\partial n} u + a_m u = 0 & \text{on } K^m, \\ \frac{\partial}{\partial n} u = 0 & \text{on } \Gamma_3. \end{array} \right. \quad (5.66)$$

According to [19], the interest in mixed Dirichlet-Robin problems for the operator $-\Delta$ arises, for example, from the study of current flowing through an electrochemical cell, where the working electrode K presents an irregular geometry or in the study of reactive molecules toward catalytic surfaces.

We have again the equivalent of Theorems 5.4.1, 5.4.2, 5.4.3, 5.4.4 and 5.4.6 replacing $\partial\Omega^{(\xi),m}$ by K^m and $\partial\Omega^{(\xi)}$ by K . As we are in a particular case of Theorem 4.1 in [19] we can write

Theorem 5.4.8. [19] For all $f \in L^2(\Omega^m)$, for every $a_m \geq 0$ there exists one and only one solution u_m of the following problem

$$\left\{ \begin{array}{l} \text{Find } u_m \in V(\Omega^m) := \{u_m \in H^1(\Omega^m) : u_m = 0 \text{ on } \Gamma_0\} \text{ such that} \\ \forall v \in V(\Omega^m), \quad \int_{\Omega^m} \nabla u_m \nabla v \, dx + a_m \int_{K^m} \text{Tr}_{K^m} u_m \text{Tr}_{K^m} v \, ds = \int_{\Omega^m} f v \, dx. \end{array} \right. \quad (5.67)$$

For any $f \in L^2(\Omega^m)$, for every $n \in \mathbb{N}$, the weak solution u_m of the prefractal problem (5.67) solves the following problem

$$\left\{ \begin{array}{ll} -\Delta u = f & \text{in } L^2(\Omega^m), \\ u = 0 & \text{in } C(\Gamma_0), \\ \frac{\partial}{\partial n} u = 0 & \text{in } L^2(\Gamma_1), \\ \frac{\partial}{\partial n} u + a_m u = 0 & \text{in } L^2(K^m), \\ \frac{\partial}{\partial n} u = 0 & \text{in } L^2(\Gamma_3). \end{array} \right. \quad (5.68)$$

Moreover,

$$\|u_m\|_{H^1(\Omega^m)} \leq C \left(\|f\|_{L^2(\Omega^m)} \right), \quad (5.69)$$

where $C > 0$ is a constant independent of m .

Remark 5.4.2. The key points of the proof are the Lax-Milgram theorem and the Poincaré inequality that we can apply on $V(\Omega^m)$ as we have an homogeneous Dirichlet boundary condition on one side of Ω^m . As the constant in the Poincaré inequality depends only of $\lambda(\Omega^m)$ which is bounded, we have the independence on n in the estimate.

Using these results we can treat the case of the strongly damped wave equation on Ω^m with the same boundary conditions. This time an acoustic wave propagates in a thermo-elastic medium, bounded by boundaries of different physical natures (in a part reflexive (Robin boundary condition), isolated (the Neumann boundary condition) and fixed (Dirichlet boundary condition)).

Definition 5.4.4. On Ω^m , for $a_m > 0$ we take $V(\Omega^m)$ defined in (5.67). For

$$f \in L^2([0, T]; L^2(\Omega^m)), \quad u_0 \in V(\Omega^m) \quad \text{and} \quad u_1 \in L^2(\Omega),$$

we say that a function with the regularities

$$u_m \in L^2([0, T]; V(\Omega^m)), \quad \partial_t u_m \in L^2([0, T]; V(\Omega^m)) \quad \text{and} \quad \partial_t^2 u_m \in L^2([0, T]; H^{-1}(\Omega))$$

is a weak solution of the hyperbolic initial/boundary value problem with $\nu > 0$

$$\left\{ \begin{array}{l} \partial_t^2 u_m - c^2 \Delta u_m - \nu \Delta \partial_t u_m = f \quad \text{on }]0, T] \times \Omega^m, \\ \frac{\partial}{\partial n} u + a_m u = 0 \quad \text{on } [0, T] \times K^m, \\ u = 0 \quad \text{on } [0, T] \times \Gamma_0, \\ \frac{\partial}{\partial n} u = 0 \quad \text{on } [0, T] \times \Gamma_1, \\ \frac{\partial}{\partial n} u = 0 \quad \text{on } [0, T] \times \Gamma_3, \\ u(0) = u_0, \quad u_t(0) = u_1, \end{array} \right. \quad (5.70)$$

if it satisfies for all $v \in V(\Omega^m)$ and a.e. time $0 \leq t \leq T$

$$\begin{aligned} \langle u_{tt}, v \rangle_{(H^{-1}(\Omega^m), V(\Omega^m))} + c^2 \left(\int_{\Omega^m} \nabla u_m \nabla v \, dx + a_m \int_{K^m} \text{Tr}_{K^m} u_m \, \text{Tr}_{K^m} v \, ds \right) \\ + \nu \left(\int_{\Omega^m} \nabla \partial_t u_m \nabla v \, dx + a_m \int_{K^m} \text{Tr}_{K^m} \partial_t u_m \, \text{Tr}_{K^m} v \, ds \right) = (f, v)_{L^2(\Omega^m)} \end{aligned} \quad (5.71)$$

with u_m satisfying the desired initial boundary conditions.

In the same way as in Theorem 4.3.5 we have the following theorem. The main tool is a Galerkin method that we can apply on Ω^m as we have an equivalent of Theorem 4.1.14 hence its proof is omitted.

Theorem 5.4.9. *Let us consider Ω^m , for $a_m > 0$ we take $V(\Omega^m)$ defined in (5.67), then assume*

$$u_0 \in V(\Omega^m), \quad u_1 \in V(\Omega^m), \quad f \in L^2([0, +\infty[; L^2(\Omega^m))$$

with

$$\Delta u_0 \in L^2(\Omega^m)$$

in the weak sense (5.67) of the Poisson problem (5.66), then we have a unique weak solution u_m of the linear problem (5.70) in the sense of the formulation (5.71)

$$\begin{aligned} u_m &\in L^\infty([0, +\infty[; V(\Omega^m)) \cap L^2([0, +\infty[; V(\Omega^m)), \\ \partial_t u_m &\in L^\infty([0, +\infty[; V(\Omega^m)) \cap L^2([0, +\infty[; V(\Omega^m)), \\ \partial_t^2 u_m &\in L^2([0, +\infty[; L^2(\Omega^m)), \\ \Delta u_m &\in L^\infty([0, +\infty[; L^2(\Omega^m)) \cap L^2([0, +\infty[; L^2(\Omega^m)), \\ \Delta \partial_t u_m &\in L^2([0, +\infty[; L^2(\Omega^m)) \end{aligned}$$

with the estimates

$$\begin{aligned} \text{ess sup}_{0 \leq t} (\|\Delta u(t)\|_{L^2(\Omega^m)}^2 + \|\nabla \partial_t u(t)\|_{L^2(\Omega^m)}^2) + \int_0^\infty \|\Delta \partial_t u(s)\|_{L^2(\Omega^m)}^2 \, ds \\ \leq C(\|f\|_{L^2([0, +\infty[; L^2(\Omega^m))}^2 + \|\Delta u_0\|_{L^2(\Omega^m)}^2 + \|\nabla u_1\|_{L^2(\Omega^m)}^2) \end{aligned} \quad (5.72)$$

and

$$\int_0^{+\infty} \|\Delta u(s)\|_{L^2(\Omega^m)}^2 \, ds \leq C(\|f\|_{L^2([0, +\infty[; L^2(\Omega^m))}^2 + \|\Delta u_0\|_{L^2(\Omega^m)}^2 + \|\nabla u_1\|_{L^2(\Omega^m)}^2), \quad (5.73)$$

where in the estimates (5.72) and (5.73) $C > 0$ is a constant independent of m .

Now we give a result on the well-posedness in $C(\overline{\Omega^m})$ of the Poisson problem (5.68) in a prefractal domain.

Theorem 5.4.10. *For all $f \in L^2(\Omega^m)$, for every $n \in \mathbb{N}$ and $0 \leq a_m \leq C$ with C fixed, the weak solution u_m of the prefractal problem (5.67) solves the Poisson problem (5.68) in the prefractal domain Ω^m with $u_m \in C(\overline{\Omega^m})$ and we have the estimate*

$$\|u_m\|_{L^\infty(\Omega^m)} \leq C\|f\|_{L^2(\Omega^m)}, \tag{5.74}$$

where $C > 0$ is a constant independent of m .

Proof. The proof is essentially based on the method used in [75] on Lipschitz domains with only Neumann or Robin boundary conditions. First we consider the case $a_m = 0$. For $n \in \mathbb{N}$, Ω^m is a Lipschitz domain in \mathbb{R}^2 . By definition for every $z \in \partial\Omega^m$ we can choose an orthogonal matrix \mathcal{O} , a radius $r > 0$, a Lipschitz continuous function $\psi : \mathbb{R} \rightarrow \mathbb{R}$, and

$$G = \{(y, \psi(y) + s) : y \in]-r, r[, s \in]-r, r[\},$$

such that

$$\mathcal{O}(\Omega^m - z) \cap \{(y, \psi(y) + s) : y \in]-r, r[, s \in]0, r[\}.$$

For local consideration we assume $\mathcal{O} = Id$ and $z = 0$. We define $T(y, s) := (y, \psi(y) + s)$ and the reflection $S : G \rightarrow G$ at the boundary $\partial\Omega$ by $S(T(y, s)) = T(y, -s)$.

We write $U = G \cap \Omega^m$ and $V = S(U) = G \setminus \overline{\Omega^m}$. For a function w on U , we define \tilde{w} (almost everywhere) on G by

$$\tilde{w}(x) := \begin{cases} w(x), & x \in U, \\ w(S(x)), & x \in V. \end{cases}$$

Considering $u_m|_U$ we show as in [75] that \tilde{u}_m solves a strictly elliptic type problem on G . Using the properties of interior regularity as in [33] with Theorem 8.24 we have for $G_0 \subset\subset G$ an estimate of the kind

$$\|\tilde{u}_m\|_{L^\infty(G_0)} \leq C\|f\|_{L^2(\Omega^m)}$$

with $C > 0$ depending only on $\lambda(G)$. Thus u_m is in $C(\Omega^m \cap G_0)$ and satisfies an estimate

$$\|u_m\|_{L^\infty(\Omega^m \cap G_0)} \leq C\|f\|_{L^2(\Omega^m)}$$

with $C > 0$ depending only on $\lambda(G)$.

Since $\partial\Omega$ is compact, we can cover $\partial\Omega$ by finitely many such sets. Thus u is continuous in an interior neighborhood of $\partial\Omega$ and its L^∞ norm can be controlled as in (5.74). Finally, we use the result about interior regularity once again to control u_m in the interior of Ω^m . As a result we have $u_m \in C(\overline{\Omega^m})$ with an estimate similar to (5.74), where the constant $C > 0$ depends only on $\lambda(\Omega^m)$ and as it is bounded it can be chosen independent on n .

The key point in the fact that the constants in the estimates depend only on the surface is the Poincaré inequality that we can apply on Ω^m considered with an homogeneous Dirichlet boundary condition on one side of Ω^m . We have considered the case $a_m = 0$, but the case $K \geq a_m \geq 0$ independently of n can be treated in the same way using [75] what implies the desired result. \square

Let us come back to the well posedness of the non linear Westervelt equation on Ω^m .

Theorem 5.4.11. *We take Ω^m our prefractal domain for all $n \in \mathbb{N}$ with $a_m \geq 0$ bounded. Let $\nu > 0$ and $\mathbb{R}^+ = [0, +\infty[$. Considering the problem for the Westervelt equation*

$$\left\{ \begin{array}{l} \partial_t^2 u - c^2 \Delta u - \nu \varepsilon \Delta \partial_t u = \alpha \varepsilon u \partial_t^2 u + \alpha \varepsilon (\partial_t u)^2 + f \quad \text{in } [0, T] \times \Omega^m, \\ \frac{\partial}{\partial n} u + a_m u = 0 \quad \text{on } [0, T] \times K^m, \\ u = 0 \quad \text{on } [0, T] \times \Gamma_0, \\ \frac{\partial}{\partial n} u = 0 \quad \text{on } [0, T] \times \Gamma_1, \\ \frac{\partial}{\partial n} u = 0 \quad \text{on } [0, T] \times \Gamma_3, \\ u(0) = u_0, \quad \partial_t u(0) = u_1, \end{array} \right. \quad (5.75)$$

where the Laplacian is considered in the weak sense of the Poisson problem (5.68), the following result holds: let the initial data

$$u_0 \in V(\Omega^m) \quad \text{and} \quad u_1 \in V(\Omega^m)$$

be such that

$$\Delta u_0 \in L^2(\Omega^m),$$

with the non homogeneous datum $f \in L^2(\mathbb{R}^+; L^2(\Omega^m))$, let $C_1 = O(1)$ be the minimal constant such that the weak solution u_m^* of the corresponding non homogeneous linear Cauchy problem (5.70) satisfies

$$\begin{aligned} & \|\partial_t^2 u_m^*\|_{L^2(\mathbb{R}^+; L^2(\Omega^m))} + \|\Delta \partial_t u_m^*\|_{L^2(\mathbb{R}^+; L^2(\Omega^m))} + \|\Delta u_m^*\|_{L^2(\mathbb{R}^+; L^2(\Omega^m))} \\ & \leq \frac{C_1}{\nu \varepsilon} (\|f\|_{L^2(\mathbb{R}^+; L^2(\Omega^m))} + \|\Delta u_0\|_{L^2(\Omega^m)} + \|u_1\|_{V(\Omega^m)}). \end{aligned}$$

Then there exists $r_* > 0$ independent of m with $r_* = O(1)$ such that for all $r \in [0, r_*[$ and all data satisfying

$$\|f\|_{L^2(\mathbb{R}^+; L^2(\Omega^m))} + \|\Delta u_0\|_{L^2(\Omega^m)} + \|u_1\|_{V(\Omega^m)} \leq \frac{\nu \varepsilon}{C_1} r,$$

there exists the unique weak solution u_m of problem (5.75) for the Westervelt equation and

$$\|\partial_t^2 u_m\|_{L^2(\mathbb{R}^+; L^2(\Omega^m))} + \|\Delta \partial_t u_m\|_{L^2(\mathbb{R}^+; L^2(\Omega^m))} + \|\Delta u_m\|_{L^2(\mathbb{R}^+; L^2(\Omega^m))} \leq 2r.$$

Proof. The proof is essentially the same as for Theorem 5.1.2 and is thus omitted. The independence on m of r^* comes from the independence on m in the estimates (5.72), (5.73) and (5.74). \square

Now, we return to the problem with a fractal boundary on Ω and recall a direct consequence of Theorem 4.2 in [19]:

Theorem 5.4.12. [19] For any $f \in L^2(\Omega)$, for every $a \geq 0$ there exists one and only one solution u of the following problem

$$\begin{cases} \text{Find } u \in V(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0\} \text{ such that} \\ \forall v \in V(\Omega), \quad \int_{\Omega} \nabla u \nabla v \, dx + a \int_K \text{Tr}_K u \text{Tr}_K v \, ds = \int_{\Omega} f v \, dx. \end{cases} \quad (5.76)$$

For any $f \in L^2(\Omega)$, for every $n \in \mathbb{N}$, the weak solution u of the fractal problem (5.76) solves the Poisson problem (5.65). Moreover,

$$\|u\|_{H^1(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)} \right). \quad (5.77)$$

As a consequence we have an equivalent of Theorem 5.4.9 replacing Ω^m by Ω and a_m by a . Then let us treat the convergence of the solution u_m of the Poisson problem (5.68) on prefractal domain Ω^m to the solution u of the Poisson problem (5.68) on the fractal Ω .

Theorem 5.4.13. Let $f \in L^2(\Omega)$ and $a \geq 0$ with for $n \in \mathbb{N}$ $a_m = a\sigma_m$, where σ_m defined in (5.24) associated to Ω^m . For u_m the solution of the Poisson problem (5.68) on prefractal domain Ω^m and u the solution of the Poisson problem (5.65) on the fractal domain Ω we have (see Theorem 5.3 in [19])

$$(E_{\mathbb{R}^2} u_m)|_{\Omega} \rightarrow u, \quad \text{in } H^1(\Omega)$$

and $u \in C(\overline{\Omega})$ with the following estimate

$$\|u\|_{L^\infty(\Omega)} \leq C \|f\|_{L^2(\Omega)}. \quad (5.78)$$

Proof. The convergence comes from Theorem 5.3 in [19] and the L^∞ -estimate is a consequence of the convergence and of estimate (5.74) on u_m which is independent on m . \square

As a result we have an equivalent of Theorem 5.4.11 replacing Ω^m by Ω and a_m by a as we have an equivalent of Theorem 5.4.9 on Ω and the estimate (5.78). Hence its proof is omitted.

Theorem 5.4.14. Let us take take Ω , domain with the fractal boundary K and $a \geq 0$. Let $\nu > 0$, and $\mathbb{R}^+ = [0, +\infty[$. Considering the problem for the Westervelt equation

$$\begin{cases} \partial_t^2 u - c^2 \Delta u - \nu \varepsilon \Delta \partial_t u = \alpha \varepsilon u \partial_t^2 u + \alpha \varepsilon (\partial_t u)^2 + f & \text{in } [0, T] \times \Omega, \\ \frac{\partial}{\partial n} u + au = 0 & \text{on } [0, T] \times K, \\ u = 0 & \text{on } [0, T] \times \Gamma_0, \\ \frac{\partial}{\partial n} u = 0 & \text{on } [0, T] \times \Gamma_1, \\ \frac{\partial}{\partial n} u = 0 & \text{on } [0, T] \times \Gamma_3, \\ u(0) = u_0, \quad \partial_t u(0) = u_1 \end{cases} \quad (5.79)$$

with the Laplacian considered in the weak sense of the Poisson problem (5.65) the following result holds: let the initial data

$$u_0 \in V(\Omega) \quad \text{and} \quad u_1 \in V(\Omega),$$

such that

$$\Delta u_0 \in L^2(\Omega)$$

with the source term $f \in L^2(\mathbb{R}^+; L^2(\Omega))$, $C_1 = O(1)$ be the minimal constant such that the weak solution u^* of the corresponding non homogeneous linear Cauchy problem satisfies

$$\begin{aligned} & \|\partial_t^2 u^*\|_{L^2(\mathbb{R}^+; L^2(\Omega))} + \|\Delta \partial_t u^*\|_{L^2(\mathbb{R}^+; L^2(\Omega))} + \|\Delta u^*\|_{L^2(\mathbb{R}^+; L^2(\Omega))} \\ & \leq \frac{C_1}{\nu \epsilon} (\|f\|_{L^2(\mathbb{R}^+; L^2(\Omega))} + \|\Delta u_0\|_{L^2(\Omega)} + \|u_1\|_{V(\Omega)}). \end{aligned}$$

Then there exists $r_* > 0$ with $r_* = O(1)$ such that for all $r \in [0, r_*[$ and all data satisfying

$$\|f\|_{L^2(\mathbb{R}^+; L^2(\Omega))} + \|\Delta u_0\|_{L^2(\Omega)} + \|u_1\|_{V(\Omega)} \leq \frac{\nu \epsilon}{C_1} r,$$

there exists the unique weak solution u of problem (5.75) for the Westervelt equation and

$$\|\partial_t^2 u\|_{L^2(\mathbb{R}^+; L^2(\Omega))} + \|\Delta \partial_t u\|_{L^2(\mathbb{R}^+; L^2(\Omega))} + \|\Delta u\|_{L^2(\mathbb{R}^+; L^2(\Omega))} \leq 2r.$$

We consider the problem (5.79) on Ω associated to the Westervelt equation with the compatibility condition

$$\begin{aligned} \left(\frac{\partial}{\partial n} u_0 + a u_0\right)|_K &= 0, & \left(\frac{\partial}{\partial n} u_1 + a u_1\right)|_K &= 0, \\ \left(\frac{\partial}{\partial n} u_0\right)|_{\Gamma_1 \cup \Gamma_3} &= 0, & \left(\frac{\partial}{\partial n} u_1\right)|_{\Gamma_1 \cup \Gamma_3} &= 0, \\ (u_0)|_{\Gamma_0} &= 0, & (u_1)|_{\Gamma_0} &= 0. \end{aligned}$$

Let Ω^* be a bounded possibly regular open set such that $\overline{\Omega}^m \subset \Omega^*$ for all n and $\overline{\Omega} \subset \Omega^*$.

Definition 5.4.5. For an open set U and $T > 0$ we define the Hilbert space

$$\tilde{H}(U) = \{u \in H^1([0, T]; V(U)) \mid \partial_t u(0) \text{ and } \partial_t u(T) \in L^2(U)\}. \quad (5.80)$$

For $\phi \in C^1([0, T] \times \mathcal{D}(\Omega^*))$ and $u \in H(\Omega^*)$ we define $\tilde{F}^m[u, \phi]$ as $F^m[u, \phi]$ in (5.29) replacing $\partial\Omega^m$ by K^m with $a_m \geq 0$ and $b_m = 0$. In the same way, for $a \geq 0$ and $b = 0$ we define $\tilde{F}[u, \phi]$ as $F[u, \phi]$ in (5.30) replacing $\partial\Omega$ by K .

Definition 5.4.6. We shall say that u is a weak solution of problem (5.79) for the Westervelt equation on $[0, T] \times \Omega$ if

- $u \in \tilde{H}(\Omega)$ defined in (5.80) with $u|_{\Gamma_0} = 0$.
- for every $\phi \in C^1([0, T] \times \mathcal{D}(\overline{\Omega}))$ with $\phi|_{\Gamma_0} = 0$

$$\tilde{F}[u, \phi] = 0.$$

- $u(0) = u_0$ and $u_t(0) = u_1$ on Ω .

The expression $\tilde{F}[u, \phi] = 0$ can be obtained multiplying the Westervelt equation from system (5.79) by $\phi \in C^1([0, T] \times \mathcal{D}(\bar{\Omega}))$ with $\phi|_{\Gamma_0} = 0$ integrating on $[0, T] \times \Omega$ and doing integration by parts. In the same way with $\tilde{F}^m[u, \phi]$ we can define the weak solution of the Westervelt problem (5.75). We have an equivalent of Theorem (5.4.7) for the M -convergence of Mosco type concerning $\tilde{F}[u, \phi] = 0$ and $\tilde{F}^m[u, \phi] = 0$. Now we can give an approximation result of the solution of the Westervelt equation on Ω by the solutions of the Westervelt equation on Ω^m .

Theorem 5.4.15. *Let $a_m = a\sigma_m$ with $a \geq 0$, $u_0 \in V(\Omega)$, $u_1 \in V(\Omega)$ and $\Delta u_0 \in L^2(\Omega)$. For the weak solution u_m of the problem (5.75) on Ω^m in the sense of Theorem 5.4.11 and the weak solution u of problem (5.79) on Ω in the sense of Theorem 5.4.14, we have that u and u_m are also weak solutions in the sense of Definition 5.4.6. Moreover*

1. $\forall \phi \in C^1([0, T] \times \mathcal{D}(\bar{\Omega}))$ with $\phi|_{\Gamma_0} = 0$ we have $\tilde{F}^m[(E_{\mathbb{R}^2}u_m)|_{\Omega}, \phi] \xrightarrow{m \rightarrow +\infty} \tilde{F}[u, \phi]$,

2. $(E_{\mathbb{R}^2}u_m)|_{\Omega} \rightharpoonup u$ in $\tilde{H}(\Omega)$,

where $\tilde{H}(\Omega)$ is defined in (5.80).

Proof. By the definition of u_m and u in Theorems 5.4.11 and 5.4.14 respectively we have as a direct consequence u_m and u weak solutions in the sense of Definition 5.4.6. Therefore for all $\phi \in C^1([0, T] \times \mathcal{D}(\bar{\Omega}))$ with $\phi|_{\Gamma_0} = 0$

$$\tilde{F}^m[u_m, \phi] = 0 \quad \text{and} \quad \tilde{F}[u, \phi] = 0.$$

By Theorem 5.4.3 we have

$$\|(E_{\mathbb{R}^2}u_m)|_{\Omega}\|_{\tilde{H}(\Omega)} \leq C\|u_m\|_{\tilde{H}(\Omega^m)}$$

with a constant $C > 0$ independent of m . But we also have by the independence on m of r^* in Theorem 5.4.11

$$\|u_m\|_{\tilde{H}(\Omega^m)} \leq K$$

with a constant $K > 0$ independent on m .

Therefore, there exists a subsequence still denoted $(E_{\mathbb{R}^2}u_m)|_{\Omega}$ that weakly converges to u^* in $\tilde{H}(\Omega)$. And in the same way that in Theorem 5.4.7 we have for all $\phi \in C^1([0, T] \times \mathcal{D}(\bar{\Omega}))$ with $\phi|_{\Gamma_0} = 0$

$$0 = \tilde{F}^m[(E_{\mathbb{R}^2}u_m)|_{\Omega}, \phi] \rightarrow \tilde{F}[u^*, \phi].$$

So $\tilde{F}[u^*, \phi] = \tilde{F}[u, \phi] = 0$ for all $\phi \in C^1([0, T] \times \mathcal{D}(\bar{\Omega}))$ with $\phi|_{\Gamma_0} = 0$. Thus we deduce $u^* = u$ which allows to conclude. \square

Remark 5.4.3. *The results presented in this Subsection can be generalized to a domain with boundaries of Koch mixtures type everywhere on the condition that we impose an homogeneous Dirichlet boundary condition on a part of the boundary.*

We will conclude with a few remarks on the case of Robin boundary conditions everywhere.

Remark 5.4.4. In Subsection 5.4.3 we show a Mosco type convergence result. To go further on a domain with homogeneous Robin boundary conditions everywhere we need to know if the solutions u_m on the prefractals domains have a weak limit and for this they need to be bounded in a certain norm independently of m so that we can say that a subsequence has a weak limit.

Remark 5.4.5. By Theorem 5.3.6 we know that we have a certain r^* which bounds the norm of u_m , the question is whether it is independent of m and does not tend to 0. For this, we need to know the dependence on m of the constant in the L^p -estimate in Theorem 5.3.4 and in the L^∞ -estimate 3.3 according to Theorem 4.1.15.

Remark 5.4.6. If we consider Theorem 4.1.15 on Ω^m we have the constant in the L^∞ -estimates which depends of the Robin coefficient $\frac{1}{a_m} = \frac{1}{a\sigma_m} \xrightarrow{m \rightarrow \infty} +\infty$.

Moreover the existence of the constant in the L^p estimate in Theorem 5.3.4 comes from the closed graph theorem and then it is difficult to determine its dependence on m for $p > 2$.

In the case $p = 2$ the dependence on m of this constant can be determined. It relies on the dependence on m the constant C in the estimate

$$\|w\|_{H^1(\Omega^m)} \leq C \|\Delta w\|_{L^2(\Omega^m)},$$

where Δ is the Laplacian operator with homogeneous Robin boundary conditions. The work of [69] Theorem 3.6.3 leads us to a constant which if it should be optimal would also depend on $\frac{1}{a_m} = \frac{1}{a\sigma_m} \xrightarrow{m \rightarrow \infty} +\infty$. The fact is that compared to the case with the mixed boundary conditions we do not have the Poincaré inequality as we do not have an homogeneous Dirichlet boundary condition on one side and have to take into account the boundary terms in the expression of the variational form

$$(u, v)_{\tilde{H}^1(\Omega)} = \int_{\Omega} \nabla u \nabla v \, dx + a \int_{\partial\Omega} \text{Tr}_{\partial\Omega} u \text{Tr}_{\partial\Omega} v \, dm_d = \int_{\Omega} f v \, dx = (f, v)_{L^2(\Omega)}.$$

Applying the Lax-Milgram theorem, constants depending of the Robin constant a could appear in the proof of continuity or coerciveness.

Therefore, we can not prove that in Theorem 5.3.6, r^* does not tend to 0 when $m \rightarrow +\infty$ if we consider the solution u_m on Ω^m .

Appendix A

Expressions of the remainder terms

The expression of H , the profile of ρ_2 , in the paraxial variables of the KZK *ansatz* is:

$$\begin{aligned}
 H(\tau, z, y) = & -\frac{\rho_0(\gamma-1)}{2c^4}(\partial_\tau\Phi)^2 - \frac{\nu}{c^4}\partial_\tau^2\Phi \\
 & + \varepsilon \left[-\frac{\rho_0}{2c^2}[(\nabla_y\Phi)^2 - \frac{2}{c}\partial_z\Phi\partial_\tau\Phi] - \frac{\nu}{c^2}[\Delta_y\Phi - \frac{2}{c}\partial_{z\tau}^2\Phi] \right] \\
 & + \varepsilon^2 \left[-\frac{\rho_0}{2c^2}(\partial_z\Phi)^2 - \frac{\nu}{c^2}\partial_z^2\Phi \right], \tag{A.1}
 \end{aligned}$$

If we consider (2.80)-(2.81) the expressions of R_1^{NS-KZK} and \mathbf{R}_2^{NS-KZK} are written with the terms I and J defined by (2.57) and (2.58) respectively.

$$\begin{aligned}
 \varepsilon^3 R_1^{NS-KZK} = & \\
 & \varepsilon^3 \left[-\rho_0\partial_z^2\Phi + \frac{1}{c}\partial_z I\partial_\tau\Phi + \frac{1}{c}\partial_\tau I\partial_z\Phi - \nabla_y I \cdot \nabla_y \Phi \right. \\
 & \left. + \frac{2}{c}I\partial_{\tau z}^2\Phi - I\Delta_y\Phi - \frac{1}{c^2}\partial_\tau J\partial_\tau\Phi - \frac{1}{c^2}J\partial_\tau^2\Phi \right] \\
 & + \varepsilon^4 \left[-\partial_z I\partial_z\Phi - I\partial_z^2\Phi + \frac{1}{c}\partial_z J\partial_\tau J + \frac{1}{c}\partial_\tau J\partial_z\Phi \right. \\
 & \left. - \nabla_y J \cdot \nabla_y \Phi + \frac{2}{c}J\partial_{\tau z}^2\Phi - J\Delta_y\Phi \right] \\
 & + \varepsilon^5 [-\partial_z J\partial_z\Phi - J\partial_z^2\Phi].
 \end{aligned}$$

Among the x_1 axis

$$\begin{aligned}
& \varepsilon^3 \mathbf{R}_2^{NS-KZK} \cdot \vec{e}_1 = \\
& \varepsilon^3 \left[-\frac{\rho_0}{2c} \partial_\tau \left[-\frac{2}{c} \partial_z \Phi \partial_\tau \Phi + (\nabla_y \Phi)^2 \right] - \frac{\nu}{c} \partial_\tau \left[-\frac{2}{c} \partial_{\tau z}^2 \Phi + \Delta_y \Phi \right] \right. \\
& \quad \left. - \frac{I}{2c} \partial_\tau \left[\frac{1}{c^2} (\partial_\tau \Phi)^2 \right] + \frac{J}{c} \partial_\tau^2 \Phi \right] \\
& + \varepsilon^4 \left[\frac{\rho_0}{2} \partial_z \left[-\frac{2}{c} \partial_z \Phi \partial_\tau \Phi + (\nabla_y \Phi)^2 \right] + \nu \partial_z \left[-\frac{2}{c} \partial_{\tau z}^2 \Phi + \Delta_y \Phi \right] \right. \\
& \quad \left. - \frac{I}{2c} \partial_\tau \left[-\frac{2}{c} \partial_z \Phi \partial_\tau \Phi + (\nabla_y \Phi)^2 \right] + \frac{I}{2} \partial_z \left[\frac{1}{c^2} (\partial_\tau \Phi)^2 \right] - J \partial_{\tau z}^2 \Phi \right. \\
& \quad \left. - \frac{J}{2c} \partial_\tau \left[\frac{1}{c^2} (\partial_\tau \Phi)^2 \right] - \frac{\rho_0}{2c} \partial_\tau \left[(\partial_z \Phi)^2 \right] - \frac{\nu}{c} \partial_\tau \partial_z^2 \Phi \right] \\
& + \varepsilon^5 \left[-\frac{I}{2c} \partial_\tau \left[(\partial_z \Phi)^2 \right] + \frac{I}{2} \partial_z \left[-\frac{2}{c} \partial_z \Phi \partial_\tau \Phi + (\nabla_y \Phi)^2 \right] \right. \\
& \quad \left. + \frac{J}{2} \partial_z \left[\frac{1}{c^2} (\partial_\tau \Phi)^2 \right] - \frac{J}{2c} \partial_\tau \left[-\frac{2}{c} \partial_z \Phi \partial_\tau \Phi + (\nabla_y \Phi)^2 \right] \right. \\
& \quad \left. + \frac{\rho_0}{2} \partial_z \left[(\partial_z \Phi)^2 \right] + \nu \partial_z^3 \Phi \right] \\
& + \varepsilon^6 \left[\frac{I}{2} \partial_z \left[(\partial_z \Phi)^2 \right] - \frac{J}{2c} \partial_\tau \left[(\partial_z \Phi)^2 \right] + \frac{J}{2} \left[-\frac{2}{c} \partial_z \Phi \partial_\tau \Phi + (\nabla_y \Phi)^2 \right] \right] \\
& + \varepsilon^7 \left[\frac{J}{2} \partial_z \left[(\partial_z \Phi)^2 \right] \right]
\end{aligned}$$

and in the hyperplane orthogonal to the x_1 axis

$$\begin{aligned}
& \sum_{i=2}^n (\mathbf{R}_2^{NS-KZK} \cdot \vec{e}_i) \vec{e}_i = \\
& \varepsilon^{\frac{7}{2}} \left[\frac{\rho_0}{2} \nabla_y \left[-\frac{2}{c} \partial_z \Phi \partial_\tau \Phi + (\nabla_y \Phi)^2 \right] + \nu \nabla_y \left[-\frac{2}{c} \partial_{\tau z}^2 \Phi + \Delta_y \Phi \right] \right. \\
& \quad \left. + \frac{I}{2} \nabla_y \left[\frac{1}{c^2} (\partial_\tau \Phi)^2 \right] - J \nabla_y [\partial_\tau \Phi] \right] \\
& + \varepsilon^{\frac{9}{2}} \left[\frac{I}{2} \nabla_y \left[-\frac{2}{c} \partial_z \Phi \partial_\tau \Phi + (\nabla_y \Phi)^2 \right] + \frac{J}{2} \nabla_y \left[\frac{1}{c^2} (\partial_\tau \Phi)^2 \right] \right. \\
& \quad \left. + \frac{\rho_0}{2} \nabla_y \left[(\partial_z \Phi)^2 \right] + \nu \nabla_y [\partial_z^2 \Phi] \right] \\
& + \varepsilon^{\frac{11}{2}} \left[\frac{I}{2} \nabla_y \left[(\partial_z \Phi)^2 \right] + \frac{J}{2} \nabla_y \left[-\frac{2}{c} \partial_z \Phi \partial_\tau \Phi + (\nabla_y \Phi)^2 \right] \right] \\
& + \varepsilon^{\frac{13}{2}} \left[\frac{J}{2} \nabla_y \left[(\partial_z \Phi)^2 \right] \right]
\end{aligned}$$

Appendix B

Résumé français

L'étude de la propagation d'ondes non-linéaires suscite un intérêt, en particulier à cause de récentes applications à l'imagerie ultrason (*i.e.* HIFU) ou des applications techniques et médicales comme la lithotripsie ou la thérapie. Ces nouvelles techniques reposent fortement sur la capacité à modéliser avec précision la propagation non-linéaire d'une pulsation sonore d'amplitude finie dans un milieu élastique thermo-visqueux. Les modèles les plus connus d'acoustique non linéaire, que nous considérerons dans cette thèse sont

1. l'équation de Kuznetsov qui se lit pour $\alpha = \frac{\gamma-1}{c^2}$, $\beta = 2$ comme

$$u_{tt} - c^2 \Delta u - \frac{\nu}{\rho_0} \varepsilon \Delta u_t = \alpha \varepsilon u_t u_{tt} + \beta \varepsilon \nabla u \cdot \nabla u_t, \quad x \in \mathbb{R}^n, \quad (\text{B.1})$$

où c , ρ_0 , γ , ν sont la vitesse du son, la densité, le ratio des chaleurs spécifiques et la viscosité du milieu respectivement. Le coefficient ε représente un petit paramètre sans dimension apparaissant dans la dérivation de l'équation. Dans ce qui suit, nous pouvons juste supposer que α et β sont des constantes positives. C'est en fait une équation d'onde quasi-linéaire (amortie), initialement introduite par Kuznetsov [60] pour le potentiel de vitesse, voir aussi les Réfs. [37, 50, 55, 63] pour d'autres variations de sa dérivation;

2. l'équation de Khokhlov-Zabolotskaya-Kuznetsov (KZK)

$$c \partial_{\tau z}^2 I - \frac{(\gamma+1)}{4\rho_0} \partial_\tau^2 I^2 - \frac{\nu}{2c^2 \rho_0} \partial_\tau^3 I - \frac{c^2}{2} \Delta_y I = 0, \quad (\text{B.2})$$

qui peut être écrite pour les perturbations de la densité ou de la pression (voir les études physiques systématiques dans le livre [13] par Bakhvalov, Zhileïkin, et Zabolotskaya);

3. l'Équation d'onde Non-linéaire Progressive (NPE)

$$\partial_{\tau z}^2 \xi + \frac{(\gamma+1)c}{4\rho_0} \partial_z^2 [(\xi)^2] - \frac{\nu}{2\rho_0} \partial_z^3 \xi + \frac{c}{2} \Delta_y \xi = 0, \quad (\text{B.3})$$

dérivée par McDonald et Kuperman dans la Réf. [70];

4. l'équation de Westervelt

$$\partial_t^2 \Pi - c^2 \Delta \Pi = \varepsilon \partial_t \left(\frac{\nu}{\rho_0} \Delta \Pi + \frac{\gamma+1}{2c^2} (\partial_t \Pi)^2 \right), \quad (\text{B.4})$$

qui est similaire à l'équation de Kuznetsov avec seulement un de ses deux termes non-linéaires, dérivée initialement par Westervelt [91] et plus tard par d'autres auteurs [1, 89].

L'équation de Kuznetsov (B.1) décrit l'évolution du potentiel de vitesse, c'est une équation d'onde quasi linéaire amortie, qui décrit la propagation d'une onde de grande amplitude dans un fluide. Elle est un des modèles dérivés du système de Navier-Stokes, et elle est appropriée pour les ondes planes, cylindriques et sphériques dans un fluide (voir [37] de Hamilton et Blackstock). La plupart des travaux sur l'équation de Kuznetsov (B.1) sont traités dans une dimension d'espace [50] ou dans un domaine borné de \mathbb{R}^n [55, 52, 53, 71]. Pour le cas visqueux, Kaltenbacher et Lasiecka [53] ont considéré le problème avec conditions de Dirichlet au bord et prouvé, pour des données initiales suffisamment petites, le caractère bien posé global pour $n \leq 3$. Meyer et Wilke [71] l'ont prouvé pour tout n . Dans [52], Kaltenbacher et Lasiecka ont prouvé le caractère bien posé local du problème avec conditions au bord de Neumann pour $n \leq 3$. Le travail des Réf. [52, 53] utilise des estimations d'énergie a priori, et la Réf. [71] la notion de régularité maximale.

L'équation de Westervelt (B.4) est aussi une approximation de l'équation de Kuznetsov, mais cette fois par une perturbation non-linéaire. De fait la seule différence entre ces deux modèles est que l'équation de Westervelt ne conserve qu'un des deux termes non-linéaires de l'équation de Kuznetsov, produisant des effets cumulatifs dans une propagation d'onde progressive selon Aanonsen, Barkve, Tjøtta et Tjøtta [1].

L'équation NPE (B.3) est habituellement utilisée pour décrire les vibrations en temps court et la propagation sur de longues distances, par exemple dans un guide d'onde océanique, où les phénomènes de réfractions sont importants, alors que l'équation de KZK (B.2) modélise typiquement la propagation d'ultrasons avec de forts phénomènes de diffraction, combinée avec des effets d'amplitude finie (voir Rozanova-Pierrat avec la Réf. [81] et les références utilisées). Bien que le contexte et l'utilisation physique des équations de KZK et NPE soient différents, il y a une bijection entre les variables de ces deux modèles et ils peuvent être représentés par le même type d'opérateur différentiel avec des coefficients constants positifs:

$$Lu = 0, \quad L = \partial_{tx}^2 - c_1 \partial_x (\partial_x \cdot)^2 - c_2 \partial_x^3 \pm c_3 \Delta_y, \quad \text{pour } t \in \mathbb{R}^+, x \in \mathbb{R}, y \in \mathbb{R}^{n-1}.$$

Ainsi, les résultats de la Réf. [80] sur les solutions de l'équation de KZK sont valides pour l'équation NPE. Voir aussi la Réf. [42] par Ito pour la décroissance exponentielle des solutions de ces modèles dans le cas visqueux.

Tous les modèles de Kuznetsov, KZK, NPE et Westervelt ont été dérivés jusqu'à de petits termes négligeables à partir de systèmes non-linéaires de Navier-Stokes (pour le milieu visqueux) et d'Euler (pour le cas non visqueux) compressibles et isentropiques. Mais toutes les dérivations physiques citées de ces modèles ne permettent pas de dire que leurs solutions approchent la solution du système de Navier-Stokes ou d'Euler. Les résultats sur le caractère bien posé des équations de KZK et NPE sont déjà connus, le premier travail l'expliquant pour l'équation de KZK est la Réf. [81] par Rozanova-Pierrat.

Nous nous sommes dès lors focalisés dans le Chapitre 1 sur le caractère bien posé du problème de Cauchy associé à l'équation de Kuznetsov dans \mathbb{R}^n pour les cas visqueux et non visqueux avec des données initiales suffisamment petites. Ces résultats correspondent à notre article [26] proposé avec Rozanova-Pierrat.

Dans le Chapitre 2, nous commençons à présenter le contexte initial du système de

Navier-Stokes isentropique

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0, \quad (\text{B.5})$$

$$\rho[\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}] = -\nabla p(\rho) + \varepsilon \nu \Delta \mathbf{v}, \quad (\text{B.6})$$

(en fait, c'est aussi une approximation du système de Navier-Stokes compressible), qui décrit le mouvement d'une onde acoustique dans un milieu thermo-élastique homogène [13, 37, 65]. Nous systématisons dans le Chapitre 2 la dérivation de tous ces modèles en utilisant les idées de Rozanova-Pierrat dans la Réf. [81], consistant à utiliser des correcteurs dans les expansions de type Hilbert des *ansatzs* physiques correspondants.

Plus précisément, nous montrons que tous ces modèles sont des approximations du système de Navier-Stokes ou d'Euler jusqu'aux termes d'ordre trois en un petit paramètre sans dimension $\epsilon > 0$ mesurant la taille des perturbations de la pression, de la densité et de la vitesse par rapport à leur état constant $(p_0, \rho_0, 0)$ (voir Fig. 1).

A l'aide des résultats connus sur le caractère bien posé des modèles, nous validons ensuite dans le Chapitre 2 ces approximations en obtenant des estimations en norme L^2 entre les solutions des modèles exacts et approchés considérés en étudiant d'abord l'approximation du système de Navier-Stokes puis l'approximation de l'équation de Kuznetsov. Il est à noter que pour le modèle exact nous pouvons considérer une solution faible peu régulière qui sera approchée par la solution régulière du modèle approché.

Ainsi nous avons été amenés dans la Partie II à étudier les solutions faibles d'équations d'ondes sur des domaines à bords fractals afin de considérer les domaines les plus généraux possibles sur lesquels de telles solutions faibles existent.

Pour en revenir au Chapitre 1 nous étudions le caractère bien posé du problème de Cauchy associé à l'équation (B.1). Dans le cas non visqueux pour $\nu = 0$, le problème de Cauchy pour l'équation de Kuznetsov est un cas particulier du système général quasi linéaire hyperbolique du second ordre considéré par Hughes, Kato et Marsden [40]. Le résultat de caractère bien posé local, prouvé dans [40], n'utilise pas des techniques d'estimations a priori et est fondé sur la théorie des semi-groupes. Alors, grâce à [40], nous avons le caractère bien posé de (B.1) dans l'espace de Sobolev H^s avec un réel $s > \frac{n}{2} + 1$. De plus, pour étendre le caractère bien posé local au cas global (pour $n \geq 4$) et pour estimer l'intervalle de temps maximal sur lequel il existe une solution régulière, John [44] a développé des estimations a priori pour le problème de Cauchy associé à une équation d'onde quasi linéaire générale à l'aide d'une énergie de la forme

$$E_m[u](t) = \|\nabla u(t)\|_{H^m(\mathbb{R}^n)}^2 + \sum_{i=1}^{m+1} \|\partial_t^i u(t)\|_{H^{m+1-i}(\mathbb{R}^n)}^2.$$

Cette fois, à cause des non linéarités $u_t u_{tt}$ et $\nabla u \nabla u_t$ incluant les dérivées en temps, pour avoir une estimation a priori pour l'équation de Kuznetsov nous avons besoin de travailler avec les espaces de Sobolev H^s caractérisés par un entier s . Si nous appliquons directement les résultats généraux par John de la Réf. [44] à notre cas pour l'équation de Kuznetsov, nous obtenons le caractère bien posé pour des données initiales très régulières. Nous améliorons ce résultat et obtenons les résultats de John pour l'équation de Kuznetsov avec une régularité minimale des données initiales correspondant à la régularité obtenue par Hughes, Kato et Marsden [40]. Les estimations d'énergie nous permettent d'évaluer le temps d'existence maximal. Dans \mathbb{R}^2 et \mathbb{R}^3 l'optimalité des estimations obtenues pour le temps d'existence maximal est assurée par les résultats d'Alinhac [5]. Dans la Réf. [5] un

blow-up géométrique pour les données petites est prouvé pour $\partial_t^2 u$ et Δu en temps fini et pour le même ordre que prédit par les estimations a priori.

Pour $n \geq 4$ et $\nu = 0$, nous améliorons aussi les résultats de John [44]. La petitesse des données initiales assure directement l'hyperbolicité de l'équation de Kuznetsov pour tout temps, *i.e.* elle assure que $1 - \alpha \epsilon u_t$ est strictement positif et borné pour tout temps. La preuve utilise les dérivées généralisées pour les équations d'ondes [44] et une estimation a priori de Klainerman [58, 59].

En présence du terme Δu_t pour le cas visqueux $\nu > 0$, la régularité des dérivées en temps d'ordre supérieur de u est différente (en comparaison au cas non visqueux), et la manière de contrôler les non linéarités change. Comme il a été montré dans [83] par Shibata, ce terme dissipatif change une vitesse finie de propagation pour l'équation d'onde en une vitesse infinie. En effet, la partie linéaire de l'équation (B.1) peut être vue comme deux compositions de l'opérateur de la chaleur $\partial_t - \Delta$ de la manière suivante:

$$u_{tt} - c^2 \Delta u - \nu \epsilon \Delta u_t = \partial_t (\partial_t u - \epsilon \nu \Delta u) - c^2 \Delta u.$$

Pour le cas visqueux nous prouvons les résultats sur le caractère bien posé global dans \mathbb{R}^n pour les données initiales suffisamment petites, dont nous spécifions la taille. Pour $n \geq 3$ nous établissons une estimation a priori qui nous donne aussi une condition suffisante pour l'existence de solutions globales avec une énergie initiale suffisamment petite. En considérant les espaces de Sobolev H^s caractérisés par un entier $s = m$ pair on contrôle l'énergie

$$E_{\frac{m}{2}}[u](t) = \|\nabla u(t)\|_{H^m(\mathbb{R}^n)}^2 + \sum_{i=1}^{\frac{m}{2}+1} \|\partial_t^i u(t)\|_{H^{m-2(i-1)}(\mathbb{R}^n)}^2.$$

Les mêmes résultats sont vrais dans $(\mathbb{R}/L\mathbb{Z}) \times \mathbb{R}^{n-1}$ pour $n \geq 2$ avec la périodicité et la valeur moyenne nulle selon une variable.

Intéressons nous dès lors au Chapitre 2. Comme il est montré dans la Fig. 1, l'équation de Kuznetsov vient du système de Navier-Stokes ou d'Euler seulement par de petites perturbations, mais pour obtenir les équations KZK et NPE nous avons besoin d'utiliser un changement de variables paraxial en plus des petites perturbations. En outre, les équations de KZK et NPE peuvent aussi être obtenues à partir de l'équation de Kuznetsov juste en pratiquant le changement de variable paraxial correspondant. Nous pouvons noter que l'équation de Kuznetsov est une équation d'onde non-linéaire contenant des termes d'ordres différents en ϵ . Mais les approximations paraxiales pour KZK et NPE permettent d'avoir les équations approchées avec tous les termes de même ordre, *i.e.* les équations de KZK et NPE.

Portons notre attention sur le fait que l'*ansatz*, proposé initialement par Bakhvalov, Zhileïkin, et Zabolotskaya dans la Réf. [13] et utilisé par Rozanova-Pierrat dans la Réf. [81] pour obtenir l'équation de KZK à partir des systèmes de Navier-Stokes ou d'Euler, est différent de l'*ansatz* que nous utilisons. De plus, cette nouvelle approximation des systèmes de Navier-Stokes et d'Euler est une amélioration en comparaison à la dérivation développée dans la Réf. [81], car dans cette référence le système de Navier-Stokes/Euler pouvait seulement être approchées jusqu'aux termes d'ordre $O(\epsilon^{\frac{5}{2}})$ (comparé à l'ordre $O(\epsilon^3)$ dans notre cas).

Les hypothèses principales pour la dérivation de tous ces modèles sont les suivantes:

- le mouvement est potentiel;

- l'état constant du milieu donné par $(p_0, \rho_0, 0)$ (0 pour la vitesse) est perturbé proportionnellement à un paramètre sans dimension $\epsilon > 0$ (par exemple, égal à 10^{-5} dans l'eau avec une puissance initiale de l'ordre de 0.3 W/cm^2);
- toutes les viscosités sont petites (d'ordre ϵ).

Pour garder le sens physique des problèmes d'approximation, nous considérerons particulièrement les cas bidimensionnel et tridimensionnel, *i.e.* \mathbb{R}^n avec $n = 2$ ou 3 , et dans la suite nous utiliserons la notation $x = (x_1, x') \in \mathbb{R}^n$ avec un axe $x_1 \in \mathbb{R}$ et la variable transversale $x' \in \mathbb{R}^{n-1}$.

Nous validons ainsi les approximations du système de Navier-Stokes compressible par les différents modèles : par l'équation de Kuznetsov, l'équation de KZK et l'équation NPE.

Puis nous faisons de même pour le système d'Euler dans le cas non visqueux. Les différences principales entre les cas visqueux et non visqueux sont le temps d'existence et la régularité des solutions. Typiquement dans le cas non visqueux, les solutions des modèles et aussi du système d'Euler lui-même (solutions fortes) peuvent entraîner la formation de fronts de choc en temps finis à cause de leurs non-linéarités [4, 26, 80, 84, 93]. Ainsi, elles sont seulement localement bien posées, alors que dans le cas visqueux tous les modèles d'approximations sont globalement bien posés pour des données initiales suffisamment petites [26, 67, 80].

Nous notons par \mathbf{U}_ϵ une solution du système de Navier-Stokes/Euler "exact" (voir l'Eq. (2.31))

$$\partial_t \mathbf{U}_\epsilon + \sum_{i=1}^n \partial_{x_i} \mathbf{G}_i(\mathbf{U}_\epsilon) - \epsilon \nu \begin{bmatrix} 0 \\ \Delta \mathbf{v}_\epsilon \end{bmatrix} = 0,$$

et par $\overline{\mathbf{U}}_\epsilon$ une solution approchée, construite par l'*ansatz* de dérivation à partir d'une solution régulière de l'un des modèles approchés (typiquement les équations de Kuznetsov, KZK et NPE), *i.e.* une fonction qui résout le système de Navier-Stokes/Euler jusqu'aux termes d'ordre ϵ^3 , désignés par $\epsilon^3 \mathbf{R}$ (voir l'Eq. (2.32)):

$$\partial_t \overline{\mathbf{U}}_\epsilon + \sum_{i=1}^n \partial_{x_i} \mathbf{G}_i(\overline{\mathbf{U}}_\epsilon) - \epsilon \nu \begin{bmatrix} 0 \\ \Delta \overline{\mathbf{v}}_\epsilon \end{bmatrix} = \epsilon^3 \mathbf{R}.$$

Pour avoir le terme de reste $\mathbf{R} \in C([0, T], L^2(\Omega))$ nous devons assurer que le terme de gauche de cette équation est dans $C([0, T], L^2(\Omega))$, *i.e.* nous avons besoin d'une solution $\overline{\mathbf{U}}_\epsilon$ suffisamment régulière. La régularité minimale des données initiales pour avoir un tel $\overline{\mathbf{U}}_\epsilon$ est donnée dans le Tableau 2.1 (voir aussi le Tableau 2.2 pour l'approximation de l'équation de Kuznetsov).

En choisissant pour le système exact les mêmes données initiales et au bord trouvées par l'*ansatz* pour $\overline{\mathbf{U}}_\epsilon$ (le cas régulier) ou les données initiales prises dans un petit voisinage L^2 , *i.e.*

$$\|\mathbf{U}_\epsilon(0) - \overline{\mathbf{U}}_\epsilon(0)\|_{L^2(\Omega)} \leq \delta \leq \epsilon,$$

avec $\mathbf{U}_\epsilon(0)$ non nécessairement régulier, mais assurant l'existence d'une solution faible admissible d'énergie bornée, nous prouvons l'existence de constantes $C > 0$ et $K > 0$ indépendantes de ϵ , δ et du temps t telles que

$$\text{pour tout } 0 \leq t \leq \frac{C}{\epsilon} \quad \|(\mathbf{U}_\epsilon - \overline{\mathbf{U}}_\epsilon)(t)\|_{L^2(\Omega)}^2 \leq K(\epsilon^3 t + \delta^2) e^{K\epsilon t} \leq 9\epsilon^2$$

avec Ω un domaine où les deux solutions \mathbf{U}_ε et $\overline{\mathbf{U}}_\varepsilon$ existent. Il devient ainsi possible d'approcher une solution faible exacte peu régulière par une solution approchée régulière.

Comme les équations de KZK et NPE peuvent être vues comme des approximations de l'équation de Kuznetsov au vu de leur dérivation (voir la Figure 1), nous validons aussi l'approximation de l'équation de Kuznetsov par les équations de KZK et NPE, et aussi par l'équation de Westervelt (voir le Tableau 2.2).

Pour être capable de considérer l'approximation de l'équation de Kuznetsov par l'équation de KZK, nous établissons d'abord des résultats sur le caractère globalement bien posé de l'équation de Kuznetsov dans le demi espace, similaires au cadre précédent pour l'équation de KZK et le système de Navier-Stokes. Nous étudions deux cas : le problème périodique en temps purement aux bords dans les variables (z, τ, y) se déplaçant avec l'onde et le problème avec conditions initiales et au bord pour l'équation de Kuznetsov dans les variables initiales (t, x_1, x') avec des données venant de la solution de l'équation de KZK. Nous validons ces deux types d'approximations pour les cas visqueux et non visqueux.

Finalement nous validons l'approximation entre les équations de Kuznetsov et NPE et les équations de Kuznetsov et Westervelt respectivement (voir le Tableau 2.2). Nous pouvons les résumer de la manière suivante: si u est une solution de l'équation de Kuznetsov et \bar{u} est une solution de l'équation de NPE ou de KZK (pour le problème avec conditions initiales et aux bords) ou de Westervelt trouvée pour des données initiales assez proches

$$\|\nabla_{t,\mathbf{x}}(u(0) - \bar{u}(0))\|_{L^2(\Omega)} \leq \delta \leq \epsilon,$$

alors il existe $K > 0$, $C_1 > 0$, $C_2 > 0$ et $C > 0$ constantes indépendantes de ϵ , δ et du temps, telles que pour tout $t \leq \frac{C}{\epsilon}$ il est vérifié

$$\|\nabla_{t,\mathbf{x}}(u - \bar{u})\|_{L^2(\Omega)} \leq C_1(\epsilon^2 t + \delta)e^{C_2 \epsilon t} \leq K\epsilon.$$

Comme les estimations de la stabilité obtenues sont valables entre une solution régulière et une solution faible de Kuznetsov nous pouvons de nouveau approcher une solution moins régulière d'un modèle exact par la solution régulière d'un modèle approché.

Dans la Partie II, nous nous intéressons à la question des solutions faibles d'équation d'ondes. On se place dans le contexte des domaines bornés et on cherche la classe des bords la plus large pour que le problème soit bien posé faiblement. Ces équations incluent:

- l'équation des ondes avec des conditions de Dirichlet homogène en utilisant Evans [30],
- l'équation des ondes fortement amortie avec des conditions de Dirichlet homogènes et non homogènes ou des conditions de Robin homogènes,
- l'équation non-linéaire de Westervelt avec des conditions de Dirichlet homogènes et non homogènes ou des conditions de Robin homogènes.

La régularité des solutions de ces équations sur des domaines réguliers, typiquement avec un bord C^2 est bien connue, notamment le fait que, plus les données initiales sont régulières, plus la solution est régulière et ce jusqu'au bord. Nous pouvons citer Evans et la Réf. [30] pour l'équation des ondes ou les Réfs. [51, 52, 53, 54, 71] pour l'équation des ondes fortement amortie ou l'équation de Westervelt ainsi que leurs références utilisées. Nous pouvons nous demander si, sur des domaines moins réguliers, on peut avoir une solution faible continue ou C^1 jusqu'au bord. Les exemples de Arendt et Elst dans la Réf. [10] montrent l'apparition de problèmes pour la définition de la trace dès que le bord n'est plus C^1 . De

plus si, pour un domaine au bord C^1 ou lipschitzien, on peut définir une normale intérieure presque partout, la question des conditions de Neumann ou Robin sur un bord moins régulier est plus délicate. Par ailleurs le fait de considérer un bord régulier C^2 comme dans [51, 52, 53, 54, 71] est une conséquence de ce que les dérivées spatiales sont au plus d'ordre 2 et peuvent ainsi être plus naturellement définies au bord. Dans le passé, les mathématiques se sont largement focalisées sur des domaines réguliers. Des ensembles comme celui de Von Koch ont principalement été considérés comme "pathologiques" et utilisés seulement pour produire des contre-exemples. Néanmoins, il y a eu un changement d'attitude lorsque les mathématiciens et les physiciens ont découvert que des structures semblable à celle de Von Koch apparaissaient dans la nature, comme par exemple la micro-structure des électrodes ou les côtes de l'Angleterre.

Un point clé pour résoudre les équations que nous étudierons sur des domaines à bords fractals est la compréhension du problème de Poisson sur ces domaines avec des conditions aux bords de Dirichlet

$$\begin{cases} -\Delta u = f \text{ sur } \Omega, \\ u|_{\Omega} = g \text{ sur } \partial\Omega \end{cases} \quad (\text{B.7})$$

ou des conditions de Robin homogènes

$$\begin{cases} -\Delta u = f \text{ sur } \Omega, \\ \frac{\partial}{\partial n} u + au = 0 \text{ avec } a > 0 \text{ sur } \partial\Omega. \end{cases} \quad (\text{B.8})$$

Pour le système (B.7) une approche générale passe par la formulation faible du problème de Dirichlet. Si u et $\partial\Omega$ sont suffisamment régulières on peut multiplier l'équation de Poisson dans le problème (B.7) par $v \in C_0^\infty(\Omega)$ et utiliser la formule de Green pour obtenir

$$\begin{cases} \int_{\Omega} \nabla u \nabla v \, dx = \int_{\Omega} f v \, dx \text{ for all } v \in C_0^\infty(\Omega), \\ u|_{\partial\Omega} = g, \end{cases}$$

qui est appelée une formulation faible du problème de Dirichlet. En introduisant les espaces de Sobolev $H^1(\Omega)$ et $H_0^1(\Omega)$ et en supposant qu'il existe $g^* \in H^1(\Omega)$ tel que la trace de g^* sur $\partial\Omega$ est g (une attention particulière doit être portée à la définition de la trace), on peut prouver, à l'aide du théorème de représentation de Riesz, qu'étant donné $f \in L^2(\Omega)$, $g^* \in H^1(\Omega)$, il existe un unique $u \in H^1(\Omega)$ tel que $-\Delta u = f$ au sens des distributions et $u - g^* \in H_0^1(\Omega)$.

Ceci soulève plusieurs questions:

- Comment définir la trace, habituellement définie pour des fonctions continues?
- Comment définir une extension g^* vérifiant $u = g$ au bord?

La réponse aux deux premières questions est connue si $\partial\Omega$ est assez régulier, on peut citer par exemple Raviart-Thomas [79], ou même lipschitzien avec le travail de Marschall [66].

- Peut-on utiliser la formule de Green? Dans le cas lipschitzien on a

$$\int_{\Omega} v \Delta u \, dx = \langle u, v \rangle_{(H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega))} - \int_{\Omega} \nabla u \nabla v \, dx.$$

- Est ce que u dépend uniquement ou continûment de f et g ?

Dans la Réf. [49] Jonsson et Wallin ont pu répondre à ces questions dans le cas où Ω est un (ϵ, δ) -domaine avec un bord $\partial\Omega$ qui est un d -ensemble pour la mesure de Hausdorff préservant l'inégalité de Markov. En se basant sur le travail de Lancia [62] on trouve un équivalent de la formule de Green faisant intervenir les espaces de Besov pour le terme de bord. Les résultats de Jonsson et Wallin sont à notre connaissance les premiers de ce type établis sur des domaines fractals. Les résultats de Jones [46] sur les d -ensembles et les domaines admettant des extensions W_p^k permettent de dire qu'en dimension 2 les (ϵ, δ) -domaines sont les domaines les plus généraux sur lesquels on peut définir des traces et des extensions des espaces de Sobolev et ainsi résoudre le problème de Poisson. Dans la Réf. [11], Arfi et Rozanova-Pierrat ont introduit un nouveau type de domaine à bords fractals dits les domaines admissibles. Ces domaines contiennent les (ϵ, δ) -domaines et sont plus généraux, ils forment la classe la plus large des domaines sur lesquels on peut définir des traces et des extensions aux espaces de Sobolev pour $\Omega \subset \mathbb{R}^n$ avec $n \geq 2$, et ainsi trouver une solution faible au problème de Poisson dépendant de manière unique et continue des données initiales.

En conséquence nous travaillerons principalement sur les domaines admissibles et résumons les résultats connus sur ces domaines. Il est à noter que le travail de la Réf. [30] par Evans nous fournit les propriétés spectrales ainsi que la régularité intérieure de la solution du problème de Poisson (B.7), *i.e.* le fait que pour un sous ensemble V inclus de manière compacte dans Ω , $V \subset\subset \Omega$, la solution sur Ω a sur V la même régularité que pour un domaine aux bords réguliers. La Réf. [11] par Arfi et Rozanova-Pierrat permet de donner des résultats similaires pour le problème de Poisson (B.8) et la Réf. [30] par Evans nous fournit encore les propriétés spectrales.

Une autre question importante est de savoir si les solutions des problèmes de Poisson (B.7) et (B.8) appartiennent à $C(\overline{\Omega})$ avec une estimation de la forme:

$$\|u\|_{L^\infty(\Omega)} \leq C \|f\|_{L^p(\Omega)}.$$

Pour le problème de Poisson (B.7) avec des conditions au bord de Dirichlet homogènes les travaux des Réfs. [77] par Nyström et [92] par Xie permettent de donner une réponse positive à cette questions en dimension $n = 2$ et 3 respectivement pour $p = 2$. Le travail de Daners dans la Réf. [25] nous donne aussi une réponse positive pour le problème de Poisson (B.8) si $p > n$. Ces estimations sont essentielles pour montrer que les solutions de nos modèles de type ondulatoires étudiés sont dans $C(\overline{\Omega})$ mais aussi pour traiter la non-linéarité de l'équation de Westervelt.

En utilisant une méthode de Galerkin comme dans la Réf. [30] par Evans nous obtenons la régularité de l'équation des ondes et de l'équation des ondes fortement amortie avec des conditions de Dirichlet homogènes avec l'aide d'une base de fonctions propres de $-\Delta$. Avec ces résultats de régularité nous traitons le caractère bien posé de l'équation de Westervelt avec des conditions de Dirichlet de la même façon que dans la preuve dans le Chapitre 1 du caractère bien posé global de l'équation de Kuznetsov sur \mathbb{R}^n . Les propriétés de la trace et de l'extension pour les domaines admissibles rappelées nous ont permis de traiter le cas

des conditions de Dirichlet non homogènes. Ces résultats reposent sur des estimations dans des espaces où la solution et certaines de ses dérivées sont dans L^2 . Notons que nous avons utilisé une méthode similaire pour les problèmes avec conditions de Robin homogènes et obtenu le caractère bien posé et des estimations L^2 pour l'équation des ondes fortement amortie sur un domaine admissible, avec une méthode de Galerkin fondée sur une base de fonctions propres de $-\Delta$, ou pour l'équation de Westervelt sur un domaine lipschitzien de la même façon que dans la preuve dans le Chapitre 1 du caractère bien posé global de l'équation de Kuznetsov sur \mathbb{R}^n . Le cas de l'équation de Westervelt sur un domaine admissible avec des conditions de Robin homogènes a été traité à l'aide d'estimations L^p avec $p > n$ de la même manière.

En conclusion de cette Partie II, nous considérons un ensemble à bord fractal de type mixture de Koch, construit par récurrence à l'aide de familles de similitudes contractantes induisant ainsi une famille de domaines à bords pré-fractals et lipschitziens convergeant vers le domaine à bords fractals. En utilisant différents travaux par Capitanelli [19], Capitanelli et Vivaldi [20] ou Lancia [62] nous avons pu considérer la convergence asymptotique de type Mosco des solutions de l'équation de Westervelt avec conditions de Robin sur les domaines à bords pré-fractals qui approximent la solution sur le domaine à bords fractal de type mixture de Koch, une démarche souvent utilisée dans le cadre de l'optimisation de forme.

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Titre : Analyse mathématique de l'équation de Kuznetsov: problème de Cauchy, questions d'approximations et problèmes aux bords fractals.

Mots Clefs : acoustique non linéaire, système de Navier-Stokes, équation de Kuznetsov, approximation, bords fractals

Résumé : Dans le contexte de l'acoustique on a systématisé la dérivation de modèles non-linéaires (l'équation de Kuznetsov, l'équation KZK et la NPE). On a estimé le temps pour lequel des solutions régulières de ces modèles restent proches des solutions des systèmes de Navier-Stokes/Euler compressibles isentropiques (en précisant leur plus faible régularité) et établi les résultats analogues entre les solutions des équations de KZK, NPE et Westervelt par rapport à la solution de l'équation de Kuznetsov. Pour ce faire, on a étudié l'équation de Kuznetsov en commençant par le problème de Cauchy dans les cas visqueux (stabilité, unicité et existence globale des solutions régulières) et non-visqueux (caractère bien posé avec les estimations optimales du temps d'existence maximale des solutions régulières) et également dans un demi espace avec des conditions au limites périodiques en temps ou dans un espace périodique dans une direction. On a aussi obtenu l'existence et l'unicité des solutions faibles pour l'équation des ondes fortement amortie et l'équation de Westervelt sur la plus large classe de domaines aux bords irréguliers, ainsi que la convergence asymptotique des solutions de l'équation de Westervelt avec conditions de Robin sur les bords préfractals approximant un bord fractal de type mixture de Koch.

Title : Mathematical analysis of the Kuznetsov equation: Cauchy problem, approximation questions and problems with fractal boundaries.

Keys words : Nonlinear acoustic, Navier-Stokes system, Kuznetsov equation, approximation, fractals boundaries

Abstract : In the framework of acoustic we systematize the derivation of nonlinear models (the Kuznetsov equation, the KZK equation and the NPE). We estimate the time for which the regular solutions of these models stay close of the solutions of the compressible isentropic Navier-Stokes/Euler systems (pointing out their weakest regularity) and establish similar results between the solutions of the KZK, NPE and Westervelt equations with respect to the solutions of the Kuznetsov equation. To do so, we study the Kuznetsov equation beginning by the Cauchy problem in the viscous case (stability, global well posedness of regular solutions) and inviscid case (well posedness with optimal estimations of the maximal existence time for regular solutions) and also in the half space with time periodic boundary conditions or in a periodic in one direction space. We also obtain the existence and unicity of weak solutions for the strongly damped wave equation and the Westervelt equation in the largest class of domains with irregular boundaries, along with the asymptotic convergence of the solutions of the Westervelt equation with Robin boundary conditions on prefractal boundaries approximating a Koch mixture as fractal boundary.