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Hui Zhu

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Hui ZHU

Contrôle, stabilisation et propagation des singularités
pour des EDP dispersives

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Résumé

Dans cette thèse, nous étudions les théories étroitement liées du contrôle, de la stabilisation et de la propagation des singularités, pour des équations aux dérivées partielles dispersives linéaires et non-linéaires. Les résultats principaux proviennent des travaux de l'auteur :

- [1] Zhu, H., 2016. *Stabilization of damped waves on spheres and Zoll surfaces of revolution*. ESAIM : Control, Optimisation and Calculus of Variations (ESAIM : COCV), à paraître.
- [2] Zhu, H., 2017. *Control of three dimensional water waves*. arXiv preprint arXiv :1712.06130.
- [3] Zhu, H., 2018. *Propagation of singularities for gravity-capillary water waves*. arXiv preprint arXiv :1810.09339.

Dans [1], nous avons étudié la stabilisation des ondes amorties sur les surfaces de révolution de Zoll. Nous avons donné un exemple où la région d'amortissement est à la limite de la condition du contrôle géométrique, alors que les ondes amorties présentent une décroissance exponentielle uniforme de l'énergie. Cet exemple généralise un résultat de Lebeau.

Dans [2], nous avons étudié la contrôlabilité du système des ondes de surface avec tension superficielle. Nous avons démontré, en dimension arbitraire, la contrôlabilité exacte pour des petites données spatialement périodiques en faisant l'hypothèse de contrôle géométrique. Ce résultat généralise le travail de Alazard, Baldi et Han-Kwan en dimension deux.

Dans [3], nous avons étudié la propagation des singularités pour des ondes de surface avec tension superficielle. Nous avons défini le front d'onde quasi-homogène, généralisant le front d'onde de Hörmander et le front d'onde homogène de Nakamura et démontré des résultats de propagation des fronts d'onde quasi-homogènes par le système des ondes de surface avec tension superficielle. Comme corollaires, nous avons obtenu des effets régularisants locaux et micro-locaux pour les données initiales présentant une décroissance spatiale suffisante.

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Chapitre 1

Introduction

Une équation aux dérivées partielles s'appelle une équation *dispersive* si ses solutions ondes planes avec des fréquences différentes se propagent à des vitesses différentes. Soit $v : \mathbb{R}^d \rightarrow \mathbb{R}^d$ une injection et définissons $\omega(\xi) = \langle v(\xi), \xi \rangle$, où $\langle \cdot, \cdot \rangle$ dénote le produit scalaire dans \mathbb{R}^d , on obtient une équation dispersive

$$(1.0.1) \quad i\partial_t u - \omega(-i\nabla)u = 0,$$

qui admet pour toute fréquence $\xi_0 \in \mathbb{R}^d$ une solution onde plane de vitesse $v(\xi_0)$,

$$u(t, x) = e^{i\langle x, \xi_0 \rangle - it\omega(\xi_0)} = e^{i\langle x - tv(\xi_0), \xi_0 \rangle}.$$

On appelle ω la *relation de dispersion*, et v la *vitesse de phase*. En variant ω , on obtient — au moins les linéarisations de — diverses équations dispersives, e.g.,

Equation	ω	Remarques
(Demi-)onde	$ \xi $	$v(\xi) = \frac{\xi}{ \xi }$ est injectif lorsqu'elle est limitée à \mathbb{S}^{d-1}
Klein–Gordon	$\sqrt{1 + \xi ^2}$	
Schrödinger	$\frac{1}{2} \xi ^2$	
Korteweg–de Vries	ξ^3	$d = 1$
Benjamin–Ono	$\operatorname{sgn}(\xi)\xi^2$	$d = 1$
Ondes de surface sans tension	$\sqrt{g \xi }$	g est la gravité de la Terre
Ondes de surface avec tension	$\sqrt{g \xi + \kappa \xi ^3}$	$\kappa > 0$ est le coefficient de la tension superficielle

Ces équations proviennent de diverses branches de la physique théorique, y compris la mécanique classique, la mécanique quantique, la mécanique des fluides et la relativité générale. Leurs natures et comportements physiques se distinguent selon leurs relations de dispersion et/ou leurs non-linéarités différentes, et selon les cadres géométriques différents dans lesquels elles sont posées. Pourtant, la nature ondulatoire des équations dispersives assure une certaine uniformité dans les méthodes qu'on emploie pour les étudier.

Dans cette thèse, nous nous intéressons aux théories étroitement liées du *contrôle*, de la *stabilisation* et de la *propagation des singularités* pour les équations dispersives. Les résultats principaux proviennent des travaux de l'auteur [203, 204, 205].

[1] Zhu, H., 2016. *Stabilization of damped waves on spheres and Zoll surfaces of revolu-*

tion. ESAIM : Control, Optimisation and Calculus of Variations (ESAIM : COCV), à paraître.

[2] Zhu, H., 2017. *Control of three dimensional water waves*. arXiv preprint arXiv :1712.06130.

[3] Zhu, H., 2018. *Propagation of singularities for gravity-capillary water waves*. arXiv preprint arXiv :1810.09339.

1.1 Contrôle et stabilisation

Soit $\Omega \subset \mathbb{R}^d$ un sous-ensemble ouvert, $T > 0$ et χ_Ω la fonction caractéristique de Ω . Considérons l'équation non homogène

$$(1.1.1) \quad i\partial_t u - \omega(-i\nabla)u = \chi_\Omega F.$$

Définition 1.1.1. On dit que (1.1.1) est *exactement contrôlable* de $[0, T] \times \Omega$ si $\forall (u_0, u_1) \in L^2 \times L^2$, $\exists F \in L^2([0, T], L^2)$ telle que $\exists! u \in C([0, T], L^2)$ qui résout (1.1.1) avec $u(0) = u_0$, $u(T) = u_1$. On dit que (1.1.1) est *contrôlable à zéro* de $[0, T] \times \Omega$ si l'énoncé est vrai $\forall u_0 \in L^2$ et $u_1 = 0$.

Remarque 1.1.2. La contrôlabilité exacte et la contrôlabilité à zéro pour (1.1.1) sont souvent essentiellement équivalentes comme cette équation est réversible en temps. En effet, si on peut montrer que — ce qui est souvent le cas — les équations

$$i\partial_t u \mp \omega(-i\nabla)u = \chi_\Omega F$$

sont tous les deux contrôlables à zéro de $[0, T] \times \Omega$, alors (1.1.1) est exactement contrôlable de $[0, 2T] \times \Omega$. On utilisera le mot *contrôlabilité* quand il n'y a pas d'ambiguïté.

Le problème de la stabilisation concerne l'équation *amortie*,

$$(1.1.2) \quad i\partial_t u - \omega(-i\nabla)u = -i\chi_\Omega u.$$

Les normes- L^2 de ses solutions décroissent de façon monotone, car

$$(1.1.3) \quad \frac{d}{dt} \|u\|_{L^2}^2 = -2 \int_\Omega |u|^2 dx \leq 0.$$

Définition 1.1.3. On dit que χ_Ω *stabilise (faiblement)* (1.1.2) si $\forall u \in C(\mathbb{R}_{\geq 0}, L^2)$ qui résout (1.1.2),

$$\lim_{t \rightarrow +\infty} \|u(t)\|_{L^2} = 0.$$

On dit que χ_Ω *stabilise uniformément (ou fortement)* (1.1.2) si $\exists f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, $\lim_{t \rightarrow +\infty} f(t) = 0$, telle que $\forall u \in C(\mathbb{R}_{\geq 0}, L^2)$ qui résout (1.1.2) et $\forall t \geq 0$,

$$\|u(t)\|_{L^2} \leq f(t) \|u(0)\|_{L^2}.$$

Des problèmes se posent naturellement, e.g. :

Contrôlabilité

- (C.1) *Condition du contrôle.* Quelles conditions pour (Ω, T) sont nécessaires et/ou suffisantes pour la contrôlabilité ?
- (C.2) *Contrôle optimal.* Quel est l'infimum pour le coût du contrôle ? C'est-à-dire, quel est l'infimum de $\|F\|_{L^2([0,T],L^2)}$ parmi tous les F possibles ? Cet infimum peut-il être atteint ?
- (C.3) *Contrôle contraint.* Que faire si F est à valeur réelle/positive ? etc.

Stabilisation

- (S.1) *Condition de la stabilisation.* Quelles conditions pour Ω sont nécessaires et/ou suffisantes pour la stabilisation faible/stabilisation uniforme ?
- (S.2) *Taux de la stabilisation.* À quel taux de décroissance (uniforme) peut-on s'attendre ?

1.1.1 Méthode d'unicité de Hilbert

Nos intérêts principaux portent sur les problèmes (C.1) et (S.1). La réponse affirmative à la contrôlabilité à zéro ou à la stabilisation uniforme est, selon Lions [137, 136], équivalente à l'inégalité de l'observabilité.

Définition 1.1.4. On dit que (1.0.1) est *observable* de $[0, T] \times \Omega$, si $\exists C > 0$, tel que $\forall u \in C([0, T], L^2)$ qui résout (1.0.1),

$$(1.1.4) \quad \|u(0)\|_{L^2} \leq C \|u\|_{L^2([0,T] \times \Omega)}.$$

Théorème 1.1.5 (Lions). *Les énoncés suivants sont vrais.*

- (1) (1.1.1) est contrôlable à zéro de $[0, T] \times \Omega$ si et seulement si (1.0.1) est observable de $[0, T] \times \Omega$.
- (2) χ_Ω stabilise uniformément (1.1.2) si et seulement si (1.0.1) est observable de $[0, T] \times \Omega$ pour un certain $T > 0$.

Pour le premier énoncé, l'idée de Lions s'appelle *la méthode d'unicité de Hilbert (HUM)*, qui établit la dualité entre la contrôlabilité à zéro et l'observabilité, et par conséquent montre leur équivalence. ⁽ⁱ⁾ Plus précisément : notons que la contrôlabilité à zéro est la surjectivité de l'opérateur d'image,

$$R : L^2([0, T], L^2) \ni F \mapsto u(0) \in L^2,$$

où u résout (1.1.1) avec $u(T) = 0$; alors que l'observabilité est la coercivité de l'opérateur de solution,

$$S : L^2 \ni u(0) \mapsto \chi_\Omega u \in L^2([0, T], L^2),$$

où u résout (1.0.1) ⁽ⁱⁱ⁾. Le Théorème 1.1.5 suit de la dualité $R^* = -S$ qui est facile à vérifier, et le corollaire suivant du théorème de l'application ouverte : Soit A une application linéaire et bornée entre deux espaces hilbertiens, alors A est surjective si et seulement si A^* est coercitive.

(i). Dans la théorie du contrôle en dimension finie, cette dualité était connue depuis longtemps par Kalman [112, 113].

(ii). En effet, u devrait résoudre l'équation duale de (1.0.1), qui est ici identique à (1.0.1) car $\omega(-i\nabla)$ et $i\partial_t$ sont symétriques par rapport au produit scalaire \mathbb{C} -bilinéaire de L^2 .

Remarque 1.1.6. La coercivité de S implique son injectivité, c'est-à-dire, un résultat de la *continuation unique* de (1.0.1) : si u résout (1.0.1) et s'annule dans $[0, T] \times \Omega$, alors $u \equiv 0$. Bien qu'en dimension finie, l'injectivité de S est équivalente à la surjectivité de S^* , cela n'est plus vraie en dimension infinie. Voir, e.g., le Chapitre 1 de [136] pour des contre-exemples.

Quant au deuxième énoncé, on peut facilement le montrer en utilisant (1.1.3) et un argument itératif. Voir, e.g., [44].

1.1.2 Condition du contrôle géométrique

On sait qu'un paquet d'onde — une enveloppe d'ondes qui transporte de l'énergie et des informations — se propage à la vitesse de groupe, voir, e.g., [78],

$$v_g = \partial_\xi \omega.$$

Par conséquent, si on souhaite *observer* de $[0, T] \times \Omega$ un paquet d'onde de la fréquence ξ , il semble raisonnable d'exiger que $\forall x \in \mathbb{R}^d, \exists t \in [0, T]$, tel que

$$(1.1.5) \quad x + tv_g(\xi) \in \Omega.$$

On n'a besoin que de (1.1.5) pour ξ grand, c'est-à-dire, pour le *régime des hautes fréquences*, car les paquets d'onde de basse fréquence s'étendent largement et sont plus faciles à observer. Les intuitions physiques nous obligent à suivre la variable de *position* x et la variable de *moment* (c'est-à-dire, la fréquence) ξ en même temps dans l'espace de *phase* $\mathbb{R}^{2d} = \mathbb{R}_x^d \times \mathbb{R}_\xi^d$, voir, e.g., §1.2.1.1. Définissons le flot dans l'espace de phase,

$$\Phi(t, x, \xi) = (x + tv_g(\xi), \xi),$$

alors (1.1.5) est équivalente à une condition pour le flot : $\Phi(t, x, \xi) \in \Omega \times \{\xi\}$.

Considérons un cadre géométrique plus général où les ondes se propagent sur une variété riemannienne (M, g) sans bord. Maintenant l'espace de phase correspond au *fibré cotangent* T^*M , la relation de dispersion ω est une fonction sur T^*M , le flot Φ devient le *flot hamiltonien* de ω ,

$$\partial_t \Phi = X_\omega(\Phi), \quad \Phi|_{t=0} = \text{Id}_{T^*M},$$

où $X_\omega = (\partial_\xi \omega, -\partial_x \omega)$ est le *champ de vecteur hamiltonien* de ω . L'équation (1.0.1) s'écrit maintenant comme :

$$i\partial_t u + \omega(x, D_x)u = 0, \quad D_x = -i\nabla,$$

en utilisant des opérateurs pseudo-différentiels, voir, e.g., §1.2.1.1.

La condition suivante généralise (1.1.5). Nous verrons que c'est une condition naturelle — et dans plusieurs cas *presque équivalente* — pour observer le régime des hautes fréquences des ondes dispersives.

Définition 1.1.7. Soit (M, g) une variété riemannienne sans bord, $\omega \in C^\infty(T^*M)$, et Φ le flot hamiltonien de ω . Soit $\Omega \subset M$ un sous-ensemble ouvert et $T > 0$. On dit que (Ω, T) satisfait la *condition du contrôle géométrique (CCG)* si $\forall (x, \xi) \in T^*M$ avec $g_x^{-1}(\xi, \xi)$ suffisamment grand, $\exists t \in [0, T]$ tel que $\Phi(t, x, \xi) \in T^*\Omega$.

1.1.3 Propriété de la continuation unique

Comme nous l'avons vu, la propriété de la continuation unique n'implique pas la contrôlabilité en dimension infinie. Ce n'est plus un problème dans le régime des basses fréquences. En effet, sur une variété riemannienne compacte, les valeurs propres du laplacien étant discrètes, le régime des basses fréquences est engendré par un nombre fini de fonctions propres du laplacien. Par le théorème de Rellich–Kondrachov, on obtient une compacité supplémentaire. C'est essentiellement l'idée de l'argument *unicité-compacité* de Bardos–Lebeau–Rauch [24]. Cet argument ramène l'observabilité du régime des basses fréquences à la propriété de la continuation unique des fonctions propres de $\omega(x, D_x)$. C'est-à-dire, si u satisfait l'équation,

$$(1.1.6) \quad \omega(x, D_x)u = \lambda u, \quad \lambda \in \mathbb{C},$$

et si $u|_{\Omega} = 0$, a-t-on $u \equiv 0$?

Lorsque M est analytique et que $\omega(x, D_x)$ est un opérateur différentiel à coefficients analytiques, Cauchy [53] et Kovalevskaya [188] ont montré l'unicité de solution parmi les fonctions analytiques, Holmgren [95] a ensuite montré l'unicité parmi les distributions. Le premier effort pour supprimer l'analyticité est dû à Carleman [52] qui a montré l'unicité en supposant que les caractéristiques de l'équation sont simples. La technique qu'il a utilisée, maintenant appelée *l'inégalité de Carleman*, est devenue par la suite un outil efficace pour monter l'unicité lorsque Ω satisfait certaines conditions de la convexité — *la pseudo-convexité*, grâce aux contributions de nombreuses mathématiciens, y compris, Müller [152], de Giorgi [63], Hartman–Whitner [89], Heinz [92], Aronszajn [21], Mizohata [151], Pederson [161], Calderón [49], Hörmander [96, 97], Nirenberg [158], Alinhac [14], Lerner [133], Lerner–Robbiano [134], Zuily [207], Tataru [185], etc.

Pour les applications dans §2 et §3, rappelons deux résultats classiques. Le premier résultat, dû à Aronszajn [21], indique que les équations elliptiques du second ordre ont la propriété de la continuation unique sans aucune condition supplémentaire de Ω .

Théorème 1.1.8 (Aronszajn). *Soit $A \in \text{Lip}_{\text{loc}}(\mathbb{R}^d, M_{d \times d})$ qui est partout strictement définie positive, et $b \in L_{\text{loc}}^{\infty}(\mathbb{R}^d)$. Supposons que $u \in H_{\text{loc}}^1(\mathbb{R}^d)$ résout l'équation*

$$(\nabla \cdot A \nabla + b)u = 0,$$

et s'annule dans un domaine non-vide, alors $u \equiv 0$.

Une conséquence du Théorème 1.1.8 est une estimation très grossière de *ensembles nodaux*⁽ⁱⁱⁱ⁾ des fonctions propres du laplacien : ils n'ont pas de points intérieurs. Des estimations beaucoup plus fines sont connues, voir, e.g., [202] et les références y compris.

Le deuxième résultat concerne le cas où $M = \mathbb{T}^d$ et $\omega(x, \xi) = \omega(\xi)$ est elliptique de degré positif. L'ellipticité de ω implique que toute solution de (1.1.6) n'admet qu'un nombre fini de modes de Fourier et ne peut donc s'annuler dans aucun ouvert non vide. Ce résultat apparemment trivial peut être utile pour traiter des équations avec des relations de dispersion non polynomiales telles que $|\xi|^\alpha$ avec $\alpha > 0$, qui incluent, e.g., l'équation de Schrödinger fractionnaire, le système des ondes de surface avec ou sans tension superficielle.

(iii). Un ensemble nodal d'une fonction propre du laplacien est l'ensemble des points où elle s'annule.

1.1.4 Stabilisation de l'équation des ondes

Sur une variété riemannienne compacte M sans bord, soit $0 \leq \varphi \in L^\infty(M)$ et considérons la stabilisation de l'équation des ondes amorties

$$(1.1.7) \quad \partial_t^2 u - \Delta u + \varphi \partial_t u = 0,$$

qui est bien posé dans $C^1(\mathbb{R}, L^2) \cap C^0(\mathbb{R}, H^1)$. Définissons pour ses solutions l'énergie

$$E(u, t) = \frac{1}{2} \|\nabla u(t)\|_{L^2(M)}^2 + \frac{1}{2} \|\partial_t u(t)\|_{L^2(M)}^2.$$

Il est classique (voir, e.g., [44]) que les conditions suivantes sont équivalentes :

- (i) *La stabilisation uniforme.* Il existe $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ avec $\lim_{t \rightarrow +\infty} f(t) = 0$, telle que $\forall u \in C^1(\mathbb{R}, L^2) \cap C^0(\mathbb{R}, H^1)$ qui résout (1.1.7) et $\forall T \geq 0$,

$$E(u, T) \leq f(T)E(u, 0).$$

- (ii) *L'observabilité.* Il existe $C > 0$ et $T > 0$ tels que $\forall u \in C^1(\mathbb{R}, L^2) \cap C^0(\mathbb{R}, H^1)$ qui résout l'équation des ondes non-amorties, i.e., (1.1.7) sans le terme $\varphi \partial_t u$,

$$(1.1.8) \quad E(u, 0) \leq C \int_0^T \int_M \varphi |\partial_t u|^2 dx dt.$$

De plus, si les conditions ci-dessus sont vérifiées, on peut choisir $f(t) = Ce^{-\alpha t}$ pour certains $C > 0$ et $\alpha > 0$, c'est-à-dire qu'on a une *décroissance exponentielle uniforme de l'énergie*.

1.1.4.1 Amortissement continu

La situation est plus simple lorsque φ est continue : Rauch–Taylor [164] et Bardos–Lebeau–Rauch [24] ont montré que la CCG de $\{\varphi > 0\}$ est équivalente à la stabilisation uniforme. Plus précisément, comme pour l'équation de la demi-onde, le flot hamiltonien de sa relation de dispersion

$$\omega(x, \xi) = \sqrt{g_x^{-1}(\xi, \xi)},$$

est le *flot co-géodésique*. Dire que $\{\varphi > 0\}$ satisfait la CCG revient à dire que *toutes les géodésiques rencontrent $\{\varphi > 0\}$* .

Théorème 1.1.9 (Rauch–Taylor, Bardos–Lebeau–Rauch). *La stabilisation uniforme de (1.1.7) est équivalente à la CCG de $\{\varphi > 0\}$.*

Chaque fois que $\{\varphi > 0\}$ satisfait la CCG, la compacité de M implique l'existence d'un certain $T > 0$, tel que toutes les géodésiques de longueur $\geq T$ rencontrent $\{\varphi > 0\}$. L'observabilité (1.1.8) est vraie avec ce T .

1.1.4.2 Amortissement non-continu

Pour le cas général où $\varphi \in L^\infty$, la situation est plus compliquée, même dans le cas apparemment plus facile où $\varphi = \chi_\Omega$ est la fonction caractéristique d'un certain ensemble ouvert Ω , car les paquets d'onde peuvent se concentrer le long de $\partial\Omega$ quand il contient une géodésique. La preuve du Théorème 1.1.9 implique que :

CCG de $\Omega \Rightarrow$ stabilisation uniforme \Rightarrow CCG d'un voisinage de $\bar{\Omega}$,

et pourtant l'écart entre la première condition et la dernière condition reste un-connu. Seuls quelques exemples intéressants existent.

Théorème 1.1.10 (Lebeau, voir [203] pour une démonstration). *Soit $M = \mathbb{S}^d$ et $\Omega = \mathbb{S}_+^d = \{x_1 > 0\}$, alors χ_Ω stabilise uniformément (1.1.7).*

Dans cet exemple, les géodésiques n'entrant pas \mathbb{S}_+^d sont contenues dans *le plan équatorial* $\{x_1 = 0\}$. Lebeau a montré la stabilisation uniforme en exprimant les solutions sous la forme d'une superposition des *harmoniques sphériques* multipliées par des phases temporelles. Ses observations clés sont premièrement la localisation très précise du spectre du laplacien sphériques, qui implique l'orthogonalité des phases temporelles sur $[0, 2\pi]$; et deuxièmement la symétrie des harmoniques sphériques par rapport au plan équatorial, qui implique que leurs masses sont également distribuées de chaque côté de celui-ci.

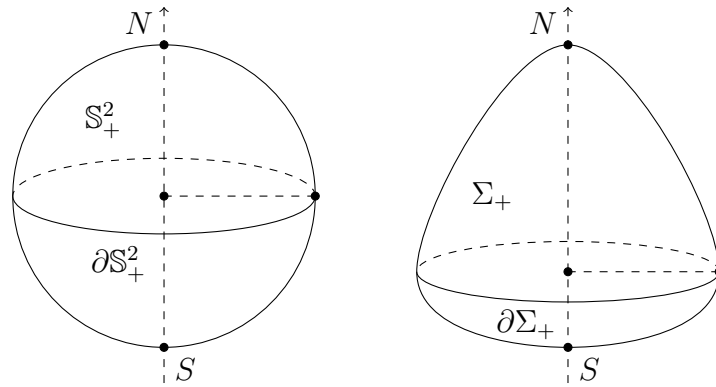


FIGURE 1.1.1 – \mathbb{S}^2 et une surface de révolution de Zoll

Dans [203], nous généralisons cet exemple à certaines surfaces ressemblant à \mathbb{S}^2 : *les surfaces de révolution de Zoll*. Par définition, une variété riemannienne s'appelle une *variété de Zoll* si elle admet un flot géodésique périodique, les sphères étant les exemples les plus simples. Une surface de révolution de Zoll est une variété de Zoll avec deux dimensions qui est en même temps une surface de révolution. La géométrie d'une surface de révolution de Zoll ressemble beaucoup à celle de \mathbb{S}^2 , car on peut définir les pôles nord et sud, les méridiens (les lignes de longitude), les parallèles (les cercles de latitude), et en particulier les héli-surfaces nord et sud, et l'équateur. Voir la Figure 1.1.1 pour une comparaison entre \mathbb{S}^2 et une surface de révolution de Zoll. Voir le livre [29] de Besse pour plus d'informations sur les variétés de Zoll et autres variétés avec des géodésiques fermées, et une caractérisation des surfaces de révolution de Zoll. Plus précisément, le Corollaire 4.16 du [29] a donné une correspondance entre les surfaces de révolution de Zoll et les fonctions impaires $\varphi : [-1, 1] \rightarrow [-1, 1]$ avec $\varphi(1) = \varphi(-1) = 0$, qui implique qu'il y a un nombre indénombrable de surfaces de révolution de Zoll.

Théorème 1.1.11 (Zhu). *Soit Σ une surface de révolution de Zoll, et Σ_+ l'héli-surface nord, alors χ_{Σ_+} stabilise uniformément (1.1.7).*

La démonstration simple de Lebeau ne marche plus sur les surfaces de révolution de Zoll car aucune correspondance d'harmoniques sphériques n'est encore connue sur les

variétés de Zoll. La preuve dans [203] examine de plus près les fonctions propres d'un opérateur laplacien modifié qui se concentrent le long de l'équateur $\partial\Sigma_+$, et montre que les masses de ces fonctions propres sont réparties de manière comparable de chaque côté de l'équateur. Voir Burq–Gérard [45] pour la stabilisation de l'équation des ondes avec un amortissement non-continu sur les tores, et pour la comparaison entre différents cadres géométriques.

1.1.5 Contrôle de l'équation de Schrödinger

Le problème du contrôle de l'équation de Schrödinger

$$(1.1.9) \quad i\partial_t u + \frac{1}{2}\Delta u = \chi_\Omega F$$

est plus compliqué que l'équation des ondes, en raison de sa vitesse infinie de propagation,

$$\lim_{\xi \rightarrow \infty} |v_g(\xi)| = \infty.$$

Sur les variétés avec une géométrie non-captante — toutes les géodésiques s'échappent vers l'infini, typiquement les espaces euclidiens (voir, e.g., la Définition 1.2.6) — le régime des hautes fréquences sera instantanément propagé hors de toute région bornée, ce qui implique un effet régularisant local, comme nous le verrons dans §1.2.2.1. Cela signifie que les géométries globales seront très importantes pour le problème du contrôle sur les variétés avec des géométries captantes, typiquement les variétés compactes.

Par Haraux [88], Jaffard [109] et Komornik [120], on sait que la CCG n'est pas nécessaire pour la contrôlabilité de (1.1.9) sur les tores. Leurs démonstrations sont basées sur la méthode de séries de Fourier, en particulier la théorie des séries lacunaires par Kahané [111]. Pour l'équation de Schrödinger avec potentiel, la contrôlabilité sans CCG a été obtenue par Burq–Zworski [48], Bourgain–Burq–Zworski [36], Anantharaman–Marciá [18], Bourgain [35], etc., à l'aide des techniques micro-locales. Récemment, en utilisant les résultats et les techniques de Dyatlov–Jin [74], Jin [110] a montré la contrôlabilité de (1.1.9) sur des surfaces hyperboliques sans CCG. Les tores et les surfaces hyperboliques sont les seuls exemples connus où la CCG n'est pas nécessaire. Sur les variétés générales, la contrôlabilité nécessite la CCG. En effet, la CCG est nécessaire sur \mathbb{S}^d ou plus généralement les variétés de Zoll, voir, e.g., Macià [140] et Macià–Rivière [141]; la CCG est suffisante sur les variétés générales, par Lebeau [131] via une approche semi-classique.

1.1.5.1 Contrôle de l'équation de Schrödinger fractionnaire

Qu'en est-il des équations de Schrödinger fractionnaires ?

$$(1.1.10) \quad i\partial_t u + |D_x|^\alpha u = \chi_\Omega F, \quad 1 < \alpha < 2.$$

Lorsque $\alpha = 1$, on a l'équation de la demi-onde, pour laquelle la CCG est nécessaire. Quand $\alpha = 2$, on a l'équation de Schrödinger. Au moins sur les tores, la CCG n'est pas nécessaire. Quand $1 < \alpha < 2$, même si on a toujours la vitesse infinie de la propagation, on a encore besoin de la CCG, au moins sur \mathbb{T}^2 , d'après notre démonstration dans [204].

Théorème 1.1.12 (Zhu). *La contrôlabilité de (1.1.10) sur \mathbb{T}^2 implique la CCG.*

1.2 Propagation des singularités

Le mot *singularité* a été largement utilisé pour décrire des états physiques au-delà de la description des lois physiques. En dehors des singularités, le monde semble évoluer de manière régulière, décrite par des équations aux dérivées partielles.

Afin de comprendre complètement le monde, il faut répondre à des questions sur les singularités. Que sont les singularités? Comment décrire les singularités? Comment évoluent les singularités? etc. Des mathématiciens de différentes branches ont essayé de répondre à ces questions depuis des décennies, dans différents contextes et avec différents outils mathématiques.

Dans l'analyse mathématique et la physique, les singularités apparaissent jadis sous formes différentes, e.g., les points discontinus des fonctions, la masse de Dirac (voir Dirac [70]), etc. Ce n'est qu'à partir du développement de *la théorie des distributions* par Sobolev [177] et Schwartz [173] que la définition des singularités — au moins du point de vue de l'analyse — est claire.

Définition 1.2.1. Soit u une distribution sur une variété M , $x \in M$ s'appelle une singularité de u si u n'est pas une fonction C^∞ lorsqu'elle est limitée à un voisinage quelconque de x . L'ensemble de toutes les singularités de u s'appelle le support singulier de u , noté par $\text{sing supp } u$.

1.2.1 Singularité micro-locale et propagation des ondes

Les solutions d'équations différentielles peuvent contenir des singularités. L'étude de la propagation des singularités par des équations différentielles a une longue histoire. Le modèle le plus simple est peut-être l'équation des ondes en dimension un,

$$\partial_t^2 u - c^2 \partial_x^2 u = 0, \quad (u, \partial_t u)|_{t=0} = (f, g),$$

dont la solution est donnée par la formule de d'Alembert [62],

$$u(t, x) = \frac{f(x + ct) + f(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds.$$

Quand $g = 0$, on voit donc que les singularités de f se propagent le long du cône de lumière $\{x = \pm ct\}$, à la vitesse c de lumière. Dans cette direction, pour les équations hyperboliques plus générales, on a les travaux de Lax [129] et de Courant–Lax [56], etc. Cependant, on ne peut toujours pas dire dans quelle direction du cône de lumière — $\{x = ct\}$ ou $\{x = -ct\}$ — se propagent les singularités, basées uniquement sur les informations données par $\text{sing supp } u$. La situation devient considérablement plus compliquée en dimension supérieure car on aura une infinité de directions possibles pour la propagation des singularités.

1.2.1.1 Mécanique quantique et analyse micro-locale

Les intuitions physiques peuvent être utiles pour résoudre ce problème. Selon la mécanique classique, le mouvement d'une particule suit la loi du mouvement de Newton [156]. Sa trajectoire est déterminée une fois que sa position initiale et son moment initial sont donnés. Parallèlement, le principe de Huygens–Fresnel, [104, 80] nous explique

comment le front d'onde d'une onde plane se propage en fonction de son oscillation. Avec le développement de la mécanique quantique, *la dualité* entre les deux descriptions ci-dessus devient plus claire. Pour décrire complètement le comportement d'une entité quantique, on doit la considérer simultanément comme une particule et comme une onde.

Il semble que nous devions utiliser parfois une théorie et parfois l'autre, alors qu'on peut parfois utiliser l'une ou l'autre . . . Nous avons deux images contradictoires de la réalité ; séparément, ni l'un ni l'autre n'expliquent complètement les phénomènes de lumière, mais ensemble, elles le font.

— Albert Einstein

Une théorie réussie qui illustre bien cette dualité onde-particule est l'équation de Schrödinger [170, 171],

$$i\hbar \frac{d}{dt} \psi = \left[-\frac{\hbar^2}{2\mu} \nabla^2 + V \right] \psi,$$

où ψ est *la fonction d'onde* de la particule, μ est la masse de la particule et

$$\hbar = 6.626070150(81) \times 10^{-34} \text{ J} \cdot \text{s}$$

est *la constante de Planck* [163]. Les grandeurs physiques classiques telles que la position et le moment doivent maintenant être exprimées en forme *d'observables quantiques* $\hat{x} = x$ et $\hat{p} = -i\hbar \partial_x$. Lorsque on les mesure, les valeurs moyennes des résultats sont données par les espérances quantiques,

$$(1.2.1) \quad \langle \hat{x} \rangle = (\psi, \hat{x}\psi)_{L^2}, \quad \langle \hat{p} \rangle = (\psi, \hat{p}\psi)_{L^2}.$$

D'après le principe d'incertitude de Heisenberg [93], les écarts-types de \hat{x} et \hat{p} ne peuvent pas être arbitrairement petits en même temps — leur produit admet une borne inférieure strictement positive,

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}.$$

Par conséquent, on ne peut pas mesurer avec précision la position et le moment d'une particule quantique en même temps. Ceci est différent de ce qui se passe pour les particules classique. Mathématiquement, comme \hat{p} est le multiplicateur de Fourier $\hbar\xi$, ce principe d'incertitude énonce simplement le fait que ψ et sa transformation de Fourier $\hat{\psi}$ ne peuvent pas être simultanément suffisamment localisées. En effet, comme on le sait, si ψ est à support compact, alors $\hat{\psi}$ est une fonction analytique, qui ne peut jamais s'annuler dans n'importe quel ouvert non-vide. Dans cet état d'esprit, on trouve naturel de considérer simultanément ψ et sa transformation de Fourier, c'est-à-dire, qu'il faut comprendre comment ψ se comporte dans l'espace de phase $\mathbb{R}_x^d \times \mathbb{R}_\xi^d$. Cela s'appelle *l'analyse micro-locale*.

Les techniques principales de l'analyse micro-locale consistent en diverses méthodes de *la micro-localisation* — la localisation dans l'espace de phase. Le plus courant est peut-être *le calcul pseudo-différentiel* : pour des fonctions appropriées a sur l'espace de phase, qui s'appellent *les symboles*, définissons des opérateurs $\text{Op}(a)$, tels que premièrement, $\text{Op}(x) = x$ et $\text{Op}(\xi) = -i\partial_x$; deuxièmement, $\text{Op}(a)u$ révèle le comportement de u dans le support de a ; troisièmement, la correspondance $a \mapsto \text{Op}(a)$, qui s'appelle une *quantification*, établit une correspondance algébrique ou géométrique entre l'espace de symbole et l'espace d'opérateur. La première de cette quantification a été donnée par Weyl [191],

$$\text{Op}^W(a)u(x) = \iint e^{i\langle x-y, \xi \rangle} a\left(\frac{1}{2}(x+y), \xi\right) u(y) dy d\xi,$$

et la deuxième par Kohn–Nirenberg [121],

$$\text{Op}^{\text{KN}}(a)u(x) = \iint e^{i\langle x-y, \xi \rangle} a(x, \xi)u(y) \, dy \, d\xi.$$

Ces deux méthodes sont essentiellement les mêmes si on ne considère que les termes principaux. Cependant, la correspondance de Weyl envoie des symboles réels à des opérateurs symétriques, alors que la correspondance de Kohn–Nirenberg envoie des polynômes $\sum a_\alpha(x)\xi^\alpha$ aux opérateurs différentiels $\sum a_\alpha(x)D_x^\alpha$, qui la rend souvent plus facile à utiliser dans l’analyse des EDP. Nous n’utiliserons que la correspondance Kohn–Nirenberg dans cette thèse, et notons $a(x, D_x) = \text{Op}^{\text{KN}}(a)$.

1.2.1.2 Front d’onde

Revenons à la discussion sur les singularités. Comme on le sait, le régime des basses fréquences a tendance à être régulier. Les singularités dépendent entièrement du régime des hautes fréquences. Cela coïncide avec l’intuition physique que les particules classiques ont tendance à avoir des fréquences plus hautes et des longueurs d’onde plus courtes. Par conséquent, on peut espérer que les singularités se comportent comme des particules classiques et que leurs directions de propagation soit déterminées par leurs oscillations dans le régime des hautes fréquences.

Dans son célèbre article [98] où la théorie des *opérateurs intégraux de Fourier* a été introduite, Hörmander a donné une définition plus fine des singularités. Il a introduit *le front d’onde* ce qui est un relèvement du support singulier dans l’espace de phase, permettant ainsi de voir comment les singularités oscillent et se propagent.

Définition 1.2.2 (Hörmander). Soit u une distribution, le front d’onde $\text{WF}(u)$ est un sous-ensemble de $\mathbb{R}_x^d \times (\mathbb{R}_\xi^d \setminus 0)$, défini comme suit. On dit que $(x_0, \xi_0) \notin \text{WF}(u)$, si $\exists \varphi \in C_c^\infty(\mathbb{R}^d)$, $\varphi(x_0) \neq 0$, telle que la transformation de Fourier $\widehat{\varphi u}$ décroît rapidement dans un voisinage conique de ξ_0 .

Si $(x_0, \xi_0) \in \text{WF}(u)$, on l’appelle une *singularité micro-locale* de u . On voit que $x_0 \in \text{sing supp } u$ si et seulement si pour un certain $\xi_0 \in \mathbb{R}^d \setminus 0$, $(x_0, \xi_0) \in \text{WF}(u)$. Ce ξ_0 décrit l’oscillation de la singularité micro-locale et sa direction de propagation.

Théorème 1.2.3 (Hörmander). Soit P un opérateur pseudo-différentiel d’ordre n , admettant un symbole principal réel $\omega = \sigma(P)$. Soit Φ le flot hamiltonien de ω . Soit u une solution de l’équation

$$i\partial_t u + Pu = 0,$$

alors $\text{WF}(u)$ est propagé par Φ , c’est-à-dire que $\forall t \in \mathbb{R}$,

$$(x_0, \xi_0) \in \text{WF}(u(0)) \text{ si et seulement si } \Phi(t, x_0, \xi_0) \in \text{WF}(u(t)).$$

En particulier pour l’équation de la demi-onde, c’est-à-dire, lorsque $P = \sqrt{-\Delta}$, les singularités se propagent le long des géodésiques à vitesse 1.

Pour la propagation des singularités pour l’équation des ondes non-linéaire, voir, e.g., Bony [34] et Lebeau [130]. Pour les progrès récents de la propagation des singularités pour l’équation des ondes, voir, e.g., Vasy [187] et Melrose–Vasy–Wunsch [145].

En utilisant le calcul pseudo-différentiel, il y a une définition équivalente du front d’onde, essentiellement dû à Guillemin–Sternberg [87]. Voir aussi Gérard [82], Sjöstrand–Zworski [176], Alexandrova [13], etc.

Théorème 1.2.4. $(x_0, \xi_0) \notin \text{WF}(u)$ si et seulement si $\exists a \in C_c^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)$, tel que $a(x_0, \xi_0) \neq 0$ et $a(x, hD_x)u = \mathcal{O}(h^\infty)_{L^2}$ lorsque $h \rightarrow 0$.

Ce petit paramètre h , appelé le paramètre *semi-classique*, joue le rôle de la constante de Planck. Lorsque $h \rightarrow 0$, on récupère non seulement le comportement du régime des hautes fréquences, mais également la correspondance entre la mécanique quantique et la mécanique classique.

1.2.1.3 Mesures de défaut

Le Théorème 1.2.3 est une justification de la CCG. Une autre manière d'étudier la propagation des singularités et de justifier la CCG utilise les *mesures de défaut*. Inspiré par (1.2.1), on souhaite comprendre comment la quantité $(\psi, a(x, hD_x)\psi)_{L^2}$ se comporte lorsque $h \rightarrow 0$.

Définition 1.2.5. Soit M une variété riemannienne et $\{u_h\}_{0 < h < 1}$ bornée dans $L^2_{\text{loc}}(M)$, alors il existe une suite $h_n \rightarrow 0$ et une mesure de Radon positive sur définie sur T^*M , telle que $\forall a \in C_c^\infty(T^*M)$,

$$\lim_{n \rightarrow \infty} (u_{h_n}, a(x, h_n D_x)u_{h_n})_{L^2(M)} = \langle \mu, a \rangle_{T^*M}.$$

Ce μ s'appelle une *mesure de défaut semi-classique* de $\{u_h\}_{0 < h < 1}$.

Pour les mesures de défaut semi-classiques, voir Gérard [84], Gérard–Leichtnam [86], Lions–Paul [138]. Voir Tartar [184] et Gérard [85] pour les *mesures de défaut microlocales*. Voir aussi [40].

La mesure de défaut semi-classique néglige la masse de taille $o(1)$, ce qui la rend moins précise que le front d'onde qui néglige la masse de taille $O(h^\infty)$. Cependant, elle et la mesure micro-locale sont déjà utiles dans la théorie du contrôle et de la stabilisation, voir, e.g., Lebeau [132], Burq–Gérard [43], Burq–Zworski [48], voir aussi [204].

1.2.2 Effet régularisant et formation des singularités

Le Théorème 1.2.3 n'est plus vrai lorsque P est d'ordre deux ou supérieur. Considérons l'équation de Schrödinger sur une variété riemannienne (M, g) ,

$$(1.2.2) \quad i\partial_t u + \frac{1}{2}\Delta u = 0, \quad u(0) = u_0.$$

Lorsque $(M, g) = (\mathbb{R}^d, \langle \cdot, \cdot \rangle)$, la solution fondamentale de (1.2.2) est

$$(1.2.3) \quad u(t, x) = \frac{e^{-\pi i d/4}}{(2\pi t)^{d/2}} e^{i|x|^2/2t},$$

qui est régulière lorsque $t \neq 0$, bien que $\text{WF}(\delta_0) = \{0\} \times (\mathbb{R}^d \setminus \{0\})$. Cela implique notamment que les solutions avec des données initiales à supports compacts sont régulières dès que $t \neq 0$, ce phénomène s'appelle un *effet régularisant*. A l'inverse, les solutions avec des données initiales régulières peuvent former des singularités, et on l'appelle *la formation des singularités*. La cause de ces deux phénomènes est la vitesse infinie de propagation, qui permet aux oscillations de hautes fréquences de se propager à l'infini ou de revenir de l'infini instantanément.

1.2.2.1 Effet régularisant

L'étude du lien entre les propriétés de localisation des données initiales et la régularité des solutions remonte à Kato [116], qui a montré que pour une équation de KdV généralisée, les données initiales avec une décroissance exponentielle ont des solutions instantanément régulières. Ce travail a été suivi par de nombreux mathématiciens dans des divers contextes dispersifs linéaires ou non-linéaires. Voir, e.g., les références dans [57].

Nous nous concentrons sur l'équation de Schrödinger, lorsque (M, g) est asymptotiquement euclidienne, plus précisément lorsque $M = \mathbb{R}^d$ et

$$(1.2.4) \quad (g_x)_{ij} - \delta_{ij} = \mathcal{O}(|x|^{-1-\epsilon}), \quad \epsilon > 0.$$

Kapitanski–Safarov [115] a d'abord montré l'effet régularisant avec des géométries *non-captantes*. Leur résultat a été généralisé micro-localement par Craig–Kappeler–Strauss [57].

Définition 1.2.6. On dit que $(x_0, \xi_0) \in T^*M$ est non-capté dans le futur (resp. dans le passé) si la co-géodésique $\{(x_t, \xi_t)\}_{t \in \mathbb{R}}$ qui le traverse s'échappe de tout ensemble compact de T^*M lorsque $t \rightarrow \infty$ (resp. $t \rightarrow -\infty$).

Théorème 1.2.7 (Craig–Kappeler–Strauss). *Supposons que g satisfait (1.2.4), et que (x_0, ξ_0) est non-capté dans le futur (resp. dans le passé), alors pour toute donnée initiale u_0 satisfaisant $|x|^k u_0 \in L^2, \forall k \in \mathbb{N}$, on a $\forall t < 0$ (resp. $\forall t > 0$),*

$$(x_0, \xi_0) \notin \text{WF}(u(t)).$$

L'idée du Théorème 1.2.7 est simple : les singularités se disperseront le long de géodésiques non-captées, à condition qu'il n'y ait pas de masse à l'infini pour compenser cette dispersion. L'hypothèse que la géométrie soit non-captante est nécessaire, voir, e.g., Doi [71, 72]. Pour d'autres résultats dans cette direction, voir Robbiano–Zuily [165] pour l'effet régularisant analytique micro-local avec des géométries non-captantes ; voir Burq [42] pour l'effet régularisant pour les problèmes de Dirichlet à l'extérieur ; voir Szeftel [181, 182] pour des effets régularisants microlocaux pour des équations de Schrödinger *semi-linéaires*, et les réflexions des singularités ; etc.

1.2.2.2 Où vont les singularités ?

L'équation de Schrödinger est réversible en temps, ainsi les singularités peuvent disparaître et émerger, comme nous l'avons vu. On est donc amenés à croire que les singularités sont plutôt transformées en quelque chose de plus traçable plutôt que de simplement disparaître dans le vide. Des travaux antérieurs, notamment Lascar [125, 126] et Boutet-de-Monvel [37] ont montré que les singularités se propagent à une vitesse infinie le long des géodésiques, mais n'ont pas décrit en fonction du temps la manière dont les singularités se propagent.

Deux décennies plus tard, en observant que la masse de Dirac est transformée par le flot de Schrödinger, lorsque $t \neq 0$, en une oscillation quadratique $e^{i|x|^2/2t}$ à l'infini, comme indiqué par (1.2.3), Wunsch [195] a défini *le front d'onde de scattering quadratique* $\text{WF}_{\text{qsc}}(u)$ dans un cadre géométrique plus général : les variétés riemanniennes équipées d'une *métrique de scattering*, qui lui permettaient de relier les singularités et les oscillations quadratiques. Indépendamment, Nakamura [153] a introduit le front d'onde homogène et a montré des résultats similaires avec des géométries asymptotiques euclidiennes.

Définition 1.2.8. Soit u une distribution tempérée, le front d'onde homogène $\text{HWF}(u)$ est un sous-ensemble de $\mathbb{R}_x^d \times \mathbb{R}_\xi^d$ défini comme suit. On dit que $(x_0, \xi_0) \notin \text{HWF}(u)$ si $\exists \varphi \in C_c^\infty(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$ telle que $\varphi(x_0, \xi_0) \neq 0$ et $\varphi(hx, hD_x)u = \mathcal{O}(h^\infty)_{L^2}$ lorsque $h \rightarrow 0$.

Théorème 1.2.9 (Nakamura). *Supposons que g satisfait (1.2.4), et soit u une solution de (1.2.2).*

- (1) *Soit $(x_0, \xi_0) \in \text{HWF}(u_0)$, $t_0 > 0$ (resp. $t_0 < 0$). Supposons que $\forall t \in [0, t_0]$ (resp. $\forall t \in [t_0, 0]$), $x_0 + t\xi_0 \neq 0$. Alors $(x_0 + t_0\xi_0, \xi_0) \in \text{HWF}(u(t_0))$.*
- (2) *Soit $(x_0, \xi_0) \in \text{WF}(u_0)$ non-capté dans le futur (resp. dans le passé), et ξ_+ (resp. ξ_-) la direction asymptotique, lorsque $t \rightarrow +\infty$ (resp. $t \rightarrow -\infty$), de la co-géodésique $\{(x_t, \xi_t)\}_{t \in \mathbb{R}}$ émise à partir de (x_0, ξ_0) , c'est-à-dire, $\lim_{t \rightarrow +\infty} \xi_t = \xi_+$ (resp. $\lim_{t \rightarrow -\infty} \xi_t = \xi_-$). Alors, $\forall t_0 > 0$ (resp. $t_0 < 0$), $(t_0\xi_+, \xi_+) \in \text{HWF}(u(t_0))$ (resp. $(t_0\xi_-, \xi_-) \in \text{HWF}(u(t_0))$).*

En comparant avec le Théorème 1.2.4, la définition de $\text{HWF}(u)$ est analogue à celle de $\text{WF}(u)$. La seule différence est la quantification $\text{Op}(a) \mapsto a(hx, hD_x)$, qui vise à extraire des informations de u à l'infini lorsque $x_0 \neq 0$ et $h \rightarrow 0$. L'intuition d'une telle quantification est double. Premièrement, on observe que pour l'équation de Schrödinger, la vitesse de groupe est $v_g = \xi$. Ainsi, un paquet d'onde de fréquence $|\xi| \sim h^{-1}$ se propage à la vitesse de groupe $|v_g| \sim h^{-1}$. Cette quantification homogène nous permet de suivre la propagation des paquets d'onde par le flot de Schrödinger. Deuxièmement, $a(hx, hD_x)$ peut également être vu comme une micro-localisation dans la région d'espace de phase où $|x| \sim h^{-1}$ et $|\xi| \sim h^{-1}$, c'est-à-dire, où la taille de x et la violence de l'oscillation sont comparables, ou en d'autres termes, la région d'oscillation quadratique. Ainsi, on peut s'attendre — au moins dans les espaces euclidiens — à ce que $\text{WF}_{\text{qsc}}(u)$ et $\text{HWF}(u)$ soient essentiellement équivalents. Cela a été montré par Ito [108].

Le premier énoncé du Théorème 1.2.9 étudie la propagation à l'infini. La quantification homogène nous permet de suivre la vitesse infinie de propagation et d'obtenir un résultat de propagation analogue au résultat classique, i.e., le Théorème 1.2.3. Le deuxième énoncé est plus fin que le Théorème 1.2.7, car il exige seulement qu'il n'y ait pas de masse dans un voisinage conique de la co-géodésique.

Il convient également de noter qu'une autre contrepartie essentiellement équivalente au front d'onde de scattering quadratique et au front d'onde homogène a été définie par Hörmander [102], où une définition similaire à celle de Nakamura a été donnée. Il a également donné, en suivant Sjöstrand [175], une caractérisation utilisant la transformation de *Fourier–Bros–Iagolnitzer (FBI)* qui lui a permis d'étudier les singularités analytiques. Dans cette direction, les travaux de Hörmander ont été complétés et commentés par Rodino–Wahlberg [166] et Schulz–Wahlberg [172], où une preuve de l'équivalence au front d'onde homogène a été fournie. Ils ont également donné le nom *le front d'onde de Gabor* à la définition de Hörmander, ce qui implique le lien étroit entre différents sujets : la transformation de Gabor, la transformation de Segal–Bargmann, la transformation en paquets d'onde, la transformation de FBI, les opérateurs intégraux de Fourier, etc. Voir Gabor [81], Bargmann [25], Segal [174], Hörmander [98], Bros–Iagolnitzer [39], Sato–Kawai–Kashiwara [169], Córdoba–Fefferman [55], voir aussi Folland [79].

1.2.2.3 Front d'onde quasi-homogène

Peut-on suivre l'échappement des singularités pour d'autres équations dispersives avec une vitesse infinie de propagation ? Dans [205], nous avons introduit *le front d'onde quasi-*

homogène qui généralise le front d'onde de Hörmander et le front d'onde homogènes de Nakamura, et étend les résultats de propagation de Wunsch [195] et de Nakamura [153] — au moins dans le cas de coefficients constants — à certaines équations du type de Schrödinger,

$$(1.2.5) \quad i\partial_t u + |D_x|^\gamma u = 0, \quad \gamma \geq 1,$$

qui incluent notamment l'équation de la demi-onde, les équations de Schrödinger et Schrödinger fractionnaires, (approximativement dans le régime des hautes fréquences) la linéarisation du système des ondes de surface avec tension superficielle.

Définition 1.2.10. Soit u une distribution tempérée, $\delta \geq 0$, $\rho \geq 0$ avec $\delta + \rho > 0$ et $\mu \in \mathbb{R} \cup \{\infty\}$. Le front d'onde quasi-homogène $\text{WF}_{\delta,\rho}^\mu(u)$ est un sous-ensemble de $\mathbb{R}_x^d \times \mathbb{R}_\xi^d$ défini comme suit. On dit que $(x_0, \xi_0) \notin \text{WF}_{\delta,\rho}^\mu(u)$ si $\exists a \in C_c^\infty(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$ avec $a(x_0, \xi_0) \neq 0$, telle que $a(h^\delta x, h^\rho D_x)u = \mathcal{O}(h^\mu)_{L^2}$ lorsque $h \rightarrow 0$.

Théorème 1.2.11 (Zhu). *Soit u une solution de (1.2.5) avec la donnée initiale u_0 .*

(1) *Si $\rho\gamma = \delta + \rho$, alors $\forall t_0 \in \mathbb{R}$,*

$$(x_0, \xi_0) \in \text{WF}_{\delta,\rho}^\mu(u_0) \setminus \{\xi = 0\} \iff (x_0 + t_0\gamma|\xi_0|^{\gamma-2}\xi_0, \xi_0) \in \text{WF}_{\delta,\rho}^\mu(u(t_0)) \setminus \{\xi = 0\}.$$

(2) *Si $\gamma > 1$, $\rho\gamma > \delta + \rho$, $t_0 \neq 0$, $(x_0, \xi_0) \in \text{WF}_{\delta,\rho}^\mu(u_0) \setminus \{\xi = 0\}$, alors*

$$(t_0\gamma|\xi_0|^{\gamma-2}\xi_0, \xi_0) \in \text{WF}_{\rho(\gamma-1),\rho}^\mu(u(t_0)) \setminus \{\xi = 0\}.$$

Les deux énoncés ci-dessus étendent respectivement les deux énoncés du Théorème 1.2.9, comme

$$\text{WF}_{0,1}^\infty(u) \cap \{\xi \neq 0\} = \text{WF}(u), \quad \text{WF}_{1,1}^\infty(u) = \text{HWF}(u).$$

De plus, on vérifie que $\text{WF}_{1,0}^\infty(u)$ est essentiellement *le front d'onde de scattering* $\text{WF}_{\text{sc}}(u)$ de Melrose [143]. En effet, dans \mathbb{R}^d , $(x, \xi) \in \text{WF}_{\text{sc}}(u)$ si et seulement si $(x, -\xi) \in \text{WF}_{1,0}^\infty(u)$.

On doit distinguer le front d'onde quasi-homogène et le front d'onde introduit par Lascar [125], appelé aussi parfois le front d'onde quasi-homogène. La quasi-homogénéité de la Définition 1.2.10 concerne à la fois les variables x et ξ , tandis que la quasi-homogénéité de Lascar ne concerne que la variable ξ . Nous avons décidé de garder le nom “front d'onde quasi-homogène”, car il ne crée aucune ambiguïté et correspond bien aux définitions de Hörmander et de Nakamura.

1.2.2.4 Formation des singularités

Comment la masse à l'infini forme-t-elle des singularités ? Pour l'équation de Schrödinger, par le Théorème 1.2.9, on peut au moins dire que : étant donné $(x_0, \xi_0) \in \text{HWF}(u(-t_0))$, elle peut éventuellement former des singularités micro-locales en temps $t = 0$ seulement si $x_0 + t_0\xi_0 = 0$. Toutes les singularités micro-locales formées de cette manière doivent osciller dans la direction ξ_0 .

Cependant, ni le front d'onde de scattering quadratique ni le front d'onde homogène ne sont en mesure de déterminer les positions de ces singularités, car des singularités différentes ayant la même direction d'oscillation forment la même oscillation quadratique. Les informations sur les positions peuvent se trouver dans le front d'onde de scattering après avoir annulé des oscillations quadratiques, comme l'a montré par Hassell–Wunsch [91], voir aussi [90]. Leur résultat a été énoncé sur les variétés riemanniennes

équipées d'une métrique de scattering. Nous ne l'énoncerons que pour l'équation de Schrödinger pour des particules libres dans les espaces euclidiens.

Théorème 1.2.12 (Hassell–Wunsch). *Si u résout (1.2.2) où g est la métrique euclidienne, alors $\forall t \neq 0$,*

$$(x_0, \xi_0) \in \text{WF}(u_0) \text{ si et seulement si } (\xi_0, x_0/t) \in \text{WF}_{\text{sc}}(u(t)e^{-i|x|^2/2t}).$$

Il existe d'autres caractérisations quand on a des géométries asymptotiquement euclidiennes, qui utilisent des outils de la théorie du scattering, voir, e.g., Nakamura [154], ou la transformation en paquets d'onde, voir, e.g., Kato–Kobayashi–Ito [117, 118]. Ces derniers utilisent la caractérisation du front d'onde par la transformation en paquets d'onde, due à Folland [79] et Ōkaji [159], voir aussi Gérard [83].

Un autre exemple important est l'oscillateur harmonique quantique, c'est l'équation de Schrödinger avec un potentiel quadratique. Dans ce cas, les singularités réapparaissent de façon périodique, d'après Zelditch [201]. Voir aussi Weinstein [190], Wunsch [196], Mao–Nakamura [142], etc.

1.3 Ondes de surface

Nous n'avons jusqu'à présent envisagé que des équations dispersives linéaires. Il est intéressant de se demander comment étudier les théories du contrôle, de la stabilisation et de la propagation des singularités pour des équations non-linéaires. Des travaux ont été effectués pour des équations semi-linéaires, pour n'en nommer que quelques-uns : pour des équations d'onde semi-linéaires, voir Bony [34], Lebeau [130] et Dehman–Lebeau–Zuazua [65]; pour des équations de Schrödinger semi-linéaires, voir Dehman–Gérard–Lebeau [66] et Szeftel [181]; pour l'équation de Benjamin–Ono, voir Linares–Rosier [135] et Laurent–Linares–Rosier [128]; pour l'équation de KdV, voir Rosier [167] et Laurent–Rosier–Zhang [127]; etc.

Cependant, pour les équations dispersives quasi-linéaires, les résultats restent peu nombreux. Nous nous concentrerons sur le système des ondes de surface avec tension superficielle et les travaux de Alazard–Baldi–Han–Kwan [5] et Zhu [204, 205].

Le système des ondes de surface avec tension superficielle décrit l'évolution du fluide *non visqueux, incompressible et irrotationnel* avec une *surface libre*, en présence d'un *champ gravitationnel* et de *tension superficielle*. En réalité, il modélise, e.g., un réservoir d'eau sur la Terre. En raison de son importance évidente, l'étude des ondes de surface a une histoire extrêmement longue, qui remonte à Newton, Euler, Laplace, Lagrange, Bernoulli et d'autres grands mathématiciens. On peut se référer à la revue historique [61] due à Craik pour les progrès de la théorie des ondes de surface avant le milieu du XIXe siècle. Voir le livre [122] de Lamb pour une introduction à l'hydrodynamique classique, et le livre [124] de Lannes pour les progrès modernes de la théorie des ondes de surface.

1.3.1 Formulations de système des ondes de surface

Il existe différentes formulations mathématiques du système des ondes de surface, voir, e.g., [124], mais nous nous concentrerons sur la formulation eulérienne, voir, e.g., [122], et la formulation de Zakharov [199] / Craig–Sulem [60].

1.3.1.1 Formulation eulérienne

Soit Ω le domaine occupé par le fluide, Σ la *surface libre* et Γ le *fond*. Voir la Figure 1.3.1. À cause du mouvement du fluide, Ω et Σ dépendent du temps. La *formulation eulérienne* décrit le mouvement du fluide par l'équation d'Euler incompressible dans Ω avec des conditions aux bords Σ et Γ .

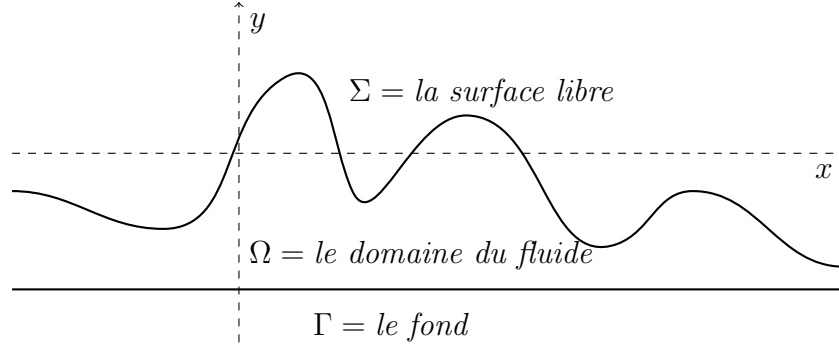


FIGURE 1.3.1 – Ondes de surface

Plus précisément, mettons le domaine du fluide dans $\mathbb{R}^{d+1} = \mathbb{R}_x^d \times \mathbb{R}_y$, où x est la coordonnée horizontale et y est la coordonnée verticale. Supposons que la surface libre soit le graphe d'une fonction dépendante du temps $\eta : \mathbb{R}_x^d \rightarrow \mathbb{R}_y$, alors, $\Sigma = \{y = \eta(t, x)\}$; tandis que le fond est plat et de profondeur $b < \infty$, c'est-à-dire, $\Gamma = \{y = -b\}$. De plus, nous nous limiterons à la *région profonde du fluide*, c'est-à-dire, où il'y existe une distance positive entre la surface libre et le fond,

$$\text{dist}(\Sigma, \Gamma) > 0.$$

Par conséquent, le domaine du fluide est simplement $\Omega = \{-b < y < \eta(t, x)\}$.

Le mouvement du fluide peut être décrit par l'évolution de Σ et l'hydrodynamique dans Ω , c'est-à-dire, l'évolution du champ de vecteur *eulérien* $v : \Omega \rightarrow \mathbb{R}^d$. L'interaction entre η et v nécessite la *pression* $P : \Omega \rightarrow \mathbb{R}$. Soit $g \in \mathbb{R}$ la *gravité de la Terre*, $\mathbf{e}_y = (0, \dots, 0, 1)$ la direction (inverse) de la gravité, $\mathbf{n} : \partial\Omega \rightarrow \mathbb{S}^d$ le champ de vecteur normal vers l'extérieur de $\partial\Omega$, $\kappa > 0$ le *coefficient de la tension superficielle* et

$$H(\eta) = \nabla \cdot \left(\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right)$$

le courbure moyenne de Σ , le système des ondes de surface avec tension superficielle peut se formuler comme suit.

$$(1.3.1) \quad \begin{cases} \partial_t v + v \cdot \nabla_{xy} v + \nabla_{xy} P = -g \mathbf{e}_y & \text{l'équation d'Euler} \\ \nabla_{xy} \cdot v = 0 & \text{l'incompressibilité} \\ \nabla_{xy} \times v = 0 & \text{l'irrotationalité} \\ \partial_t \eta = \sqrt{1 + |\nabla \eta|^2} v|_{\Sigma} \cdot \mathbf{n} & \text{la condition cinétique} \\ -P|_{\Sigma} = \kappa H(\eta) & \text{la condition dynamique} \\ v|_{\Gamma} \cdot \mathbf{n} = 0 & \text{la condition au fond} \end{cases}$$

La condition cinétique implique que les particules du fluide se trouvant initialement sur Σ resteront sur Σ ; la condition dynamique indique l'équilibre entre la pression et la *tension superficielle* $\kappa H(\eta)$; la condition du fond implique que le fond est impénétrable.

Remarque 1.3.1. Quand $g = 0$ et $\kappa > 0$, (1.3.1) modélise les ondes de capillarité, qui sont les ondes de petite longueur d’onde dont la dynamique est dominée par la tension superficielle. Lorsque $\kappa = 0$ et $g > 0$, (1.3.1) modélise les ondes de grande longueur d’onde, y compris les tsunamis et les marées océaniques, dont la dynamique est dominée par la gravité de la Terre. Voir la Figure 1.3.1 qui provient du livre *Hamonshū* (le livre des formes des ondes), contenant des illustrations artistiques japonaises des ondes de surface — de petite longueur d’onde et de grande longueur d’onde. Quand $\kappa = 0$, on a besoin que

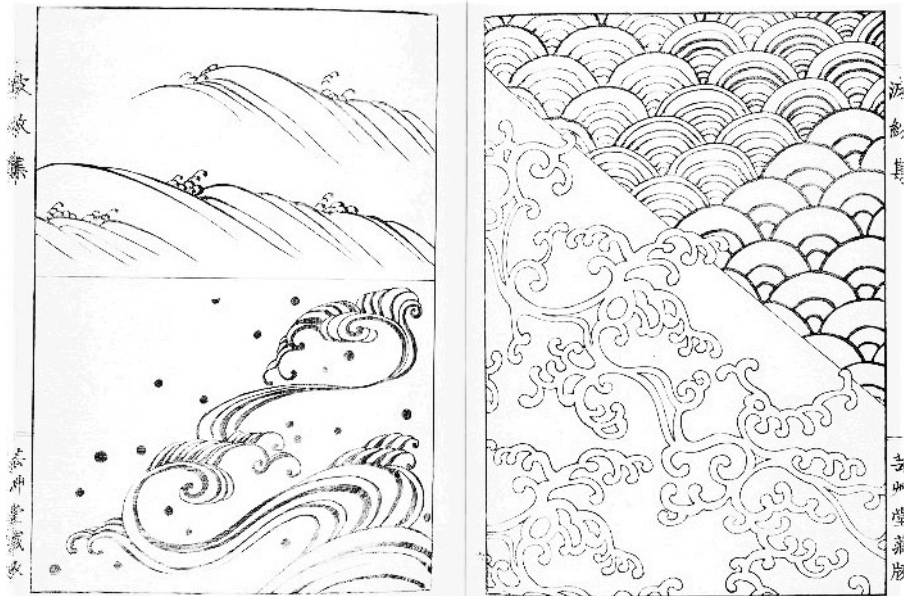


FIGURE 1.3.2 – Mori Yūzan, 1903, *Hamonshū*, Yamada Geisōdō, Kyōto-shi

$g > 0$ pour maintenir ensemble le fluide. En effet, dans ce cas, Wu [192, 193] a montré la condition de signe de Taylor,

$$-\inf_{\Sigma} \partial_{\mathbf{n}} P > 0,$$

qui est ce dont on a besoin pour que le système des ondes de surface soit bien posé. Cependant, en présence de la tension superficielle, la condition de signe de Taylor est sans importance, car la tension superficielle elle-même est capable de maintenir ensemble le fluide, ce qui coïncide avec notre intuition physique. Naturellement, comme l’ont montré Ambrose–Masmoudi [16, 17], le système des ondes de surface sans tension superficielle est la limite, en tant que $\kappa \rightarrow 0$, du système des ondes de surface avec tension superficielle.

1.3.1.2 Formulation de Zakharov / Craig–Sulem

L’une des difficultés principales dans l’étude de (1.3.1) est la dépendance temporelle du domaine Ω . D’après Zakharov et Craig–Sulem, on peut reformuler (1.3.1) en un système sur \mathbb{R}^d . Le prix à payer, cependant, est que le système qu’on obtient sera un système *pseudo-différentiel*, comme nous le verrons dans §1.3.3.

Par l’incompressibilité et l’irrationalité de v et par l’annulation de la vitesse verticale au fond, on peut trouver un potentiel $\phi : \Omega \rightarrow \mathbb{R}$ de v , c’est-à-dire, $\nabla_{xy}\phi = v$, qui satisfait

$$\Delta_{xy}\phi = 0, \quad \partial_y\phi|_{\Gamma} = 0.$$

Soit $\psi(t, x) = \phi(t, x, \eta(t, x))$, et définissons

$$G(\eta)\psi = \sqrt{1 + |\nabla\eta|^2}v|_{\Sigma} \cdot \mathbf{n} = \sqrt{1 + |\nabla\eta|^2}\partial_{\mathbf{n}}\phi|_{\Sigma}.$$

Cet opérateur $G(\eta)$ s'appelle l'opérateur de *Dirichlet–Neumann* car il envoie la condition de Dirichlet de ϕ sur la condition de Neumann de ϕ , à un facteur multiplicatif près. Enfin, en laissant $\kappa = 1$, on peut reformuler (1.3.1) en termes de (η, ψ) , qui s'appelle la formulation de Zakharov / Craig–Sulem.

$$(1.3.2) \quad \begin{cases} \partial_t \eta - G(\eta)\psi = 0, \\ \partial_t \psi + g\eta - H(\eta) + \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2} \frac{(\nabla\eta \cdot \nabla\psi + G(\eta)\psi)^2}{1 + |\nabla\eta|^2} = 0. \end{cases}$$

1.3.2 Théorie de Cauchy

La littérature sur la théorie de Cauchy du système (1.3.1) est énorme. Pour en nommer quelques-uns, Nalimov [155], Kano–Nishida [114] et Yosihara [197, 198] ont initié l'étude de la théorie de Cauchy. Dans le cadre des espaces de Sobolev, sans hypothèse de petitesse des données initiales, Wu [192, 193] a montré que le système sans tension superficielle est bien posé ; Beyer–Günther [30] a montré que le système avec tension superficielle est également bien posé. Lannes [123] a étudié le système en utilisant la formulation d'Euler. Alazard–Métivier [12] ont introduit des techniques para-différentielles dans l'étude, suivi par Alazard–Burq–Zuily [6, 8, 9] qui ont pu établir que le système — avec ou sans tension superficielle — est bien posé, avec des régularités de Sobolev faibles. Voir aussi Rousset–Tzvetkov [168] et Ming–Rousset–Tzvetkov [150] pour les ondes de surface solitaires.

Pour les progrès sur les solutions globales et les solutions aux régularités faibles, voir, e.g., Alazard–Delort [10], Berti–Delort [28], Ifrim–Tataru [105], Ionescu–Pusateri [107], Wang [189], de Poyferré–Nguyen [64], Deng–Ionescu–Pausader–Pusateri [68], Wu [194], etc. Alazard–Burq–Zuily [9] ont montré que le système sans tension superficielle est bien posée dans les espaces de Sobolev uniformément locaux, suivis par Nguyen [157], qui a obtenu des estimations dans des espaces de Sobolev pondérés uniformément locaux. Nous avons montré dans [205] l'existence de solutions dans des espaces de Sobolev pondérés, pour le système avec tension superficielle.

Définition 1.3.2. Soit $(\mu, m) \in \mathbb{R} \times \mathbb{N}$, on dit que $u \in \mathcal{H}_m^\mu$ si $\forall j \in [0, m] \cap \mathbb{N}$,

$$\langle x \rangle^j \langle D_x \rangle^{\mu-j/2} u \in L^2, \quad \langle \cdot \rangle = \sqrt{1 + |\cdot|^2}.$$

Théorème 1.3.3 (Zhu). *Soit $\mu > 3 + d/2$, $m \leq 2\mu - 6 - d$, $(\eta_0, \psi_0) \in \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu$, alors $\exists T > 0$, $\exists !(\eta, \psi) \in C([-T, T], \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu)$ qui résout (1.3.2) avec les données initiales (η_0, ψ_0) .*

Lorsque $m > 0$, on obtient des ondes de surface asymptotiquement plates, pour lesquelles les géométries des surfaces libres sont asymptotiquement euclidiennes.

1.3.3 Analyse micro-locales des ondes de surface

D'après Calderón [50, 51], au moins quand $\eta \in C^\infty$, $G(\eta)$ est un opérateur pseudo-différentiel d'ordre un, c'est-à-dire, $G(\eta)\psi = \text{Op}(\lambda)\psi$. Le symbole λ admet le développement

asymptotique

$$(1.3.3) \quad \lambda \sim \lambda^{(1)} + \lambda^{(0)} + \lambda^{(-1)} + \dots$$

où le symbole principal est donné par

$$\lambda^{(1)}(x, \xi) = \sqrt{(1 + |\nabla\eta(x)|^2)|\xi|^2 - (\nabla\eta(x) \cdot \xi)^2}.$$

Les termes de reste peuvent se calculer par récurrence, voir, e.g., [19]. Lorsque η a une régularité limitée, il existe différentes méthodes pour obtenir des estimations et la structure de $G(\eta)$, et par conséquent la structure du système des ondes de surface, voir, e.g., [59, 75, 123, 58, 30, 192, 193, 6, 8], parmi lesquels l'approche para-différentielle est efficace et robuste.

1.3.3.1 Calcul para-différentiel

L'approche para-différentielle est basée sur *le calcul para-différentiel* introduit par Bony [33], voir aussi Meyer [148] et Métivier [146]. Le but initial du calcul para-différentiel était d'étudier la propagation des singularités pour les équations non-linéaires, voir, e.g., [33, 34]. Une des limitations du calcul pseudo-différentiel était la régularité des symboles, ce qui a, pour l'étude des EDP non-linéaires, motivé l'introduction du calcul para-différentiel. En effet, étant donné une fonction a , pour que l'application $u \mapsto au$ soit bornée dans H^m , on a besoin que $a \in W^{m, \infty}$. Cependant, la composition de deux opérateurs différentiels, ou l'adjoint d'un opérateur différentiel a généralement des coefficients moins réguliers.

Pour résoudre ce problème, Bony a défini *le para-produit* $T_a u$, en ne gardant que l'interaction de "basse fréquence-haute fréquence" entre a et u . Plus précisément, par la décomposition de Littlewood–Paley $\text{Id} = \sum_{p \geq 0} \Delta_p$, $T_a u$ peut être définie essentiellement comme

$$T_a u = \sum_{p \geq 0} \sum_{q \geq p+N} \Delta_p a \Delta_q u,$$

pour un certain $N \in \mathbb{N}$ suffisamment grand. De cette façon, l'oscillation de $\Delta_p a$ est comparativement insignifiante par rapport à celle de $\Delta_q u$, ce qui rend la norme- L^∞ de a suffisante dans l'estimation de $T_a u$: si $a \in L^\infty$, alors $\forall s \in \mathbb{R}$,

$$\|T_a\|_{H^s \rightarrow H^s} \lesssim \|a\|_{L^\infty}.$$

Par symétrie, le produit au admet la décomposition $au = T_a u + T_u a + R(a, u)$. Si $a \in H^\alpha$, $u \in H^\beta$, avec $\alpha, \beta > d/2$, alors

$$\|R(a, u)\|_{H^{\alpha+\beta-d/2}} \lesssim \|a\|_{H^\alpha} \|u\|_{H^\beta}.$$

La définition du para-produit s'étend naturellement aux *opérateurs para-différentiels* pour les symboles $a(x, \xi)$ avec des régularités limitées. Ces opérateurs bénéficient d'un calcul symbolique suffisamment bon.

La *para-linéarisation* est un autre outil puissant pour traiter les non-linéarités.

Théorème 1.3.4 (Bony). *Soit $f \in C^\infty(\mathbb{R})$, $f(0) = 0$ et $u \in H^s$, $s > d/2$, alors*

$$f(u) - T_{f'(u)} u \in H^{2s-d/2}.$$

Le terme $T_{f'(u)}u$ peut être vu comme une linéarisation tangentielle de $f(u)$, ce qui réduit les estimations de $f(u)$ à l'estimation- L^∞ de $f'(u)$. Pour une généralisation d'ordre supérieur du Théorème 1.3.4, voir Bailleul–Bernicot [23].

Outre son objectif initial pour la propagation des singularités, le calcul para-différentiel s'avère également extrêmement utile pour la théorie de Cauchy des équations non-linéaires, voir, e.g., Métivier [146], Delort [67], Bienaimé [31], Bienaimé–Boulkhemair [32], Feola–Iandoli [77], Stingo [180].

De plus, avec des modifications légères, le calcul para-différentiel peut être étendu aux espaces de Sobolev pondérés, voir Zhu [205]. Pour le calcul para-différentiel semi-classique, voir Métivier–Zumbrun [147] et [205].

1.3.3.2 Para-linéarisation du système des ondes de surface

Dans l'étude des ondes de surface, le calcul para-différentiel a été utilisé pour la première fois par Alazard–Métivier [12], et encore accompli par Alazard–Burq–Zuily [6, 8]. Notons en particulier le résultat suivant d'une para-linéarisation de l'opérateur Dirichlet–Neumann.

Proposition 1.3.5 (Alazard–Métivier). *Soit $(\eta, \psi) \in H^{\mu+1/2} \times H^\mu$, avec $\mu > 3 + d/2$. Soit $\lambda = \lambda^{(1)} + \lambda^{(0)} + \dots + \lambda^{[s-2-d/2]}$ défini par (1.3.3), alors*

$$G(\eta)\psi = T_\lambda(\psi - T_B\eta) - \nabla \cdot T_V\eta + R(\eta, \psi),$$

où

$$B = \frac{\nabla\eta \cdot \nabla\psi + G(\eta)\psi}{1 + |\nabla\eta|^2}, \quad V = \nabla\psi - B\nabla\eta,$$

sont respectivement la trace de la vitesse verticale $\partial_y\phi$ et la trace de la vitesse horizontale $\nabla\phi$ à la surface libre Σ , et $R(\eta, \psi) \in H^{2\mu-3-d/2}$.

La para-linéarisation du système entier et une symétrisation révèlent la structure du système des ondes de surface. Soit $(\eta, \psi) \in H^{s+1/2} \times H^s$ une solution de (1.3.2), alors on peut définir une nouvelle variable

$$u = T_p\eta - iT_q(\psi - T_B\eta)$$

avec des opérateurs para-différentiels T_p et T_q bien choisis, respectivement des ordres 1/2 et 0, tels que u satisfait l'équation para-différentielle

$$(1.3.4) \quad \partial_t u + iT_\gamma u + T_V \cdot \nabla u + \text{termes de reste} = 0,$$

où T_γ est d'ordre 3/2, avec le symbole principal

$$\gamma^{(3/2)}(x, \xi) = \left(|\xi|^2 - \frac{(\nabla\eta(x) \cdot \xi)^2}{1 + |\nabla\eta(x)|^2} \right)^{3/4}.$$

Observant que $\gamma^{(3/2)} = [(g_\Sigma)_x^{-1}(\xi, \xi)]^{3/4}$, où g_Σ est la métrique riemannienne de l'hypersurface Σ héritée de \mathbb{R}^{d+1} , le système des ondes de surface avec tension superficielle peut donc être considéré comme une équation de Schrödinger fractionnaire sur Σ . Cette équation est quasi-linéaire, car la géométrie de Σ dépend de η .

1.3.4 Contrôle et stabilisation

Le contrôle et la stabilisation des ondes de surface ont un intérêt à la fois dans la théorie et dans l'application (voir, e.g., Alazard [2]). Nous nous intéressons principalement au contrôle des ondes de surface avec tension superficielle par une pression extérieure P_{ext} appliquée à la surface libre (voir la Figure 1.3.3). Plus précisément, nous souhaitons générer des surfaces libres et des champs de vitesse arbitraires en soufflant de l'air au-dessus de la surface libre. Nous souhaitons également savoir si cette pression extérieure peut se limiter à un domaine ω qui est petit. Des questions similaires peuvent absolument se poser pour le problème de la stabilisation, pour lequel on peut se référer à Alazard [3, 4].

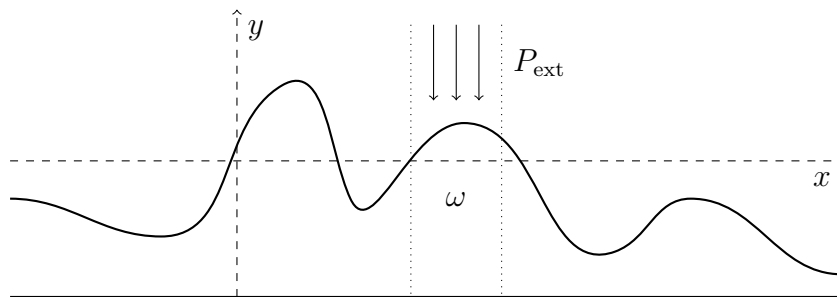


FIGURE 1.3.3 – Ondes de surface sous la pression extérieure

Nous nous limiterons aux ondes de surface *spatialement périodiques*, qui peut être générés, par exemple, en périodisant un réservoir rectangulaire du fluide. De manière équivalente, cela signifie étudier l'équation des ondes de surface sur \mathbb{T}^d , c'est-à-dire, $(x, y) \in \mathbb{T}^d \times \mathbb{R}$. L'existence de P_{ext} change la condition dynamique en

$$-P|_{\Sigma} = \kappa H(\eta) + P_{\text{ext}}.$$

Linéarisant (1.3.4) en $(\eta, \psi) = (0, 0)$, on arrive au problème du contrôle linéaire

$$\partial_t u + i|D_x|^{3/2}u = P_{\text{ext}}.$$

Sous la CCG, la contrôlabilité de cette équation peut être prouvée de la même manière que l'équation de Schrödinger, en utilisant l'approche de Lebeau [131], en faisant attention à ce que P_{ext} soit à valeur réelle!

Traitant l'équation quasi-linéaire originale, Alazard–Baldi–Han–Kwan [5] ont utilisé un schéma itératif qui réduit le problème du contrôle non-linéaire à une série de problèmes du contrôle linéaires. Comme ils étudient le système en dimension deux, c'est-à-dire, $d = 1$, ils appliquent la méthode des séries de Fourier et concluent par l'inégalité d'Ingham (voir Ingham [106]) que ces équations linéaires, et par conséquent le système des ondes de surface, sont contrôlables. En dimensions supérieures, nous avons montré dans [204] la contrôlabilité avec la CCG — qui est toujours satisfaite quand $d = 1$ — en implémentant une approche semi-classique dans le régime des hautes fréquences et en prouvant une propriété de la continuation unique dans le régime des basses fréquences.

Théorème 1.3.6 (Alazard–Baldi–Han–Kwan ($d = 1$), Zhu ($d > 1$)). *Soit $d \geq 1$, $T > 0$, s suffisamment grand, ω satisfait la CCG, alors $\exists \varepsilon_0 > 0$, $\forall (\eta_i, \psi_i) \in H^{s+1/2} \times H^s$ avec $\|(\nabla \eta_i, \nabla \psi_i)\|_{H^{s-1/2} \times H^{s-1}} < \varepsilon_0$ ($i = 0, 1$), et $\int \eta_0 dx = \int \eta_1 dx = 0$, $\exists P_{\text{ext}} \in C([0, T], H^s)$, telle que*

- (1) P_{ext} est à valeur réelle et $\text{supp } P_{\text{ext}}(t, \cdot) \subset \omega, \forall t \in [0, T]$;
 (2) il existe une solution unique de (1.3.2), $(\eta, \psi) \in C([0, T], H^{s+1/2} \times H^s)$, telle que $(\eta, \psi)|_{t=0} = (\eta_0, \psi_0)$ et $(\eta, \psi)|_{t=T} = (\eta_1, \psi_1)$.

1.3.5 Propagation des singularités

Des phénomènes tels que l'effet régularisant et la formation des singularités peuvent éventuellement être attendus pour des ondes de surface avec tension superficielle. Mathématiquement, cela est dû à la vitesse infinie de propagation de l'équation de Schrödinger fractionnaire (quasi-linéaire) (1.3.4) ; physiquement, c'est parce que la tension superficielle a tendance à lisser instantanément la surface libre.

Des effets régularisants locaux pour les ondes de surface avec tension superficielle ont été prouvés quand $d = 1$, voir Christianson–Hur–Staffilani [54] et Alazard–Burq–Zuily [6]. Voir aussi Alazard–Ifrim–Tataru [11] pour une inégalité de Morawetz pour les ondes de surface sans tension superficielle, en dimension deux. Cependant, aucun résultat concernant les effets régularisants locaux en dimensions supérieures, ou les effets régularisants microlocaux n'était connu auparavant.

Dans [205], nous avons obtenu deux résultats de propagation, correspondant respectivement aux deux énoncés de Théorème 1.2.9. Selon le Théorème 1.2.11 et la nature Schrödinger-fractionnaire du système des ondes de surface avec tension superficielle, ces résultats devraient impliquer l'utilisation des fronts d'onde quasi-homogènes avec les paramètres $(\delta, \rho) = (1/2, 1)$. De plus, ces résultats sont obtenus pour les ondes de surface avec une décroissance spatiale suffisante, dont les existences sont promises par le Théorème 1.3.3. Plus précisément, la géométrie de la surface libre sera une *perturbation à courte portée* de la géométrie euclidienne. Par conséquent, la dépendance temporelle de la géométrie n'affecte pas le comportement du régime des hautes fréquences, grâce à la vitesse infinie de propagation.

Soit $(\eta, \psi) \in C([-T, T], \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu)$ la solution de (1.3.2) avec les données initiales (η_0, ψ_0) , où μ et m sont à déterminer ci-dessous. Notre premier résultat est la propagation à l'infini.

Théorème 1.3.7 (Zhu). *Supposons que $\mu > 3 + d/2$, $3 \leq m \leq 2\mu - 6 - d$. Soit*

$$(x_0, \xi_0) \in \text{WF}_{1/2,1}^{\mu+1/2+\sigma}(\eta_0) \cup \text{WF}_{1/2,1}^{\mu+\sigma}(\psi_0), \quad \xi_0 \neq 0,$$

où $0 \leq \sigma \leq m/2 - 3/2$. Soit $t_0 \in [0, T]$ (resp. $t_0 \in [-T, 0]$) tel que

$$x_0 + \frac{3}{2}t|\xi_0|^{-1/2}\xi_0 \neq 0, \quad \forall t \in [0, t_0] \quad (\text{resp. } \forall t \in [t_0, 0]).$$

Alors

$$\left(x_0 + \frac{3}{2}t_0|\xi_0|^{-1/2}\xi_0, \xi_0\right) \in \text{WF}_{1/2,1}^{\mu+1/2+\sigma}(\eta(t_0)) \cup \text{WF}_{1/2,1}^{\mu+\sigma}(\psi(t_0)).$$

Notre deuxième résultat concerne l'effet régularisant microlocal. Soit \mathcal{G} le flot co-géodésique sur la surface libre initiale $\Sigma_0 = \{y = \eta_0(x)\}$, qui peut être identifié en tant que le flot hamiltonien sur $\mathbb{R}_x^d \times \mathbb{R}_\xi^d$ du symbole

$$G(x, \xi) = |\xi|^2 - \frac{(\nabla\eta_0(x) \cdot \xi)^2}{1 + |\nabla\eta_0(x)|^2}.$$

Théorème 1.3.8 (Zhu). *Soit $d \geq 1$, $\mu > 3 + d/2$, $3 \leq m \leq \frac{2}{3}(\mu - 3 - d/2)$,*

$$(x_0, \xi_0) \in \text{WF}_{0,1}^{\mu+1/2+\sigma}(\eta_0) \cup \text{WF}_{0,1}^{\mu+\sigma}(\psi_0), \quad \xi_0 \neq 0,$$

où $0 \leq \sigma \leq \min\{\mu/2 - 3d/4, 3m/2\}$. Supposons que la co-géodésique $\{(x_t, \xi_t) := \mathcal{G}_t(x_0, \xi_0)\}_{t \in \mathbb{R}}$ soit non-captée dans le futur (resp. dans le passé). Alors, $\exists \xi_+ \in \mathbb{R}_\xi^d \setminus \{0\}$ (resp. $\exists \xi_- \in \mathbb{R}_\xi^d \setminus \{0\}$) tel que,

$$\lim_{t \rightarrow \infty} \xi_t = \xi_+, \quad \text{resp.} \quad \lim_{t \rightarrow \infty} \xi_{-t} = \xi_-,$$

et pour tout $0 < t_0 \leq T$ (resp. $-T \leq t_0 < 0$),

$$\begin{aligned} & \left(\frac{3}{2} t_0 |\xi_+|^{-1/2} \xi_+, \xi_+ \right) \in \text{WF}_{1/2,1}^{\mu+1/2+\sigma}(\eta(t_0)) \cup \text{WF}_{1/2,1}^{\mu+\sigma}(\psi(t_0)), \\ \text{resp.} & \left(\frac{3}{2} t_0 |\xi_-|^{-1/2} \xi_-, \xi_- \right) \in \text{WF}_{1/2,1}^{\mu+1/2+\sigma}(\eta(t_0)) \cup \text{WF}_{1/2,1}^{\mu+\sigma}(\psi(t_0)). \end{aligned}$$

Observons que ξ_+ et ξ_- sont déterminés par la géométrie initiale donnée par η_0 , en raison de la vitesse infinie de propagation. Au moins dans deux cas, la géométrie de Σ_0 est (partout) non-captante : soit $d = 1$, soit $\|\langle x \rangle \nabla^2 \eta_0\|_{L^\infty}$ est suffisamment petit. Dans ces deux cas, nous avons obtenu des effets régularisant locaux lorsque les données initiales présentent une décroissance spatiale suffisante, en particulier lorsqu'elles sont aux supports compacts.

Corollaire 1.3.9 (Zhu). *Supposons que les deux conditions suivantes soit satisfaites :*

- (1) *soit $d = 1$, soit $\|\langle x \rangle \nabla^2 \eta_0\|_{L^\infty}$ est suffisamment petit ;*
- (2) $\text{WF}_{1/2,1}^{\mu+1/2+\sigma}(\eta_0) \cup \text{WF}_{1/2,1}^{\mu+\sigma}(\psi_0) \subset \{x = 0\} \cup \{\xi = 0\}$.

Alors $\forall t_0 \in [-T, T] \setminus \{0\}$ et $\forall \epsilon > 0$, $(\eta(t_0), \psi(t_0)) \in H_{\text{loc}}^{\mu+1/2+\sigma-\epsilon} \times H_{\text{loc}}^{\mu+\sigma-\epsilon}$.

Chapitre 2

Stabilization of damped waves on \mathbb{S}^d and Zoll surfaces of revolution

2.1 Introduction

2.1.1 Problem of stabilization and main result

Consider the Cauchy problem of the damped wave equation on a compact Riemannian manifold (M, g) without boundary.

$$(2.1.1) \quad \begin{cases} (\partial_t^2 - \Delta + a\partial_t)u = 0 & \text{in } \mathcal{D}'(\mathbb{R} \times M), \\ (u, \partial_t u)_{t=0} = (u_0, u_1) & \in H^1(M) \times L^2(M). \end{cases}$$

Here $\Delta = \Delta_g$ is the Laplace–Beltrami operator with respect to the metric g . The function $a \in L^\infty(M)$ is non-negative, and $a\partial_t u$ is called the damping term, as it causes decay in energy (defined below). There is a unique solution $u \in C^1(\mathbb{R}, L^2(M)) \cap C(\mathbb{R}, H^1(M))$ to (2.1.1) by the theorem of Hille–Yosida. The energy defined by

$$(2.1.2) \quad E(u, T) = \frac{1}{2} \|\nabla u(T)\|_{L^2(M)}^2 + \frac{1}{2} \|\partial_t u(T)\|_{L^2(M)}^2$$

decays monotonically as T increases, due to the non-negativity of a and the identity

$$(2.1.3) \quad E(u, T) = E(u, 0) - \int_0^T \int_M a(x) |\partial_t u(t, x)|^2 dx dt.$$

A natural question to ask is whether, as a consequence of the damping effect,

$$\lim_{T \rightarrow +\infty} E(u, T) = 0$$

for every solution u to (2.1.1). If this is true, we say that a *weakly stabilizes* (2.1.1). When such a stabilization is uniform for all solutions, or more precisely, if for some function $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ with $\lim_{T \rightarrow +\infty} f(T) = 0$ and every solution u to (2.1.1), we have for all $T \geq 0$,

$$E(u, T) \leq E(u, 0) \times f(T),$$

then we say that a *strongly stabilizes* (2.1.1). It is well known that whenever the strong stabilization holds, the function f could be chosen of the form

$$f(T) = Ce^{-\beta T}, \quad C > 0, \beta > 0,$$

so that we have in fact a uniform exponential decay of energy (see for example [44] for an elementary proof).

When $a \in C(M)$, Rauch–Taylor gave in [164] a sufficient condition (a *geometric control condition*, to be stated as condition (2) in the following Theorem 2.1.1) for strong stabilization, followed by Bardos–Lebeau–Rauch [24], who showed that this is in fact an equivalent condition (even for the similar problem of stabilization on manifolds with boundaries, which will not be elaborated here).

Theorem 2.1.1 (Bardos–Lebeau–Rauch). *Let (M, g) be a compact Riemannian manifold without boundary, and $0 \leq a \in C(M)$, then the following two statements are equivalent.*

- (1) *a strongly stabilizes (2.1.1) ;*
- (2) *All geodesics of M enter the open set $\{a > 0\}$. That is, for $x \in M$, let γ be a geodesic starting from x (i.e. $\gamma(0) = x$), then for some $t \geq 0$, $\gamma(t) \in \{a > 0\}$.*

The proof of Theorem 2.1.1 in [24] used the propagation theorem developed by Melrose–Sjöstrand [144]. Lebeau [132] managed to use microlocal defect measures (which is due to Gérard [85] and Tartar [184], see also [40]) and an argument by contradiction to give a new and much simpler proof. However, when $a \in L^\infty(M)$, it remains an open problem to give an equivalent condition for strong stabilization, even though the following necessary condition and sufficient condition are known to be classical, which follow by analyzing the proof of Theorem 2.1.1.

Proposition 2.1.2. *Let (M, g) be a compact Riemannian manifold without boundary, and let $0 \leq a \in L^\infty(M)$,*

- (1) *if a strongly stabilizes (2.1.1), then all geodesics of M intersect with $\text{supp } a$;*
- (2) *if all geodesics of M enter the open set $U(a) = \bigcup_{\epsilon > 0} \text{Int}\{x : a(x) > \epsilon\}$, then a strongly stabilizes (2.1.1).*

When $a \in C(M)$, condition (2) is also necessary because in this case $U(a) = \{a > 0\}$, and we conclude by Theorem 2.1.1. However, for general $a \in L^\infty(M)$, these two conditions are not sharp. Typical examples are as follows. Let $M = \mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\}$, define the equator $\Gamma = \mathbb{S}^2 \cap \{z = 0\}$ and the hemispheres $\mathbb{S}_\pm^2 = \mathbb{S}^2 \cap \{\pm z > 0\}$. Let $0 \leq a \in C(\mathbb{S}^2)$ be zero exactly on the equator, while $a > 0$ elsewhere. Theorem 2.1.1 says that a does not strongly stabilize (2.1.1), for the equator Γ , as a geodesic, does not enter $\{a > 0\} = \mathbb{S}_+^2 \cup \mathbb{S}_-^2$, even though all geodesics enter $\text{supp } a = \mathbb{S}^2$. On the other hand, let $a = 1_{\mathbb{S}_+^2}$ be the indicator function of the upper hemisphere, then the equator does not enter $U(a) = \text{Int } \mathbb{S}_+^2$. However, the following unpublished result due to Gilles Lebeau shows that a indeed strongly stabilizes (2.1.1).

Theorem 2.1.3 (Lebeau, unpublished). *For $d \geq 1$, let*

$$\mathbb{S}^d = \{x = (x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1} : x_1^2 + \dots + x_{d+1}^2 = 1\}$$

be the d -dimensional unit sphere, which inherits the Riemannian metric from \mathbb{R}^{d+1} . Let $\mathbb{S}_+^d = \mathbb{S}^d \cap \{x_{d+1} > 0\}$ denote the upper hemisphere, then $a(x) = 1_{\mathbb{S}_+^d}(x)$ strongly stabilizes (2.1.1).

We will first give a simple proof of this theorem (see §2.1.2) using the spectral distribution of the spherical Laplacian, and the symmetries of spherical harmonics. Then we extend this result, on dimension 2, to Zoll surfaces of revolution.

Definition 2.1.4. A Zoll manifold is a Riemannian manifold whose geodesic flow is periodic. A Zoll surface of revolution is a 2 dimensional Zoll manifold, on which the group \mathbb{S}^1 acts smoothly, faithfully, and isometrically.

We refer to Besse [29] for an introduction of Zoll manifold. Some fundamental geometric properties and examples are stated below. In particular \mathbb{S}^d ($d \geq 1$) are Zoll manifolds, and \mathbb{S}^2 is a Zoll surface of revolution. The geometry of a Zoll surface of revolution resembles much that of \mathbb{S}^2 , which makes it natural for the generalization of Theorem 2.1.3. (However, on general Zoll manifolds, such resemblance is not yet clear to the author.) Indeed, we will use the following two aspects of resemblance for our generalization.

- Local Geometry : On Zoll surfaces of revolution, the geometric objects such as the equator, and the upper and lower hemi-surfaces are well defined. Moreover, the local geometry near the equator is similar to that near a great circle of \mathbb{S}^2 .
On a general Zoll manifold, such resemblance is not clear to the knowledge of the author. That is why we will restrict ourselves to Zoll surfaces of revolution.
- Global geometry : Spectral distribution of the Laplacian–Beltrami operator. See Proposition 2.1.5. This works for Zoll manifolds of arbitrary dimension, and states that the Laplacian spectrum on Zoll manifolds of dimension d is similar to that of the spherical Laplacian on \mathbb{S}^d .

It is worth comparing to the work of Burq–Gérard [45] of a similar stabilization problem on tori, where only the local geometry is consulted.

Proposition 2.1.5 (Duistermaat–Guillemin [73]). *Let Δ be the Laplacian–Beltrami operator on a Zoll manifold of dimension d , then*

$$\text{Spec}(-\Delta) \subset \bigsqcup_{n \geq 0} I_n,$$

where $\{I_n\}_{n \geq 0}$ are mutually disjoint intervals, such that for some $\beta > 0$, $A > 0$,

$$(2.1.4) \quad I_n \subset](n + \beta/4)^2 - A, (n + \beta/4)^2 + A[.$$

Remark 2.1.6. When $d = 2$, we have $\beta = 2$. See Proposition 4.35 of [29].

Example 2.1.7. In particular, let Δ_d denote the spherical Laplacian on \mathbb{S}^d , then (see Lemma 2.1.13)

$$\text{Spec}(-\Delta_d) = \{(n + \frac{d-1}{2})^2 - \frac{(d-1)^2}{4} : n \in \mathbb{N}\}.$$

We simply let $\beta/4 = (d - 1)/2$, and let A be strictly larger than $(d - 1)^2/4$.

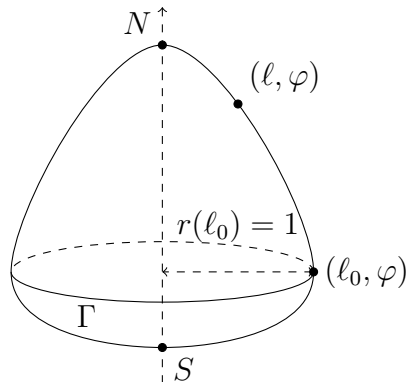
Let Σ denote a Zoll surface of revolution, we state some local geometries of Σ . More details could be found in [29]. For an intuitive understanding, see Figure 2.1.1. It is known that Σ is automatically diffeomorphic to \mathbb{S}^2 , and there exists exactly two distinct points, respectively called the north pole and the south pole, denoted by N and S , which are invariant under the actions of \mathbb{S}^1 . We then parametrize the surfaces by coordinates

$$(\ell, \varphi) \in [0, \text{dist}(N, S)] \times \mathbb{S}^1,$$

where ℓ is the arc-length parameter of one (and consequently every) geodesic from N to S , and φ is the rotational angle corresponding to the actions of \mathbb{S}^1 , so that the Riemannian metric on Σ is of the form

$$g = d\ell^2 + r(\ell)^2 d\varphi^2,$$

FIGURE 2.1.1 – Zoll surface of revolution



where $r(\ell)$ is the distance from the point (ℓ, φ) to the axis of rotation. By Lemma 4.9 of [29], there exists a unique ℓ_0 such that $r(\ell)$ attains its maximum at $\ell = \ell_0$. There is no loss of generality by assuming that $r(\ell_0) = 1$. Moreover we have $r'(\ell_0) = 0$, $r''(\ell_0) < 0$. The curve $\Gamma = \{(\ell_0, \varphi) : \varphi \in \mathbb{S}^1\}$ defines a closed geodesic of period 2π (because $r(\ell_0) = 1$) called the *equator*, while the regions $\Sigma^+ = \{(\ell, \varphi) : \ell > \ell_0\}$ and $\Sigma^- = \{(\ell, \varphi) : \ell < \ell_0\}$ are called the *upper and lower hemi-surfaces* respectively. Similarly to \mathbb{S}^2 , all geodesics on Σ enter Σ^+ except for the equator Γ .

Remark 2.1.8. If we denote $c = -r''(\ell_0)/2 > 0$, then

$$r(\ell) = 1 - c(\ell - \ell_0)^2 + O((\ell - \ell_0)^3).$$

This local geometry will be essential in performing a microlocal analysis near Γ that proves our main theorem (Theorem 2.1.10). In particular, if $\Sigma = \mathbb{S}^2$, then we take $r(\ell) = \cos \ell$, such that $(\ell, \varphi) \in [-\pi/2, \pi/2] \times \mathbb{S}^1$ parametrizes \mathbb{S}^2 . In this case $\ell_0 = 0$, and

$$r(\ell) = \cos \ell = 1 - \frac{1}{2}\ell^2 + O(\ell^3).$$

Remark 2.1.9. Using the change of variable $r(\ell) = \sin \theta$, to describe a Zoll surface of revolution, it is equivalent to give a Riemannian metric to \mathbb{S}^2 . By an abuse of notation, we still use g to denote the metric on \mathbb{S}^d obtained by this isometry. If we parametrize \mathbb{S}^2 by (θ, φ) , where θ is the latitude while φ is the longitude of \mathbb{S}^2 , then by Corollary 4.16 of [29], (\mathbb{S}^2, g) is a Zoll surface of revolution if and only if

$$g = (1 + h(\cos \theta))^2 d\theta^2 + \sin^2 \theta d\varphi^2,$$

for some smooth odd function h from $[-1, 1]$ to $(-1, 1)$ with $h(1) = h(-1) = 0$.

Now we state the main result of this paper.

Theorem 2.1.10. *Let Σ be a Zoll surface of revolution, then $a = 1_{\Sigma^+}$ strongly stabilizes (2.1.1).*

Remark 2.1.11. As a direct consequence of our proof, in order for a to strongly stabilize (2.1.1), it suffices for a to be bounded from below by a positive constant in a half-neighborhood of the equator. To be precise, this means that there exists some $\varepsilon > 0$, $\delta > 0$, such that

$$a(\ell, \varphi) \geq \delta \cdot 1_{\ell_0 < \ell < \ell_0 + \varepsilon}(\ell, \varphi).$$

However, we will only prove the case when $a = 1_{\Sigma^+}$ for simplicity.

2.1.2 Stabilization of damped waves on \mathbb{S}^d

In this section we prove Theorem 2.1.3. First we recall the following classical result, due to J.-L. Lions [137].

Proposition 2.1.12. *Let (M, g) be a compact Riemannian manifold without boundary, and let $0 \leq a \in L^\infty(M)$, then the following two statements are equivalent.*

- (1) *a strongly stabilizes (2.1.1).*
- (2) *For some $T > 0$, $C > 0$, and for every solution u to the Cauchy problem of the undamped wave equation*

$$(2.1.5) \quad \begin{cases} (\partial_t^2 - \Delta)u = 0, & \text{in } \mathcal{D}'(\mathbb{R} \times M); \\ (u, \partial_t u)_{t=0} = (u_0, u_1), & \in H^1(M) \times L^2(M), \end{cases}$$

the following observability inequality holds,

$$(2.1.6) \quad E(u, 0) \leq C \int_0^T \int_M a |\partial_t u|^2 dx dt.$$

Therefore it remains to establish this observability inequality. Coming back to \mathbb{S}^d , we recall some basic properties of the spherical Laplacian and spherical harmonics (see for example Chapter IV, Section 2 of [179]).

Lemma 2.1.13. *Let Δ_d denote the spherical Laplacian on \mathbb{S}^d , then*

- (1) *$\text{Spec}(-\Delta_d) = \{\lambda_n^2 = n(n+d-1) = (n + \frac{d-1}{2})^2 - \frac{(d-1)^2}{4} : n \in \mathbb{N}\}$.*
- (2) *The eigenspace E_n to $-\Delta_d$ of eigenvalue λ_n^2 consists of spherical harmonics of degree n , which are restrictions to \mathbb{S}^d of harmonic polynomials of $d+1$ variables, homogeneous of degree n . In particular, if $u \in E_n$, then $u(-x) = (-1)^n u(x)$.*

As a consequence, each $u \in H^s(\mathbb{S}^d)$ with $s \in \mathbb{R}$ admits a unique decomposition in distributional sense of the following form,

$$u = \sum_{n \geq 0} u_n, \quad \text{with } u_n \in E_n.$$

This allows us to specify the $H^s(\mathbb{S}^d)$ norm in terms of this decomposition by setting

$$\|u\|_{H^s}^2 = \|(1 + \Delta)^{s/2} u\|_{L^2}^2 = \sum_{n \geq 0} \langle \lambda_n \rangle^{2s} \|u_n\|_{L^2}^2, \quad \text{with } \langle \lambda_n \rangle = \sqrt{1 + \lambda_n^2}.$$

We then introduce a new differential operator as a perturbation of $-\Delta_d$,

$$(2.1.7) \quad \mathcal{L} = -\Delta_d + \frac{(d-1)^2}{4}.$$

The advantage of \mathcal{L} to $-\Delta_d$ is that the spectrum of \mathcal{L} consists of *exact* squares of arithmetic sequence, $\text{Spec}(\mathcal{L}) = \{(n + \frac{d-1}{2})^2 : n \in \mathbb{N}\}$, so that

$$(2.1.8) \quad \text{Spec}(\sqrt{\mathcal{L}}) = \{n + \frac{d-1}{2} : n \in \mathbb{N}\}.$$

Solving the following Cauchy problem

$$(2.1.9) \quad \begin{cases} (\partial_t^2 + \mathcal{L})u = 0, & \text{in } \mathcal{D}'(\mathbb{R} \times \mathbb{S}^d); \\ (u, \partial_t u)_{t=0} = (u_0, u_1), & \in H^1(\mathbb{S}^d) \times L^2(\mathbb{S}^d), \end{cases}$$

by using Fourier series,

$$\begin{aligned}
 (2.1.10) \quad u(t) &= \cos(t\sqrt{\mathcal{L}})u_0 + \sqrt{\mathcal{L}}^{-1} \sin(t\sqrt{\mathcal{L}})u_1 \\
 &= \begin{cases} \sum_{n \geq 0} (e^{it(n+\frac{d-1}{2})}u_n^+ + e^{-it(n+\frac{d-1}{2})}u_n^-), & d \geq 2, \\ u_0^0 + u_0^1 t + \sum_{n \geq 1} (e^{itn}u_n^+ + e^{-itn}u_n^-), & d = 1, \end{cases}
 \end{aligned}$$

where we write $u_0 = \sum_{n \geq 0} u_n^0$, $u_1 = \sum_{n \geq 0} u_n^1$, with $u_n^i \in E_n$, and by an explicit calculation, we have for $n \geq 0$ when $d \geq 2$ and $n \geq 1$ when $d = 1$,

$$u_n^+ + u_n^- = u_n^0, \quad i(n + \frac{d-1}{2})(u_n^+ - u_n^-) = u_n^1.$$

If we assume $(u_0, u_1) \in H^s \times H^{s-1}$ for some $s \in \mathbb{R}$, then this expression gives an a priori bound for $\|u\|_{L_{\text{loc}}^\infty H^s}$. Indeed, for $d \geq 2$ ($d = 1$ is similar), by the characterization of the H^s norm, and the triangular inequality,

$$\begin{aligned}
 (2.1.11) \quad \|u(t)\|_{H^s}^2 &= \sum_{n \geq 0} \langle \lambda_n \rangle^{2s} \|e^{it(n+\frac{d-1}{2})}u_n^+ + e^{-it(n+\frac{d-1}{2})}u_n^-\|_{L^2}^2 \\
 &= \sum_{n \geq 0} \langle \lambda_n \rangle^{2s} \|\cos(t(n + \frac{d-1}{2}))u_n^0 + (n + \frac{d-1}{2})^{-1} \sin(t(n + \frac{d-1}{2}))u_n^1\|_{L^2}^2 \\
 &\lesssim \sum_{n \geq 0} \langle \lambda_n \rangle^{2s} \|u_n^0\|_{L^2}^2 + \sum_{n \geq 0} \langle \lambda_n \rangle^{2s} (n + \frac{d-1}{2})^{-2} \|u_n^1\|_{L^2}^2 \\
 &\lesssim \|u_0\|_{H^s}^2 + \|u_1\|_{H^{s-1}}^2.
 \end{aligned}$$

When $d \geq 2$, we obtain $\|u\|_{L^\infty H^s} \lesssim \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}$, while for $d = 1$, the same estimate holds after replacing $\|u\|_{L^\infty H^s}$ with $\|u\|_{L_{\text{loc}}^\infty H^s}$, due to the linear growth in time of the term $u_0^1 t$.

Observe that in the expression of the solution, the factors $\{e^{\pm it(n+\frac{d-1}{2})}\}_{n \in \mathbb{N}}$ are orthogonal in $L^2([0, 2\pi])$. This fact makes the observability of (2.1.5) easier to prove, due to the following two reduction lemmas.

Definition 2.1.14. We say that a observes (2.1.9) if for some constants $T > 0$, $C > 0$ and every solution u to (2.1.5), the observability inequality (2.1.6) holds. We say that a observes the spherical harmonics, if for some $C > 0$, and every spherical harmonic $v \in \cup_{n \in \mathbb{N}} E_n$,

$$(2.1.12) \quad \|a^{1/2}v\|_{L^2(\mathbb{S}^d)} \geq C\|v\|_{L^2(\mathbb{S}^d)}$$

Lemma 2.1.15. For $M = \mathbb{S}^d$, let $0 \leq a \in L^\infty(\mathbb{S}^d)$, if a observes (2.1.9), then a observes (2.1.5).

Démonstration. Let u solve (2.1.5). We decompose $u = v + w$ such that

$$\begin{cases} (\partial_t^2 + \mathcal{L})v = 0, & (v, \partial_t v)_{t=0} = (u_0, u_1); \\ (\partial_t^2 + \mathcal{L})w = \frac{(d-1)^2}{4}u, & (w, \partial_t w)_{t=0} = (0, 0). \end{cases}$$

Now that a observes (2.1.9), for some $T > 0$,

$$E(u, 0) = E(v, 0) \lesssim \int_0^T \int_{\mathbb{S}^d} a |\partial_t v|^2 dx dt$$

$$\lesssim \int_0^T \int_{\mathbb{S}^d} a |\partial_t u|^2 dx dt + \int_0^T \int_{\mathbb{S}^d} a |\partial_t w|^2 dx dt.$$

By Duhamel's formula, $\partial_t w(t) = \frac{(d-1)^2}{4} \int_0^t \cos((t-s)\sqrt{\mathcal{L}}) u(s) ds$. Then we use the boundedness $\|\cos(t\sqrt{\mathcal{L}})\|_{L^2 \rightarrow L^2} \leq 1$, and the a priori estimate (2.1.11), to obtain

$$\|\partial_t w(t)\|_{L^2}^2 \lesssim \int_0^T \|u(s)\|_{L^2}^2 ds \lesssim \|(u_0, u_1)\|_{L^2 \times H^{-1}}^2.$$

Combine the inequalities above, we obtain a weak observability,

$$E(u, 0) \lesssim \int_0^T \int_{\mathbb{S}^d} a |\partial_t u|^2 dx dt + \|(u_0, u_1)\|_{L^2 \times H^{-1}}^2.$$

Then it is a classical argument of uniqueness-compactness due to Bardos–Lebeau–Rauch [24] which allows us to remove the compact remainder term $\|(u_0, u_1)\|_{L^2 \times H^{-1}}^2$ and obtain the (strong) observability. This amounts to prove by contradiction and extract a subsequence of solutions of (2.1.5) which violates the observability, but converges strongly in the energy norm due to the compactness given by the weak observability. This gives us a solution to (2.1.5) with non vanishing energy (the energy is now conserved in time because there is no damping term in (2.1.5)), say v , such that $a \partial_t v = 0$. Then we conclude by showing that, for $a \not\equiv 0$, such solution does not exist (the only solution to (2.1.5) with $a \partial_t v = 0$ must be constant, and hence with zero energy). For more details, see the proof of Lemma 2.2.10. \square

Lemma 2.1.16. *If a observes the spherical harmonics, then a observes (2.1.9).*

Démonstration. We only prove the lemma for $d \geq 2$, the proof for $d = 1$ is almost the same.

We set $T = 2\pi$, and use Fubini's theorem, (2.1.10), the orthogonality of the family $\{e^{\pm it(n + \frac{d-1}{2})}\}_{n \in \mathbb{N}}$ in $L^2([0, 2\pi])$, the observability (2.1.12), and the characterization of Sobolev norms by spherical harmonics,

$$\begin{aligned} & \int_0^{2\pi} \int_{\mathbb{S}^d} a |\partial_t u|^2 dx dt \\ &= \int_{\mathbb{S}^d} a(x) \int_0^{2\pi} \left| \sum_{n \geq 0} \left(n + \frac{d-1}{2} \right) \left(e^{it(n + \frac{d-1}{2})} u_n^+(x) - e^{-it(n + \frac{d-1}{2})} u_n^-(x) \right) \right|^2 dt dx \\ &= \int_{\mathbb{S}^d} a(x) \sum_{n \geq 0} \left(n + \frac{d-1}{2} \right)^2 (|u_n^+(x)|^2 + |u_n^-(x)|^2) dx \\ &\gtrsim \int_{\mathbb{S}^d} \sum_{n \geq 0} \left(n + \frac{d-1}{2} \right)^2 (|u_n^+(x)|^2 + |u_n^-(x)|^2) dx \\ &\gtrsim \int_{\mathbb{S}^d} \sum_{n \geq 0} \left(n + \frac{d-1}{2} \right)^2 (|u_n^+(x) + u_n^-(x)|^2 + |u_n^+(x) - u_n^-(x)|^2) dx \\ &\gtrsim \int_{\mathbb{S}^d} \sum_{n \geq 0} \left(n + \frac{d-1}{2} \right)^2 |u_n^0(x)|^2 + \int_{\mathbb{S}^d} \sum_{n \geq 0} |u_n^1(x)|^2 dx \\ &\geq E(u, 0). \end{aligned}$$

\square

Then we finish the proof of Theorem 2.1.3 by showing that $a(x) = 1_{\mathbb{S}_+^d}(x)$ observes the spherical harmonics.

Proposition 2.1.17. *On \mathbb{S}^d , $a(x) = 1_{\mathbb{S}_+^d}(x)$ observes the spherical harmonics.*

Démonstration. This comes easily from the symmetry properties of spherical harmonics stated in Lemma 2.1.13. Indeed, if $v \in E_n$, then $v(-x) = (-1)^n v(x)$ implies that $\|v\|_{L^2(\mathbb{S}_+^d)} = \|v\|_{L^2(\mathbb{S}_-^d)}$, whence the observability

$$\|a^{1/2}v\|_{L^2(\mathbb{S}^d)} = \|v\|_{L^2(\mathbb{S}_+^d)} = \frac{1}{\sqrt{2}}\|v\|_{L^2(\mathbb{S}^d)}.$$

□

2.1.3 Strategy of proof

2.1.3.1 Proof of Theorem 2.1.3

We first analyse the proof of Theorem 2.1.3 presented above, which consists of the following 4 steps.

Step 1. Reduce the strong stabilization of the damped wave equation (2.1.1) to the observability (2.1.6) of the undamped wave equation (2.1.5). This is a classical argument.

Step 2. Reduce the observability of the undamped wave equation (2.1.5) to the observability of the perturbed wave equation (2.1.9). This perturbation uses essentially the fact that the spectrum of the spherical Laplacian is distributed near squares of an arithmetic sequence $\{(n + \frac{d-1}{2})^2\}_{n \in \mathbb{N}}$. In fact, the spectrum is exactly of distance $\frac{(d-1)^2}{4}$ away from this sequence. Therefore, by adding to $-\Delta_d$ the constant $\frac{(d-1)^2}{4}$, we obtain an operator \mathcal{L} , the spectrum of whose square root is exactly the arithmetic sequence $\{n + \frac{d-1}{2}\}_{n \in \mathbb{N}}$.

Step 3. Reduce the observability of the perturbed wave equation (2.1.9) to the observability of spherical harmonics, that is (2.1.12). To do so, we solve (2.1.9) explicitly with Fourier series (that is, decomposition in spherical harmonics), and use the orthogonality of the time factors

$$\{e^{\pm it\lambda}\}_{\lambda \in \text{Spec}(\sqrt{\mathcal{L}})} = \{e^{\pm it(n + \frac{d-1}{2})}\}_{n \in \mathbb{N}}$$

in $L^2([0, 2\pi])$ to decouple the space and time variables. In this way, the time variable can be omitted, and we are left only to consider the spherical harmonics.

Step 4. Prove the observability of spherical harmonics. We use the symmetry of spherical harmonics to show that the L^2 norm of a spherical harmonic is equally distributed on upper and lower hemispheres.

2.1.3.2 Proof of Theorem 2.1.10

We will follow this strategy to prove Theorem 2.1.10, but with the following modifications.

Step 1. Same as above.

Step 2. The only (slight) difference is the definition of the perturbed wave equation, because the perturbation \mathcal{L} of the Laplacian–Beltrami operator $-\Delta$ on a Zoll surface of revolution Σ can not be so simply defined as (2.1.7). To define \mathcal{L} in this situation, we recall Proposition 2.1.5 and Remark 2.1.6. For $\lambda \geq 0$ such that $\lambda^2 \in \text{Spec}(-\Delta)$, we let E_λ

denote the (minus) Laplacian eigenspace of eigenvalue λ^2 , and set for $n \geq 0$ the linear space

$$\tilde{E}_n = \bigoplus_{\lambda^2 \in I_n} E_\lambda.$$

Then \mathcal{L} is defined by prescribing its action on each \tilde{E}_n ,

$$\mathcal{L}|_{\tilde{E}_n} = (n + 1/2)^2 \text{Id}_{\tilde{E}_n}.$$

Therefore \tilde{E}_n are eigenspaces of \mathcal{L} , whose elements will be called \mathcal{L} -eigenfunctions, and

$$\text{Spec}(\sqrt{\mathcal{L}}) \subset \{n + 1/2 : n \in \mathbb{N}\}.$$

Moreover, by (2.1.4), if we set $K = \Delta + \mathcal{L}$, then $\|K\|_{\tilde{E}_n \rightarrow \tilde{E}_n} \leq A$, where \tilde{E}_n is equipped with the $L^2(\Sigma)$ norm. Consequently, by the orthogonal direct sum decomposition $L^2(\Sigma) = \bigoplus_{n \geq 0} \tilde{E}_n$, we show that K is a bounded operator on $L^2(\Sigma)$,

$$(2.1.13) \quad \|K\|_{L^2(\Sigma) \rightarrow L^2(\Sigma)} \leq A,$$

which plays the same role as the constant $\frac{(d-1)^2}{4}$ in the spherical case. Then the same argument shows that the observability for (2.1.5) can be deduced from the observability of the following perturbed wave equation,

$$(2.1.14) \quad \begin{cases} (\partial_t^2 + \mathcal{L})u = 0, & \text{in } \mathcal{D}'(\mathbb{R} \times \Sigma); \\ (u, \partial_t u)_{t=0} = (u_0, u_1), & \in H^1(\Sigma) \times L^2(\Sigma). \end{cases}$$

Step 3. Reduce the observability of (2.1.14) to the observability of \mathcal{L} -eigenfunctions, that is to say, for some $C > 0$, and every $u \in \bigcup_{n \geq 0} \tilde{E}_n$,

$$(2.1.15) \quad \|a^{1/2}u\|_{L^2(\Sigma)} \geq C\|u\|_{L^2(\Sigma)}.$$

Recall that in our case, $a(x) = 1_{\Sigma^+}(x)$. To do this, we use the orthogonality of the time factors $\{e^{\pm it(n+1/2)}\}_{n \in \mathbb{N}}$ in $L^2([0, 2\pi])$, which comes with luck from the fact that $\beta = 2$ on dimension 2 (recall Remark 2.1.6), so that $n + \beta/4 = n + 1/2$. However, this fact is not necessary, for we can always use Ingham's inequality (see the original work of Ingham [106], see also [206] for its application in the theory of control).

Step 4. Prove the observability of \mathcal{L} -eigenfunctions (2.1.15). Unfortunately, the simple proof for the observability of spherical harmonics does not apply, because neither the \mathcal{L} -eigenfunctions nor the Laplacian eigenfunctions on Σ share such strong symmetries as the spherical harmonics. However, we observe that, by the definition of \mathcal{L} , the \mathcal{L} -eigenfunctions are quasi-modes. Indeed, let $u \in \tilde{E}_n$, normalized in L^2 norm, that is, $\|u\|_{L^2(\Sigma)} = 1$; introduce the semiclassical parameter $h = (n + 1/2)^{-1}$, then by (2.1.13)

$$(-h^2\Delta + 1)u = -h^2Ku = O(h^2)_{L^2}.$$

This suggests a proof by contradiction and analyzing the semiclassical defect measures (see Gérard [84], Gérard–Leichtnam [86], Lions–Paul [138], see also [40]) of a sequence of \mathcal{L} -eigenfunctions, which violates the observability, that is, $\|1_{\Sigma^+}u\|_{L^2(\Sigma)} = o(1)$. Such argument is originally due to G. Lebeau, dating back to his work [132] which uses the

propagation of (classical) defect measures; see for example [41, 208] for the semiclassical setting.

A classical argument shows that such semiclassical defect measure, say μ , is supported on the unit cotangent bundle $S^*\Sigma$, vanishes on $T^*\Sigma^+$, and is invariant by the (co-)geodesic flow. Therefore μ carries no mass on the union of geodesics which enter Σ^+ . Recall that on Σ , every geodesic enter Σ^+ within the period of the geodesic flow (which is, in our case, 2π , by the normalization $r(\ell_0) = 1$), except for a rogue one, the equator Γ . We are thus unable to close the routine argument as the \mathcal{L} -eigenfunctions may concentrate on Γ (a simple example is $\Sigma = \mathbb{S}^2$, where the spherical harmonics $u_n(x, y, z) = (x + iy)^n$ will concentrate on the equator $z = 0$ as $n \rightarrow \infty$); but to conclude that

$$\text{supp } \mu \subset S^*\Sigma \cap \{\ell = \ell_0, \xi = 0\} = \{(\ell_0, \varphi, 0, \pm 1) : \varphi \in \mathbb{S}^1\}.$$

To deal with this problem, we take a closer look at the concentration behavior near the equator. It suffices to show that the speed of concentration from each side of the equator is comparable, so that the L^2 norm of this sequence of \mathcal{L} -eigenfunctions must be comparably distributed on each side as well, which contradicts to our hypothesis that the observability from the upper hemi-surface is violated by this sequence. Such idea is achieved by some proper scalings of the latitude coordinate ℓ , and is closely related to the *second microlocalization* along the equator, as illustrated by [45]. It is explicitly carried out as follows :

- (1) First, to simplify some calculations, we will work on an isothermal coordinate on Σ . There exists a strictly increasing $f \in C^\infty(\mathbb{R})$ such that

$$f'(x) = r(f(x)), \quad f(0) = \ell_0.$$

Then under the change of variable $\ell = f(x)$, the north pole N , the south pole S and the equator Γ now respectively corresponds to $x = -\infty$, $x = \infty$ and $x = 0$. Denoting for simplicity $\rho = f'$, the metric g now writes under the coordinates (x, φ) as

$$g = \rho(x)(dx^2 + d\varphi^2) = (1 - cx^2 + O(x^3))(dx^2 + d\varphi^2),$$

where the positive constant c is the same as in Remark 2.1.8; and the Laplacian-Beltrami operator writes

$$(2.1.16) \quad \Delta = \rho(x)^{-1}(\partial_x^2 + \partial_\varphi^2) = (1 + cx^2 + O(x^3))(\partial_x^2 + \partial_\varphi^2).$$

We also remark that under these coordinates,

$$L^2(M) \simeq L^2(\rho^2 dx, \mathbb{R}) \otimes L^2(d\varphi, \mathbb{S}^1).$$

- (2) On a general compact surface of revolution, $-\Delta$ is invariant under rotation, and commutes with the infinitesimal generator of rotation, that is, $D_\varphi = \frac{1}{i}\partial_\varphi$. We expect each Laplacian eigenspace to be a direct sum of D_φ -eigenspaces. Indeed, on Σ , for $\lambda^2 \in \text{Spec}(-\Delta)$, the following decomposition holds,

$$E_\lambda = \bigoplus_{k \in \mathbb{Z}} e^{ik\varphi} A_{\lambda, k},$$

where $A_{\lambda,k}$ consists of smooth functions of variable x , such that, whenever $w \in A_{\lambda,k}$, we have $w \in L^2(\rho^2 dx, \mathbb{R})$; and $u(\varphi, x) = e^{ik\varphi}w(x) \in L^2(M)$ is a common eigenfunction of $-\Delta$ and D_φ ,

$$-\Delta u = \lambda^2 u, \quad D_\varphi u = ku.$$

By (2.1.16), we have a second order differential equation for w ,

$$(2.1.17) \quad -\partial_x^2 w + k^2 w = \lambda^2 \rho^2 w = \lambda^2 (1 - cx^2 + O(x^3))w.$$

It is known that the Laplacian eigenfunctions are smooth, in particular at the poles N and S . This gives a boundary condition for w ,

$$\lim_{|x| \rightarrow \infty} \partial_x^n w(x) = 0, \quad \text{when } k \neq 0, n \in \mathbb{N}.$$

Consequently, up to a multiplicative constant, there exists at most one solution to (2.1.17), which means

$$\dim A_{\lambda,k} \leq 1, \quad \text{if } k \neq 0.$$

The case $k = 0$ poses no problem because as we have seen, $\text{supp } \mu \subset \{\theta = 1\}$, therefore the terms with $1 - h^2 k^2 \rightarrow 0$ (therefore $k \sim h^{-1} \rightarrow \infty$) contribute to almost all of the total mass. Now we set

$$\tilde{A}_{n,k} = \bigoplus_{\lambda^2 \in I_n} A_{\lambda,k},$$

and obtain the decomposition for \mathcal{L} -eigenspaces,

$$\tilde{E}_n = \bigoplus_{k \in \mathbb{Z}} e^{ik\varphi} \tilde{A}_{n,k}.$$

- (3) Due to the orthogonality of the family $\{e^{ik\varphi}\}_{k \in \mathbb{Z}}$ in $L^2(d\varphi, \mathbb{S}^1)$, we are left to prove the following observability, that for any sequence $\{\tilde{w}_{n,k} \in \tilde{A}_{n,k}\}_{n \in \mathbb{N}, k \in \mathbb{Z}}$, where the indexes appearing in the sequence satisfy $1 - h^2 k^2 = o(1)$ as $n \rightarrow \infty$ (recall that $h = (n + 1/2)^{-1}$; such a sequence will be called *admissible*, see Definition 2.2.15), there exists some $C > 0$, such that for any $\tilde{w}_{n,k} \in \tilde{A}_{n,k}$ in the sequence, we have

$$(2.1.18) \quad \|\mathbf{1}_{x>0} \tilde{w}_{n,k}\|_{L^2(\rho^2 dx)} \geq C \|\tilde{w}_{n,k}\|_{L^2(\rho^2 dx)}.$$

The weight ρ^2 is of no importance as $\tilde{w}_{n,k}$ concentrates on $x = 0$ (For a rigorous argument, we will use an Lithner-Agmon type estimate). In order to prove (2.1.18), we observe that $\tilde{w}_{n,k}$ satisfies a 1-dimensional stationary semiclassical Schrödinger equation,

$$(-h^2 \partial_x^2 + V) \tilde{w}_{n,k} = E \tilde{w}_{n,k} + O(h^2)_{L^2 \rightarrow L^2} \tilde{w}_{n,k},$$

where $V = 1 - \rho^2 = cx^2 + O(x^3)$ near $x = 0$, and $E = 1 - h^2 k^2$. Then we argue by contradiction and extract a sequence $k = k(n)$ and set $\tilde{w}_n = \tilde{w}_{n,k}$ which violates the observability, and treat separately two cases, $E = O(h)$ and $E \gg h$ (we can show that $E \gtrsim -h^2$).

- (a) If $E = O(h)$, then we use the scaling $z = c^{1/4}h^{-1/2}x$ to obtain a classical Schrödinger equation,

$$(-\partial_z^2 + z^2 + O(h^{1/2}))\tilde{w}_n = (F + o(1))\tilde{w}_n,$$

for some $0 \leq F \in \mathbb{R}$, and show that \tilde{w}_n is close to an eigenfunction of the harmonic oscillator $-\partial_z^2 + z^2$, which is either an even function or an odd function, whose mass are thus equally distributed on each side of the origin $z = 0$.

- (b) If $E \gg h$, then we use another scaling $z = c^{1/2}E^{-1/2}$, $\hat{h} = c^{1/2}E^{-1}h$, and obtain a semiclassical Schrödinger equation, with a semiclassical parameter $\hat{h} = o(1)$,

$$(-\hat{h}\partial_z^2 + z^2 + o(1))\tilde{w}_n = \tilde{w}_n + o(\hat{h})_{L^2 \rightarrow L^2}\tilde{w}_n.$$

The $(\hat{h}$ -)semiclassical measure of \tilde{w}_n will be supported on the circle

$$\{(z, \zeta) \in T^*\mathbb{R}_z : z^2 + \zeta^2 = 1\},$$

and is invariant by rotation (which is induced by the Hamiltonian flow generated by the principal symbol $z^2 + \zeta^2$). So the mass of \tilde{w}_n are also asymptotically equally distributed on each side of the origin $z = 0$.

2.2 Proof of Theorem 2.1.10

2.2.1 Geometry of Zoll surfaces of revolution

Let Σ be a Zoll surface of revolution, we recall some of its basic geometric properties, referring to the monograph [29] of Besse.

2.2.1.1 Coordinates and geodesics

Σ is diffeomorphic to \mathbb{S}^2 , and admits a parametrization by local coordinates described as follows. Recall that \mathbb{S}^1 acts smoothly, faithfully, and isometrically on Σ , leaving exactly two points fixed, which are called the *north pole* and the *south pole*, denoted respectively by N and S . Fix a geodesic γ_0 from N to S , parametrized by arc length. We assume that the total length of γ_0 is equal to π , after a proper normalization. Then as φ varies in \mathbb{S}^1 , $\gamma_\varphi = \varphi\gamma_0$ varies among all geodesics joining N and S , which are called the *meridians*. The coordinates on $U = \Sigma \setminus \{N, S\}$ is given by

$$U \ni \gamma_\varphi(\ell) \mapsto (\ell, \varphi) \in]0, \pi[\times \mathbb{S}^1.$$

The coordinate patches near N and S are respectively $U_N = \{N\} \cup \{(\ell, \varphi) : 0 \leq \ell < \pi\}$, and $U_S = \{S\} \cup \{(\ell, \varphi) : 0 < \ell \leq \pi\}$. They are diffeomorphic to the 2-dimensional open ball $B(0, \pi)$ via the usual polar coordinates. The Riemannian metric on U has the form

$$g = d\ell^2 + r(\ell)^2 d\varphi^2,$$

where $r(\ell)$ is the distance from the axis of rotation (recall Figure 2.1.1). Then Σ being a Zoll surface of revolution means that the criteria stated in Remark 2.1.9 is satisfied. There is a well defined differential operator D_φ on Σ . It is the differential operator with respect

to the the vector fields X on Σ defined as follows : To each point $m = (\ell, \varphi) \in \Sigma \setminus \{N, S\}$, we associate the unit tangent vector $Y(m) \in T_m \Sigma$, tangent to the *parallel* $\mathbb{S}^1 m$ (that is, the orbit of the point m generated by the actions of \mathbb{S}^1), with direction given by the positive orientation of \mathbb{S}^1 . Letting $X(m) = r(\ell)Y(m)$, $X(N) = 0$, $X(S) = 0$, then X defines a smooth tangent vector field on Σ . For $u \in C^\infty(\Sigma)$, we define

$$D_\varphi u = \frac{1}{i} \langle du, X \rangle.$$

On U , we simply have $D_\varphi = \frac{1}{i} \partial_\varphi$, with ∂_φ being the differentiation with respect to φ . Therefore D_φ is symmetric and commutes with Δ , at least in a formal way,

$$[\Delta, D_\varphi] = 0.$$

Then we state a proposition concerning the geodesics of Σ .

Proposition 2.2.1. *Let Σ be a Zoll surface of revolution, and r be as above.*

- (1) *Then $r : [0, \pi] \rightarrow [0, 1]$ is smooth, with $r(0) = r(\pi) = 0$, $r'(0) = 1$, $r'(\pi) = -1$, $r''(0) = r''(\pi) = 0$. There exists a unique $\ell_0 \in]0, \pi[$ such that $r(\ell_0) = 1$. Furthermore, $r'(\ell_0) = 0$, $r''(\ell_0) < 0$. The curve $\Gamma = \{(\ell_0, \varphi) : \varphi \in \mathbb{S}^1\}$ is a geodesic called the equator.*
- (2) *Apart from the equator, every geodesic is contained between a pair of parallels $\{\ell = \ell_1\}$ and $\{\ell = \ell_2\}$ for some $\ell_1 < \ell_0 < \ell_2$, and contacts each of the parallel exactly once.*

Corollary 2.2.2. *From this proposition, every geodesic of Σ except for the equator Γ enters the upper hemi-surface $\Sigma^+ = \{\ell > \ell_0\}$.*

To simplify later calculations, we will work on an isothermal coordinate defined on U as follows. Let $f \in C^\infty(\mathbb{R})$ be the solution to the following first order ordinary differential equation,

$$f'(x) = r(f(x)), \quad f(0) = \ell_0.$$

It is not difficult to see that (we refer to [20])

$$0 < f < \pi, \quad \lim_{x \rightarrow -\infty} f(x) = 0, \quad \lim_{x \rightarrow \infty} f(x) = \pi.$$

Therefore f defines a diffeomorphism $\mathbb{R} \simeq]0, \pi[$, with the equator now being $\Gamma = \{x = 0\}$. Set $x = f^{-1}(\ell)$, then the coordinates (x, φ) are isothermal, indeed,

$$g = f'(x)^2 dx^2 + r(f(x))^2 d\varphi^2 = \rho(x)^2 (dx^2 + d\varphi^2),$$

where $\rho(x) := r(f(x)) = f'(x)$. We have $\rho \in]0, 1]$, and $\rho(x) < 1$ except for $x = 0$, where $\rho(0) = 1$, $\rho'(0) = 0$, $\rho''(0) < 0$. We also have, under these coordinates,

$$(2.2.1) \quad L^2(\Sigma) = L^2(\mathbb{R} \times \mathbb{S}, \rho^2 dx \otimes d\varphi),$$

and the Laplacian-Beltrami operator takes a simple form,

$$(2.2.2) \quad \Delta = \frac{1}{\rho(x)^2} (\partial_x^2 + \partial_\varphi^2).$$

2.2.1.2 Laplacian spectrum and eigenfunctions

Recall that for some $A > 0$,

$$\text{Spec}(-\Delta) \subset \bigsqcup_{n \geq 0} I_n, \quad \text{with } I_n \subset](n + 1/2)^2 - A, (n + 1/2)^2 + A[.$$

For $\lambda \geq 0$ such that $\lambda^2 \in \text{Spec}(-\Delta)$, we let E_λ denote the (minus) Laplacian eigenspace of the eigenvalue λ^2 , and set for $n \geq 0$ the linear space

$$\tilde{E}_n = \bigoplus_{\lambda^2 \in I_n} E_\lambda.$$

We define a linear (unbounded) operator \mathcal{L} by a compact perturbation of $-\Delta$ such that

$$\text{Spec}(\mathcal{L}) \subset \{(n + 1/2)^2 : n \in \mathbb{N}\}.$$

Indeed, let $\Pi_n : L^2(\Sigma) \rightarrow \tilde{E}_n$ denote the orthogonal projection, then we formally define

$$(2.2.3) \quad \mathcal{L} = \sum_{n \geq 0} (n + 1/2)^2 \Pi_n.$$

Next we study the structure of E_λ . Since $-\Delta$ commutes with D_φ , it is natural to expect an orthogonal decomposition of E_λ into D_φ eigenspaces. The following proposition is inspired by Beekmann [26].

Proposition 2.2.3. *On each E_λ , we have a direct sum decomposition,*

$$E_\lambda = \bigoplus_{k \in \mathbb{Z}} e^{ik\varphi} A_{\lambda,k},$$

where $A_{\lambda,k} \subset C^\infty(\mathbb{R})$ is the solution space to

$$(2.2.4) \quad -\partial_x^2 w + k^2 w = \lambda^2 \rho^2 w,$$

with boundary conditions $\lim_{|x| \rightarrow \infty} \partial_x^n w(x) = 0$ for $n \in \mathbb{N}$ and $k \neq 0$. In particular,

$$\dim A_{\lambda,k} \leq 1, \quad \text{if } k \neq 0.$$

If $u(x, \varphi) = e^{ik\varphi} w(x) \in e^{ik\varphi} A_{\lambda,k}$, then

$$-\Delta u = \lambda^2 u, \quad D_\varphi u = ku.$$

That is, $e^{ik\varphi} A_{\lambda,k}$ are eigenspaces of D_φ , and the decomposition is thus orthogonal.

Démonstration. The group action of \mathbb{S}^1 on Σ induces naturally a group action on function spaces by $\varphi f = f \circ \varphi^{-1}$. Now that \mathbb{S}^1 commutes with $-\Delta$, E_λ is stable under \mathbb{S}^1 . It is known that the irreducible complex representations of \mathbb{S}^1 are all one-dimensional of the form

$$\tau_k : \mathbb{S}^1 \rightarrow U(1) \quad \varphi \mapsto e^{ik\varphi}, \quad k \in \mathbb{Z}.$$

Therefore E_λ can be decomposed into τ_k -invariant subspaces, consisting of functions $u(x, \varphi)$ satisfying

$$u(x, \varphi) = \varphi^{-1} u(x, 0) = e^{-ik\varphi} u(x, 0),$$

which also shows that $D_\varphi u = ku$.

To obtain the equation satisfied by $w \in A_{\lambda,k}$, it suffices to plug $u(x, \varphi) = e^{ik\varphi}w(x)$ into the equation $-\Delta u = \lambda^2 u$. The boundary condition for $k \neq 0$ comes evidently from the continuity of $D_\varphi^n u = k^n u$ at N and S . To show that $\dim A_{\lambda,k} \leq 1$, let w_1 and w_2 be two solutions to (2.2.4), then their Wronskian $W(w_1, w_2)$, which is a constant by a direct calculation, vanishes at infinity by the boundary conditions. So these two solutions are linearly dependent. \square

Corollary 2.2.4. *If $0 \neq |k| \geq \lambda$, then $A_{\lambda,k} = \{0\}$.*

Démonstration. Suppose $w \in A_{\lambda,k}$ with $0 \neq |k| \geq \lambda$, then for $n \in \mathbb{N}$,

$$-\partial_x^2 w + k^2 w = \lambda^2 \rho^2 w, \quad \lim_{|x| \rightarrow \infty} \partial_x^n w(x) = 0.$$

We will show that $w \in H^1(\mathbb{R}) \subset C(\mathbb{R})$ (see Corollary 2.2.25), so it is legitimate to take $L^2(\mathbb{R})$ inner product between w and the equation to get

$$0 \leq \|\partial_x w\|_{L^2(\mathbb{R})}^2 = (-\partial_x^2 w, w)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} (\lambda^2 \rho^2(x) - k^2) |w(x)|^2 dx.$$

However, $0 \leq \rho \leq 1$, and that $\rho(x) < 1$ except for $x = 0$, we see that $\lambda^2 \rho(x)^2 - k^2 < 0$ except for $x = 0$. Therefore $w(x) \equiv 0$ since it is continuous. \square

Corollary 2.2.5. *For $k \in \mathbb{Z}$, and $\lambda_1 \neq \lambda_2$,*

$$A_{\lambda_1,k} \perp A_{\lambda_2,k} \quad \text{with respect to } L^2(\rho^2 dx, \mathbb{R}).$$

Démonstration. For $w_i \in A_{\lambda_i,k}$ with $i = 1, 2$, set $u_i(x, \varphi) = e^{ik\varphi} w_i(x)$, then by (2.2.1),

$$0 = (u_1, u_2)_{L^2(\Sigma)} = (e^{ik\varphi} w_1, e^{ik\varphi} w_2)_{L^2(\rho^2 dx, \mathbb{R}) \otimes L^2(d\varphi, \mathbb{S}^1)} = 2\pi (w_1, w_2)_{L^2(\rho^2 dx, \mathbb{R})}.$$

\square

Remark 2.2.6. For $n \in \mathbb{N}$, we set

$$\tilde{A}_{n,k} = \bigoplus_{\lambda^2 \in I_n} A_{\lambda,k},$$

then it is an orthogonal direct sum with respect to $L^2(\rho^2 dx, \mathbb{R})$. And we have

$$\tilde{E}_n = \bigoplus_{k \in \mathbb{Z}} e^{ik\varphi} \tilde{A}_{n,k}.$$

2.2.2 Reduction to observability of \mathcal{L} -eigenfunctions

Since Σ has no boundary, the energy of a solution u to (2.1.1) does not control its zero frequency. In order to deal with this problem, we introduce the quotient Sobolev spaces

$$\mathcal{H}^s(\Sigma) = H^s(\Sigma)/\mathbb{C} = \{[u] = u + \mathbb{C} : u \in H^s(\Sigma)\},$$

equipped with the quotient norms. We set in particular,

$$\|[u]\|_{\mathcal{H}^1(\Sigma)} = \|\nabla u\|_{L^2(\Sigma)},$$

so that, for $u \in C(\mathbb{R}, H^1(\Sigma)) \cap C^1(\mathbb{R}, L^2(\Sigma))$,

$$E(u, t) = \frac{1}{2} \|(u(t), \partial_t u(t))\|_{\mathcal{H}^1 \times L^2}^2.$$

By the theorem of Hille–Yosida, we have

Proposition 2.2.7. *Define the quotient Laplacian by $[\Delta][u] = [\Delta u]$ for $[u] \in D([\Delta]) = \{[u] \in \mathcal{H}^1(\Sigma) : \Delta u \in L^2(\Sigma)\}$. Set*

$$[A] = \begin{pmatrix} 0 & -[\text{Id}] \\ -j \circ [\Delta] & a \end{pmatrix}$$

with $D([A]) = D([\Delta]) \times H^1(\Sigma)$ where $[\text{Id}] : H^1(\Sigma) \rightarrow \mathcal{H}^1(\Sigma)$ is the canonical projection, while $j : \mathcal{H}^0(\Sigma) \rightarrow L^2(\Sigma)$ associates each $[w] \in \mathcal{H}^0(\Sigma)$ a representative w such that $\int_{\Sigma} w \, dx = 0$. Then for all $([u_0], u_1) \in \mathcal{H}^1(\Sigma) \times L^2(\Sigma)$, there exists a unique solution $([u], v) \in C(\mathbb{R}, \mathcal{H}^1(\Sigma)) \times C^1(\mathbb{R}, L^2(\Sigma))$ of the equation

$$(2.2.5) \quad \begin{cases} \partial_t \begin{pmatrix} [u] \\ v \end{pmatrix} + [A] \begin{pmatrix} [u] \\ v \end{pmatrix} = 0, \\ ([u], v)_{t=0} = ([u_0], u_1). \end{cases}$$

Moreover, if u is the solution of (2.1.1) with initial data (u_0, u_1) , then $([u], \partial_t u)$ is the solution to (2.2.5).

Proposition 2.2.8. *If $0 \leq a \in L^\infty(\Sigma)$ and $a \not\equiv 0$, then a weakly stabilizes (2.1.1).*

Démonstration. The idea of the proof comes from [44]. By a density argument, it suffices to suppose that $(u_0, u_1) \in D(A)$, so that $([u_0], u_1) \in D([A])$. Let u denote the corresponding solution to (2.1.1). Observe that $E(u, t) = \frac{1}{2} \|([u], \partial_t u)\|_{\mathcal{H}}^2$ is non-increasing, and that $[A]$ commutes with the evolution of (2.2.5),

$$(2.2.6) \quad \begin{aligned} \left\| \begin{pmatrix} [u] \\ \partial_t u \end{pmatrix} \right\|_{[A]} &:= \left\| \begin{pmatrix} [u] \\ \partial_t u \end{pmatrix} \right\|_{\mathcal{H}} + \left\| [A] \begin{pmatrix} [u] \\ \partial_t u \end{pmatrix} \right\|_{\mathcal{H}} \\ &\leq \left\| \begin{pmatrix} [u_0] \\ u_1 \end{pmatrix} \right\|_{\mathcal{H}} + \left\| A \begin{pmatrix} [u_0] \\ u_1 \end{pmatrix} \right\|_{\mathcal{H}} < \infty. \end{aligned}$$

We claim that $(D([A]), \|\cdot\|_{[A]}) \hookrightarrow \mathcal{H}$ is compact. Indeed, if $(\begin{smallmatrix} [u_n] \\ v_n \end{smallmatrix})$ is bounded in $D([A])$, then $u_n, \Delta u_n, v_n, \nabla v_n$ are bounded in $L^2(\Sigma)$. Up to a subsequence, $u_n - \int_M u_n \rightarrow u_0$ in $H^1(\Sigma)$, so $[u_n] \rightarrow [u_0]$ in $\mathcal{H}^1(\Sigma)$, and $v_n \rightarrow v_0$ in $L^2(\Sigma)$. By (2.2.6), there exists a sequence $t_k \rightarrow +\infty$ such that $([u(t_k)], \partial_t u(t_k)) \rightharpoonup ([v_0], v_1)$ weakly in $D([A])$; and strongly in $\mathcal{H}^1(\Sigma)$ by compactness. Let v be the solution to (2.1.1) with initial data (v_0, v_1) , where v_0 is the representative of $[v_0]$ such that $\int_{\Sigma} v_0 \, dx = 0$, then

$$\begin{aligned} E(v, t) &= \left\| \begin{pmatrix} [v(t)] \\ \partial_t v \end{pmatrix} \right\|_{\mathcal{H}} = \left\| e^{-t[A]} \begin{pmatrix} [v_0] \\ v_1 \end{pmatrix} \right\|_{\mathcal{H}} = \left\| e^{-t[A]} \lim_{k \rightarrow \infty} e^{-t_k[A]} \begin{pmatrix} [u_0] \\ u_1 \end{pmatrix} \right\|_{\mathcal{H}} \\ &= \left\| \lim_{k \rightarrow \infty} e^{-(t+t_k)[A]} \begin{pmatrix} [u_0] \\ u_1 \end{pmatrix} \right\|_{\mathcal{H}} = \left\| \begin{pmatrix} [v_0] \\ v_1 \end{pmatrix} \right\|_{\mathcal{H}} = E(v, 0). \end{aligned}$$

So v satisfies the undamped wave equation (2.1.5) as well.

We decompose the initial data as $v_0 = \sum_{\lambda} v_{\lambda}^0$, $v_1 = \sum_{\lambda} v_{\lambda}^1$, where λ varies in $\text{Spec}(\sqrt{-\Delta})$ and $v_{\lambda}^i \in E_{\lambda}$. Then $v_0^0 = 0$, and

$$v(t) = \cos(t\sqrt{-\Delta})v_0 + \sqrt{-\Delta}^{-1} \sin(t\sqrt{-\Delta})v_1 = v_0^1 t + \sum_{\lambda \neq 0} (e^{it\lambda} v_{\lambda}^+ + e^{-it\lambda} v_{\lambda}^-),$$

where for $\lambda \neq 0$,

$$v_{\lambda}^+ + v_{\lambda}^- = v_{\lambda}^0, \quad i\lambda(v_{\lambda}^+ - v_{\lambda}^-) = v_{\lambda}^1.$$

Now fix $\lambda' \neq 0$, and set $w_{\lambda'}(T, x) = \frac{1}{T} \int_0^T \partial_t v(t, x) e^{-it\lambda'} dt$. The fact that $a\partial_t v = 0$ implies $aw_{\lambda'} = 0$. An explicit calculation shows

$$\begin{aligned} w_{\lambda'}(T) &= i\lambda' v_{\lambda'}^+ + \sum_{\lambda \neq \lambda'} \frac{i\lambda}{iT(\lambda - \lambda')} (e^{iT(\lambda - \lambda')} - 1) v_{\lambda}^+ \\ &\quad - \sum_{\lambda} \frac{i\lambda}{iT(\lambda + \lambda')} (e^{-iT(\lambda + \lambda')} - 1) v_{\lambda}^-. \end{aligned}$$

This implies that, as $T \rightarrow \infty$, $w_{\lambda'}(T) \rightarrow i\lambda' v_{\lambda'}^+$ in $L^2(\Sigma)$. Since $aw_{\lambda'} = 0$ and $\lambda' \neq 0$, we must have $av_{\lambda'}^+ = 0$. Therefore $v_{\lambda'}^+ = 0$ because as a classical result, the nodal set $\{v_{\lambda'}^+ = 0\}$ is of zero measure. The same argument shows that $v_{\lambda'}^- = 0$ for $\lambda' \neq 0$. And similarly, since $\frac{1}{T} \int_0^T \partial_t v(t, x) dt = v_0^1$, we have $T \rightarrow \infty$, $0 \equiv av_0^1$, whence $v_0^1 = 0$. Therefore $v \equiv 0$, and $E(u, t_k) \rightarrow E(v, 0) = 0$. \square

Let \mathcal{L} be defined by (2.2.3). Recall the undamped wave equation (2.1.5)

$$(2.1.5) \quad \begin{cases} (\partial_t^2 - \Delta)u = 0, & \text{in } \mathcal{D}'(\mathbb{R} \times \Sigma); \\ (u, \partial_t u)_{t=0} = (u_0, u_1), & \in H^1(\Sigma) \times L^2(\Sigma), \end{cases}$$

and the perturbed wave equation (2.1.14),

$$(2.1.14) \quad \begin{cases} (\partial_t^2 + \mathcal{L})u = 0, & \text{in } \mathcal{D}'(\mathbb{R} \times \Sigma); \\ (u, \partial_t u)_{t=0} = (u_0, u_1), & \in H^1(\Sigma) \times L^2(\Sigma). \end{cases}$$

Definition 2.2.9. Let $0 \leq a \in L^\infty(\Sigma)$, we say that a observes (2.1.5) (resp. (2.1.14)), if for some constant $C > 0$, $T > 0$, and every solution u to (2.1.5) (resp. (2.1.14)), the observability (2.1.6) holds. We say that a observes \mathcal{L} -eigenfunctions, if for some constant $C > 0$ and every \mathcal{L} -eigenfunction $u \in \cup_n \tilde{E}_n$, the observability (2.1.15) holds.

We will reduce the observability of (2.1.5) to the observability of (2.1.14), and then to the observability of \mathcal{L} -eigenfunctions. We first state some preliminaries as those used in proving Theorem 2.1.3. For $u \in H^s(\Sigma)$ with $s \in \mathbb{R}$, there exists a unique decomposition into sums of \mathcal{L} -eigenfunctions,

$$u = \sum_{n \geq 0} u_n, \quad \text{with } u_n \in \tilde{E}_n.$$

Then we specify the $H^s(\Sigma)$ norm of u by setting

$$\|u\|_{H^s(\Sigma)}^2 = \sum_{n \geq 0} (n + 1/2)^{2s} \|u_n\|_{L^2}^2.$$

If we decompose the initial data $u_i = \sum_{n \geq 0} u_n^i$, ($i = 0, 1$), with $u_n^i \in \tilde{E}_n$, then the solution to (2.1.14) is

$$u(t) = \cos(t\sqrt{\mathcal{L}})u_0 + \sqrt{\mathcal{L}}^{-1} \sin(t\sqrt{\mathcal{L}})u_1 = \sum_{n \geq 0} (e^{it(n+1/2)}u_n^+ + e^{-it(n+1/2)}u_n^-),$$

where for $n \geq 0$

$$u_n^+ + u_n^- = u_n^0, \quad i(n+1/2)(u_n^+ - u_n^-) = u_n^1,$$

and satisfies the a priori estimate $\|u\|_{L^\infty H^s(\Sigma)} \lesssim \|(u_0, u_1)\|_{H^s(\Sigma) \times H^{s-1}(\Sigma)}$.

Lemma 2.2.10. *Let $0 \leq a \in L^\infty(\Sigma)$, if a observes (2.1.14), then a observes (2.1.5).*

Démonstration. The proof is a mimic of that of Lemma 2.1.15. Write $K = \Delta + \mathcal{L}$, then by the definition of \mathcal{L} , K is bounded on $L^2(\Sigma)$, with $\|K\|_{L^2 \rightarrow L^2} \leq A$. Let u be the solution to (2.1.5), with initial data (u_0, u_1) . There is no harm in assuming that $\int_\Sigma u_0 dx = 0$. Decompose $u = w + v$ with

$$\begin{cases} (\partial_t^2 + \mathcal{L})v = 0, & (v, \partial_t v)_{t=0} = (u_0, u_1); \\ (\partial_t^2 + \mathcal{L})w = Ku, & (w, \partial_t w)_{t=0} = (0, 0). \end{cases}$$

Now that a observes (2.1.14),

$$E(u, 0) = E(v, 0) \lesssim \int_0^T \int_\Sigma a |\partial_t v|^2 dx dt \lesssim \int_0^T \int_\Sigma a |\partial_t u|^2 dx dt + \int_0^T \int_\Sigma a |\partial_t w|^2 dx dt.$$

By Duhamel's formula,

$$\partial_t w(t) = \int_0^t \cos((t-s)\sqrt{\mathcal{L}})Ku(s) ds;$$

and we have

$$\|\partial_t w(t)\|_{L^2}^2 \lesssim \int_0^T \|u(s)\|_{L^2}^2 ds \lesssim \|(u_0, u_1)\|_{L^2 \times H^{-1}}^2 \lesssim \|(u_0, u_1)\|_{\mathcal{H}^0 \times H^{-1}}^2,$$

where the last inequality is because u_0 has no zero frequency. Combine the estimates above, we obtain a weak observability, with a compact remainder term on the right hand side

$$\frac{1}{2} \|([u_0], u_1)\|_{\mathcal{H}^1 \times L^2}^2 = E(u, 0) \lesssim \int_0^T \int_M a |\partial_t u|^2 dx dt + \|([u_0], u_1)\|_{\mathcal{H}^0 \times H^{-1}}^2.$$

To remove the remainder term and prove the (strong) observability, we appeal to the uniqueness-compactness argument originally due to Bardos–Lebeau–Rauch [24]. It is an argument by contradiction that carries out as follows. Suppose that the observability of (2.1.5) does not hold, then there exists a sequence of initial data $(u_0^n, u_1^n) \in H^1(\Sigma) \times L^2(\Sigma)$ such that, $\int_\Sigma u_0^n dx = 0$, and as $n \rightarrow \infty$

$$E(u^n, 0) = \frac{1}{2} \|([u_0^n], u_1^n)\|_{\mathcal{H}^1 \times L^2}^2 = 1, \quad \int_0^n \int_\Sigma a |\partial_t u^n|^2 dx dt = o(1).$$

where u^n are the corresponding solutions (2.1.5). By Rellich's compact injection theorem, up to a subsequence, we assume that, for some $([u_0], u_1) \in \mathcal{H}^1 \times L^2$,

- (1) $([u_0^n], u_1^n) \rightharpoonup ([u_0], u_1)$ weakly in $\mathcal{H}^1(\Sigma) \times L^2(\Sigma)$;
- (2) $([u_0^n], u_1^n) \rightarrow ([u_0], u_1)$ strongly in $\mathcal{H}^0(\Sigma) \times H^{-1}(\Sigma)$.

Passing $n \rightarrow \infty$ in the weak observability,

$$1 = E(u^n, 0) \leq o(1) + \|([u_0^n], u_1^n)\|_{\mathcal{H}^0 \times H^{-1}}^2 \rightarrow \|([u_0], u_1)\|_{\mathcal{H}^0 \times H^{-1}}^2.$$

Therefore, we will get a contradiction by showing that the right hand side vanishes.

To show this, we observe that

$$E(u^n, t) = E(u^n, 0) = \frac{1}{2} \|([u_0^n], u_1^n)\|_{\mathcal{H}^1 \times L^2}^2$$

is uniformly bounded in t and n . Therefore $[u^n]$ is bounded in $L^\infty(\mathbb{R}, \mathcal{H}^1(\Sigma))$ and $\partial_t u^n$ is bounded in $L^\infty(\mathbb{R}, L^2(\Sigma))$. Moreover $\int_\Sigma u^n(t, x) dx$ is bounded in $L^\infty_{\text{loc}}(\mathbb{R}_t)$ (and is of order $O(t)$). Consequently u^n is bounded in $L^\infty_{\text{loc}}(\mathbb{R}, H^1(\Sigma))$. The theorem of Ascoli and the compact injection theorem of Rellich show that, up to a subsequence, there exists a $(u, v) \in C(\mathbb{R}, L^2(\Sigma)) \times L^\infty(\mathbb{R}, L^2(\Sigma))$, such that

- (1) $u^n \rightarrow u$ strongly in $L^\infty_{\text{loc}}(\mathbb{R}, L^2(\Sigma))$;
- (2) $u^n \rightharpoonup u$ respect to the weak-* topology of $L^\infty_{\text{loc}}(\mathbb{R}, H^1(\Sigma))$;
- (3) $\partial_t u^n \rightharpoonup v$ with respect to the weak-* topology of $L^\infty(\mathbb{R}, L^2(\Sigma))$.

Passing to the limit in the sense of distribution, we see that u satisfies (2.1.5), with in particular $v = \partial_t u$. Therefore $\partial_t(\partial_t u^n) = \Delta u^n$ is bounded in $L^\infty(\mathbb{R}, H^{-1}(\Sigma))$, so that $\partial_t u \in C(\mathbb{R}, H^{-1}(\Sigma))$, and $u \in C(\mathbb{R}, L^2(\Sigma)) \cap C^1(\mathbb{R}, H^{-1}(\Sigma))$. However, since $(u_0, u_1) \in H^1(\Sigma) \times L^2(\Sigma)$, there exists a solution in the $C(\mathbb{R}, H^1(\Sigma)) \cap C^1(\mathbb{R}, L^2(\Sigma))$. By the uniqueness of the solution in $C(\mathbb{R}, L^2(M)) \cap C^1(\mathbb{R}, H^{-1}(M))$, these two solutions must coincide. Therefore, it is legitimate to talk about the energy of u , which is conserved $E(u, t) \equiv E(u, 0)$. On the other hand, since $a \partial_t u = 0$ in $\mathcal{D}'(\Sigma)$, u should also satisfy the damped wave equation (2.1.1). Then Proposition 2.2.8 shows that the energy $E(u, t)$ must decay to zero as $t \rightarrow +\infty$. Hence $E(u, 0) = 0$, i.e. $([u_0], u_1) = (0, 0)$. \square

Lemma 2.2.11. *Let $0 \leq a \in L^\infty(\Sigma)$, if a observes \mathcal{L} -eigenfunctions, then a observes (2.1.14).*

Démonstration. Recall that a solution to (2.1.14) is of the form

$$u(t) = \sum_{n \geq 0} (e^{it(n+1/2)} u_n^+ + e^{-it(n+1/2)} u_n^-),$$

where $u_n^\pm \in \tilde{E}_n$. Now that a observes \mathcal{L} -eigenfunctions, which implies

$$\|a^{1/2} u_n^\pm\|_{L^2(\Sigma)} \gtrsim \|u_n^\pm\|_{L^2(\Sigma)},$$

we have, by the orthogonality of $\{e^{\pm i(n+1/2)t}\}_{n \in \mathbb{N}}$ in $L^2([0, 2\pi])$, and a similar argument to that of Lemma 2.1.16,

$$\begin{aligned} & \int_0^{2\pi} \int_\Sigma a |\partial_t u|^2 dx dt \\ &= \int_\Sigma a(x) \int_0^{2\pi} \left| \sum_{n \geq 0} (n+1/2) (e^{it(n+1/2)} u_n^+ - e^{-it(n+1/2)} u_n^-) \right|^2 dt dx \\ &= 2\pi \int_\Sigma a(x) \sum_{n \geq 0} |(n+1/2) u_n^+|^2 + |(n+1/2) u_n^-|^2 dx \end{aligned}$$

$$\begin{aligned}
 &= 2\pi \sum_{n \geq 0} (n + 1/2)^2 \int_{\Sigma} a(x) (|u_n^+|^2 + |u_n^-|^2) dx \\
 &\gtrsim \sum_{n \geq 0} (n + 1/2)^2 \int_{\Sigma} (|u_n^+|^2 + |u_n^-|^2) dx \\
 &\gtrsim E(u, 0).
 \end{aligned}$$

□

2.2.3 Observability of \mathcal{L} -eigenfunctions

This sections aims to prove the observability of \mathcal{L} -eigenfunctions, which concludes Theorem 2.1.10. In order to avoid ambiguity, we denote by $X = (x, \varphi)$ the coordinate on Σ .

Proposition 2.2.12. *Let Σ be a Zoll surface of revolution, then $a(X) = 1_{\Sigma^+}(X)$ observes \mathcal{L} -eigenfunctions.*

We prove this proposition with an argument by contradiction. If the observability of \mathcal{L} -eigenfunctions does not hold, then there exists a sequence of \mathcal{L} -eigenfunctions $u_{n_m} \in \tilde{E}_{n_m}$ such that, as $m \rightarrow \infty$,

$$\|u_{n_m}\|_{L^2(\Sigma)} = 1, \quad \|1_{\Sigma^+} u_{n_m}\|_{L^2(\Sigma)} = o(1).$$

If $\{n_m\}_{m \geq 0}$ is bounded, then $\tilde{E} := \bigoplus_{m \geq 0} \tilde{E}_{n_m}$ is a finite dimensional vector subspace of $L^2(\Sigma)$, consisting only of low frequencies, on which the estimate holds, for any $N > 0$,

$$\|u\|_{L^2(\Sigma)} \lesssim \|u\|_{H^{-N}(\Sigma)}.$$

Therefore, $(\tilde{E}, \|\cdot\|_{L^2(\Sigma)})$ is relatively compact, and the bounded sequence $\{u_{n_m}\}_{m \geq 0}$ admits a limit point $u \in \tilde{E}$, that is, $u_{n_m} \rightarrow u$ in $L^2(\Sigma)$, and hence $1_{\Sigma^+} u_{n_m} \rightarrow 1_{\Sigma^+} u$ in $L^2(\Sigma)$. Consequently,

$$\|u\|_{L^2(\Sigma)} = 1, \quad \|1_{\Sigma^+} u\|_{L^2(\Sigma)} = 0.$$

However this is impossible, for u is a finite sum of Laplacian eigenfunctions, which does not vanish only on a set of zero measure.

We are left to consider the case where $\{n_m\}_{m \geq 0}$ is unbounded. Up to a subsequence, we may assume that n_m increases to ∞ . For simplicity of notation, we drop the m subindex, and write $n = n_m$, and introduce the semiclassical parameter

$$h = (n + 1/2)^{-1}.$$

We then write $u(h) = u_n$, which satisfies $\mathcal{L}u(h) = h^{-2}u(h)$, and consequently

$$(2.2.7) \quad (-h^2 \Delta - 1)u(h) = -h^2 K u(h) = O(h^2)_{L^2(\Sigma)}.$$

2.2.3.1 Concentration of \mathcal{L} -eigenfunctions

We study the semiclassical measures of the sequence $u(h)$ and show that it concentrates on the equator. This argument is rather standard, we refer to, for example [41], see also [208]. We recall the definition of the semiclassical measure and some of its basic properties in Appendix 2.3.

We extract a subsequence if necessary, and assume in addition that $u(h)$ is pure (see Remark 2.3.3 for the definition).

Proposition 2.2.13. *Let μ be the h -semiclassical measure of $u(h)$, then*

$$(2.2.8) \quad \text{supp } \mu \subset S^*\Sigma \cap \{x = 0, \xi = 0\} = \{(0, \varphi, 0, \pm 1) : \varphi \in \mathbb{S}^1\}.$$

Recall that the coordinate on Σ is $X = (x, \varphi)$, with the dual coordinate being $\Xi = (\xi, \theta)$.

Démonstration. Recall that $u(h)$ satisfies the equation

$$(-h^2\Delta - 1)u(h) = O(h^2)_{L^2(\Sigma)}.$$

The principal symbol of $-h^2\Delta - 1$ (in the semiclassical sense) is

$$p(X, \Xi) = g_X^{-1}(\Xi, \Xi) - 1,$$

where g^{-1} is the inverse matrix of g . By Theorem 2.3.5,

$$\text{supp } \mu \subset T^*M \cap \{p(X, \Xi) = 0\} = S^*M, \quad H_p\mu = 0.$$

Now that H_p generates the (co)-geodesic flow on S^*M , we see that μ is invariant by the geodesic flow. Moreover, our hypothesis $\|1_{\Sigma^+}u(h)\| = o(1)_{L^2(\Sigma)}$ implies that

$$\text{supp } \mu \cap T^*\Sigma^+ = \emptyset.$$

Recall that all geodesics enter Σ^+ , except for the equator,

$$\text{supp } \mu \subset S^*M \setminus \bigcup_{t \in \mathbb{R}} e^{tH_p} S^*\Sigma^+ = S^*\Sigma \cap \{x = 0, \xi = 0\}.$$

We conclude by a direct calculation, using $g|_{\Gamma} = dx^2 + d\varphi^2$. □

Corollary 2.2.14. *Let $\epsilon > 0$, and $\chi_\epsilon \in C_c^\infty(\mathbb{R})$ be such that $1_{[-\epsilon, \epsilon]} \leq \chi_\epsilon \leq 1_{[-2\epsilon, 2\epsilon]}$. Then*

$$u(h) = \chi_\epsilon(1 - h^2 D_\varphi^2)u(h) + o(1)_{L^2(\Sigma)},$$

where $\chi_\epsilon(1 - h^2 D_\varphi^2)$ is defined by functional calculus, and is thus of (semiclassical) principal symbol $\chi_\epsilon(1 - \theta^2)$ (see for example [69]).

Démonstration. Let $v(h) = u(h) - \chi_\epsilon(1 - h^2 D_\varphi^2)u(h)$, which is pure. Now that D_φ commute with $-\Delta$, by (2.2.7) we see that $v(h)$ satisfies $(-h^2\Delta - 1)v(h) = O(h^2)_{L^2}$. Therefore $v(h)$ is h -oscillating by Example 2.3.9. And by Proposition 2.3.8, to conclude, it suffices to show that the semiclassical measure ν of $v(h)$ vanishes. Indeed, $\nu = (1 - \chi_\epsilon(1 - \theta^2))^2 \mu = 0$, since μ is supported in $1 - \theta^2 = 0$. □

As a consequence, in particular, for any $\epsilon > 0$, when h is sufficiently small,

$$\|u(h) - \chi_\epsilon(1 - h^2 D_\varphi^2)u(h)\|_{L^2(\Sigma)} \leq \epsilon.$$

Fixing a sequence of $\epsilon \rightarrow 0$, we can find a sequence of $h = h_\epsilon \rightarrow 0$, so that

$$(2.2.9) \quad u(h) = \sum_{k \in Z_n(\epsilon)} e^{ik\varphi} \tilde{w}_{n,k} + O(\epsilon)_{L^2(\Sigma)},$$

where $Z_n(\epsilon) = \{k \in \mathbb{Z} : |1 - h^2 k^2| \leq \epsilon\}$, and $\tilde{w}_{n,k} \in \tilde{A}_{n,k}$.

For later convenience, we introduce the notion of admissible sequences.

Definition 2.2.15. A 4-tuple $(\epsilon, h, k, \tilde{w})$ is called admissible if

- (1) $\epsilon > 0$, $h = (n + 1/2)^{-1}$ for some $n \in \mathbb{N}$;
- (2) $k \in Z_n(\epsilon)$, $\tilde{w} \in \tilde{A}_{n,k}$.

A sequence of 4-tuple $(\epsilon, h, k, \tilde{w})$ (where by an abuse of notation, we omit the index of the sequence for simplicity) is called admissible if

- (1) each term of the sequence is an admissible 4-tuple;
- (2) $\epsilon \rightarrow 0$, $h \rightarrow 0$.

2.2.3.2 Reduction to observability of 1D Schrödinger equation

Proposition 2.2.16. *There exists $\epsilon_0 > 0$, $h_0 > 0$ and $C > 0$, such that for $0 < \epsilon < \epsilon_0$, $0 < h < h_0$, if a 4-tuple $(\epsilon, h, k, \tilde{w})$ is admissible, then we have the following observability,*

$$\|1_{x>0}\tilde{w}\|_{L^2(\rho^2 dx)} \geq C\|\tilde{w}\|_{L^2(\rho^2 dx)}.$$

If this proposition is proven, then we can finish the proof of Proposition 2.2.12, and thus prove Theorem 2.1.10. Indeed, we use the decomposition (2.2.9), (2.2.1), and the orthogonality of $\{e^{ik\varphi}\}_{k \in \mathbb{Z}}$ in $L^2(d\varphi, \mathbb{S}^1)$, when ϵ and $h = h_\epsilon$ are sufficiently small,

$$\begin{aligned} \|1_{\Sigma^+}u(h)\|_{L^2(\Sigma)}^2 &\gtrsim \|1_{\Sigma^+} \sum_{k \in Z_n(\epsilon)} e^{ik\varphi} \tilde{w}_{n,k}\|_{L^2(\Sigma)}^2 - \epsilon^2 \\ &\gtrsim \int_{\mathbb{R}} 1_{x>0} \int_{\mathbb{S}^1} \left| \sum_{k \in Z_n(\epsilon)} e^{ik\varphi} \tilde{w}_{n,k} \right|^2 d\varphi \rho^2 dx - \epsilon^2 \\ &\gtrsim \int_{\mathbb{R}} 1_{x>0} \sum_{k \in Z_n(\epsilon)} |\tilde{w}_{n,k}|^2 \rho^2 dx - \epsilon^2 \\ &\gtrsim \int_{\mathbb{R}} \sum_{k \in Z_n(\epsilon)} |\tilde{w}_{n,k}|^2 \rho^2 dx - \epsilon^2 \\ &\gtrsim \int_{\mathbb{R}} \int_{\mathbb{S}^1} \left| \sum_{k \in Z_n(\epsilon)} e^{ik\varphi} \tilde{w}_{n,k} \right|^2 d\varphi \rho^2 dx - \epsilon^2 \\ &\gtrsim \left\| \sum_{k \in Z_n(\epsilon)} e^{ik\varphi} \tilde{w}_{n,k} \right\|_{L^2(\Sigma)}^2 - \epsilon^2 \\ &\gtrsim \|u(h)\|_{L^2(\Sigma)}^2 - \epsilon^2 \\ &\gtrsim 1 - \epsilon^2, \end{aligned}$$

which contradicts to our hypothesis that $\|1_{\Sigma^+}u(h)\|_{L^2(\Sigma)} = o(1)$ as $h \rightarrow 0$.

Before proving Proposition 2.2.16, we observe that if $\tilde{w} \in \tilde{A}_{n,k}$, then \tilde{w} satisfies a one dimensional semiclassical stationary Schrödinger equation,

$$(2.2.10) \quad (-h^2\partial_x^2 + V)\tilde{w} = E\tilde{w} + O(h^2)_{L^2 \rightarrow L^2}\tilde{w},$$

where the potential $V = 1 - \rho^2$ satisfies $0 \leq V < 1 = \lim_{|x| \rightarrow \infty} V(x)$, and

$$(2.2.11) \quad V = cx^2 + O(x^3) \quad \text{near } x = 0,$$

recalling that $c = -r''(\ell_0)/2 > 0$; while the energy

$$E = 1 - h^2k^2,$$

satisfies by Corollary 2.2.4 and Proposition 2.1.5 the estimate

$$E = 1 - \lambda^{-2}k^2 - h^2(\lambda^2 - h^{-2})\lambda^{-2}k^2 \gtrsim -h^2.$$

To obtain (2.2.10), we write $\tilde{w} = \sum_{\lambda^2 \in I_n} w_{\lambda,k}$ with $w_{\lambda,k} \in A_{\lambda,k}$, then by Proposition 2.2.3, $w_{\lambda,k}$ satisfies

$$(2.2.12) \quad (-h^2\partial_x^2 + V)w_{\lambda,k} = Ew_{\lambda,k} + h^2(\lambda^2 - h^{-2})\rho^2w_{\lambda,k} = Ew_{\lambda,k} + O(h^2)\rho^2w_{\lambda,k}.$$

It remains to sum up $w_{\lambda,k}$, and use the orthogonality by Corollary 2.2.5 to obtain the estimate for the remainder term (be careful that the constant $O(h^2)$ varies for different $w_{\lambda,k}$, and cannot be moved to the front of the summation)

$$\begin{aligned} \left\| \sum_{\lambda^2 \in I_n} O(h^2)\rho^2w_{\lambda,k} \right\|_{L^2}^2 &\leq \|\rho\|_{L^\infty}^2 \left\| \sum_{\lambda^2 \in I_n} O(h^2)w_{\lambda,k} \right\|_{L^2(\rho^2 dx)}^2 \\ &\lesssim \sum_{\lambda^2 \in I_n} \|O(h^2)w_{\lambda,k}\|_{L^2(\rho^2 dx)}^2 \\ &\lesssim h^4 \sum_{\lambda^2 \in I_n} \|w_{\lambda,k}\|_{L^2(\rho^2 dx)}^2 \\ &\lesssim h^4 \left\| \sum_{\lambda^2 \in I_n} w_{\lambda,k} \right\|_{L^2(\rho^2 dx)}^2 \lesssim h^4 \|\tilde{w}\|_{L^2}^2 \end{aligned}$$

Proof of Proposition 2.2.16. A first consequence of (2.2.10) is that, by an Lithner-Agmon type estimate, \tilde{w} decays exponentially at infinity, so that the weight ρ^2 can be dropped (which will be done by Corollary 2.2.26), and we are left to prove the observability,

$$\|1_{x>0}\tilde{w}\|_{L^2} \gtrsim \|\tilde{w}\|_{L^2}.$$

Then we proceed with an argument by contradiction. Suppose that this observability is not true, then we can find an admissible sequence of $(\epsilon, h, k, \tilde{w})$ which violates the observability in the sense that

$$\|1_{x>0}\tilde{w}\|_{L^2} / \|\tilde{w}\|_{L^2} = o(1).$$

Now that \tilde{w} satisfies (2.2.10), and as we have seen, since $k \in Z_n(\epsilon)$, the energy E satisfies

$$-h^2 \lesssim E \leq \epsilon = o(1),$$

we may assume that, up to a subsequence, either $E = O(h)$, or $E \gg h$. We will show that, by Proposition 2.2.27 and Proposition 2.2.28, neither of these two cases is possible. This contradiction then finishes the proof. \square

2.2.3.3 Some Lithner–Agmon type estimates

In this section we prove some estimates of Lithner–Agmon Type, originally due to Lithner [139] and Agmon [1]. The argument we used here comes from [69, 94]. Let

$$P(\tau) = -h(\tau)^2 \partial_x^2 + V(x; \tau)$$

be a Schrödinger operator on \mathbb{R} , where the parameter $h(\tau)$ and the potential $V(\cdot; \tau) \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ both depend on $\tau \rightarrow 0$. We will consider the following two cases :

- (1) $h(\tau) \equiv 1$ does not depend on τ , then we get a classical Schrödinger operator ;
- (2) $h(\tau) \equiv \tau \rightarrow 0$, and we get a semiclassical Schrödinger operator.

We will estimate the solution u to the equation

$$(2.2.13) \quad P(\tau)u = E(\tau)u + f(\tau),$$

where $E(\tau) \in \mathbb{R}$, $f \in C(\mathbb{R}) \cap L^2(\mathbb{R})$. To do this, we define the Lithner–Agmon distance, for $x_1, x_2 \in \mathbb{R}$,

$$d(x_1, x_2; \tau) = \left| \int_{x_1}^{x_2} (V(x; \tau) - E(\tau))_+^{1/2} dx \right|.$$

For $\varepsilon > 0$, $R > 0$, let

$$\Phi^\varepsilon(x; \tau) = (1 - \varepsilon)d(x, 0; \tau), \quad \Phi_R^\varepsilon(x; \tau) = \chi_R(\Phi^\varepsilon(x; \tau)),$$

where $\chi_R(t) = 1_{t \leq R}(t)t + 1_{t > R}(t)R$.

We make the following assumption.

Assumption For all $\varepsilon > 0$, there exist $\tau_\varepsilon > 0$, $\delta_\varepsilon > 0$, $R_\varepsilon > 0$, $C_\varepsilon > 0$, such that for $0 < \tau < \tau_\varepsilon$, if $|x| \geq R_\varepsilon$, then $V(x; \tau) \geq E(\tau) + \delta_\varepsilon$; if $|x| \leq R_\varepsilon$, then $|V(x; \tau) - E(\tau)| < C_\varepsilon$, and $\Phi^\varepsilon(x; \tau) \leq \varepsilon$.

This assumption implies that $\Phi^\varepsilon(x; \tau) \rightarrow \infty$ as $|x| \rightarrow \infty$, uniformly for τ sufficiently small. Therefore Φ_R^ε is constant, equaling to R , for $|x|$ sufficiently large.

We will drop the parameter τ for simplicity. The following proposition comes from [69].

Proposition 2.2.17. *Let $u \in C_c^2(\mathbb{R})$ and let $\Phi \in \text{Lip}_{\text{loc}}(\mathbb{R})$ be real valued, then the following identity holds.*

$$(2.2.14) \quad h^2 \int_{\mathbb{R}} |(e^{\Phi/h} u)'|^2 dx + \int_{\mathbb{R}} (V - |\Phi'|^2) e^{2\Phi/h} |u|^2 dx = \text{Re} \int_{\mathbb{R}} e^{2\Phi/h} P u \bar{u} dx.$$

Suppose now that the phase Φ is constant for $|x|$ large, and suppose $u \in C^2(\mathbb{R}) \cap D(P)$ with $D(P) = \{w \in L^2(\mathbb{R}) : Vw \in L^2(\mathbb{R}), w'' \in L^2(\mathbb{R})\}$. Set $u_R(x) = \chi(x/R)u(x)$, with $\chi \in C_c^\infty(\mathbb{R})$. Therefore $u_R \in C_c^2(\mathbb{R})$, and the previous proposition applies.

$$h^2 \int_{\mathbb{R}} |(e^{\Phi/h} u_R)'|^2 dx + \int_{\mathbb{R}} (V - |\Phi'|^2) e^{2\Phi/h} |u_R|^2 dx = \text{Re} \int_{\mathbb{R}} e^{2\Phi/h} P u_R \overline{u_R} dx.$$

By the Lebesgue dominated convergence theorem, $u_R \rightarrow u$ and $V u_R \rightarrow V u$ both strongly in $L^2(\mathbb{R})$ as $R \rightarrow \infty$. Now that $u \in D(P)$, $P u_R \rightarrow P u$ in $L^2(\mathbb{R})$. Now that Φ being constant for large $|x|$, we can pass to the limit on each side of the identity above, and prove the following corollary.

Corollary 2.2.18. *If $u \in C^2(\mathbb{R}) \cap D(P)$, and Φ is constant for large $|x|$, then the identity (2.2.14) holds.*

Now let $u \in C(\mathbb{R}) \cap L^2(\mathbb{R})$ be a solution to (2.2.13), then $u \in C^2(\mathbb{R}) \cap D(P)$, and the corollary applies,

$$(2.2.15) \quad h^2 \int_{\mathbb{R}} |(e^{\Phi_R^\varepsilon/h} u)'|^2 dx + \int_{\mathbb{R}} (V - E - |(\Phi_R^\varepsilon)'|^2) e^{2\Phi_R^\varepsilon/h} |u|^2 dx \\ = \operatorname{Re} \int_{\mathbb{R}} e^{2\Phi_R^\varepsilon/h} f \bar{u} dx \leq A_\varepsilon \|e^{\Phi_R^\varepsilon/h} u\|_{L^2}^2 + C_\varepsilon \|e^{\Phi_R^\varepsilon/h} f\|_{L^2}^2.$$

where $A_\varepsilon = (1 - (1 - \varepsilon)^2) \delta_\varepsilon$. For $0 < \tau < \tau_\varepsilon$ and $|x| \geq R_\varepsilon$, by the definition of Φ_R^ε ,

$$V(x) - E - |(\Phi_R^\varepsilon)'|^2 \geq (1 - (1 - \varepsilon)^2)(V(x) - E) \geq (1 - (1 - \varepsilon)^2) \delta_\varepsilon = A_\varepsilon.$$

Spitting the second integral in (2.2.15) into two parts, $\int = \int_{|x| \geq R_\varepsilon} + \int_{|x| < R_\varepsilon}$, we get

$$h^2 \int_{\mathbb{R}} |(e^{\Phi_R^\varepsilon/h} u)'|^2 dx + A_\varepsilon \int_{|x| \geq R_\varepsilon} e^{2\Phi_R^\varepsilon/h} |u|^2 dx - C_\varepsilon \|e^{\Phi_R^\varepsilon/h} f\|_{L^2}^2 \\ \leq (\|V(x) - E + (\Phi_R^\varepsilon)'\|_{L^\infty(|x| \leq R_\varepsilon)} + A_\varepsilon) \int_{|x| \leq R_\varepsilon} e^{2\Phi_R^\varepsilon/h} |u|^2 dx \\ \leq C_\varepsilon \int_{|x| \leq R_\varepsilon} e^{2\Phi_R^\varepsilon/h} |u|^2 dx.$$

Adding $A_\varepsilon \int_{|x| \leq R_\varepsilon} e^{2\Phi_R^\varepsilon/h} |u|^2 dx$ to each side of the inequality, we get

$$h^2 \int_{\mathbb{R}} |(e^{\Phi_R^\varepsilon/h} u)'|^2 dx + A_\varepsilon \int_{\mathbb{R}} e^{2\Phi_R^\varepsilon/h} |u|^2 dx - C_\varepsilon \|e^{\Phi_R^\varepsilon/h} f\|_{L^2}^2 \\ \leq C_\varepsilon \int_{|x| \leq R_\varepsilon} e^{2\Phi_R^\varepsilon/h} |u|^2 dx \\ \leq C_\varepsilon \sup_{|x| \leq R_\varepsilon} (e^{2\Phi_R^\varepsilon/h}) \|u\|_{L^2}^2 \leq C_\varepsilon e^{2\varepsilon/h} \|u\|_{L^2}^2.$$

This proves the following proposition.

Proposition 2.2.19 (Inhomogeneous Lithner-Agmon Estimate). *Under the assumptions above, for each $\varepsilon > 0$, there exists $\tau_\varepsilon > 0$ and $C_\varepsilon > 0$, such that for $0 < \tau < \tau_\varepsilon$, and $R > 0$, the following estimate holds*

$$\|h(e^{\Phi_R^\varepsilon/h} u)'\|_{L^2}^2 + \|e^{\Phi_R^\varepsilon/h} u\|_{L^2}^2 \leq C_\varepsilon (e^{2\varepsilon/h} \|u\|_{L^2}^2 + \|e^{\Phi_R^\varepsilon/h} f\|_{L^2}^2).$$

The following two corollaries are important.

Corollary 2.2.20 (Homogeneous Lithner-Agmon Estimate). *If $f = 0$, then we obtain the usual (homogeneous) Lithner-Agmon estimate,*

$$\|h(e^{\Phi_R^\varepsilon/h} u)'\|_{L^2}^2 + \|e^{\Phi_R^\varepsilon/h} u\|_{L^2}^2 \leq C_\varepsilon e^{2\varepsilon/h} \|u\|_{L^2}^2.$$

Observe that, the right hand side of this estimate does not depend on R , we are thus allowed to let $R \rightarrow \infty$, and get a finer estimate,

$$\|h(e^{\Phi^\varepsilon/h} u)'\|_{L^2}^2 + \|e^{\Phi^\varepsilon/h} u\|_{L^2}^2 \leq C_\varepsilon e^{2\varepsilon/h} \|u\|_{L^2}^2.$$

Corollary 2.2.21. *Let $\chi \in L^\infty(\mathbb{R})$ be supported in the interior of $\{x \in \mathbb{R} : \Phi_R^\varepsilon(x) = R\}$, such that $0 \leq \chi \leq 1$, then*

$$(2.2.16) \quad \|\chi hu'\|_{L^2}^2 + \|\chi u\|_{L^2}^2 \leq C_\varepsilon (e^{-2(R-\varepsilon)/h} \|u\|_{L^2}^2 + \|f\|_{L^2}^2).$$

Remark 2.2.22. For any $\delta > 0$, we could modify the phase function Φ_R^ε to some $\tilde{\Phi}_R$, so that $\tilde{\Phi}_R \equiv R$ for $|x| \geq R_\varepsilon + \delta$, while $\tilde{\Phi}_R = \Phi_R^\varepsilon$ for $|x| \leq R_\varepsilon$.

Remark 2.2.23. This is a classical estimate by reversing the operator $-h^2\partial_x^2 + V(x)$ in the classical forbidden region, when V is independent of $\tau \equiv h$. For our application in §2.2.3.5, where the potential is V_E , it is believed that such a semiclassical analysis suffices. However, we decide to use the approach above for simplicity to avoid technique problems caused by the behavior of V_E at faraway from the origin.

Démonstration. Simply notice that

$$\chi(e^{\Phi_R^\varepsilon/h}u)' = e^{R/h}\chi u', \quad \chi e^{\Phi_R^\varepsilon/h}u = e^{R/h}\chi u, \quad \|e^{\Phi_R^\varepsilon/h}f\|_{L^2} \leq e^{R/h}\|f\|_{L^2}.$$

The rest of the proof is a straightforward application of the previous proposition. \square

We want to apply the discussion above to an admissible 4-tuple $(\varepsilon, h, k, \tilde{w})$ for sufficiently small ε and h . So that $\tau = h$, $P = -h^2\partial_x^2 + V$, and $E = o(1)$, and $f = O(h^2)_{L^2 \rightarrow L^2}\tilde{w}$. We are left to verify that $\tilde{w} \in D(P)$. This requires the following proposition from [160].

Proposition 2.2.24. *Let $I = (a_-, a_+) \subset \mathbb{R}$ be a finite or infinite interval, $f \in C^2(\bar{I})$ be real valued and positive, and $g \in C(\bar{I})$ be a continuous and complex valued. Let*

$$F(x) = \int \{f^{-1/4}(f^{-1/4})'' - gf^{-1/2}\} dx$$

be a primitive function of the integrand. Then in I the differential equation

$$u'' = (f + g)u$$

has twice continuously differentiable solutions of the form

$$u_\pm(x) = f^{-1/4}(x) \exp \left\{ \pm \int f^{1/2}(x) dx \right\} (1 + \varepsilon_\pm(x))$$

with estimates

$$\max \left\{ |\varepsilon_\pm(x)|, \frac{1}{2} f^{-1/2}(x) |\varepsilon'_\pm(x)| \right\} \leq \exp \left\{ \frac{1}{2} \mathcal{V}_{a_\pm, x}(F) \right\} - 1$$

provided the total variation $\mathcal{V}_{a_\pm, x}(F)$ of F on the interval (a_\pm, x) being finite. If g is real, then the solutions are real.

Corollary 2.2.25. *Let $w \in A_{\lambda, k}$, with $\lambda > 0$, $k \neq 0$, then on the interval (R_0, ∞) , w is, up to a multiplicative constant, of the form*

$$w(x) = [V(x) - E]^{-1/4} \exp \left\{ -h^{-1} \int_0^x [V(t) - E]^{1/2} dt \right\} (1 + \varepsilon(x))$$

with estimates $|\varepsilon(y)| + |\varepsilon'(y)| = O(h)$. We can do the same on $(-\infty, -R_0)$, and consequently $w \in H^1(\mathbb{R})$. Since $V \in L^\infty(\mathbb{R})$, we deduce that $w \in D(P)$. Now that \tilde{w} is a finite sum of such $w_{\lambda, k}$, we deduce that $\tilde{w} \in H^1(\mathbb{R})$.

Démonstration. By definition, $1 - h^2k^2 = E < 1 = \lim_{x \rightarrow \pm\infty} V(x)$ as $V(x) = 1 - \rho^2(x)$. Therefore, for R_0 sufficiently large, $V - E$ has a strictly positive lower bound $\delta > 0$ on (R_0, ∞) . Apply the previous proposition with $f = h^{-2}(V - E)$, $g = O(1)\rho^2$. Then

$$F(x) = h \int_c^x [V(t) - E]^{-1/4} \partial_t^2 [V(t) - E]^{-1/4} dt,$$

from which, for $x > R_0$,

$$\mathcal{V}_{x,\infty}(F) \leq Ch\delta^{-5/2} (\|r'\|_{L^\infty}^2 + \|r''\|_{L^\infty}) \int_{\mathbb{R}} \rho^2(t) dt = O(h)$$

since $V(x) = 1 - \rho^2(x) = 1 - r^2(f(x))$, $f'(x) = r(f(x))$, and that

$$\int \rho^2(t) dt = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_{\mathbb{R}} \rho^2(x) dx = \frac{1}{2\pi} \text{vol}(M) < \infty.$$

□

Corollary 2.2.26. *There exists $\epsilon_0 > 0$, $h_0 > 0$, $C > 0$, such that for $0 < \epsilon < \epsilon_0$, $0 < h < h_0$, if $(\epsilon, h, k, \tilde{w})$ is an admissible 4-tuple, then*

$$C^{-1} \|\tilde{w}\|_{L^2} \leq \|\tilde{w}\|_{L^2(\rho^2 dx)} \leq C \|\tilde{w}\|_{L^2};$$

Suppose there exists $\epsilon_0 > 0$, $h_0 > 0$, $C > 0$, such that for $0 < \epsilon < \epsilon_0$, $0 < h < h_0$, if $(\epsilon, h, k, \tilde{w})$ is an admissible 4-tuple, then

$$\|1_{x>0}\tilde{w}\|_{L^2} \geq C \|\tilde{w}\|_{L^2},$$

then there exists $\epsilon'_0 > 0$, $h'_0 > 0$, $C' > 0$, such that for $0 < \epsilon < \epsilon'_0$, $0 < h < h'_0$, if $(\epsilon, h, k, \tilde{w})$ is an admissible 4-tuple, then

$$\|1_{x>0}\tilde{w}\|_{L^2(\rho^2 dx)} \geq C' \|\tilde{w}\|_{L^2(\rho^2 dx)}.$$

Démonstration. There is no harm in assuming $\|\tilde{w}\|_{L^2} = 1$, and apply Corollary 2.2.21 with

$$f = O(h^2)_{L^2 \rightarrow L^2} \tilde{w} = O(h^2).$$

To do this, we fix $0 < \epsilon < 1$ (please do not get confused with ϵ), and fix $R > 2\epsilon$, then for some $R_0 > 0$, $\chi = 1_{|x|>R_0}$ is supported in $\{\Phi_R^\epsilon = R\}$. Then Corollary 2.2.21 implies that, for some constant $C_\epsilon > 0$,

$$\|1_{|x|>R_0}\tilde{w}\|_{L^2(\rho^2 dx)} \leq \|1_{|x|>R_0}\tilde{w}\|_{L^2} \leq C_\epsilon h^2$$

Let $\delta = \inf_{|x|<R_0} \rho(x) > 0$, then

$$\begin{aligned} 1 = \|\tilde{w}\|_{L^2} &\geq \|\tilde{w}\|_{L^2(\rho^2 dx)} \geq \|1_{|x|<R_0}\tilde{w}\|_{L^2(\rho^2 dx)} \geq \delta \|1_{|x|<R_0}\tilde{w}\|_{L^2} \\ &\geq \delta(1 - \|1_{|x|>R_0}\tilde{w}\|_{L^2}) \geq \delta(1 - C_\epsilon h^2) \geq \frac{1}{2}\delta, \end{aligned}$$

when h is sufficiently small. This proves the first statement. To prove the second statement,

$$\|1_{x>0}\tilde{w}\|_{L^2(\rho^2 dx)} / \|\tilde{w}\|_{L^2(\rho^2 dx)} \geq \|1_{R_0>x>0}\tilde{w}\|_{L^2(\rho^2 dx)} / \|\tilde{w}\|_{L^2}$$

$$\begin{aligned}
 &\geq \delta \|1_{R_0 > x > 0} \tilde{w}\|_{L^2} / \|\tilde{w}\|_{L^2} \\
 &\geq \delta (\|1_{x > 0} \tilde{w}\|_{L^2} - \|1_{x > R_0} \tilde{w}\|_{L^2}) / \|\tilde{w}\|_{L^2} \\
 &\geq \delta (C - C_\varepsilon h^2) \\
 &\geq \frac{1}{2} \delta C,
 \end{aligned}$$

when h is sufficiently small. □

2.2.3.4 Case $E = O(h)$

Proposition 2.2.27. *Let $(\varepsilon, h, k, \tilde{w})$ be an admissible sequence such that $E = O(h)$, then for some $C > 0$ and ε, h sufficiently small,*

$$\|1_{x > 0} \tilde{w}\|_{L^2} \geq C \|\tilde{w}\|_{L^2}.$$

Démonstration. We first study Laplacian eigenfunctions, rather than \mathcal{L} -eigenfunctions for simplicity, for the latter are finite sums of the former. To do this, we let $\lambda^2 \in I_n$, $k \in Z_n(\varepsilon)$, and $w \in A_{\lambda, k}$. Recall that w satisfies

$$(-h^2 \partial_x^2 + V)w = Ew + O(h^2)_{L^\infty} w.$$

Up to a subsequence, we may assume that $c^{-1/2} h^{-1} E = F + o(1)$ (recalling the definition of c by (2.2.11)) for some $F \geq 0$, and use the following scaling,

$$z = c^{1/4} h^{-1/2} x, \quad V_h(z) = c^{-1/2} h^{-1} V(x),$$

and work under the coordinate z , and with the measure dz . We normalize w so that $\|w\|_{L^2} = 1$, and observe that it satisfies the equation

$$(-\partial_z^2 + V_h)w = Fw + o(1)_{L^\infty} w.$$

Notice that $V_h(z) = z^2 + h^{1/2} O(z^3)$ for $|z| \lesssim h^{-1/2}$, we apply Corollary 2.2.20 with

$$\tau(h) \equiv 1, \quad P(\tau) = -\partial_z^2 + V_h(z), \quad \Phi^\varepsilon(z) = (1 - \varepsilon) \left| \int_0^z (V_h(t) - F - o(1)_{L^\infty})_+^{1/2} dt \right|,$$

and get

$$\|(e^{\Phi^\varepsilon} w)'\|_{L^2}^2 + \|e^{\Phi^\varepsilon} w\|_{L^2}^2 \leq C_\varepsilon \|w\|_{L^2}^2,$$

which implies

$$\|e^{\Phi^\varepsilon} w\|_{L^2} + \|h^{1/2} e^{\Phi^\varepsilon} w'\|_{L^2} \leq C_\varepsilon \|w\|_{L^2}.$$

Indeed,

$$\begin{aligned}
 \|h^{1/2} e^{\Phi^\varepsilon} w'\|_{L^2} &\leq \|h^{1/2} (e^{\Phi^\varepsilon} w)'\|_{L^2} + \|h^{1/2} (\Phi^\varepsilon)' e^{\Phi^\varepsilon} w\|_{L^2} \\
 &\leq h^{1/2} C_\varepsilon \|w\|_{L^2} + (1 - \varepsilon) (\|V\|_{L^\infty} + O(1))^{1/2} \|e^{\Phi^\varepsilon} w\|_{L^2} \\
 &\leq C_\varepsilon \|w\|_{L^2}.
 \end{aligned}$$

Since $\Phi^\varepsilon(z) \geq \alpha(|z| - M)$ for some $\alpha > 0$, $M > 0$ and is uniform for all small ε, h , we then have

$$\|w\|_{L^2(|z| \geq R)} + \|h^{1/2} \partial_z w\|_{L^2(|z| \geq R)} = O(e^{-\alpha R}) \|w\|_{L^2}.$$

Fix some $0 < \delta < 1/6$, and let $w_\chi = \chi(h^\delta z)w(z)$ where $\chi \in C_0^\infty$ is a cut-off function equaling to 1 near the origin. Therefore

$$w = w_\chi + O(h^{-1/2}e^{-\alpha h^{-\delta}})_{H^1} = w_\chi + O(h^\infty)_{H^1}.$$

Observing that on the support of w_χ , $V_h(z) = z^2 + O(h^{1/2-3\delta})$, we have,

$$\begin{aligned} (-\partial_z^2 + z^2 - F)w_\chi &= o(1)_{L^\infty}w - [\partial_z^2, \chi(h^\delta z)]w \\ (2.2.17) \qquad \qquad \qquad &= o(1)_{L^\infty}w - 2h^\delta \chi'(h^\delta z)w' - h^{2\delta} \chi''(h^\delta z)w \\ &= o(1)_{L^2}. \end{aligned}$$

Let $\{v_i\}_{i \in \mathbb{N}}$ be the complete set of normalized eigenfunctions of the classical harmonic oscillator, $-\partial_z^2 + z^2$, that is, $\|v_i\|_{L^2} = 1$, and

$$(-\partial_z^2 + z^2)v_i = (2i + 1)v_i.$$

We know that $v_i(z) = c_i H_i(z)e^{-z^2/2}$, where c_i is a constant of normalization, and H_i is the i^{th} Hermite polynomial. We will only use the fact that H_i is either an odd function (when i is odd), or an even function (when i is even).

We write $w_\chi = \sum \alpha_i v_i$, and have

$$(2.2.18) \qquad \sum_{i \geq 0} |\alpha_i|^2 = \|w_\chi\|_{L^2}^2 = \|w\|_{L^2}^2 + o(1) = 1 + o(1).$$

By (2.2.17),

$$o(1)_{L^2} = (-\partial_z^2 + z^2 - F)w_\chi = \sum_{i \geq 0} (2i + 1 - F)\alpha_i v_i,$$

which gives

$$(2.2.19) \qquad \sum_{i \geq 0} (2i + 1 - F)^2 |\alpha_i|^2 = o(1).$$

Let $i_0 \in \mathbb{N}$ be such that $|2i_0 + 1 - F|$ attains the minimum among all $|2i + 1 - F|$. Then for any integer $i \neq i_0$, $|2i + 1 - F| \geq |i - i_0|$, and hence,

$$\left\| \sum_{i \neq i_0} \alpha_i v_i \right\|_{L^2}^2 = \sum_{i \neq i_0} |\alpha_i|^2 = o(1).$$

Combining with (2.2.18), we have $\alpha_{i_0} = 1 + o(1)$. And by consequence,

$$w = w_\chi + o(1)_{L^2} = \alpha_{i_0} v_{i_0} + \sum_{i \neq i_0} \alpha_i v_i + o(1)_{L^2} = \alpha_{i_0} v_{i_0} + o(1)_{L^2} = v_{i_0} + o(1)_{L^2}.$$

Moreover, we have by (2.2.19), that $(2i_0 + 1 - F)|\alpha_{i_0}|^2 = o(1)$, which implies

$$F = 2i_0 + 1.$$

In particular i_0 depends only on F , not on λ . As a consequence, we claim that, for this admissible subsequence, which satisfies $E = O(h)$, when ϵ and h are sufficiently small, there can be at most one $\lambda^2 \in I_n$, such that $A_{\lambda,k} \neq \{0\}$. Therefore, $\tilde{A}_{n,k} = A_{\lambda,k}$. So if

$\tilde{w} \in \tilde{A}_{n,k}$, then $\tilde{w} = v_{i_0} + o(1)_{L^2}$. This concludes the proof, since v_{i_0} is either an odd function, or an even function, whose L^2 norm is thus equally distributed on each side of the origin.

To prove the claim, we argue by contradiction and use the orthogonality given by Corollary 2.2.5. Indeed, suppose we can find for arbitrary small ϵ and h two distinct λ_1, λ_2 such that $\lambda_i^2 \in I_n$, ($i = 1, 2$), and that $A_{\lambda_i,k} \neq \{0\}$, then we can choose $w_i \in A_{\lambda_i,k}$, such that $\|w_i\|_{L^2} = 1$. By the analysis above, we see that $w_i = v_{i_0} + o(1)_{L^2}$. Using the orthogonality of w_1 and w_2 with respect to $L^2(\rho^2 dz)$,

$$0 = (w_1, w_2)_{L^2(\rho^2 dz)} = (v_{i_0}, v_{i_0})_{L^2(\rho^2 dz)} + o(1) \rightarrow (v_{i_0}, v_{i_0})_{L^2(\rho^2 dz)} \neq 0,$$

we obtain a contradiction. \square

2.2.3.5 Case $E \gg h$

Proposition 2.2.28. *Let $(\epsilon, h, k, \tilde{w})$ be an admissible sequence such that $E \gg h$, then for some $C > 0$, and ϵ, h sufficiently small,*

$$\|1_{x>0}\tilde{w}\|_{L^2} \geq C\|\tilde{w}\|_{L^2}.$$

Démonstration. We use the scaling

$$z = c^{1/2}E^{-1/2}x, \quad \hat{h} = c^{1/2}E^{-1}h, \quad V_E(z) = E^{-1}V(x),$$

and work under the z coordinate and the measure dz . We normalize \tilde{w} by $\|\tilde{w}\|_{L^2} = 1$, and observe that it satisfies the equation

$$(2.2.20) \quad (-\hat{h}^2 \partial_z^2 + V_E)\tilde{w} = \tilde{w} + O(h^2/E)\rho^2 \tilde{w} = \tilde{w} + o(\hat{h})_{L^2}.$$

Let $\chi \in C_c^\infty(\mathbb{R})$ be equal to 1 in a neighbourhood of $|z| \leq 1$, and $0 \leq \chi \leq 1$, then we apply Corollary 2.2.21 and the remark after it,

$$\|(1 - \chi)\hat{h}\tilde{w}'\|_{L^2}^2 + \|(1 - \chi)\tilde{w}\|_{L^2}^2 = O(\hat{h}^\infty)\|\tilde{w}\|_{L^2}^2 + o(h^2) = o(\hat{h}^2),$$

which implies

$$(2.2.21) \quad \|\chi\tilde{w}\|_{L^2} = 1 + o(\hat{h}).$$

In order to conclude, it suffices to prove that, for some $\delta > 0$, and \hat{h} sufficiently small,

$$(2.2.22) \quad \|1_{z>0}\chi\tilde{w}\|_{L^2} \geq \delta.$$

Let $\tilde{\chi} \in C_c^\infty(\mathbb{R})$ be such that $\chi\tilde{\chi} = \chi$, then $\chi\tilde{w}$ satisfies the equation

$$(2.2.23) \quad \begin{aligned} (-\hat{h}^2 \partial_z^2 + \tilde{\chi}V_E)(\chi\tilde{w}) &= \chi\tilde{w} + o(\hat{h})_{L^2} - [\hat{h}^2 \partial_z^2, \tilde{\chi}]\tilde{w} \\ &= \chi\tilde{w} + o(\hat{h})_{L^2} - 2\hat{h}^2 \chi' \tilde{w}' - \hat{h}^2 \chi'' \tilde{w} = \chi\tilde{w} + o(\hat{h})_{L^2}. \end{aligned}$$

The bottom line comes from Corollary 2.2.21 and the fact that $\text{supp } \chi' \subset \{|z| > 1\}$. This equation first implies that $\chi\tilde{w}$ is \hat{h} -oscillating (see Example 2.3.9), whose \hat{h} -semiclassical measure μ will thus not vanish, for we have (2.2.21). Now that $\chi\tilde{w}$ is supported in $\text{supp } \chi$, we have evidently,

$$\text{supp } \mu \subset \text{supp } \chi \times \mathbb{R}_\zeta.$$

By the fact that $\tilde{\chi}(z)V_E(z) \rightarrow \tilde{\chi}(z)z^2$ in $C_c^\infty(\mathbb{R})$, and Corollary 2.3.4, we see that

$$\text{supp } \mu \subset \{(z, \zeta) : \zeta^2 + \tilde{\chi}(z)z^2 = 1\}.$$

Combing the results above,

$$\text{supp } \mu \subset \text{supp } \chi \times \mathbb{R}_\zeta \cap \{(z, \zeta) : \zeta^2 + \tilde{\chi}(z)z^2 = 1\} \subset \{(z, \zeta) : \zeta^2 + z^2 = 1\},$$

because $\tilde{\chi} = 1$ on $\text{supp } \chi$. Moreover μ is invariant by the Hamiltonian flow generated by the Hamiltonian vector field

$$H_{\zeta^2 + \tilde{\chi}(z)z^2} = (-2\zeta, 2\tilde{\chi}(z)z + \tilde{\chi}'(z)z^2),$$

which, when restricted to $\text{supp } \mu$, is $(-2\zeta, 2z)$, and generates the rotation of the circle $\zeta^2 + z^2 = 1$. Therefore (2.2.22) must be satisfied, because otherwise $\mu|_{z>0} = 0$, and by the invariance under flow, $\mu = 0$, which is a contradiction. \square

2.3 Appendix : semiclassical measure

In this section we recall some basic properties of semiclassical measures. For details we refer to [84, 86, 138, 40, 208]. In what follows (M, g) will either be the flat Euclidean space \mathbb{R}^d or a compact Riemannian manifold without boundary. The theory of semiclassical measure works for general Riemannian manifolds, we restrict ourselves to these simple cases so that we can give some simple proofs for some of the following results, which already satisfies our needs.

Theorem 2.3.1. *Let $u(h)$ ($0 < h < h_0$) be bounded in $L^2(M)$. Then there exists a sequence $h_n \rightarrow 0$ and a positive Radon measure μ on T^*M (which is called an h -semiclassical measure of $u(h)$) such that for all $a \in C_c^\infty(T^*M)$,*

$$\lim_{n \rightarrow \infty} (a_n(x, h_n D)u(h_n), u(h_n))_{L^2(M)} = \int_{T^*M} a(x, \xi) d\mu(x, \xi).$$

Remark 2.3.2. We call μ an h -semiclassical measure to emphasize the importance of the parameter h , for different parameter can be used. When there is no ambiguity, we simply call μ a semiclassical measure.

Remark 2.3.3. When there is no need to extract a subsequence, we say $u(h)$ is pure, and μ is “the” semiclassical measure of $u(h)$.

We also need the following corollary.

Corollary 2.3.4. *Let $u(h)$ ($0 < h < h_0$) be pure, with semiclassical measure μ . Suppose $\{a_n\}_n$ and a are functions in $C_c^\infty(T^*M)$ such that $a_n \rightarrow a$ in $C^\infty(T^*M)$, then*

$$\lim_{n \rightarrow \infty} (a(x, h_n D)u(h_n), u(h_n))_{L^2(M)} = \int_{T^*M} a(x, \xi) d\mu.$$

This corollary is a simple consequence of Theorem 2.3.1 and the following L^2 -estimate (we refer to [208], Theorem 5.1) that, for some $N > 0$, and all $a \in C_c^\infty(T^*M)$,

$$\|a(x, hD)\|_{L^2 \rightarrow L^2} \leq C \sup_{|\alpha| \leq Nd} h^{|\alpha|/2} \|\partial_{x, \xi}^\alpha a\|_{L^\infty}.$$

Theorem 2.3.5. *Let $u(h)$ ($0 < h < h_0$) be pure, with semiclassical measure μ . Let $p \in S^m(T^*M)$ (where $S^m(T^*M)$ is the Hörmander class, see [100]).*

$$\begin{aligned} p(x, hD)u(h) = o(1)_{L^2} &\Rightarrow \operatorname{supp} \mu \subset \{p = 0\} \\ p(x, hD)u(h) = o(h)_{L^2} &\Rightarrow H_p \mu = 0, \end{aligned}$$

where $H_p = (-\frac{\partial H}{\partial \xi} \frac{\partial}{\partial x}, \frac{\partial H}{\partial x} \frac{\partial}{\partial \xi})$.

Remark 2.3.6. By consequence of Corollary 2.3.4, Theorem 2.3.5 could be improved a little bit. Instead of a single symbol p , we can consider a sequence of symbols $\{p_n\}_{n \geq 0} \subset S^m(T^*M)$, such that $p_n \rightarrow p$ in $C_{\text{loc}}^\infty(M)$. Then

$$\begin{aligned} p_n(x, h_n D)u(h_n) = o(1)_{L^2} &\Rightarrow \operatorname{supp} \mu \subset \{p = 0\} \\ p_n(x, h_n D)u(h_n) = o(h_n)_{L^2} &\Rightarrow H_p \mu = 0. \end{aligned}$$

For a pure sequence $u(h)$, even if its semiclassical measure $\mu = 0$, we do not generally have $u(h) \rightarrow 0$ strongly in $L^2(M)$. However, this is the case if $u(h)$ is in addition h -oscillating.

Definition 2.3.7. A sequence $u(h)$ is called h -oscillating if for some $\chi \in C^\infty(\mathbb{R})$ such that $0 \leq \chi \leq 1$, $\chi = 0$ in a neighborhood of the origin, and $\chi = 1$ outside a neighborhood of the origin, then

$$\lim_{R \rightarrow +\infty} \limsup_{h \rightarrow 0} \|\chi(-h^2 \Delta / R)u(h)\|_{L^2} = 0.$$

Proposition 2.3.8. *Let $u(h)$ be a pure and h -oscillating sequence with vanishing semiclassical measure. Suppose that there exists a compact set K such that for h sufficiently small, $\operatorname{supp} u(h) \subset K$, then $u(h) \rightarrow 0$ strongly in $L^2(M)$.*

Démonstration. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $\varphi u(h) = u(h)$ when h is sufficiently small. Let χ be chose as in the definition above. Write $\chi_R(\cdot) = \chi(\cdot/R)$ for simplicity, and decompose

$$\|u(h)\|_{L^2}^2 = (\varphi(1 - \chi_R^2(-h^2 \Delta))\varphi u(h), u(h))_{L^2} + (\chi_R^2(-h^2 \Delta)u(h), u(h))_{L^2}.$$

Observe that $\varphi(1 - \chi_R^2(-h^2 \Delta))\varphi$ is a semiclassical pseudodifferential operator with principal symbol $(1 - \chi_R^2(g_x^{-1}(\xi, \xi)))\varphi(x) \in C_c^\infty(T^*M)$, therefore, since $\mu = 0$,

$$\lim_{h \rightarrow 0} ((1 - \chi_R^2(-h^2 \Delta))\varphi u(h), u(h))_{L^2} = \int_{T^*M} (1 - \chi_R^2(g_x^{-1}(\xi, \xi))) d\mu(x, \xi) = 0.$$

While for the second term, by our hypothesis of h -oscillation,

$$\lim_{R \rightarrow +\infty} \limsup_{h \rightarrow 0} (\chi_R^2(-h^2 \Delta)u(h), u(h))_{L^2} = \lim_{R \rightarrow +\infty} \limsup_{h \rightarrow 0} \|\chi_R(-h^2 \Delta)u(h)\|_{L^2}^2 = 0.$$

Combine these two limit behaviors,

$$\limsup_{h \rightarrow 0} \|u(h)\|_{L^2}^2 = \lim_{R \rightarrow +\infty} \limsup_{h \rightarrow 0} \|u(h)\|_{L^2}^2 = 0.$$

□

Example 2.3.9. Suppose that $u(h)$ is a pure sequence satisfying

$$(-h^2\Delta + V)u(h) = o(1)_{L^2},$$

where $V \in C_c^\infty(M)$ then $u(h)$ is h -oscillating. Indeed, by adding to V some constant, we may assume that $V \geq 1$. So that $-h^2\Delta + V$ is a self-adjoint operator with uniformly (in h) bounded resolvent $\|(-h^2\Delta + V)^{-1}\|_{L^2 \rightarrow L^2} \leq 1$. After this modification $u(h)$ satisfies an equation of the form

$$(-h^2\Delta + V)u(h) = Eu(h) + o(1)_{L^2},$$

for some constant $E \in \mathbb{R}$. Denote $\psi(z) = z^{-1}\chi(z) \in C^\infty(\mathbb{R})$, and $\psi_R(z) = \psi(z/R)$, then apply each side of the equation above by $R^{-1}\psi_R(-h^2\Delta + V)$, we obtain

$$\chi_R(-h^2\Delta + V)u(h) = ER^{-1}\psi_R(-h^2\Delta + V)u(h) + o(1)_{L^2}.$$

Now that M is either compact or Euclidean, we have a uniform elliptic estimate $g_x^{-1}(\xi, \xi) \gtrsim |\xi|^2$, whence for R sufficiently large and for $(x, \xi) \in T^*M$,

$$\chi_{R/2}(g_x^{-1}(\xi, \xi))\chi_R(g_x^{-1}(\xi, \xi) + V(x)) = \chi_{R/2}(g_x^{-1}(\xi, \xi)).$$

Therefore, apply the equation above by $\chi_{R/2}(-h^2\Delta)$, and by a symbolic calculus, we have

$$\chi_{R/2}(-h^2\Delta)u(h) = ER^{-1}O(1)_{L^2} + o(1)_{L^2},$$

which implies that, for R sufficiently large,

$$\limsup_{h \rightarrow 0} \|\chi_{R/2}(-h^2\Delta)u(h)\|_{L^2} = O(R^{-1}).$$

We also remark that this argument works when V depends on h , but remains bounded in $C_c^\infty(M)$.

Chapitre 3

Control of three dimensional gravity-capillary water waves

3.1 Introduction

We consider the Zakharov [199] / Craig–Sulem [60] formulation of the gravity water wave system with surface tension. It is defined as follows on the torus $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$,

$$(3.1.1) \quad \begin{cases} \partial_t \eta - G(\eta)\psi = 0, \\ \partial_t \psi + g\eta - H(\eta) + \frac{1}{2}|\nabla_x \psi|^2 - \frac{1}{2} \frac{(\nabla_x \eta \cdot \nabla_x \psi + G(\eta)\psi)^2}{1+|\nabla_x \eta|^2} = P_{\text{ext}}. \end{cases}$$

Here g is the gravitational acceleration, $H(\eta) = \nabla_x \cdot \left(\frac{\nabla_x \eta}{\sqrt{1+|\nabla_x \eta|^2}} \right)$ is the mean curvature of the surface

$$\Sigma_t = \{(x, y) \in \mathbb{T}^d \times \mathbb{R} : y = \eta(t, x)\},$$

and $G(\eta)$ is the Dirichlet–Neumann operator, defined below by (3.1.4), of the domain

$$\Omega_t = \{(x, y) \in \mathbb{T}^d \times \mathbb{R} : -b < y < \eta(t, x)\},$$

with depth $b \in]0, +\infty[$. Our main theorem states that, any sufficiently small data can be generated by a suitable localized exterior pressure P_{ext} .

Definition 3.1.1. We say that an open set $\omega \subset \mathbb{T}^d$ satisfies the *geometric control condition* if every geodesic of \mathbb{T}^d (which are straight lines, for we endow \mathbb{T}^d with the standard metric) eventually enters ω . More precisely, this means that for every $(x, \xi) \in \mathbb{T}^d \times \mathbb{S}^{d-1}$, there exists some $t \in]0, \infty[$, such that $x + t\xi \in \omega$.

Theorem 3.1.2. *Suppose that $d \geq 1$, $T > 0$, s is sufficiently large, and $\omega \subset \mathbb{T}^d$ satisfies the geometric control condition, then for some $\varepsilon_0 > 0$ sufficiently small and for all $(\eta_i, \psi_i) \in H^{s+1/2}(\mathbb{T}^d) \times H^s(\mathbb{T}^d)$ satisfying*

$$\|(\nabla_x \eta_i, \nabla_x \psi_i)\|_{H^{s-1/2} \times H^{s-1}} < \varepsilon_0, \quad (i = 0, 1)$$

and $\int_{\mathbb{T}^d} \eta_0 dx = \int_{\mathbb{T}^d} \eta_1 dx = 0$, there exists $P_{\text{ext}} \in C([0, T], H^s(\mathbb{T}^d))$, such that

- (1) P_{ext} is real valued, and for all $t \in [0, T]$, $\text{supp } P_{\text{ext}}(t, \cdot) \subset \omega$;
- (2) there exists a unique solution to (3.1.1), $(\eta, \psi) \in C([0, T], H^{s+1/2}(\mathbb{T}^d) \times H^s(\mathbb{T}^d))$, such that $(\eta, \psi)|_{t=0} = (\eta_0, \psi_0)$, and $(\eta, \psi)|_{t=T} = (\eta_1, \psi_1)$.

Remark 3.1.3. Theorem 3.1.2 remains valid for infinite depth, that is $b = +\infty$, with exactly the same proof. However, we restrict ourselves to finite depth for simplicity.

Remark 3.1.4. The same result for 2-D water waves, that is $d = 1$, was previously obtained by Alazard, Baldi, and Han-Kwan [5], where the geometric control condition is implicit, as it is always satisfied on \mathbb{T}^1 . We show in Appendix 3.8 that on \mathbb{T}^2 , for the exact controllability of the linearized equation around the flat surface (that is $\eta = 0$) of (3.1.1) with infinite depth, the geometric control condition is necessary, and consequently it is natural for the non-linear equation.

Remark 3.1.5. The condition for the mean values of η_0 and η_1 is necessary since the zero frequency of η is conserved in time.

Remark 3.1.6. We shall only prove the existence of the solution (η, ψ) , by an iterative construction performed in Section 3.6, because the uniqueness is a consequence of [6].

This is a natural control result for a quasi-linear physical equation. Although many works have been done in the control theory of nonlinear equations, including equations describing water waves in some asymptotic regions, like the Benjamin–Ono equation, the Korteweg–de Vries equation, the nonlinear Schrödinger equation, etc., the only work so far for the full water wave model is done by Alazard, Baldi and Han-Kwan [5], who proves the exact controllability for the system (3.1.1) on \mathbb{T}^1 . Our Theorem 3.1.2 extends their result to higher dimensions.

A main ingredient in [5] is Ingham’s inequality, which is a tool specific to $d = 1$. To tackle the lack of such estimate in higher dimensions, we have to distinguish the high frequency regime and the low frequency regime. The high frequency regime requires to implement in this quasi-linear framework the semi-classical approach (see Lebeau [131], Burq–Zworski [48]), while the low frequency regime is studied by the uniqueness-compactness argument (see Bardos–Lebeau–Rauch [24]). It has to be noticed that the usual Carleman estimates do not seem to apply to the paradifferential context we are working with, and this uniqueness result has to be proven by a different method.

3.1.1 From Euler to Zakharov / Craig–Sulem

We present here briefly the Eulerian formulation of the water wave system with surface tension to give some physical intuitions. Then we define the Dirichlet–Neumann operator, and derive the Zakharov / Craig–Sulem formulation (3.1.1).

Let Σ_t and Ω_t be defined as above, and let $\Gamma = \mathbb{T}^d \times \{-b\}$ be the flat bottom. Denote by $v : \Omega_t \rightarrow \mathbb{R}^d$ the Eulerian vector field, by $P : \Omega_t \rightarrow \mathbb{R}$ the internal pressure of the fluid, and by $\mathbf{n} : \partial\Omega_t \rightarrow \mathbb{S}^d$ the exterior unit normal vector to the boundary $\partial\Omega_t = \Sigma_t \cup \Gamma$. Then the Eulerian formulation of the water wave system is the following system of (η, v) ,

$$(3.1.2) \quad \begin{cases} \partial_t v + v \cdot \nabla_{x,y} v + \nabla_{x,y} P = -g \mathbf{e}_y & \text{in } \Omega_t, \\ \nabla_{x,y} \cdot v = 0 & \text{in } \Omega_t, \\ \partial_t \eta = \sqrt{1 + |\nabla_x \eta|^2} v|_{\Sigma_t} \cdot \mathbf{n}, & \text{on } \Sigma_t, \\ v \cdot \mathbf{n} = 0 & \text{on } \Gamma, \\ -P|_{\Sigma_t} = \kappa H(\eta) + P_{\text{ext}} & \text{on } \Sigma_t, \end{cases}$$

where $\mathbf{e}_y = (0, 1)$ is the unit vector in the y -direction, and $\kappa > 0$ is the surface tension coefficient. The physical interpretation of (3.1.2) is that

- (1) v satisfies the incompressible Euler equation in the domain Ω_t ,
- (2) fluid particles which are initially on the surface will stay on the surface,
- (3) the bottom is impenetrable by fluid particles,
- (4) the internal pressure, the exterior pressure, and the surface tension balance out on the surface.

We suppose furthermore that the vector field v admits a scalar potential $\phi : \Omega_t \rightarrow \mathbb{R}$, that is $v = \nabla_{x,y}\phi$, which implies furthermore that v is irrotational. Denote by ψ the trace of ϕ to the free surface, in the sense that,

$$\psi(t, x) = \phi(t, x, \eta(t, x)),$$

then ϕ satisfies the Laplace equation, with a mixed boundary condition,

$$(3.1.3) \quad \Delta_{x,y}\phi = 0, \quad \phi|_{\Sigma_t} = \psi, \quad \partial_{\mathbf{n}}\phi|_{\Gamma} = 0.$$

Define the Dirichlet–Neumann operator $G(\eta)$ by

$$(3.1.4) \quad \begin{aligned} G(\eta)\psi(t, x) &= \sqrt{1 + |\nabla_x \eta(t, x)|^2} \partial_{\mathbf{n}}\phi|_{y=\eta(t,x)} \\ &= (\partial_y \phi)(t, x, \eta(x)) - \nabla_x \eta(t, x) \cdot (\nabla_x \phi)(t, x, \eta(x)). \end{aligned}$$

Combining (3.1.2), (3.1.3) and (3.1.4), and assume $\kappa = 1$, we obtain the Zakharov / Craig–Sulem formulation (3.1.1) in variables (η, ψ) .

The study of the Cauchy problem of (3.1.1), initiated by Kano–Nishida [114] and Yosihara [197, 198], has greatly progressed over decades. To name a few, the local well-posedness in the framework of Sobolev spaces, without smallness assumptions of initial data, has been established by Beyer–Günther [30] in the case with surface tension, and by Wu [192, 193] in the case without surface tension. For recent progresses, we refer to Ifrim–Tataru [105], Ionescu–Pusateri [107], Wang [189], de Poyferré–Nguyen [64] and the references therein. Here we are influenced by the paradifferential approach developed by Alazard–Métivier [12], Alazard–Burq–Zuily [6, 8, 9], which allows them to prove the local well-posedness in low Sobolev regularity, without regularity assumptions for the bottom.

3.1.2 Outline of paper

In Section 3.2, we outline the method of our proof. In Section 3.3, we reformulate the problem by the paradifferential calculus. In Section 3.4, we prove the null controllability in L^2 for linearized control problems of (3.1.1). In Section 3.5, we prove that for H^s initial data, the control obtained by Section 3.4 is also of regularity H^s . In Section 3.6, we prove with an iterative construction the null controllability in H^s of the quasilinear paradifferential equation obtained in Section 3.3. Finally in Section 3.7, we prove the exact controllability of (3.1.1) and conclude the proof of Theorem 3.1.2.

In Appendix 3.8, we show that on \mathbb{T}^2 , when $b = \infty$, the geometric control condition is necessary for the controllability of the linearized equation of (3.1.1) around the flat surface. In Appendix 3.9, we recall the basics of the paradifferential calculus. In Appendix 3.10, we prove the well-posedness of some linear evolution equations, which are used in the study of the linearized equations of (3.1.1).

3.2 Strategy of proof and some notations

The general strategy of our proof of Theorem 3.1.2 is to combine the iterative scheme of [5] with Lebeau's method for the linear control problems, where we adapt the semi-classical approach in the high frequency regime, and use a perturbative argument to prove the unique continuation property for the low frequency regime. We explain in the following some more details.

3.2.1 Paralinearization and reduction to null controllability

By the time reversibility of (3.1.1), the procedure of paralinearization and symmetrization systematically developed in [12, 6, 8, 9] by Alazard, Burq, Métivier and Zuily, we can reduce the exact controllability of (3.1.1) to the *null controllability* of a paralinearized equation. By the null controllability, we mean the exact controllability with null final data. For the paradifferential calculus, we refer to Bony [33], Métivier [146], and Hörmander [103], see also Appendix 3.9 where some basic results are presented.

Recall that the zero frequency of η is preserved in time, and observe that the zero frequency of ψ is of no physical importance. Therefore we define the Sobolev spaces of functions with null zero frequencies,

$$(3.2.1) \quad \dot{H}^\sigma(\mathbb{T}^d) := \{f \in H^\sigma(\mathbb{T}^d) : \pi(D_x)f = f\}, \quad \sigma \in \mathbb{R},$$

where $\pi(D_x)$ is a Fourier multiplier that projects to Fourier modes of nonzero frequencies (see Appendix 3.9 for details). We equip $\dot{H}^\sigma(\mathbb{T}^d)$ with the usual Sobolev norm inherited from $H^s(\mathbb{T}^d)$. Observe that for $\sigma \geq 0$, $f \in \dot{H}^\sigma(\mathbb{T}^d)$ means that $\int_{\mathbb{T}^d} f \, dx = 0$. We also use the notation $\dot{L}^2(\mathbb{T}^d) = \dot{H}^0(\mathbb{T}^d)$.

Following [6, 5], we paralinearize (3.1.1) to obtain a paradifferential equation for the complex variable $u = u(\psi, \eta)$,

$$(3.2.2) \quad u = T_q \omega - iT_p \eta.$$

Here $\omega = \psi - T_B \eta$ is called *the good unknown of Alinhac*⁽ⁱ⁾, where B is the trace to the free surface Σ of the vertical velocity $\partial_y \phi$ (see (3.3.1) and (3.3.2)); while T_q and T_p are paradifferential operators depending solely on η , of orders 0 and 1/2 respectively, so that $u \in \dot{H}^s(\mathbb{T}^d)$ whenever $(\eta, \psi) \in H^{s+1/2}(\mathbb{T}^d) \times H^s(\mathbb{T}^d)$. We show that this transform $(\eta, \psi) \mapsto u$ is invertible except for the zero frequencies. So we first establish the null controllability for u , and then recover the zero frequencies to obtain the exact controllability for (η, ψ) . To do this, we seek a null control for u of the following form

$$(3.2.3) \quad P_{\text{ext}}(t, x) = \chi_T(t) \varphi_\omega(x) \text{Re } F(t, x),$$

where

- (1) $F \in C([0, T], \dot{H}^s(\mathbb{T}^d))$ is complex valued;
- (2) $\chi_T(\cdot) = \chi_1(\cdot/T) \in C^\infty(\mathbb{R})$ where $\chi_1(t) = 1$ for $t \leq 1/2$ and $\chi_1(t) = 0$ for $t \geq 3/4$;
- (3) $0 \leq \varphi_\omega \in C^\infty(\mathbb{T}^d)$ satisfies $1_{\omega'} \leq \varphi_\omega \leq 1_\omega$ where ω' satisfies the geometric control condition, and $\overline{\omega'} \subset \omega$. Such ω' exists because \mathbb{T}^d is compact.

(i). We use the same notation ω for the good unknown of Alinhac and the domain of control for it causes no ambiguity, and it is a standard notation in both cases in the literature.

We fix χ_T and φ_ω and seek F , so that with P_{ext} of the form (3.2.3), u satisfies the following nonlinear paradifferential equation,

$$(3.2.4) \quad (\partial_t + P(u) + R(u))u = (\mathcal{B}(u) + \beta(u))F.$$

Here $P(u)$ is a paradifferential operator of order $3/2$,

$$P(u) = iT_{\gamma(u)} + \nabla_x \cdot T_{V(u)} + \text{lower order terms},$$

with $\gamma(u)$ being a symbol of degree $3/2$ that depends on $\nabla_x \eta$ (hence depends on u), $V(u)$ being the trace to the free surface Σ of the horizontal velocity $\nabla_x \phi$ (see (3.3.1)); while

$$\mathcal{B}(u)F = \chi_T T_q \varphi_\omega \text{Re}F.$$

Under the smallness condition $u = O(\varepsilon_0)_{H^s}$, we have

$$T_{\gamma(u)} - |D_x|^{3/2} = O(\varepsilon_0)_{\mathcal{L}(\dot{H}^s, \dot{H}^{s-3/2})}, \quad V(u) = O(\varepsilon_0)_{H^{s-1}};$$

while remainders $R(u)$ and $\beta(u)$ satisfy for some $\vartheta > 0$,

$$(3.2.5) \quad \|R(u)\|_{\mathcal{L}(\dot{H}^s, \dot{H}^s)} \lesssim \|u\|_{H^s}^\vartheta, \quad \|\beta(u)\|_{\mathcal{L}(\dot{H}^s, \dot{H}^{s+1/2})} \lesssim \|u\|_{H^s}.$$

Therefore perturbation arguments can be used to simplify the situation.

Remark 3.2.1. The reason to redo the parilinearization of (3.1.1) rather than borrowing directly the results from [6] is due to two considerations. Firstly [6] studies the Cauchy problem, to which only the regularity of the remainder $R(u)u$ is important, while in this paper, the smallness is also required to treat it as a perturbation. Secondly, due to the existence of the exterior pressure P_{ext} (in [6] $P_{\text{ext}} = 0$), the same estimates for $\partial_t \psi$ no longer apply, as they now for $\partial_t \psi - P_{\text{ext}}$, resulting in the appearance of the term $\beta(u)F$.

We will first prove the null controllability of (3.2.4) (that is, Theorem 3.2.2 below), then prove that it implies the exact controllability of (3.1.1) (that is, Theorem 3.1.2) in Section 3.7.

Theorem 3.2.2. *Suppose that $d \geq 1$, $\omega \subset \mathbb{T}^d$ satisfies the geometric control condition, $T > 0$, and s sufficiently large, then for some $\varepsilon_0 > 0$ sufficiently small and for all initial data $u_0 \in \dot{H}^s(\mathbb{T}^d)$ with $\|u_0\|_{H^s} < \varepsilon_0$, there exists an $F \in C([0, T], \dot{H}^s(\mathbb{T}^d))$, such that the unique solution $u \in C([0, T], \dot{H}^s(\mathbb{T}^d))$ to (3.2.4) with initial data $u(0) = u_0$ vanishes at time T , that is $u(T) = 0$. Moreover F can be so chosen that*

$$(3.2.6) \quad \|u\|_{C([0, T], H^s) \cap W^{1, \infty}([0, T], H^{s-3/2})} \lesssim \varepsilon_0, \quad \|F\|_{C([0, T], H^s)} \lesssim \varepsilon_0.$$

Remark 3.2.3. In the statement of Theorem 3.2.2, and throughout this article, the relation $X \lesssim Y$ is used to simplify the relation $X \leq C(d, \omega, T, b)Y$ for some constant C which, whenever ε_0 (and h in the semiclassical setting) is sufficiently small, depends only on d, ω, T, b , and can be treated as a universal constant.

3.2.2 Iterative scheme.

To prove Theorem 3.2.2, we adapt the iterative scheme of [5] which reduces the study to the control problem of linear equations. To simplify the notation, we first introduce the following spaces.

Definition 3.2.4. Let $\sigma \in \mathbb{R}$, $\varepsilon_0 > 0$, $T > 0$, we say $u \in \mathcal{C}^{1,\sigma}(T, \varepsilon_0)$, resp. $F \in \mathcal{C}^{0,\sigma}(T, \varepsilon_0)$, if

$$\|u\|_{C([0,T],H^\sigma)} + \|\partial_t u\|_{L^\infty([0,T],H^{\sigma-3/2})} < \varepsilon_0, \quad \text{resp.} \quad \|F\|_{C([0,T],H^\sigma)} < \varepsilon_0.$$

For s sufficiently large and $\varepsilon_0 > 0$, we fix $\underline{u} \in \mathcal{C}^{1,s}(T, \varepsilon_0)$ and consider the null controllability of the linear equation

$$(3.2.7) \quad (\partial_t + P(\underline{u}) + R(\underline{u}))u = (\mathcal{B}(\underline{u}) + \beta(\underline{u}))F.$$

We show that for ε_0 sufficiently small, there exists a linear operator

$$(3.2.8) \quad \Phi(\underline{u}) : \dot{H}^s(\mathbb{T}^d) \rightarrow C([0, T], \dot{H}^s(\mathbb{T}^d)),$$

which *null-controls* (3.2.7), that is, for any $u_0 \in \dot{H}^s(\mathbb{T}^d)$,

$$F = \Phi(\underline{u})u_0 \in C([0, T], \dot{H}^s(\mathbb{T}^d))$$

sends the initial data $u(0) = u_0$ at time $t = 0$ to final data $u(T) = 0$ at time $t = T$ by (3.2.7). Then the iterative scheme proceeds as follows. Let $u_0 \in \dot{H}^s(\mathbb{T}^d)$ such that $\|u_0\|_{H^s} < \varepsilon_0$. We set $u^0 \equiv 0$, $F^0 \equiv 0$, and for $n \geq 0$ set $(u^{n+1}, F^{n+1}) \in C([0, T], \dot{H}^s(\mathbb{T}^d)) \times C([0, T], \dot{H}^s(\mathbb{T}^d))$ by letting

$$F^{n+1} = \Phi(u^n)u_0,$$

and letting u^{n+1} be the solution to the equation

$$(\partial_t + P(u^n) + R(u^n))u^{n+1} = \mathcal{B}(u^n)F^{n+1}, \quad u^{n+1}(0) = u_0, \quad u^{n+1}(T) = 0.$$

Be careful that for F^{n+1} to be well defined, we must check that for some constant $C > 0$, independent of n and small ε_0 , $u^n \in \mathcal{C}^{1,s}(T, C\varepsilon_0)$. To prove the convergence of this scheme, we need the following contraction estimate for the control operator,

$$\|\Phi(u^{n+1}) - \Phi(u^n)\|_{\mathcal{L}(\dot{H}^s, C([0,T], \dot{H}^{s-3/2}))} \lesssim \|u^{n+1} - u^n\|_{C([0,T], H^{s-3/2})},$$

which shows that $\{(u^n, F^n)\}_{n \geq 0}$ is a Cauchy sequence in

$$C([0, T], \dot{H}^{s-3/2}(\mathbb{T}^d)) \cap W^{1,\infty}([0, T], \dot{H}^{s-3}(\mathbb{T}^d)) \times C([0, T], \dot{H}^{s-3/2}(\mathbb{T}^d)),$$

and converges to some (u, F) that satisfies (3.2.4) in the distributional sense. To recover the H^s -regularity of u , we study (3.2.4) by treating it as the linear equation (3.2.10) with $\underline{u} = u$. The H^s -regularity of F comes from (3.2.8) after proving that $u \in \mathcal{C}^{1,s}(T, C\varepsilon_0)$, and that $F = \Phi(u)u_0$.

The construction of $\Phi(\underline{u})$ is the main effort of this paper. Treating $R(\underline{u})$ and $\beta(\underline{u})$ as perturbations, it suffices to study the following unperturbed equation,

$$(3.2.9) \quad (\partial_t + P(\underline{u}))u = \mathcal{B}(\underline{u})F.$$

Indeed, we show that there exists a linear control operator,

$$\Theta(\underline{u}) : \dot{L}^2(\mathbb{T}^d) \rightarrow C([0, T], \dot{L}^2(\mathbb{T}^d))$$

which null-controls (3.2.9) for L^2 initial data, and moreover satisfies

$$\Theta(\underline{u})|_{\dot{H}^s} : \dot{H}^s(\mathbb{T}^d) \rightarrow C([0, T], \dot{H}^s(\mathbb{T}^d)),$$

Then for some $\mathcal{E}(\underline{u}) = O(\varepsilon_0^\vartheta)_{\mathcal{L}(\dot{H}^s, \dot{H}^s)}$ with $\vartheta > 0$,

$$\Phi(\underline{u}) = \Theta(\underline{u})(1 + \mathcal{E}(\underline{u}))^{-1}.$$

We first apply the *Hilbert uniqueness method* to construct $\Theta(\underline{u})$, and use a commutator estimate to prove its H^s -regularity, with details explained below.

3.2.3 Hilbert uniqueness method

The Hilbert uniqueness method is a purely functional analysis argument, due to Lions [137] which, when applied to our situation, establishes the equivalence between the L^2 -null controllability of (3.2.9) and the L^2 -observability (see below) of its dual equation,

$$(3.2.10) \quad (\partial_t - P(\underline{u})^*)u = 0,$$

where $P(\underline{u})^*$ is the formal adjoint operator of $P(\underline{u})$ with respect to the scalar product $\operatorname{Re}(\cdot, \cdot)_{L^2}$, such that for all $f, g \in C^\infty(\mathbb{T}^d)$,

$$\operatorname{Re}(P(\underline{u})f, g)_{L^2} = \operatorname{Re}(f, P(\underline{u})^*g)_{L^2}.$$

The reason to use $\operatorname{Re}(\cdot, \cdot)_{L^2}$ instead of $(\cdot, \cdot)_{L^2}$, and thus consider the real Hilbert space $(\dot{L}^2(\mathbb{T}^d), \operatorname{Re}(\cdot, \cdot)_{L^2})$ is that $P(\underline{u})$ is not \mathbb{C} -linear, but only \mathbb{R} -linear (see the exact expression of $P(\underline{u})$ in Proposition 3.3.14), due to our pursuit of a real valued control, which results in the appearance of the \mathbb{R} -linear operator Re in (3.2.3).

The (strong) L^2 -observability of (3.2.10) is the following inequality, that for all its solution $u \in C([0, T], \dot{L}^2(\mathbb{T}^d))$,

$$(3.2.11) \quad \|u(0)\|_{L^2}^2 \lesssim \int_0^T \|\mathcal{B}(\underline{u})^*u(t)\|_{L^2}^2 dt,$$

where $\mathcal{B}(\underline{u})^*$ is the adjoint of $\mathcal{B}(\underline{u})$ with respect to $\operatorname{Re}(\cdot, \cdot)_{L^2}$. To prove it, we consider (3.2.10) as a perturbation of a pseudodifferential equation,

$$(3.2.12) \quad (D_t + Q(\underline{u}))u = 0,$$

where $D_t = \frac{1}{i}\partial_t$, and $Q(\underline{u})$ is the pseudodifferential operator with the same symbol as $iP(\underline{u})^*$, so that it is of principal symbol $\sim |\xi|^{3/2}$. Then we derive (3.2.11) from the following L^2 -observability of (3.2.12) by Duhamel's formula,

$$(3.2.13) \quad \|u(0)\|_{L^2}^2 \lesssim \int_0^T \|\varphi_\omega \operatorname{Re} u\|_{L^2}^2 dt.$$

To deal with this problem caused by the lack of \mathbb{C} -linearity of the equation, which occurs especially in the proof of the unique continuation property that deals with the low frequency regime, we consider simultaneously u and its complex conjugate \bar{u} . The pair $\vec{w} = \begin{pmatrix} u \\ \bar{u} \end{pmatrix}$ then satisfies the \mathbb{C} -linear system of pseudodifferential equation,

$$(3.2.14) \quad D_t \vec{w} + \mathcal{A}(\underline{u}) \vec{w} = 0,$$

where \mathcal{A} is of principal symbol $\sim \begin{pmatrix} |\xi|^{3/2} & 0 \\ 0 & -|\xi|^{3/2} \end{pmatrix}$. Then (3.2.13) is a consequence of the following L^2 -observability of (3.2.14),

$$(3.2.15) \quad \|\vec{w}(0)\|_{L^2}^2 \lesssim \int_0^T \|\varphi_\omega \vec{e} \cdot \vec{w}\|_{L^2}^2 dt,$$

where $\vec{e} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so that for $\vec{w} = \begin{pmatrix} w^+ \\ w^- \end{pmatrix}$, $\vec{e} \cdot \vec{w} = w^+ + w^-$.

Remark 3.2.5. Here we write $\|\vec{w}(0)\|_{L^2} = \|\vec{w}(0)\|_{L^2 \times L^2}$ for simplicity. This will be our convention for the Sobolev norms of the pair \vec{w} . In this spirit, we will also write $H^s = H^s \times H^s$ when there is no ambiguity.

3.2.4 Semiclassical observability

We adapt Lebeau's approach [131] to prove (3.2.15). This approach performs a dyadic decomposition in frequencies of \vec{w} , then proves a semiclassical observability (see below) for each dyadic part by studying the propagation of mass (L^2 -norm), and finally recovers the observability of \vec{w} using Littlewood-Paley's theory. The idea behind this approach is that, the mass of solutions of a dispersive equation, is not well propagated unless for those localized in frequencies, because group velocities for different frequencies vary, and we are not expected to have a uniform propagation for all frequencies. To be more precise, for $h = 2^{-j}$, with $j \in \mathbb{N}$, we define

$$\vec{w}_h = \Pi_h \Delta_h \vec{w},$$

where Δ_h is a semiclassical pseudodifferential operator, and Π_h is defined by a semiclassical functional calculus (see Section 3.4.3 for the exact expression; to set \vec{w}_h in this way is purely out of a technical concern), so that the frequency of \vec{w}_h is localized in $Ah^{-1} \leq |\xi| \leq Bh^{-1}$ for some $0 < A < B$. The semiclassical observability states that, for $h > 0$ sufficiently small,

$$(3.2.16) \quad \|\vec{w}_h(0)\|_{L^2}^2 \lesssim h^{-1/2} \int_0^{h^{1/2}T} \|\varphi_\omega \vec{e} \cdot \vec{w}_h\|_{L^2}^2 dt + \text{remainder terms}.$$

The reason why (3.2.16) holds on a semiclassical time interval of length $\sim h^{1/2}$ can be intuitively explained as follows. The dual equation (3.4.2) with $\underline{u} = 0$, $g = 0$ and $b = \infty$ (recall that b is the depth of the water) is

$$(3.2.17) \quad (D_t + |D_x|^{3/2})u = 0,$$

with its dispersion relation and group velocity being

$$(3.2.18) \quad w(k) = |k|^{3/2}, \quad v_g(k) = \frac{\partial w}{\partial k} = \frac{3}{2} \cdot \frac{k}{|k|^{1/2}}.$$

Due to the variation of frequencies on the interval $[Ah^{-1}, Bh^{-1}]$ of size $\sim h^{-1}$, the variation of group velocity is then $\sim h^{-1/2}$, hence the mass is only “well propagated” on a semiclassical time interval of size $\sim h^{1/2}$. That is why Lebeau introduced in [131] a semiclassical time variable s by a scaling $s = h^{-1/2}t$ (see also Staffilani–Tataru [178] and Burq–Gérard–Tzvetkov [46] where this same scaling is used to prove the Strichartz estimates of Schrödinger equations with variable coefficients), so that the semiclassical time interval is of size ~ 1 in s variable.

As for a technical explanation, equation (3.2.17) writes in variables (s, x) as a zero order semiclassical differential equation,

$$(3.2.19) \quad (\hbar D_s + |\hbar D_x|^{3/2})u = 0.$$

The vague word “well propagated” above can be quantitatively characterized by the semiclassical defect measures of solutions to (3.2.19). For semiclassical defect measures, we refer to Gérard [84], Gérard–Leichtnam [86], Lions–Paul [138], see also the survey by Burq [40]. Then the proof of (3.2.16) argues by contradiction and studies the propagation of the semiclassical defect measures, where the geometric control condition is required. This argument dates back to Lebeau [132].

3.2.5 Weak L^2 -observability

By an energy estimate, we can show that (3.2.16) remains valid after replacing the interval of integration $[0, h^{1/2}T]$ by $I_k = [h^{1/2}kT, h^{1/2}(k+1)T]$ for $k = 0, 1, \dots, h^{-1/2} - 1$ (we may of course assume that $h = 2^{2j}$), with a slight change of the remainder term. Once it is proven, we patch $\{I_k\}_{0 \leq k < h^{-1/2}}$ up and use Littlewood–Paley’s theory to obtain a weak observability, that for any $N > 0$,

$$(3.2.20) \quad \|\vec{w}(0)\|_{L^2}^2 \lesssim \int_0^T \|\varphi_\omega \vec{e} \cdot \vec{w}\|_{L^2}^2 dt + \|\vec{w}(0)\|_{H^{-N}}^2.$$

The remainder term $\|\vec{w}(0)\|_{H^{-N}}^2$ comes from the fact that (3.2.16) holds only for small h , that is, for high frequencies.

3.2.6 Unique continuation and strong L^2 -observability

We use the uniqueness-compactness argument of Bardos–Lebeau–Rauch [24] to eliminate the remainder term and obtain the strong L^2 -observability. This argument derives the strong observability from the weak observability and the unique continuation property of the linear equation, which is, in our case equation (3.2.14). However, this equation has a fractional (hence non-analytical with respect to ξ) symbol, so Carleman’s estimate does not seem to apply directly. We then use a perturbative argument (by contradiction), and reduce the problem to proving only the unique continuation property of the equation linearized around the equilibrium state, that is $\eta = 0$, which is a constant coefficient equation,

$$(3.2.21) \quad D_t \vec{w} + \mathcal{A}(0) \vec{w} = 0.$$

To be more precise, suppose that the strong observability does not hold for arbitrarily small ε_0 , then we can find a sequence of solutions \vec{w}_n to (3.2.14) with $\underline{u} = \underline{u}_n \in \mathcal{C}^{1,s}(T, \varepsilon_n)$,

$\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, such that

$$\|\vec{w}_n(0)\|_{L^2} = 1, \quad \int_0^T \|\varphi_\omega \vec{e} \cdot \vec{w}_n\|_{L^2}^2 dt = o(1).$$

The solutions \vec{w}_n converge in distributional sense to a solution \vec{w} to (3.2.21), such that

$$(3.2.22) \quad \varphi_\omega \vec{e} \cdot \vec{w}|_{0 \leq t \leq T} = 0.$$

Moreover, by the weak observability (3.2.20) and Rellich-Kondrachov's compact injection theorem, $\|\vec{w}(0)\|_{H^{-N}} > 0$, hence $\vec{w} \not\equiv 0$. We will obtain a contradiction by the unique continuation property of (3.2.21), that is, such a non-zero solution to (3.2.21) never exists. To prove this, we consider the \mathbb{C} -linear vector space \mathcal{N} of initial data $\vec{w}_0 \in \dot{L}^2(\mathbb{T}^d)$ whose corresponding solution satisfies (3.2.22). Then the weak observability (3.2.20) implies that \mathcal{N} is a compact metric space with respect to the L^2 -norm, and hence of finite dimension. It is not difficult to see that $\mathcal{A}(0)$ defines a \mathbb{C} -linear operator on \mathcal{N} , and thus admits an eigenfunction $\vec{w}_0 = \begin{pmatrix} w_0^+ \\ w_0^- \end{pmatrix} \in \dot{L}^2(\mathbb{T}^d)$ such that

$$(3.2.23) \quad \mathcal{A}(0)\vec{w}_0 = \lambda \vec{w}_0, \quad \lambda \in \mathbb{C},$$

and that

$$(w_0^+ + w_0^-)|_\omega = \vec{e} \cdot \vec{w}_0|_\omega = 0.$$

We obtain from (3.2.23) an elliptic equation for $w_0^+ + w_0^-$, and show that $w_0^+ + w_0^-$ has only a finite Fourier modes. Therefore it can never vanish on a non-empty open set unless $w_0^+ + w_0^- \equiv 0$. Again easily from (3.2.23) we also obtain $w_0^+ - w_0^- \equiv 0$. This contradicts the fact that \vec{w} is an eigenfunction and can not be identically zero.

3.2.7 H^s -controllability

Now we prove the H^s -regularity of $\Theta = \Theta(\underline{u})$, that is, whenever the initial data $u_0 \in \dot{H}^s(\mathbb{T}^d)$, the control $F = \Theta u_0 \in C([0, T], \dot{H}^s(\mathbb{T}^d))$. The Hilbert uniqueness method implies that, at the L^2 level,

$$\Theta = -\mathcal{B}^* \mathcal{S} \mathcal{K}^{-1}, \quad \mathcal{K} = -\mathcal{R} \mathcal{B} \mathcal{B}^* \mathcal{S},$$

where the definitions of the solution operator \mathcal{S} and the range operator \mathcal{R} will be made clear in Section 3.4.1. The main motivation to proving the strong L^2 -observability is that, it is a rephrasing of the coercivity of the following \mathbb{R} -bilinear form on $\dot{L}^2(\mathbb{T}^d)$,

$$\varpi(f, g) = \operatorname{Re}(\mathcal{K}f, g)_{L^2}.$$

Therefore the strong L^2 -observability implies, by Lax-Milgram's theorem, the invertibility of \mathcal{K} , and consequently the well-definedness of Θ .

Coming back to the H^s level, observing that, by the definition of \mathcal{K} , it is easily verified that \mathcal{K} sends $\dot{H}^s(\mathbb{T}^d)$ to itself. Therefore, to prove the H^s -regularity of Θ , we only need to show that $\mathcal{K}|_{\dot{H}^s} : \dot{H}^s(\mathbb{T}^d) \rightarrow \dot{H}^s(\mathbb{T}^d)$ defines an isomorphism. We use again Lax-Milgram's theorem, and consider the following \mathbb{R} -bilinear form on the semiclassical Sobolev space $\dot{H}_h^s(\mathbb{T}^d) = \dot{H}^s(\mathbb{T}^d) \cap H_h^s(\mathbb{T}^d)$ which inherits the semiclassical Sobolev norm from $H_h^s(\mathbb{T}^d)$,

$$\varpi_h^s(f, g) = \operatorname{Re}(\Lambda_h^s|_{t=0} \mathcal{K}f, \Lambda_h^s|_{t=0} g)_{L^2}.$$

where $\Lambda_h^s = 1 + h^s T_{(\gamma^{(3/2)})^{2s/3}}$. In order to conclude, the key point is the following commutator estimate, that for ε_0 and h sufficiently small,

$$(3.2.24) \quad [\mathcal{K}, \Lambda_h^s|_{t=0}] \Lambda_h^{-s}|_{t=0} = O(\varepsilon_0 + h)_{\mathcal{L}(L^2, L^2)}.$$

Indeed, if this is proven, we obtain the coercivity of ϖ_h^s on $\dot{H}_h^s(\mathbb{T}^d)$,

$$\varpi_h^s(f, f) = \varpi(\Lambda_h^s|_{t=0} f, \Lambda_h^s|_{t=0} f) + O(\varepsilon_0 + h) \|\Lambda_h^s|_{t=0} f\|_{L^2}^2 \gtrsim \|f\|_{H_h^s}^2.$$

The trick of introducing the semiclassical parameter h has already been used in [5], but our proof of the H^s -regularity is different.

3.3 Reformulation of problem

This section performs the parilinearization of (3.1.1) as developed in [12, 6, 8], but with two differences : first a more careful treatment to the remainder terms, so that they are not only of sufficient regularity, but also “super-linear” (see Remark 2.6 of [5]); and second, a modification concerning the existence of the exterior pressure P_{ext} .

The estimates below are carried out for each fixed time t , so the time variable will be temporarily omitted for simplicity.

3.3.1 Parilinearization of Dirichlet–Neumann operator

Recall the following notations which are standard in literatures. Let

$$B(x) = (\partial_y \phi)(x, \eta(x)), \quad V(x) = (\nabla_x \phi)(x, \eta(x))$$

be the traces of the vertical and horizontal components of the Eulerian vector field to the free surface, respectively. They admit the following expressions which could be seen as linear operators depending on η , applying on ψ .

$$(3.3.1) \quad B = B(\eta)\psi = \frac{\nabla_x \eta \cdot \nabla_x \psi + G(\eta)\psi}{1 + |\nabla_x \eta|^2}, \quad V = V(\eta)\psi = \nabla_x \psi - B \nabla_x \eta,$$

as well as the good unknown of Alinhac,

$$(3.3.2) \quad \omega = \omega(\eta)\psi = \psi - T_B \eta.$$

Remark 3.3.1. When there is no ambiguity, we simply write B , V and ω for short. The notation $B(\eta)$ will be used later in Lemma 3.3.11 to denote the operator

$$B(\eta) = \frac{\nabla_x \eta \cdot \nabla_x + G(\eta)}{1 + |\nabla_x \eta|^2}.$$

Lemma 3.3.2. *Suppose $s > 1/2 + d/2$ and $1 + d/2 < \sigma \leq s + 1/2$, then*

$$\|(B, V)\|_{H^{\sigma-1} \times H^{\sigma-1}} \leq C(\|\eta\|_{H^{s+1/2}}) \|\nabla_x \psi\|_{H^{\sigma-1}}.$$

Démonstration. By Theorem 3.12 of [8],

$$\|G(\eta)\psi\|_{H^{\sigma-1}} \leq C(\|\eta\|_{H^{s+1/2}}) \|\nabla_x \psi\|_{H^{\sigma-1}}.$$

We conclude by $\|uv\|_{H^{\sigma-1}} \lesssim \|u\|_{H^{\sigma-1}} \|v\|_{H^{\sigma-1}}$ and (3.9.7). \square

The following proposition follows by a careful study of Proposition 3.14 of [6].

Proposition 3.3.3 (Alazard–Burq–Zuily). *Let $s > 3+d/2$, $(\eta, \psi) \in H^{s+1/2}(\mathbb{T}^d) \times H^s(\mathbb{T}^d)$. Let the symbol*

$$\lambda = \lambda^{(1)} + \lambda^{(0)},$$

be defined by

$$\begin{aligned} \lambda^{(1)}(x, \xi) &= \sqrt{(1 + |\nabla_x \eta|^2)|\xi|^2 - (\nabla_x \eta \cdot \xi)^2}, \\ \lambda^{(0)}(x, \xi) &= \frac{1 + |\nabla_x \eta|^2}{2\lambda^{(1)}} \left\{ \nabla_x \cdot (\alpha^{(1)} \nabla_x \eta) + i \partial_\xi \lambda^{(1)} \cdot \nabla_x \alpha^{(1)} \right\}, \end{aligned}$$

where $\alpha^{(1)}(x, \xi) = \frac{\lambda^{(1)} + i \nabla_x \eta \cdot \xi}{1 + |\nabla_x \eta|^2}$. Then

$$G(\eta)\psi = T_\lambda \omega - \nabla_x \cdot T_V \eta + f(\eta, \psi),$$

where for any $1/2 < \delta < 1$,

$$(3.3.3) \quad \|f(\eta, \psi)\|_{H^{s+\delta}} \leq C(\|\eta\|_{H^{s+1/2}}) \|\nabla_x \psi\|_{H^{s-1}}.$$

Démonstration. Now that we have a larger regularity, $s > 3 + d/2$, than that that in [6], which is $s > 2+d/2$, we have no need to deal with the paraproduct T_a for some $a \in H^{d/2-\varepsilon}$, which is of positive order. The proof in [6] then implies that

$$G(\eta)\psi = T_\lambda \omega - T_V \cdot \nabla_x \eta - T_{\nabla_x \cdot V} \eta + f(\eta, \psi),$$

where for any $1/2 < \delta < 1$, $\|f(\eta, \psi)\|_{H^{s+\delta}} \leq C(\|\eta\|_{H^{s+1/2}}) \|\nabla_x \psi\|_{H^{s-1}}$. Instead of considering $-T_{\nabla_x \cdot V} \eta + f(\eta, \psi)$ together as the remainder term of regularity $H^{s+1/2}$ (which already suffices in [6] for the Cauchy problem), we observe here that

$$T_V \cdot \nabla_x \eta + T_{\nabla_x \cdot V} \eta = \nabla_x \cdot T_V \eta,$$

and conclude. □

Remark 3.3.4. By the definition of λ , V and ω , the remainder $f(\eta, \psi)$ is of the form,

$$f(\eta, \psi) = R_G(\eta)\psi + M_b \psi,$$

where $R_G(\eta)$ is a linear operator depending on η , and following (3.3.3),

$$(3.3.4) \quad \|R_G(\eta)\psi\|_{H^{s+\delta}} \leq C(\|\eta\|_{H^{s+1/2}}) \|\nabla_x \psi\|_{H^{s-1}},$$

while $M_b = G(0) - |D_x| = m_b(D_x)$ is a smoothing Fourier multiplier, with symbol

$$m_b(\xi) = |\xi|(\tanh(b|\xi|) - 1).$$

Indeed, $G(0) = |D_x| \tanh(bD_x)$ is a Fourier multiplier which could be calculated directly.

We aim to prove the following estimate.

Proposition 3.3.5. *For $s > 3 + d/2$ and some $\vartheta > 0$,*

$$\|R_G(\eta)\psi\|_{H^{s+1/2}} \leq C(\|\eta\|_{H^{s+1/2}}) \|\eta\|_{H^{s+1/2}}^\vartheta \|\nabla_x \psi\|_{H^{s-1}}.$$

Démonstration. This follows by interpolating between (3.3.4), and the following Lemma 3.3.6. \square

Lemma 3.3.6. *For $s > 3 + d/2$, $\|R_G(\eta)\psi\|_{H^{s-2}} \leq C(\|\eta\|_{H^{s+1/2}})\|\eta\|_{H^{s+1/2}}\|\nabla_x\psi\|_{H^{s-1}}$.*

Démonstration. Write

$$R_G(\eta)\psi = (G(\eta)\psi - G(0)\psi) - (T_\lambda\omega - |D_x|\psi) + \nabla_x \cdot T_V\eta,$$

and estimate the three terms separately. By Lemma 3.3.2 and Theorem 3.9.3,

$$\|\nabla_x \cdot T_V\eta\|_{H^{s-1/2}} \lesssim \|V\|_{L^\infty} \|\eta\|_{H^{s+1/2}} \leq C(\|\eta\|_{H^{s+1/2}})\|\eta\|_{H^{s+1/2}}\|\nabla_x\psi\|_{H^{s-1}}.$$

For the middle term, write $T_\lambda\omega - |D_x|\psi = T_{\lambda-|\xi|}\psi - T_\lambda T_B\eta$, and by Theorem 3.9.3,

$$\begin{aligned} \|T_{\lambda-|\xi|}\psi\|_{H^{s-1}} &\lesssim M_{0,d/2+1}^1(\lambda - |\xi|)\|\nabla_x\psi\|_{H^{s-1}} \leq C(\|\eta\|_{H^{s+1/2}})\|\eta\|_{H^{s+1/2}}\|\nabla_x\psi\|_{H^{s-1}}, \\ \|T_\lambda T_B\eta\|_{H^{s-1/2}} &\lesssim M_{0,d/2+1}^1(\lambda)\|B\|_{L^\infty}\|\eta\|_{H^{s+1/2}} \leq C(\|\eta\|_{H^{s+1/2}})\|\eta\|_{H^{s+1/2}}\|\nabla_x\psi\|_{H^{s-1}}. \end{aligned}$$

The first term is estimated by the following lemma. \square

Lemma 3.3.7. $\|G(\eta)\psi - G(0)\psi\|_{H^{s-2}} \leq C(\|\eta\|_{H^{s+1/2}})\|\eta\|_{H^{s+1/2}}\|\nabla_x\psi\|_{H^{s-1}}$

Démonstration. Recall the shape derivation formula of the Dirichlet–Neumann operator, see [123], see also [6].

Proposition 3.3.8. *Let $s > 2 + d/2$, and $\eta \in H^{s+1/2}(\mathbb{T}^d)$, $\psi \in H^\sigma(\mathbb{T}^d)$ for $1 \leq \sigma \leq s$. Then there exists a neighborhood of η in $H^{s+1/2}(\mathbb{T}^d)$, in which the mapping $\tilde{\eta} \mapsto G(\tilde{\eta})\psi$ is differentiable. Moreover, for $h \in H^{s+1/2}(\mathbb{T}^d)$,*

$$(3.3.5) \quad d_\eta G(\eta)\psi \cdot h := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \{G(\eta + \varepsilon h)\psi - G(\eta)\psi\} = -G(\eta)(Bh) - \nabla_x \cdot (Vh).$$

Then by Newton-Leibniz formula,

$$G(\eta)\psi - G(0)\psi = \int_0^1 dG(t\eta)\psi \cdot \eta dt = - \int_0^1 \left(G(t\eta)(B_t\eta) + \nabla_x \cdot (V_t\eta) \right) dt,$$

where $B_t = B(t\eta)\psi$, $V_t = V(t\eta)\psi$. Therefore,

$$\begin{aligned} \|G(\eta)\psi - G(0)\psi\|_{H^{s-2}} &\leq \int_0^1 \left(C(\|t\eta\|_{H^{s+1/2}})\|B_t\|_{H^{s-1}} + \|V_t\|_{H^{s-1}} \right) \|\eta\|_{H^{s-1}} dt \\ &\leq C(\|\eta\|_{H^{s+1/2}})\|\eta\|_{H^{s+1/2}}\|\nabla_x\psi\|_{H^{s-1}}. \end{aligned}$$

\square

3.3.2 Parilinearization of surface tension

Proposition 3.3.9. *Let $\eta \in H^{s+1/2}(\mathbb{T}^d)$ with $s > 3/2 + d/2$, then*

$$(3.3.6) \quad H(\eta) = -T_\ell\eta + R_H(\eta)\eta,$$

where the symbol

$$\ell = \ell^{(2)} + \ell^{(1)}$$

is explicitly defined by

$$\ell^{(2)} = \frac{1}{\sqrt{1 + |\nabla_x \eta|^2}} \left(|\xi|^2 - \frac{(\nabla_x \eta \cdot \xi)^2}{1 + |\nabla_x \eta|^2} \right), \quad \ell^{(1)} = \frac{1}{2} \partial_\xi \cdot D_x \ell^{(2)},$$

and $R_H(\eta)$ is a linear operator satisfying the estimate,

$$\|R_H(\eta)\|_{\mathcal{L}(H^{s+1/2}, H^{2s-2-d/2})} \leq C(\|\eta\|_{H^{s+1/2}}) \|\eta\|_{H^{s+1/2}}.$$

Démonstration. Letting $L(v) = \frac{1}{\sqrt{1+|v|^2}}$ for $v \in \mathbb{R}^d$, then $L(0) - 1 = 0$, and by (3.9.6),

$$\begin{aligned} \frac{1}{\sqrt{1 + |\nabla_x \eta|^2}} &= A(\nabla_x \eta) = 1 + (L(\nabla_x \eta) - 1) \\ &= 1 + T_{(\nabla L)(\nabla_x \eta)} \cdot \nabla_x \eta + R_{L-1}(\nabla_x \eta), \end{aligned}$$

where $(\nabla L)(\nabla_x \eta) = -\frac{\nabla_x \eta}{\sqrt{1+|\nabla_x \eta|^2}^3}$, and the remainder satisfies the estimate

$$\|R_{L-1}(\nabla_x \eta)\|_{H^{2s-1-d/2}} \leq C(\|\eta\|_{H^{s+1/2}}) \|\eta\|_{H^{s+1/2}}.$$

Therefore, using the fact that $T_{\nabla_x \eta} 1 = 0$, we have

$$\begin{aligned} \frac{\nabla_x \eta}{\sqrt{1 + |\nabla_x \eta|^2}} &= \nabla_x \eta L(\nabla_x \eta) \\ &= T_{\nabla_x \eta} L(\nabla_x \eta) + T_{L(\nabla_x \eta)} \nabla_x \eta + R(\nabla_x \eta, L(\nabla_x \eta)); \\ &= T_{\nabla_x \eta} T_{(\nabla L)(\nabla_x \eta)} \cdot \nabla_x \eta + T_{L(\nabla_x \eta)} \nabla_x \eta \\ &\quad + R(\nabla_x \eta, L(\nabla_x \eta)) + T_{\nabla_x \eta} R_{L-1}(\nabla_x \eta) \\ &= T_M \nabla_x \eta + \tilde{R}(\eta) \eta, \end{aligned}$$

with the matrix valued symbol

$$M = L(\nabla_x \eta) + \nabla_x \eta \otimes (\nabla L)(\nabla_x \eta) = \frac{1}{\sqrt{1 + |\nabla_x \eta|^2}} I_d - \frac{\nabla_x \eta \otimes \nabla_x \eta}{\sqrt{1 + |\nabla_x \eta|^2}^3}.$$

As for the remainder, $\tilde{R}(\eta)$ is a linear operator depending on η defined by

$$\begin{aligned} \tilde{R}(\eta)v &= R(\nabla_x v, L(\nabla_x \eta)) + T_{\nabla_x v} R_{L-1}(\nabla_x \eta) \\ &\quad + (T_{\nabla_x \eta} T_{(\nabla L)(\nabla_x \eta)} \cdot \nabla_x - T_{\nabla_x \eta \otimes (\nabla L)(\nabla_x \eta)} \nabla_x)v, \end{aligned}$$

satisfying by (3.9.4) the estimate

$$\|\tilde{R}(\eta)v\|_{H^{2s-1-d/2}} \leq C(\|\eta\|_{H^{s+1/2}}) \|\eta\|_{H^{s+1/2}} \|v\|_{H^{s+1/2}}.$$

Indeed, the only problem is for the first term, for which we write

$$R(\nabla_x v, L(\nabla_x \eta)) = R(\nabla_x v, 1) + R(\nabla_x v, L(\nabla_x \eta) - 1),$$

where

$$R(\nabla_x v, 1) = \nabla_x v - T_{\nabla_x v} 1 - T_1 \nabla_x v = \nabla_x v - 0 - \nabla_x v = 0.$$

Consequently,

$$H(\eta) = \nabla_x \cdot \left(\frac{\nabla_x \eta}{\sqrt{1 + |\nabla_x \eta|^2}} \right) = T_{-M\xi \cdot \xi + i \nabla_x \cdot M\xi} \eta + R_H(\eta) \eta,$$

with $\ell^{(2)} = M\xi \cdot \xi$, $\ell^{(1)} = \frac{1}{i} \nabla_x \cdot M\xi$, and $R_H(\eta)$ defined by $R_H(\eta)v = \nabla_x \cdot (\tilde{R}(\eta)v)$ which satisfies the desired estimate. \square

3.3.3 Equation for the good unknown

Proposition 3.3.10. *Let $\eta \in H^{s+1/2}(\mathbb{T}^d)$ and $\psi \in H^s(\mathbb{T}^d)$ with $s > 1 + d/2$, then*

$$(3.3.7) \quad \begin{aligned} & \frac{1}{2} \frac{(\nabla_x \eta \cdot \nabla_x \psi + G(\eta)\psi)^2}{1 + |\nabla_x \eta|^2} - \frac{1}{2} |\nabla_x \psi|^2 \\ & = T_B G(\eta)\psi + T_B T_V \cdot \nabla_x \eta - T_V \cdot \nabla_x \psi + R_S(\eta, \psi)\psi, \end{aligned}$$

where $R_S(\eta, \psi)$ is a linear operator satisfying the estimate

$$\|R_S(\eta, \psi)\|_{\mathcal{L}(H^s, H^{2s-2-d/2})} \leq C(\|\eta\|_{H^{s+1/2}}) \|\nabla_x \psi\|_{H^{s-1}}.$$

Démonstration. By the definition of B , V and $G(\eta)$, we easily verify the identity.

$$G(\eta)\psi = B - \nabla_x \eta \cdot V.$$

Plug this into the left hand side of (3.3.7),

$$\begin{aligned} & \frac{1}{2} \frac{(\nabla_x \eta \cdot \nabla_x \psi + G(\eta)\psi)^2}{1 + |\nabla_x \eta|^2} - \frac{1}{2} |\nabla_x \psi|^2 \\ & = \frac{1}{2} B(\nabla_x \eta \cdot \nabla_x \psi + G(\eta)\psi) - \frac{1}{2} \nabla_x \psi \cdot (V + B\nabla_x \eta) \\ & = BG(\eta)\psi + \frac{1}{2} (BG(\eta)\psi - \nabla_x \psi \cdot V) \\ & = BG(\eta)\psi + \frac{1}{2} (B(B - V \cdot \nabla_x \eta) - (V + B\nabla_x \eta) \cdot V) \\ & = BG(\eta)\psi - \frac{1}{2} B^2 - \frac{1}{2} |V|^2. \end{aligned}$$

Then by (3.9.4),

$$\begin{aligned} & BG(\eta)\psi - \frac{1}{2} B^2 - \frac{1}{2} |V|^2 \\ & = T_B G(\eta)\psi + T_{G(\eta)\psi} B - T_B B - T_V \cdot V + R_1(\eta, \psi)\psi \\ & = T_B G(\eta)\psi + T_{G(\eta)\psi - B} B - T_V \cdot (\nabla_x \psi - B\nabla_x \eta) + R_1(\eta, \psi)\psi \\ & = T_B G(\eta)\psi - T_{V \cdot \nabla_x \eta} B - T_V \cdot \nabla_x \psi + T_V \cdot (B\nabla_x \eta) + R_1(\eta, \psi)\psi \\ & = T_B G(\eta)\psi + T_B T_V \cdot \nabla_x \eta - T_V \cdot \nabla_x \psi + R_1(\eta, \psi)\psi + R_2(\eta, \psi)\psi, \end{aligned}$$

with the two operators defined by

$$\begin{aligned} R_1(\eta, \psi)v^1 &= R(B, G(\eta)v^1) - \frac{1}{2} R(B, B(\eta)v^1) - \frac{1}{2} R(V, V(\eta)v^1) \\ R_2(\eta, \psi)v^2 &= T_V \cdot R(B(\eta)v^2, \nabla_x \eta) + (T_V \cdot T_{\nabla_x \eta} - T_{V \cdot \nabla_x \eta}) B(\eta)v^2 + [T_V, T_{B(\eta)v^2}] \nabla_x \eta. \end{aligned}$$

Hence, by (3.9.4), (3.9.5) and Lemma 3.3.2, $R_S(\eta, \psi) = R_1(\eta, \psi) + R_2(\eta, \psi)$ satisfies the desired estimate. \square

Lemma 3.3.11. *With P_{ext} of the form (3.2.3), the good unknown ω satisfies the equation,*

$$\partial_t \omega + \nabla_x \cdot T_V \omega + T_\ell \eta + g\eta + R_\omega^1(\psi, \eta)\psi + R_\omega^2(\psi, \eta)\eta = \chi_T (\varphi_\omega \text{Re} F - T_{B(\eta)(\varphi_\omega \text{Re} F)} \eta),$$

where $R^i(\psi, \eta)$ ($i = 1, 2$) are linear operators such that, for $s > 2 + d/2$,

$$\begin{aligned} \|R_\omega^1(\eta, \psi)\|_{\mathcal{L}(H^s, H^s)} &\leq C(\|(\eta, \nabla_x \psi)\|_{H^{s+1/2} \times H^{s-1}}) \|\nabla_x \psi\|_{H^{s-1}}, \\ \|R_\omega^2(\eta, \psi)\|_{\mathcal{L}(H^{s+1/2}, H^{s+1/2})} &\leq C(\|(\eta, \nabla_x \psi)\|_{H^{s+1/2} \times H^{s-1}}) (\|\nabla_x \psi\|_{H^{s-1}} + \|\eta\|_{H^{s+1/2}}). \end{aligned}$$

Démonstration. Combining the parilinearization results above, ω satisfies the equation,

$$\partial_t \omega + \nabla_x \cdot T_V \omega + T_\ell \eta + g\eta + \tilde{R} = \chi_T \varphi_\omega \operatorname{Re} F,$$

with the remainder \tilde{R} being,

$$\tilde{R} = T_{\partial_t B} \eta + (\nabla_x T_V T_B - T_B T_V \cdot \nabla_x) \eta + R_H(\eta) \eta + R_S(\eta, \psi) \psi + (T_V \cdot \nabla_x - \nabla_x \cdot T_V) \psi,$$

where $R_x(\eta, \psi)$ is defined in Proposition 3.3.10. The term $T_{\partial_t B} \eta$ should be treated with attention, where $\partial_t \psi$ is involved.

$$\begin{aligned} \partial_t B &= \partial_t (B(\eta) \psi) = B(\eta) \partial_t \psi + [\partial_t, B(\eta)] \psi \\ &= \chi_T B(\eta) (\varphi_\omega \operatorname{Re} F) + B(\eta) (\partial_t \psi - \chi_T \varphi_\omega \operatorname{Re} F) + [\partial_t, B(\eta)] \psi, \end{aligned}$$

where, using the equation,

$$\begin{aligned} \partial_t \psi - \chi_T \varphi_\omega \operatorname{Re} F &= -g\eta + H(\eta) + \frac{1}{2} \frac{(\nabla_x \eta \cdot \nabla_x \psi + G(\eta) \psi)^2}{1 + |\nabla_x \eta|^2} - \frac{1}{2} |\nabla_x \psi|^2, \\ [\partial_t, B(\eta)] \psi &= \partial_t \left(\frac{1}{1 + |\nabla_x \eta|^2} \right) (\nabla_x \eta \cdot \nabla_x \psi + G(\eta) \psi) \\ &\quad + \frac{1}{1 + |\nabla_x \eta|^2} (\nabla_x \partial_t \eta \cdot \nabla_x \psi + (\partial_t G(\eta)) \psi). \end{aligned}$$

Therefore,

$$\|\partial_t \psi - \chi_T \varphi_\omega \operatorname{Re} F\|_{H^{s-3/2}} \leq C(\|\eta, \nabla_x \psi\|_{H^{s+1/2} \times H^{s-1}})(\|\nabla_x \psi\|_{H^s} + \|\eta\|_{H^{s+1/2}}).$$

And by (3.3.5),

$$(\partial_t G(\eta)) \psi = -G(\eta) (B \partial_t \eta) - \nabla_x \cdot (V \partial_t \eta),$$

from which, replacing $\partial_t \eta$ with $G(\eta) \psi$ by the equation,

$$\|[\partial_t, B(\eta)] \psi\|_{H^{s-5/2}} \leq C(\|(\eta, \nabla_x \psi)\|_{H^{s+1/2} \times H^{s-1}})(\|\nabla_x \psi\|_{H^s} + \|\eta\|_{H^{s+1/2}}).$$

As for an exact formula, it suffices to set

$$\begin{aligned} R_\omega^1(\eta, \psi) v^1 &= R_S(\eta, \psi) v^1 + (T_V \cdot \nabla_x - \nabla_x \cdot T_V) v^1 \\ R_\omega^2(\eta, \psi) v^2 &= T_{A(\eta, \psi)} v^2 + (\nabla_x T_V T_B - T_B T_V \cdot \nabla_x) v^2 + R_H(\eta) v^2, \end{aligned}$$

where $R_H(\eta)$ is defined in Proposition 3.3.9, and by expanding $\partial_t B - \chi_T B(\eta) (\varphi_\omega \operatorname{Re} F)$ thoroughly using the identities above,

$$\begin{aligned} A(\eta, \psi) &= -g\eta + H(\eta) + \frac{1}{2} \frac{(\nabla_x \eta \cdot \nabla_x \psi + G(\eta) \psi)^2}{1 + |\nabla_x \eta|^2} - \frac{1}{2} |\nabla_x \psi|^2 \\ &\quad - \frac{2 \nabla_x \eta \cdot \nabla_x G(\eta) \psi}{(1 + |\nabla_x \eta|^2)^2} (\nabla_x \eta \cdot \nabla_x \psi + G(\eta) \psi) \\ &\quad - \frac{1}{1 + |\nabla_x \eta|^2} (\nabla_x G(\eta) \psi \cdot \nabla_x \psi - G(\eta) (B G(\eta) \psi) - \nabla_x \cdot (V G(\eta) \psi)). \end{aligned}$$

□

3.3.4 Symmetrization

Now by Proposition 3.3.3 and Lemma 3.3.11, for $s > 3 + d/2$, the system of water waves can be reformulated as

$$\begin{cases} \partial_t \eta - T_\lambda \omega + \nabla_x \cdot T_V \eta - M_b \psi - R_G(\eta) \psi = 0, \\ \partial_t \omega + T_\ell \eta + \nabla_x \cdot T_V \omega + g\eta + R_\omega^1(\psi, \eta) \psi + R_\omega^2(\psi, \eta) \eta \\ = \chi_T (\varphi_\omega \operatorname{Re} F - T_{B(\eta)(\varphi_\omega \operatorname{Re} F)} \eta). \end{cases}$$

Following [6], let symbols which depend solely on η ,

$$q = q^{(0)}, \quad p = p^{(1/2)} + p^{(-1/2)}, \quad \gamma = \gamma^{(3/2)} + \gamma^{(1/2)},$$

be defined by

$$\begin{aligned} q^{(0)} &= (1 + |\nabla_x \eta|^2)^{1/4}, \\ \gamma^{(3/2)} &= \sqrt{\ell^{(2)} \lambda^{(1)}}, \\ \gamma^{(1/2)} &= \sqrt{\frac{\ell^{(2)} \operatorname{Re} \lambda^{(0)}}{\lambda^{(1)}}} - \frac{1}{2} \partial_\xi \cdot D_x \sqrt{\ell^{(2)} \lambda^{(1)}}, \\ p^{(1/2)} &= (1 + |\nabla_x \eta|^2)^{-1/2} \sqrt{\lambda^{(1)}}, \\ p^{(-1/2)} &= \frac{1}{\gamma^{(3/2)}} (q^{(0)} \ell^{(1)} - \gamma^{(1/2)} p^{(1/2)} + i \partial_\xi \gamma^{(3/2)} \cdot \partial_x p^{(1/2)}). \end{aligned}$$

Then $T_p \eta$ and $T_q \omega$ satisfies the equation

$$(3.3.8) \quad \begin{cases} \partial_t (T_p \eta) - T_\gamma T_q \omega + \nabla_x \cdot T_V T_p \eta - T_p M_b \psi + R_1(\eta) \psi + R_2(\eta) \eta = 0, \\ \partial_t (T_q \omega) + T_\gamma T_p \eta + \nabla_x \cdot T_V T_q \omega + g T_q \eta + R_3(\eta, \psi) \psi + R_4(\eta, \psi) \eta \\ = \chi_T (T_q \varphi_\omega \operatorname{Re} F - T_q T_{B(\eta) \varphi_\omega \operatorname{Re} F} \eta), \end{cases}$$

where the remainders are defined by

$$\begin{aligned} R_1(\eta) v^1 &= -(T_p T_\lambda - T_\gamma T_q) \omega(\eta) v^1 - T_p R_G(\eta) v^1 - [\nabla_x \cdot T_V(\eta) v^1, T_p] \eta, \\ R_2(\eta) v^2 &= -T_{\partial_t p} v^2, \\ R_3(\eta, \psi) v^3 &= -T_{\partial_t q} \omega(\eta) v^3 - [\nabla_x \cdot T_V, T_q] \omega(\eta) v^3 + T_q R_\omega^1(\eta, \psi) v^3, \\ R_4(\eta, \psi) v^4 &= -(T_\gamma T_p - T_q T_\ell) v^4 + T_q R_\omega^2(\eta, \psi) v^4. \end{aligned}$$

By our choice of symbols, the principal and sub-principal symbols in the symbolic calculus of $T_p T_\lambda - T_\gamma T_q$ vanishes. We also have Proposition 3.3.5, deducing that

$$\|R_1(\eta)\|_{\mathcal{L}(H^s, H^s)} \leq C(\|\eta\|_{H^{s+1/2}}) \|\eta\|_{H^{s+1/2}}^\vartheta, \quad \vartheta > 0.$$

By the definition of p , we plug $\partial_t \eta = G(\eta) \psi$ into $\partial_t p$ to deduce that

$$\|R_2(\eta)\|_{\mathcal{L}(H^{s+1/2}, H^s)} \leq C(\|\eta\|_{H^{s+1/2}}) \|\eta\|_{H^{s+1/2}}.$$

Then we use the definition of the good unknown ω , Lemma 3.3.2 and Lemma 3.3.11 to deduce that

$$\|R_3(\eta, \psi)\|_{\mathcal{L}(H^s, H^s)} \leq C(\|(\eta, \nabla_x \psi)\|_{H^{s+1/2} \times H^{s-1}}) (\|\nabla_x \psi\|_{H^s} + \|\eta\|_{H^{s+1/2}}).$$

Finally, by the vanishing of the principal and sub-principal symbols of $T_\gamma T_p - T_q T_\ell$ and Lemma 3.3.11,

$$\|R_4(\eta, \psi)\|_{\mathcal{L}(H^{s+1/2}, H^s)} \leq C(\|(\eta, \nabla_x \psi)\|_{H^{s+1/2} \times H^{s-1}})(\|\nabla_x \psi\|_{H^s} + \|\eta\|_{H^{s+1/2}}).$$

We introduce the new variable

$$(3.3.9) \quad u = T_q \omega - iT_p \eta.$$

Then $u \in \dot{H}^s(\mathbb{T}^d)$, and

$$\|u\|_{H^s} \leq C(\|\eta\|_{H^{s+1/2}})(\|\nabla_x \psi\|_{H^{s-1}} + \|\eta\|_{H^{s+1/2}}).$$

Proposition 3.3.12. *For $s > 3 + d/2$ and ε_0 sufficiently small, if $u \in \dot{H}^s(\mathbb{T}^d)$ such that $\|u\|_{H^s} < \varepsilon_0$, then there exists a unique $(\psi, \eta) \in \dot{H}^s(\mathbb{T}^d) \times \dot{H}^{s+1/2}(\mathbb{T}^d)$ satisfying simultaneously the estimate*

$$\|(\psi, \eta)\|_{H^s \times H^{s+1/2}} \leq 2\|u\|_{H^s},$$

and the identity (3.3.9). Moreover, if $u \in \mathcal{C}^{1,s}(T, \varepsilon_0)$, then $\eta \in \mathcal{C}^{1,s+1/2}(T, 2\varepsilon_0)$, with

$$\|\partial_t \eta\|_{L^\infty([0,T], H^{s-1})} \leq 2\|\partial_t u\|_{L^\infty([0,T], H^{s-3/2})}.$$

Démonstration. Observe that q is a real symbol and an even function in ξ , and the real part of p is even in ξ , while its imaginary part is odd in ξ , so by Lemma 3.9.7, T_p and T_q preserves the real part of a function. Therefore, by the definition of u ,

$$(3.3.10) \quad T_{p(\eta)} \eta = -\text{Im } u,$$

where we write $p = p(\eta)$ to emphasize the fact that p depends on η . Equivalently,

$$\eta = \Psi(\eta),$$

where $\Psi : \dot{H}^{\mu+1/2}(\mathbb{T}^d) \rightarrow \dot{H}^{\mu+1/2}(\mathbb{T}^d)$ for any $s - 3/2 \leq \mu \leq s$, is defined by

$$\Psi(\eta) := -|D_x|^{-1/2} \text{Im } u + |D_x|^{-1/2} T_{|\xi|^{1/2} - p(\eta)} \eta.$$

A direct application of the Banach fixed point theorem is not possible because Ψ is a contraction only when $\|\eta\|_{W^{2,\infty}}$ is sufficiently small. We need to apply the same iterative scheme while proving that each element of the sequence of iteration is sufficiently small. Let $\eta_0 = 0$ and $\eta_{n+1} = \Psi(\eta_n)$ for $n \in \mathbb{N}$. Denote $p_n = p(\eta_n)$ for simplicity. We claim that for all $n \in \mathbb{N}$,

$$\|\eta_n\|_{H^{s+1/2}} \leq 2\|\text{Im } u\|_{H^s}.$$

We prove this by induction. Clearly it is true for $n = 0$. Suppose that it is proven for n , then

$$\begin{aligned} \|\eta_{n+1}\|_{H^{s+1/2}} &= \|\Psi(\eta_n)\|_{H^{s+1/2}} \leq \|\text{Im } u\|_{H^s} + CM_{0,d/2+1}^{1/2} (|\xi|^{1/2} - p_n) \|\eta_n\|_{H^{s+1/2}} \\ &\leq \|\text{Im } u\|_{H^s} + C(\|\eta_n\|_{H^{s+1/2}}) \|\eta_n\|_{H^{s+1/2}} \|\text{Im } u\|_{H^s} \\ &\leq 2\|\text{Im } u\|_{H^s}, \end{aligned}$$

whenever ε_0 is sufficiently small. In particular $\|\eta_n\|_{H^{s+1/2}} \leq 2\varepsilon_0$ for $n \in \mathbb{N}$, and

$$\begin{aligned} \|\eta_{n+1} - \eta_n\|_{H^{\mu+1/2}} &\leq \|T_{|\xi|^{1/2-p_n}}(\eta_n - \eta_{n-1})\|_{H^\mu} + \|T_{p_n-p_{n-1}}\eta_{n-1}\|_{H^\mu} \\ &\leq C(\|\eta_n\|_{H^{s+1/2}})\|\eta_n\|_{H^{s+1/2}}\|\eta_n - \eta_{n-1}\|_{H^{\mu+1/2}} \\ &\quad + C(\|(\eta_{n-1}, \eta_n)\|_{H^{s+1/2} \times H^{s+1/2}})\|\eta_n - \eta_{n-1}\|_{H^{\mu+1/2}}\|\eta_{n-1}\|_{H^{\mu+1/2}} \\ &\leq C\varepsilon_0\|\eta_n - \eta_{n-1}\|_{H^{\mu+1/2}}. \end{aligned}$$

Hence when ε_0 is so small that $C\varepsilon_0 < 1$, $\{\eta_n\}_n$ is a Cauchy sequence in $\dot{H}^{s+1/2}(\mathbb{T}^d)$, and converges to some $\eta \in \dot{H}^{s+1/2}(\mathbb{T}^d)$ such that

$$\eta = \Psi(\eta), \quad \|\eta\|_{H^{s+1/2}} \leq 2\|\text{Im } u\|_{H^s}.$$

The uniqueness of such $\eta \in \dot{H}^{\mu+1/2}(\mathbb{T}^d)$ for any $s - 3/2 \leq \mu \leq s$ comes from the same contraction estimate. Therefore η is independent of the choice of the space $\dot{H}^{\mu+1/2}(\mathbb{T}^d)$ whenever $s - 3/2 \leq \mu \leq s$.

Write $T_q = \pi(D_x) + T_{q-1}$, $T_p = (\pi(D_x) + T_{p/|\xi|^{1/2-1}})|D_x|^{1/2}$, where

$$\begin{aligned} \|T_{q-1}\|_{\mathcal{L}(\dot{H}^s, \dot{H}^s)} &\leq C\|q - 1\|_{L^\infty} \leq C(\|\eta\|_{H^{s+1/2}})\|\eta\|_{H^{s+1/2}} \leq C\varepsilon_0. \\ \|T_{p/|\xi|^{1/2-1}}\|_{\mathcal{L}(\dot{H}^s, \dot{H}^s)} &\leq CM_{0,d/2+1}^0(p/|\xi|^{1/2} - 1) \leq C(\|\eta\|_{H^{s+1/2}})\|\eta\|_{H^{s+1/2}} \leq C\varepsilon_0. \end{aligned}$$

Observe that $\pi(D_x) = \text{Id}_{\dot{H}^\sigma} : \dot{H}^\sigma \rightarrow \dot{H}^\sigma$ for $\sigma \in \mathbb{R}$ (see Proposition 3.9.4), hence by means of Neumann series, $T_p : \dot{H}^{s+1/2}(\mathbb{T}^d) \rightarrow \dot{H}^s(\mathbb{T}^d)$ and $T_q : \dot{H}^s(\mathbb{T}^d) \rightarrow \dot{H}^s(\mathbb{T}^d)$ are invertible with their inverses being

$$T_q^{-1} = (\text{Id} + T_{q-1})^{-1}, \quad T_p^{-1} = |D_x|^{-1/2}(\text{Id} + T_{p/|\xi|^{1/2-1}})^{-1}.$$

and satisfying the estimates,

$$\|T_q^{-1}\|_{\mathcal{L}(\dot{H}^s, \dot{H}^s)} \leq 1 + C\varepsilon_0, \quad \|T_p^{-1}\|_{\mathcal{L}(\dot{H}^s, \dot{H}^{s+1/2})} \leq 1 + C\varepsilon_0.$$

Therefore there is a unique $\psi \in \dot{H}^s(\mathbb{T}^d)$ such that (3.3.9) is satisfied, which is given by the formula

$$\psi = (T_q^{-1}\text{Re} - T_B T_p^{-1}\text{Im})u,$$

from which the estimate

$$\|\psi\|_{H^s} \leq (1 + C\varepsilon_0)\|u\|_{H^s} \leq 2\|u\|_{H^s},$$

for sufficiently small ε_0 .

As for the estimate of time derivative, observe that the symbol p is a function of $\nabla_x \eta$, $\nabla_x^2 \eta$ and ξ . More precisely, one may write

$$p(t, x, \xi) = f(\nabla_x \eta(t, x), \nabla_x^2 \eta(t, x), \xi)$$

with some $f = f(v, M, \xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^{d \times d} \times \mathbb{R}_*^d)$. Denote $\nabla = \nabla_{v,M}$, then

$$\begin{aligned} \partial_t p &= \langle \nabla_v f(\nabla_x \eta, \nabla_x^2 \eta, \xi), \nabla_x \partial_t \eta \rangle + \langle \nabla_M f(\nabla_x \eta, \nabla_x^2 \eta, \xi), \nabla_x^2 \partial_t \eta \rangle \\ &= \langle \nabla f(\nabla_x \eta, \nabla_x^2 \eta, \xi), (\nabla_x \partial_t \eta, \nabla_x^2 \partial_t \eta) \rangle. \end{aligned}$$

Therefore a formal differentiation of the formula (3.3.10) gives,

$$(3.3.11) \quad T_p \partial_t \eta + T_{\langle \nabla f(\nabla_x \eta, \nabla_x^2 \eta, \xi), (\nabla_x \partial_t \eta, \nabla_x^2 \partial_t \eta) \rangle} \eta = -\text{Im } \partial_t u,$$

which leads to the consideration of the map $\tilde{\Psi}_t : \dot{H}^{s-1}(\mathbb{T}^d) \rightarrow \dot{H}^{s-1}(\mathbb{T}^d)$,

$$\tilde{\Psi}_t(\zeta) = |D_x|^{-1/2} T_{|\xi|^{1/2-p}} \zeta - |D_x|^{-1/2} T_{\langle \nabla f(\nabla_x \eta, \nabla_x^2 \eta, \xi), (\nabla_x \zeta, \nabla_x^2 \zeta) \rangle} \eta - |D_x|^{-1/2} \text{Im } \partial_t u,$$

defined for almost every time $t \in [0, T]$, such that, whenever $\partial_t \eta$ is defined,

$$\partial_t \eta = \tilde{\Psi}_t(\partial_t \eta).$$

We show that $\tilde{\Psi}_t$ is a contraction for ε_0 sufficiently small, indeed,

$$\begin{aligned} \|\tilde{\Psi}_t(\zeta_1) - \tilde{\Psi}_t(\zeta_2)\|_{H^{s-1}} &\leq \|T_{|\xi|^{1/2-p}}(\zeta_1 - \zeta_2)\|_{H^{s-3/2}} \\ &\quad + \|T_{\langle \nabla f(\nabla_x \eta, \nabla_x^2 \eta, \xi), (\nabla_x(\zeta_1 - \zeta_2), \nabla_x^2(\zeta_1 - \zeta_2)) \rangle} \eta\|_{H^{s-3/2}} \\ &\leq C(\|\eta\|_{H^{s+1/2}}) \|\eta\|_{H^{s+1/2}} \|\zeta_1 - \zeta_2\|_{H^{s-1}} \\ &\leq C\varepsilon_0 \|\zeta_1 - \zeta_2\|_{H^{s-1}}. \end{aligned}$$

Therefore, by Banach fixed point theorem, for almost every $t \in [0, T]$, there exists a unique $\zeta(t) \in \dot{H}^{s-1}(\mathbb{T}^d)$ such that

$$\tilde{\Psi}_t(\zeta(t)) = \zeta(t), \quad \|\zeta(t)\|_{H^{s-1}} \leq 2\|\text{Im } \partial_t u(t)\|_{H^{s-1}}.$$

We claim that $\partial_t \eta = \zeta$. Indeed, define

$$\tilde{\eta}(t) = \eta(0) + \int_0^t \zeta(s) \, ds,$$

Then $\tilde{\eta}(0) = \eta(0)$, $\partial_t \tilde{\eta} = \zeta$, $\tilde{\eta} \in W^{1,\infty}([0, T], \dot{H}^{s-1}(\mathbb{T}^d))$, and

$$\|\tilde{\eta}\|_{C([0, T], H^{s-1})} \leq (1 + T)C\varepsilon_0.$$

To show that $\eta \equiv \tilde{\eta}$, consider the function $g \in W^{1,\infty}([0, T], \dot{H}^{s-3/2}(\mathbb{T}^d))$ defined by

$$g(t) = T_{p(\tilde{\eta}(t))} \tilde{\eta}(t) + \text{Im } u(t).$$

Then $g(0) = 0$ by the definition of $\eta(0)$. By the definition of $\zeta(t)$ which is a solution of (3.3.11), we immediately verify that $\partial_t g \equiv 0$. Therefore $g \equiv 0$. That implies, by the uniqueness of the η in $\dot{H}^{s-3/2}(\mathbb{T}^d)$, for ε_0 sufficiently small, and for each time $t \in [0, T]$, $\eta(t) = \tilde{\eta}(t)$. Consequently, $\eta = \tilde{\eta} \in W^{1,\infty}([0, T], \dot{H}^{s-1}(\mathbb{T}^d))$. The continuity $\eta \in C([0, T], \dot{H}^{s+1/2}(\mathbb{T}^d))$, is a direct consequence of Lemma 3.5.11 to be proven later. \square

Remark 3.3.13. All the symbols that depends on η , say $a = a(\eta) \in \dot{\Gamma}_\rho^m$, that appear in this article are functions of $\nabla_x \eta$ and $\nabla_x^2 \eta$, that is, following the proof of the previous lemma,

$$a(t, x, \xi) = f(\nabla_x \eta(t, x), \nabla_x^2 \eta(t, x), \xi),$$

for some $f = f(v, M, \xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^{d \times d} \times \mathbb{R}_*^d)$.

As a consequence of this lemma, when $u \in \mathcal{C}^{1,s}(T, \varepsilon_0)$ is defined by (3.2.2) with ε_0 being sufficiently small, we can express $\eta = \eta(u) \in \dot{H}^{s+1/2}(\mathbb{T}^d)$, and thus consider $a = a(u)$ as a symbol depending on u ,

$$a(u) = f(\nabla_x \eta(u), \nabla_x^2 \eta(u), \xi).$$

When considering time derivatives of a , we use the formula,

$$\partial_t a(u) = \langle (\nabla_{v,M} f)(\nabla_x \eta(u), \nabla_x^2 \eta(u), \xi), (\nabla_x \partial_t \eta(u), \nabla_x^2 \partial_t \eta(u)) \rangle.$$

Consequently, we write $p = p(u)$, $q = q(u)$. Expressing also $\dot{\psi} = \dot{\psi}(u) \in \dot{H}^s(\mathbb{T}^d)$ where $\dot{\psi} = \pi(D_x)\psi$, so that $B = B(u)$, because B does not depend on the zero frequency of ψ . Then we have explicitly the following formula,

$$\pi(D_x)\eta = -T_{p(u)}^{-1} \text{Im}u, \quad \pi(D_x)\psi = (T_{q(u)}^{-1} \text{Re} - T_{B(u)} T_{p(u)}^{-1} \text{Im})u,$$

which will be used in Proposition 3.3.14 to derive an equation for u .

For \mathbb{R} -linear operators of the form $R_i(\eta, \psi)$, ($i = 1, 2$) that depend on nonzero frequencies of (η, ψ) , such that $R_1 : H^s(\mathbb{T}^d) \rightarrow H^s(\mathbb{T}^d)$, $R_2 : H^{s+1/2}(\mathbb{T}^d) \rightarrow H^s(\mathbb{T}^d)$, and $R_i = R_i \pi(D_x)$, we can correspond to them \mathbb{R} -linear operators of the form $\tilde{R}_i(u) : H^s(\mathbb{T}^d) \rightarrow H^s(\mathbb{T}^d)$, such that

$$R_1(\eta, \psi)\psi = \tilde{R}_1(u)u, \quad R_2(\eta, \psi)\eta = \tilde{R}_2(u)u.$$

Indeed, we simply let

$$\begin{aligned} \tilde{R}_1(u) &= R_1(\eta(u), \dot{\psi}(u))(T_{q(u)}^{-1} \text{Re} - T_{B(u)} T_{p(u)}^{-1} \text{Im}), \\ \tilde{R}_2(u) &= -R_2(\eta(u), \dot{\psi}(u)) T_{p(u)}^{-1} \text{Im}. \end{aligned}$$

For simplicity, we write henceforth by an abuse of notation $R_i(u) = \tilde{R}_i(u)$, for this correspondence.

Proposition 3.3.14. *Let $s > 3 + d/2$, and $u \in \mathcal{C}^{0,s}(T, \varepsilon_0)$ for $T > 0$ and ε_0 sufficiently small, then u satisfies the equation*

$$(3.3.12) \quad \partial_t u + P(u)u + R(u)u = \mathcal{B}(u)F + \beta(u)F,$$

with $P(u)$ being a paradifferential operator defined by,

$$(3.3.13) \quad P(u) = iT_{r(u)} + \nabla_x \cdot T_{V(u)} - gT_{r(u)}^{-1} \text{Im} + iT_{r(u)} M_b \text{Re},$$

where the symbol $r(u) = p^{(1/2)}(u)/q(u)$ is homogeneous in ξ of order $1/2$. The operators $R(u)$, $\mathcal{B}(u)$ and $\beta(u)$ depend on u , the latter two being explicitly defined as follows,

$$\mathcal{B}(u)F = \chi_T T_{q(u)} \varphi_\omega \text{Re}F, \quad \beta(u)F = \chi_T T_{q(u)} T_{B(u)} (\varphi_\omega \text{Re}F) T_{p(u)}^{-1} \text{Im}u.$$

Furthermore,

$$(3.3.14) \quad \begin{aligned} \mathcal{B}(u) &\in C([0, T], \mathcal{L}(\dot{H}^\sigma, \dot{H}^\sigma)), \quad \forall \sigma \geq 0, \\ \beta(u) &\in L^\infty([0, T], \mathcal{L}(\dot{H}^s, \dot{H}^{s+1/2})) \cap C([0, T], \mathcal{L}(\dot{H}^s, \dot{H}^{s-1})), \\ R(u) &\in L^\infty([0, T], \mathcal{L}(\dot{H}^s, \dot{H}^s)) \cap C([0, T], \mathcal{L}(\dot{H}^s, \dot{H}^{s-3/2})), \end{aligned}$$

and they satisfy the following estimates

$$(3.3.15) \quad \begin{aligned} \|\mathcal{B}(u)\|_{C([0,T],\mathcal{L}(\dot{H}^\sigma,\dot{H}^\sigma))} &\lesssim 1, \\ \|\beta(u)\|_{L^\infty([0,T],\mathcal{L}(\dot{H}^s,\dot{H}^{s+1/2}))} &\lesssim \varepsilon_0, \\ \|R(u)\|_{L^\infty([0,T],\mathcal{L}(\dot{H}^s,\dot{H}^s))} &\lesssim \varepsilon_0^\vartheta, \end{aligned}$$

for some $\vartheta > 0$.

Démonstration. By (3.3.8) and the definition of u ,

$$\partial_t u + \tilde{P}(u)u + \tilde{R}(u)u = \mathcal{B}(u)F + \beta(u)F,$$

with $\tilde{R}(u) = -iR_1(u) - iR_2(u) + R_3(u) + R_4(u)$ and

$$\tilde{P}(u) = iT_\gamma + \nabla_x \cdot T_V - gT_q T_{p-1} \text{Im} + iT_p M_b (T_{q-1} \text{Re} - T_B T_{p-1} \text{Im}).$$

Therefore it suffices to put

$$R(u) = \tilde{R}(u) + g(T_{r-1} - T_q T_p^{-1}) \text{Im} + i(T_p M_b T_q^{-1} - T_{r-1} M_b) \text{Re} - iT_p M_b T_B T_p^{-1} \text{Im}.$$

Now the estimate for $R(u)$ comes from that of $\tilde{R}(u)$ and a symbolic calculus. The estimate for $\mathcal{B}(u)$ comes directly from Theorem 3.9.3. The estimate of $\beta(u)$ uses Lemma 3.3.2,

$$\|\beta(u)F\|_{H^{s+1/2}} \lesssim \|B(u)F\|_{L^\infty} \|u\|_{H^s} \lesssim C(\|u\|_{H^s}) \|F\|_{H^s} \|u\|_{H^s}.$$

By Lemma 3.5.12, Lemma 3.5.22 and Lemma 3.5.19 that is to be proven later, we obtain the continuity of these three operators. \square

3.4 L^2 linear control

Let s be sufficiently large, $T > 0$, $\varepsilon_0 > 0$, fix $\underline{u} \in \mathcal{C}^{1,s}(T, \varepsilon_0)$, and denote for simplicity

$$P = P(\underline{u}), \quad \mathcal{B} = \mathcal{B}(\underline{u}).$$

The purpose of this section is to prove the L^2 -null controllability, when ε_0 is sufficiently small, of the following equation without the perturbation terms,

$$(3.4.1) \quad (\partial_t + P)u = \mathcal{B}F.$$

Proposition 3.4.1. *Suppose that ω satisfies the geometric control condition, s is sufficiently large, $T > 0$, $\varepsilon_0 > 0$, $\underline{u} \in \mathcal{C}^{1,s}(T, \varepsilon_0)$, then when ε_0 is sufficiently small, the L^2 -null controllability of (3.4.1) holds. More precisely, there exists a linear operator*

$$\Theta = \Theta(\underline{u}) : \dot{L}^2(\mathbb{T}^d) \rightarrow C([0, T], \dot{L}^2(\mathbb{T}^d)),$$

satisfying the estimate

$$\|\Theta\|_{\mathcal{L}(\dot{L}^2, C([0,T], \dot{L}^2))} \lesssim 1,$$

such that, for $u_0 \in \dot{L}^2(\mathbb{T}^d)$ and

$$F = \Theta u_0 \in C([0, T], \dot{L}^2(\mathbb{T}^d)),$$

the unique solution $u \in C([0, T], \dot{L}^2(\mathbb{T}^d))$ of (3.4.1) with initial data $u(0) = u_0$, vanishes at time T , that is $u(T) = 0$.

We use the *Hilbert Uniqueness Method*, or HUM for short, to prove this proposition.

3.4.1 Hilbert uniqueness method

The HUM establishes by a duality argument on a well-chosen Hilbert space, the equivalence between the null controllability of the original equation (that is in our case (3.4.1)) and an *observability inequality* to its dual equation (see (3.4.2)), with respect to the Hilbertian structure. Because P and \mathcal{B} are not \mathbb{C} -linear but only \mathbb{R} -linear, we then choose the real Hilbert space $\dot{L}^2(\mathbb{T}^d, \mathbb{C})$ equipped with the scalar product $\operatorname{Re}(\cdot, \cdot)_{L^2}$. Therefore our dual equation is

$$(3.4.2) \quad \partial_t u - P^* u = 0,$$

where P^* is the formal adjoint of P , with respect to $(\dot{L}^2(\mathbb{T}^d), \operatorname{Re}(\cdot, \cdot)_{L^2})$, in the sense that, for all $f, g \in C^\infty(\mathbb{T}^d)$,

$$\operatorname{Re}(Pf, g)_{L^2} = \operatorname{Re}(f, P^*g)_{L^2}.$$

Proposition 3.4.2. $P^* = -iT_\gamma^* - T_V^* \cdot \nabla_x - giT_{r-1}^* \operatorname{Re} + M_b T_r^* \operatorname{Im}$, where for a paradifferential operator T_a , T_a^* denotes its formal adjoint with respect to $(\cdot, \cdot)_{L^2}$.

Démonstration. For a paradifferential operator T_a , its formal adjoint T_a^* with respect to the scalar product $(\cdot, \cdot)_{L^2}$ is evidently at the same time its formal adjoint with respect to the scalar product $\operatorname{Re}(\cdot, \cdot)_{L^2}$. Moreover, by a direct verification, $\operatorname{Im}^* = i\operatorname{Re}$. Therefore,

$$\begin{aligned} P^* &= T_\gamma^* i^* - T_V^* \cdot \nabla_x - g \operatorname{Im}^* T_{r-1}^* + (i\operatorname{Re})^* M_b T_r^* \\ &= -iT_\gamma^* - T_V^* \cdot \nabla_x - gi\operatorname{Re} T_{r-1}^* + \operatorname{Im} M_b T_r^*. \end{aligned}$$

It remains to show that for $a = r^{\pm 1}$ or m_b , $[T_a^*, \operatorname{Re}] = [T_a^*, \operatorname{Im}] = 0$. Equivalently $[T_a, \operatorname{Re}] = 0$. This is a consequence of Lemma 3.9.7, as in either case a is real valued, and is an even function of ξ . \square

Now HUM proceeds as follows. Define the range operator $\mathcal{R} = \mathcal{R}(\underline{u})$ by

$$\mathcal{R} : L^2([0, T], \dot{L}^2(\mathbb{T}^d)) \rightarrow \dot{L}^2(\mathbb{T}^d), \quad G \mapsto u(0),$$

where $u \in C([0, T], \dot{L}^2(\mathbb{T}^d))$ is the unique solution to the Cauchy problem

$$(\partial_t + P)u = G, \quad u(T) = 0.$$

Also define the solution operator $\mathcal{S} = \mathcal{S}(\underline{v})$ by

$$\mathcal{S} : \dot{L}^2(\mathbb{T}^d) \rightarrow C([0, T], \dot{L}^2(\mathbb{T}^d)) \subset L^2([0, T], \dot{L}^2(\mathbb{T}^d)), \quad v_0 \mapsto v,$$

where $v \in C([0, T], \dot{L}^2(\mathbb{T}^d))$ is the unique solution to the dual equation (3.4.2) with initial data $v(0) = v_0$.

Remark 3.4.3. For the well-posedness of these equations, we refer to Appendix 3.10. Moreover, for all $\mu \geq 0$, we have

$$\mathcal{R}|_{L^2([0, T], \dot{H}^\mu)} : L^2([0, T], \dot{H}^\mu(\mathbb{T}^d)) \rightarrow \dot{H}^\mu(\mathbb{T}^d), \quad \mathcal{S}|_{\dot{H}^\mu} : \dot{H}^\mu(\mathbb{T}^d) \rightarrow C([0, T], \dot{H}^\mu(\mathbb{T}^d)).$$

Remark 3.4.4. In classical literatures, the operator \mathcal{RB} is called the range operator, while $\mathcal{B}^* \mathcal{S}$ is called the solution operator. We isolate the operator \mathcal{B} for later simplicity.

Proposition 3.4.5. *Let $F \in L^2([0, T], \dot{L}^2(\mathbb{T}^d))$, and $v_0 \in \dot{L}^2(\mathbb{T}^d)$, then the duality holds*

$$(3.4.3) \quad -\operatorname{Re}(\mathcal{R}\mathcal{B}F, v_0)_{L^2} = \operatorname{Re}(F, \mathcal{B}^* \mathcal{S}v_0)_{L^2([0, T], L^2)},$$

or formally, with respect to $\operatorname{Re}(\cdot, \cdot)_{L^2}$, $\mathcal{R}\mathcal{B} = -(\mathcal{B}^* \mathcal{S})^*$.

Démonstration. By a density argument, it suffice to prove the identity for $F \in C([0, T], \dot{H}^\infty(\mathbb{T}^d))$, and $v_0 \in \dot{H}^\infty(\mathbb{T}^d)$. Then $v = \mathcal{S}v_0 \in C^1([0, T], \dot{H}^\infty(\mathbb{T}^d))$. And the solution u to (3.4.1) with $u(T) = 0$ belongs to $C^1([0, T], \dot{H}^\infty(\mathbb{T}^d))$. By the Newton-Leibniz formula,

$$\begin{aligned} \operatorname{Re}(F, \mathcal{B}^*v)_{L^2([0, T], L^2)} &= \operatorname{Re}(\mathcal{B}F, v)_{L^2([0, T], L^2)} = \operatorname{Re}((\partial_t + P)u, v)_{L^2([0, T], L^2)} \\ &= \operatorname{Re}(u(t), v(t))_{L^2} \Big|_0^T + \operatorname{Re}(u, (-\partial_t + P^*)v)_{L^2([0, T], L^2)} \\ &= \operatorname{Re}(u(T), v(T))_{L^2} - \operatorname{Re}(u(0), v(0))_{L^2}. \end{aligned}$$

Now that $u(T) = 0$, the duality relation (3.4.3) follows. \square

Proposition 3.4.6. *Suppose that ω satisfies the geometric control condition, s is sufficiently large, $T > 0$, $\varepsilon_0 > 0$, and $\underline{u} \in \mathcal{C}^{1, s}(T, \varepsilon_0)$. Then for ε_0 sufficiently small, the L^2 -observability of (3.4.2) holds. That is, for all its solution u , with initial data in $\dot{L}^2(\mathbb{T}^d)$,*

$$(3.4.4) \quad \|u(0)\|_{L^2}^2 \lesssim \int_0^T \|\mathcal{B}^*u(t)\|_{L^2}^2 dt.$$

Here $\mathcal{B}^* = \chi_T \varphi_\omega T_q^* \operatorname{Re}$ is the dual operator of \mathcal{B} with respect to $\operatorname{Re}(\cdot, \cdot)_{L^2}$.

Remark 3.4.7. Now that $\chi_T(t) = 1$ for $t \leq T/2$, and that Proposition 3.4.6 is stated for all $T > 0$, replacing T with $2T$ in (3.4.4), then for (3.4.4) to be satisfied, we only need to prove the following observability for all $T > 0$,

$$\|u(0)\|_{L^2}^2 \lesssim \int_0^T \|\varphi_\omega T_q^* \operatorname{Re} u(t)\|_{L^2}^2 dt.$$

Therefore, we may simply omit the factor χ_T in (3.4.4).

The effort of the rest of the section will be devoted to proving this observability, which states the coercivity of the operator $\mathcal{B}^* \mathcal{S}$ because (3.4.4) writes in a compact form as

$$\|u_0\|_{L^2} \lesssim \|\mathcal{B}^* \mathcal{S}u_0\|_{L^2([0, T], L^2)}.$$

Once this is proven, define the \mathbb{R} -linear operator

$$(3.4.5) \quad \mathcal{K} := -\mathcal{R}\mathcal{B}\mathcal{B}^* \mathcal{S} : \dot{L}^2(\mathbb{T}^d) \rightarrow \dot{L}^2(\mathbb{T}^d), \quad \forall \mu \geq 0.$$

Remark 3.4.8. By Proposition 3.4.5, $\mathcal{K} = (\mathcal{B}^* \mathcal{S})^* \mathcal{B}^* \mathcal{S}$. By an energy estimate, we have for all $\mu \geq 0$, $\mathcal{K}|_{\dot{H}^\mu} : \dot{H}^\mu(\mathbb{T}^d) \rightarrow \dot{H}^\mu(\mathbb{T}^d)$.

Consider the continuous \mathbb{R} -bilinear form on $\dot{L}^2(\mathbb{T}^d)$,

$$\varpi(f_0, v_0) := \operatorname{Re}(\mathcal{K}f_0, v_0)_{L^2}.$$

It is coercive by (3.4.4),

$$\varpi(v_0, v_0) = \operatorname{Re}((\mathcal{B}^* \mathcal{S})^* \mathcal{B}^* \mathcal{S}v_0, v_0)_{L^2} = \operatorname{Re}(\mathcal{B}^* \mathcal{S}v_0, \mathcal{B}^* \mathcal{S}v_0)_{L^2([0, T], L^2)} \gtrsim \|v_0\|_{L^2}^2.$$

By Lax-Milgram's theorem, for $u_0 \in \dot{L}^2(\mathbb{T}^d)$, there exists a unique $f_0 \in \dot{L}^2(\mathbb{R}^d)$, such that

$$\varpi(f_0, v_0) = \operatorname{Re}(u_0, v_0)_{L^2}, \quad \forall v_0 \in \dot{L}^2(\mathbb{T}^d),$$

and consequently, we have the invertibility of \mathcal{K} ,

$$(3.4.6) \quad \mathcal{K}f_0 = u_0, \quad \|f_0\|_{L^2} \lesssim \|u_0\|_{L^2}.$$

Therefore, we have proven the following proposition, as a consequence of the L^2 -observability.

Proposition 3.4.9. *Suppose that ω satisfies the geometric control condition, s is sufficiently large, $T > 0$, $\varepsilon_0 > 0$, and $\underline{u} \in \mathcal{C}^{1,s}(T, \varepsilon_0)$. Then for ε_0 sufficiently small, \mathcal{K} defines an isomorphism on $\dot{L}^2(\mathbb{T}^d)$ with*

$$\|\mathcal{K}\|_{\mathcal{L}(\dot{L}^2, \dot{L}^2)} + \|\mathcal{K}^{-1}\|_{\mathcal{L}(\dot{L}^2, \dot{L}^2)} \lesssim 1.$$

To construct Θ and thus prove Proposition 3.4.1, we set

$$F = -\mathcal{B}^* \mathcal{S} f_0 = -\mathcal{B}^* \mathcal{S} \mathcal{K}^{-1} u_0 \in C([0, T], \dot{L}^2(\mathbb{T}^d)),$$

then by (3.4.6) and the definition of \mathcal{K} , $\mathcal{R} \mathcal{B} F = u_0$. Therefore,

$$(3.4.7) \quad \Theta := -\mathcal{B}^* \mathcal{S} \mathcal{K}^{-1} : \dot{L}^2(\mathbb{T}^d) \rightarrow C([0, T], \dot{L}^2(\mathbb{R}^d))$$

defines a desired control operator.

It remains to prove Proposition 3.4.6.

3.4.2 Reduction to pseudodifferential equation

This sections shows that, the observability of the paradifferential equation (3.4.2) can be reduced to that of a pseudodifferential equation. To do this, write

$$-P^* = iQ + R_Q,$$

where the pseudodifferential operator $Q = Q(\underline{u})$ is

$$(3.4.8) \quad Q = \pi(D_x) \operatorname{Op}(\gamma\pi) + \pi(D_x)(V \cdot D_x + g \operatorname{Op}(r^{-1}\pi) \operatorname{Re} + \operatorname{Op}(r \cdot m_b) i \operatorname{Im}).$$

Lemma 3.4.10. *Suppose that s is sufficiently large, $T > 0$, $\varepsilon_0 > 0$, and $\underline{u} \in \mathcal{C}^{0,s}(T, \varepsilon_0)$. Then for ε_0 sufficiently small,*

$$R_Q \in L^\infty([0, T], \mathcal{L}(\dot{L}^2, \dot{L}^2)) \cap C([0, T], \mathcal{L}(\dot{L}^2, \dot{H}^{-3/2})),$$

with the estimate

$$\|R_Q\|_{L^\infty([0, T], \mathcal{L}(\dot{L}^2, \dot{L}^2))} \lesssim \varepsilon_0.$$

Démonstration. Lemma 3.5.13 later shows that $P^* \in C([0, T], \mathcal{L}(\dot{L}^2, \dot{H}^{-3/2}))$. To prove that $Q \in C([0, T], \mathcal{L}(\dot{L}^2, \dot{H}^{-3/2}))$, the main estimate is a similar contraction estimate for the principal term. Let $\underline{u}_i \in \dot{H}^s(\mathbb{T}^d)$ ($i = 0, 1$) be such that $\|\underline{u}_i\|_{H^s} < \varepsilon_0$, then

$$\|\operatorname{Op}(\gamma(\underline{u}_1)^{(3/2)}\pi) - \operatorname{Op}(\gamma(\underline{u}_2)^{(3/2)}\pi)\|_{\mathcal{L}(\dot{L}^2, \dot{H}^{-3/2})} \lesssim \|\underline{u}_1 - \underline{u}_2\|_{H^s}.$$

To estimate the L^∞ -norm, write $-R_Q = P^* + iQ = (P^* + P) + (P - iQ)$. For $P^* + P$, we estimate $T_\gamma^* - T_\gamma$. By the definition of $\gamma = \gamma^{(3/2)} + \gamma^{(1/2)}$,

$$(3.4.9) \quad i \operatorname{Im} \gamma^{(1/2)} = \frac{1}{2} \partial_\xi \cdot D_x \gamma^{(3/2)}.$$

Therefore, a symbolic calculus shows,

$$T_\gamma^* = T_\gamma + O(\varepsilon_0)_{\mathcal{L}(L^2, L^2)}.$$

Remark 3.4.11. The remainder is of size $O(\varepsilon_0)_{\mathcal{L}(\dot{L}^2, \dot{L}^2)}$. To prove this, we write

$$T_\gamma^* = T_{\gamma^{(3/2)} - |\xi|^{3/2}}^* + T_{|\xi|^{3/2}}^* + T_{\gamma^{(1/2)}}^*,$$

and proceeds with the symbolic calculus, using the estimates that

$$M_{3/2, d/2+1}^{3/2}(\gamma^{(3/2)} - |\xi|^{3/2}) \lesssim \varepsilon_0, \quad M_{1/2, d/2+1}^{1/2}(\gamma^{(1/2)}) \lesssim \varepsilon_0.$$

The idea is that the symbols do not differ much from some Fourier multipliers, in the sense that their differences are of size ε_0 . This idea will be frequently used in later estimates of remainders, and will not again be explained in detail.

To estimate $P - iQ$, write

$$\begin{aligned} P - iQ &= i\pi(D_x)(T_\gamma - \operatorname{Op}(\gamma\pi)) + i\pi(D_x)(T_{V \cdot \xi} - \operatorname{Op}(V \cdot \xi)) \\ &\quad + ig\pi(D_x)(T_{r^{-1}} - \operatorname{Op}(r^{-1}\pi)) \operatorname{Re} - \pi(D_x)(T_{r \cdot m_b} - \operatorname{Op}(r \cdot m_b)) \operatorname{Im}. \end{aligned}$$

We conclude with Proposition 3.9.6 that $P - iQ = O(\varepsilon_0)_{\mathcal{L}(\dot{L}^2, \dot{L}^2)}$. \square

Consider the following pseudodifferential equation,

$$(3.4.10) \quad D_t u + Qu = 0.$$

Proposition 3.4.12. *Suppose that ω satisfies the geometric control condition, s is sufficiently large, $T > 0$, $\varepsilon_0 > 0$, and $\underline{u} \in \mathcal{C}^{1, s}(T, \varepsilon_0)$. Then for ε_0 sufficiently small, the L^2 -observability of (3.4.10) holds. That is, for all its solution with initial data in $\dot{L}^2(\mathbb{T}^d)$,*

$$(3.4.11) \quad \|u(0)\|_{L^2}^2 \lesssim \int_0^T \|\varphi_\omega \operatorname{Re} u(t)\|_{L^2}^2 dt.$$

By Duhamel's formula and Lemma 3.4.10, we can deduce Proposition 3.4.6 from Proposition 3.4.12. Indeed, for every solution $u \in C([0, T], \dot{L}^2(\mathbb{T}^d))$ of equation (3.4.2), write $u = v + w$ with $v, w \in C([0, T], \dot{L}^2(\mathbb{T}^d))$ such that

$$(\partial_t + iQ)v = 0, \quad v(0) = u(0); \quad (\partial_t + iQ + R_Q)w = -R_Q v, \quad w(0) = 0.$$

Then by Proposition 3.10.1,

$$\begin{aligned} \|v\|_{C([0, T], L^2)} &\lesssim \|u(0)\|_{L^2}, \\ \|w\|_{C([0, T], L^2)} &\lesssim \|R_Q v\|_{L^1([0, T], L^2)} \lesssim \varepsilon_0 \|v\|_{L^1([0, T], L^2)} \lesssim \varepsilon_0 \|u(0)\|_{L^2}. \end{aligned}$$

By Proposition 3.9.4,

$$\|(\mathcal{B}^* - \varphi_\omega \text{Re})u\|_{L^2} = \|\varphi_\omega \text{Re}(T_q^* - \pi)u\|_{L^2} \lesssim \varepsilon_0 \|u\|_{L^2}.$$

Therefore by Proposition 3.4.12,

$$\begin{aligned} \|u(0)\|_{L^2}^2 &\lesssim \int_0^T \|\varphi_\omega \text{Re} v(t)\|_{L^2}^2 dt \\ &\lesssim \int_0^T \|\mathcal{B}^* u(t)\|_{L^2}^2 dt + \int_0^T \|(\mathcal{B}^* - \varphi_\omega \text{Re})u(t)\|_{L^2}^2 dt \\ &\quad + \int_0^T \|\varphi_\omega \text{Re} w(t)\|_{L^2}^2 dt \\ &\lesssim \int_0^T \|\mathcal{B}^* u(t)\|_{L^2}^2 dt + \varepsilon_0 \|u(0)\|_{L^2}^2. \end{aligned}$$

The observability for u then follows for ε_0 sufficiently small.

To deal with the problem caused by the \mathbb{R} -linearity of the equation, we are going to consider u and \bar{u} simultaneously, and study the equation satisfied by the pair $\begin{pmatrix} u \\ \bar{u} \end{pmatrix}$. First we derive the equation for \bar{u} with the help of the following observation. For a pseudo-differential operator with symbol a , we have

$$\text{Op}(a)u = \overline{\text{Op}(\bar{a})\bar{u}}, \quad \bar{a}(x, \xi) = \overline{a(x, -\xi)}.$$

If a is a real even function or pure imaginary odd function of ξ , then $\bar{a} = a$. In particular $\tilde{\gamma} = \gamma$, because $\gamma^{(3/2)}$ and $\text{Re} \gamma^{(1/2)}$ are real and even functions of ξ , while $i \text{Im} \gamma^{(1/2)}$ is a pure imaginary odd function of ξ . Taking the complex conjugation of (3.4.10), we have

$$(D_t - \tilde{Q})\bar{u} = 0,$$

with (using the identity $\pi^2|_{\mathbb{Z}^d} = \pi|_{\mathbb{Z}^d}$)

$$\tilde{Q} = \pi(D_x)\text{Op}(\gamma\pi) + \pi(D_x)(-V \cdot D_x + g\text{Op}(r^{-1}\pi)\text{Re} + \text{Op}(r \cdot m_b)i\text{Im}).$$

This suggest the consideration of the following system of equations,

$$(3.4.12) \quad D_t \vec{w} + \mathcal{A} \vec{w} = 0,$$

where $\vec{w} = \begin{pmatrix} w_+ \\ w_- \end{pmatrix} \in \dot{L}^2(\mathbb{T}^d) \times \dot{L}^2(\mathbb{T}^d)$, and $\mathcal{A} = \pi(D_x)\text{Op}(A\pi)$ with the symbol

$$(3.4.13) \quad A = \gamma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + V \cdot \xi + \frac{g}{2r} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + \frac{r \cdot m_b}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Indeed, $\begin{pmatrix} u \\ \bar{u} \end{pmatrix}$ satisfies (3.4.12) whenever u satisfies (3.4.10). Then the L^2 -observability of (3.4.10) is a simple consequence of that of (3.4.12).

Proposition 3.4.13. *Suppose that ω satisfies the geometric control condition, s is sufficiently large, $T > 0$, $\varepsilon_0 > 0$, and $\underline{u} \in \mathcal{C}^{1,s}(T, \varepsilon_0)$. Then for ε_0 sufficiently small, the L^2 -observability of (3.4.12) holds, that is, for all solution \vec{w} with initial data in $\dot{L}^2(\mathbb{T}^d) \times \dot{L}^2(\mathbb{T}^d)$,*

$$(3.4.14) \quad \|\vec{w}(0)\|_{L^2}^2 \lesssim \int_0^T \|\varphi_\omega \vec{e} \cdot \vec{w}\|_{L^2}^2 dt,$$

where $\vec{e} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ defines a linear map $\vec{e} \cdot : \dot{L}^2(\mathbb{T}^d) \times \dot{L}^2(\mathbb{T}^d) \rightarrow \dot{L}^2(\mathbb{T}^d)$ $\vec{w} \mapsto w^+ + w^-$.

The rest of the section is devoted to proving this proposition.

3.4.3 Reduction to semiclassical equation

This section derives the equation satisfied by frequency localized quasi-modes of (3.4.12). By the hypothesis that ω satisfies the geometric control condition, for fixed $T > 0$, there exists some $v \in]0, \infty[$ such that

$$(3.4.15) \quad \frac{3}{2}v^{1/2} \cdot T > \min\{L \geq 0 : \forall (x, \xi) \in \mathbb{T}^d \times \mathbb{S}^{d-1}, [x, x + L\xi] \cap \omega \neq \emptyset\},$$

where $[x, x + L\xi] = \{x + t\xi : t \in [0, L]\}$. Indeed the right hand side is finite by the geometric control condition and the compactness of \mathbb{T}^d . The expression of the left hand side is in accordance with the group velocity (3.2.18), so that wave packets of the linearized equation around frequencies with modulus $\geq v$ will travel into ω within time T . We define a class of cutoff functions

$$\Xi(v) = \left\{ \chi \in C_c^\infty(\mathbb{R} \setminus \{0\}) : \begin{array}{l} 0 \leq \chi \leq 1, \quad \chi|_{2 \leq |z|/v^{3/2} \leq 4} \equiv 1 \\ \text{supp } \chi \subset \{1 \leq |z|/v^{3/2} \leq 5\} \end{array} \right\}.$$

Fix $\chi \in \Xi(v)$. Let $\phi \in C^\infty(\mathbb{R})$ be such that $0 \leq \phi \leq 1$, $\phi(z) = 1$ for $z \geq v^{3/2}/2$, and $\phi(z) = 0$ for $z \leq v^{3/2}/4$. In particular, $\phi\chi = \chi$. Then set $\varphi(\xi) = \phi(|\xi|^{3/2})$.

Now for s sufficiently large, $T > 0$, $\varepsilon_0 > 0$, $\underline{u} \in \mathcal{C}^{1,s}(T, \varepsilon_0)$, set

$$h = 2^{-j}$$

as a semiclassical parameter, and define the operator,

$$\mathcal{Z}_h = \text{Op}_h(\varphi)\text{Op}_h(\gamma_h\varphi), \quad \gamma_h = \gamma^{(3/2)} + h\gamma^{(1/2)},$$

so that $\gamma_h(x, h\xi) = h^{3/2}\gamma_1(x, \xi) = h^{3/2}\gamma(x, \xi)$. For ε_0 sufficiently small, \mathcal{Z}_h is elliptic of order $3/2$, and the symmetric operator

$$\text{Re}\mathcal{Z}_h := \frac{1}{2}(\mathcal{Z}_h + \mathcal{Z}_h^*), \quad D(\text{Re}\mathcal{Z}_h) = C^\infty(\mathbb{T}^d)$$

has a Friedrichs extension. We still denote by $\text{Re}\mathcal{Z}_h$ this extension, and define

$$\Delta_h = \text{Op}_h(\chi(\gamma^{(3/2)})), \quad \Pi_h = \chi(\text{Re}\mathcal{Z}_h)$$

where the latter is defined by the functional calculus of $\text{Re}\mathcal{Z}_h$. Finally set

$$(3.4.16) \quad \vec{w}_h = \Pi_h \Delta_h \vec{w}.$$

We will derive the equation satisfied by \vec{w}_h , under the coordinates (s, x) , where

$$(3.4.17) \quad s = h^{-1/2}t,$$

so that $D_s = h^{1/2}D_t$. The equation (3.4.12) is equivalent to

$$(3.4.18) \quad hD_s \vec{w} + h^{3/2}\mathcal{A}\vec{w} = 0.$$

The equation for \vec{w}_h will be obtained by commuting successively Δ_h and Π_h with (3.4.18). A careful study of the operator Π_h is needed.

3.4.3.1 Semiclassical functional calculus for Π_h

We use the notations $[a, b]_s$ and $[a, b]_t$ to denote the time intervals $\{s : a \leq s \leq b\}$ and $\{t : a \leq t \leq b\}$ to avoid ambiguity.

Lemma 3.4.14. *Suppose that s is sufficiently large, $T > 0$, $\varepsilon_0 > 0$, and $\underline{u} \in \mathcal{C}^{0,s}(T, \varepsilon_0)$. Then for ε_0 sufficiently small, $\Delta_h \in C([0, T]_t, \mathcal{L}(L^2, L^2))$ and*

$$\|\Pi_h - \Delta_h\|_{L^\infty([0, T]_t, \mathcal{L}(L^2, L^2))} \lesssim h.$$

Moreover, if $\underline{u} \in \mathcal{C}^{1,s}(T, \varepsilon_0)$, then for ε_0 sufficiently small,

$$\|D_s \Pi_h\|_{L^\infty([0, T]_t, \mathcal{L}(L^2, L^2))} \lesssim (\varepsilon_0 + h)h^{1/2},$$

and in particular $\Pi_h \in W^{1,\infty}([0, T]_t, \mathcal{L}(L^2, L^2))$.

Démonstration. We omit the time variable t for simplicity. The idea of the proof is to use Helffer-Sjöstrand's formula (see for example [69, 46, 208]),

$$(3.4.19) \quad \Pi_h = -\frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{\chi}(z) (z - \operatorname{Re} \mathcal{Z}_h)^{-1} dz,$$

where $\tilde{\chi}$ is an almost analytic extension of χ , such that for some $n \in \mathbb{N}$ to be fixed later,

$$\tilde{\chi}(z) = \varrho(\operatorname{Im} z) \sum_{k=0}^n \frac{\chi^{(k)}(\operatorname{Re} z)}{k!} (i \operatorname{Im} z)^k,$$

with $\varrho \in C_c^\infty(\mathbb{R})$ such that $\varrho = 1$ near zero. Notice that

$$(3.4.20) \quad |\bar{\partial} \tilde{\chi}(z)| \lesssim |\operatorname{Im} z|^{n+1}.$$

In order to perform a semiclassical parametrix construction for $z - \operatorname{Re} \mathcal{Z}_h$, we first determine the symbol for $\operatorname{Re} \mathcal{Z}_h$. By a semiclassical symbolic calculus,

$$(3.4.21) \quad \begin{aligned} \mathcal{Z}_h &= \operatorname{Op}_h(\gamma^{(3/2)} \varphi^2) + h \operatorname{Op}_h(\gamma^{(1/2)} \varphi^2) \\ &\quad + h \operatorname{Op}_h(\varphi \partial_\xi \varphi \cdot D_x \gamma^{(3/2)}) + O(h^2)_{\mathcal{L}(L^2, L^2)} \end{aligned}$$

$$(3.4.22) \quad \begin{aligned} \mathcal{Z}_h^* &= \operatorname{Op}_h(\gamma_h \varphi)^* \operatorname{Op}_h(\varphi) \\ &= (\operatorname{Op}_h(\gamma^{(3/2)} \varphi) + h \operatorname{Op}_h(\overline{\gamma^{(1/2)}} \varphi) + h \operatorname{Op}_h(\partial_\xi \cdot D_x (\gamma^{(3/2)} \varphi))) \operatorname{Op}_h(\varphi) \\ &\quad + O(h^2)_{\mathcal{L}(L^2, L^2)} \\ &= \operatorname{Op}_h(\gamma^{(3/2)} \varphi^2) + h \operatorname{Op}_h(\overline{\gamma^{(1/2)}} \varphi^2) + h \operatorname{Op}_h(\varphi \partial_\xi \varphi \cdot D_x \gamma^{(3/2)}) \\ &\quad + h \operatorname{Op}_h(\partial_\xi \cdot D_x \gamma^{(3/2)} \varphi^2) + O(h^2)_{\mathcal{L}(L^2, L^2)}. \end{aligned}$$

Consequently,

$$\operatorname{Re} \mathcal{Z}_h = \operatorname{Op}_h(\gamma^{(3/2)} \varphi^2) + h \operatorname{Op}_h(\zeta^{(1/2)} \varphi) + O(h^2)_{\mathcal{L}(L^2, L^2)},$$

where

$$\zeta^{(1/2)} = \left(\frac{1}{2} \partial_\xi \cdot D_x \gamma^{(3/2)} + \operatorname{Re} \gamma^{(1/2)}\right) \varphi^2 + \varphi \partial_\xi \varphi \cdot D_x \gamma^{(3/2)}.$$

Now let $q_0(z, x, \xi) = (z - \gamma^{(3/2)}\varphi^2)^{-1}$ for $z \in \mathbb{C} \setminus \mathbb{R}$, then for some $N > 0$,

$$\text{Op}_h(q_0(z)) \circ (z - \text{Re}\mathcal{Z}_h) = 1 + O(h|\text{Im}z|^{-N})_{\mathcal{L}(L^2, L^2)}.$$

Apply both sides to $(z - \text{Re}\mathcal{Z}_h)^{-1}$, and use the resolvent estimate

$$\|(z - \text{Re}\mathcal{Z}_h)^{-1}\|_{\mathcal{L}(L^2, L^2)} \leq |\text{Im}z|^{-1},$$

we have,

$$(z - \text{Re}\mathcal{Z}_h)^{-1} = \text{Op}_h(q_0(z)) + O(h|\text{Im}z|^{-(N+1)})_{\mathcal{L}(L^2, L^2)}.$$

Plug this into (3.4.19) with $n \geq N$. Use Cauchy's integral formula and (3.4.20),

$$\begin{aligned} \Pi_h &= -\frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{\chi}(z) (\text{Op}_h(q_0(z)) + O(h|\text{Im}z|^{-(N+1)})_{\mathcal{L}(L^2, L^2)}) dz \\ &= \text{Op}_h\left(-\frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{\chi}(z) (z - \gamma^{(3/2)}\varphi^2)^{-1} dz\right) \\ &\quad - \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{\chi}(z) O(h|\text{Im}z|^{-(N+1)})_{\mathcal{L}(L^2, L^2)} dz \\ &= \text{Op}_h(\chi(\gamma^{(3/2)}\varphi^2)) + O(h)_{\mathcal{L}(L^2, L^2)} \\ &= \Delta_h + O(h)_{\mathcal{L}(L^2, L^2)}, \end{aligned}$$

where the last equality uses $\chi(\gamma^{(3/2)}\varphi^2) = \chi(\gamma^{(3/2)})$, for small ε_0 .

As for $D_s \Pi_h$, we apply the identity

$$D_t(z - A)^{-1} = (z - A)^{-1}(D_t A)(z - A)^{-1}$$

to (3.4.19) and deduce

$$D_s \Pi_h = -\frac{h^{1/2}}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{\chi}(z) (z - \text{Re}\mathcal{Z}_h)^{-1} (D_t \text{Re}\mathcal{Z}_h) (z - \text{Re}\mathcal{Z}_h)^{-1} dz,$$

where

$$D_t \text{Re}\mathcal{Z}_h = \text{Re}(D_t \mathcal{Z}_h) = \text{Op}_h(D_t \gamma^{(3/2)}\varphi^2) + h \text{Op}_h(D_t \zeta^{(1/2)}\varphi) + O(h^2)_{\mathcal{L}(L^2, L^2)}.$$

Therefore, use again Cauchy's integral formula,

$$\begin{aligned} D_s \Pi_h &= \text{Op}_h\left(-\frac{h^{1/2}}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{\chi}(z) D_t \gamma^{(3/2)}\varphi^2 (z - \gamma^{(3/2)}\varphi^2)^{-2}\right) dz + O(h^{3/2})_{\mathcal{L}(L^2, L^2)} \\ &= -h^{1/2} \text{Op}_h(\partial_z \chi(\gamma^{(3/2)}\varphi^2) D_t \gamma^{(3/2)}\varphi^2) + O(h^{3/2})_{\mathcal{L}(L^2, L^2)} \\ (3.4.23) \quad &= O((\varepsilon_0 + h)h^{1/2})_{\mathcal{L}(L^2, L^2)}. \end{aligned}$$

□

Corollary 3.4.15. *Suppose that s is sufficiently large, $T > 0$, $\varepsilon_0 > 0$, and $\underline{u} \in \mathcal{C}^{1,s}(T, \varepsilon_0)$. Let $\chi' \in \Xi(v)$ be such that $\chi'\chi = \chi$, then, for ε_0 sufficiently small,*

$$(3.4.24) \quad \vec{w}_h = \Delta'_h \vec{w}_h + R_{\Delta, h} \vec{w}_h,$$

where $\Delta'_h = \text{Op}_h(\chi'(\gamma^{(3/2)}))$, and

$$\|R_{\Delta, h}\|_{C([0, T]_t, \mathcal{L}(L^2, L^2))} \lesssim h.$$

Démonstration. Let $\Pi'_h = \chi'(\text{Re}\mathcal{Z}_h)$, then Lemma 3.4.14 shows that

$$\Pi'_h = \Delta'_h + O(h)_{C([0,T]_t, \mathcal{L}(L^2, L^2))}.$$

We conclude with the functional calculus $\Pi'_h \vec{w}_h = \Pi'_h \Pi_h \Delta_h \vec{w} = \Pi_h \Delta_h \vec{w} = \vec{w}_h$. \square

Corollary 3.4.16. *Suppose that s is sufficiently large, $T > 0$, $\varepsilon_0 > 0$, and $\underline{u} \in \mathcal{C}^{1,s}(T, \varepsilon_0)$. Then for ε_0 sufficiently small, for all $t \in [0, T]$, $\mathfrak{D}_h = \Pi_h$, Δ_h or $\Pi_h \Delta_h$, where we omit the time variable t in $\mathfrak{D}_h = \mathfrak{D}_h(t)$, for $u \in L^2(\mathbb{T}^d)$,*

$$\sum_{\substack{h=2^{-j} \\ j \geq 0}} \|\mathfrak{D}_h u\|_{L^2}^2 \lesssim \|u\|_{L^2}^2.$$

Moreover, for $j_0 > 0$ sufficiently large, and all $N > 0$,

$$\sum_{\substack{h=2^{-j} \\ j \geq j_0}} \|\mathfrak{D}_h u\|_{L^2}^2 \gtrsim \|u\|_{L^2}^2 - \|u\|_{H^{-N}}^2.$$

Démonstration. We only prove the estimates for $\mathfrak{D}_h = \Pi_h \Delta_h$. The rest is similar. By Lemma 3.4.14,

$$\Pi_h \Delta_h = (\Delta_h + O(h)_{\mathcal{L}(L^2, L^2)}) \Delta_h = \text{Op}_h(\chi(\gamma^{(3/2)})^2) + O(h)_{\mathcal{L}(L^2, L^2)}.$$

Let $\chi', \chi'' \in \Xi(v)$ be such that $\chi\chi' = \chi'$, $\chi''\chi = \chi$, then

$$\begin{aligned} \Pi_h \Delta_h &= \Pi_h \Delta_h \chi''(|hD_x|^{3/2}) + \Pi_h \Delta_h (1 - \chi''(|hD_x|^{3/2})) \\ &= O(1)_{\mathcal{L}(L^2, L^2)} \chi''(|hD_x|^{3/2}) + O(h)_{\mathcal{L}(L^2, L^2)}. \\ \chi'(|hD_x|^{3/2}) &= \chi'(|hD_x|^{3/2}) \Pi_h \Delta_h + \chi'(|hD_x|^{3/2}) (1 - \Pi_h \Delta_h) \\ &= O(1)_{\mathcal{L}(L^2, L^2)} \Pi_h \Delta_h + O(h)_{\mathcal{L}(L^2, L^2)}. \end{aligned}$$

Therefore, by Littlewood-Paley's theory,

$$\begin{aligned} \sum_{\substack{h=2^{-j} \\ j \geq 0}} \|\Pi_h \Delta_h u\|_{L^2}^2 &\lesssim \sum_{\substack{h=2^{-j} \\ j \geq 0}} \|\chi''(|hD_x|^{3/2})u\|_{L^2}^2 + h^2 \|u\|_{L^2}^2 \lesssim \|u\|_{L^2}^2. \\ \sum_{\substack{h=2^{-j} \\ j \geq j_0}} \|\Pi_h \Delta_h u\|_{L^2}^2 + h^2 \|u\|_{L^2}^2 &\gtrsim \sum_{\substack{h=2^{-j} \\ j \geq j_0}} \|\chi'(|hD_x|^{3/2})u\|_{L^2}^2 \gtrsim \|u\|_{L^2}^2 - \|u\|_{H^{-N}}^2. \end{aligned}$$

We conclude for j_0 sufficiently large. \square

3.4.3.2 Equation for $\Delta_h \vec{w}$

Proposition 3.4.17. *Suppose that s is sufficiently large, $T > 0$, $\varepsilon_0 > 0$, $\underline{u} \in \mathcal{C}^{1,s}(T, \varepsilon_0)$. Then for ε_0 sufficiently small, $\Delta_h \vec{w}$ satisfies the equation*

$$(3.4.25) \quad (hD_s + \mathcal{A}_h) \Delta_h \vec{w} + \tilde{R}_h \vec{w} = 0,$$

where $\mathcal{A}_h = h^{3/2} \text{Op}_h(\varphi) \mathcal{A} \text{Op}_h(\varphi)$, and

$$(3.4.26) \quad \|\tilde{R}_h\|_{L^\infty([0,T]_t, \mathcal{L}(L^2, L^2))} \lesssim (\varepsilon_0 + h^{1/2}) h^{3/2}.$$

Remark 3.4.18. \mathcal{A}_h is of the form $\mathcal{A}_h = \begin{pmatrix} \mathcal{Z}_h & 0 \\ 0 & -\mathcal{Z}_h \end{pmatrix} +$ small lower order terms.

Démonstration. The remainder \tilde{R}_h is explicitly written as follows,

$$\begin{aligned} \tilde{R}_h &= [\Delta_h, hD_s] + h^{3/2}[\Delta_h, \mathcal{A}] \\ &\quad + h^{3/2}\mathcal{A}(1 - \text{Op}_h(\varphi))\Delta_h + h^{3/2}(1 - \text{Op}_h(\varphi))\mathcal{A}\text{Op}_h(\varphi)\Delta_h, \\ &= \text{(I)} + \text{(II)} + \text{(III)} + \text{(IV)}. \end{aligned}$$

Estimate the four terms respectively.

Estimate of (I). Use $D_s = h^{1/2}D_t$, and bound $D_t\gamma^{(3/2)}$ by ε_0 ,

$$\text{(I)} = -h^{3/2}\text{Op}_h(\partial_z\chi(\gamma^{(3/2)})) \cdot D_t\gamma^{(3/2)} = O(\varepsilon_0 h^{3/2})_{\mathcal{L}(L^2, L^2)}.$$

Estimate of (II). The estimate of the commutator $[\pi(D_x)\text{Op}(\gamma^{(3/2)}\pi), \Delta_h]$ is the main difficulty. By a symbolic calculus, and Remark 3.4.11,

$$\begin{aligned} \pi(D_x)\text{Op}(\gamma^{(3/2)}\pi) &= \text{Op}(\gamma^{(3/2)}\pi) + \text{Op}(\pi\partial_\xi\pi \cdot D_x\gamma^{(3/2)}) + O(\varepsilon_0)_{\mathcal{L}(L^2, L^2)} \\ &= \text{Op}(\gamma^{(3/2)}\pi) + O(\varepsilon_0)_{\mathcal{L}(L^2, L^2)}, \end{aligned}$$

where we use $\pi\partial_\xi\pi = 0$ on \mathbb{Z}^d , whence $\text{Op}(\pi\partial_\xi\pi \cdot D_x\gamma^{(3/2)}) = 0$. Therefore,

$$\begin{aligned} [\Delta_h, \pi(D_x)\text{Op}(\gamma^{(3/2)}\pi)] &= [\Delta_h, \text{Op}(\gamma^{(3/2)}\pi)] + O(\varepsilon_0)_{\mathcal{L}(L^2, L^2)} \\ &= \frac{1}{i}\text{Op}(\{\chi(h^{3/2}\gamma^{(3/2)}), \gamma^{(3/2)}\pi\}) + O(\varepsilon_0)_{\mathcal{L}(L^2, L^2)}. \end{aligned}$$

Observe that the Poisson bracket vanishes,

$$\{\chi(h^{3/2}\gamma^{(3/2)}), \gamma^{(3/2)}\pi\} = \{\chi(h^{3/2}\gamma^{(3/2)}), \gamma^{(3/2)}\}\pi + \{\chi(h^{3/2}\gamma^{(3/2)}), \pi\}\gamma^{(3/2)} = 0.$$

Indeed, the first term vanishes because $\chi(h^{3/2}\gamma^{(3/2)})$ is a function of $\gamma^{(3/2)}$. The singularity at 0 is not a problem when ε_0 is sufficiently small. The second term vanishes because the supports of $\chi(h^{3/2}\gamma^{(3/2)})$ and π are disjointed, for h and ε_0 sufficiently small. Combining the estimates of commutators of lower orders, $\text{(II)} = O((\varepsilon_0 + h^{1/2})h^{3/2})_{\mathcal{L}(L^2, L^2)}$.

Estimate of (III). Write

$$\text{(III)} = \mathcal{A}\langle D_x \rangle^{-3/2} h^{3/2} \langle D_x \rangle^{3/2} (1 - \text{Op}_h(\varphi)) \Delta_h.$$

Recall that $\phi\chi = \chi$, so for ε_0 sufficiently small, $(1 - \text{Op}_h(\varphi))\Delta_h = O(h^2)_{\mathcal{L}(L^2, H_h^2)}$. And $h^{3/2}\langle D_x \rangle^{3/2} : H_h^2 \rightarrow H_h^{1/2} \hookrightarrow L^2$, $\mathcal{A}\langle D_x \rangle^{-3/2} : L^2 \rightarrow L^2$. Therefore, $\text{(III)} = O(h^2)_{\mathcal{L}(L^2, L^2)}$.

Estimate of (IV). Observe that $h^{3/2}\mathcal{A}\text{Op}_h(\varphi)$ is a semiclassical pseudodifferential operator with principle symbol $\begin{pmatrix} \gamma^{(3/2)} & 0 \\ 0 & -\gamma^{(3/2)} \end{pmatrix} \varphi$. It suffices therefore to estimate $(1 - \text{Op}_h(\varphi))\text{Op}_h(\gamma^{(3/2)}\varphi)\Delta_h$. By the symbolic calculus, this is of order $O(h^2)_{\mathcal{L}(L^2, H_h^2)}$. Therefore, $\text{(IV)} = O(h^2)_{\mathcal{L}(L^2, L^2)}$. \square

3.4.3.3 Equation for $\vec{w}_h = \Pi_h \Delta_h \vec{w}$

Proposition 3.4.19. *Suppose that s is sufficiently large, $T > 0$, $\varepsilon_0 > 0$, $\underline{u} \in \mathcal{C}^{1,s}(T, \varepsilon_0)$. Then for ε_0 sufficiently small, \vec{w}_h satisfies the equation*

$$(3.4.27) \quad (hD_s \vec{w}_h + \mathcal{A}_h) \vec{w}_h + R_h \vec{w} = 0,$$

Let $\Pi'_h = \chi'(\text{Re} \mathcal{Z}_h)$ with $\chi' \in \Xi(\eta)$ and $\chi' \chi = \chi$, then for some operators R_h^i ($i = 1, 2, 3$) satisfying the estimates

$$(3.4.28) \quad \|R_h^i\|_{L^\infty([0,T]_t, \mathcal{L}(L^2, L^2))} \lesssim (\varepsilon_0 + h^{1/2}) h^{3/2},$$

the following decomposition holds,

$$(3.4.29) \quad R_h = \Pi_h R_h^1 + R_h^2 \Pi'_h + R_h^3 \Delta_h.$$

Démonstration. Commuting (3.4.25) with Π_h , using the functional calculus $\Pi_h = \Pi_h \Pi'_h$, the remainder R_h writes explicitly

$$\begin{aligned} R_h &= \Pi_h \tilde{R}_h + [\Pi_h, hD_s] + [\Pi_h, \mathcal{A}_h] \Delta_h \\ &= \Pi_h (\tilde{R}_h - hD_s \Pi'_h) + (-hD_s \Pi_h) \Pi'_h + [\Pi_h, \mathcal{A}_h] \Delta_h. \end{aligned}$$

Therefore $R_h^1 = \tilde{R}_h - hD_s \Pi'_h$, $R_h^2 = -hD_s \Pi_h$, $R_h^3 = [\Pi_h, \mathcal{A}_h]$. The estimates of R_h^1 and R_h^2 follow from Lemma 3.4.14 and (3.4.26). In view of Remark 3.4.18, the main estimate of R_h^3 is that of the commutator $[\Pi_h, \mathcal{Z}_h]$. To do this, write

$$\mathcal{Z}_h = \text{Re} \mathcal{Z}_h + \frac{1}{2}(\mathcal{Z}_h - \mathcal{Z}_h^*),$$

where by a functional calculus $[\Pi_h, \text{Re} \mathcal{Z}_h] = 0$, while by (3.4.21) and (3.4.9),

$$(3.4.30) \quad \mathcal{Z}_h - \mathcal{Z}_h^* = h \text{Op}((2i \text{Im} \gamma^{(1/2)} - \partial_\xi \cdot D_x \gamma^{(3/2)}) \varphi^2) + O(h^2)_{\mathcal{L}(L^2, L^2)} = O(h^2)_{\mathcal{L}(L^2, L^2)}.$$

Therefore $[\Pi_h, \mathcal{Z}_h] = O(h^2)_{\mathcal{L}(L^2, L^2)}$. To estimate the commutator with the lower order terms, we use Lemma 3.4.14, and the fact that the first order term in \mathcal{A}_h has a coefficient $O(h)$ and is scalar. Therefore, the commutator with \mathcal{Z}_h should be of size $O(h^2)$. The other lower order terms are no longer scalar, but their coefficients are at most of size $O(h^{3/2})$. So their commutators with \mathcal{Z}_h are of size $O(\varepsilon_0 h^{3/2})$. We conclude that, $(R_h^3) = O((\varepsilon_0 + h^{1/2}) h^{3/2})_{\mathcal{L}(L^2, L^2)}$. \square

Corollary 3.4.20. *Suppose that s is sufficiently large, $T > 0$, $\varepsilon_0 > 0$, $\underline{u} \in \mathcal{C}^{1,s}(T, \varepsilon_0)$. Then for ε_0 sufficiently small, for almost every $t \in [0, T]$ and for all $j_0 \in \mathbb{N}$,*

$$\sum_{\substack{h=2^{-j} \\ j \geq j_0}} \|h^{-3/2} R_h u\|_{L^2}^2 \lesssim (\varepsilon_0^2 + 2^{-j_0}) \|u\|_{L^2}^2.$$

Démonstration. We omit the time variable t for simplicity. By Corollary 3.4.16 and the estimates that $h^{-3/2} R_h^i = O(\varepsilon_0 + h^{1/2})_{\mathcal{L}(L^2, L^2)}$,

$$\begin{aligned} &\sum_{\substack{h=2^{-j} \\ j \geq j_0}} \|h^{-3/2} R_h^2 \Pi'_h u\|_{L^2}^2 + \sum_{\substack{h=2^{-j} \\ j \geq j_0}} \|h^{-3/2} R_h^3 \Delta_h u\|_{L^2}^2 \\ &\lesssim (\varepsilon_0^2 + 2^{-j_0}) \left(\sum_{\substack{h=2^{-j} \\ j \geq j_0}} \|\Pi'_h u\|_{L^2}^2 + \sum_{\substack{h=2^{-j} \\ j \geq j_0}} \|\Delta_h u\|_{L^2}^2 \right) \lesssim (\varepsilon_0^2 + 2^{-j_0}) \|u\|_{L^2}^2. \end{aligned}$$

The main difficulty is to estimate the square sum for $\Pi_h R_h^1$. Recall that

$$R_h^1 = \tilde{R}_h - hD_s \Pi'_h.$$

By Lemma 3.4.14, in particular (3.4.23), and apply the same proof as Corollary 3.4.16,

$$\sum_{\substack{h=2^{-j} \\ j \geq j_0}} \|h^{-3/2} \Pi_h hD_s \Pi'_h u\|_{L^2}^2 \lesssim \sum_{\substack{h=2^{-j} \\ j \geq j_0}} \|h^{-1/2} D_s \Pi'_h u\|_{L^2}^2 \lesssim (\varepsilon_0^2 + 2^{-j_0}) \|u\|_{L^2}^2.$$

Indeed, we use the identity

$$h^{-1/2} D_s \Pi'_h = -\text{Op}_h(\partial_z \chi'(\gamma^{(3/2)} \phi^2) D_t \gamma^{(3/2)} \varphi^2) + O(h)_{\mathcal{L}(L^2, L^2)},$$

and apply Littlewood-Paley's theory.

It remains to estimate $\Pi_h \tilde{R}_h$. Use the decomposition for \tilde{R}_h (Proposition (3.4.17)), the terms (I), (III) and (IV) pose no problem because they each ends with Δ_h or $hD_s \Delta_h$. A similar argument as above works. However (II) should be treated with care. Write

$$(II) = (II) \chi'(|hD_x|^{3/2}) + (II)(1 - \chi'(|hD_x|^{3/2})).$$

By $\Pi_h(II) = O((\varepsilon_0 + h^{1/2})h^{3/2})_{\mathcal{L}(L^2, L^2)}$, the estimate of $\Pi_h(II) \chi'(|hD_x|^{3/2})$ is exactly the same as before,

$$\sum_{\substack{h=2^{-j} \\ j \geq j_0}} \|h^{-3/2} \Pi_h(II) \chi''(|hD_x|^{3/2}) u\|_{L^2}^2 \lesssim (\varepsilon_0^2 + 2^{-j_0}) \|u\|_{L^2}^2.$$

As for the second term, separate it into two halves by expanding the commutator,

$$(II)(1 - \chi'(|hD_x|^{3/2})) = h^{3/2} \Delta_h \mathcal{A}(1 - \chi'(|hD_x|^{3/2})) - h^{3/2} \mathcal{A} \Delta_h (1 - \chi'(|hD_x|^{3/2})),$$

and estimate them separately by semiclassical symbolic calculus. It suffices to show that each half is of order $O(h^2)_{\mathcal{L}(L^2, L^2)}$. For the second half,

$$\Delta_h(1 - \chi'(|hD_x|^{3/2})) = O(h^2)_{\mathcal{L}(L^2, H_h^2)} = O(h^2)_{\mathcal{L}(L^2, H_h^{3/2})} = O(h^{1/2})_{\mathcal{L}(L^2, H^{3/2})}.$$

Therefore $h^{3/2} \mathcal{A} \Delta_h(1 - \chi'(|hD_x|^{3/2})) = O(h^2)_{\mathcal{L}(L^2, L^2)}$. As for the first half, we only do the estimate for the principal term. However, the basic semiclassical symbolic calculus cannot be directly applied here, for

$$h^{3/2} \text{Op}(\gamma \pi)(1 - \chi'(|hD_x|^{3/2})) = \text{Op}_h(\gamma_h(1 - \chi'(|\xi|^{3/2})) \pi_h),$$

with $\pi_h(\xi) = \pi(\xi/h)$, whose ξ -derivative is unbounded as h to 0. However, we have the following Lemma 3.4.21, which shows that no problems can be caused by the low frequency, where the estimate of the ξ -derivatives of the symbols are not required. Therefore the first term is also of order $O(h^2)_{\mathcal{L}(L^2, L^2)}$, as long as s is sufficiently large. \square

Lemma 3.4.21. *Let $\varphi \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and let $\{a_h\}_{h>0} \subset \dot{\Gamma}_\rho^m$ (see Definition 3.9.1) be a family of symbols depending on h . Suppose that there exists $0 < C_1 < C_2$, such that for h sufficiently small,*

$$\text{supp } \varphi \subset \{\xi : |\xi| > C_2\}, \quad \text{supp } a_h \subset \mathbb{T}^d \times \{\xi : |\xi| < C_1 h^{-1}\}.$$

Then if $\rho = 2N$, with $N \ni N > d$,

$$\|\text{Op}_h(\varphi) \text{Op}(a_h)\|_{\mathcal{L}(L^2, L^2)} \lesssim h^{\rho/2-m} M_{\rho,0}^m(a_h) \|u\|_{L^2}.$$

Démonstration. Set $A_h = \text{Op}_h(\varphi)\text{Op}(a_h)$, $\Lambda^\mu = \langle D_x \rangle^\mu$ for $\mu \in \mathbb{R}$. Let \widehat{a}_h denote the Fourier transform of a_h with respect to the x . Then for $\xi \in \mathbb{Z}^d \cap \text{supp } \varphi(h\cdot)$,

$$\begin{aligned} \widehat{A_h u}(\xi) &= \varphi(h\xi) \widehat{\text{Op}(a_h)u}(\xi) = (2\pi)^{-d} \sum_{\eta \in \mathbb{Z}^d} \varphi(h\xi) \widehat{a}_h(\xi - \eta, \eta) \widehat{u}(\eta) \\ &= (2\pi)^{-d} \sum_{|\eta| < C_1 h^{-1}} \frac{1}{\langle \xi - \eta \rangle^{2N}} \varphi(h\xi) \widehat{\Lambda^{2N} a_h}(\xi - \eta, \eta) \widehat{u}(\eta). \end{aligned}$$

Observe that by our hypothesis, $|\xi - \eta| \gtrsim h^{-1}$ whenever $\xi \in \text{supp } \varphi(h\cdot) \subset \{\xi : |\xi| > C_2 h^{-1}\}$ and $|\eta| < C_1 h^{-1}$, and use the fact that the Fourier transform defines a bounded operator from $L^\infty(\mathbb{T}^d) \subset L^1(\mathbb{T}^d)$ to $\ell^\infty(\mathbb{Z}^d)$, we have

$$\begin{aligned} |\widehat{A_h u}(\xi)| &\lesssim h^N \sum_{|\eta| < C_1 h^{-1}} \frac{1}{\langle \xi - \eta \rangle^N} |\varphi(h\xi) \widehat{\Lambda^{2N} a_h}(\xi - \eta, \eta) \widehat{u}(\eta)| \\ &\lesssim h^N \|\varphi\|_{L^\infty} \sum_{|\eta| < C_1 h^{-1}} \|a_h(\cdot, \eta) / \langle \eta \rangle^m\|_{W^{2N, \infty}} \frac{\langle \eta \rangle^m}{\langle \xi - \eta \rangle^N} |\widehat{u}(\eta)| \\ &\lesssim h^{N-m} M_{2N, 0}^m(a_h) \sum_{\eta \in \mathbb{Z}^d} \frac{|\widehat{u}(\eta)|}{\langle \xi - \eta \rangle^N}. \end{aligned}$$

Consequently, for $\rho/2 = N > d$,

$$\begin{aligned} \|A_h u\|_{L^2(\mathbb{T}^d)} &\lesssim \|\widehat{A_h u}\|_{\ell^2(\mathbb{Z}^d)} \lesssim h^{N-m} M_{\rho, 0}^m(a_h) \|\langle \cdot \rangle^{-N} * |\widehat{u}|\|_{\ell^2(\mathbb{Z}^d)} \\ &\lesssim h^{N-m} M_{\rho, 0}^m(a_h) \|\langle \cdot \rangle^{-N}\|_{\ell^1(\mathbb{Z}^d)} \|\widehat{u}\|_{\ell^2(\mathbb{Z}^d)} \lesssim h^{N-m} M_{\rho, 0}^m(a_h) \|u\|_{L^2(\mathbb{T}^d)}. \end{aligned}$$

□

3.4.4 Semiclassical observability

In this section, unless otherwise specified, the time variable will be s , and thus we write $[0, T] = [0, T]_s$ for simplicity. The purpose is to prove the following Proposition 3.4.23, by carefully adapting [84] to our case.

Remark 3.4.22. We will deal with both the symbols defined on $T^*\mathbb{T}^d$ and $T^*(\mathbb{R} \times \mathbb{T}^d)$. To void ambiguity, the semiclassical quantification operator $\overline{\text{Op}}_h$ will be applied to symbols defined on $T^*(\mathbb{R} \times \mathbb{T}^d) \simeq \mathbb{R}_s \times \mathbb{T}_x^d \times \mathbb{R}_\sigma \times \mathbb{R}_\xi^d$, that is, for $a = a(s, x, \sigma, \xi)$

$$\overline{\text{Op}}_h(a) := a(s, x, hD_s, hD_x);$$

while Op_h will only be applied to symbols defined on $T^*\mathbb{T}^d \simeq \mathbb{T}_x^d \times \mathbb{R}_\xi^d$, that is, for $a = a(x, \xi)$,

$$\text{Op}_h(a) = a(x, hD_x).$$

Proposition 3.4.23. *Suppose that ω satisfies the geometric control condition, s is sufficiently large, $T > 0$, then for some $\varepsilon_0 > 0$ and $h_0 > 0$ sufficiently small, for all $0 < h < h_0$ and for $\underline{u} \in \mathcal{C}^{0, s}(h^{1/2}T, \varepsilon_0)$ (we only put $h^{1/2}T$ here because we are dealing with the time variable $s = h^{-1/2}t$), the solution $\vec{w}_h = \begin{pmatrix} w_h^+ \\ w_h^- \end{pmatrix} \in C([0, T], L^2(\mathbb{T}^d))$ to the equation*

$$(3.4.31) \quad (hD_s + \mathcal{A}_h(\underline{u}))\vec{w}_h = \vec{f}_h,$$

where $\vec{f}_h \in L^2([0, T], L^2(\mathbb{T}^d))$, satisfies a semiclassical observability

$$(3.4.32) \quad \|\vec{w}_h(0)\|_{L^2}^2 \lesssim \int_0^T \|\varphi_\omega \vec{e} \cdot \vec{w}_h\|_{L^2}^2 ds + h^{-2} \|\vec{f}_h\|_{L^2([0, T], L^2)}^2,$$

provided that

$$(3.4.33) \quad \vec{w}_h = \Pi'_h(\underline{u})\vec{w}_h + o(1)_{C([0, T], \mathcal{L}(L^2, L^2))}\vec{w}_h,$$

where $\Pi'_h(\underline{u})$ is defined as in Corollary 3.4.15.

We prove this proposition by contradiction. If (3.4.32) is not true, then we can find sequences of $\varepsilon_n > 0$, $h_n > 0$, $\underline{u}_n \in \mathcal{C}^{0,s}(T, \varepsilon_n)$, and

$$(\vec{w}_n, \vec{f}_n) \in C([0, T], L^2(\mathbb{T}^d)) \times L^2([0, T], L^2(\mathbb{T}^d)),$$

such that \vec{w}_n satisfies the condition (3.4.33) and

$$(h_n D_s + \mathcal{A}_n)\vec{w}_n = \vec{f}_n,$$

where we denote $\mathcal{A}_n = \mathcal{A}_{h_n}(\underline{u}_n)$. And as $n \rightarrow \infty$, $\varepsilon_n = o(1)$, $h_n = o(1)$,

$$\|\vec{w}_n(0)\|_{L^2} = 1, \quad \|\varphi_\omega \vec{e} \cdot \vec{w}_n\|_{L^2([0, T], L^2)} = o(1), \quad \|\vec{f}_n\|_{L^2([0, T], L^2)} = o(h_n).$$

We perform an energy estimate to show that $\{\vec{w}_n\}_n$ is bounded in $L^2([0, T], L^2(\mathbb{T}^d))$ so that its space-time semiclassical defect measure is well defined. Indeed, recall that $\mathcal{Z}_{h_n}(\underline{u}_n) - \mathcal{Z}_{h_n}(\underline{u}_n)^* = O(h_n^2)_{L^\infty([0, T], \mathcal{L}(L^2, L^2))}$ (see (3.4.30)), combining with the estimates of the lower order terms,

$$(3.4.34) \quad \mathcal{A}_n - \mathcal{A}_n^* = O((\varepsilon_n + h_n^{1/2})h_n^{3/2})_{L^\infty([0, T], \mathcal{L}(L^2, L^2))}.$$

Using only a weaker estimate $\mathcal{A}_n - \mathcal{A}_n^* = O(h_n)_{\mathcal{L}(L^2, L^2)}$,

$$(3.4.35) \quad \begin{aligned} h_n \partial_s \|\vec{w}_n\|_{L^2}^2 &= (i(\mathcal{A}_n - \mathcal{A}_n^*)\vec{w}_n, \vec{w}_n)_{L^2} + 2\operatorname{Re}(\vec{f}_n, \vec{w}_n)_{L^2} \\ &\lesssim h_n \|\vec{w}_n\|_{L^2}^2 + h_n^{-1} \|\vec{f}_n\|_{L^2}^2. \end{aligned}$$

By Gronwall's inequality, and the hypothesis that $\|\vec{f}_n\|_{L^2([0, T], L^2)} = o(h_n)$,

$$(3.4.36) \quad \|\vec{w}_n\|_{C([0, T], L^2)}^2 \lesssim \|\vec{w}_n(0)\|_{L^2}^2 + h_n^{-2} \|\vec{f}_n\|_{L^2([0, T], L^2)}^2 \lesssim 1,$$

proving that $\{\vec{w}_n\}_n$ is bounded in $C([0, T], L^2(\mathbb{T}^d))$. Therefore, up to a subsequence, we may suppose that $\{\vec{w}_n|_{]0, T[\times \mathbb{T}^d}\}_n$ is pure, and let

$$\mu = \mu(s, x, \sigma, \xi) \in M_{2 \times 2}(\mathcal{M}(T^*(]0, T[\times \mathbb{T}^d), \mathbb{C}))$$

be its space-time semiclassical defect measure, such that

$$(3.4.37) \quad \lim_{n \rightarrow \infty} (\overline{\mathcal{O}p}_{h_n}(X)(\psi \vec{w}_n), \psi \vec{w}_n)_{L^2(]0, T[, L^2)} = \int_{T^*(]0, T[\times \mathbb{T}^d)} \operatorname{tr}(X d\mu),$$

for all $\psi \in C_c^\infty(]0, T[, \mathbb{R})$ and $X \in M_{2 \times 2}(C_c^\infty(T^*(]0, T[\times \mathbb{T}^d)))$, such that $\psi X = X$. It is classical that μ is hermitian in the sense that, for all $c \in \mathbb{C}^2$, $c^t \mu c$ is a non-negative Radon measure. It admits the following form,

$$\mu = \begin{pmatrix} \mu_+ & \mu_* \\ \overline{\mu_*} & \mu_- \end{pmatrix},$$

where $\mu_\pm = \mu[w_n^\pm] \geq 0$ are respectively semiclassical defect measures of the pure sequences $\{w_n^\pm|_{]0, T[\times \mathbb{T}^d}\}_n$. And moreover,

$$(3.4.38) \quad \mu_* \ll \sqrt{\mu_+ \mu_-}.$$

Lemma 3.4.24. *The following properties hold.*

- (1) $\text{supp } \mu \subset \{|\xi| \geq \nu\}$,
- (2) $\text{supp } \mu_\pm \subset \Sigma^\pm = \{\sigma \pm |\xi|^{3/2} = 0\}$,
- (3) $\mu_* = 0$, therefore μ is diagonal.

Démonstration. The first statement comes from (3.4.33). Indeed, let $\psi \in C_c^\infty(]0, T[, \mathbb{R})$ and $X \in M_{2 \times 2}(C_c^\infty(T^*(]0, T[\times \mathbb{T}^d)))$ be such that $\psi X = X$. Denote $\Pi'_n = \Pi'_{h_n}(\underline{u}_n)$, and $\Delta'_n = \Delta'_{h_n}(\underline{u}_n)$, then by Lemma 3.4.14

$$\Pi'_n = \Delta'_n + o(1)_{L^\infty(]0, T[, \mathcal{L}(\dot{L}^2, \dot{L}^2))} = \text{Op}_{h_n}(\chi'(|\xi|^{3/2})) + o(1)_{L^\infty(]0, T[, \mathcal{L}(L^2, L^2))}.$$

The second equality comes from the fact that $\chi'(\gamma^{3/2}) - \chi'(|\xi|^{3/2})$ is bounded by ε_0 . Consequently, by our hypothesis

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} (\overline{\text{Op}}_{h_n}(X)(1 - \Pi'_n)\psi\vec{w}_n, \psi\vec{w}_n)_{L^2(]0, T[, L^2)} \\ &= \int_{T^*(]0, T[\times \mathbb{T}^d)} \text{tr}(X(1 - \chi'(|\xi|^{3/2})) d\mu), \end{aligned}$$

implying that $\text{supp } \mu \subset \text{supp } \chi'(|\xi|^{3/2}) \subset \{|\xi| \geq \nu\}$.

To prove the second statement, let $a \in C_c^\infty(T^*(]0, T[\times \mathbb{T}^d))$ be such that $\psi a = a$, and set $X^+ = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, and $X^- = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$. Then write

$$(hD_s \pm \mathcal{A}_n^*)\overline{\text{Op}}_{h_n}(X^\pm) = (hD_s \pm \mathcal{A}_{h_n}(0))\overline{\text{Op}}_{h_n}(X^\pm) \pm (\mathcal{A}_n^* - \mathcal{A}_{h_n}(0))\overline{\text{Op}}_{h_n}(X^\pm).$$

For the first term,

$$\mathcal{A}_{h_n}(0) = h_n^{3/2} \text{Op}_{h_n}(\varphi)\mathcal{A}(0)\text{Op}_{h_n}(\varphi),$$

where $\mathcal{A}(0) = \mathcal{A}(\underline{u} = 0)$ whose exact expression will be explicitly written in (3.4.54), such that the principal symbol of $\mathcal{A}_{h_n}(0)$ is $\varphi(\xi)^2 \begin{pmatrix} |\xi|^{3/2} & 0 \\ 0 & -|\xi|^{3/2} \end{pmatrix}$, and, considering ψ as a multiplication operator,

$$\begin{aligned} (hD_s \pm \mathcal{A}_{h_n}(0))\overline{\text{Op}}_{h_n}(X^\pm)\psi &= \overline{\text{Op}}_{h_n} \left(\begin{pmatrix} \sigma + \varphi^2 |\xi|^{3/2} & 0 \\ 0 & \sigma - \varphi^2 |\xi|^{3/2} \end{pmatrix} X^\pm \right) \psi \\ &\quad + O(h_n)_{\mathcal{L}(L^2(]0, T[, L^2), L^2(]0, T[, L^2))}. \end{aligned}$$

While for the second term, by (3.4.34),

$$\begin{aligned} (\mathcal{A}_n^* - \mathcal{A}_{h_n}(0)) \langle h_n D_x \rangle^{-3/2} &= (\mathcal{A}_{h_n}(\underline{u}_n)^* - \mathcal{A}_{h_n}(\underline{u}_n)) \langle h_n D_x \rangle^{-3/2} \\ &\quad + (\mathcal{A}_{h_n}(\underline{u}_n) - \mathcal{A}_{h_n}(0)) \langle h_n D_x \rangle^{-3/2} \\ &= O(\varepsilon_n + h_n^{1/2})_{L^\infty([0,T], \mathcal{L}(L^2, L^2))}. \end{aligned}$$

and

$$\langle h D_x \rangle^{3/2} \text{Op}_{h_n}(X^\pm) : L^2([0, T], L^2(\mathbb{T}^d)) \rightarrow L^2([0, T], L^2(\mathbb{T}^d)).$$

Therefore

$$\pm(\mathcal{A}_n^* - \mathcal{A}_{h_n}(0)) \overline{\text{Op}}_{h_n}(X^\pm) = O(\varepsilon_n)_{\mathcal{L}(L^2([0,T], L^2), L^2([0,T], L^2))}.$$

Consequently, since $\varphi \equiv 1$ on $\{|\xi| \geq v\} \supset \text{supp } \mu$,

$$\begin{aligned} (3.4.39) \quad \lim_{n \rightarrow \infty} ((h D_s \pm \mathcal{A}_n^*) \overline{\text{Op}}_{h_n}(X^\pm)(\psi \vec{w}_n), \psi \vec{w}_n)_{L^2([0,T], L^2)} \\ = \int_{T^*(]0, T[\times \mathbb{T}^d)} (\sigma \pm |\xi|^{3/2}) a(s, x, \sigma, \xi) \, d\mu_\pm, \end{aligned}$$

On the other hand,

$$\begin{aligned} (3.4.40) \quad ((h D_s \pm \mathcal{A}_n^*) \overline{\text{Op}}_{h_n}(X^\pm)(\psi \vec{w}_n), \psi \vec{w}_n)_{L^2([0,T], L^2)} \\ = (\overline{\text{Op}}_{h_n}(X^\pm)(\psi \vec{w}_n), (h D_s \pm \mathcal{A}_n)(\psi \vec{w}_n))_{L^2([0,T], L^2)} \\ = (\overline{\text{Op}}_{h_n}(X^\pm)(\psi \vec{w}_n), [h D_s, \psi] \vec{w}_n + \psi \vec{f}_n)_{L^2([0,T], L^2)}, \end{aligned}$$

where, since the support of $[h D_s, \psi] \vec{w}_n = (h D_s \psi) \vec{w}_n$ and $\overline{\text{Op}}_{h_n}(X^\pm) \vec{w}_n$ are disjointed, and by the hypothesis that $\vec{f}_n = o(h_n)_{L^2([0,T], L^2)}$,

$$\begin{aligned} (3.4.41) \quad (\overline{\text{Op}}_{h_n}(X^\pm)(\psi \vec{w}_n), [h D_s, \psi] \vec{w}_n)_{L^2([0,T], L^2)} &= 0, \\ (\overline{\text{Op}}_{h_n}(X^\pm)(\psi \vec{w}_n), \psi \vec{f}_n)_{L^2([0,T], L^2)} &= o(h_n). \end{aligned}$$

Combining (3.4.39), (3.4.40) and (3.4.41),

$$\int_{T^*(]0, T[\times \mathbb{T}^d)} (\sigma \pm |\xi|^{3/2}) a(s, x, \sigma, \xi) \, d\mu_\pm = 0,$$

which implies that $\text{supp } \mu_\pm \subset \{\sigma \pm |\xi|^{3/2} = 0\}$, and by the first statement,

$$\text{supp } \mu_\pm \subset \Sigma^\pm \cap \{|\xi| > v\}.$$

In particular $\text{supp } \mu_+ \cap \text{supp } \mu_- = \emptyset$, so

$$(3.4.42) \quad \mu_+ \perp \mu_-,$$

whence by (3.4.38), $\mu_* \ll \sqrt{\mu_+ \mu_-} = 0$. \square

The following corollary states that the formula (3.4.37) of the semiclassical defect measure for the sequence $\{\vec{w}_n\}_n$ remains valid for some symbols which are not of compact support in the σ variable. The main idea is that on the supports of μ_\pm , which are contained in the hypersurfaces Σ^\pm , σ is automatically bounded whenever ξ is bounded.

Corollary 3.4.25. *Define the projection*

$$\kappa : T^*(]0, T[\times \mathbb{T}^d) \rightarrow]0, T[\times T^*\mathbb{T}^d, \quad (s, \sigma, x, \xi) \mapsto (s, x, \xi),$$

and its corresponding pullback

$$\kappa^* : C_c^\infty(]0, T[\times T^*\mathbb{T}^d) \rightarrow C^\infty(T^*(]0, T[\times \mathbb{T}^d)), \quad a \mapsto \kappa^* a = a \circ \kappa.$$

Let $a \in C_c^\infty(]0, T[\times T^*\mathbb{T}^d)$, and $u \in L^2([0, T], L^2(\mathbb{T}^d))$, then

$$(3.4.43) \quad \text{Op}(a)u = \overline{\text{Op}}(\kappa^* a)u.$$

Moreover, if $a \in C_c^\infty(]0, T[\times T^*\mathbb{T}^d)$, and $\psi \in C_c^\infty(]0, T[)$, such that $\psi a = a$, then

$$(3.4.44) \quad \lim_{n \rightarrow \infty} (\text{Op}_{h_n}(a)(\psi w_n^\pm), \psi w_n^\pm)_{L^2([0, T], L^2)} = \int_{\Sigma^\pm} \kappa^* a \, d\mu_\pm.$$

Démonstration. The identity (3.4.43) follows by a direct verification of the definitions. To prove (3.4.44), observe that w_n^\pm satisfies the equation

$$(3.4.45) \quad (h_n D_s \pm \mathcal{Z}_n + h_n^{1/2} \text{Op}_{h_n}(V_n \cdot \xi \varphi^2)) w_n^\pm = o(h_n)_{L^2([0, T], L^2)}.$$

Apply both sides by ψ , then ψw_n^\pm satisfies

$$(\overline{\text{Op}}_{h_n}(\sigma \pm \gamma^{(3/2)} \varphi^2) + O(h^{1/2}) \times \text{lower order terms})(\psi w_n^\pm) = o(1)_{L^2(\mathbb{R} \times \mathbb{T}^d)}.$$

Let $\zeta \in C_c^\infty(\mathbb{R})$ be such that $\zeta(z) = 1$ in a neighborhood of 0, and apply both sides by $\overline{\text{Op}}_{h_n}(\frac{1 - \zeta(\sigma \pm \gamma^{(3/2)} \varphi^2)}{\sigma \pm \gamma^{(3/2)} \varphi^2})$, we see that

$$(3.4.46) \quad (1 - \overline{\text{Op}}_{h_n}(b^\pm))(\psi w_n^\pm) = o(1)_{L^2(\mathbb{R} \times \mathbb{T}^d)}.$$

where $b^\pm = \zeta(\sigma \pm \gamma^{(3/2)} \varphi^2)$. Now let $\theta \in C_c^\infty(\mathbb{R}_s \times \mathbb{T}_x^d \times \mathbb{R}_\xi^d)$ be such that $\theta \equiv 1$ on the support of a , then by a symbolic calculus and (3.4.46),

$$\begin{aligned} \text{Op}_{h_n}(a)(\psi w_n^\pm) &= \overline{\text{Op}}_{h_n}(\kappa^* a)(\psi w_n^\pm) = \overline{\text{Op}}_{h_n}(b^\pm \kappa^* a)(\psi w_n^\pm) + \overline{\text{Op}}_{h_n}((1 - b^\pm) \kappa^* a)(\psi w_n^\pm) \\ &= \overline{\text{Op}}_{h_n}(\theta b^\pm \kappa^* a)(\psi w_n^\pm) + o(1)_{L^2(\mathbb{R} \times \mathbb{T}^d)}. \end{aligned}$$

Notice that here the symbol $\theta b^\pm \kappa^* a$ is of compact support in $T^*(]0, T[\times \mathbb{T}^d)$, and that $\theta b^\pm \kappa^* a|_{\Sigma^\pm} = \kappa^* a|_{\Sigma^\pm}$. Therefore

$$\begin{aligned} \lim_{n \rightarrow \infty} (\text{Op}_{h_n}(a)(\psi w_n^\pm), \psi w_n^\pm)_{L^2([0, T], L^2)} &= \lim_{n \rightarrow \infty} (\overline{\text{Op}}_{h_n}(\theta b^\pm \kappa^* a)(\psi w_n^\pm), \psi w_n^\pm)_{L^2([0, T], L^2)} \\ &= \int_{\Sigma^\pm} \theta b^\pm \kappa^* a \, d\mu_\pm = \int_{\Sigma^\pm} \kappa^* a \, d\mu_\pm. \end{aligned}$$

□

Now that by (3.4.36), $\{\vec{w}_n\}_n$ is bounded in $C([0, T], L^2(\mathbb{T}^d))$, for all $s \in [0, T]$, up to a subsequence, $\{\vec{w}_n(s)\}_n$ is pure, and admits a semiclassical measure. The next proposition proves that this subsequence can be so chosen that $\{\vec{w}_n(s)\}_n$ is pure for all $s \in [0, T]$.

Proposition 3.4.26. *Up to a subsequence, for all $s \in [0, T]$, the sequence $\{\vec{w}_n(s)\}_n$ is pure in $L^2(\mathbb{T}^d)$, and thus admits a semiclassical measure*

$$\nu_s = \nu_s(x, \xi) \in M_{2 \times 2}(\mathcal{M}(T^*\mathbb{T}^d, \mathbb{C})),$$

which is positive definite and has the following form

$$\nu_s = \begin{pmatrix} \nu_{s,+} & \nu_{s,*} \\ \nu_{s,*} & \nu_{s,-} \end{pmatrix},$$

where $\nu_{s,\pm}$ are semiclassical measures of the sequences $\{w_n^\pm(s)\}_n$, we denote this by $\nu_{s,\pm} = \nu[w_n^\pm(s)]$. Moreover, the distribution valued function $\nu : s \rightarrow \nu_s$ is continuous in time, that is, it belongs to $C([0, T], M_{2 \times 2}(\mathcal{D}'(T^*\mathbb{T}^d)))$.

Démonstration. By a diagonal argument, we assume that $\{\vec{w}_n(s)\}_n$ is pure for each s in a countable and dense subset of $[0, T]$, for example $\mathbb{Q} \cap [0, T]$. In order to conclude, it suffices to show that for each $X \in C_c^\infty(T^*\mathbb{T}^d)$, the following family of functions is equicontinuous,

$$g_n^X : [0, T] \ni s \mapsto (\text{Op}_{h_n}(X)\vec{w}_n(s), \vec{w}_n(s))_{L^2}.$$

We prove this by showing that $\{\partial_s g_n^X\}_n$ is bounded in $L^1([0, T])$.

$$\begin{aligned} h_n \partial_s g_n^X(s) &= (i(\mathcal{A}_n^* \text{Op}_{h_n}(X) - \text{Op}_{h_n}(X)\mathcal{A}_n)\vec{w}_n(s), \vec{w}_n(s))_{L^2} \\ &\quad + 2\text{Re}(i(\text{Op}_{h_n}(X) + \text{Op}_{h_n}(X)^*)\vec{f}_n(s), \vec{w}_n(s))_{L^2} \\ &\lesssim h_n \|\vec{w}_n(s)\|_{L^2}^2 + h_n^{-1} \|\vec{f}_n(s)\|_{L^2}^2, \end{aligned}$$

where we use the estimate

$$\begin{aligned} \mathcal{A}_n^* \text{Op}_{h_n}(X) - \text{Op}_{h_n}(X)\mathcal{A}_n &= [\mathcal{A}_n, \text{Op}_{h_n}(X)] + (\mathcal{A}_n^* - \mathcal{A}_n)\text{Op}_{h_n}(X) \\ &= O(h_n)_{L^\infty([0, T], \mathcal{L}(L^2, L^2))}. \end{aligned}$$

Therefore, take the integration over $[0, T]$,

$$\|\partial_s g_n^X\|_{L^1([0, T])} \lesssim \|\vec{w}_n\|_{L^2([0, T], L^2)}^2 + h_n^{-2} \|\vec{f}_n\|_{L^2([0, T], L^2)}^2 \lesssim 1.$$

□

Now consider the following distribution valued continuous functions,

$$\nu_\pm : [0, T] \rightarrow \mathcal{D}'(T^*\mathbb{T}^d), \quad s \mapsto \nu_{s,\pm}.$$

Then ν_\pm define two distributions on $]0, T[\times T^*\mathbb{T}^d$ such that, for $\phi \in C_c^\infty(]0, T[\times T^*\mathbb{T}^d)$,

$$\langle \nu_\pm, \phi \rangle_{\mathcal{D}', \mathcal{D}([0, T] \times T^*\mathbb{T}^d)} := \int_0^T \int_{T^*\mathbb{T}^d} \phi(s, x, \xi) d\nu_{s,\pm}(x, \xi) ds.$$

Proposition 3.4.27. *The following properties of ν_\pm holds.*

- (1) $\text{supp } \nu_\pm \subset \{|\xi| \geq \nu\}$.

(2) Consider the two sections of κ ,

$$\zeta^\pm :]0, T[\times T^*\mathbb{T}^d \rightarrow \Sigma^\pm \subset T^*(]0, T[\times \mathbb{T}^d), \quad (s, x, \xi) \mapsto (s, x, \mp|\xi|^{-3/2}, \xi).$$

Then for $\phi = \phi_1 \otimes \phi_2$ with $\phi_1 \in C_c^\infty(]0, T[)$ and $\phi_2 \in C_c^\infty(\mathbb{R}_\sigma) \otimes C_c^\infty(T^*\mathbb{T}^d)$,

$$(3.4.47) \quad \langle \mu_\pm, \phi \rangle_{\mathcal{D}', \mathcal{D}(T^*(]0, T[\times \mathbb{T}^d))} = \langle \nu_\pm, (\zeta^\pm)^* \phi \rangle_{\mathcal{D}', \mathcal{D}(]0, T[\times T^*\mathbb{T}^d)},$$

where $(\zeta^\pm)^* \phi = \phi(\zeta^\pm)$; or equivalently,

$$\int_{\Sigma^\pm} \phi(s, x, \sigma, \xi) d\mu_\pm(s, x, \sigma, \xi) = \int_0^T \int_{T^*\mathbb{T}^d} \phi(s, x, \mp|\xi|^{3/2}, \xi) d\nu_{s, \pm}(x, \xi) ds.$$

(3) ν_\pm is propagated via the following transportation equation,

$$(3.4.48) \quad (\partial_s \pm \frac{3}{2}|\xi|^{-1/2}\xi \cdot \nabla_x) \nu_\pm = 0.$$

Démonstration. The first statement is by the same reason as that for μ (see Proposition 3.4.27). To prove the second, use the uniform convergence following from the equicontinuity of g_n^X (here $X = \phi_2 \circ \zeta^\pm$) proved in Proposition 3.4.26, and Corollary 3.4.25. Let $\psi \in C_c^\infty(]0, T[)$ be such that $\psi\phi = \psi$, then

$$\begin{aligned} \int_0^T \int_{T^*\mathbb{T}^d} (\zeta^\pm)^* \phi d\nu_{s, \pm} ds &= \lim_{n \rightarrow \infty} \int_0^T (\text{Op}_{h_n}((\zeta^\pm)^* \phi) \psi w_n^\pm, \psi w_n^\pm)_{L^2} ds \\ &= \int_{\Sigma^\pm} \kappa^*((\zeta^\pm)^* \phi) d\mu_\pm = \int_{\Sigma^\pm} \phi d\mu_\pm, \end{aligned}$$

where the last equality is due to $\kappa^*((\zeta^\pm)^* \phi)|_{\Sigma^\pm} = \phi|_{\Sigma^\pm}$.

To prove the propagation property, we use the equation (3.4.45) satisfied by w_n^\pm , and omit the factor ψ above for simplicity,

$$\begin{aligned} \langle \partial_s \nu_\pm, \phi \rangle_{\mathcal{D}', \mathcal{D}} &= -\langle \nu_\pm, \partial_s \phi \rangle_{\mathcal{D}', \mathcal{D}} = -\int_0^T \lim_{n \rightarrow \infty} (\text{Op}_{h_n}(\partial_s \phi) w_n^\pm, w_n^\pm)_{L^2} ds \\ &= -\int_0^T \lim_{n \rightarrow \infty} (\partial_s (\text{Op}_{h_n}(\phi) w_n^\pm) - \text{Op}_{h_n}(\phi) \partial_s w_n^\pm, w_n^\pm)_{L^2} ds \\ &= \int_0^T \lim_{n \rightarrow \infty} (\text{Op}_{h_n}(\phi) w_n^\pm, \partial_s w_n^\pm)_{L^2} + (\text{Op}_{h_n}(\phi) \partial_s w_n^\pm, w_n^\pm)_{L^2} ds \\ &= \int_0^T \mp \lim_{n \rightarrow \infty} \frac{1}{h_n} (\text{Op}_{h_n}(\phi) w_n^\pm, i \mathcal{Z}_n w_n^\pm)_{L^2} + (\text{Op}_{h_n}(\phi) i \mathcal{Z}_n w_n^\pm, w_n^\pm)_{L^2} ds \\ &= \int_0^T \pm \lim_{n \rightarrow \infty} \frac{i}{h_n} ((\mathcal{Z}_n^* \text{Op}_{h_n}(\phi) - \text{Op}_{h_n}(\phi) \mathcal{Z}_n) w_n^\pm, w_n^\pm)_{L^2} ds. \end{aligned}$$

To continue, we use the explicit calculus of \mathcal{Z}_h and \mathcal{Z}_h^* (see (3.4.21))

$$\begin{aligned} \mathcal{Z}_n^* \text{Op}_{h_n}(\phi) - \text{Op}_{h_n}(\phi) \mathcal{Z}_n &= [\mathcal{Z}_n, \text{Op}_{h_n}(\phi)] + (\mathcal{Z}_n^* - \mathcal{Z}_n) \text{Op}_{h_n}(\phi) \\ &= \frac{h_n}{i} \text{Op}_{h_n} \{ \gamma^{(3/2)} \varphi^2, \phi \} + O(h_n^2)_{L^\infty(]0, T[, \mathcal{L}(L^2, L^2))}. \end{aligned}$$

Plug this into the limit above, and use the fact that $\varphi \equiv 1$ on a neighborhood of $\text{supp } \nu_{\pm}$, whence $\{|\xi|^{3/2}\varphi^2, \phi\} = \{|\xi|^{3/2}, \phi\}$ on $\text{supp } \nu_{\pm}$,

$$\begin{aligned} \langle \partial_s \nu_{\pm}, \phi \rangle_{\mathcal{D}', \mathcal{D}} &= \pm \langle \nu_{\pm}, \{|\xi|^{3/2}\varphi^2, \phi\} \rangle_{\mathcal{D}', \mathcal{D}} \\ &= \pm \langle \nu_{\pm}, \{|\xi|^{3/2}, \phi\} \rangle_{\mathcal{D}', \mathcal{D}} = \mp \langle \{|\xi|^{3/2}, \nu_{\pm}\}, \phi \rangle_{\mathcal{D}', \mathcal{D}}. \end{aligned}$$

Therefore, $\partial_s \nu_{\pm} \pm \{|\xi|^{3/2}, \nu_{\pm}\} = 0$ in the sense of distribution, which is the desired transportation equation for ν_{\pm} . \square

We continue with the proof of the semiclassical observability.

Proof of Proposition 3.4.23. By the hypothesis that $\|\varphi_{\omega} \vec{e} \cdot \vec{w}_n\|_{L^2([0, T], L^2)} = o(1)$, the semiclassical measure for the sequence $\vec{e} \cdot \vec{w}_n = w_n^+ + w_n^-$ (which we denote by $\mu[\vec{e} \cdot \vec{w}_n]$) vanishes on $T^*(]0, T[\times \omega)$. Combining this with the orthogonality (3.4.42),

$$0 = \mu[\vec{e} \cdot \vec{w}_n]|_{T^*(]0, T[\times \omega)} = \mu_+|_{T^*(]0, T[\times \omega)} + \mu_-|_{T^*(]0, T[\times \omega)},$$

which implies $\mu_{\pm}|_{T^*(]0, T[\times \omega)} = 0$ because $\mu_{\pm} \geq 0$. Then by the identity (3.4.47),

$$\nu_{\pm}|_{]0, T[\times T^* \omega} = 0.$$

By the geometric control condition, the propagation law (3.4.48), and the condition for propagation speed (3.4.15),

$$\nu_{\pm}(0) = \nu_{0, \pm} = 0.$$

We then conclude by contradiction. Since by (3.4.33), $\{\vec{w}_n(0)\}_n$ is h_n -oscillating, and by hypothesis $\|\vec{w}_n(0)\|_{L^2} = 1$, whence by Lemma 3.4.14,

$$\int_{T^* \mathbb{T}^d} \chi'(\gamma^{(3/2)} \varphi^2)|_{s=0} \text{tr } d\nu_0 = \lim_{n \rightarrow \infty} (\Pi'_n \vec{w}_n, \vec{w}_n)_{L^2}|_{s=0} = \lim_{n \rightarrow \infty} (\vec{w}_n, \vec{w}_n)_{L^2}|_{s=0} = 1,$$

and $\nu_{0,+} + \nu_{0,-} = \text{tr } \nu_0 \neq 0$. \square

Corollary 3.4.28. *Suppose that ω satisfies the geometric control condition, s is sufficiently large, $T > 0$, $\varepsilon_0 > 0$, and $\underline{u} \in \mathcal{C}^{1,s}(T, \varepsilon_0)$. Let \vec{w} satisfy (3.4.12), and let \vec{w}_h be defined by (3.4.16). Then for ε_0 and $h_0 > 0$ sufficiently small, and all $0 < h = 2^{-j} < h_0$ with $j \in 2\mathbb{N}$, the following observability holds for all $k = 0, 1, \dots, h^{-1/2} - 1$,*

$$(3.4.49) \quad h^{1/2} \|\vec{w}_h(t = kh^{1/2}T)\|_{L^2}^2 \lesssim \int_{I_h^k} \|\varphi_{\omega} \vec{e} \cdot \vec{w}_h(t)\|_{L^2}^2 dt + h^{-2} \|R_h \vec{w}\|_{L^2(I_h^k, L^2)}^2,$$

where $I_h^k := [kh^{1/2}T, (k+1)h^{1/2}T]_t$.

Démonstration. The condition (3.4.33) of Proposition 3.4.23 is verified by (3.4.24), therefore, by Proposition 3.4.23, for some $\varepsilon_0 > 0$ and $h_0 > 0$, uniformly for $k = 0, 1, \dots, h^{-1/2} - 1$,

$$\|\vec{w}_h(s = kT)\|_{L^2}^2 \lesssim \int_{kT}^{(k+1)T} \|\varphi_{\omega} \vec{e} \cdot \vec{w}_h(s)\|_{L^2}^2 ds + h^{-2} \|R_h \vec{w}\|_{L^2_s([kT, (k+1)T], L^2)}^2.$$

We conclude by $\|\cdot\|_{L^2([a,b]_s, L^2)}^2 = h^{-1/2} \|\cdot\|_{L^2([h^{1/2}a, h^{1/2}b]_t, L^2)}^2$, following from the change of time variable, $s = h^{-1/2}t$. \square

3.4.5 Weak observability

Proposition 3.4.29. *Suppose that ω satisfies the geometric control condition, s is sufficiently large, $T > 0$, $\varepsilon_0 > 0$, and $\underline{u} \in \mathcal{C}^{1,s}(T, \varepsilon_0)$. Then for $\varepsilon_0 > 0$ sufficiently small, and for any $N > 0$, the following weak observability of (3.4.12) holds, that for all its solution \vec{w} with L^2 initial data,*

$$(3.4.50) \quad \|\vec{w}(0)\|_{L^2}^2 \lesssim \int_0^T \|\varphi_\omega \vec{e} \cdot \vec{w}\|_{L^2}^2 dt + \|\vec{w}(0)\|_{H^{-N}}^2.$$

Démonstration. The idea is to first sum up (3.4.49) to obtain an observability for \vec{w}_h on the whole interval $[0, T]$, and then use Littlewood-Paley's theory to conclude. However, the left hand side of (3.4.49) is $\|\vec{w}_h(t = kh^{1/2}T)\|_{L^2}^2$, an energy estimate of the equation (3.4.27) should be performed to bound it from below by $\|\vec{w}_h(0)\|_{L^2}^2$. As in (3.4.35), but we use the estimate $\mathcal{A}_h - \mathcal{A}_h^* = O(h^{3/2})_{\mathcal{L}(L^2, L^2)}$ (the size $h^{3/2}$ is crucial here to obtain a uniform energy estimate, independent of h , on the interval $[0, T]_t = [0, h^{-1/2}T]_s$),

$$\begin{aligned} h\partial_s \|\vec{w}_h\|_{L^2}^2 &= (i(\mathcal{A}_h - \mathcal{A}_h^*)\vec{w}_h, \vec{w}_h)_{L^2} + 2\operatorname{Re}(R_h \vec{w}, \vec{w}_h)_{L^2} \\ &\lesssim h^{3/2} \|\vec{w}_h\|_{L^2}^2 + h^{-3/2} \|R_h \vec{w}\|_{L^2}^2. \end{aligned}$$

Therefore, for some constant $C > 0$,

$$\partial_s \|\vec{w}_h\|_{L^2}^2 - Ch^{1/2} \|\vec{w}_h\|_{L^2}^2 \lesssim h^{-5/2} \|R_h \vec{w}\|_{L^2}^2,$$

and consequently,

$$\partial_s (e^{-Ch^{1/2}s} \|\vec{w}_h\|_{L^2}^2) = e^{-Ch^{1/2}s} (\partial_s \|\vec{w}_h\|_{L^2}^2 - Ch^{1/2} \|\vec{w}_h\|_{L^2}^2) \lesssim e^{-Ch^{1/2}s} h^{-5/2} \|R_h \vec{w}\|_{L^2}^2.$$

Now that $h^{1/2}s$ is bounded for $s \in [0, h^{-1/2}T]$, by Newton-Leibniz's rule,

$$\|\vec{w}_h(0)\|_{L^2}^2 \lesssim \|\vec{w}_h(s)\|_{L^2}^2 + h^{-5/2} \|R_h \vec{w}\|_{L^2([0, h^{-1/2}T]_s, L^2)}^2.$$

Or equivalently, for $t \in [0, T]$,

$$h^{1/2} \|\vec{w}_h(0)\|_{L^2}^2 \lesssim h^{1/2} \|\vec{w}_h(t)\|_{L^2}^2 + h^{-5/2} \|R_h \vec{w}\|_{L_t^2([0, T]_t, L^2)}^2.$$

Set $t = kh^{1/2}T$ for $h = 2^{-j}$ with $j \in 2\mathbb{N}$ sufficiently large, and $k = 0, 1, \dots, h^{-1/2} - 1$, and use (3.4.49) by absorbing $h^{-2} \|R_h \vec{w}\|_{L_t^2(I_h^k, L^2)}^2$ into $h^{-5/2} \|R_h \vec{w}\|_{L_t^2([0, T]_t, L^2)}^2$, we have

$$h^{1/2} \|\vec{w}_h(0)\|_{L^2}^2 \lesssim \int_{I_h^k} \|\varphi_\omega \vec{e} \cdot \vec{w}_h(t)\|_{L^2}^2 dt + h^{-5/2} \|R_h \vec{w}\|_{L_t^2([0, T]_t, L^2)}^2.$$

Sum up for $k = 0, 1, \dots, h^{-1/2} - 1$,

$$(3.4.51) \quad \|\vec{w}_h(0)\|_{L^2}^2 \lesssim \int_0^T \|\varphi_\omega \vec{e} \cdot \vec{w}_h(t)\|_{L^2}^2 dt + h^{-3} \|R_h \vec{w}\|_{L_t^2([0, T]_t, L^2)}^2.$$

For the integrand, write

$$\varphi_\omega \vec{e} \cdot \vec{w}_h = \Pi_h \Delta_h \varphi_\omega \vec{e} \cdot \vec{w} + O(h)_{\mathcal{L}(L^2, L^2)} \vec{w}.$$

Then sum up for $h = 2^{-j} < h_0 = 2^{-j_0}$ sufficiently small, using Littlewood-Paley's theory for $\Pi_h \Delta_h$ and R_h , i.e., by Corollary 3.4.16 and Corollary 3.4.20, we obtain

$$\|\vec{w}(0)\|_{L^2}^2 \lesssim \int_0^T \|\varphi_\omega \vec{e} \cdot \vec{w}(t)\|_{L^2}^2 dt + (\varepsilon_0^2 + 2^{-2j_0}) \|\vec{w}\|_{L^2([0, T]_t, L^2)}^2 + \|\vec{w}(0)\|_{H^{-N}}^2.$$

It suffices to bound $\|\vec{w}\|_{L^2([0, T]_t, L^2)} \lesssim \|\vec{w}(0)\|_{L^2}$ and then absorb $(\varepsilon_0^2 + 2^{-2j_0}) \|\vec{w}(0)\|_{L^2}^2$ into the left hand side. \square

3.4.6 Unique continuation and strong observability

We remove the remainder in (3.4.50) by the uniqueness-compactness argument to finish the proof of Proposition 3.4.13.

Proof of Proposition 3.4.13. We proceed by contradiction. Suppose that the strong observability does not hold, then there exists a sequence $\{\varepsilon_n, \underline{u}_n, \vec{w}_n\}_n$, with $\varepsilon_n > 0$, $\underline{u}_n \in \mathcal{C}^{1,s}(T, \varepsilon_n)$, and $\vec{w}_n \in C([0, T], \dot{L}^2(\mathbb{T}^d))$ satisfying the equation

$$(3.4.52) \quad (D_t + \mathcal{A}(\underline{u}_n))\vec{w}_n = 0,$$

such that, as $n \rightarrow \infty$,

$$\varepsilon_n = o(1), \quad \|\vec{w}_n(0)\|_{L^2} = 1, \quad \int_0^T \|\varphi_\omega \vec{e} \cdot \vec{w}_n\|_{L^2}^2 dt = o(1).$$

By an energy estimate, $\{\vec{w}_n\}_n$ is bounded in $C([0, T], \dot{L}^2(\mathbb{T}^d))$, $\{\partial_t \vec{w}_n\}_n$ is bounded in $L^\infty([0, T], \dot{H}^{-3/2}(\mathbb{T}^d))$. Therefore, by Arzelà-Ascoli's theorem, we may assume that, up to a subsequence,

- $\vec{w}_n \rightarrow \vec{w}$ strongly in $C([0, T], \dot{H}^{-3/2}(\mathbb{T}^d))$,
- $\vec{w}_n \rightharpoonup \vec{w}$ weakly in $L^2([0, T], \dot{L}^2(\mathbb{T}^d))$,
- $\vec{w}_n(0) \rightharpoonup \vec{w}(0)$ weakly in $\dot{L}^2(\mathbb{T}^d)$.

Now that $\underline{u}_n \rightarrow 0$ in $C([0, T], \dot{H}^s(\mathbb{T}^d))$, we also have $\mathcal{A}(\underline{u}_n)\vec{w}_n \rightarrow \mathcal{A}(0)\vec{w}$ strongly in $C([0, T], \dot{H}^{-3}(\mathbb{T}^d))$. Therefore, passing to the limit $n \rightarrow \infty$ of (3.4.52) in the sense of distribution, we see that $\vec{w} \in C([0, T], \dot{L}^2(\mathbb{T}^d))$ as it satisfies the following equation,

$$(3.4.53) \quad D_t \vec{w} + \mathcal{A}(0)\vec{w} = 0,$$

where $\mathcal{A}(0) = \text{Op}(A(0)\pi)$ with

$$(3.4.54) \quad A(0) = |\xi|^{3/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{g}{2|\xi|^{1/2}} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + \frac{|\xi|^{1/2} m_b}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

By the weak convergence, $\varphi_\omega \vec{e} \cdot \vec{w}_n \rightharpoonup \varphi_\omega \vec{e} \cdot \vec{w}$ in $L^2([0, T], L^2(\mathbb{T}^d))$, we have

$$\int_0^T \|\varphi_\omega \vec{e} \cdot \vec{w}\|_{L^2}^2 dt \leq \liminf_{n \rightarrow \infty} \int_0^T \|\varphi_\omega \vec{e} \cdot \vec{w}_n\|_{L^2}^2 dt = 0,$$

implying that $\vec{e} \cdot \vec{w}|_{[0, T] \times \omega} = 0$ in the sense of distribution. Then by the weak observability (3.4.50) and Rellich-Kondrachov's compact injection theorem,

$$\|\vec{w}(0)\|_{H^{-N}}^2 = \lim_{n \rightarrow \infty} \|\vec{w}_n(0)\|_{H^{-N}}^2 \gtrsim \limsup_{n \rightarrow \infty} \left(\|\vec{w}_n(0)\|_{L^2}^2 - \int_0^T \|\varphi_\omega \vec{e} \cdot \vec{w}_n\|_{L^2}^2 dt \right) = 1,$$

whence $\vec{w}(0) \neq 0$. To conclude, it suffices to prove the unique continuation property of (3.4.53) and obtain a contradiction. This is done in the following lemma. \square

Lemma 3.4.30. *Under the hypothesis of Proposition 3.4.13, let $\vec{w} \in C([0, T], \dot{L}^2)$ satisfy (3.4.53) and that $\vec{e} \cdot \vec{w}|_{I \times \omega} = 0$ for some interval $I \subset [0, T]$ with non-empty interior, then $\vec{w} \equiv 0$.*

Démonstration. There is no harm in assuming that $I = [0, T]$. For any $0 \leq \delta < T$, define the following \mathbb{C} -linear subspace space of $\dot{L}^2(\mathbb{T}^d)$

$$\mathcal{N}_\delta = \{\vec{w}_0 \in \dot{L}^2(\mathbb{T}^d) : \vec{e} \cdot \exp\{-it\mathcal{A}(0)\}\vec{w}_0|_{[0, T-\delta] \times \omega} = 0\},$$

where $e^{-it\mathcal{A}(0)}\vec{w}_0 \in C([0, T], \dot{L}^2(\mathbb{T}^d))$ denotes the solution to equation (3.4.53) with initial data \vec{w}_0 . It suffices to show that for some $0 \leq \delta_0 < T$, $\mathcal{N}_{\delta_0} = \{0\}$.

Applying the weak observability (3.4.50) with $\varepsilon_0 = 0$, $\underline{u} = 0$, $N > 0$, for time $T - \delta > 0$, and for $\vec{w}_0 \in \mathcal{N}_\delta$,

$$(3.4.55) \quad \|\vec{w}_0\|_{L^2} \leq C(T - \delta)\|\vec{w}_0\|_{H^{-N}},$$

where the constant $C(T - \delta)$ is uniformly bounded as long as $T - \delta$ stays away from 0. This implies that, by the compact injection theorem, the closed unit ball of $(\mathcal{N}_\delta, \|\cdot\|_{L^2})$ is compact, and thus

$$\dim \mathcal{N}_\delta < \infty, \quad \forall \delta \in [0, T].$$

Moreover, by definition

$$\delta < \delta' \Rightarrow \mathcal{N}_\delta \subset \mathcal{N}_{\delta'},$$

which implies that the family $\{\mathcal{N}_\delta\}_{0 \leq \delta < T}$ is totally ordered by the inclusion relation \subset . If $\dim \mathcal{N}_0 = 0$, then the proof is closed. Otherwise, there exists a $\delta_0 > 0$, such that for all $0 < \delta \leq \delta_0$,

$$\dim \mathcal{N}_\delta = \dim \mathcal{N}_{\delta_0} \geq \dim \mathcal{N}_0 > 0,$$

or equivalently,

$$\mathcal{N} := \mathcal{N}_\delta = \mathcal{N}_{\delta_0} \supset \mathcal{N}_0 \neq \{0\}.$$

We will show that $\mathcal{N} = \{0\}$ to obtain a contradiction. Let $\vec{w}_0 \in \mathcal{N}$ and set $\vec{w}(t) = e^{-it\mathcal{A}(0)}\vec{w}_0$. Then for $0 < \varepsilon < \delta_0$, by the identity $\vec{w}(t) = e^{-i(t-\varepsilon)\mathcal{A}(0)}\vec{w}(\varepsilon)$, we see that $\vec{w}(\varepsilon) \in \mathcal{N}$. And since \mathcal{N} is a linear vector space, $\frac{1}{i\varepsilon}(\vec{w}(\varepsilon) - \vec{w}(0)) \in \mathcal{N}$. Moreover, apply the compactness (3.4.55) with $N = 3/2$,

$$\begin{aligned} \left\| \frac{1}{i\varepsilon}(\vec{w}(\varepsilon) - \vec{w}(0)) \right\|_{L^2} &\lesssim \left\| \frac{1}{i\varepsilon}(\vec{w}(\varepsilon) - \vec{w}(0)) \right\|_{H^{-3/2}} \lesssim \sup_{0 \leq t \leq \varepsilon} \|D_t \vec{w}(t)\|_{H^{-3/2}} \\ &\lesssim \sup_{0 \leq t \leq \varepsilon} \|\mathcal{A}(0)\vec{w}(t)\|_{H^{-3/2}} \lesssim \sup_{0 \leq t \leq \varepsilon} \|\vec{w}(t)\|_{L^2} \lesssim \|\vec{w}_0\|_{L^2}. \end{aligned}$$

So the family $\{\frac{1}{i\varepsilon}(\vec{w}(\varepsilon) - \vec{w}(0))\}_{0 < \varepsilon < \delta}$ is bounded in $(\mathcal{N}, \|\cdot\|_{L^2})$, and consequently relatively compact. Therefore, up to a subsequence $\varepsilon_n \rightarrow 0$,

$$\frac{1}{i\varepsilon_n}(\vec{w}(\varepsilon_n) - \vec{w}(0)) \rightarrow D_t \vec{w}(0) = -\mathcal{A}(0)\vec{w}_0$$

strongly in $(\mathcal{N}, \|\cdot\|_{L^2})$, and we have a well defined \mathbb{C} -linear map on \mathcal{N} ,

$$\vec{w}_0 \mapsto \mathcal{A}(0)\vec{w}_0,$$

which admits an eigenfunction, say $\mathcal{N} \ni \vec{w}_0 = \begin{pmatrix} w_0^+ \\ w_0^- \end{pmatrix} \neq 0$, with $\mathcal{A}(0)\vec{w} = \lambda\vec{w}$ for some $\lambda \in \mathbb{C}$. By the definition of $\mathcal{A}(0)$,

$$|D_x|^{3/2}w_0^+ + \frac{g}{2}|D_x|^{-1/2}(w_0^+ + w_0^-) + \frac{1}{2}|D_x|^{1/2}M_b(w_0^+ - w_0^-) = \lambda w_0^+,$$

$$-|D_x|^{3/2}w_0^- - \frac{g}{2}|D_x|^{-1/2}(w_0^+ + w_0^-) - \frac{1}{2}|D_x|^{1/2}M_b(w_0^+ - w_0^-) = \lambda w_0^-,$$

Taking respectively the sum and the difference of the above two equations,

$$\begin{aligned} |D_x|^{3/2}(w_0^+ - w_0^-) &= \lambda(w_0^+ + w_0^-), \\ (|D_x|^{3/2} + g|D_x|^{-1/2})(w_0^+ + w_0^-) + |D_x|^{1/2}M_b(w_0^+ - w_0^-) &= \lambda(w_0^+ - w_0^-). \end{aligned}$$

Apply $|D_x|^{3/2}$ to the second equation, and use the first one to eliminate $w_0^+ - w_0^-$,

$$(|D_x|^3 + g|D_x| + \lambda|D_x|^{1/2}M_b)(w_0^+ + w_0^-) = \lambda^2(w_0^+ + w_0^-).$$

Now that M_b is of order $-\infty$, it is an elliptic equation, implying that $w_0^+ + w_0^-$ has only a finite number of Fourier modes. So it is analytic and can never vanish on a nonempty open set unless it is identically zero. Hence by the definition of \mathcal{N} , $(w_0^+ + w_0^-)|_\omega = \vec{e} \cdot \vec{w}_0|_\omega = 0$, we have $w_0^+ + w_0^- \equiv 0$. Then by the first equation, $|D_x|^{3/2}(w_0^+ - w_0^-) = \lambda(w_0^+ + w_0^-) = 0$, we have $w_0^+ - w_0^- \equiv 0$, for \vec{w}_0 has no zero frequency. Therefore $\vec{w}_0 = 0$, which is a contradiction. \square

3.5 H^s linear control

3.5.1 Sobolev regularity of HUM control operator

Proposition 3.5.1. *Suppose that ω satisfies the geometric control condition, s is sufficiently large, $\mu \geq 0$, $T > 0$, $\varepsilon_0 > 0$, and $\underline{u} \in \mathcal{C}^{1,s}(T, \varepsilon_0)$. Then for ε_0 sufficiently small, $\Theta|_{\dot{H}^\mu} = \Theta(\underline{u})|_{\dot{H}^\mu}$ defines a bounded \mathbb{R} -linear operator from $\dot{H}^\mu(\mathbb{T}^d)$ to $C([0, T], \dot{H}^\mu(\mathbb{T}^d))$, such that*

$$\|\Theta|_{\dot{H}^\mu}\|_{\mathcal{L}(\dot{H}^\mu, C([0, T], \dot{H}^\mu))} \lesssim 1.$$

To prove this, recall that $\Theta = -\mathcal{B}^* \mathcal{S} \mathcal{K}^{-1}$, with $\mathcal{K} = -\mathcal{R} \mathcal{B} \mathcal{B}^* \mathcal{S}$, where by the Hilbert uniqueness method (Proposition 3.4.9), \mathcal{K} defines an isomorphism on $\dot{L}^2(\mathbb{T}^d)$; by Theorem 3.9.3 and Corollary 3.10.5, $\mathcal{K}|_{\dot{H}^\mu}$ sends $\dot{H}^\mu(\mathbb{T}^d)$ to itself. Therefore, to prove Proposition 3.5.1, it remains to show that $\mathcal{K}|_{\dot{H}^\mu}$ defines an isomorphism on $\dot{H}^\mu(\mathbb{T}^d)$. Our idea is to use the Lax-Milgram's theorem as in the Hilbert uniqueness method. However, for technical reasons, we will work in the semiclassical Sobolev spaces $\dot{H}_h^\mu(\mathbb{T}^d) = (\dot{H}^\mu(\mathbb{T}^d), \|\cdot\|_{H_h^\mu})$, equipped with a real scalar product,

$$\operatorname{Re}(u, v)_{H_h^\mu} = \operatorname{Re}(\langle hD_x \rangle^\mu u, \langle hD_x \rangle^\mu v)_{L^2}.$$

For each fixed h , $\dot{H}_h^\mu(\mathbb{T}^d)$ and $\dot{H}^\mu(\mathbb{T}^d)$ are isomorphic as Banach spaces with equivalent norms, even though not uniformly in h . Therefore, Proposition 3.5.1 is a consequence of the following Proposition 3.5.2.

Proposition 3.5.2. *Under the hypothesis of Proposition 3.5.1, for h and ε_0 sufficiently small, $\mathcal{K}|_{\dot{H}_h^\mu} : \dot{H}_h^\mu(\mathbb{T}^d) \rightarrow \dot{H}_h^\mu(\mathbb{T}^d)$ defines an isomorphism.*

Some preliminary results will be proven before the proof of this proposition.

Lemma 3.5.3. *Under the hypothesis of Proposition 3.5.1, define the symbol $\hat{\gamma}^{(\mu)} = (\gamma^{(3/2)})^{2\mu/3}$ for $\mu \geq 0$, and the paradifferential operator*

$$(3.5.1) \quad \Lambda_h^\mu = 1 + h^\mu T_{\hat{\gamma}^{(\mu)}}.$$

Then for ε_0 sufficiently small, $\Lambda_h^\mu : \dot{H}_h^\mu(\mathbb{T}^d) \rightarrow \dot{L}^2(\mathbb{T}^d)$ is invertible (whose inverse will be denoted by $\Lambda_h^{-\mu} := (\Lambda_h^\mu)^{-1}$). Moreover, we have the following estimate, uniformly in h ,

$$(3.5.2) \quad \Lambda_h^\mu - \langle hD_x \rangle^\mu = O(\varepsilon_0)_{L^\infty([0,T], \mathcal{L}(\dot{H}_h^\mu, \dot{L}^2))},$$

which in particular implies the norm equivalence, uniformly for $t_0 \in [0, T]$,

$$(3.5.3) \quad \|\cdot\|_{H_h^\mu} \sim \|\Lambda_h^\mu|_{t=t_0} \cdot\|_{L^2}.$$

Démonstration. We omit the time variable in the proof, and write

$$(3.5.4) \quad \Lambda_h^\mu = (1 + B_h)(1 + h^\mu T_{|\xi|^\mu})$$

with $B_h = h^\mu T_{\hat{\gamma}^{(\mu)} - |\xi|^\mu} (1 + h^\mu T_{|\xi|^\mu})^{-1}$ when h is sufficiently small. Then by Theorem 3.9.3, $B_h = O(\varepsilon_0)_{\mathcal{L}(\dot{L}^2, \dot{L}^2)}$, uniformly in h . For ε_0 sufficiently small, $\text{Id} + B_h : \dot{L}^2(\mathbb{T}^d) \rightarrow \dot{L}^2(\mathbb{T}^d)$ is invertible. The norm equivalence follows as $(1 + h^\mu T_{|\xi|^\mu}) \langle hD_x \rangle^{-\mu}$ and $(1 + h^\mu T_{|\xi|^\mu})^{-1} \langle hD_x \rangle^\mu$ are both bounded on L^2 as they both have Fourier multipliers which are bounded independently of h . \square

The key point to the proof of Proposition 3.5.2 is the following commutator estimate.

Lemma 3.5.4. *Under the hypothesis of Proposition 3.5.1, for $\mu \geq 1$, and h and ε_0 sufficiently small, the following commutator estimate holds,*

$$(3.5.5) \quad [\mathcal{K}, \Lambda_h^\mu|_{t=0}] \Lambda_h^{-\mu}|_{t=0} = O(\varepsilon_0 + h)_{\mathcal{L}(\dot{L}^2, \dot{L}^2)},$$

Démonstration. By the definition of \mathcal{K} , write

$$\begin{aligned} -[\mathcal{K}, \Lambda_h^\mu|_{t=0}] \Lambda_h^{-\mu}|_{t=0} &= (\mathcal{R}T_{\hat{\gamma}^{(\mu)}} - T_{\hat{\gamma}^{(\mu)}|_{t=0}} \mathcal{R}) \mathcal{B} \mathcal{B}^* \mathcal{S}(h^\mu \Lambda_h^{-\mu}|_{t=0}) \\ &\quad + h \mathcal{R} (\mathcal{B} \mathcal{B}^* T_{\hat{\gamma}^{(\mu)}} - T_{\hat{\gamma}^{(\mu)}} \mathcal{B} \mathcal{B}^*) \mathcal{S}(h^{\mu-1} \Lambda_h^{-\mu}|_{t=0}) \\ &\quad + \mathcal{R} \mathcal{B} \mathcal{B}^* (\mathcal{S} T_{\hat{\gamma}^{(\mu)}|_{t=0}} - T_{\hat{\gamma}^{(\mu)}} \mathcal{S})(h^\mu \Lambda_h^{-\mu}|_{t=0}). \end{aligned}$$

By Lemma 3.5.3, for $0 \leq \sigma \leq \mu$,

$$\Lambda_h^{-\mu}|_{t=0} = O(1)_{\mathcal{L}(\dot{L}^2, \dot{H}_h^\mu)} = O(1)_{\mathcal{L}(\dot{L}^2, \dot{H}_h^\sigma)} = O(h^{-\sigma})_{\mathcal{L}(\dot{L}^2, \dot{H}^\sigma)},$$

from which

$$\|h^\mu \Lambda_h^{-\mu}|_{t=0}\|_{\mathcal{L}(\dot{L}^2, \dot{H}^\mu)} \lesssim 1, \quad \|h^{\mu-1} \Lambda_h^{-\mu}|_{t=0}\|_{\mathcal{L}(\dot{L}^2, \dot{H}^{\mu-1})} \lesssim 1.$$

Then (3.5.5) results from the following estimates,

$$\begin{aligned} \|\mathcal{B} \mathcal{B}^* T_{\hat{\gamma}^{(\mu)}} - T_{\hat{\gamma}^{(\mu)}} \mathcal{B} \mathcal{B}^*\|_{C([0,T], \mathcal{L}(\dot{H}^{\mu-1}, \dot{L}^2))} &\lesssim 1, \\ \|\mathcal{R} T_{\hat{\gamma}^{(\mu)}} - T_{\hat{\gamma}^{(\mu)}|_{t=0}} \mathcal{R}\|_{\mathcal{L}(L^2([0,T], \dot{H}^\mu), \dot{L}^2)} &\lesssim \varepsilon_0, \\ \|\mathcal{S} T_{\hat{\gamma}^{(\mu)}|_{t=0}} - T_{\hat{\gamma}^{(\mu)}} \mathcal{S}\|_{\mathcal{L}(\dot{H}^\mu, C([0,T], \dot{L}^2))} &\lesssim \varepsilon_0, \end{aligned}$$

which will be treated separately in the following lemmas. \square

Lemma 3.5.5. $\|\mathcal{R}T_{\hat{\gamma}(\mu)} - T_{\hat{\gamma}(\mu)|_{t=0}}\mathcal{R}\|_{\mathcal{L}(L^2([0,T],\dot{H}^\mu),\dot{L}^2)} \lesssim \varepsilon_0$.

Démonstration. Let $G \in L^2([0, T], \dot{H}^\mu(\mathbb{T}^d))$, and let $u \in C([0, T], \dot{H}^\mu(\mathbb{T}^d))$, $v \in C([0, T], \dot{L}^2(\mathbb{T}^d))$ solve respectively the following equations

$$(\partial_t + P)u = G, \quad u(T) = 0; \quad (\partial_t + P)v = T_{\hat{\gamma}(\mu)}G, \quad v(T) = 0.$$

Then $w := v - T_{\hat{\gamma}(\mu)}u \in C([0, T], \dot{L}^2(\mathbb{T}^d))$ satisfies

$$(\partial_t + P)w = [T_{\hat{\gamma}(\mu)}, \partial_t + P]u, \quad w(T) = 0.$$

Observe that

$$[T_{\hat{\gamma}(\mu)}, \partial_t + P] = -T_{\partial_t \hat{\gamma}(\mu)} + [T_{\hat{\gamma}(\mu)}, P]$$

where $T_{\partial_t \hat{\gamma}(\mu)} = O(\varepsilon_0)_{\mathcal{L}(\dot{H}^\mu, \dot{L}^2)}$ by Theorem 3.9.3. The main estimate for $[T_{\hat{\gamma}(\mu)}, P]$ is $[T_{\hat{\gamma}(\mu)}, T_{\gamma(3/2)}]$. By Remark 3.4.11,

$$[T_{\hat{\gamma}(\mu)}, T_{\gamma(3/2)}] = T_{\{\hat{\gamma}(\mu), \gamma(3/2)\}/i} + O(\varepsilon_0)_{\mathcal{L}(\dot{H}^\mu, \dot{L}^2)} = O(\varepsilon_0)_{\mathcal{L}(\dot{H}^\mu, \dot{L}^2)},$$

because $\{\hat{\gamma}(\mu), \gamma(3/2)\} = 0$ for $\xi \neq 0$. Combining the lower order terms, we then have

$$[T_{\hat{\gamma}(\mu)}, \partial_t + P] = O(\varepsilon_0)_{\mathcal{L}(\dot{H}^\mu, \dot{L}^2)}.$$

Finally by an energy estimate,

$$\begin{aligned} \|(\mathcal{R}T_{\hat{\gamma}(\mu)} - T_{\hat{\gamma}(\mu)|_{t=0}}\mathcal{R})G\|_{L^2} &= \|w(0)\|_{L^2} \lesssim \|[T_{\hat{\gamma}(\mu)}, \partial_t + P]u\|_{L^1([0,T], L^2)} \\ &\lesssim \varepsilon_0 \|u\|_{L^1([0,T], H^\mu)} \lesssim \varepsilon_0 \|G\|_{L^2([0,T], H^\mu)}. \end{aligned}$$

□

Lemma 3.5.6. $\|\mathcal{S}T_{\hat{\gamma}(\mu)|_{t=0}} - T_{\hat{\gamma}(\mu)}\mathcal{S}\|_{\mathcal{L}(\dot{H}^\mu, C([0,T], \dot{L}^2))} \lesssim \varepsilon_0$.

Démonstration. Let $u_0 \in \dot{H}^\mu(\mathbb{T}^d)$, and let $u \in C([0, T], \dot{H}^\mu(\mathbb{T}^d))$, $v \in C([0, T], \dot{L}^2(\mathbb{T}^d))$ solve respectively the following equations,

$$(\partial_t - P^*)u = 0, \quad u(0) = u_0, \quad (\partial_t - P^*)v = 0, \quad v(0) = T_{\hat{\gamma}(\mu)|_{t=0}}u_0.$$

Then $w = v - T_{\hat{\gamma}(\mu)}u \in C([0, T], \dot{L}^2(\mathbb{T}^d))$ satisfies

$$(\partial_t - P^*)w = [T_{\hat{\gamma}(\mu)}, \partial_t - P^*]u, \quad w(0) = 0.$$

Recall that, for $\sigma \geq 0$, $P - P^* = O(\varepsilon_0)_{\mathcal{L}(\dot{H}^\sigma, \dot{H}^\sigma)}$, we then have

$$[T_{\hat{\gamma}(\mu)}, \partial_t - P^*] = -T_{\partial_t \hat{\gamma}(\mu)} - [T_{\hat{\gamma}(\mu)}, P] + [T_{\hat{\gamma}(\mu)}, P - P^*] = O(\varepsilon_0)_{\mathcal{L}(L^2, L^2)}.$$

Again by an energy estimate,

$$\begin{aligned} \|(\mathcal{S}T_{\hat{\gamma}(\mu)|_{t=0}} - T_{\hat{\gamma}(\mu)}\mathcal{S})u_0\|_{L^2([0,T], L^2)} &= \|w\|_{L^2([0,T], L^2)} \lesssim \|[T_{\hat{\gamma}(\mu)}, \partial_t - P^*]u\|_{L^1([0,T], L^2)} \\ &\lesssim \varepsilon_0 \|u\|_{L^1([0,T], H^\mu)} \lesssim \varepsilon_0 \|u_0\|_{H^\mu}. \end{aligned}$$

□

Lemma 3.5.7. $\|\mathcal{B}\mathcal{B}^*T_{\hat{\gamma}(\mu)} - T_{\hat{\gamma}(\mu)}\mathcal{B}\mathcal{B}^*\|_{C([0,T],\mathcal{L}(\dot{H}^{\mu-1},\dot{L}^2))} \lesssim 1$.

Démonstration. By symbolic calculus $[T_q, T_{\hat{\gamma}(\mu)}]$ and $[T_q^*, T_{\hat{\gamma}(\mu)}]$ are of order $\mu - 1$, it suffices to show that $[\varphi_\omega, T_{\hat{\gamma}(\mu)}]$ is also of order $\mu - 1$. Indeed, write $\varphi_\omega = T_{\varphi_\omega} + (\varphi_\omega - T_{\varphi_\omega})$, then $[T_{\varphi_\omega}, T_{\hat{\gamma}(\mu)}]$ is of order $\mu - 1$, while $\varphi_\omega - T_{\varphi_\omega}$ is of order $-\infty$ since φ_ω is smooth. \square

Proof of Proposition 3.5.2. Consider the \mathbb{R} -bilinear form on $\dot{H}_h^\mu(\mathbb{T}^d)$,

$$\varpi_h^\mu(f_0, g_0) = \operatorname{Re}(\Lambda_h^\mu|_{t=0}\mathcal{K}f_0, \Lambda_h^\mu|_{t=0}g_0)_{L^2}.$$

Then by Lemma 3.5.4, and the L^2 -coercivity of \mathcal{K} , for h and ε_0 sufficiently small,

$$\begin{aligned} \varpi_h^\mu(f_0, f_0) &= \operatorname{Re}(\mathcal{K}\Lambda_h^\mu|_{t=0}f_0, \Lambda_h^\mu|_{t=0}f_0)_{L^2} + \operatorname{Re}([\Lambda_h^\mu|_{t=0}, \mathcal{K}]\Lambda_h^{-\mu}|_{t=0}\Lambda_h^\mu|_{t=0}f_0, \Lambda_h^\mu|_{t=0}f_0) \\ &\gtrsim \|\Lambda_h^\mu|_{t=0}f_0\|_{L^2}^2 - \|[\Lambda_h^\mu|_{t=0}, \mathcal{K}]\Lambda_h^{-\mu}|_{t=0}\|_{\mathcal{L}(\dot{L}^2, \dot{L}^2)}\|\Lambda_h^\mu|_{t=0}f_0\|_{L^2}^2 \\ &\gtrsim \|f_0\|_{H_h^\mu}^2 - (\varepsilon_0 + h)\|f_0\|_{H_h^\mu}^2 \\ &\gtrsim \|f_0\|_{H_h^\mu}^2. \end{aligned}$$

Therefore, ϖ_h^μ is coercive on $\dot{H}_h^\mu(\mathbb{T}^d)$, and we conclude by Lax-Milgram's theorem. \square

3.5.2 H^s -controllability

The HUM control operator Θ solves the linear control problem (3.4.1) without the perturbation terms Ru and βF . This section constructs a control operator Φ for the linear control problem

$$(3.5.6) \quad (\partial_t + P + R)u = (\mathcal{B} + \beta)F,$$

where $P = P(\underline{u})$, $R = R(\underline{u})$, $\mathcal{B} = \mathcal{B}(\underline{u})$, $\beta = \beta(\underline{u})$ with $\underline{u} \in \mathcal{C}^{1,s}(T, \varepsilon_0)$.

Proposition 3.5.8. *Suppose that ω satisfies the geometric control condition, s is sufficiently large, $T > 0$, $\varepsilon_0 > 0$, and $\underline{u} \in \mathcal{C}^{1,s}(T, \varepsilon_0)$. Then for $\varepsilon_0 > 0$ sufficiently small, there exists an operator*

$$\Phi = \Phi(\underline{u}) : \dot{H}^s(\mathbb{T}^d) \rightarrow C([0, T], \dot{H}^s(\mathbb{T}^d))$$

satisfying

$$\|\Phi\|_{\mathcal{L}(\dot{H}^s, C([0,T], \dot{H}^s))} \lesssim 1,$$

such that, for $u_0 \in \dot{H}^s(\mathbb{T}^d)$, setting $F = \Phi u_0$, the solution u to (3.5.6) with initial data $u(0) = u_0$ vanishes at time T , that is, $u(T) = 0$.

Démonstration. First we define a new solution operator $\mathcal{S}_\Theta = \mathcal{S}_\Theta(\underline{u})$.

Lemma 3.5.9. *For $v_0 \in \dot{L}^2(\mathbb{T}^d)$, set $F = \Theta v_0 \in C([0, T], \dot{L}^2(\mathbb{T}^d))$. Let $v \in C([0, T], \dot{L}^2)$ be the solution to*

$$(\partial_t + P)v = \mathcal{B}F, \quad v(0) = v_0, \quad v(T) = 0.$$

We set $\mathcal{S}_\Theta v_0 = v$, then for any $\mu \geq 0$, $\mathcal{S}_\Theta|_{\dot{H}^\mu}$ is a bounded \mathbb{R} -linear operator from $\dot{H}^\mu(\mathbb{T}^d)$ to $C([0, T], \dot{H}^\mu(\mathbb{T}^d))$.

Démonstration. For $v_{0,i} \in \dot{L}^2(\mathbb{T}^d)$ and $\lambda_i \in \mathbb{R}$ ($i = 1, 2$), set $v_i = \mathcal{S}_\Theta v_{0,i}$ and $v_\lambda = \lambda_1 v_1 + \lambda_2 v_2$. Then they satisfy the equations

$$\begin{aligned} (\partial_t + P)v_i &= \mathcal{B}\Theta v_{0,i}, & v_i(0) &= v_{0,i}, & v_i(T) &= 0. \\ (\partial_t + P)v_\lambda &= \mathcal{B}(\lambda_1 \Theta v_{0,1} + \lambda_2 \Theta v_{0,2}), & v_\lambda(0) &= \lambda_1 v_{0,1} + \lambda_2 v_{0,2}, & v_\lambda(T) &= 0. \end{aligned}$$

By the \mathbb{R} -linearity of Θ , $\lambda_1 \Theta v_{0,1} + \lambda_2 \Theta v_{0,2} = \Theta(\lambda_1 v_{0,1} + \lambda_2 v_{0,2})$, whence

$$v_\lambda = \mathcal{S}_\Theta(\lambda_1 v_{0,1} + \lambda_2 v_{0,2}),$$

that is, the \mathbb{R} -linearity of \mathcal{S}_Θ . By Proposition 3.5.1 and Proposition 3.10.4, we have

$$(3.5.7) \quad \|\mathcal{S}_\Theta v_0\|_{C([0,T],H^\mu)} \lesssim \|\mathcal{B}\Theta v_0\|_{L^1([0,T],H^\mu)} \lesssim \|v_0\|_{H^\mu}.$$

□

Then we define a new range operator $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}(\underline{u})$.

Lemma 3.5.10. *For $G \in L^2([0, T], \dot{H}^s(\mathbb{T}^d))$, let $w \in C([0, T], \dot{H}^s(\mathbb{T}^d))$ be the solution to the equation*

$$(\partial_t + P + R)w = G, \quad w(T) = 0.$$

We set $\tilde{\mathcal{R}}G = w(0)$, then $\tilde{\mathcal{R}}$ defines a bounded \mathbb{R} -linear operator from $L^2([0, T], \dot{H}^s(\mathbb{T}^d))$ to $\dot{H}^s(\mathbb{T}^d)$.

Démonstration. This is a consequence of Proposition 3.10.4 for the backwards equation. □

Now we start constructing Φ by perturbing Θ . For $v_0 \in \dot{H}^s(\mathbb{T}^d)$, set $v = \mathcal{S}_\Theta v_0$, $F = \Theta v_0$, and let w be the solution to the equation

$$(\partial_t + P + R)w = -Rv + \beta F, \quad w(T) = 0,$$

then $u = v + w$ satisfies the equation

$$(\partial_t + P + R)u = (\mathcal{B} + \beta)F, \quad u(0) = v_0 + w_0, \quad u(T) = 0,$$

where

$$w_0 := w(0) = \tilde{\mathcal{R}}(-Rv + \beta F) = \tilde{\mathcal{R}}(-R\mathcal{S}_\Theta + \beta\Theta)v_0.$$

Define the perturbation operator,

$$\mathcal{E} = \tilde{\mathcal{R}}(-R\mathcal{S}_\Theta + \beta\Theta|_{\dot{H}^s}) : \dot{H}^s(\mathbb{T}^d) \mapsto \dot{H}^s(\mathbb{T}^d),$$

then $u_0 = (1 + \mathcal{E})v_0$. By (3.3.15), (3.5.7), Proposition 3.4.1 and Corollary 3.10.5, \mathcal{E} is small in the sense that,

$$\|\mathcal{E}\|_{\mathcal{L}(\dot{H}^s, \dot{H}^s)} \lesssim \varepsilon_0^\vartheta, \quad \vartheta > 0.$$

Therefore, for ε_0 sufficiently small, $1 + \mathcal{E} : \dot{H}^s(\mathbb{T}^d) \rightarrow \dot{H}^s(\mathbb{T}^d)$ is invertible, and

$$(3.5.8) \quad \Phi := \Theta|_{\dot{H}^s}(1 + \mathcal{E})^{-1}$$

is the desired control operator. □

3.5.3 Contraction estimate of control operator

In the next section, we use an iterative scheme to solve the nonlinear control problem (3.3.12). Contraction estimates of some operators, especially of the control operator, will be of great importance.

To do this, let $\underline{u}_i \in \mathcal{C}^{1,s}(T, \varepsilon_0)$, ($i = 1, 2$). And let $\psi_i = \psi(\underline{u}_i) \in \dot{H}^s(\mathbb{T}^d)$ and $\eta_i = \eta(\underline{u}_i) \in \dot{H}^{s+1/2}(\mathbb{T}^d)$ be determined by Proposition 3.3.12. For any symbol depending on η , $a = a(\eta)$, set $a_i = a(\eta_i)$. For any operator $\mathcal{L} = \mathcal{L}(\underline{u})$, set $\mathcal{L}_i = \mathcal{L}(\underline{u}_i)$.

Lemma 3.5.11. *Suppose that $s > 3/2 + d/2$, then for ε_0 sufficiently small,*

$$\|\eta_i\|_{H^{s+1/2}} \lesssim \varepsilon_0, \quad \|\eta_1 - \eta_2\|_{H^{s+1/2}} \lesssim \|\underline{u}_1 - \underline{u}_2\|_{H^s}.$$

Démonstration. By Proposition 3.3.12, $\|\eta_i\|_{H^{s+1/2}} \leq 2\varepsilon_0$. Then write

$$\eta_1 - \eta_2 = -T_{p_1}^{-1} \text{Im}(\underline{u}_1 - \underline{u}_2) + T_{p_2}^{-1}(T_{p_1} - T_{p_2})T_p^{-1} \text{Im} \underline{u}_2,$$

from which

$$\|\eta_1 - \eta_2\|_{H^{s+1/2}} \lesssim \|\underline{u}_1 - \underline{u}_2\|_{H^s} + \varepsilon_0 \|\eta_1 - \eta_2\|_{H^{s+1/2}}.$$

We conclude for ε_0 sufficiently small. \square

Lemma 3.5.12. *Suppose that $s > 3/2 + d/2$, $\sigma \in \mathbb{R}$, then for ε_0 sufficiently small,*

$$\|\mathcal{B}_1^* - \mathcal{B}_2^*\|_{\mathcal{L}(H^\sigma, H^\sigma)} = \|\mathcal{B}_1 - \mathcal{B}_2\|_{\mathcal{L}(H^\sigma, H^\sigma)} \lesssim \|\underline{u}_1 - \underline{u}_2\|_{H^s}.$$

Démonstration. By the definition of \mathcal{B} , Theorem 3.9.3 and Lemma 3.5.11, this comes from the estimate

$$\|T_{q_1} - T_{q_2}\|_{\mathcal{L}(H^\sigma, H^\sigma)} \lesssim \|\eta_1 - \eta_2\|_{H^{s+1/2}} \lesssim \|\underline{u}_1 - \underline{u}_2\|_{H^s}.$$

\square

Lemma 3.5.13. *Suppose that $s > 3/2 + d/2$, $\sigma \in \mathbb{R}$, then for ε_0 sufficiently small,*

$$\|P_1 - P_2\|_{\mathcal{L}(\dot{H}^{\sigma+3/2}, \dot{H}^\sigma)} + \|P_1^* - P_2^*\|_{\mathcal{L}(\dot{H}^{\sigma+3/2}, \dot{H}^\sigma)} \lesssim \|\underline{u}_1 - \underline{u}_2\|_{H^s}.$$

Démonstration. The main estimate is, by Theorem 3.9.3 and Lemma 3.5.11,

$$\|T_{\gamma_1^{(3/2)}} - T_{\gamma_2^{(3/2)}}\|_{\mathcal{L}(\dot{H}^{\sigma+3/2}, \dot{H}^\sigma)} \lesssim \|\eta_1 - \eta_2\|_{H^{s+1/2}} \lesssim \|\underline{u}_1 - \underline{u}_2\|_{H^s}.$$

\square

Lemma 3.5.14. *Suppose that $s > 3/2 + d/2$, $\sigma \geq 0$, then for ε_0 sufficiently small,*

$$\|\mathcal{S}_1 - \mathcal{S}_2\|_{\mathcal{L}(\dot{H}^{\sigma+3/2}, C([0, T], \dot{H}^\sigma))} \lesssim \|\underline{u}_1 - \underline{u}_2\|_{L^\infty([0, T], H^s)}.$$

Démonstration. For $v_0 \in \dot{H}^{\sigma+3/2}(\mathbb{T}^d)$, set $v^i = \mathcal{S}_i v_0 \in C([0, T], \dot{H}^{\sigma+3/2}(\mathbb{T}^d))$ for $i = 1, 2$. Then $w = v^1 - v^2$ satisfies the equation

$$(\partial_t - P_1^*)w = (P_1^* - P_2^*)v^2, \quad w(0) = 0,$$

whence the energy estimate

$$\|w\|_{C([0, T], H^\sigma)} \lesssim \|(P_1^* - P_2^*)v^2\|_{L^1([0, T], H^\sigma)} \lesssim \|\underline{u}_1 - \underline{u}_2\|_{L^\infty([0, T], H^s)} \|v_0\|_{\dot{H}^{\sigma+3/2}}.$$

\square

Lemma 3.5.15. *Suppose that $s > 3/2 + d/2$, $\sigma \geq 0$, then for ε_0 sufficiently small,*

$$\|\mathcal{R}_1 - \mathcal{R}_2\|_{\mathcal{L}(L^2([0,T], \dot{H}^{\sigma+3/2}), \dot{H}^\sigma)} \lesssim \|\underline{u}_1 - \underline{u}_2\|_{L^\infty([0,T], H^s)}.$$

Démonstration. Let $G \in L^2([0, T], \dot{H}^{\sigma+3/2}(\mathbb{T}^d))$, and let $v^i \in C([0, T], \dot{H}^{\sigma+3/2}(\mathbb{T}^d))$ for $i = 1, 2$ be solutions to the equations

$$(\partial_t + P_i)v^i = G, \quad v^i(T) = 0.$$

Then $w = v^1 - v^2$ satisfies

$$(\partial_t + P_1)w = (P_2 - P_1)v^2, \quad w(T) = 0,$$

whence the estimate

$$\begin{aligned} \|(\mathcal{R}_1 - \mathcal{R}_2)G\|_{H^\sigma} &= \|w(0)\|_{H^\sigma} \lesssim \|(P_2 - P_1)v^2\|_{L^1([0,T], H^\sigma)} \\ &\lesssim \|\underline{u}_1 - \underline{u}_2\|_{L^\infty([0,T], H^s)} \|v^2\|_{L^1([0,T], H^{\sigma+3/2})} \\ &\lesssim \|\underline{u}_1 - \underline{u}_2\|_{L^\infty([0,T], H^s)} \|G\|_{L^2([0,T], H^{\sigma+3/2})}. \end{aligned}$$

□

Lemma 3.5.16. *Suppose that $s > 3/2 + d/2$, $\sigma \geq 0$, then for ε_0 sufficiently small,*

$$\|\mathcal{K}_1 - \mathcal{K}_2\|_{\mathcal{L}(\dot{H}^{\sigma+3/2}, \dot{H}^\sigma)} \lesssim \|\underline{u}_1 - \underline{u}_2\|_{L^\infty([0,T], H^s)}.$$

Démonstration. By the definition (3.4.5) of \mathcal{K} , we have the identity

$$\begin{aligned} \mathcal{K}_2 - \mathcal{K}_1 &= (\mathcal{R}_1 - \mathcal{R}_2)\mathcal{B}_1\mathcal{B}_1^*\mathcal{S}_1 + \mathcal{R}_2(\mathcal{B}_1 - \mathcal{B}_2)\mathcal{B}_1^*\mathcal{S}_1 \\ &\quad + \mathcal{R}_2\mathcal{B}_2(\mathcal{B}_1^* - \mathcal{B}_2^*)\mathcal{S}_1 + \mathcal{R}_2\mathcal{B}_2\mathcal{B}_2^*(\mathcal{S}_1 - \mathcal{S}_2). \end{aligned}$$

And we conclude by Lemma 3.5.15, Lemma 3.5.12 and Lemma 3.5.14. □

Lemma 3.5.17. *Suppose that s is sufficiently large, $\sigma \geq 0$, then for ε_0 sufficiently small,*

$$\|\Theta_1 - \Theta_2\|_{\mathcal{L}(\dot{H}^{\sigma+3/2}, C([0,T], \dot{H}^\sigma))} \lesssim \|\underline{u}_1 - \underline{u}_2\|_{L^\infty([0,T], H^s)}.$$

Démonstration. By the invertibility of \mathcal{K} (Proposition 3.5.2) and the definition (3.4.7) of Θ , we have the following identity

$$\Theta_1 - \Theta_2 = (\mathcal{B}_2^* - \mathcal{B}_1^*)\mathcal{S}_2\mathcal{K}_2^{-1} + \mathcal{B}_1^*(\mathcal{S}_2 - \mathcal{S}_1)\mathcal{K}_2^{-1} + \mathcal{B}_1^*\mathcal{S}_1\mathcal{K}_2^{-1}(\mathcal{K}_1 - \mathcal{K}_2)\mathcal{K}_1^{-1}.$$

And we conclude by Lemma 3.5.12, Lemma 3.5.14 and Lemma 3.5.16. □

Lemma 3.5.18. *Suppose that s is sufficiently large, $\sigma \geq 0$, then for ε_0 sufficiently small,*

$$\|(\mathcal{S}_\Theta)_1 - (\mathcal{S}_\Theta)_2\|_{\mathcal{L}(\dot{H}^{\sigma+3/2}, C([0,T], \dot{H}^\sigma))} \lesssim \|\underline{u}_1 - \underline{u}_2\|_{L^\infty([0,T], H^s)}.$$

Démonstration. For $v_0 \in \dot{H}^{\sigma+3/2}(\mathbb{T}^d)$, let $v^i = (\mathcal{S}_\Theta)_i v_0 \in C([0, T], \dot{H}^{\sigma+3/2}(\mathbb{T}^d))$, ($i = 1, 2$), then $w = v^1 - v^2$ satisfies the equation

$$(\partial_t + P_1)w = (\mathcal{B}_1\Theta_1 - \mathcal{B}_2\Theta_2)v_0 + (P_2 - P_1)v^2, \quad w(0) = 0.$$

Therefore, by an energy estimate,

$$\begin{aligned} \|v^1 - v^2\|_{L^2([0,T], H^\sigma)} &\lesssim \|(\mathcal{B}_1\Theta_1 - \mathcal{B}_2\Theta_2)v_0\|_{L^1([0,T], H^\sigma)} + \|(P_2 - P_1)v^2\|_{L^1([0,T], H^\sigma)} \\ &\lesssim \|\underline{u}_1 - \underline{u}_2\|_{L^\infty([0,T], H^s)} \|v_0\|_{\dot{H}^{\sigma+3/2}}, \end{aligned}$$

where we have used Lemma 3.5.13, Lemma 3.5.12 and Lemma 3.5.17. □

Lemma 3.5.19. *Suppose $s > 2 + d/2$, then for ε_0 sufficiently small,*

$$\|R_1 - R_2\|_{\mathcal{L}(\dot{H}^s, \dot{H}^{s-3/2})} \lesssim \|\underline{u}_1 - \underline{u}_2\|_{H^{s-3/2}}.$$

Démonstration. The main difficulty is to estimate the $R_G(\eta_1) - R_G(\eta_2)$, as by the definition of R , the other terms in R can easily be estimated by the estimates of paradifferential calculus. By Lemma 6.8 of [6],

$$\|d_\eta(R_G(\eta)\psi) \cdot \delta\eta\|_{H^{s-1}} \lesssim \|\delta\eta\|_{H^{s-1}}$$

for $(\psi, \eta) \in H^s(\mathbb{T}^d) \times H^{s+1/2}(\mathbb{T}^d)$ and $\delta\eta \in H^{s+1/2}(\mathbb{T}^d)$, from which

$$\|R_G(\eta_1)\psi - R_G(\eta_2)\psi\|_{H^{s-1}} \lesssim \|\eta_1 - \eta_2\|_{H^{s-1}} \|\nabla_x \psi\|_{H^{s-1}} \lesssim \|\underline{u}_1 - \underline{u}_2\|_{H^{s-3/2}} \|\nabla_x \psi\|_{H^{s-1}}.$$

□

Lemma 3.5.20. *Suppose that $s > 2 + d/2$, then for ε_0 sufficiently small,*

$$\|\tilde{\mathcal{R}}_1 - \tilde{\mathcal{R}}_2\|_{\mathcal{L}(L^2([0,T], \dot{H}^s), \dot{H}^{s-3/2})} \lesssim \|\underline{u}_1 - \underline{u}_2\|_{L^\infty([0,T], H^{s-3/2})}.$$

Démonstration. Let $G \in L^2([0, T], \dot{H}^s(\mathbb{T}^d))$, and let $v^i \in C([0, T], \dot{H}^s(\mathbb{T}^d))$ for $i = 1, 2$ be solutions to the equations

$$(\partial_t + P_i + R_i)v^i = G, \quad v^i(T) = 0.$$

Then $w = v^1 - v^2$ satisfies

$$(\partial_t + P_1 + R_1)w = (P_2 - P_1)v^2 + (R_2 - R_1)v^2, \quad w(T) = 0,$$

whence the estimate, by Lemma 3.5.13, Lemma 3.5.19 and Proposition 3.10.4,

$$\begin{aligned} \|(\tilde{\mathcal{R}}_1 - \tilde{\mathcal{R}}_2)G\|_{H^{s-3/2}} &= \|w(0)\|_{H^{s-3/2}} \\ &\lesssim \|(P_2 - P_1)v^2\|_{L^1([0,T], H^{s-3/2})} + \|(R_2 - R_1)v^2\|_{L^1([0,T], H^{s-3/2})} \\ &\lesssim \|\underline{u}_1 - \underline{u}_2\|_{L^\infty([0,T], H^{s-3/2})} \|v^2\|_{L^1([0,T], H^s)} \\ &\lesssim \|\underline{u}_1 - \underline{u}_2\|_{L^\infty([0,T], H^{s-3/2})} \|G\|_{L^2([0,T], H^s)}. \end{aligned}$$

□

Lemma 3.5.21. *Suppose $s > 2 + d/2$, then for ε_0 sufficiently small,*

$$\|B(\eta_1) - B(\eta_2)\|_{\mathcal{L}(\dot{H}^s, L^\infty)} \lesssim \|\underline{u}_1 - \underline{u}_2\|_{H^{s-3/2}}.$$

Démonstration. This is merely a consequence of the contraction property for the Dirichlet–Neumann operator, see Theorem 5.2 of [8],

$$\|(G(\eta_1) - G(\eta_2))\psi\|_{H^{s-2}} \lesssim \|\eta_1 - \eta_2\|_{H^{s-1}} \|\nabla_x \psi\|_{H^{s-1}}.$$

□

Lemma 3.5.22. *Suppose $s > 2 + d/2$, then for ε_0 sufficiently small,*

$$\|\beta_1 - \beta_2\|_{\mathcal{L}(\dot{H}^s, \dot{H}^{s-1})} \lesssim \|\underline{u}_1 - \underline{u}_2\|_{H^{s-3/2}}.$$

Démonstration. It suffices to write

$$\begin{aligned} \beta_1 F - \beta_2 F &= T_{q_1 - q_2} T_{B(\eta_1)F} T_{p_1}^{-1} \text{Im} \underline{u}_1 + T_{q_2} T_{(B(\eta_1) - B(\eta_2))F} T_{p_1}^{-1} \text{Im} \underline{u}_1 \\ &\quad + T_{q_2} T_{B(\eta_2)F} (T_{p_1}^{-1} - T_{p_2}^{-1}) \text{Im} \underline{u}_1 + T_{q_2} T_{B(\eta_2)F} T_{p_2}^{-1} \text{Im}(\underline{u}_1 - \underline{u}_2), \end{aligned}$$

and conclude by Theorem 3.9.3, Lemma 3.5.11 and Lemma 3.5.21. \square

Lemma 3.5.23. *Suppose that s is sufficiently large, then for ε_0 sufficiently small,*

$$\|\mathcal{E}_1 - \mathcal{E}_2\|_{\mathcal{L}(\dot{H}^s, \dot{H}^{s-3/2})} \lesssim \|\underline{u}_1 - \underline{u}_2\|_{L^\infty([0, T], H^{s-3/2})}.$$

Démonstration. We use the identity,

$$\begin{aligned} \mathcal{E}_1 - \mathcal{E}_2 &= (\tilde{\mathcal{R}}_1 - \tilde{\mathcal{R}}_2)(-R_1(\mathcal{S}_\Theta)_1 + \beta_1 \Theta_1|_{\dot{H}^s}) \\ &\quad + \tilde{\mathcal{R}}_2(R_2 - R_1)(\mathcal{S}_\Theta)_1 + \tilde{\mathcal{R}}_2 R_2((\mathcal{S}_\Theta)_2 - (\mathcal{S}_\Theta)_1) \\ &\quad + \tilde{\mathcal{R}}_2(\beta_1 - \beta_2) \Theta_1|_{\dot{H}^s} + \tilde{\mathcal{R}}_2 \beta_2 (\Theta_1|_{\dot{H}^s} - \Theta_2|_{\dot{H}^s}). \end{aligned}$$

And conclude by Lemma 3.5.20, Lemma 3.5.19, Lemma 3.5.18, Lemma 3.5.22 and Lemma 3.5.17. \square

Proposition 3.5.24. *Suppose that s is sufficiently large, then for ε_0 sufficiently small,*

$$\|\Phi_1 - \Phi_2\|_{\mathcal{L}(\dot{H}^s, C([0, T], \dot{H}^{s-3/2}))} \lesssim \|\underline{u}_1 - \underline{u}_2\|_{L^\infty([0, T], H^{s-3/2})}.$$

Démonstration. We use the identity,

$$\Phi_1 - \Phi_2 = (\Theta_1|_{\dot{H}^s} - \Theta_2|_{\dot{H}^s})(\text{Id} + \mathcal{E}_1)^{-1} + \Theta_2|_{\dot{H}^s}(\text{Id} + \mathcal{E}_1)^{-1}(\mathcal{E}_2 - \mathcal{E}_1)(\text{Id} + \mathcal{E}_2)^{-1}.$$

And conclude by Lemma 3.5.17 and Lemma 3.5.23. \square

3.6 Iterative scheme

We adapt the iterative scheme of [5] to construct a solution for the nonlinear control problem (3.3.12) and thus close the proof of Theorem 3.2.2. We have to be careful about the constants and will no longer use the notation \lesssim in this section.

Suppose that s is sufficiently large, fix $0 < \varepsilon_0 < 1$ sufficiently small, such that the results of previous sections apply. Let $C > 10$ be a constant such that the following conditions are satisfied.

- For all $\underline{u} \in \mathcal{C}^{1,s}(T, \varepsilon_0)$, and all $\sigma \in \mathbb{R}$ with $s \geq \sigma \geq 0$,

$$\begin{aligned} &\|\Phi(\underline{u})\|_{\mathcal{L}(\dot{H}^s, C([0, T], \dot{H}^s))} + \|P(\underline{u})\|_{L^\infty([0, T], \mathcal{L}(\dot{H}^\sigma, \dot{H}^{\sigma-3/2}))} \\ &\quad + \|\mathcal{B}(\underline{u})\|_{L^\infty([0, T], \mathcal{L}(\dot{H}^\sigma, \dot{H}^\sigma))} + \|\beta(\underline{u})\|_{L^\infty([0, T], \mathcal{L}(\dot{H}^s, \dot{H}^{s+1/2}))} \leq C. \end{aligned}$$

- For $s \geq \mu \geq s - 3/2$, $G \in L^\infty([0, T], \dot{H}^\mu(\mathbb{T}^d))$, if u satisfies the equation

$$(\partial_t + P(\underline{u}) + R(\underline{u}))u = G, \quad u(0) = 0 \text{ or } u(T) = 0,$$

then we have the energy estimate (by Proposition 3.10.4),

$$(3.6.1) \quad \|u\|_{C([0, T], H^\mu)} + \|\partial_t u\|_{L^\infty([0, T], H^{\mu-3/2})} \leq C \|G\|_{L^\infty([0, T], H^\mu)}.$$

— In all the lemmas and propositions of Section 3.5.3, the statements remain valid after replacing the relations l.h.s \lesssim r.h.s with l.h.s $\leq C \times$ r.h.s.

Now fix $K > C^{10}$, and let $u_0 \in \dot{H}^s(\mathbb{T}^d)$ be such that

$$\|u_0\|_{H^s} < \varepsilon_0/K.$$

We will define a sequence of functions $\{u^n, F^n\}_{n \geq 0}$ by inductively solving a sequence a linear control problems. Once u^n is defined, for any operator $\mathcal{L}(u^n)$ that depends on u^n , we denote for simplicity $\mathcal{L}_n = \mathcal{L}(u^n)$. The induction proceeds as follows. Let $u^0 = F^0 \equiv 0$, and for $n \geq 0$, let

$$F^{n+1} = \Phi_n(u_0) \in C([0, T], \dot{H}^s(\mathbb{T}^d)),$$

and let $u^{n+1} \in C([0, T], \dot{H}^s(\mathbb{T}^d))$ be the solution to the equation

$$(3.6.2) \quad (\partial_t + P_n + R_n)u^{n+1} = (\mathcal{B}_n + \beta_n)F^{n+1}, \quad u^{n+1}(0) = u_0, \quad u^{n+1}(T) = 0.$$

In order for F^{n+1} to be well defined, we have to verify that $u^n \in \mathcal{C}^{1,s}(T, \varepsilon_0)$, this is justified by the following lemma.

Lemma 3.6.1. *The sequence of functions $\{u^n, F^n\}_{n \geq 0}$, formally defined as above, is well defined, and satisfies furthermore the following estimates, for all $n \geq 0$,*

$$\begin{aligned} u^n &\in \mathcal{C}^{1,s}(T, \varepsilon_0), & u^{n+1} - u^n &\in \mathcal{C}^{1,s-3/2}(T, \varepsilon_0^{n+1}), \\ F^n &\in \mathcal{C}^{0,s}(T, \varepsilon_0), & F^{n+1} - F^n &\in \mathcal{C}^{0,s-3/2}(T, \varepsilon_0^{n+1}). \end{aligned}$$

Démonstration. In the following estimates, we keep in mind that $C^{10}K^{-1} < 1$. For $n = 0$, the estimate for u^0 and F^0 are clearly satisfied. As for the differences, we use $F^1 = \Theta_0(u_0)$, Proposition 3.5.8, and the energy estimate (3.6.1)

$$\begin{aligned} \|F^1 - F^0\|_{C([0,T], H^{s-3/2})} &= \|F^1\|_{C([0,T], H^{s-3/2})} \leq C\|u_0\|_{H^s} \leq CK^{-1}\varepsilon_0. \\ \|u^1 - u^0\|_{C([0,T], H^{s-3/2})} &= \|u^1\|_{C([0,T], H^{s-3/2})} \leq C\|F^1\|_{C([0,T], H^{s-3/2})} \leq C^2K^{-1}\varepsilon_0. \end{aligned}$$

Then by the equation and (3.6.1),

$$\|\partial_t(u^1 - u^0)\|_{L^\infty([0,T], H^{s-3})} = \|\partial_t u^1\|_{L^\infty([0,T], H^{s-3})} \leq C^2\|F^1\|_{C([0,T], H^{s-3/2})} \leq C^3K^{-1}\varepsilon_0.$$

Suppose by now the estimates are proven for n . By Proposition 3.5.8,

$$\|F^{n+1}\|_{C([0,T], H^s)} \leq C\|u_0\|_{H^s} \leq CK^{-1}\varepsilon_0,$$

and then by (3.6.1),

$$(3.6.3) \quad \|u^{n+1}\|_{C([0,T], H^s)} + \|\partial_t u^{n+1}\|_{L^\infty([0,T], H^{s-3/2})} \leq C^2\|F^{n+1}\|_{C([0,T], H^s)} \leq C^3K^{-1}\varepsilon_0.$$

For the difference, we use Proposition 3.5.24,

$$\|F^{n+1} - F^n\|_{C([0,T], H^{s-3/2})} \leq C\|u^n - u^{n-1}\|_{C([0,T], H^{s-3/2})}\|u_0\|_{H^s} \leq CK^{-1}\varepsilon_0^{n+1}.$$

Observe that $(\delta u)_n := u^{n+1} - u^n$ satisfies the equation

$$\begin{aligned} (\partial_t + P_n + R_n)(\delta u)_n &= -(P_n - P_{n-1})u^n - (R_n - R_{n-1})u^n \\ &\quad + (\mathcal{B}_n F^{n+1} - \mathcal{B}_{n-1} F^n) + (\beta_n F^{n+1} - \beta_{n-1} F^n), \end{aligned}$$

with $(\delta u)_n(0) = (\delta u)_n(T) = 0$. By (3.6.1),

$$\begin{aligned} & \|u^{n+1} - u^n\|_{C([0,T],H^{s-3/2})} + \|\partial_t(u^{n+1} - u^n)\|_{L^\infty([0,T],H^{s-3})} \\ & \leq C\|(P_n - P_{n-1})u^n\|_{L^\infty([0,T],H^{s-3/2})} + C\|(R_n - R_{n-1})u^n\|_{L^\infty([0,T],H^{s-3/2})} \\ & \quad + C\|\mathcal{B}_n F^{n+1} - \mathcal{B}_{n-1} F^n\|_{L^\infty([0,T],H^{s-3/2})} + C\|\beta_n F^{n+1} - \beta_{n-1} F^n\|_{L^\infty([0,T],H^{s-3/2})}, \end{aligned}$$

where, by Lemma 3.5.13, and (3.6.3),

$$\begin{aligned} \|(P_n - P_{n-1})u^n\|_{L^\infty([0,T],H^{s-3/2})} & \leq C\|u^n - u^{n-1}\|_{L^\infty([0,T],H^{s-3/2})}\|u^n\|_{L^\infty([0,T],H^s)} \\ & \leq C \times \varepsilon_0^n \times C^3 K^{-1} \varepsilon_0 \leq C^4 K^{-1} \varepsilon_0^{n+1}. \end{aligned}$$

The same estimate, using Lemma 3.5.19, gives

$$\begin{aligned} \|(R_n - R_{n-1})u^n\|_{L^\infty([0,T],H^{s-3/2})} & \leq C\|u^n - u^{n-1}\|_{L^\infty([0,T],H^{s-3/2})}\|u^n\|_{L^\infty([0,T],H^s)} \\ & \leq C \times \varepsilon_0^n \times C^3 K^{-1} \varepsilon_0 \leq C^4 K^{-1} \varepsilon_0^{n+1}. \end{aligned}$$

Similarly, by Lemma 3.5.12 and Lemma 3.5.22, and the triangular inequality, we show that

$$\|\mathcal{B}_n F^{n+1} - \mathcal{B}_{n-1} F^n\|_{L^\infty([0,T],H^{s-3/2})} + \|\beta_n F^{n+1} - \beta_{n-1} F^n\|_{L^\infty([0,T],H^{s-3/2})} \leq C^3 K^{-1} \varepsilon_0^{n+1}.$$

In summary, we have

$$\|u^{n+1} - u^n\|_{C([0,T],H^{s-3/2})} + \|\partial_t(u^{n+1} - u^n)\|_{L^\infty([0,T],H^{s-3})} \leq C^6 K^{-1} \varepsilon_0^{n+1},$$

which closes the proof. \square

Corollary 3.6.2. $\{u^n, F^n\}_{n \geq 0}$ is a Cauchy sequence in

$$C([0, T], \dot{H}^{s-3/2}(\mathbb{T}^d)) \cap W^{1,\infty}([0, T], \dot{H}^{s-3}(\mathbb{T}^d)) \times C([0, T], \dot{H}^{s-3/2}(\mathbb{T}^d)),$$

whose limit will be denoted by

$$(u, F) = \lim_{n \rightarrow \infty} (u^n, F^n) \in \mathcal{C}^{1,s-3/2}(T, \varepsilon_0) \times \mathcal{C}^{0,s-3/2}(T, \varepsilon_0).$$

Corollary 3.6.3. The following convergence holds,

- (1) $P(u^n)u^{n+1} \rightarrow P(u)u$, strongly in $C([0, T], \dot{H}^{s-3})$;
- (2) $R(u^n)u^{n+1} \rightarrow R(u)u$, strongly in $C([0, T], \dot{H}^{s-3})$, weakly in $L^2([0, T], H^s)$;
- (3) $\mathcal{B}(u^n)F^{n+1} \rightarrow \mathcal{B}(u)F$, strongly in $C([0, T], \dot{H}^{s-3/2})$, weakly in $L^2([0, T], H^s)$;
- (4) $\beta(u^n)F^{n+1} \rightarrow \beta(u)F$, strongly in $C([0, T], \dot{H}^{s-3/2})$, weakly in $L^2([0, T], H^s)$.

In particular, (u, F) satisfies the equation

$$(3.6.4) \quad (\partial_t + P(u) + R(u))u = (\mathcal{B}(u) + \beta(u))F, \quad u(0) = u_0, \quad u(T) = 0,$$

in sense of distribution. Moreover,

$$\|R(u)u\|_{L^2([0,T],H^s)} + \|\mathcal{B}(u)F\|_{L^2([0,T],H^s)} + \|\beta(u)F\|_{L^2([0,T],H^s)} \lesssim \varepsilon_0.$$

Démonstration. The convergences follow from our construction, Corollary 3.6.2, the triangular inequality, and Lemma 3.5.13, Lemma 3.5.19, Lemma 3.5.12, Lemma 3.5.22 and (3.3.15). Therefore we pass to the limit $n \rightarrow \infty$ in (3.6.2), and obtain (3.6.4). The last estimate comes from the weak convergence. \square

Now we rewrite (3.6.4) as

$$(3.6.5) \quad (\partial_t + P(\underline{u}))u = G, \quad u(0) = u_0, \quad u(T) = 0,$$

where $\underline{u} = u$, and

$$G = -R(u)u + \mathcal{B}(u)F + \beta(u)F$$

with $\|G\|_{L^2([0,T],H^s)} \lesssim \varepsilon_0$. By Proposition 3.10.4, the equation (3.6.5) admits a unique solution in $C([0, T], \dot{H}^s(\mathbb{T}^d))$. Therefore, $u \in C([0, T], \dot{H}^s(\mathbb{T}^d))$ with the energy estimate

$$\|u\|_{C([0,T],H^s)} \lesssim \|u_0\|_{H^s} + \|G\|_{L^2([0,T],H^s)} \lesssim \varepsilon_0.$$

That is, $u \in \mathcal{C}^{0,s}(T, C'\varepsilon_0)$ for some $C' > 0$. Therefore, for ε_0 sufficiently small, by (3.3.15), $G \in L^\infty([0, T], \dot{H}^{s-3/2}(\mathbb{T}^d))$, with $\|G\|_{L^\infty([0,T],H^{s-3/2})} \lesssim \varepsilon_0$. This implies that

$$\|\partial_t u\|_{L^\infty([0,T],H^{s-3/2})} \lesssim \|u_0\|_{H^s} + \|G\|_{L^\infty([0,T],H^{s-3/2})} \lesssim \varepsilon_0.$$

Consequently $u \in \mathcal{C}^{1,s}(T, C''\varepsilon_0)$ for some $C'' > 0$. When ε_0 is sufficiently small, $\Phi(u)$ is well defined, and as $n \rightarrow \infty$,

$$\|\Phi(u) - \Phi(u^n)\|_{\mathcal{L}(\dot{H}^s, C([0,T], \dot{H}^{s-3/2}))} \lesssim \|u - u^n\|_{C([0,T],H^{s-3/2})} \rightarrow 0.$$

Therefore in the sense of distribution,

$$F = \lim_{n \rightarrow \infty} F^n = \lim_{n \rightarrow \infty} \Phi(u^n)u_0 = \Phi(u)u_0 \in C([0, T], \dot{H}^s(\mathbb{T}^d)).$$

This finishes the proof of Theorem 3.2.2.

3.7 Theorem 3.2.2 implies Theorem 3.1.2

Now we deduce Theorem 3.1.2 from Theorem 3.2.2. Observe that the null controllability holds for the time reversed equation (that is, the equation obtained by the change of variable $t \mapsto -t$) of (3.2.4) as well, with the same proof. Let (η_i, ψ_i) ($i = 0, 1$) satisfy the hypotheses of Theorem 3.1.2, and let $u_i = u(\eta_i, \psi_i) \in \dot{H}^s(\mathbb{T}^d)$ be defined by (3.2.2). Then $\|u_i\|_{H^s} \lesssim \varepsilon_0$, and for ε_0 sufficiently small, there exist $\dot{F}^i \in C([0, T/2], \dot{H}^s(\mathbb{T}^d))$ by Theorem 3.2.2, such that \dot{F}^0 sends initial data u_0 at time $t = 0$ to final data 0 at time $t = T/2$ by (3.2.4), while \dot{F}^1 sends initial data u_1 at time $t = 0$ to final data 0 at time $t = T/2$ by the time reversed equation of (3.2.4). Moreover, the estimates (3.2.6) are verified by \dot{F}^i and the corresponding solutions $u^i \in C([0, T/2], \dot{H}^s(\mathbb{T}^d))$. Now define $u \in C([0, T], \dot{H}^s(\mathbb{T}^d))$ and $\dot{F} \in C([0, T], \dot{H}^s(\mathbb{T}^d))$ by

$$u(t) = \begin{cases} u^0(t), & t \in [0, T/2], \\ u^1(T-t), & t \in (T/2, T]. \end{cases} \quad \dot{F}(t) = \begin{cases} \chi_{T/2}(t)\dot{F}^0(t), & t \in [0, T/2], \\ \chi_{T/2}(T-t)\dot{F}^1(T-t), & t \in (T/2, T]. \end{cases}$$

Indeed \dot{F} is continue in time, for the truncation functions vanish near $t = T/2$. Then u satisfies (3.2.4) (with F replaced by \dot{F}), and $u(0) = u_0$, $u(T) = u_1$. By Proposition 3.3.12, we can find

$$\dot{\eta} = \dot{\eta}(u) \in C([0, T], \dot{H}^{s+1/2}(\mathbb{T}^d)) \cap W^{1,\infty}([0, T], \dot{H}^{s-1}(\mathbb{T}^d)),$$

$$\dot{\psi} = \dot{\psi}(u) \in C([0, T], \dot{H}^s(\mathbb{T}^d)) \cap W^{1, \infty}([0, T], \dot{H}^{s-3/2}(\mathbb{T}^d)),$$

such that $u = T_{q(\dot{\eta})}(\dot{\psi} - T_{B(\dot{\eta})\dot{\psi}}\dot{\eta}) - iT_{p(\dot{\eta})}\dot{\eta}$ by (3.3.9), and

$$\dot{\eta}(0) = \eta_0, \quad \dot{\psi}(0) = \pi(D_x)\psi_0, \quad \dot{\eta}(T) = \eta_1, \quad \dot{\psi}(T) = \pi(D_x)\psi_1.$$

Next we look for (η, ψ, F) by adding zero frequencies to $(\dot{\eta}, \dot{\psi}, \dot{F})$ in such way that (η, ψ) satisfies (3.1.1), as well as the initial and final conditions, with the exterior pressure being

$$P_{\text{ext}}(t, x) = \varphi_\omega(x) \text{Re } F(t, x).$$

More precisely, we look for (η, ψ, F) of the following form,

$$(3.7.1) \quad \eta(t, x) = \dot{\eta}(t, x), \quad \psi(t, x) = \dot{\psi}(t, x) + \frac{1}{(2\pi)^d} \alpha(t), \quad F(t, x) = \dot{F}(t, x) + c_0,$$

where $c_0 \in \mathbb{R}$ is a constant, and α is a C^1 function of t . By reversing the parilinearization process, $\partial_t \dot{\eta} = G(\dot{\eta})\dot{\psi}$, hence $\partial_t \eta = G(\eta)\psi$. In order for ψ to meet the initial and final data, α should satisfy the boundary conditions,

$$\alpha(0) = \int_{\mathbb{T}^d} \psi_0(x) \, dx, \quad \alpha(T) = \int_{\mathbb{T}^d} \psi_1(x) \, dx.$$

Plugging (3.7.1) into (3.1.1), and integrating it over \mathbb{T}^d , we obtain an ordinary differential equation for α ,

$$\frac{d}{dt} \alpha(t) + \int_{\mathbb{T}^d} \left(\frac{1}{2} |\nabla_x \dot{\psi}|^2 - \frac{1}{2} \frac{(\nabla_x \dot{\eta} \cdot \nabla_x \dot{\psi} + G(\dot{\eta})\dot{\psi})^2}{1 + |\nabla_x \dot{\eta}|^2} \right) dx = \int_{\mathbb{T}^d} \varphi_\omega \text{Re}(\dot{F}(t, x) + c_0) \, dx$$

Solving this equation by integrating it over $[0, t]$ and using the initial condition for $\alpha(0)$,

$$(3.7.2) \quad \begin{aligned} \alpha(t) = & \int_{\mathbb{T}^d} \psi_0 \, dx + \int_0^t \int_{\mathbb{T}^d} \varphi_\omega \text{Re}(\dot{F} + c_0) \, dx \, dt \\ & + \int_0^t \int_{\mathbb{T}^d} \left(\frac{1}{2} \frac{(\nabla_x \dot{\eta} \cdot \nabla_x \dot{\psi} + G(\dot{\eta})\dot{\psi})^2}{1 + |\nabla_x \dot{\eta}|^2} - \frac{1}{2} |\nabla_x \dot{\psi}|^2 \right) dx \, dt. \end{aligned}$$

Observe that $c_0 \mapsto \alpha(T)$ is an affine function of c_0 , with the coefficient of c_0 being $T \times \int_{\mathbb{T}^d} \varphi_\omega(x) \, dx \neq 0$, so there exists a unique $c_0 \in \mathbb{R}$ such that $\alpha(T) = \int_{\mathbb{T}^d} \psi_1(x) \, dx$.

Remark 3.7.1. In the solution (3.7.2) of α the integrals for $g\eta$ and $H(\eta)$ do not appear because they have no zero frequencies. The former is due to our assumption, while the latter is by the divergence theorem. If we have infinite depth, that is, $b = \infty$, then

$$\int_{\mathbb{T}^d} \left(\frac{1}{2} \frac{(\nabla_x \dot{\eta} \cdot \nabla_x \dot{\psi} + G(\dot{\eta})\dot{\psi})^2}{1 + |\nabla_x \dot{\eta}|^2} - \frac{1}{2} |\nabla_x \dot{\psi}|^2 \right) dx = 0.$$

This can be proven by a direct computation using Green's identity, and is related to a conserved quantity of (3.1.1) by its Hamiltonian structure. For more on this subject, see Benjamin–Olver [27].

3.8 Appendix : necessity of GCC

We prove that the geometric control condition is necessary for the controllability of the three dimensional water wave equation (that is $d = 2$) with infinite depth (that is $b = \infty$) linearized around the flat surface (that is $\eta = 0$). We believe that similar arguments are suffice to prove the same results in arbitrary dimensions and for finite depth, however, we do not attempt to generalize the result in this direction for the sake of simplicity.

Now this linearized equation is a fractional Schrödinger equation

$$\partial_t u + i|D|^{3/2}u = T_1 \text{Re}\varphi_\omega F.$$

We will consider more generally the following control problem in $L^2(\mathbb{T}^2)$,

$$(3.8.1) \quad \partial_t u + i|D|^\alpha u = B\varphi F, \quad 1 \leq \alpha \leq 2.$$

Here $\varphi \in C^\infty(\mathbb{T}^2)$ and B is a bounded operator on $L^2(\mathbb{T}^2)$. If $\alpha = 1$, we have the (half) wave equation. When $B = \text{Id}$, it is exactly controllable, if and only if the geometric control condition is satisfied, see [24, 43]. If $\alpha = 2$, we have the Schrödinger equation. When $B = \text{Id}$, it is always exactly controllable (on tori) whether or not under the geometric control condition, see [88, 109, 18, 48, 36]. We are now in the middle of the two typical cases $1 < \alpha < 2$ where we show that the geometric control condition is necessary to exactly control (3.8.1) on \mathbb{T}^2 .

Definition 3.8.1. We say that (3.8.1) is exactly controllable, if there exists $T > 0$, such that for all $u_0, u_1 \in L^2(\mathbb{T}^2)$, there exists $F \in C([0, T], L^2(\mathbb{T}^2))$, and a solution $u \in C([0, T], L^2(\mathbb{T}^2))$ to (3.8.1), satisfying $u(0) = u_0$, $u(T) = u_1$.

Proposition 3.8.2. Suppose that $1 < \alpha < 2$, and (3.8.1) is exactly controllable, then

$$\omega := \{z = (x, y) \in \mathbb{T}^2 : \varphi(z) \neq 0\}$$

satisfies the geometric control condition.

Démonstration. The idea is to prove by contradiction by using the following lemma due to Burq–Zworski [47], and Miller [149].

Lemma 3.8.3. The equation (3.8.1) is exactly controllable, if and only if, for some $C > 0$, for all $\lambda \in \mathbb{R}$, and for all $u \in C^\infty(\mathbb{T}^2)$,

$$(3.8.2) \quad C\|u\|_{L^2} \leq \|(|D_z|^\alpha - \lambda)u\|_{L^2} + \|B\varphi u\|_{L^2}.$$

We may assume that $\varphi \not\equiv 0$, so that $\omega \neq \emptyset$, for the case will be trivial otherwise. Suppose that ω does not satisfy the geometric control condition, we will show that (3.8.2) does not hold for any fixed $C > 0$. By hypothesis, modulo some necessary translation, there exists some $\gamma \in \mathbb{R}^2 \setminus \{0\}$ such that the geodesic $\Gamma_\gamma = \{\gamma t : t \in \mathbb{R}\}$ does not enter ω . Now that $\omega \neq \emptyset$, Γ_γ cannot be dense. So we may further more assume that $\gamma \in \mathbb{Z}^2$.

Consider Γ_γ as a Lie group acting on \mathbb{T}^2 , $\Gamma_\gamma \ni \gamma t : z \mapsto z + \gamma t$, which defines a quotient manifold $\kappa : \mathbb{T}^2 \rightarrow \mathbb{T}^2/\Gamma_\gamma$, $z \mapsto z + \Gamma_\gamma$. Choose $\delta > 0$ sufficiently small such that

$$\|\varphi\|_{L^\infty(\mathcal{N}_\delta)} < \frac{C}{2}(1 + \|B\|_{\mathcal{L}(L^2, L^2)})^{-1},$$

where $\mathcal{N}_\delta = \{z \in \mathbb{T}^2 : \text{dist}(z, \Gamma_\gamma) < \delta\}$. Observe that $\kappa(\mathcal{N}_\delta)$ is open (with respect to the canonical quotient topology) for $\kappa^{-1}(\kappa(\mathcal{N}_\delta)) = \mathcal{N}_\delta$ is open. Fix

$$0 \neq \psi \in C_c^\infty(\kappa(\mathcal{N}_\delta)) \subset C^\infty(\mathbb{T}^2/\Gamma_\gamma),$$

and set for $n \in \mathbb{N}$,

$$\chi = \psi \circ \kappa \in C_c^\infty(\mathcal{N}_\delta) \subset C^\infty(\mathbb{T}^2), \quad u^n(z) = e^{in\gamma \cdot z} \chi(z).$$

Expanding χ in Fourier series, we write $\chi(z) = \sum_{k \in \mathbb{Z}^2} c_k e^{ik \cdot z}$, and claim that $c_k = 0$ unless $k \in \gamma^\perp := \{\ell \in \mathbb{Z}^d : \ell \cdot \gamma = 0\}$. Indeed, if $k \notin \gamma^\perp$, then there exists $w = \gamma t \in \Gamma_\gamma$ such that, $k \cdot w \not\equiv 0$ modulo 2π . Observe that

$$\chi(z + w) = \psi(\kappa(z + w)) = \psi(\kappa(z)) = \chi(z),$$

we have

$$c_k = \frac{1}{4\pi^2} \int_{\mathbb{T}^2} \chi(z + w) e^{-ik \cdot z} dz = e^{ik \cdot w} \frac{1}{4\pi^2} \int_{\mathbb{T}^2} \chi(z) e^{-ik \cdot z} dz = e^{ik \cdot w} c_k,$$

which implies that $c_k = 0$. Therefore, $u^n(z) = \sum_{k \in \gamma^\perp} c_k e^{i(n\gamma+k) \cdot z}$, and

$$|D_z|^\alpha u^n = \sum_{k \in \gamma^\perp} c_k |n\gamma + k|^\alpha e^{i(n\gamma+k) \cdot z} = \sum_{k \in \gamma^\perp} c_k (n^2 |\gamma|^2 + |k|^2)^{\alpha/2} e^{i(n\gamma+k) \cdot z}.$$

Let $\lambda_n = n^\alpha |\gamma|^\alpha$, then

$$(|D_z|^\alpha - \lambda_n) u^n = \sum_{0 \neq k \in \gamma^\perp} c_k |k|^\alpha f\left(\frac{|k|}{n|\gamma|}\right) e^{i(n\gamma+k) \cdot z},$$

where

$$f(t) = \frac{(1+t^2)^{\alpha/2} - 1}{t^\alpha}.$$

By an integration by part, for any $N \geq 1$, $|k|^\alpha |c_k| \lesssim |k|^{-N}$. Observe that f is continuous on $]0, +\infty[$, with $\lim_{t \rightarrow +\infty} f(t) = 1$, and $f(t) = O(t^{2-\alpha})$ as $t \rightarrow 0^+$. We have therefore the estimate,

$$f\left(\frac{|k|}{n|\gamma|}\right) \lesssim \begin{cases} \frac{1}{n^{(2-\alpha)/2}}, & 0 < |k| \leq \sqrt{n}; \\ 1, & |k| > \sqrt{n}. \end{cases}$$

To conclude, we show that the sequence (u^n, λ_n) violates (3.8.2). Indeed, $\|u^n\|_{L^2} = \|\chi\|_{L^2}$, and

$$\|B\varphi u^n\|_{L^2} \leq \|B\|_{\mathcal{L}(L^2, L^2)} \|\varphi\|_{L^\infty(\mathcal{N}_\delta)} \|\chi\|_{L^2} \leq \frac{C}{2} \|\chi\|_{L^2},$$

and for any $N \geq 2$,

$$\|(|D_z|^\alpha - \lambda_n) u^n\|_{L^2}^2 \lesssim \frac{1}{n^{(2-\alpha)/2}} \sum_{|k| \leq \sqrt{n}} \frac{1}{|k|^N} + \sum_{|k| > \sqrt{n}} \frac{1}{|k|^N} = o(1), \quad \text{as } n \rightarrow \infty.$$

□

3.9 Appendix : paradifferential calculus

For results of this section, we refer to [146, 9].

3.9.1 Paradifferential operators

For $\infty \geq \rho \geq 0$, denote by $W^{\rho, \infty}(\mathbb{T}^d)$ the space of Hölderian functions of regularity ρ on \mathbb{T}^d .

Definition 3.9.1. For $m \in \mathbb{R}$, $\rho \geq 0$, let $\dot{\Gamma}_\rho^m(\mathbb{T}^d)$ denote the space of locally bounded functions $a(x, \xi)$ on $\mathbb{T}_x^d \times (\mathbb{R}_\xi^d \setminus 0)$, which are C^∞ with respect to $\xi \in \mathbb{R}^d \setminus 0$, such that for all $\alpha \in \mathbb{N}^d$ and $\xi \neq 0$, the function $x \mapsto \partial_\xi^\alpha a(x, \xi)$ belongs to $W^{\rho, \infty}(\mathbb{T}^d)$, and that for some constant C_α ,

$$\|\partial_\xi^\alpha a(\cdot, \xi)\|_{W^{\rho, \infty}} \leq C_\alpha \langle \xi \rangle^{m - |\alpha|}, \quad \forall |\xi| \geq \frac{1}{2},$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.

Define on $\dot{\Gamma}_\rho^m(\mathbb{T}^d)$ the semi-norms,

$$M_{\rho, n}^m(a) = \sup_{|\alpha| \leq n} \sup_{|\xi| \geq 1/2} \|\langle \xi \rangle^{|\alpha| - m} \partial_\xi^\alpha a(\cdot, \xi)\|_{W^{\rho, \infty}}.$$

Definition 3.9.2. $\chi = \chi(\theta, \eta)$ is called an admissible cutoff function, if

- (1) $\chi \in C^\infty(\mathbb{R}_\theta^d \times \mathbb{R}_\eta^d)$ is an even function, that is, $\chi(-\theta, -\eta) = \chi(\theta, \eta)$;
- (2) it satisfies the following spectral condition : for some $0 < \epsilon_1 < \epsilon_2 < 1/2$,

$$(3.9.1) \quad \begin{cases} \chi(\theta, \eta) = 1, & |\theta| \leq \epsilon_1 \langle \eta \rangle, \\ \chi(\theta, \eta) = 0, & |\theta| \geq \epsilon_2 \langle \eta \rangle; \end{cases}$$

- (3) for all $(\alpha, \beta) \in \mathbb{N}^d \times \mathbb{N}^d$, and some $C_{\alpha\beta} > 0$,

$$|\partial_\theta^\alpha \partial_\eta^\beta \chi(\theta, \eta)| \leq C_{\alpha\beta} \langle \eta \rangle^{-|\alpha| - |\beta|}.$$

Let χ be an admissible cutoff function, and let $\pi \in C^\infty(\mathbb{R}^d)$ be an even function such that $0 \leq \pi \leq 1$, $\pi(\xi) = 0$ for $|\xi| \leq 1/4$, and $\pi(\xi) = 1$ for $|\xi| \geq 3/4$. Now given a symbol $a \in \dot{\Gamma}_\rho^m(\mathbb{T}^d)$, the paradifferential operator T_a is formally defined by

$$(3.9.2) \quad \widehat{T_a u}(\xi) = (2\pi)^{-d} \sum_{\eta \in \mathbb{Z}^d} \chi(\xi - \eta, \eta) \hat{a}(\xi - \eta, \eta) \pi(\eta) \hat{u}(\eta),$$

where $\hat{a}(\theta, \eta) = (\mathcal{F}_{x \rightarrow \theta} a)(\theta, \eta) = \int_{\mathbb{T}^d} e^{-ix \cdot \theta} a(x, \eta) dx$. Alternatively, set

$$a^x(\cdot, \xi) = \chi(D_x, \xi) a(\cdot, \xi),$$

then by definition, $T_a u = a^x(x, D_x) \pi(D_x) u$.

Theorem 3.9.3. Let $a \in \dot{\Gamma}_0^m$, then T_a is of order m such that for all $s \in \mathbb{R}$, T_a defines a bounded operator from $H^{s+m}(\mathbb{T}^d)$ to $\dot{H}^s(\mathbb{T}^d)$, such that

$$(3.9.3) \quad \|T_a\|_{\mathcal{L}(H^{s+m}, \dot{H}^s)} \lesssim M_{0, d/2+1}^m(a).$$

In particular, $T_a = T_a \pi(D_x) = \pi(D_x) T_a$.

Démonstration. For the estimate we refer to [146]. It remains to show that for $u \in H^{s+m}(\mathbb{T}^d)$, $T_a u$ has no zero frequency, or equivalently, $\widehat{T_a u}(0) = 0$. Indeed, by definition

$$\widehat{T_a u}(0) = (2\pi)^{-d} \sum_{0 \neq \eta \in \mathbb{Z}^d} \chi(-\eta, \eta) \hat{a}(-\eta, \eta) \hat{u}(\eta) = 0,$$

since $\chi(-\eta, \eta) = 0$ for all $\eta \neq 0$ by (3.9.1). \square

Proposition 3.9.4. *For all $s \in \mathbb{R}$, $T_1 = \pi(D_x) = \text{Id}_{\dot{H}^s}$.*

Démonstration. Observe that $\hat{a}(\theta, \eta) = 1_{\theta=0}(\theta, \eta)$ if $a \equiv 1$, therefore by definition,

$$\widehat{T_1 u}(\xi) = \sum_{\eta \in \mathbb{Z}^d} \chi(\xi - \eta, \eta) 1_{\xi=\eta} \pi(\eta) \hat{u}(\eta) = \chi(0, \xi) \pi(\xi) \hat{u}(\xi) = \pi(\xi) \hat{u}(\xi),$$

since $\chi(0, \xi) = 1$ for all $\xi \in \mathbb{Z}^d$. \square

Lemma 3.9.5. *Let $a \in \dot{\Gamma}_\rho^m(\mathbb{T}^d)$, and $\alpha \in \mathbb{N}^d$, with $|\alpha| \leq \rho$, then $\partial_x^\alpha(a - a^x) \in \dot{\Gamma}_0^{m-\rho+|\alpha|}(\mathbb{T}^d)$ with estimates that for all $n \in \mathbb{N}$, $M_{0,n}^{m-\rho+|\alpha|}(\partial_x^\alpha(a - a^x)) \lesssim M_{\rho,n}^m(a)$.*

Démonstration. See [146]. \square

Proposition 3.9.6. *Let $a \in \dot{\Gamma}_\rho^m(\mathbb{T}^d)$ with $m \geq 0$ and $\rho > m + 1 + d/2$, then*

$$\|\text{Op}(a\pi) - T_a\|_{\mathcal{L}(L^2, L^2)} \lesssim M_{\rho, d/2+1}^m(a).$$

Démonstration. By the theorem Calderón–Vaillancourt, it suffices to show that

$$M_{d/2+1, d/2+1}^0(a - a^x) \lesssim M_{\rho, d/2+1}^m(a).$$

Indeed, for $|\alpha| \leq d/2 + 1$, by the previous lemma,

$$M_{0, d/2+1}^0(\partial_x^\alpha(a - a^x)) \lesssim M_{0, d/2+1}^{m-\rho+|\alpha|}(\partial_x^\alpha(a - a^x)) \lesssim M_{\rho, d/2+1}^m(a).$$

\square

Lemma 3.9.7. *Let $a \in S^m(\mathbb{T}^d)$ be in the Hörmander class. If it is either a real valued even function of ξ , or a pure imaginary valued odd function of ξ , then for $u \in C^\infty(\mathbb{T}^d, \mathbb{C})$,*

$$\text{Op}(a)\text{Re } u = \text{Re } \text{Op}(a)u, \quad T_a \text{Re } u = \text{Re } T_a u.$$

Démonstration. We first prove the case of pseudodifferential operators $\text{Op}(a)$. Let $\tilde{a}(x, \xi) = a(x, -\xi)$, then by our hypothesis $a = \tilde{a}$. Therefore, for any real valued function u ,

$$\text{Op}(a)u = \overline{\text{Op}(\tilde{a})\bar{u}} = \overline{\text{Op}(a)u},$$

which implies that $\text{Op}(a)u$ is real valued. Then for a complex function u ,

$$\text{Op}(a)u = \text{Op}(a)(\text{Re } u + i\text{Im } u) = \text{Op}(a)\text{Re } u + i\text{Op}(a)\text{Im } u.$$

We conclude that $\text{Re } \text{Op}(a)u = \text{Op}(a)\text{Re } u$, $\text{Im } \text{Op}(a)u = \text{Op}(a)\text{Im } u$.

As for the paradifferential case, $T_a = \text{Op}(a^x\pi)$ with $(a^x\pi)(x, \xi) = \chi(D_x, \xi)a(\cdot, \xi)$. Now that χ is even, by the pseudodifferential case, $\chi(D_x, \xi)$ commute with Re . This implies that $a^x\pi$ remains to be a real symbol, and an even function of ξ , or a pure imaginary symbol and an odd function of ξ . So the case of paradifferential operators follows. \square

3.9.2 Symbolic calculus

Theorem 3.9.8. *Let $a \in \dot{\Gamma}_\rho^m(\mathbb{T}^d)$, $b \in \dot{\Gamma}_\rho^{m'}(\mathbb{T}^d)$, with $m, m' \in \mathbb{R}$, $0 \leq \rho < \infty$. Set*

$$a \sharp b = \sum_{|\alpha| < \rho} \frac{1}{\alpha!} \partial_\xi^\alpha a D_x^\alpha b.$$

Then $T_a T_b - T_{a \sharp b}$ is of order $m + m' - \rho$, and

$$\|T_a T_b - T_{a \sharp b}\|_{\mathcal{L}(H^{s+m+m'-\rho}, H^s)} \lesssim M_{0,d/2+1+\rho}^m(a) M_{\rho,d/2+1}^{m'}(b) + M_{0,d/2+1+\rho}^{m'}(b) M_{\rho,d/2+1}^m(a).$$

Theorem 3.9.9. *Let $a \in \dot{\Gamma}_\rho^m(\mathbb{T}^d)$, with $m \in \mathbb{R}$ and $0 \leq \rho < \infty$. Set*

$$a^* = \sum_{|\alpha| < \rho} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \bar{a}.$$

Denote by T_a^ the formal adjoint of T_a , then $T_a^* - T_{a^*}$ is of order $m - \rho$, and*

$$\|T_a^* - T_{a^*}\|_{\mathcal{L}(H^{s+m-\rho}, H^s)} \lesssim M_{\rho,d/2+1+\rho}^m(a).$$

3.9.3 Paraproducts and parilinearization

Theorem 3.9.10. *Let $a \in H^\alpha(\mathbb{T}^d)$ and $b \in H^\beta(\mathbb{T}^d)$ with $\alpha > d/2$, $\beta > d/2$. Then*

(1) *$T_a T_b - T_{ab}$ is of order $-\rho$ with $\rho = \min\{\alpha, \beta\} - d/2$, that is, for $s \in \mathbb{R}$,*

$$(3.9.4) \quad \|T_a T_b - T_{ab}\|_{\mathcal{L}(H^{s-\rho}, H^s)} \lesssim \|a\|_{H^\alpha} \|b\|_{H^\beta};$$

(2) *$T_a^* - T_{\bar{a}}$ is of order $-\rho$ with $\rho = \alpha - d/2$, that is, for $s \in \mathbb{R}$,*

$$\|T_a^* - T_{\bar{a}}\|_{\mathcal{L}(H^{s-\rho}, H^s)} \lesssim \|a\|_{H^\alpha};$$

(3) *Define the bilinear form,*

$$(3.9.5) \quad R(a, b) = ab - T_a b - T_b a,$$

then $R(a, b) \in H^{\alpha+\beta-d/2}(\mathbb{T}^d)$,

$$\|R(a, b)\|_{H^{\alpha+\beta-d/2}} \lesssim \|a\|_{H^\alpha} \|b\|_{H^\beta};$$

(4) *Let $F \in C^\infty$ with $F(0) = 0$, then $F(a) = T_{F'(a)} a + R_F(a)$ with*

$$(3.9.6) \quad \|R_F(a)\|_{H^{2\alpha-d/2}} \lesssim C(\|a\|_{H^\alpha}) \|a\|_{H^\alpha}.$$

In particular,

$$(3.9.7) \quad \|F(a)\|_{H^\alpha} \leq C(\|a\|_{H^\alpha}) \|a\|_{H^\alpha}.$$

3.10 Appendix : some linear equations

Proposition 3.10.1. *Let $\underline{u} \in \mathcal{C}^{0,s}(T, \varepsilon_0)$ for s sufficiently large, $T > 0$, and ε_0 sufficiently small. Let $Q = Q(\underline{u})$ be defined by (3.4.8), and suppose*

$$R \in L^\infty([0, T], \mathcal{L}(\dot{L}^2, \dot{L}^2)) \cap C([0, T], \mathcal{L}(\dot{L}^2, \dot{H}^{-\mu}))$$

for some $\mu \geq 0$, $f \in L^1([0, T], \dot{L}^2(\mathbb{T}^d))$. Then the Cauchy problem

$$(3.10.1) \quad (\partial_t + iQ + R)u = f, \quad u(0) = u_0 \in \dot{L}^2(\mathbb{T}^d),$$

admits a unique solution $u \in C([0, T], \dot{L}^2(\mathbb{T}^d))$, which moreover satisfies the estimate

$$\|u\|_{C([0, T], H^s)} \lesssim \|u_0\|_{L^2} + \|f\|_{L^1([0, T], L^2)}$$

Démonstration. Let

$$j_\varepsilon = \exp(-\varepsilon\gamma^{(3/2)}) + \frac{1}{2}\partial_\xi \cdot D_x \exp(-\varepsilon\gamma^{(3/2)}),$$

and set $J_\varepsilon = \pi(D_x)\text{Op}(j_\varepsilon\pi)$. Consider the regularized Cauchy problem

$$(\partial_t + iQJ_\varepsilon + RJ_\varepsilon)u^\varepsilon = f, \quad u^\varepsilon(0) = u_0,$$

which admits a unique solution $u^\varepsilon \in C([0, T_\varepsilon], \dot{L}^2(\mathbb{T}^d))$ for some $T_\varepsilon > 0$, by Cauchy-Lipschitz theorem, as $QJ_\varepsilon, RJ_\varepsilon \in C([0, T], \mathcal{L}(\dot{L}^2, \dot{L}^2))$. Following a routine method, see for example [146], to prove the existence of a solution, on the whole interval $[0, T]$, it suffice to prove a uniform *a priori* bound for u^ε and its time derivative in the energy space over the time interval $[0, T]$.

By the choice of the symbol j_ε , we have

$$\| [Q, J_\varepsilon] \|_{\mathcal{L}(\dot{L}^2, \dot{L}^2)} \lesssim 1, \quad \| Q - Q^* \|_{\mathcal{L}(\dot{L}^2, \dot{L}^2)} \lesssim 1, \quad \| J_\varepsilon - J_\varepsilon^* \|_{\mathcal{L}(\dot{L}^2, \dot{H}^2)} \lesssim 1,$$

from which the a priori estimate, that for almost every $t \in [0, T]$,

$$(3.10.2) \quad \frac{d}{dt} \|u^\varepsilon(t)\|_{L^2}^2 \lesssim \|u^\varepsilon(t)\|_{L^2}^2 + (u^\varepsilon(t), f(t))_{L^2}.$$

Moreover, by Gronwall's inequality, we have

$$(3.10.3) \quad \|u^\varepsilon\|_{C([0, T], L^2)} \lesssim \|u_0\|_{L^2} + \|f\|_{L^1([0, T], L^2)}.$$

Plugging it into (3.10.2), we obtain

$$(3.10.4) \quad \left\| \frac{d}{dt} \|u^\varepsilon(t)\|_{L^2}^2 \right\|_{L^1([0, T])} \lesssim \|u_0\|_{L^2}^2 + \|f\|_{L^1([0, T], L^2)}^2.$$

The energy estimate (3.10.3), the hypothesis on R , and Arzela-Ascoli's theorem imply the existence of a weak solution

$$u \in L^2([0, T], \dot{L}^2(\mathbb{T}^d)) \cap C([0, T], \dot{H}^{-\mu}(\mathbb{T}^d))$$

to (3.10.1). Then (3.10.4) and Arzela-Ascoli's theorem again implies that $t \mapsto \|u\|_{L^2}^2$ is continuous in time. Therefore $u \in C([0, T], \dot{L}^2(\mathbb{T}^d))$. The energy estimate for u follows by Gronwall's inequality as in (3.10.3), and the uniqueness follows from the energy estimate. \square

Using the same method, we have the following corollary.

Corollary 3.10.2. *For $\vec{w}_0 \in \dot{L}^2(\mathbb{T}^d)$, there exists a unique solution to the Cauchy problem (3.4.12), $\vec{w} \in C([0, T], \dot{L}^2(\mathbb{T}^d))$ with $\vec{w}(0) = \vec{w}_0$.*

Corollary 3.10.3. *For $\vec{w}_{h,0} \in \dot{L}^2(\mathbb{T}^d)$, $f \in L^1([0, T]_s, \dot{L}^2(\mathbb{T}^d))$, there exists a unique solution to the Cauchy problem (3.4.31), $\vec{w}_h \in C([0, T]_s, \dot{L}^2(\mathbb{T}^d))$ with $\vec{w}_h(0) = \vec{w}_{h,0}$.*

Similar results hold for the paradifferential equations.

Proposition 3.10.4. *Suppose $s \geq s' \geq 0$, $\mu \geq 3 + d/2$, $T > 0$, $\underline{u} \in \mathcal{C}^{0,\mu}(T, \varepsilon_0)$ for some $\varepsilon_0 > 0$ sufficiently small. Let $P = P(\underline{u})$ be defined by (3.3.13),*

$$R \in L^\infty([0, T], \mathcal{L}(\dot{H}^s, \dot{H}^s)) \cap C([0, T], \mathcal{L}(\dot{H}^s, \dot{H}^{s'})),$$

and let $F \in L^1([0, T], \dot{H}^s(\mathbb{T}^d))$. Then for $u_0 \in \dot{H}^s(\mathbb{T}^d)$, the following Cauchy problem

$$(3.10.5) \quad (\partial_t + P + R)u = F, \quad u(0) = u_0,$$

admits a unique solution $u \in C([0, T], \dot{H}^s(\mathbb{T}^d))$, which moreover satisfies the estimate

$$(3.10.6) \quad \|u\|_{C([0, T], H^s)} \lesssim \|u_0\|_{H^s} + \|F\|_{L^1([0, T], H^s)}.$$

Démonstration. The proof is almost the same as above, but here we choose $J_\varepsilon = T_{j_\varepsilon}$ with j_ε defined as above, and use the following estimates

$$\|[P, J_\varepsilon]\|_{\mathcal{L}(\dot{H}^s, \dot{H}^s)} \lesssim 1, \quad \|P - P^*\|_{\mathcal{L}(\dot{H}^s, \dot{H}^s)} \lesssim 1, \quad \|J_\varepsilon - J_\varepsilon^*\|_{\mathcal{L}(\dot{H}^s, \dot{H}^{s+2})} \lesssim 1.$$

□

Corollary 3.10.5. *Let the Range operator \mathcal{R} and the solution operator \mathcal{S} be formally defined in Section 3.4.1, then for all $\mu \geq 0$,*

$$\mathcal{R} : L^2([0, T], \dot{H}^\mu(\mathbb{T}^d)) \rightarrow \dot{H}^\mu(\mathbb{T}^d), \quad \mathcal{S} : \dot{H}^\mu(\mathbb{T}^d) \rightarrow C([0, T], \dot{H}^\mu(\mathbb{T}^d)),$$

and satisfies the estimates

$$\|\mathcal{R}\|_{\mathcal{L}(L^2([0, T], \dot{H}^\mu), \dot{H}^\mu)} \lesssim 1, \quad \|\mathcal{S}\|_{\mathcal{L}(\dot{H}^\mu, C([0, T], \dot{H}^\mu))} \lesssim 1.$$

Chapitre 4

Propagation of singularities for gravity-capillary water waves

4.1 Introduction

We are interested in the propagation of singularities for the gravity-capillary water wave equation, which is a quasilinear dispersive equation to be defined later. We shall first revisit some classical results about propagation of singularities for simpler linear dispersive equations, and see how they lead to a more generalized definition of singularities that is adaptive for gravity-capillary water waves.

4.1.1 Half wave equation

By the classical definition, $x_0 \in \mathbb{R}^d$ is called a singularity of $u \in \mathcal{D}'(\mathbb{R}^d)$, if u is not C^∞ in any neighborhood of x_0 ; the singular support of u , denoted by $\text{sing supp } u$, is the set of all singularities of u . To study the propagation of $\text{sing supp } u$ when u solves some PDEs, the information given by $\text{sing supp } u$ alone is usually insufficient, as the direction of propagation for a singularity is not determined by its position, but rather by its “direction of oscillation”.

More precisely, in [98], Hörmander introduced the wavefront set $\text{WF}(u)$, lifting $\text{sing supp } u$ to the phase space $\mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$. By definition, $(x_0, \xi_0) \notin \text{WF}(u)$ if for some $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\varphi(x_0) \neq 0$, $\widehat{\varphi u}$ decays rapidly within some conical neighborhood of ξ_0 . If $(x_0, \xi_0) \in \text{WF}(u)$, then it is called a *microlocal singularity* of u ; we have $x_0 \in \text{sing supp } u$ and ξ_0 is called the *frequency* of this microlocal singularity. By Guillemin–Sternberg [87], $(x_0, \xi_0) \notin \text{WF}(u)$ if and only if for some $a \in C_c^\infty(\mathbb{R}^{2d})$ with $a(x_0, \xi_0) \neq 0$, $a(x, hD_x)u = \mathcal{O}(h^\infty)_{L^2}$, $0 < h < 1$. Therefore, $\text{WF}(u)$ describes the mass accumulation of u in the phase space.

A classical result says that for solutions to the half wave equation

$$\partial_t u + i\sqrt{-\Delta} u = 0,$$

singularities travel at finite speeds along geodesics.

Theorem 4.1.1 (Lax [129], Courant–Lax [56], Hörmander [98, 99], etc). *Let u solve the half wave equation with initial data $u_0 \in \mathcal{D}'(\mathbb{R}^d)$, and $t_0 \in \mathbb{R}$, then $(x_0, \xi_0) \in \text{WF}(u_0)$ if and only if $(x_0 + t_0 \xi_0 / |\xi_0|, \xi_0) \in \text{WF}(u(t_0))$, i.e., $\text{WF}(u)$ is propagated by the Hamiltonian flow of $|\xi|$.*

More generally, if u solves $\partial_t u + iPu = 0$ where P is a first order pseudodifferential operator admitting a real principal symbol p , then $\text{WF}(u)$ is propagated by the Hamiltonian flow of p .

As for semilinear wave equations, we refer to Bony [34], Lebeau [130], and the references therein, for classical results of propagation of singularities.

4.1.2 Schrödinger equation

For dispersive equations with an infinite speed of propagation, singularities can disappear and emerge. For example, if u solves the linear Schrödinger equation,

$$\partial_t u - i\Delta u = 0,$$

with $u(0) \in \mathcal{E}'(\mathbb{R}^d)$ (the space of compactly supported distributions), then $u(t) \in C^\infty(\mathbb{R}^d)$ for $t \neq 0$. Conversely, if $u(0, x) = e^{-i|x|^2}$ with a quadratic oscillation at infinity, then u develops a delta-function singularity when $t = 1$.

Early works, including Lascar [125] and Boutet-de-Monvel [37], proved that singularities travel at an infinite speed along geodesics, but did not give time-dependent information in terms of initial data. Wunsch [195] however, revealed the time-dependent propagation of singularities by tracking the transformation between singularities and quadratic oscillations, using the quadratic-scattering wavefront set, $\text{WF}_{\text{qsc}}(u)$. Similar results were later obtained, independently, by Nakamura [153], using the homogeneous wavefront set, $\text{HWF}(u)$. These two wavefront sets, $\text{WF}_{\text{qsc}}(u)$ and $\text{HWF}(u)$, were proven to be essentially equivalent by Ito [108]. We shall recall the definition of $\text{HWF}(u)$, because the similarity it shares with $\text{WF}(u)$ inspires us to define the quasi-homogeneous wavefront set that is adaptive to the gravity-capillary water wave equation.

By definition, $(x_0, \xi_0) \notin \text{HWF}(u)$ if for some $a \in C_c^\infty(\mathbb{R}^{2d})$ with $a(x_0, \xi_0) \neq 0$, $a(hx, hD_x)u = \mathcal{O}(h^\infty)_{L^2}$, $0 < h < 1$. Please allow us to call (x_0, ξ_0) a *homogeneous singularity* of u if it belongs to $\text{HWF}(u)$. Therefore, $\text{HWF}(u)$ describes the phase space distribution of u at the spatial infinity. Observe that the dispersion relation of the Schrödinger equation is $\omega = |\xi|^2$, so a wave packet near the frequency $\xi \sim h^{-1}$ travels at the group velocity $v = \frac{d\omega}{d\xi} = 2\xi \sim h^{-1}$. That explains Nakamura's choice of the homogeneously scaled quantization $a \mapsto a(hx, hD_x)$.

Theorem 4.1.2 (Nakamura [153]). *Let u solve the Schrödinger equation with initial data $u_0 \in L^2(\mathbb{R}^d)$, and let $t_0 \in \mathbb{R}$, then*

(S.1) $(x_0, \xi_0) \in \text{HWF}(u_0)$ if and only if $(x_0 + 2t_0\xi_0, \xi_0) \in \text{HWF}(u(t_0))$, i.e., $\text{HWF}(u)$ is propagated by the Hamiltonian flow of $|\xi|^2$.

(S.2) if $(x_0, \xi_0) \in \text{HWF}(u_0)$ with $\xi_0 \neq 0$, and $t_0 \neq 0$, then $(2t_0\xi_0, \xi_0) \in \text{HWF}(u(t_0))$.

In [153], Nakamura gave the credit of (S.1) to S. Doi, and extended Theorem 4.1.2 to asymptotically Euclidean manifolds. Theorem 4.1.2 states that, homogeneous singularities of u travel at a finite speed along geodesics. $(x_0, \xi_0) \in \text{HWF}(u_0)$ may develop singularities, but only when $x_0 + 2t\xi_0 = 0$. By [153], if $(0, \xi_0) \notin \text{HWF}(u)$, then $(x_0, \xi_0) \notin \text{WF}(u)$, $\forall x_0 \in \mathbb{R}^d$. Therefore, although we do not know where the newly developed singularities are, their frequencies must be ξ_0 . Moreover, $(x_0, \xi_0) \in \text{WF}(u_0)$ of u_0 instantaneously creates $(2t_0\xi_0, \xi_0) \in \text{HWF}(u(t_0))$ whenever $t_0 \neq 0$, due to the infinite speed of propagation. Conversely, if $(2t_0\xi_0, \xi_0) \notin \text{HWF}(u(t_0))$, then $\text{WF}(u_0) \cap \{\xi = \xi_0\} = \emptyset$. Particularly,

if $\text{HWF}(u(t_0)) = \emptyset$, then $\text{WF}(u_0) = \emptyset$. These microlocal smoothing effects, when extended to asymptotic Euclidean manifolds, generalize the pioneer work of Craig–Kappeler–Strauss [57]. We also refer to Robbiano–Zuily [165] for a microlocal analytic smoothing effect of linear Schrödinger equations, and Szeftel [181] for microlocal smoothing effects of semilinear Schrödinger equations.

4.1.3 Gravity-capillary water wave equation

The gravity-capillary water wave equation describes the evolution of inviscid, incompressible, and irrotational fluid with a free surface, in the presence of a gravitational field and the surface tension.

4.1.3.1 Eulerian formulation

Let $\eta = \eta(t, x)$, $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, be real valued, and let $0 < b < \infty$. Define

$$\Omega = \{-b < y < \eta(t, x)\},$$

which is a time-dependent domain in $\mathbb{R}^{d+1} = \mathbb{R}_x^d \times \mathbb{R}_y$. Then $\partial\Omega$ consists of a free surface

$$\Sigma = \{y = \eta(t, x)\}$$

and a flat bottom

$$\Gamma = \{y = -b\}.$$

Let $v : \Omega \rightarrow \mathbb{R}^d$ be the Eulerian vector field, $P : \Omega \rightarrow \mathbb{R}$ be the internal pressure, $g \in \mathbb{R}$ be the gravitational acceleration, and $\mathbf{e}_y = (0, \dots, 0, 1)$. Then

$$(4.1.1) \quad \nabla_{xy} \cdot v = 0, \quad \nabla_{xy} \times v = 0,$$

where $\nabla_{xy} = (\nabla, \partial_y)$, $\nabla = (\partial_{x_1}, \dots, \partial_{x_d})$; (v, P) satisfies the Euler equation,

$$(4.1.2) \quad \partial_t v + v \cdot \nabla_{xy} v + \nabla_{xy} P = -g \mathbf{e}_y.$$

Let $\mathbf{n} : \partial\Omega \rightarrow \mathbb{S}^d$ be the exterior unit normal vector field of $\partial\Omega$, then

$$(4.1.3) \quad \partial_t \eta = \sqrt{1 + |\nabla \eta|^2} v|_{\Sigma} \cdot \mathbf{n},$$

implying that fluid particles that were initially on Σ will stay on Σ ; and

$$(4.1.4) \quad v|_{\Gamma} \cdot \mathbf{n} = 0,$$

meaning that Γ is impenetrable. Finally, let $\kappa > 0$ be the constant of surface tension, and let

$$(4.1.5) \quad H(\eta) = \nabla \cdot \left(\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right)$$

be the mean curvature of the free surface, then $\kappa H(\eta)$ is the surface tension on Σ , which should be balanced out by $P|_{\Sigma}$, i.e.,

$$(4.1.6) \quad -P|_{\Sigma} = \kappa H(\eta).$$

The equations (4.1.1), (4.1.2), (4.1.3), (4.1.4), (4.1.5) and (4.1.6) give the Eulerian formulation of the system of gravity-capillary water waves.

4.1.3.2 Zakharov [199] / Craig–Sulem [60] formulation

By (4.1.1) and (4.1.4), there exists a real valued $\phi : \Omega \rightarrow \mathbb{R}$ such that

$$\nabla_{xy}\phi = v, \quad \Delta_{xy}\phi = 0, \quad \partial_y\phi|_{\Gamma} = 0.$$

Denote $\psi(t, x) = \phi(t, x, \eta(t, x))$, then

$$\sqrt{1 + |\nabla\eta|^2} v|_{\Sigma} \cdot \mathbf{n} = G(\eta)\psi,$$

where $G(\eta)$, called the Dirichlet–Neumann operator, is defined by

$$G(\eta)\psi(t, x) = \partial_y\phi(t, x, \eta(t, x)) - \nabla\eta(t, x) \cdot \nabla\phi(t, x, \eta(t, x)).$$

Letting $\kappa = 1$, the gravity-capillary water wave equation can be formulated in terms of (η, ψ) , which is called the Zakharov / Craig–Sulem formulation,

$$(4.1.7) \quad \begin{cases} \partial_t\eta - G(\eta)\psi = 0, \\ \partial_t\psi + g\eta - H(\eta) + \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2} \frac{(\nabla\eta \cdot \nabla\psi + G(\eta)\psi)^2}{1 + |\nabla\eta|^2} = 0. \end{cases}$$

The purpose of this paper is to present the propagation of singularities for solutions to (4.1.7), including a microlocal smoothing effect. To the best of our knowledge, these results are the first of this type for quasilinear dispersive equations.

4.1.3.3 Quasi-homogeneous wavefront set and model equation

Recall that the linearization of (4.1.7) at the stationary state $(\eta, \psi) = (0, 0)$ is

$$(4.1.8) \quad \begin{cases} \partial_t\eta - |D_x| \tanh(b|D_x|)\psi = 0, \\ \partial_t\psi + g\eta - \Delta\eta = 0. \end{cases}$$

Let us oversimplify (4.1.8) by setting $g = 0$, $b = \infty$, then $u := \psi - i\sqrt{|D_x|}\eta$ satisfies the fractional Schrödinger equation

$$\partial_t u + i|D_x|^{3/2}u = 0,$$

for which neither $\text{WF}(u)$ nor $\text{HWF}(u)$ seems to suitably define singularities. Let us consider the more general model equation

$$(4.1.9) \quad \partial_t u + i|D_x|^\gamma u = 0, \quad \gamma \geq 1,$$

which includes the half wave equation ($\gamma = 1$), the fractional Schrödinger equations ($1 < \gamma < 2$), the Schrödinger equation ($\gamma = 2$), the fourth order Schrödinger equation ($\gamma = 4$), etc., as special cases. A wave packet of (4.1.9) near the frequency $\xi \sim h^{-1}$ travels at the group velocity $v = \frac{d|\xi|^\gamma}{d\xi} = \gamma|\xi|^{\gamma-2}\xi \sim h^{-(\gamma-1)}$, so the quasi-homogeneously scaled quantization $a \mapsto a(h^{\gamma-1}x, hD_x)$ seems to be one of the rightful choice. By a more general scaling, we define the quasi-homogeneous wavefront set.

Definition 4.1.3. Let $u \in \mathcal{S}'$, $\delta \geq 0$, $\rho \geq 0$ with $\delta + \rho > 0$, and $\mu \in \mathbb{R} \cup \{\infty\}$. The quasi-homogeneous wavefront set $\text{WF}_{\delta,\rho}^\mu(u)$ is a subset of \mathbb{R}^{2d} such that $(x_0, \xi_0) \notin \text{WF}_{\delta,\rho}^\mu(u)$ if and only if for some $a \in C_c^\infty(\mathbb{R}^{2d})$ with $a(x_0, \xi_0) \neq 0$, $a(h^\delta x, h^\rho D_x)u = \mathcal{O}(h^\mu)_{L^2}$, $0 < h < 1$. Here,

$$a(h^\delta x, h^\rho D_x)u(x) = (2\pi)^{-d} \iint e^{i(x-y)\cdot\xi} a(h^\delta x, h^\rho \xi) u(y) dy d\xi.$$

If $(x_0, \xi_0) \in \text{WF}_{\delta,\rho}^\mu(u)$, we shall call it a quasi-homogeneous singularity of u , or more precisely, a (δ, ρ) -singularity of u (of order μ).

Clearly the quasi-homogeneous wavefront set generalizes $\text{WF}(u)$ and $\text{HWF}(u)$. It allows us to extend Theorem 4.1.1 and Theorem 4.1.2 to the model equation (4.1.9).

Theorem 4.1.4. Let u solve (4.1.9) with initial data $u_0 \in H^{-\infty}(\mathbb{R}^d)$, $\mu \in \mathbb{R} \cup \{\infty\}$ and $t_0 \in \mathbb{R}$.

(M.1) If $\rho\gamma = \delta + \rho$, $(x_0, \xi_0) \in \text{WF}_{\delta,\rho}^\mu(u_0) \setminus \{\xi = 0\}$, then

$$(x_0 + t_0\gamma|\xi_0|^{\gamma-2}\xi_0, \xi_0) \in \text{WF}_{\delta,\rho}^\mu(u(t_0)) \setminus \{\xi = 0\}.$$

(M.2) If $\gamma > 1$, $\rho\gamma > \delta + \rho$, $t_0 \neq 0$, $(x_0, \xi_0) \in \text{WF}_{\delta,\rho}^\mu(u_0) \setminus \{\xi = 0\}$, then

$$(t_0\gamma|\xi_0|^{\gamma-2}\xi_0, \xi_0) \in \text{WF}_{\rho(\gamma-1),\rho}^\mu(u(t_0)) \setminus \{\xi = 0\}.$$

Indeed, letting $(\delta, \rho, \gamma, \mu) = (0, 1, 1, \infty)$ or $(1, 1, 2, \infty)$ in (M.1), we recover respectively Theorem 4.1.1 and (S.1) of Theorem 4.1.2; letting $(\delta, \rho, \gamma, \mu) = (0, 1, 2, \infty)$ in (M.2), we recover (S.2) of Theorem 4.1.2.

4.1.3.4 Existence in weighted Sobolev spaces

Instead of the linearization at $(\eta, \psi) = (0, 0)$, if we parilinearize and symmetrize (4.1.7) as Alazard–Burq–Zuily [6], we obtain a fractional Schrödinger equation (with lower order terms) on Σ . The geometry of Σ is time dependent and is given by the solution itself, as (4.1.7) is quasilinear. We need this geometry to be asymptotically Euclidean to avoid the mess caused by the infinite speed of propagation, but the existence of such geometry is not cheap. We shall prove it by establishing the existence of asymptotically “flat” gravity-capillary water waves.

Definition 4.1.5. For $(\nu, k) \in \mathbb{R}^2$, H_k^μ consists of those $u \in \mathcal{S}'$ with

$$\|u\|_{H_k^\nu} := \|\langle x \rangle^k \langle D_x \rangle^\nu u\|_{L^2} < \infty.$$

Here and throughout this paper, $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$. Moreover, given $(\mu, m) \in \mathbb{R} \times \mathbb{N}$, denote

$$\mathcal{H}_m^\mu = \bigcap_{j=0}^m H_j^{\mu-j/2}.$$

Theorem 4.1.6. Let $d \geq 1$, $\mu > 3 + d/2$, $m \leq 2\mu - 6 - d$, and $(\eta_0, \psi_0) \in \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu$, then for some $T > 0$, there exists a unique solution

$$(\eta, \psi) \in C([-T, T], \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu)$$

to the Cauchy problem of (4.1.7) with initial data (η_0, ψ_0) .

For the Cauchy problem of water waves, we refer to the initial works of Kano–Nishida [114] and Yosihara [197, 198], the breakthroughs of Wu [192, 193] and Beyer–Günther [30] for the local well-posedness in Sobolev spaces with general initial data. Using the paradifferential calculus, Alazard–Burq–Zuily [6, 8, 9] proved the local well-posedness with low Sobolev regularity. We shall prove Theorem 4.1.6 by combining the analysis of [6] and a spatial dyadic decomposition. More precisely, we define the dyadic paradifferential operators

$$\mathcal{P}_a = \sum_{j \in \mathbb{N}} \psi_j T_{\psi_j a} \psi_j,$$

where (i) $\psi_j, \underline{\psi}_j \in C_c^\infty(\mathbb{R}^d)$ are supported in the dyadic annulus $C^{-1}2^j \leq |x| \leq C2^j$, (ii) $\psi_j \underline{\psi}_j = \psi_j$, (iii) $\{\psi_j\}_{j \in \mathbb{N}}$ is a partition of unity of \mathbb{R}^d , and (iv) $T_{\psi_j a}$ is the usual paradifferential operator of Bony [33], which we shall review in §4.4.1. We show that the dyadic paradifferential calculus naturally extends Bony’s parilinearization to weighted Sobolev spaces.

We do not attempt to lower μ to $> 2 + d/2$, as it was the case in [6]. The range of m is so chosen that $\mu - m/2 > 3 + d/2$, enabling us to parilinearize (4.1.7) in \mathcal{H}_m^μ . We should mention that the well-posedness of gravity water waves, i.e., without surface tension, in uniformly local weighted Sobolev spaces was obtained by Nguyen [157] using a periodic spatial decomposition from [9].

4.1.3.5 Propagation at spatial infinity

Our first main result concerns about the propagation of $(1/2, 1)$ -singularities at the spatial infinity, corresponding to (M.1) of Theorem 4.1.4.

Theorem 4.1.7. *Let $d \geq 1$, $\mu > 3 + d/2$, $3 \leq m \leq 2\mu - 6 - d$. Suppose that for some $T > 0$, $(\eta, \psi) \in C([-T, T], \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu)$ solves (4.1.7) with initial data (η_0, ψ_0) . Let*

$$(x_0, \xi_0) \in \text{WF}_{1/2,1}^{\mu+1/2+\sigma}(\eta_0) \cup \text{WF}_{1/2,1}^{\mu+\sigma}(\psi_0), \quad \xi_0 \neq 0,$$

where $0 \leq \sigma \leq m/2 - 3/2$. Let $t_0 \in [-T, T]$ such that

$$x_0 + \frac{3}{2}t|\xi_0|^{-1/2}\xi_0 \neq 0$$

for all $t \in [0, t_0]$ if $t_0 \geq 0$, or respectively for all $t \in [t_0, 0]$ if $t_0 \leq 0$. Then

$$\left(x_0 + \frac{3}{2}t_0|\xi_0|^{-1/2}\xi_0, \xi_0\right) \in \text{WF}_{1/2,1}^{\mu+1/2+\sigma}(\eta(t_0)) \cup \text{WF}_{1/2,1}^{\mu+\sigma}(\psi(t_0)).$$

We will see that, by Lemma 4.2.16, as $(\eta, \psi) \in \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu$,

$$\text{WF}_{1/2,1}^{\mu+1/2}(\eta) \cup \text{WF}_{1/2,1}^\mu(\psi) \subset \{x = 0\} \cup \{\xi = 0\}.$$

By Alazard–Métivier [12], we expect σ to be at most $\mu - \alpha - d/2$ for some $\alpha > 0$, corresponding to the extra gain of regularity by the remainder of the parilinearization procedure. Although Theorem 4.1.7 does not give the optimal upper bound for σ , as it is not our priority, but when $m = 2\mu - 6 - d$, σ can still be as big as $\mu - 9/2 - d/2$.

4.1.3.6 Microlocal smoothing effect

Our second main result shows that a $(0, 1)$ -singularity creates instantaneously a $(1/2, 1)$ -singularity at the spatial infinity. Corresponding to (M.2) of Theorem 4.1.4. To state the result, observe that, given $\eta_0 \in C^3(\mathbb{R}^d)$, the initial free surface $\Sigma_0 = \Sigma|_{t=0}$ is isometric to $(\mathbb{R}^d, \varrho_0)$, with

$$\varrho_0 = \begin{pmatrix} \text{Id} + {}^t\nabla\eta_0\nabla\eta_0 & {}^t\nabla\eta_0 \\ \nabla\eta_0 & 1 \end{pmatrix}.$$

We identify the co-geodesic flow \mathcal{G}_t on $T^*\Sigma_0$ with the Hamiltonian flow on \mathbb{R}^{2d} of the symbol

$$G(x, \xi) = (\varrho_0)_x^{-1}(\xi, \xi) = |\xi|^2 - \frac{(\nabla\eta_0 \cdot \xi)^2}{1 + |\nabla\eta_0|^2}.$$

That is,

$$\partial_t \mathcal{G}_t = X_G(\mathcal{G}_t), \quad \mathcal{G}_0 = \text{Id}_{\mathbb{R}^{2d}},$$

where $X_G = (\partial_\xi G, -\partial_x G)$.

Theorem 4.1.8. *Let $d \geq 1$, $\mu > 3 + d/2$, $3 \leq m \leq \frac{2}{3}(\mu - 3 - d/2)$. Suppose that for some $T > 0$, $(\eta, \psi) \in C([-T, T], \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu)$ solves (4.1.7) with initial data (η_0, ψ_0) . Let*

$$(x_0, \xi_0) \in \text{WF}_{0,1}^{\mu+1/2+\sigma}(\eta_0) \cup \text{WF}_{0,1}^{\mu+\sigma}(\psi_0), \quad \xi_0 \neq 0,$$

where $0 \leq \sigma \leq \min\{\mu/2 - 3 - d/4, 3m/2\}$, and suppose that the co-geodesic $\{(x_t, \xi_t) := \mathcal{G}_t(x_0, \xi_0)\}_{t \in \mathbb{R}}$ is forwardly resp. backwardly non-trapping, i.e., for any compact set $K \subset \mathbb{R}^d$, $x_t \notin K$, resp. $x_{-t} \notin K$ whenever $t > 0$ is sufficiently large. Then there exists $\xi_{+\infty} \in \mathbb{R}^d \setminus \{0\}$, resp. $\xi_{-\infty} \in \mathbb{R}^d \setminus \{0\}$ such that,

$$\lim_{t \rightarrow \infty} \xi_t = \xi_{+\infty}, \quad \text{resp.} \quad \lim_{t \rightarrow \infty} \xi_{-t} = \xi_{-\infty},$$

and for all $0 < t_0 \leq T$, resp. $-T \leq t_0 < 0$,

$$\begin{aligned} & \left(\frac{3}{2}t_0 |\xi_{+\infty}|^{-1/2} \xi_{+\infty}, \xi_{+\infty} \right) \in \text{WF}_{1/2,1}^{\mu+1/2+\sigma}(\eta(t_0)) \cup \text{WF}_{1/2,1}^{\mu+\sigma}(\psi(t_0)), \\ \text{resp.} & \left(\frac{3}{2}t_0 |\xi_{-\infty}|^{-1/2} \xi_{-\infty}, \xi_{-\infty} \right) \in \text{WF}_{1/2,1}^{\mu+1/2+\sigma}(\eta(t_0)) \cup \text{WF}_{1/2,1}^{\mu+\sigma}(\psi(t_0)). \end{aligned}$$

We remark that the asymptotic directions, $\xi_{\pm\infty}$, are determined solely by the initial geometry given by η_0 , due to the infinite speed of propagation.

One may wonder whether the non-trapping assumption in Theorem 4.1.8 is necessary. We are tempted to believe that the co-geodesic flow on Σ_0 is everywhere non-trapping, both forwardly and backwardly, because Σ_0 is the graph of a function from \mathbb{R}^d to \mathbb{R} . However, there are only two cases that are known to us be true : either when $d = 1$, or when $\nabla\eta_0 \in L^\infty$ and $\|\langle x \rangle \nabla^2 \eta_0\|_{L^\infty}$ is sufficiently small, see §4.6.4. In both cases we obtain the following local smoothing effect.

Corollary 4.1.9. *Let $d \geq 1$, $\mu > 3 + d/2$, $3 \leq m \leq \frac{2}{3}(\mu - 3 - d/2)$. Suppose that for some $T > 0$, $(\eta, \psi) \in C([-T, T], \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu)$ solves (4.1.7) with initial data (η_0, ψ_0) . Suppose that the following two conditions are satisfied,*

- (1) either $d = 1$ or $\|\langle x \rangle \nabla^2 \eta_0\|_{L^\infty}$ is sufficiently small;

(2) $\text{WF}_{1/2,1}^{\mu+1/2+\sigma}(\eta_0) \cup \text{WF}_{1/2,1}^{\mu+\sigma}(\psi_0) \subset \{x=0\} \cup \{\xi=0\}$,
 then $\forall t_0 \in [-T, T] \setminus \{0\}$ and $\forall \epsilon > 0$,

$$(\eta(t_0), \psi(t_0)) \in H_{\text{loc}}^{\mu+1/2+\sigma-\epsilon} \times H_{\text{loc}}^{\mu+\sigma-\epsilon}.$$

We remark that the second condition is satisfied if $(\eta_0, \psi_0) \in H_{2k}^{\mu+1/2+\sigma-k} \times H_{2k'}^{\mu+\sigma-k'}$ for some $(k, k') \in \mathbb{R}^2$, which is particularly the case if $(\eta_0, \psi_0) \in \mathcal{E}' \times \mathcal{E}'$.

We refer to Christianson–Hur–Staffilani [54] and Alazard–Burq–Zuily [6] for local smoothing effects of 2D capillary-gravity water waves, and Alazard–Ifrim–Tataru [11] for a Morawetz inequality of 2D gravity water waves.

4.1.4 Outline of paper

In §4.2, we present basic properties of weighted Sobolev spaces and the quasi-homogeneous wavefront set. In §4.3, we prove Theorem 4.1.4 by extending the idea of Nakamura [153]. In §4.4, we review the paradifferential calculus of Bony, and extend it to weighted Sobolev spaces by a spatial dyadic decomposition. We also develop a quasi-homogeneous semiclassical paradifferential calculus, and study its relations with the quasi-homogeneous wavefront set. In §4.5, we study the Dirichlet–Neumann operator in weighted Sobolev spaces and prove the existence of asymptotically flat gravity-capillary water waves, i.e., Theorem 4.1.6. In §4.6, we prove our main results, Theorem 4.1.7, Theorem 4.1.8 and Corollary 4.1.9, by extending the proof of Theorem 4.1.4 to the quasilinear equation using the paradifferential calculus.

4.2 Quasi-homogeneous microlocal analysis

4.2.1 Quasi-homogeneous semiclassical calculus

Definition 4.2.1. For $(\mu, k) \in \mathbb{R}^2$, set $m_k^\mu(x, \xi) = \langle x \rangle^k \langle \xi \rangle^\mu$. Let $a_h \in C^\infty(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$, we say that $a_h \in S_k^\mu$ if $\forall \alpha, \beta \in \mathbb{N}^d$, $\exists C_{\alpha\beta} > 0$, such that $\forall (x, \xi) \in \mathbb{R}_x^d \times \mathbb{R}_\xi^d$,

$$\sup_{0 < h < 1} |\partial_x^\alpha \partial_\xi^\beta a_h(x, \xi)| \leq C_{\alpha\beta} m_{k-|\alpha|}^{\mu-|\beta|}(x, \xi).$$

We say that $a_h \in S_k^\mu$ is (μ, k) -elliptic if $\exists R > 0, C > 0$ such that for $|x| + |\xi| \geq R$,

$$\inf_{0 < h < 1} |a_h(x, \xi)| \geq C m_k^\mu(x, \xi).$$

Denote

$$S_\infty^\infty = \bigcup_{(\mu, k) \in \mathbb{R}^2} S_k^\mu, \quad S_{-\infty}^{-\infty} = \bigcap_{(\mu, k) \in \mathbb{R}^2} S_k^\mu.$$

We say that $a_h \in S_{-\infty}^{-\infty}$ is elliptic at (x_0, ξ_0) if for some neighborhood Ω of (x_0, ξ_0) ,

$$\inf_{0 < h < 1} \inf_{(x, \xi) \in \Omega} |a_h(x, \xi)| > 0.$$

Definition 4.2.2. Given $\delta \geq 0, \rho \geq 0$, for $h > 0$, define

$$\theta_h^{\delta, \rho} : \mathbb{R}_x^d \times \mathbb{R}_\xi^d \rightarrow \mathbb{R}_x^d \times \mathbb{R}_\xi^d \quad (x, \xi) \mapsto (h^\delta x, h^\rho \xi),$$

which induces a pullback $\theta_{h,*}^{\delta,\rho}$ on S_∞^∞ , $\theta_{h,*}^{\delta,\rho} a_h = a_h \circ \theta_h^{\delta,\rho}$. Then set

$$\text{Op}_h^{\delta,\rho}(a_h) = \text{Op}(\theta_{h,*}^{\delta,\rho} a_h),$$

where

$$\text{Op}(a)u(x) := (2\pi)^{-d} \iint e^{i(x-y)\cdot\xi} a(x, \xi) u(y) dy d\xi.$$

By the formula $(\vartheta_h^\delta)^{-1} \text{Op}_h^{\delta,\rho}(a) \vartheta_h^\delta = \text{Op}_h^{0,\delta+\rho}(a)$, where $\vartheta_h^\delta u(x) := h^{\delta d/2} u(h^\delta x)$ is an isometry of $L^2(\mathbb{R}^d)$, we induce the following results from the usual semiclassical calculus for which we refer to the book [208] of Zworski.

Proposition 4.2.3. *There exists $K > 0$ such that, if $a \in C^\infty(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$ with*

$$M = \sum_{|\alpha|+|\beta| \leq d} \|\partial_x^\alpha \partial_\xi^\beta a\|_{L^\infty} < \infty,$$

then $\|\text{Op}_h^{\delta,\rho}(a)\|_{L^2 \rightarrow L^2} \leq KM$.

Proposition 4.2.4 (Sharp Gårding Inequality). *If $\delta + \rho > 0$ and $a_h \in S_0^0$ such that $\text{Re } a_h \geq 0$, then for some $C > 0$, all $u \in L^2(\mathbb{R}^d)$, and $0 < h < 1$,*

$$\text{Re}(\text{Op}_h^{\delta,\rho}(a_h)u, u)_{L^2} \geq -Ch^{\delta+\rho} \|u\|_{L^2}^2.$$

Proposition 4.2.5. *There exists a bilinear operator*

$$\sharp_h^{\delta,\rho} : S_\infty^\infty \times S_\infty^\infty \rightarrow S_\infty^\infty,$$

such that

$$\text{Op}_h^{\delta,\rho}(a_h) \text{Op}_h^{\delta,\rho}(b_h) = \text{Op}_h^{\delta,\rho}(a_h \sharp_h^{\delta,\rho} b_h).$$

If $a_h \in S_k^\mu$, $b_h \in S_\ell^\nu$, then $a_h \sharp_h^{\delta,\rho} b_h \in S_{k+\ell}^{\mu+\nu}$. We denote through this article that $D_x = \frac{1}{i} \nabla_x$. For $\gamma > 0$, define

$$a_h \sharp_{h,\gamma}^{\delta,\rho} b_h = \sum_{|\alpha| < \gamma} \frac{h^{|\alpha|(\delta+\rho)}}{\alpha!} \partial_\xi^\alpha a_h D_x^\alpha b_h,$$

then $a_h \sharp_h^{\delta,\rho} b_h - a_h \sharp_{h,\gamma}^{\delta,\rho} b_h = \mathcal{O}(h^{\gamma(\delta+\rho)})_{S_{k+\ell-\gamma}^{\mu+\nu}}$. If $\delta + \rho > 0$ and either $a_h \in S_{-\infty}^-$ or $b_h \in S_{-\infty}^-$, then $a_h \sharp_h^{\delta,\rho} b_h - a_h \sharp_{h,\gamma}^{\delta,\rho} b_h = \mathcal{O}(h^{\gamma(\delta+\rho)})_{S_{-\infty}^-}$.

Proposition 4.2.6. *Then there exists a linear operator*

$$\zeta_h^{\delta,\rho} : S_\infty^\infty \rightarrow S_\infty^\infty$$

such that

$$\text{Op}_h^{\delta,\rho}(a_h)^* = \text{Op}_h^{\delta,\rho}(\zeta_h^{\delta,\rho} a_h).$$

If $a_h \in S_k^\mu$, then $\zeta_h^{\delta,\rho} a_h \in S_k^\mu$. For $\gamma > 0$, define

$$\zeta_{h,\gamma}^{\delta,\rho} a_h = \sum_{|\alpha| < \gamma} \frac{h^{|\alpha|(\delta+\rho)}}{\alpha!} \partial_\xi^\alpha D_x^\alpha \overline{a_h},$$

then $\zeta_h^{\delta,\rho} a_h - \zeta_{h,\gamma}^{\delta,\rho} a_h = \mathcal{O}(h^{\gamma(\delta+\rho)})_{S_{k-\gamma}^{\mu-\gamma}}$. If $\delta + \rho > 0$ and $a_h \in S_{-\infty}^-$, then $\zeta_h^{\delta,\rho} a_h - \zeta_{h,\gamma}^{\delta,\rho} a_h = \mathcal{O}(h^{\gamma(\delta+\rho)})_{S_{-\infty}^-}$.

4.2.2 Weighted Sobolev spaces

Let us recall that $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space, and $\mathcal{S}'(\mathbb{R}^d)$ is the space of tempered distributions.

Definition 4.2.7. We say that a linear operator $\mathcal{A} : \mathcal{S} \rightarrow \mathcal{S}'$ is of order $(\nu, \ell) \in \mathbb{R}^2$, and denote $\mathcal{A} \in \mathcal{O}_\ell^\nu$ if $\forall (\mu, k) \in \mathbb{R}^2$, there exists $C > 0$ such that $\forall u \in \mathcal{S}$, $\|\mathcal{A}u\|_{H_{k-\ell}^{\mu-\nu}} \leq C\|u\|_{H_k^\mu}$. Therefore, \mathcal{A} extends to a bounded linear operator from H_k^μ to $H_{k-\ell}^{\mu-\nu}$. We denote $\mathcal{A} \in \mathcal{O}_{-\infty}^\nu$ if $\mathcal{A} \in \mathcal{O}_\ell^\nu$, $\forall (\nu, \ell) \in \mathbb{R}^2$. Let $\mathcal{A}_\alpha : \mathcal{S} \rightarrow \mathcal{S}'$ and $C_\alpha > 0$ be indexed by $\alpha \in \mathcal{A}$. We say that $\mathcal{A}_\alpha = \mathcal{O}(C_\alpha)_{\mathcal{O}_\ell^\nu}$, if $\forall (\mu, k) \in \mathbb{R}^2$, $\exists K > 0$, such that $\forall \alpha \in \mathcal{A}$, $\|\mathcal{A}_\alpha\|_{H_k^\mu \rightarrow H_{k-\ell}^{\mu-\nu}} \leq KC_\alpha$.

By Proposition 4.2.3 and Proposition 4.2.5, we obtain

Proposition 4.2.8. *If $a_h \in S_\ell^\nu$ with $(\nu, \ell) \in \mathbb{R}^2$, then $\text{Op}(a_h) \in \mathcal{O}_\ell^\nu$.*

Proposition 4.2.9. *$u \in H_k^\mu$ if and only if for some $a_h \in S_k^\mu$ which is (μ, k) -elliptic, $\text{Op}(a_h)u \in L^2$.*

Proposition 4.2.10. $\mathcal{S} = \bigcap_{(\mu, k) \in \mathbb{R}^2} H_k^\mu$, $\mathcal{S}' = \bigcup_{(\mu, k) \in \mathbb{R}^2} H_k^\mu$.

Démonstration. Clearly $\mathcal{S} \subset \bigcap_{(\mu, k) \in \mathbb{R}^2} H_k^\mu$. The converse follows by Sobolev injection. As for the second statement, clearly $\bigcup_{(\mu, k) \in \mathbb{R}^2} H_k^\mu \subset \mathcal{S}'$. Conversely, let $u \in \mathcal{S}'$, then $\exists N > 0$, such that $\forall \varphi \in \mathcal{S}$,

$$\langle u, \varphi \rangle_{\mathcal{S}', \mathcal{S}} \lesssim \sum_{|\alpha|+|\beta| \leq N} \|x^\alpha \partial_x^\beta \varphi\|_{L^\infty} \lesssim \|\text{Op}(m_M^M) \varphi\|_{L^2}$$

with M sufficiently large, implying that $u \in H_{-M}^{-M}$. \square

Lemma 4.2.11. *Let $u \in \mathcal{S}'$, then $u = h^{-M} \text{Op}_h^{\delta, \rho}(m_{-M}^{-M}) \mathcal{O}(1)_{L^2}$ for some $M > 0$. Therefore, if $\delta + \rho > 0$, and $a_h \in \mathcal{O}(h^\infty)_{S_{-\infty}^\infty}$, then $\text{Op}_h^{\delta, \rho}(a_h)u_h = \mathcal{O}(h^\infty)_{\mathcal{S}}$.*

Démonstration. By the proof of Proposition 4.2.10, $\exists M, N > 0$, such that $\forall \varphi \in \mathcal{S}$,

$$\langle u, \varphi \rangle_{\mathcal{S}', \mathcal{S}} \lesssim \sum_{|\alpha|+|\beta| \leq N} \|x^\alpha \partial_x^\beta \varphi\|_{L^\infty} \lesssim h^{-M} \|\text{Op}_h^{\delta, \rho}(m_M^M) \varphi\|_{L^2}.$$

\square

Next, we characterize weighted Sobolev spaces by a dyadic decomposition.

Definition 4.2.12. Let $\psi : \mathbb{N} \rightarrow C_c^\infty(\mathbb{R}^d)$, $j \mapsto \psi_j$, we denote $\psi \in \mathcal{P}$ if

- (i) $\text{supp } \psi_j \subset \{C^{-1}2^j \leq |x| \leq C2^j\}$ for some $C > 1$ and all $j \geq 1$;
- (ii) $\psi_j \geq 0$ for all $j \in \mathbb{N}$ and $C^{-1} \leq \sum_{j \in \mathbb{N}} \psi_j \leq C$ for some $C > 1$;
- (iii) $\|\partial_x^\alpha \psi_j\|_{L^\infty} \leq C_\alpha 2^{-j|\alpha|}$ for all $\alpha \in \mathbb{N}$, $j \in \mathbb{N}$, and some $C_\alpha > 0$ depending solely on α .

Given $\psi \in \mathcal{P}$, we denote $\psi \in \mathcal{P}_*$ if

- (i) $\sum_{j \in \mathbb{N}} \psi_j = 1$;
- (ii) $\text{supp } \psi_j \cap \text{supp } \psi_k = \emptyset$ whenever $|j - k| > 2$.

Proposition 4.2.13. *Let $(\mu, k) \in \mathbb{R}^2$, $\psi \in \mathcal{P}$, and $u \in \mathcal{S}'$, then $u \in H_k^\mu$ if and only if $\sum_{j \in \mathbb{N}} 2^{2jk} \|\psi_j u\|_{H^\mu}^2 < \infty$. Moreover, $\exists C > 1$, such that $\forall u \in H_k^\mu$,*

$$C^{-1} \|u\|_{H_k^\mu}^2 \leq \sum_{j \in \mathbb{N}} 2^{2jk} \|\psi_j u\|_{H^\mu}^2 \leq C \|u\|_{H_k^\mu}^2.$$

Démonstration. We may assume that $\psi \in \mathcal{P}_*$, because $\forall \phi^1, \phi^2 \in \mathcal{P}$,

$$\sum_{j \in \mathbb{N}} 2^{2jk} \|\phi_j^1 u\|_{H^\mu}^2 \simeq \sum_{j \in \mathbb{N}} 2^{2jk} \|\phi_j^2 u\|_{H^\mu}^2,$$

Define $\tilde{\psi} \in \mathcal{P}$ by setting $\tilde{\psi}_j = \sum_{|k-j| \leq 2} \psi_k$, then $\tilde{\psi}_j \psi_j = \psi_j$, $\forall j$. For $u \in H_k^\mu$,

$$2^{2jk} \|\psi_j u\|_{H^\mu}^2 \lesssim \|\tilde{\psi}_j \langle D_x \rangle^\mu \psi_j \langle x \rangle^k u\|_{L^2}^2 + \|(1 - \tilde{\psi}_j) \langle D_x \rangle^\mu \psi_j \langle x \rangle^k u\|_{L^2}^2,$$

Apply Proposition 4.2.5 with $(\delta, \rho) = (1, 0)$, we have

$$(1 - \tilde{\psi}_j) \langle D_x \rangle^\mu \psi_j \langle D_x \rangle^{-\mu} = \mathcal{O}(2^{-jN})_{L^2 \rightarrow L^2}, \quad \forall N > 0.$$

Therefore,

$$\sum_{j \in \mathbb{N}} \|(1 - \tilde{\psi}_j) \langle D_x \rangle^\mu \psi_j \langle x \rangle^k u\|_{L^2}^2 \lesssim \sum_{j \in \mathbb{N}} 2^{-2jN} \|u\|_{H_k^\mu}^2 \lesssim \|u\|_{H_k^\mu}^2.$$

For $r = 0, 1, \dots, 9$, set

$$a_r = \sum_{j \in 10\mathbb{N}+r} \tilde{\psi}_j \langle \xi \rangle^\mu \# (\langle x \rangle^k \psi_j) \in S_k^\mu.$$

Observe that for $0 \neq j - j' \in 10\mathbb{N}$, $\text{supp } \tilde{\psi}_j \cap \text{supp } \tilde{\psi}_{j'} = \emptyset$, therefore,

$$\begin{aligned} \sum_{j \in \mathbb{N}} \|\tilde{\psi}_j \langle D_x \rangle^\mu \psi_j \langle x \rangle^k u\|_{L^2}^2 &= \sum_{r=0}^9 \sum_{j \in 10\mathbb{N}+r} \|\tilde{\psi}_j \langle D_x \rangle^\mu \psi_j \langle x \rangle^k u\|_{L^2}^2 \\ &= \sum_{r=0}^9 \|\text{Op}(a_r) u\|_{L^2}^2 \lesssim \|u\|_{H_k^\mu}^2, \end{aligned}$$

And we proved that

$$\sum_{j \in \mathbb{N}} 2^{2jk} \|\psi_j u\|_{H^\mu}^2 \lesssim \|u\|_{H_k^\mu}^2.$$

Conversely, observe that $a := \sum_{r=0}^9 a_r$ is (μ, k) -elliptic. So for some $r \in S_{-\infty}^-$,

$$\|u\|_{H_k^s}^2 \lesssim \|\text{Op}(a) u\|_{L^2}^2 + \|\text{Op}(r) u\|_{L^2}^2.$$

Similarly as above, we have

$$\begin{aligned} \|\text{Op}(a) u\|_{L^2}^2 &\lesssim \sum_{r=0}^9 \|\text{Op}(a_r) u\|_{L^2}^2 \lesssim \sum_{r=0}^9 \sum_{j \in 10\mathbb{N}+r} \|\tilde{\psi}_j \langle D_x \rangle^\mu \psi_j \langle x \rangle^k u\|_{L^2}^2 \\ &\lesssim \sum_{j \in \mathbb{N}} 2^{2jk} \|\langle D_x \rangle^\mu \psi_j u\|_{L^2}^2; \end{aligned}$$

while for the second term, we denote $\sharp = \sharp_1^{0,0}$ for simplicity and write

$$\|\text{Op}(r)u\|_{L^2}^2 = (u, \text{Op}(r^*\sharp r)u)_{L^2} = \sum_{j \in \mathbb{N}} (u, \text{Op}(r^*\sharp r)\psi_j u)_{L^2}.$$

For each term in the summation, we have for all $N > 0$ and $\varepsilon > 0$,

$$\begin{aligned} (u, \text{Op}(r^*\sharp r)\psi_j u)_{L^2} &= (\text{Op}(m_k^\mu)u, \text{Op}(m_k^{-\mu}\sharp r^*\sharp m_{-N+k}^\mu)\langle D_x \rangle^\mu \langle x \rangle^{-N+k}\psi_j u)_{L^2} \\ &\lesssim \|u\|_{H_k^\mu} \|\langle D_x \rangle^\mu \langle x \rangle^{-N+k}\psi_j u\|_{L^2} \\ &\lesssim 2^{-jN} \|u\|_{H_k^\mu} 2^{jk} \|\langle D_x \rangle^\mu \psi_j u\|_{L^2} \\ &\lesssim 2^{-jN} (\varepsilon \|u\|_{H_k^\mu}^2 + \varepsilon^{-1} 2^{2jk} \|\langle D_x \rangle^\mu \psi_j u\|_{L^2}^2). \end{aligned}$$

Summing up in j ,

$$\|\text{Op}(r)u\|_{L^2}^2 \lesssim \varepsilon \|u\|_{H_k^\mu}^2 + \varepsilon^{-1} \sum_{j \in \mathbb{N}} 2^{2jk} \|\psi_j u\|_{H^\mu}^2.$$

We conclude by choosing ε sufficiently small. \square

4.2.3 Quasi-homogeneous wavefront sets

The following characterization is easy to prove by a routine construction of parametrrix.

Proposition 4.2.14. *Let $u \in \mathcal{S}'$, then $(x_0, \xi_0) \notin \text{WF}_{\delta, \rho}^\mu(u)$ if and only if for some $a_h \in S_{-\infty}^-$ which is elliptic at (x_0, ξ_0) , $\text{Op}_h^{\delta, \rho}(a_h)u = \mathcal{O}(h^\mu)_{L^2}$.*

Lemma 4.2.15. *Let $u \in \mathcal{S}'$ and $a_h \in S_{-\infty}^-$ such that*

$$\text{supp } a_h \subset K \subset \mathbb{R}_x^d \times \mathbb{R}_\xi^d \setminus \text{WF}_{\delta, \rho}^\mu(u)$$

for some compact set K and all $0 < h < 1$, then

$$\langle u, \text{Op}_h^{\delta, \rho}(a_h)u \rangle_{\mathcal{S}', \mathcal{S}} = \mathcal{O}(h^{2\mu}).$$

Consequently, $\text{Op}_h^{\delta, \rho}(a_h)u = \mathcal{O}(h^\mu)_{L^2}$.

Démonstration. Let $\{\Omega_i\}_{i \in I}$ be an open cover of K . Let $b_h^i \in S_{-\infty}^-$ be elliptic everywhere in Ω_i , such that $\text{Op}_h^{\delta, \rho}(b_h^i)u = \mathcal{O}(h^\mu)_{L^2}$. By a partition of unity, we may assume that $K \subset \Omega := \Omega_{i_0}$ for some $i_0 \in I$, and set $b_h = b_h^{i_0}$. By the ellipticity of b_h , we can find $c_h \in S_{-\infty}^-$ and $r_h = \mathcal{O}(h^\infty)_{S_{-\infty}^-}$ such that $a_h = (\zeta_h^{\delta, \rho} b_h) \sharp_h^{\delta, \rho} c_h \sharp_h^{\delta, \rho} b_h + r_h$. Therefore, by Lemma 4.2.11,

$$\begin{aligned} \langle u, \text{Op}_h^{\delta, \rho}(a_h)u \rangle_{\mathcal{S}', \mathcal{S}} &= (\text{Op}_h^{\delta, \rho}(b_h)u, \text{Op}_h^{\delta, \rho}(c_h)\text{Op}_h^{\delta, \rho}(b_h)u)_{L^2} \\ &\quad + \langle u, \text{Op}_h^{\delta, \rho}(r_h)u \rangle_{\mathcal{S}', \mathcal{S}} = \mathcal{O}(h^\mu)^2 + \mathcal{O}(h^\infty) = \mathcal{O}(h^{2\mu}). \end{aligned}$$

\square

Lemma 4.2.16. *Let $u \in \mathcal{S}'$, then*

- (1) $\text{WF}_{\delta, \rho}^\mu(u)$ is a closed (δ, ρ) -cone, i.e., $\theta_\lambda^{\delta, \rho} \text{WF}_{\delta, \rho}^\mu(u) = \text{WF}_{\delta, \rho}^\mu(u)$, $\forall \lambda > 0$;

- (2) $\text{WF}_{\delta,\rho}^\mu(u) = \text{WF}_{\frac{\delta}{\gamma}, \frac{\rho}{\gamma}}^\mu(u)$, $\forall \gamma > 0$;
 (3) $(x_0, \xi_0) \in \text{WF}_{\delta,\rho}^\mu(u) \Leftrightarrow (\xi_0, -x_0) \in \text{WF}_{\rho,\delta}^\mu(\hat{u})$;
 (4) $(x_0, \xi_0) \in \text{WF}_{\delta,\rho}^\mu(u) \Leftrightarrow (x_0, -\xi_0) \in \text{WF}_{\delta,\rho}^\mu(\bar{u})$;
 (5) Denote $\text{WF}_{\delta,\rho}^\mu(u)^\circ = \text{WF}_{\delta,\rho}^\mu(u) \setminus \mathcal{N}_{\delta,\rho}$, where

$$(4.2.1) \quad \mathcal{N}_{\delta,\rho} = \begin{cases} \{x=0\} \times \mathbb{R}_\xi^d, & \delta > 0, \rho = 0; \\ \mathbb{R}_x^d \times \{\xi=0\}, & \delta = 0, \rho > 0; \\ \{x=0\} \times \mathbb{R}_\xi^d \cup \mathbb{R}_x^d \times \{\xi=0\}, & \delta > 0, \rho > 0. \end{cases}$$

If $u \in H_k^\mu$ with $(\mu, k) \in \mathbb{R}^2$, and $a_h \in S_{-\infty}^-$ such that

$$\mathcal{N}_{\delta,\rho} \cap \overline{\bigcup_{0 < h < 1} \text{supp } a_h} = \emptyset,$$

then $\text{Op}_h^{\delta,\rho}(a_h)u = \mathcal{O}(h^{\delta\mu+\rho k})_{L^2}$. In particular, $\text{WF}_{\delta,\rho}^{\delta\mu+\rho k}(u)^\circ = \emptyset$.

Démonstration. ((1)) and (2) are easy. To prove (3), we use $\mathcal{F}^{-1}\text{Op}_h^{\rho,\delta}(a_h)\mathcal{F} = \text{Op}_h^{\delta,\rho}(\zeta_h^{\delta,\rho}b_h)$ where \mathcal{F} is the Fourier transform and $b_h(x, \xi) = a_h(\xi, -x)$. To prove (4), we use $\text{Op}(a_h)u_h = \text{Op}(b_h)\bar{u}_h$, where $b_h(x, \xi) = \bar{a}_h(x, -\xi)$. To prove the case where $\delta > 0, \rho > 0$ of (5), we simply observe that,

$$(\theta_{h,*}^{\delta,\rho}a_h)\langle \xi \rangle^{-\mu} \#_1^{0,0} \langle x \rangle^{-k} = \mathcal{O}(h^{\delta\mu+\rho k})_{S_0^0}.$$

The other two cases are similar. □

4.3 Model equation

We prove Theorem 4.1.4 by combining the ideas of Nakamura [153] and simple methods of scaling. There is no harm in assuming that $u_0 \in L^2(\mathbb{R}^d)$.

4.3.1 Proof of (M.1)

For $a \in W_{\text{loc}}^{1,\infty}(\mathbb{R}_t \times \mathbb{R}_x^d \times \mathbb{R}_\xi^d)$ and $\mathcal{A} \in W_{\text{loc}}^{1,\infty}(\mathbb{R}_t, L^2 \rightarrow L^2)$ (which means that $t \mapsto \|\mathcal{A}(t)\|_{L^2 \rightarrow L^2} \in L_{\text{loc}}^\infty$, $t \mapsto \|\partial_t \mathcal{A}(t)\|_{L^2 \rightarrow L^2} \in L_{\text{loc}}^\infty$), define

$$\mathcal{L}_t a = \partial_t a + \{|\xi|^\gamma, a\}, \quad \mathcal{L}_t \mathcal{A} = \partial_t \mathcal{A} + i[|D_x|^\gamma, \mathcal{A}].$$

Here $\{\cdot, \cdot\}$ is the Poisson bracket, i.e. $\{f, g\} = \nabla_\xi f \cdot \nabla_x g - \nabla_x f \cdot \nabla_\xi g$.

Lemma 4.3.1. For $a_h \in W_{\text{loc}}^{1,\infty}(\mathbb{R}, S_{-\infty}^-)$ with

$$\overline{\bigcup_{0 < h < 1} \text{supp } a_h} \cap \{\xi = 0\} = \emptyset,$$

there exists $b_h \in L_{\text{loc}}^\infty(\mathbb{R}, S_{-\infty}^-)$ with $\text{supp } b_h \subset \text{supp } a_h$, such that

$$\mathcal{L}_t \text{Op}_h^{\delta,\rho}(a_h) = \text{Op}_h^{\delta,\rho}(\mathcal{L}_t a_h) + h^{\delta+\rho} \text{Op}_h^{\delta,\rho}(b_h) + \mathcal{O}(h^\infty)_{L_{\text{loc}}^\infty(\mathbb{R}, L^2 \rightarrow L^2)}.$$

Démonstration. $\forall T > 0, \exists \epsilon > 0$ such that

$$\overline{\bigcup_{t \in [-T, T]} \bigcup_{0 < h < 1} \text{supp } a_h(t, \cdot)} \cap \{|\xi| \leq \epsilon\} = \emptyset.$$

Let $\pi \in C^\infty(\mathbb{R}^d)$ such that $0 \leq \pi \leq 1$, $\pi(\xi) = 0$ for $|\xi| \leq \epsilon/3$, $\pi(\xi) = 1$ for $|\xi| \geq 2\epsilon/3$. Then

$$i[|D_x|^\gamma, \text{Op}_h^{\delta, \rho}(a_h)] = ih^{-\rho\gamma} [|h^\rho D_x|^\gamma \pi(h^\rho D_x), \text{Op}_h^{\delta, \rho}(a_h)] + \mathcal{O}(h^\infty)_{L^\infty([-T, T], L^2 \rightarrow L^2)}.$$

Now that $|\xi|^\gamma \pi(\xi) \in S_0^\gamma$, we conclude by Proposition 4.2.5 and the hypothesis $\rho\gamma = \delta + \rho$. \square

Assume that $\mu = \infty$, as the proof is similar for $\mu < \infty$. Let $(x_0, \xi_0) \notin \text{WF}_{\delta, \rho}^\mu(u_0)$ with $\xi_0 \neq 0$. We aim to find $a_h \in W_{\text{loc}}^{1, \infty}(\mathbb{R}, S_{-\infty}^-)$ of the asymptotic expansion

$$a_h \sim \sum_{j \in \mathbb{N}} h^{j(\delta + \rho)} a_h^j$$

where $a_h^j \in W_{\text{loc}}^{1, \infty}(\mathbb{R}, S_{-\infty}^-)$, such that $\forall t \in \mathbb{R}$, $a_h(t, \cdot)$ is elliptic at $(x_0 + t\gamma|\xi_0|^{\gamma-2}\xi_0, \xi_0)$, and

$$\mathcal{L}_t \text{Op}_h^{\delta, \rho}(a_h) = \mathcal{O}(h^\infty)_{L_{\text{loc}}^\infty(\mathbb{R}, L^2 \rightarrow L^2)}.$$

If such a_h is found, we set $\mathcal{A}_h = \text{Op}_h^{\delta, \rho}(a_h)$, then

$$\frac{d}{dt} \|\mathcal{A}_h u\|_{L^2}^2 = 2\text{Re}(\mathcal{L}_t \mathcal{A}_h u, \mathcal{A}_h u)_{L^2} + 2\text{Re}(i|D_x|^\gamma \mathcal{A}_h u, \mathcal{A}_h u)_{L^2} \leq \mathcal{O}(h^\infty) \|\mathcal{A}_h u\|_{L^2} + 0.$$

We deduce by Gronwall's inequality that $\mathcal{A}_h u = \mathcal{O}(h^\infty)_{L_{\text{loc}}^\infty(\mathbb{R}, L^2)}$ and conclude.

To construct a_h , first by Lemma 4.2.15, $\exists \varphi \in C_c^\infty(\mathbb{R}_x^d \times (\mathbb{R}_\xi^d \setminus \{0\}))$ with $\varphi(x_0, \xi_0) \neq 0$, such that $\text{Op}_h^{\delta, \rho}(\varphi)u = \mathcal{O}(h^\infty)_{L^2}$. Then we can construct a_h with $a_h|_{t=0} = \varphi$ with $a_h^j \in W_{\text{loc}}^{\infty, \infty}(\mathbb{R}, S_{-\infty}^-)$ solving iteratively the transportation equations

$$\begin{cases} \mathcal{L}_t a_h^0 = 0, & \mathcal{L}_t a_h^j + b_h^{j-1} = 0, \\ a_h^0|_{t=0} = \varphi; & a_h^j|_{t=0} = 0, \quad j \geq 1, \end{cases}$$

where $b_h^j \in W_{\text{loc}}^{\infty, \infty}(\mathbb{R}, S_{-\infty}^-)$ satisfies, by Lemma 4.3.1,

$$\mathcal{L}_t \text{Op}_h^{\delta, \rho}(a_h^j) = \text{Op}_h^{\delta, \rho}(\mathcal{L}_t a_h^j) + h^{\delta + \rho} \text{Op}_h^{\delta, \rho}(b_h^j) + \mathcal{O}(h^\infty)_{L_{\text{loc}}^\infty(\mathbb{R}, L^2 \rightarrow L^2)}.$$

\square

4.3.2 Proof of (M.2)

Let $\beta = \rho\gamma - (\delta + \rho) > 0$, introduce the semiclassical time variable $s = h^{-\beta}t$, and rewrite (4.1.9) as

$$(4.3.1) \quad \partial_s u + ih^\beta |D_x|^\gamma u = 0.$$

For $a \in W_{\text{loc}}^{1, \infty}(\mathbb{R}_s \times \mathbb{R}_x^d \times \mathbb{R}_\xi^d)$ and $\mathcal{A} \in W_{\text{loc}}^{1, \infty}(\mathbb{R}_s, L^2 \rightarrow L^2)$, define

$$\mathcal{L}_s a = \partial_s a + \{|\xi|^\gamma, a\}, \quad \mathcal{L}_s \mathcal{A} = \partial_s \mathcal{A} + ih^\beta [|D_x|^\gamma, \mathcal{A}].$$

Lemma 4.3.2. *Let $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $\phi \geq 0$, $\phi(0) > 0$, and $x \cdot \nabla \phi(x) \leq 0$, $\forall x \in \mathbb{R}^d$. Let $\epsilon > 0$, $(x_0, \xi_0) \in \mathbb{R}_x^d \times (\mathbb{R}_\xi^d \setminus \{0\})$. For $s \geq 0$ and $(x, \xi) \in \mathbb{R}_x^d \times \mathbb{R}_\xi^d$, set*

$$\chi(s, x, \xi) = \phi\left(\frac{x - s\gamma|\xi|^{\gamma-2}\xi - x_0}{1+s}\right)\phi\left(\frac{\xi - \xi_0}{\epsilon}\right).$$

Then $\chi \in W^{\infty, \infty}(\mathbb{R}_{\geq 0}, S_0^{-\infty})$, $\mathcal{L}_s \chi \in W^{\infty, \infty}(\mathbb{R}_{\geq 0}, S_{-1}^{-\infty})$, and

$$(4.3.2) \quad \mathcal{L}_s \chi \geq 0.$$

Let $t_0 > 0$, and set $(\tau u)(s, x, \xi) = u(s, \frac{s}{t_0}x, \xi)$, then $\tau \chi \in W^{\infty, \infty}(\mathbb{R}_{\geq 0}, S_{-\infty}^{-\infty})$. Let ϵ be sufficiently small, then $(\tau \chi)(s, \cdot)$ is elliptic at $(t_0\gamma|\xi_0|^{\gamma-2}\xi_0, \xi_0)$ for s sufficiently large.

Démonstration. Each time we differentiate χ with respect to x , we get a multiplicative factor $(1+s)^{-1}$, which is of size $\langle x \rangle^{-1}$ on $\text{supp } \chi$, as $\text{supp } \chi \subset \{C^{-1}s \leq |x| \leq Cs\}$ for some $C > 0$ when $|s|$ is sufficiently large. Therefore $\chi \in W^{\infty, \infty}(\mathbb{R}_{\geq 0}, S_0^{-\infty})$. It is easy to see that $\tau \chi(s, \cdot)$ is bounded in $C_c^\infty(\mathbb{R}^{2d})$. We write

$$(4.3.3) \quad (\tau \chi)(s, x, \xi) = \phi\left(\frac{x - t_0\gamma|\xi_0|^{\gamma-2}\xi_0}{t_0(1+s)/s} - \frac{\gamma|\xi|^{\gamma-2}\xi - \gamma|\xi_0|^{\gamma-2}\xi_0}{(1+s)/s} - \frac{x_0}{1+s}\right)\phi\left(\frac{\xi - \xi_0}{\epsilon}\right),$$

where $|\xi|^{\gamma-2}\xi - |\xi_0|^{\gamma-2}\xi_0 = o(1)$ as $\epsilon \rightarrow 0$, whence $\tau \chi(s, \cdot)$ elliptic at $(t_0\gamma|\xi_0|^{\gamma-2}\xi_0, \xi_0)$ for sufficiently large s . To estimate $\mathcal{L}_s \chi$, we perform an explicit computation,

$$\begin{aligned} \partial_s \chi &= -(\nabla \phi)\left(\frac{x - s\gamma|\xi|^{\gamma-2}\xi - x_0}{1+s}\right)\phi\left(\frac{\xi - \xi_0}{\epsilon}\right) \\ &\quad \times \frac{(x - s\gamma|\xi|^{\gamma-2}\xi - x_0) + (1+s)\gamma|\xi|^{\gamma-2}\xi}{(1+s)^2}, \\ \{|\xi|^\gamma, \chi\} &= \gamma|\xi|^{\gamma-2}\xi \cdot (\nabla \phi)\left(\frac{x - s\gamma|\xi|^{\gamma-2}\xi - x_0}{1+s}\right)\phi\left(\frac{\xi - \xi_0}{\epsilon}\right) \times \frac{1}{1+s}. \end{aligned}$$

Therefore,

$$\mathcal{L}_s \chi = -(\nabla \phi)\left(\frac{x - s\gamma|\xi|^{\gamma-2}\xi - x_0}{1+s}\right)\phi(\xi - \xi_0) \cdot \frac{x - s\gamma|\xi|^{\gamma-2}\xi - x_0}{(1+s)^2} \geq 0.$$

Similarly as above, we prove that $\mathcal{L}_s \chi \in W^{\infty, \infty}(\mathbb{R}_{\geq 0}, S_{-1}^{-\infty})$. \square

We assume $t_0 > 0$ and $\mu = \infty$ as the other cases are similar. Let $\epsilon > 0$ be sufficiently small and let $\{\lambda_j\}_{j \in \mathbb{N}} \subset [1, 1 + \epsilon[$ be strictly increasing. Let ϕ be as in Lemma 4.3.2, and set $\forall j \in \mathbb{N}$,

$$\chi_j(s, x, \xi) = \phi\left(\frac{x - s\gamma|\xi|^{\gamma-2}\xi - x_0}{\lambda_j(1+s)}\right)\phi\left(\frac{\xi - \xi_0}{\lambda_j \epsilon}\right),$$

Then $\text{supp } \chi_j \subset \{\chi_{j+1} > 0\}$. We aim to construct $a_h \in W^{\infty, \infty}(\mathbb{R}_{\geq 0}, S_0^{-\infty})$, such that

- (i) $\text{supp } a_h \subset \cup_{j \in \mathbb{N}} \text{supp } \chi_j$;
- (ii) $a_h(0, \cdot) - (\zeta_h^{\delta, \rho} \chi_0) \#_h^{\delta, \rho} \chi_0(0, \cdot) = \mathcal{O}(h^\infty)_{S_{-\infty}^{-\infty}}$;
- (iii) $\tau a_h \in W^{\infty, \infty}(\mathbb{R}_{\geq 0}, S_{-\infty}^{-\infty})$;
- (iv) $\tau a_h(s, \cdot)$ is elliptic at $(t_0\gamma|\xi_0|^{\gamma-2}\xi_0, \xi_0)$ for sufficiently large s ;
- (v) $\mathcal{L}_s^h \text{Op}_h^{\delta, \rho}(a_h) \geq \mathcal{O}(h^\infty)_{L^\infty(\mathbb{R}_{\geq 0}, L^2 \rightarrow L^2)}$.

If such a_h is found, and assume that

$$(t_0\gamma|\xi_0|^{\gamma-2}\xi_0, \xi_0) \notin \text{WF}_{\rho(\gamma-1),\rho}^\mu(u(t=t_0)).$$

By (i) and (4.3.3), if we replace ϕ with $\phi(\lambda\cdot)$ for some $\lambda > 1$ sufficiently large, then for some compact set K , and sufficiently small $h > 0$,

$$\text{supp } \theta_{1/h,*}^{\beta,0} a_h|_{s=h^{-\beta}t_0} \subset K \subset \mathbb{R}_x^d \times \mathbb{R}_\xi^d \setminus \text{WF}_{\rho(\gamma-1),\rho}^\infty(u(t=t_0)).$$

By (iv), $\theta_{1/h,*}^{\beta,0} a_h|_{s=h^{-\beta}t_0} \in S_{-\infty}^-$ is elliptic at $(t_0\gamma|\xi_0|^{\gamma-2}\xi_0, \xi_0)$. Therefore, by Lemma 4.2.15,

$$(u, \text{Op}_h^{\delta,\rho}(a_h)u)_{L^2}|_{s=h^{-\beta}t_0} = (u, \text{Op}_h^{\rho(\gamma-1),\rho}(\theta_{1/h,*}^{\beta,0} a_h)u)_{L^2}|_{s=h^{-\beta}t_0} = \mathcal{O}(h^\infty).$$

By (4.3.1),

$$\frac{d}{ds}(u, \text{Op}_h^{\delta,\rho}(a_h)u)_{L^2} = (u, \mathcal{L}_s^h \text{Op}_h^{\delta,\rho}(a_h)u)_{L^2},$$

which implies by (v) that

$$\begin{aligned} (u, \text{Op}_h^{\delta,\rho}(a_h)u)_{L^2}|_{s=0} &= (u, \text{Op}_h^{\delta,\rho}(a_h)u)_{L^2}|_{s=h^{-\beta}t_0} - \int_0^{h^{-\beta}t_0} (u, \mathcal{L}_s^h \text{Op}_h^{\delta,\rho}(a_h)u)_{L^2} ds \\ &\leq \mathcal{O}(h^\infty) + \mathcal{O}(h^{-\beta} \times h^\infty) = \mathcal{O}(h^\infty). \end{aligned}$$

Therefore, by (ii),

$$\|\text{Op}_h^{\delta,\rho}(\chi_0)u|_{s=0}\|_{L^2}^2 = (u, \text{Op}_h^{\delta,\rho}(a_h)u)_{L^2}|_{s=0} + \mathcal{O}(h^\infty) = \mathcal{O}(h^\infty).$$

And we conclude that $(x_0, \xi_0) \notin \text{WF}_{\delta,\rho}^\infty(u_0)$.

We shall construct a_h in the following form of asymptotic expansion

$$a_h(s, x, \xi) \sim \sum_{j \in \mathbb{N}} h^{j(\delta+\rho)} \varphi^j(s) a_h^j(s, x, \xi),$$

where $a_h^j \in W^{\infty,\infty}(\mathbb{R}_{\geq 0}, S_0^{-\infty})$, with $\text{supp } a_h^j \subset \text{supp } \chi_j$, and $\varphi^j \in P_j$, with

$$(4.3.4) \quad P_j = \left\{ f(\ln(1+s)) : f(X) = \sum_{k=0}^j c_k X^k; c_k \geq 0, \forall k \right\}.$$

All functions belonging to P_j are smooth and non-negative for $s \geq 0$. Moreover, if $\psi \in P_j$, then

$$((1+s)\partial_s)^{-1}\psi(s) := \int_0^s (1+\sigma)^{-1}\psi(\sigma) d\sigma \in P_{j+1}.$$

Indeed,

$$((1+s)\partial_s)^{-1}(\ln(1+\cdot))^n = (n+1)^{-1}(\ln(1+\cdot))^{n+1}, \quad \forall n \in \mathbb{N}.$$

The above asymptotic expansion is in the weak sense that, for some $\epsilon' > 0$, and all $N \in \mathbb{N}$,

$$a_h - \sum_{j < N} h^{j(\delta+\rho)} \varphi^j a_h^j \in \mathcal{O}(h^{N(\delta+\rho-\epsilon')})_{W^{\infty,\infty}([0, h^{-\beta}T], S_0^{-\infty})}.$$

We begin by setting $\varphi^0 \equiv 1$ and choosing a_h^0 satisfying

$$\begin{aligned} a_h^0 - (\zeta_h^{\delta,\rho} \chi_0) \sharp_h^{\delta,\rho} \chi_0 &= \mathcal{O}(h^\infty)_{W^{\infty,\infty}(\mathbb{R}_{\geq 0}, S_0^{-\infty})}, \\ a_h^0(0, \cdot) - (\zeta_h^{\delta,\rho} \chi_0) \sharp_h^{\delta,\rho} \chi_0(0, \cdot) &= \mathcal{O}(h^\infty)_{S_{-\infty}^{-\infty}}. \end{aligned}$$

By the definition of β and Proposition 4.2.5, Proposition 4.2.6,

$$\mathcal{L}_s^h \text{Op}_h^{\delta,\rho}(a_h^0) = 2\text{Op}_h^{\delta,\rho}(\chi_0 \mathcal{L}_s \chi_0) + h^{\delta+\rho} \text{Op}_h^{\delta,\rho}(r_h^0) + \mathcal{O}(h^\infty)_{L^\infty(\mathbb{R}_{\geq 0}, L^2 \rightarrow L^2)},$$

where $r_h^0 \in L^\infty(\mathbb{R}_{\geq 0}, S_{-1}^{-\infty})$, with $\text{supp } r_h^0 \subset \text{supp } \chi_0$. So $\langle s \rangle r_h^0 \in L^\infty(\mathbb{R}_{\geq 0}, S_0^{-\infty})$. Similarly, $\langle s \rangle \chi_0 \mathcal{L}_s \chi_0 \in L^\infty(\mathbb{R}_{\geq 0}, S_0^{-\infty})$. By Lemma 4.3.2, $\chi_0 \mathcal{L}_s \chi_0 \geq 0$. So by the sharp Gårding inequality, $\exists b_h^0 \in L^\infty(\mathbb{R}_{\geq 0}, S_0^{-\infty})$, with $\text{supp } b_h^0 \subset \{\chi_1 > 0\}$, such that

$$\mathcal{L}_s^h \text{Op}_h^{\delta,\rho}(a_h^0) \geq -\langle s \rangle^{-1} h^{\delta+\rho} \text{Op}_h^{\delta,\rho}(b_h^0) + \mathcal{O}(h^\infty)_{L^\infty(\mathbb{R}_{\geq 0}, L^2 \rightarrow L^2)}.$$

Suppose that we have found $\varphi^j \in P_j$, a_h^j for $j = 0, \dots, \ell - 1$ and $\psi^{\ell-1} \in P_{\ell-1}$, $b_h^{\ell-1} \in L^\infty(\mathbb{R}_{\geq 0}, S_0^{-\infty})$, with $\text{supp } b_h^{\ell-1} \subset \{\chi_\ell > 0\}$, such that

(4.3.5)

$$\mathcal{L}_s^h \text{Op}_h^{\delta,\rho} \left(\sum_{j=0}^{\ell-1} h^{j(\delta+\rho)} \varphi^j a_h^j \right) \geq -\langle s \rangle^{-1} \psi^{\ell-1} h^{\ell(\delta+\rho)} \text{Op}_h^{\delta,\rho}(b_h^{\ell-1}) + \mathcal{O}(h^\infty)_{L^\infty(\mathbb{R}_{\geq 0}, L^2 \rightarrow L^2)}.$$

Then we set

$$\varphi^\ell = ((1+s)\partial_s)^{-1} \psi^{\ell-1}, \quad a_h^\ell(s, x, \xi) = B_\ell \chi_\ell(s, x, \xi),$$

for some constant $B_\ell > 0$ sufficiently large, such that

$$\mathcal{L}_s(\varphi^\ell a_h^\ell) = B_\ell (1+s)^{-1} \psi^{\ell-1} \chi_\ell + B_\ell \varphi^\ell \mathcal{L}_s \chi_\ell \geq B_\ell (1+s)^{-1} \psi^{\ell-1} \chi_\ell \geq \langle s \rangle^{-1} \psi^{\ell-1} b_h^{\ell-1}.$$

Observe that

$$\mathcal{L}_s(\varphi^\ell a_h^\ell) = \mathcal{O}(\langle s \rangle^{-1} (\psi^{\ell-1} + \varphi^\ell))_{S_0^{-\infty}}, \quad \langle s \rangle^{-1} \psi^{\ell-1} b_h^{\ell-1} = \mathcal{O}(\langle s \rangle^{-1} \psi^{\ell-1})_{S_0^{-\infty}}.$$

By the sharp Gårding inequality, $\exists b_h^\ell \in L^\infty(\mathbb{R}_{\geq 0}, S_0^{-\infty})$, with $\text{supp } b_h^\ell \subset \{\chi_{\ell+1} > 0\}$, such that

$$(4.3.6) \quad \mathcal{L}_s^h \text{Op}_h^{\delta,\rho}(\varphi^\ell a_h^\ell) - \langle s \rangle^{-1} \psi^{\ell-1} \text{Op}_h^{\delta,\rho}(b_h^{\ell-1}) \geq -\langle s \rangle^{-1} \psi^\ell h^{\delta+\rho} \text{Op}_h^{\delta,\rho}(b_h^\ell) + \mathcal{O}(h^\infty)_{L^2 \rightarrow L^2},$$

with $\psi^\ell = \psi^{\ell-1} + \varphi^\ell \in P_\ell$. Summing up (4.3.5) and $h^{\ell(\delta+\rho)} \times$ (4.3.6), we close the induction procedure.

We prove the asymptotic expansion by observing that, for all $\epsilon' > 0$,

$$\|\varphi^\ell\|_{L^\infty([0, h^{-\beta T}])} = \mathcal{O}(|\log h|^\ell) = \mathcal{O}(h^{-\epsilon' \ell}).$$

□

4.4 Paradifferential calculus

We develop a paradifferential calculus on weighted Sobolev spaces, and a semiclassical paradifferential calculus.

4.4.1 Classical paradifferential calculus

We recall some classical results of the paradifferential calculus. We refer to the original work of Bony [33], and the books [103, 146, 22]. The proofs below are mainly based on [146], so we shall only sketch them. In the meanwhile, we shall also make some refinements regarding the estimates of the remainder terms, for the sake of the semiclassical paradifferential calculus that will be developed later.

Definition 4.4.1. For $m \in \mathbb{R}$, $r \geq 0$, $\Gamma^{m,r}$ is the space of locally bounded functions $a(x, \xi)$ on $\mathbb{R}_x^d \times (\mathbb{R}_\xi^d \setminus 0)$ which are C^∞ with respect to ξ , and $\forall \alpha \in \mathbb{N}^d$, $\exists C_\alpha > 0$ such that

$$\|\partial_\xi^\alpha a(\cdot, \xi)\|_{W^{r,\infty}} \leq C_\alpha \langle \xi \rangle^{m-|\alpha|}, \quad \forall |\xi| \geq 1/2.$$

Moreover we denote

$$M^{m,r}(a) = \sup_{|\alpha| \leq 2(d+2)+r} \sup_{|\xi| \geq 1/2} \langle \xi \rangle^{|\alpha|-m} \|\partial_\xi^\alpha a(\cdot, \xi)\|_{W^{r,\infty}}.$$

Definition 4.4.2. $(\chi, \pi) \in C^\infty(\mathbb{R}_\theta^d \times \mathbb{R}_\eta^d \setminus \{(0,0)\}) \times C^\infty(\mathbb{R}_\eta^d)$ is called admissible if

- (1) $\pi(\eta) = 1$ for $|\eta| \geq 1$, $\pi(\eta) = 0$ for $|\eta| \leq 1/2$;
- (2) χ is an even function, homogeneous of degree 0, and for some $0 < \epsilon_1 < \epsilon_2 < 1$,

$$(4.4.1) \quad \begin{cases} \chi(\theta, \eta) = 1, & |\theta| \leq \epsilon_1 |\eta|, \\ \chi(\theta, \eta) = 0, & |\theta| \geq \epsilon_2 |\eta|. \end{cases}$$

Definition 4.4.3. For $a \in \Gamma^{m,0}$, $m \in \mathbb{R}$, the paradifferential operator T_a is defined by

$$(4.4.2) \quad \widehat{T_a u}(\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} \chi(\xi - \eta, \eta) \pi(\eta) \hat{a}(\xi - \eta, \eta) \hat{u}(\eta) d\eta,$$

where (χ, π) is admissible and

$$\hat{a}(\theta, \xi) = \int e^{-ix \cdot \theta} a(x, \xi) dx.$$

In other words, $T_a = \text{Op}(\sigma_a)$ with

$$\sigma_a(\cdot, \xi) = \pi(\xi) \chi(D_x, \xi) a(\cdot, \xi).$$

Proposition 4.4.4. Let $a \in \Gamma^{m,0}$, $m \in \mathbb{R}$, then $T_a = \mathcal{O}(M^{m,0}(a))_{\mathcal{O}_0^m}$.

Lemma 4.4.5. Let $a \in \Gamma^{m,r}$ with $m \in \mathbb{R}$ and $r \geq 0$, then $M^{m,r}(\sigma_a) \lesssim M^{m,r}(a)$. If $r \in \mathbb{N}$, then $\forall \beta \in \mathbb{N}^d$ with $|\beta| \leq r$,

$$(4.4.3) \quad M^{m-r+|\beta|,0}(\partial_x^\beta(\sigma_a - a\pi)) \lesssim M^{m,0}(\nabla^r a).$$

Démonstration. The first statement is proven in [146]. We only prove the second statement. There is no harm in assuming that $\beta = 0$. By [146], for $\xi \neq 0$,

$$(\sigma_a - a\pi)(x, \xi) = \pi(\xi) \int \rho(x, y, \xi) \Phi(y, \xi) dy,$$

where

$$\rho(x, y, \xi) = \sum_{|\gamma|=r} \frac{(-y)^\gamma}{\gamma!} \int_0^1 r(1-t)^{r-1} \partial_x^\gamma a(x-ty, \xi) dt$$

satisfies for $|\xi| \geq 1/2$ and $|\alpha| \leq 2(d+2) + r$ the estimates

$$\|\partial_\xi^\alpha \rho(\cdot, y, \xi)\|_{L^\infty} \lesssim |y|^r \|\partial_\xi^\alpha \nabla^r a(\cdot, \xi)\|_{L^\infty} \lesssim |y|^r |\xi|^{m-|\alpha|} M^{m,0}(\nabla^r a),$$

and $\Phi(\cdot, \xi) = \mathcal{F}^{-1} \chi(\cdot, \xi)$. We conclude with simple integral inequalities as in [146]. \square

Corollary 4.4.6. *Let $a \in \Gamma_0^{m,r}$ with $m \in \mathbb{R}$ and $r \in \mathbb{N}$. Let (χ, π) and (χ', π') be admissible. Denote by T_a and T'_a the paradifferential operators respectively defined by these two admissible pairs. Then*

$$T_a - T'_a = \mathcal{O}(M^{m,0}(\nabla^r a))_{\mathcal{O}_0^{m-r}} + \mathcal{O}(M^{m,r}(a))_{\mathcal{O}_0^{-\infty}}.$$

Démonstration. Let T''_a be the paradifferential operator defined with respect to (χ', π) , then by Lemma 4.4.5, $T_a - T''_a = \mathcal{O}(M^{m,0}(\nabla^r a))_{\mathcal{O}_0^{m-r}}$, while we easily verify that $T''_a - T'_a = \mathcal{O}(M^{m,r}(a))_{\mathcal{O}_0^{-\infty}}$. \square

Corollary 4.4.7. *Let $\psi \in W^{\infty,\infty}(\mathbb{R}_x^d)$, then $T_\psi - \psi \in \mathcal{O}_0^{-\infty}$.*

Proposition 4.4.8. *Let $a \in \Gamma_0^{m,r}$, $b \in \Gamma_0^{m',r}$ with $r \in \mathbb{N}$, $m \in \mathbb{R}$, $m' \in \mathbb{R}$, then*

$$\begin{aligned} T_a T_b - T_{a\sharp b} &= \mathcal{O}(M^{m,r}(a)M^{m',0}(\nabla^r b) + M^{m,0}(\nabla^r a)M^{m',r}(b))_{\mathcal{O}_0^{m+m'-r}} \\ &\quad + \mathcal{O}(M^{m,r}(a)M^{m',r}(b))_{\mathcal{O}_0^{-\infty}}. \end{aligned}$$

where $a\sharp b = \sum_{\alpha \in \mathbb{N}^d}^{\alpha < r} \frac{1}{\alpha!} \partial_\xi^\alpha a D_x^\alpha b$.

Remark 4.4.9. If $a\pi = a$ and $b\pi = b$, then by the proof below,

$$T_a T_b - T_{a\sharp b} = \mathcal{O}(M^{m,r}(a)M^{m',0}(\nabla^r b) + M^{m,0}(\nabla^r a)M^{m',r}(b))_{\mathcal{O}_0^{m+m'-r}}.$$

Démonstration. By Corollary 4.4.6, we may fix χ such that $\epsilon_2 < 1/4$. We decompose

$$T_a T_b - T_{a\sharp b} = \text{(I)} + \text{(II)},$$

where

$$\text{(I)} = \text{Op}(\sigma_a)\text{Op}(\sigma_b) - \text{Op}(\sigma_a\sharp\sigma_b), \quad \text{(II)} = \text{Op}(\sigma_a\sharp\sigma_b) - \text{Op}(\sigma_{a\sharp b}).$$

Let θ be admissible such that $\theta\chi = \chi$. Then by [146], $\text{Op}(\sigma_a)\text{Op}(\sigma_b) = \text{Op}(\sigma)$,

$$\begin{aligned} \sigma(x, \xi) &= \frac{1}{(2\pi)^d} \iint e^{i(x-y)\cdot\eta} \sigma_a(x, \xi + \eta) \theta(\eta, \xi) \sigma_b(y, \xi) dy d\eta \\ &= \sigma_a\sharp\sigma_b(x, \xi) + \sum_{|\alpha|=r} q_\alpha(x, \xi). \end{aligned}$$

where

$$q_\alpha(x, \xi) = \int R_\alpha(x, x-y, \xi) (D_x^\alpha \sigma_b)(y, \xi) dy,$$

with R_α satisfying

$$\|\partial_\xi^\beta R_\alpha(x, \cdot, \xi)\|_{L^1} \lesssim M^{m,r}(a) \langle \xi \rangle^{m-|\alpha|-|\beta|}.$$

Use $D_x^\alpha \sigma_b = \sigma_{D_x^\alpha b}$, we easily verify that

$$\|\partial_\xi^\beta q_\alpha(\cdot, \xi)\|_{L^\infty} \lesssim M^{m,0}(a) M^{m',0}(\nabla^r b) \langle \xi \rangle^{m+m'-|\alpha|-|\beta|},$$

and consequently, as in [146],

$$\|(I)\|_{H^s \rightarrow H^{s-m-m'+r}} \lesssim \sum_{|\alpha|=r} M^{m+m'-|\alpha|,0}(q_\alpha) \lesssim M^{m,r}(a) M^{m',0}(\nabla^r b).$$

To Estimate (II), for each $|\alpha| < r$, we decompose

$$\partial_\xi^\alpha \sigma_a D_x^\alpha \sigma_b - \sigma_{\partial_\xi^\alpha a D_x^\alpha b} = \text{(i)} + \text{(ii)} + \text{(iii)} + \text{(iv)},$$

where

$$\begin{aligned} \text{(i)} &= \partial_\xi^\alpha (\sigma_a - a\pi) D_x^\alpha \sigma_b, & \text{(ii)} &= \partial_\xi^\alpha (a\pi) D_x^\alpha (\sigma_b - b\pi), \\ \text{(iii)} &= \partial_\xi^\alpha (a\pi) D_x^\alpha (b\pi) - \sigma_{\partial_\xi^\alpha (a\pi) D_x^\alpha (b\pi)}, & \text{(iv)} &= \sigma_{\partial_\xi^\alpha (a\pi) D_x^\alpha (b\pi)} - \sigma_{\partial_\xi^\alpha a D_x^\alpha b}. \end{aligned}$$

By Lemma 4.4.5,

$$\begin{aligned} M^{m+m'-r,0} \text{(i)} &\lesssim M^{m-r,0} (\sigma_a - a\pi) M^{m',0} (D_x^\alpha \sigma_b) \lesssim M^{m,0} (\nabla^r a) M^{m',r}(b), \\ M^{m+m'-r,0} \text{(ii)} &\lesssim M^{m-|\alpha|,0} (\partial_\xi^\alpha a) M^{m'-r+|\alpha|,0} (D_x^\alpha (\sigma_b - b\pi)) \lesssim M^{m,0}(a) M^{m',0}(\nabla^r b), \end{aligned}$$

By Lemma 4.4.5, Leibniz's rule and interpolation,

$$\begin{aligned} M^{m+m'-r,0} \text{(iii)} &\lesssim M^{m+m'-|\alpha|,0} (\nabla^{r-|\alpha|} (\partial_\xi^\alpha a D_x^\alpha b)) \\ &\lesssim M^{m-|\alpha|,0} (\nabla^r \partial_\xi^\alpha a) M^{m',0}(b) + M^{m-|\alpha|,0} (\partial_\xi^\alpha a) M^{m',0}(\nabla^r b) \\ &\lesssim M^{m,0} (\nabla^r a) M^{m',r}(b) + M^{m,0}(a) M^{m',0}(\nabla^r b), \end{aligned}$$

And we easily verify that,

$$M^{-N,0} \text{(iv)} \lesssim M^{m,r}(a) M^{m',r}(b), \quad \forall N > 0.$$

These estimates implies that,

$$\begin{aligned} \text{(II)} &= \mathcal{O}(M^{m,r}(a) M^{m',0}(\nabla^r b) + M^{m,0}(\nabla^r a) M^{m',r}(b))_{\mathcal{O}_0^{m+m'-r}} \\ &\quad + \mathcal{O}(M^{m,r}(a) M^{m',r}(b))_{\mathcal{O}_0^{-\infty}}. \end{aligned}$$

□

Proposition 4.4.10. *Let $a \in \Gamma^{m,r}$ with $r \in \mathbb{N}$ and $m \in \mathbb{R}$, then*

$$T_a^* - T_{a^*} = \mathcal{O}(M^{m,0}(\nabla^r a))_{\mathcal{O}_0^{m-r}},$$

where $a^* = \sum_{\alpha \in \mathbb{N}^d} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \bar{a}$.

Démonstration. Let (θ, π) be admissible such that $\theta\chi = \chi$, then $T_a^* = \text{Op}(\sigma_a^*)$ with

$$\sigma_a^*(x, \xi) = (2\pi)^{-d} \int e^{-ix\cdot\eta} \overline{\sigma_a}(x+y, \xi+\eta) d\eta dy = a^*(x, \xi) + \sum_{|\alpha|=r} r_\alpha(x, \xi),$$

where by [146],

$$r_\alpha(x, \xi) = \frac{2\pi}{\alpha!} \iiint_{\mathbb{R}^{2d} \times [0,1]} r(1-t)^{r-1} e^{-iy\cdot\eta} D_x^\alpha \partial_\xi^\alpha \overline{\sigma_a}(x, \xi+t\eta) \theta(\eta, \xi) dt d\eta dy,$$

Similarly, observing the term $D_x^\alpha \partial_\xi^\alpha \overline{\sigma_a}(x, \xi+t\eta)$ in the integral, the analysis in [146] implies that

$$M^{m-r,0}(\sigma_a^* - \sigma_{a^*}) \leq \sum_{|\alpha|=r} M^{m-r,0}(r_\alpha) + M^{m-r,0}(a^* - \sigma_{a^*}) \lesssim M^{m,0}(\nabla^r a).$$

□

Proposition 4.4.11. *Let $a \in H^\alpha$ and $b \in H^\beta$ with $\alpha > d/2$, and $\beta > d/2$, then*

$$\|ab - T_a b - T_b a\|_{H^{\alpha+\beta-d/2}} \lesssim \|a\|_{H^\alpha} \|b\|_{H^\beta}.$$

Proposition 4.4.12. *Let $F \in C^\infty(\mathbb{R})$ such that $F(0) = 0$, then $\forall \mu > d/2$, there exists a monotonically increasing function $C : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, such that $\forall u \in H^\mu$,*

$$\|F(u)\|_{H^s} + \|F(u) - T_{F'(u)}u\|_{H^{2s-d/2}} \leq C(\|u\|_{H^s})\|u\|_{H^s}.$$

4.4.2 Dyadic paradifferential calculus

We develop the theory of paradifferential calculus on weighted Sobolev spaces.

Definition 4.4.13. Let $r \in \mathbb{N}$, $k \in \mathbb{R}$, and $0 \leq \delta \leq 1$. Given $u \in \mathcal{S}'$, we say that $u \in W_{k,\delta}^{r,\infty}$ if for all $\alpha \in \mathbb{N}$ with $|\alpha| \leq r$, $\langle x \rangle^{k+\delta|\alpha|} \partial_x^\alpha u \in L^\infty(\mathbb{R}^d)$.

Definition 4.4.14. Let $m \in \mathbb{R}$, $k \in \mathbb{R}$, $r \in \mathbb{N}$, $0 \leq \delta \leq 1$, $\Gamma_{k,\delta}^{m,r}$ is the space of locally bounded functions $a(x, \xi)$ on $\mathbb{R}_x^d \times (\mathbb{R}_\xi^d \setminus 0)$ which are C^∞ respect to ξ , and $\forall \alpha \in \mathbb{N}^d$, $\exists C_\alpha > 0$, such that

$$\|\partial_\xi^\alpha a(\cdot, \xi)\|_{W_{-k,\delta}^{r,\infty}} \leq C_\alpha \langle \xi \rangle^{m-|\alpha|}, \quad \forall |\xi| \geq 1/2.$$

Moreover, we denote

$$M_{k,\delta}^{m,r}(a) = \sup_{|\alpha| \leq 2(d+2)+r} \sup_{|\xi| \geq 1/2} \langle \xi \rangle^{|\alpha|-m} \|\partial_\xi^\alpha a(\cdot, \xi)\|_{W_{-k,\delta}^{r,\infty}}.$$

Let

$$\Gamma_{k,\delta}^{-\infty,r} = \bigcap_{m \in \mathbb{R}} \Gamma_{k,\delta}^{m,r}, \quad \Gamma_{-\infty,\delta}^{m,r} = \bigcap_{k \in \mathbb{R}} \Gamma_{k,\delta}^{m,r}.$$

Then for $(m, k) \in (\mathbb{R} \cup \{-\infty\})^2$ and $h > 0$, define

$$h \Sigma_{k,\delta}^{m,r} = \sum_{0 \leq j \leq r} h^j \Gamma_{k-\delta j,\delta}^{m-j,r-j}.$$

Let $f(h) > 0$ be a positive function of $0 < h < 1$, we denote $a_h \in f(h) \Sigma_{k,\delta}^{m,r}$ if $a_h = \mathcal{O}(1)_{f(h) \Sigma_{k,\delta}^{m,r}}$ for $0 < h < 1$.

Remark 4.4.15. Observe that $W_{0,0}^{r,\infty} = W^{r,\infty}$, $\Gamma_{0,0}^{m,r} = \Gamma^{m,r}$. We shall denote for simplicity

$$W_k^{r,\infty} = W_{k,0}^{r,\infty} \quad \Sigma_{k,\delta}^{m,r} = {}_1\Sigma_{k,\delta}^{m,r}, \quad {}_h\Sigma^{m,r} = {}_h\Sigma_{0,0}^{m,r}, \quad \Sigma^{m,r} = {}_1\Sigma_{0,0}^{m,r}.$$

Lemma 4.4.16. *Let $\mathcal{A} : \mathcal{S} \rightarrow \mathcal{S}'$, $\psi, \phi \in \mathcal{P}$, $(m, k) \in \mathbb{R}^2$. If $\forall (\mu, \ell) \in \mathbb{R}^2$, $\exists C > 0$, such that $\forall u \in \mathcal{S}$, $\|\psi_j \mathcal{A}u\|_{H^{\mu-m}} \leq C 2^{jk} \|\phi_j u\|_{H^\mu}$, then $\mathcal{A} \in \mathcal{O}_k^m$. Particularly, let $\{\mathcal{A}_j\}_{j \in \mathbb{N}} \in \ell^\infty(\mathcal{O}_0^m)$, then*

$$\mathcal{A} := \sum_{j \in \mathbb{N}} 2^{2jk} \psi_j \mathcal{A}_j \phi_j \in \mathcal{O}_k^m.$$

Definition 4.4.17. Fix $\psi \in \mathcal{P}_*$, and define $\underline{\psi} \in \mathcal{P}$ by setting $\underline{\psi}_j = \sum_{|j-k| \leq 10} \psi_k$. Let $a \in \Gamma_{k,\delta}^{m,r}$, the dyadic paradifferential operator \mathcal{P}_a is defined by

$$\mathcal{P}_a = \sum_{j \in \mathbb{N}} \underline{\psi}_j T_{\psi_j a} \underline{\psi}_j.$$

Proposition 4.4.18. *Let $a \in \Gamma_{k,\delta}^{m,r}$, then $\mathcal{P}_a = \mathcal{O}(M_{k,0}^{m,0}(a))_{\mathcal{O}_k^m}$.*

Démonstration. Observe that $\|T_{\psi_j a}\|_{H^\nu \rightarrow H^{\nu-m}} \lesssim M^{m,0}(\psi_j a) \lesssim 2^{jk} M_{k,0}^{m,0}(a)$. \square

Proposition 4.4.19. *Let $a \in \Gamma_{k,\delta}^{m,r}$, $b \in \Gamma_{k',\delta}^{m',r}$, $r \in \mathbb{N}$, $(m, k), (m', k') \in \mathbb{R}^2$, $0 \leq \delta \leq 1$, then*

$$\mathcal{P}_a \mathcal{P}_b - \mathcal{P}_{a\sharp b} = \mathcal{O}(M_{k,\delta}^{m,r}(a) M_{k',\delta}^{m',r}(b))_{\mathcal{O}_{k+k'-\delta r}^{m+m'-r} + \mathcal{O}_{k+k'}^{-\infty}}.$$

where $a\sharp b = \sum_{|\alpha| < r} \frac{1}{\alpha!} \partial_\xi^\alpha a D_x^\alpha b \in \Sigma_{k+k',\delta}^{m+m',r}$.

Démonstration. Let $\tilde{\psi}_j : \mathbb{N} \rightarrow C_c^\infty$, $\tilde{\psi}_j = \sum_{|j-j'| \leq 50} \psi_{j'}$, so $\underline{\psi}_j \tilde{\psi}_j = \underline{\psi}_j$ if $|j - j'| \leq 20$. Then write,

$$\mathcal{P}_a \mathcal{P}_b = \sum_{(j,j') \in \mathbb{N}^2} \sum_{|j-j'| \leq 20} \underline{\psi}_j T_{\psi_j a} \underline{\psi}_j \underline{\psi}_{j'} T_{\psi_{j'} b} \underline{\psi}_{j'} = \sum_{(j,j') \in \mathbb{N}^2} \sum_{|j-j'| \leq 20} \tilde{\psi}_j T_{\psi_j a} T_{\psi_{j'} b} \tilde{\psi}_j + \tilde{\psi}_j R_{j,j'} \tilde{\psi}_j,$$

the remainder being

$$\begin{aligned} R_{j,j'} &= \underline{\psi}_j T_{\psi_j a} \underline{\psi}_j \underline{\psi}_{j'} T_{\psi_{j'} b} \underline{\psi}_{j'} - T_{\psi_j a} T_{\psi_{j'} b} \\ &= \mathcal{O}(2^{j(k+k'-\delta r)} M_{k,\delta}^{m,r}(a) M_{k',\delta}^{m',r}(b))_{\mathcal{O}_0^{m+m'-r}} + \mathcal{O}(2^{j(k+k')} M_{k,\delta}^{m,r}(a) M_{k',\delta}^{m',r}(b))_{\mathcal{O}_0^{-\infty}} \end{aligned}$$

by Proposition 4.4.4, Proposition 4.4.8 and Corollary 4.4.7. More precisely, when composing $T_{\psi_j a}$ and $T_{\underline{\psi}_j}$, we use $\psi_j \underline{\psi}_j = \psi_j$ and have

$$\begin{aligned} T_{\psi_j a} T_{\underline{\psi}_j} &= T_{\psi_j a} + \mathcal{O}(M^{m,r}(\psi_j a) M^{0,0}(\nabla^r \underline{\psi}_j))_{\mathcal{O}_0^{m-r}} \\ &\quad + \mathcal{O}(M^{m,0}(\nabla^r(\psi_j a)) M^{0,r}(\underline{\psi}_j))_{\mathcal{O}_0^{m-r}} \\ &\quad + \mathcal{O}(M^{m,r}(\psi_j a) M^{0,r}(\underline{\psi}_j))_{\mathcal{O}_0^{-\infty}}, \end{aligned}$$

where $M^{0,0}(\nabla^r \underline{\psi}_j) = \mathcal{O}(2^{-jr})$, $M^{m,r}(\psi_j a) = \mathcal{O}(2^{jk})$, and we use $0 \leq \delta \leq 1$ to induce that

$$M^{m,0}(\nabla^r(\psi_j a)) = \mathcal{O}\left(\max_{0 \leq n \leq r} \{2^{-j(r-n)+j(k-\delta n)}\}\right) = \mathcal{O}(2^{j(k-\delta r)}).$$

Similar arguments work for the composition $T_{\underline{\psi}_j} T_{\psi_j a}$.

Next observe that

$$\sum_{j': |j-j'| \leq 20} (\psi_j a) \# \psi_{j'} b = (\psi_j a) \# b.$$

Hence

$$\sum_{j': |j-j'| \leq 20} T_{\psi_j a} T_{\psi_{j'} b} = \underline{\psi}_j T_{\psi_j (a \# b)} \underline{\psi}_j + R_j,$$

where the remainder can be estimated similarly as above,

$$R_j = \mathcal{O}(2^{j(k+k'-\delta r)} M_{k,\delta}^{m,r}(a) M_{k',\delta}^{m',r}(b)) \mathcal{O}_0^{m+m'-\delta r} + \mathcal{O}(2^{j(k+k')} M_{k,\delta}^{m,r}(a) M_{k',\delta}^{m',r}(b)) \mathcal{O}_0^{-\infty}.$$

We conclude by Lemma 4.4.16. \square

Proposition 4.4.20. *Let $a \in \Gamma_{k,\delta}^{m,r}$ with $(m, k) \in \mathbb{R}^2$, and $r \in \mathbb{N}$, $0 \leq \delta \leq 1$, then*

$$(4.4.4) \quad \mathcal{P}_a^* - \mathcal{P}_{a^*} = \mathcal{O}(M_{k,\delta}^{m,r}(a)) \mathcal{O}_{k-\delta r}^{m-r} + \mathcal{O}_k^{-\infty},$$

where $a^* = \sum_{\alpha \in \mathbb{N}^d} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \bar{a} \in \Sigma_{k,\delta}^{m,r}$.

Démonstration. Observe that for any real valued $\psi \in C_c^\infty(\mathbb{R}^d)$,

$$(4.4.5) \quad (\psi a)^* = a^* \# \psi.$$

More precisely, this means that,

$$\begin{aligned} (\psi a)^* &= \sum_{|\gamma| < r} \frac{1}{\gamma!} \partial_\xi^\gamma D_x^\gamma (\psi \bar{a}) = \sum_{|\gamma| < r} \frac{1}{\gamma!} \sum_{\alpha + \beta = \gamma} \frac{\gamma!}{\alpha! \beta!} D_x^\alpha \psi \partial_\xi^\beta D_x^\beta \bar{a} \\ &= \sum_{|\alpha| + |\beta| < r} \frac{1}{\alpha!} \partial_\xi^\alpha \left(\frac{1}{\beta!} \partial_\xi^\beta D_x^\beta \bar{a} \right) D_x^\alpha \psi = \sum_{|\beta| < r} \sum_{|\alpha| < r - |\beta|} \frac{1}{\alpha!} \partial_\xi^\alpha \left(\frac{1}{\beta!} \partial_\xi^\beta D_x^\beta \bar{a} \right) D_x^\alpha \psi \\ &= \sum_{|\beta| < r} \left(\frac{1}{\beta!} \partial_\xi^\beta D_x^\beta \bar{a} \right) \# \psi = a^* \# \psi. \end{aligned}$$

Then write

$$\mathcal{P}_a^* - \mathcal{P}_{a^*} = \sum_{j \in \mathbb{N}} \underline{\psi}_j (R_j^1 + R_j^2) \underline{\psi}_j,$$

where by (4.4.5),

$$R_j^1 = T_{\psi_j a}^* - T_{(\psi_j a)^*}, \quad R_j^2 = T_{(\psi_j a)^*} - T_{\psi_j a^*} = T_{a^* \# \psi_j - \psi_j a^*}.$$

For R_j^1 we use Proposition 4.4.10,

$$R_j^1 = \mathcal{O}(M^{m,0}(\nabla_x^r(\psi_j a))) \mathcal{O}_0^{m-r} = \mathcal{O}(2^{j(k-\delta r)} M_{k,\delta}^{m,r}(a)) \mathcal{O}_0^{m-r}.$$

By Lemma 4.4.16,

$$\sum_{j \in \mathbb{N}} \underline{\psi}_j R_j^1 \underline{\psi}_j = \mathcal{O}(M_{k,\delta}^{m,r}(a)) \mathcal{O}_{k-\delta r}^{m-r} + \mathcal{O}_k^{-\infty}.$$

Using $\sum_{j \in \mathbb{N}} \psi_j \equiv 1$, we induce that

$$(4.4.6) \quad \sum_{j \in \mathbb{N}} \partial_x^\alpha \psi_j \equiv 0, \forall \alpha \in \mathbb{N}^d \setminus 0; \quad \sum_{j \in \mathbb{N}} a^* \# \psi_j - \psi_j a^* = 0.$$

Then we write

$$a^* \# \psi_j - \psi_j a^* = \sum_{\substack{\alpha \neq 0 \\ |\alpha| + |\beta| < r}} D_x^\alpha \psi_j \cdot w_{\alpha\beta}, \quad w_{\alpha\beta} \in \Gamma_{k-|\beta|\delta, \delta}^{m-|\alpha|-|\beta|, r-|\beta|},$$

where the symbols $w_{\alpha\beta}$ are independent of j . Set

$$R_{\alpha\beta} = \sum_{j \in \mathbb{N}} \underline{\psi}_j T_{D_x^\alpha \psi_j \cdot w_{\alpha\beta}} \underline{\psi}_j,$$

then

$$\sum_{j \in \mathbb{N}} \underline{\psi}_j R_j^2 \underline{\psi}_j = \sum_{\alpha, \beta} R_{\alpha\beta}.$$

By (4.4.6), we prove similarly as in Proposition 4.4.19 that

$$\begin{aligned} \psi_j R_{\alpha\beta} &= \psi_j \sum_{|j-j'| \leq 20} \underline{\psi}_{j'} T_{D_x^\alpha \psi_{j'} \cdot w_{\alpha\beta}} \underline{\psi}_{j'} \\ &= \mathcal{O}(2^{j(-|\alpha|+k-|\beta|\delta-(r-|\beta|)\delta)} M_{k-|\beta|\delta, \delta}^{m-|\alpha|-|\beta|, r-|\beta|}(w_{\alpha\beta})) \mathcal{O}_0^{m-r-|\alpha|} \\ &\quad + \mathcal{O}(2^{j(-|\alpha|+k-|\beta|\delta)} M_{k-|\beta|\delta, \delta}^{m-|\alpha|-|\beta|, r-|\beta|}(w_{\alpha\beta})) \mathcal{O}_0^{-\infty} \\ &= \mathcal{O}(2^{j(k-\delta r)} M_{k, \delta}^{m, r}(a)) \mathcal{O}_0^{m-r} + \mathcal{O}(2^{jk} M_{k, \delta}^{m, r}(a)) \mathcal{O}_0^{-\infty}. \end{aligned}$$

Setting $\psi'_j = \sum_{|j'-j| \leq 100} \psi_{j'}$. We again conclude by Lemma 4.4.16, and the identity

$$R_{\alpha\beta} = \sum_{j \in \mathbb{N}} \psi_j R_{\alpha\beta} \psi'_j,$$

that $R_{\alpha\beta} = \mathcal{O}(M_{k, \delta}^{m, r}(a)) \mathcal{O}_{k-\delta r}^{m-r} + \mathcal{O}_0^{-\infty}$. □

Proposition 4.4.21. *Let $a \in H_k^\alpha$, $b \in H_\ell^\beta$, with $\alpha > d/2$, $\beta > d/2$, $k \in \mathbb{R}$, $\ell \in \mathbb{R}$, then $\forall \epsilon > 0$,*

$$\|ab - \mathcal{P}_a b - \mathcal{P}_b a\|_{H_{k+\ell}^{\alpha+\beta-d/2-\epsilon}} \lesssim \|a\|_{H_k^\alpha} \|b\|_{H_\ell^\beta}.$$

Démonstration. Decompose the product ab as follows,

$$ab = \sum_{j \in \mathbb{N}} \psi_j (\underline{\psi}_j a) (\underline{\psi}_j b) = \mathcal{P}_a b + \mathcal{P}_b a + R_j^1 + R_j^2,$$

where

$$\begin{aligned} R_j^1 &= \psi_j (\underline{\psi}_j a \cdot \underline{\psi}_j b - T_{\underline{\psi}_j a}(\underline{\psi}_j b) - T_{\underline{\psi}_j b}(\underline{\psi}_j a)), \\ R_j^2 &= \underline{\psi}_j (\psi_j T_{\underline{\psi}_j a} - T_{\psi_j a}) \underline{\psi}_j b + \underline{\psi}_j (\psi_j T_{\underline{\psi}_j b} - T_{\psi_j b}) \underline{\psi}_j a. \end{aligned}$$

By Proposition 4.4.11,

$$\|R_j^1\|_{H^{\alpha+\beta-d/2}} \lesssim \|\underline{\psi}_j a\|_{H^\alpha} \|\underline{\psi}_j b\|_{H^\beta} \lesssim 2^{-j(k+\ell)} \|a\|_{H_k^\alpha} \|b\|_{H_\ell^\beta}.$$

By Proposition 4.4.8 and Corollary 4.4.7 and Sobolev injection,

$$\begin{aligned} \psi_j T_{\underline{\psi}_j a} \underline{\psi}_j - T_{\psi_j a} &= 2^{-jk} \mathcal{O}(\|a\|_{H_k^\alpha}) \sigma_0^{\alpha-d/2-\epsilon}, \\ \psi_j T_{\underline{\psi}_j b} \underline{\psi}_j - T_{\psi_j b} &= 2^{-j\ell} \mathcal{O}(\|b\|_{H_\ell^\beta}) \sigma_0^{\beta-d/2-\epsilon}, \quad \forall \epsilon > 0. \end{aligned}$$

We conclude by Proposition 4.2.13. \square

Proposition 4.4.22. *Suppose $F \in C^\infty(\mathbb{R})$ with $F(0) = 0$, then for $\mu > d/2$, there exists some monotonically increasing function $C : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, such that for $u \in H_k^\mu$, with $k \geq 0$,*

$$\|F(u)\|_{H_k^\mu} + \|F(u) - \mathcal{P}_{F'(u)} u\|_{H_k^{2\mu-d/2}} \leq C(\|u\|_{H^\mu}) \|u\|_{H_k^\mu}.$$

Démonstration. Decompose

$$F(u) = \sum_{j \geq 0} \psi_j F(\underline{\psi}_j u).$$

By Proposition 4.4.12,

$$\|F(\underline{\psi}_j u)\|_{H^\mu} \leq C(\|\underline{\psi}_j u\|_{H^\mu}) \|\underline{\psi}_j u\|_{H^\mu} \leq C(\|u\|_{H^\mu}) \|\underline{\psi}_j u\|_{H^\mu}.$$

Then write

$$\psi_j F(\underline{\psi}_j u) = \underline{\psi}_j T_{\psi_j F'(u)} \underline{\psi}_j u + \underline{\psi}_j R_j,$$

with

$$R_j = \psi_j (F(\underline{\psi}_j u) - T_{F'(\underline{\psi}_j u)} \underline{\psi}_j u) + \underline{\psi}_j (\psi_j T_{F'(\underline{\psi}_j u)} - T_{\psi_j F'(\underline{\psi}_j u)}) \underline{\psi}_j u.$$

By Proposition 4.4.12, Proposition 4.4.11 and Corollary 4.4.7,

$$\|R_j\|_{H^{2\mu-d/2}} \leq C(\|u\|_{H^\mu}) \|\underline{\psi}_j u\|_{H^\mu}.$$

We conclude with Proposition 4.2.13. \square

4.4.3 Semiclassical paradifferential calculus

We develop a semiclassical dyadic paradifferential calculus, and a quasi-homogeneous semiclassical paradifferential calculus, using scaling arguments inspired by Métivier–Zumbrun [147].

Definition 4.4.23. Define the scaling operator $\tau_h : u(\cdot) \mapsto h^d u(h\cdot)$. For $b \in \Gamma^{m,r}$, $h > 0$, we define the semiclassical paradifferential operator

$$T_b^h = \tau_h^{-1} T_{\theta_{h,*}^{1,0} b} \tau_h.$$

For $a \in \Gamma_{k,\delta}^{m,r}$, $h > 0$, we define the semiclassical dyadic paradifferential operator

$$\mathcal{P}_a^h = \sum_{j \in \mathbb{N}} \underline{\psi}_j T_{\psi_j a}^h \underline{\psi}_j.$$

For $\epsilon \geq 0$, we define the quasi-homogeneous semiclassical paradifferential operator

$$\mathcal{P}_a^{h,\epsilon} = \mathcal{P}_{\theta_{h,*}^{\epsilon,0} a}^h.$$

Proposition 4.4.24. *If $m \leq 0$, and $k \leq 0$, then $\sup_{0 < h < 1} \|\mathcal{P}_a^{h,\epsilon}\|_{L^2 \rightarrow L^2} < \infty$.*

Démonstration. Observe that $\theta_{h,*}^{1+\epsilon,0} a = \mathcal{O}(1)_{\Gamma^{0,0}}$. We conclude with Lemma 4.4.16. \square

Definition 4.4.25. For $\epsilon \geq 0$, $a_h \in \mathcal{D}'(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$, we say $a_h \in \sigma_\epsilon$ if

$$\overline{\bigcup_{0 < h < 1} \text{supp } a_h} \cap \mathcal{N}_{\epsilon,1} = \emptyset,$$

recalling the definition (4.2.1) of $\mathcal{N}_{\delta,\rho}$.

Proposition 4.4.26. *Let $(m, k), (m', k') \in (\mathbb{R} \cup \{-\infty\})^2$, $r \in \mathbb{N}$, with $r \geq m + m'$, $\delta r \geq k + k'$. Let $a_h \in \Gamma_{k,\delta}^{m,r} \cap \sigma_0$, $b_h \in \Gamma_{k',\delta}^{m',r} \cap \sigma_0$, such that for some $R_h \geq 0$ depending on h ,*

$$(4.4.7) \quad \text{supp } a_h \cap \text{supp } b_h \subset \{|x| \geq R_h\} \times \mathbb{R}_\xi^d,$$

then for $h > 0$ sufficiently small,

$$\mathcal{P}_{a_h}^h \mathcal{P}_{b_h}^h - \mathcal{P}_{a_h \#_h b_h}^h = \mathcal{O}(h^r (1 + R_h)^{k+k'-\delta r})_{L^2 \rightarrow L^2},$$

where $a_h \#_h b_h = \sum_{\alpha \in \mathbb{N}^d} \frac{h^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a_h D_x^\alpha b_h \in {}_h\Sigma_{k+k',\delta}^{m+m',r}$.

Démonstration. By (4.4.7), $\psi_j a_h \neq 0$ and $\psi_{j'} b_h \neq 0$ implies that $j \gtrsim \log_2(1 + R_h)$. We claim that

$$\begin{aligned} \mathcal{P}_a^h \mathcal{P}_b^h &= \sum_{\substack{j \gtrsim \log_2(1+R_h) \\ |j'-j| \leq 20}} \underline{\psi}_j T_{\psi_j a_h}^h \underline{\psi}_{j'} \underline{\psi}_{j'} T_{\psi_{j'} b_h}^h \underline{\psi}_{j'} \\ &= \sum_{\substack{j \gtrsim \log_2(1+R_h) \\ |j'-j| \leq 20}} \underline{\psi}_j T_{(\psi_j a_h) \#_h (\psi_{j'} b_h)}^h \underline{\psi}_{j'} + \mathcal{O}(h^r (1 + R_h)^{k+k'-\delta r})_{L^2 \rightarrow L^2}. \end{aligned}$$

And then we can conclude by the identity

$$\sum_{j': |j'-j| \leq 20} (\psi_j a_h) \#_h (\psi_{j'} b_h) = \psi_j (a_h \#_h b_h).$$

Indeed, we use (4.4.1) to induce that $\mathcal{F}(T_{\theta_{h,*}^{1,0}(\psi_{j'} b_h)} u)$ vanishes in a neighborhood of $\xi = 0$. By (4.4.3), for some $\pi' \in C^\infty(\mathbb{R}^d)$ which vanishes near $\xi = 0$ and equals to 1 outside a neighborhood of $\xi = 0$, and for all $m + m' \leq N \in \mathbb{N}$,

$$\begin{aligned} \tau_h T_{\psi_j a_h}^h \underline{\psi}_j \underline{\psi}_{j'} T_{\psi_{j'} b_h}^h \tau_h^{-1} &= T_{\theta_{h,*}^{1,0}(\psi_j a_h)} \theta_{h,*}^{1,0}(\underline{\psi}_j \underline{\psi}_{j'}) \pi'(D_x) T_{\theta_{h,*}^{1,0}(\psi_{j'} b_h)} \\ &= T_{\theta_{h,*}^{1,0}(\psi_j a_h)} T_{\theta_{h,*}^{1,0}(\underline{\psi}_j \underline{\psi}_{j'})} \otimes \pi' T_{\theta_{h,*}^{1,0}(\psi_{j'} b_h)} \\ &\quad + \mathcal{O}(M^{m,0}(\psi_j a_h))_{\mathcal{O}_0^m} \mathcal{O}(2^{-jN} h^N)_{\mathcal{O}_0^{-N}} \mathcal{O}(M^{m',0}(\psi_{j'} b_h))_{\mathcal{O}_0^{m'}}. \end{aligned}$$

Then we use Proposition 4.4.8, Remark 4.4.9, and the fact that $a_h, b_h \in \sigma_0$ to deduce

$$\begin{aligned} T_{\theta_{h,*}^{1,0}(\psi_j a_h)} T_{\theta_{h,*}^{1,0}(\underline{\psi}_j \underline{\psi}_{j'})} \otimes \pi' T_{\theta_{h,*}^{1,0}(\psi_{j'} b_h)} \\ = T_{\theta_{h,*}^{1,0}(\psi_j a_h) \#_h \theta_{h,*}^{1,0}(\underline{\psi}_j \underline{\psi}_{j'}) \otimes \pi'} \#_h \theta_{h,*}^{1,0}(\psi_{j'} b_h) \end{aligned}$$

$$\begin{aligned}
 & + \mathcal{O}(M^{m,0}(\nabla^r \theta_{h,*}^{1,0}(\psi_j a_h)) M^{0,r}(\theta_{h,*}^{1,0}(\psi_j \psi_{j'})) M^{m',r}(\theta_{h,*}^{1,0}(\psi_{j'} b_h)))_{\mathcal{O}_0^{m+m'-r}} \\
 & + \mathcal{O}(M^{m,r}(\theta_{h,*}^{1,0}(\psi_j a_h)) M^{0,0}(\nabla^r \theta_{h,*}^{1,0}(\psi_j \psi_{j'})) M^{m',r}(\theta_{h,*}^{1,0}(\psi_{j'} b_h)))_{\mathcal{O}_0^{m+m'-r}} \\
 & + \mathcal{O}(M^{m,r}(\theta_{h,*}^{1,0}(\psi_j a_h)) M^{0,r}(\theta_{h,*}^{1,0}(\psi_j \psi_{j'})) M^{m',0}(\nabla^r \theta_{h,*}^{1,0}(\psi_{j'} b_h)))_{\mathcal{O}_0^{m+m'-r}} \\
 & = T_{\theta_{h,*}^{1,0}((\psi_j a_h) \sharp_h (\psi_{j'} b_h))} + \mathcal{O}(h^r (1 + R_h)^{k+k'-\delta r})_{L^2 \rightarrow L^2}.
 \end{aligned}$$

To estimate the remainders, we see that for each $\alpha \in \mathbb{N}^d$ with $|\alpha| = r$,

$$\begin{aligned}
 \partial_x^\alpha \theta_{h,*}^{1,0}(\psi_j a_h) & = \sum_{\alpha_1 + \alpha_2 = \alpha} \frac{\alpha!}{\alpha_1! \alpha_2!} \partial_x^{\alpha_1} \theta_{h,*}^{1,0} \psi_j \partial_x^{\alpha_2} \theta_{h,*}^{1,0} a_h \\
 & = \sum_{\alpha_1 + \alpha_2 = \alpha} \frac{\alpha!}{\alpha_1! \alpha_2!} \mathcal{O}(h^{|\alpha_1|} 2^{-j|\alpha_1|} \times h^{|\alpha_2|} 2^{j(k-\delta|\alpha_2|)})_{L^\infty} = \mathcal{O}(h^r 2^{j(k-\delta r)})_{L^\infty},
 \end{aligned}$$

where we use $0 \leq \delta \leq 1$. Therefore, the first term in the remainder is

$$\mathcal{O}(h^r 2^{j(k+k'-\delta r)})_{L^2 \rightarrow L^2} = \mathcal{O}(h^r (1 + R_h)^{k+k'-\delta r})_{L^2 \rightarrow L^2}.$$

Similar methods apply to the other two terms. \square

Combining the analysis of Proposition 4.4.26, Proposition 4.4.20, using Proposition 4.4.8, we obtain a similar result for the adjoint, whose proof we shall omit, as it is similar as above.

Proposition 4.4.27. *Let $(m, k) \in (\mathbb{R} \cup \{-\infty\})^2$, $r \in \mathbb{N}$, with $r \geq m$, $\delta r \geq k$. Let $a_h \in \Gamma_{k,\delta}^{m,r} \cap \sigma_0$, such that for some $R_h \geq 0$ depending on h ,*

$$\text{supp } a_h \subset \{|x| \geq R_h\} \times \mathbb{R}_\xi^d,$$

then for $h > 0$ sufficiently small,

$$(\mathcal{P}_{a_h}^h)^* - \mathcal{P}_{a_h^*}^h = \mathcal{O}(h^r (1 + R_h)^{k-\delta r})_{L^2 \rightarrow L^2},$$

where $a_h^* = \sum_{\alpha \in \mathbb{N}^d} \frac{h^{|\alpha|}}{\alpha!} \partial_\xi^\alpha D_x^\alpha a_h \in {}_h\Sigma_{k,\delta}^{m,r}$.

Corollary 4.4.28. *Let $\epsilon \geq 0$, $(m, k), (m', k') \in (\mathbb{R} \cup \{-\infty\})^2$, $r \in \mathbb{N}$ with $r \geq \max\{m + m', k'\}$, $k \leq 0$. Let $a_h \in \Gamma_{k,1}^{m,r} \cap \sigma_\epsilon$, $b_h \in \Gamma_{k',1}^{m',r} \cap \sigma_0$. Then*

$$\begin{aligned}
 \mathcal{P}_{a_h}^{h,\epsilon} \mathcal{P}_{b_h}^h - \mathcal{P}_{(\theta_{h,*}^{\epsilon,0} a_h) \sharp_h b_h}^h & = \mathcal{O}(h^{(1+\epsilon)r - \epsilon(k+k')})_{L^2 \rightarrow L^2}, \\
 \mathcal{P}_{b_h}^h \mathcal{P}_{a_h}^{h,\epsilon} - \mathcal{P}_{b_h \sharp_h (\theta_{h,*}^{\epsilon,0} a_h)}^h & = \mathcal{O}(h^{(1+\epsilon)r - \epsilon(k+k')})_{L^2 \rightarrow L^2}.
 \end{aligned}$$

Démonstration. It suffices to observe that, if $\epsilon > 0$ then $\text{supp } \theta_{h,*}^{\epsilon,0} a_h \subset \{|x| \gtrsim h^{-\epsilon}\}$, $\theta_{h,*}^{\epsilon,0} a_h = \mathcal{O}(h^{-\epsilon k})_{\Gamma_{0,1}^{m,r}}$. We conclude by Proposition 4.4.26. \square

Corollary 4.4.29. *Let $\epsilon \geq 0$, $(m, k), (m', k') \in (\mathbb{R} \cup \{-\infty\})^2$, $r \in \mathbb{N}$, with $r \geq m + m'$, $k \leq 0$, $k' \leq 0$. Let $a_h \in \Gamma_{k,1}^{m,r} \cap \sigma_\epsilon$, $b_h \in \Gamma_{k',1}^{m',r} \cap \sigma_\epsilon$. Then for $h > 0$ sufficiently small,*

$$\mathcal{P}_{a_h}^{h,\epsilon} \mathcal{P}_{b_h}^{h,\epsilon} - \mathcal{P}_{a_h \sharp_h^\epsilon b_h}^{h,\epsilon} = \mathcal{O}(h^{(1+\epsilon)r - \epsilon(k+k')})_{L^2 \rightarrow L^2},$$

where $a_h \sharp_h^\epsilon b_h = \sum_{\alpha \in \mathbb{N}^d} \frac{h^{(1+\epsilon)|\alpha|}}{\alpha!} \partial_\xi^\alpha a_h D_x^\alpha b_h \in {}_{h^{1+\epsilon}}\Sigma_{k+k',1}^{m+m',r}$.

Démonstration. It suffices to use the identity $(\theta_{h,*}^{\epsilon,0} a_h) \#_h (\theta_{h,*}^{\epsilon,0} b_h) = \theta_{h,*}^{\epsilon,0} (a_h \#_h^\epsilon b_h)$. \square

The results above only concerned about the high frequency regime. The next lemma studies the interaction of high frequencies and low frequencies.

Lemma 4.4.30. *Let $m \in \mathbb{R}$, $a_h \in \Gamma^{0,0}$, $b_h \in \Gamma^{0,0}$, such that for some $R > 0$,*

$$\text{supp } a_h \in \{|\xi| \geq R\}, \quad \text{supp } b_h \in \{|\xi| \leq h^{-1}R/4\}.$$

Then $\mathcal{P}_{a_h}^h \mathcal{P}_{b_h} = \mathcal{O}(h^\infty)_{L^2 \rightarrow L^2}$.

Démonstration. By definition

$$\widehat{T_{\psi_j b_h} u}(\xi) = (2\pi)^{-d} \int \chi(\xi - \eta, \eta) \pi(\eta) \widehat{\psi_j b_h}(\xi - \eta, \eta) \widehat{u}(\eta) d\eta.$$

The admissibility of χ implies that

$$\text{supp } \widehat{T_{\psi_j b_h} u} \subset \{|\xi| \leq h^{-1}R/3\}.$$

Therefore, for any $|j' - j| \leq 20$,

$$\begin{aligned} \underline{\psi}_j T_{\psi_j a_h} \underline{\psi}_j \underline{\psi}_{j'} T_{\psi_{j'} b_h} \underline{\psi}_{j'} &= \underline{\psi}_j T_{\psi_j a_h} \pi(hD_x/R) \underline{\psi}_j \underline{\psi}_{j'} (1 - \pi(2hD_x/R)) T_{\psi_{j'} b_h} \underline{\psi}_{j'} \\ &= \underline{\psi}_j \mathcal{O}(h^\infty)_{L^2 \rightarrow L^2} \underline{\psi}_{j'}. \end{aligned}$$

We conclude by Lemma 4.4.16. \square

Corollary 4.4.31. *Let $a \in \Gamma^{m,0}$ which is homogeneous of degree m with respect to ξ , then*

$$h^m \mathcal{P}_a = \mathcal{P}_a^h + R_h,$$

where for $b \in \Gamma^{0,0} \cap \sigma_0$ and $h > 0$ sufficiently small, $\mathcal{P}_b R_h = \mathcal{O}(h^\infty)_{L^2 \rightarrow L^2}$.

Démonstration. By a direct verification using (4.4.2), the homogeneity of a and the admissible function χ , we see that $h^m \mathcal{P}_a - \mathcal{P}_a^h = \mathcal{P}_{\tilde{a}_h}^h$ where $\tilde{a}_h \in \Gamma^{0,0}$ satisfies

$$\text{supp } \tilde{a}_h \subset \mathbb{R}_x^d \times \text{supp } \xi(1 - \pi(h \cdot)) \subset \mathbb{R}_x^d \times \{|\xi| \leq 2h^{-1}\}.$$

We conclude by Lemma 4.4.30. \square

Lemma 4.4.32. *Let $a_h \in \Gamma^{m,r} \cap \sigma_0$ with $r \geq m$, then for $h > 0$ sufficiently small,*

$$T_{a_h}^h - \text{Op}_h(a_h) = \mathcal{O}(h^r)_{L^2 \rightarrow L^2}.$$

Démonstration. By Calderón–Vaillancourt theorem, we have

$$\begin{aligned} T_{a_h}^h - \text{Op}_h(a_h) &= \tau_h^{-1} (T_{\theta_{h,*}^{1,0} a_h} - \text{Op}(\theta_{h,*}^{1,0} a_h)) \tau_h \\ &= \mathcal{O} \left(\sum_{\max\{|\alpha|, |\beta|\} \leq 1+d/2} \left\| \partial_\xi^\alpha \partial_x^\beta (\sigma_{\theta_{h,*}^{1,0} a_h} - \theta_{h,*}^{1,0} a_h) \right\|_{L^\infty} \right)_{L^2 \rightarrow L^2}. \end{aligned}$$

By hypothesis $1 + d/2 \geq m - r + |\beta|$, so we use (4.4.3) to deduce that

$$\begin{aligned} \|\partial_\xi^\alpha \partial_x^\beta (\sigma_{\theta_{h,*}^{1,0} a_h} - \theta_{h,*}^{1,0} a_h)\|_{L^\infty} &= \mathcal{O}(M^{1+d/2,0}(\partial_x^\beta(\sigma_{\theta_{h,*}^{1,0} a_h} - \theta_{h,*}^{1,0} a_h))) \\ &= \mathcal{O}(M^{m-r+|\beta|,0}(\partial_x^\beta(\sigma_{\theta_{h,*}^{1,0} a_h} - \theta_{h,*}^{1,0} a_h))) \\ &= \mathcal{O}(M^{m,0}(\nabla_x^r(\theta_{h,*}^{1,0} a_h)))_{L^2 \rightarrow L^2} \\ &= \mathcal{O}(h^r M^{m,0}(a_h))_{L^2 \rightarrow L^2}. \end{aligned}$$

□

Lemma 4.4.33. *Let $a_h \in \Gamma^{m,\infty} \cap \sigma_0$ with $m \in \mathbb{R} \cup \{-\infty\}$, then for $h > 0$ sufficiently small,*

$$\mathcal{P}_{a_h}^h - \text{Op}_h(a_h) = \mathcal{O}(h^\infty)_{L^2 \rightarrow L^2}.$$

Démonstration. By Lemma 4.4.32 and Lemma 4.4.16,

$$\mathcal{P}_{a_h}^h - \sum_{j \in \mathbb{N}} \underline{\psi}_j \text{Op}_h(\psi_j a_h) \underline{\psi}_j = \mathcal{O}(h^\infty)_{L^2 \rightarrow L^2}.$$

By the usual semiclassical symbolic calculus,

$$\underline{\psi}_j \#_h(\psi_j a_h) \#_h \underline{\psi}_j = \psi_j a_h + \underline{\psi}_j \mathcal{O}(h^\infty)_{\Gamma^{-\infty,\infty}},$$

which is uniform with respect to $j \in \mathbb{N}$. Therefore,

$$\sum_{j \in \mathbb{N}} \underline{\psi}_j \#_h(\psi_j a_h) \#_h \underline{\psi}_j = a_h + \mathcal{O}(h^\infty)_{\Gamma^{-\infty,\infty}}.$$

□

Lemma 4.4.34 (Paradifferential Gårding Inequality). *Let $a_h \in M_{n \times n}(\Gamma^{0,r}) \cap \sigma_0$ with $n \geq 0$ and $r \geq 1 + d/2$. Suppose that $\text{Re } a_h \geq 0$, then $\exists C > 0$ such that $\forall u \in L^2$,*

$$(4.4.8) \quad \text{Re}(T_{a_h}^h u, u)_{L^2} \geq -Ch \|u\|_{L^2}^2.$$

Consequently, for some $C' > 0$,

$$\text{Re}(\mathcal{P}_{a_h}^h u, u)_{L^2} \geq -C'h \|u\|_{L^2}^2.$$

Démonstration. (4.4.8) results from Lemma 4.4.32 and Proposition 4.2.4. Therefore,

$$\begin{aligned} \text{Re}(\mathcal{P}_{a_h}^h u, u)_{L^2} &= \sum_{j \in \mathbb{N}} \text{Re}(\underline{\psi}_j T_{\psi_j a_h}^h \underline{\psi}_j u, u)_{L^2} = \sum_{j \in \mathbb{N}} \text{Re}(T_{\psi_j a_h}^h \underline{\psi}_j u, \underline{\psi}_j u)_{L^2} \\ &\geq -Ch \sum_{j \in \mathbb{N}} \|\underline{\psi}_j u\|_{L^2}^2 \geq -C'h \|u\|_{L^2}^2. \end{aligned}$$

□

4.4.4 Relation with quasi-homogeneous wavefront sets

Lemma 4.4.35. *Let $a_h \in {}_h\Sigma^{-\infty,r} \cap \sigma_0$, $r > 1 + d/2$, be elliptic at $(x_0, \xi_0) \in \mathbb{R}_x^d \times (\mathbb{R}_\xi^d \setminus \{0\})$, in the sense that, for some neighborhood Ω of (x_0, ξ_0) ,*

$$\inf_{0 < h < 1} \inf_{(x, \xi) \in \Omega} |a_h(x, \xi)| > 0.$$

Let $u \in L^2$, and suppose that $T_{a_h}^h u = \mathcal{O}(h^\sigma)_{L^2}$ with $0 \leq \sigma \leq r$, then $(x_0, \xi_0) \notin \text{WF}_{0,1}^\sigma(u)$.

Démonstration. Assume that $\Omega \subset \mathbb{R}_x^d \times (\mathbb{R}_\xi^d \setminus \{0\})$. Let $b_h \in S_{-\infty}^-$ with $\text{supp } b_h \subset \Omega$. Then for some $c_h \in {}_h\Sigma^{-\infty,r}$, $T_{b_h}^h = T_{c_h}^h T_{a_h}^h + \mathcal{O}(h^r)_{L^2 \rightarrow L^2}$. Therefore, $T_{b_h}^h u = \mathcal{O}(h^\sigma)_{L^2}$, and by Lemma 4.4.32, $\text{Op}_h(b_h)u = \mathcal{O}(h^\sigma)_{L^2}$. \square

Lemma 4.4.36. *Let $\epsilon \geq 0$, $e \in \Gamma_{0,0}^{m,r}$ (if $\epsilon = 0$) resp. $\Gamma_{0,1}^{m,r}$ (if $\epsilon > 0$), and suppose that e is homogeneous of degree m with respect to ξ . Then for $f \in H^s$ and $0 \leq \sigma \leq (1 + \epsilon)r$,*

$$\text{WF}_{\epsilon,1}^{s+\sigma-m}(\mathcal{P}_e f)^\circ \subset \text{WF}_{\epsilon,1}^{s+\sigma}(f)^\circ.$$

If in addition e is elliptic, i.e., for some $C > 0$ and $|\xi|$ sufficiently large, $|e(x, \xi)| \geq C|\xi|^m$, then

$$\text{WF}_{\epsilon,1}^{s+\sigma-m}(\mathcal{P}_e f)^\circ = \text{WF}_{\epsilon,1}^{s+\sigma}(f)^\circ.$$

Recall the definition of $\text{WF}_{\delta,\rho}^\mu(u)^\circ$ in §4.2.

Démonstration. For $\mu \in \mathbb{R}$, denote $Z^\mu = \mathcal{P}_{|\xi|^\mu}$. Then $Z^{-\mu}Z^\mu - \text{Id} \in \mathcal{O}_{-\infty}^-$. Therefore,

$$f - Z^{-s}Z^s f \in H_\infty^\infty, \quad \mathcal{P}_e f - \mathcal{P}_{e|\xi|^{-s}}Z^s f \in H_{\delta r}^{s+r-m},$$

where $\delta = 0$ if $\epsilon = 0$, while $\delta = 1$ if $\epsilon > 0$. By Lemma 4.2.16 and the fact that $Z^{\pm s}$ are pseudodifferential operators with elliptic symbols in $S_0^{\pm s}$, we readily have

$$\text{WF}_{\epsilon,1}^{s+\sigma}(f)^\circ = \text{WF}_{\epsilon,1}^\sigma(Z^s f)^\circ, \quad \text{WF}_{\epsilon,1}^{s+\sigma-m}(\mathcal{P}_e f)^\circ = \text{WF}_{\epsilon,1}^{\sigma-(m-s)}(\mathcal{P}_{e|\xi|^{-s}}Z^s f)^\circ.$$

So we may assume that $s = 0$. Let $a, b \in S_{-\infty}^- \cap \sigma_\epsilon$, such that

$$\text{supp } b \subset \{a \neq 0\} \subset \text{supp } a \subset \mathbb{R}_x^d \times \mathbb{R}_\xi^d \setminus \text{WF}_{\epsilon,1}^\sigma(f),$$

then by Lemma 4.2.15, $\text{Op}_h^{\epsilon,1}(a)f = \mathcal{O}(h^\sigma)_{L^2}$. By Corollary 4.4.31, Lemma 4.4.33, Proposition 4.4.26, and Corollary 4.4.28,

$$\begin{aligned} h^m \text{Op}_h^{\epsilon,1}(b)\mathcal{P}_e f &= \text{Op}_h^{\epsilon,1}(b)\mathcal{P}_e^h f + \mathcal{O}(h^\infty)_{L^2} \\ &= \text{Op}_h^{\epsilon,1}(b)\mathcal{P}_e^h \text{Op}_h^{\epsilon,1}(a)f + \text{Op}_h^{\epsilon,1}(b)\mathcal{P}_e^h \text{Op}_h^{\epsilon,1}(1-a)f + \mathcal{O}(h^\infty)_{L^2} \\ &= \mathcal{O}(1)_{L^2 \rightarrow L^2} \text{Op}_h^{\epsilon,1}(a)f + \mathcal{O}(h^{r(1+\epsilon)})_{L^2} \\ &= \mathcal{O}(h^\sigma)_{L^2}, \end{aligned}$$

proving the first statement. The second statement follows by a construction of parametrix. \square

4.5 Asymptotically flat water waves

In this section we prove Theorem 4.1.6. The idea is to combine the analysis in [6] with the dyadic paradifferential calculus on weighted Sobolev spaces. We shall use the following notations for simplicity. Let $w \in L^\infty(\mathbb{R}^d)$ which is nowhere vanishing, then for $\mathcal{A} : \mathcal{S}' \rightarrow \mathcal{S}'$ and $f \in \mathcal{S}'$, we denote $\mathcal{A}^{(w)} = w\mathcal{A}w^{-1}$, $f^{(w)} = wf$. Particularly, $(\mathcal{A}f)^{(w)} = \mathcal{A}^{(w)}f^{(w)}$. For $k \in \mathbb{R}$, we also denote, by an abuse of notation, $\mathcal{A}^{(k)} = \mathcal{A}^{(\langle x \rangle^k)}$, $f^{(k)} = f^{(\langle x \rangle^k)}$, when there is no ambiguity. Observe that $L_k^2 = H_k^0$ is an Hilbert space with the inner product

$$(f, g)_{L_k^2} = (f^{(k)}, g^{(k)})_{L^2}.$$

4.5.1 Dirichlet–Neumann operator

We study the Dirichlet–Neumann operator on weighted Sobolev spaces and its par-linearization. The time variable will be temporarily omitted for simplicity.

4.5.1.1 Boundary flattening

Let $\eta \in W^{1,\infty}(\mathbb{R}^d)$, such that

$$(4.5.1) \quad \delta := b + \inf_{x \in \mathbb{R}^d} \eta(x) > 0,$$

and define the Lipschitzian diffeomorphism

$$\tau : \mathbb{R}_x^d \times \mathbb{R}_z \rightarrow \mathbb{R}_x^d \times \mathbb{R}_y, \quad (x, z) \mapsto (x, z + \eta(x)).$$

Set $\tilde{\Omega} = \tau^{-1}(\Omega)$, $\tilde{\Sigma} = \tau^{-1}(\Sigma)$, $\tilde{\Gamma} = \tau^{-1}(\Gamma)$, then

$$\tilde{\Omega} = \{-b - \eta(x) < z < 0\}, \quad \tilde{\Sigma} = \{z = 0\}, \quad \tilde{\Gamma} = \{z = -b - \eta(x)\}.$$

Let τ_* be the pullback induced by τ , then

$$\tau_*(dx^2 + dy^2) = (dx \ dz)\varrho \begin{pmatrix} dx \\ dz \end{pmatrix}$$

with

$$\varrho = \begin{pmatrix} \text{Id} + {}^t\nabla\eta\nabla\eta & {}^t\nabla\eta \\ \nabla\eta & 1 \end{pmatrix}, \quad \varrho^{-1} = \begin{pmatrix} \text{Id} & -{}^t\nabla\eta \\ -\nabla\eta & 1 \end{pmatrix}.$$

Let $\nabla_{xz} = (\nabla, \partial_z)$, then the divergence, gradient and Laplacian operators with respect to the metric ϱ are

$$\begin{aligned} \text{div}_\varrho u &= \nabla_{xz} \cdot u, \\ \nabla_\varrho u &= (\nabla u - \nabla\eta\partial_z u, -\nabla\eta \cdot \nabla u + (1 + |\nabla\eta|^2)\partial_z u), \\ \Delta_\varrho u &= \partial_z^2 u + (\nabla - \nabla\eta\partial_z)^2 u. \end{aligned}$$

And the exterior unit normal to $\partial\tilde{\Omega} = \tilde{\Sigma} \cup \tilde{\Gamma}$ is

$$\mathbf{n}_\varrho = \langle (D\tau)^{-1}|_{T\partial\tilde{\Omega}}, \mathbf{n} \rangle = \begin{cases} \frac{{}^t(-\nabla\eta, 1 + |\nabla\eta|^2)}{\sqrt{1 + |\nabla\eta|^2}}, & \tilde{\Sigma}; \\ {}^t(0, 1), & \tilde{\Gamma}. \end{cases}$$

Let $\psi \in H^{1/2}$, and suppose that ϕ satisfies the equation

$$\Delta_{xy}\phi = 0, \quad \phi|_{\Sigma} = \psi, \quad \partial_{\mathbf{n}}\phi|_{\Gamma} = 0,$$

then $v = (\tau|_{\tilde{\Omega}})_*\phi$ satisfies

$$(4.5.2) \quad \Delta_{\varrho}v = 0, \quad v|_{\tilde{\Sigma}} = \psi, \quad \partial_{\mathbf{n}_{\varrho}}v|_{\tilde{\Gamma}} = 0.$$

And

$$\sqrt{1 + |\nabla\eta|^2}^{-1}G(\eta)\psi = \partial_{\mathbf{n}_{\varrho}}v|_{\tilde{\Sigma}} = \mathbf{n}_{\varrho} \cdot \nabla_{xz}v|_{z=0}.$$

4.5.1.2 Elliptic estimate

Let $\chi_0 \in C^\infty(\mathbb{R}_z)$ with $\chi_0(z) = 0$ for $z \leq -\delta/2$ and $\chi_0(z) = 1$ for $z \geq 0$. Decompose $v = \tilde{v} + \underline{\psi}$, where

$$\underline{\psi}(x, z) = \chi_0(z)e^{z\langle D_x \rangle}\psi(x).$$

Lemma 4.5.1. *Let $n \geq 0$, $m \in \mathbb{R}$, $\mu \in \mathbb{R}$, $k \in \mathbb{R}$, $a \in S_0^m$, then*

$$\|\partial_z^n \text{Op}(a)\underline{\psi}\|_{L_z^2(\mathbb{R}_{\leq 0}, H_k^{\mu-n-m+1/2})} \lesssim \|\psi\|_{H_k^\mu}.$$

Démonstration. We only prove the case with $n = 0$. The general case follows with similar arguments and the identity

$$\partial_z^n \underline{\psi}(x, z) = \sum_{j=0}^n \binom{n}{j} \chi_0^{(n-j)}(z) \langle D_x \rangle^j e^{z\langle D_x \rangle} \psi(x).$$

Let $b(x, \xi) = a(x, \xi)\langle \xi \rangle^{\mu-m} \in S_0^\mu$, $\lambda(z, \xi) = \chi_0(z)e^{z\langle \xi \rangle}\langle \xi \rangle^{1/2} \in L_{z \leq 0}^\infty S_0^{1/2}$, then $\forall N \geq 0$,

$$\|\text{Op}(a)\underline{\psi}\|_{L_z^2(\mathbb{R}_{\leq 0}, H_k^{\mu-m+1/2})} \lesssim \|\text{Op}(\lambda)\text{Op}(b)\psi\|_{L_z^2(\mathbb{R}_{\leq 0}, L_k^2)} + \|\psi\|_{H_k^{-N}}.$$

Observe that,

$$\text{Op}(\lambda)^{(k)} - (\text{Op}(\lambda)^{(k)})^* \in L_{z \leq 0}^\infty \mathcal{O}_0^{-1/2}, \quad (\text{Op}(\lambda)^{(k)})^2 - \text{Op}(\lambda^2)^{(k)} \in L_{z \leq 0}^\infty \mathcal{O}_0^0.$$

and

$$\sigma(\xi) := \int_{-\infty}^0 \lambda^2(z, \xi) dz = \langle \xi \rangle \int_{-\infty}^0 \chi_0^2(z) e^{2\langle \xi \rangle z} dz \in S_0^0,$$

therefore,

$$\begin{aligned} \|\text{Op}(\lambda)\text{Op}(b)\psi\|_{L_z^2(\mathbb{R}_{\leq 0}, L_k^2)}^2 &= (\text{Op}(\lambda^2)\text{Op}(b)\psi, \text{Op}(b)\psi)_{L_z^2(\mathbb{R}_{\leq 0}, L_k^2)} + \mathcal{O}(\|\psi\|_{H_k^\mu}^2) \\ &= (\text{Op}(\sigma)\text{Op}(b)\psi, \text{Op}(b)\psi)_{L_k^2} + \mathcal{O}(\|\psi\|_{H_k^\mu}^2) \\ &= \mathcal{O}(\|\psi\|_{H_k^\mu}^2). \end{aligned}$$

□

Lemma 4.5.2. $\forall k \in \mathbb{R}$, $\|\tilde{v}\|_{H_k^1} \leq C(\|\eta\|_{W^{1,\infty}})\|\psi\|_{H_k^{1/2}}$.

Démonstration. Let $H_\varrho^{1,0}$ be the completion of the space

$$\{f \in C^\infty(\tilde{\Omega}) : f \text{ vanishes in a neighborhood of } \tilde{\Sigma}\}$$

with respect to the norm

$$\|u\|_{H_\varrho^{1,0}} := \|\nabla_\varrho u\|_{L_\varrho^2} = (\nabla_\varrho u, \nabla_\varrho u)_{L_\varrho^2}^{1/2},$$

where $(X, Y)_{L_\varrho^2} := \int_{\tilde{\Omega}} \varrho(X, Y) dx dz$. As $b < \infty$, by Poincaré inequality,

$$\|u\|_{L^2} \leq C(\|\eta\|_{L^\infty}) \|\partial_z u\|_{L^2} \leq C(\|\eta\|_{W^{1,\infty}}) \|u\|_{H_\varrho^{1,0}}, \quad \forall u \in H_\varrho^{1,0}.$$

Let $0 < \zeta \in C^\infty(\mathbb{R})$ be such that $\zeta(z) = 1$ for $|z| \leq 1$, and $\zeta(z) = z$ for $|z| \geq 2$. For some $R > 0$ sufficiently large to be determined later, set $w(x) = R \times \zeta(\langle x \rangle^k / R)$. Then $\langle x \rangle^k \lesssim w(x) \lesssim R \langle x \rangle^k$, $\text{supp } \nabla w \subset \{\langle x \rangle \gtrsim R^{1/k}\}$, and $|\nabla w(x)| \lesssim R^{(k-1)/k}$.

Because \tilde{v} satisfies the equation $\Delta_\varrho \tilde{v} = -\Delta_\varrho \underline{\psi}$, we consider $\tilde{v}^{(w)}$ as the variational solution to the equation

$$B(\tilde{v}^{(w)}, \cdot) = -L(\cdot),$$

where for $u, \varphi \in H_\varrho^{1,0}$,

$$B(u, \varphi) = (\nabla_\varrho^{(w)} u, \nabla_\varrho^{(1/w)} \varphi)_{L^2(\tilde{\Omega})}, \quad L(\varphi) = (\nabla_\varrho^{(w)} \underline{\psi}^{(w)}, \nabla_\varrho^{(1/w)} \varphi)_{L^2(\tilde{\Omega})}.$$

Observe that $\nabla_\varrho^{(w^{\pm 1})} = \nabla_\varrho \mp b_w$, where $b_w = (w^{-1} \nabla w, -\nabla \eta \cdot w^{-1} \nabla w) \in L^\infty$ satisfies

$$\|b_w\| \leq C(\|\eta\|_{W^{1,\infty}}) R^{-1/k}.$$

We easily verify that L and B are continuous linear and bilinear forms on $H_\varrho^{1,0}$. Moreover B is coercive when R is sufficiently large, indeed,

$$B(\varphi, \varphi) = \|\nabla_\varrho \varphi\|_{L_\varrho^2}^2 - \|b_w \varphi\|_{L_\varrho^2}^2 \geq (1 - C(\|\eta\|_{W^{1,\infty}}) R^{-2/k}) \|\nabla_\varrho \varphi\|_{L_\varrho^2}^2.$$

Therefore, by Lax-Milgram's Theorem and Lemma 4.5.1,

$$\|\tilde{v}\|_{H_k^1} \lesssim \|\tilde{v}^{(w)}\|_{H_\varrho^{1,0}} \lesssim \|L\|_{(H_\varrho^{1,0})^*} \lesssim \|\underline{\psi}\|_{H^1} \lesssim \|\psi\|_{H_k^{1/2}}.$$

□

Theorem 4.5.3. *Suppose that $(\eta, \psi) \in W^{1,\infty} \times H_k^{1/2}$ with $k \in \mathbb{R}$, then*

$$\|G(\eta)\psi\|_{H_k^{-1/2}} \leq C(\|\eta\|_{W^{1,\infty}}) \|\psi\|_{H_k^{1/2}}.$$

Démonstration. By Lemma 4.5.1 and Lemma 4.5.2,

$$v \in L_z^2(-\delta, 0[, H_k^1) \cap H_z^1(-\delta, 0[, L_k^2).$$

By a classical interpolation result (see, e.g., Lemma 2.19 of [8]) and using the equation satisfied by v , we induce that

$$v \in C_z^0([-\delta, 0], H_k^{1/2}) \cap C_z^1([-\delta, 0], H_k^{-1/2}).$$

□

4.5.1.3 Higher regularity

Proposition 4.5.4. *Suppose that $(\eta, \psi) \in H^{\mu+1/2} \times H_k^\sigma$ with $\mu > 1/2 + d/2$, $1/2 \leq \sigma \leq \mu$ and $k \in \mathbb{R}$, then*

$$\|G(\eta)\psi\|_{H_k^{\sigma-1}} \leq C(\|\eta\|_{H^{\mu+1/2}})\|\psi\|_{H_k^\sigma}.$$

Consequently, if $(\eta, \psi) \in H^{\mu+1/2} \times \mathcal{H}_m^\sigma$ with $\sigma - m/2 \geq 1/2$, then

$$\|G(\eta)\psi\|_{\mathcal{H}_m^{\sigma-1}} \leq C(\|\eta\|_{H^{\mu+1/2}})\|\psi\|_{\mathcal{H}_m^\sigma}.$$

Démonstration. Let $\Lambda = \langle D_x \rangle^{\sigma-1/2}$, and let $\chi, \tilde{\chi} \in C_c^\infty(]-\delta, \delta[)$, where δ is defined by (4.5.1), such that $\chi(0) \neq 0$ and $\tilde{\chi}(z) = 1$ for $z \in \text{supp } \chi$. Then we have the following equation for $\Lambda\tilde{v}$,

$$-\Delta_\varrho(\chi\Lambda\tilde{v}) + K(\tilde{\chi}\Lambda\tilde{v}) = \Delta_\varrho(\chi\Lambda\underline{\psi}) - K(\tilde{\chi}\Lambda\underline{\psi}).$$

where $K = [\Delta_\varrho, \chi\Lambda]\Lambda^{-1}$. Observe that $\Delta_\varrho = P \cdot P$ with $P = (\nabla - \nabla\eta\partial_z, \partial_z)$, and

$$K = P \cdot Q + Q \cdot P - Q \cdot \tilde{Q}, \quad Q = [P, \chi\Lambda]\Lambda^{-1}, \quad \tilde{Q} = [P, \tilde{\chi}\Lambda]\Lambda^{-1}.$$

We also denote P^* , Q^* and \tilde{Q}^* the L^2 -formal adjoints of P , Q and \tilde{Q} respectively. Then we expand Q (similarly for \tilde{Q})

$$\begin{aligned} Q &= ([\nabla - \nabla\eta\partial_z, \chi\Lambda], [\partial_z, \chi\Lambda])\Lambda^{-1} \\ &= -(\nabla\eta[\partial_z, \chi] + [\nabla\eta, \Lambda]\Lambda^{-1}\chi\partial_z, [\partial_z, \chi]). \end{aligned}$$

By the Kato–Ponce commutator estimate (see [119]), we verify that

$$\|Q^{(k)}\|_{H^1L^2 \rightarrow L^2L^2} + \|(Q^*)^{(-k)}\|_{H^1L^2 \rightarrow L^2L^2} \leq C(\|\eta\|_{H^{\mu+1/2}}).$$

Then expand

$$\begin{aligned} (Kf, g)_{L^2L_k^2} &= (Q^{(k)}f^{(k)}, (P^*)^{(-k)}g^{(k)})_{L^2L^2} \\ &\quad + (P^{(k)}f^{(k)}, (Q^*)^{(-k)}g^{(k)})_{L^2L^2} \\ &\quad - (\tilde{Q}^{(k)}f^{(k)}, (Q^*)^{(-k)}g^{(k)})_{L^2L^2}, \end{aligned}$$

and observe that $P = P^{(k)} + \alpha \cdot \nabla_{xz} = (P^*)^{(-k)} + \beta \cdot \nabla_{xz}$ with $\alpha, \beta \in L^\infty$, we obtain the estimate

$$|(Kf, g)_{L^2L_k^2}| \lesssim \|\nabla_{xz}f\|_{L^2L_k^2}\|g\|_{H^1L_k^2} + \|f\|_{H^1L_k^2}\|\nabla_{xz}g\|_{L^2L_k^2}.$$

Suppose that we have already proven that

$$\tilde{v} \in L^2(]-\delta, 0[, H_k^{\sigma-1/2}) \cap H^1(]-\delta, 0[, H_k^{\sigma-3/2}),$$

with the norm denoted by N_σ . Then,

$$\begin{aligned} |(K(\tilde{\chi}\Lambda\tilde{v}), \chi\Lambda\tilde{v})_{L^2L_k^2}| &\lesssim \|\tilde{\chi}\Lambda\tilde{v}\|_{H^1H_k^1}\|\chi\Lambda\tilde{v}\|_{H^1L_k^2} + \|\tilde{\chi}\Lambda\tilde{v}\|_{H^1L_k^2}\|\chi\Lambda\tilde{v}\|_{H^1H_k^1} \\ &\lesssim N_\sigma\|\tilde{\chi}\Lambda\tilde{v}\|_{H^1H_k^1}. \\ |(K(\tilde{\chi}\Lambda\underline{\psi}), \chi\Lambda\tilde{v})_{L^2L_k^2}| &\lesssim \|\tilde{\chi}\Lambda\underline{\psi}\|_{H^1H_k^1}\|\chi\Lambda\tilde{v}\|_{H^1L_k^2} + \|\tilde{\chi}\Lambda\underline{\psi}\|_{H^1L_k^2}\|\chi\Lambda\tilde{v}\|_{H^1H_k^1} \\ &\lesssim \|\underline{\psi}\|_{L^2H_k^{\sigma+1/2}}\|\chi\Lambda\tilde{v}\|_{H^1H_k^1}, \end{aligned}$$

$$\begin{aligned}
 (-\Delta_\varrho(\chi\Lambda\underline{\psi}), \chi\Lambda\tilde{v})_{L^2L_k^2} &\lesssim \|P(\chi\Lambda\underline{\psi})\|_{L^2L_k^2} \|P(\chi\Lambda\tilde{v})\|_{L^2L_k^2} + \|\chi\Lambda\underline{\psi}\|_{L^2L_k^2} \|\chi\Lambda\tilde{v}\|_{L^2L_k^2}, \\
 &\lesssim \|\underline{\psi}\|_{H^1H_k^{\sigma+1/2}} \|\chi\Lambda\tilde{v}\|_{H^1H_k^1} \\
 (-\Delta_\varrho(\chi\Lambda\tilde{v}), \chi\Lambda\tilde{v})_{L^2L_k^2} &\gtrsim \|P(\chi\Lambda\tilde{v})\|_{L^2L_k^2}^2 - \|\chi\Lambda\tilde{v}\|_{L^2L_k^2}^2 \gtrsim \|\chi\Lambda\tilde{v}\|_{H^1H_k^1}^2 - N_\delta^2,
 \end{aligned}$$

Combing the estimates above, by Lemma 4.5.1, we have $\forall \varepsilon > 0$

$$\|\tilde{\chi}\Lambda\tilde{v}\|_{H^1H_k^1}^2 \lesssim \varepsilon \|\tilde{\chi}\Lambda\tilde{v}\|_{H^1H_k^1}^2 + \varepsilon^{-1} N_\delta^2 + \varepsilon^{-1} \|\underline{\psi}\|_{H_k^\sigma}^2.$$

By choosing ε sufficiently small, this *a priori* estimate implies that

$$\tilde{v} \in L^2(\cdot - \delta, 0[, H_k^{\sigma+1/2}) \cap H^1(\cdot - \delta, 0[, H_k^{\sigma-1/2}).$$

Therefore

$$v = \tilde{v} + \underline{\psi} \in L^2(\cdot - \delta, 0[, H_k^{\sigma+1/2}) \cap H^1(\cdot - \delta, 0[, H_k^{\sigma-1/2}).$$

By interpolation as in Theorem 4.5.3,

$$v \in C_z^0([-\delta, 0], H_k^\sigma) \cap C_z^1([-\delta, 0], H_k^{\sigma-1}).$$

□

4.5.2 Paralinearization

Now we paralinearize the system of water waves. The following results can be proven directly by combining the analysis in [6] and our dyadic paradifferential calculus, mainly Proposition 4.4.19, Proposition 4.4.21, and Proposition 4.4.22. We shall work in the Sobolev spaces \mathcal{H}_m^μ , recalling Definition 4.1.5.

Proposition 4.5.5. *Let $(\eta, \psi) \in \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu$ with $\mu > 3 + d/2$, $m < 2\mu - 6 - d$. Set $\lambda = \lambda^{(1)} + \lambda^{(0)}$, where*

$$\begin{aligned}
 \lambda^{(1)}(x, \xi) &= \sqrt{(1 + |\nabla\eta|^2)|\xi|^2 - (\nabla\eta \cdot \xi)^2}, \\
 \lambda^{(0)}(x, \xi) &= \frac{1 + |\nabla\eta|^2}{2\lambda^{(1)}} \{ \nabla \cdot (\alpha^{(1)} \nabla\eta) + i\partial_\xi \lambda^{(1)} \cdot \nabla \alpha^{(1)} \},
 \end{aligned}$$

with $\alpha^{(1)}(x, \xi) = \frac{\lambda^{(1)} + i\nabla\eta \cdot \xi}{1 + |\nabla\eta|^2}$. Then

$$G(\eta)\psi = \mathcal{P}_\lambda(\psi - \mathcal{P}_B\eta) - \mathcal{P}_V \cdot \nabla\eta + R(\eta, \psi),$$

where

$$B = \frac{\nabla\eta \cdot \nabla\psi + G(\eta)\psi}{1 + |\nabla\eta|^2}, \quad V = \nabla\psi - B\nabla\eta.$$

and $R(\eta, \psi) \in \mathcal{H}_m^{\mu+1/2}$.

We shall denote $\omega = \psi - \mathcal{P}_B\eta$, which is called the good unknown of Alinhac.

Démonstration. We only sketch the proof, for the key ingredients are already given in [6]. Let v be defined as in §4.5.1, and set $u = v - \mathcal{P}_{\partial_z v} \eta$. By (4.5.2), Proposition 4.4.21, and [6],

$$\mathcal{P}_\alpha \partial_z^2 u + \Delta u + \mathcal{P}_\beta \cdot \nabla \partial_z u - \mathcal{P}_\gamma \partial_z u \in \mathcal{H}_m^\mu,$$

where $\alpha = 1 + |\nabla \eta|^2$, $\beta = -2\nabla \eta$, $\gamma = \Delta \eta$. Then we find symbols $a_\pm = a_\pm^{(1)} + a_\pm^{(0)}$, whose explicit expressions are given later in Proposition 4.6.2, such that

$$(\partial_z - \mathcal{P}_{a_-})(\partial_z - \mathcal{P}_{a_+})u \in \mathcal{H}_m^\mu.$$

The exact same proof as Proposition 3.19 in [6] implies that

$$(\partial_z u - \mathcal{P}_{a_+} u)|_{z=0} \in \mathcal{H}_m^{\mu+1/2}.$$

We conclude by setting $\lambda = (1 + |\nabla \eta|^2)a_+ - i\nabla \eta \cdot \xi$. \square

The proofs of following results are in the same spirit and simpler, and we shall omit them.

Proposition 4.5.6. *Let $\mathcal{H}_m^{\mu+1/2}$ with $\mu > 3 + d/2$ and $m < 2\mu - 6 - d/2$, then*

$$H(\eta) = -\mathcal{P}_\ell \eta + f(\eta),$$

where $\ell = \ell^{(2)} + \ell^{(1)}$ is defined by

$$\ell^{(2)} = \frac{(1 + |\nabla \eta|^2)|\xi|^2 - (\nabla \eta \cdot \xi)^2}{(1 + |\nabla \eta|^2)^{3/2}}, \quad \ell^{(1)} = \frac{1}{2} \partial_\xi \cdot D_x \ell^{(2)},$$

and $f(\eta) \in \mathcal{H}_m^{2\mu-2-d/2}$.

Lemma 4.5.7. *Let $(\eta, \psi) \in \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu$, with $\mu > 3 + d/2$ and $m < 2\mu - 6 - d/2$, then*

$$\frac{1}{2} |\nabla \psi|^2 - \frac{1}{2} \frac{(\nabla \eta \cdot \nabla \psi + G(\eta)\psi)^2}{1 + |\nabla \eta|^2} = \mathcal{P}_V \cdot \nabla \psi - \mathcal{P}_B \mathcal{P}_V \cdot \nabla \eta - \mathcal{P}_B G(\eta)\psi + f(\eta, \psi),$$

where $f(\eta, \psi) \in \mathcal{H}_m^{2\mu-2-d/2}$.

Proposition 4.5.8. *Let $(\eta, \psi) \in \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu$, with $\mu > 3 + d/2$ and $m < 2\mu - 6 - d/2$, then (η, ψ) solves the water wave equation if and only if*

$$(\partial_t + \mathcal{P}_V \cdot \nabla + \mathcal{L}) \begin{pmatrix} \eta \\ \psi \end{pmatrix} = f(\eta, \psi)$$

where

$$\mathcal{L} = Q^{-1} \begin{pmatrix} 0 & -\mathcal{P}_\lambda \\ \mathcal{P}_\ell & 0 \end{pmatrix} Q, \quad \text{with } Q = \begin{pmatrix} \text{Id} & 0 \\ -\mathcal{P}_B & \text{Id} \end{pmatrix},$$

and $f(\eta, \psi) = Q^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu$ is defined by

$$\begin{aligned} f_1 &= G(\eta)\psi - \{\mathcal{P}_\lambda(\psi - \mathcal{P}_B \eta) - \mathcal{P}_V \cdot \nabla \eta\}, \\ f_2 &= -\frac{1}{2} |\nabla \psi|^2 + \frac{1}{2} \frac{(\nabla \eta \cdot \nabla \psi + G(\eta)\psi)^2}{1 + |\nabla \eta|^2} + H(\eta) \\ &\quad + \mathcal{P}_V \cdot \nabla \psi - \mathcal{P}_B \mathcal{P}_V \cdot \nabla \eta - \mathcal{P}_B G(\eta)\psi + \mathcal{P}_\ell \eta - g\eta. \end{aligned}$$

4.5.3 Symmetrization

Definition 4.5.9. For $T > 0$, $\gamma \in \mathbb{R}$ and two operators $\mathcal{A}, \mathcal{B} \in L^\infty([0, T], \mathcal{O}_0^\gamma)$, we say that $\mathcal{A} \sim_\gamma \mathcal{B}$, or simply $\mathcal{A} \sim \mathcal{B}$ when there is no ambiguity of the choice of γ , if

$$\mathcal{A} - \mathcal{B} \in L^\infty([0, T], \mathcal{O}_0^{\gamma-3/2}).$$

According to [6], there exists symbols which depend solely on η ,

$$\gamma = \gamma^{(3/2)} + \gamma^{(1/2)}, \quad p = p^{(1/2)} + p^{(-1/2)}, \quad q = q^{(0)},$$

whose principal symbols being explicitly

$$\gamma^{(3/2)} = \sqrt{\ell^{(2)}\lambda^{(1)}}, \quad p^{(1/2)} = (1 + |\nabla\eta|^2)^{-1/2}\sqrt{\lambda^{(1)}}, \quad q^{(0)} = (1 + |\nabla\eta|^2)^{1/4},$$

such that

$$\mathcal{P}_p \mathcal{P}_\lambda \sim_{\frac{3}{2}} \mathcal{P}_\gamma \mathcal{P}_q, \quad \mathcal{P}_q \mathcal{P}_\ell \sim_2 \mathcal{P}_\gamma \mathcal{P}_p, \quad \mathcal{P}_\gamma \sim_{\frac{3}{2}} (\mathcal{P}_\gamma)^*.$$

Define the symmetrizer

$$S = \begin{pmatrix} \mathcal{P}_p & 0 \\ 0 & \mathcal{P}_q \end{pmatrix} Q,$$

then

$$(4.5.3) \quad S\mathcal{L} \sim \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} S,$$

where the equivalence relation is applied separately to each component of the matrices.

4.5.4 Approximate system

Set the mollifier $J_\varepsilon = \mathcal{P}_{j_\varepsilon}$ where $j_\varepsilon = j_\varepsilon^{(0)} + j_\varepsilon^{(-1)}$,

$$j_\varepsilon^{(0)} = \exp(-\varepsilon\gamma^{(3/2)}), \quad j_\varepsilon^{(-1)} = \frac{1}{2}\partial_\xi \cdot D_x j_\varepsilon^{(0)}.$$

Then we have, uniformly for $\varepsilon > 0$,

$$J_\varepsilon \mathcal{P}_\gamma \sim_{\frac{3}{2}} \mathcal{P}_\gamma J_\varepsilon, \quad J_\varepsilon^* \sim_0 J_\varepsilon.$$

Let $\tilde{p} = \tilde{p}^{(-1/2)} + \tilde{p}^{(-3/2)}$ with

$$\tilde{p}^{(-1/2)} = 1/p^{(1/2)}, \quad \tilde{p}^{(-3/2)} = -(\tilde{p}^{(-1/2)}p^{(-1/2)} + \frac{1}{i}\partial_\xi \tilde{p}^{(-1/2)} \cdot \partial_x p^{(1/2)})/p^{(1/2)},$$

then

$$\mathcal{P}_p \mathcal{P}_{\tilde{p}} \sim_0 \text{Id}, \quad \mathcal{P}_q \mathcal{P}_{1/q} \sim_0 \text{Id}.$$

Finally we define

$$\mathcal{L}_\varepsilon = \mathcal{L}Q^{-1} \begin{pmatrix} \mathcal{P}_{\tilde{p}} J_\varepsilon \mathcal{P}_p & 0 \\ 0 & \mathcal{P}_{1/q} J_\varepsilon \mathcal{P}_q \end{pmatrix} Q,$$

and the approximate system

$$(4.5.4) \quad (\partial_t + \mathcal{P}_V \cdot \nabla J_\varepsilon + \mathcal{L}_\varepsilon) \begin{pmatrix} \eta \\ \psi \end{pmatrix} = f(J_\varepsilon \eta, J_\varepsilon \psi).$$

A key identity of the operator \mathcal{L}_ε is that

$$(4.5.5) \quad S\mathcal{L}_\varepsilon \sim \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} J_\varepsilon S.$$

4.5.5 A priori estimate

Proposition 4.5.10. *Let $(\eta, \psi) \in C^1([0, T], \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu)$ with $\mu > 3 + d/2$ and $m < 2\mu - 6 - d/2$ solve the approximate system (4.5.4). Define*

$$M_T = \sup_{0 \leq t \leq T} \|(\eta, \psi)(t)\|_{\mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu}, \quad M_0 = \|(\eta, \psi)(0)\|_{\mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu}.$$

Then

$$M_T \leq C(M_0) + TC(M_T)$$

for some non-decaying function $C : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$.

Démonstration. For $0 \leq k \leq m$, set

$$M_T^k = \sup_{0 \leq t \leq T} \|(\eta, \psi)(t)\|_{H_k^{\mu+1/2-k/2} \times H_k^{\mu-k/2}}, \quad M_0^k = \|(\eta, \psi)(0)\|_{H_k^{\mu+1/2-k/2} \times H_k^{\mu-k/2}}.$$

By [6],

$$M_T^0 \leq C(M_0^0) + TC(M_T^0).$$

It remains to prove that for $1 \leq k \leq m$,

$$M_T^k \leq C(M_0^k) + TC(M_T^k).$$

To do this, let $\Lambda_k^\mu = \mathcal{P}_{m_k^{\mu-k/2}}$, and set $\Phi = \Lambda_k^\mu S \begin{pmatrix} \eta \\ \psi \end{pmatrix}$. Then

$$(\partial_t + \mathcal{P}_V \cdot \nabla J_\varepsilon) \Phi + \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} J_\varepsilon \Phi = F_\varepsilon$$

where $F_\varepsilon = F_\varepsilon^1 + F_\varepsilon^2 + F_\varepsilon^3$, with

$$\begin{aligned} F_\varepsilon^1 &= \Lambda_k^\mu S f(J_\varepsilon \eta, J_\varepsilon \psi), \\ F_\varepsilon^2 &= [\partial_t + \mathcal{P}_V \cdot \nabla J_\varepsilon, \Lambda_k^\mu S] \begin{pmatrix} \eta \\ \psi \end{pmatrix}, \\ F_\varepsilon^3 &= \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} J_\varepsilon \Lambda_k^\mu S \begin{pmatrix} \eta \\ \psi \end{pmatrix} - \Lambda_k^\mu S \mathcal{L}_\varepsilon \begin{pmatrix} \eta \\ \psi \end{pmatrix}. \end{aligned}$$

By Proposition 4.5.8, Proposition 4.5.5, Proposition 4.5.6, and Lemma 4.5.7,

$$\|f(J_\varepsilon \eta, J_\varepsilon \psi)\|_{\mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu} \leq C(\|(J_\varepsilon \eta, J_\varepsilon \psi)\|_{\mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu}) \leq C(\|(\eta, \psi)\|_{\mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu}),$$

therefore,

$$\|F_\varepsilon^1\|_{L^\infty([0, T], L^2)} \leq C(M_T).$$

As $\mathcal{P}_V \cdot \nabla J_\varepsilon$ is a scalar operator, a symbolic calculus using Proposition 4.4.19 gives the following estimate

$$\|[\partial_t + \mathcal{P}_V \cdot \nabla J_\varepsilon, \Lambda_k^\mu S]\|_{L^\infty([0, T], H_k^{\mu+1/2-k/2} \times H_k^{\mu-k/2} \rightarrow L^2 \times L^2)} \leq C(M_T),$$

which implies

$$\|F_\varepsilon^2\|_{L^\infty([0, T], L^2)} \leq C(M_T).$$

By (4.5.5), $\begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} J_\varepsilon S - S\mathcal{L}_\varepsilon$ send $H^{\mu+1/2} \times H^\mu$ to $H^\mu \times H^\mu$. This is due to our specific choice of the symbols $p, q, \gamma, j_\varepsilon$ that makes both the principal symbol and sub-principal symbol in the symbolic calculus cancel out. Unfortunately, the operator

$$\begin{aligned} R &:= \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} J_\varepsilon \Lambda_k^\mu S - \Lambda_k^\mu S \mathcal{L}_\varepsilon \\ &= \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} J_\varepsilon [\Lambda_k^\mu, S] + [S\mathcal{L}_\varepsilon, \Lambda_k^\mu] + \left(\begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} J_\varepsilon S - S\mathcal{L}_\varepsilon \right) \Lambda_k^\mu \\ &=: \text{(I)} + \text{(II)} + \text{(III)} \end{aligned}$$

does not send $H_k^{\mu+1/2-k/2} \times H_k^{\mu-k/2}$ to $L^2 \times L^2$ because the sub-principal symbol will not be canceled out in the symbolic calculus, due to the existence of Λ_k^μ . Particularly, we need to use Proposition 4.4.19 to estimate the commutators $[\Lambda_k^\mu, S]$ and $[S\mathcal{L}_\varepsilon, \Lambda_k^\mu]$, and obtain

$$\left\| R \begin{pmatrix} \eta \\ \psi \end{pmatrix} \right\|_{L^2 \times L^2} \lesssim \|(\eta, \psi)\|_{H_{k-1}^{\mu+1-k/2} \times H_{k-1}^{\mu+1/2-k/2}} + \|(\eta, \psi)\|_{H_k^{\mu+1/2-k/2} \times H_k^{\mu-k/2}}.$$

More precisely, the first term on the right hand side comes from (I) and (II) while the second term comes from (III). When $k \geq 1$,

$$H_{k-1}^{\mu+1-k/2} \times H_{k-1}^{\mu+1/2-k/2} = H_{k-1}^{\mu+1/2-(k-1)/2} \times H_{k-1}^{\mu-(k-1)/2} \supset \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu,$$

so we still have

$$\|F_\varepsilon^3\|_{L^\infty([0,T],L^2)} \leq C(M_T).$$

Finally by an exact same energy estimate as in [6], we deduce that

$$M_T^k \lesssim \|\Phi\|_{L^\infty([0,T],L^2)} \leq C(M_0^k) + TC(M_T).$$

□

4.5.6 Existence

Lemma 4.5.11. *For all $(\eta_0, \psi_0) \in \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu$ and all $\varepsilon > 0$, the Cauchy problem of the approximate system (4.5.4) has a unique maximal solution*

$$(\eta_\varepsilon, \psi_\varepsilon) \in C([0, T_\varepsilon[, \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu).$$

Moreover, $\exists T_0 > 0$ such that

$$\inf_{\varepsilon \in]0,1]} T_\varepsilon \geq T_0.$$

Démonstration. Following [6], the existence follows from the existence theory of ODEs by writing (4.5.4) in the compact form

$$\partial_t X = \mathcal{F}_\varepsilon(X),$$

where \mathcal{F}_ε is a Lipschitz map on $\mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu$. Indeed, $J_\varepsilon \in \mathcal{O}_0^{-\infty}$ is a smoothing operator. We do not need $J_\varepsilon \in \mathcal{O}_{-\infty}^{-\infty}$ because the operators such as $\mathcal{P}_V \cdot \nabla, \mathcal{L}$, etc., are all of non-positive orders with respect to the spatial decay. The estimates to proving the Lipschitz

regularity can be carried out similarly as in the proof of Proposition 4.5.10. The only nontrivial term that remains is the Dirichlet–Neumann operator, whose regularity follows by combining Proposition 4.5.4 and the shape derivative formula (which goes back to Zakharov [200]),

$$\langle dG(\eta)\psi, \varphi \rangle := \lim_{h \rightarrow 0} \frac{1}{h} (G(\eta + h\varphi) - G(\eta))\psi = -G(\eta)(B\varphi) - \nabla_x \cdot (V\varphi).$$

A standard abstract argument then shows that T_ε has a strictly positive lower bound, we refer to [6] for more details. \square

Proof of Theorem 4.1.6. By Lemma 4.5.11, we obtain a sequence $\{(\eta_\varepsilon, \psi_\varepsilon)\}_{0 < \varepsilon \leq 1}$ which satisfies (4.5.4) and is uniformly bounded in $L^\infty([0, T], \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu)$ for some $T > 0$. By (4.5.4), $\{(\partial_t \eta_\varepsilon, \partial_t \psi_\varepsilon)\}_{0 < \varepsilon \leq 1}$ is uniformly bounded in $L^\infty([0, T], \mathcal{H}_m^{\mu-1} \times \mathcal{H}_m^{\mu-3/2})$. By [6], there exists

$$(4.5.6) \quad (\eta, \psi) \in C([0, T], H^{\mu+1/2} \times H^\mu)$$

which solves (4.1.7), such that as $\varepsilon \rightarrow 0$, $(\eta_\varepsilon, \psi_\varepsilon) \rightarrow (\eta, \psi)$ weakly in $L^2([0, T], \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu)$, and strongly in $C([0, T], \mathcal{H}_m^{\mu-1} \times \mathcal{H}_m^{\mu-3/2})$. We then prove that for $1 \leq k \leq m$,

$$\Phi = \Phi(\eta, \psi) := \Lambda_k^\mu S(\eta, \psi) \begin{pmatrix} \eta \\ \psi \end{pmatrix}$$

lies in $C([0, T], L^2)$, where Λ_k^μ is defined in Proposition 4.5.10, and $S = S(\eta, \psi)$ is the symmetrizer. Up to an extraction of a subsequence, we may assume by weak convergence that

$$\begin{aligned} (\eta, \psi) &\in L^\infty([0, T], \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu), \\ (\partial_t \eta, \partial_t \psi) &\in L^\infty([0, T], \mathcal{H}_m^{\mu-1} \times \mathcal{H}_m^{\mu-3/2}), \end{aligned}$$

with

$$\|(\eta, \psi)\|_{L^\infty([0, T], \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu) \cap W^{1, \infty}([0, T], \mathcal{H}_m^{\mu-1} \times \mathcal{H}_m^{\mu-3/2})} \leq C(\|(\eta_0, \psi_0)\|_{\mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu}).$$

This already implies that (η, ψ) is weakly continuous in $\mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu$. By the analysis in the previous section,

$$(\partial_t + \mathcal{P}_V \cdot \nabla)\Phi + \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} \Phi = F,$$

with

$$\|F\|_{L^\infty([0, T], L^2)} \leq C(\|(\eta_0, \psi_0)\|_{\mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu}).$$

Let $J_h = \text{Op}_h(e^{-|x|^2 - |\xi|^2})$. Now that $e^{-h^2|x|^2 - h^2|\xi|^2} \in S_0^0$, we have the commutator estimate

$$[J_h, \mathcal{P}_V \cdot \nabla] = \mathcal{O}(1)_{\mathcal{O}_{-1}^0}, \quad [J_h, \mathcal{P}_\gamma] = \mathcal{O}(1)_{\mathcal{O}_{-1}^{1/2}}.$$

Because $k \geq 1$, by the same spirit of estimating R in Proposition 4.5.10, we obtain the following energy estimate

$$\frac{d}{dt} \|J_h \Phi(t)\|_{L^2}^2 \leq C(\|(\eta_0, \psi_0)\|_{\mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu}).$$

Therefore, $t \mapsto \|J_h \Phi(t)\|_{L^2}^2$ are uniformly Lipschitzian. Consequently, by Arzelà–Ascoli theorem, $t \mapsto \|\Phi(t)\|_{L^2}^2$ is continuous, because $J_h \Phi \rightarrow \Phi$ as $h \rightarrow 0$. Combining the weak continuity, we induce by functional analysis that $\Phi \in C([0, T], L^2)$. By (4.5.6), the paradifferential calculus, and the definition of Φ , we easily induce that

$$(\eta, \psi) \in C([0, T], \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu).$$

Thus we finish the proof of Theorem 4.1.6. \square

4.6 Propagation of singularities for water waves

4.6.1 Finer parilinearization and symmetrization

To study the propagation of singularities, we need much finer results of parilinearization and symmetrization than Proposition 4.5.5 and Proposition 4.5.8 so as to gain regularities in the remainder terms.

Proposition 4.6.1. *Let $(\eta, \psi) \in H^{\mu+1/2} \times H^\mu$, with $\mu > 3 + d/2$. Then $\exists \lambda \in \Sigma^{1, \mu-1-d/2}$, such that*

$$G(\eta)\psi = \mathcal{P}_\lambda(\psi - \mathcal{P}_B \eta) - \mathcal{P}_V \cdot \nabla \eta + R(\eta, \psi),$$

where $R(\eta, \psi) \in H^{2\mu-3-d/2}$.

Démonstration. This theorem follows by replacing the usual paradifferential calculus with the dyadic paradifferential calculus in the analysis of [12]. In [12], the explicit expression for λ is given. We write it down for the sake of later applications.

$$\lambda = (1 + |\nabla \eta|^2) a_+ - i \nabla \eta \cdot \xi,$$

where $a_\pm = \sum_{j \leq 1} a_\pm^{(j)}$ is defined as follows. Setting $c = \frac{1}{1+|\nabla \eta|^2}$, then

$$\begin{aligned} a_-^{(1)} &= ic \nabla \eta \cdot \xi - \sqrt{c|\xi|^2 - (c \nabla \eta \cdot \xi)^2}, & a_+^{(1)} &= ic \nabla \eta \cdot \xi + \sqrt{c|\xi|^2 - (c \nabla \eta \cdot \xi)^2}, \\ a_-^{(0)} &= \frac{i \partial_\xi a_-^{(1)} \cdot \partial_x a_+^{(1)} - c \Delta \eta a_-^{(1)}}{a_+^{(1)} - a_-^{(1)}}, & a_+^{(0)} &= \frac{i \partial_\xi a_-^{(1)} \cdot \partial_x a_+^{(1)} - c \Delta \eta a_+^{(1)}}{a_-^{(1)} - a_+^{(1)}}. \end{aligned}$$

Suppose that $a_\pm^{(j)}$ are defined for $m \leq j \leq 1$, then we define

$$\begin{aligned} a_-^{(m-1)} &= \frac{1}{a_-^{(1)} - a_+^{(1)}} \sum_{m \leq k \leq 1} \sum_{m \leq \ell \leq 1} \sum_{|\alpha|=k+\ell-m} \frac{1}{\alpha!} \partial_\xi^\alpha a_-^{(k)} D_x^\alpha a_+^{(\ell)} \\ a_+^{(m-1)} &= -a_-^{(m-1)}. \end{aligned}$$

The principal and sub-principal symbols of λ clearly coincide with the ones given by Proposition 4.5.5. \square

Proposition 4.6.2. *Set $w = \Lambda^\mu U S(\frac{\eta}{\psi})$, where $\Lambda^\mu = \mathcal{P}_{(\gamma(3/2))_{2\mu/3}}$ and*

$$U = \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}.$$

Then, for some $Q \in M_{2 \times 2}(\Sigma^{0, \mu-5/2})$, $\zeta \in \Sigma^{-1/2, \mu-5/2}$,

$$(4.6.1) \quad (\partial_t + \mathcal{P}_V \cdot \nabla + \mathcal{P}_Q)w + i \mathcal{P}_\gamma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} w + \frac{ig}{2} \mathcal{P}_\zeta \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \in H^{\mu-4-d/2}.$$

Remark 4.6.3. Because χ in the definition of paradifferential operators is an even function, we easily verify that $\Lambda^\mu, \mathcal{P}_p, \mathcal{P}_q, \mathcal{P}_B$ all map real-valued functions to real-valued functions. Therefore, $w = \begin{pmatrix} u \\ \bar{u} \end{pmatrix}$ with

$$(4.6.2) \quad u = \Lambda^\mu(-i, 1)S \begin{pmatrix} \eta \\ \psi \end{pmatrix} = \Lambda^\mu \mathcal{P}_p \eta - i \Lambda^\mu \mathcal{P}_q \omega,$$

recalling that $\omega = \psi - \mathcal{P}_B \eta$ is the good unknown of Alinhac.

Démonstration. Combining Proposition 4.6.1 and Proposition 4.5.8, moving the term $g\eta$ to the left hand side,

$$(\partial_t + \mathcal{P}_V \cdot \nabla + \mathcal{L}) \begin{pmatrix} \eta \\ \psi \end{pmatrix} + g \begin{pmatrix} 0 \\ \eta \end{pmatrix} = f(\eta, \psi),$$

where $f(\eta, \psi) = Q^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in H^{2\mu-7/2-d/2} \times H^{2\mu-4-d/2}$ is defined by

$$\begin{aligned} f_1 &= G(\eta)\psi - \{\mathcal{P}_\lambda(\psi - \mathcal{P}_B \eta) - \mathcal{P}_V \cdot \nabla \eta\}, \\ f_2 &= -\frac{1}{2}|\nabla \psi|^2 + \frac{1}{2} \frac{(\nabla \eta \cdot \nabla \psi + G(\eta)\psi)^2}{1 + |\nabla \eta|^2} + H(\eta) \\ &\quad + \mathcal{P}_V \cdot \nabla \psi - \mathcal{P}_B \mathcal{P}_V \cdot \nabla \eta - \mathcal{P}_B G(\eta)\psi + \mathcal{P}_\ell \eta. \end{aligned}$$

Given two time-dependent operators $\mathcal{A}, \mathcal{B} : \mathcal{S} \rightarrow \mathcal{S}'$, we say that $\mathcal{A} \sim \mathcal{B}$ if

$$\mathcal{A} - \mathcal{B} \in L^\infty([0, T], \mathcal{O}_0^{-\mu+d/2+4}).$$

By the ellipticity of $\gamma^{(3/2)}, p^{(1/2)}$ and $q^{(0)}$, we can find paradifferential operators $\tilde{\Lambda}^\mu$ and \tilde{S} by a routine construction of parametrix such that

$$\tilde{\Lambda}^\mu \Lambda^\mu \sim \text{Id}, \quad \tilde{S} S \sim \text{Id}.$$

Observe that S is a lower triangular matrix, we can also choose \tilde{S} to be lower triangular. Therefore, we can find $\zeta \in \Sigma^{-1/2, \mu-5/2}$ by the symbolic calculus, such that

$$\begin{pmatrix} 0 & 0 \\ \mathcal{P}_\zeta & 0 \end{pmatrix} \Lambda^\mu S - \Lambda^\mu S \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \sim 0.$$

Moreover, the principal symbols of S is

$$\begin{pmatrix} p^{(1/2)} & 0 \\ Bp^{(1/2)} & q^{(0)} \end{pmatrix}.$$

We verify that the principal symbol of ζ is

$$\zeta^{(-1/2)} = q^{(0)} / p^{(1/2)}.$$

Then by (4.5.3) and the fact that the Poisson bracket between the symbol of Λ^μ and γ vanishes, we find by the symbolic calculus two symbols $A, B \in M_{2 \times 2}(\Sigma^{0, \mu-5/2})$ such that

$$\mathcal{A} := [\partial_t + \mathcal{P}_V \cdot \nabla, \Lambda^\mu S] \sim [\partial_t + \mathcal{P}_V \cdot \nabla, \Lambda^\mu S] \tilde{S} \tilde{\Lambda}^\mu \Lambda^\mu S \sim \mathcal{P}_A \Lambda^\mu S,$$

$$\mathcal{B} := \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} \Lambda^\mu S - \Lambda^\mu S \mathcal{L} \sim \left(\begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} - \Lambda^\mu S \mathcal{L} \tilde{S} \tilde{\Lambda}^\mu \right) \Lambda^\mu S \sim \mathcal{P}_B \Lambda^\mu S.$$

Therefore, let $\Phi = \Lambda^\mu S \begin{pmatrix} \eta \\ \psi \end{pmatrix}$, and write

$$g \begin{pmatrix} 0 \\ \eta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ g & 0 \end{pmatrix} \begin{pmatrix} \eta \\ \psi \end{pmatrix},$$

we obtain by the analysis above that

$$(\partial_t + \mathcal{P}_V \cdot \nabla) \Phi + \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} \Phi + \begin{pmatrix} 0 & 0 \\ g \mathcal{P}_\zeta & 0 \end{pmatrix} \Phi = \mathcal{P}_A \Phi + \mathcal{P}_B \Phi + F,$$

where

$$F = (\mathcal{A} + \mathcal{B}) \begin{pmatrix} \eta \\ \psi \end{pmatrix} - \mathcal{P}_{A+B} \Phi + \begin{pmatrix} 0 & 0 \\ g \mathcal{P}_\zeta & 0 \end{pmatrix} \Phi - g \Lambda^\mu S \begin{pmatrix} 0 \\ \eta \end{pmatrix} + \Lambda^\mu S f(\eta, \psi) \in H^{\mu-4-d/2}.$$

Finally, observe that

$$U \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} U^{-1} = i \begin{pmatrix} \mathcal{P}_\gamma & 0 \\ 0 & -\mathcal{P}_\gamma \end{pmatrix}, \quad U \begin{pmatrix} 0 & 0 \\ \mathcal{P}_\zeta & 0 \end{pmatrix} U^{-1} = \frac{i}{2} \begin{pmatrix} \mathcal{P}_\zeta & -\mathcal{P}_\zeta \\ \mathcal{P}_\zeta & -\mathcal{P}_\zeta \end{pmatrix},$$

We conclude by setting $Q = -\frac{1}{2} U(A+B)U^{-1}$. \square

Remark 4.6.4. If $\eta \in \mathcal{H}_m^{\mu+1/2}$ with $m < 2\mu - 6 - d/2$, then for $0 \leq j < \mu + 1/2 - d/2$,

$$\nabla^j \eta \in \mathcal{H}_m^{\mu+1/2-j} \subset W_{\rho_j}^{0,\infty},$$

where

$$\rho_j = \min\{2(\mu + 1/2 - j - d/2), m\}.$$

By the construction of λ in the proof of Proposition 4.6.1, we can prove by induction that

$$\lambda^{(1-j)}(t, x, \xi) = f_j(\nabla \eta(t, x), \dots, \nabla^{j+1} \eta(t, x), \xi),$$

where $f_j(u_0, \dots, u_j, \xi)$ is smooth with respect to u_0, \dots, u_j and homogeneous of degree $1-j$ with respect to ξ , and moreover

$$|f_j(u_0, \dots, u_j, \xi)| \leq C(|u_0, \dots, u_j|) |u_j| |\xi|^{1-j}.$$

Indeed, this is clearly true when $j = 0$, and when $j \geq 1$, f_j is linear with respect to u_j . Therefore,

$$\lambda^{(1)} - |\xi| \in \Gamma_{m,0}^{1,0}, \quad \lambda^{(1-j)} \in \Gamma_{\rho_{j+1,0}}^{1-j,0}, \quad \forall j \geq 1.$$

Similarly, we have

$$\begin{aligned} \gamma^{(3/2)} - |\xi|^{3/2} &\in \Gamma_{m,0}^{3/2,0}, & \lambda^{(3/2-j)} &\in \Gamma_{\rho_{j+1,0}}^{3/2-j,0}, & \forall j \geq 1. \\ \zeta^{(-1/2)} - |\xi|^{-1/2} &\in \Gamma_{m,0}^{-1/2,0}, & \zeta^{(-1/2-j)} &\in \Gamma_{\rho_{j+1,0}}^{-1/2-j,0}, & \forall j \geq 1. \\ & & Q^{(-j)} &\in \Gamma_{\rho_{j+2,0}}^{-j,0}, & \forall j \geq 0. \end{aligned}$$

Lemma 4.6.5. *Let u be defined as (4.6.2). If $(\eta, \psi) \in H^{\mu+1/2} \times H^\mu$, then for $0 \leq \sigma \leq r$ where r is the largest integer such that $r < \mu - 1 - d/2$,*

$$\mathrm{WF}_{0,1}^\sigma(u)^\circ = \mathrm{WF}_{0,1}^{\mu+1/2+\sigma}(\eta)^\circ \cup \mathrm{WF}_{0,1}^{\mu+\sigma}(\psi)^\circ.$$

If $(\eta, \psi) \in \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu$, with $m < \frac{2}{3}(\mu - 1 - d/2)$, then for $0 \leq \sigma \leq \frac{3}{2}m$,

$$\mathrm{WF}_{1/2,1}^\sigma(u)^\circ = \mathrm{WF}_{1/2,1}^{\mu+1/2+\sigma}(\eta)^\circ \cup \mathrm{WF}_{1/2,1}^{\mu+\sigma}(\psi)^\circ.$$

Démonstration. If $\eta \in H^{\mu+1/2}$, then $(\gamma^{(3/2)})^{2\mu/3} \in \Gamma^{\mu,r}$, $p^{(1/2)} \in \Gamma^{1/2,r}$, $q^{(0)} \in \Gamma^{0,r}$, $B \in \Gamma^{0,r}$. If $\eta \in \mathcal{H}_m^{\mu+1/2}$, then for $0 \leq j \leq m$,

$$\nabla^j \eta \in \mathcal{H}_m^{\mu+1/2-j} \subset H_j^{\mu+1/2-3j/2} \subset W_j^{0,\infty}.$$

Therefore, $(\gamma^{(3/2)})^{2\mu/3} \in \Gamma_{0,1}^{\mu,m}$, $p^{(1/2)} \in \Gamma_{0,1}^{1/2,m}$, $q^{(0)} \in \Gamma_{0,1}^{0,m}$, $B \in \Gamma_{0,1}^{0,m}$. By Lemma 4.4.36 and (4.6.2), for either $\epsilon = 0$ or $\epsilon = 1/2$,

$$\begin{aligned} \mathrm{WF}_{\epsilon,1}^\sigma(u)^\circ &= \mathrm{WF}_{\epsilon,1}^\sigma(\Lambda^\mu \mathcal{P}_p \eta)^\circ \cup \mathrm{WF}_{\epsilon,1}^\sigma(\Lambda^\mu \mathcal{P}_q(\psi - \mathcal{P}_B \eta))^\circ \\ &= \mathrm{WF}_{\epsilon,1}^{\mu+1/2+\sigma}(\eta)^\circ \cup \mathrm{WF}_{\epsilon,1}^{\mu+\sigma}(\psi - \mathcal{P}_B \eta)^\circ \\ &\subset \mathrm{WF}_{\epsilon,1}^{\mu+1/2+\sigma}(\eta)^\circ \cup (\mathrm{WF}_{\epsilon,1}^{\mu+\sigma}(\psi)^\circ \cup \mathrm{WF}_{\epsilon,1}^{\mu+\sigma}(\mathcal{P}_B \eta)^\circ) \\ &\subset \mathrm{WF}_{\epsilon,1}^{\mu+1/2+\sigma}(\eta)^\circ \cup (\mathrm{WF}_{\epsilon,1}^{\mu+\sigma}(\psi)^\circ \cup \mathrm{WF}_{\epsilon,1}^{\mu+\sigma}(\eta)^\circ) \\ &= \mathrm{WF}_{\epsilon,1}^{\mu+1/2+\sigma}(\eta)^\circ \cup \mathrm{WF}_{\epsilon,1}^{\mu+\sigma}(\psi)^\circ. \end{aligned}$$

Conversely, as $\mathrm{WF}_{\epsilon,1}^{\mu+\sigma}(\mathcal{P}_B \eta)^\circ \subset \mathrm{WF}_{\epsilon,1}^{\mu+1/2+\sigma}(\eta)^\circ$, we have

$$\begin{aligned} &\mathrm{WF}_{\epsilon,1}^{\mu+1/2+\sigma}(\eta)^\circ \cup \mathrm{WF}_{\epsilon,1}^{\mu+\sigma}(\psi)^\circ \\ &= \mathrm{WF}_{\epsilon,1}^{\mu+1/2+\sigma}(\eta)^\circ \cup (\mathrm{WF}_{\epsilon,1}^{\mu+\sigma}(\psi)^\circ \setminus \mathrm{WF}_{\epsilon,1}^{\mu+1/2+\sigma}(\eta)^\circ) \\ &= \mathrm{WF}_{\epsilon,1}^{\mu+1/2+\sigma}(\eta)^\circ \cup (\mathrm{WF}_{\epsilon,1}^{\mu+\sigma}(\psi - \mathcal{P}_B \eta)^\circ \setminus \mathrm{WF}_{\epsilon,1}^{\mu+1/2+\sigma}(\eta)^\circ) \\ &= \mathrm{WF}_{\epsilon,1}^{\mu+1/2+\sigma}(\eta)^\circ \cup \mathrm{WF}_{\epsilon,1}^{\mu+\sigma}(\psi - \mathcal{P}_B \eta)^\circ \\ &= \mathrm{WF}_{\epsilon,1}^\sigma(u)^\circ. \end{aligned}$$

□

4.6.2 Proof of Theorem 4.1.7

By Lemma 4.6.5, it is equivalent to prove the following theorem.

Theorem 4.6.6. *Under the hypothesis of Theorem 4.1.7, let u be defined by (4.6.2), and let*

$$(x_0, \xi_0) \in \mathrm{WF}_{1/2,1}^\sigma(u_0)^\circ$$

with $0 \leq \sigma \leq \max\{m/2 - 3/2, 0\}$. Let $t_0 \in [0, T]$, and suppose that

$$x_0 + \frac{3}{2}t|\xi_0|^{-1/2}\xi_0 \neq 0, \quad \forall t \in [0, t_0],$$

then

$$\left(x_0 + \frac{3}{2}t_0|\xi_0|^{-1/2}\xi_0, \xi_0\right) \in \mathrm{WF}_{1/2,1}^\sigma(u(t_0))^\circ.$$

Démonstration. For $\nu \in \mathbb{R}$, denote

$$X^\nu = \sum_{k \in \mathbb{Z}} H_k^{\nu-k/2}.$$

By Lemma 4.2.16, if $f \in X^\nu$, then $\text{WF}_{1/2,1}^\nu(f)^\circ = \emptyset$. By Remark 4.6.4,

$$\begin{aligned} \mathcal{P}_V \cdot \nabla w &\in H_m^{-1} \subset X^{m/2-1}, \\ \mathcal{P}_{Q^{(-j)}} w &\in H_{\rho_{j+2}}^j \subset X^{\min\{j+m/2, \mu-3/2-d/2\}}, \quad j \geq 0, \\ \mathcal{P}_{\gamma^{(3/2)}} w - \mathcal{P}_{|\xi|^{3/2}} w &\in H_m^{-3/2} \subset X^{m/2-3/2}, \\ \mathcal{P}_{\gamma^{(3/2-j)}} w &\in H_{\rho_{j+1}}^{j-3/2} \subset X^{\min\{j+m/2-3/2, \mu-2-d/2\}}, \quad j \geq 1, \\ \mathcal{P}_{\zeta^{(-1/2)}} w - \mathcal{P}_{|\xi|^{-1/2}} w &\in H_m^{1/2} \subset X^{m/2+1/2}, \\ \mathcal{P}_{\zeta^{(-1/2-j)}} w &\in H_{\rho_{j+1}}^{1/2+j} \subset X^{\min\{j+m/2+1/2, \mu-d/2\}}, \quad j \geq 1. \end{aligned}$$

By the hypothesis on m , we thus obtain

$$(4.6.3) \quad \partial_t w' + i|D_x|^{3/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} w' + \frac{ig}{2}|D_x|^{-1/2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} w' \in X^{m/2-3/2},$$

where $w' = \pi(D_x)w$, and $\pi \in C^\infty(\mathbb{R}^d)$ which vanishes near the origin, and equals to 1 out side a neighborhood of the origin. Moreover, we require that $\text{supp } \pi \subset \{\tilde{\pi} = 1\}$ such that $1 - \tilde{\pi} \in C_c^\infty(\mathbb{R}^d)$ and $\tilde{\pi}(\xi) = 0$ if $|\xi|^2 \leq |g|$. Observe that the matrix

$$M = |\xi|^{3/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{g}{2} |\xi|^{-1/2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

is symmetrizable when restricted to $\text{supp } \pi$. Indeed, let

$$\theta = \sqrt{\tilde{\pi}(\xi) \cdot (g|\xi|^{-2} + 1)}$$

and set

$$P = \frac{1}{2} \begin{pmatrix} 1 + \theta & 1 - \theta \\ -(1 - \theta) & -(1 + \theta) \end{pmatrix},$$

then $P \in \mathcal{O}_0^0$, and for $\xi \in \text{supp } \pi$,

$$PMP^{-1} = |\xi|^{3/2} \theta(\xi) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\tilde{w} = P(D_x)w'$, then

$$\partial_t \tilde{w} + |D_x|^{3/2} \theta(D_x) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tilde{w} \in X^{m/2-3/2}.$$

Let $u' = \pi(D_x)u$, then

$$\tilde{w} = P(D_x) \begin{pmatrix} u' \\ u' \end{pmatrix} = \begin{pmatrix} \text{Re } u' + i\theta(D_x)\text{Im } u' \\ -\text{Re } u' + i\theta(D_x)\text{Im } u' \end{pmatrix}.$$

Let $v = \text{Re } u' + i\theta(D_x)\text{Im } u'$, then $\text{WF}_{1/2,1}^\sigma(u)^\circ = \text{WF}_{1/2,1}^\sigma(v)^\circ$, and

$$\partial_t v + |D_x|^{3/2} \theta(D_x)v \in X^{m/2-3/2}.$$

We are left to prove that if $(x_0, \xi_0) \in \text{WF}_{1/2,1}^\sigma(v(0))^\circ$, then

$$\left(x_0 + \frac{3}{2}t_0|\xi_0|^{-1/2}\xi_0, \xi_0\right) \in \text{WF}_{1/2,1}^\sigma(v(t_0)).$$

Because $\theta(\xi) \sim 1$ in the high frequency regime, similar proof as (M.1) of Theorem 4.1.4 yields the conclusion. \square

4.6.3 Proof of Theorem 4.1.8

4.6.3.1 Hamiltonian flow

Let $\Phi = \Phi_s : \mathbb{R}_x^d \times (\mathbb{R}_\xi^d \setminus 0) \rightarrow \mathbb{R}_x^d \times (\mathbb{R}_\xi^d \setminus 0)$ be the Hamiltonian flow of

$$H(x, \xi) = \gamma^{(3/2)}(0, x, \xi) = \left(|\xi|^2 - \frac{(\nabla\eta_0 \cdot \xi)^2}{1 + |\nabla\eta_0|^2}\right)^{3/4}.$$

That is

$$\partial_s \Phi_s(x, \xi) = X_H(\Phi_s(x, \xi)), \quad \Phi|_{s=0} = \text{Id}_{\mathbb{R}_x^d \times (\mathbb{R}_\xi^d \setminus 0)},$$

where $X_H = (\partial_\xi H, -\partial_x H)$. We use s to denote the time variable in accordance to the semiclassical time variable in the following section. Observe that

Lemma 4.6.7. *For $(x, \xi) \in \mathbb{R}_x^d \times (\mathbb{R}_\xi^d \setminus 0)$, we have*

$$\Phi_s(x, \xi) = \mathcal{G}_{\varphi_s(x, \xi)}(x, \xi),$$

where \mathcal{G} is the geodesic flow defined in §4.1.3.6, and

$$\varphi_s(x, \xi) = \frac{3}{4} \int_0^s G(\Phi_\sigma(x, \xi))^{-1/4} d\sigma.$$

Démonstration. We have $\mathcal{G}_{\varphi_0(x, \xi)}(x, \xi) = \mathcal{G}_0(x, \xi) = (x, \xi) = \Phi_0(x, \xi)$. Then observe that

$$H(x, \xi) = G(x, \xi)^{3/4} = \varrho_x^{-1}(\xi, \xi)^{3/4}.$$

Therefore,

$$\begin{aligned} \frac{d}{ds} \mathcal{G}_{\varphi_s(x, \xi)}(x, \xi) &= \frac{d}{ds} \varphi_s(x, \xi) \left(\frac{d}{ds} \mathcal{G}_{\varphi_s(x, \xi)}(x, \xi) \right) \\ &= \frac{3}{4} G(\mathcal{G}_{\varphi_s(x, \xi)}(x, \xi))^{-1/4} X_G(\mathcal{G}_{\varphi_s(x, \xi)}(x, \xi)) = X_H(\mathcal{G}_{\varphi_s(x, \xi)}(x, \xi)). \end{aligned}$$

We conclude by the uniqueness of solutions to Hamiltonian ODEs. \square

Lemma 4.6.8. *Suppose that for some $\epsilon > 0$, $\nabla\eta_0 \in W_{1/2+\epsilon}^{0,\infty}$, $\nabla^2\eta_0 \in W_{1+\epsilon}^{0,\infty}$. Let $(x_0, \xi_0) \in \mathbb{R}_x^d \times (\mathbb{R}_\xi^d \setminus 0)$ such that the co-geodesic $\{(x_s, \xi_s) = \Phi_s(x_0, \xi_0)\}_{s \in \mathbb{R}}$ is forwardly non-trapping. Set*

$$z_s = x_s - x_0 - \frac{3}{2} \int_0^s |\xi_\sigma|^{-1/2} \xi_\sigma d\sigma,$$

then $\exists(z_{+\infty}, \xi_{+\infty}) \in \mathbb{R}_x^d \times (\mathbb{R}_\xi^d \setminus 0)$ such that

$$\lim_{s \rightarrow +\infty} (z_s, \xi_s) = (z_{+\infty}, \xi_{+\infty}).$$

Consequently, by Lemma 4.6.7, let $(x'_s, \xi'_s) = \mathcal{G}_s(x_0, \xi_0)$, then $\lim_{s \rightarrow +\infty} \xi'_s = \xi_{+\infty}$.

Démonstration. Because $\{(x_s, \xi_s)\}_{s \in \mathbb{R}}$ is forwardly non-trapping, and we only consider the limiting behavior when $s \rightarrow +\infty$, we may assume that $\varepsilon_0 := \|\langle x \rangle \nabla^2 \eta_0\|_{L^\infty}$ is sufficiently small. As $\nabla \eta_0 \in L^\infty$, we have $H(\cdot, \xi) \simeq |\xi|^{3/2}$. Then

$$\frac{d}{ds}(x_s \cdot \xi_s) = \partial_\xi H(x_s, \xi_s) \cdot \xi_s - x_s \cdot \partial_x H(x_s, \xi_s),$$

where

$$\partial_\xi H(x_s, \xi_s) \cdot \xi_s = \frac{3}{2}H(x_s, \xi_s) = \frac{3}{2}H(x_0, \xi_0) \simeq |\xi_0|^{3/2},$$

and

$$\begin{aligned} \partial_x H(x_s, \xi_s) &= \frac{3}{4}H(x_s, \xi_s)^{-1/3} \partial_x G(x_s, \xi_s) \\ &= \frac{3}{4}H(x_s, \xi_s)^{-1/3} \left(\frac{2\nabla \eta_0 \cdot \xi_s}{1 + |\nabla \eta_0|^2} \nabla^2 \eta_0 \xi_s - \frac{2(\nabla \eta_0 \cdot \xi_s)^2}{(1 + |\nabla \eta_0|^2)^2} \nabla^2 \eta_0 \nabla \eta_0 \right) \Big|_{x=x_s}. \end{aligned}$$

Therefore

$$x_s \cdot \partial_x H(x_s, \xi_s) = \mathcal{O}(\varepsilon |\xi_s|^{3/2}) = \mathcal{O}(\varepsilon |\xi_0|^2),$$

and consequently,

$$(4.6.4) \quad \frac{d}{ds}(x_s \cdot \xi_s) \gtrsim |\xi_0|^{3/2}.$$

So for any bounded set $B \in \mathbb{R}^d$,

$$(4.6.5) \quad \begin{aligned} \lambda(s \geq 0 : x_s \in B) &\lesssim \frac{\sup\{|x \cdot \xi| : (x, \xi) \in B \times \mathbb{R}^d, H(x, \xi) = H(x_0, \xi_0)\}}{|\xi_0|^{3/2}} \\ &\lesssim \sup_{x \in B} |x| \langle \xi_0 \rangle^{-1/2}, \end{aligned}$$

where λ is the Lebesgue measure on \mathbb{R} . Let $E(x, \xi) = H(x, \xi) - |\xi|^{3/2}$, then by the hypothesis of the decay of η_0 , $E \in \Gamma_{1+\epsilon, 0}^{3/2, 1}$. By the definition of z_s , we have

$$\frac{d}{ds}(z_s, \xi_s) = (\partial_\xi E, -\partial_x E)(z_s, \xi_s) = \mathcal{O}(\langle x_s \rangle^{-1-\epsilon}).$$

By (4.6.5),

$$\int_0^\infty \langle x_s \rangle^{-1-\epsilon} ds = (1 + \epsilon) \int_0^\infty t^\epsilon \lambda(s \geq 0 : \langle x_s \rangle^{-1} > t) dt \lesssim \int_0^1 t^\epsilon \sqrt{t^{-2} - 1} dt < \infty.$$

Therefore, for any $0 < s^- < s^+$ with $s^- \rightarrow \infty$,

$$|(z_{s^+}, \xi_{s^+}) - (z_{s^-}, \xi_{s^-})| \lesssim \int_{s^-}^{s^+} \langle x_\sigma \rangle^{-1-\epsilon} d\sigma \rightarrow 0,$$

implying that (x_s, ξ_s) is a Cauchy sequence as $s \rightarrow \infty$. □

4.6.3.2 Construction of symbol

For $h \geq 0$, and $h^{1/2}s \leq T$, set

$$H_h(s, x, \xi) = \gamma^{(3/2)}(h^{1/2}s, x, \xi),$$

so in particular $H(x, \xi) \equiv H_0(s, x, \xi)$. For $h > 0$, the semiclassical time variable $s = h^{-1/2}t$ was inspired by Lebeau [131], see also Zhu [204] for an application in theory of control for water waves.

For $a \in C^\infty([0, h^{-1/2}T] \times \mathbb{R}_x^d \times \mathbb{R}_\xi^d)$, set

$$\mathcal{L}_{h,s}^\pm a = \partial_s a \pm \{H_h, a\}.$$

Lemma 4.6.9. *Suppose that for some $\epsilon > 0$, $\nabla \eta_0 \in W_{1/2+\epsilon}^{0,\infty}$, $\nabla^2 \eta_0 \in W_{1+\epsilon}^{0,\infty}$, $\nabla^3 \eta_0 \in W_{3/2+\epsilon}^{0,\infty}$. Let $(x_0, \xi_0) \in \mathbb{R}_x^d \times (\mathbb{R}_\xi^d \setminus \{0\})$ such that the co-geodesic $\{(x_s, \xi_s) = \Phi_s(x_0, \xi_0)\}_{s \in \mathbb{R}}$ is forwardly non-trapping, then for some $s_0 > 0$, there exists*

$$(4.6.6) \quad \chi^\pm \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \Gamma^{-\infty, \mu-3-d/2}) \cap W^{1,\infty}(\mathbb{R}_{\geq s_0}, S_0^{-\infty})$$

such that

1. $\chi^\pm(0, x, \xi) \in S_{-\infty}^-$ is elliptic at $(x_0, \pm \xi_0)$;
2. $\forall t_0 > 0$, $\chi^\pm(s, \frac{s}{t_0}x, \xi) \in S_{-\infty}^-$ is elliptic at $(\frac{3}{2}t_0|\xi_\infty|^{-1/2}\xi_\infty, \pm \xi_\infty)$ for sufficiently large s .

Moreover, if $(\eta, \psi) \in \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu$ with $\mu > 3 + d/2$ and $m \geq 2$, then

$$\mathcal{L}_{h,s}^\pm \chi^\pm \in L^\infty([0, h^{-1/2}T], \Gamma_{-1,0}^{-\infty, \mu-4-d/2}),$$

and

$$(4.6.7) \quad \mathcal{L}_{h,s}^\pm \chi^\pm \geq \mathcal{O}(h^{1/2})_{L^\infty([0, h^{-1/2}T], \Gamma_{-1,0}^{-\infty, \mu-4-d/2})}.$$

Démonstration. Let $\phi \in C_c^\infty(\mathbb{R}^d)$ such that

- (i) $\phi \geq 0$, $\phi(x) = 1$ for $|x| \leq 1/2$, $\phi(x) = 0$ for $|x| \geq 1$, $\text{supp } \phi = \{|x| \leq 1\}$;
- (ii) $x \cdot \nabla \phi(x) \leq 0$ for all $x \in \mathbb{R}^d$;
- (iii) $y \cdot \nabla \phi(x) = 0$ for all $x, y \in \mathbb{R}^d$ with $x \cdot y = 0$.

Such ϕ can be constructed by setting $\phi(x) = \varphi(|x|)$ where $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $0 \leq \varphi \leq 1$, $\varphi(z) = 1$ if $z \leq 1/2$, $\varphi(z) = 0$ if $z \geq 1$. For $\delta > 0$, $\lambda > 0$, $\nu > 0$ and sufficiently large $s > 0$, set

$$\tilde{\chi}^\pm(s, x, \xi) = \phi\left(\frac{x - x_s}{\lambda \delta s}\right) \phi\left(\frac{\xi \mp \xi_s}{\delta - s^{-\nu}}\right).$$

We verify that $\mathcal{L}_{0,s}^\pm \tilde{\chi}^\pm(s, \cdot) \geq 0$ for $s > 0$ sufficient large. Indeed,

$$\begin{aligned} & \mathcal{L}_{0,s}^\pm \tilde{\chi}^\pm(s, x, \xi) \\ &= \left(\pm \frac{\partial_\xi H(x, \xi) - \partial_\xi H(x_s, \mp \xi_s)}{\lambda \delta s} - \frac{x - x_s}{\lambda \delta s^2} \right) \nabla \phi\left(\frac{x - x_s}{\lambda \delta s}\right) \phi\left(\frac{\xi \mp \xi_s}{\delta - s^{-\nu}}\right) \\ &+ \left(\pm \frac{\partial_x H(x_s, \mp \xi_s) - \partial_x H(x, \xi)}{\delta - s^{-\nu}} - \nu \frac{\xi \mp \xi_s}{(\delta - s^{-\nu})^2 s^{\nu+1}} \right) \phi\left(\frac{x - x_s}{\lambda \delta s}\right) \nabla \phi\left(\frac{\xi \mp \xi_s}{\delta - s^{-\nu}}\right). \end{aligned}$$

By (i),

$$\begin{aligned} \operatorname{supp} \phi\left(\frac{\cdot - x_s}{\lambda \delta s}\right) &\subset \{x \in \mathbb{R}^d : |x - x_s| \leq \lambda \delta s\}, \\ \operatorname{supp} \phi\left(\frac{\cdot \mp \xi_s}{\delta - s^{-\nu}}\right) &\subset \{\xi \in \mathbb{R}^d : |\xi \mp \xi_s| \leq \delta - s^{-\nu}\}, \\ \operatorname{supp} \nabla \phi\left(\frac{\cdot - x_s}{\lambda \delta (1 + s)}\right) &\subset \{x \in \mathbb{R}^d : \frac{1}{2} \lambda \delta s \leq |x - x_s| \leq \lambda \delta s\}, \\ \operatorname{supp} \nabla \phi\left(\frac{\cdot \mp \xi_s}{\delta - s^{-\nu}}\right) &\subset \{\xi \in \mathbb{R}^d : \frac{1}{2}(\delta - s^{-\nu}) \leq |\xi \mp \xi_s| \leq \delta - s^{-\nu}\}. \end{aligned}$$

By Lemma 4.6.8,

$$x_s = x_0 + \frac{3}{2} \int_0^s |\xi_\sigma|^{-1/2} \xi_\sigma \, d\sigma + z_s = \frac{3}{2} s |\xi_\infty|^{-1/2} \xi_\infty + o(s).$$

There, by writing

$$\tilde{\chi}^\pm(s, \frac{s}{t_0} x, \xi) = \phi\left(\frac{x - \frac{3}{2} t_0 |\xi_\infty|^{-1/2} \xi_\infty + o(1)}{\lambda \delta t_0}\right) \phi\left(\frac{\xi \mp \xi_\infty + o(1)}{\delta - s^{-\nu}}\right),$$

we see that $\tilde{\chi}^\pm(s, \frac{s}{t_0} x, \xi)$ is elliptic at $(\frac{3}{2} t_0 |\xi_\infty|^{-1/2} \xi_\infty, \pm \xi_\infty)$ for sufficiently large s . Moreover, if $\lambda \delta$ is sufficiently small and s is sufficiently large, then

$$\operatorname{supp} \phi\left(\frac{\cdot - x_s}{\lambda \delta s}\right) \subset \{x \in \mathbb{R}^d : |x| \gtrsim s\}.$$

Therefore, by the hypothesis on η_0 , we have for $(x, \xi) \in \operatorname{supp} \tilde{\chi}^\pm(s, \cdot)$,

$$\nabla_{xy}^2 H(x, \xi) = \begin{pmatrix} \nabla_x^2 H & \nabla_x \nabla_\xi H \\ \nabla_\xi \nabla_x H & \nabla_\xi^2 H \end{pmatrix} (x, \xi) = \begin{pmatrix} \mathcal{O}(s^{-2-\epsilon}) & \mathcal{O}(s^{-3/2-\epsilon}) \\ \mathcal{O}(s^{-3/2-\epsilon}) & \mathcal{O}(1) \end{pmatrix},$$

and consequently, by the finite increment formula,

$$\begin{aligned} |\partial_\xi H(x_s, \mp \xi_s) - \partial_\xi H(x, \xi)| &\lesssim s^{-3/2-\epsilon} |x - x_s| + |\xi \mp \xi_s| \lesssim s^{-1/2-\epsilon} \lambda \delta + \delta; \\ |\partial_x H(x_s, \mp \xi_s) - \partial_x H(x, \xi)| &\lesssim s^{-2-\epsilon} |x - x_s| + s^{-3/2-\epsilon} |\xi \mp \xi_s| \lesssim \lambda \delta s^{-1-\epsilon} + \delta s^{-3/2-\epsilon}. \end{aligned}$$

By (iii),

$$\begin{aligned} &(\partial_\xi H(x, \xi) - \partial_\xi H(x_s, \mp \xi_s)) \cdot \nabla \phi\left(\frac{x - x_s}{\lambda \delta s}\right) \\ &= (\partial_\xi H(x, \xi) - \partial_\xi H(x_s, \mp \xi_s)) \cdot \frac{x - x_s}{|x - x_s|^2} (x - x_s) \cdot \nabla \phi\left(\frac{x - x_s}{\lambda \delta s}\right) \\ &= \mathcal{O}(s^{-3/2-\epsilon} + \lambda^{-1} s^{-1}) (x - x_s) \cdot \nabla \phi\left(\frac{x - x_s}{\lambda \delta s}\right); \\ &(\partial_x H(x_s, \mp \xi_s) - \partial_x H(x, \xi)) \cdot \nabla \phi\left(\frac{\xi \mp \xi_s}{\delta - s^{-\nu}}\right) \\ &= (\partial_x H(x_s, \mp \xi_s) - \partial_x H(x, \xi)) \cdot \frac{\xi \mp \xi_s}{|\xi \mp \xi_s|^2} (\xi \mp \xi_s) \cdot \nabla \phi\left(\frac{\xi \mp \xi_s}{\delta - s^{-\nu}}\right) \\ &= \mathcal{O}(\lambda s^{-1-\epsilon} + s^{-3/2-\epsilon}) (\xi \mp \xi_s) \cdot \nabla \phi\left(\frac{\xi \mp \xi_s}{\delta - s^{-\nu}}\right). \end{aligned}$$

Finally, we fix $0 < \nu < \epsilon$, $\delta > 0$. Then, when λ is sufficiently large, and $s \geq s_0 - 1 > 0$ with s_0 being sufficiently large, by (ii),

$$(4.6.8) \quad \begin{aligned} \mathcal{L}_{0,s}^\pm \tilde{\chi}^\pm &= -\frac{1 + \mathcal{O}(s^{-1/2-\epsilon} + \lambda^{-1})}{\lambda \delta s^2} (x - x_s) \cdot \nabla \phi \left(\frac{x - x_s}{\lambda \delta s} \right) \phi \left(\frac{\xi \mp \xi_s}{\delta - s^{-\nu}} \right) \\ &\quad - \frac{\nu(\delta - s^{-\nu}) - \mathcal{O}(\lambda) s^{\nu-\epsilon}}{(\delta - s^{-\nu})^2 s^{\nu+1}} (\xi \mp \xi_s) \cdot \phi \left(\frac{x - x_s}{\lambda \delta s} \right) \nabla \phi \left(\frac{\xi \mp \xi_s}{\delta - s^{-\nu}} \right) \geq 0. \end{aligned}$$

We verify as in Lemma 4.3.2 that

$$\tilde{\chi}^\pm \in W^{\infty,\infty}(\mathbb{R}_{\geq s_0}, S_0^{-\infty}), \quad \mathcal{L}_{0,s}^\pm \tilde{\chi}^\pm \in W^{\infty,\infty}(\mathbb{R}_{\geq s_0}, \Gamma_{-1,0}^{-\infty, -(\mu-4-d/2)}).$$

Next, we set for $s \geq s_0$,

$$\chi^\pm(s, x, \xi) = \tilde{\chi}^\pm(s, x, \xi).$$

To define χ^\pm for $s \leq s_0$, we choose $\rho \in C^\infty(\mathbb{R})$ such that $0 \leq \rho \leq 1$, $\rho(s) = 1$ for $s \geq s_0$, and $\rho(s) = 0$ for $s \leq s_0 - \alpha$ for some small $\alpha > 0$ to be specified later, and solve the transport equation on $[0, s_0]$,

$$\mathcal{L}_{0,s}^\pm \chi^\pm(s, x, \xi) = \rho(s) \mathcal{L}_{0,s}^\pm \tilde{\chi}^\pm(s, x, \xi), \quad \chi^\pm(s_0, x, \xi) = \tilde{\chi}^\pm(s_0, x, \xi).$$

Clearly χ^\pm satisfies (4.6.6), and

$$(4.6.9) \quad \mathcal{L}_{0,s}^\pm \chi^\pm \geq 0.$$

Moreover, because

$$\chi^\pm(s, x, \xi) = \tilde{\chi}^\pm(s_0, \Phi_{\pm(s_0-s)}(x, \xi)) - \int_s^{s_0} \rho(\sigma) \mathcal{L}_{0,\sigma}^\pm \tilde{\chi}^\pm(\sigma, \Phi_{\pm(\sigma-s)}(x, \xi)) d\sigma,$$

if we choose $\alpha > 0$ sufficiently small, then

$$\begin{aligned} \chi^\pm(0, x_0, \pm \xi_0) &= \tilde{\chi}^\pm(s_0, x_{s_0}, \pm \xi_{s_0}) - \int_{s_0-\alpha}^{s_0} \rho(\sigma) \mathcal{L}_{0,\sigma}^\pm \tilde{\chi}^\pm(\sigma, x_\sigma, \pm \xi_\sigma) d\sigma \\ &\geq 1 - \|\mathcal{L}_{0,s}^\pm \tilde{\chi}^\pm(\sigma, x_\sigma, \pm \xi_\sigma)\|_{L^1_\sigma([s_0-\alpha, s_0])} > 0. \end{aligned}$$

Therefore, $\chi^\pm(0, \cdot)$ is elliptic at $(x_0, \pm \xi_0)$.

To estimate $\mathcal{L}_{h,s}^\pm \chi^\pm$, we use

$$\begin{aligned} H_h(s, x, \xi) - H_0(s, x, \xi) &= H_h(s, x, \xi) - H_h(0, x, \xi) \\ &= \int_0^s (\partial_s H_h)(\sigma, x, \xi) d\sigma = h^{1/2} \int_0^s (\partial_t \gamma^{(3/2)})(h^{1/2} \sigma, x, \xi) d\sigma, \end{aligned}$$

and write

$$\begin{aligned} \mathcal{L}_{h,s}^\pm \chi^\pm(s, \cdot) - \mathcal{L}_{0,s}^\pm \chi^\pm(s, \cdot) &= \pm \{H_h - H_0, \chi^\pm\}(s, \cdot) \\ &= \pm h^{1/2} \int_0^s \{\partial_t \gamma^{(3/2)}(h^{1/2} \sigma, \cdot), \chi^\pm(s, \cdot)\} d\sigma. \end{aligned}$$

Observe that

$$\partial_t \gamma^{(3/2)} = -\frac{3}{2} \left(|\xi|^2 - \frac{(\nabla \eta \cdot \xi)^2}{1 + |\nabla \eta|^2} \right)^{-1/4} \left(\frac{\nabla \eta \cdot \xi}{1 + |\nabla \eta|^2} \nabla G(\eta) \psi \cdot \xi - \frac{(\nabla \eta \cdot \xi)^2}{(1 + |\nabla \eta|^2)^2} \nabla G(\eta) \psi \cdot \nabla \eta \right).$$

By hypothesis and Proposition 4.5.4, $\nabla G(\eta)\psi \in \mathcal{H}_m^{\mu-2} \subset H_2^{\mu-3}$ as $m \geq 2$. Therefore,

$$\partial_t \gamma^{(3/2)}(h^{1/2}\cdot, \cdot) \in L^\infty([0, h^{-1/2}T], \Gamma_{-2,0}^{3/2,\mu-7/2-d/2}).$$

Using $|x| \sim s$ on $\text{supp } \chi^\pm(s, \cdot)$, we have, uniformly for all $s \in [0, h^{-1/2}T]$,

$$\langle s \rangle \{ \partial_t \gamma^{(3/2)}(h^{1/2}\sigma, \cdot), \chi^\pm(s, \cdot) \} \in L^\infty_\sigma([0, h^{-1/2}T], \Gamma_{-1,0}^{-\infty,\mu-4-d/2}).$$

Therefore,

$$\begin{aligned} \mathcal{L}_{h,s}^\pm \chi^\pm(s, \cdot) - \mathcal{L}_{0,s}^\pm \chi^\pm(s, \cdot) &= \pm h^{1/2} \langle s \rangle^{-1} \int_0^s \mathcal{O}(1)_{L^\infty([0, h^{-1/2}T], \Gamma_{-1,0}^{-\infty,\mu-4-d/2})} d\sigma \\ &= \pm h^{1/2} \langle s \rangle^{-1} \mathcal{O}(s)_{\Gamma_{-1,0}^{-\infty,\mu-4-d/2}} \\ &= \mathcal{O}(h^{1/2})_{\Gamma_{-1,0}^{-\infty,\mu-4-d/2}}, \end{aligned}$$

which, together with (4.6.9), proves (4.6.7). \square

4.6.3.3 Propagation

Now we prove Theorem 4.1.8. By Lemma 4.6.5 and Lemma 4.6.7, it suffices to prove the following theorem.

Theorem 4.6.10. *Under the hypothesis of Theorem 4.1.8, let u be defined as (4.6.2). Let*

$$(x_0, \xi_0) \in \text{WF}_{0,1}^\sigma(u_0)^\circ,$$

with $0 \leq \sigma < \min\{\mu/2-3-d/4, 3m/2\}$, such that the co-geodesic $\{(x_s, \xi_s) = \Phi_s(x_0, \xi_0)\}_{s \in \mathbb{R}}$ is forwardly non-trapping. Set

$$\xi_\infty = \lim_{s \rightarrow +\infty} \xi_s,$$

then $\forall t_0 \in]0, T]$,

$$\left(\frac{3}{2} t_0 |\xi_\infty|^{-1/2} \xi_\infty, \xi_\infty \right) \in \text{WF}_{1/2,1}^\sigma(u(t_0)).$$

Under the semiclassical time variable $s = h^{-1/2}t$, (4.6.1) becomes

$$\begin{aligned} (\partial_s + h^{1/2} \mathcal{P}_V \cdot \nabla + h^{1/2} \mathcal{P}_Q)w + ih^{1/2} \begin{pmatrix} \mathcal{P}_\gamma & 0 \\ 0 & -\mathcal{P}_\gamma \end{pmatrix} w + \frac{ih^{1/2}g}{2} \mathcal{P}_\zeta \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} w \\ = F_h = \mathcal{O}(h^{1/2})_{H^{\mu-4-d/2}}. \end{aligned}$$

We define \mathcal{L}_s^h which applies to time dependent operators $\mathcal{A} : \mathcal{S}' \rightarrow \mathcal{S}'$,

$$\mathcal{L}_s^h \mathcal{A} = \partial_s \mathcal{A} + h^{1/2} \left[\mathcal{P}_V \cdot \nabla + \mathcal{P}_Q + i\mathcal{P}_\gamma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{ig}{2} \mathcal{P}_\zeta \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \mathcal{A} \right].$$

We also define \mathcal{L}_s^h which applies to symbols of the diagonal form $A = \begin{pmatrix} A^+ & 0 \\ 0 & A^- \end{pmatrix}$,

$$\mathcal{L}_s^h A = \begin{pmatrix} \mathcal{L}_{h,s}^+ A^+ & 0 \\ 0 & \mathcal{L}_{h,s}^- A^- \end{pmatrix}.$$

Proof of Theorem 4.6.10. We shall from now on denote $\rho = [\mu - 4 - d/2]$, $I_h = [0, h^{-1/2}T]$ and

$$Y_h^\rho = L^\infty(I_h, M_{2 \times 2}(h\Sigma^{-\infty, \rho}))$$

for simplicity. Choose a strictly increasing sequence $\{\lambda_j\}_{j \geq 0} \subset [1, 1 + \epsilon]$ with $\epsilon > 0$ being sufficiently small. Define χ_j^\pm as in Lemma 4.6.9 where we replace ϕ with $\phi(\cdot/\lambda_j)$. Then

$$\text{supp } \chi_j^\pm \subset \{\chi_{j+1}^\pm > 0\}, \quad \forall j \in \mathbb{N}.$$

And we set

$$\chi_j = \begin{pmatrix} \chi_j^+ & 0 \\ 0 & \chi_j^- \end{pmatrix},$$

We shall construct an operator $\mathcal{A}_h \in L^\infty(I_h, L^2 \rightarrow L^2)$ satisfying the following properties :

(i) \mathcal{A}_h is a paradifferential operator, more precisely, there exists

$$A_h^\pm \in W^{1, \infty}(\mathbb{R}_{\geq 0}, \Gamma^{-\infty, \rho+1}) \cap W^{1, \infty}(\mathbb{R}_{\geq s_0}, S_0^{-\infty})$$

for some $s_0 > 0$, such that

$$\mathcal{A}_h - \mathcal{P}_{A_h}^h = \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \rightarrow L^2)}, \quad A_h = \begin{pmatrix} A_h^+ & 0 \\ 0 & A_h^- \end{pmatrix}.$$

Moreover,

$$\text{supp } A_h^\pm \subset \bigcup_{j \geq 0} \text{supp } \chi_j^\pm.$$

- (ii) $A_h^\pm(0, x, \xi)$ is elliptic at $(x_0, \pm\xi_0)$;
- (iii) $A_h^\pm(s, \frac{s}{t_0}x, \xi) \in S_{-\infty}^{-\infty}$ is elliptic at $(\frac{3}{2}t_0|\xi_\infty|^{-1/2}\xi_\infty, \xi_\infty)$ for $s > 0$ sufficiently large ;
- (iv) $\mathcal{L}_s^h \mathcal{A}_h \geq \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \rightarrow L^2)}$.

We shall construct \mathcal{A}_h of the form

$$\mathcal{A}_h = \sum_{j \geq 0}^{2\rho} h^{j/2} \varphi^j \mathcal{A}_h^j,$$

where $\varphi \in P_j$, recalling the definition (4.3.4), and $\mathcal{A}_h^j \in L^\infty(I_h, L^2 \rightarrow L^2)$. We begin by setting

$$\mathcal{A}_h^0 = (\mathcal{P}_{\chi_0}^h)^* \mathcal{P}_{\chi_0}^h, \quad \varphi^0 \equiv 1.$$

Therefore, by the symbolic calculus, Lemma 4.6.9 and Corollary 4.4.31 (observe that the symbol of \mathcal{A}_h^0 belongs to σ_0 , and that γ is a sum of homogeneous symbols),

$$\partial_s \mathcal{A}_h^0 + h^{1/2} \left[i \mathcal{P}_\gamma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathcal{A}_h^0 \right] = 2\mathcal{P}_{\chi_0 \mathcal{L}_s^h \chi_0}^h + h\mathcal{P}_{b_h^0}^h + \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \rightarrow L^2)},$$

for some $b_h^0 \in L^\infty(I_h, h\Sigma_{-1,0}^{-\infty, \rho})$, with $\text{supp } b_h^0 \subset \text{supp } \chi_0$. Therefore $\langle s \rangle b_h^0 \in Y_h^\rho$. Similarly,

$$h^{1/2} [\mathcal{P}_V \cdot \nabla, \mathcal{A}_h^0] = h^{1/2} \mathcal{P}_{b_h^1}^h + \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \rightarrow L^2)},$$

where $\langle s \rangle b_h^1 \in Y_h^\rho$, with $\text{supp } b_h^1 \subset \text{supp } \chi_0$. Be careful that, because Q and $\mathcal{P}_\zeta \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ are not diagonal matrices, their commutators with \mathcal{A}_h^0 do not gain an extra h , for the principal symbols do not cancel each other. So,

$$h^{1/2} [\mathcal{P}_Q, \mathcal{A}_h^0] = h^{1/2} \mathcal{P}_{b_h^2}^h + \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \rightarrow L^2)},$$

$$h^{1/2} \left[\mathcal{P}_\zeta \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \mathcal{A}_h^0 \right] = h\mathcal{P}_{b_h^3}^h + \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \rightarrow L^2)},$$

where $\langle s \rangle b_h^2, \langle s \rangle b_h^3 \in Y_h^\rho$, with $\text{supp } b_h^2 \cup \text{supp } b_h^3 \subset \text{supp } \chi_0$. By Lemma 4.6.9,

$$\chi_0 \mathcal{L}_s^h \chi_0 \geq h^{1/2} b_h^4,$$

where $\langle s \rangle b_h^4 \in L^\infty(I_h, \Gamma^{-\infty, \rho}) \subset Y_h^\rho$, with $\text{supp } b_h^4 \subset \text{supp } \chi_0$. Therefore, by the paradifferential Gårding inequality, i.e., Lemma 4.4.34,

$$\mathcal{P}_{\chi_0 \mathcal{L}_s^h \chi_0}^h - h^{1/2} \mathcal{P}_{b_h^4}^h \geq h\mathcal{P}_{b_h^5}^h + \mathcal{O}(h^\rho)_{L^2 \rightarrow L^2},$$

for some $b_h^5 \in Y_h^\rho$ with $\text{supp } b_h^5 \subset \{\chi_1 > 0\}$. Set

$$\alpha_h^0 = \langle s \rangle (b_h^1 + b_h^2 + 2b_h^4) \in Y_h^\rho, \quad \beta_h^0 = \langle s \rangle (b_h^0 + b_h^3 + 2b_h^5) \in Y_h^\rho.$$

Then

$$\mathcal{L}_s^h \mathcal{A}_h^0 \geq h^{1/2} \langle s \rangle^{-1} \mathcal{P}_{\alpha_h^0 + h^{1/2} \beta_h^0}^h + \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \rightarrow L^2)}.$$

Suppose that we have found $\mathcal{A}_h^j \in L^\infty(I_h, L^2 \rightarrow L^2)$, $\varphi^j \in P_j$ for $j = 0, \dots, \ell - 1$, and $\psi^{\ell-1} \in P_{\ell-1}$, $\alpha_h^{\ell-1}, \beta_h^{\ell-1} \in Y_h^\rho$ with $\text{supp } \alpha_h^{\ell-1} \cup \text{supp } \beta_h^{\ell-1} \subset \{\chi_\ell > 0\}$, such that

$$(4.6.10) \quad \mathcal{L}_s^h \left(\sum_{j=0}^{\ell-1} h^{j/2} \varphi^j \mathcal{A}_h^j \right) \geq h^{\ell/2} \langle s \rangle^{-1} \psi^{\ell-1} \mathcal{P}_{\alpha_h^{\ell-1} + h^{1/2} \beta_h^{\ell-1}}^h + \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \rightarrow L^2)}.$$

Then as in the proof of (M.2) of Theorem 4.1.4, we set

$$\varphi^\ell(s) = \int_0^s (1 + \sigma)^{-1} \psi^{\ell-1}(\sigma) d\sigma, \quad \mathcal{A}_h^\ell = C_\ell \varphi^\ell \mathcal{P}_{\chi_\ell}^h,$$

where the constant C_ℓ is sufficiently large, such that by Lemma 4.6.9, in the sense of positivity of matrices,

$$C_\ell \mathcal{L}_s^s(\varphi^\ell \chi_\ell) = C_\ell (1 + s)^{-1} \psi^{\ell-1} \chi_\ell + C_\ell \varphi^\ell \mathcal{L}_s^s \chi_\ell \geq \langle s \rangle^{-1} \psi^{\ell-1} \alpha_h^{\ell-1} + \varphi^\ell h^{1/2} \langle s \rangle^{-1} \tilde{\beta}_h^\ell.$$

for some $\tilde{\beta}_h^\ell \in Y_h^\rho$. By the paradifferential Gårding inequality, and a routine construction of parametrix, we find $\tilde{\alpha}_h^\ell \in Y_h^\rho$, with $\text{supp } \tilde{\alpha}_h^\ell \subset \{\chi_{\ell+1} > 0\}$, such that

$$\mathcal{P}_{C_\ell \mathcal{L}_s^s(\varphi^\ell \chi_\ell)}^h - \langle s \rangle^{-1} \mathcal{P}_{\psi^{\ell-1} \alpha_h^{\ell-1} + h^{1/2} \varphi^\ell \tilde{\beta}_h^\ell}^h \geq h \langle s \rangle^{-1} \mathcal{P}_{(\psi^{\ell-1} + \varphi^\ell) \tilde{\alpha}_h^\ell}^h + \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \rightarrow L^2)}.$$

Similarly as in the estimate of \mathcal{A}_h^0 , by a symbolic calculus, we find $\underline{\alpha}_h^\ell, \underline{\beta}_h^\ell \in Y_h^\rho$, with $\text{supp } \underline{\alpha}_h^\ell \cup \text{supp } \underline{\beta}_h^\ell \subset \text{supp } \chi_\ell$, such that

$$\mathcal{L}_s^h \mathcal{A}_h^\ell = \mathcal{P}_{C_\ell \mathcal{L}_s^s(\varphi^\ell \chi_\ell)}^h + h^{1/2} \langle s \rangle^{-1} \varphi^\ell \mathcal{P}_{\underline{\alpha}_h^\ell + h^{1/2} \underline{\beta}_h^\ell}^h + \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \rightarrow L^2)}.$$

Summing up the two inequalities above,

$$(4.6.11) \quad \mathcal{L}_s^h \mathcal{A}_h^\ell - \langle s \rangle^{-1} \psi^{\ell-1} \mathcal{P}_{\alpha_h^{\ell-1}}^h \geq h^{1/2} \langle s \rangle^{-1} \mathcal{P}_{\varphi^\ell (\underline{\alpha}_h^\ell + \tilde{\beta}_h^\ell) + h^{1/2} (\psi^{\ell-1} + \varphi^\ell) \tilde{\alpha}_h^\ell + h^{1/2} \varphi^\ell \underline{\beta}_h^\ell}^h + \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \rightarrow L^2)}.$$

Therefore, combining (4.6.10) and (4.6.11),

$$\mathcal{L}_s^h \left(\sum_{j=0}^{\ell} h^{j/2} \varphi^j \mathcal{A}_h^j \right) \geq h^{(\ell+1)/2} \langle s \rangle^{-1} \psi^\ell \mathcal{P}_{\alpha_h^\ell + h^{1/2} \beta_h^\ell}^h + \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \rightarrow L^2)},$$

with

$$\psi^\ell = 1 + \psi^{\ell-1} + \varphi^\ell, \quad \alpha_h^\ell = \frac{\psi^{\ell-1}}{\psi^\ell} \beta_h^{\ell-1} + \frac{\phi^\ell}{\psi^\ell} (\underline{\alpha}_h^\ell + \tilde{\beta}_h^\ell), \quad \beta_h^\ell = \frac{\psi^{\ell-1} + \varphi^\ell}{\psi^\ell} \tilde{\alpha}_h^\ell + \frac{\varphi^\ell}{\psi^\ell} \beta_h^\ell.$$

And we close the induction procedure.

To finish the proof, suppose that

$$\left(\frac{3}{2} t_0 |\xi_\infty|^{-1/2} \xi_\infty, \xi_\infty \right) \notin \text{WF}_{1/2,1}^\sigma(u(t_0)),$$

then by Lemma 4.2.16,

$$\left(\frac{3}{2} t_0 |\xi_\infty|^{-1/2} \xi_\infty, -\xi_\infty \right) \notin \text{WF}_{1/2,1}^\sigma(\overline{u(t_0)}).$$

By (i) and (iii), if we replace ϕ with $\phi(\lambda \cdot)$ for some sufficiently large $\lambda > 0$, then for sufficiently small $h > 0$,

$$\begin{aligned} \text{supp } \theta_{1/h,*}^{1/2,0} \chi_j^+ |_{s=h^{-1/2}t_0} &\subset \mathbb{R}_x^d \times \mathbb{R}_\xi^d \setminus \text{WF}_{1/2,1}^\sigma(u(t_0)), \\ \text{supp } \theta_{1/h,*}^{1/2,0} \chi_j^- |_{s=h^{-1/2}t_0} &\subset \mathbb{R}_x^d \times \mathbb{R}_\xi^d \setminus \text{WF}_{1/2,1}^\sigma(\overline{u(t_0)}). \end{aligned}$$

So by Proposition 4.4.33 and Lemma 4.2.15,

$$(\mathcal{A}_h w, w)_{L^2} |_{s=h^{-1/2}t_0} = \mathcal{O}(h^{2\sigma}).$$

Because by our construction, $\varphi^\ell(0) = 0, \forall \ell \geq 1$,

$$\mathcal{A}_h |_{s=0} = \mathcal{A}_h^0 |_{s=0} = (\mathcal{P}_{\chi_0}^h)^* \mathcal{P}_{\chi_0}^h |_{s=0}.$$

Because $F_h = \mathcal{O}(h^{1/2})_{H^\rho}$, we have, by Lemma 4.2.16,

$$\mathcal{A}_h F_h = \mathcal{O}(h^{\rho+1/2})_{L^2}.$$

Therefore,

$$\begin{aligned} \|\mathcal{P}_{\chi_0}^h w |_{s=0}\|_{L^2}^2 &= \text{Re}(\mathcal{A}_h w, w)_{L^2} |_{s=h^{-1/2}t_0} - \int_0^{h^{-1/2}t_0} \text{Re}(\mathcal{L}_s^h \mathcal{A}_h w, w)_{L^2} ds \\ &\quad - \int_0^{h^{-1/2}t_0} \text{Re}(\mathcal{A}_h F_h, w)_{L^2} ds \\ &\leq \mathcal{O}(h^{2\sigma}) + \mathcal{O}(h^{\rho-1/2}) = \mathcal{O}(h^{2\sigma}). \end{aligned}$$

Observe that $\chi_0 |_{s=0}$ is of compact support with respect to x , we have

$$\mathcal{P}_{\chi_0 |_{s=0}}^h = T_{\beta_h}^h + \mathcal{O}(h^\rho)_{L^2 \rightarrow L^2},$$

where

$$\beta_h = \sum_{j \geq 0} \psi_j \chi_0 |_{s=0} \#_h \psi_j \in {}_h \Sigma^{-\infty, \rho}$$

is a finite summation. We conclude by Lemma 4.4.35 that $(x_0, \xi_0) \notin \text{WF}_{0,1}^\sigma(u_0)$. \square

4.6.4 Proof of Corollary 4.1.9

The case when $d = 1$ is trivial. For the second case, we shall prove that on any co-geodesic $\{(x_t, \xi_t)\}_{t \in \mathbb{R}}$,

$$(4.6.12) \quad \lim_{t \rightarrow +\infty} x_t \cdot \xi_t = \infty,$$

so no geodesics can be trapped. The proof of (4.6.12) is almost finished by the proof of Lemma 4.6.8. Indeed, similar calculations imply that

$$\frac{d}{dt}(x_t \cdot \xi_t) \gtrsim |\xi_0|^2.$$

□

Chapitre 5

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Titre : Contrôle, stabilisation et propagation des singularités pour des EDP dispersives

Mots Clefs : EDP dispersive, théorie du contrôle, analyse microlocale, mécanique des fluides

Résumé :

Dans cette thèse, nous étudions les théories étroitement liées du contrôle, de la stabilisation et de la propagation des singularités, pour des équations aux dérivées partielles dispersives linéaires et non-linéaires. Les résultats principaux proviennent des travaux de l'auteur :

- [1] Zhu, H., 2016. *Stabilization of damped waves on spheres and Zoll surfaces of revolution*. ESAIM : Control, Optimisation and Calculus of Variations (ESAIM : COCV), à paraître.
- [2] Zhu, H., 2017. *Control of three dimensional water waves*. arXiv preprint arXiv :1712.06130.
- [3] Zhu, H., 2018. *Propagation of singularities for gravity-capillary water waves*. arXiv preprint arXiv :1810.09339.

Dans [1], nous avons étudié la stabilisation des ondes amorties sur les surfaces de révolution de Zoll. Nous avons donné un exemple où la région d'amortissement est à la limite de la condition du contrôle géométrique, alors que les ondes amorties présentent une décroissance exponentielle uniforme de l'énergie. Cet exemple généralise un résultat de Lebeau.

Dans [2], nous avons étudié la contrôlabilité du système des ondes de surface avec tension superficielle. Nous avons démontré, en dimension arbitraire, la contrôlabilité exacte pour des petites données spatialement périodiques en faisant l'hypothèse de contrôle géométrique. Ce résultat généralise le travail de Alazard, Baldi et Han-Kwan en dimension deux.

Dans [3], nous avons étudié la propagation des singularités pour des ondes de surface avec tension superficielle. Nous avons défini le front d'onde quasi-homogène, généralisant le front d'onde de Hörmander et le front d'onde homogène de Nakamura et démontré des résultats de propagation des fronts d'onde quasi-homogènes par le système des ondes de surface avec tension superficielle. Comme corollaires, nous avons obtenu des effets régularisants locaux et micro-locaux pour les données initiales présentant une décroissance spatiale suffisante.



Title : Control, stabilization and propagation of singularities for dispersive PDEs

Keys words : Dispersive PDE, control theory, microlocal analysis, fluid mechanics

Abstract :

In this thesis, we study the closely related theories of control, stabilization and propagation of singularities for some linear and nonlinear dispersive partial differential equations. Main results come from the author's works :

- [1] Zhu, H., 2016. *Stabilization of damped waves on spheres and Zoll surfaces of revolution*. ESAIM : Control, Optimisation and Calculus of Variations (ESAIM : COCV), to appear.
- [2] Zhu, H., 2017. *Control of three dimensional water waves*. arXiv preprint arXiv :1712.06130.
- [3] Zhu, H., 2018. *Propagation of singularities for gravity-capillary water waves*. arXiv preprint arXiv :1810.09339.

In [1] we studied the stabilization of the damped wave equation on Zoll surfaces of revolution. We gave an example where the region of damping is at the borderline of the geometric control condition, yet the damped waves exhibit a uniform exponential decay of energy, generalizing an example of Lebeau.

In [2] we studied the controllability of the gravity-capillary water wave equation. Under the geometric control condition, we proved in arbitrary spatial dimension the exact controllability for spatially periodic small data. This generalizes a result of Alazard, Baldi and Han-Kwan for the 2D gravity-capillary water wave equation.

In [3] we studied the propagation of singularities for the gravity-capillary water wave equation. We defined the quasi-homogeneous wavefront set, generalizing the wavefront set of Hörmander and the homogeneous wavefront set of Nakamura, and proved propagation results for quasi-homogeneous wavefront sets by the gravity-capillary water wave equation. As corollaries, we obtained local and microlocal smoothing effects for gravity-capillary water waves with sufficient spatial decay.