



**HAL**  
open science

## Estimation of the jump processes

Thi Thu Huong Nguyen

► **To cite this version:**

Thi Thu Huong Nguyen. Estimation of the jump processes. General Mathematics [math.GM]. Université Paris-Est, 2018. English. NNT : 2018PESC1124 . tel-02127797

**HAL Id: tel-02127797**

**<https://theses.hal.science/tel-02127797>**

Submitted on 13 May 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



École Doctorale Mathématiques et Sciences et Technologie de l'Information et de la  
Communication  
UNIVERSITÉ PARIS-EST

## THÈSE DE DOCTORAT

Spécialité : **MATHÉMATIQUES APPLIQUÉES**

---

---

# Estimation de processus de sauts

---

---

Présentée par

NGUYEN THI THU HUONG

Thèse dirigée par Emmanuelle Clément et Arnaud Gloter au LAMA, UPEM

MEMBRES DU JURY :

**Directeur de thèse** : Emmanuelle CLÉMENT, Université Paris-Est, Marne La Vallée

**Directeur de thèse** : Arnaud GLOTER, Université d'Évry Val d'Essonne

**Examineur** : Aurélien ALFONSI, Ecole des Ponts ParisTech

**Examineur** : Fabienne COMTE, Université Paris Descartes

**Rapporteur** : Mohamed BEN ALAYA, Université de Rouen Normandie

**Rapporteur** : Laurent DENIS, Université du Maine

Date de soutenance: 06-12-2018



---

# Résumé

---

Dans cette thèse, on considère une équation différentielle stochastique gouvernée par un processus de Lévy de saut pur dont l'indice d'activité des sauts  $\alpha \in (0, 2)$  et on observe des données haute fréquence de ce processus sur un intervalle de temps fixé. Cette thèse est consacrée tout d'abord à l'étude du comportement de la densité du processus en temps petit. Ces résultats permettent ensuite de montrer la propriété LAMN (Local Asymptotic Mixed Normality) pour les paramètres de dérive et d'échelle. Enfin, on étudie des estimateurs de l'indice  $\alpha$  du processus.

La première partie traite du comportement asymptotique de la densité en temps petit du processus. Le processus est supposé dépendre d'un paramètre  $\beta = (\theta, \sigma)^T$  et on étudie, dans cette partie, la sensibilité de la densité par rapport à ce paramètre. Cela étend les résultats de [17] qui étaient restreints à l'indice  $\alpha \in (1, 2)$  et ne considéraient que la sensibilité par rapport au paramètre de dérive. En utilisant le calcul de Malliavin, on obtient la représentation de la densité, de sa dérivée et de sa dérivée logarithmique comme une espérance et une espérance conditionnelle. Ces formules de représentation font apparaître des poids de Malliavin dont les expressions sont données explicitement, ce qui permet d'analyser le comportement asymptotique de la densité en temps petit, en utilisant la propriété d'autosimilarité du processus stable.

La deuxième partie de cette thèse concerne la propriété LAMN (Local Asymptotic Mixed Normality) pour les paramètres. Le coefficient de dérive et le coefficient d'échelle dépendent tous les deux de paramètres inconnus et on étend les résultats de [17]. On identifie l'information de Fisher asymptotique ainsi que les vitesses optimales de convergence. Ces quantités dépendent de l'indice  $\alpha$ .

La troisième partie propose des estimateurs pour l'indice d'activité des sauts  $\alpha \in (0, 2)$  basés sur des méthodes de moments qui généralisent les résultats de Masuda [53]. On montre la consistance et la normalité asymptotique des estimateurs et on illustre les résultats par des simulations numériques.



---

# Résumé substantiel

---

## Modèle

Dans cette thèse, nous considérons un processus  $X = (X_t)_{t \geq 0}$  solution d'une équation différentielle stochastique de la forme

$$(0.0.1) \quad X_t = x_0 + \int_0^t b(X_s, \theta) ds + \sigma L_t \quad t \in [0, 1]$$

où  $(L_t)_{t \in [0, 1]}$  est un processus de Lévy de saut pur dont l'indice d'activité des sauts  $\alpha \in (0, 2)$ ,  $\beta = (\theta, \sigma)^T \in \mathbb{R} \times (0, \infty)$  est un paramètre inconnu et  $b$  est une fonction à valeur réelle.

Plus précisément, nous supposons que les hypothèses suivantes sont remplies pour le processus  $\alpha$ -stable tronqué  $(L_t)_{t \in [0, 1]}$ .

(i) Le processus de Lévy  $(L_t)_{t \in [0, 1]}$  est donné par

$$L_t = \int_0^t \int_{[-1, 1]} z \{ \bar{\mu}(ds, dz) - \bar{\nu}(ds, dz) \} + \int_0^t \int_{[-1, 1]^c} z \bar{\mu}(ds, dz),$$

où  $\bar{\mu}$  est une mesure aléatoire de Poisson, de compensateur  $\bar{\nu}(dt, dz) = dt \times F(z) dz$  où  $F(z)$  est donnée sur  $\mathbb{R}$  par  $F(z) = \frac{1}{|z|^{\alpha+1}} 1_{|z| \neq 0} \tau(z)$ ,  $\alpha \in (0, 2)$ . Nous supposons que  $\tau$  est une fonction lisse non négative, égale à 1 sur  $[-1, 1]$ , s'annulant sur  $[-2, 2]^c$  telle que  $0 \leq \tau \leq 1$ .

(ii) nous supposons que  $\forall p \geq 1$ ,

$$(0.0.2) \quad \int_{\mathbb{R}} \left| \frac{\tau'(u)}{\tau(u)} \right|^p \tau(u) du < \infty,$$

$$(0.0.3) \quad \int_{\mathbb{R}} \left| \frac{\tau''(u)}{\tau(u)} \right|^p \tau(u) du < \infty.$$

Nous observons ce processus à des temps discrets, équidistants, sur l'intervalle fini fixé  $[0, 1]$ , ce qui signifie que  $X$  est observé aux instants  $t_i^n = \frac{i}{n} := i \Delta_n$ ,  $i = 0, 1, \dots, n$ , dans l'intervalle  $[0, 1]$ . Nous

---

nous intéressons aux propriétés asymptotiques lorsque  $\Delta_n = \frac{1}{n}$ , tend vers 0. En pratique, cela signifie que nous sommes dans le contexte des données à haute fréquence. On sait que, dans le cas de la diffusion, on ne peut pas estimer le paramètre de dérive avec un temps d'observation fixé. Au contraire, pour un processus de saut pur, on peut estimer le paramètre de dérive dans les deux cas: une période d'observation fixée ou un temps d'observation tendant vers l'infini. Sans perte de généralité, nous supposons ici que l'intervalle d'observation est  $[0, 1]$ .

En raison du rôle essentiel de la propriété LAN (LAMN) dans les problèmes d'estimation paramétrique et des applications importantes des processus de saut pur, l'un des buts de cette thèse est d'étudier la propriété LAMN pour l'équation différentielle stochastique donnée par (0.0.1).

Le calcul de Malliavin est le principal outil pour prouver la propriété LAMN et il explique partiellement les hypothèses spécifiques dans (i), (ii). Premièrement, puisque notre méthode est basée sur le calcul de Malliavin, les hypothèses d'intégrabilité pour les queues du processus de Lévy sont cruciales pour s'assurer que le processus (0.0.1) appartient à l'espace de Malliavin. Deuxièmement, le comportement stable de la mesure Lévy autour de zéro est également nécessaire pour que le processus mis à l'échelle  $(n^{1/\alpha}L_{t/n})$  soit proche du processus  $\alpha$ -stable  $(L_t^\alpha)$ . Alors la fonction de troncation  $\tau$  assure à la fois l'intégrabilité de  $|L_t|^p, \forall p \geq 1$ , et le comportement  $\alpha$ -stable exact autour de zéro ( $\tau = 1$ ). De plus, pour obtenir le développement asymptotique de la log-vraisemblance dans l'étude de la propriété LAMN, il nous faut connaître le comportement asymptotique de la densité du processus et de sa dérivée par rapport aux paramètres. En utilisant le calcul de Malliavin, nous obtenons des formules de représentation pour la densité et sa dérivée. Ces formules de représentation dépendent de poids de Malliavin dont les expressions sont données explicitement. Ensuite, nous observons que l'introduction de la fonction de troncation  $\tau$  permet d'assurer que le processus  $(n^{1/\alpha}L_{t/n})$  n'a pas de saut de taille supérieure à  $2n^{1/\alpha}$  et par conséquent il facilite le contrôle du comportement asymptotique des poids de Malliavin afin d'établir la convergence en temps petit de la densité. Plus précisément, les hypothèses (0.0.2) et (0.0.3) sont des outils techniques pour prouver les convergences de la densité et de sa dérivée, respectivement.

## Contexte

Il existe une littérature importante concernant la propriété LAN(LAMN) pour les processus de saut pur basée sur des observations à haute fréquence, par exemple, les travaux de Aït-Sahalia et Jacod [2] [3], Kawai et Masuda [44], Masuda [53] [54], Ivanenko, Kulik et Masuda [35], Clément et Gloter [17].

En particulier, Masuda [53] propose un estimateur des paramètres du processus de Lévy stable

$$(0.0.4) \quad X_t = \theta t + \sigma L_t^\alpha$$

basé sur des observations en temps discret  $(X_{i/n})_{i=1,\dots,n}$ . Il prouve la propriété LAN avec la vitesse  $u_n = \text{diag}\left(\frac{1}{n^{-1/2+1/\alpha}}, \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n} \log(n)}\right)$  pour l'estimation de  $\beta = (\theta, \sigma, \alpha)$  avec une matrice d'information de Fisher singulière. Par conséquent, les théorèmes de convolution et de minimax ne peuvent pas être appliqués et une borne inférieure pour la variance asymptotique des estimateurs ne peut pas être déduite dans cette expérience statistique. Concernant les estimateurs du paramètre  $\alpha$  du processus, l'auteur a construit deux estimateurs  $\hat{\alpha}_{\log,n}$  (basé sur les moments logarithmiques) et  $\hat{\alpha}_{r,n}$  (basé sur les moments) pour  $\alpha$  avec la vitesse de convergence  $\sqrt{n}$ .

Récemment, pour résoudre le problème avec la matrice d'information de Fisher singulière et connaître la vitesse optimale d'estimation pour les paramètres dans (0.0.4), Brouste et Masuda [13] établissent la propriété LAN avec une vitesse non-diagonale, par exemple

$$u_n = \frac{1}{\sqrt{n}} \begin{pmatrix} n^{1-1/\alpha} & 0 & 0 \\ 0 & 1 & -\alpha^{-2}\sigma \log(n) \\ 0 & 0 & 1 \end{pmatrix}$$

et une matrice d'information de Fisher non-dégénérée. Les théorèmes de convolution et du minimax peuvent être appliqués dans ce cas. La vitesse optimale de convergence de l'estimateur pour l'indice  $\alpha$  lorsque le paramètre d'échelle  $\sigma$  est connu est  $\sqrt{n} \log(n)$  et est  $\sqrt{n}$  lorsque le paramètre d'échelle  $\sigma$  est inconnu.

Dans Aït-Sahalia et Jacod [3] et [2], les auteurs établissent le comportement de l'information de Fisher d'un processus  $X$  décomposé comme la somme de deux processus de Lévy indépendants

$$X_t = \sigma L_t^\alpha + \theta U_t$$



---

où  $L$  est un processus stable symétrique standard avec l'indice  $\alpha \in (0, 2]$ ,  $(U_t)$  est un autre processus de Lévy, indépendant de  $(L_t^\alpha)$  et dominé par  $(L_t^\alpha)$ . Alors ils déduisent qu'avec le paramètre d'échelle  $\sigma$  connu et la vraie valeur  $\alpha < 2$ , la vitesse de convergence pour l'estimation de  $\alpha$  est  $\sqrt{n} \log(n)$ . La propriété LAN est également prouvée avec la vitesse  $n^{-\frac{1}{2}}$  pour le paramètre  $\sigma$ .

Plus récemment, Ivanenko, Kulik et Masuda [35] considèrent le modèle

$$X_t = \theta t + \sigma L_t^\alpha + Z_t$$

où  $L$  est un processus non-symétrique localement  $\alpha$ -stable et  $Z$  est un processus indépendant et moins actif. Lorsque  $X$  est observé à haute fréquence, ils prouvent la propriété LAN pour le paramètre  $\beta = (\theta, \sigma) \in \mathbb{R}^2$  avec la vitesse  $u_n = \frac{1}{\sqrt{n}} \begin{pmatrix} n^{1-1/\alpha} & c_n n \\ 0 & 1 \end{pmatrix}$  où  $c_n = \frac{1}{n} \int_{n^{-1/\alpha} < |u| < 1} u \mu(du)$  est identiquement nul si la mesure Lévy  $\mu$  est symétrique. Dans les travaux susmentionnés, à partir de la linéarité des modèles, les accroissements  $(X_{\frac{i}{n}} - X_{\frac{i-1}{n}})_{i \leq n}$  sont indépendants et ont la même loi que  $X_{1/n}$ . La densité de transition du processus en temps discret  $(X_{\frac{i}{n}})_{1 \leq i \leq n}$  est presque explicite donc on peut étudier la théorie statistique en utilisant l'expression explicite de la densité de  $X_t$ .

En revenant au modèle (0.0.1), la difficulté dans l'étude statistique du modèle (0.0.1) est que la densité de transition de la chaîne de Markov  $(X_{\frac{i}{n}})_{1 \leq i \leq n}$  est inconnue et le lien entre la densité de  $L_t$  et la densité de  $X_t$  n'est pas clair. Dans cette configuration, nous pouvons citer les articles récents de Clément et Gloter [18], Masuda [54] où certains estimateurs des paramètres de dérive et d'échelle sont proposés pour l'équation générale

$$X_t = x_0 + \int_0^t b(X_s, \theta) ds + \int_0^t c(X_{s-}, \sigma) dL_s$$

où  $L$  est un processus de saut pur localement stable. Cependant, dans ce cas, l'efficacité asymptotique des estimateurs n'est pas encore établie. Le seul résultat sur la propriété LAMN, à notre connaissance, est donné dans Clément et Gloter [17] où la propriété LAMN est prouvée pour l'estimation du paramètre de dérive  $\theta$  pour la solution de (0.0.1) (avec  $\sigma = 1$  et  $\alpha \in (1, 2]$ ). Dans [17], ils montrent que la propriété LAMN est satisfaite avec la vitesse  $r_n = n^{1/2-1/\alpha}$  et

l'information  $\mathcal{I}_\theta = \int_0^1 \partial_\theta b(X_s^\beta, \theta)^2 ds \int_{\mathbb{R}} \frac{\varphi'_\alpha(u)^2}{\varphi_\alpha(u)} du$ , où  $\varphi_\alpha$  est la densité de la distribution  $\alpha$ -stable de fonction caractéristique  $u \rightarrow e^{-C(\alpha)|u|^\alpha}$ .

Le premier but de cette thèse est d'étendre les résultats de Clément et de Gloter [17] où l'indice  $\alpha \in (1, 2)$  et le paramètre n'apparaît que dans le coefficient de dérive. C'est-à-dire nous étendons dans cette thèse ces résultats pour le cas  $\alpha \in (0, 2)$  et prouvons que la propriété LAMN est satisfaite pour les deux paramètres  $(\theta, \sigma)$  lorsque le paramètre d'indice  $\alpha$  est connu.

La preuve est principalement basée sur le théorème 1 de [57] et ensuite utilise le comportement asymptotique de la densité de  $(X_t)$  et de sa dérivée par rapport aux paramètres. Pour cette raison, nous discutons, au chapitre 2, l'asymptotique en temps petit pour la densité de  $(X_t)$ .

Un autre but de cette thèse est d'estimer le paramètre d'indice de la solution de (0.0.1), en supposant que le paramètre  $\theta$  est connu.

Dans la dernière partie de cette thèse, nous construisons deux estimateurs avec la vitesse de convergence  $\sqrt{n}$  pour l'indice de Blumenthal-Gettoor  $\alpha \in (0, 2)$  lorsque le paramètre d'échelle  $\sigma$  est inconnu, basés sur les méthodes de moments introduites dans Masuda [53]. Nous prouvons la consistance et la normalité asymptotique des estimateurs et les résultats d'estimation sont illustrés par des simulations numériques. Si le paramètre d'échelle est connu, nous construisons des estimateurs du paramètre d'indice avec la vitesse  $\sqrt{n} \log(n)$ , également basés sur les moments et moments logarithmiques.

En plus des références ci-dessus, nous listons ici d'autres travaux concernant l'estimation de l'indice d'activité des sauts à partir d'observations à haute fréquence. Dans Aït-Sahalia et Jacod [4] et Jing, Kong et Mykland [42], les auteurs considèrent le processus

$$X_t = x_0 + \int_0^t b_s ds + \int_0^t \sigma_s W_s + \int_0^t \int_{|x| \leq 1} x(\mu - \nu)(ds, dx) + \int_0^t \int_{|x| \geq 1} x \mu(ds, dx)$$

où  $W_t$  est un mouvement brownien standard,  $b$  et  $\sigma$  sont des processus d'Itô et  $\mu$  est une mesure aléatoire avec compensateur  $\nu$ . Basés sur la  $p$ -variation, ils proposent des estimateurs pour le paramètre d'indice  $\alpha$  à haute fréquence.

Todorov et Tauchen [63] proposent une estimation de l'indice d'activité des sauts dans un contexte de semi-martingales de saut pur pour laquelle la vitesse de convergence est  $\sqrt{n}$  pour  $\sqrt{2} \leq \alpha < 2$  (le cas  $\alpha < \sqrt{2}$  nécessite une hypothèse de dérive nulle).

---

Récemment, Todorov [62] considère l'estimation de l'indice d'activité de sauts  $\alpha \in (1, 2]$  du processus

$$dX_t = b_t dt + \sigma_t dL_t + dY_t$$

où  $L$  est un processus de Lévy de saut pur localement stable dont la mesure de Lévy autour de zéro se comporte comme un processus stable et  $Y$  est un processus de saut pur qui est dominé aux hautes fréquences par  $L$ . Il obtient la vitesse de convergence  $\sqrt{n}$  si  $\alpha \in (1, 2)$  et  $\sqrt{n} u_n^{-2}$  pour une suite  $u_n$  qui tend vers zéro si  $\alpha = 2$ , en utilisant la fonction caractéristique empirique des accroissements normalisés.

## Résumé des principaux résultats

Cette thèse est divisée en quatre chapitres.

**Chapitre 2:** Nous nous concentrons sur le comportement asymptotique, en temps petit, de la densité de  $(X_t^\beta)$  avec l'indice  $\alpha \in (0, 2)$  pour la solution de (0.0.1), ainsi que sa dérivée par rapport au paramètre  $\beta = (\theta, \sigma)^T$ . Ce problème joue un rôle important dans les statistiques asymptotiques basées sur des observations à haute fréquence. En effet, en considérant l'estimation de  $\beta$  à partir des observations en temps discret  $(X_{i/n}^\beta)_{0 \leq i \leq n}$ , et en notant par  $p_{1/n}^\beta(x, y)$  la densité de transition du processus en temps discret, la vitesse d'estimation du paramètre  $\beta$  repose fortement sur le comportement asymptotique de la dérivée  $\nabla_\beta p_{1/n}^\beta(x, y)$ , quand  $n$  tend vers l'infini.

En utilisant le calcul de Malliavin, nous obtenons des formules de représentation pour la densité, sa dérivée et sa dérivée logarithmique, comme une espérance et une espérance conditionnelle. Ces formules de représentation font apparaître des poids de Malliavin dont les expressions sont données explicitement, ce qui permet d'analyser le comportement asymptotique de la densité en temps petit, en utilisant la propriété d'autosimilarité du processus stable. Nos résultats sont établis grâce à une étude attentive de chaque terme apparaissant dans les poids de Malliavin, ce qui est compliqué par la non-intégrabilité du processus  $\alpha$ -stable quand  $\alpha \leq 1$ . De plus, lorsque nous étudions le comportement asymptotique, pour traiter toute valeur de l'indice  $\alpha \in (0, 2)$ , nous introduisons la solution de l'équation différentielle ordinaire

$$(0.0.5) \quad \zeta_t^{n, \theta, x} = x + \frac{1}{n} \int_0^t b(\zeta_s^{n, \theta, x}, \theta) ds \quad t \in (0, 1], \quad x \in \mathbb{R}$$

et alors nous obtenons que  $n^{1/\alpha} \left\{ X_{\frac{1}{n}}^\beta(x) - \zeta_1^{n,\theta,x} \right\}$  est proche d'un processus de Lévy stable.

On note  $\varphi_\alpha$  la densité de  $L_1^\alpha$  où  $(L_t^\alpha)$  est un processus  $\alpha$ -stable de mesure de Lévy  $\nu(dz) = \frac{1}{|z|^{\alpha+1}} \mathbf{1}_{z \neq 0} dz$ . Nous avons les résultats suivants pour le comportement asymptotique de la densité et de sa dérivée:

**Theorem 0.0.1** (Clément, Gloter and Nguyen [19]). *Soit  $(\zeta_t^{n,\theta,x_0})$  la solution de l'équation différentielle ordinaire (0.0.5) et soit  $(\beta_n)_{n \geq 1} = ((\theta_n, \sigma_n)^T)_{n \geq 1}$  une suite telle que  $\beta_n \xrightarrow{n \rightarrow \infty} \beta$ . Pour tout  $(x_0, u) \in \mathbb{R}^2$ ,*

1.  $\frac{\sigma_n}{n^{1/\alpha}} p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0}) \xrightarrow{n \rightarrow \infty} \varphi_\alpha(u),$
2.  $\frac{\sigma_n^2}{n^{\frac{2}{\alpha}-1}} \partial_\theta p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0}) \xrightarrow{n \rightarrow \infty} -\partial_\theta b(x_0, \theta) \times \varphi'_\alpha(u),$   
 $\frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0}) \xrightarrow{n \rightarrow \infty} -\varphi_\alpha(u) - u\varphi'_\alpha(u),$

Le chapitre 2 est publié dans l'article suivant:

\* *Emmanuelle Clément, Arnaud Gloter, and Huong Nguyen. Asymptotics in small time for the density of a stochastic differential equation driven driven by a stable Lévy process. ESAIM: P/S, 2018.*

**Chapitre 3:** Nous discutons de la propriété Local Asymptotic Mixed Normality pour les paramètres de dérive et d'échelle à partir des observations à haute fréquence d'une solution de l'équation différentielle stochastique gouvernée par un processus  $\alpha$ -stable tronqué avec l'indice  $\alpha \in (0, 2)$  donnée par l'équation (0.0.1). Ceci étend les résultats de [17] où l'indice  $\alpha \in (1, 2)$  et seulement la propriété Local Asymptotic Mixed Normality pour le coefficient de dérive est considérée. A partir de l'expression des dérivés logarithmiques de la densité par rapport à  $\beta$  comme une espérance conditionnelle, des poids de Malliavin et des résultats sur le comportement asymptotique de la densité établis au chapitre 2, nous déduisons le comportement asymptotique de la matrice d'information de Fisher en temps petit pour une observation du processus. En outre, nous établissons un développement asymptotique de la log-vraisemblance (en utilisant le calcul de Malliavin pour les processus de saut) et prouvons que la propriété LAMN est vérifiée pour les paramètres de dérive et de volatilité avec la vitesse  $u_n = \begin{pmatrix} n^{\frac{1}{2}-\frac{1}{\alpha}} & 0 \\ 0 & n^{-\frac{1}{2}} \end{pmatrix}$  et la matrice

---

d'information  $I = \begin{pmatrix} \mathcal{I}_{11} & 0 \\ 0 & \mathcal{I}_{22} \end{pmatrix}$  où

$$\begin{cases} \mathcal{I}_{11} = \frac{1}{\sigma^2} \int_0^1 \partial_\theta b(X_s^\beta, \theta)^2 ds \int_{\mathbb{R}} \frac{\varphi'_\alpha(u)^2}{\varphi_\alpha(u)} du \\ \mathcal{I}_{22} = \frac{1}{\sigma^2} \int_{\mathbb{R}} \frac{(\varphi_\alpha(u) + u\varphi'_\alpha(u))^2}{\varphi_\alpha(u)} du. \end{cases}$$

La preuve de la propriété LAMN est basée sur le théorème 1 dans [40]. Cette partie contient également un résultat indépendant et intéressant qui établit la continuité par rapport au conditionnement dans une espérance conditionnelle.

Le chapitre 3 est publié dans l'article suivant:

\* *Emmanuelle Clément, Arnaud Gloter, and Huong Nguyen. LAMN property for the drift and volatility parameters of a SDE driven by a stable Lévy process. ESAIM: P/S, 2018.*

**Chapitre 4:** Le but de ce chapitre est d'estimer le paramètre  $\alpha$  d'une équation différentielle stochastique gouvernée par un processus  $\alpha$ -stable tronqué avec l'indice  $\alpha \in (0, 2)$  donnée par l'équation (0.0.1) et avec le paramètre  $\theta$  connu à partir d'observations à haute fréquence sur un intervalle de temps donné. Masuda [53] a construit des estimateurs basés sur des méthodes de moments pour l'indice  $\alpha$  avec la vitesse de convergence  $\sqrt{n}$ , quand le paramètre d'échelle  $\sigma$  est inconnu. Basé sur les idées du Masuda [53], dans ce chapitre, nous adaptons la procédure de l'estimation à l'équation (0.0.1) et fournissons les distributions asymptotiques.

Les principaux outils pour établir les résultats statistiques sont la loi des grands nombres et le théorème de la limite centrale pour les sommes

$$\frac{1}{n} \sum_{j=1}^{n-1} f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \zeta_{1/n}(X_{t_j}^\alpha)\} \right)$$

avec  $f$  fonction appropriée et  $\zeta_{1/n}$  solution de

$$\zeta_t(x) = x + \int_0^t b(\zeta_s(x)) ds \quad t \in (0, 1/n], \quad x \in \mathbb{R}.$$

L'introduction de cette équation différentielle ordinaire est nécessaire pour traiter n'importe quelle valeur de  $\alpha$ , cependant, cela conduit à une limitation dans la pratique puisque la fonction de dérive  $b$  doit être connue. Si nous connaissons la vraie valeur du paramètre d'échelle, alors

nous construisons des estimateurs du paramètre d'indice avec la vitesse  $\sqrt{n} \log(n)$ . De plus, dans ce chapitre, nous illustrons les résultats par des simulations numériques.



---

# Abstract

---

In this thesis, we consider a stochastic differential equation driven by a truncated pure jump Lévy process with index  $\alpha \in (0, 2)$  and observe high frequency data of the process on a fixed observation time. We first study the behavior of the density of the process in small time. Next, we prove the Local Asymptotic Mixed Normality (LAMN) property for the drift and scaling parameters from high frequency observations. Finally, we propose some estimators of the index parameter of the process.

The first part deals with the asymptotic behavior of the density in small time of the process. The process is assumed to depend on a parameter  $\beta = (\theta, \sigma)^T$  and we study, in this part, the sensitivity of the density with respect to this parameter. This extends the results of [17] which were restricted to the index  $\alpha \in (1, 2)$  and considered only the sensitivity with respect to the drift coefficient. By using Malliavin calculus, we obtain the representation of the density, its derivative and its logarithm derivative as an expectation and a conditional expectation. These representation formulas involve some Malliavin weights whose expressions are given explicitly and this permits to analyze the asymptotic behavior in small time of the density, using the self-similarity property of the stable process.

The second part of this thesis concerns the Local Asymptotic Mixed Normality property for the parameters. Both the drift coefficient and scale coefficient depend on the unknown parameters. Extending the results of [17], we compute the asymptotic Fisher information and find that the rate in the Local Asymptotic Mixed Normality property depends on the index  $\alpha$ .

The third part proposes some estimators of the jump activity index  $\alpha \in (0, 2)$  based on the method of moments as in Masuda [53]. We prove the consistency and asymptotic normality of the estimators and give some simulations to illustrate the finite-sample behaviors of the estimators.





---

## Remerciements

---

**M**es travaux de thèse, présentés dans ce manuscrit, n'auraient pu aboutir sans l'aide et la présence de nombreuses personnes que je tiens à remercier sincèrement ici.

Je voudrais tout d'abord remercier mes deux directeurs de thèse Emmanuelle Clément et Arnaud Gloter, non seulement pour m'avoir guidé pas à pas dans le domaine de *Processus de sauts* et soutenue durant ces trois années, mais aussi pour leur patience de m'écouter. Ils m'ont bien donné de grands pistes d'études ou de preuves lors de discussions passionnantes qu'en répondant à mes questions naïves sur tel ou tel détail technique. Cette thèse n'aurait jamais vu le jour sans le grand soin - pas toujours plaisant - qu'ils ont pris à la correction des premières versions de chacun de mes travaux et leur conseils sur mes travaux. De plus, j'ai beaucoup appris de leur façon de travail. C'est un grand honneur pour moi d'avoir été leur élève et je suis assurément encore loin d'avoir mis à profit tout ce que j'ai reçu d'eux.

Je tiens à remercier Laurent Denis et Mohamed Ben Alaya pour avoir accepté de relire mon manuscrit en tant que rapporteurs ma thèse. Leurs commentaires sur mes travaux m'ont permis d'améliorer la qualité globale de ce manuscrit.

Je remercie également Fabienne Comte et Aurélien Alfonsi qui m'ont fait l'honneur d'accepter de faire partie de mon jury.

Je souhaite remercier les membres du LAMA pour leur accueil, leur écoute, leurs conseils et en particulier, Christiane Lafargue et Audrey Patout pour s'être occupées de toutes les démarches administratives. Également un grand merci à Sylvie Cach pour avoir pris le temps de répondre à bon nombre de mes questions et m'avoir aidé pour le procédure de la soutenance. Merci mes amis du LAMA Yushun, Giulia, Hoai-Duc, Evgenii, Peng pour tous les moments agréables et toutes les discussions mathématiques et non mathématiques que j'ai pu partager avec eux.

Merci mes proche amis vietnamiens Ngoc, Hong, Nghia, Trung, An pour partager les bon moments à Paris et m'aider à intégrer dans la vie quotidienne en France. Merci les membres du group "Hoi dinh hai hoa qua" pour les sorties et repas ensemble. Une mention spéciale pour Nhut d'avoir devenu mon maître d'IT et puis mon hôte à Antibes pendant les vacances d'été, un grand merci à Dang, Quang, Mehdi, Revekka, Krisztián, Hieu, Tu pour tous les moments de rire, d'émotions et de complicité que l'on a partagés.

Merci aux membres de ma famille et à Bruno et Patricia, parrain - marraine et puis devenu ma famille parisienne, qui m'ont toujours soutenu et ont essayé, avec beaucoup de persévérance

---

et de courage, de comprendre en quoi consiste le contenu de cette thèse. Je terminerai avec un grand merci à mon amour pour ta présence à mes côtés, pour ton soutien jour après jour. Je comprends que je ne fais rien sans toi.

---

# Table of Contents

---

Modèle . . . . .		iii
Contexte . . . . .		v
Résumé des principaux résultats . . . . .		viii
		<b>Page</b>
<b>1 Introduction</b>		<b>1</b>
1.1 Asymptotic statistics and Local Asymptotic (Mixed) Normality property . . . . .		1
1.2 Lévy processes . . . . .		8
1.3 Motivation and Overview . . . . .		12
1.3.1 Model . . . . .		12
1.3.2 Background . . . . .		14
1.3.3 Overview of the main results . . . . .		17
<b>2 Asymptotics in small time for the density of a stochastic differential equation driven by a stable Lévy process</b>		<b>21</b>
2.1 Introduction . . . . .		21
2.2 Asymptotics for the density and its derivative . . . . .		23
2.3 Rescaling and representation of the density in small time . . . . .		27
2.3.1 Rescaling . . . . .		27
2.3.2 Representation of the density in small time and first approximation . . . . .		29
2.4 Proof of Theorems 2.3.1 and 2.2.1 . . . . .		32
2.4.1 Proof of Theorem 2.3.1 . . . . .		32
2.4.2 Proof of Theorem 2.2.1 . . . . .		35
2.5 Proof of Theorem 2.2.2 . . . . .		39
2.5.1 Representation of $\nabla_{\beta} p_{\frac{1}{n}}^{\beta_n}$ and computation of the iterated Malliavin weights		39
2.5.2 Convergence of the iterated weights . . . . .		41
2.5.3 Proof of Theorem 2.2.2 . . . . .		47
2.5.4 Proofs of the intermediate lemmas . . . . .		50
2.6 Appendix. Representation of the transition density via Malliavin calculus . . . . .		58
2.6.1 Integration by parts setting . . . . .		58

2.6.2	Representation of the density of $Y_1^\beta$ and its derivative . . . . .	60
<b>3</b>	<b>LAMN property for the drift and volatility parameters of a sde driven by a stable Lévy process</b>	<b>67</b>
3.1	Introduction . . . . .	67
3.2	Main results . . . . .	70
3.3	The asymptotic Fisher information matrix in small time . . . . .	75
3.3.1	The asymptotic properties of the Fisher information matrix . . . . .	75
3.3.2	Proof of Theorem 3.3.1 . . . . .	76
3.4	Proof of the asymptotic expansion of the likelihood (Theorems 3.2.1 - 3.2.2) . . . .	81
3.4.1	Proof of the condition A1 (the $L^2$ -regularity condition). . . . .	82
3.4.2	Proof of the conditions A2 and A3 (Theorem 3.2.2) . . . . .	85
3.5	Proof of Theorem 3.2.3 (Stable central limit theorem) . . . . .	86
3.6	Proofs of Lemmas 3.3.2-3.3.5 . . . . .	95
3.6.1	Malliavin calculus and preliminary lemmas . . . . .	95
3.6.2	Regularity of the conditional expectation . . . . .	100
3.6.3	Proofs of Lemma 3.3.2, Lemma 3.3.3, Lemma 3.3.4 and Lemma 3.3.5 . . . .	101
3.6.4	Lemmas 3.6.4 and 3.6.5 . . . . .	105
3.7	Appendix. Checking the conditions for $\mathcal{H} = \mathcal{H}_{L^\alpha}(1)$ and $\mathcal{H} = L_1^\alpha \mathcal{H}_{L^\alpha}(1)$ in Proposition 3.6.3 . . . . .	112
3.7.1	The case $\mathcal{H} = \mathcal{H}_{L^\alpha}(1)$ . . . . .	113
3.7.2	The case $\mathcal{H} = L_1^\alpha \mathcal{H}_{L^\alpha}(1)$ . . . . .	117
<b>4</b>	<b>ESTIMATION OF THE INDEX PARAMETER FOR A STOCHASTIC DIFFERENTIAL EQUATION DRIVEN BY A STABLE LÉVY PROCESS</b>	<b>125</b>
4.1	Introduction . . . . .	125
4.2	Model and notations . . . . .	128
4.3	Construction of the estimator and main results . . . . .	130
4.3.1	Construction of the estimator . . . . .	130
4.3.2	Main results . . . . .	134
4.4	Limit theorems . . . . .	138
4.4.1	Law of Large Numbers . . . . .	138
4.4.2	Central Limit Theorem . . . . .	145
4.5	Proof of Main results (Theorems 4.3.1-4.3.4) . . . . .	158
4.6	A simulation study . . . . .	162
4.6.1	Comparison of asymptotic variances . . . . .	162
4.6.2	Monte-Carlo Simulations . . . . .	163
4.6.3	The drift function $b(x) = -x + \sin(2\pi x)$ . . . . .	163
4.6.4	The drift function $b(x) = -x$ . . . . .	168

4.7	Estimation of the index parameter for the unknown drift function when $\alpha \in (1, 2)$ .	171
4.8	Appendix. Python Code . . . . .	178
4.8.1	Python code for the figure 4.1 . . . . .	178
4.8.2	Python code for the simulation of the process (4.6.1) and the estimators of $\alpha_0$	179
<b>Bibliography</b>		<b>185</b>



## CHAPTER 1

---

# INTRODUCTION

---

The goal of this chapter is to provide a brief review of definitions and properties on asymptotic statistical inference and on Lévy processes, that are used in the sequel. We give the motivations and overviews of the main results achieved in this thesis as well. None of the statements and theorems, that we recall, will be proved here since most of them are well-known from the asymptotic statistics and probability theories.

### 1.1 Asymptotic statistics and Local Asymptotic (Mixed) Normality property

The concept of statistical experiment, also called a statistical model, is introduced by Blackwell [10] as a triplet

$$\mathcal{E} = (\Omega, \mathcal{F}, (\mathbf{P}^\theta : \theta \in \Theta))$$

where  $(\Omega, \mathcal{F})$  is a measurable space,  $\Theta$  is a set called the parameter space and  $(\mathbf{P}^\theta : \theta \in \Theta)$  is a family of probability measures on  $(\Omega, \mathcal{F})$  depending on an unknown parameter  $\theta \in \Theta \subset \mathbb{R}^d$ ,  $d \geq 1$ . It is used for the mathematical description of observed data. If the size or dimension of the data is  $n$ , with  $n$  increasing, then a sequence of statistical experiments is defined as

$$\mathcal{E}^n = (\Omega^n, \mathcal{F}^n, \mathbf{P}_n^\theta : \theta \in \Theta \subset \mathbb{R}^d), n \geq 1.$$

In the second half of the 20th century, Lucien Le Cam introduced the so-called Local Asymptotic Normality (LAN) property of a sequence of statistical models.

**Definition 1.1.1** (A.W. van der Vaart [64]). *The sequence of statistical experiments  $(\Omega^n, \mathcal{F}^n, \mathbf{P}_n^\theta : \theta \in \Theta \subset \mathbb{R}^d)$ ,  $n \geq 1$ , is locally asymptotically normal (LAN) at  $\theta$  if there exist  $d \times d$  matrices  $u_n$*



( $\|u_n\|$  tends to zero as  $n$  goes to infinity) and a non random information matrix  $I(\theta)$  and random vectors  $(\Delta_{n,\theta})$  such that  $(\Delta_{n,\theta})$  converges in law under  $\mathbf{P}_n^\theta$  to  $\mathcal{N}(0, I_\theta)$  and for every  $h \in \mathbb{R}^d$

$$\log \frac{d\mathbf{P}_n^{\theta+u_n h}}{d\mathbf{P}_n^\theta} = h^T \Delta_{n,\theta} - \frac{1}{2} h^T I(\theta) h + o_{\mathbf{P}_n^\theta}(1).$$

In other words, an experiment is called locally asymptotically normal when its likelihood converges to the likelihood of the Gaussian shift experiment.

The LAN property is an important concept in asymptotically optimal statistical analysis. This property allows the sequence of statistical models to be asymptotically approximated by a normal location model, after a rescaling of the parameter.

We give now a number of simplest examples, where the LAN property holds true.

*Example 1* ([58]). Let  $\mathbf{P}_n^\theta$  be the law of  $(X_1, X_2, \dots, X_n)$  where  $(X_n)_{n \geq 1}$  is a sequence of independent and identically distributed (i.i.d.) random variables from  $\mathcal{N}(\mu, \sigma^2)$ . Here  $\theta = (\mu, \sigma)^T \in \mathbb{R} \times (0, +\infty)$  is an unknown parameter to be estimated. Then the LAN property holds at  $\theta$  with the rate  $u_n = \begin{pmatrix} n^{-1/2} & 0 \\ 0 & n^{-1/2} \end{pmatrix}$ , and the information matrix  $I(\theta) = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^4} \end{pmatrix}$ .

*Example 2* ([1]). We consider the Ornstein-Uhlenbeck process given by

$$dX_t = -\theta X_t dt + dW_t, \theta > 0, \quad X_0 = x_0$$

where  $\theta$  is an unknown parameter and satisfies  $\theta > 0$ ,  $(W_t)_{t \geq 0}$  is a standard Wiener process. We consider here, the mathematically simplest case where the whole path is observed over an interval  $[0, T]$ . If  $\theta > 0$  then the process is ergodic and, in this case, the LAN property holds true with rate  $T^{-1/2}$  at each point  $\theta > 0$  and with asymptotic Fisher information  $I(\theta) = 1/2\theta$ .

In general case (see Kutoyants [48]), we consider the diffusion process

$$dX_t = b(\theta, X_t) dt + \sigma(X_t) dW_t, \quad X_0 = x_0, 0 \leq t \leq T,$$

where  $b(\cdot, \cdot)$  is some known function,  $x_0$  is a starting point and  $\theta \in \Theta \subset \mathbb{R}^d$  is a finite-dimensional parameter. Under appropriate assumptions on the coefficients, we have the LAN property with rate  $1/\sqrt{T}$  and with information matrix

$$I(\theta) = \mathbb{E}_\theta \left( \frac{\partial_\theta b(\theta, \vartheta) \partial_\theta b(\theta, \vartheta)^T}{\sigma(\vartheta)^2} \right).$$

Here  $\vartheta$  is the random variable with stationary density function  $f(\theta, x)$  given by

$$f(\theta, x) = G(\theta)^{-1} \sigma(x)^{-2} \exp\left(2 \int_0^x \frac{b(\theta, v)}{\sigma(v)^2} dv\right)$$

where  $G(\theta) = \int_{-\infty}^{+\infty} \sigma(y)^{-2} \exp\left(2 \int_0^y \frac{a(\theta, v)}{\sigma^2(v)} dv\right) dy$ .

However, it was observed that there exists a number of processes, where LAN condition is not satisfied. Coming back to Example 2, if  $\theta < 0$  then the process is non-ergodic, the LAN property does not hold true at  $\theta$  (see [1]). The explosive auto-regressive process of first order and the Super-critical Galton-Watson branching process with geometric off-spring distribution are also situations where the Fisher information matrix must be replaced by a random matrix (see Le Cam and Yang [51]). As a consequence, the concept of Local Asymptotic Mixed Normality (LAMN) property has been introduced by Jeganathan [[39], [40]] to extend to random Fisher information the LAN property.

**Definition 1.1.2** (Jeganathan [39]). *For each  $n \geq 1$ , let  $(\Omega^n, \mathcal{F}^n, \mathbf{P}_n^\theta : \theta \in \Theta \subset \mathbb{R}^d)$  be a statistical experiment. The sequence of families  $\{\mathbf{P}_n^\theta : \theta \in \Theta\}, n \geq 1$  satisfies the LAMN property at  $\theta$  (an interior point of  $\Theta$ ), at rate  $u_n$  (a  $d \times d$  positive definite matrix such that  $\|u_n\|$  tends to zero as  $n$  goes to infinity) with the random information matrix  $I(\theta) > 0$  if for every  $h \in \mathbb{R}^d$*

$$(1.1.1) \quad \log \frac{d\mathbf{P}^{\theta+u_n h}}{d\mathbf{P}_n^\theta} = h^T I_n(\theta)^{1/2} N_n - \frac{1}{2} h^T I_n(\theta) h + o_{\mathbf{P}_n^\theta(1)}$$

where  $(N_n, I_n(\theta))$  converges in law (under  $\mathbf{P}_n^\theta$ ) to  $(N, I(\theta))$ , where  $N$  is a standard  $d$ -variate normal distribution independent of  $I(\theta)$ .

We can remark that when the information matrix  $I(\theta)$  is deterministic, the LAN property holds. The matrix  $u_n$  is often called the rate matrix.

We present sufficient conditions stated by Jeganathan [39] to establish the LAMN property that are used in the sequel.

Let  $(X_i)_{i \geq 1}$  be a sequence of real random variables defined on a probability space  $(\Omega, \mathcal{F}, \mathbf{P}^\theta : \theta \in \Theta \subset \mathbb{R}^d)$  and define  $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$  the  $\sigma$ -field induced by the random vector  $(X_1, X_2, \dots, X_n)$ . We assume that, for  $j \geq 2$ , there exists a  $\sigma$ -finite measure  $\mu_j$  such that a version of the regular conditional probability measure of the distribution of  $X_j$  given  $(X_1, X_2, \dots, X_{j-1})$ , is absolutely

continuous with respect to the measure  $\mu_j$  with a corresponding density

$$p_j^\theta(X_j|X_1, \dots, X_{j-1}) := p_j^\theta, j \geq 2 \quad \text{and} \quad p_1^\theta(X_1) = p_1^\theta.$$

We now introduce the following assumptions

**A1.**  $L^2$ -regularity

We assume that there exist a sequence of  $d$ -dimensional random vectors  $(\chi_j(\theta))_{j \geq 1}$  and a sequence of  $d \times d$  positive definite matrices  $u_n, n \geq 1$  with  $\|u_n\| \rightarrow 0$  such that for every  $h \in \mathbb{R}^d$

$$\sum_{j=1}^n \mathbb{E} \left[ \int_{\mathbb{R}} \left\{ \sqrt{p_j^{\theta+u_n h}} - \sqrt{p_j^\theta} - \frac{1}{2} h^T u_n \chi_j(\theta) \right\}^2 d\mu_j \right] \xrightarrow{n \rightarrow \infty} 0.$$

**A2.** For every  $j \geq 1$

$$\mathbb{E} [\xi_j(\theta) | \mathcal{F}_{j-1}] = 0$$

where  $\xi_j(\theta) = (\xi_j^i(\theta))_{i=1, \dots, d}$  with  $\xi_j(\theta) = \begin{cases} \frac{\chi_j(\theta)}{\sqrt{p_j^\theta}} & \text{if } p_j^\theta \neq 0, \\ 0 & \text{otherwise.} \end{cases}$

**A3.** There exists a measurable function  $I(\theta) > 0$  mapping  $\Omega$  to the set of  $d \times d$  symmetric matrices

$$u_n \sum_{j=1}^n \mathbb{E} \left[ \xi_j(\theta) \xi_j^T(\theta) | \mathcal{F}_{j-1} \right] u_n^T \xrightarrow{n \rightarrow \infty} I(\theta) \quad (> 0 \text{ a.e.}), \quad \text{in probability.}$$

**A4.**  $\forall \epsilon > 0,$

$$\sum_{j=1}^n \mathbb{E} \left[ |u_n \xi_j(\theta)|^2 \mathbf{1}_{\{|u_n \xi_j(\theta)| \geq \epsilon\}} \right] \xrightarrow{n \rightarrow \infty} 0.$$

**A5.**

$$\sup_{n \geq 1} \sum_{j=1}^n \mathbb{E} \left[ |u_n \xi_j(\theta) \xi_j^T(\theta) u_n^T| \right] \leq C, \quad \text{for a strictly positive constant } C.$$

Let us remark that as soon as  $\theta \mapsto p^\theta$  is a smooth function then we can set  $\chi_j$  in **A1** such

that  $\chi_j(\theta) = (\chi_j^i(\theta))_{i=1, \dots, d}$  with  $\chi_j^i(\theta) = \begin{cases} \frac{\partial_{\theta_i} p_j^\theta}{\sqrt{p_j^\theta}} & \text{if } p_j^\theta \neq 0 \\ 0 & \text{otherwise} \end{cases}$  and we get  $\xi_j(\theta) = (\xi_j^i(\theta))_{i=1, \dots, d}$  with

$$\xi_j^i(\theta) = \begin{cases} \frac{\partial_{\theta_i} p_j^\theta}{p_j^\theta} & \text{if } p_j^\theta \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

If  $X$  is a homogeneous Markov process then the assumption **A2** is natural. Indeed, for every  $j \geq 1$ , if  $p_j^\theta \neq 0$  we have

$$\mathbb{E} [\xi_j(\theta) | \mathcal{F}_{j-1}] = \mathbb{E} \left[ \frac{\partial_{\theta} p_j^\theta}{p_j^\theta} (X_j | X_{j-1}) | \mathcal{F}_{j-1} \right] = \mathbb{E} \left[ \frac{\partial_{\theta} p_j^\theta}{p_j^\theta} (X_j | X_{j-1}) | X_{j-1} \right].$$

But we can write

$$\mathbb{E} \left[ \frac{\partial_\theta p_j^\theta}{p_j^\theta}(X_j|X_{j-1}) | X_{j-1} = x \right] = \mathbb{E} \left[ \frac{\partial_\theta p_1^\theta}{p_1^\theta}(X_1|x) \right] = \int \partial_\theta p_1^\theta(y|x) dy.$$

Moreover, suppose that we can exchange the order of differentiation and integration then we obtain

$$\int \partial_\theta p_1^\theta(y|x) dy = \partial_\theta \int p_1^\theta(y|x) dy = 0.$$

**Theorem 1.1.1** (Jeganathan [39]). *Assume that the assumptions **A1-A5** are satisfied. Then the sequence of families  $(\mathbf{P}_n^\theta : \theta \in \Theta \subset \mathbb{R}^d), n \geq 1, d \geq 1$  satisfies the LAMN property at  $\theta$ , at rate  $u_n$  with*

$$I_n(\theta) = u_n \sum_{j=1}^n \mathbb{E} \left[ \xi_j(\theta) \xi_j^T(\theta) | \mathcal{F}_{j-1} \right] u_n^T$$

and

$$N_n = I_n(\theta)^{1/2} u_n \sum_{j=1}^n \xi_j(\theta).$$

Moreover, under  $\mathbf{P}_n^\theta$ ,  $(N_n, I_n(\theta))$  converges in law to a limit  $(N, I(\theta))$ , with  $N = I(\theta)^{1/2} W$  where  $W$  is independent of  $I(\theta)$  and distributed according to  $\mathcal{N}(0, I_d)$  where  $I_d$  is the  $d \times d$  identity matrix.

The LAMN (LAN) property has been playing an important role in parametric estimation problems because if it holds true and if the information matrix is non degenerate, the Minimax Hájek-Le Cam Theorem and the Convolution Theorem can be applied and then a lower bound for the asymptotic variance of estimators can be obtained via minimax theorems. Hence, once LAMN (LAN) is proved, the asymptotic optimality of estimation for the parameters is known. Remark that the nondegeneracy of the rate  $u_n$  and the information matrix  $I(\theta)$  is essential in the minimax theorem due to Hájek [30] and Le Cam [49].

We are now in a position to state the convolution theorem. The general references on the convolution theorem are Jeganathan [39] for the LAMN case and Hájek [30], Ibragimov and Has'minskii [32] for the LAN case.

We begin with the definitions of a regular parameter estimator and a stochastic kernel.

**Definition 1.1.3** (Regular Estimator).

(a) (Hájek [29]) *Let a family  $(\mathbf{P}_n^\theta)$  satisfy the LAN property with the normalizing matrix  $u_n$  at point  $\theta$ . An estimator  $T_n$  (possibly a randomized one) of parameter  $\theta$  is called regular at the point*

$\theta$  if for some random variable  $H$  the convergence in law is valid

$$(u_n^{-1}(T_n - (\theta + u_n h))) \Rightarrow H \quad \text{under } \mathbf{P}_n^{\theta + u_n h}$$

as  $n \rightarrow \infty$  for any  $h \in \mathbb{R}^d$ . The distribution of  $H$  does not depend on the value of the local parameter  $h \in \mathbb{R}^d$ .

(b) (Jeganathan [39]) If LAMN property holds at point  $\theta$ ,  $(T_n)_n$  an estimator of  $\theta$  is termed regular at the point  $\theta$  if there is some random variable  $\tilde{H}$  on  $\mathbb{R}^{d \times d} \times \mathbb{R}^d$  such that for every  $h \in \mathbb{R}^d$  the convergence in law

$$(I_n(\theta), u_n^{-1}(T_n - (\theta + u_n h))) \Rightarrow \tilde{H} \quad \text{under } \mathbf{P}_n^{\theta + u_n h}$$

as  $n \rightarrow \infty$  and the limiting law  $\tilde{H}$  does not depend on  $h \in \mathbb{R}^d$ .

**Definition 1.1.4** (Stochastic kernel). Let  $(\Omega, \mathcal{F})$  and  $(\Omega^*, \mathcal{F}^*)$  be two measurable spaces. A stochastic (Markov) kernel from  $(\Omega, \mathcal{F})$  to  $(\Omega^*, \mathcal{F}^*)$  is a function  $K = K(\omega, A^*) : \Omega \times \mathcal{F}^* \rightarrow [0, 1]$  such that

- (1)  $K(\cdot, A^*)$  is  $\mathcal{F}$ -measurable function for each  $A^* \in \mathcal{F}^*$
- (2)  $K(\omega, \cdot)$  is a probability measure on  $(\Omega^*, \mathcal{F}^*)$  for each  $\omega \in \Omega$ .

In the LAMN (LAN) case, the following theorem is known as the convolution theorem.

**Theorem 1.1.2.** (Convolution Theorem) Suppose that the sequence of families  $\{\mathbf{P}_n^\theta : \theta \in \Theta\}$ ,  $n \geq 1$  satisfies the LAMN or LAN property at  $\theta$ .

(a) (Hájek [29]) When LAN holds at  $\theta$ ,  $(T_n)_{n \geq 1}$  is a sequence of regular estimators of  $\theta$  then any limit distribution of  $H$  arising in Definition 1.1.3(a) can be written as a convolution representation

$$\mathcal{L}(H) = \mathcal{N}(0, I(\theta)^{-1}) * v(\theta)$$

for some distribution  $v(\theta)$  on  $\mathbb{R}^d$ .

(b) (Jeganathan [39]) When LAMN holds at  $\theta$ ,  $(T_n)_{n \geq 1}$  is a sequence of estimators regular at  $\theta$ , i.e. for every  $h \in \mathbb{R}^d$ ,

$$(I_n(\theta), u_n^{-1}(T_n - \theta - u_n h)) \Rightarrow (I(\theta), T(\theta)) \quad \text{under } \mathbf{P}_n^{\theta + u_n h}$$

for some random  $d$ -vector  $T(\theta)$  independent of  $h$ . Let  $\mathcal{L}_{I(\theta)}$  be a regular conditional probability measure of  $T(\theta)$  given  $I(\theta)$ . Then there exists a stochastic kernel  $K_{I(\theta)}$  such that  $\mathcal{L}_{I(\theta)}$  is a convolution between the Gaussian law  $\mathcal{N}(0, I(\theta)^{-1})$  and  $K_{I(\theta)}$ , that is,

$$\mathcal{L}_{I(\theta)} = K_{I(\theta)} * \mathcal{N}(0, I(\theta)^{-1}) \quad a.s.$$

In other words, the convolution theorem states that any regular estimator in a parametric model has a conditional asymptotic distribution equal to a sum of two independent random variables, one of which is normal with asymptotic variance equal to the inverse of Fisher information, and the other having arbitrary distribution. If we have a sequence of estimators  $(T_n)_{n \geq 1}$  satisfying the condition in the convolution theorem, then we have the following proposition.

**Proposition 1.1.1** (Jeganathan [39]). *Let  $(\mathbf{P}_n^\theta : \theta \in \Theta), n \geq 1$  be LAMN at  $\theta \in \Theta$ . Let  $(T_n)_{n \geq 1}$  be a sequence of estimator satisfying the condition in Theorem 1.1.2. Let  $Z$  be a random vector with standard normal distribution on  $\mathbb{R}^d$ . Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a loss function of the form  $f(0) = 0, f(x) = f(|x|)$  and  $f(|x|) \leq f(|y|)$  if  $|x| \leq |y|$  then*

$$\liminf_{n \rightarrow \infty} \mathbb{E}_\theta [f(u_n^{-1}(T_n - \theta))] \geq \mathbb{E} \left[ f \left( I(\theta)^{1/2} Z \right) \right].$$

We state a local asymptotic minimax bound that Hájek [30] introduced for estimation in LAMN (LAN) parametric models.

**Theorem 1.1.3.** (Minimax Theorem) *Suppose that the model  $(\mathbf{P}_n^\theta)$  is LAMN or LAN at  $\theta \in \Theta \subset \mathbb{R}^d$ . We consider an arbitrary sequence of estimators  $(T_n)_n$  for the unknown parameter and an arbitrary loss function  $f : \mathbb{R}^d \rightarrow [0, \infty)$  which is continuous, symmetric and subconvex. Let  $Z$  be a random vector with standard normal distribution on  $\mathbb{R}^d$ , independent of  $I(\theta)$ . Then we have*

$$\lim_{r \rightarrow \infty} \liminf_{n \rightarrow \infty} \inf_{T_n} \sup_{|h| \leq r} \mathbb{E}_{\theta + u_n h} [f(u_n^{-1}(T_n - (\theta + u_n h)))] \geq \mathbb{E} \left[ f(I(\theta)^{-1/2} Z) \right]$$

the infimum being taken over all estimators of  $\theta$ .

In particular, if we choose the quadratic function  $f(x) = x^2$  then we obtain from the above theorem that an asymptotic lower bound for the covariance matrix of any sequence of unbiased estimators is given by  $I(\theta)^{-1}$ . A sequence of estimators that attains this asymptotic bound is called asymptotically efficient.

In the context of Local Asymptotic Mixed Normality (Local Asymptotic Normality), we cannot fail to mention the definitions of rate optimal and efficiency of estimators.

**Definition 1.1.5.** *Let  $(\Omega^n, \mathcal{F}^n, \mathbf{P}_n^\theta : \theta \in \Theta \subset \mathbb{R}^d)$ ,  $n \geq 1$ , be a statistical experiment which is LAMN (LAN) at  $\theta$ , with rate  $u_n$  and (random) Fisher information  $I(\theta)$ . Then a sequence of estimators  $(T_n)$  for  $\theta$  is rate optimal if there exists a non degenerate random variable  $H$  under  $\mathbf{P}^\theta$  such that*

$$u_n^{-1}(T_n - \theta) \Rightarrow H.$$

*Additionally, the sequence is asymptotically efficient if  $H$  has the distribution of  $I(\theta)^{-1/2}Z$ , with  $Z$  standard normally distributed and independent of  $I(\theta)$ .*

In brief, when the Local Asymptotic Mixed Normality (Local Asymptotic Normality) holds at  $\theta$  with the rate  $u_n$  and the information matrix  $I(\theta) > 0$ , the convolution and minimax theorems can be applied. Moreover, the asymptotically efficient estimators of the parameter  $\theta$  are defined in terms of the optimal asymptotic variance  $I(\theta)^{-1}$  and the optimal rate of convergence  $u_n^{-1}$ . We can obtain the lower bound for the asymptotic variance of estimators given by  $I(\theta)^{-1}$  via minimax theorem.

## 1.2 Lévy processes

In this section, we recall the useful definitions and properties of real Lévy processes. For additional details on Lévy processes, we refer to Sato [59] or Applebaum [6].

**Definition 1.2.1.** *(Lévy Process) A càdlàg stochastic process  $L \equiv (L_t)_{t \geq 0}$  defined on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$  is said to be a Lévy process if it possesses the following properties:*

- (1)  $L_0 = 0$  a.s.
- (2)  $L$  has independent increments. That is, for any choice of  $n \geq 1$  and  $0 \leq t_0 < t_1 < \dots < t_n$ ,  $(L_{t_j} - L_{t_{j-1}}, 1 \leq j \leq n)$  are independent.
- (3)  $L$  has stationary increments. That is, for all  $0 \leq s < t < \infty$ , the distribution of  $L_t - L_s$  is equal to  $L_{t-s}$ .

That is to say, a Lévy process is a process with stationary and independent increments whose sample paths are right continuous with left limits. The properties of stationary and independent increments implies that a Lévy process is a Markov process and the distribution of a Lévy process has the property of infinite divisibility. Any linear combination of two independent Lévy processes is again a Lévy process.

**Definition 1.2.2.** (*Stable Distribution*) A random variable  $X$  is called stable if for any integer  $n \geq 2$ , there exist real-valued sequences  $(c_n)$  and  $(d_n)$  with each  $c_n > 0$  such that the distribution of  $X_1 + X_2 + \dots + X_n$  is equal to  $c_n X + d_n$ , where  $X_1, X_2, \dots, X_n$  are independent copies of  $X$ . In particular,  $X$  is said to be strictly stable if each  $d_n = 0$  and  $X$  is said to be symmetric stable if  $X$  and  $-X$  have the same distribution.

It can be shown (see Feller [65], p.166) that the only possible choice of  $c_n$  is of the form  $\sigma n^{1/\alpha}$  where  $\sigma > 0$  and  $0 < \alpha \leq 2$ . The parameter  $\alpha$  plays a key role in the investigation of stable random variables and is called the index of stability or the exponent of the process.

**Definition 1.2.3.** (*Stable Process*) A stable process is a Lévy process  $(L_t)_{t \geq 0}$  such that  $L_1$  has the stable distribution.

The Brownian motion and the Poisson process are two typical examples of Lévy processes. The Brownian motion is the only (non-deterministic) Lévy process with continuous paths and it is a stable process with exponent  $\alpha = 2$ . The Poisson process is not stable.

**Definition 1.2.4.** (*Self-similarity*) Let  $(L_t : t \geq 0)$  be a stochastic process on  $\mathbb{R}$ . It is called selfsimilar if, for any  $a > 0$ , there is  $b > 0$  such that  $(L_{at})_{t \geq 0}$  has the same law as  $(bL_t)_{t \geq 0}$ .

A process  $L$  is said to be a strictly stable process, if it is a Lévy process which also satisfies the self-similarity property with  $b = a^{1/\alpha}$ . It is known that  $\alpha \in (0, 2]$ , and the case  $\alpha = 2$  corresponds to Brownian motion. A symmetric stable process is easily seen to be also strictly stable.

When  $L$  is a stable process for  $\alpha \in (0, 2)$  then, for any  $t$ , we have

$$\mathbb{E}(|L_t|^p) < \infty \quad \text{for any } 0 < p < \alpha$$

$$\mathbb{E}(|L_t|^p) = \infty \quad \text{for any } p \geq \alpha.$$



In particular, when  $\alpha \leq 1$  the expected value of  $L_t$  is not defined. For more properties of  $\alpha$ -stable process, we refer to Sato [59].

The following theorem is of the most fundamental importance in the theory of Lévy process which allows to show that the law of a Lévy process is uniquely determined by its drift, volatility and Lévy measure.

**Theorem 1.2.1.** (*Lévy-Khintchine formula for Lévy processes*) *Let  $L$  be a one-dimensional Lévy process. Then there exists (unique)  $b \in \mathbb{R}$ ,  $\sigma \geq 0$  and  $\nu$  a measure defined on  $\mathbb{R} \setminus \{0\}$  with  $\int_{\mathbb{R} \setminus \{0\}} \min(1, x^2) \nu(dx) < \infty$  such that  $\forall u \in \mathbb{R}, \forall t \geq 0$ ,*

$$\mathbb{E}(e^{i(u, L_t)}) = e^{t\phi(u)} = \exp\left(t\left(ib u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R} \setminus \{0\}} \left(e^{iuz} - 1 - iuz \mathbf{1}_{|z| \leq 1}\right) \nu(dz)\right)\right).$$

*The triplet  $(b, \sigma^2, \nu)$  is called the Lévy or characteristic triplet and  $\phi(u)$  is called the Lévy or characteristic exponent. Moreover,  $b$  is called the drift term,  $\sigma$  the diffusion coefficient and  $\nu$  the Lévy measure.*

Note that a Borel measure  $\nu$  defined on  $\mathbb{R} \setminus \{0\}$  is called a Lévy measure if

$$\int_{\mathbb{R} \setminus \{0\}} \min(1, x^2) \nu(dx) < \infty$$

and it characterizes the size and frequency of the jumps. If this measure is infinite, then the process has an infinite number of jumps of every small sizes in any small interval.

In particular, if  $L$  is a stable real-valued random variable for  $\alpha \in (0, 2]$ , then its characteristics must take one of the following forms:

- (1) when  $\alpha = 2$ , the Lévy measure  $\nu = 0$ , so  $L$  corresponds to Brownian motion with mean  $b$  and variance  $\sigma^2$ .
- (2) when  $\alpha \neq 2$ , the Gaussian component  $\sigma^2 = 0$  and the Lévy measure  $\nu$  is absolutely continuous with density given by

$$c_+ x^{-(\alpha+1)} \mathbf{1}_{x>0} + c_- |x|^{-(\alpha+1)} \mathbf{1}_{x<0}, \quad x \in \mathbb{R}$$

where  $c_+ \geq 0, c_- \geq 0, c_+ + c_- > 0$ .

Moreover, in the case  $\alpha \in (0, 2)$ , there exist  $c > 0$ ,  $-1 \leq \beta \leq 1$  and  $b \in \mathbb{R}$  such that for  $u \in \mathbb{R}$ ,  $L$  has the characteristic exponent of the form:

$$\phi(u) = \begin{cases} iub - c|u|^\alpha(1 - i\beta \tan(\frac{\pi\alpha}{2}) \operatorname{sgn}(u)) & \alpha \in (0, 2) \setminus \{1\} \\ iub - c|u|(1 + i\beta \frac{2}{\pi} \operatorname{sgn}(u) \log(|u|)) & \alpha = 1. \end{cases}$$

Note that if a stable random variable is symmetric then its characteristic exponent is given by

$$\phi(u) = -C_\alpha |u|^\alpha \quad \text{for all } 0 < \alpha \leq 2,$$

where  $C_\alpha > 0$ .

A proof can be found in Sato [59] and we can also extend the result for the random variables taking values in  $\mathbb{R}^d$ , see more details in Sato [59] p.83 and Applebaum [6] p.35. In the following, we consider the jump activity index or the index parameter of a Lévy process, a non-random parameter which is independent of time and determines the strength of the jump process at high frequencies.

**Definition 1.2.5.** (*Jump Activity Index*)

$$\alpha := \inf \left\{ r \in [0, 2]; \int_{|x| \leq 1} |x|^r \nu(dx) < +\infty \right\}$$

where  $\nu(dx)$  is the Lévy measure.

The index parameter of a Lévy process is thus the Blumenthal-Gettoor index which was first introduced by Blumenthal and Gettoor [11]. Moreover, for a stable process, it is the so-called stability index of the process. The interest in identifying the jump activity index stays in the fact that the index classifies the Lévy process according to the "degree of activity" of jumps: when  $\alpha$  increases from 0 to 2, the small jumps tend to be more and more frequent.

Every Lévy process can be represented as the sum of three independent processes: a linear drift, a Brownian motion and a pure jump process. This result is stated as follows:

**Theorem 1.2.2.** (*Lévy Itô Decomposition*) *Let  $L$  be an  $\mathbb{R}$ -valued Lévy process with Lévy measure  $\nu$  such that  $\int_{\mathbb{R} \setminus \{0\}} \min(x^2, 1) \nu(dx) < \infty$ . Then  $L$  has the decomposition*

$$(1.2.1) \quad L_t = \mu t + \sigma B_t + Z_t$$

where  $\mu \in \mathbb{R}$ ,  $B$  is a standard one-dimensional Brownian motion,  $Z_t$  is a pure jump process corresponding to the jump part of  $L$ . Moreover, for a Poisson random measure  $N$  independent of  $(B_t)$  on  $\mathbb{R}^+ \times \mathbb{R} \setminus \{0\}$  with intensity  $dt \times \nu$ , we can represent  $Z$  as

$$\begin{aligned} Z_t &= Z_t^{(0)} + Z_t^{(1)}, \quad \text{with} \\ Z_t^{(0)} &= \int_0^t \int_{|z| \leq 1} z \{N(ds, dz) - \nu(dz)ds\} \quad \text{and} \\ Z_t^{(1)} &= \int_0^t \int_{|z| > 1} z N(ds, dz) = \sum_{s \leq t} (L_s - L_{s-}) \mathbf{1}_{\{|L_s - L_{s-}| > 1\}}. \end{aligned}$$

Remark that since  $Z$  is a semi-martingale and a standard Brownian motion is a martingale then  $L$  is a semi-martingale.

From the above theorems, we realize that a pure-jump Lévy process is a Lévy process with the characteristic triplet  $(0, 0, \nu)$ . In recent years, pure jump models have been widely used, especially in view of applications to finance (see Schoutens [60], Cont and Tankov [61]), network traffic (see Mikosch et al [55]) and climate dynamics (see [23]). The reason for this attention is that pure-jump processes provide a sufficiently rich class of models with jumps to describe more realistic pictures of empirically observed situations in practice, for example, Cont and Tankov [61], and Carr et al. [15]. On the other hand, with the progress made in understanding jump processes, many computations can be handled analytically (see Sato [59], Applebaum [6]).

## 1.3 Motivation and Overview

### 1.3.1 Model

Throughout this thesis, we consider a process  $X = (X_t)_{t \geq 0}$  solving a stochastic differential equation of the form

$$(1.3.1) \quad X_t = x_0 + \int_0^t b(X_s, \theta) ds + \sigma L_t \quad t \in [0, 1]$$

where  $(L_t)_{t \in [0, 1]}$  is a truncated pure jump  $\alpha$ -stable Lévy process with exponent  $\alpha \in (0, 2)$ ,  $\beta = (\theta, \sigma)^T \in \mathbb{R} \times (0, \infty)$  is an unknown parameter and  $b$  is a real valued function. More precisely, we assume that the following assumptions are fulfilled for the truncated  $\alpha$ -stable process  $(L_t)_{t \in [0, 1]}$ .

(i) The Lévy process  $(L_t)_{t \in [0,1]}$  is given by

$$L_t = \int_0^t \int_{[-1,1]} z \{\bar{\mu}(ds, dz) - \bar{\nu}(ds, dz)\} + \int_0^t \int_{[-1,1]^c} z \bar{\mu}(ds, dz),$$

where  $\bar{\mu}$  is a Poisson random measure, with compensator  $\bar{\nu}(dt, dz) = dt \times F(z)dz$  where  $F(z)$  is given on  $\mathbb{R}$  by  $F(z) = \frac{1}{|z|^{\alpha+1}} 1_{|z| \neq 0} \tau(z)$ ,  $\alpha \in (0, 2)$ . We assume that  $\tau$  is a non negative smooth function equal to 1 on  $[-1,1]$ , vanishing on  $[-2, 2]^c$  such that  $0 \leq \tau \leq 1$ .

(ii) We assume that  $\forall p \geq 1$ ,

$$(1.3.2) \quad \int_{\mathbb{R}} \left| \frac{\tau'(u)}{\tau(u)} \right|^p \tau(u) du < \infty,$$

$$(1.3.3) \quad \int_{\mathbb{R}} \left| \frac{\tau''(u)}{\tau(u)} \right|^p \tau(u) du < \infty.$$

We observe this process at discrete times, equally spaced, over the fixed finite interval  $[0,1]$ , that means that  $X$  is observed at  $n + 1$  discrete, equidistant time-points  $t_i^n = \frac{i}{n} := i\Delta_n, i = 0, 1, \dots, n$ , in the interval  $[0,1]$ . We are concerned with asymptotic properties as the time lag  $\Delta_n = \frac{1}{n}$ , goes to 0. In practice, this means that we are in the context of high-frequency data. It is known that, in the diffusion case, we can not estimate the drift parameter in finite time horizon. On the contrary, for a pure jump process, we can estimate the drift parameter in both cases: a fixed observation period or a long time asymptotic. Without loss of generality, we can suppose that the observation interval is  $[0, 1]$ .

Because of the essential role of the LAN (LAMN) property in parametric estimation problems and the important applications of pure jump Lévy processes, one of the purposes of this thesis is to study the LAMN property for the stochastic differential equation given by (1.3.1).

Malliavin calculus is the principal tool to prove the LAMN property and it explains partially the specific assumptions in (i), (ii). First, since our method is based on Malliavin calculus, the integrability assumptions for the tails of the Lévy process are crucial to ensure that the process (1.3.1) belongs to the Malliavin space. Secondly, the stable behavior of the Lévy measure around zero is also required to make the rescaled process  $(n^{1/\alpha} L_{t/n})$  close to the  $\alpha$ -stable process  $(L_t^\alpha)$ . Then the truncation function  $\tau$  ensures both the integrability of  $|L_t|^p, \forall p \geq 1$ , and the exact  $\alpha$ -stable behavior around zero ( $\tau = 1$ ). Moreover, in order to obtain the asymptotic expansion of the log-likelihood ratio in the study of LAMN property, we need to know about the asymptotic

behavior of the density of the process and its derivative with respect to the parameters. Using Malliavin calculus, we get the representation formulas for the density and its derivative. These representation formulas involve some Malliavin weights whose expressions are given explicitly. Then we observe that the introduction of the truncation function  $\tau$  permits to ensure that the process  $(n^{1/\alpha}L_{t/n})$  has no jump of size larger than  $2n^{1/\alpha}$  and consequently it makes easier the control of the asymptotic behavior of the Malliavin weights in order to establish the convergence in small time of the density. More precisely, the assumptions (1.3.2) and (1.3.3) are technical tools to prove the convergences of the density and its derivative, respectively.

### 1.3.2 Background

There is a large literature concerning the LAN (LAMN) property for pure jump Lévy processes based on high frequency observations, for example, the works of Aït-Sahalia and Jacod [2] [3], Kawai and Masuda [44], Masuda [53][54], Ivanenko, Kulik and Masuda [35], Clément and Gloter [17].

In particular, Masuda [53] proposes a joint estimator for a stable Lévy process

$$(1.3.4) \quad X_t = \theta t + \sigma L_t^\alpha$$

based on discrete time observations  $(X_{i/n})_{i=1,\dots,n}$ . He proves the Local Asymptotic Normality property with the rate  $u_n = \text{diag}\left(\frac{1}{n^{-1/2+1/\alpha}}, \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n} \log(n)}\right)$  for the estimation of  $\beta = (\theta, \sigma, \alpha)$  with a singular Fisher information matrix. Consequently, the conventional convolution and minimax theorems can not be applied and then a lower bound for the asymptotic variance of estimators can not be deduced in this statistical experiment. Concerning the estimators of the index parameter  $\alpha$  of the process, the author has constructed two estimators  $\hat{\alpha}_{\log,n}$  based on the logarithmic moments and  $\hat{\alpha}_{r,n}$  based on the lower-order fractional moments for  $\alpha$  with the convergence rate  $\sqrt{n}$ .

Recently, to fix the problem with singular Fisher information matrix and know the optimal rate of estimators for the parameters in (1.3.4), Brouste and Masuda [13] develop the Local

Asymptotic Normality property with a non-diagonal norming rate, for example

$$u_n = \frac{1}{\sqrt{n}} \begin{pmatrix} n^{1-1/\alpha} & 0 & 0 \\ 0 & 1 & -\alpha^{-2}\sigma \log(n) \\ 0 & 0 & 1 \end{pmatrix}$$

and a non-degeneracy Fisher information matrix. The convolution and minimax theorems can be applied in this case and then the optimal rate of convergence of estimator for the index parameter  $\alpha$  when the scale parameter  $\sigma$  known is  $\sqrt{n} \log(n)$  and is  $\sqrt{n}$  when the scale parameter  $\sigma$  unknown.

In Ait-Sahalia and Jacod [3] and [2], the authors exhibit the behavior of the Fisher information of a process  $X$  decomposed as the sum of two independent Lévy processes

$$X_t = \sigma L_t^\alpha + \theta U_t$$

where  $L$  is a standard symmetric stable process with index  $\alpha \in (0, 2]$ ,  $(U_t)$  is a another Lévy process, independent of  $(L_t^\alpha)$  and dominated by  $(L_t^\alpha)$ . Then they derive that with the scale parameter  $\sigma$  known and the true value  $\alpha < 2$ , they can hope for estimators converging to  $\alpha$  at the rate  $\sqrt{n} \log(n)$ . The LAN property is also proved with rate  $n^{-\frac{1}{2}}$  for the parameter  $\sigma$ .

More recently, Ivanenko, Kulik and Masuda [35] consider the model

$$X_t = \theta t + \sigma L_t^\alpha + Z_t$$

where  $L$  is a non-symmetric locally  $\alpha$ -stable process and  $Z$  is an independent and less active process. When  $X$  is observed at high-frequency, they prove the LAN property for the parameter  $\beta = (\theta, \sigma) \in \mathbb{R}^2$  with the rate  $u_n = \frac{1}{\sqrt{n}} \begin{pmatrix} n^{1-1/\alpha} & c_n n \\ 0 & 1 \end{pmatrix}$  where  $c_n = \frac{1}{n} \int_{n^{-1/\alpha} < |u| < 1} u \mu(du)$  which is identically zero if the Lévy measure  $\mu$  is symmetric.

In above works, from the linearity of the models then the increments  $(X_{\frac{i}{n}} - X_{\frac{i-1}{n}})_{i \leq i \leq n}$  are independent and have the same law as  $X_{1/n}$ . The transition density of the discrete time process  $(X_{\frac{i}{n}})_{1 \leq i \leq n}$  is almost explicit therefore the statistical study can be done using the explicit expression of the density of  $X_t$ .

Coming back to the model (1.3.1), the difficulty in the statistical study of model (1.3.1) is that the transition density of the Markov chain  $(X_{\frac{i}{n}})_{1 \leq i \leq n}$  is unknown and the link between the

density of  $L_t$  and the density of  $X_t$  is not clear. In this setup, we can mention the recent papers by Clément and Gloter [18], Masuda [54] where some estimators of the drift and scale parameters are proposed for the general equation

$$X_t = x_0 + \int_0^t b(X_s, \theta) ds + \int_0^t c(X_{s-}, \sigma) dL_s$$

where  $L$  is a locally stable pure-jump Lévy process. However, in that case, the asymptotic efficiency of the estimators is not yet established. The only result about LAMN property, to our knowledge, in this direction is given in Clément and Gloter [17] where the LAMN property is proved for the estimation of the drift parameter  $\theta$  for the process solution of (1.3.1) (with  $\sigma = 1$  and  $\alpha \in (1, 2)$ ). In [17], they show that the LAMN property is satisfied with rate  $r_n = n^{1/2-1/\alpha}$  and information  $\mathcal{I}_\theta = \int_0^1 \partial_\theta b(X_s^\beta, \theta)^2 ds \int_{\mathbb{R}} \frac{\varphi'_\alpha(u)^2}{\varphi_\alpha(u)} du$ , where  $\varphi_\alpha$  is the density of the standard  $\alpha$ -stable distribution with characteristic function  $u \rightarrow e^{-C(\alpha)|u|^\alpha}$ .

The first goal of this thesis is to extend the results of Clément and Gloter [17] where the index  $\alpha \in (1, 2)$  and the parameter appears only in the drift coefficient. Namely, we extend in this thesis these results to  $\alpha \in (0, 2)$  and prove that the LAMN property holds for the two parameters  $(\theta, \sigma)$  when the index parameter  $\alpha$  is known.

The proof is mainly based on Theorem 1.1.1 and then remark that the asymptotic behavior of the density of  $(X_t)$  and its derivative with respect to the parameters need to be known to obtain the asymptotic of Fisher information matrix in small time, as well as the asymptotic expansion of the log-likelihood ratio. Because of this reason, we discuss in Chapter 2 the asymptotic in small time for the density of  $(X_t)$ .

Another purpose of this thesis is to estimate the index parameter (see definition 1.2.5) of the process solving (1.3.1) and assuming that the parameter  $\theta$  is known.

In the last part of this thesis, we construct two estimators with the convergence rate  $\sqrt{n}$  for the Blumenthal-Gettoor index  $\alpha \in (0, 2)$  when the scale parameter  $\sigma$  is unknown, based on the two moment-fitting procedures (the logarithmic moments and the lower-order fractional moments) introduced in Masuda [53]. We prove the consistency and asymptotic normality for the estimators and the estimation results will be supported by the simulation experiments. If the scale parameter is known, we construct some estimators of the index parameter with the rate

$\sqrt{n} \log(n)$  also based on the logarithmic moments and the lower-order fractional moments.

Besides the above references, we list here some others works concerning the estimation of the jump activity index from high frequency observations. In Aït-Sahalia and Jacod [4] and Jing, Kong and Mykland [42] the authors consider the process with the dominant component being a Brownian motion given by

$$X_t = x_0 + \int_0^t b_s ds + \int_0^t \sigma_s W_s + \int_0^t \int_{|x| \leq 1} x(\mu - \nu)(ds, dx) + \int_0^t \int_{|x| \geq 1} x \mu(ds, dx)$$

where  $W_t$  is a standard Brownian motion,  $b$  and  $\sigma$  are Itô processes and  $\mu$  is a random measure with compensator  $\nu$ . Based on truncated power variation, they propose estimators for the index parameter  $\alpha$  in high-frequency.

Todorov and Tauchen [63] propose estimation of the jump activity index in pure-jump semi-martingales setting for which the rate of convergence is  $\sqrt{n}$  for  $\sqrt{2} \leq \alpha < 2$  (the case  $\alpha < \sqrt{2}$  needs an assumption of zero drift) based on adaptively chosen optimal power for which the realized power variation does not explode.

Recently, Todorov [62] considers the estimation of the jump-activity index  $\alpha \in (1, 2]$  of the process

$$dX_t = b_t dt + \sigma_t dL_t + dY_t$$

where  $L$  is locally stable pure jump Lévy process whose Lévy measure around zero behaves like that of a stable process and  $Y$  is a pure jump process which is dominated at high frequencies by  $L$ . He obtains the rate of convergence  $\sqrt{n}$  if  $\alpha \in (1, 2)$  and  $\sqrt{n} u_n^{-2}$  for some sequence  $u_n$  converging to zero if  $\alpha = 2$ , by using the empirical characteristic function of the scaled differenced increments.

### 1.3.3 Overview of the main results

This thesis is divided into four chapters.

**Chapter 2:** We focus on the asymptotic behavior, in small time, of the density of  $(X_t^\beta)$  with index  $\alpha \in (0, 2)$  for the solution of (1.3.1), as well as its derivative with respect to the parameter  $\beta = (\theta, \sigma)^T$ . This problem plays an important role in asymptotic statistics based on high frequency observations. Indeed, considering the estimation of  $\beta$  from the discrete time observations  $(X_{i/n}^\beta)_{0 \leq i \leq n}$ , and denoting by  $p_{1/n}^\beta(x, y)$  the transition density of the discrete time



process, the estimation rate of the parameter  $\beta$  strongly relies on the asymptotic behavior of the derivative  $\nabla_{\beta} p_{1/n}^{\beta}(x, y)$ , as  $n$  goes to infinity.

By using Malliavin calculus, we obtain the representation formulas for the density, its derivative and its logarithm derivative, as an expectation and a conditional expectation. These representation formulas involve some Malliavin weights whose expressions are given explicitly (see Theorem 2.3.1) and this permits to analyze the asymptotic behavior in small time of the density by using the time rescaling property of the stable process. Our results are established through a careful study of each terms appearing in the Malliavin weights, which is complicated by the non integrability of the  $\alpha$ -stable process as if  $\alpha \leq 1$ . Moreover, when we study the asymptotic behavior, in order to deal with any value of the index  $\alpha \in (0, 2)$ , we introduce the solution of the ordinary differential equation

$$(1.3.5) \quad \zeta_t^{n, \theta, x} = x + \frac{1}{n} \int_0^t b(\zeta_s^{n, \theta, x}, \theta) ds \quad t \in (0, 1], \quad x \in \mathbb{R}$$

and then we obtain that  $n^{1/\alpha} \left\{ X_{\frac{1}{n}}^{\beta}(x) - \zeta_1^{n, \theta, x} \right\}$  is close to a stable Lévy process.

We denote by  $\varphi_{\alpha}$  the density of  $L_1^{\alpha}$  where  $(L_t^{\alpha})$  is an  $\alpha$ -stable process with Lévy measure  $\nu(dz) = \frac{1}{|z|^{\alpha+1}} 1_{z \neq 0} dz$ . We have the following results about the asymptotic behavior of the density and its derivative:

**Theorem 1.3.1** (Clément, Gloter and Nguyen [19]). *Let  $(\zeta_t^{n, \theta, x_0})$  be the solution to the ordinary differential equation (1.3.5) and let  $(\beta_n)_{n \geq 1} = ((\theta_n, \sigma_n)^T)_{n \geq 1}$  be a sequence such that  $\beta_n \xrightarrow{n \rightarrow \infty} \beta$ . For all  $(x_0, u) \in \mathbb{R}^2$ ,*

1.  $\frac{\sigma_n}{n^{1/\alpha}} p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n, \theta_n, x_0}) \xrightarrow{n \rightarrow \infty} \varphi_{\alpha}(u),$
2.  $\frac{\sigma_n^2}{n^{\frac{2}{\alpha}-1}} \partial_{\theta} p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n, \theta_n, x_0}) \xrightarrow{n \rightarrow \infty} -\partial_{\theta} b(x_0, \theta) \times \varphi'_{\alpha}(u),$   
 $\frac{\sigma_n^2}{n^{1/\alpha}} \partial_{\sigma} p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n, \theta_n, x_0}) \xrightarrow{n \rightarrow \infty} -\varphi_{\alpha}(u) - u\varphi'_{\alpha}(u),$

Chapter 2 is published as the following article:

- \* *Emmanuelle Clément, Arnaud Gloter, and Huong Nguyen. Asymptotics in small time for the density of a stochastic differential equation driven by a stable Lévy process. ESAIM: P/S, 2018.*

**Chapter 3:** We discuss the Local Asymptotic Mixed Normality property for the drift and scaling parameters from high frequency observations, of a continuous time process solution of a stochastic differential equation driven by a truncated  $\alpha$ -stable process with index  $\alpha \in (0, 2)$  given by the equation (1.3.1). This extends the results of [17] where the index  $\alpha \in (1, 2)$  and only the Local Asymptotic Mixed Normality property for the drift coefficient is considered. Based on the expression for the logarithm derivatives of the density with respect to  $\beta$  in terms of a conditional expectation, Malliavin weights and the results about the asymptotic behavior of the density established in Chapter 2, we derive the asymptotic behavior of Fisher information matrix in small time for one observation of the process. Furthermore, we prove an asymptotic expansion of the log-likelihood ratio (using Malliavin calculus for jump processes) and prove that the LAMN property holds for the drift and volatility parameters with rate  $u_n = \begin{pmatrix} n^{\frac{1}{2}-\frac{1}{\alpha}} & 0 \\ 0 & n^{-\frac{1}{2}} \end{pmatrix}$

and information matrix  $I = \begin{pmatrix} \mathcal{I}_{11} & 0 \\ 0 & \mathcal{I}_{22} \end{pmatrix}$  where

$$\begin{cases} \mathcal{I}_{11} = \frac{1}{\sigma^2} \int_0^1 \partial_\theta b(X_s^\beta, \theta)^2 ds \int_{\mathbb{R}} \frac{\varphi'_\alpha(u)^2}{\varphi_\alpha(u)} du \\ \mathcal{I}_{22} = \frac{1}{\sigma^2} \int_{\mathbb{R}} \frac{(\varphi_\alpha(u) + u\varphi'_\alpha(u))^2}{\varphi_\alpha(u)} du. \end{cases}$$

The proof of the LAMN property is based on Theorem 1.1.1. This part contains also an independent and interesting result stating a continuity property with respect to the conditioning variable in a conditional expectation (see Proposition 3.6.3).

Chapter 3 is published as the following article:

\* *Emmanuelle Clément, Arnaud Gloter, and Huong Nguyen. LAMN property for the drift and volatility parameters of a SDE driven by a stable Lévy process. ESAIM: P/S, 2018.*

**Chapter 4:** The aim of this chapter is to estimate the index parameter  $\alpha$  of a stochastic differential equation driven by a truncated  $\alpha$ -stable process with index  $\alpha \in (0, 2)$  given by the equation (1.3.1) with known parameter  $\theta$  based on high frequency observations on a fixed time period. Masuda [53] has constructed the estimators based on the logarithmic moments and the lower-order fractional moments for the index  $\alpha$  with the convergence rate  $\sqrt{n}$ , when the scale parameter  $\sigma$  is

unknown. Based on the ideas of Masuda [53], in this chapter, we adapt the estimation procedure to the equation (1.3.1) and provide the asymptotic distributions.

The main tools for establishing the statistical results are Law of Large Numbers and Central Limit Theorem for

$$\frac{1}{n} \sum_{j=1}^{n-1} f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \zeta_{1/n}(X_{t_j}^\alpha)\} \right)$$

with  $f$  some suitable functions and  $\zeta_{1/n}$  solving

$$\zeta_t(x) = x + \int_0^t b(\zeta_s(x)) ds \quad t \in (0, 1/n], \quad x \in \mathbb{R}.$$

The introduction of this ordinary differential equation is convenient for dealing with any value of  $\alpha$ , however, this leads to a limitation in practice since the drift function  $b$  has to be known. If we know the true value of the scale parameter, then we construct some estimators of the index parameter with the rate  $\sqrt{n} \log(n)$ . Moreover, in this chapter, we give some simulations to illustrate the finite-sample behaviors of our estimators.

---

**ASYMPTOTICS IN SMALL TIME FOR THE  
DENSITY OF A STOCHASTIC DIFFERENTIAL  
EQUATION DRIVEN BY A STABLE LÉVY  
PROCESS**

---

This work focuses on the asymptotic behavior of the density in small time of a stochastic differential equation driven by a truncated  $\alpha$ -stable process with index  $\alpha \in (0, 2)$ . We assume that the process depends on a parameter  $\beta = (\theta, \sigma)^T$  and we study the sensitivity of the density with respect to this parameter. This extends the results of [17] which was restricted to the index  $\alpha \in (1, 2)$  and considered only the sensitivity with respect to the drift coefficient. By using Malliavin calculus, we obtain the representation of the density and its derivative as an expectation and a conditional expectation. This permits to analyze the asymptotic behavior in small time of the density, using the time rescaling property of the stable process.

## 2.1 Introduction

We consider the following stochastic differential equation (SDE)

$$(2.1.1) \quad X_t^\beta = x_0 + \int_0^t b(X_s^\beta, \theta) ds + \sigma L_t$$

for  $t \in [0, 1]$ , where  $(L_t)_{t \in [0, 1]}$  is a truncated  $\alpha$ -stable process with exponent  $0 < \alpha < 2$  and our aim is to study the asymptotic behavior, in small time, of the density of  $(X_t^\beta)$ , the solution of

(2.1.1), as well as its derivative with respect to the parameter  $\beta = (\theta, \sigma)^T$ . This problem plays an important role in asymptotic statistics based on high frequency observations. Indeed, considering the estimation of  $\beta$  from the discrete time observations  $(X_{i/n}^\beta)_{0 \leq i \leq n}$ , and denoting by  $p_{1/n}^\beta(x, y)$  the transition density of the discrete time process, the estimation rate of the parameter  $\beta$  strongly relies on the asymptotic behavior of the derivative  $\nabla_\beta p_{1/n}^\beta(x, y)$ , as  $n$  goes to infinity. Based on the results established in the present paper, we derive, in [20], an asymptotic expansion of the log-likelihood ratio and we prove the LAMN property for the parameter  $\beta$ .

In the last decades, a large literature has been devoted to the existence and regularity of the density to the solution  $(X_t)_t$ , for  $t > 0$ , of a general stochastic equation driven by pure jump Lévy processes. We can mention the works of Bichteler, Gravereaux and Jacod [8], Picard [56], Denis [22], Ishikawa-Kunita [33], Fournier-Printems [24] and more recently the works of Debussche-Fournier [21] and Kulik [47], under Hölder continuity assumptions on the coefficients of the equation and assuming that the equation is driven by an  $\alpha$ -stable process.

In this paper, the main contributions are obtained by using the Malliavin calculus for jump processes developed by Bichteler, Gravereaux and Jacod [8] and adapted to the particular case of equation (2.1.1) by Clément-Gloter [17]. Although it requires some strong derivability assumptions on the coefficients of the equation, it leads to some explicit representation formulas for the density and its derivative (see also Ivanenko - Kulik [34]). Let us mention that alternative representations for the density can be obtained by other methods, for example the method proposed by Bouleau-Denis [12] based on Dirichlet forms or the parametrix method used by Kulik [47].

To study the asymptotic behavior of the transition density of  $X_t^\beta$  and its derivative, in small time, we establish some representation formulas. This extends the results of Clément-Gloter [17] where only the derivative with respect to the drift parameter  $\theta$  was considered, with the restriction  $\alpha > 1$ . These representation formulas involve some Malliavin weights whose expressions are given explicitly. This permits first to identify in the Malliavin weights a main part and a negligible part in small time asymptotics and then to derive the asymptotics for the density stated in Theorem 2.2.1 and Theorem 2.2.2. In contrast to [17], the exposition now involves the solution of the ordinary differential equation defined by the deterministic part of (2.1.1). Moreover, the study

of each terms appearing in the Malliavin weights is complicated by the non integrability of the  $\alpha$ -stable process as  $\alpha \leq 1$ .

The present paper is organized as follows. Section 2.2 contains the main results (Theorem 2.2.1 and Theorem 2.2.2). Section 2.3 presents the methodology consisting first in a representation of the density by Malliavin calculus where the Malliavin weights can be decomposed into a main part and a negligible part and then in the study of their asymptotic behavior. We prove our main results in Sections 2.4 and 2.5. Finally, in Section 2.6, we recall the Malliavin integration by parts setting developed by [8] and used in [17], and give some representations of the transition density, its derivative, as well as its logarithm derivative. We also explicit the iterated Malliavin weights appearing in the expression of the derivative of the density.

## 2.2 Asymptotics for the density and its derivative

We consider the process  $(X_t^\beta)_{t \in [0,1]}$  solution to the stochastic equation (2.1.1) where  $(L_t)_{t \in [0,1]}$  is a pure jump Lévy process defined on a filtered probability space  $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \in [0,1]}, \mathbb{P})$ ,  $b$  is a real valued function and the parameter  $\beta = (\theta, \sigma)^T$  belongs to  $\mathbb{R} \times (0, \infty)$ . We assume that the following assumptions are fulfilled.

**H<sub>1</sub>**: (a) The function  $b$  has bounded derivatives up to order five with respect to both variables.

(b<sub>i</sub>) The Lévy process  $(L_t)_{t \in [0,1]}$  is given by

$$L_t = \int_0^t \int_{[-1,1]} z \{\bar{\mu}(ds, dz) - \bar{\nu}(ds, dz)\} + \int_0^t \int_{[-1,1]^c} z \bar{\mu}(ds, dz)$$

where  $\bar{\mu}$  is a Poisson random measure, with compensator  $\bar{\nu}(dt, dz) = dt \times F(z)dz$  where  $F(z)$  is given on  $\mathbb{R}^*$  by  $F(z) = \frac{1}{|z|^{\alpha+1}} \tau(z)$ ,  $\alpha \in (0, 2)$ . Moreover, we assume that  $\tau$  is a non negative smooth function equal to 1 on  $[-1,1]$ , vanishing on  $[-2,2]^c$  such that  $0 \leq \tau \leq 1$ .

(b<sub>ii</sub>) We assume that  $\forall p \geq 1$ ,

$$\int_{\mathbb{R}} \left| \frac{\tau'(u)}{\tau(u)} \right|^p \tau(u) du < \infty,$$

$$\int_{\mathbb{R}} \left| \frac{\tau''(u)}{\tau(u)} \right|^p \tau(u) du < \infty.$$

Under these assumptions,  $X_t^\beta$  admits a smooth density for  $t > 0$  (see Section 2.6), and we denote by  $p_t^\beta(x, y)$  the transition density of the Markov process  $(X_t^\beta)$ .

Throughout the paper, we will use the following notation. For a vector  $h \in \mathbb{R}^2$ ,  $h^T$  denotes the transpose of  $h$ , and  $|h|$  denotes the euclidean norm. For a function  $f$  defined on  $\mathbb{R} \times \mathbb{R}^2$  depending on both variables  $(x, \beta)$ , here  $\beta = (\theta, \sigma)^T \in \mathbb{R} \times (0, +\infty)$ , we denote by  $f'$  the derivative of  $f$  with respect to the variable  $x$ , by  $\partial_\theta f$  the derivative of  $f$  with respect to the parameter  $\theta$ , by  $\partial_\sigma f$  the derivative of  $f$  with respect to the parameter  $\sigma$ , and  $\nabla_\beta f = \begin{pmatrix} \partial_\theta f \\ \partial_\sigma f \end{pmatrix}$ .

The regularity assumption  $\mathbf{H}_1(a)$  on the drift coefficient  $b$  is a sufficient condition to obtain the representations of the density and its derivative. This assumption could be weakened but our methodology is based on the Malliavin calculus developed in [8] that requires strong regularity assumptions on the coefficients. From Theorem 10-3 in [8], the computation of the Malliavin operators for a stochastic differential equation needs coefficients with derivatives up to order three. As we iterate the Malliavin operators we need derivatives up to order five (see Lemma 2.6.2). Note that we relax the boundedness assumption on  $b$  assumed in [17].

**Remark 2.2.1.** *The assumptions on the Lévy measure are restrictive and one may expect that our results hold with a more general Lévy measure  $F(z) = \frac{1}{|z|^{\alpha+1}}g(z)$ , where  $g$  satisfies  $(b_{ii})$  and  $g(0) = 1$ . However in our approach the integrability assumptions for the tails of the Lévy process are crucial to ensure that our process belongs to the Malliavin space. Moreover, the exact  $\alpha$ -stable behavior of the Lévy measure around zero is also largely used (see Lemma 2.3.1) to study the Malliavin weights asymptotics. The truncation function  $\tau$  ensures both the integrability of  $|L_t|^p, \forall p \geq 1$ , and the exact  $\alpha$ -stable behavior around zero ( $\tau = 1$ ). It permits the careful study of each Malliavin terms appearing in the representation formulas (2.3.12) and (2.5.1). All these terms are not yet being in control without these restrictions on the Lévy measure.*

Our aim is to study the asymptotic behavior of  $p_{\frac{1}{n}}^\beta(x_0, u)$  (the density of  $X_{\frac{1}{n}}^\beta$ ) and its derivative with respect to the parameter  $\beta$ . To this end, we introduce the solution to the ordinary differential equation

$$(2.2.1) \quad \zeta_t^{n, \theta, x_0} = x_0 + \frac{1}{n} \int_0^t b(\zeta_s^{n, \theta, x_0}, \theta) ds \quad t \in [0, 1].$$

Heuristically,  $n^{1/\alpha}(X_{1/n}^\beta - \zeta_1^{n, \theta, x_0})$  is close to  $\sigma n^{1/\alpha} L_{1/n}$  and from assumption  $\mathbf{H}_1(b_i)$ , the rescaled process  $(n^{1/\alpha} L_{t/n})_{t \in [0, 1]}$  converges in distribution to an  $\alpha$ -stable process  $(L_t^\alpha)_{t \in [0, 1]}$  (see Section

2.3.1). Our first result shows that the density of  $\frac{n^{1/\alpha}}{\sigma}(X_{1/n}^\beta - \zeta_1^{n,\theta,x_0})$  converges to the density of  $L_1^\alpha$ , as  $n$  goes to infinity.

In what follows, we denote by  $\varphi_\alpha$  the density of  $L_1^\alpha$  where  $(L_t^\alpha)$  is an  $\alpha$ -stable process with Lévy measure  $\nu(dz) = \frac{1}{|z|^{\alpha+1}} \mathbf{1}_{z \neq 0} dz$ .

With these notations, we can state our main results. In view of statistical applications, we need some uniformity with respect to the parameter around the true value  $\beta = (\theta, \sigma)^T$  and consequently we study the asymptotic behavior of  $p_{\frac{1}{n}}^{\beta_n}$  where  $(\beta_n)_{n \geq 1} = ((\theta_n, \sigma_n)^T)_{n \geq 1}$  is a sequence converging to  $\beta$ .

**Theorem 2.2.1.** *Let  $(\zeta_t^{n,\theta,x_0})$  be the solution to the ordinary differential equation (2.2.1) and let  $(\beta_n)_{n \geq 1}$  be a sequence such that  $\beta_n \xrightarrow{n \rightarrow \infty} \beta$ . For all  $(x_0, u) \in \mathbb{R}^2$ ,*

1.  $\frac{\sigma_n}{n^{1/\alpha}} p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0}) \xrightarrow{n \rightarrow \infty} \varphi_\alpha(u)$ ,
2.  $\sup_{u \in \mathbb{R}} \sup_n \frac{\sigma_n}{n^{1/\alpha}} p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0}) < \infty$ ,

where  $\varphi_\alpha$  is the density of  $L_1^\alpha$ .

If the solution to the ordinary equation (2.2.1) is not given explicitly, we can approximate it by a numerical scheme. The previous convergence will be preserved if the order of the numerical scheme is sufficiently high. This is explain in the next remark.

**Remark 2.2.2.** *If we assume that the function  $b$  is of class  $\mathcal{C}^{1+k}$  with respect to  $x$  ( $k \geq 0$ ) and setting  $A(f) = bf'$  (and  $A^0(f) = f$ ) such that  $f(\zeta_t^{n,\theta,x_0}, \theta) = f(\zeta_0^{n,\theta,x_0}, \theta) + \int_0^t (Af)(\zeta_s^{n,\theta,x_0}, \theta) ds$ , we obtain*

$$\begin{aligned} \zeta_t^{n,\theta,x_0} &= x_0 + \frac{1}{n} \int_0^t A^0(b)(\zeta_s^{n,\theta,x_0}, \theta) ds \\ &= x_0 + \frac{t(A^0 b)(x_0, \theta)}{n} + \frac{1}{n} \int_0^t \int_0^{t_1} \frac{(Ab)}{n}(\zeta_{t_2}^{n,\theta,x_0}, \theta) dt_2 dt_1 \\ &= \zeta_t^{(k),n,\theta,x_0} + \frac{1}{n^{k+1}} \int_0^t \int_0^{t_1} \dots \int_0^{t_k} (A^k b)(\zeta_{t_k}^{n,\theta,x_0}, \theta) dt_{k+1} \dots dt_{t_1} \end{aligned}$$

with  $\zeta_t^{(k),n,\theta,x_0} = x_0 + \frac{t(A^0 b)(x_0, \theta)}{n} + \frac{t^2 (Ab)(x_0, \theta)}{2n^2} + \dots + \frac{t^k (A^{k-1} b)(x_0, \theta)}{k! n^k}$ , for  $k \geq 1$ , and  $\zeta_t^{(0),n,\theta,x_0} = x_0$ .

Assuming moreover that the function  $b$  has bounded derivatives, we deduce that

$$\left| \zeta_t^{n,\theta,x_0} - \zeta_t^{(k),n,\theta,x_0} \right| \leq \frac{C}{n^{k+1}}.$$



Then from the proof of Theorem 2.2.1, if  $n^{1/\alpha}/n^{k+1}$  goes to zero, we can replace  $\zeta_1^{n,\theta_n,x_0}$  by  $\zeta_1^{(k),n,\theta_n,x_0}$  in the statement of the theorem and we obtain

$$\frac{\sigma_n}{n^{1/\alpha}} p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{(k),n,\theta_n,x_0}) \xrightarrow{n \rightarrow \infty} \varphi_\alpha(u), \quad \text{if } k > \frac{1}{\alpha} - 1.$$

In particular, if  $\alpha > 1$ , the choice  $\zeta_1^{(0),n,\theta_n,x_0} = x_0$  is convenient as established in [17].

**Remark 2.2.3.** The results of Theorem 2.2.1 have been obtained by Kulik [47], using the parametric method.

The next result gives the asymptotic behavior of the derivatives of the density with respect to the parameters  $\theta$  and  $\sigma$ .

**Theorem 2.2.2.** Let  $(\beta_n)_{n \geq 1}$  be a sequence such that  $\beta_n \xrightarrow{n \rightarrow \infty} \beta$ . For all  $(x_0, u) \in \mathbb{R}^2$ ,

$$i) \quad \begin{aligned} \frac{\sigma_n^2}{n^{\frac{2}{\alpha}-1}} \partial_\theta p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0}) &\xrightarrow{n \rightarrow \infty} -\partial_\theta b(x_0, \theta) \times \varphi'_\alpha(u), \\ \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0}) &\xrightarrow{n \rightarrow \infty} -\varphi_\alpha(u) - u\varphi'_\alpha(u), \end{aligned}$$

$$ii) \quad \begin{aligned} \sup_{u \in \mathbb{R}} \sup_n \left| \frac{\sigma_n^2}{n^{\frac{2}{\alpha}-1}} \partial_\theta p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0}) \right| &< \infty, \\ \sup_{u \in \mathbb{R}} \sup_n \left| \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0}) \right| &< \infty. \end{aligned}$$

Considering the statistical experiment  $(\mathbb{R}^n, \mathcal{B}_n, \mathbb{P}_n^\beta)$  corresponding to the observation of  $(X_{i/n}^\beta)_{1 \leq i \leq n}$ , Theorems 2.2.1 and 2.2.2 permit to prove in [20] the  $L^2$ -regularity property of the transition density  $p_{1/n}^\beta(x, y)$ :

$$\sum_{i=1}^n \mathbb{E} \left[ \int_{\mathbb{R}} \left\{ p_{\frac{1}{n}}^{\beta+r_n h} \left( X_{\frac{i-1}{n}}^\beta, y \right)^{1/2} - p_{\frac{1}{n}}^\beta \left( X_{\frac{i-1}{n}}^\beta, y \right)^{1/2} - \frac{1}{2} h^T r_n \frac{\nabla_\beta p_{\frac{1}{n}}^\beta \left( X_{\frac{i-1}{n}}^\beta, y \right)}{\left( p_{\frac{1}{n}}^\beta \right)^{1/2} \left( X_{\frac{i-1}{n}}^\beta, y \right)} \right\}^2 dy \right] \xrightarrow{n \rightarrow \infty} 0,$$

$$\text{with rate } r_n = \begin{pmatrix} n^{\frac{1}{2}-\frac{1}{\alpha}} & 0 \\ 0 & n^{-\frac{1}{2}} \end{pmatrix}.$$

In this application, the sequence  $(\beta_n = (\theta_n, \sigma_n)^T)$  is  $(\beta + r_n h)$ . The  $L^2$ -regularity property (related to the  $L^2$ -differentiability of  $\beta \mapsto (p_{\frac{1}{n}}^\beta)^{1/2}$ ) is the first step to obtain an asymptotic expansion of the log-likelihood ratio  $\log \frac{d\mathbb{P}_n^{\beta+r_n h}}{d\mathbb{P}_n^\beta}(X_{1/n}^\beta, \dots, X_1^\beta)$  (see Theorem 2.1 in [20]) and to deduce the Local Asymptotic Mixed Normality property (see Jeganathan [39]). We proved that the LAMN property holds (Corollary 2.4 in [20]) for the parameter  $\beta$  with rate  $r_n$  and information matrix

$\mathcal{I} = \begin{pmatrix} \mathcal{I}_{11} & 0 \\ 0 & \mathcal{I}_{22} \end{pmatrix}$  where  $\mathcal{I}_{11} = \frac{1}{\sigma^2} \int_0^1 \partial_\theta b(X_s^\beta, \theta)^2 ds \int_{\mathbb{R}} \frac{\varphi'_\alpha(u)^2}{\varphi_\alpha(u)} du$  and  $\mathcal{I}_{22} = \frac{1}{\sigma^2} \int_{\mathbb{R}} \frac{(\varphi_\alpha(u) + u\varphi'_\alpha(u))^2}{\varphi_\alpha(u)} du$ . As

a consequence, we deduce that the estimation rate for  $\theta$  is  $n^{\frac{1}{2} - \frac{1}{\alpha}}$  and that the estimation rate for  $\sigma$  is the usual one  $n^{-\frac{1}{2}}$ . Moreover, the best asymptotic variance of any regular estimator of  $\beta$  (that converges in distribution with rate  $r_n$ ) is the inverse of the information matrix  $\mathcal{I}$ .

The proofs of Theorems 2.2.1 and 2.2.2 are based on the representations of the density and its derivative obtained by using Malliavin calculus and on the study of the asymptotic behavior of the Malliavin weights. This is given in the next sections.

## 2.3 Rescaling and representation of the density in small time

In this section, we give a representation of the density and identify in this representation the main terms and the remainder terms. This decomposition is a key step for the convergence study and is mainly based on the rescaling described in the next subsection.

### 2.3.1 Rescaling

We can observe that the process  $(n^{1/\alpha} L_{t/n})$  equals in law to a centered Lévy process with Lévy measure

$$(2.3.1) \quad F_n(z) = \frac{1}{|z|^{1+\alpha}} \tau\left(\frac{z}{n^{1/\alpha}}\right).$$

As mentioned previously, this clearly suggests that when  $n$  grows, the process  $(n^{1/\alpha} L_{t/n})$  converges to an  $\alpha$ -stable process. In the sequel, it will be convenient to construct a family of Lévy processes  $(L_t^n)_{n \geq 1}$  with the same law as  $(n^{1/\alpha} L_{t/n})$ , on a common probability space where the limiting  $\alpha$ -stable process exists as well, and where the convergence holds true in a path-wise sense, as done in [17].

Let us consider  $\mu^e(dt, dz, du)$  a Poisson measure on  $[0, \infty) \times \mathbb{R}^* \times [0, 1]$  with compensating measure  $v^e(dt, dz, du) = dt \frac{dz}{|z|^{1+\alpha}} du$  and we denote by  $\tilde{\mu}^e(dt, dz, du) = \mu^e(dt, dz, du) - v^e(dt, dz, du)$  the compensated Poisson random measure. This measure corresponds to the jump measure of an  $\alpha$ -stable process, where each jump is marked with an uniform variable on  $[0, 1]$ .

We define the Poisson measures  $\mu^{(n)}$ , for all  $n \geq 1$ , and  $\mu$  by setting :

$$\forall A \subset [0, \infty) \times \mathbb{R}, \quad \mu^{(n)}(A) = \int_{[0, \infty)} \int_{\mathbb{R}} \int_{[0, 1]} 1_A(t, z) 1_{\{u \leq \tau(\frac{z}{n^{1/\alpha}})\}} \mu^e(dt, dz, du),$$

$$\forall A \subset [0, \infty) \times \mathbb{R}, \quad \mu(A) = \int_{[0, \infty)} \int_{\mathbb{R}} \int_{[0, 1]} 1_A(t, z) \mu^e(dt, dz, du).$$

By simple computation, one can check that the compensator of the measure  $\mu^{(n)}(dt, dz)$  is  $v^{(n)}(dt, dz) = dt \times \tau(\frac{z}{n^{1/\alpha}}) \frac{dz}{|z|^{1+\alpha}} = dt \times F_n(z) dz$  and the compensator of  $\mu(dt, dz)$  is  $v(dt, dz) = dt \times \frac{dz}{|z|^{1+\alpha}}$ . Moreover, we note  $\tilde{\mu}^{(n)}(dt, dz) = \mu^{(n)}(dt, dz) - v^{(n)}(dt, dz)$  and  $\tilde{\mu}(dt, dz) = \mu(dt, dz) - v(dt, dz)$  the compensated Poisson random measures. Remark that since  $\tau(z) = 1$  for  $|z| \leq 1$ , the measures  $\mu^{(n)}(dt, dz)$  and  $\mu(dt, dz)$  coincide on the set  $\{(t, z) | t \in [0, 1], |z| \leq n^{1/\alpha}\}$ .

Now we define the stochastic processes associated to these random measures,

$$(2.3.2) \quad L_t^\alpha = \int_0^t \int_{[-1, 1]} z \tilde{\mu}(ds, dz) + \int_0^t \int_{[-1, 1]^c} z \mu(ds, dz),$$

$$(2.3.3) \quad L_t^n = \int_0^t \int_{[-n^{1/\alpha}, n^{1/\alpha}]} z \tilde{\mu}^{(n)}(ds, dz) + \int_0^t \int_{[-n^{1/\alpha}, n^{1/\alpha}]^c} z \mu^{(n)}(ds, dz).$$

By construction, the process  $(L_t^\alpha)$  is a centered  $\alpha$ -stable process and the process  $(L_t^n)$  is equal in law to the process  $(n^{1/\alpha} L_{t/n})_{t \in [0, 1]}$ , since they are based on random measures with the same compensator. Remark that the jumps of  $L_t^n$  with size smaller than  $n^{1/\alpha}$  exactly coincide with the jumps of  $L^\alpha$  with size smaller than  $n^{1/\alpha}$ . On the other hand, the process  $L^n$  has no jump with a size greater than  $2n^{1/\alpha}$ .

Using that the measures  $\mu$  and  $\mu^{(n)}$  coincide on the subsets of  $\{(t, z); |z| \leq n^{1/\alpha}\}$ , and the function  $\tau(\frac{z}{n^{1/\alpha}}) \frac{1}{|z|^{1+\alpha}} = \frac{1}{|z|^{1+\alpha}}$  is symmetric on  $|z| \leq n^{1/\alpha}$ , we can rewrite:

$$(2.3.4) \quad L_t^n = \int_0^t \int_{[-1, 1]} z \tilde{\mu}(ds, dz) + \int_0^t \int_{1 < |z| < n^{1/\alpha}} z \mu(ds, dz) + \int_0^t \int_{n^{1/\alpha} \leq |z| \leq 2n^{1/\alpha}} z \mu^{(n)}(ds, dz).$$

The following simple lemma gives a connection between  $L^n$  and the stable process  $L^\alpha$ .

**Lemma 2.3.1.** *On the event  $A_n = \{\mu(\{(t, z) | 0 \leq t \leq 1, |z| \geq n^{1/\alpha}\}) = 0\}$ , we have*

$$(2.3.5) \quad \mu^{(n)} = \mu, \quad L_t^n = L_t^\alpha,$$

and

$$(2.3.6) \quad \mathbb{P}(A_n) = 1 + O(1/n).$$

Furthermore, let  $(f_n)_{n \in \mathbb{N}}$  and  $f$  be measurable functions from  $\Omega \times [0, 1] \times \mathbb{R}$  to  $\mathbb{R}$  such that there exists  $C$  with  $\mathbb{P}(C) = 1$  and  $\forall \omega \in C, \forall s \in [0, 1], \forall |z| > 1$   $f_n(\omega, s, z) \xrightarrow{n \rightarrow \infty} f(\omega, s, z)$ . Then

$$(2.3.7) \quad \int_0^1 \int_{|z|>1} f_n(\omega, s, z) \mu^{(n)}(ds, dz) \xrightarrow[n.s.]{n \rightarrow \infty} \int_0^1 \int_{|z|>1} f(\omega, s, z) \mu(ds, dz).$$

Moreover, we have  $\sup_{t \in [0, 1]} |L_t^n - L_t^\alpha| \xrightarrow[n.s.]{n \rightarrow \infty} 0$ .

**Proof.** We know that the measures  $\mu^{(n)}$  and  $\mu$  coincide on the set  $\{(s, z) | s \in [0, 1], |z| \leq n^{1/\alpha}\}$ , and by comparison of the representations (2.3.2) and (2.3.4), it is clear that equation (2.3.5) holds true on the event that the supports of the random measure  $\mu$  and  $\mu^{(n)}$  do not intersect  $\{(t, z) | 0 \leq t \leq 1, |z| \geq n^{1/\alpha}\}$ . On the other hand, the support of  $\mu^{(n)}$  is included in the support of  $\mu$ , and thus (2.3.5) is true on the event

$$A_n = \{\mu(\{(t, z) | 0 \leq t \leq 1, |z| \geq n^{1/\alpha}\}) = 0\}.$$

The probability of the latter event is  $e^{-2/\alpha n}$  which converges to 1 at rate  $1/n$  as stated. Then we also get (2.3.6).

Let  $A = \cup_{n=1}^{\infty} A_n$ , we get that  $P(A) = 1$  since  $A_n \subseteq A_{n+1}$  for each  $n \in \mathbb{N}$  and (2.3.6) holds. Thus, for all  $\omega \in A \cap C, \exists n_0(\omega) \geq 1, \forall n \geq n_0(\omega), \mu^{(n)} = \mu$  and  $f_n(\omega, s, z) \rightarrow f(\omega, s, z) \forall s \in [0, 1], \forall |z| > 1$ . Then we deduce that

$$\int_0^1 \int_{|z|>1} f_n(\omega, s, z) \mu^{(n)}(ds, dz) \xrightarrow[n.s.]{n \rightarrow \infty} \int_0^1 \int_{|z|>1} f(\omega, s, z) \mu(ds, dz).$$

We also get  $\sup_{t \in [0, 1]} |L_t^n - L_t^\alpha| \xrightarrow{n \rightarrow \infty} 0$ . ■

### 2.3.2 Representation of the density in small time and first approximation

We introduce the process  $(\bar{Y}_t^{n, \beta, x_0})_{t \in [0, 1]}$  given by

$$(2.3.8) \quad \bar{Y}_t^{n, \beta, x_0} = x_0 + \frac{1}{n} \int_0^t b(\bar{Y}_s^{n, \beta, x_0}, \theta) ds + \frac{\sigma}{n^{1/\alpha}} L_t^n \quad t \in [0, 1],$$

where  $(L_t^n)$  is defined by (2.3.4) and is such that  $\frac{1}{n^{1/\alpha}}(L_t^n)$  equals in law to  $(L_{t/n})$ . By construction, the process  $(X_{\frac{t}{n}}^\beta)_{t \in [0, 1]}$  equals in law  $(\bar{Y}_t^{n, \beta, x_0})_{t \in [0, 1]}$ . Let  $q^{n, \beta, x_0}$  be the density of  $\bar{Y}_1^{n, \beta, x_0}$  then the connection between the densities of  $X_{\frac{1}{n}}^\beta$  and  $\bar{Y}_1^{n, \beta, x_0}$  is given by

$$(2.3.9) \quad p_{1/n}^\beta(x_0, x) = q^{n, \beta, x_0}(x).$$

Using the Malliavin calculus for jump processes recalled in Section 2.6, we get a representation of the density of  $X_{\frac{1}{n}}^\beta$  involving some random variable  $\mathcal{H}_{\bar{Y}_1^{n,\beta,x_0}}(1)$  (Malliavin weight). Moreover, we obtain an asymptotic expansion of this weight that leads to the decomposition of the density into a main part and a remainder part.

To state our next result we define  $(\epsilon_s^n)_{s \in [0,1]}$  by

$$(2.3.10) \quad \epsilon_s^n = \exp\left(\frac{1}{n} \int_0^s b'(\bar{Y}_u^{n,\beta,x_0}, \theta) du\right)$$

and the function  $\rho^n$

$$(2.3.11) \quad \rho^n(z) = \begin{cases} z^4 & \text{if } |z| < 1 \\ \zeta(z) & \text{if } 1 \leq |z| \leq 2 \\ z^2 \tau\left(\frac{z}{2n^{1/\alpha}}\right) & \text{if } |z| > 2, \end{cases}$$

where  $\tau$  is defined in assumption  $\mathbf{H}_1(b_i)$ , and  $\zeta$  is a non negative function belonging to  $\mathbf{C}^\infty$  such that  $\rho^n$  belongs to  $\mathbf{C}^\infty$ . The function  $\rho^n$  is an auxiliary function related the the Malliavin calculus developed in [8]. In our setting, the above choice (not unique) is convenient (see Remark 2.4.1).

**Theorem 2.3.1.** *Under the assumption  $\mathbf{H}_1$ , we have*

$$(2.3.12) \quad p_{\frac{1}{n}}^\beta(x_0, u) = q^{n,\beta,x_0}(u) = \mathbb{E}(1_{\{\bar{Y}_1^{n,\beta,x_0} \geq u\}} \mathcal{H}_{\bar{Y}_1^{n,\beta,x_0}}(1)),$$

with

$$(2.3.13) \quad \mathcal{H}_{\bar{Y}_1^{n,\beta,x_0}}(1) = \frac{1}{\sigma} n^{1/\alpha} \left[ \hat{\mathcal{H}}_{1,\beta}^n(1) + \hat{\mathcal{H}}_{2,\beta}^n(1) \right] + \mathcal{R}_{1,\beta}^n(1) + \mathcal{R}_{2,\beta}^n(1) + \mathcal{R}_{3,\beta}^n(1).$$

The main terms  $\hat{\mathcal{H}}_{1,\beta}^n(1), \hat{\mathcal{H}}_{2,\beta}^n(1)$  are given by

$$(2.3.14) \quad \hat{\mathcal{H}}_{1,\beta}^n(1) = \left[ \frac{\int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-3} \rho^n(z) (\rho^n)'(z) \mu^{(n)}(ds, dz)}{\epsilon_1^n \left[ \int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-2} \rho^n(z) \mu^{(n)}(ds, dz) \right]^2} \right],$$

$$(2.3.15) \quad \hat{\mathcal{H}}_{2,\beta}^n(1) = \left[ - \frac{\int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-1} \left[ (\rho^n)'(z) - \frac{1+\alpha}{z} \rho^n(z) \right] \mu^{(n)}(ds, dz)}{\epsilon_1^n \int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-2} \rho^n(z) \mu^{(n)}(ds, dz)} \right],$$

and the remainder terms satisfy for any compact subset  $\mathbf{Q} \subset \mathbb{R} \times (0, \infty)$

$$(2.3.16) \quad \forall p \geq 2, \quad \mathbb{E} \sup_{\beta \in \mathbf{Q}} \left| \mathcal{R}_{1,\beta}^n(1) \right|^p \leq \frac{C}{n}, \quad \sup_{\beta \in \mathbf{Q}} |\mathcal{R}_{2,\beta}^n(1)| \leq \frac{C}{n}, \quad \sup_{\beta \in \mathbf{Q}} |\mathcal{R}_{3,\beta}^n(1)| \leq \frac{C}{n},$$

where  $C$  is some deterministic constant.

Let us heuristically explain how this decomposition permits to establish the result of Theorem 2.2.1. Let  $(\beta_n)$  be a sequence converging to  $\beta$ . First, from Lemma 2.4.1,  $n^{1/\alpha}(\bar{Y}_1^{n,\beta_n,x_0} - \zeta_1^{n,\theta_n,x_0})$  converges almost surely to  $\sigma L_1^\alpha$ , this permits to check that  $1_{\{n^{1/\alpha}(\bar{Y}_1^{n,\beta_n,x_0} - \zeta_1^{n,\theta_n,x_0}) \geq \sigma_n v\}}$  converges to  $1_{\{L_1^\alpha \geq v\}}$ . From Theorem 2.3.1, we deduce that  $\frac{\sigma_n}{n^{1/\alpha}} p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{\sigma_n}{n^{1/\alpha}} u + \zeta_1^{n,\theta_n,x_0})$  is close to  $\mathbb{E}1_{\{L_1^\alpha \geq u\}}[\hat{\mathcal{H}}_{1,\beta_n}^n(1) + \hat{\mathcal{H}}_{2,\beta_n}^n(1)]$  and it remains to study the limit of the main terms.

We can see from the definition of  $\rho^n$  that  $\rho^n(z) \xrightarrow{n \rightarrow \infty} \rho(z)$  where

$$(2.3.17) \quad \rho(z) = \begin{cases} z^4 & \text{if } |z| < 1 \\ \zeta(z) & \text{if } 1 \leq |z| \leq 2, \\ z^2 & \text{if } |z| > 2. \end{cases}$$

Combining this with Lemma 2.3.1, it permits to establish the almost sure convergence of the main terms:

$$(2.3.18) \quad \hat{\mathcal{H}}_{1,\beta_n}^n(1) \xrightarrow[a.s.]{n \rightarrow \infty} \mathcal{H}_{1,L^\alpha}(1),$$

$$(2.3.19) \quad \hat{\mathcal{H}}_{2,\beta_n}^n(1) \xrightarrow[a.s.]{n \rightarrow \infty} \mathcal{H}_{2,L^\alpha}(1),$$

where  $\mathcal{H}_{1,L^\alpha}(1), \mathcal{H}_{2,L^\alpha}(1)$  are given by

$$(2.3.20) \quad \mathcal{H}_{1,L^\alpha}(1) = \frac{\int_0^1 \int_{\mathbb{R}} \rho(z) \rho'(z) \mu(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho(z) \mu(ds, dz) \right]^2},$$

$$(2.3.21) \quad \mathcal{H}_{2,L^\alpha}(1) = - \frac{\int_0^1 \int_{\mathbb{R}} \left[ \rho'(z) - \frac{1+\alpha}{z} \rho(z) \right] \mu(ds, dz)}{\int_0^1 \int_{\mathbb{R}} \rho(z) \mu(ds, dz)}.$$

Moreover, the limit weight  $\mathcal{H}_{1,L^\alpha}(1) + \mathcal{H}_{2,L^\alpha}(1)$  can be interpreted as a Malliavin weight (see (2.4.29)) and we have the following representation for the density of  $L_1^\alpha$

$$\varphi_\alpha(u) = \mathbb{E}1_{[u,\infty)}(L_1^\alpha) [\mathcal{H}_{1,L^\alpha}(1) + \mathcal{H}_{2,L^\alpha}(1)].$$

This suggests that, as  $n$  goes to infinity,  $\frac{\sigma_n}{n^{1/\alpha}} p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{\sigma_n}{n^{1/\alpha}} u + \zeta_1^{n,\theta_n,x_0})$  is close to  $\varphi_\alpha$ .

This is rigorously established in the next section.

## 2.4 Proof of Theorems 2.3.1 and 2.2.1

### 2.4.1 Proof of Theorem 2.3.1

The proof is based on the Malliavin calculus developed in Section 2.6. We recall that  $q^{n,\beta,x_0}$  is the density of  $\bar{Y}_1^{n,\beta,x_0}$  and that the connection between the densities of  $X_{\frac{1}{n}}^\beta$  and  $\bar{Y}_1^{n,\beta,x_0}$  is given by  $p_{1/n}^\beta(x_0, x) = q^{n,\beta,x_0}(x)$ .

We use the framework of Sections 2.6.1 and 2.6.2, with  $g(z) := F_n(z) = \frac{1}{|z|^{1+\alpha}} \tau(\frac{z}{n^{1/\alpha}})$  and with the auxiliary function  $\rho^n$  defined by (2.3.11) such that it satisfies all conditions of Section 2.6.1. From the assumptions on  $\tau$ , we can easily deduce that

$$z^2 \tau\left(\frac{z}{2n^{1/\alpha}}\right) = \begin{cases} z^2 & \text{if } 2 \leq |z| \leq 2n^{1/\alpha} \\ 0 & \text{if } |z| > 4n^{1/\alpha}. \end{cases}$$

Moreover, we recall that  $\rho^n(z) \xrightarrow{n \rightarrow \infty} \rho(z)$  where  $\rho$  is defined by (2.3.17). Note that from the definitions of  $\rho^n$  and  $\rho$ , we can easily see that  $\rho^n(z) = \rho(z)$  if  $|z| \leq 2n^{1/\alpha}$ .

**Remark 2.4.1.** *The choice of the auxiliary function  $\rho^n$  for  $|z| < 1$  ensures that the non-degeneracy condition (2.6.6) is satisfied. It will appear later that the choice of the auxiliary function  $\rho^n$  for  $|z| > 2$  permits to obtain Malliavin weights sufficiently integrable to compensate the lack of integrability of  $L_1^\alpha$  (see Remark 2.5.1 below).*

The equation (2.3.8) (defining  $(\bar{Y}_t^{n,\beta,x_0})$ ) is a particular case of (2.6.1) with the function  $a$  and the constant  $c$  given explicitly as

$$(2.4.1) \quad a(x, \theta) = \frac{1}{n} b(x, \theta), \quad c = \frac{1}{n^{1/\alpha}}.$$

Under the assumptions  $\mathbf{H}_1$ , we can apply the results of Theorem 2.6.1 to  $\bar{Y}_1^{n,\beta,x_0}$ . The non-degeneracy assumption is verified by the choice of  $\rho^n(z)$  near zero (see (2.3.11)). Let us denote by  $U_t^{n,\beta} = \Gamma[\bar{Y}_t^{n,\beta,x_0}, \bar{Y}_t^{n,\beta,x_0}]$  and  $W_t^{n,\beta} = \Gamma[\bar{Y}_t^{n,\beta,x_0}, U_t^{n,\beta}]$ , then we obtain:

$$p_{\frac{1}{n}}^\beta(x_0, u) = q^{n,\beta,x_0}(u) = \mathbb{E}(1_{\{\bar{Y}_1^{n,\beta,x_0} \geq u\}} \mathcal{H}_{\bar{Y}_1^{n,\beta,x_0}}(1));$$

with

$$(2.4.2) \quad \mathcal{H}_{\bar{Y}_1^{n,\beta,x_0}}(1) = \frac{W_1^{n,\beta}}{(U_1^{n,\beta})^2} - 2 \frac{L\bar{Y}_1^{n,\beta,x_0}}{U_1^{n,\beta}}.$$

Applying the results of Theorem 2.6.1 and solving the linear equations (2.6.8)-(2.6.10) (with  $a$  and  $c$  given by (2.4.1)) we get,

$$(2.4.3) \quad U_1^{n,\beta} = \frac{(\epsilon_1^n)^2 \sigma^2}{n^{2/\alpha}} \int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-2} \rho^n(z) \mu^{(n)}(ds, dz),$$

$$(2.4.4) \quad L(\bar{Y}_1^{n,\beta,x_0}) = \frac{\epsilon_1^n}{2n} \int_0^1 b''(\bar{Y}_s^{n,\beta,x_0}, \theta)(U_s^{n,\beta})(\epsilon_s^n)^{-1} ds \\ + \frac{\sigma \epsilon_1^n}{2n^{1/\alpha}} \int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-1} [(\rho^n)'(z) + \frac{F_n'(z)}{F_n(z)} \rho^n(z)] \mu^{(n)}(ds, dz),$$

$$(2.4.5) \quad W_1^{n,\beta} = \frac{\sigma^3 (\epsilon_1^n)^3}{n^{3/\alpha}} \int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-3} (\rho^n)'(z) \rho^n(z) \mu^{(n)}(ds, dz) + \frac{2(\epsilon_1^n)^3}{n} \int_0^1 b''(\bar{Y}_s^{n,\beta,x_0}, \theta)(U_s^{n,\beta})^2 (\epsilon_s^n)^{-3} ds.$$

Recalling that  $F_n(z) = \frac{1}{|z|^{1+\alpha}} \tau\left(\frac{z}{n^{1/\alpha}}\right)$  (see (2.3.1)), then  $\frac{F_n'(z)}{F_n(z)} = -\frac{1+\alpha}{z} + \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \frac{1}{n^{1/\alpha}}$  if  $|z| \leq 2n^{1/\alpha}$ . Based on these expressions and (2.4.2) we deduce, after some calculus, the decomposition (2.3.13), where the remainder terms are given by,

$$(2.4.6) \quad \mathcal{R}_{1,\beta}^n(1) = -\frac{\int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-1} \rho^n(z) \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \mu^{(n)}(ds, dz)}{\sigma \epsilon_1^n \int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-2} \rho^n(z) \mu^{(n)}(ds, dz)},$$

$$(2.4.7) \quad \mathcal{R}_{2,\beta}^n(1) = \frac{2(\epsilon_1^n)^3 \int_0^1 b''(\bar{Y}_s^{n,\beta,x_0}, \theta)(U_s^{n,\beta})^2 (\epsilon_s^n)^{-3} ds}{n(U_1^{n,\beta})^2},$$

$$(2.4.8) \quad \mathcal{R}_{3,\beta}^n(1) = -\frac{(\epsilon_1^n) \int_0^1 b''(\bar{Y}_s^{n,\beta,x_0}, \theta)(U_s^{n,\beta})(\epsilon_s^n)^{-1} ds}{2n(U_1^{n,\beta})}.$$

We can establish the following bounds for the remainder terms.

For  $\mathcal{R}_{1,\beta}^n(1)$ , since  $(\epsilon_s^n)$  is lower and upper bounded uniformly with respect to  $\beta$  (recall (2.3.10)), and since  $\tau'(z) = 0$  on  $[-1, 1]$  then for  $M$  a positive constant we have

$$(2.4.9) \quad \sup_{\beta \in \mathcal{Q}} \left| \mathcal{R}_{1,\beta}^n(1) \right| \leq M \left( \frac{\int_0^1 \int_{|z|>2} z^2 \left| \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right| \mu^{(n)}(ds, dz)}{\int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz)} \right).$$

Assume that there exists a jump of the Lévy process  $L_1^n$  in  $[-2n^{1/\alpha}, -n^{1/\alpha}] \cup (n^{1/\alpha}, 2n^{1/\alpha}]$ , then we get  $\int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz) > n^{2/\alpha}$ . Thus,

$$(2.4.10) \quad \frac{\int_0^1 \int_{|z|>2} z^2 \left| \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right| \mu^{(n)}(ds, dz)}{\int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz)} \leq \int_0^1 \int_{|z|>2} \left( \frac{z}{n^{1/\alpha}} \right)^2 \left| \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right| \mu^{(n)}(ds, dz).$$



Assume that there are no jumps in  $[-2n^{1/\alpha}, -n^{1/\alpha}) \cup (n^{1/\alpha}, 2n^{1/\alpha}]$ , since  $\tau(z/n^{1/\alpha}) = 1$  if  $|z| \leq n^{1/\alpha}$ , then  $\tau'(z/n^{1/\alpha}) = 0$  and as a consequence, the right-hand side of (2.4.9) equals zero in this case.

In both cases, for any  $p \geq 1$

$$(2.4.11) \quad \mathbb{E} \left[ \frac{\int_0^1 \int_{|z|>2} z^2 \left| \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right| \mu^{(n)}(ds, dz)}{\int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz)} \right]^p \leq \mathbb{E} \left[ \int_0^1 \int_{|z|>2} \left( \frac{z}{n^{1/\alpha}} \right)^2 \left| \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right| \mu^{(n)}(ds, dz) \right]^p.$$

Now from  $\mu^{(n)}(ds, dz) = \tilde{\mu}^{(n)}(ds, dz) + v^{(n)}(ds, dz)$ , by convexity inequality, we have for  $C(p)$  a positive constant

$$(2.4.12) \quad \begin{aligned} & \mathbb{E} \left[ \int_0^1 \int_{|z|>2} \left( \frac{z}{n^{1/\alpha}} \right)^2 \left| \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right| \mu^{(n)}(ds, dz) \right]^p \\ & \leq C(p) \mathbb{E} \left[ \int_0^1 \int_{|z|>2} \left( \frac{z}{n^{1/\alpha}} \right)^2 \left| \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right| \tilde{\mu}^{(n)}(ds, dz) \right]^p \\ & \quad + C(p) \left[ \int_0^1 \int_{|z|>2} \left( \frac{z}{n^{1/\alpha}} \right)^2 \left| \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right| v^{(n)}(ds, dz) \right]^p. \end{aligned}$$

Using Kunita's first inequality (see Theorem 4.4.23 in [6]) for  $p \geq 2$ , there exists a constant  $D(p) > 0$  such that

$$\begin{aligned} & \mathbb{E} \left[ \int_0^1 \int_{|z|>2} \left( \frac{z}{n^{1/\alpha}} \right)^2 \left| \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right| \tilde{\mu}^{(n)}(ds, dz) \right]^p \\ & \leq D(p) \left[ \int_0^1 \int_{|z|>2} \left( \frac{z}{n^{1/\alpha}} \right)^4 \left| \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right|^2 v^{(n)}(ds, dz) \right]^{p/2} \\ & \quad + D(p) \left[ \int_0^1 \int_{|z|>2} \left( \frac{z}{n^{1/\alpha}} \right)^{2p} \left| \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right|^p v^{(n)}(ds, dz) \right] \\ & = \frac{D(p)}{n^{p/2}} \left[ \int_0^1 \int_1^2 \left( \frac{1}{u^{\alpha-3}} \left| \frac{\tau'(u)}{\tau(u)} \right|^2 \tau(u) \right) dud s \right]^{p/2} + \left[ \frac{D(p)}{n} \int_0^1 \int_1^2 \left( \frac{1}{u^{\alpha+1-2p}} \left| \frac{\tau'(u)}{\tau(u)} \right|^p \tau(u) \right) dud s \right], \end{aligned}$$

where we have used that  $v^{(n)}(ds, dz) = ds \frac{1}{|z|^{1+\alpha}} \tau(z/n^{1/\alpha}) dz$  and the change of variable  $u = \frac{z}{n^{1/\alpha}}$ .

Moreover, we have

$$(2.4.13) \quad \begin{aligned} \left[ \int_0^1 \int_{|z|>2} \left( \frac{z}{n^{1/\alpha}} \right)^2 \left| \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right| v^{(n)}(ds, dz) \right]^p &= \left[ \int_0^1 \int_{|z|>2} \left( \frac{z}{n^{1/\alpha}} \right)^2 \left| \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right| \frac{1}{|z|^{1+\alpha}} \tau(z/n^{1/\alpha}) dz ds \right]^p \\ &= \left[ \frac{1}{n} \int_0^1 \int_1^2 \frac{1}{u^{\alpha-1}} \left| \frac{\tau'(u)}{\tau(u)} \right| \tau(u) dud s \right]^p. \end{aligned}$$

Under the assumption  $\mathbf{H}_1(b_{ii})$ , we deduce  $\mathbb{E} \sup_{\beta \in Q} \left| \mathcal{R}_{1,\beta}^n(1) \right|^p \leq C/n, \forall p \geq 2$ .

Using that  $b$  has bounded derivatives and that  $\sup_{\beta \in Q} \sup_{0 \leq s \leq 1} \frac{U_s^{n,\beta}}{U_1^{n,\beta}}$  is bounded, the remainder

terms  $R_{2,\beta}^n(1), R_{3,\beta}^n(1)$  satisfy the upper bound

$$\sup_{\beta \in Q} |R_{2,\beta}^n(1)| \leq \frac{C}{n}, \quad \sup_{\beta \in Q} |R_{3,\beta}^n(1)| \leq \frac{C}{n},$$

where  $C$  is some deterministic constant.

### 2.4.2 Proof of Theorem 2.2.1

We first prove that  $n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})$  is close to a stable Lévy process.

**Lemma 2.4.1.** *Let  $(\zeta_t^{n,\theta,x_0})$  be the solution to the ordinary differential equation (2.2.1), then*

$$(2.4.14) \quad n^{1/\alpha}(\bar{Y}_1^{n,\beta_n,x_0} - \zeta_1^{n,\theta_n,x_0}) \xrightarrow[a.s.]{n \rightarrow \infty} \sigma L_1^\alpha,$$

for any sequence  $(\beta_n)$  converging to  $\beta$ , and this convergence is uniform with respect to  $x_0$ .

**Proof.** Using (2.3.8) and the boundedness of  $b'$ , we have for  $t \in [0, 1]$

$$\begin{aligned} \left| n^{1/\alpha}(\bar{Y}_t^{n,\beta_n,x_0} - \zeta_t^{n,\theta_n,x_0}) - \sigma L_t^\alpha \right| &= \left| \frac{1}{n} \int_0^t n^{1/\alpha} \left[ b(\bar{Y}_s^{n,\beta_n,x_0}, \theta_n) - b(\zeta_s^{n,\theta_n,x_0}, \theta_n) \right] ds + [\sigma_n L_t^n - \sigma L_t^\alpha] \right| \\ &\leq \frac{1}{n} \int_0^t \|b'\|_\infty \left[ \left| n^{1/\alpha}(\bar{Y}_s^{n,\beta_n,x_0} - \zeta_s^{n,\theta_n,x_0}) - \sigma L_s^\alpha \right| \right] ds + \frac{\sigma \|b'\|_\infty}{n} \int_0^1 |L_s^\alpha| ds + \sup_{t \in [0,1]} |\sigma_n L_t^n - \sigma L_t^\alpha|, \end{aligned}$$

where  $\|b'\|_\infty = \sup_{x,\theta} |b'(x,\theta)|$ . Applying Gronwall's Lemma, we get

$$(2.4.15) \quad \sup_{x_0} \left| n^{1/\alpha}(\bar{Y}_1^{n,\beta_n,x_0} - \zeta_1^{n,\theta_n,x_0}) - \sigma L_1^\alpha \right| \leq C \left[ \frac{\sigma}{n} \int_0^1 |L_s^\alpha| ds + \sup_{t \in [0,1]} |\sigma_n L_t^n - \sigma L_t^\alpha| \right],$$

where  $C$  is a positive constant.

From Lemma 2.3.1, we have  $\sup_{t \in [0,1]} |L_t^n - L_t^\alpha| \xrightarrow{a.s.} 0$  and we deduce  $\sup_{t \in [0,1]} |\sigma_n L_t^n - \sigma L_t^\alpha| \xrightarrow{a.s.} 0$ . Since  $t \mapsto L_t^\alpha$  is càdlàg, we get  $\int_0^1 |L_s^\alpha| ds < \infty$  a.s., then  $\frac{\sigma}{n} \int_0^1 |L_s^\alpha| ds \xrightarrow{a.s.} 0$  and we get the result of Lemma 2.4.1.  $\blacksquare$

We now proceed to the proof of Theorem 2.2.1. From (2.3.9) and Theorem 2.3.1, we have the representation

$$(2.4.16) \quad \frac{\sigma_n}{n^{1/\alpha}} p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0}) = \frac{\sigma_n}{n^{1/\alpha}} q^{n,\beta_n,x_0} \left( \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0} \right) = \mathbb{E} \left( \mathbf{1}_{\{\bar{Y}_1^{n,\beta_n,x_0} \geq \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0}\}} \frac{\sigma_n}{n^{1/\alpha}} \mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}^n(1) \right),$$

where  $\frac{\sigma_n}{n^{1/\alpha}} \mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}^n(1) = \hat{\mathcal{H}}_{1,\beta_n}^n(1) + \hat{\mathcal{H}}_{2,\beta_n}^n(1) + \frac{\sigma_n}{n^{1/\alpha}} \mathcal{R}_{1,\beta_n}^n(1) + \frac{\sigma_n}{n^{1/\alpha}} \mathcal{R}_{2,\beta_n}^n(1) + \frac{\sigma_n}{n^{1/\alpha}} \mathcal{R}_{3,\beta_n}^n(1)$ , with  $\hat{\mathcal{H}}_{1,\beta_n}^n(1), \hat{\mathcal{H}}_{2,\beta_n}^n(1)$  given by (2.3.14), (2.3.15) and  $\mathcal{R}_{1,\beta_n}^n(1), \mathcal{R}_{2,\beta_n}^n(1), \mathcal{R}_{3,\beta_n}^n(1)$  satisfy the bounds (2.3.16).

Since  $b'$  is bounded and  $\rho^n(z) \geq 0$ , we deduce the upper bounds

$$(2.4.17) \quad \left| \hat{\mathcal{H}}_{1,\beta_n}^n(1) \right| \leq C^* \left[ \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) |(\rho^n)'(z)| \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^2} \right],$$

$$(2.4.18) \quad \left| \hat{\mathcal{H}}_{2,\beta_n}^n(1) \right| \leq C^* \left[ \frac{\int_0^1 \int_{\mathbb{R}} \left[ |\rho^{n'}(z)| + \frac{1+\alpha}{|z|} \rho^n(z) \right] \mu^{(n)}(ds, dz)}{\int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz)} \right],$$

for some constant  $C^* > 0$ .

We now show that  $\sup_n \left| \hat{\mathcal{H}}_{1,\beta_n}^n(1) \right|^p$  and  $\sup_n \left| \hat{\mathcal{H}}_{2,\beta_n}^n(1) \right|^p$  are integrable  $\forall p \geq 1$ . The proof will be divided into the two following steps.

**Step 1:** We show that the right-hand side of (2.4.17) is bounded by a random variable independent of  $n$  and belonging to  $\cap_{p \geq 1} \mathbf{L}^p$ . In fact, since the measures  $\mu^{(n)}$  and  $\mu$  coincide on the set  $\{(s, z) | s \in [0, 1], |z| \leq n^{1/\alpha}\}$ , and  $\rho^n(z) = \rho(z)$  on the support of the Poisson measure  $\mu^{(n)}$ , we have

$$(2.4.19) \quad \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) |(\rho^n)'(z)| \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^2} \leq \left( \frac{\int_0^1 \int_{|z| \leq 2} |\rho'(z)| \rho(z) \mu(ds, dz)}{\left( \int_0^1 \int_{|z| \leq 2} \rho(z) \mu(ds, dz) \right)^2} + \frac{\int_0^1 \int_{|z| > 2} 2|z|^3 \mu^{(n)}(ds, dz)}{\left( \int_0^1 \int_{|z| > 2} z^2 \mu^{(n)}(ds, dz) \right)^2} \right).$$

We first consider the first term in the right-hand side of (2.4.19). Using that  $\rho, \rho'$  belongs to  $\cap_{p \geq 1} \mathbf{L}^p(1_{|z| \leq 2} |z|^{-1-\alpha} dz)$ , we get

$$(2.4.20) \quad \mathbb{E} \left[ \left( \int_0^1 \int_{|z| \leq 2} |\rho'(z)| \rho(z) \mu(ds, dz) \right)^p \right] < \infty, \quad \forall p \geq 1.$$

On the other hand, since  $\rho$  satisfies the non degeneracy assumption (2.6.6),  $[\int_0^1 \int_{|z| \leq 2} \rho(z) \mu(ds, dz)]^{-1}$  belongs to  $\cap_{p \geq 1} \mathbf{L}^p$  [see [17, Theorem 4 p.2323]], we deduce that the first term of (2.4.19) belongs to  $\cap_{p \geq 1} \mathbf{L}^p$ , moreover, it does not depend on  $n$ .

Turning to the second term in the right-hand side of (2.4.19), since  $v^{(n)}(\{(t, z) | 0 \leq t \leq 1, |z| > 2\}) < \infty$ , we have the following representation [see Chapter VI in [16]]

$$(2.4.21) \quad \begin{aligned} \int_0^1 \int_{|z| > 2} 2|z|^3 \mu^{(n)}(ds, dz) &= \sum_{i=1}^{N_1} 2|Z_i|^3 \quad \text{a.s.}, \\ \int_0^1 \int_{|z| > 2} z^2 \mu^{(n)}(ds, dz) &= \sum_{i=1}^{N_1} Z_i^2 \quad \text{a.s.}, \end{aligned}$$

where  $N = (N_t)_{1 \geq t \geq 0}$  is a Poisson process with intensity  $\lambda_n = \int_{|z| > 2} F_n(z) dz < \infty$ , and  $(Z_i)_{i \geq 0}$  are i.i.d. random variables independent of  $N$  with probability measure  $\frac{F_n(z) 1_{|z| > 2} dz}{\lambda_n}$ . Thus,

$$\frac{\int_0^1 \int_{|z| > 2} 2|z|^3 \mu^{(n)}(ds, dz)}{\left( \int_0^1 \int_{|z| > 2} z^2 \mu^{(n)}(ds, dz) \right)^2} = \frac{\sum_{i=1}^{N_1} 2|Z_i|^3}{\left( \sum_{i=1}^{N_1} Z_i^2 \right)^2} \leq \frac{\sum_{i=1}^{N_1} 2|Z_i|^3}{\sum_{i=1}^{N_1} Z_i^4} \leq 1.$$

where we used  $Z_i^2 \geq 0$  and  $|Z_i| > 2$ . We deduce that  $\sup_n \left| \hat{\mathcal{H}}_{1,\beta_n}^n(1) \right|^p$  is integrable  $\forall p \geq 1$ .

**Step 2:** We show that  $\sup_n \left| \hat{\mathcal{H}}_{2,\beta_n}^n(1) \right|^p$  is integrable.

Using the definitions of  $\rho^n$  (recall (2.3.11)),  $\rho$  (recall (2.3.17)) and  $\rho^n = \rho$  on the support of the Poisson measure  $\mu^{(n)}$  (see Section 2.3.2), we have

$$(2.4.22) \quad \frac{\int_0^1 \int_{\mathbb{R}} \left[ |\rho^{n'}(z)| + \frac{1+\alpha}{|z|} \rho^n(z) \right] \mu^{(n)}(ds, dz)}{\int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz)} \leq \frac{\int_0^1 \int_{|z| \leq 2} \left( |\rho(z)'| + \rho(z) \frac{1+\alpha}{|z|} \right) \mu(ds, dz)}{\int_0^1 \int_{|z| \leq 2} \rho(z) \mu(ds, dz)} + \frac{\int_0^1 \int_{|z| > 2} (3+\alpha) |z| \mu^{(n)}(ds, dz)}{\int_0^1 \int_{|z| > 2} z^2 \mu^{(n)}(ds, dz)}$$

where we used  $\int_0^1 \int_{|z| \leq 2} \rho(z) \mu(ds, dz) \geq 0$ ,  $\int_0^1 \int_{|z| > 2} \rho^n(z) \mu^{(n)}(ds, dz) \geq 0$ , and the equality of  $\mu^{(n)}$  and  $\mu$  on the set  $\{(s, z) | s \in [0, 1], |z| \leq n^{1/\alpha}\}$ .

Proceeding as for the first term in the right-hand side of (2.4.19), we also get that the first term of (2.4.22) belongs to  $\cap_{p \geq 1} \mathbf{L}^p$ .

On the other hand, for the second term of (2.4.22) we have:

$$\frac{\int_0^1 \int_{|z| > 2} (3+\alpha) |z| \mu^{(n)}(ds, dz)}{\int_0^1 \int_{|z| > 2} z^2 \mu^{(n)}(ds, dz)} \leq \frac{\int_0^1 \int_{|z| > 2} (3+\alpha) z^2 \mu^{(n)}(ds, dz)}{\int_0^1 \int_{|z| > 2} z^2 \mu^{(n)}(ds, dz)} = 3 + \alpha.$$

This completes the proof of **Step 2**.

We finally deduce (with additionally some uniformity with respect to  $x_0$ )

$$(2.4.23) \quad \forall p \geq 1, \quad \mathbb{E} \left( \sup_{n, \beta, x_0} \left| \hat{\mathcal{H}}_{1,\beta}^n(1) \right|^p + \sup_{n, \beta, x_0} \left| \hat{\mathcal{H}}_{2,\beta}^n(1) \right|^p \right) < \infty.$$

Recalling the almost sure convergences (2.3.18) and (2.3.19), we get from the dominated convergence theorem the  $\mathbf{L}^p$ -convergences

$$(2.4.24) \quad \hat{\mathcal{H}}_{1,\beta_n}^n(1) \xrightarrow[\mathbf{L}^p]{n \rightarrow \infty} \mathcal{H}_{1,L^\alpha}(1), \quad \forall p \geq 1.$$

$$(2.4.25) \quad \hat{\mathcal{H}}_{2,\beta_n}^n(1) \xrightarrow[\mathbf{L}^p]{n \rightarrow \infty} \mathcal{H}_{2,L^\alpha}(1), \quad \forall p \geq 1,$$

where  $\mathcal{H}_{1,L^\alpha}(1)$  and  $\mathcal{H}_{2,L^\alpha}(1)$  are defined respectively by (2.3.20) and (2.3.21).

On the other hand, Lemma 2.4.1 implies that  $n^{1/\alpha}(\bar{Y}_1^{n,\beta_n,x_0} - \zeta_1^{n,\theta_n,x_0})$  converges almost surely to  $\sigma L_1^\alpha$ . Then, using  $P(L_1^\alpha = u) = 0$ , we deduce the almost sure convergence

$$(2.4.26) \quad \mathbf{1}_{\{\bar{Y}_1^{n,\beta_n,x_0} \geq \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0}\}} = \mathbf{1}_{[u,\infty)} \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta_n,x_0} - \zeta_1^{n,\theta_n,x_0})}{\sigma_n} \right) \xrightarrow[a.s.]{n \rightarrow \infty} \mathbf{1}_{[u,\infty)}(L_1^\alpha).$$

Applying the dominated convergence theorem, we get the latter convergence in  $\mathbf{L}^p$ ,  $\forall p \geq 1$ . This gives finally :

$$(2.4.27) \quad \frac{\sigma_n}{n^{1/\alpha}} q^{n, \beta_n, x_0} \left( \frac{u \sigma_n}{n^{1/\alpha}} + \zeta_1^{n, \theta_n, x_0} \right) \xrightarrow{n \rightarrow \infty} \mathbb{E}[1_{[u, \infty)}(L_1^\alpha) \mathcal{H}_{L^\alpha}(1)],$$

where  $\mathcal{H}_{L^\alpha}(1) = \mathcal{H}_{1, L^\alpha}(1) + \mathcal{H}_{2, L^\alpha}(1)$  and  $\mathcal{H}_{1, L^\alpha}(1), \mathcal{H}_{2, L^\alpha}(1)$  are given by (2.3.20), (2.3.21), respectively. Remark that, we also get from (2.3.12), (2.3.13), (2.3.16) and (2.4.24), (2.4.25)

$$(2.4.28) \quad \sup_{u \in \mathbb{R}} \sup_n \frac{\sigma_n}{n^{1/\alpha}} q^{n, \beta_n, x_0} \left( \frac{u \sigma_n}{n^{1/\alpha}} + \zeta_1^{n, \theta_n, x_0} \right) < \infty.$$

To finish the proof of the convergence, it remains to show that

$$(2.4.29) \quad \varphi_\alpha(u) = \mathbb{E}[1_{[u, \infty)}(L_1^\alpha) \mathcal{H}_{L^\alpha}(1)].$$

Let us denote by  $\varphi^n(u)$  the density of the variable  $L_1^n$ . We consider the situation where the drift function  $b \equiv 0$  and  $x_0 = 0$  for which  $n^{1/\alpha} \bar{Y}_1^{n, \beta, x_0} = \sigma L_1^n$ . Then (2.4.27), (2.4.28) yield

$$(2.4.30) \quad \varphi^n(u) \xrightarrow{n \rightarrow \infty} \mathbb{E}[1_{[u, \infty)}(L_1^\alpha) \mathcal{H}_{L^\alpha}(1)] := \psi(u),$$

$$(2.4.31) \quad \sup_{u \in \mathbb{R}} \sup_n \varphi^n(u) < \infty.$$

Assume by contradiction that, for some  $u$ , we have  $\psi(u) \neq \varphi_\alpha(u)$ . Since  $P(L_1^\alpha = u) = 0$ , it can be seen that  $\psi$  is continuous at the point  $u$ . Hence, one can find a continuous, compactly supported, function  $f$  such that

$$(2.4.32) \quad \int f(x) \psi(x) dx \neq \int f(x) \varphi_\alpha(x) dx.$$

On the one hand we have,  $\mathbb{E}[f(L_1^n)] = \int f(x) \varphi^n(x) dx \xrightarrow{n \rightarrow \infty} \int f(x) \psi(x) dx$  where we have used the dominated convergence theorem with (2.4.30)-(2.4.31). On the other hand, we can write

$$(2.4.33) \quad \mathbb{E}[f(L_1^n)] = \mathbb{E}[f(L_1^n) 1_{\{L_1^n = L_1^\alpha\}}] + \mathbb{E}[f(L_1^n) 1_{\{L_1^n \neq L_1^\alpha\}}].$$

By Lemma 2.3.1, we have  $\mathbb{P}(L_1^n = L_1^\alpha) \xrightarrow{n \rightarrow \infty} 1$ . We deduce that,

$$(2.4.34) \quad \mathbb{E}[f(L_1^n)] \xrightarrow{n \rightarrow \infty} \mathbb{E}[f(L_1^\alpha)] = \int f(x) \varphi_\alpha(x) dx.$$

This last convergence result clearly contradicts (2.4.32) and we get (2.4.29).

Combining the preceding results with (2.4.27), we can get the results of Theorem 2.2.1.

## 2.5 Proof of Theorem 2.2.2

The proof is divided into three steps. We first give a representation of the derivative of the density and explicit the iterated Malliavin weights based on the calculus given in Section 2.6. We then study the convergence of these weights and proceed to the proof of Theorem 2.2.2.

### 2.5.1 Representation of $\nabla_{\beta} p_{\frac{1}{n}}^{\beta_n}$ and computation of the iterated Malliavin weights

We intensively use the results of Section 2.6. From Theorem 2.6.3, we have the representation

$$(2.5.1) \quad \nabla_{\beta} p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0}) = \nabla_{\beta} Q^{n,\beta_n,x_0}(\frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0}) \\ = \mathbb{E} \left[ \mathbf{1}_{\{\bar{Y}_1^{n,\beta_n,x_0} \geq \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0}\}} \mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(\nabla_{\beta} \bar{Y}_1^{n,\beta_n,x_0})) \right].$$

Using (2.6.7), (2.6.14) and (2.6.18), (2.6.19), by some simple calculus, we get the explicit formula for the iterated Malliavin weight

$$(2.5.2) \quad \mathcal{H}_{\bar{Y}_1^{n,\beta,x_0}}(\mathcal{H}_{\bar{Y}_1^{n,\beta,x_0}}(\nabla_{\beta} \bar{Y}_1^{n,\beta,x_0})) \\ = \begin{pmatrix} \partial_{\theta} \bar{Y}_1^{n,\beta,x_0} \\ \partial_{\sigma} \bar{Y}_1^{n,\beta,x_0} \end{pmatrix} \mathcal{H}_{\bar{Y}_1^{n,\beta,x_0}}(1)^2 - \begin{pmatrix} V_1^{n,\theta} \\ V_1^{n,\sigma} \end{pmatrix} \frac{2\mathcal{H}_{\bar{Y}_1^{n,\beta,x_0}}(1)}{U_1^{n,\beta}} + \begin{pmatrix} \partial_{\theta} \bar{Y}_1^{n,\beta,x_0} \\ \partial_{\sigma} \bar{Y}_1^{n,\beta,x_0} \end{pmatrix} \frac{\mathcal{H}_{\bar{Y}_1^{n,\beta,x_0}}(1)W_1^{n,\beta}}{(U_1^{n,\beta})^2} + \begin{pmatrix} \partial_{\theta} \bar{Y}_1^{n,\beta,x_0} \\ \partial_{\sigma} \bar{Y}_1^{n,\beta,x_0} \end{pmatrix} \frac{(W_1^{n,\beta})^2}{(U_1^{n,\beta})^4} \\ - \begin{pmatrix} V_1^{n,\theta} \\ V_1^{n,\sigma} \end{pmatrix} \frac{W_1^{n,\beta}}{(U_1^{n,\beta})^3} + \begin{pmatrix} \partial_{\theta} \bar{Y}_1^{n,\beta,x_0} \\ \partial_{\sigma} \bar{Y}_1^{n,\beta,x_0} \end{pmatrix} \frac{2D_1^{n,\beta}}{(U_1^{n,\beta})^2} - \begin{pmatrix} \partial_{\theta} \bar{Y}_1^{n,\beta,x_0} \\ \partial_{\sigma} \bar{Y}_1^{n,\beta,x_0} \end{pmatrix} \frac{Q_1^{n,\beta}}{(U_1^{n,\beta})^3} + \begin{pmatrix} T_1^{n,\theta} \\ T_1^{n,\sigma} \end{pmatrix} \frac{1}{(U_1^{n,\beta})^2}$$

where  $\mathcal{H}_{\bar{Y}_1^{n,\beta,x_0}}^n(1)$ ,  $U_1^{n,\beta}$ ,  $W_1^{n,\beta}$  are given by (2.3.13), (2.4.3), (2.4.5), respectively.

Moreover the processes  $(\partial_{\theta} \bar{Y}_t^{n,\beta,x_0})_t$  and  $(\partial_{\sigma} \bar{Y}_t^{n,\beta,x_0})_t$  are respectively solution to

$$(2.5.3) \quad \partial_{\theta} \bar{Y}_t^{n,\beta,x_0} = \frac{1}{n} \int_0^t b'(\bar{Y}_s^{n,\beta,x_0}, \theta) \partial_{\theta} \bar{Y}_s^{n,\beta,x_0} ds + \frac{1}{n} \int_0^t \partial_{\theta} b(\bar{Y}_s^{n,\beta,x_0}, \theta) ds,$$

$$(2.5.4) \quad \partial_{\sigma} \bar{Y}_t^{n,\beta,x_0} = \frac{1}{n} \int_0^t b'(\bar{Y}_s^{n,\beta,x_0}, \theta) \partial_{\sigma} \bar{Y}_s^{n,\beta,x_0} ds + \frac{L_t^n}{n^{1/\alpha}}.$$

For the computations of  $V_1^{n,\theta} = \Gamma(\bar{Y}_1^{n,\beta,x_0}, \partial_\theta \bar{Y}_1^{n,\beta,x_0})$  and  $V_1^{n,\sigma} = \Gamma(\bar{Y}_1^{n,\beta,x_0}, \partial_\sigma \bar{Y}_1^{n,\beta,x_0})$ , using (2.6.15), (2.6.16) we have

(2.5.5)

$$V_1^{n,\theta} = \frac{1}{n}(\epsilon_1^n)^2 \int_0^1 (\epsilon_s^n)^{-2} \left( U_s^{n,\beta} \left[ (\partial_\theta b)'(\bar{Y}_s^{n,\beta,x_0}, \theta) + b''(\bar{Y}_s^{n,\beta,x_0}, \theta) \partial_\theta \bar{Y}_s^{n,\beta,x_0} \right] \right) ds,$$

(2.5.6)

$$V_1^{n,\sigma} = \frac{1}{n}(\epsilon_1^n)^2 \int_0^1 (\epsilon_s^n)^{-2} \left( b''(\bar{Y}_s^{n,\beta,x_0}, \theta) \partial_\sigma \bar{Y}_s^{n,\beta,x_0} U_s^{n,\beta} \right) ds + \frac{\sigma}{n^{2/\alpha}} (\epsilon_1^n)^2 \int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-2} \rho^n(z) \mu^{(n)}(ds, dz).$$

Finally from ((2.6.20) - (2.6.23)) we compute explicitly  $D_1^{n,\beta} = \Gamma(\bar{Y}_1^{n,\beta,x_0}, L\bar{Y}_1^{n,\beta,x_0})$ ,  $Q_1^{n,\beta} = \Gamma(\bar{Y}_1^{n,\beta,x_0}, W_1^{n,\beta})$ ,  $T_1^{n,\theta} = \Gamma(\bar{Y}_1^{n,\beta,x_0}, V_1^{n,\theta})$  and  $T_1^{n,\sigma} = \Gamma(\bar{Y}_1^{n,\beta,x_0}, V_1^{n,\sigma})$ . We get:

(2.5.7)

$$D_1^{n,\beta} = \frac{(\epsilon_1^n)^2}{n} \int_0^1 (\epsilon_s^n)^{-2} b''(\bar{Y}_s^{n,\beta,x_0}, \theta) L\bar{Y}_s^{n,\beta,x_0} U_s^{n,\beta} ds + \frac{(\epsilon_1^n)^2}{2n} \int_0^1 (\epsilon_s^n)^{-2} b''(\bar{Y}_s^{n,\beta,x_0}, \theta) W_s^{n,\beta} ds + \frac{(\epsilon_1^n)^2}{2n} \int_0^1 (\epsilon_s^n)^{-2} b'''(\bar{Y}_s^{n,\beta,x_0}, \theta) (U_s^{n,\beta})^2 ds + \frac{\sigma^2 (\epsilon_1^n)^2}{2n^{2/\alpha}} \int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-2} \rho^n(z) \left( (\rho^n)'(z) + \rho^n(z) \frac{F_n'(z)}{F_n(z)} \right)' \mu^{(n)}(ds, dz),$$

(2.5.8)

$$Q_1^{n,\beta} = \frac{7(\epsilon_1^n)^4}{n} \int_0^1 (\epsilon_s^n)^{-4} b''(\bar{Y}_s^{n,\beta,x_0}, \theta) W_s^{n,\beta} U_s^{n,\beta} ds + \frac{2(\epsilon_1^n)^4}{n} \int_0^1 (\epsilon_s^n)^{-4} b'''(\bar{Y}_s^{n,\beta,x_0}, \theta) (U_s^{n,\beta})^3 ds + \frac{\sigma^4 (\epsilon_1^n)^4}{n^{4/\alpha}} \int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-4} \rho^n(z) \left[ ((\rho^n)'(z))^2 + \rho^n(z) (\rho^n)''(z) \right] \mu^{(n)}(ds, dz),$$

(2.5.9)

$$T_1^{n,\theta} = \frac{3(\epsilon_1^n)^3}{n} \int_0^1 (\epsilon_s^n)^{-3} b''(\bar{Y}_s^{n,\beta,x_0}, \theta) V_s^{n,\theta} U_s^{n,\beta} ds + \frac{(\epsilon_1^n)^3}{n} \int_0^1 (\epsilon_s^n)^{-3} (\partial_\theta b)'(\bar{Y}_s^{n,\beta,x_0}, \theta) W_s^{n,\beta} ds + \frac{(\epsilon_1^n)^3}{n} \int_0^1 (\epsilon_s^n)^{-3} b''(\bar{Y}_s^{n,\beta,x_0}, \theta) \partial_\theta \bar{Y}_s^{n,\beta,x_0} W_s^{n,\beta} ds + \frac{(\epsilon_1^n)^3}{n} \int_0^1 (\epsilon_s^n)^{-3} (\partial_\theta b)''(\bar{Y}_s^{n,\beta,x_0}, \theta) (U_s^{n,\beta})^2 ds + \frac{(\epsilon_1^n)^3}{n} \int_0^1 (\epsilon_s^n)^{-3} b'''(\bar{Y}_s^{n,\beta,x_0}, \theta) \partial_\theta \bar{Y}_s^{n,\beta,x_0} (U_s^{n,\beta})^2 ds,$$

(2.5.10)

$$T_1^{n,\sigma} = \frac{3(\epsilon_1^n)^3}{n} \int_0^1 (\epsilon_s^n)^{-3} b''(\bar{Y}_s^{n,\beta,x_0}, \theta) V_s^{n,\sigma} U_s^{n,\beta} ds + \frac{(\epsilon_1^n)^3}{n} \int_0^1 (\epsilon_s^n)^{-3} b''(\bar{Y}_s^{n,\beta,x_0}, \theta) \partial_\sigma \bar{Y}_s^{n,\beta,x_0} W_s^{n,\beta} ds + \frac{(\epsilon_1^n)^3}{n} \int_0^1 (\epsilon_s^n)^{-3} b'''(\bar{Y}_s^{n,\beta,x_0}, \theta) \partial_\sigma \bar{Y}_s^{n,\beta,x_0} (U_s^{n,\beta})^2 ds + \frac{\sigma^2 (\epsilon_1^n)^3}{n^{3/\alpha}} \int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-3} \rho^n(z) (\rho^n)'(z) \mu^{(n)}(ds, dz).$$

From the above calculus and combining with (2.5.1) and (2.5.2) we have an explicit representation for the derivative of the density with respect to parameter  $\beta$  that allows to analyze its asymptotic behavior in small time. To obtain the results of Theorem 2.2.2, we will study the convergence

of each term appearing in the decomposition (2.5.2). This is based on the preceding explicit expressions that permit to identify some main terms and some remainder terms.

In the sequel, we prove that all the terms involving the derivatives of  $b$  with respect to  $x$  are remainder terms.

### 2.5.2 Convergence of the iterated weights

In this section, we study the convergence of the iterated Malliavin weight  $\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(\nabla_{\beta} \bar{Y}_1^{n,\beta_n,x_0}))$  which is the cornerstone of the proof for the convergence of  $\nabla_{\beta} p_{\frac{1}{n}}^{\beta_n}$  later. Firstly, we state some technical lemmas useful for our aim. The proofs of these lemmas are postponed to the end of the section.

We recall that  $(\partial_{\theta} \bar{Y}_t^{n,\beta,x_0})_t$  and  $(\partial_{\sigma} \bar{Y}_t^{n,\beta,x_0})_t$  are respectively solution to (2.5.3) and (2.5.4).

**Lemma 2.5.1.** *We have for all compact subset  $Q \subset \mathbb{R} \times (0, \infty)$*

$$i) \sup_{\beta \in Q} |\partial_{\theta} \bar{Y}_1^{n,\beta,x_0}| \leq \frac{C}{n}, \quad \text{where } C \text{ is some deterministic constant.}$$

$$ii) \sup_{\beta \in Q} \sup_{s \in [0,1]} \left| \partial_{\sigma} \bar{Y}_s^{n,\beta,x_0} \right| \xrightarrow{L^p} 0, \quad \forall p \geq 1.$$

**Lemma 2.5.2.** *Let  $(\beta_n)$  be a sequence converging to  $\beta$  and  $Q$  be a compact subset of  $\mathbb{R} \times (0, \infty)$ , the following decompositions and estimates hold,*

$$i) \frac{1}{n^{1/\alpha}} \frac{D_1^{n,\beta}}{(U_1^{n,\beta})^2} = \frac{n^{1/\alpha}}{2\sigma^2} \hat{\mathcal{H}}_{3,\beta}^n(1) + \mathcal{R}_{4,\beta}^n(1) + \mathcal{R}_{5,\beta}^n(1) + \mathcal{R}_{6,\beta}^n(1).$$

$$ii) \frac{1}{n^{1/\alpha}} \frac{Q_1^{n,\beta}}{(U_1^{n,\beta})^3} = \frac{n^{1/\alpha}}{\sigma^2} \hat{\mathcal{H}}_{4,\beta}^n(1) + \mathcal{R}_{7,\beta}^n(1) + \mathcal{R}_{8,\beta}^n(1).$$

$$iii) \sup_{\beta \in Q} \frac{1}{n^{1/\alpha+1}} \frac{\sup_{s \in [0,1]} |\partial_{\sigma} \bar{Y}_s^{n,\beta,x_0} W_s^{n,\beta}|}{(U_1^{n,\beta})^2} \xrightarrow{L^p} 0, \quad \forall p \geq 1.$$

The main terms  $\hat{\mathcal{H}}_{3,\beta}^n(1), \hat{\mathcal{H}}_{4,\beta}^n(1)$  are given by

$$(2.5.11) \quad \hat{\mathcal{H}}_{3,\beta}^n(1) = \frac{\int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-2} \rho^n(z) \left[ (\rho^n)''(z) - (\rho^n)'(z) \frac{(1+\alpha)}{z} + \rho^n(z) \frac{(1+\alpha)}{z^2} \right] \mu^{(n)}(ds, dz)}{(\epsilon_1^n)^2 \left( \int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-2} \rho^n(z) \mu^{(n)}(ds, dz) \right)^2},$$

$$(2.5.12) \quad \hat{\mathcal{H}}_{4,\beta}^n(1) = \frac{\int_0^1 \int_{\mathbb{R}} (\epsilon_1^n)^{-4} \rho^n(z) \left[ ((\rho^n)'(z))^2 + \rho^n(z) (\rho^n)''(z) \right] \mu^{(n)}(ds, dz)}{(\epsilon_1^n)^2 \left( \int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-2} \rho^n(z) \mu^{(n)}(ds, dz) \right)^3}.$$



where  $(\epsilon_s^n)_{s \in [0,1]}$  is given by (2.3.10). Moreover we have for  $p \geq 1$  and some deterministic constant  $C$

$$(2.5.13) \quad \hat{\mathcal{H}}_{3,\beta_n}^n(1) \xrightarrow[\mathbf{L}^p]{n \rightarrow \infty} \mathcal{H}_{3,L^\alpha}(1), \quad \hat{\mathcal{H}}_{4,\beta_n}^n(1) \xrightarrow[\mathbf{L}^p]{n \rightarrow \infty} \mathcal{H}_{4,L^\alpha}(1),$$

$$(2.5.14) \quad \sup_{\beta \in Q} |\mathcal{R}_{4,\beta}^n(1)| \xrightarrow[\mathbf{L}^p]{n \rightarrow \infty} 0, \quad \sup_{\beta \in Q} |\mathcal{R}_{5,\beta}^n(1)| \xrightarrow[\mathbf{L}^p]{n \rightarrow \infty} 0, \quad \sup_{\beta \in Q} |\mathcal{R}_{6,\beta}^n(1)| \leq \frac{C}{2n^{1+1/\alpha}},$$

$$(2.5.15) \quad \sup_{\beta \in Q} |\mathcal{R}_{7,\beta}^n(1)| \xrightarrow[\mathbf{L}^p]{n \rightarrow \infty} 0, \quad \sup_{\beta \in Q} |\mathcal{R}_{8,\beta}^n(1)| \leq \frac{C}{n^{1+1/\alpha}},$$

where

$$(2.5.16) \quad \mathcal{H}_{3,L^\alpha}(1) = \frac{\int_0^1 \int_{\mathbb{R}} \left( \rho(z) \rho''(z) - \rho(z) \rho'(z) \frac{(1+\alpha)}{z} + (\rho(z))^2 \frac{(1+\alpha)}{z^2} \right) \mu(ds, dz)}{\left( \int_0^1 \int_{\mathbb{R}} \rho(z) \mu(ds, dz) \right)^2},$$

$$(2.5.17) \quad \mathcal{H}_{4,L^\alpha}(1) = \frac{\int_0^1 \int_{\mathbb{R}} \rho(z) \left[ (\rho'(z))^2 + \rho(z) \rho''(z) \right] \mu(ds, dz)}{\left( \int_0^1 \int_{\mathbb{R}} \rho(z) \mu(ds, dz) \right)^3}.$$

**Lemma 2.5.3.** Let  $(\beta_n)$  be a sequence converging to  $\beta$  and  $Q$  be a compact subset of  $\mathbb{R} \times (0, \infty)$ , the following estimates hold:

$$i) \sup_{\beta \in Q} \left| \frac{V_1^{n,\theta}}{U_1^{n,\beta}} \right| \leq \frac{C}{n},$$

$$ii) \sup_{\beta \in Q} \frac{1}{n^{2/\alpha-1}} \frac{|T_1^{n,\theta}|}{(U_1^{n,\beta})^2} \xrightarrow[\mathbf{L}^p]{n \rightarrow \infty} 0, \quad \forall p \geq 1,$$

$$iii) \frac{V_1^{n,\sigma}}{U_1^{n,\beta}} = \frac{1}{\sigma} + \mathcal{R}_{9,\beta}^n(1),$$

$$iv) \frac{1}{n^{1/\alpha}} \frac{T_1^{n,\sigma}}{(U_1^{n,\beta})^2} = \frac{1}{\sigma^2} \hat{\mathcal{H}}_{5,\beta}^n(1) + \mathcal{R}_{10,\beta}^n(1) + \mathcal{R}_{11,\beta}^n(1) + \mathcal{R}_{12,\beta}^n(1),$$

where  $C$  is some deterministic constant. The main term  $\hat{\mathcal{H}}_{5,\beta}^n(1)$  is given by

$$(2.5.18) \quad \hat{\mathcal{H}}_{5,\beta}^n(1) = \frac{\int_0^1 \int_{\mathbb{R}} (\epsilon_1^n)^{-3} \rho^n(z) (\rho^n)'(z) \mu^{(n)}(ds, dz)}{\epsilon_1^n \left( \int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-2} \rho^n(z) \mu^{(n)}(ds, dz) \right)^2}$$

with  $(\epsilon_s^n)$  given by (2.3.10). Moreover,  $(\sup_{\beta \in Q} |\mathcal{R}_{i,\beta}^n(1)|)_{9 \leq i \leq 12}$  converge to zero as  $n \rightarrow \infty$  in  $\mathbf{L}^p$ ,  $\forall p \geq 1$  and  $\hat{\mathcal{H}}_{5,\beta_n}^n(1) \xrightarrow[\mathbf{L}^p, \forall p \geq 1]{n \rightarrow \infty} \mathcal{H}_{5,L^\alpha}(1)$ , with

$$(2.5.19) \quad \mathcal{H}_{5,L^\alpha}(1) = \frac{\int_0^1 \int_{\mathbb{R}} \rho(z) \rho'(z) \mu(ds, dz)}{\left( \int_0^1 \int_{\mathbb{R}} \rho(z) \mu(ds, dz) \right)^2}.$$

**Lemma 2.5.4.** *Let  $(\beta_n)$  be a sequence converging to  $\beta$ . For all  $p \geq 1$ , the following convergences hold uniformly with respect to  $x_0$ :*

(2.5.20)

$$n\partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \left( \hat{\mathcal{H}}_{\beta_n}^n(1) \right)^q \xrightarrow[\mathbf{L}^p]{n \rightarrow \infty} \partial_\theta b(x_0, \theta) (\mathcal{H}_{L^\alpha}(1))^q, \quad \forall q \geq 1,$$

(2.5.21)

$$n\partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \hat{\mathcal{H}}_{\beta_n}^n(1) \hat{\mathcal{H}}_{1,\beta_n}^n(1) \xrightarrow[\mathbf{L}^p]{n \rightarrow \infty} \partial_\theta b(x_0, \theta) \mathcal{H}_{L^\alpha}(1) \mathcal{H}_{1,L^\alpha}(1),$$

(2.5.22)

$$n^{1/\alpha} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \left( \hat{\mathcal{H}}_{\beta_n}^n(1) \right)^q \xrightarrow[\mathbf{L}^p]{n \rightarrow \infty} L_1^\alpha (\mathcal{H}_{L^\alpha}(1))^q, \quad \forall q \geq 1,$$

(2.5.23)

$$n^{1/\alpha} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \hat{\mathcal{H}}_{\beta_n}^n(1) \hat{\mathcal{H}}_{1,\beta_n}^n(1) \xrightarrow[\mathbf{L}^p]{n \rightarrow \infty} L_1^\alpha \mathcal{H}_{L^\alpha}(1) \mathcal{H}_{1,L^\alpha}(1),$$

where  $\hat{\mathcal{H}}_\beta^n(1) = \hat{\mathcal{H}}_{1,\beta}^n(1) + \hat{\mathcal{H}}_{2,\beta}^n(1)$  with  $\hat{\mathcal{H}}_{1,\beta}^n(1), \hat{\mathcal{H}}_{2,\beta}^n(1)$  given by (2.3.14), (2.3.15);  $\mathcal{H}_{L^\alpha}(1) = \mathcal{H}_{1,L^\alpha}(1) + \mathcal{H}_{2,L^\alpha}(1)$  where  $\mathcal{H}_{1,L^\alpha}(1), \mathcal{H}_{2,L^\alpha}(1)$  are defined by (2.3.20), (2.3.21).

**Lemma 2.5.5.** *Let  $(\beta_n)$  be a sequence converging to  $\beta$ . For all  $p \geq 1$  the following convergences hold uniformly with respect to  $x_0$ :*

$$i) \quad n\partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \hat{\mathcal{H}}_{3,\beta_n}^n(1) \xrightarrow[\mathbf{L}^p]{n \rightarrow \infty} \partial_\theta b(x_0, \theta) \mathcal{H}_{3,L^\alpha}(1),$$

$$ii) \quad n\partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \hat{\mathcal{H}}_{4,\beta_n}^n(1) \xrightarrow[\mathbf{L}^p]{n \rightarrow \infty} \partial_\theta b(x_0, \theta) \mathcal{H}_{4,L^\alpha}(1),$$

$$iii) \quad n^{1/\alpha} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \hat{\mathcal{H}}_{3,\beta_n}^n(1) \xrightarrow[\mathbf{L}^p]{n \rightarrow \infty} L_1^\alpha \mathcal{H}_{3,L^\alpha}(1),$$

$$iv) \quad n^{1/\alpha} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \hat{\mathcal{H}}_{4,\beta_n}^n(1) \xrightarrow[\mathbf{L}^p]{n \rightarrow \infty} L_1^\alpha \mathcal{H}_{4,L^\alpha}(1),$$

where  $\hat{\mathcal{H}}_{3,\beta}^n(1), \hat{\mathcal{H}}_{4,\beta}^n(1)$  are given by (2.5.11), (2.5.12), and  $\mathcal{H}_{3,L^\alpha}(1), \mathcal{H}_{4,L^\alpha}(1)$ , are defined by (2.5.16), (2.5.17).

The uniform convergence with respect to  $x_0$  is not required in this paper but will be useful in [20].

**Remark 2.5.1.** *We observe that although  $L_1^\alpha$  does not belong to  $\mathbf{L}^p$ , the choice of the auxiliary function  $\rho$  permits to prove that  $L_1^\alpha (\mathcal{H}_{L^\alpha}(1))^2, L_1^\alpha \mathcal{H}_{3,L^\alpha}(1)$  and  $L_1^\alpha \mathcal{H}_{4,L^\alpha}(1)$  belong to  $\mathbf{L}^p, \forall p \geq 1$ .*

Based on the preceding lemmas, we can prove the following convergence result.

**Proposition 2.5.1.** *Let  $(\beta_n)_{n \geq 1}$  be a sequence such that  $\beta_n \xrightarrow{n \rightarrow \infty} \beta$  then for all  $p \geq 1$*

$$(2.5.24) \quad \frac{\sigma_n^2}{n^{1/\alpha}} \mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}} (\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}} (\partial_\sigma \bar{Y}_1^{n,\beta_n,x_0})) \xrightarrow[n \rightarrow \infty]{L^p} \mathcal{H}^{(2)},$$

$$(2.5.25) \quad \frac{\sigma_n^2}{n^{2/\alpha-1}} \mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}} (\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}} (\partial_\theta \bar{Y}_1^{n,\beta_n,x_0})) \xrightarrow[n \rightarrow \infty]{L^p} \partial_\theta b(x_0, \theta) \mathcal{H}_1^{(2)},$$

where  $\mathcal{H}^{(2)}$  and  $\mathcal{H}_1^{(2)}$  are some random variables whose expressions do not depend on  $\beta$  and  $b$ .

**Proof.** From the equation (2.5.2), we have

$$(2.5.26) \quad \begin{aligned} & \left( \frac{\sigma_n^2}{n^{2/\alpha-1}} \mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}} (\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}} (\partial_\theta \bar{Y}_1^{n,\beta_n,x_0})) \right) = \left( \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \right) \mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(1)^2 - \left( \frac{\sigma_n^2}{n^{2/\alpha-1}} V_1^{n,\theta_n} \right) \frac{2\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(1)}{(U_1^{n,\beta_n})} \\ & + \left( \frac{\sigma_n^2}{n^{1/\alpha}} \mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}} (\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}} (\partial_\sigma \bar{Y}_1^{n,\beta_n,x_0})) \right) = \left( \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \right) \frac{(W_1^{n,\beta_n})^2}{(U_1^{n,\beta_n})^4} - \left( \frac{\sigma_n^2}{n^{1/\alpha}} V_1^{n,\sigma_n} \right) \frac{W_1^{n,\beta_n}}{(U_1^{n,\beta_n})^3} \\ & + \left( \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \right) \frac{\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(1) W_1^{n,\beta_n}}{(U_1^{n,\beta_n})^2} + \left( \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \right) \frac{(W_1^{n,\beta_n})^2}{(U_1^{n,\beta_n})^4} - \left( \frac{\sigma_n^2}{n^{2/\alpha-1}} V_1^{n,\theta_n} \right) \frac{W_1^{n,\beta_n}}{(U_1^{n,\beta_n})^3} \\ & + \left( \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \right) \frac{2D_1^{n,\beta_n}}{(U_1^{n,\beta_n})^2} - \left( \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \right) \frac{Q_1^{n,\beta_n}}{(U_1^{n,\beta_n})^3} + \left( \frac{\sigma_n^2}{n^{2/\alpha-1}} T_1^{n,\theta_n} \right) \frac{1}{(U_1^{n,\beta_n})^2}. \end{aligned}$$

We will prove the convergence of each term in the right-hand side of (2.5.26)

**Term 1:** Recall (2.3.13) and set  $\hat{\mathcal{H}}_{\beta_n}^n(1) = \hat{\mathcal{H}}_{1,\beta_n}^n(1) + \hat{\mathcal{H}}_{2,\beta_n}^n(1)$ ,  $\mathcal{R}_{\beta_n}^n(1) = \mathcal{R}_{2,\beta_n}^n(1) + \mathcal{R}_{3,\beta_n}^n(1)$ .

Remark that by (2.3.16), we have  $|\mathcal{R}_{\beta_n}^n(1)| \leq \frac{C}{n}$  where C is some deterministic constant. Moreover,

we can rewrite the first term as

$$(2.5.27) \quad \begin{aligned} & \left( \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \right) \left[ \mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(1) \right]^2 \\ & = \left( \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \right) \left[ \frac{n^{1/\alpha}}{\sigma_n} \left[ \hat{\mathcal{H}}_{1,\beta_n}^n(1) + \hat{\mathcal{H}}_{2,\beta_n}^n(1) \right] + \mathcal{R}_{1,\beta_n}^n(1) + \mathcal{R}_{2,\beta_n}^n(1) + \mathcal{R}_{3,\beta_n}^n(1) \right]^2 \\ & = \left( \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \right) \left[ \frac{n^{1/\alpha}}{\sigma_n} \left[ \hat{\mathcal{H}}_{\beta_n}^n(1) \right]^2 + \left( \frac{2\sigma_n}{n^{1/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \right) \hat{\mathcal{H}}_{\beta_n}^n(1) \mathcal{R}_{\beta_n}^n(1) + \left( \frac{2\sigma_n}{n^{1/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \right) \hat{\mathcal{H}}_{\beta_n}^n(1) \mathcal{R}_{1,\beta_n}^n(1) \right. \\ & \quad \left. + \left( \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \right) \mathcal{R}_{1,\beta_n}^n(1)^2 + \left( \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \right) \left( \mathcal{R}_{\beta_n}^n(1) \right)^2 + \left( \frac{2\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \right) \mathcal{R}_{1,\beta_n}^n(1) \mathcal{R}_{\beta_n}^n(1) \right. \\ & \quad \left. + \left( \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \right) \mathcal{R}_{1,\beta_n}^n(1)^2 + \left( \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \right) \left( \mathcal{R}_{\beta_n}^n(1) \right)^2 + \left( \frac{2\sigma_n^2}{n^{1/\alpha}} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \right) \mathcal{R}_{1,\beta_n}^n(1) \mathcal{R}_{\beta_n}^n(1) \right]. \end{aligned}$$

where  $\mathcal{R}_{1,\beta_n}^n(1)$  is given by (2.4.6). We can deduce from (2.3.16), (2.4.24), (2.4.25), Lemma 2.5.4 and Lemma 2.5.1 that

$$\left( \frac{\frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0}}{\frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0}} \right) \left( \mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(1) \right)^2 \xrightarrow[\mathbf{L}^p, \forall p \geq 1]{n \rightarrow \infty} \begin{pmatrix} \partial_\theta b(x_0, \theta) \\ L_1^\alpha \end{pmatrix} (\mathcal{H}_{L^\alpha}(1))^2.$$

**Term 2:** From (2.3.13) and Lemma 2.5.3 part *i*) and part *iii*), we can estimate the second term as

$$\begin{aligned} & \left( \frac{\frac{\sigma_n^2}{n^{2/\alpha-1}} V_1^{n,\theta_n}}{\frac{\sigma_n^2}{n^{1/\alpha}} V_1^{n,\sigma_n}} \right) \frac{2\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}^n(1)}{(U_1^{n,\beta_n})} = \begin{pmatrix} O\left(\frac{1}{n^{2/\alpha}}\right) \\ \frac{2\sigma_n^2}{n^{1/\alpha}} \left[ \mathcal{R}_{9,\beta_n}^n(1) + \frac{1}{\sigma_n} \right] \end{pmatrix} \left[ \frac{1}{\sigma_n} n^{1/\alpha} \hat{\mathcal{H}}_{\beta_n}^n(1) + \mathcal{R}_{1,\beta_n}^n(1) + \mathcal{R}_{\beta_n}^n(1) \right] \\ & = \begin{pmatrix} O\left(\frac{1}{n^{1/\alpha}}\right) \hat{\mathcal{H}}_{\beta_n}^n(1) + O\left(\frac{1}{n^{2/\alpha}}\right) \mathcal{R}_{1,\beta_n}^n(1) + O\left(\frac{1}{n^{2/\alpha}}\right) \mathcal{R}_{\beta_n}^n(1) \\ 2\hat{\mathcal{H}}_{\beta_n}^n(1) + \frac{2\sigma_n \mathcal{R}_{1,\beta_n}^n(1)}{n^{1/\alpha}} + \frac{2\sigma_n \mathcal{R}_{\beta_n}^n(1)}{n^{1/\alpha}} + 2\sigma_n \mathcal{R}_{9,\beta_n}^n(1) \hat{\mathcal{H}}_{\beta_n}^n(1) + \frac{2\sigma_n^2 \mathcal{R}_{1,\beta_n}^n(1) \mathcal{R}_{9,\beta_n}^n(1)}{n^{1/\alpha}} + \frac{2\sigma_n^2 \mathcal{R}_{\beta_n}^n(1) \mathcal{R}_{9,\beta_n}^n(1)}{n^{1/\alpha}} \end{pmatrix} \end{aligned}$$

where  $C$  is some deterministic constant and  $O\left(\frac{1}{n^{2/\alpha}}\right)$  is a random variable bounded by  $\frac{C}{n^{2/\alpha}}$ . From (2.3.16), (2.4.24), (2.4.25) and Lemma 2.5.3, we also conclude that

$$\left( \frac{\frac{\sigma_n^2}{n^{2/\alpha-1}} V_1^{n,\theta_n}}{\frac{\sigma_n^2}{n^{1/\alpha}} V_1^{n,\sigma_n}} \right) \frac{2\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(1)}{(U_1^{n,\beta_n})} \xrightarrow[\mathbf{L}^p, \forall p \geq 1]{n \rightarrow \infty} \begin{pmatrix} 0 \\ 2\mathcal{H}_{L^\alpha}(1) \end{pmatrix}$$

**Term 3:** From (2.3.13) and  $\frac{W_1^{n,\beta_n}}{(U_1^{n,\beta_n})^2} = \frac{n^{1/\alpha}}{\sigma_n} \hat{\mathcal{H}}_{1,\beta_n}^n(1) + \mathcal{R}_{2,\beta_n}^n(1)$  where  $\hat{\mathcal{H}}_{1,\beta_n}^n(1)$  and  $\mathcal{R}_{2,\beta_n}^n(1)$  are given by (2.3.14) and (2.4.7), we have

$$\begin{aligned} & \left( \frac{\frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0}}{\frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0}} \right) \frac{\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(1) W_1^{n,\beta_n}}{(U_1^{n,\beta_n})^2} \\ & = \left( \frac{\frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0}}{\frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0}} \right) \left[ \frac{n^{1/\alpha} \hat{\mathcal{H}}_{\beta_n}^n(1)}{\sigma_n} + \mathcal{R}_{1,\beta_n}^n(1) + \mathcal{R}_{\beta_n}^n(1) \right] \left[ \frac{n^{1/\alpha}}{\sigma_n} \hat{\mathcal{H}}_{1,\beta_n}^n(1) + \mathcal{R}_{2,\beta_n}^n(1) \right] \\ & = \begin{pmatrix} n \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \\ n^{1/\alpha} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \hat{\mathcal{H}}_{\beta_n}^n(1) \hat{\mathcal{H}}_{1,\beta_n}^n(1) + \begin{pmatrix} \frac{\sigma_n}{n^{1/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \\ \sigma_n \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \hat{\mathcal{H}}_{\beta_n}^n(1) \mathcal{R}_{2,\beta_n}^n(1) \\ & + \begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \mathcal{R}_{1,\beta_n}^n(1) \mathcal{R}_{2,\beta_n}^n(1) + \begin{pmatrix} \frac{\sigma_n}{n^{1/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \\ \sigma_n \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \mathcal{R}_{1,\beta_n}^n(1) \hat{\mathcal{H}}_{1,\beta_n}^n(1) \\ & + \begin{pmatrix} \frac{\sigma_n}{n^{1/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \\ \sigma_n \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \mathcal{R}_{\beta_n}^n(1) \hat{\mathcal{H}}_{1,\beta_n}^n(1) + \begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \mathcal{R}_{\beta_n}^n(1) \mathcal{R}_{2,\beta_n}^n(1). \end{aligned}$$

From (2.3.16), (2.4.24), (2.4.25), Lemma 2.5.1 and Lemma 2.5.4, we also conclude that

$$\left( \begin{array}{c} \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \end{array} \right) \frac{\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(1) W_1^{n,\beta_n}}{(U_1^{n,\beta_n})^2} \xrightarrow[n \rightarrow \infty]{\mathbf{L}^p, \forall p \geq 1} \left( \begin{array}{c} \partial_\theta b(x_0, \theta) \mathcal{H}_{1,L^\alpha}(1) \mathcal{H}_{L^\alpha}(1) \\ L_1^\alpha \mathcal{H}_{1,L^\alpha}(1) \mathcal{H}_{L^\alpha}(1) \end{array} \right)$$

**Term 4:** Using  $\frac{W_1^{n,\beta_n}}{(U_1^{n,\beta_n})^2} = \frac{n^{1/\alpha}}{\sigma_n} \hat{\mathcal{H}}_{1,\beta_n}^n(1) + \mathcal{R}_{2,\beta_n}^n(1)$  again, we can rewrite

$$\begin{aligned} & \left( \begin{array}{c} \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \end{array} \right) \frac{(W_1^{n,\beta_n})^2}{(U_1^{n,\beta_n})^4} = \left( \begin{array}{c} \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \end{array} \right) \left[ \frac{n^{1/\alpha}}{\sigma_n} \hat{\mathcal{H}}_{1,\beta_n}^n(1) + \mathcal{R}_{2,\beta_n}^n(1) \right]^2 \\ & = \left( \begin{array}{c} n \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \\ n^{1/\alpha} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \end{array} \right) (\hat{\mathcal{H}}_{1,\beta_n}^n(1))^2 + \left( \begin{array}{c} \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \end{array} \right) (\mathcal{R}_{2,\beta_n}^n(1))^2 + \left( \begin{array}{c} \frac{2\sigma_n}{n^{1/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \\ 2\sigma_n \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \end{array} \right) \mathcal{R}_{2,\beta_n}^n(1) \hat{\mathcal{H}}_{1,\beta_n}^n(1). \end{aligned}$$

From (2.3.16), (2.4.24), (2.4.25), Lemma 2.5.4 and Lemma 2.5.1, we also conclude that

$$\left( \begin{array}{c} \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \end{array} \right) \frac{(W_1^{n,\beta_n})^2}{(U_1^{n,\beta_n})^4} \xrightarrow[n \rightarrow \infty]{\mathbf{L}^p, \forall p \geq 1} \left( \begin{array}{c} \partial_\theta b(x_0, \theta) (\mathcal{H}_{1,L^\alpha}(1))^2 \\ L_1^\alpha (\mathcal{H}_{1,L^\alpha}(1))^2 \end{array} \right)$$

**Term 5:** From Lemma 2.5.3 we can estimate the fifth term as

$$\begin{aligned} & \left( \begin{array}{c} \frac{\sigma_n^2}{n^{2/\alpha-1}} V_1^{n,\theta_n} \\ \frac{\sigma_n^2}{n^{1/\alpha}} V_1^{n,\sigma_n} \end{array} \right) \frac{W_1^{n,\beta_n}}{(U_1^{n,\beta_n})^3} = \left( \begin{array}{c} O(\frac{1}{n^{2/\alpha}}) \\ \frac{\sigma_n^2}{n^{1/\alpha}} (\mathcal{R}_{9,\beta_n}^n(1) + \frac{1}{\sigma_n}) \end{array} \right) \left( \begin{array}{c} n^{1/\alpha} \hat{\mathcal{H}}_{1,\beta_n}^n(1) + \mathcal{R}_{2,\beta_n}^n(1) \\ \sigma_n \mathcal{R}_{9,\beta_n}^n(1) \hat{\mathcal{H}}_{1,\beta_n}^n(1) + \frac{\sigma_n^2}{n^{1/\alpha}} \mathcal{R}_{9,\beta_n}^n(1) \mathcal{R}_{2,\beta_n}^n(1) + \hat{\mathcal{H}}_{1,\beta_n}^n(1) + \frac{\sigma_n}{n^{1/\alpha}} \mathcal{R}_{2,\beta_n}^n(1) \end{array} \right) \\ & = \left( \begin{array}{c} O(\frac{1}{n^{1/\alpha}}) \hat{\mathcal{H}}_{1,\beta_n}^n(1) + O(\frac{1}{n^{2/\alpha}}) \mathcal{R}_{2,\beta_n}^n(1) \\ \sigma_n \mathcal{R}_{9,\beta_n}^n(1) \hat{\mathcal{H}}_{1,\beta_n}^n(1) + \frac{\sigma_n^2}{n^{1/\alpha}} \mathcal{R}_{9,\beta_n}^n(1) \mathcal{R}_{2,\beta_n}^n(1) + \hat{\mathcal{H}}_{1,\beta_n}^n(1) + \frac{\sigma_n}{n^{1/\alpha}} \mathcal{R}_{2,\beta_n}^n(1) \end{array} \right) \end{aligned}$$

where  $C$  is some deterministic constant. From (2.3.16), (2.4.24), Lemma 2.5.3, we also conclude that

$$\left( \begin{array}{c} \frac{\sigma_n^2}{n^{2/\alpha-1}} V_1^{n,\theta_n} \\ \frac{\sigma_n^2}{n^{1/\alpha}} V_1^{n,\sigma_n} \end{array} \right) \frac{W_1^{n,\beta_n}}{(U_1^{n,\beta_n})^3} \xrightarrow[n \rightarrow \infty]{\mathbf{L}^p, \forall p \geq 1} \left( \begin{array}{c} 0 \\ \mathcal{H}_{1,L^\alpha}(1) \end{array} \right)$$

**Term 6:** Using Lemma 2.5.2 we write the sixth term as

$$\begin{aligned} & \left( \begin{array}{c} \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \end{array} \right) \frac{D_1^{n,\beta_n}}{(U_1^{n,\beta_n})^2} \\ & = \left( \begin{array}{c} \frac{n}{2} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \\ \frac{n^{1/\alpha}}{2} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \end{array} \right) \hat{\mathcal{H}}_{3,\beta_n}^n(1) + \left( \begin{array}{c} \frac{\sigma_n^2}{n^{1/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \\ \sigma_n^2 \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \end{array} \right) (\mathcal{R}_{4,\beta_n}^n(1) + \mathcal{R}_{5,\beta_n}^n(1) + \mathcal{R}_{6,\beta_n}^n(1)). \end{aligned}$$

Applying Lemma 2.5.2, Lemma 2.5.5 and Lemma 2.5.1 we obtain that

$$\left( \begin{array}{c} \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \end{array} \right) \frac{2D_1^{n,\beta_n}}{(U_1^{n,\beta_n})^2} \xrightarrow[n \rightarrow \infty]{\mathbf{L}^p, \forall p \geq 1} \left( \begin{array}{c} \partial_\theta b(x_0, \theta) \mathcal{H}_{3,L^\alpha}(1) \\ L_1^\alpha \mathcal{H}_{3,L^\alpha}(1) \end{array} \right)$$

where  $\mathcal{H}_{3,L^\alpha}(1)$  is defined in Lemma 2.5.2.

**Term 7:** From Lemma 2.5.2, we can rewrite the seventh term as

$$\begin{aligned} & \left( \begin{array}{c} \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \end{array} \right) \frac{Q_1^{n,\beta_n}}{(U_1^{n,\beta_n})^3} \\ &= \left( \begin{array}{c} n \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \\ n^{1/\alpha} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \end{array} \right) \hat{\mathcal{H}}_{4,\beta_n}^n(1) + \left( \begin{array}{c} \frac{\sigma_n^2}{n^{1/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \\ \sigma_n^2 \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \end{array} \right) (\mathcal{R}_{7,\beta_n}^n(1) + \mathcal{R}_{8,\beta_n}^n(1)). \end{aligned}$$

Applying Lemma 2.5.2, Lemma 2.5.5 and Lemma 2.5.1 we obtain that

$$\left( \begin{array}{c} \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \bar{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma \bar{Y}_1^{n,\beta_n,x_0} \end{array} \right) \frac{Q_1^{n,\beta_n}}{(U_1^{n,\beta_n})^3} \xrightarrow[n \rightarrow \infty]{\mathbf{L}^p, \forall p \geq 1} \left( \begin{array}{c} \partial_\theta b(x_0, \theta) \mathcal{H}_{4,L^\alpha}(1) \\ L_1^\alpha \mathcal{H}_{4,L^\alpha}(1) \end{array} \right)$$

where  $\mathcal{H}_{4,L^\alpha}(1)$  is defined in Lemma 2.5.2.

**Term 8:** From Lemma 2.5.3, we have

$$\left( \begin{array}{c} \frac{\sigma_n^2}{n^{2/\alpha-1}} T_1^{n,\theta_n} \\ \frac{\sigma_n^2}{n^{1/\alpha}} T_1^{n,\sigma_n} \end{array} \right) \frac{1}{(U_1^{n,\beta_n})^2} = \left( \begin{array}{c} \frac{\sigma_n^2}{n^{2/\alpha-1}} \frac{T_1^{n,\theta_n}}{(U_1^{n,\beta_n})^2} \\ \hat{\mathcal{H}}_{5,\beta_n}^n(1) + \sigma_n^2 \mathcal{R}_{10,\beta_n}^n(1) + \sigma_n^2 \mathcal{R}_{11,\beta_n}^n(1) + \sigma_n^2 \mathcal{R}_{12,\beta_n}^n(1) \end{array} \right)$$

Using the results of Lemma 2.5.3, we easily deduce that

$$\left( \begin{array}{c} \frac{\sigma_n^2}{n^{2/\alpha-1}} T_1^{n,\theta_n} \\ \frac{\sigma_n^2}{n^{1/\alpha}} T_1^{n,\sigma_n} \end{array} \right) \frac{1}{(U_1^{n,\beta_n})^2} \xrightarrow[n \rightarrow \infty]{\mathbf{L}^p, \forall p \geq 1} \left( \begin{array}{c} 0 \\ \mathcal{H}_{5,L^\alpha}(1) \end{array} \right)$$

where  $\mathcal{H}_{5,L^\alpha}(1)$  is defined in Lemma 2.5.3.

Finally from the above convergences, we can deduce the result of Proposition 2.5.1. ■

### 2.5.3 Proof of Theorem 2.2.2

We will first prove part *ii*) and then give a proof for part *i*).

*ii*) Remark that from (2.5.24), (2.5.25)

$$\sup_{u \in \mathbb{R}} \sup_n \mathbb{E} \left| \mathbf{1}_{\{\bar{Y}_1^{n,\beta_n,x_0} \geq \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0}\}} \frac{\sigma_n^2}{n^{2/\alpha-1}} \mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(\partial_\theta \bar{Y}_1^{n,\beta_n,x_0})) \right| < \infty$$

and

$$\sup_{u \in \mathbb{R}} \sup_n \mathbb{E} \left| \mathbf{1}_{\{\bar{Y}_1^{n,\beta_n,x_0} \geq \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0}\}} \frac{\sigma_n^2}{n^{1/\alpha}} \mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}} (\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}} (\partial_\sigma \bar{Y}_1^{n,\beta_n,x_0})) \right| < \infty.$$

By representation (2.5.1) this leads to

$$\sup_{u \in \mathbb{R}} \sup_n \left| \frac{\sigma_n^2}{n^{\frac{2}{\alpha}-1}} \partial_\theta p_{\frac{1}{n}}^{\beta_n} \left( x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0} \right) \right| < \infty \text{ and } \sup_{u \in \mathbb{R}} \sup_n \left| \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma p_{\frac{1}{n}}^{\beta_n} \left( x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0} \right) \right| < \infty.$$

i) From (2.5.1), (2.4.26) and Proposition 2.5.1, we easily deduce that

$$(2.5.28) \quad \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma p_{\frac{1}{n}}^{\beta_n} \left( x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0} \right) = \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma q^{n,\beta_n,x_0} \left( \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0} \right) \xrightarrow{n \rightarrow \infty} \mathbb{E}[\mathbf{1}_{[u,\infty)}(L_1^\alpha) \mathcal{H}^{(2)}],$$

(2.5.29)

$$\frac{\sigma_n^2}{n^{\frac{2}{\alpha}-1}} \partial_\theta p_{\frac{1}{n}}^{\beta_n} \left( x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0} \right) = \frac{\sigma_n^2}{n^{\frac{2}{\alpha}-1}} \partial_\theta q^{n,\beta_n,x_0} \left( \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0} \right) \xrightarrow{n \rightarrow \infty} \partial_\theta b(x_0, \theta) \times \mathbb{E}[\mathbf{1}_{[u,\infty)}(L_1^\alpha) \mathcal{H}_1^{(2)}],$$

where  $\mathcal{H}^{(2)}$  and  $\mathcal{H}_1^{(2)}$  are defined in Proposition 2.5.1.

To finish the proof of Theorem 2.2.2, it remains to show that  $\mathbb{E}[\mathbf{1}_{[u,\infty)}(L_1^\alpha) \mathcal{H}_1^{(2)}] = -\varphi'_\alpha(u)$  and  $\mathbb{E}[\mathbf{1}_{[u,\infty)}(L_1^\alpha) \mathcal{H}^{(2)}] = -[\varphi_\alpha(u) + u\varphi'_\alpha(u)]$ . This is done in Lemma 2.5.6 below.

**Lemma 2.5.6.** *We have*

$$\begin{aligned} \varphi'_\alpha(u) &= -\mathbb{E}[\mathbf{1}_{[u,\infty)}(L_1^\alpha) \mathcal{H}_1^{(2)}], \\ -[\varphi_\alpha(u) + u\varphi'_\alpha(u)] &= \mathbb{E}[\mathbf{1}_{[u,\infty)}(L_1^\alpha) \mathcal{H}^{(2)}], \end{aligned}$$

where  $\varphi_\alpha$  is the density of  $L_1^\alpha$  and  $\mathcal{H}^{(2)}$  and  $\mathcal{H}_1^{(2)}$  are defined in Proposition 2.5.1.

**Proof.** Let us consider the situation where  $b(x, \theta) = \theta$  and  $x_0 = 0$ . In that case, we have  $\bar{Y}_1^{n,\beta,x_0} = \frac{\theta}{n} + \frac{\sigma}{n^{1/\alpha}} L_1^n$  and thus the density of  $\bar{Y}_1^{n,\beta,x_0}$  is related to the density of  $L_1^n$  by the relation,

$$q^{n,\beta,x_0}(u) = \frac{n^{1/\alpha}}{\sigma} \varphi^n \left( \frac{n^{1/\alpha}}{\sigma} \left( u - \frac{\theta}{n} \right) \right).$$

Then,

$$\begin{aligned} \partial_\theta q^{n,\beta,x_0}(u) &= -\frac{n^{2/\alpha-1}}{\sigma^2} (\varphi^n)' \left( \frac{n^{1/\alpha}}{\sigma} \left( u - \frac{\theta}{n} \right) \right), \\ \partial_\sigma q^{n,\beta,x_0}(u) &= -\frac{n^{1/\alpha}}{\sigma^2} \varphi^n \left( \frac{n^{1/\alpha}}{\sigma} \left( u - \frac{\theta}{n} \right) \right) - \frac{(n^{1/\alpha})^2}{\sigma^3} \left( u - \frac{\theta}{n} \right) (\varphi^n)' \left( \frac{n^{1/\alpha}}{\sigma} \left( u - \frac{\theta}{n} \right) \right). \end{aligned}$$

By a change of variables, we get

$$\begin{aligned}\partial_\theta q^{n,\beta,x_0}\left(\frac{u\sigma}{n^{1/\alpha}} + \frac{\theta}{n}\right) &= -\frac{n^{2/\alpha-1}}{\sigma^2}(\varphi^n)'(u) \\ \partial_\sigma q^{n,\beta,x_0}\left(\frac{u\sigma}{n^{1/\alpha}} + \frac{\theta}{n}\right) &= -\frac{n^{1/\alpha}}{\sigma^2}[\varphi^n(u) + u(\varphi^n)'(u)].\end{aligned}$$

Hence, we can apply the results of part *ii*) of Theorem 2.2.2 and (2.5.28), (2.5.29) in this specific setting. This yields

$$(2.5.30) \quad \forall u, \quad (\varphi^n)'(u) \xrightarrow{n \rightarrow \infty} -\mathbb{E}[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}_1^{(2)}],$$

$$(2.5.31) \quad \forall u, \quad [\varphi^n(u) + u(\varphi^n)'(u)] \xrightarrow{n \rightarrow \infty} -\mathbb{E}[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}^{(2)}],$$

$$(2.5.32) \quad \sup_{u,n} |(\varphi^n)'(u)| < \infty,$$

$$(2.5.33) \quad \sup_{u,n} |\varphi^n(u) + u(\varphi^n)'(u)| < \infty.$$

Let us denote  $\mathcal{X}(u) = -\mathbb{E}[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}_1^{(2)}]$  and assume by contradiction that  $\mathcal{X} \neq \varphi'_\alpha$ . Using the continuity of  $u \mapsto \mathcal{X}(u)$ , there exists a smooth, compactly supported function  $f$ , such that  $\int \mathcal{X}(u)f(u)du \neq \int \varphi'_\alpha(u)f(u)du$ . Now, on the one hand we have

$$(2.5.34) \quad \int (\varphi^n)'(u)f(u)du \xrightarrow{n \rightarrow \infty} \int \mathcal{X}(u)f(u)du,$$

where we have used the dominated convergence theorem, together with (2.5.30), (2.5.32).

On the other hand, we can write,

$$\begin{aligned}(2.5.35) \quad \int (\varphi^n)'(u)f(u)du &= -\int \varphi^n(u)f'(u)du \\ &= -\mathbb{E}[f'(L_1^n)] \xrightarrow{n \rightarrow \infty} -\mathbb{E}[f'(L_1^\alpha)]\end{aligned}$$

$$(2.5.36) \quad = -\int \varphi_\alpha(u)f'(u)du = \int \varphi'_\alpha(u)f(u)du$$

where the convergence (2.5.35) is obtained in the same way as (2.4.34). Clearly (2.5.36) contradicts (2.5.34), and we get  $\mathbb{E}[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}_1^{(2)}] = -\varphi'_\alpha(u)$ .

By the same method, let us denote  $\mathcal{X}_1(u) = -\mathbb{E}[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}^{(2)}]$  and assume by contradiction that  $u \mapsto \mathcal{X}_1(u)$  is different from  $u \mapsto [\varphi_\alpha(u) + u(\varphi_\alpha)'(u)]$ . Using the continuity of  $u \mapsto \mathcal{X}_1(u)$ , there exists a smooth, compactly supported function  $f$ , such that  $\int \mathcal{X}_1(u)f(u)du \neq \int [\varphi_\alpha(u) + u(\varphi_\alpha)'(u)]f(u)du$ .



Now, we have

$$(2.5.37) \quad \int [\varphi^n(u) + u(\varphi^n)'(u)] f(u) du \xrightarrow{n \rightarrow \infty} \int \mathcal{X}_1(u) f(u) du,$$

where we have used the dominated convergence theorem, together with (2.5.31), (2.5.33).

On the other hand, letting  $g(u) = uf(u)$  and using the integration by parts formula, we can write,

$$(2.5.38) \quad \begin{aligned} \int [\varphi^n(u) + u(\varphi^n)'(u)] f(u) du &= \int \varphi^n(u) f(u) du + \int (\varphi^n)'(u) g(u) du \\ &= \mathbb{E}[f(L_1^n)] - \int \varphi^n(u) g'(u) du = \mathbb{E}[f(L_1^n)] - \mathbb{E}[g'(L_1^n)] \xrightarrow{n \rightarrow \infty} \mathbb{E}[f(L_1^\alpha)] - \mathbb{E}[g'(L_1^\alpha)] \end{aligned}$$

$$(2.5.39) \quad = \int \varphi_\alpha(u) f(u) du - \int \varphi_\alpha(u) g'(u) du = \int \varphi_\alpha(u) f(u) du + \int \varphi_\alpha'(u) g(u) du$$

where the convergence (2.5.38) is obtained in the same way as (2.4.34). Clearly (2.5.39) contradicts (2.5.37), and the lemma is proved.  $\blacksquare$

## 2.5.4 Proofs of the intermediate lemmas

In this subsection, we give the proofs of Lemmas 2.5.1 - 2.5.5 of Section 2.5.2.

**Proof of Lemma 2.5.1:** *i)* Since  $b$  has bounded derivatives, we obtain from (2.5.3)

$$\sup_{\beta \in Q} |\partial_\theta \bar{Y}_1^{n, \beta, x_0}| \leq \frac{C}{n}.$$

*ii)* From (2.5.4) and Gronwall's Lemma, we get

$$(2.5.40) \quad \begin{aligned} \sup_{\beta \in Q} \sup_{s \in [0, 1]} \left| \partial_\sigma \bar{Y}_s^{n, \beta, x_0} \right| &\leq \frac{C}{n^{1/\alpha}} \sup_{s \in [0, 1]} |L_s^n| \\ &\leq \frac{C}{n^{1/\alpha}} \sup_{s \in [0, 1]} \left| \int_0^s \int_{|z| \leq 1} z \tilde{\mu}(du, dz) \right| + \frac{C}{n^{1/\alpha}} \int_0^1 \int_{|z| > 1} |z| \mu^{(n)}(du, dz) \end{aligned}$$

We now consider the first term of (2.5.40).

Using Kunita's first inequality (see Theorem 4.4.23 in [6])  $\forall p \geq 2$ , we have  $\mathbb{E} \sup_{s \in [0, 1]} \left| \int_0^s \int_{|z| \leq 1} z \tilde{\mu}(du, dz) \right|^p < \infty$ , and the first term of (2.5.40) converges to zero in  $\mathbf{L}^p$ ,  $\forall p \geq 1$ .

We now consider the second term of (2.5.40).

From  $\mu^{(n)}(ds, dz) = \tilde{\mu}^{(n)}(ds, dz) + v^{(n)}(ds, dz)$  then for  $C_2(p)$  a positive constant, we have

$$\begin{aligned} & \frac{1}{n^{p/\alpha}} \mathbb{E} \left( \int_0^1 \int_{|z|>1} |z| \mu^{(n)}(du, dz) \right)^p \\ & \leq \frac{C_2(p)}{n^{p/\alpha}} \left[ \mathbb{E} \left( \int_0^1 \int_{|z|>1} |z| \tilde{\mu}^{(n)}(du, dz) \right)^p + \mathbb{E} \left( \int_0^1 \int_{|z|>1} |z| v^{(n)}(du, dz) \right)^p \right]. \end{aligned}$$

Using again Kunita's first inequality (see Theorem 4.4.23 in [6]), for  $p \geq 2$ , there exists a positive constant  $C_3(p)$  such that

$$\begin{aligned} (2.5.41) \quad & \frac{1}{n^{p/\alpha}} \mathbb{E} \left[ \left| \int_0^1 \int_{|z|>1} |z| \tilde{\mu}^{(n)}(ds, dz) \right| \right]^p \leq \frac{C_3(p)}{n^{p/\alpha}} \left[ \left( \int_0^1 \int_{|z|>1} z^2 v^{(n)}(ds, dz) \right)^{p/2} + \left( \int_0^1 \int_{|z|>1} |z|^p v^{(n)}(ds, dz) \right) \right] \\ & = \frac{C_3(p)}{n^{p/\alpha}} \left( \int_0^1 \int_{|z|>1} z^2 \frac{1}{|z|^{1+\alpha}} \tau(z/n^{1/\alpha}) dz ds \right)^{p/2} + \frac{C_3(p)}{n^{p/\alpha}} \left[ \int_0^1 \int_{|z|>1} |z|^p \frac{1}{|z|^{1+\alpha}} \tau(z/n^{1/\alpha}) dz ds \right] \\ & \leq \frac{2C_3(p)}{n^{p/\alpha}} \left[ \int_0^1 \int_1^{2n^{1/\alpha}} \frac{1}{z^{\alpha-1}} dz ds \right]^{p/2} + \left[ \frac{2C_3(p)}{n^{p/\alpha}} \int_0^1 \int_1^{2n^{1/\alpha}} \frac{1}{z^{\alpha+1-p}} dz ds \right] \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

where we used that  $0 \leq \tau \leq 1$  and  $\tau = 0$  on  $[-2, 2]^c$ . Hence, we get that the second term of (2.5.40) also converges to zero in  $\mathbf{L}^p$ ,  $\forall p \geq 1$ . This finishes the proof of *ii*.  $\blacksquare$

**Proof of Lemma 2.5.2:** Recall that  $D_1^{n,\beta}$  and  $U_1^{n,\beta}$  are given by (2.5.7) and (2.4.3). The part *i*) is proved by decomposing  $\frac{D_1^{n,\beta}}{(U_1^{n,\beta})^2}$ , then we obtain that the main term is (2.5.11) and the remainder terms are

$$\begin{aligned} \mathcal{R}_{4,\beta}^n(1) &= \frac{\int_0^1 \int_{|z|>2} (\epsilon_s^n)^{-2} \rho^n(z) \rho^{n'}(z) \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \mu^{(n)}(ds, dz)}{2\sigma^2 n^{1/\alpha} (\epsilon_1^n)^2 \left( \int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-2} \rho^n(z) \mu^{(n)}(ds, dz) \right)^2} + \\ & \quad + \frac{\int_0^1 \int_{|z|>2} (\epsilon_s^n)^{-2} (\rho^n(z))^2 \left[ \frac{\tau''(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} - \left( \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right)^2 \right] \mu^{(n)}(ds, dz)}{2\sigma^2 n^{1/\alpha} (\epsilon_1^n)^2 \left( \int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-2} \rho^n(z) \mu^{(n)}(ds, dz) \right)^2}, \\ \mathcal{R}_{5,\beta}^n(1) &= \frac{(\epsilon_1^n)^2 \int_0^1 (\epsilon_s^n)^{-2} b''(\bar{Y}_s^{n,\beta,x_0}, \theta) \left( 2L \bar{Y}_s^{n,\beta,x_0} U_s^{n,\beta} + W_s^{n,\beta} \right) ds}{2n^{1+1/\alpha} (U_1^{n,\beta})^2}, \\ \mathcal{R}_{6,\beta}^n(1) &= \frac{(\epsilon_1^n)^2 \int_0^1 (\epsilon_s^n)^{-2} b'''(\bar{Y}_s^{n,\beta,x_0}, \theta) (U_s^{n,\beta})^2 ds}{2n^{1+1/\alpha} (U_1^{n,\beta})^2}. \end{aligned}$$

The part *ii*) is proved by decomposing  $\frac{Q_1^{n,\beta}}{(U_1^{n,\beta})^3}$ , then we obtain that the main term is (2.5.12) and the remainder terms are

$$\begin{aligned}\mathcal{R}_{7,\beta}^n(1) &= \frac{7(\epsilon_1^n)^4 \int_0^1 (\epsilon_1^n)^{-4} b''(\bar{Y}_s^{n,\beta,x_0}, \theta) W_s^{n,\beta} U_s^{n,\beta} ds}{n^{1+1/\alpha} (U_1^{n,\beta})^3} \\ \mathcal{R}_{8,\beta}^n(1) &= \frac{2(\epsilon_1^n)^4 \int_0^1 (\epsilon_1^n)^{-4} b'''(\bar{Y}_s^{n,\beta,x_0}, \theta) (U_s^{n,\beta})^3 ds}{n^{1+1/\alpha} (U_1^{n,\beta})^3}.\end{aligned}$$

We now study the convergence of the main terms.

From (2.3.10), the boundedness of  $b'$  and Lemma 2.3.1, it is clear that  $\hat{\mathcal{H}}_{3,\beta_n}^n(1)$  converges almost surely to  $\mathcal{H}_{3,L^\alpha}(1)$ . Moreover, using again the boundedness of  $b'$ , the upper and lower bounds of  $(\epsilon_s^n)_{s \in [0,1]}$  and the fact that  $\rho^n(z)$  is a non negative function, we deduce the upper bound, for some constant  $C > 0$ ,

(2.5.42)

$$\left| \hat{\mathcal{H}}_{3,\beta}^n(1) \right| \leq C \left[ \frac{\int_0^1 \int_{\mathbb{R}} \left( \rho^n(z) |(\rho^n)''(z)| + \rho^n(z) |\rho^n'(z)| \frac{(1+\alpha)}{|z|} + (\rho^n(z))^2 \frac{(1+\alpha)}{z^2} \right) \mu^{(n)}(ds, dz)}{\left( \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right)^2} \right].$$

Proceeding as in **Step 1** in the proof of Theorem 2.2.1, we show that  $\sup_n \left| \hat{\mathcal{H}}_{3,\beta_n}^n(1) \right|^p$  is integrable. Then applying the dominated convergence theorem, we get  $\hat{\mathcal{H}}_{3,\beta_n}^n(1) \xrightarrow[\mathbf{L}^p, \forall p \geq 1]{n \rightarrow \infty} \mathcal{H}_{3,L^\alpha}(1)$ . In the same way we prove that  $\hat{\mathcal{H}}_{4,\beta_n}^n(1) \xrightarrow[\mathbf{L}^p, \forall p \geq 1]{n \rightarrow \infty} \mathcal{H}_{4,L^\alpha}(1)$ .

Turning to the remainder terms, since  $b$  has bounded derivatives and  $\sup_\beta \sup_s \frac{U_s^{n,\beta}}{U_1^{n,\beta}}$  is bounded, we obtain

(2.5.43)

$$\sup_{\beta \in Q} |\mathcal{R}_{5,\beta}^n(1)| \leq C \sup_{\beta \in Q} \left[ \frac{\sup_{s \in [0,1]} |L \bar{Y}_s^{n,\beta,x_0}|}{n^{1+1/\alpha} U_1^{n,\beta}} + \frac{\sup_{s \in [0,1]} |W_s^{n,\beta}|}{n^{1+1/\alpha} (U_1^{n,\beta})^2} \right], \quad \sup_{\beta \in Q} |\mathcal{R}_{6,\beta}^n(1)| \leq \frac{C}{2n^{1+1/\alpha}},$$

(2.5.44)

$$\sup_{\beta \in Q} |\mathcal{R}_{7,\beta}^n(1)| \leq C \sup_{\beta \in Q} \left[ \frac{\sup_{s \in [0,1]} |W_s^{n,\beta}|}{n^{1+1/\alpha} (U_1^{n,\beta})^2} \right], \quad \sup_{\beta \in Q} |\mathcal{R}_{8,\beta}^n(1)| \leq \frac{C}{n^{1+1/\alpha}}.$$

Moreover from (2.4.3), (2.4.4) and (2.4.5), we get that

$$(2.5.45) \quad \sup_{\beta \in \mathcal{Q}} \frac{\sup_{s \in [0,1]} |L\bar{Y}_s^{n,\beta,x_0}|}{n^{1+1/\alpha} U_1^{n,\beta}} \leq C \left[ \frac{1}{n^{2+1/\alpha}} + \frac{\int_0^1 \int_{\mathbb{R}} \left[ |\rho^{n'}(z)| + \frac{1+\alpha}{|z|} \rho^n(z) \right] \mu^{(n)}(ds, dz)}{n \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz)} \right] \\ + C \left[ \frac{\int_0^1 \int_{|z|>2} z^2 \left| \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right| \mu^{(n)}(ds, dz)}{n^{1+1/\alpha} \int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz)} \right],$$

$$(2.5.45) \quad \sup_{\beta \in \mathcal{Q}} \frac{\sup_{s \in [0,1]} |W_s^{n,\beta}|}{n^{1+1/\alpha} (U_1^{n,\beta})^2} \leq C \left[ \frac{1}{n^{2+1/\alpha}} + \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) |(\rho^n)'(z)| \mu^{(n)}(ds, dz)}{n \left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^2} \right].$$

From the results of **Step 1**, **Step 2** in the proof of Theorem 2.2.1, and the control given in the proof of Theorem 2.3.1 for (2.4.11) we can deduce that  $\sup_{\beta \in \mathcal{Q}} \frac{\sup_{s \in [0,1]} |L\bar{Y}_s^{n,\beta,x_0}|}{n^{1+1/\alpha} U_1^{n,\beta}}$  and  $\sup_{\beta \in \mathcal{Q}} \frac{\sup_{s \in [0,1]} |W_s^{n,\beta}|}{n^{1+1/\alpha} (U_1^{n,\beta})^2}$  converge to zero in  $\mathbf{L}^p$ ,  $\forall p \geq 1$  and we deduce the convergence of  $\sup_{\beta \in \mathcal{Q}} |\mathcal{R}_{5,\beta}^n(1)|$  and  $\sup_{\beta \in \mathcal{Q}} |\mathcal{R}_{7,\beta}^n(1)|$ . It remains to study the convergence of  $\sup_{\beta \in \mathcal{Q}} |\mathcal{R}_{4,\beta}^n(1)|$ .

From the boundedness of  $(c_t^n)_{t \in [0,1]}$ , the definition of  $\rho^n$  (see (2.3.11)), and since  $\mu^{(n)}$  is a positive measure, we have

$$(2.5.46) \quad \sup_{\beta \in \mathcal{Q}} |\mathcal{R}_{4,\beta}^n(1)| \leq C \left[ \frac{\int_0^1 \int_{|z|>2} \left[ \frac{|z|^3}{n^{1/\alpha}} \left| \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right| + \frac{z^4}{n^{1/\alpha}} \left( \left| \frac{\tau''(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right| + \left( \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right)^2 \right) \right] \mu^{(n)}(ds, dz)}{\left( \int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz) \right)^2} \right]$$

$$(2.5.47) \quad \leq C \left[ \frac{\int_0^1 \int_{|z|>2} \left[ \frac{|z|^3}{n^{1/\alpha}} \left| \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right| + \frac{z^4}{n^{1/\alpha}} \left( \left| \frac{\tau''(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right| + \left( \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right)^2 \right) \right] \mu^{(n)}(ds, dz)}{n^{4/\alpha}} \right]$$

where we used  $\int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz) > n^{2/\alpha}$ , if there exists a jump of the Lévy process in  $[-2n^{1/\alpha}, -n^{1/\alpha}) \cup (n^{1/\alpha}, 2n^{1/\alpha}]$ . If there are no jumps in  $[-2n^{1/\alpha}, -n^{1/\alpha}) \cup (n^{1/\alpha}, 2n^{1/\alpha}]$ , since  $\tau(z/n^{1/\alpha}) = 1$  if  $|z| \leq n^{1/\alpha}$ , we have  $\tau'(z/n^{1/\alpha}) = 0$  and  $\tau''(z/n^{1/\alpha}) = 0$ . Thus for  $M(p)$  a positive constant we have

$$\mathbb{E} \sup_{\beta \in \mathcal{Q}} |\mathcal{R}_{4,\beta}^n(1)|^p \leq \frac{C^p}{n^{p/\alpha}} \mathbb{E} \left[ \int_0^1 \int_{|z|>2} \left[ \frac{1}{n^{1/\alpha}} \left( \left| \frac{z}{n^{1/\alpha}} \right| \right)^3 \left| \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right| \right. \right. \\ \left. \left. + \left( \frac{z}{n^{1/\alpha}} \right)^4 \left( \left| \frac{\tau''(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right| + \left( \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right)^2 \right) \right] \mu^{(n)}(ds, dz) \right]^p \\ \leq \frac{M(p)}{n^{p/\alpha}} \left[ \mathbb{E} \left[ \int_0^1 \int_{|z|>2} \frac{1}{n^{1/\alpha}} \left( \left| \frac{z}{n^{1/\alpha}} \right| \right)^3 \left| \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right| \mu^{(n)}(ds, dz) \right]^p + \right. \\ \left. \mathbb{E} \left[ \int_0^1 \int_{|z|>2} \left( \frac{z}{n^{1/\alpha}} \right)^4 \left| \frac{\tau''(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right| \mu^{(n)}(ds, dz) \right]^p + \mathbb{E} \left[ \int_0^1 \int_{|z|>2} \left( \frac{z}{n^{1/\alpha}} \right)^4 \left( \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right)^2 \mu^{(n)}(ds, dz) \right]^p \right]$$

Similarly to the proof of Theorem 2.3.1, we show that under assumption  $\mathbf{H}_1(b_{ii})$ ,  $\sup_{\beta \in Q} |\mathcal{R}_{4,\beta}^n(1)|$  converges to zero as  $n \rightarrow \infty$  in  $\mathbf{L}^p$  for all  $p \geq 1$  and this completes the proof of *ii*).

The result of *iii*) follows from Lemma 2.5.1 *ii*) and the estimation (2.5.45).  $\blacksquare$

**Proof of Lemma 2.5.3:** *i*) From (2.5.5), the fact that  $b$  has bounded derivatives,  $\sup_{\beta} \sup_{0 \leq s \leq 1} \frac{U_s^{n,\beta}}{U_1^{n,\beta}}$  is bounded, the upper and lower bounds of  $((\epsilon_t^n))_{t \in [0,1]}$ , we deduce *i*).

*ii*) From (2.4.3), (2.5.9) we have

$$\begin{aligned} \frac{T_1^{n,\theta}}{n^{\frac{2}{\alpha}-1} \left(U_1^{n,\beta}\right)^2} &= \frac{3(\epsilon_1^n)^3 \int_0^1 (\epsilon_1^n)^{-3} b''(\bar{Y}_s^{n,\beta,x_0}, \theta) V_s^{n,\theta} U_s^{n,\beta} ds}{n^{\frac{2}{\alpha}} \left(U_1^{n,\beta}\right)^2} + \frac{(\epsilon_1^n)^3 \int_0^1 (\epsilon_1^n)^{-3} (\partial_\theta b)'(\bar{Y}_s^{n,\beta,x_0}, \theta) W_s^{n,\beta} ds}{n^{\frac{2}{\alpha}} \left(U_1^{n,\beta}\right)^2} \\ &+ \frac{(\epsilon_1^n)^3 \int_0^1 (\epsilon_1^n)^{-3} b''(\bar{Y}_s^{n,\beta,x_0}, \theta) \partial_\theta \bar{Y}_s^{n,\beta,x_0} W_s^{n,\beta} ds}{n^{\frac{2}{\alpha}} \left(U_1^{n,\beta}\right)^2} + \frac{(\epsilon_1^n)^3 \int_0^1 (\epsilon_1^n)^{-3} (\partial_\theta b)''(\bar{Y}_s^{n,\beta,x_0}, \theta) (U_s^{n,\beta})^2 ds}{n^{\frac{2}{\alpha}} \left(U_1^{n,\beta}\right)^2} \\ &+ \frac{(\epsilon_1^n)^3 \int_0^1 (\epsilon_1^n)^{-3} b'''(\bar{Y}_s^{n,\beta,x_0}, \theta) \partial_\theta \bar{Y}_s^{n,\beta,x_0} (U_s^{n,\beta})^2 ds}{n^{\frac{2}{\alpha}} \left(U_1^{n,\beta}\right)^2}. \end{aligned}$$

We deduce, using Lemma 2.5.1 *i*) and Lemma 2.5.3 *i*) that

$$(2.5.48) \quad \sup_{\beta \in Q} \left| \frac{T_1^{n,\theta}}{n^{\frac{2}{\alpha}-1} \left(U_1^{n,\beta}\right)^2} \right| \leq \frac{C_1}{n^{\frac{2}{\alpha}}} + C_2 \sup_{\beta \in Q} \frac{\sup_{s \in (0,1]} |W_s^{n,\beta}|}{n^{\frac{2}{\alpha}} \left(U_1^{n,\beta}\right)^2},$$

where  $C_1, C_2$  are some deterministic constants. Now from the estimation (2.5.45), we easily deduce that  $\sup_{\beta \in Q} \frac{\sup_{s \in (0,1]} |W_s^{n,\beta}|}{n^{\frac{2}{\alpha}} \left(U_1^{n,\beta}\right)^2}$  tends to zero as  $n \rightarrow \infty$  and then we get *ii*).

*iii*) and *iv*) From (2.4.3), (2.5.6), (2.5.10), an easy computation shows the decomposition of  $\frac{V_1^{n,\sigma}}{U_1^{n,\beta}}$  and  $\frac{1}{n^{1/\alpha}} \frac{T_1^{n,\sigma}}{\left(U_1^{n,\beta}\right)^2}$ , where the leading term is (2.5.18) and the remainder terms are given by

$$\begin{aligned} \mathcal{R}_{9,\beta}^n(1) &= \frac{(\epsilon_1^n)^2 \int_0^1 (\epsilon_s^n)^{-2} \left( b''(\bar{Y}_s^{n,\beta,x_0}, \theta) \partial_\sigma \bar{Y}_s^{n,\beta,x_0} U_s^{n,\beta} \right) ds}{n U_1^{n,\beta}}, \\ \mathcal{R}_{10,\beta}^n(1) &= \frac{3(\epsilon_1^n)^3 \int_0^1 (\epsilon_1^n)^{-3} b''(\bar{Y}_s^{n,\beta,x_0}, \theta) V_s^{n,\sigma} U_s^{n,\beta} ds}{n^{1+1/\alpha} \left(U_1^{n,\beta}\right)^2}, \\ \mathcal{R}_{11,\beta}^n(1) &= \frac{(\epsilon_1^n)^3 \int_0^1 (\epsilon_1^n)^{-3} b''(\bar{Y}_s^{n,\beta,x_0}, \theta) \partial_\sigma \bar{Y}_s^{n,\beta,x_0} W_s^{n,\beta} ds}{n^{1+1/\alpha} \left(U_1^{n,\beta}\right)^2}, \\ \mathcal{R}_{12,\beta}^n(1) &= \frac{(\epsilon_1^n)^3 \int_0^1 (\epsilon_1^n)^{-3} b'''(\bar{Y}_s^{n,\beta,x_0}, \theta) \partial_\sigma \bar{Y}_s^{n,\beta,x_0} (U_s^{n,\beta})^2 ds}{n^{1+1/\alpha} \left(U_1^{n,\beta}\right)^2}. \end{aligned}$$

Moreover, using that  $b$  has bounded derivatives and  $\sup_{\beta} \sup_{0 \leq s \leq 1} \frac{U_s^{n,\beta}}{U_1^{n,\beta}}$  is bounded, the remainder terms satisfy the upper bounds

$$\begin{aligned} \sup_{\beta \in Q} |\mathcal{R}_{9,\beta}^n(1)| &\leq \frac{C}{n} \sup_{\beta \in Q} \sup_{s \in [0,1]} |\partial_{\sigma} \bar{Y}_s^{n,\beta,x_0}|, \quad \sup_{\beta \in Q} |\mathcal{R}_{10,\beta}^n(1)| \leq \frac{C_1}{n^{1+1/\alpha}} \sup_{\beta \in Q} \left[ \frac{\sup_{s \in [0,1]} |V_s^{n,\sigma}|}{U_1^{n,\beta}} \right], \\ \sup_{\beta \in Q} |\mathcal{R}_{11,\beta}^n(1)| &\leq \frac{C_2}{n^{1+1/\alpha}} \sup_{\beta \in Q} \left[ \frac{\sup_{s \in [0,1]} |\partial_{\sigma} \bar{Y}_s^{n,\beta,x_0} W_s^{n,\beta}|}{(U_1^{n,\beta})^2} \right], \quad \sup_{\beta \in Q} |\mathcal{R}_{12,\beta}^n(1)| \leq \frac{C_3}{n^{1+1/\alpha}} \sup_{\beta \in Q} \sup_{s \in [0,1]} |\partial_{\sigma} \bar{Y}_s^{n,\beta,x_0}| \end{aligned}$$

where  $C, C_1, C_2, C_3$  are deterministic constants.

We observe that from Lemma 2.5.1 and Lemma 2.5.2 *iii*), we can deduce immediately the convergences to zero in  $\mathbf{L}^p, \forall p \geq 1$  of the remainder terms  $\sup_{\beta \in Q} |\mathcal{R}_{9,\beta}^n(1)|, \sup_{\beta \in Q} |\mathcal{R}_{11,\beta}^n(1)|$  and  $\sup_{\beta \in Q} |\mathcal{R}_{12,\beta}^n(1)|$ .

For  $\sup_{\beta \in Q} |\mathcal{R}_{10,\beta}^n(1)|$ , the proof follows from Lemma 2.5.1 *ii*), (2.5.6), (2.4.3), the boundedness of  $(e_s^n)$ , the fact that  $b$  has bounded derivatives and  $\sup_{\beta} \sup_s \frac{U_s^{n,\beta}}{U_1^{n,\beta}}$  is bounded.

The convergence of  $\hat{\mathcal{H}}_{5,\beta_n}^n(1)$  is proved as the convergence of  $\hat{\mathcal{H}}_{3,\beta_n}^n(1)$  in the proof of Lemma 2.5.2.

This completes the proof Lemma 2.5.3. ■

**Proof of Lemma 2.5.4:** We first prove (2.5.20). From (2.5.3) we have (we omit the details)

$$\sup_{x_0} |n \partial_{\theta} \bar{Y}_1^{n,\beta_n,x_0} - \partial_{\theta} b(x_0, \theta)| \xrightarrow[n \rightarrow \infty]{a.s.} 0.$$

From the expressions (2.3.14), (2.3.15), using  $\sup_{x_0} \sup_{s \in [0,1]} |e_s^n - 1| \rightarrow 0$  and Lemma 2.3.1, it can be seen that

$$(2.5.49) \quad \sup_{x_0} |\hat{\mathcal{H}}_{\beta_n}^n(1)^q - \mathcal{H}_{L^a}(1)^q| \xrightarrow[n \rightarrow \infty]{a.s.} 0, \quad \forall q \geq 1.$$

We deduce that almost surely, one has the convergence

$$(2.5.50) \quad \forall q \geq 1, \quad \sup_{x_0} \left| n \partial_{\theta} \bar{Y}_1^{n,\beta_n,x_0} \left( \hat{\mathcal{H}}_{\beta_n}^n(1) \right)^q - \partial_{\theta} b(x_0, \theta) (\mathcal{H}_{L^a}(1))^q \right| \xrightarrow[n \rightarrow \infty]{} 0.$$

Using (2.4.23) and  $\sup_{\beta, x_0} |\partial_{\theta} \bar{Y}_1^{n,\beta,x_0}| \leq \frac{C}{n}$ , we can apply the dominated convergence theorem and see that the convergence (2.5.50) holds in  $\mathbf{L}^p$ -norm for all  $p \geq 1$ .

For (2.5.21): the proof is similar to (2.5.20).

For (2.5.22): using (2.5.4), Gronwall's Lemma and Lemma 2.3.1, we can prove

$$\sup_{x_0} |n^{1/\alpha} \partial_{\sigma} \bar{Y}_1^{n,\beta_n,x_0} - L_1^{\alpha}| \xrightarrow[n \rightarrow \infty]{a.s.} 0,$$

and from (2.5.49) we deduce

$$\forall q \geq 1, \quad \sup_{x_0} \left| n^{1/\alpha} \partial_\sigma \bar{Y}_1^{n, \beta_n, x_0} \left( \hat{\mathcal{H}}_{\beta_n}^n(1) \right)^q - L_1^\alpha \left( \mathcal{H}_{L^\alpha}(1) \right)^q \right| \xrightarrow{n \rightarrow \infty} 0.$$

Consequently, to prove the convergence in  $L^p$ -norm, it remains to check

$$(2.5.51) \quad \forall p, q \geq 1, \quad \mathbb{E} \sup_{n, \beta, x_0} \left| n^{1/\alpha} \partial_\sigma \bar{Y}_1^{n, \beta_n, x_0} \left( \hat{\mathcal{H}}_\beta^n(1) \right)^q \right|^p < \infty.$$

Using again (2.5.4) and Gronwall's Lemma, we have

$$\sup_{x_0, \beta} |n^{1/\alpha} \partial_\sigma \bar{Y}_1^{n, \beta, x_0}| \leq C \sup_{t \in [0, 1]} |L_t^n|,$$

and (2.5.51) reduces to

$$(2.5.52) \quad \forall p, q \geq 1, \quad \mathbb{E} \sup_{n, \beta, x_0} \left| \sup_{t \in [0, 1]} |L_t^n| \left( \hat{\mathcal{H}}_\beta^n(1) \right)^q \right|^p < \infty.$$

Let us recall that  $L_t^n = \int_0^t \int_{|z| \leq 1} z \tilde{\mu}(ds, dz) + \int_0^t \int_{|z| > 1} z \mu^{(n)}(ds, dz)$ . Then we have

$$\sup_{t \in [0, 1]} |L_t^n| \leq \sup_{t \in [0, 1]} \left| \int_0^t \int_{|z| \leq 1} z \tilde{\mu}(ds, dz) \right| + \int_0^1 \int_{|z| > 1} |z| \mu^{(n)}(ds, dz).$$

From this decomposition and since  $\sup_{t \in [0, 1]} \left| \int_0^t \int_{|z| \leq 1} z \tilde{\mu}(ds, dz) \right|^p$  is integrable, for all  $p \geq 1$ , we see that (2.5.52) is a consequence of (2.4.23) and the following bound

$$(2.5.53) \quad \forall p \geq 1, \quad \mathbb{E} \sup_{n, \beta, x_0} \left| \hat{\mathcal{H}}_\beta^n(1) \int_0^1 \int_{|z| > 1} |z| \mu^{(n)}(ds, dz) \right|^p < \infty.$$

To prove (2.5.53), we first remark from (2.4.17) that for  $C$  a positive constant,

$$(2.5.54) \quad \left| \hat{\mathcal{H}}_\beta^n(1) \int_0^1 \int_{|z| > 1} |z| \mu^{(n)}(ds, dz) \right| \\ \leq C \left[ \left( \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) |(\rho^n)'(z)| \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^2} + \frac{\int_0^1 \int_{\mathbb{R}} \left[ |\rho^{n'}(z)| + \frac{1+\alpha}{|z|} \rho^n(z) \right] \mu^{(n)}(ds, dz)}{\int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz)} \right) \int_0^1 \int_{2 \geq |z| > 1} |z| \mu^{(n)}(ds, dz) \right] \\ + C \left[ \left( \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) |(\rho^n)'(z)| \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^2} + \frac{\int_0^1 \int_{\mathbb{R}} \left[ |\rho^{n'}(z)| + \frac{1+\alpha}{|z|} \rho^n(z) \right] \mu^{(n)}(ds, dz)}{\int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz)} \right) \int_0^1 \int_{|z| > 2} |z| \mu^{(n)}(ds, dz) \right]$$

Considering the first term in the right-hand side of (2.5.54), from the proofs of **Step 1** and **Step 2** in Theorem 2.2.1, we deduce that it is bounded by a random variable independent of  $n, \beta$  and

$x_0$  and belonging to  $\cap_{p \geq 1} \mathbf{L}^p$ .

We now consider the second term in the right-hand side of (2.5.54). From (2.3.11), we have

(2.5.55)

$$\begin{aligned} & \left| \left( \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) |(\rho^n)'(z)| \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^2} + \frac{\int_0^1 \int_{\mathbb{R}} \left[ |\rho^n'(z)| + \frac{1+\alpha}{|z|} \rho^n(z) \right] \mu^{(n)}(ds, dz)}{\int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz)} \right) \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(ds, dz) \right| \\ & \leq \left[ \frac{\int_0^1 \int_{|z| \leq 2} \rho(z) |\rho'(z)| \mu(ds, dz)}{\left( \int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz) \right)^2} + \frac{\int_0^1 \int_{|z| \leq 2} \left( |\rho'(z)| + \rho(z) \frac{1+\alpha}{|z|} \right) \mu(ds, dz)}{\int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz)} \right] \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \\ & \quad + \left[ \frac{\int_0^1 \int_{|z|>2} 2|z|^3 \mu^{(n)}(ds, dz)}{\left( \int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz) \right)^2} + \frac{\int_0^1 \int_{|z|>2} (3+\alpha) |z| \mu^{(n)}(ds, dz)}{\int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz)} \right] \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \end{aligned}$$

Using the Cauchy-Schwarz inequality

$$\int_0^1 \int_{|z|>2} \mu^{(n)}(dt, dz) \times \int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz) \geq \left( \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right)^2$$

we get:

(2.5.56)

$$\begin{aligned} & \frac{\int_0^1 \int_{|z|>2} 2|z|^3 \mu^{(n)}(ds, dz)}{\left( \int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz) \right)^2} \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \leq \frac{\int_0^1 \int_{|z|>2} 2|z|^3 \mu^{(n)}(ds, dz)}{\left( \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right)^3} \left( \int_0^1 \int_{|z|>2} \mu^{(n)}(dt, dz) \right)^2 \\ & = \frac{\sum_{i=1}^{N_1} 2|Z_i|^3}{\left( \sum_{i=1}^{N_1} |Z_i| \right)^3} \left( \int_0^1 \int_{|z|>2} \mu^{(n)}(dt, dz) \right)^2 \leq 2 \left( \int_0^1 \int_{|z|>2} \mu^{(n)}(dt, dz) \right)^2 \end{aligned}$$

and

$$(2.5.57) \quad \frac{\int_0^1 \int_{|z|>2} (3+\alpha) |z| \mu^{(n)}(ds, dz)}{\int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz)} \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \leq (3+\alpha) \int_0^1 \int_{|z|>2} \mu^{(n)}(dt, dz).$$

Combining (2.5.56), (2.5.57) with (2.5.55), it follows that the second term in the right-hand side of (2.5.54) is also bounded by a random variable independent of  $n, \beta$  and  $x_0$  and belonging to  $\cap_{p \geq 1} \mathbf{L}^p$ . Consequently, we get (2.5.53) and this achieves the proof of (2.5.22).

For (2.5.23), the proof is similar to (2.5.22). ■

**Proof of Lemma 2.5.5:** From (2.5.13), we prove *i*) and *ii*) proceeding as in the proof of (2.5.20), and *iii*) and *iv*) proceeding as in the proof of (2.5.22). ■



## 2.6 Appendix. Representation of the transition density via Malliavin calculus

The aim of this section is to represent the density of a pure jump Lévy process as well as its derivative and its logarithm derivative as an expectation, using the Malliavin calculus for jump processes developed by Bichteler, Gravereaux and Jacod [8] and used by Clément-Gloter [17]. We are not exactly in the context of [8], where the compensator of the Poisson measure is uniform on  $[0, 1] \times E$ . In fact, in our context the compensator of the Poisson measure is  $dt \times g(z)dz$ , where  $g$  is the density of the Lévy measure and we need to adapt slightly the definitions of Malliavin operators given in [8]. We recall here the appropriate integration by parts setting developed in [17] for the reader convenience.

### 2.6.1 Integration by parts setting

We consider a filtered probability space  $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \in [0,1]}, \mathbb{P})$  endowed with a Poisson random measure  $\mu$  on  $[0, 1] \times E$ , where  $E$  is an open subset of  $\mathbb{R}$ , with compensator  $\nu$  on  $[0, 1] \times E$  and with compensated measure  $\tilde{\mu} = \mu - \nu$ . We now consider the process  $(Y_t^\beta)_{t \in [0,1]}$ , the solution of

$$(2.6.1) \quad Y_t^\beta = y_0 + \int_0^t a(Y_s^\beta, \theta) ds + c\sigma \int_0^t \int_E z \tilde{\mu}(ds, dz),$$

where the parameter  $\beta = (\theta, \sigma)^T$  belongs to  $\mathbb{R} \times (0, \infty)$ ,  $a$  is a real valued function and  $c$  is a constant.

This is the framework of Clément-Gloter [17] and our aim is to give some explicit representation formulas for the density of  $Y_1^\beta$  and its derivative with respect to  $\beta$ .

We assume that the following assumptions are fulfilled.

- H:** (a) The function  $a$  has bounded derivatives up to order five with respect to both variables.  
 (b) The compensator of the Poisson random measure  $\mu$  is given by  $\nu(dt, dz) = dt \times g(z)dz$  with  $g \geq 0$  on  $E$ ,  $\mathcal{C}^1$  on  $E$  and such that

$$\forall p \geq 2, \int_E |z|^p g(z) dz < \infty.$$

We now recall the Malliavin operators  $L$  and  $\Gamma$  and their basic properties (see Bichteler, Gravereaux, Jacod [8], Chapter IV, Section 8-9-10). For a test function  $f : [0, 1] \times E \rightarrow \mathbb{R}$  ( $f$  is measurable,

$\mathcal{C}^2$  with respect to the second variable, with bounded derivative, and  $f \in \cap_{p \geq 1} \mathbf{L}^p(\nu)$  we set  $\mu(f) = \int_0^1 \int_E f(t, z) \mu(dt, dz)$ . We introduce an auxiliary function  $\rho : E \mapsto (0, \infty)$  such that  $\rho$  admits a derivative and  $\rho, \rho'$  and  $\rho \frac{g'}{g}$  belong to  $\cap_{p \geq 1} \mathbf{L}^p(g(z) dz)$ . With these notations, we define the Malliavin operator  $L$ , on a simple functional  $\mu(f)$ , in the same way as in [17] by the following equations :

$$L(\mu(f)) = \frac{1}{2} \mu \left( \rho' f' + \rho \frac{g'}{g} f' + \rho f'' \right),$$

where  $f'$  and  $f''$  are the derivatives with respect to the second variable. For  $\Phi = F(\mu(f_1), \dots, \mu(f_k))$ , with  $F$  of class  $\mathcal{C}^2$ , we set

$$L\Phi = \sum_{i=1}^k \frac{\partial F}{\partial x_i}(\mu(f_1), \dots, \mu(f_k)) L(\mu(f_i)) + \frac{1}{2} \sum_{i,j=1}^k \frac{\partial^2 F}{\partial x_i \partial x_j}(\mu(f_1), \dots, \mu(f_k)) \mu(\rho f'_i f'_j).$$

These definitions permit to construct a linear operator  $L$  on a space  $D \subset \cap_{p \geq 1} \mathbf{L}^p$  with the same basic properties as in [17, equations (i)-(iii), p.2322].

We associate to  $L$ , the symmetric bilinear operator  $\Gamma$ :

$$\Gamma(\Phi, \Psi) = L(\Phi\Psi) - \Phi L\Psi - \Psi L\Phi.$$

Moreover, if  $f$  and  $h$  are two test functions, we have:

$$\Gamma(\mu(f), \mu(h)) = \mu(\rho f' h').$$

These operators satisfy the following properties (see [8, equation (8-3)])

$$(2.6.2) \quad LF(\Phi) = F'(\Phi)L\Phi + \frac{1}{2}F''(\Phi)\Gamma(\Phi, \Phi),$$

$$\Gamma(F(\Phi), \Psi) = F'(\Phi)\Gamma(\Phi, \Psi),$$

$$\Gamma(F(\Phi_1, \Phi_2), \Psi) = \partial_{\Phi_1} F(\Phi_1, \Phi_2)\Gamma(\Phi_1, \Psi) + \partial_{\Phi_2} F(\Phi_1, \Phi_2)\Gamma(\Phi_2, \Psi).$$

The operator  $L$  and the operator  $\Gamma$  permit to establish the following integration by parts formula (see [8, Propositions 8-10, p.103]).

**Proposition 2.6.1.** *For  $\Phi$  and  $\Psi$  in  $D$ , and  $f$  bounded with bounded derivatives up to order two, we have*

$$\mathbb{E} f'(\Phi) \Psi \Gamma(\Phi, \Phi) = \mathbb{E} f(\Phi) (-2\Psi L\Phi - \Gamma(\Phi, \Psi)).$$

Moreover, if  $\Gamma(\Phi, \Phi)$  is invertible and  $\Gamma^{-1}(\Phi, \Phi) \in \cap_{p \geq 1} \mathbf{L}^p$ , we have

$$(2.6.3) \quad \mathbb{E}f'(\Phi)\Psi = \mathbb{E}f(\Phi)\mathcal{H}_\Phi(\Psi),$$

with

$$(2.6.4) \quad \mathcal{H}_\Phi(\Psi) = -2\Psi\Gamma^{-1}(\Phi, \Phi)L\Phi - \Gamma(\Phi, \Psi\Gamma^{-1}(\Phi, \Phi))$$

$$(2.6.5) \quad = -2\Psi\Gamma^{-1}(\Phi, \Phi)L\Phi - \frac{1}{\Gamma(\Phi, \Phi)}\Gamma(\Phi, \Psi) + \frac{\Psi}{\Gamma(\Phi, \Phi)^2}\Gamma(\Phi, \Gamma(\Phi, \Phi)).$$

## 2.6.2 Representation of the density of $Y_1^\beta$ and its derivative

The integration by parts setting of the preceding section permits to derive the existence of the density of  $Y_1^\beta$  given by (2.6.1), and gives a representation of this density as an expectation. From Bichteler, Gravereaux, Jacod [8, Section 10, p.130], we know that  $\forall t > 0$ , the variable  $Y_t^\beta$ , the solution of (2.6.1), belongs to the domain of the operator  $L$ , and we can compute  $LY_t^\beta$  and  $\Gamma(Y_t^\beta, Y_t^\beta)$  as in [17]. We recall the representation formula for the density of  $Y_1^\beta$  (see [17]).

**Theorem 2.6.1.** [Clément-Gloter [17]]: *Let us denote by  $q^\beta$  the density of  $Y_1^\beta$ . We assume that  $\mathbf{H}$  holds and that the auxiliary function  $\rho$  satisfies:*

$$(2.6.6) \quad \liminf_{u \rightarrow \infty} \frac{1}{\ln u} \int_E 1_{\{\rho(z) \geq 1/u\}} g(z) dz = +\infty.$$

Then,

$$q^\beta(u) = \mathbb{E}(1_{\{Y_1^\beta \geq u\}} \mathcal{H}_{Y_1^\beta}(1)),$$

with,

$$(2.6.7) \quad \mathcal{H}_{Y_1^\beta}(1) = \frac{\Gamma(Y_1^\beta, \Gamma(Y_1^\beta, Y_1^\beta))}{\Gamma(Y_1^\beta, Y_1^\beta)^2} - 2 \frac{LY_1^\beta}{\Gamma(Y_1^\beta, Y_1^\beta)} = \frac{W_1^\beta}{(U_1^\beta)^2} - 2 \frac{LY_1^\beta}{U_1^\beta},$$

where the processes  $(LY_t^\beta)$  and  $(U_t^\beta) = \Gamma(Y_t^\beta, Y_t^\beta)$  are solutions of the linear equations:

$$(2.6.8) \quad LY_t^\beta = \int_0^t a'(Y_s^\beta, \theta) LY_s^\beta ds + \frac{1}{2} \int_0^t a''(Y_s^\beta, \theta) U_s^\beta ds + \frac{c\sigma}{2} \int_0^t \int_E \left( \rho'(z) + \rho(z) \frac{g'(z)}{g(z)} \right) \mu(ds, dz),$$

$$(2.6.9) \quad U_t^\beta = 2 \int_0^t a'(Y_s^\beta, \theta) U_s^\beta ds + c^2 \sigma^2 \int_0^1 \int_E \rho(z) \mu(ds, dz).$$

The process  $(W_t^\beta) = \Gamma(Y_t^\beta, U_t^\beta)$  is the solution of the linear equation:

$$(2.6.10) \quad W_t^\beta = 3 \int_0^t a'(Y_s^\beta, \theta) W_s^\beta ds + 2 \int_0^t a''(Y_s^\beta, \theta) (U_s^\beta)^2 ds + c^3 \sigma^3 \int_0^t \int_E \rho(z) \rho'(z) \mu(ds, dz).$$

In [17], the authors studied the derivative of  $q^\beta$  with respect to the drift parameter  $\theta$  only. Here, we intend to study the derivative of  $q^\beta$  with respect to both parameters  $\theta$  and  $\sigma$ . We first remark that  $(Y_t^\beta)_t$  admits derivatives with respect to  $\theta$  and  $\sigma$  (see [8, Theorem 5.24 p.51]), denoted by  $(\partial_\theta Y_t^\beta)_t$  and  $(\partial_\sigma Y_t^\beta)_t$  respectively. Moreover,  $(\partial_\theta Y_t^\beta)_t, (\partial_\sigma Y_t^\beta)_t$  are respectively the unique solutions of

$$(2.6.11) \quad \partial_\theta Y_t^\beta = \int_0^t a'(Y_s^\beta, \theta) \partial_\theta Y_s^\beta ds + \int_0^t \partial_\theta a(Y_s^\beta, \theta) ds,$$

$$(2.6.12) \quad \partial_\sigma Y_t^\beta = \int_0^t a'(Y_s^\beta, \theta) \partial_\sigma Y_s^\beta ds + c \int_0^t \int_E z \tilde{\mu}(ds, dz).$$

By iterating the integration by parts formula, since  $Y_1^\beta$  admits derivatives with respect to  $\theta$  and  $\sigma$ , one can prove, under the assumption **H**, the existence and the continuity in  $\beta$  of  $\nabla_\beta q^\beta$  (see Theorem 4-21 in [8]), moreover, we will represent it as an expectation in Theorem 2.6.3. The next result extends the result of Theorem 5 in [17], by giving an expression for the logarithm derivatives of the density w.r.t.  $(\theta, \sigma)$  in terms of a conditional expectation.

**Theorem 2.6.2.** *Under the assumptions of Theorem 2.6.1,*

$$(2.6.13) \quad \frac{\nabla_\beta q^\beta}{q^\beta}(u) = \begin{pmatrix} \frac{\partial_\theta q^\beta}{q^\beta}(u) \\ \frac{\partial_\sigma q^\beta}{q^\beta}(u) \end{pmatrix} = \mathbb{E}(\mathcal{H}_{Y_1^\beta}(\nabla_\beta Y_1^\beta) | Y_1^\beta = u),$$

where

$$(2.6.14) \quad \mathcal{H}_{Y_1^\beta}(\nabla_\beta Y_1^\beta) := \begin{pmatrix} \mathcal{H}_{Y_1^\beta}(\partial_\theta Y_1^\beta) \\ \mathcal{H}_{Y_1^\beta}(\partial_\sigma Y_1^\beta) \end{pmatrix} = -2 \begin{pmatrix} \partial_\theta Y_1^\beta \\ \partial_\sigma Y_1^\beta \end{pmatrix} \frac{LY_1^\beta}{U_1^\beta} + \begin{pmatrix} \partial_\theta Y_1^\beta \\ \partial_\sigma Y_1^\beta \end{pmatrix} \frac{W_1^\beta}{(U_1^\beta)^2} - \frac{1}{U_1^\beta} \begin{pmatrix} \Gamma(Y_1^\beta, \partial_\theta Y_1^\beta) \\ \Gamma(Y_1^\beta, \partial_\sigma Y_1^\beta) \end{pmatrix},$$

$LY_1^\beta, U_1^\beta$  and  $W_1^\beta$  are given in Theorem 2.6.1, the process  $(V_t^\theta) = \Gamma(Y_t^\beta, \partial_\theta Y_t^\beta)$  is the solution of

$$(2.6.15) \quad V_t^\theta = 2 \int_0^t a'(Y_s^\beta, \theta) V_s^\theta ds + \int_0^t U_s^\beta \left[ (\partial_\theta a)'(Y_s^\beta, \theta) + a''(Y_s^\beta, \theta) \partial_\theta Y_s^\beta \right] ds,$$

and the process  $(V_t^\sigma) = \Gamma(Y_t^\beta, \partial_\sigma Y_t^\beta)$  is the solution of

$$(2.6.16) \quad V_t^\sigma = 2 \int_0^t a'(Y_s^\beta, \theta) V_s^\sigma ds + \int_0^t a''(Y_s^\beta, \theta) \partial_\sigma Y_s^\beta U_s^\beta ds + c^2 \sigma \int_0^t \int_E \rho(z) \mu(ds, dz).$$

**Proof.** Theorem 2.6.2 is an extension of Theorem 5 in [17] where the main novelty is the expression for  $\frac{\partial_\sigma q^\beta}{q^\beta}$ . For the computation of the new term  $\mathcal{H}_{Y_1^\beta}(\partial_\sigma Y_1^\beta)$ , we apply Theorem 10-3 in

[8] to the stochastic differential equation satisfied by the vector  $(Y_t^\beta, U_t^\beta, \partial_\sigma Y_t^\beta)^T$ , this gives the above expression for  $(V_t^\sigma)$ . ■

We end this subsection with an explicit representation of  $\nabla_\beta q^\beta(u)$  which gives a computation of the iterated Malliavin weight  $\mathcal{H}_{Y_1^\beta}(\mathcal{H}_{Y_1^\beta}(\nabla_\beta Y_1^\beta))$ .

**Theorem 2.6.3.** *Under the assumptions of Theorem 2.6.1,*

$$(2.6.17) \quad \nabla_\beta q^\beta(u) = \begin{pmatrix} \partial_\theta q^\beta(u) \\ \partial_\sigma q^\beta(u) \end{pmatrix} = \mathbb{E} \left[ \mathbf{1}_{\{Y_1^\beta \geq u\}} \mathcal{H}_{Y_1^\beta}(\mathcal{H}_{Y_1^\beta}(\nabla_\beta Y_1^\beta)) \right],$$

where

$$(2.6.18) \quad \mathcal{H}_{Y_1^\beta}(\mathcal{H}_{Y_1^\beta}(\nabla_\beta Y_1^\beta)) = -2\mathcal{H}_{Y_1^\beta}(\nabla_\beta Y_1^\beta) \frac{LY_1^\beta}{U_1^\beta} + \mathcal{H}_{Y_1^\beta}(\nabla_\beta Y_1^\beta) \frac{W_1^\beta}{(U_1^\beta)^2} - \left( \frac{\Gamma(Y_1^\beta, \mathcal{H}_{Y_1^\beta}(\partial_\theta Y_1^\beta))}{\Gamma(Y_1^\beta, \mathcal{H}_{Y_1^\beta}(\partial_\sigma Y_1^\beta))} \right) \frac{1}{U_1^\beta},$$

where  $\partial_\theta Y_1^\beta, \partial_\sigma Y_1^\beta$  are respectively given by equations (2.6.11), (2.6.12) and  $U_1^\beta, W_1^\beta$  are computed in Theorem 2.6.1,  $\mathcal{H}_{Y_1^\beta}(\nabla_\beta Y_1^\beta)$  is given in Theorem 2.6.2.

**Proof.** Let  $f$  be a smooth functions with compact support. Then,

$$\nabla_\beta \mathbb{E} \left[ f(Y_1^\beta) \right] = \int_{\mathbb{R}} du \nabla_\beta q^\beta(u) f(u).$$

On the other hand, using the integration by parts formula of the Malliavin calculus, we have

$$\begin{aligned} \nabla_\beta \mathbb{E} \left[ f(Y_1^\beta) \right] &= \mathbb{E} \left[ f'(Y_1^\beta) \nabla_\beta Y_1^\beta \right] \\ &= \mathbb{E} \left[ f(Y_1^\beta) \mathcal{H}_{Y_1^\beta}(\nabla_\beta Y_1^\beta) \right] \\ &= \mathbb{E} \left[ F(Y_1^\beta) \mathcal{H}_{Y_1^\beta} \left( \mathcal{H}_{Y_1^\beta}(\nabla_\beta Y_1^\beta) \right) \right] \end{aligned}$$

where  $F$  denotes a primitive function of  $f$ . If  $f$  converges to Dirac mass at some point  $u$ , from the estimates above, we can deduce (2.6.17). Moreover, from (2.6.5) we also get (2.6.18). ■

To complete the result of Theorem 2.6.3, we give the expressions for  $\Gamma(Y_1^\beta, \mathcal{H}_{Y_1^\beta}(\partial_\theta Y_1^\beta))$  and  $\Gamma(Y_1^\beta, \mathcal{H}_{Y_1^\beta}(\partial_\sigma Y_1^\beta))$ .

**Lemma 2.6.1.** *Under the assumptions of Theorem 2.6.1,*

(2.6.19)

$$\begin{aligned} \begin{pmatrix} \Gamma(Y_1^\beta, \mathcal{H}_{Y_1^\beta}(\partial_\theta Y_1^\beta)) \\ \Gamma(Y_1^\beta, \mathcal{H}_{Y_1^\beta}(\partial_\sigma Y_1^\beta)) \end{pmatrix} &= \begin{pmatrix} V_1^\theta \\ V_1^\sigma \end{pmatrix} \mathcal{H}_{Y_1^\beta}(1) - \begin{pmatrix} \partial_\theta Y_1^\beta \\ \partial_\sigma Y_1^\beta \end{pmatrix} \frac{2D_1^\beta}{U_1^\beta} - \begin{pmatrix} \partial_\theta Y_1^\beta \\ \partial_\sigma Y_1^\beta \end{pmatrix} \frac{\mathcal{H}_{Y_1^\beta}(1)W_1^\beta}{U_1^\beta} + \begin{pmatrix} \partial_\theta Y_1^\beta \\ \partial_\sigma Y_1^\beta \end{pmatrix} \frac{Q_1^\beta}{(U_1^\beta)^2} - \\ &\quad - \begin{pmatrix} T_1^\theta \\ T_1^\sigma \end{pmatrix} \frac{1}{U_1^\beta} + \begin{pmatrix} V_1^\theta \\ V_1^\sigma \end{pmatrix} \frac{W_1^\beta}{(U_1^\beta)^2}, \end{aligned}$$

where  $\partial_\theta Y_1^\beta, \partial_\sigma Y_1^\beta$  are respectively given in (2.6.11), (2.6.12),  $U_1^\beta, W_1^\beta$  are computed in Theorem 2.6.1,  $V_1^\theta, V_1^\sigma$  are computed in Theorem 2.6.2,  $\mathcal{H}_{Y_1^\beta}(1)$  is given in (2.6.7) and  $D_1^\beta = \Gamma(Y_1^\beta, LY_1^\beta)$ ,  $Q_1^\beta = \Gamma(Y_1^\beta, W_1^\beta)$ ,  $T_1^\theta = \Gamma(Y_1^\beta, V_1^\theta)$  and  $T_1^\sigma = \Gamma(Y_1^\beta, V_1^\sigma)$ .

**Proof.** From the basic properties of the operators  $L$  and  $\Gamma$  (linearity and the chain rule property) stated in Section 2.6.1, we get that

$$\begin{aligned} \Gamma(Y_1^\beta, \mathcal{H}_{Y_1^\beta}(\partial_\theta Y_1^\beta)) &= \Gamma \left[ Y_1^\beta, -2\partial_\theta Y_1^\beta \frac{LY_1^\beta}{U_1^\beta} \right] + \Gamma \left[ Y_1^\beta, \partial_\theta Y_1^\beta \frac{W_1^\beta}{(U_1^\beta)^2} \right] + \Gamma \left[ Y_1^\beta, -\frac{\Gamma(Y_1^\beta, \partial_\theta Y_1^\beta)}{U_1^\beta} \right], \\ \Gamma(Y_1^\beta, \mathcal{H}_{Y_1^\beta}(\partial_\sigma Y_1^\beta)) &= \Gamma \left[ Y_1^\beta, -2\partial_\sigma Y_1^\beta \frac{LY_1^\beta}{U_1^\beta} \right] + \Gamma \left[ Y_1^\beta, \partial_\sigma Y_1^\beta \frac{W_1^\beta}{(U_1^\beta)^2} \right] + \Gamma \left[ Y_1^\beta, -\frac{\Gamma(Y_1^\beta, \partial_\sigma Y_1^\beta)}{U_1^\beta} \right], \end{aligned}$$

where

$$\begin{aligned} \Gamma \left[ Y_1^\beta, -2\partial_\theta Y_1^\beta \frac{LY_1^\beta}{U_1^\beta} \right] &= -2 \frac{LY_1^\beta}{U_1^\beta} \Gamma(Y_1^\beta, \partial_\theta Y_1^\beta) - 2 \frac{\partial_\theta Y_1^\beta}{U_1^\beta} \Gamma(Y_1^\beta, LY_1^\beta) + 2\partial_\theta Y_1^\beta \frac{LY_1^\beta}{(U_1^\beta)^2} \Gamma(Y_1^\beta, U_1^\beta) \\ &= -2 \frac{LY_1^\beta}{U_1^\beta} V_1^\theta - 2 \frac{\partial_\theta Y_1^\beta}{U_1^\beta} D_1^\beta + 2\partial_\theta Y_1^\beta \frac{LY_1^\beta}{(U_1^\beta)^2} W_1^\beta. \\ \Gamma \left[ Y_1^\beta, \partial_\theta Y_1^\beta \frac{W_1^\beta}{(U_1^\beta)^2} \right] &= \frac{W_1^\beta}{(U_1^\beta)^2} \Gamma(Y_1^\beta, \partial_\theta Y_1^\beta) + \frac{\partial_\theta Y_1^\beta}{(U_1^\beta)^2} \Gamma(Y_1^\beta, W_1^\beta) - \frac{2\partial_\theta Y_1^\beta W_1^\beta}{(U_1^\beta)^3} \Gamma(Y_1^\beta, U_1^\beta) \\ &= \frac{W_1^\beta}{(U_1^\beta)^2} V_1^\theta + \frac{\partial_\theta Y_1^\beta}{(U_1^\beta)^2} Q_1^\beta - \frac{2\partial_\theta Y_1^\beta W_1^\beta}{(U_1^\beta)^3} W_1^\beta. \\ \Gamma \left[ Y_1^\beta, -\frac{\Gamma(Y_1^\beta, \partial_\theta Y_1^\beta)}{U_1^\beta} \right] &= -\frac{\Gamma(Y_1^\beta, \Gamma(Y_1^\beta, \partial_\theta Y_1^\beta))}{U_1^\beta} + \frac{\Gamma(Y_1^\beta, \partial_\theta Y_1^\beta)}{(U_1^\beta)^2} \Gamma(Y_1^\beta, U_1^\beta) = -\frac{T_1^\theta}{U_1^\beta} + \frac{V_1^\theta}{(U_1^\beta)^2} W_1^\beta. \end{aligned}$$

Similarly, we have

$$\Gamma \left[ Y_1^\beta, -2\partial_\sigma Y_1^\beta \frac{LY_1^\beta}{U_1^\beta} \right] = -2 \frac{LY_1^\beta}{U_1^\beta} V_1^\sigma - 2 \frac{\partial_\sigma Y_1^\beta}{U_1^\beta} D_1^\beta + 2\partial_\sigma Y_1^\beta \frac{LY_1^\beta}{(U_1^\beta)^2} W_1^\beta.$$

$$\Gamma \left[ Y_1^\beta, \partial_\sigma Y_1^\beta \frac{W_1^\beta}{(U_1^\beta)^2} \right] = \frac{W_1^\beta}{(U_1^\beta)^2} V_1^\sigma + \frac{\partial_\sigma Y_1^\beta}{(U_1^\beta)^2} Q_1^\beta - \frac{2\partial_\sigma Y_1^\beta W_1^\beta}{(U_1^\beta)^3} W_1^\beta.$$

$$\Gamma \left[ Y_1^\beta, -\frac{\Gamma(Y_1^\beta, \partial_\sigma Y_1^\beta)}{U_1^\beta} \right] = -\frac{T_1^\sigma}{U_1^\beta} + \frac{V_1^\sigma}{(U_1^\beta)^2} W_1^\beta.$$

Then, from (2.6.7) and the above estimates, we get the formula (2.6.19), after some calculus and the proof is complete.  $\blacksquare$

**Lemma 2.6.2.** *Under the assumptions of Theorem 2.6.1, there are versions of the processes  $(D_t^\beta) = (\Gamma(Y_t^\beta, LY_t^\beta))_t$ ,  $(Q_t^\beta) = \Gamma(Y_t^\beta, W_t^\beta)_t$ ,  $(T_t^\theta)_t = (\Gamma(Y_t^\beta, V_t^\theta))_t$  and  $(T_t^\sigma)_t = (\Gamma(Y_t^\beta, V_t^\sigma))_t$  that are solutions of the linear equations:*

(2.6.20)

$$D_t^\beta = 2 \int_0^t a'(Y_s^\beta, \theta) D_s^\beta ds + \int_0^t a''(Y_s^\beta, \theta) LY_s^\beta U_s^\beta ds + \frac{1}{2} \int_0^t a''(Y_s^\beta, \theta) W_s^\beta ds + \frac{1}{2} \int_0^t a'''(Y_s^\beta, \theta) (U_s^\beta)^2 ds + \frac{c^2 \sigma^2}{2} \int_0^t \int_E \rho(z) \left( \rho'(z) + \rho(z) \frac{g'(z)}{g(z)} \right)' \mu(ds, dz),$$

(2.6.21)

$$Q_t^\beta = 4 \int_0^t a'(Y_s^\beta, \theta) Q_s^\beta ds + 7 \int_0^t a''(Y_s^\beta, \theta) W_s^\beta U_s^\beta ds + 2 \int_0^t a'''(Y_s^\beta, \theta) (U_s^\beta)^3 ds + c^4 \sigma^4 \int_0^t \int_E \rho(z) [(\rho(z)')^2 + \rho(z) \rho(z)''] \mu(ds, dz),$$

(2.6.22)

$$T_t^\theta = 3 \int_0^t a'(Y_s^\beta, \theta) T_s^\theta ds + 3 \int_0^t a''(Y_s^\beta, \theta) V_s^\theta U_s^\beta ds + \int_0^t (\partial_\theta a)'(Y_s^\beta, \theta) W_s^\beta ds + \int_0^t a''(Y_s^\beta, \theta) \partial_\theta Y_s^\beta W_s^\beta ds + \int_0^t (\partial_\theta a)''(Y_s^\beta, \theta) (U_s^\beta)^2 ds + \int_0^t a'''(Y_s^\beta, \theta) \partial_\theta Y_s^\beta (U_s^\beta)^2 ds,$$

(2.6.23)

$$T_t^\sigma = 3 \int_0^t a'(Y_s^\beta, \theta) T_s^\sigma ds + 3 \int_0^t a''(Y_s^\beta, \theta) V_s^\sigma U_s^\beta ds + \int_0^t a''(Y_s^\beta, \theta) \partial_\sigma Y_s^\beta W_s^\beta ds + \int_0^t a'''(Y_s^\beta, \theta) \partial_\sigma Y_s^\beta (U_s^\beta)^2 ds + c^3 \sigma^2 \int_0^t \int_E \rho(z) \rho(z)' \mu(ds, dz).$$

**Proof.** The proof of Lemma 2.6.2 is a direct consequence of Theorem 10-3 in [8]. Indeed, considering the stochastic differential equation satisfied by the vector  $(Y_t^\beta, LY_t^\beta, U_t^\beta, W_t^\beta, V_t^\theta, V_t^\sigma, \partial_\theta Y_t^\beta, \partial_\sigma Y_t^\beta)^T$

and using Theorem 10-3 in [8], we prove that the processes  $(D_t^\beta) = (\Gamma(Y_t^\beta, LY_t^\beta))_t$ ,  $(Q_t^\beta) = \Gamma(Y_t^\beta, W_t^\beta)_t$ ,  $(T_t^\theta)_t = (\Gamma(Y_t^\beta, V_t^\theta))_t$  and  $(T_t^\sigma)_t = (\Gamma(Y_t^\beta, V_t^\sigma))_t$  are solutions of linear equations, respectively, given by (2.6.20)-(2.6.23). ■





---

**LAMN PROPERTY FOR THE DRIFT AND  
VOLATILITY PARAMETERS OF A SDE DRIVEN  
BY A STABLE LÉVY PROCESS**

---

This work focuses on the Local Asymptotic Mixed Normality (LAMN) property from high frequency observations, of a continuous time process solution of a stochastic differential equation driven by a truncated  $\alpha$ -stable process with index  $\alpha \in (0, 2)$ . The process is observed on the fixed time interval  $[0, 1]$  and the parameters appear in both the drift coefficient and scale coefficient. This extends the results of [17] where the index  $\alpha \in (1, 2)$  and the parameter appears only in the drift coefficient. We compute the asymptotic Fisher information and find that the rate in the LAMN property depends on the behavior of the Lévy measure near zero. The proof relies on the small time asymptotic behavior of the transition density of the process obtained in [19].

### 3.1 Introduction

Modeling with pure jump Lévy processes plays an important role in many fields, especially in mathematical finance (see for example [7], [41], [46]) and parametric inference for such processes is a currently active subject.

In this paper, we are concerned with parametric estimation of a stochastic differential equation driven by a pure jump Lévy process, from high-frequency observations on a fixed observation time. More precisely we consider the statistical experiment  $(\mathbb{R}^n, \mathcal{B}_n, (\mathbb{P}_n^\beta)_{\beta \in \Theta \subset \mathbb{R}^2})$  corresponding to

the observation of a Lévy driven stochastic equation at discrete times  $(X_{\frac{i}{n}}^\beta)_{1 \leq i \leq n}$ , solution of

$$X_t^\beta = x_0 + \int_0^t b(X_s^\beta, \theta) ds + \sigma L_t, \quad t \in [0, 1],$$

where  $(L_t)_{t \in [0,1]}$  is a truncated  $\alpha$ -stable process with exponent  $\alpha \in (0, 2)$  and  $\beta = (\theta, \sigma)^T \in \mathbb{R} \times (0, \infty)$  is an unknown parameter to be estimated. We prove in this work that the Local Asymptotic Mixed Normality property (LAMN) holds for the parameter  $\beta$ . The LAMN property has been introduced by Jeganathan [39] [40] to extend to the markovian case the LAN property introduced in the pioneer works by Lecam and Hájek (see [50], [30]) in the i.i.d. case. This property permits in particular to identify the optimal estimation rate for the parameters  $\theta$  and  $\sigma$  and the asymptotic Fisher information.

Parametric inference and LAN property for pure jump Lévy processes based on high frequency observations have been investigated in several papers, see for example Aït-Sahalia and Jacod [2] [3], Kawai and Masuda [43] [44], Masuda [53]. In particular, in [53], the LAN property is established and estimators are proposed for the parameters  $(\theta, \sigma, \alpha)$  in the model

$$X_t = \theta t + \sigma L_t^\alpha,$$

where  $(L_t^\alpha)$  is an  $\alpha$ -stable process. Aït-Sahalia and Jacod [2] [3] considered the model

$$X_t = \sigma L_t^\alpha + \theta Y_t,$$

where  $(Y_t)$  is a Lévy process, independent of  $(L_t^\alpha)$  and dominated by  $(L_t^\alpha)$ . More recently, Ivanenko, Kulik and Masuda [35] proved the LAN property for the parameter  $(\theta, \sigma)$  in the model

$$X_t = \theta t + \sigma Z_t + U_t,$$

where  $Z$  is a locally  $\alpha$ -stable process and  $U$  is an independent and less active process. In all these works, the increments  $(X_{\frac{i}{n}} - X_{\frac{i-1}{n}})_{1 \leq i \leq n}$  are independent and the transition density of the discrete time process  $(X_{\frac{i}{n}})_{1 \leq i \leq n}$  is almost explicit. Extensions to stochastic equations driven by pure jump Lévy processes are not immediate and require a different approach since the transition density of the Markov chain  $(X_{\frac{i}{n}})_{1 \leq i \leq n}$  is unknown. Moreover they involve a random asymptotic Fisher information and lead to the LAMN property.

Concerning the parametric estimation of a stochastic differential equation driven by a pure jump Lévy process from high frequency observations on a fixed observation time, we can mention the recent paper by Masuda [54] where some estimators of the parameters  $(\theta, \sigma)$  are proposed for the general equation

$$X_t = x_0 + \int_0^t b(X_s, \theta) ds + \int_0^t c(X_{s-}, \sigma) dL_s,$$

where  $L$  is a locally  $\alpha$ -stable process, with  $\alpha \in [1, 2)$ . However in that case the asymptotic efficiency of the estimators is not yet established and to our knowledge, the only result in that direction is given in Clément and Gloter [17], where the LAMN property is proved for the estimation of the drift parameter  $\theta$  for the process solution of (3.2.1) (with  $\sigma = 1$ ), in the case  $\alpha \in (1, 2)$ . They show that the LAMN property is satisfied with rate  $r_n = n^{\frac{1}{2} - \frac{1}{\alpha}}$  and information  $\mathcal{I}_\theta = \int_0^1 \partial_\theta b(X_s^\beta, \theta)^2 ds \int_{\mathbb{R}} \frac{\varphi'_\alpha(u)^2}{\varphi_\alpha(u)} du$ , where  $\varphi_\alpha$  is the density of the standard  $\alpha$ -stable distribution with characteristic function  $u \mapsto e^{-C(\alpha)|u|^\alpha}$ .

Based on the main ideas of [17] and using the results of [19], we extend in the present paper these results to  $\alpha \in (0, 2)$  and prove that the LAMN property holds for the parameters  $(\theta, \sigma)$  with rate  $r_n = \begin{pmatrix} n^{\frac{1}{2} - \frac{1}{\alpha}} & 0 \\ 0 & n^{-\frac{1}{2}} \end{pmatrix}$  and information matrix  $\mathcal{I} = \begin{pmatrix} \mathcal{I}_{11} & 0 \\ 0 & \mathcal{I}_{22} \end{pmatrix}$  where  $\mathcal{I}_{11} = \frac{1}{\sigma^2} \mathcal{I}_\theta$  and  $\mathcal{I}_{22} = \frac{1}{\sigma^2} \int_{\mathbb{R}} \frac{(\varphi_\alpha(u) + u\varphi'_\alpha(u))^2}{\varphi_\alpha(u)} du$ . The proof is mainly based on the  $L^2$ -regularity property of the transition density (see Jeganathan [39]) and on Malliavin calculus (see for example Gobet [27] for the use of Malliavin calculus in the case of a diffusion process). The  $L^2$ -regularity property is established here by using the asymptotic behavior of the density of the process solution of (3.2.1) in small time as well as its derivative with respect to the parameter, given in [19] and based on the Malliavin calculus for jump processes developed by Bichteler, Gravereaux and Jacod [8]. It also requires a careful study of the asymptotic behavior of the information matrix based on one observation of the process, this is the subject of Section 3.3. This paper contains also an independent and interesting result stating a continuity property with respect to the conditioning variable in a conditional expectation (see Proposition 3.6.3).

This paper is organized as follows. The main results (asymptotic expansion of the log-likelihood function and LAMN property) are stated in Section 3.2. Section 3.3 studies the asymptotic behavior of the Fisher information matrix based on the observation of  $X_{\frac{1}{n}}^\beta$  (as  $n$  goes

to infinity). The proof of the main results are given in Section 3.4 and Section 3.5. Section 3.6 contains some additional technical proof required to establish the results of Section 3.3. Finally, Section 3.7 checks the conditions given in Proposition 3.6.3 for the choices  $\mathcal{H} = \mathcal{H}_{L^\alpha}(1)$  and  $\mathcal{H} = L_1^\alpha \mathcal{H}_{L^\alpha}(1)$ .

## 3.2 Main results

We consider the process  $(X_t^\beta)_{t \in [0,1]}$  solution to

$$(3.2.1) \quad X_t^\beta = x_0 + \int_0^t b(X_s^\beta, \theta) ds + \sigma L_t \quad t \in [0, 1],$$

where  $(L_t)_{t \in [0,1]}$  is a pure jump Lévy process defined on some probability space  $(\Omega, \mathcal{A}, \mathcal{P})$  and we are interested in the statistical properties of the process  $(X_t^\beta)$ , based on the discrete time observations  $(X_{i/n}^\beta)_{i=0, \dots, n}$ . We assume that the following assumptions are fulfilled.

**H<sub>1</sub>**: (a) The function  $b$  has bounded derivatives up to order five with respect to both variables.  
 (b<sub>i</sub>) The Lévy process  $(L_t)_{t \in [0,1]}$  is given by  $L_t = \int_0^t \int_{[-1,1]} z \{\bar{\mu}(ds, dz) - \bar{\nu}(ds, dz)\} + \int_0^t \int_{[-1,1]^c} z \bar{\mu}(ds, dz)$  where  $\bar{\mu}$  is a Poisson random measure, with compensator  $\bar{\nu}(dt, dz) = dt \times F(z) dz$  where  $F(z)$  is given on  $\mathbb{R}$  by  $F(z) = \frac{1}{|z|^{\alpha+1}} \mathbb{1}_{|z| \neq 0} \tau(z)$ ,  $\alpha \in (0, 2)$ . We assume that  $\tau$  is a non negative smooth function equal to 1 on  $[-1, 1]$ , vanishing on  $[-2, 2]^c$  such that  $0 \leq \tau \leq 1$ .

(b<sub>ii</sub>) We assume that  $\forall p \geq 1$ ,  $\int_{\mathbb{R}} \left| \frac{\tau'(u)}{\tau(u)} \right|^p \tau(u) du < \infty$ ,  $\int_{\mathbb{R}} \left| \frac{\tau''(u)}{\tau(u)} \right|^p \tau(u) du < \infty$ .

These assumptions are sufficient to ensure that (3.2.1) has an unique solution belonging to  $L^p$ ,  $\forall p \geq 1$ , and that  $X_t^\beta$  admits a density, for  $t > 0$  (see [56]). Moreover, it is proved in [19] that this density is differentiable with respect to  $\beta$ .

**Remark 3.2.1.** *Our assumptions on the Lévy measure  $F$  are quite restrictive and reduce the generality of our results but simplify the proofs which nevertheless remain still technical. There are mainly two important properties required on the Lévy measure in our approach. First, since our method is based on Malliavin calculus, the  $L^p$ -bounds for the tails of the Lévy process are crucial to ensure that our process belongs to the Malliavin space. Secondly, the stable behavior of the Lévy measure around zero is also required to make the rescaled process  $(n^{1/\alpha} L_{t/n})$  close to the  $\alpha$ -stable process  $(L_t^\alpha)$ . The introduction of the truncation function  $\tau$  permits to address both issues*

and to avoid more technical proofs. In particular it permits to ensure that the process  $(n^{1/\alpha}L_{t/n})$  has no jump of size larger than  $2n^{1/\alpha}$  and consequently makes easier the control of the asymptotic behavior of the Malliavin weights (mainly studied in [19]). Moreover the exact stable behavior of the Lévy measure around zero ( $\tau = 1$ ) gives the equality between the rescaled process  $(n^{1/\alpha}L_{t/n})$  and the  $\alpha$ -stable process  $(L_t^\alpha)$ , and also the equality of the corresponding Poisson measures, on a set  $A_n$  whose complementary has small probability ( $\mathbb{P}(A_n^c) \leq C/n$ , see Lemma 3.3.1 below). This property is repeatedly used in our proofs (see for example the proof of Theorem 3.2.3) and is also essential to study the limit of the Malliavin weights in [19].

However, since the information matrix obtained in the LAMN property (established in Corollary 3.2.4) does not depend on the function  $\tau$ , this suggests that the same result probably holds for a more general Lévy measure even with no integrability conditions on the large jumps and that the truncation or integrability assumptions should only be introduced in the proof sections. To that end, a possible extension of our paper (and also of [19]) could be to replace  $\tau$  by a more general function  $g$  such that  $g(0) = 1$  and satisfying  $(b_{ii})$ , but up to now we do not know how to obtain the key results established in [19] in this more general context.

Before stating our main results, we introduce some notations which are used throughout this paper. For a vector  $h \in \mathbb{R}^2$ ,  $h^T$  denotes the transpose of  $h$ , and  $|h|$  denotes the euclidean norm. For a function  $f$  defined on  $\mathbb{R} \times \mathbb{R}^2$  depending on both variables  $(x, \beta)$ , here  $\beta = (\theta, \sigma)^T \in \mathbb{R} \times (0, +\infty)$ , we denote by  $f'$  the derivative of  $f$  with respect to the variable  $x$ , by  $\partial_\theta f$  the derivative of  $f$  with respect to the parameter  $\theta$ , by  $\partial_\sigma f$  the derivative of  $f$  with respect to the parameter  $\sigma$ , and

$$\nabla_\beta f = \begin{pmatrix} \partial_\theta f \\ \partial_\sigma f \end{pmatrix}.$$

We denote by  $p_{i/n}^\beta(x, y)$  the transition density of the homogeneous Markov chain  $(X_{i/n}^\beta)_{i=0, \dots, n}$ , by  $(\mathcal{F}_{i/n})_i$  the  $\sigma$ -field such that  $\mathcal{F}_{i/n} = \sigma(X_s^\beta, s \leq i/n) = \sigma(L_s, s \leq i/n)$  and by  $\mathbb{P}_n^\beta$  the law of the vector  $(X_{\frac{1}{n}}^\beta, \dots, X_1^\beta)$  on  $\mathbb{R}^n$ .

Our first result is an asymptotic expansion of the log-likelihood ratio.

**Theorem 3.2.1.** *We assume that  $\mathbf{H}_1$  holds. Let  $r_n = \begin{pmatrix} n^{\frac{1}{2}-\frac{1}{\alpha}} & 0 \\ 0 & n^{-\frac{1}{2}} \end{pmatrix}$ , then for every  $h \in \mathbb{R}^2$*

$$(3.2.2) \quad \log \frac{d\mathbb{P}_n^{\beta+r_n h}}{d\mathbb{P}_n^\beta}(X_{\frac{1}{n}}^\beta, \dots, X_1^\beta) = h^T J_n(\beta)^{\frac{1}{2}} N_n(\beta) - \frac{1}{2} h^T J_n(\beta) h + o_{\mathcal{P}}(1),$$

with

$$J_n(\beta) = r_n \sum_{i=0}^{n-1} \mathbb{E} \left[ \xi_{i,n}(\beta) \xi_{i,n}^T(\beta) | \mathcal{F}_{i/n} \right] r_n,$$

$$N_n(\beta) = J_n(\beta)^{\frac{1}{2}} r_n \sum_{i=0}^{n-1} \xi_{i,n}(\beta),$$

$$\xi_{i,n}(\beta) = \begin{pmatrix} \frac{\partial_\theta p_{\frac{1}{n}}^\beta}{p_{\frac{1}{n}}^\beta} \left( X_{\frac{i}{n}}^\beta, X_{\frac{i+1}{n}}^\beta \right) \\ \frac{\partial_\sigma p_{\frac{1}{n}}^\beta}{p_{\frac{1}{n}}^\beta} \left( X_{\frac{i}{n}}^\beta, X_{\frac{i+1}{n}}^\beta \right) \end{pmatrix}.$$

We can precise the asymptotic behavior of  $J_n(\beta)$  and  $N_n(\beta)$ . Let  $\varphi_\alpha$  be the density of  $L_1^\alpha$ , where  $(L_1^\alpha)$  is a centered  $\alpha$ -stable Lévy process whose Lévy measure is  $\frac{dz}{|z|^{1+\alpha}} 1_{|z| \neq 0}$ . We define the following quantity which will be the random asymptotic information of the statistical model:

$$(3.2.3) \quad \mathcal{I}(\beta) = \begin{pmatrix} \mathcal{I}_{11} & 0 \\ 0 & \mathcal{I}_{22} \end{pmatrix}$$

where

$$\mathcal{I}_{11} = \frac{1}{\sigma^2} \int_0^1 \partial_\theta b(X_s^\beta, \theta)^2 ds \times \int_{\mathbb{R}} \frac{\varphi'_\alpha(u)^2}{\varphi_\alpha(u)} du, \quad \mathcal{I}_{22} = \frac{1}{\sigma^2} \times \int_{\mathbb{R}} \frac{(\varphi_\alpha(u) + u\varphi'_\alpha(u))^2}{\varphi_\alpha(u)} du.$$

**Remark 3.2.2.** *i) From [3] and [35], we know that the parameter  $\theta$  of the process  $X_t^\theta = \theta t + L_t$  is estimated with rate  $n^{\frac{1}{2}-\frac{1}{\alpha}}$  and Fisher information  $\int_{\mathbb{R}} \frac{\varphi'_\alpha(u)^2}{\varphi_\alpha(u)} du$  and that the parameter  $\sigma$  of the process  $X_t^\sigma = \sigma L_t$  is estimated with the usual rate  $n^{-1/2}$  and Fisher information  $\int_{\mathbb{R}} \frac{(\varphi_\alpha(u) + u\varphi'_\alpha(u))^2}{\varphi_\alpha(u)} du$ .  
ii) It is worth to notice that the information does not depend on the truncation function  $\tau$ , but depends on  $\alpha$  through the Fisher information of the translated  $\alpha$ -stable process and multiplicative  $\alpha$ -stable process.*

**Theorem 3.2.2.** *With the notations and assumptions of Theorem 3.2.1, the following convergences hold:*

$$(3.2.4) \quad J_n(\beta) \xrightarrow{n \rightarrow \infty} \mathcal{I}(\beta) \quad \text{in probability,}$$

$$(3.2.5) \quad \forall \epsilon > 0, \quad \sum_{i=0}^{n-1} \mathbb{E} [ |r_n \xi_{i,n}(\beta)|^2 \mathbf{1}_{\{|r_n \xi_{i,n}(\beta)| \geq \epsilon\}} ] \xrightarrow{n \rightarrow \infty} 0.$$

**Theorem 3.2.3.** *We have the convergence in law*

$$(3.2.6) \quad J_n(\beta)^{\frac{1}{2}} N_n(\beta) = r_n \sum_{i=0}^{n-1} \xi_{i,n}(\beta) \Rightarrow \mathcal{N}(0, \mathcal{I}(\beta))$$

where the limit variable is conditionally Gaussian and the convergence is stable with respect to the  $\sigma$ -field  $\sigma(L_s, s \leq 1)$ .

The stable convergence in law (3.2.6) and the convergence in probability (3.2.4) yield the convergence in law of the couple  $(J_n(\beta), N_n(\beta))$ :

$$(J_n(\beta), N_n(\beta)) \Rightarrow (\mathcal{I}(\beta), N),$$

where  $N$  is a standard Gaussian variable independent of  $\mathcal{I}(\beta)$ . As a consequence of the asymptotic expansion given in Theorem 3.2.1 and the preceding limit theorems, we deduce the LAMN property.

**Corollary 3.2.4.** *The family  $(\mathbb{P}_n^\beta)$  satisfies the LAMN property with rate  $r_n = \begin{pmatrix} n^{\frac{1}{2} - \frac{1}{\alpha}} & 0 \\ 0 & n^{-\frac{1}{2}} \end{pmatrix}$ , and information  $\mathcal{I}(\beta)$  given by (3.2.3).*

The rate of estimation of the drift parameter depends on  $\alpha$  : when  $\alpha$  tends to 2, the rate is extremely low, however, when  $\alpha$  goes to zero, it becomes high, especially for  $\alpha < 1$  where it is faster than the usual one  $n^{-1/2}$ . On the other hand, the rate of estimation of the volatility parameter  $\sigma$  is  $n^{-1/2}$  and does not depend on  $\alpha$ .

Before proceeding to the proof of these results, we discuss some extensions of our model that are not addressed in this paper.

- The Malliavin calculus used in this paper allows to consider the more general process

$$X_t^\beta = x_0 + \int_0^t b(X_s^\beta, \theta) ds + \int_0^t c(X_{s-}^\beta, \sigma) dL_s,$$



and based on the results given in [8] the Malliavin operators have explicit expressions. But the difficulty relies on the control of the Malliavin weights. Although explicit, these weights contain a lot of cumbersome terms especially the iterated weights involving the derivatives of the process with respect to the parameters  $\theta$  and  $\sigma$ . These iterated weights (and their asymptotic behavior) are crucial to obtain the asymptotic behavior of the derivative of the transition density in small time (see [19]). The restriction to a constant coefficient  $c$ , assumed in this paper, permits to handle all these terms successfully. The theoretically possible extension to a general coefficient is still open.

- Unlike the papers [2], [3] and [35], our model does not contain an additional noise. Based on the structure of an additive model, the key point in these papers is that the density of the observed process can be written explicitly as a convolution between the Lévy process and the additional noise. Since we are dealing with a stochastic equation, this approach does not work anymore in our case and the introduction of an additional noise complicates significantly our model study.
- In contrast to the diffusion or jump-diffusion case, the interesting particularity of a pure jump process is that we can estimate the drift coefficient observing the process on a fixed time period  $[0, T]$ . It is important to stress that the estimation of  $\theta$  is impossible without letting  $T$  go to infinity if  $(L_t)$  has a Brownian component. This is why we focus in this paper on the estimation of  $(\theta, \sigma)$  from high frequency observations on the time interval  $[0, 1]$ . The long time estimation problem ( $nh_n \rightarrow \infty$ , where  $n$  is the number of observations and  $h_n$  the step between two consecutive observations) is also an interesting problem, but substantially different, that can certainly be treated with our methodology under ergodicity assumptions.

The remainder of the paper is devoted to the proofs of the main theorems above. The first step of our approach consists in studying the asymptotic Fisher information matrix by using Malliavin calculus techniques.

### 3.3 The asymptotic Fisher information matrix in small time

#### 3.3.1 The asymptotic properties of the Fisher information matrix

Our main concern in this section is to study the asymptotic properties of the Fisher information carried by the observation of the random variable  $X_{1/n}^\beta$ . We recall the definition of the Fisher information matrix:

$$(3.3.1) \quad \mathcal{I}^{n,\beta,x_0} = \begin{pmatrix} \mathcal{I}_{11}^{n,\beta,x_0} & \mathcal{I}_{12}^{n,\beta,x_0} \\ \mathcal{I}_{12}^{n,\beta,x_0} & \mathcal{I}_{22}^{n,\beta,x_0} \end{pmatrix}$$

where

$$\mathcal{I}_{11}^{n,\beta,x_0} = \mathbb{E} \left[ \left( \frac{\partial_\theta p_{\frac{1}{n}}^\beta(x_0, X_{1/n}^\beta)}{p_{\frac{1}{n}}^\beta(x_0, X_{1/n}^\beta)} \right)^2 \right], \quad \mathcal{I}_{12}^{n,\beta,x_0} = \mathbb{E} \left[ \left( \frac{\partial_\theta p_{\frac{1}{n}}^\beta(x_0, X_{1/n}^\beta)}{p_{\frac{1}{n}}^\beta(x_0, X_{1/n}^\beta)} \frac{\partial_\sigma p_{\frac{1}{n}}^\beta(x_0, X_{1/n}^\beta)}{p_{\frac{1}{n}}^\beta(x_0, X_{1/n}^\beta)} \right) \right],$$

$$\text{and} \quad \mathcal{I}_{22}^{n,\beta,x_0} = \mathbb{E} \left[ \left( \frac{\partial_\sigma p_{\frac{1}{n}}^\beta(x_0, X_{1/n}^\beta)}{p_{\frac{1}{n}}^\beta(x_0, X_{1/n}^\beta)} \right)^2 \right].$$

The following proposition gives the asymptotic behavior of the Fisher information based on the observation of the random variable  $X_{1/n}^\beta$  as  $\frac{1}{n}$  tends to zero.

**Theorem 3.3.1.** *Let  $(\beta_n)$  be a sequence such that  $\beta_n \xrightarrow{n \rightarrow \infty} \beta$ ,  $Q \subset \mathbb{R} \times (0, \infty)$  a compact set and*

$$r_n = \begin{pmatrix} n^{\frac{1}{2}-\frac{1}{\alpha}} & 0 \\ 0 & n^{-\frac{1}{2}} \end{pmatrix} \text{ then}$$

$$i) \quad nr_n \mathcal{I}^{n,\beta_n,x_0} r_n \xrightarrow{n \rightarrow \infty} \begin{pmatrix} \frac{1}{\sigma^2} \partial_\theta b(x_0, \theta)^2 \int_{\mathbb{R}} \frac{\varphi'_\alpha(u)^2}{\varphi_\alpha(u)} du & 0 \\ 0 & \frac{1}{\sigma^2} \int_{\mathbb{R}} \frac{[\varphi_\alpha(u) + u\varphi'_\alpha(u)]^2}{\varphi_\alpha(u)} du \end{pmatrix}$$

and this convergence is uniform with respect to  $x_0$ .

$$ii) \quad \forall p \geq 1, \quad \sup_{n,\beta \in Q, x_0} \mathbb{E} \left| n^{1/2} r_n \frac{\nabla_\beta p_{\frac{1}{n}}^\beta(x_0, X_{1/n}^\beta)}{p_{\frac{1}{n}}^\beta(x_0, X_{1/n}^\beta)} \right|^p < \infty.$$

As a consequence of ii) with  $p = 2$ , we deduce immediately  $\sup_{n,\beta \in Q, x_0} n^{2-2/\alpha} \mathcal{I}_{11}^{n,\beta,x_0} < \infty$ ,  $\sup_{n,\beta \in Q, x_0} \mathcal{I}_{22}^{n,\beta,x_0} < \infty$  and from Cauchy-Schwarz inequality  $\sup_{n,\beta \in Q, x_0} n^{1-1/\alpha} \mathcal{I}_{12}^{n,\beta,x_0} < \infty$ .

**Remark 3.3.1.** *From Theorem 3.3.1, we see that the Fisher information contained in one observation is of magnitude  $n^{-1}r_n^{-2}$  and the Fisher information based on  $n$  observations is of magnitude*

$r_n^{-2}$ . This is consistent with the fact that  $r_n$  is the rate in the LAMN property stated in Corollary 3.2.4.

The rest of this section is devoted to the proof of Theorem 3.3.1.

### 3.3.2 Proof of Theorem 3.3.1

The proof of Theorem 3.3.1 relies on a representation of the score function using Malliavin calculus initiated by Gobet (see [27] and [28]) and adapted to our context in [19].

This representation is established after a rescaling that we describe in the next subsection.

#### 3.3.2.1 Rescaling and representation of the score function using Malliavin calculus

We consider  $\mu^e(dt, dz, du)$  a Poisson measure on  $[0, \infty) \times \mathbb{R} \times [0, 1]$  with compensating measure  $v^e(dt, dz, du) = dt 1_{|z| \neq 0} \frac{dz}{|z|^{1+\alpha}} du$  and for  $n \geq 1$ , we define the Poisson random measure  $\mu^{(n)}$  by

$$\forall A \subset [0, \infty) \times \mathbb{R}, \quad \mu^{(n)}(A) = \int_{[0, \infty)} \int_{\mathbb{R}} \int_{[0, 1]} 1_A(t, z) 1_{\{u \leq \tau(\frac{z}{n^{1/\alpha}})\}} \mu^e(dt, dz, du).$$

We note that the compensator of  $\mu^{(n)}(dt, dz)$  is  $v^{(n)}(dt, dz) = dt \times \tau(\frac{z}{n^{1/\alpha}}) 1_{|z| \neq 0} \frac{dz}{|z|^{1+\alpha}} := dt \times F_n(z) dz$  and the compensated Poisson random measure  $\tilde{\mu}^{(n)}(dt, dz) = \mu^{(n)}(dt, dz) - v^{(n)}(dt, dz)$ .

We define the process  $(L_t^n)$  by:

$$(3.3.2) \quad L_t^n = \int_0^t \int_{[-n^{1/\alpha}, n^{1/\alpha}]} z \tilde{\mu}^{(n)}(ds, dz) + \int_0^t \int_{[-n^{1/\alpha}, n^{1/\alpha}]^c} z \mu^{(n)}(ds, dz).$$

We observe that the process  $(L_{t/n})$  (recall  $\mathbf{H}_1(b_i)$ ) equals in law  $(\frac{1}{n^{1/\alpha}} L_t^n)$  since the associated Poisson measures have the same compensator. Moreover, when  $n$  grows, we can show that the process  $(L_t^n)$  converges almost surely to an  $\alpha$ -stable process defined by

$$(3.3.3) \quad L_t^\alpha = \int_0^t \int_{[-1, 1]} z \tilde{\mu}(ds, dz) + \int_0^t \int_{[-1, 1]^c} z \mu(ds, dz),$$

where  $\mu$  is the Poisson random measure defined by,

$$\forall A \subset [0, \infty) \times \mathbb{R}, \quad \mu(A) = \int_{[0, \infty)} \int_{\mathbb{R}} \int_{[0, 1]} 1_A(t, z) \mu^e(dt, dz, du).$$

The compensator of  $\mu(dt, dz)$  is  $v(dt, dz) = dt \times 1_{|z| \neq 0} \frac{dz}{|z|^{1+\alpha}}$  and we note the compensated Poisson random measure  $\tilde{\mu}(dt, dz) = \mu(dt, dz) - v(dt, dz)$ .

It is important to note that  $L^n$  and the  $\alpha$ -stable process  $L^\alpha$  are defined on the same probability space (this property is crucial in our method to study the convergence of the Fisher information  $\mathcal{I}^{n,\beta,x_0}$ ). The connection between  $L^n$  and  $L^\alpha$  is given more clearly by the following lemma.

**Lemma 3.3.1.** [lemma 3.1 in [19]] *On the event  $A_n = \{\mu(\{(t,z)|0 \leq t \leq 1, |z| \geq n^{1/\alpha}\}) = 0\}$ , we have*

$$\mu^{(n)} = \mu, \quad L_t^n = L_t^\alpha,$$

and  $\mathbb{P}(A_n) = 1 + O(1/n)$ .

Furthermore, let  $(f_n)_{n \in \mathbb{N}}$  and  $f$  be measurable functions from  $\Omega \times [0, 1] \times \mathbb{R}$  to  $\mathbb{R}$  such that there exists  $C$  with  $\mathbb{P}(C) = 1$  and  $\forall \omega \in C, \forall s \in [0, 1], \forall |z| > 1$   $f_n(\omega, s, z) \xrightarrow{n \rightarrow \infty} f(\omega, s, z)$ . Then

$$\int_0^1 \int_{|z|>1} f_n(\omega, s, z) \mu^{(n)}(ds, dz) \xrightarrow{a.s.} \int_0^1 \int_{|z|>1} f(\omega, s, z) \mu(ds, dz).$$

Moreover, we have  $\sup_{t \in [0,1]} |L_t^n - L_t^\alpha| \xrightarrow{a.s.} 0$ .

We now consider the process  $(\bar{Y}_t^{n,\beta,x_0})$  solution to the equation

$$(3.3.4) \quad \bar{Y}_t^{n,\beta,x_0} = x_0 + \frac{1}{n} \int_0^t b(\bar{Y}_s^{n,\beta,x_0}, \theta) ds + \frac{\sigma}{n^{1/\alpha}} L_t^n \quad t \in [0, 1].$$

From the construction of  $L^n$ ,  $(X_{\frac{t}{n}}^\beta)_{t \in [0,1]}$  equals in law  $(\bar{Y}_t^{n,\beta,x_0})_{t \in [0,1]}$ . Let  $q^{n,\beta,x_0}$  be the density of  $\bar{Y}_1^{n,\beta,x_0}$  then the connection between the densities of  $X_{\frac{1}{n}}^\beta$  and  $\bar{Y}_1^{n,\beta,x_0}$  is given by

$$(3.3.5) \quad p_{1/n}^\beta(x_0, x) = q^{n,\beta,x_0}(x).$$

We remark also that  $(\bar{Y}_t^{n,\beta,x_0})_t$  admits derivatives with respect to the parameters  $\theta$  and  $\sigma$ , denoted by  $(\partial_\theta Y_t^\beta)_t$  and  $(\partial_\sigma Y_t^\beta)_t$ , respectively. With these notations, we have the following representation.

**Proposition 3.3.1.** [Theorem 6.2 in [19]]

Let  $q^{n,\beta,x_0}$  be the density of  $\bar{Y}_1^{n,\beta,x_0}$  then we have the representation of the logarithmic derivative of the density as a conditional expectation:

$$(3.3.6) \quad \frac{\nabla_\beta p_{\frac{1}{n}}^\beta(x_0, u)}{p_{\frac{1}{n}}^\beta(x_0, u)} = \frac{\nabla_\beta q^{n,\beta,x_0}(u)}{q^{n,\beta,x_0}(u)} = \begin{pmatrix} \frac{\partial_\theta q^{n,\beta,x_0}(u)}{q^{n,\beta,x_0}(u)} \\ \frac{\partial_\sigma q^{n,\beta,x_0}(u)}{q^{n,\beta,x_0}(u)} \end{pmatrix} = \mathbb{E}(\mathcal{H}_{\bar{Y}_1^{n,\beta,x_0}}(\nabla_\beta \bar{Y}_1^{n,\beta,x_0}) | \bar{Y}_1^{n,\beta,x_0} = u)$$

with

$$\mathcal{H}_{\bar{Y}_1^{n,\beta,x_0}}(\nabla_\beta \bar{Y}_1^{n,\beta,x_0}) = \begin{pmatrix} \mathcal{H}_{\bar{Y}_1^{n,\beta,x_0}}(\partial_\theta \bar{Y}_1^{n,\beta,x_0}) \\ \mathcal{H}_{\bar{Y}_1^{n,\beta,x_0}}(\partial_\sigma \bar{Y}_1^{n,\beta,x_0}) \end{pmatrix}.$$

The Malliavin weight  $\mathcal{H}_{\bar{Y}_1^{n,\beta,x_0}}(\nabla_{\beta}\bar{Y}_1^{n,\beta,x_0})$  depends on the derivatives of  $\bar{Y}_1^{n,\beta,x_0}$  with respect to  $\theta$  and  $\sigma$  and on Malliavin operators. Its explicit expression will be given in Section 3.6 (see (3.6.11)) after some recalling on Malliavin calculus.

### 3.3.2.2 Intermediate lemmas

In this section, we study the convergence of the Malliavin weight appearing in the representation of the score function. The limit of this Malliavin weight brings out an other weight denoted by  $\mathcal{H}_{L^\alpha}(1)$  (given explicitly in (3.6.16)) that permits to represent  $\varphi'_\alpha/\varphi_\alpha$ , where  $\varphi_\alpha$  is the density of  $L_1^\alpha$ , as an expectation. This representation is not immediate since  $L_1^\alpha$  does not belong to the domain of our Malliavin operators (see Section 3.6).

**Lemma 3.3.2.** *We have the representation*

$$(3.3.7) \quad \frac{\varphi'_\alpha(u)}{\varphi_\alpha(u)} = -\mathbb{E}[\mathcal{H}_{L^\alpha}(1)|L_1^\alpha = u].$$

The connection between the weights  $\mathcal{H}_{\bar{Y}_1^{n,\beta,x_0}}(\nabla_{\beta}\bar{Y}_1^{n,\beta,x_0})$  and  $\mathcal{H}_{L^\alpha}(1)$  is established in the next lemmas. The first lemma shows the convergence of the normalized iterated Malliavin weight  $\mathcal{H}_{\bar{Y}_1^{n,\beta,x_0}}(\nabla_{\beta}\bar{Y}_1^{n,\beta,x_0})$ .

**Lemma 3.3.3.** *Let  $(\beta_n)_{n \geq 1}$  be a sequence such that  $\beta_n \xrightarrow{n \rightarrow \infty} \beta$ . Then, the following convergence holds uniformly with respect to  $x_0$*

$$(3.3.8) \quad n^{1/2}r_n\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(\nabla_{\beta}\bar{Y}_1^{n,\beta_n,x_0}) = \begin{pmatrix} n^{1-1/\alpha}\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(\partial_{\theta}\bar{Y}_1^{n,\beta_n,x_0}) \\ \mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(\partial_{\sigma}\bar{Y}_1^{n,\beta_n,x_0}) \end{pmatrix} \xrightarrow[n \rightarrow \infty]{L^p, \forall p \geq 1} \begin{pmatrix} \frac{1}{\sigma}\partial_{\theta}b(x_0, \theta)\mathcal{H}_{L^\alpha}(1) \\ \frac{1}{\sigma}(L_1^\alpha\mathcal{H}_{L^\alpha}(1) - 1) \end{pmatrix},$$

where  $L_1^\alpha$  is defined by (3.3.3). Moreover, for any compact subset  $\mathcal{Q} \subset \mathbb{R} \times (0, \infty)$ ,

$$\forall p \geq 1, \quad \sup_{n, \beta \in \mathcal{Q}, x_0} \mathbb{E} \left| n^{1/2}r_n\mathcal{H}_{\bar{Y}_1^{n,\beta,x_0}}(\nabla_{\beta}\bar{Y}_1^{n,\beta,x_0}) \right|^p < \infty.$$

The next two lemmas are related to a continuity property with respect to the conditioning variable, in a conditional expectation.

**Lemma 3.3.4.** *Let  $(\beta_n)_{n \geq 1}$  be a sequence such that  $\beta_n \xrightarrow{n \rightarrow \infty} \beta$ . Then, the following convergence holds uniformly with respect to  $x_0$ .*

$$i) \quad n^{2-2/\alpha} \mathbb{E}[\mathbb{E}[\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(\partial_\theta \bar{Y}_1^{n,\beta_n,x_0}) | \bar{Y}_1^{n,\beta_n,x_0}]^2] \xrightarrow{n \rightarrow \infty} \frac{1}{\sigma^2} [\partial_\theta b(x_0, \theta)]^2 \mathbb{E}[\mathbb{E}[\mathcal{H}_{L_1^\alpha}(1) | L_1^\alpha]^2],$$

$$ii) \quad \mathbb{E}[\mathbb{E}[\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(\partial_\sigma \bar{Y}_1^{n,\beta_n,x_0}) | \bar{Y}_1^{n,\beta_n,x_0}]^2] \xrightarrow{n \rightarrow \infty} \frac{1}{\sigma^2} \mathbb{E}[\mathbb{E}[L_1^\alpha \mathcal{H}_{L_1^\alpha}(1) - 1 | L_1^\alpha]^2],$$

**Lemma 3.3.5.** *Let  $(\beta_n)_{n \geq 1}$  be a sequence such that  $\beta_n \xrightarrow{n \rightarrow \infty} \beta$ . Then, the following convergence holds uniformly with respect to  $x_0$ ,*

$$(3.3.9) \quad n^{1-1/\alpha} \mathbb{E}[\mathbb{E}[\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(\partial_\theta \bar{Y}_1^{n,\beta_n,x_0}) | \bar{Y}_1^{n,\beta_n,x_0}] \mathbb{E}[\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(\partial_\sigma \bar{Y}_1^{n,\beta_n,x_0}) | \bar{Y}_1^{n,\beta_n,x_0}]] \xrightarrow{n \rightarrow \infty} \frac{1}{\sigma^2} \mathbb{E}[\mathbb{E}[\mathcal{H}_{L_1^\alpha}(1) | L_1^\alpha] \mathbb{E}[L_1^\alpha \mathcal{H}_{L_1^\alpha}(1) - 1 | L_1^\alpha]].$$

The proofs of the above lemmas are very technical. They are postponed to Section 3.6. Admitting these intermediate results, we can proceed to the proof of Theorem 3.3.1.

### 3.3.2.3 Proof of Theorem 3.3.1

**Proof.** *i)* We need to prove that for  $(\beta_n)$  a sequence such that  $\beta_n \xrightarrow{n \rightarrow \infty} \beta$  and  $r_n = \begin{pmatrix} n^{\frac{1}{2}-\frac{1}{\alpha}} & 0 \\ 0 & n^{-\frac{1}{2}} \end{pmatrix}$

we have

$$nr_n \mathcal{G}^{n,\beta_n,x_0} r_n \xrightarrow{n \rightarrow \infty} \begin{pmatrix} \frac{1}{\sigma^2} \partial_\theta b(x_0, \theta)^2 \int_{\mathbb{R}} \frac{\varphi'_\alpha(u)^2}{\varphi_\alpha(u)} du & 0 \\ 0 & \frac{1}{\sigma^2} \int_{\mathbb{R}} \frac{[\varphi_\alpha(u) + u\varphi'_\alpha(u)]^2}{\varphi_\alpha(u)} du \end{pmatrix}$$

and that this convergence is uniform with respect to  $x_0$ .

Since  $nr_n \mathcal{G}^{n,\beta_n,x_0} r_n = \begin{pmatrix} n^{2-\frac{2}{\alpha}} \mathcal{G}_{11}^{n,\beta_n,x_0} & n^{1-\frac{1}{\alpha}} \mathcal{G}_{12}^{n,\beta_n,x_0} \\ n^{1-\frac{1}{\alpha}} \mathcal{G}_{12}^{n,\beta_n,x_0} & \mathcal{G}_{22}^{n,\beta_n,x_0} \end{pmatrix}$ , the proof of the above convergence reduces

to prove the convergence of each entries of the matrix.

**Convergence of  $n^{2-2/\alpha} \mathcal{G}_{11}^{n,\beta_n,x_0}$ .** From (3.3.6) in Proposition 3.3.1, we have

$$\begin{aligned} n^{2-2/\alpha} \mathcal{G}_{11}^{n,\beta_n,x_0} &= n^{2-2/\alpha} \mathbb{E} \left[ \mathbb{E} \left[ \mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(\partial_\theta \bar{Y}_1^{n,\beta_n,x_0}) | \bar{Y}_1^{n,\beta_n,x_0} \right]^2 \right] \\ &\xrightarrow{n \rightarrow \infty} \frac{1}{\sigma^2} \partial_\theta b(x_0, \theta)^2 \mathbb{E} \left[ \mathbb{E}[\mathcal{H}_{L_1^\alpha}(1) | L_1^\alpha]^2 \right], \quad \text{uniformly with respect to } x_0, \text{ from Lemma 3.3.4 i),} \\ &= \frac{1}{\sigma^2} \partial_\theta b(x_0, \theta)^2 \mathbb{E} \left[ \frac{\varphi'_\alpha(L_1^\alpha)^2}{\varphi_\alpha(L_1^\alpha)^2} \right] \quad \text{from (3.3.7) in Lemma 3.3.2.} \end{aligned}$$

**Convergence of  $\mathcal{G}_{22}^{n,\beta_n,x_0}$ .** We remark that from the representation (3.3.7), we can deduce the following representation

$$(3.3.10) \quad \frac{\varphi_\alpha(u) + u\varphi'_\alpha(u)}{\varphi_\alpha(u)} = -u \mathbb{E}[\mathcal{H}_{L_1^\alpha}(1) | L_1^\alpha = u] + 1 = -\mathbb{E}[(L_1^\alpha \mathcal{H}_{L_1^\alpha}(1) - 1) | L_1^\alpha = u].$$

Furthermore, combining (3.3.6) and Lemma 3.3.4 *ii*), we have

$$\begin{aligned} \mathcal{I}_{22}^{n,\beta_n,x_0} &= \mathbb{E} \left[ \mathbb{E} \left[ \mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(\partial_\sigma \bar{Y}_1^{n,\beta_n,x_0}) | \bar{Y}_1^{n,\beta_n,x_0} \right]^2 \right], \\ &\xrightarrow{n \rightarrow \infty} \frac{1}{\sigma^2} \mathbb{E} \left[ \mathbb{E} \left[ (L_1^\alpha \mathcal{H}_{L^\alpha}(1) - 1) | L_1^\alpha \right]^2 \right], \quad \text{uniformly with respect to } x_0, \\ &= \frac{1}{\sigma^2} \mathbb{E} \left[ \frac{(\varphi_\alpha(L_1^\alpha) + L_1^\alpha \varphi'_\alpha(L_1^\alpha))^2}{\varphi_\alpha(L_1^\alpha)^2} \right] \quad \text{from (3.3.10)}. \end{aligned}$$

**Convergence of  $n^{1-1/\alpha} \mathcal{I}_{12}^{n,\beta_n,x_0}$ .** We first recall the expression for the Fisher information

$$\mathcal{I}_{12}^{n,\beta_n,x_0} = \mathbb{E} \left[ \left( \frac{\partial_\theta p_{\frac{1}{n}}^{\beta_n}(x_0, X_{1/n}^{\beta_n})}{p_{\frac{1}{n}}^{\beta_n}(x_0, X_{1/n}^{\beta_n})} \partial_\sigma p_{\frac{1}{n}}^{\beta_n}(x_0, X_{1/n}^{\beta_n}) \right) \right],$$

then from (3.3.6) in Proposition 3.3.1 and Lemma 3.3.5 we have

$$(3.3.11) \quad n^{1-1/\alpha} \mathcal{I}_{12}^{n,\beta_n,x_0} = n^{1-1/\alpha} \mathbb{E}[\mathbb{E}[\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(\partial_\theta \bar{Y}_1^{n,\beta_n,x_0}) | \bar{Y}_1^{n,\beta_n,x_0}] \mathbb{E}[\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(\partial_\sigma \bar{Y}_1^{n,\beta_n,x_0}) | \bar{Y}_1^{n,\beta_n,x_0}]] \\ \xrightarrow{n \rightarrow \infty} \frac{1}{\sigma^2} \mathbb{E}[\mathbb{E}[\mathcal{H}_{L^\alpha}(1) | L_1^\alpha] \mathbb{E}[L_1^\alpha \mathcal{H}_{L^\alpha}(1) - 1 | L_1^\alpha]].$$

On the other hand, from (3.3.7) and (3.3.10) we get

$$(3.3.12) \quad \frac{\varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)} \left( \frac{\varphi_\alpha(L_1^\alpha) + L_1^\alpha \varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)} \right) = \mathbb{E}[\mathcal{H}_{L^\alpha}(1) | L_1^\alpha] \mathbb{E}[(L_1^\alpha \mathcal{H}_{L^\alpha}(1) - 1) | L_1^\alpha].$$

Combining (3.3.11) with (3.3.12) gives

$$(3.3.13) \quad n^{1-1/\alpha} \mathcal{I}_{12}^{n,\beta_n,x_0} \xrightarrow{n \rightarrow \infty} \frac{1}{\sigma^2} \int_{\mathbb{R}} \frac{\varphi'_\alpha(u) [\varphi_\alpha(u) + u \varphi'_\alpha(u)]}{\varphi_\alpha(u)} du = \frac{1}{\sigma^2} \int_{\mathbb{R}} \varphi'_\alpha(u) du + \frac{1}{\sigma^2} \int_{\mathbb{R}} \frac{u (\varphi'_\alpha(u))^2}{\varphi_\alpha(u)} du = 0,$$

where we used the fact that  $\varphi_\alpha$  is a symmetric function, and that the functions under the integral are odd. This completes the proof of part *i*).

*ii*) Using (3.3.6) in Proposition 3.3.1 again and Jensen's inequality, we get:

$$\mathbb{E} \left[ \left| n^{1/2} r_n \frac{\nabla_\beta p_{\frac{1}{n}}^\beta(x_0, X_{\frac{1}{n}}^\beta)}{p_{\frac{1}{n}}^\beta(x_0, X_{\frac{1}{n}}^\beta)} \right|^p \right] \leq \mathbb{E} |n^{1/2} r_n \mathcal{H}_{\bar{Y}_1^{n,\beta,x_0}}(\nabla_\beta \bar{Y}_1^{n,\beta,x_0})|^p,$$

and the result follows from Lemma 3.3.3. This achieves the proof of Theorem 3.3.1. ■

### 3.4 Proof of the asymptotic expansion of the likelihood

#### (Theorems 3.2.1 - 3.2.2)

The aim of this section is to prove the asymptotic expansion of the log-likelihood function, stated in Theorem 3.2.1 as well as the convergence given in Theorem 3.2.2. The proof of Theorem 3.2.1 is based essentially on the  $L^2$ -regularity property of the transition density  $p_{1/n}^\beta(x, y)$ . From Jeganathan's article [39], the following four conditions **A1-A4** are sufficient to get the expansion (3.2.2) of Theorem 3.2.1.

We recall the notation  $\xi_{i,n}(\beta) = \begin{pmatrix} \frac{\partial_\theta p_{\frac{1}{n}}^\beta}{p_{\frac{1}{n}}^\beta} \left( X_{\frac{i}{n}}^\beta, X_{\frac{i+1}{n}}^\beta \right) \\ \frac{\partial_\sigma p_{\frac{1}{n}}^\beta}{p_{\frac{1}{n}}^\beta} \left( X_{\frac{i}{n}}^\beta, X_{\frac{i+1}{n}}^\beta \right) \end{pmatrix}$  and we denote  $\chi_n(\beta, x, y) = \begin{pmatrix} \frac{\partial_\theta p_{\frac{1}{n}}^\beta(x, y)}{p_{\frac{1}{n}}^\beta(x, y)^{1/2}} \\ \frac{\partial_\sigma p_{\frac{1}{n}}^\beta(x, y)}{p_{\frac{1}{n}}^\beta(x, y)^{1/2}} \end{pmatrix}$ .

#### A1. $L^2$ -regularity

$$\sum_{j=1}^n \mathbb{E} \left[ \int_{\mathbb{R}} \left\{ p_{\frac{1}{n}}^{\beta+r_n h} \left( X_{\frac{j-1}{n}}^\beta, y \right)^{1/2} - p_{\frac{1}{n}}^\beta \left( X_{\frac{j-1}{n}}^\beta, y \right)^{1/2} - \frac{1}{2} h^T r_n \chi_n(\beta, X_{\frac{j-1}{n}}^\beta, y) \right\}^2 dy \right] \xrightarrow{n \rightarrow \infty} 0.$$

#### A2.

$$J_n(\beta) = r_n \sum_{i=0}^{n-1} \mathbb{E} \left[ \xi_{i,n}(\beta) \xi_{i,n}^T(\beta) | \mathcal{F}_{i/n} \right] r_n \xrightarrow{n \rightarrow \infty} \mathcal{J}(\beta) \quad (> 0 \text{ a.e.}), \text{ in probability.}$$

#### A3.

$$\forall \epsilon > 0, \quad \sum_{i=0}^{n-1} \mathbb{E} \left[ |r_n \xi_{i,n}(\beta)|^2 \mathbf{1}_{\{|r_n \xi_{i,n}(\beta)| \geq \epsilon\}} \right] \xrightarrow{n \rightarrow \infty} 0.$$

#### A4.

$$\sup_n \sum_{i=0}^{n-1} \mathbb{E} \left( \left| r_n \xi_{i,n}(\beta) \xi_{i,n}^T(\beta) r_n \right| \right) \leq C, \quad \text{for a strictly positive constant } C.$$

The condition **A1** is proved in Section 3.4.1 and **A2** and **A3** are proved in Section 3.4.2. The condition **A4** is immediate from Theorem 3.3.1 *ii*) since

$$n \mathbb{E} (r_n \xi_{i,n}(\beta) \xi_{i,n}^T(\beta) r_n) = \mathbb{E} \begin{pmatrix} n^{2-\frac{2}{\alpha}} \mathcal{G}_{11}^{n, \beta, X_{i/n}^\beta} & n^{1-\frac{1}{\alpha}} \mathcal{G}_{12}^{n, \beta, X_{i/n}^\beta} \\ n^{1-\frac{1}{\alpha}} \mathcal{G}_{12}^{n, \beta, X_{i/n}^\beta} & \mathcal{G}_{22}^{n, \beta, X_{i/n}^\beta} \end{pmatrix}.$$

Note that these conditions do not imply the stable convergence in law (3.2.6) since in our framework the filtration  $(\mathcal{F}_i)_i$  does not satisfy the "nested condition" (see Theorem 3.2 in [31]). The proof of the stable convergence in law will be given in Section 3.5.



### 3.4.1 Proof of the condition A1 (the $L^2$ -regularity condition).

Following [17], the crucial point of the proof is the asymptotic behavior of the transition density of  $X_t^\beta$  established in [19] and recalled below.

**Theorem 3.4.1.** [Theorem 2.1 in [19]]

We assume that  $\mathbf{H}_1$  holds. Let  $(\zeta_t^{n,\theta,x_0})$  be the solution to the ordinary differential equation

$$(3.4.1) \quad \zeta_t^{n,\theta,x_0} = x_0 + \frac{1}{n} \int_0^t b(\zeta_s^{n,\theta,x_0}, \theta) ds \quad t \in [0, 1].$$

Let  $(\beta_n)_{n \geq 1}$  be a sequence such that  $\beta_n \xrightarrow{n \rightarrow \infty} \beta$  then for all  $(x_0, u) \in \mathbb{R}^2$ ,

- i)  $\frac{\sigma_n}{n^{1/\alpha}} p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0}) \xrightarrow{n \rightarrow \infty} \varphi_\alpha(u)$ ,
- ii)  $\sup_{u \in \mathbb{R}} \sup_n \frac{\sigma_n}{n^{1/\alpha}} p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0}) < \infty$ ,

where  $\varphi_\alpha$  is the density of  $L_1^\alpha$ , a centered  $\alpha$ -stable Lévy process.

**Theorem 3.4.2.** [Theorem 2.2 in [19]]

Under the assumptions of Theorem 3.4.1,

- i)  $\frac{\sigma_n^2}{n^{\frac{2}{\alpha}-1}} \partial_\theta p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0}) \xrightarrow{n \rightarrow \infty} -\partial_\theta b(x_0, \theta) \times \varphi'_\alpha(u)$ ,  
 $\frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0}) \xrightarrow{n \rightarrow \infty} -\varphi_\alpha(u) - u\varphi'_\alpha(u)$ ,
- ii)  $\sup_{u \in \mathbb{R}} \sup_n \left| \frac{\sigma_n^2}{n^{\frac{2}{\alpha}-1}} \partial_\theta p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0}) \right| < \infty$ ,  
 $\sup_{u \in \mathbb{R}} \sup_n \left| \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \zeta_1^{n,\theta_n,x_0}) \right| < \infty$ .

**Proof of A1** By the change of variable  $y = \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n,\theta, X_{\frac{j-1}{n}}^\beta}$  proving **A1** is equivalent to show:

$$(3.4.2) \quad \frac{1}{n} \sum_{j=1}^n \int_{\mathbb{R}} \mathbb{E} \left[ \left\{ f_n(X_{\frac{j-1}{n}}^\beta, u) - g_n(X_{\frac{j-1}{n}}^\beta, u) \right\}^2 \right] du \xrightarrow{n \rightarrow \infty} 0,$$

where

$$f_n(x, u) = \sqrt{\sigma} n^{1/2-1/2\alpha} \left[ p_{\frac{1}{n}}^{\beta+r_n h} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n,\theta,x} \right)^{1/2} - p_{\frac{1}{n}}^\beta \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n,\theta,x} \right)^{1/2} \right],$$

$$g_n(x, u) = \frac{\sqrt{\sigma}}{2} n^{1/2-1/2\alpha} (r_n h)^T \chi_n \left( \beta, x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n,\theta,x} \right).$$

Following the proof of Proposition 8 in [17], the next three properties are sufficient to prove (3.4.2).

1. There exists a function  $f$  such that,

$$(3.4.3) \quad \begin{aligned} \forall x, u, \quad f_n(x, u) &\xrightarrow{n \rightarrow \infty} f(x, u), \\ g_n(x, u) &\xrightarrow{n \rightarrow \infty} f(x, u). \end{aligned}$$

2. We have for all  $x$ ,

$$(3.4.4) \quad \begin{aligned} \limsup_n \int_{\mathbb{R}} f_n(x, u)^2 du &\leq \int_{\mathbb{R}} f(x, u)^2 du, \\ \limsup_n \int_{\mathbb{R}} g_n(x, u)^2 du &\leq \int_{\mathbb{R}} f(x, u)^2 du. \end{aligned}$$

3. We have

$$(3.4.5) \quad \begin{aligned} \sup_{x, n} \int_{\mathbb{R}} f_n(x, u)^2 du &< \infty, \\ \sup_{x, n} \int_{\mathbb{R}} g_n(x, u)^2 du &< \infty. \end{aligned}$$

We now need to check the validity of the conditions (3.4.3), (3.4.4) and (3.4.5).

We start with the proof of the condition (3.4.3).

We recall that  $r_n = \begin{pmatrix} n^{\frac{1}{2} - \frac{1}{\alpha}} & 0 \\ 0 & n^{-\frac{1}{2}} \end{pmatrix}$  and  $h = (h_1, h_2)^T \in \mathbb{R}^2$  then by a simple computation we have,

$$g_n(x, u) = \frac{\sqrt{\sigma}}{2} n^{1 - \frac{3}{2\alpha}} h_1 \frac{\partial_{\theta} p_{\frac{1}{n}}^{\beta} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)}{p_{\frac{1}{n}}^{\beta} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)^{1/2}} + \frac{\sqrt{\sigma}}{2} n^{\frac{-1}{2\alpha}} h_2 \frac{\partial_{\sigma} p_{\frac{1}{n}}^{\beta} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)}{p_{\frac{1}{n}}^{\beta} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)^{1/2}}.$$

From Theorems 3.4.1 and 3.4.2, we see that

$$g_n(x, u) \xrightarrow{n \rightarrow \infty} f(x, u) := -\frac{h_1}{2\sigma} \partial_{\theta} b(x, \theta) \frac{\varphi'_{\alpha}(u)}{\varphi_{\alpha}(u)^{1/2}} - \frac{h_2}{2\sigma} \frac{(\varphi_{\alpha}(u) + u\varphi'_{\alpha}(u))}{\varphi_{\alpha}(u)^{1/2}}.$$

Let  $m_t = p_{\frac{1}{n}}^{\beta + r_n h t} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)$ ,  $t \in [0, 1]$ , then we can rewrite  $f_n(x, u)$  as

$$f_n(x, u) = \sqrt{\sigma} n^{1/2 - 1/2\alpha} \left[ m_1^{1/2} - m_0^{1/2} \right].$$

Using the mean value theorem, we get for some  $s(x, u) \in (0, 1)$

$$f_n(x, u) = \frac{\sqrt{\sigma}}{2} n^{1/2 - 1/2\alpha} \frac{m'_{s(x, u)}}{\sqrt{m_{s(x, u)}}} = \frac{\sqrt{\sigma}}{2} n^{1/2 - 1/2\alpha} (r_n h)^T \frac{\nabla_{\beta} p_{\frac{1}{n}}^{\beta} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)}{p_{\frac{1}{n}}^{\beta} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)^{1/2}},$$

where  $\beta_n = \beta + r_n s(x, u) h$ .

From Theorems 3.4.1 and 3.4.2, we also get that  $f_n(x, u) \xrightarrow{n \rightarrow \infty} f(x, u)$ .

Now we prove the condition (3.4.4).

We have

(3.4.6)

$$\begin{aligned} \int_{\mathbb{R}} g_n(x, u)^2 du &= \frac{\sigma h_1^2}{4} n^{2-3/\alpha} \int_{\mathbb{R}} \frac{\partial_{\theta} p_{\frac{1}{n}}^{\beta} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)^2}{p_{\frac{1}{n}}^{\beta} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)} du + \frac{\sigma h_2^2}{4} n^{-1/\alpha} \int_{\mathbb{R}} \frac{\partial_{\sigma} p_{\frac{1}{n}}^{\beta} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)^2}{p_{\frac{1}{n}}^{\beta} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)} du \\ &+ \frac{\sigma h_1 h_2}{2} n^{1-2/\alpha} \int_{\mathbb{R}} \frac{\partial_{\theta} p_{\frac{1}{n}}^{\beta} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right) \partial_{\sigma} p_{\frac{1}{n}}^{\beta} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)}{p_{\frac{1}{n}}^{\beta} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)^{1/2} p_{\frac{1}{n}}^{\beta} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)^{1/2}} du. \end{aligned}$$

From Theorem 3.3.1, we get

$$(3.4.7) \quad \int_{\mathbb{R}} g_n(x, u)^2 du \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}} f(x, u)^2 du, \quad \forall x.$$

Using

$$f_n(x, u) = \frac{\sqrt{\sigma}}{2} n^{1/2-1/2\alpha} \int_0^1 (r_n h)^T \frac{\nabla_{\beta} p_{\frac{1}{n}}^{\beta+r_n h s} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)}{p_{\frac{1}{n}}^{\beta+r_n h s} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)^{1/2}} ds,$$

we write:

$$\begin{aligned} \int_{\mathbb{R}} f_n(x, u)^2 du &= \|f_n(x, \cdot)\|_2^2 = \frac{\sigma n^{1-1/\alpha}}{4} \left\| \int_0^1 (r_n h)^T \frac{\nabla_{\beta} p_{\frac{1}{n}}^{\beta+r_n h s} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)}{p_{\frac{1}{n}}^{\beta+r_n h s} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)^{1/2}} ds \right\|_2^2 \\ &\leq \frac{\sigma n^{1-1/\alpha}}{4} \left( \int_0^1 \left\| (r_n h)^T \frac{\nabla_{\beta} p_{\frac{1}{n}}^{\beta+r_n h s} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)}{p_{\frac{1}{n}}^{\beta+r_n h s} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)^{1/2}} \right\|_2 ds \right)^2 \\ &= \frac{\sigma n^{1-1/\alpha}}{4} \left( \int_0^1 \left\| n^{\frac{1}{2}-\frac{1}{\alpha}} h_1 \frac{\partial_{\theta} p_{\frac{1}{n}}^{\beta+r_n h s} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)}{p_{\frac{1}{n}}^{\beta+r_n h s} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)^{1/2}} + n^{-\frac{1}{2}} h_2 \frac{\partial_{\sigma} p_{\frac{1}{n}}^{\beta+r_n h s} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)}{p_{\frac{1}{n}}^{\beta+r_n h s} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)^{1/2}} \right\|_2 ds \right)^2 \\ &= \frac{\sigma n^{1-1/\alpha}}{4} \left[ \int_0^1 \left( \int_{\mathbb{R}} n^{1-\frac{2}{\alpha}} h_1^2 \frac{\partial_{\theta} p_{\frac{1}{n}}^{\beta+sr_n h} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)^2}{p_{\frac{1}{n}}^{\beta+sr_n h} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)} du + \int_{\mathbb{R}} n^{-1} h_2^2 \frac{\partial_{\sigma} p_{\frac{1}{n}}^{\beta+sr_n h} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)^2}{p_{\frac{1}{n}}^{\beta+sr_n h} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)} du \right. \right. \\ &\quad \left. \left. + \int_{\mathbb{R}} n^{-\frac{1}{\alpha}} 2h_1 h_2 \frac{\partial_{\theta} p_{\frac{1}{n}}^{\beta+sr_n h} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right) \partial_{\sigma} p_{\frac{1}{n}}^{\beta+sr_n h} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)}{p_{\frac{1}{n}}^{\beta+sr_n h} \left( x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} \right)} du \right)^2 ds \right]^2 \end{aligned}$$

$$\begin{aligned}
&= \frac{n^{1-1/\alpha}}{4} \left[ \int_0^1 \left( \int_{\mathbb{R}} n^{1-\frac{1}{\alpha}} h_1^2 \frac{\partial_{\theta} p_{\frac{1}{n}}^{\beta+sr_n h}(x, v)^2}{p_{\frac{1}{n}}^{\beta+sr_n h}(x, v)} dv + \int_{\mathbb{R}} n^{\frac{1}{\alpha}-1} h_2^2 \frac{\partial_{\sigma} p_{\frac{1}{n}}^{\beta+sr_n h}(x, v)^2}{p_{\frac{1}{n}}^{\beta+sr_n h}(x, v)} dv \right. \right. \\
&\quad \left. \left. + \int_{\mathbb{R}} 2h_1 h_2 \frac{\partial_{\theta} p_{\frac{1}{n}}^{\beta+sr_n h}(x, v) \partial_{\sigma} p_{\frac{1}{n}}^{\beta+sr_n h}(x, v)}{p_{\frac{1}{n}}^{\beta+sr_n h}(x, v)} dv \right)^{1/2} ds \right]^2 \text{ by the change of variable } \frac{u\sigma}{n^{1/\alpha}} + \zeta_1^{n, \theta, x} = v, \\
&= \frac{1}{4} \left[ \int_0^1 \left( n^{2-\frac{2}{\alpha}} h_1^2 \mathcal{J}_{11}^{n, \beta+sr_n h, x} + h_2^2 \mathcal{J}_{22}^{n, \beta+sr_n h, x} + n^{1-1/\alpha} 2h_1 h_2 \mathcal{J}_{12}^{n, \beta+sr_n h, x} \right)^{1/2} ds \right]^2 \\
(3.4.8) \quad &\xrightarrow{n \rightarrow \infty} \frac{1}{\sigma^2} \frac{h_1^2}{4} (\partial_{\theta} b(x, \theta))^2 \int_{\mathbb{R}} \frac{(\varphi'_{\alpha}(u))^2}{\varphi_{\alpha}(u)} du + \frac{h_1 h_2}{2\sigma^2} \int_{\mathbb{R}} \frac{\varphi'_{\alpha}(u) [\varphi_{\alpha}(u) + u\varphi'_{\alpha}(u)]}{\varphi_{\alpha}(u)} du \\
&\quad + \frac{1}{\sigma^2} \frac{h_2^2}{4} \int_{\mathbb{R}} \frac{(\varphi_{\alpha}(u) + u\varphi'_{\alpha}(u))^2}{\varphi_{\alpha}(u)} du = \int_{\mathbb{R}} f^2(x, u) du.
\end{aligned}$$

Where, in the last line, we have used Theorem 3.3.1 for the convergences of  $\mathcal{J}_{11}^{n, \beta+sr_n h, x}$ ,  $\mathcal{J}_{22}^{n, \beta+sr_n h, x}$ ,  $\mathcal{J}_{12}^{n, \beta+sr_n h, x}$ , respectively and the application of the dominated convergence theorem. From (3.4.7) and (3.4.8) we get (3.4.4). (3.4.5) is deduced directly from Theorem 3.3.1.  $\blacksquare$

### 3.4.2 Proof of the conditions A2 and A3 (Theorem 3.2.2)

From the Markov property and (3.3.1) we have:

$$\mathcal{J}^{n, \beta, X_{i/n}^{\beta}} = \mathbb{E} \left[ \xi_{i,n}(\beta) \xi_{i,n}(\beta)^T | \mathcal{G}_{i/n} \right] = \begin{pmatrix} \mathcal{J}_{11}^{n, \beta, X_{i/n}^{\beta}} & \mathcal{J}_{12}^{n, \beta, X_{i/n}^{\beta}} \\ \mathcal{J}_{12}^{n, \beta, X_{i/n}^{\beta}} & \mathcal{J}_{22}^{n, \beta, X_{i/n}^{\beta}} \end{pmatrix}.$$

From Theorem 3.3.1 we know that the quantities

$$\sup_{0 \leq i \leq n-1} \left| nr_n \mathcal{J}^{n, \beta, X_{i/n}^{\beta}} r_n - \begin{pmatrix} \frac{1}{\sigma^2} [\partial_{\theta} b(X_{i/n}^{\beta})]^2 \int_{\mathbb{R}} \frac{\varphi'(u)^2}{\varphi(u)} du & 0 \\ 0 & \frac{1}{\sigma^2} \int_{\mathbb{R}} \frac{[\varphi_{\alpha}(u) + u\varphi'_{\alpha}(u)]^2}{\varphi_{\alpha}(u)} du \end{pmatrix} \right|$$

converge to zero as  $n \rightarrow \infty$ . Then the convergence **A2** is a consequence of the convergence of a Riemann sum.

To prove **A3**, from the Markov property, we get:  $\mathbb{E} \left[ |r_n \xi_{i,n}(\beta)|^k | X_{i/n}^{\beta} = x \right] = \mathbb{E} \left[ \left| r_n \frac{\nabla_{\beta} p_{\frac{1}{n}}^{\beta}(x, X_{\frac{1}{n}}^{\beta})}{p_{\frac{1}{n}}^{\beta}(x, X_{\frac{1}{n}}^{\beta})} \right|^k \right]$ , for any  $k \geq 1$ . But from Theorem 3.3.1 *ii*) we have  $\sup_{n,x} \mathbb{E} \left[ \left| n^{1/2} r_n \frac{\nabla_{\beta} p_{\frac{1}{n}}^{\beta}(x, X_{\frac{1}{n}}^{\beta})}{p_{\frac{1}{n}}^{\beta}(x, X_{\frac{1}{n}}^{\beta})} \right|^k \right] < \infty$ ,  $\forall k \geq 1$ . This control, for instance with  $k = 4$ , is sufficient to imply the Lindeberg's condition **A3**.

### 3.5 Proof of Theorem 3.2.3 (Stable central limit theorem)

The aim of this section is to prove the stable convergence in law stated in Theorem 3.2.3. We first recall the following result established in [19] where  $\bar{Y}_1^{n,\beta,x_0}$  is defined by (3.3.4) and is equal in law to  $X_{\frac{1}{n}}^\beta$ .

**Lemma 3.5.1.** [Lemma 4.1 in [19]] Let  $(\zeta_t^{n,\theta,x_0})$  be the solution to the ordinary differential equation (3.4.1), then

$$(3.5.1) \quad n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) \xrightarrow[n \rightarrow \infty]{a.s.} \sigma L_1^\alpha,$$

and this convergence is uniform with respect to  $x_0$ .

**Proof of Theorem 3.2.3** Since  $r_n = \begin{pmatrix} n^{\frac{1}{2}-\frac{1}{\alpha}} & 0 \\ 0 & n^{-\frac{1}{2}} \end{pmatrix}$  we have

$$r_n \sum_{i=0}^{n-1} \xi_{i,n}(\beta) = \sum_{i=0}^{n-1} \begin{pmatrix} n^{\frac{1}{2}-\frac{1}{\alpha}} \frac{\partial_\theta p_{\frac{1}{n}}^\beta}{p_{\frac{1}{n}}^\beta} \left( X_{\frac{i}{n}}^\beta, X_{\frac{i+1}{n}}^\beta \right) \\ n^{-\frac{1}{2}} \frac{\partial_\sigma p_{\frac{1}{n}}^\beta}{p_{\frac{1}{n}}^\beta} \left( X_{\frac{i}{n}}^\beta, X_{\frac{i+1}{n}}^\beta \right) \end{pmatrix}.$$

Theorem 3.2.3 is an immediate consequence of Lemmas 3.5.2-3.5.3-3.5.4 below. ■

**Lemma 3.5.2.** We consider

$$\omega_{i,n} = n^{1-1/\alpha} \frac{\partial_\theta p_{\frac{1}{n}}^\beta}{p_{\frac{1}{n}}^\beta} \left( X_{\frac{i}{n}}^\beta, X_{\frac{i+1}{n}}^\beta \right) + \frac{1}{\sigma} \partial_\theta b \left( X_{\frac{i}{n}}^\beta, \theta \right) \frac{\varphi'_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)}{\varphi_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)}$$

then we have  $n^{-1/2} \sum_{i=0}^{n-1} \omega_{i,n} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$ .

**Proof.** Using Lemma 9 in [26], it is sufficient to show that:

$$(3.5.2) \quad n^{-1/2} \sum_{i=0}^{n-1} |\mathbb{E}[\omega_{i,n} | \mathcal{F}_{i/n}]| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0,$$

$$(3.5.3) \quad n^{-1} \sum_{i=0}^{n-1} |\mathbb{E}[\omega_{i,n}^2 | \mathcal{F}_{i/n}]| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0,$$

We start by the proof of (3.5.2). Since a score function has an expectation equal to zero, and  $L_{\frac{i+1}{n}} - L_{\frac{i}{n}}$  is independent of  $\mathcal{F}_{i/n}$ , we deduce that

$$\mathbb{E}[\bar{\omega}_{i,n} | \mathcal{F}_{i/n}] = \frac{1}{\sigma} \partial_{\theta} b(X_{\frac{i}{n}}^{\beta}, \theta) \mathbb{E} \left[ \frac{\varphi'_{\alpha} \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)}{\varphi_{\alpha} \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)} \right].$$

Since  $(L_t)_t$  has stationary increments, the law of  $n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}})$  is the same as the law of  $L_1^n$ . Moreover, we know from Lemma 3.3.1, that  $\mathbb{P}(L_1^n \neq L_1^{\alpha}) = O(1/n)$ , thus

$$\mathbb{E}[\bar{\omega}_{i,n} | \mathcal{F}_{i/n}] = \frac{1}{\sigma} \partial_{\theta} b(X_{\frac{i}{n}}^{\beta}, \theta) \mathbb{E} \left[ \frac{\varphi'_{\alpha}(L_1^{\alpha})}{\varphi_{\alpha}(L_1^{\alpha})} \right] + \left\| \frac{\varphi'_{\alpha}}{\varphi_{\alpha}} \right\|_{\infty} O(n^{-1}),$$

where we used the fact that  $\frac{\varphi'_{\alpha}}{\varphi_{\alpha}}$  is bounded (see e.g. Theorem 7.3.2 in [45]). Using  $\mathbb{E} \left[ \frac{\varphi'_{\alpha}(L_1^{\alpha})}{\varphi_{\alpha}(L_1^{\alpha})} \right] = \int_{\mathbb{R}} \varphi'_{\alpha}(u) du = 0$ , we deduce  $|\mathbb{E}[\bar{\omega}_{i,n} | \mathcal{G}_{i/n}]| \leq Cn^{-1}$  for some constant  $C$  and (3.5.2) follows.

We now prove (3.5.3). We have

$$(3.5.4) \quad \begin{aligned} \mathbb{E}[\bar{\omega}_{i,n}^2 | \mathcal{F}_{i/n}] &= n^{2-2/\alpha} \mathcal{G}_{11}^{n,\beta, X_{\frac{i}{n}}^{\beta}} + \frac{1}{\sigma^2} \left[ \partial_{\theta} b(X_{\frac{i}{n}}^{\beta}, \theta) \right]^2 \mathbb{E} \left[ \frac{\varphi'_{\alpha} \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)^2}{\varphi_{\alpha} \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)^2} \right] \\ &+ 2\mathbb{E} \left[ n^{1-1/\alpha} \frac{\partial_{\theta} p_{\frac{1}{n}}^{\beta}(X_{\frac{i}{n}}^{\beta}, X_{\frac{i+1}{n}}^{\beta})}{p_{\frac{1}{n}}^{\beta}(X_{\frac{i}{n}}^{\beta}, X_{\frac{i+1}{n}}^{\beta})} \frac{1}{\sigma} \partial_{\theta} b(X_{\frac{i}{n}}^{\beta}, \theta) \frac{\varphi'_{\alpha} \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)}{\varphi_{\alpha} \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)} \middle| \mathcal{F}_{\frac{i}{n}} \right]. \end{aligned}$$

With a method analogous to the proof of (3.5.2), we can show that

$\mathbb{E} \left[ \frac{\varphi'_{\alpha} \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)^2}{\varphi_{\alpha} \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)^2} \right] = \mathbb{E} \left[ \frac{\varphi'_{\alpha}(L_1^{\alpha})^2}{\varphi_{\alpha}(L_1^{\alpha})^2} \right] + o(1)$ . From Theorem 3.3.1, it appears that the first two terms in the right-hand side of (3.5.4) are asymptotically close to the same quantities, and that (3.5.3)

is proved as soon as we show that the following control holds, uniformly with respect to  $i$ ,

$$(3.5.5) \quad \begin{aligned} &\mathbb{E} \left[ n^{1-1/\alpha} \frac{\partial_{\theta} p_{\frac{1}{n}}^{\beta}(X_{\frac{i}{n}}^{\beta}, X_{\frac{i+1}{n}}^{\beta})}{p_{\frac{1}{n}}^{\beta}(X_{\frac{i}{n}}^{\beta}, X_{\frac{i+1}{n}}^{\beta})} \frac{1}{\sigma} \partial_{\theta} b(X_{\frac{i}{n}}^{\beta}, \theta) \frac{\varphi'_{\alpha} \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)}{\varphi_{\alpha} \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)} \middle| \mathcal{F}_{\frac{i}{n}} \right] \\ &= -\frac{1}{\sigma^2} \left[ \partial_{\theta} b(X_{\frac{i}{n}}^{\beta}, \theta) \right]^2 \mathbb{E} \left[ \frac{\varphi'_{\alpha}(L_1^{\alpha})^2}{\varphi_{\alpha}(L_1^{\alpha})^2} \right] + o(1). \end{aligned}$$

Using the notations of Section 3.3.2.1, we define  $d^{n,\theta,x_0} = \mathbb{E} \left[ n^{1-1/\alpha} \frac{\partial_{\theta} q^{n,\beta,x_0}(\bar{Y}_1^{n,\beta,x_0})}{q^{n,\beta,x_0}(\bar{Y}_1^{n,\beta,x_0})} \frac{1}{\sigma} \partial_{\theta} b(x_0, \theta) \frac{\varphi'_{\alpha}(L_1^n)}{\varphi_{\alpha}(L_1^n)} \right]$ , so that the left-hand side of (3.5.5) reduces, from the Markov property, (3.3.5) and the fact that

$\bar{Y}_1^{n,\beta,x_0}$  equals in law  $X_{\frac{1}{n}}^\beta$ , to  $d^{n,\theta,X_{\frac{1}{n}}^\beta}$ . On the other hand, we can rewrite  $d^{n,\theta,x_0}$  as

$$(3.5.6) \quad \begin{aligned} d^{n,\theta,x_0} = & \mathbb{E} \left[ n^{1-1/\alpha} \frac{\partial_\theta q^{n,\beta,x_0}(\bar{Y}_1^{n,\beta,x_0})}{q^{n,\beta,x_0}(\bar{Y}_1^{n,\beta,x_0})} \frac{1}{\sigma} \partial_\theta b(x_0, \theta) \frac{\varphi'_\alpha \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \right)}{\varphi_\alpha \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \right)} \right] \\ & + \mathbb{E} \left[ n^{1-1/\alpha} \frac{\partial_\theta q^{n,\beta,x_0}(\bar{Y}_1^{n,\beta,x_0})}{q^{n,\beta,x_0}(\bar{Y}_1^{n,\beta,x_0})} \frac{1}{\sigma} \partial_\theta b(x_0, \theta) \left( \frac{\varphi'_\alpha(L_1^n)}{\varphi_\alpha(L_1^n)} - \frac{\varphi'_\alpha \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \right)}{\varphi_\alpha \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \right)} \right) \right]. \end{aligned}$$

Using the Cauchy-Schwarz inequality for the second term in the right-hand side of (3.5.6), we get

$$(3.5.7) \quad \begin{aligned} & \mathbb{E} \left[ \left| n^{1-1/\alpha} \frac{\partial_\theta q^{n,\beta,x_0}(\bar{Y}_1^{n,\beta,x_0})}{q^{n,\beta,x_0}(\bar{Y}_1^{n,\beta,x_0})} \frac{1}{\sigma} \partial_\theta b(x_0, \theta) \left( \frac{\varphi'_\alpha(L_1^n)}{\varphi_\alpha(L_1^n)} - \frac{\varphi'_\alpha \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \right)}{\varphi_\alpha \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \right)} \right) \right| \right] \\ & \leq \left[ \mathbb{E} \left( n^{1-1/\alpha} \frac{\partial_\theta q^{n,\beta,x_0}(\bar{Y}_1^{n,\beta,x_0})}{q^{n,\beta,x_0}(\bar{Y}_1^{n,\beta,x_0})} \frac{1}{\sigma} \partial_\theta b(x_0, \theta) \right)^2 \right]^{1/2} \left[ \mathbb{E} \left( \frac{\varphi'_\alpha(L_1^n)}{\varphi_\alpha(L_1^n)} - \frac{\varphi'_\alpha \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \right)}{\varphi_\alpha \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \right)} \right)^2 \right]^{1/2}. \end{aligned}$$

Furthermore,  $\forall \epsilon > 0$  we have

$$\begin{aligned} & \mathbb{E} \left( \frac{\varphi'_\alpha(L_1^n)}{\varphi_\alpha(L_1^n)} - \frac{\varphi'_\alpha \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \right)}{\varphi_\alpha \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \right)} \right)^2 \\ & = \mathbb{E} \left( \frac{\varphi'_\alpha(L_1^n)}{\varphi_\alpha(L_1^n)} - \frac{\varphi'_\alpha \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \right)}{\varphi_\alpha \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \right)} \right)^2 \mathbf{1} \left\{ \left| \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} - L_1^n \right| \leq \epsilon \right\} \\ & + \mathbb{E} \left( \frac{\varphi'_\alpha(L_1^n)}{\varphi_\alpha(L_1^n)} - \frac{\varphi'_\alpha \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \right)}{\varphi_\alpha \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \right)} \right)^2 \mathbf{1} \left\{ \left| \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} - L_1^n \right| > \epsilon \right\} \\ & \leq C_1 \epsilon^2 + 2C_2 \mathbb{E} \left[ \mathbf{1} \left\{ \left| \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} - L_1^n \right| > \epsilon \right\} \right] \\ & = C_1 \epsilon^2 + 2C_2 \mathbb{P} \left[ \left| \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} - L_1^n \right| > \epsilon \right] \xrightarrow{n \rightarrow \infty} C_1 \epsilon^2, \end{aligned}$$

where we used the fact that  $\frac{\varphi'_\alpha}{\varphi_\alpha}$  is bounded with a bounded derivative (see e.g. Theorem 7.3.2 in [45]) and Lemma 3.5.1. From Theorem 3.3.1 *ii*), and the estimates above we deduce that (3.5.7)

converges to zero as  $n \rightarrow \infty$ . Then,

$$(3.5.8) \quad d^{n,\theta,x_0} = \mathbb{E} \left[ n^{1-1/\alpha} \frac{\partial_\theta q^{n,\beta,x_0}(\bar{Y}_1^{n,\beta,x_0})}{q^{n,\beta,x_0}(\bar{Y}_1^{n,\beta,x_0})} \frac{1}{\sigma} \partial_\theta b(x_0, \theta) \frac{\varphi'_\alpha \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \right)}{\varphi_\alpha \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \right)} \right] + o(1),$$

where the  $o(1)$  term is uniform with respect to  $x_0$ . Now, using Proposition 3.3.1, we get

$$d^{n,\theta,x_0} = \mathbb{E} \left[ n^{1-1/\alpha} \mathcal{H}_{\bar{Y}_1^{n,\beta,x_0}}(\partial_\theta \bar{Y}_1^{n,\beta,x_0}) \frac{1}{\sigma} \partial_\theta b(x_0, \theta) \frac{\varphi'_\alpha \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \right)}{\varphi_\alpha \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \right)} \right] + o(1).$$

From Lemma 3.3.3, we also have

$$\sup_{x_0} \left| d^{n,\theta,x_0} - \frac{1}{\sigma^2} \partial_\theta b(x_0, \theta)^2 \mathbb{E} \left[ \mathcal{H}_{L^\alpha}(1) \frac{\varphi'_\alpha \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \right)}{\varphi_\alpha \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \right)} \right] \right| \xrightarrow{n \rightarrow \infty} 0.$$

From Lemma 3.5.1 we can deduce that

$$d^{n,\theta,x_0} \xrightarrow{n \rightarrow \infty} \frac{1}{\sigma^2} [\partial_\theta b(x_0, \theta)]^2 \mathbb{E} \left[ \mathcal{H}_{L^\alpha}(1) \frac{\varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)} \right],$$

uniformly with respect to  $x_0$ . Then, the relation (3.3.7) enables to rewrite this convergence as,

$$d^{n,\theta,x_0} \xrightarrow{n \rightarrow \infty} -\frac{1}{\sigma^2} [\partial_\theta b(x_0, \theta)]^2 \mathbb{E} \left[ \frac{\varphi'_\alpha(L_1^\alpha)^2}{\varphi_\alpha(L_1^\alpha)^2} \right], \text{ uniformly with respect to } x_0.$$

This result implies (3.5.5) and hence (3.5.3). ■

**Lemma 3.5.3.** *We consider*

$$\varrho_{i,n} = \frac{\partial_\sigma p_{\frac{1}{n}}^\beta(X_{\frac{i}{n}}^\beta, X_{\frac{i+1}{n}}^\beta)}{p_{\frac{1}{n}}^\beta} + \frac{1}{\sigma} \frac{\varphi_\alpha \left( n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right) + n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \varphi'_\alpha \left( n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)}{\varphi_\alpha \left( n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)}$$

then we have  $n^{-1/2} \sum_{i=0}^{n-1} \varrho_{i,n} \xrightarrow[n]{\mathbb{P}} 0$ .

**Proof.** We proceed as in the proof of Lemma 3.5.2 and check that

$$(3.5.9) \quad n^{-1/2} \sum_{i=0}^{n-1} |\mathbb{E}[\varrho_{i,n} | \mathcal{F}_{i/n}]| \xrightarrow[n]{\mathbb{P}} 0,$$

$$(3.5.10) \quad n^{-1} \sum_{i=0}^{n-1} |\mathbb{E}[\varrho_{i,n}^2 | \mathcal{F}_{i/n}]| \xrightarrow[n]{\mathbb{P}} 0.$$



We start by the proof of (3.5.9). Since a score function has an expectation equal to zero, and  $L_{\frac{i+1}{n}} - L_{\frac{i}{n}}$  is independent of  $\mathcal{F}_{i/n}$ , we deduce that

$$\mathbb{E}[\varrho_{i,n} | \mathcal{F}_{i/n}] = \frac{1}{\sigma} \mathbb{E} \left[ \frac{\varphi_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right) + n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \varphi'_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)}{\varphi_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)} \right].$$

Since  $(L_t)_t$  has stationary increments, the law of  $n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}})$  is the same as the law of  $L_1^n$ .

Moreover, we know from Lemma 3.3.1, that  $\mathbb{P}(L_1^n \neq L_1^\alpha) = O(1/n)$ , thus

$$\mathbb{E}[\varrho_{i,n} | \mathcal{F}_{i/n}] = \frac{1}{\sigma} \mathbb{E} \left[ \frac{\varphi_\alpha(L_1^\alpha) + L_1^\alpha \varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)} \right] + \left\| \frac{\varphi_\alpha(u) + u \varphi'_\alpha(u)}{\varphi_\alpha(u)} \right\|_\infty O(n^{-1}),$$

where we used the fact that  $\frac{u \varphi'_\alpha(u)}{\varphi_\alpha(u)}$  is bounded (see e.g. Theorem 7.3.2 in [45]).

Using  $\mathbb{E} \left[ \frac{L_1^\alpha \varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)} \right] = \int_{\mathbb{R}} u \varphi'_\alpha(u) du = -1$ , we deduce  $|\mathbb{E}[\varrho_{i,n} | \mathcal{F}_{i/n}]| \leq Cn^{-1}$  for some constant  $C$  and (3.5.9) follows.

We now prove (3.5.10). We have

$$(3.5.11) \quad \mathbb{E}[\varrho_{i,n}^2 | \mathcal{F}_{i/n}] = \mathcal{I}_{22}^{n,\beta, X_{\frac{i}{n}}^\beta} + \frac{1}{\sigma^2} \mathbb{E} \left[ \frac{\left[ \varphi_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right) + n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \varphi'_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right) \right]^2}{\varphi_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)^2} \right] \\ + 2 \mathbb{E} \left[ \frac{\partial_\sigma p_{\frac{1}{n}}^\beta(X_{\frac{i}{n}}^\beta, X_{\frac{i+1}{n}}^\beta)}{p_{\frac{1}{n}}^\beta(X_{\frac{i}{n}}^\beta, X_{\frac{i+1}{n}}^\beta)} \frac{1}{\sigma} \frac{\varphi_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right) + n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \varphi'_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)}{\varphi_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)} \middle| \mathcal{F}_{\frac{i}{n}} \right].$$

With a method analogous to the proof of (3.5.9), we can show that

$$\mathbb{E} \left[ \frac{\left[ \varphi_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right) + n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \varphi'_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right) \right]^2}{\varphi_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)^2} \right] = \mathbb{E} \left[ \frac{[\varphi_\alpha(L_1^\alpha) + L_1^\alpha \varphi'_\alpha(L_1^\alpha)]^2}{\varphi_\alpha(L_1^\alpha)^2} \right] + o(1).$$

Proceeding as in the proof of (3.5.4), then (3.5.9) is proved as soon as we show that the following control holds uniformly with respect to  $i$ ,

$$(3.5.12) \quad \mathbb{E} \left[ \frac{\partial_\sigma p_{\frac{1}{n}}^\beta(X_{\frac{i}{n}}^\beta, X_{\frac{i+1}{n}}^\beta)}{p_{\frac{1}{n}}^\beta(X_{\frac{i}{n}}^\beta, X_{\frac{i+1}{n}}^\beta)} \frac{1}{\sigma} \frac{\varphi_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right) + n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \varphi'_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)}{\varphi_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)} \middle| \mathcal{F}_{\frac{i}{n}} \right] \\ = -\frac{1}{\sigma^2} \mathbb{E} \left[ \frac{(\varphi_\alpha(L_1^\alpha) + L_1^\alpha \varphi'_\alpha(L_1^\alpha))^2}{\varphi_\alpha(L_1^\alpha)^2} \right] + o(1).$$

We define  $d_1^{n,\sigma, x_0} = \mathbb{E} \left[ \frac{\partial_\sigma q(\bar{Y}_1^{n,\beta, x_0})}{q(\bar{Y}_1^{n,\beta, x_0})} \frac{1}{\sigma} \frac{\varphi_\alpha(L_1^n) + L_1^n \varphi'_\alpha(L_1^n)}{\varphi_\alpha(L_1^n)} \right]$ , so that the left-hand side of (3.5.12) reduces, from the Markov property, to  $d_1^{n,\sigma, X_{i/n}^\beta}$ .

Proceeding as in the proof of (3.5.8), noting that  $\frac{u\varphi'_\alpha(u)}{\varphi_\alpha(u)}$  is bounded with a bounded derivative (see e.g. Theorem 7.3.2 in [45]), then we also get that

$$d_1^{n,\sigma,x_0} = \mathbb{E} \left[ \frac{\partial_\sigma q(\bar{Y}_1^{n,\beta,x_0})}{q(\bar{Y}_1^{n,\beta,x_0})} \frac{1}{\sigma} \right] + \mathbb{E} \left[ \frac{\partial_\sigma q(\bar{Y}_1^{n,\beta,x_0})}{q(\bar{Y}_1^{n,\beta,x_0})} \frac{1}{\sigma} \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \frac{\varphi'_\alpha \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \right)}{\varphi_\alpha \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \right)} \right] + o(1)$$

where the  $o(1)$  term is uniform with respect to  $x_0$ . Now, using Proposition 3.3.1, we get

$$d_1^{n,\sigma,x_0} = \mathbb{E} \left[ \frac{\mathcal{H}_{\bar{Y}_1^{n,\beta,x_0}}(\partial_\sigma \bar{Y}_1^{n,\beta,x_0})}{\sigma} \right] + \mathbb{E} \left[ \frac{\mathcal{H}_{\bar{Y}_1^{n,\beta,x_0}}(\partial_\sigma \bar{Y}_1^{n,\beta,x_0})}{\sigma} \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \frac{\varphi'_\alpha \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \right)}{\varphi_\alpha \left( \frac{n^{1/\alpha}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\sigma} \right)} \right] + o(1).$$

From Lemma 3.5.1 and the convergence result (3.3.8) we can deduce that

$$d_1^{n,\sigma,x_0} \xrightarrow{n \rightarrow \infty} \frac{1}{\sigma^2} \mathbb{E} \left[ \left( L_1^\alpha \mathcal{H}_{L_1^\alpha}(1) - 1 \right) \frac{\varphi_\alpha(L_1^\alpha) + L_1^\alpha \varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)} \right],$$

uniformly with respect to  $x_0$ . Then, the relation (3.3.10) enables to rewrite this convergence as,

$$d_1^{n,\sigma,x_0} \xrightarrow{n \rightarrow \infty} -\frac{1}{\sigma^2} \mathbb{E} \left[ \frac{(\varphi_\alpha(L_1^\alpha) + L_1^\alpha \varphi'_\alpha(L_1^\alpha))^2}{\varphi_\alpha(L_1^\alpha)^2} \right].$$

This result implies (3.5.12) and hence (3.5.10). ■

**Lemma 3.5.4.** *We have the convergence in law,*

$$(3.5.13) \quad \left( \begin{array}{c} -n^{-1/2} \sum_{i=0}^{n-1} \frac{\varphi'_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)}{\varphi_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)} \frac{1}{\sigma} \partial_\theta b(X_{\frac{i}{n}}^\beta, \theta) \\ -n^{-1/2} \sum_{i=0}^{n-1} \frac{1}{\sigma} \frac{\varphi_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right) + n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \varphi'_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)}{\varphi_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)} \end{array} \right) \Rightarrow \mathcal{N}(0, \mathcal{I}(\beta))$$

where the convergence is stable with respect to the  $\sigma$ -field  $\sigma(L_s, s \leq 1)$ .

**Proof.** We define the following processes:

$$Z_t^n = \sum_{i=0}^{\lfloor nt \rfloor} \left( L_{\frac{i+1}{n}} - L_{\frac{i}{n}} \right),$$

$$\Gamma_t^n = \begin{pmatrix} \Gamma_t^{n,1} \\ \Gamma_t^{n,2} \end{pmatrix} = n^{-1/2} \begin{pmatrix} \sum_{i=0}^{\lfloor nt \rfloor} \frac{\varphi'_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)}{\varphi_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)} \frac{1}{\sigma} \partial_\theta b(X_{\frac{i}{n}}^\beta, \theta) \\ \sum_{i=0}^{\lfloor nt \rfloor} \frac{\varphi_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right) + n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \varphi'_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)}{\varphi_\alpha \left( n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)} \frac{1}{\sigma} \end{pmatrix},$$

$$\Gamma_t^n = \begin{pmatrix} \Gamma_t^{n,1} \\ \Gamma_t^{n,2} \end{pmatrix} = n^{-1/2} \begin{pmatrix} \sum_{i=0}^{[nt]} \frac{\varphi'_\alpha(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}}))}{\varphi_\alpha(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}}))} \\ \sum_{i=0}^{[nt]} \frac{\varphi_\alpha(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}})) + n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}})\varphi'_\alpha(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}}))}{\varphi_\alpha(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}}))} \end{pmatrix}.$$

We will apply Lemma 2.8 in [36] to prove (3.5.13). Indeed, we will show that there exists a

Gaussian random vector  $\gamma$  with  $var(\gamma) = \begin{pmatrix} \mathbb{E} \left[ \frac{\varphi'_\alpha(L_1^\alpha)^2}{\varphi_\alpha(L_1^\alpha)^2} \right] & 0 \\ 0 & \mathbb{E} \left[ \frac{(\varphi_\alpha(L_1^\alpha) + L_1^\alpha \varphi'_\alpha(L_1^\alpha))^2}{\varphi_\alpha(L_1^\alpha)^2} \right] \end{pmatrix}$ , independent of  $L_1$

such that one has the convergence in law

$$(3.5.14) \quad (\Gamma_1^n, Z_1^n) \Rightarrow (\gamma, L_1).$$

Then, by application of Lemma 2.8 in [36], there exists a bi-dimensional Brownian motion  $(\Gamma'_t)$  independent of  $(L_t)_t$  such that one has the convergence in law  $(Z^n, \Gamma^n, \Gamma'^n) \Rightarrow (L, \Gamma, \Gamma')$  where

$$\Gamma_t = \int_0^t \begin{pmatrix} \frac{1}{\sigma} \partial_\theta b(X_s^\beta, \theta) & 0 \\ 0 & \frac{1}{\sigma} \end{pmatrix} d\Gamma'_s \text{ and } var(\Gamma'_1) = var(\gamma).$$

Let us focus on the derivation of the convergence (3.5.14). For  $(u, v, w) \in \mathbb{R}^3$ , we set

$$\mathcal{X}_n(u, v, w) = \mathbb{E} \left[ \exp \left( i \frac{u}{n^{1/2}} \frac{\varphi'_\alpha(n^{1/\alpha} L_{\frac{1}{n}})}{\varphi_\alpha(n^{1/\alpha} L_{\frac{1}{n}})} + i \frac{v}{n^{1/2}} \frac{(\varphi_\alpha(n^{1/\alpha} L_{\frac{1}{n}}) + n^{1/\alpha} L_{\frac{1}{n}} \varphi'_\alpha(n^{1/\alpha} L_{\frac{1}{n}}))}{\varphi_\alpha(n^{1/\alpha} L_{\frac{1}{n}})} + iw L_{\frac{1}{n}} \right) \right].$$

Using the i.i.d. structure of the increments of the Lévy process  $L$ , we easily get the following expression about the characteristic function of  $(\Gamma_1^n, Z_1^n)$

$$(3.5.15) \quad \log \mathbb{E} \left[ \exp(iu \Gamma_1^{n,1} + iv \Gamma_1^{n,2} + iw Z_1^n) \right] = n \log \mathcal{X}_n(u, v, w).$$

Let us study the asymptotic behavior of  $\mathcal{X}_n(u, v, w)$ . Using the expansion of the exponential function near zero and that  $\frac{\varphi'_\alpha}{\varphi_\alpha}$  and  $\frac{x \varphi'_\alpha(x)}{\varphi_\alpha(x)}$  are bounded we get

$$\begin{aligned} \mathcal{X}_n(u, v, w) = & \mathbb{E} \left\{ \left[ e^{iw L_{\frac{1}{n}}} \right] \left[ 1 + \left( \frac{i u}{n^{1/2}} \frac{\varphi'_\alpha(n^{1/\alpha} L_{\frac{1}{n}})}{\varphi_\alpha(n^{1/\alpha} L_{\frac{1}{n}})} + \frac{i v}{n^{1/2}} \frac{(\varphi_\alpha(n^{1/\alpha} L_{\frac{1}{n}}) + n^{1/\alpha} L_{\frac{1}{n}} \varphi'_\alpha(n^{1/\alpha} L_{\frac{1}{n}}))}{\varphi_\alpha(n^{1/\alpha} L_{\frac{1}{n}})} \right) \right. \right. \\ & \left. \left. + \frac{1}{2} \left( \frac{i u}{n^{1/2}} \frac{(\varphi'_\alpha(n^{1/\alpha} L_{\frac{1}{n}}))}{\varphi_\alpha(n^{1/\alpha} L_{\frac{1}{n}})} + \frac{i v}{n^{1/2}} \frac{(\varphi_\alpha(n^{1/\alpha} L_{\frac{1}{n}}) + n^{1/\alpha} L_{\frac{1}{n}} \varphi'_\alpha(n^{1/\alpha} L_{\frac{1}{n}}))}{\varphi_\alpha(n^{1/\alpha} L_{\frac{1}{n}})} \right)^2 \right] \right\} + O(n^{-3/2}) \end{aligned}$$

$$\begin{aligned}
 &= \mathbb{E} \left[ e^{iwL_{\frac{1}{n}}} \right] + \frac{i u}{n^{1/2}} \mathbb{E} \left[ \frac{\varphi'_\alpha(n^{1/\alpha} L_{\frac{1}{n}})}{\varphi_\alpha(n^{1/\alpha} L_{\frac{1}{n}})} e^{iwL_{\frac{1}{n}}} \right] + \frac{i v}{n^{1/2}} \mathbb{E} \left[ \frac{\left( \varphi_\alpha(n^{1/\alpha} L_{\frac{1}{n}}) + n^{1/\alpha} L_{\frac{1}{n}} \varphi'_\alpha(n^{1/\alpha} L_{\frac{1}{n}}) \right)}{\varphi_\alpha(n^{1/\alpha} L_{\frac{1}{n}})} e^{iwL_{\frac{1}{n}}} \right] \\
 &- \frac{u^2}{2n} \mathbb{E} \left[ \left( \frac{\varphi'_\alpha(n^{1/\alpha} L_{\frac{1}{n}})}{\varphi_\alpha(n^{1/\alpha} L_{\frac{1}{n}})} \right)^2 e^{iwL_{\frac{1}{n}}} \right] - \frac{u v}{n} \mathbb{E} \left[ \frac{\varphi'_\alpha(n^{1/\alpha} L_{\frac{1}{n}}) \left( \varphi_\alpha(n^{1/\alpha} L_{\frac{1}{n}}) + n^{1/\alpha} L_{\frac{1}{n}} \varphi'_\alpha(n^{1/\alpha} L_{\frac{1}{n}}) \right)}{\varphi_\alpha(n^{1/\alpha} L_{\frac{1}{n}})^2} e^{iwL_{\frac{1}{n}}} \right] \\
 &- \frac{v^2}{2n} \mathbb{E} \left[ \left( \frac{\varphi_\alpha(n^{1/\alpha} L_{\frac{1}{n}}) + n^{1/\alpha} L_{\frac{1}{n}} \varphi'_\alpha(n^{1/\alpha} L_{\frac{1}{n}})}{\varphi_\alpha(n^{1/\alpha} L_{\frac{1}{n}})} \right)^2 e^{iwL_{\frac{1}{n}}} \right] + O(n^{-3/2})
 \end{aligned}$$

(3.5.16)

$$\begin{aligned}
 &:= \mathcal{X}_n^{(1)}(u, v, w) + \frac{i u}{n^{1/2}} \mathcal{X}_n^{(2)}(u, v, w) + \frac{i v}{n^{1/2}} \mathcal{X}_n^{(3)}(u, v, w) - \frac{u^2}{2n} \mathcal{X}_n^{(4)}(u, v, w) - \frac{u v}{n} \mathcal{X}_n^{(5)}(u, v, w) \\
 &\quad - \frac{v^2}{2n} \mathcal{X}_n^{(6)}(u, v, w) + O(n^{-3/2}).
 \end{aligned}$$

First, we have

$$(3.5.17) \quad \mathcal{X}_n^{(1)}(u, v, w) = e^{\psi(w)/n} = 1 + \psi(w)/n + O(n^{-2})$$

where  $\psi(w)$  is the Lévy Khintchine exponent of  $L_1$ .

We now focus on the term  $\mathcal{X}_n^{(2)}(u, v, w)$ . Using the results of Lemma 3.3.1, and the fact that  $n^{1/\alpha} L_{1/n}$  has the same law as  $L_1^n$ , we get

$$\begin{aligned}
 (3.5.18) \quad \mathcal{X}_n^{(2)}(u, v, w) &= \mathbb{E} \left[ \frac{\varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)} e^{i \frac{w L_1^\alpha}{n^{1/\alpha}}} \right] + O(n^{-1}) \\
 &= \int_{\mathbb{R}} \varphi'_\alpha(s) e^{i \frac{w s}{n^{1/\alpha}}} ds + O(n^{-1}) \\
 &= -\frac{i w}{n^{1/\alpha}} \int_{\mathbb{R}} \varphi_\alpha(s) e^{i \frac{w s}{n^{1/\alpha}}} ds + O(n^{-1}) \quad \text{using integration by parts formula} \\
 &= O(\max(n^{-1/\alpha}, n^{-1})).
 \end{aligned}$$

For the term  $\mathcal{X}_n^{(3)}(u, v, w)$ , using Lemma 3.3.1 again, it is easy to see that

$$\begin{aligned}
 (3.5.19) \quad \mathcal{X}_n^{(3)}(u, v, w) &= \mathbb{E} \left[ \left( \frac{\varphi_\alpha(L_1^\alpha) + L_1^\alpha \varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)} \right) e^{i \frac{w L_1^\alpha}{n^{1/\alpha}}} \right] + O(n^{-1}) \\
 &= \int_{\mathbb{R}} \varphi_\alpha(s) e^{i \frac{w s}{n^{1/\alpha}}} ds + \int_{\mathbb{R}} s \varphi'_\alpha(s) e^{i \frac{w s}{n^{1/\alpha}}} ds + O(n^{-1}).
 \end{aligned}$$

Using integration by parts formula, we have

$$\mathbb{E} \left( e^{i \frac{w L_1^\alpha}{n^{1/\alpha}}} \right) = \int_{\mathbb{R}} \varphi_\alpha(s) e^{i \frac{w s}{n^{1/\alpha}}} ds = -\frac{n^{1/\alpha}}{i w} \int_{\mathbb{R}} \varphi'_\alpha(s) e^{i \frac{w s}{n^{1/\alpha}}} ds.$$

Then, we deduce that

$$(3.5.20) \quad \int_{\mathbb{R}} \varphi'_\alpha(s) e^{\frac{iws}{n^{1/\alpha}}} ds = -\frac{iw}{n^{1/\alpha}} \mathbb{E}(e^{i w L_1^\alpha / n}) = -\frac{iw}{n^{1/\alpha}} \mathbb{E}(e^{\frac{iw}{n^{1/\alpha}} L_1^\alpha}).$$

Since  $L_1^\alpha$  is a symmetric  $\alpha$ -stable process then we have for some constant  $C(\alpha) > 0$

$$(3.5.21) \quad \mathbb{E}(e^{\frac{iw}{n^{1/\alpha}} L_1^\alpha}) = e^{-C(\alpha) \left| \frac{w}{n^{1/\alpha}} \right|^\alpha}.$$

Combining (3.5.20) with (3.5.21), we have

$$(3.5.22) \quad \int_{\mathbb{R}} \varphi'_\alpha(s) e^{\frac{iws}{n^{1/\alpha}}} ds = -\frac{iw}{n^{1/\alpha}} e^{-C(\alpha) \left| \frac{w}{n^{1/\alpha}} \right|^\alpha}.$$

Now, since  $\int |s \varphi'_\alpha(s)| ds < \infty$  and  $w \mapsto w e^{-C(\alpha) \left| \frac{w}{n^{1/\alpha}} \right|^\alpha}$  admits a derivative on  $\mathbb{R}$ , we obtain by taking the derivative with respect to  $w$  of the both sides of (3.5.22)

$$(3.5.23) \quad \begin{aligned} \int_{\mathbb{R}} s \varphi'_\alpha(s) e^{\frac{iws}{n^{1/\alpha}}} ds &= -e^{-C(\alpha) \left| \frac{w}{n^{1/\alpha}} \right|^\alpha} + \alpha C(\alpha) e^{-C(\alpha) \left| \frac{w}{n^{1/\alpha}} \right|^\alpha} \frac{|w|^\alpha}{n} \\ &= -\int_{\mathbb{R}} \varphi_\alpha(s) e^{\frac{iws}{n^{1/\alpha}}} ds + \alpha C(\alpha) e^{-C(\alpha) \left| \frac{w}{n^{1/\alpha}} \right|^\alpha} \frac{|w|^\alpha}{n}. \end{aligned}$$

From (3.5.19) and (3.5.23) we can deduce that

$$(3.5.24) \quad \mathcal{X}_n^{(3)}(u, v, w) = O(n^{-1}).$$

For the term  $\mathcal{X}_n^{(4)}(u, v, w)$ , using Lemma 3.3.1 again, we have

$$(3.5.25) \quad \begin{aligned} \mathcal{X}_n^{(4)}(u, v, w) &= \mathbb{E} \left[ \frac{\varphi'_\alpha(L_1^\alpha)^2 e^{i \frac{w L_1^\alpha}{n^{1/\alpha}}}}{\varphi_\alpha(L_1^\alpha)} \right] + O(n^{-1}) \\ &\xrightarrow{n \rightarrow \infty} \mathbb{E} \left[ \frac{\varphi'_\alpha(L_1^\alpha)^2}{\varphi_\alpha(L_1^\alpha)} \right]. \end{aligned}$$

For the term  $\mathcal{X}_n^{(5)}(u, v, w)$  we have

$$(3.5.26) \quad \begin{aligned} \mathcal{X}_n^{(5)}(u, v, w) &= \mathbb{E} \left[ \frac{\varphi'_\alpha(L_1^\alpha) (\varphi_\alpha(L_1^\alpha) + L_1^\alpha \varphi'_\alpha(L_1^\alpha))}{\varphi_\alpha(L_1^\alpha) \varphi_\alpha(L_1^\alpha)} e^{\frac{i w L_1^\alpha}{n^{1/\alpha}}} \right] + O(n^{-1}) \\ &\xrightarrow{n \rightarrow \infty} \mathbb{E} \left[ \frac{\varphi'_\alpha(L_1^\alpha) (\varphi_\alpha(L_1^\alpha) + L_1^\alpha \varphi'_\alpha(L_1^\alpha))}{\varphi_\alpha(L_1^\alpha) \varphi_\alpha(L_1^\alpha)} \right] = 0 \quad \text{from (3.3.13)}. \end{aligned}$$

For the term  $\mathcal{X}_n^{(6)}(u, v, w)$ , we see that

$$(3.5.27) \quad \begin{aligned} \mathcal{X}_n^{(6)}(u, v, w) &= \mathbb{E} \left[ \left( \frac{\varphi'_\alpha(L_1^\alpha) + L_1^\alpha \varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)} \right)^2 e^{\frac{i w L_1^\alpha}{n^{1/\alpha}}} \right] + O(n^{-1}) \\ &\xrightarrow{n \rightarrow \infty} \mathbb{E} \left[ \left( \frac{\varphi'_\alpha(L_1^\alpha) + L_1^\alpha \varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)} \right)^2 \right]. \end{aligned}$$

Collecting the convergence of  $(\mathcal{X}_n^{(i)}(u, v, w))_{1 \leq i \leq 6}$ , we deduce the convergence

$$\log \mathbb{E} \left[ \exp(iu\Gamma_1^{n,1} + iv\Gamma_1^{n,2} + iwZ_1^n) \right] \xrightarrow{n \rightarrow \infty} \psi(v) - \frac{u^2}{2} \mathbb{E} \left[ \frac{\varphi'_\alpha(L_1^\alpha)^2}{\varphi_\alpha(L_1^\alpha)} \right] - \frac{v^2}{2} \mathbb{E} \left[ \left( \frac{\varphi'_\alpha(L_1^\alpha) + L_1^\alpha \varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)} \right)^2 \right]$$

and thus the convergence in law of this lemma.  $\blacksquare$

## 3.6 Proofs of Lemmas 3.3.2-3.3.5

The proof of these lemmas is very technical and requires many intermediate results. We first recall the Malliavin calculus for jump processes used in [19] and some properties of the Malliavin weights. Next we will establish a regularity property of a conditional expectation with respect to the conditioning variable. Then we will proceed to the proof of the lemmas.

### 3.6.1 Malliavin calculus and preliminary lemmas

We recall the Malliavin calculus on the Poisson space associated to the measure  $\mu^{(n)}$  (defined in Section 3.3.2.1) and the basic properties of the Malliavin operators (see Bichteler, Gravereaux, Jacod [8], Chapter IV, Section 8-9-10). For a test function  $f : [0, 1] \times E \rightarrow \mathbb{R}$  ( $f$  is measurable,  $\mathcal{C}^2$  with respect to the second variable, with bounded derivative, and  $f \in \cap_{p \geq 1} \mathbf{L}^p(\nu)$ ) we set  $\mu^{(n)}(f) = \int_0^1 \int_E f(t, z) \mu^{(n)}(dt, dz)$ . We introduce an auxiliary function  $\rho^n$  as

$$(3.6.1) \quad \rho^n(z) = \begin{cases} z^4 & \text{if } |z| < 1 \\ \zeta(z) & \text{if } 1 \leq |z| \leq 2 \\ z^2 \tau\left(\frac{z}{2n^{1/\alpha}}\right) & \text{if } |z| > 2 \end{cases}$$

where  $\tau$  is defined in the assumption  $\mathbf{H}_1(b_i)$ , and  $\zeta$  is a non negative function belonging to  $\mathbf{C}^\infty$  such that the function  $\rho^n$  belongs to  $\mathbf{C}^\infty$ . Note that  $\zeta$  is defined such that  $\rho^n(z)$  admits a derivative and  $\rho^n, (\rho^n)', \rho^n \frac{F'_n(z)}{F_n(z)}$  belong to  $\cap_{p \geq 1} \mathbf{L}^p(F_n(z) dz)$ . From the conditions on  $\tau$ , we can easily deduce

$$\text{that } z^2 \tau\left(\frac{z}{2n^{1/\alpha}}\right) = \begin{cases} z^2 & \text{if } 2 \leq |z| \leq 2n^{1/\alpha} \\ 0 & \text{if } |z| > 4n^{1/\alpha}. \end{cases}$$

Moreover, we can see that  $\rho^n(z) \xrightarrow{n \rightarrow \infty} \rho(z)$  where

$$(3.6.2) \quad \rho(z) = \begin{cases} z^4 & \text{if } |z| < 1 \\ \zeta(z) & \text{if } 1 \leq |z| \leq 2 \\ z^2 & \text{if } |z| > 2. \end{cases}$$

Note that from the definition of  $\rho^n$  and  $\rho$ , we can easily see that  $\rho^n(z) = \rho(z)$  if  $|z| \leq 2n^{1/\alpha}$ .

With these notations, we define the Malliavin operator  $L$ , on a simple functional  $\mu^{(n)}(f)$ , in the same way as in [17] by the following equations :

$$L(\mu^{(n)}(f)) = \frac{1}{2} \mu^{(n)} \left( (\rho^n)' f' + \rho^n \frac{F_n'}{F_n} f' + \rho^n f'' \right),$$

where  $f'$  and  $f''$  are the derivatives with respect to the second variable. For  $\Phi = F(\mu^{(n)}(f_1), \dots, \mu^{(n)}(f_k))$ , with  $F$  of class  $\mathcal{C}^2$ , we set

$$L\Phi = \sum_{i=1}^k \frac{\partial F}{\partial x_i}(\mu^{(n)}(f_1), \dots, \mu^{(n)}(f_k)) L(\mu^{(n)}(f_i)) + \frac{1}{2} \sum_{i,j=1}^k \frac{\partial^2 F}{\partial x_i \partial x_j}(\mu^{(n)}(f_1), \dots, \mu^{(n)}(f_k)) \mu^{(n)}(\rho^n f_i' f_j').$$

These definitions permit to construct a linear operator  $L$  on a space  $D \subset \cap_{p \geq 1} \mathbf{L}^p$  whose basic properties are the following.

i)  $L$  is self-adjoint:  $\forall \Phi, \Psi \in D$ , we have  $\mathbb{E}\Phi L\Psi = \mathbb{E}L\Phi\Psi$ .

ii)  $L\Phi^2 \geq 2\Phi L\Phi$ .

iii)  $\mathbb{E}L\Phi = 0$ .

We associate to  $L$ , the symmetric bilinear operator  $\Gamma$ :

$$(3.6.3) \quad \Gamma(\Phi, \Psi) = L(\Phi\Psi) - \Phi L\Psi - \Psi L\Phi.$$

This operator satisfies the following properties (see [8, equation (8-3)])

$$(3.6.4) \quad \Gamma(F(\Phi), \Psi) = F'(\Phi)\Gamma(\Phi, \Psi),$$

$$(3.6.5) \quad \Gamma(F(\Phi_1, \Phi_2), \Psi) = \partial_{\Phi_1} F(\Phi_1, \Phi_2)\Gamma(\Phi_1, \Psi) + \partial_{\Phi_2} F(\Phi_1, \Phi_2)\Gamma(\Phi_2, \Psi),$$

$$(3.6.6) \quad |\Gamma(\Phi, \Psi)| \leq \Gamma(\Phi, \Phi)^{1/2} \Gamma(\Psi, \Psi)^{1/2}.$$

**Remark 3.6.1.** *The operators  $L$  and  $\Gamma$  depend on  $n$  through the functions  $\rho^n$  and  $F_n$  but to simplify the notation we omit the dependence in  $n$ .*

The operator  $L$  and the operator  $\Gamma$  permit to establish the following integration by parts formula (see [8, Propositions 8-10, p.103]).

**Proposition 3.6.1.** *For  $\Phi$  and  $\Psi$  in  $D$ , and  $f$  bounded with bounded derivative up to order two, if  $\Gamma(\Phi, \Phi)$  is invertible and  $\Gamma^{-1}(\Phi, \Phi) \in \cap_{p \geq 1} L^p$  then we have*

$$(3.6.7) \quad \mathbb{E}f'(\Phi)\Psi = \mathbb{E}f(\Phi)\mathcal{H}_\Phi(\Psi),$$

with

$$(3.6.8) \quad \mathcal{H}_\Phi(\Psi) = -2\Psi\Gamma^{-1}(\Phi, \Phi)L\Phi - \Gamma(\Phi, \Psi\Gamma^{-1}(\Phi, \Phi))$$

$$(3.6.9) \quad = -2\Psi\Gamma^{-1}(\Phi, \Phi)L\Phi - \frac{1}{\Gamma(\Phi, \Phi)}\Gamma(\Phi, \Psi) + \frac{\Psi}{\Gamma(\Phi, \Phi)^2}\Gamma(\Phi, \Gamma(\Phi, \Phi))$$

$$(3.6.10) \quad = \Phi L \left( \frac{\Psi}{\Gamma(\Phi, \Phi)} \right) - \frac{\Psi L \Phi}{\Gamma(\Phi, \Phi)} - L \left( \frac{\Psi \Phi}{\Gamma(\Phi, \Phi)} \right).$$

With these notations, we can explicit the Malliavin weight  $\mathcal{H}_{\bar{Y}_1^{n, \beta, x_0}}(\nabla_\beta \bar{Y}_1^{n, \beta, x_0})$  appearing in the representation of the score function given in Proposition 3.3.1.

**Proposition 3.6.2.** *[Theorem 3.1 and Theorem 6.2 in [19]]*

$$(3.6.11) \quad \begin{aligned} \mathcal{H}_{\bar{Y}_1^{n, \beta, x_0}}(\nabla_\beta \bar{Y}_1^{n, \beta, x_0}) &= \begin{pmatrix} \mathcal{H}_{\bar{Y}_1^{n, \beta, x_0}}(\partial_\theta \bar{Y}_1^{n, \beta, x_0}) \\ \mathcal{H}_{\bar{Y}_1^{n, \beta, x_0}}(\partial_\sigma \bar{Y}_1^{n, \beta, x_0}) \end{pmatrix} \\ &= \begin{pmatrix} \partial_\theta \bar{Y}_1^{n, \beta, x_0} \\ \partial_\sigma \bar{Y}_1^{n, \beta, x_0} \end{pmatrix} \mathcal{H}_{\bar{Y}_1^{n, \beta, x_0}}(1) - \frac{1}{\Gamma(\bar{Y}_1^{n, \beta, x_0}, \bar{Y}_1^{n, \beta, x_0})} \begin{pmatrix} \Gamma(\bar{Y}_1^{n, \beta, x_0}, \partial_\theta \bar{Y}_1^{n, \beta, x_0}) \\ \Gamma(\bar{Y}_1^{n, \beta, x_0}, \partial_\sigma \bar{Y}_1^{n, \beta, x_0}) \end{pmatrix} \end{aligned}$$

and

$$(3.6.12) \quad \begin{aligned} \mathcal{H}_{\bar{Y}_1^{n, \beta, x_0}}(1) &= \frac{\Gamma(\bar{Y}_1^{n, \beta, x_0}, \Gamma(\bar{Y}_1^{n, \beta, x_0}, \bar{Y}_1^{n, \beta, x_0}))}{(\Gamma(\bar{Y}_1^{n, \beta, x_0}, \bar{Y}_1^{n, \beta, x_0}))^2} - 2 \frac{L\bar{Y}_1^{n, \beta, x_0}}{\Gamma(\bar{Y}_1^{n, \beta, x_0}, \bar{Y}_1^{n, \beta, x_0})} \\ &= \frac{1}{\sigma} n^{1/\alpha} \hat{\mathcal{H}}_\beta^n(1) + \mathcal{R}_{1, \beta}^n(1) + \mathcal{R}_{2, \beta}^n(1) + \mathcal{R}_{3, \beta}^n(1). \end{aligned}$$

The main term  $\hat{\mathcal{H}}_\beta^n(1)$  is given by

$$(3.6.13) \quad \hat{\mathcal{H}}_\beta^n(1) = \frac{\int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-3} \rho^n(z) (\rho^n)'(z) \mu^{(n)}(ds, dz)}{\epsilon_1^n \left[ \int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-2} \rho^n(z) \mu^{(n)}(ds, dz) \right]^2} - \frac{\int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-1} [(\rho^n)'(z) - \frac{1+\alpha}{z} \rho^n(z)] \mu^{(n)}(ds, dz)}{\epsilon_1^n \int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-2} \rho^n(z) \mu^{(n)}(ds, dz)}$$



with

$$(3.6.14) \quad c_s^n = \exp\left(\frac{1}{n} \int_0^s b'(\bar{Y}_u^{n,\beta,x_0}, \theta) du\right).$$

The remainder terms satisfy for all compact set  $Q \subset \mathbb{R} \times (0, \infty)$

$$(3.6.15) \quad \forall p \geq 2, \quad \mathbb{E} \sup_{\beta \in Q} \left| \mathcal{R}_{1,\beta}^n(1) \right|^p \leq \frac{C}{n}, \quad \sup_{\beta \in Q} |\mathcal{R}_{2,\beta}^n(1)| \leq \frac{C}{n}, \quad \sup_{\beta \in Q} |\mathcal{R}_{3,\beta}^n(1)| \leq \frac{C}{n},$$

where  $C$  is some deterministic constant.

**Remark 3.6.2.** *i) It is proved in [19] (see (4.23)) that  $\hat{\mathcal{H}}_\beta^n(1)$  is bounded by a random variable independent of  $n, \beta$  and  $x_0$  and belonging to  $\cap_{p \geq 1} \mathbf{L}^p$  and that it converges in  $\mathbf{L}^p, \forall p \geq 1$ , uniformly with respect to  $x_0$  (see (4.23) and (5.49) in [19]) to  $\mathcal{H}_{L^\alpha}(1)$  given by*

$$(3.6.16) \quad \mathcal{H}_{L^\alpha}(1) = \frac{\int_0^1 \int_{\mathbb{R}} \rho(z) \rho'(z) \mu(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho(z) \mu(ds, dz) \right]^2} - \frac{\int_0^1 \int_{\mathbb{R}} \left[ \rho'(z) - \frac{1+\alpha}{z} \rho(z) \right] \mu(ds, dz)}{\int_0^1 \int_{\mathbb{R}} \rho(z) \mu(ds, dz)}.$$

Moreover,  $\mathcal{H}_{L^\alpha}(1)$  and  $L_1^\alpha \mathcal{H}_{L^\alpha}(1)$  belong to  $\mathbf{L}^p, \forall p \geq 1$ .

*ii) In the case  $b \equiv 0$  and  $\sigma = 1$ , we have  $c_s^n = 1$  and the remainder terms  $\mathcal{R}_{2,\beta}^n(1), \mathcal{R}_{3,\beta}^n(1)$  are equal to zero (see (4.7) and (4.8) in [19]). Moreover, we see that (3.6.12) can be rewritten as  $\hat{\mathcal{H}}_\beta^n(1) + \frac{1}{n^{1/\alpha}} \mathcal{R}_{1,\beta}^n(1) = \mathcal{H}_{L_1^n}(1)$ , then we can deduce that  $\mathcal{H}_{L_1^n}(1) \xrightarrow[p \geq 1]{\mathbf{L}^p} \mathcal{H}_{L^\alpha}(1)$ .*

Before studying the Malliavin weight  $\mathcal{H}_{\bar{Y}_1^{n,\beta,x_0}}(\nabla_\beta \bar{Y}_1^{n,\beta,x_0})$ , we give some control on the processes  $(\partial_\theta Y_t^\beta)_t$  and  $(\partial_\sigma Y_t^\beta)_t$ , respectively solution to the equations

$$(3.6.17) \quad \partial_\theta \bar{Y}_t^{n,\beta,x_0} = \frac{1}{n} \int_0^t b'(\bar{Y}_s^{n,\beta,x_0}, \theta) \partial_\theta \bar{Y}_s^{n,\beta,x_0} ds + \frac{1}{n} \int_0^t \partial_\theta b(\bar{Y}_s^{n,\beta,x_0}, \theta) ds,$$

$$(3.6.18) \quad \partial_\sigma \bar{Y}_t^{n,\beta,x_0} = \frac{1}{n} \int_0^t b'(\bar{Y}_s^{n,\beta,x_0}, \theta) \partial_\sigma \bar{Y}_s^{n,\beta,x_0} ds + \frac{L_t^n}{n^{1/\alpha}}.$$

We have the following properties.

**Lemma 3.6.1.** *[Lemma 5.1 in [19]] Let  $Q \subset \mathbb{R} \times (0, \infty)$  be a compact subset. We have*

$$i) \sup_{\beta \in Q} |\partial_\theta \bar{Y}_1^{n,\beta,x_0}| \leq \frac{C}{n},$$

$$ii) \sup_{\beta \in Q} \sup_{s \in [0,1]} \left| \partial_\sigma \bar{Y}_s^{n,\beta,x_0} \right| \xrightarrow[p]{n \rightarrow \infty} 0, \quad \forall p \geq 1.$$

We now proceed to the decomposition of the Malliavin weight  $\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(\nabla_{\beta}\bar{Y}_1^{n,\beta_n,x_0})$  defined in Proposition 3.6.2 into some main parts and some remainder parts. From (3.6.11), (3.6.12), we can rewrite  $\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(\nabla_{\beta}\bar{Y}_1^{n,\beta_n,x_0})$  as,

$$(3.6.19) \quad \mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(\nabla_{\beta}\bar{Y}_1^{n,\beta_n,x_0}) = \left( \begin{array}{c} \frac{1}{\sigma_n} n^{1/\alpha} \partial_{\theta} \bar{Y}_1^{n,\beta_n,x_0} \hat{\mathcal{H}}_{\beta_n}^n(1) \\ \frac{1}{\sigma_n} \left( n^{1/\alpha} \partial_{\sigma} \bar{Y}_1^{n,\beta_n,x_0} \hat{\mathcal{H}}_{\beta_n}^n(1) - 1 \right) \end{array} \right) + \mathcal{R}_{\beta_n}^n(\nabla_{\beta}\bar{Y}_1^{n,\beta_n,x_0})$$

where  $\hat{\mathcal{H}}_{\beta}^n(1)$  is given by (3.6.13) and

$$(3.6.20) \quad \mathcal{R}_{\beta_n}^n(\nabla_{\beta}\bar{Y}_1^{n,\beta_n,x_0}) = \begin{pmatrix} \partial_{\theta} \bar{Y}_1^{n,\beta_n,x_0} \\ \partial_{\sigma} \bar{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \left[ \mathcal{R}_{1,\beta_n}^n(1) + \mathcal{R}_{2,\beta_n}^n(1) + \mathcal{R}_{3,\beta_n}^n(1) \right] - \begin{pmatrix} \frac{V_1^{n,\theta n}}{U_1^{n,\beta n}} \\ \frac{V_1^{n,\sigma n}}{U_1^{n,\beta n}} - \frac{1}{\sigma_n} \end{pmatrix}$$

with  $U_1^{n,\beta} = \Gamma(\bar{Y}_1^{n,\beta,x_0}, \bar{Y}_1^{n,\beta,x_0})$ ,  $V_1^{n,\theta} = \Gamma(\bar{Y}_1^{n,\beta,x_0}, \partial_{\theta} \bar{Y}_1^{n,\beta,x_0})$  and  $V_1^{n,\sigma} = \Gamma(\bar{Y}_1^{n,\beta,x_0}, \partial_{\sigma} \bar{Y}_1^{n,\beta,x_0})$  given by

$$(3.6.21) \quad U_1^{n,\beta} = \frac{(\epsilon_1^n)^2 \sigma^2}{n^{2/\alpha}} \int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-2} \rho^n(z) \mu^{(n)}(ds, dz),$$

$$(3.6.22) \quad V_1^{n,\theta} = \frac{1}{n} (\epsilon_1^n)^2 \int_0^1 (\epsilon_s^n)^{-2} \left( U_s^{n,\beta} \left[ (\partial_{\theta} b)'(\bar{Y}_s^{n,\beta,x_0}, \theta) + b''(\bar{Y}_s^{n,\beta,x_0}, \theta) \partial_{\theta} \bar{Y}_s^{n,\beta,x_0} \right] \right) ds,$$

$$(3.6.23) \quad V_1^{n,\sigma} = \frac{1}{n} (\epsilon_1^n)^2 \int_0^1 (\epsilon_s^n)^{-2} \left( b''(\bar{Y}_s^{n,\beta,x_0}, \theta) \partial_{\sigma} \bar{Y}_s^{n,\beta,x_0} U_s^{n,\beta} \right) ds + \frac{\sigma}{n^{2/\alpha}} (\epsilon_1^n)^2 \int_0^t \int_{\mathbb{R}} (\epsilon_s^n)^{-2} \rho^n(z) \mu^{(n)}(ds, dz)$$

and  $(\epsilon_s^n)_{s \in [0,1]}$  given by (3.6.14).

Now we recall two technical lemmas given in [19] useful to study the convergence of the Malliavin weight  $\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}}(\nabla_{\beta}\bar{Y}_1^{n,\beta_n,x_0})$  in the proof of Lemmas 3.3.4 - 3.3.5 later.

**Lemma 3.6.2.** [Lemma 5.4 in [19]]

Let  $(\beta_n)$  be a sequence converging to  $\beta$ . For all  $p \geq 1$ , the following convergences hold uniformly with respect to  $x_0$

$$i) \quad n \partial_{\theta} \bar{Y}_1^{n,\beta_n,x_0} \hat{\mathcal{H}}_{\beta_n}^n(1) \xrightarrow[L^p]{n \rightarrow \infty} \partial_{\theta} b(x_0, \theta) \mathcal{H}_{L^{\alpha}}(1),$$

$$ii) \quad n^{1/\alpha} \partial_{\sigma} \bar{Y}_1^{n,\beta_n,x_0} \hat{\mathcal{H}}_{\beta_n}^n(1) \xrightarrow[L^p]{n \rightarrow \infty} L_1^{\alpha} \mathcal{H}_{L^{\alpha}}(1),$$

where  $\hat{\mathcal{H}}_{\beta}^n(1)$  and  $\mathcal{H}_{L^{\alpha}}(1)$  are respectively given by (3.6.13) and (3.6.16).

**Lemma 3.6.3.** [Lemma 5.3 in [19]]

Let  $Q \subset \mathbb{R} \times (0, \infty)$  be a compact subset. The following estimates hold:

$$\begin{aligned} i) \quad & \sup_{\beta \in Q} \left| \frac{V_1^{n,\theta}}{U_1^{n,\beta}} \right| \leq \frac{C}{n}, \\ ii) \quad & \frac{V_1^{n,\sigma}}{U_1^{n,\beta}} = \frac{1}{\sigma} + \mathcal{R}_{9,\beta}^n(1), \end{aligned}$$

where  $C$  is some deterministic constant and  $\sup_{\beta \in Q} |\mathcal{R}_{9,\beta}^n(1)|$  converges to zero as  $n \rightarrow \infty$  in  $L^p, \forall p \geq 1$ .

### 3.6.2 Regularity of the conditional expectation

In this section, we prove a regularity property of the conditional expectation with respect to the conditioning variable.

**Proposition 3.6.3.** Let  $\mathcal{H}$  be a random variable such that  $\mathbb{E}(\mathcal{H})^2 < \infty$ . We assume that there exists a sequence of random variables  $(\mathcal{H}^n)_{n \geq 1}$  with  $\mathbb{E}(\mathcal{H}^n)^2 < \infty$  and such that  $\mathcal{H}^n \xrightarrow[n \rightarrow \infty]{L^2} \mathcal{H}$  and  $\sup_n \|\Gamma(\mathcal{H}^n, \mathcal{H}^n)\|_2 < \infty$ . Then,

$$(3.6.24) \quad \mathbb{E} \left[ \mathbb{E}[\mathcal{H} | \bar{Y}_1^{n,\beta_n,x_0}]^2 \right] - \mathbb{E} \left[ \mathbb{E}[\mathcal{H} | L_1^\alpha]^2 \right] \xrightarrow[n \rightarrow \infty]{} 0$$

and this convergence is uniform with respect to  $x_0$ .

**Remark 3.6.3.** Note that if the random variable  $\mathcal{H}$  depends on all the measure  $\mu$  then the Malliavin calculus of Section 3.3.2.1 is not defined. So we need to introduce the sequence of random variables  $(\mathcal{H}^n)$ , for which the Malliavin calculus of Section 3.6.1 is defined, such that  $\Gamma(\mathcal{H}^n, \mathcal{H}^n)$  is also well defined. It is the case, for instance, if  $\mathcal{H}^n$  is a simple functional of  $\mu^{(n)}$ .

**Proof of Proposition 3.6.3** First we reduce the situation to the case where the random variable in the expectation is bounded. Let  $K > 1$  and denote by  $x \mapsto \mathcal{X}_K(x)$  a smooth truncation function with

$$(3.6.25) \quad \begin{cases} \mathcal{X}_K(x) = 0 & \text{for } |x| > K \\ \mathcal{X}_K(x) = 1 & \text{for } |x| \leq K/2 \\ 0 \leq \mathcal{X}_K(x) \leq 1 & \text{for } K/2 \leq |x| \leq K. \end{cases}$$

For all  $\epsilon > 0$ , we can choose  $K$  large enough such that  $\|\mathcal{H} - \mathcal{H}\mathcal{X}_K(\mathcal{H})\|_2^2 < \epsilon$  and then, one can see that (3.6.24) is implied by the following convergence,  $\forall K > 2$

$$\sup_{x_0} \left| \mathbb{E} \left[ \mathbb{E}[\mathcal{H}\mathcal{X}_K(\mathcal{H})|\bar{Y}_1^{n,\beta_n,x_0}]^2 \right] - \mathbb{E} \left[ \mathbb{E}[\mathcal{H}\mathcal{X}_K(\mathcal{H})|L_1^\alpha]^2 \right] \right| \xrightarrow{n \rightarrow \infty} 0.$$

Now since  $\mathbb{E}(\mathcal{H}^n - \mathcal{H})^2 \xrightarrow{n \rightarrow \infty} 0$ , it is sufficient to prove that,  $\forall K > 2$

$$(3.6.26) \quad \sup_{x_0} \left| \mathbb{E} \left[ \mathbb{E}[\mathcal{H}^n \mathcal{X}_K(\mathcal{H}^n)|\bar{Y}_1^{n,\beta_n,x_0}]^2 \right] - \mathbb{E} \left[ \mathbb{E}[\mathcal{H}^n \mathcal{X}_K(\mathcal{H}^n)|L_1^\alpha]^2 \right] \right| \xrightarrow{n \rightarrow \infty} 0.$$

We now prove (3.6.26). First, we define  $\eta^{\mathcal{H}^n,K}$  and  $\bar{\eta}^{\mathcal{H}^n,K}$  as follows

$$(3.6.27) \quad \begin{aligned} \eta^{\mathcal{H}^n,K}(\bar{Y}_1^{n,\beta_n,x_0} - \zeta_1^{n,\theta_n,x_0}) &= \mathbb{E} \left[ \mathcal{H}^n \mathcal{X}_K(\mathcal{H}^n) |\bar{Y}_1^{n,\beta_n,x_0} - \zeta_1^{n,\theta_n,x_0} \right] = \mathbb{E} \left[ \mathcal{H}^n \mathcal{X}_K(\mathcal{H}^n) |\bar{Y}_1^{n,\beta_n,x_0} \right], \\ \bar{\eta}^{\mathcal{H}^n,K} \left( \frac{\sigma L_1^\alpha}{n^{1/\alpha}} \right) &= \mathbb{E} \left[ \mathcal{H}^n \mathcal{X}_K(\mathcal{H}^n) \left| \frac{\sigma L_1^\alpha}{n^{1/\alpha}} \right. \right] = \mathbb{E} \left[ \mathcal{H}^n \mathcal{X}_K(\mathcal{H}^n) | L_1^\alpha \right]. \end{aligned}$$

With these notations, we can rewrite (3.6.26) as

$$(3.6.28) \quad \sup_{x_0} \left| \mathbb{E} \left[ \eta^{\mathcal{H}^n,K}(\bar{Y}_1^{n,\beta_n,x_0} - \zeta_1^{n,\theta_n,x_0})^2 \right] - \mathbb{E} \left[ \bar{\eta}^{\mathcal{H}^n,K} \left( \frac{\sigma L_1^\alpha}{n^{1/\alpha}} \right)^2 \right] \right| \xrightarrow{n \rightarrow \infty} 0.$$

Using Lemma 3.6.5 in Section 3.6.4, we know that:

$$\sup_{x_0} \mathbb{E} \left[ \left| \eta^{\mathcal{H}^n,K}(\bar{Y}_1^{n,\beta_n,x_0} - \zeta_1^{n,\theta_n,x_0}) - \bar{\eta}^{\mathcal{H}^n,K}(\bar{Y}_1^{n,\beta_n,x_0} - \zeta_1^{n,\theta_n,x_0}) \right| \right] \xrightarrow{n \rightarrow \infty} 0$$

and since  $|\eta^{\mathcal{H}^n,K}|$  and  $|\bar{\eta}^{\mathcal{H}^n,K}|$  are bounded by the constant  $K$ , we deduce

$$\sup_{x_0} \left| \mathbb{E}[\eta^{\mathcal{H}^n,K}(\bar{Y}_1^{n,\beta_n,x_0} - \zeta_1^{n,\theta_n,x_0})^2] - \mathbb{E}[\bar{\eta}^{\mathcal{H}^n,K}(\bar{Y}_1^{n,\beta_n,x_0} - \zeta_1^{n,\theta_n,x_0})^2] \right| \xrightarrow{n \rightarrow \infty} 0.$$

Now, applying Lemma 3.6.4 in Section 3.6.4, with the choice  $\mathcal{H}^n = 1$  with the bounded function  $(\bar{\eta}^{\mathcal{H}^n,K})^2$  we get (3.6.28) and the proposition is proved.  $\blacksquare$

We can now prove Lemma 3.3.2, Lemma 3.3.3, Lemma 3.3.4 and Lemma 3.3.5.

### 3.6.3 Proofs of Lemma 3.3.2, Lemma 3.3.3, Lemma 3.3.4 and Lemma 3.3.5

**Proof of Lemma 3.3.2** First we remark that although  $L_1^\alpha$  does not belong to the domain of Malliavin operators  $D$  we can establish a representation for  $\phi'_\alpha/\phi_\alpha$ .

Indeed, since  $L_1^n$  belongs to the Malliavin space  $D$ , the integration by parts formula (3.6.7) gives for any test function  $f$  ( $f$  is bounded, compactly supported and  $f'$  is bounded),

$$(3.6.29) \quad \mathbb{E}[f'(L_1^n)] = \mathbb{E}[f(L_1^n)\mathcal{H}_{L_1^n}(1)].$$

Now from Lemma 3.3.1, we have  $\mathbb{P}(L_1^n = L_1^\alpha) \xrightarrow{n \rightarrow \infty} 1$  and from Remark 3.6.2 *ii*) we have  $\mathcal{H}_{L_1^n}^p(1) \xrightarrow[p \geq 1]{\mathbf{L}^p} \mathcal{H}_{L_1^\alpha}(1)$ . Letting  $n$  go to infinity in (3.6.29), we deduce

$$(3.6.30) \quad \mathbb{E}[f'(L_1^\alpha)] = \mathbb{E}[f(L_1^\alpha)\mathcal{H}_{L_1^\alpha}(1)].$$

Observing that  $\int \varphi_\alpha(u)f'(u)du = -\int \varphi'_\alpha(u)f(u)du$ , we get  $\int f(u)\varphi'_\alpha(u)du = -\mathbb{E}[f(L_1^\alpha)\mathcal{H}_{L_1^\alpha}(1)]$  and we deduce the representation (3.3.7). ■

**Proof of Lemma 3.3.3** The proof of this lemma is based on the results in [19] recalled in Section 3.6. From (3.6.20), Lemma 3.6.1, Lemma 3.6.3 and (3.6.15), we easily deduce that  $\sup_{\beta \in Q} \mathcal{R}_\beta^n(\nabla_\beta \bar{Y}_1^{n, \beta_n, x_0})$  converges to zero in  $\mathbf{L}^p$ ,  $\forall p \geq 1$  (uniformly with respect to  $x_0$ ).

From Lemma 3.6.2 *i - ii*) and (3.6.19) we can deduce (3.3.8).

The uniform control of  $\mathbb{E} \left| n^{1/2} r_n \mathcal{H}_{\bar{Y}_1^{n, \beta_n, x_0}}(\nabla_\beta \bar{Y}_1^{n, \beta_n, x_0}) \right|^p$  is immediate. ■

**Proof of Lemma 3.3.4** From Jensen's inequality, we have

$$\begin{aligned} & \mathbb{E} \left[ \left| \mathbb{E} \left[ n^{1-1/\alpha} \mathcal{H}_{\bar{Y}_1^{n, \beta_n, x_0}}(\partial_\theta \bar{Y}_1^{n, \beta_n, x_0}) | \bar{Y}_1^{n, \beta_n, x_0} \right] - \mathbb{E} \left[ \frac{1}{\sigma} \partial_\theta b(x_0, \theta) \mathcal{H}_{L_1^\alpha}(1) | \bar{Y}_1^{n, \beta_n, x_0} \right] \right|^2 \right] \\ & \leq \mathbb{E} \left[ \mathbb{E} \left[ \left| n^{1-1/\alpha} \mathcal{H}_{\bar{Y}_1^{n, \beta_n, x_0}}(\partial_\theta \bar{Y}_1^{n, \beta_n, x_0}) - \frac{1}{\sigma} \partial_\theta b(x_0, \theta) \mathcal{H}_{L_1^\alpha}(1) \right|^2 | \bar{Y}_1^{n, \beta_n, x_0} \right] \right] \\ & = \mathbb{E} \left[ \left| n^{1-1/\alpha} \mathcal{H}_{\bar{Y}_1^{n, \beta_n, x_0}}(\partial_\theta \bar{Y}_1^{n, \beta_n, x_0}) - \frac{1}{\sigma} \partial_\theta b(x_0, \theta) \mathcal{H}_{L_1^\alpha}(1) \right|^2 \right] \end{aligned}$$

From Lemma 3.3.3, the last term converges to zero uniformly with respect to  $x_0$ . In turn, it gives the uniform convergence

$$\sup_{x_0} \left| n^{2-2/\alpha} \mathbb{E} \left[ \mathbb{E} \left[ \mathcal{H}_{\bar{Y}_1^{n, \beta_n, x_0}}(\partial_\theta \bar{Y}_1^{n, \beta_n, x_0}) | \bar{Y}_1^{n, \beta_n, x_0} \right]^2 \right] - \frac{1}{\sigma^2} \partial_\theta b(x_0, \theta)^2 \mathbb{E} \left[ \mathbb{E} \left[ \mathcal{H}_{L_1^\alpha}(1) | \bar{Y}_1^{n, \beta_n, x_0} \right]^2 \right] \right| \xrightarrow{n \rightarrow \infty} 0.$$

By the same method as above, we also get the uniform convergence

$$\sup_{x_0} \left| \mathbb{E} \left[ \mathbb{E} \left[ \mathcal{H}_{\bar{Y}_1^{n, \beta_n, x_0}}(\partial_\sigma \bar{Y}_1^{n, \beta_n, x_0}) | \bar{Y}_1^{n, \beta_n, x_0} \right]^2 \right] - \frac{1}{\sigma^2} \mathbb{E} \left[ \mathbb{E} \left[ (L_1^\alpha \mathcal{H}_{L_1^\alpha}(1) - 1) | \bar{Y}_1^{n, \beta_n, x_0} \right]^2 \right] \right| \xrightarrow{n \rightarrow \infty} 0.$$

Hence, this lemma will be proved as soon as we show that

$$(3.6.31) \quad \sup_{x_0} \left| \mathbb{E} \left[ \mathbb{E} \left[ \mathcal{H}_{L_1^\alpha}(1) | \bar{Y}_1^{n, \beta_n, x_0} \right]^2 \right] - \mathbb{E} \left[ \mathbb{E} \left[ \mathcal{H}_{L_1^\alpha}(1) | L_1^\alpha \right]^2 \right] \right| \xrightarrow{n \rightarrow \infty} 0,$$

$$(3.6.32) \quad \sup_{x_0} \left| \mathbb{E} \left[ \mathbb{E} \left[ (L_1^\alpha \mathcal{H}_{L_1^\alpha}(1) - 1) | \bar{Y}_1^{n, \beta_n, x_0} \right]^2 \right] - \mathbb{E} \left[ \mathbb{E} \left[ (L_1^\alpha \mathcal{H}_{L_1^\alpha}(1) - 1) | L_1^\alpha \right]^2 \right] \right| \xrightarrow{n \rightarrow \infty} 0.$$

To prove (3.6.31), we apply Proposition 3.6.3 with the choice  $\mathcal{H} = \mathcal{H}_{L^\alpha}(1)$  and

$$(3.6.33) \quad \mathcal{H}^n = \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) (\rho^n)'(z) \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^2} - \frac{\int_0^1 \int_{\mathbb{R}} \left[ (\rho^n)'(z) - \frac{1+\alpha}{z} \rho^n(z) \right] \mu^{(n)}(ds, dz)}{\int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz)} := \tilde{\mathcal{H}}^n.$$

From Remark 3.6.2 *ii*) we get that  $\tilde{\mathcal{H}}^n = \mathcal{H}_{L_1^n}(1) - \frac{1}{n^{1/\alpha}} \mathcal{R}_{1,\beta}^n$ , moreover  $\mathbb{E}(\mathcal{H}_{L^\alpha}(1))^2 < \infty$  and  $\tilde{\mathcal{H}}^n \xrightarrow[n \rightarrow \infty]{\mathbf{L}^2} \mathcal{H}_{L^\alpha}(1)$ . The computation of  $\Gamma(\tilde{\mathcal{H}}^n, \tilde{\mathcal{H}}^n)$  is omitted but reduces to the computation of the  $\Gamma$ -bracket between simple functionals. After some calculus (similar to those in the proof of Theorem 2.1 in [19]) we get that  $\Gamma(\tilde{\mathcal{H}}^n, \tilde{\mathcal{H}}^n)$  is bounded by a random variable independent of  $n$  and belonging to  $\cap_{p \geq 1} \mathbf{L}^p$  (see the detail verification in Section 3.7). Turning to 3.6.32, we proceed similarly with the choices  $\mathcal{H} = L_1^\alpha \mathcal{H}_{L^\alpha}(1)$  and  $\mathcal{H}^n = L_1^n \tilde{\mathcal{H}}^n$ . Note that using Lemma 3.6.2 *ii*) with  $b(x, \theta) = 0$  and  $\sigma = 1$  we deduce that  $L_1^n \tilde{\mathcal{H}}^n \xrightarrow[n \rightarrow \infty]{\mathbf{L}^2} L_1^\alpha \mathcal{H}_{L^\alpha}(1)$  moreover we can prove that  $\sup_n \|\Gamma(L_1^n \tilde{\mathcal{H}}^n, L_1^n \tilde{\mathcal{H}}^n)\|_2 < \infty$  (see the detail verification in Section 3.7).  $\blacksquare$

**Proof of Lemma 3.3.5** It is easy to see that

$$(3.6.34) \quad \mathbb{E} \left[ \left| n^{1-1/\alpha} \mathbb{E}[\mathcal{H}_{\tilde{Y}_1^{n,\beta_n,x_0}}(\partial_\theta \tilde{Y}_1^{n,\beta_n,x_0}) | \tilde{Y}_1^{n,\beta_n,x_0}] \mathbb{E}[\mathcal{H}_{\tilde{Y}_1^{n,\beta_n,x_0}}(\partial_\sigma \tilde{Y}_1^{n,\beta_n,x_0}) | \tilde{Y}_1^{n,\beta_n,x_0}] \right. \right. \\ \left. \left. - \mathbb{E} \left[ \frac{1}{\sigma} (L_1^\alpha \mathcal{H}_{L^\alpha}(1) - 1) | \tilde{Y}_1^{n,\beta_n,x_0} \right] \mathbb{E} \left[ \frac{1}{\sigma} \partial_\theta b(x_0, \theta) \mathcal{H}_{L^\alpha}(1) | \tilde{Y}_1^{n,\beta_n,x_0} \right] \right| \right] \\ \leq \mathbb{E} \left[ \left| n^{1-1/\alpha} \mathbb{E}[\mathcal{H}_{\tilde{Y}_1^{n,\beta_n,x_0}}(\partial_\theta \tilde{Y}_1^{n,\beta_n,x_0}) | \tilde{Y}_1^{n,\beta_n,x_0}] \right. \right. \\ \left. \left. \left( \mathbb{E}[\mathcal{H}_{\tilde{Y}_1^{n,\beta_n,x_0}}(\partial_\sigma \tilde{Y}_1^{n,\beta_n,x_0}) | \tilde{Y}_1^{n,\beta_n,x_0}] - \mathbb{E} \left[ \frac{(L_1^\alpha \mathcal{H}_{L^\alpha}(1) - 1)}{\sigma} | \tilde{Y}_1^{n,\beta_n,x_0} \right] \right) \right| \right] \\ + \mathbb{E} \left[ \left| \mathbb{E} \left[ \frac{(L_1^\alpha \mathcal{H}_{L^\alpha}(1) - 1)}{\sigma} | \tilde{Y}_1^{n,\beta_n,x_0} \right] \right. \right. \\ \left. \left. \left( n^{1-1/\alpha} \mathbb{E}[\mathcal{H}_{\tilde{Y}_1^{n,\beta_n,x_0}}(\partial_\theta \tilde{Y}_1^{n,\beta_n,x_0}) | \tilde{Y}_1^{n,\beta_n,x_0}] - \mathbb{E} \left[ \frac{\partial_\theta b(x_0, \theta) \mathcal{H}_{L^\alpha}(1)}{\sigma} | \tilde{Y}_1^{n,\beta_n,x_0} \right] \right) \right| \right].$$

Then using Cauchy-Schwarz inequality

$$\mathbb{E} \left[ \left| n^{1-1/\alpha} \mathbb{E}[\mathcal{H}_{\tilde{Y}_1^{n,\beta_n,x_0}}(\partial_\theta \tilde{Y}_1^{n,\beta_n,x_0}) | \tilde{Y}_1^{n,\beta_n,x_0}] \times \right. \right. \\ \left. \left. \times \left( \mathbb{E}[\mathcal{H}_{\tilde{Y}_1^{n,\beta_n,x_0}}(\partial_\sigma \tilde{Y}_1^{n,\beta_n,x_0}) | \tilde{Y}_1^{n,\beta_n,x_0}] - \mathbb{E} \left[ \frac{(L_1^\alpha \mathcal{H}_{L^\alpha}(1) - 1)}{\sigma} | \tilde{Y}_1^{n,\beta_n,x_0} \right] \right) \right| \right]$$

$$\begin{aligned}
& \leq \left[ \mathbb{E} \left( \left( n^{1-1/\alpha} \mathbb{E} [\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}} (\partial_\theta \bar{Y}_1^{n,\beta_n,x_0}) | \bar{Y}_1^{n,\beta_n,x_0}] \right)^2 \right) \right]^{1/2} \times \\
& \quad \times \left[ \mathbb{E} \left( \left( \mathbb{E} [\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}} (\partial_\sigma \bar{Y}_1^{n,\beta_n,x_0}) | \bar{Y}_1^{n,\beta_n,x_0}] - \mathbb{E} \left[ \frac{(L_1^\alpha \mathcal{H}_{L^\alpha}(1) - 1)}{\sigma} | \bar{Y}_1^{n,\beta_n,x_0} \right] \right)^2 \right) \right]^{1/2} \\
(3.6.35) \quad & \leq \left[ \mathbb{E} \left( n^{2-2/\alpha} \mathbb{E} \left[ \left| \mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}} (\partial_\theta \bar{Y}_1^{n,\beta_n,x_0}) \right|^2 | \bar{Y}_1^{n,\beta_n,x_0} \right] \right) \right]^{1/2} \times \\
& \quad \times \left[ \mathbb{E} \left( \mathbb{E} \left[ \left| \mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}} (\partial_\sigma \bar{Y}_1^{n,\beta_n,x_0}) - \frac{(L_1^\alpha \mathcal{H}_{L^\alpha}(1) - 1)}{\sigma} \right|^2 | \bar{Y}_1^{n,\beta_n,x_0} \right] \right) \right]^{1/2} \\
& = \left[ \mathbb{E} \left( n^{2-2/\alpha} \left| \mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}} (\partial_\theta \bar{Y}_1^{n,\beta_n,x_0}) \right|^2 \right) \right]^{1/2} \left[ \mathbb{E} \left( \left| \mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}} (\partial_\sigma \bar{Y}_1^{n,\beta_n,x_0}) - \frac{(L_1^\alpha \mathcal{H}_{L^\alpha}(1) - 1)}{\sigma} \right|^2 \right) \right]^{1/2}.
\end{aligned}$$

Furthermore, from (3.3.8) we easily deduce that (3.6.35) converges to zero uniformly with respect to  $x_0$ . Similarly, we also get that

$\mathbb{E} \left[ \left| \mathbb{E} \left[ \frac{(L_1^\alpha \mathcal{H}_{L^\alpha}(1) - 1)}{\sigma} | \bar{Y}_1^{n,\beta_n,x_0} \right] \left( n^{1-1/\alpha} \mathbb{E} [\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}} (\partial_\theta \bar{Y}_1^{n,\beta_n,x_0}) | \bar{Y}_1^{n,\beta_n,x_0}] - \mathbb{E} \left[ \frac{\partial_\theta b(x_0, \theta) \mathcal{H}_{L^\alpha}(1)}{\sigma} | \bar{Y}_1^{n,\beta_n,x_0} \right] \right) \right| \right]$  tends to zero uniformly with respect to  $x_0$ . And then, we can conclude that (3.6.34) converges to zero uniformly with respect to  $x_0$ . In turn, it gives the uniform convergence

$$\begin{aligned}
(3.6.36) \quad & \sup_{x_0} \left| \mathbb{E} \left[ n^{1-1/\alpha} \mathbb{E} [\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}} (\partial_\theta \bar{Y}_1^{n,\beta_n,x_0}) | \bar{Y}_1^{n,\beta_n,x_0}] \mathbb{E} [\mathcal{H}_{\bar{Y}_1^{n,\beta_n,x_0}} (\partial_\sigma \bar{Y}_1^{n,\beta_n,x_0}) | \bar{Y}_1^{n,\beta_n,x_0}] - \right. \right. \\
& \quad \left. \left. - \mathbb{E} \left[ \mathbb{E} \left[ \frac{1}{\sigma} (L_1^\alpha \mathcal{H}_{L^\alpha}(1) - 1) | \bar{Y}_1^{n,\beta_n,x_0} \right] \mathbb{E} \left[ \frac{1}{\sigma} \partial_\theta b(x_0, \theta) \mathcal{H}_{L^\alpha}(1) | \bar{Y}_1^{n,\beta_n,x_0} \right] \right] \right| \right] \xrightarrow{n \rightarrow \infty} 0.
\end{aligned}$$

On the other hand, we can rewrite

$$\begin{aligned}
(3.6.37) \quad & \mathbb{E} \left[ \mathbb{E} \left[ \frac{1}{\sigma} (L_1^\alpha \mathcal{H}_{L^\alpha}(1) - 1) | \bar{Y}_1^{n,\beta_n,x_0} \right] \mathbb{E} \left[ \frac{1}{\sigma} \partial_\theta b(x_0, \theta) \mathcal{H}_{L^\alpha}(1) | \bar{Y}_1^{n,\beta_n,x_0} \right] \right] \\
& = \frac{1}{4} \mathbb{E} \left\{ \left[ \mathbb{E} \left( \frac{1}{\sigma} (L_1^\alpha \mathcal{H}_{L^\alpha}(1) - 1) + \frac{1}{\sigma} \partial_\theta b(x_0, \theta) \mathcal{H}_{L^\alpha}(1) | \bar{Y}_1^{n,\beta_n,x_0} \right) \right]^2 \right. \\
& \quad \left. - \left[ \mathbb{E} \left( \frac{1}{\sigma} (L_1^\alpha \mathcal{H}_{L^\alpha}(1) - 1) - \frac{1}{\sigma} \partial_\theta b(x_0, \theta) \mathcal{H}_{L^\alpha}(1) | \bar{Y}_1^{n,\beta_n,x_0} \right) \right]^2 \right\}.
\end{aligned}$$

Then, the lemma will be proved as soon as we show that

$$\begin{aligned}
(3.6.38) \quad & \mathbb{E} \left\{ \left[ \mathbb{E} \left( \frac{1}{\sigma} (L_1^\alpha \mathcal{H}_{L^\alpha}(1) - 1) + \frac{1}{\sigma} \partial_\theta b(x_0, \theta) \mathcal{H}_{L^\alpha}(1) | \bar{Y}_1^{n,\beta_n,x_0} \right) \right]^2 \right. \\
& \quad \left. - \left[ \mathbb{E} \left( \frac{1}{\sigma} (L_1^\alpha \mathcal{H}_{L^\alpha}(1) - 1) - \frac{1}{\sigma} \partial_\theta b(x_0, \theta) \mathcal{H}_{L^\alpha}(1) | \bar{Y}_1^{n,\beta_n,x_0} \right) \right]^2 \right\}
\end{aligned}$$

is uniformly convergent with respect to  $x_0$  to

$$(3.6.39) \quad \mathbb{E} \left\{ \left[ \mathbb{E} \left( \frac{1}{\sigma} (L_1^\alpha \mathcal{H}_{L^\alpha}(1) - 1) + \frac{1}{\sigma} \partial_\theta b(x_0, \theta) \mathcal{H}_{L^\alpha}(1) \mid L_1^\alpha \right) \right]^2 - \left[ \mathbb{E} \left( \frac{1}{\sigma} (L_1^\alpha \mathcal{H}_{L^\alpha}(1) - 1) - \frac{1}{\sigma} \partial_\theta b(x_0, \theta) \mathcal{H}_{L^\alpha}(1) \mid L_1^\alpha \right) \right]^2 \right\}.$$

We end the proof by using Proposition 3.6.3 with  $\mathcal{H} = L_1^\alpha \mathcal{H}_{L^\alpha}(1) \pm \partial_\theta b(x_0, \theta) \mathcal{H}_{L^\alpha}(1)$  and  $\mathcal{H}^n = L_1^n \tilde{\mathcal{H}}^n \pm \partial_\theta b(x_0, \theta) \tilde{\mathcal{H}}^n$  where  $\tilde{\mathcal{H}}^n$  given by (3.6.33).  $\blacksquare$

### 3.6.4 Lemmas 3.6.4 and 3.6.5

The aim of this section is to show that the functions  $\eta^{\mathcal{H}^{n,K}}$  and  $\bar{\eta}^{\mathcal{H}^{n,K}}$  defined by (3.6.27) are close in some sense. The idea is mainly based on [17, Proposition 9, p.2348], however we need a more technical study since  $\alpha \in (0, 2)$  and the function  $b$  is not assumed to be bounded. Our first result is the following.

**Lemma 3.6.4.** *Under the assumptions of Proposition 3.6.3, for all bounded function  $h$ ,  $\forall K > 2$ , there exists a constant  $C_K > 0$  such that*

$$|\mathbb{E}[\mathcal{H}^n \mathcal{X}_K(\mathcal{H}^n) h(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})] - \mathbb{E}[\mathcal{H}^n \mathcal{X}_K(\mathcal{H}^n) h(\frac{\sigma L_1^\alpha}{n^{1/\alpha}})]| \leq \frac{C_K}{n} \|h\|_\infty$$

and the above estimate is uniform with respect to  $x_0 \in \mathbb{R}$  and  $\beta \in \mathcal{Q}$ , for any compact set  $\mathcal{Q} \subset \mathbb{R} \times (0, \infty)$ .

**Proof.** Since  $\mathcal{H}^n \mathcal{X}_K(\mathcal{H}^n)$  is bounded and  $\mathbb{P}(L_1^n \neq L_1^\alpha) \leq \frac{C}{n}$  (see Lemma 3.3.1) it is sufficient to show that

$$(3.6.40) \quad \left| \mathbb{E}[\mathcal{H}^n \mathcal{X}_K(\mathcal{H}^n) h(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})] - \mathbb{E}[\mathcal{H}^n \mathcal{X}_K(\mathcal{H}^n) h(\frac{\sigma L_1^n}{n^{1/\alpha}})] \right| \leq \frac{C_K}{n} \|h\|_\infty.$$

We now prove (3.6.40).

Let us denote  $\mathcal{H}^{n,K} = \mathcal{H}^n \mathcal{X}_K(\mathcal{H}^n)$  and  $H$  any primitive function of  $h$ . Using the integration by parts formula (3.6.7), we have

$$(3.6.41) \quad \mathbb{E} \left[ h \left( \frac{\sigma L_1^n}{n^{1/\alpha}} \right) \mathcal{H}^{n,K} \right] = \mathbb{E} \left[ H \left( \frac{\sigma L_1^n}{n^{1/\alpha}} \right) \mathcal{H} \left( \frac{\sigma L_1^n}{n^{1/\alpha}} \right) (\mathcal{H}^{n,K}) \right]$$



where  $\mathcal{H}_{\left(\frac{\sigma L_1^n}{n^{1/\alpha}}\right)}(\mathcal{H}^{n,K})$  is given by (3.6.10), namely here

$$\mathcal{H}_{\left(\frac{\sigma L_1^n}{n^{1/\alpha}}\right)}(\mathcal{H}^{n,K}) = L\left(\frac{\mathcal{H}^{n,K}}{\Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}}\right)}\right) \frac{\sigma L_1^n}{n^{1/\alpha}} - \frac{L\left(\frac{\sigma L_1^n}{n^{1/\alpha}}\right)\mathcal{H}^{n,K}}{\Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}}\right)} - L\left(\frac{\mathcal{H}^{n,K} \frac{\sigma L_1^n}{n^{1/\alpha}}}{\Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}}\right)}\right).$$

On the other hand, we have for  $t \in [0, 1]$

$$\begin{aligned} & \left| \bar{Y}_t^{n,\beta,x_0} - \zeta_t^{n,\theta,x_0} - \frac{\sigma L_t^n}{n^{1/\alpha}} \right| = \left| \frac{1}{n} \int_0^t \left[ b(\bar{Y}_s^{n,\beta,x_0}, \theta) - b(\zeta_s^{n,\theta,x_0}, \theta) \right] ds \right| \\ & \leq \frac{1}{n} \int_0^t \|b'\|_\infty |\bar{Y}_s^{n,\beta,x_0} - \zeta_s^{n,\theta,x_0}| ds \leq \frac{1}{n} \int_0^t \|b'\|_\infty |\bar{Y}_s^{n,\beta,x_0} - \zeta_s^{n,\theta,x_0} - \frac{\sigma L_s^n}{n^{1/\alpha}}| ds + \frac{1}{n^{1+1/\alpha}} \int_0^1 |\sigma L_s^n| ds \end{aligned}$$

Applying the Gronwall's inequality, for  $C$  a positive constant, independent of  $n$  and  $K$ ,

$$(3.6.42) \quad \left| \bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0} - \frac{\sigma L_1^n}{n^{1/\alpha}} \right| \leq \frac{C}{n^{1+1/\alpha}} \int_0^1 |\sigma L_s^n| ds.$$

Using that the function  $H$  is globally Lipschitz with a Lipschitz constant  $\|h\|_\infty$ , we deduce from (3.6.41) that

$$(3.6.43) \quad \begin{aligned} & \left| \mathbb{E} \left[ \mathcal{H}^{n,K} h\left(\frac{\sigma L_1^n}{n^{1/\alpha}}\right) \right] - \mathbb{E} \left[ H(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) \mathcal{H}_{\left(\frac{\sigma L_1^n}{n^{1/\alpha}}\right)}(\mathcal{H}^{n,K}) \right] \right| \\ & \leq \left| \frac{\sigma C}{n^{1+1/\alpha}} \|h\|_\infty \mathbb{E} \left[ \left| \int_0^1 |L_s^n| ds \mathcal{H}_{\left(\frac{\sigma L_1^n}{n^{1/\alpha}}\right)}(\mathcal{H}^{n,K}) \right| \right] \right|. \end{aligned}$$

Now we compute  $\mathbb{E}[H(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) \mathcal{H}_{\left(\frac{\sigma L_1^n}{n^{1/\alpha}}\right)}(\mathcal{H}^{n,K})]$  using successively the self-adjoint property of the operator  $L$ , (3.6.3) and (3.6.10), to obtain an integration by part formula in a reverse direction:

(3.6.44)

$$\begin{aligned} & \mathbb{E}[H(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) \mathcal{H}_{\left(\frac{\sigma L_1^n}{n^{1/\alpha}}\right)}(\mathcal{H}^{n,K})] \\ & = \mathbb{E} \left[ H(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) \left\{ L \left( \frac{\mathcal{H}^{n,K}}{\Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}}\right)} \right) \frac{\sigma L_1^n}{n^{1/\alpha}} - \frac{L\left(\frac{\sigma L_1^n}{n^{1/\alpha}}\right)\mathcal{H}^{n,K}}{\Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}}\right)} - L \left( \frac{\mathcal{H}^{n,K} \frac{\sigma L_1^n}{n^{1/\alpha}}}{\Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}}\right)} \right) \right\} \right] \\ & = \mathbb{E} \left[ \frac{\{L(H(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) \frac{\sigma L_1^n}{n^{1/\alpha}}) - H(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) L\left(\frac{\sigma L_1^n}{n^{1/\alpha}}\right) - L(H(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) \frac{\sigma L_1^n}{n^{1/\alpha}})\} \mathcal{H}^{n,K}}{\Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}}\right)} \right] \\ & = \mathbb{E} \left[ \mathcal{H}^{n,K} \frac{\Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}}, H(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})\right)}{\Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}}\right)} \right] = \mathbb{E} \left[ \mathcal{H}^{n,K} h(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) \frac{\Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}}, \bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}\right)}{\Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}}\right)} \right]. \end{aligned}$$

Putting together (3.6.43) and (3.6.44) we deduce,

$$\begin{aligned}
 & \left| \mathbb{E}[\mathcal{H}^{n,K} h(\frac{\sigma L_1^n}{n^{1/\alpha}})] - \mathbb{E}[\mathcal{H}^{n,K} h(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})] \right| \\
 & \leq (\frac{\sigma C}{n^{1+1/\alpha}}) \|h\|_\infty \left\| \mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K}) \int_0^1 |L_s^n| ds \right\|_1 \\
 & \quad + \mathbb{E} \left\| \mathcal{H}^{n,K} h(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) \left\{ \frac{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}}, \bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}})} - 1 \right\} \right\| \\
 (3.6.45) \quad & \leq (\frac{\sigma C}{n^{1+1/\alpha}}) \|h\|_\infty \left\| \mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K}) \int_0^1 |L_s^n| ds \right\|_1 + \|\mathcal{H}^{n,K}\|_\infty \|h\|_\infty \left\| \frac{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}}, \bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}})} - 1 \right\|_1 \\
 & \leq \frac{C}{n} \|h\|_\infty \int_0^1 \underbrace{\left\| \frac{\sigma}{n^{1/\alpha}} \mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K}) |L_s^n| \right\|_1}_{I_1^{(n,s)}} ds + \|\mathcal{H}^{n,K}\|_\infty \|h\|_\infty \underbrace{\left\| \frac{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}}, \bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}})} - 1 \right\|_1}_{I_2^n}.
 \end{aligned}$$

Hence the lemma will be proved if we show that  $\sup_n \sup_{s \in [0,1]} I_1^{(n,s)} < \infty$  and  $\sup_n n I_2^n < \infty$ .

**Step 1:** we show that  $\sup_n \sup_{s \in [0,1]} I_1^{(n,s)} < \infty$ .

We can write from (3.6.9)

$$\begin{aligned}
 (3.6.46) \quad \mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K}) &= \frac{-2L(\frac{\sigma L_1^n}{n^{1/\alpha}}) \mathcal{H}^{n,K}}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}})} + \frac{\mathcal{H}^{n,K}}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}})^2} \Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}}, \Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}}\right)\right) - \frac{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}}, \mathcal{H}^{n,K})}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}})} \\
 &= \frac{n^{1/\alpha}}{\sigma} \left[ \frac{-2L(L_1^n) \mathcal{H}^{n,K}}{\Gamma(L_1^n, L_1^n)} + \frac{\mathcal{H}^{n,K}}{\Gamma(L_1^n, L_1^n)^2} \Gamma(L_1^n, \Gamma(L_1^n, L_1^n)) - \frac{\Gamma(L_1^n, \mathcal{H}^{n,K})}{\Gamma(L_1^n, L_1^n)} \right].
 \end{aligned}$$

Now, let us recall that from (3.3.2)

$$L_s^n = \int_0^s \int_{|z| \leq 1} z \tilde{\mu}^{(n)}(dt, dz) + \int_0^s \int_{|z| > 1} z \mu^{(n)}(dt, dz),$$

then,

$$(3.6.47) \quad \left| \frac{\sigma}{n^{1/\alpha}} \mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K}) |L_s^n| \right| \leq \left| \frac{\sigma}{n^{1/\alpha}} \mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K}) \int_0^s \int_{|z| \leq 1} z \tilde{\mu}^{(n)}(dt, dz) \right|$$

$$(3.6.48) \quad + \left| \frac{\sigma}{n^{1/\alpha}} \mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K}) \int_0^1 \int_{|z| > 1} |z| \mu^{(n)}(dt, dz) \right|$$

First, we consider the expectation of the right-hand side term in (3.6.47), we have:

$$\begin{aligned}
 & \mathbb{E} \left[ \left| \frac{\sigma}{n^{1/\alpha}} \mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K}) \int_0^s \int_{|z| \leq 1} z \tilde{\mu}^{(n)}(dt, dz) \right| \right] \\
 & \leq \left[ \mathbb{E} \left( \int_0^s \int_{|z| \leq 1} z \tilde{\mu}^{(n)}(dt, dz) \right)^2 \right]^{1/2} \left[ \mathbb{E} \left( \frac{\sigma}{n^{1/\alpha}} \mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K}) \right)^2 \right]^{1/2} \text{ by Cauchy-Schwarz inequality,}
 \end{aligned}$$

(3.6.49)

$$= \left[ \int_0^1 \int_{|z| \leq 1} z^2 \frac{1}{|z|^{1+\alpha}} dz dt \right]^{1/2} \left[ \mathbb{E} \left( \frac{\sigma}{n^{1/\alpha}} \mathcal{H}_{\left(\frac{\sigma L_1^n}{n^{1/\alpha}}\right)}(\mathcal{H}^{n,K}) \right)^2 \right]^{1/2} \leq M \left[ \mathbb{E} \left( \frac{\sigma}{n^{1/\alpha}} \mathcal{H}_{\left(\frac{\sigma L_1^n}{n^{1/\alpha}}\right)}(\mathcal{H}^{n,K}) \right)^2 \right]^{1/2}$$

where  $M$  is a deterministic constant.

Furthermore, from (3.6.6) we have  $|\Gamma(L_1^n, \mathcal{H}^{n,K})| \leq \Gamma(\mathcal{H}^{n,K}, \mathcal{H}^{n,K})^{1/2} \Gamma(L_1^n, L_1^n)^{1/2}$  and from (3.6.46), convexity inequality, we get

(3.6.50)

$$\left[ \left| \frac{\sigma}{n^{1/\alpha}} \mathcal{H}_{\left(\frac{\sigma L_1^n}{n^{1/\alpha}}\right)}(\mathcal{H}^{n,K}) \right| \right]^2 \leq 2 \left[ \left( \frac{-2L(L_1^n) \mathcal{H}^{n,K}}{\Gamma(L_1^n, L_1^n)} + \frac{\mathcal{H}^{n,K}}{\Gamma(L_1^n, L_1^n)^2} \Gamma(L_1^n, \Gamma(L_1^n, L_1^n)) \right)^2 + \frac{\Gamma(\mathcal{H}^{n,K}, \mathcal{H}^{n,K})}{\Gamma(L_1^n, L_1^n)} \right].$$

Then, we can deduce that

$$\begin{aligned} & \mathbb{E} \left( \left| \frac{\sigma}{n^{1/\alpha}} \mathcal{H}_{\left(\frac{\sigma L_1^n}{n^{1/\alpha}}\right)}(\mathcal{H}^{n,K}) \right| \right)^2 \\ (3.6.51) \quad & \leq 2 \mathbb{E} \left[ \underbrace{\left( \frac{-2L(L_1^n) \mathcal{H}^{n,K}}{\Gamma(L_1^n, L_1^n)} + \frac{\mathcal{H}^{n,K}}{\Gamma(L_1^n, L_1^n)^2} \Gamma(L_1^n, \Gamma(L_1^n, L_1^n)) \right)^2}_{I_{1.1}^n} \right] + 2 \mathbb{E} \left[ \underbrace{\frac{\Gamma(\mathcal{H}^{n,K}, \mathcal{H}^{n,K})}{\Gamma(L_1^n, L_1^n)}}_{I_{1.2}^n} \right]. \end{aligned}$$

Our aim is to prove that  $I_{1.1}^n$  and  $I_{1.2}^n$  are bounded independently of  $n$ .

For  $I_{1.1}^n$ , we see from (3.6.8) and Remark 3.6.2,

$$(3.6.52) \quad \frac{\Gamma(L_1^n, \Gamma(L_1^n, L_1^n))}{\Gamma(L_1^n, L_1^n)^2} - \frac{2L(L_1^n)}{\Gamma(L_1^n, L_1^n)} = \mathcal{H}_{L_1^n}(1) = \hat{\mathcal{H}}_\beta^n(1) + \frac{1}{n^{1/\alpha}} \mathcal{R}_{1,\beta}^n.$$

From the crucial fact  $\|\mathcal{H}^{n,K}\|_\infty \leq K$  and from (3.6.12), (3.6.15), Remark 3.6.2 we can deduce that  $I_{1.1}^n$  is bounded by a random variable independent of  $n$  (but depending on  $K$ ).

For  $I_{1.2}^n$ , from (3.6.1) and (3.6.21), we have

$$I_{1.2}^n = \mathbb{E} \left( \frac{\Gamma(\mathcal{H}^{n,K}, \mathcal{H}^{n,K})}{\Gamma(L_1^n, L_1^n)} \right) = \mathbb{E} \left( \frac{\Gamma(\mathcal{H}^{n,K}, \mathcal{H}^{n,K})}{\int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(dt, dz)} \right) \leq \mathbb{E} \left( \frac{\Gamma(\mathcal{H}^{n,K}, \mathcal{H}^{n,K})}{\int_0^1 \int_{|z| < 1} z^4 \mu(dt, dz)} \right).$$

Now since  $\mathcal{H}^{n,K}$  is a smooth Malliavin functional, using the chain rule property (3.6.4) we have

$$(3.6.53) \quad \Gamma(\mathcal{H}^{n,K}, \mathcal{H}^{n,K}) \leq c_K^2 \Gamma(\mathcal{H}^n, \mathcal{H}^n)$$

where  $c_K$  is any upper bound of the derivative of  $x \mapsto x \mathcal{X}_K(x)$ .

Then we deduce that

$$I_{1.2}^n \leq c_K^2 \mathbb{E} \left( \frac{\Gamma(\mathcal{H}^n, \mathcal{H}^n)}{\int_0^1 \int_{|z| < 1} z^4 \mu(dt, dz)} \right).$$

From the assumption on  $\Gamma(\mathcal{H}^n, \mathcal{H}^n)$  in Proposition 3.6.3 and since  $\left(\int_0^1 \int_{|z|<1} z^4 \mu(dt, dz)\right)^{-1}$  belongs to  $\cap_{p \geq 1} \mathbf{L}^p$  (see [17, the proof of Theorem 4]), we can deduce that  $I_{1,2}^n$  is bounded independently of  $n$ . Thus, we get that the expectation of the right-hand side term in (3.6.47) is finite.

Turning to the expectation of (3.6.48), we have:

$$\begin{aligned} & \mathbb{E} \left[ \left| \frac{\sigma}{n^{1/\alpha}} \mathcal{H}_{\left(\frac{\sigma L_1^n}{n^{1/\alpha}}\right)}(\mathcal{H}^{n,K}) \int_0^1 \int_{|z|>1} |z| \mu^{(n)}(dt, dz) \right| \right] \\ &= \mathbb{E} \left[ \left| \frac{\sigma}{n^{1/\alpha}} \mathcal{H}_{\left(\frac{\sigma L_1^n}{n^{1/\alpha}}\right)}(\mathcal{H}^{n,K}) \left( \int_0^1 \int_{1 \leq |z| \leq 2} |z| \mu^{(n)}(dt, dz) + \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right) \right| \right] \\ &\leq \mathbb{E} \left[ \left| \frac{\sigma}{n^{1/\alpha}} \mathcal{H}_{\left(\frac{\sigma L_1^n}{n^{1/\alpha}}\right)}(\mathcal{H}^{n,K}) \int_0^1 \int_{1 \leq |z| \leq 2} |z| \mu^{(n)}(dt, dz) \right| \right] + \mathbb{E} \left[ \left| \frac{\sigma}{n^{1/\alpha}} \mathcal{H}_{\left(\frac{\sigma L_1^n}{n^{1/\alpha}}\right)}(\mathcal{H}^{n,K}) \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right| \right]. \end{aligned}$$

By a similar estimation technique as for the bound of (3.6.47), we get that

$$\sup_n \mathbb{E} \left[ \left| \frac{\sigma}{n^{1/\alpha}} \mathcal{H}_{\left(\frac{\sigma L_1^n}{n^{1/\alpha}}\right)}(\mathcal{H}^{n,K}) \int_0^1 \int_{1 \leq |z| \leq 2} |z| \mu^{(n)}(dt, dz) \right| \right] \leq C_K < +\infty.$$

We now show that

$$(3.6.54) \quad \sup_n \mathbb{E} \left[ \left| \frac{\sigma}{n^{1/\alpha}} \mathcal{H}_{\left(\frac{\sigma L_1^n}{n^{1/\alpha}}\right)}(\mathcal{H}^{n,K}) \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right| \right] \leq C_K < +\infty.$$

In fact, from (3.6.46) we have

$$\begin{aligned} & \mathbb{E} \left[ \left| \frac{\sigma}{n^{1/\alpha}} \mathcal{H}_{\left(\frac{\sigma L_1^n}{n^{1/\alpha}}\right)}(\mathcal{H}^{n,K}) \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right| \right] \\ &\leq \mathbb{E} \left[ \left[ \left| \frac{-2L(L_1^n) \mathcal{H}^{n,K}}{\Gamma(L_1^n, L_1^n)} + \frac{\mathcal{H}^{n,K}}{\Gamma(L_1^n, L_1^n)^2} \Gamma(L_1^n, \Gamma(L_1^n, L_1^n)) \right| + \left| \frac{\Gamma(L_1^n, \mathcal{H}^{n,K})}{\Gamma(L_1^n, L_1^n)} \right| \times \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right] \right] \\ &= \mathbb{E} \left[ \underbrace{\left[ \left| \frac{-2L(L_1^n) \mathcal{H}^{n,K}}{\Gamma(L_1^n, L_1^n)} + \frac{\mathcal{H}^{n,K}}{\Gamma(L_1^n, L_1^n)^2} \Gamma(L_1^n, \Gamma(L_1^n, L_1^n)) \right| \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right]}_{I_{1,3}^n} \right] \\ (3.6.55) \quad & + \mathbb{E} \left[ \underbrace{\left[ \left| \frac{\Gamma(L_1^n, \mathcal{H}^{n,K})}{\Gamma(L_1^n, L_1^n)} \right| \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right]}_{I_{1,4}^n} \right] \end{aligned}$$

To prove (3.6.54), we just have to show that  $I_{1,3}^n$  and  $I_{1,4}^n$  are bounded independently of  $n$ . For  $I_{1,3}^n$ , we recall here (3.6.52),

$$\frac{\Gamma(L_1^n, \Gamma(L_1^n, L_1^n))}{\Gamma(L_1^n, L_1^n)^2} - \frac{2L(L_1^n)}{\Gamma(L_1^n, L_1^n)} = \hat{\mathcal{H}}_\beta^n(1) + \frac{1}{n^{1/\alpha}} \mathcal{R}_{1,\beta}^n(1).$$

Then from the boundedness of  $\mathcal{H}^{n,K}$ , we get

$$I_{1.3}^n \leq K\mathbb{E} \left[ \left| \hat{\mathcal{H}}_\beta^n(1) \right| \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right] + K\mathbb{E} \left[ \left| \mathcal{R}_{1,\beta}^n(1) \right| \frac{1}{n^{1/\alpha}} \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right]$$

From the proof of Lemma 5.4 in [19], we can deduce that  $\left| \hat{\mathcal{H}}_\beta^n(1) \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right|$  is bounded by a random variable independent of  $n$  and belonging to  $\cap_{p \geq 1} \mathbf{L}^p$ .

Using Cauchy-Schwarz inequality and (3.6.15), we get

$$(3.6.56) \quad \mathbb{E} \left[ \left| \mathcal{R}_{1,\beta}^n(1) \right| \frac{1}{n^{1/\alpha}} \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right] \leq \frac{C}{\sqrt{n}} \left( \mathbb{E} \left[ \int_0^1 \int_{|z|>2} \frac{|z|}{n^{1/\alpha}} \mu^{(n)}(dt, dz) \right]^2 \right)^{1/2}$$

Now from  $\mu^{(n)}(ds, dz) = \tilde{\mu}^{(n)}(ds, dz) + v^{(n)}(ds, dz)$ , by convexity inequality, we have

$$(3.6.57) \quad \begin{aligned} \mathbb{E} \left[ \int_0^1 \int_{|z|>2} \frac{|z|}{n^{1/\alpha}} \mu^{(n)}(ds, dz) \right]^2 &\leq 2\mathbb{E} \left[ \int_0^1 \int_{|z|>2} \frac{|z|}{n^{1/\alpha}} \tilde{\mu}^{(n)}(ds, dz) \right]^2 + 2 \left[ \int_0^1 \int_{|z|>2} \frac{|z|}{n^{1/\alpha}} v^{(n)}(ds, dz) \right]^2 \\ &\leq 2\mathbb{E} \left[ \int_0^1 \int_{2<|z|\leq 2n^{1/\alpha}} \frac{|z|^2}{n^{2/\alpha}} \frac{1}{|z|^{1+\alpha}} dz ds \right] + 2 \left[ \int_0^1 \int_{2<|z|\leq 2n^{1/\alpha}} \frac{|z|}{n^{1/\alpha}} \frac{1}{|z|^{1+\alpha}} dz ds \right]^2 \\ &\leq \frac{C}{n^{2/\alpha}} \left( \frac{n^{2/\alpha}}{n} + \frac{n^{1/\alpha}}{n} \right) \quad \text{where } C \text{ is a deterministic constant.} \end{aligned}$$

From (3.6.56) and (3.6.57), we deduce that the left-hand side of (3.6.56) is bounded by  $\frac{C}{n}$ . Then we get that  $\sup_n I_{1.3}^n < +\infty$ .

For  $I_{1.4}^n$ , from the boundedness of  $\mathcal{H}^{n,K}$ , (3.6.21), and the fact that

$\Gamma(L_1^n, \mathcal{H}^{n,K}) \leq \Gamma(\mathcal{H}^{n,K}, \mathcal{H}^{n,K})^{1/2} \Gamma(L_1^n, L_1^n)^{1/2}$  we have:

$$\begin{aligned} I_{1.4}^n &\leq \mathbb{E} \left[ \left( \left| \frac{\Gamma(\mathcal{H}^{n,K}, \mathcal{H}^{n,K})}{\Gamma(L_1^n, L_1^n)} \right| \right)^{1/2} \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right] \\ &= \mathbb{E} \left[ \frac{|\Gamma(\mathcal{H}^{n,K}, \mathcal{H}^{n,K})|^{1/2}}{\left| \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right|^{1/2}} \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right] \\ &\leq \mathbb{E} \left[ \frac{|\Gamma(\mathcal{H}^{n,K}, \mathcal{H}^{n,K})|^{1/2}}{\left| \int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz) \right|^{1/2}} \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right]. \end{aligned}$$

Applying the Cauchy-Schwarz inequality, we have

$$(3.6.58) \quad \int_0^1 \int_{|z|>2} \mu^{(n)}(dt, dz) \times \int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz) \geq \left( \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right)^2.$$

We deduce

$$\begin{aligned}
 (3.6.59) \quad & \mathbb{E} \left[ \frac{|\Gamma(\mathcal{H}^{n,K}, \mathcal{H}^{n,K})|^{1/2}}{\left| \int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz) \right|^{1/2}} \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right] \\
 & \leq \mathbb{E} \left[ \left| \Gamma(\mathcal{H}^{n,K}, \mathcal{H}^{n,K}) \right|^{1/2} \left\{ \int_0^1 \int_{|z|>2} \mu^{(n)}(dt, dz) \right\}^{1/2} \right] \\
 & \leq \left\{ \mathbb{E} \left[ \left| \Gamma(\mathcal{H}^{n,K}, \mathcal{H}^{n,K}) \right| \right] \right\}^{1/2} \left\{ \mathbb{E} \left[ \int_0^1 \int_{|z|>2} \mu^{(n)}(dt, dz) \right] \right\}^{1/2}.
 \end{aligned}$$

From  $\left\{ \mathbb{E} \left[ \int_0^1 \int_{|z|>2} \mu^{(n)}(dt, dz) \right] \right\}^{1/2} \leq C_4$  where  $C_4$  is a constant and the fact that  $\Gamma(\mathcal{H}^{n,K}, \mathcal{H}^{n,K})$  admits finite moment, independently of  $n$  (but depending of  $K$ ). Then, (3.6.59) is bounded independently of  $n$  and  $I_{1.4}^n$  is proved. Hence (3.6.54) follows.

**Step 2:** We now prove  $\sup_n n I_2^n < C^*$  where  $C^*$  is a positive constant.

We have

$$\begin{aligned}
 & \left| \frac{\Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}}, \bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}\right)}{\Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}}\right)} - 1 \right| = \frac{1}{n} \left| \int_0^1 \frac{\Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}}, b(\bar{Y}_s^{n,\beta,x_0}, \theta) - b(\zeta_s^{n,\theta,x_0}, \theta)\right)}{\Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}}\right)} ds \right| \\
 & \leq \frac{1}{n} \frac{\|b'\|_\infty}{\Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}}\right)} \int_0^1 \left| \Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}}, \bar{Y}_s^{n,\beta,x_0}\right) \right| ds.
 \end{aligned}$$

Using that

$$\begin{cases} \Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}}, \bar{Y}_s^{n,\beta,x_0}\right) \leq \Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}}\right)^{1/2} \Gamma(\bar{Y}_s^{n,\beta,x_0}, \bar{Y}_s^{n,\beta,x_0})^{1/2} \\ \Gamma(\bar{Y}_s^{n,\beta,x_0}, \bar{Y}_s^{n,\beta,x_0})^{1/2} \leq M \Gamma(\bar{Y}_1^{n,\beta,x_0}, \bar{Y}_1^{n,\beta,x_0})^{1/2} \\ \Gamma(\bar{Y}_1^{n,\beta,x_0}, \bar{Y}_1^{n,\beta,x_0})^{1/2} \leq M_1 \Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}}\right)^{1/2} \end{cases}$$

for some constants  $M$  and  $M_1$ , we can easily deduce that

$$\left| \frac{\Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}}, \bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}\right)}{\Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}}\right)} - 1 \right| \leq \frac{C^*}{n},$$

for a positive constant  $C^*$ . ■

**Lemma 3.6.5.** *Under the assumptions of Proposition 3.6.3, for any compact set  $Q \subset \mathbb{R} \times (0, \infty)$ ,  $\forall K > 2$ , there exists a constant  $c_K > 0$  such that*

$$(3.6.60) \quad \sup_{x_0, \beta \in Q} \left\| \bar{\eta}^{\mathcal{H}^{n,K}}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) - \bar{\eta}^{\mathcal{H}^{n,K}}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) \right\|_1 \leq \frac{c_K}{n}.$$

**Proof.** We estimate the  $\mathbf{L}^1$ - norm appearing in (3.6.60) by duality. Let  $\bar{\beta} : \mathbb{R} \rightarrow [-1, 1]$  be a measurable function, we evaluate:

$$\begin{aligned} & \left| \mathbb{E} \left[ (\eta^{\mathcal{H}^{n,K}}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) - \bar{\eta}^{\mathcal{H}^{n,K}}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})) \bar{\beta}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) \right] \right| \\ & \leq \left| \mathbb{E} \left[ \eta^{\mathcal{H}^{n,K}}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) \bar{\beta}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) \right] - \mathbb{E} \left[ \bar{\eta}^{\mathcal{H}^{n,K}} \left( \frac{\sigma L_1^\alpha}{n^{1/\alpha}} \right) \bar{\beta} \left( \frac{\sigma L_1^\alpha}{n^{1/\alpha}} \right) \right] \right| \\ & + \left| \mathbb{E} \left[ \bar{\eta}^{\mathcal{H}^{n,K}} \left( \frac{\sigma L_1^\alpha}{n^{1/\alpha}} \right) \bar{\beta} \left( \frac{\sigma L_1^\alpha}{n^{1/\alpha}} \right) \right] - \mathbb{E} \left[ \bar{\eta}^{\mathcal{H}^{n,K}}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) \bar{\beta}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) \right] \right| \\ & \leq \left| \mathbb{E} \left[ \eta^{\mathcal{H}^{n,K}}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) \bar{\beta}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) \right] - \mathbb{E} \left[ \bar{\eta}^{\mathcal{H}^{n,K}} \left( \frac{\sigma L_1^\alpha}{n^{1/\alpha}} \right) \bar{\beta} \left( \frac{\sigma L_1^\alpha}{n^{1/\alpha}} \right) \right] \right| + \frac{C_K}{n} K \end{aligned}$$

where we have used Lemma 3.6.4 with the choice  $\mathcal{H}^n = 1, K > 2$  and the choice  $h = \bar{\eta}^{\mathcal{H}^{n,K}} \bar{\beta}$ , recalling that  $\|\bar{\eta}^{\mathcal{H}^{n,K}}\|_\infty \leq K$ . From the definition of  $\bar{\eta}^{\mathcal{H}^{n,K}} \left( \frac{\sigma L_1^\alpha}{n^{1/\alpha}} \right)$  and  $\eta^{\mathcal{H}^{n,K}}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})$  as conditional expectations, we have:

$$\begin{aligned} & \left| \mathbb{E} \left[ \eta^{\mathcal{H}^{n,K}}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) \bar{\beta}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) \right] - \mathbb{E} \left[ \bar{\eta}^{\mathcal{H}^{n,K}} \left( \frac{\sigma L_1^\alpha}{n^{1/\alpha}} \right) \bar{\beta} \left( \frac{\sigma L_1^\alpha}{n^{1/\alpha}} \right) \right] \right| \\ & = \mathbb{E} \left[ \mathcal{H}^n \mathcal{X}_K(\mathcal{H}^n) \bar{\beta}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) \right] - \mathbb{E} \left[ \mathcal{H}^n \mathcal{X}_K(\mathcal{H}^n) \bar{\beta} \left( \frac{L_1^\alpha}{n^{1/\alpha}} \right) \right] \leq \frac{C_K}{n} \end{aligned}$$

where we used Lemma 3.6.4. This gives,

$$\sup_{\|\bar{\beta}\|_\infty \leq 1} \left| \mathbb{E} \left[ (\eta^{\mathcal{H}^{n,K}}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) - \bar{\eta}^{\mathcal{H}^{n,K}}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0})) \bar{\beta}(\bar{Y}_1^{n,\beta,x_0} - \zeta_1^{n,\theta,x_0}) \right] \right| \leq (1+K) \frac{C_K}{n}$$

and we deduce the result of this lemma.  $\blacksquare$

### 3.7 Appendix. Checking the conditions for $\mathcal{H} = \mathcal{H}_{L^\alpha}(1)$ and

#### $\mathcal{H} = L_1^\alpha \mathcal{H}_{L^\alpha}(1)$ in Proposition 3.6.3

The aim of this section is to verify the conditions given in Proposition 3.6.3 for the choices  $\mathcal{H} = \mathcal{H}_{L^\alpha}(1)$  and  $\mathcal{H} = L_1^\alpha \mathcal{H}_{L^\alpha}(1)$ . Recall that we need to check the following four conditions with  $\mathcal{H} = \mathcal{H}_{L^\alpha}(1)$ ,  $\mathcal{H} = L_1^\alpha \mathcal{H}_{L^\alpha}(1)$  and the choice of  $\mathcal{H}^n$  in the hypothesis of Proposition 3.6.3 to be clarified later:

- (1)  $\mathbb{E}(\mathcal{H})^2 < \infty$
- (2)  $\mathbb{E}(\mathcal{H}^n)^2 < \infty$
- (3)  $\mathcal{H}^n \xrightarrow[\mathbf{L}^2]{n \rightarrow \infty} \mathcal{H}$

$$(4) \sup_n \|\Gamma(\mathcal{H}^n, \mathcal{H}^n)\|_2 < \infty.$$

We denote  $\mu^{(n)}(f) = \int_0^1 \int_{\mathbb{R}} f(t, z) \mu^{(n)}(dt, dz)$ .

### 3.7.1 The case $\mathcal{H} = \mathcal{H}_{L^\alpha}(1)$

We set  $\mathcal{H}^n := \tilde{\mathcal{H}}^n$  where

$$(3.7.1) \quad \begin{aligned} \tilde{\mathcal{H}}^n &= \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) (\rho^n)'(z) \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^2} - \frac{\int_0^1 \int_{\mathbb{R}} \left[ (\rho^n)'(z) - \frac{1+\alpha}{z} \rho^n(z) \right] \mu^{(n)}(ds, dz)}{\int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz)} \\ &:= \underbrace{\frac{\mu^{(n)}(\rho^n (\rho^n)')}{\mu^{(n)}(\rho^n)^2}}_{A_n} - \underbrace{\frac{\mu^{(n)} \left[ \left( (\rho^n)' - \frac{1+\alpha}{z} \rho^n \right) \right]}{\mu^{(n)}(\rho^n)}}_{B_n} \end{aligned}$$

and verify the four above conditions (1)-(4).

From Remark 3.6.2 *ii*), we get that  $\tilde{\mathcal{H}}^n = \mathcal{H}_{L_1^n}(1) - \frac{1}{n^{1/\alpha}} \mathcal{R}_{1,\beta}^n$ , moreover,  $\mathbb{E}(\mathcal{H}_{L^\alpha}(1))^2 < \infty$  and  $\tilde{\mathcal{H}}^n \xrightarrow[\mathbf{L}^2]{n \rightarrow \infty} \mathcal{H}_{L^\alpha}(1)$ . Thus, we just have to check the condition (4), namely

$$\sup_n \|\Gamma(\tilde{\mathcal{H}}^n, \tilde{\mathcal{H}}^n)\|_2 < \infty.$$

From the inequality (8.7) in [8], we have

$$(3.7.2) \quad \Gamma(\tilde{\mathcal{H}}^n, \tilde{\mathcal{H}}^n) = \Gamma(A_n - B_n, A_n - B_n) \leq 2[\Gamma(A_n, A_n) + \Gamma(B_n, B_n)]$$

then the condition (4) will be proved if we show that  $\sup_n \|\Gamma(A_n, A_n)\|_2 < \infty$  and  $\sup_n \|\Gamma(B_n, B_n)\|_2 < \infty$ .

**Step 1:** We prove that  $\sup_n \|\Gamma(A_n, A_n)\|_p < \infty, \quad \forall p \geq 1$ .

Using the chain rule property for the operator  $\Gamma$ , we get

$$(3.7.3) \quad \begin{aligned} \Gamma(A_n, A_n) &= \Gamma \left[ \frac{\mu^{(n)}(\rho^n (\rho^n)')}{\mu^{(n)}(\rho^n)^2}, \frac{\mu^{(n)}(\rho^n (\rho^n)')}{\mu^{(n)}(\rho^n)^2} \right] \\ &= \frac{1}{\mu^{(n)}(\rho^n)^4} \Gamma(\mu^{(n)}(\rho^n (\rho^n)'), \mu^{(n)}(\rho^n (\rho^n)')) - \frac{4\mu^{(n)}(\rho^n (\rho^n)')}{\mu^{(n)}(\rho^n)^5} \Gamma(\mu^{(n)}(\rho^n (\rho^n)'), \mu^{(n)}(\rho^n)) \\ &\quad + \frac{4\mu^{(n)}(\rho^n (\rho^n)')^2}{\mu^{(n)}(\rho^n)^6} \Gamma(\mu^{(n)}(\rho^n), \mu^{(n)}(\rho^n)). \end{aligned}$$



Let us consider the first term in the right-hand side of (3.7.3).

Using the definition of the operator  $\Gamma$  for a test function, we obtain

$$(3.7.4) \quad \left| \frac{\Gamma(\mu^{(n)}(\rho^n(\rho^n)'), \mu^{(n)}(\rho^n(\rho^n)'))}{\mu^{(n)}(\rho^n)^4} \right| = \left| \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) [\rho^n(z)(\rho^n)''(z) + ((\rho^n)'(z))^2]^2 \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^4} \right|$$

$$\leq \frac{\int_0^1 \int_{|z| \leq 2} \rho(z) [\rho(z)|(\rho)''(z)| + ((\rho)'(z))^2]^2 \mu(ds, dz)}{\left[ \int_0^1 \int_{|z| \leq 2} \rho(z) \mu(ds, dz) \right]^4} + \frac{\int_0^1 \int_{|z| > 2} 36z^6 \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{|z| > 2} z^2 \mu^{(n)}(ds, dz) \right]^4}$$

where we used the fact that the measures  $\mu^{(n)}(ds, dz)$  and  $\mu(ds, dz)$  coincide on the set  $\{(s, z) | s \in [0, 1], |z| \leq n^{1/\alpha}\}$ , and  $\rho^n(z) = \rho(z)$  on the support of the Poisson measure  $\mu^{(n)}$ .

We now consider (3.7.4). Using that  $\rho, \rho', \rho''$  belongs to  $\cap_{p \geq 1} \mathbf{L}^p(1_{|z| \leq 2} |z|^{-1-\alpha} dz)$ , we get

$$\mathbb{E} \left[ \left( \int_0^1 \int_{|z| < 2} \rho(z) [\rho(z)|(\rho)''(z)| + ((\rho)'(z))^2]^2 \mu(ds, dz) \right)^p \right] < \infty, \quad \forall p \geq 1.$$

We can check that  $\rho$  satisfies the non degeneracy assumption

$$\liminf_{u \rightarrow \infty} \frac{1}{\ln(u)} \int_{|z| \leq 2} 1_{\rho(z) \geq 1/u} F(z) dz$$

$$= \liminf_{u \rightarrow \infty} \frac{1}{\ln(u)} \int_{|z| \leq 2} 1_{\rho(z) \geq 1/u} \frac{1}{|z|^{1+\alpha}} 1_{|z| \neq 0} dz < +\infty.$$

Proceeding as in the proof of [ [17, Theorem 4 p.2323]], we obtain  $[\int_0^1 \int_{|z| \leq 2} \rho(z) \mu(ds, dz)]^{-1}$  belongs to  $\cap_{p \geq 1} \mathbf{L}^p$ , as a consequence we deduce that the first term of (3.7.4) belongs to  $\cap_{p \geq 1} \mathbf{L}^p$ , moreover, it does not depend on  $n$ .

For the second term of (3.7.4), from the fact that  $v^{(n)}(\{(t, z) | 0 \leq t \leq 1, |z| > 2\}) < \infty$ , we can construct the integral with respect to the random measure  $\mu^{(n)}$  as follows

$$\int_0^1 \int_{|z| > 2} |z|^6 \mu^{(n)}(ds, dz) = \sum_{i=1}^{N_1} |Z_i|^6 \quad \text{a.s.,}$$

$$\int_0^1 \int_{|z| > 2} z^2 \mu^{(n)}(ds, dz) = \sum_{i=1}^{N_1} Z_i^2 \quad \text{a.s.,}$$

where  $N = (N_t)_{1 \geq t \geq 0}$  is a Poisson process with intensity  $\lambda_n = \int_{|z| > 2} F_n(z) dz < \infty$ , and  $(Z_i)_{i \geq 0}$  are i.i.d. random variable independent of  $N$  with probability measure  $\frac{F_n(z) 1_{|z| > 2} dz}{\lambda_n}$ . Thus,

$$\frac{\int_0^1 \int_{|z| > 2} |z|^6 \mu^{(n)}(ds, dz)}{\left( \int_0^1 \int_{|z| > 2} z^2 \mu^{(n)}(ds, dz) \right)^4} = \frac{\sum_{i=1}^{N_1} |Z_i|^6}{\left( \sum_{i=1}^{N_1} Z_i^2 \right)^4} \leq \frac{\sum_{i=1}^{N_1} |Z_i|^6}{\sum_{i=1}^{N_1} Z_i^8} \leq 1$$

where we used the fact that  $Z_i^2 \geq 0$ , and  $|Z_i| > 2$ . Hence, from the above estimates, the first term in the right-hand side of (3.7.3) is bounded by a random variable independent of  $n$  and belonging to  $\cap_{p \geq 1} \mathbf{L}^p$ .

We estimate the second term in the right-hand side of (3.7.3).

Using the definition of the operator  $\Gamma$  for a test function, we have

$$(3.7.5) \quad \left| \frac{\mu^{(n)}(\rho^n(\rho^n)')}{\mu^{(n)}(\rho^n)^5} \Gamma(\mu^{(n)}(\rho^n(\rho^n)'), \mu^{(n)}(\rho^n)) \right| \\
= \left| \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z)(\rho^n)'(z) \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^5} \int_0^1 \int_{\mathbb{R}} \rho^n(z)(\rho^n)'(z) [((\rho^n)'(z))^2 + \rho^n(z)(\rho^n)''(z)] \mu^{(n)}(ds, dz) \right| \\
\leq \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) |(\rho^n)'(z)| \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^2} \times \\
\times \left( \underbrace{\frac{\int_0^1 \int_{|z| \leq 2} \rho(z) |(\rho)'(z)| [((\rho)'(z))^2 + \rho(z)|(\rho)''(z)|] \mu(ds, dz)}{\left[ \int_0^1 \int_{|z| < 2} \rho(z) \mu(ds, dz) \right]^3}}_{A_{1,n}} + \underbrace{\frac{\int_0^1 \int_{|z| > 2} 12|z|^5 \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{|z| > 2} z^2 \mu^{(n)}(ds, dz) \right]^3}}_{A_{2,n}} \right).$$

From **Step 1** in the proof of Theorem 2.1 in [19], we have

$$\mathbb{E} \left( \sup_n \left| \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) |(\rho^n)'(z)| \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^2} \right|^p \right) < \infty, \forall p \geq 1.$$

Proceeding as in the proof of (3.7.4) then we can easily deduce that  $A_{1,n}, A_{2,n}$  are bounded by a random variable independent of  $n$  and belonging to  $\cap_{p \geq 1} \mathbf{L}^p$ .

We now consider the last term in the right-hand side of (3.7.3).

Using the definition of the operator  $\Gamma$  for a test function, we obtain

$$\left| \frac{\mu^{(n)}(\rho^n(\rho^n)')^2}{\mu^{(n)}(\rho^n)^6} \Gamma(\mu^{(n)}(\rho^n), \mu^{(n)}(\rho^n)) \right| = \left| \frac{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) (\rho^n)'(z) \mu^{(n)}(ds, dz) \right]^2}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^6} \int_0^1 \int_{\mathbb{R}} \rho^n(z) (\rho^n)'(z) (\rho^n)'(z) \mu^{(n)}(ds, dz) \right| \\
= \left| \frac{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) (\rho^n)'(z) \mu^{(n)}(ds, dz) \right]^2}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^4} \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) (\rho^n)'(z) (\rho^n)'(z) \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^2} \right|.$$

We proceed as in the proof of the first term of (3.7.3), we can get the result for this term, as well.

The **Step 1** follows.

**Step 2:** We show that  $\sup_n \|\Gamma(B_n, B_n)\|_p < \infty, \forall p \geq 1$ .

Using the chain rule property for the operator  $\Gamma$ , we get

$$(3.7.6) \quad \Gamma(B_n, B_n) = \Gamma\left(\frac{\mu^{(n)}([\rho^n]' - \frac{1+\alpha}{z}\rho^n)}{\mu^{(n)}(\rho^n)}, \frac{\mu^{(n)}([\rho^n]' - \frac{1+\alpha}{z}\rho^n)}{\mu^{(n)}(\rho^n)}\right)$$

$$(3.7.7) \quad = \frac{1}{\mu^{(n)}(\rho^n)^2} \Gamma\left(\mu^{(n)}\left[\rho^n' - \frac{1+\alpha}{z}\rho^n\right], \mu^{(n)}\left[\rho^n' - \frac{1+\alpha}{z}\rho^n\right]\right)$$

$$(3.7.8) \quad - \frac{2\mu^{(n)}\left[\rho^n' - \frac{1+\alpha}{z}\rho^n\right]}{\mu^{(n)}(\rho^n)^3} \Gamma\left(\mu^{(n)}\left[\rho^n' - \frac{1+\alpha}{z}\rho^n\right], \mu^{(n)}(\rho^n)\right)$$

$$(3.7.9) \quad + \frac{\mu^{(n)}([\rho^n]' - \frac{1+\alpha}{z}\rho^n)^2}{\mu^{(n)}(\rho^n)^4} \Gamma(\mu^{(n)}(\rho^n), \mu^{(n)}(\rho^n))$$

We consider first (3.7.7). We have

$$\begin{aligned} & \left| \frac{1}{\mu^{(n)}(\rho^n)^2} \Gamma\left(\mu^{(n)}\left[\rho^n' - \frac{1+\alpha}{z}\rho^n\right], \mu^{(n)}\left[\rho^n' - \frac{1+\alpha}{z}\rho^n\right]\right) \right| \\ &= \left| \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) \left[\rho^n''(z) - \frac{1+\alpha}{z}(\rho^n)'(z) + \frac{1+\alpha}{z^2}\rho^n(z)\right]^2 \mu^{(n)}(ds, dz)}{\left(\int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz)\right)^2} \right| \\ &\leq \frac{\int_0^1 \int_{|z|\leq 2} \rho(z) \left[|\rho''(z)| + \frac{1+\alpha}{|z|}|\rho'(z)| + \frac{1+\alpha}{z^2}\rho(z)\right]^2 \mu(ds, dz)}{\left(\int_0^1 \int_{|z|\leq 2} \rho(z) \mu(ds, dz)\right)^2} + \frac{\int_0^1 \int_{|z|>2} (5+3\alpha)^2 z^2 \mu^{(n)}(ds, dz)}{\left(\int_0^1 \int_{|z|\geq 2} z^2 \mu^{(n)}(ds, dz)\right)^2} \end{aligned}$$

where we used the fact that  $\int_0^1 \int_{|z|\leq 2} \rho(z) \mu(ds, dz) \geq 0$ ,  $\int_0^1 \int_{|z|>2} \rho^n(z) \mu^{(n)}(ds, dz) \geq 0$  and the measures  $\mu^{(n)}(ds, dz)$  and  $\mu(ds, dz)$  coincide on the set  $\{(s, z) | s \in [0, 1], |z| \leq n^{1/\alpha}\}$ .

Similarly to (3.7.4), we also obtain that (3.7.7) is bounded by a random variable independent of  $n$  and belonging to  $\cap_{p \geq 1} \mathbf{L}^p$ .

Let us consider (3.7.8). We have

$$\begin{aligned} & \left| \frac{\mu^{(n)}([\rho^n]' - \frac{1+\alpha}{z}\rho^n)}{\mu^{(n)}(\rho^n)^3} \Gamma\left(\mu^{(n)}\left[\rho^n' - \frac{1+\alpha}{z}\rho^n\right], \mu^{(n)}(\rho^n)\right) \right| \\ &= \left| \frac{\int_0^1 \int_{\mathbb{R}} \left[\rho^n'(z) - \frac{1+\alpha}{z}\rho^n(z)\right] \mu^{(n)}(ds, dz)}{\left[\int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz)\right]} \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) \left[\rho^n''(z) - \frac{1+\alpha}{z}(\rho^n)'(z) + \frac{1+\alpha}{z^2}\rho^n(z)\right] (\rho^n)'(z) \mu^{(n)}(ds, dz)}{\left[\int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz)\right]^2} \right| \\ &\leq \left( \frac{\int_0^1 \int_{|z|\leq 2} \left[|\rho(z)'| + \frac{1+\alpha}{|z|}\rho(z)\right] \mu(ds, dz)}{\left[\int_0^1 \int_{|z|\leq 2} \rho(z) \mu(ds, dz)\right]} + \frac{\int_0^1 \int_{|z|>2} \left[2|z| + \frac{1+\alpha}{|z|}z^2\right] \mu^{(n)}(ds, dz)}{\left[\int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz)\right]} \right) \\ &\times \left( \frac{\int_0^1 \int_{|z|\leq 2} \rho(z) \left[|\rho''(z)| + \frac{1+\alpha}{|z|}|\rho'(z)| + \frac{1+\alpha}{z^2}\rho(z)\right] |\rho'(z)| \mu(ds, dz)}{\left[\int_0^1 \int_{|z|\leq 2} \rho(z) \mu(ds, dz)\right]^2} + \frac{\int_0^1 \int_{|z|>2} 2(5+3\alpha)|z|^3 \mu^{(n)}(ds, dz)}{\left[\int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz)\right]^2} \right) \end{aligned}$$

Proceeding as for the first term in the right-hand side of (3.7.4), we get that (3.7.8) is bounded by a random variable independent of  $n$  and belonging to  $\cap_{p \geq 1} \mathbf{L}^p$ .

For the last term (3.7.9), we have

$$\begin{aligned} & \frac{\mu^{(n)}\left([\rho^n]' - \frac{1+\alpha}{z}\rho^n\right)^2}{\mu^{(n)}(\rho^n)^4} \Gamma(\mu^{(n)}(\rho^n), \mu^{(n)}(\rho^n)) \\ &= \frac{\left[\int_0^1 \int_{\mathbb{R}} [(\rho^n)'(z) - \frac{1+\alpha}{z}\rho^n(z)] \mu^{(n)}(ds, dz)\right]^2}{\left[\int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz)\right]^2} \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) (\rho^n)'(z) (\rho^n)'(z) \mu^{(n)}(ds, dz)}{\left[\int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz)\right]^2} \end{aligned}$$

With the same technique as for the above terms, we also get that this term is bounded by a random variable independently of  $n$  and belonging to  $\cap_{p \geq 1} \mathbf{L}^p$ . This achieves the proof of **Step 2**.

We now can conclude that  $\sup_n \|\Gamma(\tilde{\mathcal{H}}^n, \tilde{\mathcal{H}}^n)\|_p < \infty, \forall p \geq 1$ . Hence the condition (4) is proved.

### 3.7.2 The case $\mathcal{H} = L_1^\alpha \mathcal{H}_{L^\alpha}(1)$

Let us recall that

$$L_1^n = \int_0^1 \int_{|z| \leq 1} z \tilde{\mu}^{(n)}(dt, dz) + \int_0^1 \int_{|z| > 1} z \mu^{(n)}(dt, dz)$$

and set  $\mathcal{H}^n = L_1^n \tilde{\mathcal{H}}^n$  where  $\tilde{\mathcal{H}}^n$  is given by (3.7.1). We verify the four conditions (1)-(4).

From Remark 3.6.2 i), we get  $\mathbb{E} (L_1^\alpha \mathcal{H}_{L^\alpha}(1))^2 < \infty$ . Using Lemma 3.6.2 ii) with  $b(x, \theta) = 0$  and  $\sigma = 1$  we deduce that  $L_1^n \tilde{\mathcal{H}}^n \xrightarrow[\mathbf{L}^2]{n \rightarrow \infty} L_1^\alpha \mathcal{H}_{L^\alpha}(1)$ . Hence, we just have to verify the condition (4).

Namely,

$$\sup_n \|\Gamma(L_1^n \tilde{\mathcal{H}}^n, L_1^n \tilde{\mathcal{H}}^n)\|_2 < \infty.$$

Using the Chain rule property for the operator  $\Gamma$ , we have,

$$(3.7.10) \quad \Gamma(L_1^n \tilde{\mathcal{H}}^n, L_1^n \tilde{\mathcal{H}}^n) = (L_1^n)^2 \Gamma(\tilde{\mathcal{H}}^n, \tilde{\mathcal{H}}^n) + (\tilde{\mathcal{H}}^n)^2 \Gamma(L_1^n, L_1^n) + 2 \tilde{\mathcal{H}}^n L_1^n \Gamma(L_1^n, \tilde{\mathcal{H}}^n).$$

So it is sufficient to prove that  $\sup_n \|(L_1^n)^2 \Gamma(\tilde{\mathcal{H}}^n, \tilde{\mathcal{H}}^n)\|_2 < \infty$ ,  $\sup_n \|(\tilde{\mathcal{H}}^n)^2 \Gamma(L_1^n, L_1^n)\|_2 < \infty$  and  $\sup \| \tilde{\mathcal{H}}^n L_1^n \Gamma(L_1^n, \tilde{\mathcal{H}}^n) \|_2 < \infty$ .

**Step 1** : we prove that  $\sup_n \|(L_1^n)^2 \Gamma(\tilde{\mathcal{H}}^n, \tilde{\mathcal{H}}^n)\|_p < \infty, \forall p \geq 2$ .

From (3.7.2), we have

$$(3.7.11) \quad (L_1^n)^2 \Gamma(\tilde{\mathcal{H}}^n, \tilde{\mathcal{H}}^n) \leq 2(L_1^n)^2 \Gamma(A_n, A_n) + 2(L_1^n)^2 \Gamma(B_n, B_n).$$

Using the convexity inequality, we get

$$\begin{aligned}
(L_1^n)^2 \Gamma(A_n, A_n) &= \left( \int_0^1 \int_{|z| \leq 1} z \tilde{\mu}^{(n)}(dt, dz) + \int_0^1 \int_{|z| > 1} z \mu^{(n)}(dt, dz) \right)^2 \Gamma(A_n, A_n) \\
&\leq 2 \left( \int_0^1 \int_{|z| \leq 1} z \tilde{\mu}(dt, dz) \right)^2 \Gamma(A_n, A_n) + 2 \left( \int_0^1 \int_{|z| > 1} z \mu^{(n)}(dt, dz) \right)^2 \Gamma(A_n, A_n) \\
&\leq 2 \left( \int_0^1 \int_{|z| \leq 1} z \tilde{\mu}(dt, dz) \right)^2 \Gamma(A_n, A_n) + 4 \left( \int_0^1 \int_{2 \geq |z| > 1} |z| \mu(dt, dz) \right)^2 \Gamma(A_n, A_n) + \\
&\quad + 4 \left( \int_0^1 \int_{|z| > 2} |z| \mu^{(n)}(dt, dz) \right)^2 \Gamma(A_n, A_n).
\end{aligned}$$

From **Step 1** in Section 3.7.1, we have  $\sup_n \|\Gamma(A_n, A_n)\|_p < \infty$  for all  $p \geq 1$ , it follows that  $\sup_n \left\| \left( \int_0^1 \int_{|z| \leq 1} z \tilde{\mu}(dt, dz) \right)^2 \Gamma(A_n, A_n) \right\|_p < \infty$  and  $\sup_n \left\| \left( \int_0^1 \int_{2 \geq |z| > 1} |z| \mu(dt, dz) \right)^2 \Gamma(A_n, A_n) \right\|_p < \infty$ .

Let us estimate  $\left( \int_0^1 \int_{|z| > 2} |z| \mu^{(n)}(dt, dz) \right)^2 \Gamma(A_n, A_n)$ . From (3.7.3), we have

$$(3.7.12) \quad \left( \int_0^1 \int_{|z| > 2} |z| \mu^{(n)}(dt, dz) \right)^2 \Gamma(A_n, A_n)$$

$$(3.7.13) \quad = \left( \int_0^1 \int_{|z| > 2} |z| \mu^{(n)}(dt, dz) \right)^2 \left( \frac{1}{\mu^{(n)}(\rho^n)^4} \Gamma(\mu^{(n)}(\rho^n(\rho^n)'), \mu^{(n)}(\rho^n(\rho^n)')) \right)$$

$$(3.7.14) \quad - \left( \int_0^1 \int_{|z| > 2} |z| \mu^{(n)}(dt, dz) \right)^2 \left( \frac{4\mu^{(n)}(\rho^n(\rho^n)')}{\mu^{(n)}(\rho^n)^5} \Gamma(\mu^{(n)}(\rho^n(\rho^n)'), \mu^{(n)}(\rho^n)) \right)$$

$$(3.7.15) \quad + \left( \int_0^1 \int_{|z| > 2} |z| \mu^{(n)}(dt, dz) \right)^2 \left( \frac{4\mu^{(n)}(\rho^n(\rho^n)')^2}{\mu^{(n)}(\rho^n)^6} \Gamma(\mu^{(n)}(\rho^n), \mu^{(n)}(\rho^n)) \right)$$

We consider (3.7.13), using the definition of the operator  $\Gamma$  for a test function, we have

$$\begin{aligned}
&\left| \left( \int_0^1 \int_{|z| > 2} |z| \mu^{(n)}(dt, dz) \right)^2 \left( \frac{1}{\mu^{(n)}(\rho^n)^4} \Gamma(\mu^{(n)}(\rho^n(\rho^n)'), \mu^{(n)}(\rho^n(\rho^n)')) \right) \right| \\
&\leq \left( \int_0^1 \int_{|z| > 2} |z| \mu^{(n)}(dt, dz) \right)^2 \left( \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) [\rho^n(z) |(\rho^n)''(z)| + ((\rho^n)'(z))^2] \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^4} \right) \\
&\leq \frac{\left( \int_0^1 \int_{|z| > 2} |z| \mu^{(n)}(dt, dz) \right)^2}{\int_0^1 \int_{|z| > 2} z^2 \mu^{(n)}(ds, dz)} \left( \frac{\int_0^1 \int_{|z| \leq 2} \rho(z) [\rho(z) |(\rho)''(z)| + ((\rho)'(z))^2] \mu(ds, dz)}{\left[ \int_0^1 \int_{|z| \leq 2} \rho(z) \mu(ds, dz) \right]^3} \right) \\
&\quad + \left( \int_0^1 \int_{|z| > 2} |z| \mu^{(n)}(dt, dz) \right)^2 \left( \frac{\int_0^1 \int_{|z| > 2} 36z^6 \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{|z| > 2} z^2 \mu^{(n)}(ds, dz) \right]^4} \right)
\end{aligned}$$

Using the Cauchy-Schwarz inequality 3.6.58, that is

$$\int_0^1 \int_{|z| > 2} \mu^{(n)}(dt, dz) \times \int_0^1 \int_{|z| > 2} z^2 \mu^{(n)}(ds, dz) \geq \left( \int_0^1 \int_{|z| > 2} |z| \mu^{(n)}(dt, dz) \right)^2,$$

we get

$$\begin{aligned} & \frac{\left(\int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz)\right)^2}{\int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz)} \left( \frac{\int_0^1 \int_{|z|\leq 2} \rho(z) [\rho(z) |(\rho)''(z)| + ((\rho)'(z))^2] \mu(ds, dz)}{\left[\int_0^1 \int_{|z|\leq 2} \rho(z) \mu(ds, dz)\right]^3} \right) \\ & \leq \int_0^1 \int_{|z|>2} \mu^{(n)}(dt, dz) \left( \frac{\int_0^1 \int_{|z|\leq 2} \rho(z) [\rho(z) |(\rho)''(z)| + ((\rho)'(z))^2] \mu(ds, dz)}{\left[\int_0^1 \int_{|z|\leq 2} \rho(z) \mu(ds, dz)\right]^3} \right) \\ & \leq \int_0^1 \int_{|z|>2} \mu(dt, dz) \left( \frac{\int_0^1 \int_{|z|\leq 2} \rho(z) [\rho(z) |(\rho)''(z)| + ((\rho)'(z))^2] \mu(ds, dz)}{\left[\int_0^1 \int_{|z|\leq 2} \rho(z) \mu(ds, dz)\right]^3} \right) \end{aligned}$$

and

$$\begin{aligned} & \left(\int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz)\right)^2 \left( \frac{\int_0^1 \int_{|z|>2} 36z^6 \mu^{(n)}(ds, dz)}{\left[\int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz)\right]^4} \right) \leq \left( \frac{\int_0^1 \int_{|z|>2} 36z^6 \mu^{(n)}(ds, dz)}{\left[\int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz)\right]^3} \right) \int_0^1 \int_{|z|>2} \mu^{(n)}(dt, dz) \\ & = \frac{\sum_{i=1}^{N_1} 36|Z_i|^6}{\left(\sum_{i=1}^{N_1} |Z_i|^2\right)^3} \int_0^1 \int_{|z|>2} \mu^{(n)}(dt, dz) \leq 36 \int_0^1 \int_{|z|>2} \mu(dt, dz) \end{aligned}$$

where  $N = (N_t)_{1 \geq t \geq 0}$  is a Poisson process with intensity  $\lambda_n = \int_{|z|>2} F_n(z) dz < \infty$ , and  $(Z_i)_{i \geq 0}$  are i.i.d. random variable independent of  $N$  with probability measure  $\frac{F_n(z) 1_{|z|>2} dz}{\lambda_n}$ .

From the above estimates, we can deduce that (3.7.13) is bounded by a random variable independent of  $n$  and belonging to  $\cap_{p \geq 1} \mathbf{L}^p$ .

For (3.7.14), we have

$$\begin{aligned} & (3.7.16) \\ & \left| \left(\int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz)\right)^2 \left( \frac{\mu^{(n)}(\rho^n(\rho^n)')}{\mu^{(n)}(\rho^n)^5} \Gamma(\mu^{(n)}(\rho^n(\rho^n)'), \mu^{(n)}(\rho^n)) \right) \right| \\ & \leq \left(\int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz)\right)^2 \times \frac{1}{\int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz)} \\ & \times \left| \left( \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) |(\rho^n)'(z)| \mu^{(n)}(ds, dz)}{\left[\int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz)\right]^4} \int_0^1 \int_{\mathbb{R}} \rho^n(z) (\rho^n)'(z) [((\rho^n)'(z))^2 + \rho^n(z) |(\rho^n)''(z)|] \mu^{(n)}(ds, dz) \right) \right| \\ & \leq \left( \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) |(\rho^n)'(z)| \mu^{(n)}(ds, dz)}{\left[\int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz)\right]^4} \int_0^1 \int_{\mathbb{R}} \rho^n(z) |(\rho^n)'(z)| [((\rho^n)'(z))^2 + \rho^n(z) |(\rho^n)''(z)|] \mu^{(n)}(ds, dz) \right) \times \\ & \times \int_0^1 \int_{|z|>2} \mu^{(n)}(dt, dz) \end{aligned}$$

where we have used the Cauchy-Schwarz inequality (3.6.58). Furthermore, we can estimate

$$\begin{aligned}
 (3.7.17) \quad & \left( \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) |(\rho^n)'(z)| \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^4} \int_0^1 \int_{\mathbb{R}} \rho^n(z) |(\rho^n)'(z)| [((\rho^n)'(z))^2 + \rho^n(z) |(\rho^n)''(z)|] \mu^{(n)}(ds, dz) \right) \times \\
 & \times \int_0^1 \int_{|z|>2} \mu^{(n)}(dt, dz) \\
 & \leq \left[ \frac{\int_0^1 \int_{|z|\leq 2} \rho(z) |(\rho)'(z)| \mu(ds, dz)}{\left[ \int_0^1 \int_{|z|\leq 2} \rho(z) \mu(ds, dz) \right]^{3/2}} + \frac{\int_0^1 \int_{|z|>2} 2|z|^3 \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz) \right]^{3/2}} \right] \times \\
 & \times \left[ \frac{\int_0^1 \int_{|z|\leq 2} \rho(z) |(\rho)'(z)| [((\rho)'(z))^2 + \rho(z) |(\rho)''(z)|] \mu(ds, dz)}{\left[ \int_0^1 \int_{|z|\leq 2} \rho(z) \mu(ds, dz) \right]^{5/2}} + \frac{\int_0^1 \int_{|z|>2} 20|z|^5 \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz) \right]^{5/2}} \right] \times \\
 & \times \int_0^1 \int_{|z|>2} \mu(dt, dz).
 \end{aligned}$$

We observe that from the fact that  $v^{(n)}(\{(t, z) | 0 \leq t \leq 1, |z| > 2\}) < \infty$ , we can estimate that

$$(3.7.18) \quad \frac{\int_0^1 \int_{|z|>2} |z|^3 \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz) \right]^{3/2}} = \frac{\sum_{i=1}^{N_1} |Z_i|^3}{\left( \sum_{i=1}^{N_1} |Z_i|^2 \right)^{3/2}} \leq \frac{\left( \sum_{i=1}^{N_1} |Z_i|^2 \right)^{3/2}}{\left( \sum_{i=1}^{N_1} |Z_i|^2 \right)^{3/2}} = 1$$

$$(3.7.19) \quad \frac{\int_0^1 \int_{|z|>2} |z|^5 \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz) \right]^{5/2}} = \frac{\sum_{i=1}^{N_1} |Z_i|^5}{\left( \sum_{i=1}^{N_1} |Z_i|^2 \right)^{5/2}} \leq \frac{\left( \sum_{i=1}^{N_1} |Z_i|^2 \right)^{5/2}}{\left( \sum_{i=1}^{N_1} |Z_i|^2 \right)^{5/2}} = 1$$

where  $N = (N_t)_{1 \geq t \geq 0}$  is a Poisson process with intensity  $\lambda_n = \int_{|z|>2} F_n(z) dz < \infty$ , and  $(Z_i)_{i \geq 0}$  are i.i.d. random variable independent of  $N$  with probability measure  $\frac{F_n(z) 1_{|z|>2} dz}{\lambda_n}$  and we have used the elementary inequality

$$(3.7.20) \quad \left( \sum_{i=1}^d |a_i|^r \right)^{1/r} \leq \sum_{i=1}^d |a_i| \quad \text{for } (a_1, \dots, a_d) \in \mathbb{R}, r > 1.$$

Proceeding as in the proof of (3.7.5) we can easily deduce a similar result for (3.7.14).

We consider (3.7.15), we have

$$\begin{aligned}
 (3.7.21) \quad & \left| \left( \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right)^2 \left( \frac{\mu^{(n)}(\rho^n(\rho^n)')^2}{\mu^{(n)}(\rho^n)^6} \Gamma(\mu^{(n)}(\rho^n), \mu^{(n)}(\rho^n)) \right) \right| \\
 & = \left| \left( \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right)^2 \left( \frac{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) (\rho^n)'(z) \mu^{(n)}(ds, dz) \right]^2}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^6} \int_0^1 \int_{\mathbb{R}} \rho^n(z) (\rho^n)'(z) (\rho^n)'(z) \mu^{(n)}(ds, dz) \right) \right| \\
 & \leq \left( \frac{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) |(\rho^n)'(z)| \mu^{(n)}(ds, dz) \right]^2}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^5} \int_0^1 \int_{\mathbb{R}} \rho^n(z) |(\rho^n)'(z)|^2 \mu^{(n)}(ds, dz) \right) \int_0^1 \int_{|z|>2} \mu(dt, dz)
 \end{aligned}$$

$$= \frac{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) |(\rho^n)'(z)| \mu^{(n)}(ds, dz) \right]^2}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^3} \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) |(\rho^n)'(z)|^2 \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^2} \int_0^1 \int_{|z|>2} \mu(dt, dz)$$

Proceeding as **Step 1** in the proof of Theorem 2.1 in [19], we have

$$\sup_n \left| \frac{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) (\rho^n)'(z) \mu^{(n)}(ds, dz) \right]^2}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^3} \right|^p \quad \text{and} \quad \sup_n \left| \frac{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) [(\rho^n)'(z)]^2 \mu^{(n)}(ds, dz) \right]^2}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^2} \right|^p \quad \text{are integrable } \forall p \geq 1.$$

From the above estimates, we can deduce that  $\mathbb{E} \left( \sup_n (L_1^n)^2 \Gamma(A_n, A_n) \right)^p < \infty$  for all  $p \geq 1$ .

We now consider the second term in the right-hand side of (3.7.11), we have

(3.7.22)

$$\begin{aligned} (L_1^n)^2 \Gamma(B_n, B_n) &= \left( \int_0^1 \int_{|z| \leq 1} z \tilde{\mu}^{(n)}(dt, dz) + \int_0^1 \int_{|z| > 1} z \mu^{(n)}(dt, dz) \right)^2 \Gamma(B_n, B_n) \\ &\leq 2 \left( \int_0^1 \int_{|z| \leq 1} z \tilde{\mu}(dt, dz) \right)^2 \Gamma(B_n, B_n) + 2 \left( \int_0^1 \int_{|z| > 1} z \mu^{(n)}(dt, dz) \right)^2 \Gamma(B_n, B_n) \\ &\leq 2 \left( \int_0^1 \int_{|z| \leq 1} z \tilde{\mu}(dt, dz) \right)^2 \Gamma(B_n, B_n) + 4 \left( \int_0^1 \int_{2 \geq |z| > 1} |z| \mu(dt, dz) \right)^2 \Gamma(B_n, B_n) + \\ &\quad + 4 \left( \int_0^1 \int_{|z| > 2} |z| \mu^{(n)}(dt, dz) \right)^2 \Gamma(B_n, B_n). \end{aligned}$$

From **Step 2** in Section 3.7.1, we have  $\sup_n \|\Gamma(B_n, B_n)\|_p < \infty, \forall p \geq 1$ , it follows that

$\left( \int_0^1 \int_{|z| \leq 1} z \tilde{\mu}(dt, dz) \right)^2 \Gamma(B_n, B_n)$  and  $\left( \int_0^1 \int_{2 \geq |z| > 1} |z| \mu(dt, dz) \right)^2 \Gamma(B_n, B_n)$  are bounded by a random variable independent of  $n$  and belonging to  $\cap_{p \geq 1} \mathbf{L}^p$ .

We now estimate  $\left( \int_0^1 \int_{|z| > 2} |z| \mu^{(n)}(dt, dz) \right)^2 \Gamma(B_n, B_n)$ . From (3.7.6)-(3.7.9), we have

$$(3.7.23) \quad \left| \left( \int_0^1 \int_{|z| > 2} |z| \mu^{(n)}(dt, dz) \right)^2 \Gamma(B_n, B_n) \right|$$

$$(3.7.24) \quad \leq \left| \left( \int_0^1 \int_{|z| > 2} |z| \mu^{(n)}(dt, dz) \right)^2 \left( \frac{1}{\mu^{(n)}(\rho^n)^2} \Gamma \left( \mu^{(n)} \left[ (\rho^n)' - \frac{1+\alpha}{z} \rho^n \right], \mu^{(n)} \left[ (\rho^n)' - \frac{1+\alpha}{z} \rho^n \right] \right) \right|$$

$$(3.7.25) \quad + \left| \left( \int_0^1 \int_{|z| > 2} |z| \mu^{(n)}(dt, dz) \right)^2 \left( \frac{2\mu^{(n)} \left[ (\rho^n)' - \frac{1+\alpha}{z} \rho^n \right]}{\mu^{(n)}(\rho^n)^3} \Gamma \left( \mu^{(n)} \left[ (\rho^n)' - \frac{1+\alpha}{z} \rho^n \right], \mu^{(n)}(\rho^n) \right) \right|$$

$$(3.7.26) \quad + \left| \left( \int_0^1 \int_{|z| > 2} |z| \mu^{(n)}(dt, dz) \right)^2 \left( \frac{\mu^{(n)} \left[ (\rho^n)' - \frac{1+\alpha}{z} \rho^n \right]^2}{\mu^{(n)}(\rho^n)^4} \Gamma(\mu^{(n)}(\rho^n), \mu^{(n)}(\rho^n)) \right) \right|.$$



We consider first (3.7.24).

$$\begin{aligned}
 (3.7.27) \quad & \left| \left( \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right)^2 \left( \frac{1}{\mu^{(n)}(\rho^n)^2} \Gamma \left( \mu^{(n)} \left[ (\rho^n)' - \frac{1+\alpha}{z} \rho^n \right], \mu^{(n)} \left[ (\rho^n)' - \frac{1+\alpha}{z} \rho^n \right] \right) \right| \\
 &= \left| \left( \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right)^2 \left( \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) \left[ (\rho^n)''(z) - \frac{1+\alpha}{z} (\rho^n)'(z) + \frac{1+\alpha}{z^2} \rho^n(z) \right]^2 \mu^{(n)}(ds, dz)}{\left( \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right)^2} \right) \right| \\
 &\leq \frac{\left( \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right)^2}{\int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(dt, dz)} \left( \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) \left[ |(\rho^n)''(z)| + \frac{1+\alpha}{|z|} |(\rho^n)'(z)| + \frac{1+\alpha}{z^2} \rho^n(z) \right]^2 \mu^{(n)}(ds, dz)}{\int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz)} \right) \\
 &\leq \left( \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) \left[ |(\rho^n)''(z)| + \frac{1+\alpha}{|z|} |(\rho^n)'(z)| + \frac{1+\alpha}{z^2} \rho^n(z) \right]^2 \mu^{(n)}(ds, dz)}{\int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz)} \right) \int_0^1 \int_{|z|>2} \mu^{(n)}(dt, dz)
 \end{aligned}$$

where we have used the inequality (3.6.58) to get the last line. Proceeding as in the proof of (3.7.7), we get a similar result for (3.7.24).

We consider (3.7.25), we have

$$\begin{aligned}
 & \left| \left( \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right)^2 \left( \frac{2\mu^{(n)} \left[ (\rho^n)' - \frac{1+\alpha}{z} \rho^n \right]}{\mu^{(n)}(\rho^n)^3} \Gamma \left( \mu^{(n)} \left[ (\rho^n)' - \frac{1+\alpha}{z} \rho^n \right], \mu^{(n)}(\rho^n) \right) \right| \\
 &= \left( \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right)^2 \times \\
 &\times \left| \frac{\int_0^1 \int_{\mathbb{R}} \left[ (\rho^n)'(z) - \frac{1+\alpha}{z} \rho^n(z) \right] \mu^{(n)}(ds, dz) \int_0^1 \int_{\mathbb{R}} \rho^n(z) \left[ (\rho^n)''(z) - \frac{1+\alpha}{z} (\rho^n)'(z) + \frac{1+\alpha}{z^2} \rho^n(z) \right] (\rho^n)'(z) \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^3} \right| \\
 &\leq \frac{\left( \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right)^2}{\int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(dt, dz)} \times \frac{\int_0^1 \int_{\mathbb{R}} \left[ |(\rho^n)''(z)| + \frac{1+\alpha}{|z|} |(\rho^n)'(z)| + \frac{1+\alpha}{z^2} \rho^n(z) \right] \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^{1/2}} \\
 &\times \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) \left[ |(\rho^n)''(z)| + \frac{1+\alpha}{|z|} |(\rho^n)'(z)| + \frac{1+\alpha}{z^2} \rho^n(z) \right] |(\rho^n)'(z)| \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^{3/2}} \\
 &\leq \frac{\left( \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right)^2}{\int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(dt, dz)} \times \left[ \frac{\int_0^1 \int_{|z|\leq 2} \left[ |(\rho^n)''(z)| + \frac{1+\alpha}{|z|} |(\rho^n)'(z)| + \frac{1+\alpha}{z^2} \rho^n(z) \right] \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{|z|\leq 2} \rho^n(z) \mu^{(n)}(ds, dz) \right]^{1/2}} + \frac{\int_0^1 \int_{|z|>2} (3+\alpha) |z| \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz) \right]^{1/2}} \right] \\
 &\times \left[ \frac{\int_0^1 \int_{|z|\leq 2} \rho^n(z) \left[ |(\rho^n)''(z)| + \frac{1+\alpha}{|z|} |(\rho^n)'(z)| + \frac{1+\alpha}{z^2} \rho^n(z) \right] |(\rho^n)'(z)| \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{|z|\leq 2} \rho^n(z) \mu^{(n)}(ds, dz) \right]^{3/2}} + \frac{\int_0^1 \int_{|z|>2} (10+6\alpha) |z|^3 \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(ds, dz) \right]^{3/2}} \right].
 \end{aligned}$$

Using the inequality (3.6.58), we get

$$(3.7.28) \quad \frac{\left(\int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz)\right)^2}{\int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(dt, dz)} \leq \int_0^1 \int_{|z|>2} \mu^{(n)}(dt, dz) \leq \int_0^1 \int_{|z|>2} \mu(dt, dz).$$

From (3.7.28), (3.7.18) and proceeding as in the proof of the first term of (3.7.4), we can deduce that (3.7.25) is bounded by a random variable independent of  $n$  and belonging to  $\cap_{p \geq 1} \mathbf{L}^p$ .

For (3.7.26), using (3.7.28), we have

$$(3.7.29) \quad \begin{aligned} & \left| \left( \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right)^2 \left( \frac{\mu^{(n)}\left(\left[(\rho^n)(z)' - \frac{1+\alpha}{z} \rho^n\right]\right)^2}{\mu^{(n)}(\rho^n)^4} \Gamma(\mu^{(n)}(\rho^n), \mu^{(n)}_n(\rho^n)) \right) \right| \\ &= \left| \left( \int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz) \right)^2 \left[ \frac{\left[ \int_0^1 \int_{\mathbb{R}} [(\rho^n)(z)' - \frac{1+\alpha}{z} \rho^n(z)] \mu^{(n)}(ds, dz) \right]^2}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^2} \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) (\rho^n)'(z) (\rho^n)'(z) \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^2} \right] \right| \\ &\leq \frac{\left(\int_0^1 \int_{|z|>2} |z| \mu^{(n)}(dt, dz)\right)^2}{\int_0^1 \int_{|z|>2} z^2 \mu^{(n)}(dt, dz)} \left[ \frac{\left[ \int_0^1 \int_{\mathbb{R}} \left[ |(\rho^n)(z)'| + \frac{1+\alpha}{|z|} \rho^n(z) \right] \mu^{(n)}(ds, dz) \right]^2}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]} \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) |(\rho^n)'(z)|^2 \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^2} \right] \\ &\leq \int_0^1 \int_{|z|>2} \mu(ds, dz) \left[ \frac{\left[ \int_0^1 \int_{\mathbb{R}} \left[ |(\rho^n)(z)'| + \frac{1+\alpha}{|z|} \rho^n(z) \right] \mu^{(n)}(ds, dz) \right]^2}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]} \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) |(\rho^n)'(z)|^2 \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^2} \right] \end{aligned}$$

Proceeding as **Step 1** in the proof of Theorem 2.1 in [19], we have

$$\mathbb{E} \left( \sup_n \left| \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) |(\rho^n)'(z)|^2 \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^2} \right|^p \right) < \infty$$

and with the same technique as for (3.7.7), we can deduce (3.7.26) is bounded by a random variable independent of  $n$  and belonging to  $\cap_{p \geq 1} \mathbf{L}^p$ .

From the above estimates, we get that  $\sup_n \|(L_1^n)^2 \Gamma(B_n, B_n)\|_p < \infty$  for all  $p \geq 1$ . Thus **Step 1** follows.

**Step 2** : we show that  $\sup_n \|(\tilde{\mathcal{H}}^n)^2 \Gamma(L_1^n, L_1^n)\|_p < \infty, \forall p \geq 1$ .

Note that we have

$$(3.7.30) \quad \Gamma(L_1^n, L_1^n) = \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz).$$

Combining with (3.7.1), we get

$$\begin{aligned}
& |(\tilde{\mathcal{H}}^n)^2 \Gamma(L_1^n, L_1^n)| \\
&= \left| \left( \frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z) (\rho^n)'(z) \mu^{(n)}(ds, dz)}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^2} - \frac{\int_0^1 \int_{\mathbb{R}} \left[ (\rho^n)'(z) - \frac{1+\alpha}{z} \rho^n(z) \right] \mu^{(n)}(ds, dz)}{\int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz)} \right)^2 \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right| \\
&\leq 2 \frac{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) |(\rho^n)'(z)| \mu^{(n)}(ds, dz) \right]^2}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^4} \left| \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right| \\
&+ 2 \left( \frac{\int_0^1 \int_{\mathbb{R}} \left[ (\rho^n)'(z) + \frac{1+\alpha}{|z|} \rho^n(z) \right] \mu^{(n)}(ds, dz)}{\int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz)} \right)^2 \left| \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right| \\
&= 2 \frac{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) |(\rho^n)'(z)| \mu^{(n)}(ds, dz) \right]^2}{\left[ \int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right]^3} + 2 \frac{\left( \int_0^1 \int_{\mathbb{R}} \left[ (\rho^n)'(z) + \frac{1+\alpha}{|z|} \rho^n(z) \right] \mu^{(n)}(ds, dz) \right)^2}{\int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz)}
\end{aligned}$$

Proceeding as in the proof of (3.7.29), we get the result.

**Step 3:** We prove that  $\sup \|\tilde{\mathcal{H}}^n L_1^n \Gamma(L_1^n, \tilde{\mathcal{H}}^n)\|_p < \infty, \forall p \geq 1$ .

Using the inequality

$$|\Gamma(L_1^n, \tilde{\mathcal{H}}^n)| \leq \Gamma(L_1^n, L_1^n)^{1/2} \Gamma(\tilde{\mathcal{H}}^n, \tilde{\mathcal{H}}^n)^{1/2}$$

then we obtain

$$|\tilde{\mathcal{H}}^n L_1^n \Gamma(L_1^n, \tilde{\mathcal{H}}^n)| \leq \left| \tilde{\mathcal{H}}^n \Gamma(L_1^n, L_1^n)^{1/2} \right| \left| L_1^n \Gamma(\tilde{\mathcal{H}}^n, \tilde{\mathcal{H}}^n)^{1/2} \right|$$

From the results of **Step 1**, **Step 2** and using the Cauchy-Schwarz inequality, we can deduce the

**Step 3.**

We now conclude that

$$\sup_n \|\Gamma(L_1^n, \tilde{\mathcal{H}}^n, L_1^n, \tilde{\mathcal{H}}^n)\|_p < \infty, \forall p \geq 1.$$

---

# ESTIMATION OF THE INDEX PARAMETER FOR A STOCHASTIC DIFFERENTIAL EQUATION DRIVEN BY A STABLE LÉVY PROCESS

---

This work focuses on the estimation of the Blumenthal-Gettoor index  $\alpha$  of a stochastic differential equation driven by a truncated  $\alpha$ -stable process with index  $\alpha \in (0, 2)$  given by  $X_t^\alpha = x_0 + \int_0^t b(X_s^\alpha) ds + \sigma Z_t^\alpha$  based on high frequency observations on a fixed time period. In [53], the author has succeeded to study a joint estimation  $(\theta, \sigma, \alpha)$  of a stable Lévy process  $X_t = \theta t + \sigma L_t^\alpha$ . Based on the ideas in [53], we derive consistency and rate of convergences for the estimators of the index parameter  $\alpha$ . We give some simulations to illustrate the finite-sample behaviors of our estimators.

## 4.1 Introduction

Pure-jump processes have attracted many attention recently, especially in view of applications to finance (see Schoutens [60]), network traffic (see Mikosch et al [55]) and climate dynamics (see [23]). In this chapter, we consider the following stochastic differential equation (SDE):

$$(4.1.1) \quad dX_t^\alpha = b(X_t^\alpha) dt + \sigma dZ_t^\alpha,$$

where  $Z^\alpha$  is a truncated  $\alpha$ -stable process with the index  $\alpha \in (0, 2)$ . In this chapter, we use the notation  $(Z_t^\alpha)$  instead of  $(L_t)$  to make clear the dependence on  $\alpha$ . All formal conditions for  $X$  are

given in Section 4.2. We assume that the stochastic process  $(X_t^\alpha)$  is observed at discrete time points  $\{t_i = i\Delta_n, i = 0, 1, \dots, n, \Delta_n = \frac{1}{n}\}$ , where  $\Delta_n$  is the time frequency for observation and  $n$  is the sample size. We aim to estimate the index of jump activity  $\alpha \in (0, 2)$  based on the sampling data  $(X_{t_i}^\alpha)_{i=0}^n$ .

An index of jump activity introduced in [11], when  $X$  is a Lévy process, coincides with the Blumenthal-Gettoor index defined in terms of the Lévy measure as

$$\alpha := \inf \left\{ r \in [0, 2]; \int_{|x| \leq 1} |x|^r \nu(dx) < +\infty \right\}$$

where  $\nu(dx)$  is the Lévy measure. For a stable process, the Blumenthal-Gettoor index is also the stability index of the process.

The topic of high frequency estimation for discretely observed pure-jump Lévy processes is a currently active topic. Estimation of the jump activity index from high frequency observations has also received a lot of attention in recent years since the jump activity index can be used for different purposes, especially in financial field (see for example [9],[14]). In Aït-Sahalia and Jacod [3], the authors studied Fisher's information for the parameters of a process  $X$  decomposed as the sum of two independent Lévy processes  $(X_t = \sigma W_t + \theta Y_t)$  and then they derive that if the scale parameter  $\sigma$  is known and the true value  $\alpha < 2$ , we can hope for estimators converging to  $\alpha$  at the rate  $\sqrt{n} \log(n)$ . Based on truncated power variation, [4] and [42] propose estimation in high-frequency setting when the process has the dominant component being a Brownian motion. Todorov and Tauchen [63] propose estimation of the jump activity index in pure-jump semi-martingales setting for which the rate of convergence is  $\sqrt{n}$  for  $\sqrt{2} \leq \alpha < 2$  (the case  $\alpha < \sqrt{2}$  needs an assumption of zero drift) based on adaptively chosen optimal power for which the realized power variation does not explode. Recently, Todorov [62] has considered the estimation of the jump-activity index  $\alpha \in (1, 2]$  for a locally-stable pure-jump Itô semi-martingale from high frequency data, by using the empirical characteristic function of the scaled differenced increments. He obtains the rate of convergence  $\sqrt{n}$  if  $\alpha \in (1, 2)$  and  $\sqrt{n} u_n^{-2}$  for some sequence  $u_n$  converging to zero if  $\alpha = 2$ .

Related to the topic, Masuda [53] and Brouste and Masuda [13] consider a stable Lévy process  $X_t = \theta t + \sigma L_t^\alpha$ . In [53], the Local Asymptotic Normality property is proved with the rate

$\left(\frac{1}{n^{1/2-1/\alpha}}, \sqrt{n}, \sqrt{n} \log(n)\right)$  for the estimation of  $\beta = (\theta, \sigma, \alpha)$  with the singular Fisher information matrix  $\mathcal{I}(\beta) = \begin{pmatrix} M_\alpha/\sigma^2 & 0 & 0 \\ 0 & H_\alpha/\sigma^2 & H_\alpha/(\sigma\alpha^2) \\ 0 & H_\alpha/(\sigma\alpha^2) & H_\alpha/\alpha^4 \end{pmatrix}$  where  $H_\alpha := \int (\varphi_\alpha(y) + y\varphi'_\alpha(y))^2 \varphi_\alpha(y)^{-1} dy$  and  $M_\alpha := \int (\varphi'_\alpha(y))^2 \varphi_\alpha(y)^{-1} dy$  with  $\varphi_\alpha$  the density of the centered  $\alpha$ -stable Lévy process. From the convergence in probability of the form

$$(4.1.2) \quad \frac{1}{n} \sum_{j=0}^{n-1} f\left(\frac{n^{1/\alpha}}{\sigma} (X_{t_{j+1}} - X_{t_j} - \theta \frac{1}{n})\right) \xrightarrow{n \rightarrow \infty} \int f(y) \varphi_\alpha(y) dy,$$

for suitable  $f$ , the author has constructed two estimators for the index  $\alpha$ :  $\hat{\alpha}_{\log, n}$  based on the logarithmic moments and  $\hat{\alpha}_{r, n}$  based on the lower-order fractional moments with the convergence rate  $\sqrt{n}$ .

The aim of this chapter is to adapt these estimation procedures to the equation (4.1.1) and to provide the asymptotic distributions. The main tool for establishing the statistical results is the convergences in probability (4.1.3) and the associated central limit theorem. To do this, we consider  $(\zeta_t(x))_{t \in (0, 1/n]}$  the solution of the ordinary differential equation

$$\zeta_t(x) = x + \int_0^t b(\zeta_s(x)) ds \quad t \in (0, 1/n], \quad x \in \mathbb{R}.$$

The introduction of this ordinary differential equation is convenient for dealing with any value of  $\alpha$ , however, this leads to a limitation in practice since the drift function  $b$  has to be known. Then, for suitable  $f$ , Proposition 4.4.1 in Section 4.3.1 shows that

$$(4.1.3) \quad \frac{1}{n} \sum_{j=0}^{n-1} f\left(\frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \zeta_{1/n}(X_{t_j}^\alpha)\}\right) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \int f(y) \varphi_\alpha(y) dy.$$

If we know the true value of the scale parameter, then we construct some estimators of the index parameter with the rate  $\sqrt{n} \log(n)$ .

The chapter is organized as follows. In Section 4.2, we describe our model and notations. We construct estimators of the index  $\alpha$  based on the two moment-fitting procedures (the logarithmic moments and the lower-order fractional moments) in Subsection 4.3.1. The main results about the consistency and asymptotic normality for the estimators are given in Subsection 4.3.2. In Section 4.4, we provide the law of large numbers and central limit theorems (Propositions 4.4.1 and 4.4.2) for our proposed statistics, and use them to prove the main results in Section 4.5. The

estimation results will be supported by the simulation experiments presented in Section 4.6. In Section 4.7, we construct some estimators of the index in the case  $\alpha \in (1, 2)$  where we replace  $\zeta_t(x)$  by  $x$  in the expressions of the estimators. As a consequence, we do not need to know the drift function  $b$  in practice but the rate of convergence in probability in this case is less than  $n^{1/2-1/2\alpha}$  (Proposition 4.7.1). Finally, we give the Python code used in numerical results in this chapter in Section 4.8.

## 4.2 Model and notations

In this work we aim to estimate the unknown index parameter  $\alpha \in (0, 2)$  based on the discrete time observations  $(X_{i/n}^\alpha)_{i=0, \dots, n}$  of the process  $X^\alpha$  given by

$$X_t^\alpha = x + \int_0^t b(X_s^\alpha) ds + \sigma Z_t^\alpha \quad t \in [0, 1]$$

where  $x$  is the initial condition,  $b$  is a real valued function, the parameter  $\sigma$  belongs to  $(0, \infty)$  and  $(Z_t^\alpha)_{t \in [0, 1]}$  is a pure-jump Lévy process defined on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ . We assume that the following assumptions are fulfilled.

**H<sub>0</sub>**  $b : \mathbb{R} \rightarrow \mathbb{R}$  is a function with bounded derivatives up to order two.

**H<sub>1</sub>**:  $(a_i)$  The Lévy process  $(Z_t^\alpha)_{t \in [0, 1]}$  is given by

$$Z_t^\alpha = \int_0^t \int_{[-1, 1]} z \{\bar{\mu}(ds, dz) - \bar{\nu}(ds, dz)\} + \int_0^t \int_{[-1, 1]^c} z \bar{\mu}(ds, dz)$$

where  $\bar{\mu}$  is a Poisson random measure, with compensator  $\bar{\nu}(dt, dz) = dt \times F(z) dz$  where  $F(z)$  is given on  $\mathbb{R}^*$  by  $F(z) = \frac{1}{|z|^{\alpha+1}} \tau(z)$ ,  $\alpha \in (0, 2)$ . Moreover, we assume that  $\tau$  is a non negative smooth function equal to 1 on  $[-1, 1]$ , vanishing on  $[-2, 2]^c$  such that  $0 \leq \tau \leq 1$ .

$(a_{ii})$  We assume that  $\forall p \geq 1, \int_{\mathbb{R}} \left| \frac{\tau'(u)}{\tau(u)} \right|^p \tau(u) du < \infty$ .

Under the above assumptions,  $X_t^\alpha$  admits a smooth density for  $t > 0$  (see [56]). The introduction of the truncation function  $\tau$  in the density of the Lévy measure and the assumption **H<sub>1</sub>**( $a_{ii}$ ) are technical tools to get (4.2.1) and (4.2.5), as showed in [19] and [20] respectively, which we will use later for the proof of the convergences in probability and in law in Section 4.4.

We will use the following notations. For a vector  $h \in \mathbb{R}^2$ ,  $h^T$  denotes the transpose of  $h$  and  $|h|$  denotes the euclidean norm. For a bounded function  $g : \mathbb{R} \rightarrow \mathbb{R}$ ,  $\|g\|_\infty = \sup_x |g(x)|$ . We denote

by  $X_{\frac{1}{n}}^{\alpha,x}$  the solution to (4.1.1) starting from  $X_0^\alpha = x$ , by  $\varphi_\alpha$  the density of  $L_1^\alpha$  where  $(L_t^\alpha)$  is an  $\alpha$ -stable process with Lévy measure  $\nu(dz) = \frac{1}{|z|^{\alpha+1}} 1_{z \neq 0} dz$ , by  $\varphi^n$  the density of the variable  $n^{1/\alpha} Z_{\frac{1}{n}}^\alpha$  and by  $p_{1/n}^\alpha(x, y)$  the transition density of the homogeneous Markov chain  $(X_{i/n}^\alpha)_{i=0}^n$ . We recall below some important results proved in [20] and [19]. The first result is the uniform boundedness of  $\frac{\sigma}{n^{1/\alpha}} p_{\frac{1}{n}}^\alpha(x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_{\frac{1}{n}}(x))$ , the density of  $\frac{n^{1/\alpha}}{\sigma} (X_{1/n}^{\alpha,x} - \zeta_{1/n}(x))$ , proved in [19]:

$$(4.2.1) \quad \sup_x \sup_{u \in \mathbb{R}} \sup_n \frac{1}{n^{1/\alpha}} p_{\frac{1}{n}}^\alpha(x, \frac{u\sigma}{n^{1/\alpha}} + \zeta_{\frac{1}{n}}(x)) < \infty,$$

where  $\zeta_{\frac{1}{n}}(x)$  is the solution to the ordinary differential equation

$$(4.2.2) \quad \zeta_t(x) = x + \int_0^t b(\zeta_s(x)) ds \quad t \in (0, 1/n], \quad x \in \mathbb{R}.$$

Moreover, if the drift function  $b \equiv 0$ ,  $\sigma = 1$  and  $x = 0$  then (4.2.1) yields

$$(4.2.3) \quad \sup_{u \in \mathbb{R}} \sup_n \varphi^n(u) < \infty.$$

We recall also Lemma 6.4 and Equation (6.40) in the case  $\mathcal{H}^n \mathcal{X}_K(\mathcal{H}^n) = 1$  given in [20]. For any bounded function  $h$ , there exists a constant  $C > 0$  such that

$$(4.2.4) \quad \sup_x \left| \mathbb{E} \left[ h \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \zeta_{\frac{1}{n}}(x) \right\} \right) \right] - \mathbb{E} [h(L_1^\alpha)] \right| \leq \frac{C \|h\|_\infty}{n},$$

$$(4.2.5) \quad \sup_x \left| \mathbb{E} \left[ h \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \zeta_{\frac{1}{n}}(x) \right\} \right) \right] - \mathbb{E} \left[ h \left( n^{1/\alpha} \left\{ Z_{\frac{1}{n}}^\alpha \right\} \right) \right] \right| \leq \frac{C \|h\|_\infty}{n}.$$

If the drift function  $b \equiv 0$ ,  $\sigma = 1$  and  $x = 0$  then (4.2.4) can be rewritten as

$$(4.2.6) \quad \left| \mathbb{E} \left[ h \left( n^{1/\alpha} \left\{ Z_{\frac{1}{n}}^\alpha \right\} \right) \right] - \mathbb{E} [h(L_1^\alpha)] \right| \leq \frac{C \|h\|_\infty}{n}.$$

Throughout the paper, for suitable functions  $f$ , we use the following notations

$$(4.2.7) \quad G_n(\alpha) := \frac{1}{n} \sum_{j=0}^{n-1} f \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{t_{j+1}}^\alpha - \zeta_{1/n}(X_{t_j}^\alpha) \right\} \right),$$

$$(4.2.8) \quad H_n(\alpha) = \frac{1}{n} \sum_{j=0}^{n-1} f \left( n^{1/\alpha} \left\{ Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha \right\} \right),$$

$$(4.2.9) \quad g(\alpha) := \int f(y) \varphi_\alpha(y) dy.$$



### 4.3 Construction of the estimator and main results

Let  $X^\alpha$  be given by (4.1.1). Suppose that we observe a finite sample  $(X_{i/n}^\alpha)_{i=0,\dots,n}$ .

The main goal of this section is to build an estimator using the method of moments based on the sample moment

$$\frac{1}{n} \sum_{j=0}^{n-1} f \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha) \right\} \right)$$

for suitable functions  $f$  and to prove the consistency and asymptotic normality of the estimator as  $n$  gets large. In Section 4.3.1 we construct the estimators of the index  $\alpha$  and in Section 4.3.2 the main results about the consistency and asymptotic normality for the estimators are given.

#### 4.3.1 Construction of the estimator

We construct now an estimator for the index parameter  $\alpha$  based on the ideas of Masuda [53] where the author has succeeded to estimate  $(\theta, \sigma, \alpha)$  of a stable Lévy process  $X_t = \theta t + \sigma L_t^\alpha$ . In [53], in view of the scaling property of  $L^\alpha$ ,  $\left( \frac{n^{1/\alpha}}{\sigma} (X_{t_{j+1}} - X_{t_j} - \theta \frac{1}{n}) \right)_{j=0,\dots,n-1}$  are independent and identically distributed and each random variable has as the  $\alpha$ -stable distribution. Based on the convergence in probability, for suitable  $f$ ,

$$\frac{1}{n} \sum_{i \leq n} f \left\{ \frac{n^{1/\alpha}}{\sigma} \left( X_{t_{j+1}} - X_{t_j} - \theta \frac{1}{n} \right) \right\} \xrightarrow{n \rightarrow \infty} \int f(y) \varphi_\alpha(y) dy,$$

Masuda [53] has constructed an estimator of the index  $\alpha$  with the convergence rate  $\sqrt{n}$  by studying the two moment-fitting procedures

- the logarithmic moments  $\int \{\log|y|\}^l \varphi_\alpha(y) dy$  for  $l = 1, 2$ ,
- the lower-order fractional moments  $\int |y|^{lr} \varphi_\alpha(y) dy$  for  $l = 1, 2$  and  $r \in (0, \alpha/4)$ .

In the rest of this paper, we consider the two moment-fitting procedures as in [53] for the model (4.1.1) and we propose two estimators of the index parameter  $\alpha$  which are consistent in probability and asymptotically normal.

##### 4.3.1.1 Logarithmic moments

Let  $X^\alpha$  be given by (4.1.1) and we denote by  $\alpha_0$  the true parameter value. We now construct an estimator for  $\alpha_0$  based on  $(X_{i/n}^{\alpha_0})_{i=0,\dots,n}$ .

We set

$$f(y) = (\log|y|)^l, \quad l = 1, 2.$$

The distribution of  $L_1^\alpha$  admits explicit finite logarithmic moments for any positive order (see [53]).

Here are the first two moments

$$(4.3.1) \quad \mathbb{E}(\log|L_1^\alpha|) = \mathcal{C} \left[ \frac{1}{\alpha} - 1 \right]$$

$$(4.3.2) \quad \mathbb{E}(\log|L_1^\alpha|)^2 = \frac{\pi^2}{6} \left[ \frac{1}{\alpha^2} + \frac{1}{2} \right] + \mathcal{C}^2 \left[ \frac{1}{\alpha} - 1 \right]^2, \quad \mathcal{C} \text{ the Euler constant.}$$

This yields

$$(4.3.3) \quad \text{Var}(\log|L_1^\alpha|) = \mathbb{E}(\log|L_1^\alpha|)^2 - (\mathbb{E}(\log|L_1^\alpha|))^2 = \frac{\pi^2}{6} \left[ \frac{1}{\alpha^2} + \frac{1}{2} \right].$$

Recalling (4.1.3), we observe that the method of moment consisting in solving the equation

$$(4.3.4) \quad \frac{1}{n} \sum_{j=0}^{n-1} \log \left| \frac{1}{\sigma} n^{1/\alpha_0} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right| = \mathbb{E}(\log(|L_1^\alpha|))$$

is unfeasible due to the appearance of the unknown factor  $\frac{n^{1/\alpha_0}}{\sigma}$  in the left-hand side of (4.3.4). In order to overcome this problem, we use the method of moments estimator based on the variance of the log-observations. Indeed, the variance is invariant with regards to translation. Under the log-transformed observations, the data points and the mean are shifted by the same amount that is the unknown factor  $\log \frac{n^{1/\alpha_0}}{\sigma}$ .

Let us consider equating the variance of  $\log|L_1^\alpha|$  with the corresponding sample moment, this leads to the following equation in  $\alpha$

$$(4.3.5) \quad \text{Var}(\log|L_1^\alpha|) = \frac{1}{n} \sum_{j=0}^{n-1} \log^2 \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right| - \left( \frac{1}{n} \sum_{j=0}^{n-1} \log \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right| \right)^2$$

Combining (4.3.3) and (4.3.5), we get an explicit solution  $\hat{\alpha}_{\log,n}$  as follows :

$$\begin{aligned}
 (4.3.6) \quad \hat{\alpha}_{\log,n} &= \left\{ \frac{6}{\pi^2} \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \left( \log \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right| \right)^2 - \frac{1}{n^2} \left( \sum_{j=0}^{n-1} \log \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right| \right)^2 \right\} - \frac{1}{2} \right\}^{-1/2} \\
 &= \left\{ \frac{6}{n\pi^2} \sum_{j=0}^{n-1} \left( \log \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right| - \frac{1}{n} \sum_{j=0}^{n-1} \log \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right| \right)^2 - \frac{1}{2} \right\}^{-1/2} \\
 &= \left\{ \frac{6}{n\pi^2} \sum_{j=0}^{n-1} \left( \log \left| \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right| - \frac{1}{n} \sum_{j=0}^{n-1} \log \left| \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right| \right)^2 - \frac{1}{2} \right\}^{-1/2},
 \end{aligned}$$

as soon as the square root is well defined. Here, observe that the unknown factor  $\frac{n^{1/\alpha_0}}{\sigma}$  involved in  $\frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\}$  is cancelled out in the computation of  $\hat{\alpha}_{\log,n}$ , making the quantities (4.3.6) usable.

### 4.3.1.2 Lower-order fractional moments

Now we set

$$f(y) = |y|^{lr} \quad \text{for } l = 1, 2 \quad \text{and } r \in (0, \alpha/4).$$

We need the condition  $r < \frac{\alpha}{4}$  in order that  $\mathbb{E}(|L_1^\alpha|^{2lr}) < \infty$ . In this section, we construct an estimator of the true parameter  $\alpha_0$  based on  $(X_{i/n}^{\alpha_0})_{i=0, \dots, n}$ .

The  $q$ -th absolute moments of  $L^\alpha$  exist for  $q \in (0, \alpha)$  (see [53] ) and

$$(4.3.7) \quad \mathbb{E}(|L_1^\alpha|^q) = \int |y|^q \varphi_\alpha(y) dy = \frac{2^q \Gamma\left(\frac{q+1}{2}\right) \Gamma\left(1 - \frac{q}{\alpha}\right)}{\sqrt{\pi} \Gamma\left(1 - \frac{q}{2}\right)} := C(\alpha, q),$$

where the gamma function, denoted  $\Gamma(t)$ , is defined, for  $t > 0$  by (see equation (2.42) in [5]):

$$\Gamma(t) = \int_0^\infty y^{t-1} e^{-y} dy.$$

We see that the main problem of using the method of moments for the lower-order fractional moments is similar to the case of the logarithmic moments, that means we can not estimate the true parameter value  $\alpha_0$  by  $\hat{\alpha}_{r,n}$  such that

$$(4.3.8) \quad \frac{1}{n} \sum_{j=0}^{n-1} \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right|^r = C(\hat{\alpha}_{r,n}, r),$$

because of the appearance of the unknown factors  $n^{1/\alpha_0}$  and the parameter  $\sigma$  in the formula in the left-hand side of (4.3.8).

Following Masuda [53], we observe that the factor  $n^{1/\alpha_0}$  and  $\sigma$  can be effectively cancelled out in the following case

$$(4.3.9) \quad \frac{\left(\frac{1}{n} \sum_{j=0}^{n-1} \left| \frac{n^{1/\alpha_0}}{\sigma} \left\{ X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0}) \right\} \right|^r \right)^2}{\frac{1}{n} \sum_{j=0}^{n-1} \left| \frac{n^{1/\alpha_0}}{\sigma} \left\{ X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0}) \right\} \right|^{2r}} = \frac{\left(\frac{1}{n} \sum_{j=0}^{n-1} \left| \left\{ X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0}) \right\} \right|^r \right)^2}{\frac{1}{n} \sum_{j=0}^{n-1} \left| \left\{ X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0}) \right\} \right|^{2r}}.$$

This yields that we can estimate  $\alpha_0$  by solving equation

$$(4.3.10) \quad \frac{\left(\frac{1}{n} \sum_{j=0}^{n-1} \left| \left\{ X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0}) \right\} \right|^r \right)^2}{\frac{1}{n} \sum_{j=0}^{n-1} \left| \left\{ X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0}) \right\} \right|^{2r}} = \frac{C(\hat{\alpha}_{r,n}, r)^2}{C(\hat{\alpha}_{r,n}, 2r)}.$$

From (4.3.7), we have

$$\frac{C(\hat{\alpha}_{r,n}, r)^2}{C(\hat{\alpha}_{r,n}, 2r)} = \frac{1}{\sqrt{\pi}} \left\{ \frac{\Gamma\left(\frac{r+1}{2}\right)^2 \Gamma(1-r)}{\Gamma\left(1-\frac{r}{2}\right)^2 \Gamma\left(\frac{1}{2}+r\right)} \right\} \frac{\Gamma(1-r/\hat{\alpha}_{r,n})^2}{\Gamma(1-2r/\hat{\alpha}_{r,n})},$$

then (4.3.10) can be replaced by the following equation

$$(4.3.11) \quad \frac{\left(\frac{1}{n} \sum_{j=0}^{n-1} \left| \left\{ X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0}) \right\} \right|^r \right)^2}{\frac{1}{n} \sum_{j=0}^{n-1} \left| \left\{ X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0}) \right\} \right|^{2r}} = \frac{1}{\sqrt{\pi}} \left\{ \frac{\Gamma\left(\frac{r+1}{2}\right)^2 \Gamma(1-r)}{\Gamma\left(1-\frac{r}{2}\right)^2 \Gamma\left(\frac{1}{2}+r\right)} \right\} \frac{\Gamma(1-r/\hat{\alpha}_{r,n})^2}{\Gamma(1-2r/\hat{\alpha}_{r,n})}.$$

Note that we can see here that for each  $r \in (0, \alpha/4)$ , the right-hand side of (4.3.11) is a constant multiple of the function  $g(\alpha)$  that we denote by

$$(4.3.12) \quad g(\alpha) := \frac{\Gamma(1-r/\alpha)^2}{\Gamma(1-2r/\alpha)}.$$

We observe that

$$g'(\alpha) = -\frac{2r\Gamma(1-r/\alpha)^2\phi(1-2r/\alpha)}{\alpha^2\Gamma(1-2r/\alpha)} + \frac{2r\Gamma(1-r/\alpha)^2\phi(1-r/\alpha)}{\alpha^2\Gamma(1-2r/\alpha)}$$

where  $\phi(z)$  (the digamma function) is the derivative of  $\log\Gamma(z)$  at  $z$  and is given by

$$\phi(z) = \int_0^\infty \left( \frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}} \right) dt \quad (\text{see equation (2.68) in [5]}).$$

It is easy to see that  $\phi'(z) > 0$ , furthermore, from the properties of  $\Gamma$ -function that is  $\Gamma(u) > 0$  for all  $u > 0$ , we easily obtain that  $g'(\alpha) > 0$ . This yields that the function  $g$  is strictly increasing in  $\alpha \in (4r, 2)$  and continuously differentiable. By the inverse function theorem, the function  $g$  is invertible and the inverse  $g^{-1}$  is continuously

differentiable. In the proof of Theorem 4.3.3 below, we show that the solution of the equation (4.3.11) exists with probability tending to 1 and the estimator  $\hat{\alpha}_{r,n}$  is uniquely determined as

$$(4.3.13) \quad \hat{\alpha}_{r,n} = g^{-1} \left( \frac{1 \left( \frac{1}{n} \sum_{j=0}^{n-1} \left| \left\{ X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0}) \right\} \right|^r \right)^2}{\mathcal{D} \frac{1}{n} \sum_{j=0}^{n-1} \left| \left\{ X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0}) \right\} \right|^{2r}} \right)$$

where

$$(4.3.14) \quad \mathcal{D} := \frac{1}{\sqrt{\pi}} \left\{ \frac{\Gamma\left(\frac{r+1}{2}\right)^2 \Gamma(1-r)}{\Gamma\left(1-\frac{r}{2}\right)^2 \Gamma\left(\frac{1}{2}+r\right)} \right\}.$$

### 4.3.2 Main results

The following theorems give the consistency and asymptotic normality results for the estimators  $\hat{\alpha}_{\log,n}$  and  $\hat{\alpha}_{r,n}$ .

**Theorem 4.3.1** (Consistency). *The estimator  $\hat{\alpha}_{\log,n}$  defined by (4.3.6) is consistent in probability:*

$$\hat{\alpha}_{\log,n} \xrightarrow{\mathbb{P}} \alpha_0, \quad n \rightarrow \infty.$$

**Theorem 4.3.2** (Asymptotic normality). *With the estimator  $\hat{\alpha}_{\log,n}$  defined by (4.3.6), we have*

$$(4.3.15) \quad \sqrt{n}(\hat{\alpha}_{\log,n} - \alpha_0) \Rightarrow \mathcal{N}(0, V^{\log}(\alpha_0))$$

where  $V^{\log}(\alpha_0) = \frac{11}{10}\alpha_0^2 + \frac{1}{2}\alpha_0^4 + \frac{13}{20}\alpha_0^6$ .

**Remark 4.3.1.** *Following [53] via the logarithmic moment fitting, if we know the true value of the volatility parameter  $\sigma$  then we can construct an estimator of  $\alpha$  with the rate  $\sqrt{n} \log(n)$ . In fact, we set  $S_n := \frac{1}{n} \sum_{j=0}^{n-1} \log \left| \left\{ X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0}) \right\} \right|$  and recall from (4.3.1) that  $\mathbb{E}(\log |L_1^{\alpha_0}|) = \mathcal{C} \left[ \frac{1}{\alpha_0} - 1 \right]$  then we can compute that*

$$\begin{aligned} & \sqrt{n} \left( \frac{1}{n} \sum_{j=0}^{n-1} \log \left| \frac{n^{1/\alpha_0}}{\sigma} \left\{ X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0}) \right\} \right| - \mathbb{E}(\log |L_1^{\alpha_0}|) \right) \\ &= \frac{\sqrt{n} \log n}{\alpha_0} - \sqrt{n} \log \sigma + \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \log \left| \left\{ X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0}) \right\} \right| - \sqrt{n} \mathbb{E}(\log |L_1^{\alpha_0}|) \\ &= \sqrt{n}(\log n) \left[ \frac{1}{\alpha_0} \left( 1 - \frac{\mathcal{C}}{\log n} \right) - \frac{\log \sigma}{\log n} + \frac{S_n}{\log n} + \frac{\mathcal{C}}{\log n} \right] \\ &= \sqrt{n}(\log n) \left( 1 - \frac{\mathcal{C}}{\log n} \right) \left( \frac{1}{\alpha_0} + \frac{S_n + \mathcal{C} - \log \sigma}{\log n - \mathcal{C}} \right) \sim^{\mathbb{P}} \sqrt{n}(\log n) \left( \frac{1}{\alpha_0} + \frac{S_n + \mathcal{C} - \log \sigma}{\log n - \mathcal{C}} \right). \end{aligned}$$

Combining this with (4.5.5), we have

$$\sqrt{n}(\log n) \left( \frac{1}{\alpha_0} + \frac{S_n + \mathcal{C} - \log \sigma}{\log n - \mathcal{C}} \right) \Rightarrow \mathcal{N}(0, v_{2, \alpha_0}).$$

where  $v_{2, \alpha_0} = \mathbb{E} \left[ \left( \log |L_1^{\alpha_0}| - \mathbb{E} [\log |L_1^{\alpha_0}|] \right)^2 \right] = \frac{\pi^2}{6} \left( \frac{1}{\alpha_0^2} + \frac{1}{2} \right)$ .

Therefore, combining with the delta method, it follows that

$$(4.3.16) \quad \tilde{\alpha}_n = \left( -\frac{S_n + \mathcal{C} - \log \sigma}{\log n - \mathcal{C}} \right)^{-1}$$

is an asymptotically normal estimator with the rate  $\sqrt{n} \log(n)$ . As expected, the estimator  $\tilde{\alpha}_n$  exhibits excellent finite-sample performance; see Tables 1 and 2 in Section 4.6.

**Theorem 4.3.3** (Consistency). *The estimator  $\hat{\alpha}_{r,n}$ , the solution of (4.3.11) exists with probability tending to 1 and it is consistent:*

$$\hat{\alpha}_{r,n} \xrightarrow{\mathbb{P}} \alpha_0 \quad n \rightarrow \infty.$$

**Theorem 4.3.4** (Asymptotic normality). *We have*

$$(4.3.17) \quad \sqrt{n}(\hat{\alpha}_{r,n} - \alpha_0) \Rightarrow \mathcal{N}(0, V^r(\alpha_0))$$

where  $V^r(\alpha_0) = \frac{\alpha_0^4}{r^2 \eta_r(\alpha_0)^2} \left\{ \frac{C(\alpha_0, 2r)}{C(\alpha_0, r)^2} - \frac{C(\alpha_0, 3r)}{C(\alpha_0, r)C(\alpha_0, 2r)} + \frac{1}{4} \left( \frac{C(\alpha_0, 4r)}{C(\alpha_0, 2r)^2} - 1 \right) \right\}$

where

$$(4.3.18) \quad \eta_r(\alpha_0) = \phi(1 - r/\alpha_0) - \phi(1 - 2r/\alpha_0)$$

with  $\phi(z)$  (the digamma function) is the derivative of  $\log \Gamma(z)$  at  $z$  (see equation (2.68) in [5]).

If we know the true value of the volatility parameter  $\sigma$  then we can construct an estimator of  $\alpha$ , via the lower-order fractional moment fitting, with the rate  $\sqrt{n} \log(n)$ .

**Proposition 4.3.1.** *Let  $p \in (0, \alpha_0/2)$  and define  $\alpha^*(p)$  the estimator of the index  $\alpha$  given by*

$$(4.3.19) \quad \alpha^*(p) = \frac{p}{M_n^{-1}(S_n)},$$

where  $M_n(u), S_n$  are given by

$$M_n(u) = -u + \frac{\log(\Gamma(1-u))}{\log(n)} \text{ for } u \in \left[ 0, \frac{1}{2} \right]$$

$$S_n = \frac{\log Q_n(p)}{\log(n)} - \log \left( \frac{2^p \Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi} \Gamma\left(1 - \frac{p}{2}\right)} \right) \frac{1}{\log(n)} - \frac{p \log(\sigma)}{\log(n)}$$

with

$$Q_n(p) := \frac{1}{n} \sum_{j=0}^{n-1} \left| \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right|^p.$$

Then we have the convergence in law

$$\sqrt{n} \log(n) (\alpha^*(p) - \alpha_0) \Rightarrow \mathcal{N} \left( 0, \frac{\alpha_0^4}{p^2} \left( \frac{\mathcal{C}(\alpha_0, 2p)}{\mathcal{C}^2(\alpha_0, p)} - 1 \right) \right).$$

**Proof.** We set

$$(4.3.20) \quad Z_n(p) = \sqrt{n} \left( \frac{n^{\frac{p}{\alpha_0}}}{\sigma^p} Q_n(p) - \mathcal{C}(\alpha_0, p) \right)$$

where  $\mathcal{C}(\alpha_0, p) = \mathbb{E}(|L_1^{\alpha_0}|^p) = \frac{2^p \Gamma\left(\frac{p+1}{2}\right) \Gamma\left(1 - \frac{p}{\alpha_0}\right)}{\sqrt{\pi} \Gamma\left(1 - \frac{p}{2}\right)}$ .

From (4.5.8), we have

$$(4.3.21) \quad Z_n(p) \Rightarrow \mathcal{N}(0, \mathcal{C}(\alpha_0, 2p) - \mathcal{C}^2(\alpha_0, p)).$$

We observe that, from (4.3.20), we can write

$$\begin{aligned} \log Q_n(p) &= \log \left[ \sigma^p n^{-\frac{p}{\alpha_0}} \left( \frac{Z_n(p)}{\sqrt{n}} + \mathcal{C}(\alpha_0, p) \right) \right] = \log \left[ \sigma^p n^{-\frac{p}{\alpha_0}} \mathcal{C}(\alpha_0, p) \left( 1 + \frac{Z_n(p)}{\mathcal{C}(\alpha_0, p) \sqrt{n}} \right) \right] \\ &= p \log(\sigma) - \frac{p}{\alpha_0} \log(n) + \log(\mathcal{C}(\alpha_0, p)) + \log \left( 1 + \frac{Z_n(p)}{\mathcal{C}(\alpha_0, p) \sqrt{n}} \right) \\ &= p \log(\sigma) - \frac{p}{\alpha_0} \log(n) + \log \left( \frac{2^p \Gamma\left(\frac{p+1}{2}\right) \Gamma\left(1 - \frac{p}{\alpha_0}\right)}{\sqrt{\pi} \Gamma\left(1 - \frac{p}{2}\right)} \right) + \log \left( 1 + \frac{Z_n(p)}{\mathcal{C}(\alpha_0, p) \sqrt{n}} \right) \end{aligned}$$

It follows that

$$(4.3.22) \quad \frac{\log Q_n(p)}{\log(n)} - \log \left( \frac{2^p \Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi} \Gamma\left(1 - \frac{p}{2}\right)} \right) \frac{1}{\log(n)} - \frac{p \log(\sigma)}{\log(n)} = -\frac{p}{\alpha_0} + \frac{\log \left( \Gamma\left(1 - \frac{p}{\alpha_0}\right) \right)}{\log(n)} + \log \left( 1 + \frac{Z_n(p)}{\mathcal{C}(\alpha_0, p) \sqrt{n}} \right) \frac{1}{\log(n)}.$$

We observe that  $M'_n(u) = -1 - \frac{\phi(1-u)}{\log(n)}$  where  $\phi(z)$  (the digamma function) is the derivative of  $\log \Gamma(z)$  at  $z$  and is given by  $\phi(z) = \int_0^\infty \left( \frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}} \right) dt$  (see equation (2.68) in [5]). For  $n$  large enough, we get that  $M'_n(u) < 0$  for  $u \in [0, 1/2]$ . This yields that the function  $M_n$  is a continuous strictly decreasing function on  $[0, 1/2]$ . By the inverse function theorem, the function  $M_n$  is

invertible and the inverse  $M_n^{-1}$  is continuously differentiable. Moreover, from (4.3.22), we can easily deduce that

$$(4.3.23) \quad S_n \xrightarrow[\mathbb{P}]{n \rightarrow \infty} \frac{-p}{\alpha_0}.$$

Then, let  $\Omega_n^*$  be the set where  $S_n$  is in the range of the function  $M_n$ , we get that  $\mathbb{P}(\Omega_n^*)$  tends to one as  $n \rightarrow \infty$  by the convergence in probability (4.3.23). Hence, in this case  $M_n^{-1}(S_n)$  exists and since the function  $M_n$  is strictly decreasing then the estimator  $\alpha^*(p)$  is uniquely determined by (4.3.19).

We now observe that, from (4.3.22), we have

$$(4.3.24) \quad S_n - M_n\left(\frac{p}{\alpha_0}\right) = \log\left(1 + \frac{Z_n(p)}{\mathcal{C}(\alpha_0, p)\sqrt{n}}\right) \frac{1}{\log(n)}.$$

From (4.3.21), we can deduce that

$$S_n - M_n\left(\frac{p}{\alpha_0}\right) = \frac{Z_n(p)}{\mathcal{C}(\alpha_0, p)\sqrt{n}} \frac{1}{\log(n)} + o_p\left(\frac{1}{\sqrt{n} \log(n)}\right).$$

It follows

$$(4.3.25) \quad \sqrt{n} \log(n) \left\{ S_n - M_n\left(\frac{p}{\alpha_0}\right) \right\} \Rightarrow \mathcal{N}\left(0, \frac{\mathcal{C}(\alpha_0, 2p)}{\mathcal{C}^2(\alpha_0, p)} - 1\right).$$

We observe that

$$M_n^{-1}(S_n) - M_n^{-1}\left(M_n\left(\frac{p}{\alpha_0}\right)\right) = \left(S_n - M_n\left(\frac{p}{\alpha_0}\right)\right) (M_n^{-1})'\left(M_n\left(\frac{p}{\alpha_0}\right)\right) + o_p\left(S_n - M_n\left(\frac{p}{\alpha_0}\right)\right)$$

where  $o_p(R_n) = Y_n R_n$  with  $Y_n \xrightarrow{\mathbb{P}} 0$ . Multiplying both sides by  $\sqrt{n} \log(n)$ , we obtain

$$\begin{aligned} & \sqrt{n} \log(n) \left( M_n^{-1}(S_n) - M_n^{-1}\left(M_n\left(\frac{p}{\alpha_0}\right)\right) \right) \\ &= \sqrt{n} \log(n) \left( S_n - M_n\left(\frac{p}{\alpha_0}\right) \right) (M_n^{-1})'\left(M_n\left(\frac{p}{\alpha_0}\right)\right) + \sqrt{n} \log(n) o_p\left(S_n - M_n\left(\frac{p}{\alpha_0}\right)\right) \end{aligned}$$

where  $(M_n^{-1})'\left(M_n\left(\frac{p}{\alpha_0}\right)\right) = \left(M_n'\left(\frac{p}{\alpha_0}\right)\right)^{-1}$  (the inverse function theorem).

Since  $\sup_u |M_n'(u) + 1| \xrightarrow{n \rightarrow \infty} 0$  then we can deduce that  $\left(M_n'\left(\frac{p}{\alpha_0}\right)\right)^{-1} \xrightarrow{n \rightarrow \infty} -1$ . Combining this with (4.3.25), we obtain that

$$\sqrt{n} \log(n) \left( S_n - M_n\left(\frac{p}{\alpha_0}\right) \right) (M_n^{-1})'\left(M_n\left(\frac{p}{\alpha_0}\right)\right) \Rightarrow \mathcal{N}\left(0, \frac{\mathcal{C}(\alpha_0, 2p)}{\mathcal{C}^2(\alpha_0, p)} - 1\right).$$



Moreover, since  $\sqrt{n} \log(n) \left( S_n - M_n \left( \frac{p}{\alpha_0} \right) \right)$  converges in law by (4.3.25) then we can deduce that  $\sqrt{n} \log(n) o_p \left( S_n - M_n \left( \frac{p}{\alpha_0} \right) \right) = o_p(1)$ . Hence, from the above estimates, we can conclude that

$$\sqrt{n} \log(n) \left( M_n^{-1}(S_n) - \frac{p}{\alpha_0} \right) \Rightarrow \mathcal{N} \left( 0, \frac{\mathcal{C}(\alpha_0, 2p)}{\mathcal{C}^2(\alpha_0, p)} - 1 \right).$$

Therefore, combining with Delta method, we get that

$$\sqrt{n} \log(n) (\alpha^*(p) - \alpha_0) \Rightarrow \mathcal{N} \left( 0, \frac{\alpha_0^4}{p^2} \left( \frac{\mathcal{C}(\alpha_0, 2p)}{\mathcal{C}^2(\alpha_0, p)} - 1 \right) \right).$$

then

$$\alpha^*(p) = \frac{p}{M_n^{-1}(S_n)}$$

is an asymptotically normal estimator with the rate  $\sqrt{n} \log(n)$ . ■

As expected, the estimator  $\alpha^*(p)$  exhibits excellent finite-sample performance; see Tables 1 in Section 4.6.

The remainder of the paper is devoted to the proofs of the main theorems above. The main statistical tools for establishing the consistency and asymptotic normality of the estimator are the convergence in probability of  $\frac{1}{n} \sum_{j=0}^{n-1} f \left( \frac{n^{1/\alpha}}{\sigma} \{ X_{t_{j+1}}^\alpha - \varsigma_{1/n} (X_{t_j}^\alpha) \} \right)$  and the convergence in law associated, for suitable functions  $f$ . For this reason, in the next section, we will study Law of Large Numbers and Central Limit Theorem which are the two principal tools for the proofs of theorems in Section 4.3.2.

## 4.4 Limit theorems

### 4.4.1 Law of Large Numbers

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be an even function and we suppose that the function  $f$  satisfies the following assumptions

**Assumption 4.4.1.**

1.  $\exists C > 0$  such that  $|f(y)| \leq C (|\log |y||^4 + 1)$  for all  $|y| \leq 1$ .
2. There exist  $0 < p < \alpha$ ,  $C > 0$  such that  $|f(y)| \leq C |y|^p$  for all  $|y| > 1$  where  $C$  is a positive constant.

**Proposition 4.4.1.** *Under Assumption 4.4.1, we have*

$$\frac{1}{n} \sum_{j=0}^{n-1} f \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha) \right\} \right) \xrightarrow{\mathbb{P}} \int f(y) \varphi_\alpha(y) dy.$$

To prove this proposition, we first prove Lemmas 4.4.1 and 4.4.2 below. From now on, we will use the following notations in the proofs. We denote by  $f_K(x) := f(x)\mathcal{X}_K(x)$  where  $x \mapsto \mathcal{X}_K(x)$  for all  $K > 2$  is a smooth truncation function with

$$(4.4.1) \quad \begin{cases} \mathcal{X}_K(x) = 0 & \text{for } \{|x| < \frac{1}{2K^2}\} \cup \{|x| > 2K^{1/p}\} \\ \mathcal{X}_K(x) = 1 & \text{for } \frac{1}{K^2} \leq |x| \leq K^{1/p} \\ 0 \leq \mathcal{X}_K(x) \leq 1 & \text{for } \{\frac{1}{2K^2} \leq |x| \leq \frac{1}{K^2}\} \cup \{K^{1/p} \leq |x| \leq 2K^{1/p}\}. \end{cases}$$

Moreover, the derivative of function  $\mathcal{X}_K$  with respect to the variable  $x$  satisfies

$$(4.4.2) \quad \begin{cases} |\mathcal{X}'_K(x)| \leq CK^2 & \text{for } \frac{1}{2K^2} \leq |x| \leq \frac{1}{K^2} \\ |\mathcal{X}'_K(x)| \leq C & \text{for } K^{1/p} \leq |x| \leq 2K^{1/p} \end{cases}$$

where  $C$  is a positive constant independent of  $K$ . Note that we will not use this condition for the proof of the convergence in probability in Proposition 4.4.1, however it is a useful condition for the convergence in law in Proposition 4.4.2.

We set

$$(4.4.3) \quad G_{n,K}(\alpha) = \frac{1}{n} \sum_{j=0}^{n-1} f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha) \right\} \right)$$

$$(4.4.4) \quad H_{n,K}(\alpha) = \frac{1}{n} \sum_{j=0}^{n-1} f_K \left( n^{1/\alpha} \left\{ Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha \right\} \right)$$

$$(4.4.5) \quad g_K(\alpha) = \int f_K(y) \varphi_\alpha(y) dy.$$

**Lemma 4.4.1.** *Under the notations (4.2.7), (4.4.3), (4.2.9) and (4.4.5) we have*

$$(1) \sup_n \mathbb{E}(|G_n(\alpha) - G_{n,K}(\alpha)|) \leq c_K \quad \text{where } c_K \text{ tends to zero as } K \rightarrow \infty.$$

$$(2) |g(\alpha) - g_K(\alpha)| \xrightarrow{K \rightarrow \infty} 0.$$

**Proof of Lemma 4.4.1** We start with the proof of the part (1). We recall that

$$\begin{aligned} & \sup_n \mathbb{E} (|G_n(\alpha) - G_{n,K}(\alpha)|) \\ &= \sup_n \mathbb{E} \left| \left[ \frac{1}{n} \sum_{j=0}^{n-1} f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) - \frac{1}{n} \sum_{j=0}^{n-1} f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \right] \right|. \end{aligned}$$

Our proof starts with the observation that

$$(4.4.6) \quad |f(x) - f_K(x)| \leq |f(x)| \mathbf{1}_{|x| \geq K^{1/p}} + |f(x)| \mathbf{1}_{|x| \leq \frac{1}{K^2}}$$

then we have

$$\begin{aligned} & \sup_n \mathbb{E} \left| \left[ \frac{1}{n} \sum_{j=0}^{n-1} f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) - \frac{1}{n} \sum_{j=0}^{n-1} f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \right] \right| \\ (4.4.7) \quad & \leq \sup_n \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{E} \left| f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \right| \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right| \geq K^{1/p}} \\ & + \sup_n \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{E} \left| f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \right| \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right| \leq \frac{1}{K^2}}. \end{aligned}$$

Let us estimate  $\mathbb{E} \left| f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \right| \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right| \geq K^{1/p}}$ .

We observe that for all  $t \in [t_j, t_{j+1}]$

$$\begin{aligned} \left| X_t^\alpha - \varsigma_{t-t_j}(X_{t_j}^\alpha) \right| &= \left| \int_{t_j}^t b(X_s^\alpha) ds - \int_{t_j}^t b(\varsigma_{s-t_j}(X_{t_j}^\alpha)) ds + \sigma \{Z_t^\alpha - Z_{t_j}^\alpha\} \right| \\ &\leq \left| \int_{t_j}^t [b(X_s^\alpha) - b(\varsigma_{s-t_j}(X_{t_j}^\alpha))] ds \right| + \sigma \sup_{t_j \leq t \leq t_{j+1}} |Z_t^\alpha - Z_{t_j}^\alpha| \\ &\leq \|b'\|_\infty \int_{t_j}^t |X_s^\alpha - \varsigma_{s-t_j}(X_{t_j}^\alpha)| ds + \sigma \sup_{t_j \leq t \leq t_{j+1}} |Z_t^\alpha - Z_{t_j}^\alpha|. \end{aligned}$$

Using the Gronwall's inequality, for  $C$  a positive constant, for all  $t \in [t_j, t_{j+1}]$ , we have

$$\left| X_t^\alpha - \varsigma_{t-t_j}(X_{t_j}^\alpha) \right| \leq C \sigma \sup_{t_j \leq t \leq t_{j+1}} |Z_t^\alpha - Z_{t_j}^\alpha|.$$

It follows that

$$(4.4.8) \quad \left| X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha) \right| \leq C \sigma \sup_{t_j \leq t \leq t_{j+1}} |Z_t^\alpha - Z_{t_j}^\alpha|.$$

Using Assumption 4.4.1(2) and combining with (4.4.8), we get

$$\begin{aligned}
 (4.4.9) \quad & \sup_n \mathbb{E} \left( \left| f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \right| \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right| > K^{1/p}} \right) \\
 & \leq \sup_n \mathbb{E} \left( \left| \frac{n^{1/\alpha}}{\sigma} (X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)) \right|^p \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right| > K} \right) \\
 & \leq C \sup_n \mathbb{E} \left( n^{p/\alpha} \sup_{t_j \leq t \leq t_{j+1}} |Z_t^\alpha - Z_{t_j}^\alpha|^p \mathbf{1}_{n^{p/\alpha} \sup_{t_j \leq t \leq t_{j+1}} |Z_t^\alpha - Z_{t_j}^\alpha|^p > K/C^p} \right) \\
 & \leq C \sup_n \mathbb{E} \left( n^{p/\alpha} \sup_{0 \leq s \leq \frac{1}{n}} |Z_s^\alpha|^p \mathbf{1}_{n^{p/\alpha} \sup_{0 \leq s \leq 1/n} |Z_s^\alpha|^p > K/C^p} \right)
 \end{aligned}$$

where we use the fact that  $Z^\alpha$  has stationary increments to get the last line. From Luschgy and Pagès [52], we have for all  $r \in (0, \alpha)$

$$(4.4.10) \quad \mathbb{E} \left( \sup_{s \leq \frac{1}{n}} |Z_s^\alpha|^r \right) \leq \frac{C}{n^{r/\alpha}}.$$

where  $C$  is a constant independent of  $n$ . Indeed, the assumptions in Theorem 2 in [52] are satisfied since  $(Z_t^\alpha)_{t \in [0,1]}$  is a Lévy process with characteristics  $(0, 0, \nu)$  such that  $\mathbb{E}|Z_1^\alpha|^r < +\infty$  for any  $r \in (0, \alpha)$  and the Lévy measure of the process  $(Z_s^\alpha)$  (recall Assumption  $\mathbf{H}_1(a_i)$ ) is bounded by  $\bar{\varphi}(z) := \frac{1}{|z|^{1+\alpha}}$  which is a regularly varying function at zero of index  $-(1+\alpha)$ . Using Theorem 2 in [52] with  $l(z) := z^{\alpha+1}\bar{\varphi}(z) = 1$ , we obtain (4.4.10). As a consequence, recalling that  $p \in (0, \alpha)$ , there exists  $\delta > 0$  such that  $p(1+\delta) < \alpha$  then

$$(4.4.11) \quad \sup_n \mathbb{E} \left( \left[ n^{p/\alpha} \sup_{0 \leq s \leq \frac{1}{n}} |Z_s^\alpha|^p \right]^{1+\delta} \right) \leq C.$$

It follows that  $(n^{p/\alpha} \sup_{0 \leq s \leq 1/n} |Z_s^\alpha|^p)_n$  is a sequence of uniformly integrable random variables and we can conclude that (4.4.9) converges to zero as  $K \rightarrow \infty$ .

We now consider  $\mathbb{E} \left| f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \right| \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right| \leq \frac{1}{K^2}}$ .

We recall that  $p_{\frac{1}{n}}^\alpha(x, y)$  is the transition density of the homogeneous Markov chain  $(X_{i/n}^\alpha)_{i=0, \dots, n}$  then we can deduce that the density of  $\frac{n^{1/\alpha}}{\sigma} \{X_{1/n}^{\alpha, x} - \varsigma_{1/n}(x)\}$  is  $\frac{\sigma}{n^{1/\alpha}} P_{\frac{1}{n}}^\alpha \left( x, \frac{u\sigma}{n^{1/\alpha}} + \varsigma_{\frac{1}{n}}(x) \right)$ .

We observe that

$$\begin{aligned}
 (4.4.12) \quad & \mathbb{E} \left[ \left| f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \right| \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right| \leq \frac{1}{K^2}} \right] \\
 &= \mathbb{E} \left[ \mathbb{E} \left[ \left| f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \right| \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right| \leq \frac{1}{K^2}} \middle| \mathcal{F}_{t_j} \right] \right] \\
 &= \mathbb{E} \left[ \mathbb{E} \left[ \left| f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \right| \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right| \leq \frac{1}{K^2}} \middle| X_{t_j}^\alpha \right] \right]
 \end{aligned}$$

where we have used the Markov property to get the last line. Furthermore, we can write that

$$\begin{aligned}
 (4.4.13) \quad & \mathbb{E} \left[ \left| f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \right| \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right| \leq \frac{1}{K^2}} \middle| X_{t_j}^\alpha = x \right] \\
 &= \int_{-\frac{1}{K^2}}^{\frac{1}{K^2}} |f(u)| \left[ \frac{\sigma}{n^{1/\alpha}} P_{\frac{1}{n}}^\alpha \left( x, \frac{u\sigma}{n^{1/\alpha}} + \varsigma_{\frac{1}{n}}(x) \right) \right] du \\
 &= 2 \int_0^{\frac{1}{K^2}} |f(u)| \left[ \frac{\sigma}{n^{1/\alpha}} P_{\frac{1}{n}}^\alpha \left( x, \frac{u\sigma}{n^{1/\alpha}} + \varsigma_{\frac{1}{n}}(x) \right) \right] du \leq 2C \int_0^{\frac{1}{K^2}} |f(u)| du
 \end{aligned}$$

where we have used (4.2.1) and the fact that  $f$  is a symmetric function. The result follows from Assumption 4.4.1(1) and the dominated convergence theorem.

We prove the part (2). We need to show that

$$(4.4.14) \quad |g(\alpha) - g_K(\alpha)| = |\mathbb{E}(f(L_1^\alpha)) - \mathbb{E}(f_K(L_1^\alpha))| \xrightarrow{K \rightarrow \infty} 0.$$

From (4.4.6), we can estimate

$$(4.4.15) \quad |\mathbb{E}(f(L_1^\alpha)) - \mathbb{E}(f_K(L_1^\alpha))| \leq \mathbb{E}|f(L_1^\alpha)| \mathbf{1}_{|L_1^\alpha| \geq K^{1/p}} + \mathbb{E}|f(L_1^\alpha)| \mathbf{1}_{|L_1^\alpha| \leq \frac{1}{K^2}}.$$

From Assumptions 4.4.1(1)(2) we obtain  $\mathbb{E}(|f(L_1^\alpha)|) < \infty$ , using the dominated convergence theorem, we can deduce that the right-hand side of (4.4.15) tends to zero as  $K \rightarrow \infty$ . Hence the result follows. ■

**Lemma 4.4.2.** *Under the notations (4.4.5) and (4.4.3) and for all  $K > 2$ , we have*

$$|G_{n,K}(\alpha) - g_K(\alpha)| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

**Proof of Lemma 4.4.2** We have

$$|G_{n,K}(\alpha) - g_K(\alpha)| \leq |G_{n,K}(\alpha) - H_{n,K}(\alpha)| + |H_{n,K}(\alpha) - g_K(\alpha)|.$$

In order to prove Lemma 4.4.2, we need to check the two following properties for all  $K > 2$ :

$$(1) |G_{n,K}(\alpha) - H_{n,K}(\alpha)| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$$

$$(2) |H_{n,K}(\alpha) - g_K(\alpha)| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$$

**We start with the proof of the property (1).** We need to prove that

$$|G_{n,K}(\alpha) - H_{n,K}(\alpha)| = \left| \frac{1}{n} \sum_{j=0}^{n-1} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) - f_K \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \right] \right| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

We will use Lemma 9 from Genon-Catalot and Jacod [25] to show the property (1). It is sufficient to show that

$$(4.4.16) \quad \frac{1}{n} \sum_{j=0}^{n-1} \left| \mathbb{E} \left[ \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) - f_K \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \right] \middle| \mathcal{F}_{t_j} \right] \right| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0,$$

$$(4.4.17) \quad \frac{1}{n^2} \sum_{j=0}^{n-1} \mathbb{E} \left[ \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) - f_K \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \right]^2 \middle| \mathcal{F}_{t_j} \right] \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

Since  $Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha$  is independent of  $\mathcal{F}_{t_j}$  and  $Z^\alpha$  has stationary increments then we get

$$(4.4.18) \quad \mathbb{E} \left[ f_K \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \middle| \mathcal{F}_{t_j} \right] = \mathbb{E} \left[ f_K \left( n^{1/\alpha} \{Z_{\frac{1}{n}}^\alpha\} \right) \right].$$

From the Markov property, we have

$$(4.4.19) \quad \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \middle| \mathcal{F}_{t_j} \right] = \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \middle| X_{t_j}^\alpha \right].$$

But we can write

$$(4.4.20) \quad \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \middle| X_{t_j}^\alpha = x \right] = \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{\frac{1}{n}}^{\alpha,x} - \varsigma_{\frac{1}{n}}(x)\} \right) \right]$$

where  $X_{\frac{1}{n}}^{\alpha,x}$  is the solution to (4.1.1) starting from  $X_0^\alpha = x$ .

From the construction of the function  $f_K$  and Assumption 4.4.1(1)(2), we can deduce that

$$(4.4.21) \quad |f_K| \leq CK.$$

Using (4.2.5) with the bounded function  $f_K$ , there exists a constant  $C > 0$  such that

$$(4.4.22) \quad \sup_x \left| \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{\frac{1}{n}}^{\alpha,x} - \varsigma_{\frac{1}{n}}(x)\} \right) \right] - \mathbb{E} \left[ f_K \left( n^{1/\alpha} \{Z_{\frac{1}{n}}^\alpha\} \right) \right] \right| \leq \frac{CK}{n} \xrightarrow[n \rightarrow \infty]{} 0$$

Combining (4.4.22) with (4.4.18) and (4.4.20), we deduce that

$$(4.4.23) \quad \left| \mathbb{E} \left[ \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) - f_K \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \right] \middle| \mathcal{F}_{t_j} \right] \right| \leq \frac{CK}{n} \xrightarrow[n \rightarrow \infty]{} 0$$

and then (4.4.16) follows.

We now prove (4.4.17). From the boundedness of  $f_K$  we get that

$$\frac{1}{n^2} \sum_{j=0}^{n-1} \left| \mathbb{E} \left[ \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) - f_K \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \right]^2 \middle| \mathcal{F}_{t_j} \right] \right| \leq \frac{4CK^2}{n}.$$

Letting  $n \rightarrow \infty$ , we get (4.4.17).

**We now prove the property (2).** We need to show that

$$(4.4.24) \quad |H_{n,K}(\alpha) - g_K(\alpha)| = \left| \frac{1}{n} \sum_{j=0}^{n-1} \left[ f_K \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) - \mathbb{E}(f_K(L_1^\alpha)) \right] \right| \xrightarrow[\mathbb{P}]{n \rightarrow \infty} 0.$$

Using again Lemma 9 from Genon-Catalot and Jacod [25] to show the property (2), it is sufficient to prove that

$$(4.4.25) \quad \frac{1}{n} \sum_{j=0}^{n-1} \left| \mathbb{E} \left[ \left[ f_K \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) - \mathbb{E}(f_K(L_1^\alpha)) \right] \middle| \mathcal{F}_{t_j} \right] \right| \xrightarrow[\mathbb{P}]{n \rightarrow \infty} 0,$$

$$(4.4.26) \quad \frac{1}{n^2} \sum_{j=0}^{n-1} \mathbb{E} \left[ \left[ f_K \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) - \mathbb{E}(f_K(L_1^\alpha)) \right]^2 \middle| \mathcal{F}_{t_j} \right] \xrightarrow[\mathbb{P}]{n \rightarrow \infty} 0.$$

Since  $Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha$  is independent of  $\mathcal{F}_{t_j}$  and  $Z^\alpha$  has stationary increments then we get

$$(4.4.27) \quad \frac{1}{n} \sum_{j=0}^{n-1} \left| \mathbb{E} \left[ \left[ f_K \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) - \mathbb{E}(f_K(L_1^\alpha)) \right] \middle| \mathcal{F}_{t_j} \right] \right| = \left| \mathbb{E} \left[ f_K \left( n^{1/\alpha} \left\{ Z_{\frac{1}{n}}^\alpha \right\} \right) - \mathbb{E}(f_K(L_1^\alpha)) \right] \right| \xrightarrow{n \rightarrow \infty} 0$$

where we have used (4.2.6) with the boundedness of  $f_K$  to get the convergence. Hence we can deduce (4.4.25). Since  $|f_K| \leq CK$ , we can deduce immediately (4.4.26) and hence the property (2) is proved. Hence the result of this lemma follows.  $\blacksquare$

**Proof of Proposition 4.4.1:** We can see that for any positive number  $\epsilon$

$$\begin{aligned} & \mathbb{P} \left( \left| \frac{1}{n} \sum_{j=0}^{n-1} f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) - \int f(y) \varphi_\alpha(y) dy \right| > \epsilon \right) = \mathbb{P}(|G_n(\alpha) - g(\alpha)| > \epsilon) \\ & \leq \mathbb{P} \left( |G_n(\alpha) - G_{n,K}(\alpha)| > \frac{\epsilon}{3} \right) + \mathbb{P} \left( |G_{n,K}(\alpha) - g_K(\alpha)| > \frac{\epsilon}{3} \right) + \mathbb{P} \left( |g_K(\alpha) - g(\alpha)| > \frac{\epsilon}{3} \right) \\ & \leq \frac{2}{\epsilon} \mathbb{E}(|G_n(\alpha) - G_{n,K}(\alpha)|) + \mathbb{P} \left( |G_{n,K}(\alpha) - g_K(\alpha)| > \frac{\epsilon}{3} \right) + \mathbb{P} \left( |g_K(\alpha) - g(\alpha)| > \frac{\epsilon}{3} \right) \\ & \leq \frac{4c_K}{\epsilon} + \mathbb{P} \left( |G_{n,K}(\alpha) - g_K(\alpha)| > \frac{\epsilon}{3} \right) + \mathbb{P} \left( |g_K(\alpha) - g(\alpha)| > \frac{\epsilon}{3} \right) \end{aligned}$$

where in the second inequality, the third inequality and the four inequality we used the Markov's inequality and Lemma 4.4.1(1) respectively. From Lemma 4.4.2, we deduce that  $\forall \epsilon > 0$

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left( \left| \frac{1}{n} \sum_{j=0}^{n-1} f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) - \int f(y) \varphi_\alpha(y) dy \right| > \epsilon \right) \leq \frac{4c_K}{\epsilon} + \mathbb{P} \left( |g_K(\alpha) - g(\alpha)| > \frac{\epsilon}{3} \right)$$

for all  $K > 2$ . Taking  $K \rightarrow \infty$ ,  $c_K \rightarrow 0$  and using Lemma 4.4.1(2), we deduce Proposition 4.4.1.  $\blacksquare$

### 4.4.2 Central Limit Theorem

In this section, we want to study the convergence in law associated with the convergence in probability in Proposition 4.4.1. At this point, we need stronger assumptions on the function  $f$  because of the exploding factor  $\sqrt{n}$ . Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a symmetric function and we suppose that the function  $f$  satisfies the following assumptions

- Assumption 4.4.2.**
1.  $\exists C > 0$  such that  $|f(y)| \leq C(|\log|y||^2 + 1)$  for all  $|y| \leq 1$ .
  2. There exists  $0 < p < \frac{\alpha}{2}$  such that  $|f(y)| \leq C|y|^p$  for all  $|y| > 1$  and  $C$  is a positive constant.
  3.  $f$  is a differentiable function such that for a constant  $C > 0$  then  $|f'(y)| \leq \frac{C|\log|y||+1}{|y|}$  for all  $|y| \leq 1$  and  $|f'(y)| \leq C|y|^p$  for all  $|y| > 1$ .

**Remark 4.4.1.** From Assumptions 4.4.2(1)(2) we can deduce that there exists  $\delta > 0$  such that  $(2 + \delta)p < \alpha$  and then  $\int |f(y)|^{(2+\delta)} \varphi_\alpha(y) dy < \infty$ .

**Proposition 4.4.2.** Let  $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$  satisfy Assumption 4.4.2. We set  $g_l = \int f_l(y) \varphi_\alpha(y) dy$  for  $l = 1, 2$ . We have the following convergence in law

$$(4.4.28) \quad \sqrt{n} \left( \begin{array}{l} \frac{1}{n} \sum_{j=0}^{n-1} f_1 \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) - \int f_1(y) \varphi_\alpha(y) dy \\ \frac{1}{n} \sum_{j=0}^{n-1} f_2 \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) - \int f_2(y) \varphi_\alpha(y) dy \end{array} \right) \Rightarrow \mathcal{N}(0, \Sigma)$$

where the asymptotic covariance matrix  $\Sigma = [\Sigma_{ik}]_{i,k=1,2}$  is given by

$$(4.4.29) \quad \Sigma_{ik} = \int (f_i(y) - g_i(\alpha))(f_k(y) - g_k(\alpha)) \varphi_\alpha(y) dy, \quad 1 \leq i, k \leq 2.$$

We observe that the left-hand side of (4.4.28) can be written as

$$(4.4.30) \quad \sqrt{n} \left( \begin{array}{l} \frac{1}{n} \sum_{j=0}^{n-1} f_1 \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) - \frac{1}{n} \sum_{j=0}^{n-1} f_1 \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \\ \frac{1}{n} \sum_{j=0}^{n-1} f_2 \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) - \frac{1}{n} \sum_{j=0}^{n-1} f_2 \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \end{array} \right)_+$$

$$(4.4.31) \quad + \sqrt{n} \left( \begin{array}{l} \frac{1}{n} \sum_{j=0}^{n-1} f_1 \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) - \int f_1(y) \varphi_\alpha(y) dy \\ \frac{1}{n} \sum_{j=0}^{n-1} f_2 \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) - \int f_2(y) \varphi_\alpha(y) dy \end{array} \right).$$

Hence Proposition 4.4.2 will be proved if we can prove the two propositions (Propositions 4.4.3-4.4.4).



**Proposition 4.4.3.** *Let  $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$  satisfy Assumption 4.4.2, then we have*

$$\sqrt{n} \begin{pmatrix} \frac{1}{n} \sum_{j=0}^{n-1} f_1 \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) - \frac{1}{n} \sum_{j=0}^{n-1} f_1 \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \\ \frac{1}{n} \sum_{j=0}^{n-1} f_2 \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) - \frac{1}{n} \sum_{j=0}^{n-1} f_2 \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \end{pmatrix} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \mathbf{0}.$$

**Proposition 4.4.4.** *Let  $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$  satisfy Assumption 4.4.2. We have the following convergence in law*

$$(4.4.32) \quad \sqrt{n} \begin{pmatrix} \frac{1}{n} \sum_{j=0}^{n-1} f_1 \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) - \int f_1(y) \varphi_\alpha(y) dy \\ \frac{1}{n} \sum_{j=0}^{n-1} f_2 \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) - \int f_2(y) \varphi_\alpha(y) dy \end{pmatrix} \Rightarrow \mathcal{N}(0, \Sigma)$$

where the asymptotic covariance matrix  $\Sigma$  is given by (4.4.29).

The proof of Proposition 4.4.3 is based on Lemmas 4.4.3-4.4.6 and the proof Proposition 4.4.4 is based on Lemmas 4.4.7-4.4.8. From now on, we choose  $K := K_n = n^{\frac{2+\delta}{4(1+\delta)}}$  where  $\delta$  is chosen such that  $(2+\delta)p < \alpha$  (in order to simplify the notation, in the rest of this proof, we shall simply write  $K$  instead of  $K_n$ ).

**Lemma 4.4.3.** *We suppose that Assumption 4.4.2 holds then, with  $K := K_n = n^{\frac{2+\delta}{4(1+\delta)}}$ , we have*

$$1) \quad \sqrt{n} \mathbb{E}(|G_n(\alpha) - G_{n,K}(\alpha)|) \xrightarrow[n \rightarrow \infty]{} 0,$$

$$2) \quad \sqrt{n} \mathbb{E}(|H_n(\alpha) - H_{n,K}(\alpha)|) \xrightarrow[n \rightarrow \infty]{} 0,$$

where  $G_n(\alpha)$ ,  $H_n(\alpha)$ ,  $G_{n,K}(\alpha)$  and  $H_{n,K}(\alpha)$  are defined by (4.2.7), (4.2.8) and (4.4.3), (4.4.4) respectively.

**Proof of Lemma 4.4.3** We first show the part 1). For  $K := K_n = n^{\frac{2+\delta}{4(1+\delta)}}$  where  $\delta$  satisfies  $(2+\delta)p < \alpha$ . From (4.4.7), we have

$$(4.4.33) \quad \begin{aligned} & \mathbb{E} |\sqrt{n} (G_n(\alpha) - G_{n,K}(\alpha))| \\ &= \mathbb{E} \left| \left| \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) - \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \right| \right| \\ &\leq \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \mathbb{E} \left( \left| f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \right| \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right| \geq K^{1/p}} \right) \end{aligned}$$

$$(4.4.34) \quad + \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \mathbb{E} \left( \left| f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \right| \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right| \leq \frac{1}{K^2}} \right).$$

**Let us estimate (4.4.33).** From (4.4.9), we have

$$(4.4.35) \quad \mathbb{E} \left( \left| f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \right| \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right| \geq K^{1/p}} \right) \\ \leq C \mathbb{E} \left( n^{p/\alpha} \sup_{0 \leq s \leq \frac{1}{n}} |Z_s^\alpha|^p \mathbf{1}_{n^{p/\alpha} \sup_{0 \leq s \leq \frac{1}{n}} |Z_s^\alpha|^p > K/C^p} \right).$$

Moreover, we can estimate (4.4.35) as follows

$$C \mathbb{E} \left( n^{p/\alpha} \sup_{0 \leq s \leq \frac{1}{n}} |Z_s^\alpha|^p \mathbf{1}_{n^{p/\alpha} \sup_{0 \leq s \leq \frac{1}{n}} |Z_s^\alpha|^p > K/C^p} \right) \leq \mathbb{E} \left( n^{p/\alpha} \sup_{0 \leq s \leq \frac{1}{n}} |Z_s^\alpha|^p \left[ \frac{n^{p/\alpha} \sup_{0 \leq s \leq \frac{1}{n}} |Z_s^\alpha|^p}{K/C^p} \right]^{1+\delta} \right) \\ = \frac{C^{p(1+\delta)}}{K^{1+\delta}} \mathbb{E} \left( \left[ n^{p/\alpha} \sup_{0 \leq t \leq \frac{1}{n}} |Z_s^\alpha|^p \right]^{2+\delta} \right) \quad \text{for } \delta \text{ is chosen such that } (2+\delta)p < \alpha.$$

From the above estimate, with the choice  $K := K_n = n^{\frac{2+\delta}{4(1+\delta)}}$ , we can deduce that (4.4.33) is bounded by

$$(4.4.36) \quad \frac{M}{n^{1+\frac{\delta}{4}}} \sum_{j=0}^{n-1} \mathbb{E} \left( \left[ n^{p/\alpha} \sup_{0 \leq t \leq \frac{1}{n}} |Z_s^\alpha|^p \right]^{2+\delta} \right)$$

where  $M$  is some positive constant. From (4.4.10) with  $r = p(2+\delta) < \alpha$  and (4.4.36), we get that (4.4.33) converges to zero as  $n \rightarrow \infty$ .

**Let us estimate (4.4.34).** We easily see that

$$\mathbb{E} \left( \left| f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \right| \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right| \leq \frac{1}{K^2}} \right) \\ = \mathbb{E} \left[ \mathbb{E} \left( \left| f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \right| \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right| \leq \frac{1}{K^2}} \middle| \mathcal{F}_{t_j} \right) \right] \\ = \mathbb{E} \left[ \mathbb{E} \left( \left| f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \right| \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right| \leq \frac{1}{K^2}} \middle| X_{t_j}^\alpha \right) \right]$$

where we have used the Markov property to get the last line. Moreover, we have

$$(4.4.37) \quad \mathbb{E} \left( \left| f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \right| \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right| \leq \frac{1}{K^2}} \middle| X_{t_j}^\alpha = x \right) \\ = \int_{-\frac{1}{K^2}}^{\frac{1}{K^2}} |f(u)| \left[ \frac{\sigma}{n^{1/\alpha}} P_{\frac{1}{n}}^\alpha \left( x, \frac{u\sigma}{n^{1/\alpha}} + \varsigma_{\frac{1}{n}}(x) \right) \right] du \leq C_1 \int_{-\frac{1}{K^2}}^{\frac{1}{K^2}} |f(u)| du \\ = 2C_1 \int_0^{\frac{1}{K^2}} |f(u)| du \leq 2C_1 C \int_0^{\frac{1}{K^2}} |\log|u||^2 du + \frac{2C_1 C}{K^2} := T$$

where we have used (4.2.1),  $f$  is a symmetric function and Assumption 4.4.2 (1). Then we can deduce that (4.4.34)  $\leq \sqrt{n}T$ . Moreover, we observe that

$$(4.4.38) \quad \begin{aligned} \sqrt{n} \int_0^{\frac{1}{K^2}} |\log|u||^2 du &= \frac{2\sqrt{n}}{K^2} + \frac{4\sqrt{n}}{K^2} \log K + \frac{4\sqrt{n}}{K^2} (\log(K))^2 \\ &= \frac{2}{n^{\frac{1}{2(1+\delta)}}} + \frac{4 \log(n^{\frac{2+\delta}{4(1+\delta)}})}{n^{\frac{1}{2(1+\delta)}}} + \frac{4 \left( \log(n^{\frac{2+\delta}{4(1+\delta)}}) \right)^2}{n^{\frac{1}{2(1+\delta)}}} \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

thus (4.4.34) tends to zeros as  $n \rightarrow \infty$ . Hence the result of this lemma follows.

Using the part 1) with the drift function  $b \equiv 0, \sigma = 1$  and  $x = 0$ , we deduce immediately the result of the part 2). ■

**Lemma 4.4.4.** *We assume that the function  $f$  satisfies Assumption 4.4.2 then*

$$\sup_n \mathbb{E} \left[ f \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - \varsigma_{1/n}(x) \right\} \right) \right]^{2+\delta} < \infty$$

where  $\delta > 0$  satisfies  $p(2+\delta) < \alpha$ .

**Proof of Lemma 4.4.4** From Assumption 4.4.2(2), we have

$$(4.4.39) \quad \begin{aligned} &\mathbb{E} \left[ f \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - \varsigma_{1/n}(x) \right\} \right) \right]^{2+\delta} \\ &= \mathbb{E} \left[ f \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - \varsigma_{1/n}(x) \right\} \right) \right]^{2+\delta} \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - \varsigma_{1/n}(x) \right\} \right| > 1} \\ &\quad + \mathbb{E} \left[ f \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - \varsigma_{1/n}(x) \right\} \right) \right]^{2+\delta} \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - \varsigma_{1/n}(x) \right\} \right| \leq 1} \\ &\leq \mathbb{E} \left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - \varsigma_{1/n}(x) \right\} \right|^{p(2+\delta)} + \mathbb{E} \left[ f \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - \varsigma_{1/n}(x) \right\} \right) \right]^{2+\delta} \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - \varsigma_{1/n}(x) \right\} \right| \leq 1}. \end{aligned}$$

Let us consider the first term of (4.4.39). Using (4.4.8) that  $Z^\alpha$  has stationary increments and (4.4.10) with  $p(2+\delta) < \alpha$ , we get the first term of (4.4.39) is bounded by a positive constant that does not depend on  $n$ .

We consider the second term of (4.4.39). We have

$$\begin{aligned} &\mathbb{E} \left( \left[ f \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - \varsigma_{1/n}(x) \right\} \right) \right]^{2+\delta} \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - \varsigma_{1/n}(x) \right\} \right| \leq 1} \right) \\ &= \int_{-1}^1 |f(u)|^{2+\delta} \left[ \frac{\sigma}{n^{1/\alpha}} P_{\frac{1}{n}}^\alpha \left( x, \frac{u\sigma}{n^{1/\alpha}} + \varsigma_{\frac{1}{n}}(x) \right) \right] du \leq C \int_{-1}^1 |f(u)|^{2+\delta} du = 2C \int_0^1 |f(u)|^{2+\delta} du < +\infty \end{aligned}$$

where we have used the boundedness of  $\frac{\sigma}{n^{1/\alpha}} P_{\frac{1}{n}}^\alpha \left( x, \frac{u\sigma}{n^{1/\alpha}} + \varsigma_{\frac{1}{n}}(x) \right)$  (see (4.2.2)), the function  $f$  is symmetric and Assumption 4.4.2(1). From the above estimates, we can deduce that (4.4.39) is bounded by a positive constant that does not depend on  $n$ . The result of this lemma follows. ■

**Lemma 4.4.5.** *Let  $X_{\frac{1}{n}}^{\alpha,x}$  be the solution to (4.1.1) starting from  $X_0^\alpha = x$  then*

$$\sup_x \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \varsigma_{1/n}(x) \right\} \right) - f_K \left( n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right) \right]^2 \xrightarrow{n \rightarrow \infty} 0,$$

recalling  $f_K(x) = f(x)\mathcal{X}_K(x)$  with  $\mathcal{X}_K$  is defined by (4.4.1) and (4.4.2),  $K := K_n = n^{\frac{2+\delta}{4(1+\delta)}}$  and  $\delta > 0$  satisfies  $p(2+\delta) < \alpha$ .

**Proof of Lemma 4.4.5** For all  $\epsilon_n > 0$ , we can write

$$(4.4.40) \quad \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \varsigma_{1/n}(x) \right\} \right) - f_K \left( n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right) \right]^2$$

$$(4.4.41) \quad = \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \varsigma_{1/n}(x) \right\} \right) - f_K \left( n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right) \right]^2 \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \varsigma_{1/n}(x) \right\} - n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right| \leq \epsilon_n}$$

$$(4.4.42) \quad + \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \varsigma_{1/n}(x) \right\} \right) - f_K \left( n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right) \right]^2 \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \varsigma_{1/n}(x) \right\} - n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right| > \epsilon_n}.$$

We observe that, for  $C$  is a deterministic constant, from Assumption 4.4.2 (1-3) we have

$$(4.4.43) \quad |f'_K(x)| = |f'(x)\mathcal{X}_K(x) + f(x)(\mathcal{X}_K(x))'| \leq C(K + K^2(\log K)^2).$$

Then we can estimate

$$(4.4.44) \quad \begin{aligned} & \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \varsigma_{1/n}(x) \right\} \right) - f_K \left( n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right) \right]^2 \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \varsigma_{\frac{1}{n}}(x) \right\} - n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right| \leq \epsilon_n} \\ & \leq (\|f'_K\|_\infty)^2 \epsilon_n^2 = C(K + K^2 \log^2(K))^2 \epsilon_n^2 = C \left[ \left( n^{\frac{2+\delta}{4(1+\delta)}} + n^{\frac{2+\delta}{2(1+\delta)}} \log^2(n^{\frac{2+\delta}{4(1+\delta)}}) \right) \epsilon_n \right]^2. \end{aligned}$$

We choose  $\epsilon_n = n^{-\frac{2-\frac{3}{2}\delta}{2(1+\delta)}}$  then we can easily deduce that  $\left( n^{\frac{2+\delta}{4(1+\delta)}} + n^{\frac{2+\delta}{2(1+\delta)}} \log^2(n^{\frac{2+\delta}{4(1+\delta)}}) \right) \epsilon_n$  goes to zero as  $n \rightarrow \infty$ . This implies that (4.4.41) tends to zero as  $n \rightarrow \infty$ .

We now consider (4.4.42). From  $|f_K| \leq |f|$ , we have that (4.4.42) is bounded by

$$(4.4.45) \quad 2\mathbb{E} \left[ f \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \varsigma_{1/n}(x) \right\} \right) \right]^2 \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \varsigma_{1/n}(x) \right\} - n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right| > \epsilon_n}$$

$$(4.4.46) \quad + 2\mathbb{E} \left[ f \left( n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right) \right]^2 \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \varsigma_{1/n}(x) \right\} - n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right| > \epsilon_n}.$$

Let us consider (4.4.45). Using Hölder's inequality, we obtain

$$\begin{aligned}
 & \mathbb{E} \left[ f \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \zeta_{1/n}(x) \right\} \right) \right]^2 \mathbb{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \zeta_{1/n}(x) \right\} - n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right| > \epsilon_n} \\
 & \leq \left[ \mathbb{E} \left[ f^2 \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \zeta_{1/n}(x) \right\} \right) \right] \right]^{\frac{2+\delta}{2}} \left[ \mathbb{E} \left[ \mathbb{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \zeta_{1/n}(x) \right\} - n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right| > \epsilon_n} \right] \right]^{\frac{\delta}{2+\delta}} \\
 (4.4.47) \quad & = \left[ \mathbb{E} \left[ f \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \zeta_{1/n}(x) \right\} \right) \right] \right]^{2+\delta} \left[ \mathbb{P} \left( \left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \zeta_{1/n}(x) \right\} - n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right| > \epsilon_n \right) \right]^{\frac{\delta}{2+\delta}}.
 \end{aligned}$$

From Lemma 4.4.4, in order to show that (4.4.47) converges to zero as  $n \rightarrow \infty$ , it remains to show that

$$(4.4.48) \quad \mathbb{P} \left( \left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \zeta_{1/n}(x) \right\} - n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right| > \epsilon_n \right) \xrightarrow{n \rightarrow \infty} 0.$$

In fact, we observe that for any  $t \in [0, \frac{1}{n}]$

$$\begin{aligned}
 & \left| n^{1/\alpha} \left\{ X_t^{\alpha,x} - \zeta_t(x) \right\} - n^{1/\alpha} \sigma Z_t^\alpha \right| = \left| \int_0^t n^{1/\alpha} [b(X_s^{\alpha,x}) - b(\zeta_s(x))] ds \right| \\
 & \leq \int_0^t \|b'\|_\infty \left[ \left| n^{1/\alpha} (X_s^{\alpha,x} - \zeta_s(x)) - n^{1/\alpha} \sigma Z_s^\alpha \right| \right] ds + \|b'\|_\infty \sup_{0 \leq t \leq 1/n} \int_0^t n^{1/\alpha} \sigma |Z_s^\alpha| ds,
 \end{aligned}$$

where  $\|b'\|_\infty = \sup_{x \in \mathbb{R}} |b'(x)|$ . Applying Gronwall's lemma and using  $\|b'\|_\infty < \infty$ , we get

$$\sup_x \left| n^{1/\alpha} \left\{ X_t^{\alpha,x} - \zeta_t(x) \right\} - n^{1/\alpha} \sigma Z_t^\alpha \right| \leq C \left[ \sup_{0 \leq t \leq 1/n} \int_0^t n^{1/\alpha} \sigma |Z_s^\alpha| ds \right],$$

where  $C$  is a positive constant and it follows that

$$(4.4.49) \quad \sup_x \left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \zeta_{\frac{1}{n}}(x) \right\} - n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right| \leq \frac{C}{n} \left( n^{1/\alpha} \sup_{0 \leq s \leq \frac{1}{n}} |Z_s^\alpha| \right).$$

Let  $0 < r < \alpha$  then, from (4.4.49), we have

$$\begin{aligned}
 (4.4.50) \quad & \mathbb{P} \left[ \left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \zeta_{1/n}(x) \right\} - n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right| > \epsilon_n \right] \\
 & \leq \mathbb{P} \left[ \left( n^{1/\alpha} \sup_{0 \leq s \leq \frac{1}{n}} |Z_s^\alpha| \right)^r > \frac{n^r \epsilon_n^r}{C^r} \right] \leq \frac{C^r}{n^r \epsilon_n^r} \mathbb{E} \left[ \left( n^{1/\alpha} \sup_{0 \leq s \leq \frac{1}{n}} |Z_s^\alpha| \right)^r \right].
 \end{aligned}$$

With the choice  $\epsilon_n = n^{-\frac{2-\frac{3}{2}\delta}{2(1+\delta)}}$ , from (4.4.10), we can easily deduce (4.4.50) goes to zero as  $n \rightarrow \infty$ .

Hence this yields that (4.4.47) tends to zero as  $n \rightarrow \infty$  and thus (4.4.45) goes to zero as  $n \rightarrow \infty$ .

Let us consider (4.4.46). Using the Hölder's inequality, for  $C$  is a positive constant, we have

$$\begin{aligned}
 (4.4.51) \quad & \mathbb{E} \left[ f \left( n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right) \right]^2 \mathbb{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \varsigma_{1/n}(x) \right\} - n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right| > \epsilon_n} \\
 & \leq \left[ \mathbb{E} \left[ f^2 \left( n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right) \right]^{\frac{2+\delta}{2}} \right]^{\frac{2}{2+\delta}} \left[ \mathbb{E} \left[ \mathbb{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \varsigma_{1/n}(x) \right\} - n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right| > \epsilon_n} \right]^{\frac{2+\delta}{\delta}} \right]^{\frac{\delta}{2+\delta}} \\
 & = \left[ \mathbb{E} \left[ f \left( n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right) \right]^{2+\delta} \right]^{\frac{2}{2+\delta}} \left[ \mathbb{P} \left( \left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \varsigma_{1/n}(x) \right\} - n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right| > \epsilon_n \right) \right]^{\frac{\delta}{2+\delta}}.
 \end{aligned}$$

We observe that

$$\begin{aligned}
 (4.4.52) \quad & \mathbb{E} \left[ f \left( n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right) \right]^{2+\delta} = \mathbb{E} \left[ f \left( n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right) \right]^{2+\delta} \mathbb{1}_{|n^{1/\alpha} Z_{\frac{1}{n}}^\alpha| > 1} + \mathbb{E} \left[ f \left( n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right) \right]^{2+\delta} \mathbb{1}_{|n^{1/\alpha} Z_{\frac{1}{n}}^\alpha| \leq 1} \\
 & \leq \mathbb{E} \left[ |n^{1/\alpha} Z_{\frac{1}{n}}^\alpha|^{p(2+\delta)} \right] + \int_{|u| \leq 1} |f(u)|^{2+\delta} \varphi^n(u) du \\
 & \leq C_{\alpha,p,\delta} + C \int_0^1 |f(u)|^{2+\delta} du < \infty
 \end{aligned}$$

where we have used (4.4.10) with  $r = p(2+\delta) < \alpha$ , the fact that  $f$  is a symmetric function, (4.2.3) and Assumption 4.4.2(1) to get the last line. From (4.4.48) and (4.4.52), we deduce that (4.4.51) tends to zero as  $n \rightarrow \infty$ . This implies that (4.4.46) tends to zero as  $n \rightarrow \infty$ . From the above estimates, we can deduce that (4.4.42) goes to zero as  $n \rightarrow \infty$ . Hence this lemma is proved.  $\blacksquare$

**Lemma 4.4.6.** *We assume that Assumption 4.4.2 holds and  $K := K_n = n^{\frac{2+\delta}{4(1+\delta)}}$  then*

$$\sqrt{n} (G_{n,K}(\alpha) - H_{n,K}(\alpha)) \xrightarrow[\mathbb{P}]{n \rightarrow \infty} 0$$

where  $G_{n,K}$  and  $H_{n,K}$  are defined by (4.4.3)-(4.4.4).

**Proof of Lemma 4.4.6** We use Lemma 9 from Genon-Catalot and Jacod [25] then it is sufficient to show that

$$(4.4.53) \quad \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left| \mathbb{E} \left[ \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha) \right\} \right) - f_K \left( n^{1/\alpha} \left\{ Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha \right\} \right) \right] \middle| \mathcal{F}_{t_j} \right] \right| \xrightarrow[\mathbb{P}]{n \rightarrow \infty} 0,$$

$$(4.4.54) \quad \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{E} \left[ \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha) \right\} \right) - f_K \left( n^{1/\alpha} \left\{ Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha \right\} \right) \right]^2 \middle| \mathcal{F}_{t_j} \right] \right| \xrightarrow[\mathbb{P}]{n \rightarrow \infty} 0.$$

From (4.4.23), we can easily deduce that for  $C$  is a positive constant then

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left| \mathbb{E} \left[ \left[ f_K \left( n^{1/\alpha} \left\{ X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha) \right\} \right) - f_K \left( n^{1/\alpha} \left\{ Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha \right\} \right) \right] \middle| \mathcal{F}_{t_j} \right] \right| \leq \frac{CK}{\sqrt{n}}.$$

With the choice  $K := K_n = n^{\frac{2+\delta}{4(1+\delta)}}$ , this yields (4.4.53).

We consider (4.4.54). We observe that

$$(4.4.55) \quad \begin{aligned} & \mathbb{E} \left[ \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) - f_K \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \right]^2 \middle| \mathcal{F}_{t_j} \right] \\ &= \mathbb{E} \left[ f_K^2 \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \middle| \mathcal{F}_{t_j} \right] + \mathbb{E} \left[ f_K^2 \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \middle| \mathcal{F}_{t_j} \right] \\ & \quad - 2\mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \times f_K \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \middle| \mathcal{F}_{t_j} \right]. \end{aligned}$$

From Markov property, we have

$$(4.4.56) \quad \mathbb{E} \left[ f_K^2 \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \middle| \mathcal{F}_{t_j} \right] = \mathbb{E} \left[ f_K^2 \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \middle| X_{t_j}^\alpha \right].$$

But we can write

$$(4.4.57) \quad \mathbb{E} \left[ f_K^2 \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \middle| X_{t_j}^\alpha = x \right] = \mathbb{E} \left[ f_K^2 \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \varsigma_{1/n}(x) \right\} \right) \right]$$

where  $X_{\frac{1}{n}}^{\alpha,x}$  is the solution to (4.1.1) starting from  $X_0^\alpha = x$ .

Using the fact that  $Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha$  is independent of  $\mathcal{F}_{t_j}$  then we can write

$$(4.4.58) \quad \mathbb{E} \left[ f_K^2 \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \middle| \mathcal{F}_{t_j} \right] = \mathbb{E} \left[ f_K^2 \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \right] = \mathbb{E} \left[ f_K^2 \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \middle| X_{t_j}^\alpha \right]$$

From  $X_{t_{j+1}}^\alpha = X_{t_j}^\alpha + \int_{t_j}^{t_{j+1}} b(X_s^\alpha) ds + \sigma (Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha)$ , we can see that

$$(4.4.59) \quad \begin{aligned} & f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \times f_K \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \\ &= f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \times f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{t_{j+1}}^\alpha - X_{t_j}^\alpha - \int_{t_j}^{t_{j+1}} b(X_s^\alpha) ds \right\} \right) \end{aligned}$$

Since  $(X_t^\alpha)_{t \geq 0}$  has the Markov property with respect to  $(\mathcal{F}_t)$  then we obtain

$$(4.4.60) \quad \begin{aligned} & \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \times f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{t_{j+1}}^\alpha - X_{t_j}^\alpha - \int_{t_j}^{t_{j+1}} b(X_s^\alpha) ds \right\} \right) \middle| \mathcal{F}_{t_j} \right] \\ &= \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha)\} \right) \times f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{t_{j+1}}^\alpha - X_{t_j}^\alpha - \int_{t_j}^{t_{j+1}} b(X_s^\alpha) ds \right\} \right) \middle| X_{t_j}^\alpha \right] \quad a.s. \end{aligned}$$

Moreover, we can write

$$\begin{aligned}
 (4.4.61) \quad & \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha) \right\} \right) \times f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{t_{j+1}}^\alpha - X_{t_j}^\alpha - \int_{t_j}^{t_{j+1}} b(X_s^\alpha) ds \right\} \right) \middle| X_{t_j}^\alpha = x \right] \\
 &= \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \varsigma_{1/n}(x) \right\} \right) \times f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - x - \int_0^{\frac{1}{n}} b(X_s^{\alpha,x}) ds \right\} \right) \right] \\
 &= \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \varsigma_{1/n}(x) \right\} \right) \times f_K \left( n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right) \right].
 \end{aligned}$$

From (4.4.55), (4.4.56), (4.4.58) and (4.4.60), we can deduce that

$$\begin{aligned}
 (4.4.62) \quad & \mathbb{E} \left[ \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha) \right\} \right) - f_K \left( n^{1/\alpha} \left\{ Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha \right\} \right) \right]^2 \middle| \mathcal{F}_{t_j} \right] \\
 &= \mathbb{E} \left[ \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha) \right\} \right) - f_K \left( n^{1/\alpha} \left\{ Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha \right\} \right) \right]^2 \middle| X_{t_j}^\alpha \right].
 \end{aligned}$$

From (4.4.57), (4.4.61) and since  $\left( n^{1/\alpha} \left\{ Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha \right\} \right)_{j=0,\dots,n-1}$  are random variables with the same distribution as  $n^{1/\alpha} Z_{\frac{1}{n}}^\alpha$  then we can write

$$\begin{aligned}
 (4.4.63) \quad & \mathbb{E} \left[ \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha) \right\} \right) - f_K \left( n^{1/\alpha} \left\{ Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha \right\} \right) \right]^2 \middle| X_{t_j}^\alpha = x \right] \\
 &= \mathbb{E} \left[ \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - \varsigma_{1/n}(x) \right\} \right) - f_K \left( n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right) \right]^2 \right].
 \end{aligned}$$

From Lemma 4.4.5, we get

$$\sup_j \sup_x \mathbb{E} \left[ \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha) \right\} \right) - f_K \left( n^{1/\alpha} \left\{ Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha \right\} \right) \right]^2 \middle| X_{t_j}^\alpha = x \right] \xrightarrow{n \rightarrow \infty} 0.$$

Combining this with (4.4.62), we can deduce that

$$\sup_j \mathbb{E} \left[ \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{t_{j+1}}^\alpha - \varsigma_{1/n}(X_{t_j}^\alpha) \right\} \right) - f_K \left( n^{1/\alpha} \left\{ Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha \right\} \right) \right]^2 \middle| \mathcal{F}_{t_j} \right] \xrightarrow{n \rightarrow \infty} 0 \quad \text{a.s.}$$

Thus (4.4.54) is proved and the proof of this lemma is complete.  $\blacksquare$

### Proof of Proposition 4.4.3

**Proof.** Since  $f_i$  for  $i = 1, 2$  satisfy Assumption 4.4.2, then Proposition 4.4.3 will be proved if we show that for  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfying Assumption 4.4.2 then  $\sqrt{n} (G_n(\alpha) - H_n(\alpha)) \xrightarrow{\mathbb{P}} 0$  where  $G_n(\alpha)$  and  $H_n(\alpha)$  are defined by (4.2.7), (4.2.8) respectively.

Under the notations (4.2.7)-(4.2.9) and (4.4.3)-(4.4.5), we observe that

$$(4.4.64) \quad |G_n(\alpha) - H_n(\alpha)| \leq |G_n(\alpha) - G_{n,K}(\alpha)| + |G_{n,K}(\alpha) - H_{n,K}(\alpha)| + |H_{n,K}(\alpha) - H_n(\alpha)|.$$



From Lemma 4.4.6, we get  $\sqrt{n}(G_{n,K}(\alpha) - H_{n,K}(\alpha)) \xrightarrow[\mathbb{P}]{n \rightarrow \infty} 0$ . From Lemma 4.4.3, we have  $\sqrt{n}\mathbb{E}(|G_n(\alpha) - G_{n,K}(\alpha)|) \xrightarrow{n \rightarrow \infty} 0$  and  $\sqrt{n}\mathbb{E}|H_n(\alpha) - H_{n,K}(\alpha)| \xrightarrow{n \rightarrow \infty} 0$ . From the above estimates, we can conclude that Proposition 4.4.3 is proved.  $\blacksquare$

Now we will prove Proposition 4.4.4. The proof of this proposition is based on the two following lemmas.

**Lemma 4.4.7.** *We suppose that the function  $f$  satisfies Assumptions 4.4.1 then*

$$\left| \mathbb{E}\left(f(n^{1/\alpha} Z_{\frac{1}{n}}^\alpha)\right) - \mathbb{E}\left(f(L_1^\alpha)\right) \right| \xrightarrow{n \rightarrow \infty} 0.$$

**Proof of Lemma 4.4.7** We recall that  $f_K(x) := f(x)\mathcal{X}_K(x)$  where  $\mathcal{X}_K(x)$  is defined by (4.4.1) and (4.4.2). Then we have for  $K$  fixed

$$(4.4.65) \quad \limsup_n \left| \mathbb{E}\left(f(n^{1/\alpha} Z_{\frac{1}{n}}^\alpha)\right) - \mathbb{E}\left(f(L_1^\alpha)\right) \right| \leq \limsup_n \left| \mathbb{E}\left(f(n^{1/\alpha} Z_{\frac{1}{n}}^\alpha)\right) - \mathbb{E}\left(f_K(n^{1/\alpha} Z_{\frac{1}{n}}^\alpha)\right) \right| + \limsup_n \left| \mathbb{E}\left(f_K(n^{1/\alpha} Z_{\frac{1}{n}}^\alpha)\right) - \mathbb{E}\left(f_K(L_1^\alpha)\right) \right| + \left| \mathbb{E}\left(f_K(L_1^\alpha)\right) - \mathbb{E}\left(f(L_1^\alpha)\right) \right|.$$

Hence the proposition will be proved if we show that each term on the right-hand side of (4.4.65) tends to 0 as  $K \rightarrow \infty$ .

We get immediately that the third term on the right-hand side of (4.4.65) converges to 0 as  $K \rightarrow \infty$  from (4.4.14). From (4.4.27), we obtain that

$$\limsup_n \left| \mathbb{E}\left(f_K(n^{1/\alpha} Z_{\frac{1}{n}}^\alpha)\right) - \mathbb{E}\left(f_K(L_1^\alpha)\right) \right| = 0.$$

We now show that  $\limsup_n \left| \mathbb{E}\left(f(n^{1/\alpha} Z_{\frac{1}{n}}^\alpha)\right) - \mathbb{E}\left(f_K(n^{1/\alpha} Z_{\frac{1}{n}}^\alpha)\right) \right| \xrightarrow{K \rightarrow \infty} 0$ .

We have, for  $\delta$  is chosen such that  $(1 + \delta)p < \alpha$ ,

$$\begin{aligned} & \limsup_n \left| \mathbb{E}\left(f(n^{1/\alpha} Z_{\frac{1}{n}}^\alpha)\right) - \mathbb{E}\left(f_K(n^{1/\alpha} Z_{\frac{1}{n}}^\alpha)\right) \right| \leq \limsup_n \mathbb{E} \left| f(n^{1/\alpha} Z_{\frac{1}{n}}^\alpha) - f_K(n^{1/\alpha} Z_{\frac{1}{n}}^\alpha) \right| \\ & \leq \limsup_n \mathbb{E} \left| f(n^{1/\alpha} Z_{\frac{1}{n}}^\alpha) \right| \mathbf{1}_{\left|n^{1/\alpha} Z_{\frac{1}{n}}^\alpha\right| \geq K^{1/p}} + \limsup_n \mathbb{E} \left| f(n^{1/\alpha} Z_{\frac{1}{n}}^\alpha) \right| \mathbf{1}_{\left|n^{1/\alpha} Z_{\frac{1}{n}}^\alpha\right| \leq \frac{1}{K^2}} \\ & \leq \limsup_n \mathbb{E} \left( \left| n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right|^p \mathbf{1}_{\left|n^{1/\alpha} Z_{\frac{1}{n}}^\alpha\right| \geq K^{1/p}} \right) + C \int_{\frac{1}{K^2}}^{\frac{1}{K^2}} |f(u)| du \\ & \leq \limsup_n \mathbb{E} \left( \left| n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right|^p \left[ \frac{\left| n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right|^p}{K} \right]^\delta \right) + 2C \int_0^{\frac{1}{K^2}} ((\log(|u|))^4 + 1) du \end{aligned}$$

$$(4.4.66) \quad = \frac{C}{K^\delta} + 2C \int_0^{\frac{1}{K^2}} ((\log(|u|))^4 + 1) du$$

where we use (4.4.6) for the second inequality, Assumption 4.4.1(2) and (4.2.3) for the third inequality, the inequality (4.4.10), Assumption 4.4.1(1) and the fact that  $f$  is a symmetric function to obtain the last line. It is easy to see that (4.4.66) goes to zeros as  $K \rightarrow \infty$ . Hence the first term on the right-hand side of (4.4.65) tends to 0 as  $K \rightarrow \infty$ . Lemma 4.4.7 is proved.  $\blacksquare$

**Lemma 4.4.8.** *We assume that Assumption 4.4.2 is satisfied then we have*

$$(4.4.67) \quad \sqrt{n} \mathbb{E}(H_n(\alpha) - g(\alpha)) \xrightarrow{n \rightarrow \infty} 0.$$

where  $H_n(\alpha)$  and  $g(\alpha)$  are defined by (3.7.22) and (4.2.9) respectively.

**Proof of Lemma 4.4.8** Under the notations (4.2.8)-(4.2.9) and (4.4.4)-(4.4.5), we observe that

$$\sqrt{n} \mathbb{E}(H_n(\alpha) - g(\alpha)) = \sqrt{n} \mathbb{E}(H_n(\alpha) - H_{n,K}(\alpha)) + \sqrt{n} \mathbb{E}(H_{n,K}(\alpha) - g_K(\alpha)) + \sqrt{n} (g_K(\alpha) - g(\alpha))$$

Using Lemma 4.4.3 2), we have  $\sqrt{n} \mathbb{E}(H_n(\alpha) - H_{n,K}(\alpha)) \xrightarrow{n \rightarrow \infty} 0$ . Using (4.2.6) with the bounded function  $f_K$  and the choice  $K = n^{\frac{2+\delta}{4(1+\delta)}}$ , we obtain  $\sqrt{n} \mathbb{E}(H_{n,K}(\alpha) - g_K(\alpha)) \xrightarrow{n \rightarrow \infty} 0$ . Hence Lemma 4.4.8 will be proved if we can show that  $\sqrt{n} (g_K(\alpha) - g(\alpha)) \xrightarrow{n \rightarrow \infty} 0$ . In fact, we observe that, from (4.4.15), we have

$$(4.4.68)$$

$$\sqrt{n} |g_K(\alpha) - g(\alpha)| = \sqrt{n} |\mathbb{E}(f_K(L_1^\alpha)) - \mathbb{E}(f(L_1^\alpha))| \leq \sqrt{n} \mathbb{E}|f(L_1^\alpha)| 1_{|L_1^\alpha| \geq K^{1/p}} + \sqrt{n} \mathbb{E}|f(L_1^\alpha)| 1_{|L_1^\alpha| \leq \frac{1}{K^2}}.$$

Moreover, using Assumption 4.4.2(1)(2), we can see that (4.4.68) is bounded by

$$(4.4.69) \quad \sqrt{n} \mathbb{E} \left( |L_1^\alpha|^p \left[ \frac{|L_1^\alpha|^p}{K} \right]^{1+\delta} \right) + \sqrt{n} \mathbb{E} \left( |f(L_1^\alpha)| 1_{|L_1^\alpha| \leq \frac{1}{K^2}} \right) \\ = \frac{\sqrt{n}}{K^{1+\delta}} \mathbb{E} \left( |L_1^\alpha|^{p(2+\delta)} \right) + \sqrt{n} \int_{\frac{-1}{K^2}}^{\frac{1}{K^2}} |f(u)| \varphi_\alpha(u) du \\ \leq \frac{\sqrt{n}}{K^{1+\delta}} \mathbb{E} \left( |L_1^\alpha|^{p(2+\delta)} \right) + 2C \sqrt{n} \int_0^{\frac{1}{K^2}} |\log |u||^2 du + \frac{2C \sqrt{n}}{K^2}$$

where  $\delta$  is chosen such that  $p(2+\delta) < \alpha$  and  $C$  is a deterministic constant. We observe that, with the choice  $K := K_n = n^{\frac{2+\delta}{4(1+\delta)}}$ , then  $\frac{\sqrt{n}}{K^{1+\delta}} = \frac{1}{n^{\delta/4}} \xrightarrow{n \rightarrow \infty} 0$ . Combining this with (4.4.38) and the fact that  $\mathbb{E}(|L_1^\alpha|^{p(2+\delta)}) < \infty$  (see Theorem 2 in [52]), we can easily deduce that (4.4.69) goes to zero as  $n \rightarrow \infty$ . Hence we can conclude that  $\sqrt{n} (g_K(\alpha) - g(\alpha)) \xrightarrow{n \rightarrow \infty} 0$ . Thus Lemma 4.4.8 is proved.  $\blacksquare$

**Proof of Proposition 4.4.4**

**Proof.** We set

$$(4.4.70) \quad \xi_j^n := \begin{pmatrix} \xi_{j,1}^n \\ \xi_{j,2}^n \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{n}} \left[ f_1 \left( n^{1/\alpha} \{ Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha \} \right) - \int f_1(y) \varphi_\alpha(y) dy \right] \\ \frac{1}{\sqrt{n}} \left[ f_2 \left( n^{1/\alpha} \{ Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha \} \right) - \int f_2(y) \varphi_\alpha(y) dy \right] \end{pmatrix}$$

then we get

$$\sqrt{n} \begin{pmatrix} \frac{1}{n} \sum_{j=0}^{n-1} f_1 \left( n^{1/\alpha} \{ Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha \} \right) - \int f_1(y) \varphi_\alpha(y) dy \\ \frac{1}{n} \sum_{j=0}^{n-1} f_2 \left( n^{1/\alpha} \{ Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha \} \right) - \int f_2(y) \varphi_\alpha(y) dy \end{pmatrix} = \sum_{j=0}^{n-1} \xi_j^n.$$

Since the variables  $(Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha : j \geq 0)$  are independent, we use Theorem 2.2.14 in [37] to prove (4.4.32). It is sufficient to show that the following properties hold

- (1)  $\sum_{j=0}^{n-1} \mathbb{E} \left( \xi_j^n \right) \xrightarrow{n \rightarrow \infty} 0$ .
- (2)  $\sum_{j=0}^{n-1} \mathbb{E} \left( \xi_{j,i}^n \xi_{j,k}^n \right) \xrightarrow{n \rightarrow \infty} \Sigma_{ik}$  for all  $i, k = 1, 2$  and  $\Sigma_{ik}$  is defined by (4.4.29).
- (3)  $\sum_{j=0}^{n-1} \mathbb{E} \left( \|\xi_j^n\|^r \right) \xrightarrow{n \rightarrow \infty} 0$  for some  $r > 2$ .

**The property (1)** can be deduced immediately from Lemma 4.4.8. For **the property (2)**, we start the proof by considering the case  $i = k$ , we have

$$(4.4.71) \quad \begin{aligned} \sum_{j=0}^{n-1} \mathbb{E} \left( \xi_{j,i}^n \xi_{j,i}^n \right) &= \sum_{j=0}^{n-1} \mathbb{E} \left( \frac{1}{\sqrt{n}} \left[ f_i \left( n^{1/\alpha} \{ Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha \} \right) - \int f_i(y) \varphi_\alpha(y) dy \right] \right)^2 \\ &= \mathbb{E} \left[ f_i \left( n^{1/\alpha} \left\{ Z_{\frac{1}{n}}^\alpha \right\} \right) - \mathbb{E} \left( f_i(L_1^\alpha) \right) \right]^2 \\ &= \mathbb{E} \left( f_i^2 \left( n^{1/\alpha} \left\{ Z_{\frac{1}{n}}^\alpha \right\} \right) \right) + \left( \mathbb{E} \left( f_i(L_1^\alpha) \right) \right)^2 - 2 \mathbb{E} \left( f_i \left( n^{1/\alpha} \left\{ Z_{\frac{1}{n}}^\alpha \right\} \right) \right) \mathbb{E} \left( f_i(L_1^\alpha) \right) \end{aligned}$$

Since  $f_1, f_2, f_1^2, f_2^2$  satisfy Assumption 4.4.1 then we can apply Lemma 4.4.7 and obtain for  $i = 1, 2$

$$(4.4.72) \quad \mathbb{E} \left( f_i^2 \left( n^{1/\alpha} \left\{ Z_{\frac{1}{n}}^\alpha \right\} \right) \right) \xrightarrow{n \rightarrow \infty} \mathbb{E} \left( f_i^2(L_1^\alpha) \right),$$

$$(4.4.73) \quad \mathbb{E} \left( f_i \left( n^{1/\alpha} \left\{ Z_{\frac{1}{n}}^\alpha \right\} \right) \right) \xrightarrow{n \rightarrow \infty} \mathbb{E} \left( f_i(L_1^\alpha) \right).$$

Using the two above results, we can estimate that (4.4.71) converges to

$$(4.4.74) \quad \mathbb{E} \left( f_i^2(L_1^\alpha) \right) - \left( \mathbb{E} \left( f_i(L_1^\alpha) \right) \right)^2 = \int (f_i(y) - g_i(\alpha))^2 \varphi_\alpha(y) dy = \Sigma_{ii}.$$

We now focus on the case  $i \neq k$ , we have

$$\begin{aligned}
 (4.4.75) \quad & \sum_{j=0}^{n-1} \mathbb{E} \left( \xi_{j,i}^n \xi_{j,k}^n \right) \\
 &= \sum_{j=0}^{n-1} \mathbb{E} \left( \frac{1}{n} \left[ f_i \left( n^{1/\alpha} \{ Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha \} \right) - \int f_i(y) \varphi_\alpha(y) dy \right] \left[ f_k \left( n^{1/\alpha} \{ Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha \} \right) - \int f_k(y) \varphi_\alpha(y) dy \right] \right) \\
 &= \mathbb{E} \left( \left[ f_i \left( n^{1/\alpha} \left\{ Z_{\frac{1}{n}}^\alpha \right\} \right) - \mathbb{E}(f_i(L_1^\alpha)) \right] \left[ f_k \left( n^{1/\alpha} \left\{ Z_{\frac{1}{n}}^\alpha \right\} \right) - \mathbb{E}(f_k(L_1^\alpha)) \right] \right) \\
 &= \mathbb{E} \left( f_i \left( n^{1/\alpha} \left\{ Z_{\frac{1}{n}}^\alpha \right\} \right) f_k \left( n^{1/\alpha} \left\{ Z_{\frac{1}{n}}^\alpha \right\} \right) \right) + \mathbb{E}(f_i(L_1^\alpha)) \mathbb{E}(f_k(L_1^\alpha)) \\
 &\quad - \mathbb{E} \left( f_i \left( n^{1/\alpha} \left\{ Z_{\frac{1}{n}}^\alpha \right\} \right) \mathbb{E}(f_k(L_1^\alpha)) \right) - \mathbb{E} \left( f_k \left( n^{1/\alpha} \left\{ Z_{\frac{1}{n}}^\alpha \right\} \right) \mathbb{E}(f_i(L_1^\alpha)) \right).
 \end{aligned}$$

Since  $(f_i + f_k)^2$  and  $(f_i - f_k)^2$  satisfy Assumption 4.4.1 then using Lemma 4.4.7 again, we get

$$\begin{aligned}
 (4.4.76) \quad & \mathbb{E}(f_i(Z_1^{n,\alpha}) f_k(Z_1^{n,\alpha})) = \frac{1}{4} \left[ \mathbb{E}(f_i(Z_1^{n,\alpha}) + f_k(Z_1^{n,\alpha}))^2 - \mathbb{E}(f_i(Z_1^{n,\alpha}) - f_k(Z_1^{n,\alpha}))^2 \right] \\
 & \xrightarrow{n \rightarrow \infty} \frac{1}{4} \left[ \mathbb{E}(f_i(L_1^\alpha) + f_k(L_1^\alpha))^2 - \mathbb{E}(f_i(L_1^\alpha) - f_k(L_1^\alpha))^2 \right] = \mathbb{E}(f_i(L_1^\alpha) f_k(L_1^\alpha)).
 \end{aligned}$$

From (4.4.73) and (4.4.76), we can deduce that, as  $n \rightarrow \infty$ , (4.4.75) tends to

$$\begin{aligned}
 & \mathbb{E}(f_i(L_1^\alpha) f_k(L_1^\alpha)) + \mathbb{E}(f_i(L_1^\alpha)) \mathbb{E}(f_k(L_1^\alpha)) - \mathbb{E}(f_i(L_1^\alpha)) \mathbb{E}(f_k(L_1^\alpha)) - \mathbb{E}(f_k(L_1^\alpha)) \mathbb{E}(f_i(L_1^\alpha)) \\
 &= \int (f_i(y) - g_i(\alpha))(f_k(y) - g_k(\alpha)) \varphi_\alpha(y) dy = \Sigma_{ik}.
 \end{aligned}$$

Thus **the property (2)** is proved.

For **the property (3)**, we need to show that  $\sum_{j=0}^{n-1} \mathbb{E} \left( \|\xi_j^n\|^r \right) \xrightarrow{\mathbb{P}} 0$  for some  $r > 2$ . This is equivalent to prove that for  $i = 1, 2$  and for some  $r > 2$

$$(4.4.77) \quad \sum_{j=0}^{n-1} \mathbb{E} \left( |\xi_{j,i}^n|^r \right) \xrightarrow{\mathbb{P}} 0$$

We have

$$\begin{aligned}
 & \sum_{j=0}^{n-1} \mathbb{E} \left( |\xi_{j,i}^n|^r \right) = \sum_{j=0}^{n-1} \mathbb{E} \left( \left| \frac{1}{\sqrt{n}} \left[ f_i \left( n^{1/\alpha} \{ Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha \} \right) - \int f_i(y) \varphi_\alpha(y) dy \right] \right|^r \right) \\
 &= \frac{1}{n^{r/2-1}} \mathbb{E} \left( \left| f_i \left( n^{1/\alpha} \left\{ Z_{\frac{1}{n}}^\alpha \right\} \right) - \int f_i(y) \varphi_\alpha(y) dy \right|^r \right) \leq \frac{C_r}{n^{r/2-1}} \left[ \mathbb{E} \left( \left| f_i \left( n^{1/\alpha} \left\{ Z_{\frac{1}{n}}^\alpha \right\} \right) \right|^r \right) + \left( \mathbb{E}(f_i(L_1^\alpha)) \right)^r \right]
 \end{aligned}$$

From  $r > 2$  and  $\mathbb{E}(f_i(L_1^\alpha)) < \infty$  then, in order to get (4.4.77), we just need to show that  $\mathbb{E} \left( \left| f_i \left( n^{1/\alpha} \left\{ Z_{\frac{1}{n}}^\alpha \right\} \right) \right|^r \right)$

is bounded by a positive constant that does not depend on  $n$ . Actually, this holds true with the choice  $r = 2 + \delta$  where  $\delta > 0$  satisfies  $p(2 + \delta) < \alpha$  and the result of (4.4.52). Hence **the property (3)** is proved and Proposition 4.4.4 follows. ■

## 4.5 Proof of Main results (Theorems 4.3.1-4.3.4)

The aim of this section is to prove the consistency and asymptotic normality results given in Section 4.3.2. We first proceed to the proof of Theorem 4.3.1.

### Proof of Theorem 4.3.1:

We recall that the expression of  $\hat{\alpha}_{\log,n}$  is

$$(4.5.1) \quad \left\{ \frac{6}{\pi^2} \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \left( \log \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right| \right)^2 - \frac{1}{n^2} \left( \sum_{j=0}^{n-1} \log \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right| \right)^2 \right\} - \frac{1}{2} \right\}^{-1/2},$$

as soon as the square root is well defined.

We see that Assumption 4.4.1 is satisfied with the choice  $f_l(y) := (\log|y|)^l, l = 1, 2$  then using the result of Proposition 4.4.1 and (4.3.1), (4.3.2), we get the two following convergence in probability

$$\begin{aligned} \frac{1}{n} \sum_{j=0}^{n-1} \log \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right| &\xrightarrow[n \rightarrow \infty]{\mathbb{P}} \mathcal{C} \left[ \frac{1}{\alpha_0} - 1 \right], \\ \frac{1}{n} \sum_{j=0}^{n-1} \log^2 \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right| &\xrightarrow[n \rightarrow \infty]{\mathbb{P}} \frac{\pi^2}{6} \left[ \frac{1}{\alpha_0^2} + \frac{1}{2} \right] + \mathcal{C}^2 \left[ \frac{1}{\alpha_0} - 1 \right]^2. \end{aligned}$$

This yields that the expression (4.5.1) is well defined with large probability and

$$\hat{\alpha}_{\log,n} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \left\{ \frac{6}{\pi^2} \left\{ \frac{\pi^2}{6} \left[ \frac{1}{\alpha_0^2} + \frac{1}{2} \right] + \mathcal{C}^2 \left[ \frac{1}{\alpha_0} - 1 \right]^2 - \left( \mathcal{C} \left[ \frac{1}{\alpha_0} - 1 \right] \right)^2 \right\} - \frac{1}{2} \right\}^{-1/2} = \alpha_0.$$

Hence the consistency result of this theorem is proved.

### Proof of Theorem 4.3.2:

We denote  $v_{1,\alpha_0} := \mathbb{E}[\log|L_1^{\alpha_0}|]$  and  $v_{k,\alpha_0} := \mathbb{E}\left[\left(\log|L_1^{\alpha_0}| - \mathbb{E}[\log|L_1^{\alpha_0}|]\right)^k\right]$  with  $k \geq 2$  then we have in particular

$$\begin{aligned} v_{2,\alpha_0} &= \frac{\pi^2}{6} \left( \frac{1}{\alpha_0^2} + \frac{1}{2} \right) \\ v_{3,\alpha_0} &= 2\zeta(3) \left( \frac{1}{\alpha_0^3} - 1 \right) \\ v_{4,\alpha_0} &= \pi^4 \left( \frac{3}{20} \frac{1}{\alpha_0^4} + \frac{1}{12} \frac{1}{\alpha_0^2} + \frac{19}{240} \right) \end{aligned}$$

where  $\zeta$  stands for Riemann's zeta function;  $\zeta(3) \approx 1.202057$  (for more information see Section 3.1 in [53]).

Denote by

$$S_n^{\alpha_0} := \frac{1}{n} \sum_{j=0}^{n-1} \log^2 \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right| - \left( \frac{1}{n} \sum_{j=0}^{n-1} \log \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right| \right)^2,$$

we can write

$$\begin{aligned} \sqrt{n} (S_n^{\alpha_0} - v_{2,\alpha_0}) &= \sqrt{n} \left( \frac{1}{n} \sum_{j=0}^{n-1} \left( \log \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right| - \mathbb{E}(\log |L_1^{\alpha_0}|) \right)^2 - v_{2,\alpha_0} \right) \\ &\quad - \sqrt{n} \left( \frac{1}{n} \sum_{j=0}^{n-1} \log \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right| - \mathbb{E}(\log |L_1^{\alpha_0}|) \right)^2. \end{aligned}$$

Our aim is to show that

$$(4.5.2) \quad \sqrt{n} (S_n^{\alpha_0} - v_{2,\alpha_0}) \Rightarrow \mathcal{N}(0, v_{4,\alpha_0} - v_{2,\alpha_0}^2).$$

The proof will be divided into two steps.

**Step 1:** We show that

$$(4.5.3) \quad \sqrt{n} \left( \frac{1}{n} \sum_{j=0}^{n-1} \left( \log \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right| - \mathbb{E}(\log |L_1^{\alpha_0}|) \right)^2 - v_{2,\alpha_0} \right) \Rightarrow \mathcal{N}(0, v_{4,\alpha_0} - v_{2,\alpha_0}^2)$$

Indeed, Assumption 4.4.2 is satisfied with the choice  $f_2(y) := (\log |y| - \mathbb{E}(\log |L_1^{\alpha_0}|))^2$  then using the result of Proposition 4.4.2 we easily obtain (4.5.3).

**Step 2:** We show that

$$(4.5.4) \quad \sqrt{n} \left( \frac{1}{n} \sum_{j=0}^{n-1} \log \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right| - \mathbb{E}(\log |L_1^{\alpha_0}|) \right)^2 \xrightarrow[\mathbb{P}]{n \rightarrow \infty} 0.$$

We observe that Assumption 4.4.2 is satisfied with the choice  $f_1(y) := \log |y|$  then using the result of Proposition 4.4.2 we have

$$(4.5.5) \quad \sqrt{n} \left( \frac{1}{n} \sum_{j=0}^{n-1} \log \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right| - \mathbb{E}(\log |L_1^{\alpha_0}|) \right) \Rightarrow \mathcal{N}(0, v_{2,\alpha_0}).$$

From (4.5.5), we can deduce the convergence (4.5.4) and **Step 2** is proved.

Let  $e : (0, 2) \rightarrow \mathbb{R}$  be the function defined by  $e(\alpha) = \text{Var}(\log |L_1^\alpha|) = \frac{\pi^2}{6} \left[ \frac{1}{\alpha^2} + \frac{1}{2} \right]$  (see (4.3.3)). Then the estimator  $\hat{\alpha}_{\log, n}$  solves the equation

$$S_n^{\alpha_0} = e(\hat{\alpha}_{\log, n}).$$

Using the Delta-method, we get

$$(4.5.6) \quad \sqrt{n}(\hat{\alpha}_{\log,n} - \alpha_0) = \sqrt{n}(e^{-1}(S_n^{\alpha_0}) - e^{-1}(v_{2,\alpha_0})) \Rightarrow \mathcal{N}(0, (e_{\alpha_0}^{-1})' \Sigma_{\alpha_0} ((e_{\alpha_0}^{-1})'))$$

where  $\Sigma_{\alpha_0} = v_{4,\alpha_0} - v_{2,\alpha_0}^2 = \frac{\pi^4}{18} \left( \frac{11}{5\alpha_0^4} + \frac{1}{\alpha_0^2} + \frac{13}{10} \right)$  and  $(e_{\alpha_0}^{-1})'$  is the derivative of  $e^{-1}$  at  $e(\alpha_0)$ . From the inverse function theorem we have the the derivative of  $e^{-1}$  at  $e(\alpha_0)$  is the inverse  $e_{\alpha_0}^{\prime-1}$  of the derivative  $e'$  of  $e$  at  $\alpha_0$ . Then by some computations, we get  $(e_{\alpha_0}^{-1})' = e_{\alpha_0}^{\prime-1} = \frac{-3\alpha_0^3}{n^2}$  and  $(e_{\alpha_0}^{-1})' \Sigma_{\alpha_0} ((e_{\alpha_0}^{-1})') := V^{\log}(\alpha_0) = \frac{11}{10}\alpha_0^2 + \frac{1}{2}\alpha_0^4 + \frac{13}{20}\alpha_0^6$ . Hence this theorem is proved.

**Proof of Theorem 4.3.3:**

Let  $\Omega_n$  be the set where  $\frac{1}{\mathcal{D}} \left( \frac{\frac{1}{n} \sum_{j=0}^{n-1} \left| \left\{ X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0}) \right\} \right|^r}{\frac{1}{n} \sum_{j=0}^{n-1} \left| \left\{ X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0}) \right\} \right|^{2r}} \right)^2$  is in the range of the function  $g$  defined by (4.3.12). Observe that Assumption 4.4.1 is satisfies with the choice  $f(y) := |y|^r$  and  $f(y) := |y|^{2r}$  for  $r \in (0, \alpha/4)$  then applying Proposition 4.4.1, we obtain

$$\begin{cases} \frac{1}{n} \sum_{j=0}^{n-1} \left| \frac{n^{1/\alpha_0}}{\sigma} \left\{ X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0}) \right\} \right|^r \xrightarrow[\mathbb{P}]{n \rightarrow \infty} \mathcal{C}(\alpha_0, r) \\ \frac{1}{n} \sum_{j=0}^{n-1} \left| \frac{n^{1/\alpha_0}}{\sigma} \left\{ X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0}) \right\} \right|^{2r} \xrightarrow[\mathbb{P}]{n \rightarrow \infty} \mathcal{C}(\alpha_0, 2r). \end{cases}$$

Combining this with (4.3.9), we can deduce that

$$(4.5.7) \quad \frac{1}{\mathcal{D}} \left( \frac{\frac{1}{n} \sum_{j=0}^{n-1} \left| \left\{ X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0}) \right\} \right|^r}{\frac{1}{n} \sum_{j=0}^{n-1} \left| \left\{ X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0}) \right\} \right|^{2r}} \right)^2 \xrightarrow[\mathbb{P}]{n \rightarrow \infty} \frac{\Gamma(1-r/\alpha_0)^2}{\Gamma(1-2r/\alpha_0)},$$

hence  $\mathbb{P}(\Omega_n)$  tends to one as  $n \rightarrow \infty$  by the convergence in probability (4.5.7). Then, in this case

$g^{-1} \left( \frac{1}{\mathcal{D}} \left( \frac{\frac{1}{n} \sum_{j=0}^{n-1} \left| \left\{ X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0}) \right\} \right|^r}{\frac{1}{n} \sum_{j=0}^{n-1} \left| \left\{ X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0}) \right\} \right|^{2r}} \right)^2 \right)$  exists. Since the function  $g$  is strictly increasing then the estimator  $\hat{\alpha}_{r,n}$  is uniquely determined as

$$\hat{\alpha}_{r,n} = g^{-1} \left( \frac{1}{\mathcal{D}} \left( \frac{\frac{1}{n} \sum_{j=0}^{n-1} \left| \left\{ X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0}) \right\} \right|^r}{\frac{1}{n} \sum_{j=0}^{n-1} \left| \left\{ X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0}) \right\} \right|^{2r}} \right)^2 \right).$$

From the continuity of  $g^{-1}$  and from the convergence (4.5.7), we can deduce the result of this theorem.

**Proof of Theorem 4.3.4:**

We will first prove the following convergence

(4.5.8)

$$\sqrt{n} \left( \left( \frac{1}{n} \sum_{j=0}^{n-1} \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right|^r \right) - \left( \begin{array}{c} \mathcal{C}(\alpha_0, r) \\ \mathcal{C}(\alpha_0, 2r) \end{array} \right) \right) \Rightarrow \mathcal{N} \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma_{11}(\alpha_0) & \Sigma_{12}(\alpha_0) \\ \Sigma_{12}(\alpha_0) & \Sigma_{22}(\alpha_0) \end{pmatrix} \right).$$

with  $\Sigma_{ik}(\alpha_0) = \mathcal{C}(\alpha_0, (i+k)r) - \mathcal{C}(\alpha_0, ir)\mathcal{C}(\alpha_0, kr)$  for  $i, k \in 1, 2$  and noting from (4.3.7) that  $\mathbb{E}(|L_1^{\alpha_0}|^r) = \mathcal{C}(\alpha_0, r)$ ,  $\mathbb{E}(|L_1^{\alpha_0}|^{2r}) = \mathcal{C}(\alpha_0, 2r)$ .

Assumption 4.4.2 is satisfied with the choice  $(f_l(y))_{l=1,2} = |y|^{lr}$  for  $r \in (0, \alpha/4)$  then using the result of Proposition 4.4.2 we can deduce (4.5.8).

Now we denote by

$$W_n^{\alpha_0} := \frac{1}{\mathcal{D}} \frac{\left( \frac{1}{n} \sum_{j=0}^{n-1} \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right|^r \right)^2}{\frac{1}{n} \sum_{j=0}^{n-1} \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right|^{2r}},$$

where  $\mathcal{D}$  is given by (4.3.14).

Then  $W_n^{\alpha_0}$  can be written as  $h \left( \frac{1}{n} \sum_{j=0}^{n-1} \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right|^r, \frac{1}{n} \sum_{j=0}^{n-1} \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right|^{2r} \right)$  for the function  $h(x, y) = \frac{x^2}{\mathcal{D}y}$ . The map  $h$  is differentiable at the point  $\theta = (\mathcal{C}(\alpha_0, r), \mathcal{C}(\alpha_0, 2r))^T$ , with derivative  $\nabla h(\theta) = \left( \frac{\partial h}{\partial x}(\theta), \frac{\partial h}{\partial y}(\theta) \right)^T = \left( \frac{2\mathcal{C}(\alpha_0, r)}{\mathcal{D}(\mathcal{C}(\alpha_0, 2r))}, \frac{-\mathcal{C}(\alpha_0, r)^2}{\mathcal{D}(\mathcal{C}(\alpha_0, 2r)^2)} \right)^T$ .

Applying the delta method we have

(4.5.9)

$$\begin{aligned} \sqrt{n} \left( h \left( \frac{1}{n} \sum_{j=0}^{n-1} \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right|^r, \frac{1}{n} \sum_{j=0}^{n-1} \left| \frac{n^{1/\alpha_0}}{\sigma} \{X_{t_{j+1}}^{\alpha_0} - \varsigma_{1/n}(X_{t_j}^{\alpha_0})\} \right|^{2r} \right) - h(\mathcal{C}(\alpha_0, r), \mathcal{C}(\alpha_0, 2r)) \right) \\ \Rightarrow \mathcal{N} \left( 0, (\nabla h(\theta))^T \begin{pmatrix} \Sigma_{11}(\alpha_0) & \Sigma_{12}(\alpha_0) \\ \Sigma_{12}(\alpha_0) & \Sigma_{22}(\alpha_0) \end{pmatrix} \nabla h(\theta) \right). \end{aligned}$$

After some computations, we obtain

$$(4.5.10) \quad (\nabla h(\theta))^T \begin{pmatrix} \Sigma_{11}(\alpha_0) & \Sigma_{12}(\alpha_0) \\ \Sigma_{12}(\alpha_0) & \Sigma_{22}(\alpha_0) \end{pmatrix} \nabla h(\theta)$$

$$(4.5.11) \quad = \frac{1}{\mathcal{D}^2} \left[ \frac{4\mathcal{C}(\alpha_0, r)^2}{\mathcal{C}(\alpha_0, 2r)} - \frac{4\mathcal{C}(\alpha_0, r)^3\mathcal{C}(\alpha_0, 3r)}{\mathcal{C}(\alpha_0, 2r)^3} + \frac{\mathcal{C}(\alpha_0, r)^4\mathcal{C}(\alpha_0, 4r)}{\mathcal{C}(\alpha_0, 2r)^4} - \frac{\mathcal{C}(\alpha_0, r)^4}{\mathcal{C}(\alpha_0, 2r)^2} \right] := \bar{V}^r(\alpha_0).$$



Let us recall that the function  $g(\alpha) := \frac{\Gamma(1-r/\alpha)^2}{\Gamma(1-2r/\alpha)} = \frac{\mathcal{C}(\alpha, r)^2}{\mathcal{D}\mathcal{C}(\alpha, 2r)}$  (see (4.3.10) and (4.3.12)). From (4.3.13) and using the delta method again we get that

$$(4.5.12) \quad \sqrt{n}(\hat{\alpha}_{r,n} - \alpha_0) = \sqrt{n} \left[ g^{-1}(W_n^{\alpha_0}) - g^{-1} \left( \frac{\mathcal{C}(\alpha_0, r)^2}{\mathcal{D}\mathcal{C}(\alpha_0, 2r)} \right) \right] \Rightarrow \mathcal{N}(0, (g_{\alpha_0}^{-1})' \bar{V}^r(\alpha_0) ((g_{\alpha_0}^{-1})'))$$

where  $\bar{V}^r(\alpha_0)$  is given by (4.5.11) and  $(g_{\alpha_0}^{-1})'$  is the derivative of  $g^{-1}$  at  $g(\alpha_0)$ .

From the inverse function theorem we have the the derivative of  $g^{-1}$  at  $g(\alpha_0)$  is the inverse  $g_{\alpha_0}^{\prime-1}$  of the derivative  $g'_{\alpha_0}$  of  $g$  at  $\alpha_0$ . Then by some computations, we get

$$(g_{\alpha_0}^{-1})' = g_{\alpha_0}^{\prime-1} = \frac{\alpha_0^2 \Gamma(1 - \frac{2r}{\alpha_0})}{2r \eta_r(\alpha_0) \Gamma(1 - \frac{r}{\alpha_0})^2},$$

where  $\eta_r(\alpha_0)$  is defined by (4.3.18) and

$$(g_{\alpha_0}^{-1})' \bar{V}^r(\alpha_0) ((g_{\alpha_0}^{-1})') := V^r(\alpha_0) = \frac{\alpha_0^4}{r^2 \eta_r(\alpha_0)^2} \left\{ \frac{C(\alpha_0, 2r)}{C(\alpha_0, r)^2} - \frac{C(\alpha_0, 3r)}{C(\alpha_0, r)C(\alpha_0, 2r)} + \frac{1}{4} \left( \frac{C(\alpha_0, 4r)}{C(\alpha_0, 2r)^2} - 1 \right) \right\}.$$

Hence this theorem is proved.

## 4.6 A simulation study

### 4.6.1 Comparison of asymptotic variances

For comparison between the asymptotic variances  $V^{\log}(\alpha_0)$  and  $V^r(\alpha_0)$  given in Theorem 4.3.2 and Theorem 4.3.4, we give, below, the graph (Figure 4.1) for the asymptotic variances of the estimators of  $\alpha_0$  (drawn by the Python code in Section 4.8.1).

From Figure 4.1, since the asymptotic variance  $V^r(\alpha_0)$  is just finite with  $r < \alpha_0/4$ , that is: we can not use the estimator  $\hat{\alpha}_{r,n}$  for estimating the true parameter  $\alpha_0$  in the case  $r \geq \alpha_0/4$ . It can be seen that the red line has been a slight increase with  $\alpha \in (0, 2)$ . This show that the asymptotic performance of  $\hat{\alpha}_{\log,n}$  gets worse for larger  $\alpha_0$  and better for smaller  $\alpha_0$ . For  $\alpha_0 < 1$ , the asymptotic variance of  $\hat{\alpha}_{\log,n}$  is the smallest thus the estimate  $\hat{\alpha}_{\log,n}$  is recommended in this case. While  $\hat{\alpha}_{r,n}$  with small  $r$  such that  $0.05 \sim 0.2$  is recommended for  $\alpha_0 > 1$ . We can see the result of Table 1 below as a demonstration of this remark.

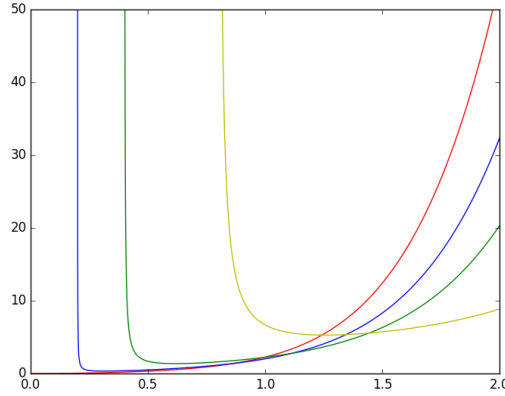


Figure 4.1: Plot of asymptotic variances of the estimators of  $\alpha_0$ . The panel containing the logarithmic moment based one (red line), the three lower-order moment based ones with  $r = 0.05$  (blue line),  $r = 0.1$  (green line) and  $r = 0.2$  (yellow line).

#### 4.6.2 Monte-Carlo Simulations

We will now focus on a simulation study to observe different finite-sample behaviors of the estimators according to the true value of  $\alpha_0$  developed in Section 4.3.2. In this section, we consider two different cases for the choice of the drift function  $b$  ( $b(x) = -x + \sin(2\pi x)$  and  $b(x) = -x$ ).

#### 4.6.3 The drift function $b(x) = -x + \sin(2\pi x)$

With this choice, we simulate and approximate  $X_t^{\alpha_0}$  by using the Euler scheme. Let  $n$  be the number of observations,  $M = 1000 \times n$  be the number of steps of Euler scheme,  $\Delta = \frac{1}{n}$  be the observation time frequency and  $\delta = \frac{1}{M}$  be the simulation step. Then in each simulation below, the Lévy process  $(L_t^{\alpha_0})_{t \in [0,1]}$  is a symmetric  $\alpha_0$ -stable Lévy process with the index  $\alpha_0 \in (0, 2)$  and the Euler scheme is

$$(4.6.1) \quad X_{t_{j+1}}^{\alpha_0} = X_{t_j}^{\alpha_0} + b(X_{t_j}^{\alpha_0})\delta + L_{t_{j+1}}^{\alpha_0} - L_{t_j}^{\alpha_0},$$

where  $t_j = j\delta$  for  $j = 0, \dots, M-1$ . The drift function in the simulation is  $b(x) = -x + \sin(2\pi x)$ , the initial value is  $X_0^\alpha = 1$  with the true parameter  $\alpha_0 = 1.5$ . In the figure 4.2, ten simulated sample paths of the process  $X_t^{\alpha_0}$  are considered in the code (see Section 4.8.2). In order to generate the Lévy sample path  $L_{t_{j+1}}^{\alpha_0} - L_{t_j}^{\alpha_0}$ , we generate first a symmetric  $\alpha_0$ -stable random sample, we use the

direct method in Janicki and Weron [38] (1994, pp.48). That is: we generate a random variable  $V$  uniformly distributed on  $(-\pi/2, \pi/2)$  and an exponential random variable  $W$  with mean 1. We compute the symmetric  $\alpha_0$ -stable random sample

$$S_{t_i} = \frac{\sin(\alpha_0 V_{t_i})}{\{\cos(V_{t_i})\}^{1/\alpha_0}} \times \left( \frac{\cos(V_{t_i} - \alpha_0 V_{t_i})}{W_{t_i}} \right)^{\frac{(1-\alpha_0)}{\alpha_0}}.$$

Last, we generate the Lévy increment by  $L_{t_{j+1}}^{\alpha_0} - L_{t_j}^{\alpha_0} = \delta^{1/\alpha_0} S_{t_j}$ .

Next, we proceed to the estimation of  $\alpha_0$  by considering the logarithmic moments and the three

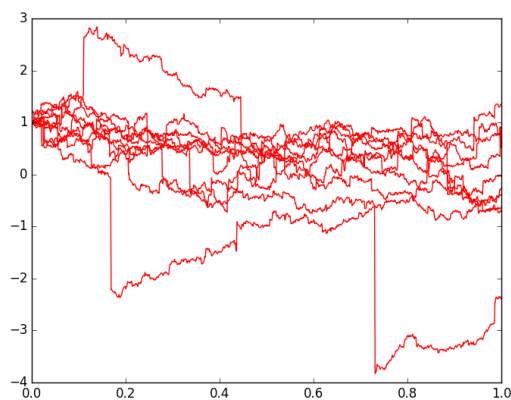


Figure 4.2: The ten simulated sample paths of the process  $X_t^{\alpha_0}$  with  $n = 1000$  and  $\alpha_0 = 1,5$ .

lower-order moments based ones with  $r = 0.05, 0.1$  and  $0.2$ . The index  $\alpha_0$  considered is  $0.8, 1, 1.5, 1.7$  while the sample size for observation  $n$  considered is  $500, 1000, 2000$ , respectively, the number of Euler scheme steps for simulation  $M = 1000 \times n$  and the simultaneously computed estimates based on 1000 replicates.

Note that in the formulas (4.3.6) and (4.3.10), the function  $\zeta$  appears in the estimators of  $\alpha_0$  therefore it is necessary to know the drift function  $b$  and this is a disadvantage of the method. We recall that the purpose of the introduction of the ordinary differential equation  $\zeta_t(x)$  is to deal with any value of  $\alpha$  belonging to  $(0, 2)$ . The simulation of the solution  $\zeta_t(x)$  is done by using the package "scipy.integrate.odeint" in python.

Table 1 below reports the results on the sample means and sample root mean-square errors (RMSEs) of the estimators

$$(4.6.2) \quad (\hat{\alpha}_{\log, n}, \hat{\alpha}_{0.05, n}, \hat{\alpha}_{0.1, n}, \hat{\alpha}_{0.2, n}, \tilde{\alpha}_n, \alpha^*(0.05), \alpha^*(0.1), \alpha^*(0.2))$$

4.6. A SIMULATION STUDY

for three sample sizes and four true index parameters which we have chosen before based on 1000 replicates.

	mean	RMSE	mean	RMSE	mean	RMSE
The true index $\alpha_0$	$\hat{\alpha}_{\log,n}$ with $n= 500$		$\hat{\alpha}_{\log,n}$ with $n= 1000$		$\hat{\alpha}_{\log,n}$ with $n = 2000$	
0.8	0.8037252	0.0455599	0.8032420	0.0345784	0.8016381	0.0237851
1.0	1.0104462	0.070802	1.0034732	0.0471853	1.0034792	0.0346555
1.5	1.5437813	0.1784205	1.5180485	0.1186146	1.51749272	0.0835110
1.7	1.8065441	0.2871730	1.7666206	0.1843406	1.7577750	0.1303583

	mean	RMSE	mean	RMSE	mean	RMSE
The true index $\alpha_0$	$\hat{\alpha}_{0.05,n}$ with $n = 500$		$\hat{\alpha}_{0.05,n}$ with $n = 1000$		$\hat{\alpha}_{0.05,n}$ with $n = 2000$	
0.8	0.8035548	0.0493085	0.80160196	0.0342533	0.8033393	0.0243324
1.0	1.0070387	0.0637764	1.0048068	0.0462132	1.0012119	0.0324376
1.5	1.5253815	0.1475756	1.5143174	0.0941305	1.5136220	0.0655977
1.7	1.6605543	0.2951940	1.6969005	0.2513790	1.7402026	0.1255374

	mean	RMSE	mean	RMSE	mean	RMSE
The true index $\alpha_0$	$\hat{\alpha}_{0.1,n}$ with $n = 500$		$\hat{\alpha}_{0.1,n}$ with $n = 1000$		$\hat{\alpha}_{0.1,n}$ with $n = 2000$	
0.8	0.8059862	0.0537238	0.8031538	0.03780156	0.8028442	0.0278225
1.0	1.0036059	0.0673361	1.0065038	0.0449074	1.0010474	0.0329233
1.5	1.5185346	0.1198106	1.5147109	0.0813918	1.5126050	0.0566897
1.7	1.6737466	0.2998005	1.7253146	0.1766232	1.7446146	0.0985915

CHAPTER 4. ESTIMATION OF THE INDEX PARAMETER FOR A STOCHASTIC DIFFERENTIAL EQUATION DRIVEN BY A STABLE LÉVY PROCESS

	mean	RMSE	mean	RMSE	mean	RMSE
The true index $\alpha_0$	$\hat{\alpha}_{0.2,n}$ with $n = 500$		$\hat{\alpha}_{0.2,n}$ with $n = 1000$		$\hat{\alpha}_{0.2,n}$ with $n = 2000$	
0.8	—	—	—	—	—	—
1.0	1.0234143	0.1016859	1.0111369	0.0747016	1.0049630	0.0526113
1.5	1.5297790	0.1082202	1.5167167	0.0723243	1.5110450	0.0526677
1.7	1.7252562	0.1986302	1.7415116	0.0927847	1.7437377	0.0566068

	mean	RMSE	mean	RMSE	mean	RMSE
The true index $\alpha_0$	$\tilde{\alpha}_n$ with $n = 500$		$\tilde{\alpha}_n$ with $n = 1000$		$\tilde{\alpha}_n$ with $n = 2000$	
0.8	0.7699008	0.0313345	0.7734386	0.0271524	0.7999666	0.0042935
1.0	1.0003541	0.0126898	1.0002279	0.0080386	1.0001365	0.0049952
1.5	1.6061061	0.1091086	1.5939569	0.0953270	1.5842965	0.0848937
1.7	1.8070126	0.1109151	1.7918403	0.0937689	1.7832002	0.0840492

	mean	RMSE	mean	RMSE	mean	RMSE
The true index $\alpha_0$	$\alpha^*(0.05)$ with $n = 500$		$\alpha^*(0.05)$ with $n = 1000$		$\alpha^*(0.05)$ with $n = 2000$	
0.8	0.7694143	0.0318793	0.7729256	0.0276266	0.7755499	0.0247046
1.0	0.9996752	0.0125434	0.9997976	0.0079439	0.9998605	0.0052414
1.5	1.6067400	0.1097934	1.5945983	0.0959445	1.5844919	0.0850574
1.7	1.8047055	0.1089453	1.7918403	0.0937689	1.7834100	0.0842196

	mean	RMSE	mean	RMSE	mean	RMSE
The true index $\alpha_0$	$\alpha^*(0.1)$ with $n = 500$		$\alpha^*(0.1)$ with $n = 1000$		$\alpha^*(0.1)$ with $n = 2000$	
0.8	0.7694723	0.0319901	0.7723942	0.0271524	0.7751606	0.0251329
1.0	0.9996210	0.0130434	0.9997550	0.0283150	1.0000276	0.0050884
1.5	1.6086234	0.1113341	1.5953294	0.0965713	1.5846691	0.0852433
1.7	1.8063842	0.1101661	1.7929472	0.0950375	1.7834307	0.0842211

	mean	RMSE	mean	RMSE	mean	RMSE
The true index $\alpha_0$	$\alpha^*(0.2)$ with n= 500		$\alpha^*(0.2)$ with n= 1000		$\alpha^*(0.2)$ with n = 2000	
0.8	0.7672141	0.0346441	0.7706410	0.0302085	0.7739378	0.0042935
1.0	0.9999466	0.0144433	0.9997626	0.0090004	0.9999333	0.0264721
1.5	1.6102755	0.1131355	1.5960832	0.0973221	1.5866909	0.0872366
1.7	1.8063371	0.1101885	1.7953793	0.0970049	1.7841309	0.0848353

In the Table 1, as  $n$  get larger, the RMSEs are getting better. The best estimates for  $n = 2000$  are  $\hat{\alpha}_{\log,n}$ ,  $\hat{\alpha}_{0.05,n}$ ,  $\hat{\alpha}_{0.2,n}$ , and  $\hat{\alpha}_{0.2,n}$ , for  $\alpha = 0.8, 1.0, 1.5$  and  $1.7$ , respectively (the red-letter elements).

#### Some practical remarks.

- We use  $\hat{\alpha}_{\log,n}$ , if we know  $\alpha_0$  is small, we still can have reasonable result even when the number of observation is small, for example 50.

	mean	RMSE	mean	RMSE	mean	RMSE
The true index $\alpha_0$	$\alpha_0 = 0.5$		$\alpha_0 = 0.6$		$\alpha_0 = 0.8$	
$\hat{\alpha}_{\log,n}$ with n=50	0.5279346	0.0854579	0.6299152	0.1077263	0.8566200	0.1801666

-As was expected from Figure 4.1, we observe that  $\hat{\alpha}_{\log,n}$  becomes more unstable for  $\alpha_0$  closer to 2. We need the number of observation enough large, for example 500, to obtain a good result for estimating  $\alpha_0 > 1.5$  (see Table 1).

- We obtain the better results for estimating  $\alpha_0 > 1$  by using  $\hat{\alpha}_{r,n}$  with small  $r$ . In practice, we can begin with a small  $r$  such as  $r = 0.1$  and then adaptively choice  $p$  according to the estimated value of  $\alpha_0$  (choosing  $r = 0.2$  if the estimate of  $\alpha_0$  is greater than 1.5). For  $\alpha_0 > 1.5$ , we need the number of observation  $n \geq 250$  for using the estimator  $\hat{\alpha}_{0.2,n}$ .

	mean	RMSE	mean	RMSE	mean	RMSE
The true index $\alpha_0$	$\alpha_0 = 1.5$		$\alpha_0 = 1.6$		$\alpha_0 = 1.7$	
$\hat{\alpha}_{0.2,n}$ with n=250	1.5233678	0.1547434	1.6323820	0.1787911	1.6728896	0.3043069
$\hat{\alpha}_{\log,n}$ with n=250	1.5691571	0.2751471	1.7049045	0.3536027	1.8491799	0.4390338

#### 4.6.4 The drift function $b(x) = -x$

In this case, the process  $(X^\alpha)$  is defined as a Lévy-driven Ornstein-Uhlenbeck process

$$dX_t^\alpha = -X_t^\alpha dt + dL_t^\alpha, t \in [0, 1]$$

which may also be written by Itô formula as

$$(4.6.3) \quad X_t^\alpha = e^{-t} X_0^\alpha + e^{-t} \int_0^t e^s dL_s^\alpha.$$

Here we consider the Lévy process  $(L_t^\alpha)_{t \in [0, 1]}$  is the symmetric  $\alpha$ -stable with the index  $\alpha \in (0, 2)$ .

For  $\Delta_n = t_{j+1} - t_j$ , we have

$$(4.6.4) \quad \begin{aligned} X_{t_{j+1}}^\alpha &= e^{-\Delta_n} X_{t_j}^\alpha + \int_{t_j}^{t_{j+1}} e^{-(t_{j+1}-s)} dL_s^\alpha \\ &\stackrel{\mathcal{L}}{=} e^{-\Delta_n} X_{t_j}^\alpha + e^{-\Delta_n} \int_0^{\Delta_n} e^u dL_u^\alpha \end{aligned}$$

where we have used the fact that  $L^\alpha$  has independent and stationary increments to get the last line. We observe that in order to simulate (4.6.4), we need to simulate  $\int_0^t e^u dL_u^\alpha$ . In order to do this, we first observe that by the Lévy-integral transform of the characteristic function (see Lemma 15.1 in [61]) and denoting by  $F^*(z) = \frac{C_\alpha}{|z|^{1+\alpha}} 1_{|z| \neq 0}$  the Lévy density of the Lévy measure of  $L_1^\alpha$  we get

$$(4.6.5) \quad \begin{aligned} \ln \mathbb{E} \left( e^{iv \int_0^t e^u dL_u^\alpha} \right) &= \int_0^t \ln \mathbb{E} \left( e^{ive^u L_1^\alpha} \right) du \\ &= \int_0^t \int_{\mathbb{R}} [\exp(ive^u z) - 1 - ive^u z 1_{|z| \leq 1}] F^*(z) dz du \\ &= \int_0^t \int_{\mathbb{R}} [\exp(ive^u z) - 1 - ive^u z 1_{|z| \leq \frac{1}{e^u}}] F^*(z) dz du \end{aligned}$$

where we have used the fact that  $F^*(z)$  is a symmetric function then  $\int_{|z| > \epsilon} z F^*(z) dz = 0$  for all  $\epsilon > 0$ .

Let us set  $z' = e^u z$  then (4.6.5) can be written by

$$\begin{aligned}
 & \int_0^t \int_{\mathbb{R}} [\exp(ivz') - 1 - ivz' 1_{|z'| \leq 1}] e^{-u} F^*(e^{-u} z') du dz' \\
 &= \int_{\mathbb{R}} [\exp(ivz') - 1 - ivz' 1_{|z'| \leq 1}] \left[ \int_0^t e^{-u} F^*(e^{-u} z') du \right] dz' \\
 &= \int_{\mathbb{R}} [\exp(ivz') - 1 - ivz' 1_{|z'| \leq 1}] \left( \int_0^t e^{-u} \frac{C_\alpha}{|e^{-u} z'|^{1+\alpha}} 1_{|z'| \neq 0} du \right) dz' \\
 &= \int_{\mathbb{R}} [\exp(ivz') - 1 - ivz' 1_{|z'| \leq 1}] \frac{C_\alpha}{|z'|^{1+\alpha}} 1_{|z'| \neq 0} \left( \int_0^t e^{u\alpha} du \right) dz' \\
 &= \ln \mathbb{E} \left( e^{ivL_m^\alpha} \right) \text{ where } m = \int_0^t e^{u\alpha} du.
 \end{aligned}$$

Hence we can deduce that  $\int_0^t e^u dL_u^\alpha = \mathcal{L} \left( \int_0^t e^{u\alpha} du \right)^{1/\alpha} L_1^\alpha$ . It follows that

$$\begin{aligned}
 (4.6.6) \quad X_{t_{j+1}}^\alpha &= \mathcal{L} e^{-\Delta_n} X_{t_j}^\alpha + e^{-\Delta_n} \left( \int_0^{\Delta_n} e^{u\alpha} du \right)^{1/\alpha} L_1^\alpha \\
 &= e^{-\Delta_n} X_{t_j}^\alpha + e^{-\Delta_n} \left( \frac{-1 + e^{\alpha \Delta_n}}{\alpha} \right)^{1/\alpha} L_1^\alpha
 \end{aligned}$$

and as a result it is possible to simulate exactly the process  $(X_{t_j}^\alpha)_j$  as soon as we simulate the stable variable.

**Algorithm for Ornstein-Uhlenbeck simulation:**

1. Simulate the symmetric  $\alpha$ -stable Lévy process  $L_1^\alpha$ :
  - 1.1. Simulate a random variable  $V$  uniformly distributed on  $(-\pi/2, \pi/2)$ .
  - 1.2. Simulate an exponential random variable  $W$  with mean 1.

We compute the symmetric  $\alpha$ -stable random sample by

$$S_{t_i} = \frac{\sin(\alpha V_{t_i})}{\{\cos(V_{t_i})\}^{1/\alpha}} \times \left( \frac{\cos(V_{t_i} - \alpha V_{t_i})}{W_{t_i}} \right)^{\frac{(1-\alpha)}{\alpha}}.$$

2. Sample the path of the Lévy driven Ornstein-Uhlenbeck process  $X^\alpha = (X_{\frac{i+1}{n}}^\alpha)_{i \in [0, n-1]}$

$$X_{\frac{i+1}{n}}^\alpha = e^{-\frac{1}{n}} X_{\frac{i}{n}}^\alpha + e^{-\frac{1}{n}} \left( \frac{-1 + e^{\frac{\alpha}{n}}}{\alpha} \right)^{1/\alpha} S_{\frac{i}{n}}.$$

Note that when  $b(x) = -x$  then the ordinary differential equation (4.2.2) can be solved explicitly, namely  $\zeta_t(x) = xe^{-t}$ .

Table 2 below reports the results on the sample means and sample root mean-square errors (RMSEs) of the estimators

$$(4.6.7) \quad (\hat{\alpha}_{\log, n}, \hat{\alpha}_{0.05, n}, \hat{\alpha}_{0.1, n}, \hat{\alpha}_{0.2, n}, \tilde{\alpha}_n)$$



CHAPTER 4. ESTIMATION OF THE INDEX PARAMETER FOR A STOCHASTIC DIFFERENTIAL EQUATION DRIVEN BY A STABLE LÉVY PROCESS

---

for three sample sizes and three true index parameters which we have chosen before based on 1000 replicates.

	mean	RMSE	mean	RMSE	mean	RMSE
The true index $\alpha_0$	$\hat{\alpha}_{\log,n}$ with $n= 500$		$\hat{\alpha}_{\log,n}$ with $n= 1000$		$\hat{\alpha}_{\log,n}$ with $n = 2000$	
0.8	0.8039770	0.0469657	0.8034170	0.0331716	0.7934735	0.0238552
1.0	1.0057761	0.0685933	1.0017601	0.0497605	0.9969568	0.0332515
1.5	1.5284446	0.1657480	1.5198140	0.1150394	1.4971832	0.0754708
1.7	1.7413834	0.2319133	1.72844916	0.1679066	1.7007702	0.1054180

	mean	RMSE	mean	RMSE	mean	RMSE
The true index $\alpha_0$	$\hat{\alpha}_{0.05,n}$ with $n = 500$		$\hat{\alpha}_{0.05,n}$ with $n = 1000$		$\hat{\alpha}_{0.05,n}$ with $n = 2000$	
0.8	0.8027989	0.0498397	0.8038067	0.0336671	0.7914927	0.0253941
1.0	1.0065823	0.0639163	1.0053305	0.0450491	0.9922054	0.0336705
1.5	1.5185111	0.1446205	1.5165603	0.0939864	1.4942323	0.0680315
1.7	1.6352202	0.3005584	1.6945244	0.1513708	1.6901503	0.1034397

	mean	RMSE	mean	RMSE	mean	RMSE
The true index $\alpha_0$	$\hat{\alpha}_{0.1,n}$ with $n = 500$		$\hat{\alpha}_{0.1,n}$ with $n = 1000$		$\hat{\alpha}_{0.1,n}$ with $n = 2000$	
0.8	0.8081127	0.0533465	0.8038097	0.0383832	0.7906515	0.0306369
1.0	1.0096465	0.0657211	1.0021536	0.0465007	0.9929123	0.0348359
1.5	1.5216986	0.1152965	1.5096984	0.0817859	1.4888127	0.0587072
1.7	1.6713295	0.2234651	1.7066020	0.1429506	1.6902396	0.07167946

4.7. ESTIMATION OF THE INDEX PARAMETER FOR THE UNKNOWN DRIFT FUNCTION  
WHEN  $\alpha \in (1, 2)$ .

	mean	RMSE	mean	RMSE	mean	RMSE
The true index $\alpha_0$	$\hat{\alpha}_{0.2,n}$ with $n = 500$		$\hat{\alpha}_{0.2,n}$ with $n = 1000$		$\hat{\alpha}_{0.2,n}$ with $n = 2000$	
0.8	—	—	—	—	—	—
1.0	1.0173278	0.0911352	1.0124521	0.0643748	0.9898929	0.060756
1.5	1.5155668	0.1033405	1.5082002	0.0734346	1.4860405	0.0545570
1.7	1.7020924	0.1411906	1.7099061	0.0793526	1.6830948	0.0633612

	mean	RMSE	mean	RMSE	mean	RMSE
The true index $\alpha_0$	$\tilde{\alpha}_n$ with $n= 500$		$\tilde{\alpha}_n$ with $n= 1000$		$\tilde{\alpha}_n$ with $n = 2000$	
0.8	0.7999835	0.0094323	0.7999789	0.0059331	0.8016138	0.0040069
1.0	1.0001276	0.0123388	0.9998303	0.0077497	1.0008398	0.0053345
1.5	1.6061061	0.1091086	1.5001159	0.0138787	1.5005670	0.0091295
1.7	1.6992440	0.0268464	1.6991394	0.0168119	1.7009518	0.0111302

**4.7 Estimation of the index parameter for the unknown drift function when  $\alpha \in (1, 2)$ .**

The introduction of the ordinary differential equation  $\zeta_t(x)$  is to deal with any value of  $\alpha$  belonging to  $(0, 2)$ . However, the appearance of  $\zeta$  in the estimators of  $\alpha$  makes necessary to know the drift function. If  $\alpha \in (1, 2)$ , we can replace  $\zeta_t(x)$  by  $x$  and then build some estimators which do not require to know the drift function  $b$ . Namely, in this case, the estimators  $\hat{\alpha}_{\log,n}$  and  $\hat{\alpha}_{r,n}$  are replaced by  $\hat{\alpha}_{\log,n}^*$  and  $\hat{\alpha}_{r,n}^*$  given respectively by

$$\hat{\alpha}_{\log,n}^* = \left\{ \frac{6}{n\pi^2} \sum_{j=0}^{n-1} \left( \log \left| \{X_{t_{j+1}}^{\alpha_0} - X_{t_j}^{\alpha_0}\} \right| - \frac{1}{n} \sum_{j=0}^{n-1} \log \left| \{X_{t_{j+1}}^{\alpha_0} - X_{t_j}^{\alpha_0}\} \right| \right)^2 - \frac{1}{2} \right\}^{-1/2},$$

$$\hat{\alpha}_{r,n}^* = g^{-1} \left( \frac{1}{\mathcal{D}} \frac{\left( \frac{1}{n} \sum_{j=0}^{n-1} \left| \{X_{t_{j+1}}^{\alpha_0} - X_{t_j}^{\alpha_0}\} \right|^r \right)^2}{\frac{1}{n} \sum_{j=0}^{n-1} \left| \{X_{t_{j+1}}^{\alpha_0} - X_{t_j}^{\alpha_0}\} \right|^{2r}} \right).$$

where  $\mathcal{D}$  and the function  $g$  are defined by (4.3.14) and (4.3.12) respectively.

We assume that the assumption  $\mathbf{H}_0$  is replaced by the new assumption  $\bar{\mathbf{H}}_0$ :

$\bar{\mathbf{H}}_0$   $b : \mathbb{R} \rightarrow \mathbb{R}$  is a bounded function with bounded derivatives up to order two.

Then we prove in this section, that under the assumptions  $\bar{\mathbf{H}}_0$  and  $\mathbf{H}_1$ , then the rate for the law of large number is at least  $n^{1/2-1/2\alpha}$  for  $\alpha \in (1, 2)$ .

**Proposition 4.7.1.** *We set  $u_n = n^{\frac{1}{2}-\frac{1}{2\alpha}}$ , under Assumption 4.4.2 then for  $\alpha \in (1, 2)$*

$$u_n \left( \frac{1}{n} \sum_{j=0}^{n-1} f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\} \right) - \int f(y) \varphi_\alpha(y) dy \right) \xrightarrow{\mathbb{P}} 0.$$

From now on, we choose  $K := K_n = n^{(1-\frac{1}{\alpha})\frac{1}{2+\delta}}$  and we denote

$$(4.7.1) \quad G_n^*(\alpha) := \frac{1}{n} \sum_{i=0}^{n-1} f \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\} \right),$$

$$(4.7.2) \quad G_{n,K}^*(\alpha) = \frac{1}{n} \sum_{j=0}^{n-1} f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\} \right).$$

We observe that from notations (4.2.8)-(4.2.9), (4.4.4) with  $u_n = n^{\frac{1}{2}-\frac{1}{2\alpha}}$ , we have

$$\begin{aligned} u_n |G_n^*(\alpha) - g(\alpha)| &\leq u_n |G_n^*(\alpha) - G_{n,K}^*(\alpha)| + u_n |G_{n,K}^*(\alpha) - H_{n,K}(\alpha)| \\ &\quad + u_n |H_{n,K}(\alpha) - H_n(\alpha)| + u_n |H_n(\alpha) - g(\alpha)| \end{aligned}$$

From Proposition 4.4.4 and the fact that  $\frac{u_n}{\sqrt{n}} \xrightarrow{n \rightarrow \infty} 0$ , we have  $u_n |H_n(\alpha) - g(\alpha)| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$ . Then

Proposition 4.7.1 will be proved if we can prove that

$$u_n \mathbb{E} \left( |G_n^*(\alpha) - G_{n,K}^*(\alpha)| \right) \xrightarrow{n \rightarrow \infty} 0, \quad u_n \mathbb{E} (|H_n(\alpha) - H_{n,K}(\alpha)|) \xrightarrow{n \rightarrow \infty} 0 \quad (\text{Lemma 4.7.1 below})$$

$$\text{and } u_n \left( |G_{n,K}^*(\alpha) - H_{n,K}(\alpha)| \right) \xrightarrow{n \rightarrow \infty} 0. \quad (\text{Lemma 4.7.3 below}).$$

**Remark 4.7.1.** *In this part, we choose  $K := K_n = n^{(1-\frac{1}{\alpha})\frac{1}{2+\delta}}$  for the proof which is different with the choice in Lemma 4.4.3. So we can not immediately obtain  $u_n \mathbb{E} (|H_n(\alpha) - H_{n,K}(\alpha)|) \xrightarrow{n \rightarrow \infty} 0$  from Lemma 4.4.3-2).*

**Lemma 4.7.1.** *Under Assumption 4.4.2, we have, with  $u_n = n^{\frac{1}{2}-\frac{1}{2\alpha}}$  and  $K := K_n = n^{(1-\frac{1}{\alpha})\frac{1}{2+\delta}}$  where  $\delta$  satisfies  $p(2+\delta) < \alpha$ ,*

$$(i) \quad u_n \mathbb{E} \left( |G_n^*(\alpha) - G_{n,K}^*(\alpha)| \right) \xrightarrow{n \rightarrow \infty} 0$$

$$(ii) \quad u_n \mathbb{E} (|H_n(\alpha) - H_{n,K}(\alpha)|) \xrightarrow{n \rightarrow \infty} 0$$

where  $G_n(\alpha)^*$ ,  $G_{n,K}^*(\alpha)$ ,  $H_n(\alpha)$  and  $H_{n,K}(\alpha)$  are defined by (4.7.1), (4.7.2), (4.2.8) and (4.4.4) respectively.

**Proof of Lemma 4.7.1** The idea of the proof is the same as Lemma 4.4.3. It is enough to prove the first assertion as using it with the drift function  $b \equiv 0, \sigma = 1$  and  $x = 0$ , we deduce immediately the result of the second one. Using  $|f(x) - f_K(x)| \leq |f(x)|1_{|x| \geq K^{1/p}} + f(x)1_{|x| \leq \frac{1}{K^2}}$  then we have

$$\begin{aligned} & u_n \mathbb{E} \left( \left| G_n^*(\alpha) - G_{n,K}^*(\alpha) \right| \right) \\ &= u_n \mathbb{E} \left[ \left| \frac{1}{n} \sum_{j=0}^{n-1} f \left( n^{1/\alpha} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\} \right) - \frac{1}{n} \sum_{i=0}^{n-1} f_K \left( n^{1/\alpha} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\} \right) \right| \right] \\ (4.7.3) \quad & \leq \frac{u_n}{n} \sum_{j=0}^{n-1} \mathbb{E} \left| f \left( n^{1/\alpha} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\} \right) \right| 1_{|n^{1/\alpha} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\}| \geq K^{1/p}} \end{aligned}$$

$$(4.7.4) \quad + \frac{u_n}{n} \sum_{j=0}^{n-1} \mathbb{E} \left| f \left( n^{1/\alpha} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\} \right) \right| 1_{|n^{1/\alpha} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\}| \leq \frac{1}{K^2}}.$$

Let us estimate  $\mathbb{E} \left| f \left( n^{1/\alpha} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\} \right) \right| 1_{|n^{1/\alpha} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\}| \geq K^{1/p}}$ .

Using  $\alpha > 1$  and the drift function  $b$  bounded then we can estimate, for  $C$  is a positive constant

$$\begin{aligned} (4.7.5) \quad & \left| n^{1/\alpha} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\} \right| = \left| n^{1/\alpha} \int_{t_j}^{t_{j+1}} b(X_s^\alpha) ds + n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right| \\ & \leq \frac{1}{n^{1-1/\alpha}} \|b\|_\infty + \left| n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right| \leq C + \left| n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right|. \end{aligned}$$

Using Assumption 4.4.2(2) and combining with the above estimate, we get

$$\begin{aligned} (4.7.6) \quad & \mathbb{E} \left( \left| f \left( n^{1/\alpha} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\} \right) \right| 1_{|n^{1/\alpha} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\}| > K^{1/p}} \right) \\ & \leq \mathbb{E} \left( \left| n^{1/\alpha} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\} \right|^p 1_{|(n^{1/\alpha} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\})| > K} \right) \\ & \leq \mathbb{E} \left( \left[ C + \left| n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right| \right]^p 1_{[C + |n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\}|]^p > K} \right). \end{aligned}$$

Similarly to the estimate of (4.4.35), we obtain that (4.7.6) is bounded by

$$\frac{M}{K^{1+\delta}} \mathbb{E} \left[ C^{(2+\delta)p} + \left| n^{1/\alpha} \{Z_{1/n}^\alpha\} \right|^{(2+\delta)p} \right] \text{ for } \delta \in (0, 2) \text{ chosen such that } (2+\delta)p < \alpha.$$

Hence, with the choice  $K := K_n = n^{(1-\frac{1}{\alpha})\frac{1}{2+\delta}}$ , we can deduce that (4.7.3) is bounded by

$$(4.7.7) \quad \frac{M u_n}{K^{1+\delta}} \mathbb{E} \left[ C^{(2+\delta)p} + \left| n^{1/\alpha} \{Z_{1/n}^\alpha\} \right|^{(2+\delta)p} \right] = \frac{M}{n^{(\frac{1}{2}-\frac{1}{2\alpha})\frac{\delta}{2+\delta}}} \mathbb{E} \left[ C^{(2+\delta)p} + \left| n^{1/\alpha} \{Z_{1/n}^\alpha\} \right|^{(2+\delta)p} \right]$$

where  $C$  and  $M$  are some positive constants. Using (4.4.10) with  $r = p(2+\delta) < \alpha$  and the fact that  $\alpha > 1$  we get that (4.7.7) converges to zero as  $n \rightarrow \infty$ . Hence (4.7.3) converges to zero as  $n \rightarrow \infty$ .

Let us estimate (4.7.4). We easily see that

$$\begin{aligned} & \mathbb{E} \left( \left| f \left( n^{1/\alpha} \{ X_{t_{j+1}}^\alpha - X_{t_j}^\alpha \} \right) \right| \mathbf{1}_{|n^{1/\alpha} \{ X_{t_{j+1}}^\alpha - X_{t_j}^\alpha \}| \leq \frac{1}{K^2}} \right) \\ &= \mathbb{E} \left[ \mathbb{E} \left( \left| f \left( n^{1/\alpha} \{ X_{t_{j+1}}^\alpha - X_{t_j}^\alpha \} \right) \right| \mathbf{1}_{|n^{1/\alpha} \{ X_{t_{j+1}}^\alpha - X_{t_j}^\alpha \}| \leq \frac{1}{K^2}} \middle| \mathcal{F}_{t_j} \right) \right] \\ &= \mathbb{E} \left[ \mathbb{E} \left( \left| f \left( n^{1/\alpha} \{ X_{t_{j+1}}^\alpha - X_{t_j}^\alpha \} \right) \right| \mathbf{1}_{|n^{1/\alpha} \{ X_{t_{j+1}}^\alpha - X_{t_j}^\alpha \}| \leq \frac{1}{K^2}} \middle| X_{t_j}^\alpha \right) \right] \end{aligned}$$

where we have used the Markov property to get the last line. Moreover, using equations (10) and (52) in [17], we have

$$(4.7.8) \quad \sup_x \sup_{u \in \mathbb{R}} \sup_n \frac{\sigma}{n^{1/\alpha}} p_{\frac{1}{n}}^\alpha \left( x, \frac{u\sigma}{n^{1/\alpha}} + x \right) < \infty,$$

then we can deduce that

$$\begin{aligned} (4.7.9) \quad & \mathbb{E} \left( \left| f \left( \frac{n^{1/\alpha}}{\sigma} \{ X_{t_{j+1}}^\alpha - X_{t_j}^\alpha \} \right) \right| \mathbf{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \{ X_{t_{j+1}}^\alpha - X_{t_j}^\alpha \} \right| \leq \frac{1}{K^2}} \middle| X_{t_j}^\alpha = x \right) \\ &= \int_{-\frac{1}{K^2}}^{\frac{1}{K^2}} |f(u)| \left[ \frac{\sigma}{n^{1/\alpha}} p_{\frac{1}{n}}^\alpha \left( x, \frac{u\sigma}{n^{1/\alpha}} + x \right) \right] du \leq C_1 \int_{-\frac{1}{K^2}}^{\frac{1}{K^2}} |f(u)| du \\ &= 2C_1 \int_0^{\frac{1}{K^2}} |f(u)| du \leq 2C_1 C \int_0^{\frac{1}{K^2}} |\log|u||^2 du + \frac{2C_1 C}{K^2} \end{aligned}$$

where we have used (4.7.8),  $f$  is a symmetric function and Assumption 4.4.2 (1). Hence, we obtain that (4.7.4) is bounded by

$$2C_1 C u_n \int_0^{\frac{1}{K^2}} |\log|u||^2 du + \frac{2C_1 C u_n}{K^2}.$$

We observe that, with the choice  $K := K_n = n^{(1-\frac{1}{\alpha})\frac{1}{2+\delta}}$  and  $u_n = n^{1/2-1/2\alpha}$ , we have

$$\begin{aligned} (4.7.10) \quad & \frac{u_n}{K^2} = \frac{1}{n^{(\frac{1}{2}-\frac{1}{2\alpha})\frac{2-\delta}{2+\delta}}} \xrightarrow{n \rightarrow \infty} 0, \\ & u_n \int_0^{\frac{1}{K^2}} |\log|u||^2 du = \frac{2u_n}{K^2} + \frac{4u_n}{K^2} \log K + \frac{4u_n}{K^2} (\log(K))^2 \\ &= \frac{2}{n^{(\frac{1}{2}-\frac{1}{2\alpha})\frac{2-\delta}{2+\delta}}} + \frac{4 \log \left( n^{(1-\frac{1}{\alpha})\frac{1}{2+\delta}} \right)}{n^{(\frac{1}{2}-\frac{1}{2\alpha})\frac{2-\delta}{2+\delta}}} + \frac{4 \left( \log \left( n^{(1-\frac{1}{\alpha})\frac{1}{2+\delta}} \right) \right)^2}{n^{(\frac{1}{2}-\frac{1}{2\alpha})\frac{2-\delta}{2+\delta}}} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

This implies that (4.7.4) goes to zeros as  $n \rightarrow \infty$ . Hence this lemma is proved. ■

In order to prove  $u_n \left( \left| G_{n,K}^*(\alpha) - H_{n,K}(\alpha) \right| \right) \xrightarrow{n \rightarrow \infty} 0$  (Lemma 4.7.3 below), we first show Lemma 4.7.2.

**Lemma 4.7.2.** *Let  $X_{\frac{1}{n}}^{\alpha, x}$  be the solution to (4.1.1) starting from  $X_0^\alpha = x$ . We suppose that Assumption 4.4.2 holds then*

$$\sup_x \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - x \right\} \right) - f_K \left( n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right) \right]^2 \xrightarrow{n \rightarrow \infty} 0.$$

where  $f_K(x) := f(x)\mathcal{X}_K(x)$  with  $\mathcal{X}_K$  is defined by (4.4.1) and (4.4.2) and with  $K := K_n = n^{(1-\frac{1}{\alpha})\frac{1}{2+\delta}}$  where  $\delta > 0$  satisfies  $p(2+\delta) < \alpha$ .

**Proof of Lemma 4.7.2** The idea of the proof is the same as Lemma 4.4.5.

For all  $\epsilon_n > 0$ , we can write

$$(4.7.11) \quad \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - x \right\} \right) - f_K \left( n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right) \right]^2$$

$$(4.7.12) \quad = \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - x \right\} \right) - f_K \left( n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right) \right]^2 \mathbb{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - x \right\} - n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right| \leq \epsilon_n}$$

$$(4.7.13) \quad + \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - x \right\} \right) - f_K \left( n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right) \right]^2 \mathbb{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - x \right\} - n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right| > \epsilon_n}.$$

Using (4.4.43), we can estimate

$$(4.7.14) \quad \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - x \right\} \right) - f_K \left( n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right) \right]^2 \mathbb{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - x \right\} - n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right| \leq \epsilon_n}$$

$$\leq (\|f'_K\|_\infty)^2 \epsilon_n^2 = C (K + K^2 \log^2(K))^2 \epsilon_n^2 = C \left[ \left( n^{(1-\frac{1}{\alpha})\frac{1}{2+\delta}} + n^{(1-\frac{1}{\alpha})\frac{2}{2+\delta}} \log^2(n^{(1-\frac{1}{\alpha})\frac{1}{2+\delta}}) \right) \epsilon_n \right]^2.$$

We choose  $\epsilon_n = n^{(\frac{1}{\alpha}-1)(\frac{1}{2}+\frac{1}{2+\delta})}$  then we can easily deduce that  $\left( n^{(1-\frac{1}{\alpha})\frac{1}{2+\delta}} + n^{(1-\frac{1}{\alpha})\frac{2}{2+\delta}} \log^2(n^{(1-\frac{1}{\alpha})\frac{1}{2+\delta}}) \right) \epsilon_n$  goes to zero as  $n \rightarrow \infty$ . This implies that (4.7.12) tends to zero as  $n \rightarrow \infty$ .

We now consider (4.7.13). From  $|f_K| \leq |f|$ , we have that (4.7.13) is bounded by

$$(4.7.15) \quad 2\mathbb{E} \left[ f \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - x \right\} \right) \right]^2 \mathbb{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - x \right\} - n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right| > \epsilon_n}$$

$$(4.7.16) \quad + 2\mathbb{E} \left[ f \left( n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right) \right]^2 \mathbb{1}_{\left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - x \right\} - n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right| > \epsilon_n}.$$

Let us consider (4.7.15). Combining (4.4.47) with the Markov's inequality, we obtain that (4.7.15) is bounded by

$$(4.7.17) \quad \left[ \mathbb{E} \left[ f \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - x \right\} \right) \right]^{2+\delta} \right]^{\frac{2}{2+\delta}} \left[ \frac{1}{\epsilon_n} \mathbb{E} \left( \left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - x \right\} - n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right| \right) \right]^{\frac{\delta}{2+\delta}}$$

We observe that

$$\left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - x \right\} - n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right| = \left| \int_0^{1/n} n^{1/\alpha} b(X_s^{\alpha,x}) ds \right| \leq \frac{\|b\|_\infty}{n^{1-1/\alpha}}$$

where  $\|b\|_\infty = \sup_{x \in \mathbb{R}} |b(x)|$ . Then (4.7.17) is bounded by

$$(4.7.18) \quad \begin{aligned} & \left( \frac{\|b\|_\infty}{n^{1-1/\alpha}} \right)^{\frac{\delta}{2+\delta}} \frac{1}{c_n^{\frac{\delta}{2+\delta}}} \left[ \mathbb{E} \left[ f \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - x \right\} \right) \right]^{2+\delta} \right]^{\frac{2}{2+\delta}} \\ &= \frac{C}{n^{\frac{\delta}{2+\delta} (1-\frac{1}{\alpha}) (\frac{1}{2} - \frac{1}{(2+\delta)})}} \left[ \mathbb{E} \left[ f \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - x \right\} \right) \right]^{2+\delta} \right]^{\frac{2}{2+\delta}}. \end{aligned}$$

Then, in order to show that (4.7.17) converges to zero as  $n \rightarrow \infty$ , we need to show that

$\mathbb{E} \left[ f \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - x \right\} \right) \right]^{2+\delta}$  is bounded by a positive constant that does not depend on  $n$ . Proceeding as in the proof of Lemma 4.4.4 and using (4.7.8), we get that

$$\sup_n \mathbb{E} \left[ f \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - x \right\} \right) \right]^{2+\delta} < \infty.$$

Hence this yields that (4.7.17) tends to zero as  $n \rightarrow \infty$  and thus (4.7.15) goes to zero as  $n \rightarrow \infty$ .

Let us consider (4.7.16). Combining (4.4.51) and the Markov's inequality, we obtain that (4.7.16) is bounded by

$$\begin{aligned} & \left[ \mathbb{E} \left[ f \left( n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right) \right]^{2+\delta} \right]^{\frac{2}{2+\delta}} \frac{1}{c_n^{\frac{\delta}{2+\delta}}} \left[ \mathbb{E} \left( \left| \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha,x} - x \right\} - n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right| \right) \right]^{\frac{\delta}{2+\delta}} \\ & \leq \frac{C}{n^{\frac{\delta}{2+\delta} (1-\frac{1}{\alpha}) (\frac{1}{2} - \frac{1}{(2+\delta)})}} \left[ \mathbb{E} \left[ f \left( n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right) \right]^{2+\delta} \right]^{\frac{2}{2+\delta}} \text{ from (4.7.18),} \end{aligned}$$

for  $\delta \in (0, 2)$  is chosen such that  $(2+\delta)p < \alpha$ . Combining this with (4.4.52), we can deduce that (4.7.16) tends to zero as  $n \rightarrow \infty$ . From the above estimates, we can deduce that (4.7.13) goes to zero as  $n \rightarrow \infty$ . Hence this lemma is proved.  $\blacksquare$

**Lemma 4.7.3.** *We assume that Assumption 4.4.2 holds and under the notations (4.4.4)-(4.7.2) then*

$$u_n \left( G_{n,K}^*(\alpha) - H_{n,K}(\alpha) \right) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

**Proof.** The idea of the proof is the same as Lemma 4.4.6. We use Lemma 9 from Genon-Catalot and Jacod [25] then it is sufficient to show that

$$(4.7.19) \quad \frac{u_n}{n} \sum_{j=0}^{n-1} \left| \mathbb{E} \left[ \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\} \right) - f_K \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \right] \middle| \mathcal{F}_{t_j} \right] \right| \xrightarrow[\mathbb{P}]{n \rightarrow \infty} 0,$$

$$(4.7.20) \quad \frac{u_n^2}{n^2} \sum_{j=0}^{n-1} \left| \mathbb{E} \left[ \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\} \right) - f_K \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \right]^2 \middle| \mathcal{F}_{t_j} \right] \right| \xrightarrow[\mathbb{P}]{n \rightarrow \infty} 0.$$

Looking at the proof of Proposition 9 in [17], we can deduce that

$$(4.7.21) \quad \sup_x \left| \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - x \right\} \right) - f_K \left( n^{1/\alpha} \left\{ Z_{\frac{1}{n}}^\alpha \right\} \right) \right] \right| \leq \frac{C \|f_K\|_\infty}{n^{1-1/\alpha}}$$

where  $C$  is a positive constant. From the Markov property, we have

$$(4.7.22) \quad \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\} \right) \middle| \mathcal{F}_{t_j} \right] = \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\} \right) \middle| X_{t_j}^\alpha \right].$$

But we can write

$$(4.7.23) \quad \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\} \right) \middle| X_{t_j}^\alpha = x \right] = \mathbb{E} \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - x \right\} \right) \right]$$

where  $X_{\frac{1}{n}}^{\alpha, x}$  is the solution to (4.1.1) starting from  $X_0^\alpha = x$ . Combining this with (4.4.18), (4.4.21) and (4.7.21), we obtain

$$\frac{u_n}{n} \sum_{j=0}^{n-1} \left| \mathbb{E} \left[ \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\} \right) - f_K \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \right] \middle| \mathcal{F}_{t_j} \right] \right| \leq \frac{CK u_n}{n^{1-1/\alpha}}.$$

With the choice  $K := K_n = n^{(1-\frac{1}{\alpha})\frac{1}{2+\delta}}$  where  $\delta \in (0, 2)$  is chosen such that  $(2+\delta)p < \alpha$  and  $u_n = n^{1/2-1/2\alpha}$ , it yields (4.7.19).

We consider (4.7.20). We observe that by the same proof as (4.4.62), we have

$$(4.7.24) \quad \begin{aligned} & \mathbb{E} \left[ \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\} \right) - f_K \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \right]^2 \middle| \mathcal{F}_{t_j} \right] \\ &= \mathbb{E} \left[ \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\} \right) - f_K \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \right]^2 \middle| X_{t_j}^\alpha \right] \end{aligned}$$

and

$$(4.7.25) \quad \begin{aligned} & \mathbb{E} \left[ \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\} \right) - f_K \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \right]^2 \middle| X_{t_j}^\alpha = x \right] \\ &= \mathbb{E} \left[ \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \left\{ X_{\frac{1}{n}}^{\alpha, x} - x \right\} \right) - f_K \left( n^{1/\alpha} Z_{\frac{1}{n}}^\alpha \right) \right]^2 \right]. \end{aligned}$$



From Lemma 4.7.2, we get

$$\sup_j \sup_x \mathbb{E} \left[ \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\} \right) - f_K \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \right]^2 \mid X_{t_j}^\alpha = x \right] \xrightarrow{n \rightarrow \infty} 0.$$

Combining this with (4.7.24), we can deduce that

$$\sup_j \mathbb{E} \left[ \left[ f_K \left( \frac{n^{1/\alpha}}{\sigma} \{X_{t_{j+1}}^\alpha - X_{t_j}^\alpha\} \right) - f_K \left( n^{1/\alpha} \{Z_{t_{j+1}}^\alpha - Z_{t_j}^\alpha\} \right) \right]^2 \mid \mathcal{F}_{t_j} \right] \xrightarrow{n \rightarrow \infty} 0 \quad \text{a.s.}$$

Thus (4.7.20) is proved and the proof of this lemma is complete. ■

## 4.8 Appendix. Python Code

In this section, we present the Python code that we used to study the asymptotic variances of the estimators of  $\alpha_0$  (Figure 4.1) and the Python code for estimating  $\alpha \in (0, 2)$  by using the estimators  $\hat{\alpha}_{n,\log}$ ,  $\hat{\alpha}_{r,n}$  and  $\tilde{\alpha}_n, \alpha_n^*$  given by (4.3.6), (4.3.11) and in Remark 4.3.1, Proposition 4.3.1, respectively. We note that the drift function  $b$  is required to be known.

### 4.8.1 Python code for the figure 4.1

```

from scipy.special import digamma, gamma
from numpy import sqrt, pi, arange
import matplotlib.pyplot as plt
#The asymptotic variance  $V^{\wedge} \log(\alpha_0)$  given in Theorem 3.2.
def equationlog(alpha):
    return (11./10)*alpha**2 + (1./2)*alpha**4 + (13./20)* (alpha**6)
# For  $V^{\wedge}\{r\}(\alpha_0)$  given by Theorem 3.4,
#We define by equationp(alpha, r) according to def equations 1-5.
def equation1(alpha, r):
    return (2**r)*gamma((r+1)/2)*gamma(1-r/alpha)/(sqrt(pi)*gamma(1-r/2))
def equation2(alpha, r):
    return digamma(1 - r/alpha) - digamma(1-2*r/alpha)
def equation3(alpha, r):
    return equation1(alpha, 2*r)/(equation1(alpha, r)**2)
def equation4(alpha, r):

```

```

    return equation1(alpha,3*r)/(equation1(alpha,r)*equation1(alpha,2*r))
def equation5(alpha, r):
    return equation1(alpha,4*r)/(equation1(alpha,2*r)**2)
def equationp(alpha, r):
    return (alpha**4)/(r**2 * equation2(alpha,r)**2)*
(equation3(alpha,r)-equation4(alpha,r)+(1./4)*(equation5(alpha,r)-1))
q_range = [0.05, 0.1, 0.2]
color = ['b-', 'g-', 'y-']
alpha_range = arange(0., 2., 0.001)
# We need the condition  $r < \alpha/4$  in order that  $V^r(\alpha_0)$  is finite.
alpha_range_1 = arange(0.2, 2., 0.0001)
alpha_range_2 = arange(0.4, 2., 0.0001)
alpha_range_3 = arange(0.8, 2., 0.0001)
plt.plot(alpha_range, [equationlog(alpha) for alpha in alpha_range], 'r-')
plt.plot(alpha_range_1, [equationp(alpha,0.05) for alpha in alpha_range_1], 'b-')
plt.plot(alpha_range_2, [equationp(alpha,0.1) for alpha in alpha_range_2], 'g-')
plt.plot(alpha_range_3, [equationp(alpha,0.2) for alpha in alpha_range_3], 'y-')

```

## 4.8.2 Python code for the simulation of the process (4.6.1) and the estimators

of  $\alpha_0$

```

#A stochastic differential equation driven by a stable Levy process
#initial value: X0
#the index parameter: alpha
#the scale coefficient of Levy: sigma
# Euler_sub_pas : le nb de pas d'Euler entre chaque observations
# the number of observations: n
#the number of simulations: nEuler= (Euler_sub_pas)*n
from scipy.special import gamma
from numpy import log
import numpy as np
from scipy.integrate import odeint
import scipy.optimize as optimize

```

```

from numba import jit
@jit
def rmse(predictions , targets):
    return np.sqrt(((predictions - targets) ** 2).mean())
@jit
def b(x):
    return -x + np.sin(2*np.pi*x)
def dy_dt(y,t):
    return b(y)
def Simulation(n, alpha , sigma ,X0, Euler_sub_pas ):
    for j in range (1,11):
        nEuler=n*Euler_sub_pas
        Xt=[float(X0)]*(nEuler+1)
        for i in range (1,nEuler+1):
            V=np.random.uniform(-np.pi/2,np.pi/2)
            W= np.random.exponential(1.)
            S=((np.sin(alpha*V)/((np.cos(V))**(1./alpha))))
            *((np.cos(V-alpha*V))/W)**((alpha-1)/alpha))
            Xt[i]= Xt[i-1] + b(Xt[i - 1])* (float(1)/nEuler)
                +sigma*((float(1)/nEuler)**(1./alpha))*S
        plt.plot([float(i*1)/nEuler for i in range(nEuler+1)], Xt)
    return 0
@jit
def SDE_Levy(n, alpha , sigma ,X0, Euler_sub_pas ):
    nEuler=n*Euler_sub_pas
    Xt=[float(X0)]*(n+1)
    for i in range(1,n+1):
        Xtemp=[Xt[i - 1]]*(Euler_sub_pas+1)
        for j in range(1,Euler_sub_pas+1):
            V=np.random.uniform(-np.pi/2,np.pi/2)
            W= np.random.exponential(1.)
            S=((np.sin(alpha*V)/((np.cos(V))**(1./alpha))))
            *((np.cos(V-alpha*V))/W)**((1-alpha)/alpha))

```

---

```

        Xtemp[j] = Xtemp[j-1] + b(Xtemp[j - 1])
        * (float(1)/nEuler)
        + sigma*((float(1)/nEuler)**(1./alpha))*S
    Xt[i]=Xtemp[Euler_sub_pas]
    return(Xt)

@jit
def ODE(n,Euler_sub_pas,Xt):
    nEuler= n*Euler_sub_pas
    ts=np.linspace(0,Euler_sub_pas/nEuler,Euler_sub_pas+20)
    x= np.copy(Xt)
    # x longueur n+1
    result= np.zeros(n)
    for i in range(0,n):
        y0= x[i]
        ys=odeint(dy_dt,y0,ts)
        ys= np.array(ys).flatten()
        result[i] = ys[-1]
    return result

@jit
def f(n,Xt,result):
    h = np.zeros(n)
    for j in range(0,n):
        h[j] = log(abs(Xt[j+1] - result[j]))
    return h

def alpha_log(n,f):
    mean=np.average([f[j] for j in range(n)])
    # alpha_hat = ((6/(n*np.pi**2))*sum([(f[j]-mean)**2
    #for j in range(n)])-1/2)**(-1/2)
    if (6/(n*np.pi**2))*sum([(f[j]-mean)**2 for j in range(n)])-1/2 > 0:
        alpha_hat = ((6/(n*np.pi**2))*sum([(f[j]-mean)**2
        for j in range(n)])-1/2)**(-1/2)
    else:
        alpha_hat = np.random.uniform(0,2)

```

```

        return alpha_hat

@jit
def g(n,Xt,result):
    k = np.zeros(n)
    for j in range(0,n):
        k[j] = (abs(Xt[j+1] - result[j]))
    return k

# r the lower-order
def alpha_r(n,r,g):
    mean1 = np.average([g[j]**r for j in range(n)])
    mean2 = np.average([g[j]**(2*r) for j in range(n)])
    alpha_r = mean1**2/mean2
    D = (1/ np.pi**(1/2))*
        (((gamma((r+1)/2))**2)*gamma(1-r))/ (((gamma(1-r/2))**2)*gamma(1/2+r))
def Sol_alpha_r(x):
    return (gamma(1-r/x)**2/gamma(1-2*r/x)*D-alpha_r
    if Sol_alpha_r(4*r)*Sol_alpha_r(2) < 0:
        alphas=optimize.bisect(Sol_alpha_r, 4*r, 2)
    else:
        alphas=np.random.uniform(0,2)
    return alphas

def S_n(n,result,Xt):
    Sn=((1)/n)*sum([log(abs(Xt[j+1]-result[j])) for j in range(n)])
    return Sn

def alpha_title(n,Sn,alpha):
    C= 0.5772
    alphasible = (float(-Sn-C)/(log(n)-C))**(-1)
    return alphasible

# l is number of the replicates.
def program(n,alpha,r,sigma,X0,Euler_sub_pas,l):
    sol = [float(alpha)]*(l)
    sol1= [float(alpha)]*(l)
    if r == 0:

```

---

```

for j in range (0,1):
    Xt = SDE_Levy(n, alpha ,sigma ,X0, Euler_sub_pas)
    result = ODE(n,Euler_sub_pas ,Xt)
    h = f(n,Xt, result)
    sol[j] = alpha_log(n,h)
    Sn= S_n(n, result ,Xt)
    sol1[j] = alpha_title(n,Sn, alpha)
meanalphalog= np.average(sol)
print("meanalphalog_is:" + str(meanalphalog))
rmse_val = rmse(np.array(sol), np.array(alpha))
print("rmslog_error_is:" + str(rmse_val))
meanalphatible = np.average(sol1)
print("meanalphatible_is:" +str(meanalphatible))
rmse_val1 = rmse(np.array(sol1), np.array(alpha))
print("rmsalphatible_error_is:" + str(rmse_val1))
else :
    for j in range (0,1):
        Xt = SDE_Levy(n, alpha ,sigma ,X0, Euler_sub_pas)
        result = ODE(n,Euler_sub_pas ,Xt)
        k = g(n,Xt, result)
        sol[j] = alpha_r(n,r,k)
    meanalphar= np.average(sol)
    rmse_val = rmse(np.array(sol), np.array(alpha))
    if rmse_val <= 0.34:
        print("meanalphar",r,"is:" + str (meanalphar))
        print("rms",r,"error:" + str(rmse_val))
    else :
        print("error_alpha",r,"(rmse_=",rmse_val,")")
program(500,1.,0.,1,1.,1000,1000)
program(500,1., 0.05,1,1.,1000,1000)
program(500,1., 0.1,1,1.,1000,1000)
program(500,1., 0.2,1,1.,1000,1000)

```



---

## Bibliography

---

- [1] Yacine Aït-Sahalia and Lars Hansen Hansen.  
*Handbook of Financial Econometrics: Applications*, volume 2 of *Handbooks in Finance*.  
Elsevier, 2010.
- [2] Yacine Aït-Sahalia and Jean Jacod.  
Volatility estimators for discretely sampled Lévy processes.  
*Ann. Statist.*, 35(1):355–392, 2007.
- [3] Yacine Aït-Sahalia and Jean Jacod.  
Fisher’s information for discretely sampled Lévy processes.  
*Econometrica*, 76(4):727–761, 2008.
- [4] Yacine Aït-Sahalia and Jean Jacod.  
Estimating the degree of activity of jumps in high frequency data.  
*Ann. Statist.*, 37(5A):2202–2244, 2009.
- [5] Larry C. Andrews.  
*Special functions of mathematics for engineers*.  
SPIE Optical Engineering Press, Bellingham, WA; Oxford University Press, Oxford, second  
edition, 1998.
- [6] David Applebaum.  
*Lévy processes and stochastic calculus*, volume 116 of *Cambridge Studies in Advanced  
Mathematics*.  
Cambridge University Press, Cambridge, second edition, 2009.
- [7] Ole E Barndorff-Nielsen, Thomas Mikosch, and Sidney I Resnick.  
*Lévy processes: theory and applications*.  
Springer Science & Business Media, 2012.
- [8] Klaus Bichteler, Jean-Bernard Gravereaux, and Jean Jacod.  
*Malliavin calculus for processes with jumps*, volume 2 of *Stochastics Monographs*.  
Gordon and Breach Science Publishers, New York, 1987.



- [9] John Birge and Vadim Linetsky.  
Modeling financial security returns using lévy processes.  
*Handbooks in Operations Research and Management Science: Financial Engineering*, 15,  
Elsevier (2008).
- [10] D. Blackwell.  
The comparison of experiments.  
*In Proceedings, Second Berkeley Symposium on Mathematical Statistics and Probability*, p.  
93102. Univ. of California Press, Berkeley, 1951.
- [11] R. M. Blumenthal and R. K. Gettoor.  
Sample functions of stochastic processes with stationary independent increments.  
*J. Math. Mech.*, 10:493–516, 1961.
- [12] Nicolas Bouleau and Laurent Denis.  
*Dirichlet forms methods for Poisson point measures and Lévy processes*, volume 76 of *Probability Theory and Stochastic Modelling*.  
Springer, Cham, 2015.
- [13] Alexandre Brouste and Hiroki Masuda.  
Efficient estimation of stable lévy process with symmetric jumps.  
*Statistical Inference for Stochastic Processes*, 2018.
- [14] Adam D. Bull.  
Near-optimal estimation of jump activity in semimartingales.  
*Ann. Statist.*, 44(1):58–86, 2016.
- [15] Peter Carr and Helyette Geman.  
The fine structure of asset returns: An empirical investigation.  
*The Journal of Business*, 75(2):305–332, 2002.
- [16] Erhan Çinlar.  
*Probability and stochastics*, volume 261 of *Graduate Texts in Mathematics*.  
Springer, New York, 2011.
- [17] Emmanuelle Clément and Arnaud Gloter.  
Local asymptotic mixed normality property for discretely observed stochastic differential  
equations driven by stable Lévy processes.  
*Stochastic Process. Appl.*, 125(6):2316–2352, 2015.
- [18] Emmanuelle Clément and Arnaud Gloter.  
Estimating functions for SDE driven by stable Lévy processes.  
*to appear AIHP*, 2018.

- 
- [19] Emmanuelle Clément, Arnaud Gloter, and Huong Nguyen.  
Asymptotics in small time for the density of a stochastic differential equation driven driven by a stable Lévy process.  
*ESAIM Probab.Stat.*, 2018.
- [20] Emmanuelle Clément, Arnaud Gloter, and Huong Nguyen.  
LAMN property for the drift and volatility parameters of a SDE driven by a stable Lévy process.  
*ESAIM Probab.Stat.*, 2018.
- [21] Arnaud Debussche and Nicolas Fournier.  
Existence of densities for stable-like driven SDE's with Hölder continuous coefficients.  
*J. Funct. Anal.*, 264(8):1757–1778, 2013.
- [22] Laurent Denis.  
A criterion of density for solutions of Poisson-driven SDEs.  
*Probab. Theory Related Fields*, 118(3):406–426, 2000.
- [23] P.D. Ditlevsen.  
Anomalous jumping in a double-well potential.  
*Phys. Rev. E*, 60:172–179, 1999.
- [24] Nicolas Fournier and Jacques Printems.  
Absolute continuity for some one-dimensional processes.  
*Bernoulli*, 16(2):343–360, 2010.
- [25] V. Genon-Catalot and J. Jacod.  
Estimation of the diffusion coefficient for diffusion processes: random sampling.  
*Scand. J. Statist.*, 21(3):193–221, 1994.
- [26] Valentine Genon-Catalot and Jean Jacod.  
On the estimation of the diffusion coefficient for multi-dimensional diffusion processes.  
*Ann. Inst. H. Poincaré Probab. Statist.*, 29(1):119–151, 1993.
- [27] Emmanuel Gobet.  
Local asymptotic mixed normality property for elliptic diffusion: a Malliavin calculus approach.  
*Bernoulli*, 7(6):899–912, 2001.
- [28] Emmanuel Gobet.  
LAN property for ergodic diffusions with discrete observations.  
*Ann. Inst. H. Poincaré Probab. Statist.*, 38(5):711–737, 2002.

- [29] J. Hájek.  
A characterization of limiting distributions for regular estimators.  
*Z. Wahrscheinlichkeitsth. Verw. Geb.* 14, 323-330, 1970.
- [30] J. Hájek.  
Local asymptotic minimax and admissibility in estimation.  
*Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Theory of Statistics*, pages 175–194, 1972.
- [31] Peter Hall and Christopher Charles Heyde.  
*Martingale limit theory and its application*.  
Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980.  
Probability and Mathematical Statistics.
- [32] I.A. Ibragimov, R.Z. Has'minskii, and S. Kotz.  
*Statistical Estimation: Asymptotic Theory*, volume 16 of *Stochastic Modelling and Applied Probability*.  
Springer-Verlag New York, 1981.
- [33] Yasushi Ishikawa and Hiroshi Kunita.  
Malliavin calculus on the Wiener-Poisson space and its application to canonical SDE with jumps.  
*Stochastic Process. Appl.*, 116(12):1743–1769, 2006.
- [34] Dmytro Ivanenko and Alexei Kulik.  
Malliavin calculus approach to statistical inference for Lévy driven SDE's.  
*Methodol. Comput. Appl. Probab.*, 17(1):107–123, 2015.
- [35] Dmytro Ivanenko, Alexey M. Kulik, and Hiroki Masuda.  
Uniform LAN property of locally stable Lévy process observed at high frequency.  
*ALEA Lat. Am. J. Probab. Math. Stat.*, 12(2):835–862, 2015.
- [36] Jean Jacod.  
The Euler scheme for Lévy driven stochastic differential equations: limit theorems.  
*Ann. Probab.*, 32(3A):1830–1872, 2004.
- [37] Jean Jacod and Philip Protter.  
*Discretization of processes*, volume 67 of *Stochastic Modelling and Applied Probability*.  
Springer, Heidelberg, 2012.
- [38] Aleksander Janicki and Aleksander Weron.  
*Simulation and chaotic behavior of  $\alpha$ -stable stochastic processes*, volume 178 of *Monographs and Textbooks in Pure and Applied Mathematics*.

- Marcel Dekker, Inc., New York, 1994.
- [39] P. Jeganathan.  
On the asymptotic theory of estimation when the limit of the log-likelihood ratios is mixed normal.  
*Sankhyā Ser. A*, 44(2):173–212, 1982.
- [40] P. Jeganathan.  
Some asymptotic properties of risk functions when the limit of the experiment is mixed normal.  
*Sankhyā Ser. A*, 45(1):66–87, 1983.
- [41] Bing-Yi Jing, Xin-Bing Kong, and Zhi Liu.  
Modeling high-frequency financial data by pure jump processes.  
*Ann. Statist.*, 40(2):759–784, 2012.
- [42] Bing-Yi Jing, Xin-Bing Kong, Zhi Liu, and Per Mykland.  
On the jump activity index for semimartingales.  
*J. Econometrics*, 166(2):213–223, 2012.
- [43] Reiichiro Kawai and Hiroki Masuda.  
On the local asymptotic behavior of the likelihood function for Meixner Lévy processes under high-frequency sampling.  
*Statist. Probab. Lett.*, 81(4):460–469, 2011.
- [44] Reiichiro Kawai and Hiroki Masuda.  
Local asymptotic normality for normal inverse Gaussian Lévy processes with high-frequency sampling.  
*ESAIM Probab. Stat.*, 17:13–32, 2013.
- [45] Vassili N. Kolokoltsov.  
*Markov processes, semigroups and generators*, volume 38 of *de Gruyter Studies in Mathematics*.  
Walter de Gruyter & Co., Berlin, 2011.
- [46] Xin-Bing Kong, Zhi Liu, and Bing-Yi Jing.  
Testing for pure-jump processes for high-frequency data.  
*Ann. Statist.*, 43(2):847–877, 2015.
- [47] Alexei Kulik.  
On weak uniqueness and distributional properties of a solution to an SDE with  $\alpha$ -stable noise.  
*arXiv: 1511.00106 .v1*, 2015.

## BIBLIOGRAPHY

---

- [48] Y.A. Kutoyants.  
*Statistical Inference for Ergodic Diffusion Processes.*  
Springer, 1998.
- [49] L. Le Cam.  
Limits of experiments.  
*Proc. 6th Berkeley Symp. Math. Statist. Prob., 1, 245-261, 1972.*
- [50] Lucien Le Cam and Grace Lo Yang.  
*Asymptotics in statistics: Some basic concepts.*  
Springer Series in Statistics. Springer-Verlag, New York, 1990.
- [51] Lucien Le Cam and Grace Lo Yang.  
*Asymptotics in statistics: some basic concepts.*  
Springer Science, Business Media, 2000.
- [52] Harald Luschgy and Gilles Pagès.  
Moment estimates for Lévy processes.  
*Electron. Commun. Probab., 13:422–434, 2008.*
- [53] Hiroki Masuda.  
Joint estimation of discretely observed stable Lévy processes with symmetric Lévy density.  
*J. Japan Statist. Soc., 39(1):49–75, 2009.*
- [54] Hiroki Masuda.  
Non-gaussian quasi-likelihood estimation of sde driven by locally stable Lévy process.  
*Preprint arXiv:1608.06758v3, 2017.*
- [55] Thomas Mikosch, Sidney Resnick, Holger Rootzén, and Alwin Stegeman.  
Is network traffic approximated by stable Lévy motion or fractional Brownian motion?  
*Ann. Appl. Probab., 12(1):23–68, 2002.*
- [56] Jean Picard.  
On the existence of smooth densities for jump processes.  
*Probab. Theory Related Fields, 105(4):481–511, 1996.*
- [57] Jean Picard.  
Density in small time at accessible points for jump processes.  
*Stochastic Process. Appl., 67(2):251–279, 1997.*
- [58] G.G. Roussas.  
*Contiguity of probability measures: some applications in statistics, volume 63.*  
Cambridge University Press London, 1972.

- [59] K. Sato.  
*Lévy Processes and Infinitely Divisible Distributions.*  
Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1999.
- [60] W. Schoutens.  
Lévy processes in finance: Pricing financial derivatives.  
*Wiley Series in Probability and Statistics, Hoboken, NJ, USA., 2003.*
- [61] P. Tankov and R. Cont.  
*Financial Modelling with Jump Processes, Second Edition.*  
Chapman and Hall/CRC Financial Mathematics Series. Taylor & Francis, 2015.
- [62] Viktor Todorov.  
Jump activity estimation for pure-jump semimartingales via self-normalized statistics.  
*Ann. Statist.*, 43(4):1831–1864, 2015.
- [63] Viktor Todorov and George Tauchen.  
Limit theorems for power variations of pure-jump processes with application to activity estimation.  
*Ann. Appl. Probab.*, 21(2):546–588, 2011.
- [64] A. W. van der Vaart.  
*Asymptotic statistics*, volume 3 of *Cambridge Series in Statistical and Probabilistic Mathematics.*  
Cambridge University Press, Cambridge, 1998.
- [65] W.Feller.  
*An Introduction to Probability Theory and its Applications.*  
vol 2 (second edition), Wiley, 1971.

