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Philippe Mathieu

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THÈSE

Pour obtenir le grade de

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Présentée par

Philippe Mathieu

Thèse dirigée par **Frank Thuillier**

Préparée au sein du **Laboratoire d'Annecy de Physique Théorique**
et de l'**École Doctorale de Physique de Grenoble**

Abelian BF Theory

Théorie BF abélienne

Thèse soutenue publiquement le **2 juillet 2018**,
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Contents

Remerciements	vii
Introduction (English version)	xiii
Introduction (version française)	xxiii
Some preliminary remarks	xxxv
I A mathematical approach	3
1 Notions on categories	5
Introduction (English version)	6
Introduction (version française)	8
1.1 The category $\mathbb{C}^{\mathbb{Z}_k}$ of representations of \mathbb{Z}_k	10
1.2 The category $\mathbb{C}^{\mathbb{Z}_k}$ as a monoidal category	11
1.3 The category $\mathbb{C}^{\mathbb{Z}_k}$ as a braided and twisted category	11
1.4 The category $\mathbb{C}^{\mathbb{Z}_k}$ as a ribbon and spherical category	13
1.5 The category $\mathbb{C}^{\mathbb{Z}_k}$ as a modular category?	14
1.6 Drinfeld center of $\mathbb{C}^{\mathbb{Z}_k}$	15
2 Abelian Reshetikhin-Turaev invariant	19
Introduction (English version)	20
Introduction (version française)	21
2.1 Generalities	22
2.2 Extension of abelian RT construction	25
2.3 Abelian RT invariant associated with $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$	27
2.4 Abelian expectation values of observables	28
2.5 Examples	30

3	Abelian Turaev-Viro invariant	35
	Introduction (English version)	36
	Introduction (version française)	37
	3.1 Generalities	38
	3.2 Abelian expectation values of observables	43
	3.3 Examples	44
	3.4 Relations between abelian RT and TV theories	47
II A physical approach		53
1	Deligne-Beilinson cohomology	55
	Introduction (English version)	56
	Introduction (version française)	57
	1.1 Definition of DB cohomology	58
	1.2 Structure of the space of DB cohomology classes	64
	1.3 Operations and duality on DB cohomology classes	66
	1.4 Decomposition of DB cohomology classes	69
2	Abelian Chern-Simons theory	75
	Introduction (English version)	76
	Introduction (version française)	79
	2.1 Computation of Z_{CS_k}	82
	2.2 Abelian expectation value of observables	84
3	Abelian BF theory	89
	Introduction (English version)	90
	Introduction (version française)	91
	3.1 Computation of Z_{BF_k}	92
	3.2 Abelian expectation value of observables	94
III Additional matters		101
1	SU(N) Chern-Simons invariant of flat connections	103
	Introduction (English version)	104
	Introduction (version française)	106
	1.1 General method	108
	1.2 Example of lens spaces	116
	1.3 Example of some genus 2 Heegaard splittings	119

Contents

2 Geometric interpretation of gauge fixing in \mathbb{R}^{4+3}	133
Introduction (English version)	134
Introduction (version française)	136
2.1 Conventions	138
2.2 Correlators	140
2.3 Limits	146
2.4 Geometric interpretation	148
Conclusion (English version)	153
Conclusion (version française)	157
Bibliography	166

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Introduction (English version)

One of the main issues in physics today is the reconciliation of general relativity (GR) and quantum mechanics (QM). An attempt to reach this goal is proposed by loop quantum gravity (LQG) whose most accurate model was defined quite recently, in 2008 in [Engle et al., 2008]. However, this study is the result of a 50-year-old development that merged considerations from mathematics and physics, the BF theory being a step arising at their intersection.

In 1961, T. Regge proposed to express GR without coordinates but using a cellular decomposition of space-time [Regge, 1961]. A while after, Regge studied with G. Ponzano the large spins limit of Racah coefficients [Ponzano and Regge, 1968]. Racah coefficients are, up to a sign, connected with the $6j$ -symbols defined by Wigner which have interesting symmetry properties that are represented in Figure 1. In this picture, the $j_{\alpha\beta}$ represent angular momenta, referring to indices of SU(2) representations related to the $6j$ -symbol, but they can be regarded as the lengths of the edges of the tetrahedron.

The relation between tetrahedra and $6j$ -symbols is much deeper than a simple mnemotechnical trick. Indeed, in their work, Ponzano and Regge showed that a $6j$ -symbol is related to the volume of its corresponding tetrahedron by:

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \frac{F}{\sqrt{12\pi V}}, \quad (1)$$

where V is the volume of the tetrahedron and F is a rapidly oscillating function of the dihedral angles of the tetrahedron and of the angular momenta so that the average of F^2 is $\frac{1}{2}$.

Moreover, in an analogous way, the $3nj$ -symbols are associated with more complicated polyhedra P_{3nj} . They can also be written as a function of $3mj$ -symbols with $m < n$. Those $3mj$ -symbols are themselves associated with polyhedra P_{3mj} and the decomposition of the $3nj$ according to the $3mj$ is connected with the geometrical decomposition of the P_{3nj} into P_{3mj} pieces. Formalizing this idea, Ponzano and Regge were led to consider sums of $6j$ -symbols that represent elementary tetrahedra that stick together to give a more complicated polyhedron associated with a

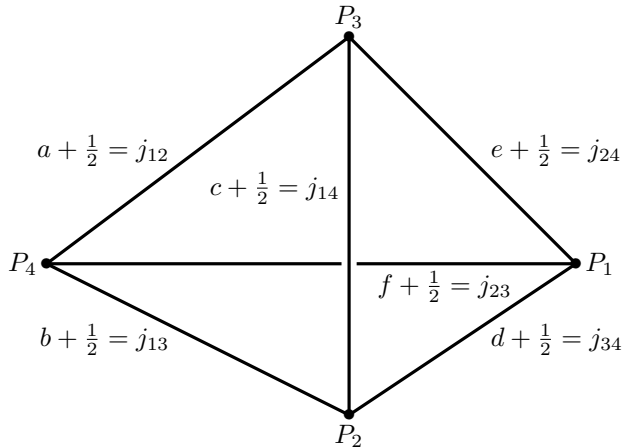


Figure 1: Symmetry of the $6j$ -symbols.

$3nj$ -symbol. The unicity of a given $3nj$ -symbol implies that these sums have to be independent of the decomposition of P_{3nj} in elementary tetrahedra.

To stick together along a face, two tetrahedra have to carry the same charge on their edges. Summing over all the configurations allowed should then produce a quantity independent of the triangulation of the polyhedron P_{3nj} . Sadly, such a sum over all representations of $SU(2)$ is infinite.

However, conducting a formal study of its large spin, limit Ponzano and Regge recognized, thanks to the result obtained by Regge a few years before, a discretization of the partition function for Euclidean GR in dimension 3. The partition function is also ill-defined since it represents a sum over the infinite dimensional space of all the cellular decompositions of space-time.

Euclidean GR in dimension 3 is trivial in the sense that, for an empty space, Euler-Lagrange (EL) equations lead to a flat space-time. Nevertheless, this was the first step towards an attempt of quantizing GR by quantizing space-time itself. This story slept nearly 20 years before emerging back under different shapes.

In 1992, V. Turaev and O. Viro defined the so-called state-sum invariant of 3-manifolds or Turaev-Viro (TV) invariant using the same idea as Ponzano and Regge of labeled edges of a triangulation of the manifold considered [Turaev and Viro, 1992]. The construction is based on modular categories but the weaker hypothesis of finite semisimple spherical categories can be considered [Barrett and Westbury, 1996]. In both finite semisimple spherical and modular categories, a constraint, called domination axiom, makes it possible to construct the invariant with a finite set of objects. The expression of the TV invariant is then formally very similar to the formula of Ponzano and Regge in the case of a realization of the category by

representations of $\mathcal{U}_q(\mathfrak{sl}_2)$ (quantum deformation of the universal enveloping algebra of \mathfrak{sl}_2 at a root of unity q). Contrary to the formula of Ponzano and Regge, the TV invariant is a well-defined quantity, since the domination axiom imposes the sum to be taken over a period of the cyclic representations of $\mathcal{U}_q(\mathfrak{sl}_2)$. It can be therefore considered as a regularization of Ponzano-Regge formula.

Modular categories are also at the root of the construction of the so-called Reshetikhin-Turaev (RT) invariant, which has been shown to be related to the TV invariant built along the same modular category \mathcal{C} :

$$\left| Z_{\text{RT}_{\mathcal{C}}} (M^{(3)}) \right|^2 = Z_{\text{TV}_{\mathcal{C}}} (M^{(3)}). \quad (2)$$

It must be borne in mind that this equality is not correct anymore with the weaker hypothesis that \mathcal{C} is only a finite semisimple spherical category. The RT invariant is built from a framed knot associated with the manifold considered via a surgery procedure. What is interesting is that in the case of $\mathcal{U}_q(\mathfrak{sl}_2)$ this invariant has been shown to be equivalent to the Jones polynomial of this surgery knot [Reshetikhin and Turaev, 1991], and thus, according to E. Witten, related to the $\text{SU}(2)$ Chern-Simons (CS) partition function as shown further.

In 1978 A. S. Schwarz was the first to introduce the idea that topological invariants could be recovered with the partition function Z of some quantum field theories founded on a gauge invariant classical action S [Schwarz, 1978], that is, a formal integral written as:

$$Z = \frac{1}{\mathcal{N}} \int_{\mathcal{H}} \mathcal{D}\varphi e^{iS(\varphi)}, \quad (3)$$

where the integration domain \mathcal{H} is the space of configurations φ of the theory and \mathcal{D} is a measure on this space. Mathematicians assert that a non-zero measure invariant under translation over \mathcal{H} cannot exist. J. Baez proposed nevertheless an alternative that consists in considering linear functionals over specific subspaces of \mathcal{H} [Baez, 1993]. However, there exists no practical method analogous to primitive computation to obtain an exact result. Such a partition function is actually written by analogy with statistical physics. A perturbative expansion, together with formal manipulations, makes it possible to find Feynman rules without using Wick contractions on field operators that live in the Fock space of quantum states associated with the system. The normalization \mathcal{N} has to enable the extraction of relevant quantities absorbing intrinsic divergency of the integral arising for example because of gauge invariance of the action (but not only).

In 1982, Witten showed relations between Morse theory and supersymmetric QM [Witten, 1982], giving a second example of a mathematical development thanks to physics tools. The year after, S. Donaldson used some considerations on gauge theories to produce some results in four dimensional topology [Donaldson, 1983]. Specially collecting the results of Witten and Donaldson, M. Atiyah showed in 1987

Introduction (English version)

that the study of the infinite dimensional manifold of flat connections of trivial bundles over a manifold $M^{(3)}$ of dimension 3 was of central interest to study the topological properties of $M^{(3)}$ itself as well as topological properties of manifolds that bound $M^{(3)}$ and manifolds that are bounded by $M^{(3)}$ [Atiyah, 1987].

More rigorously, in the mathematics literature, a connection is a global 1-form on a principal fiber bundle. Its pullback by a local section defines on the base manifold local 1-forms with coefficients in a Lie algebra. In the intersections of their respective domain of definition, two such local 1-forms stick together up to a gauge transformation. Hence, by a misuse of language a family of such local 1-forms will be called *connection* as well. Such a family modelizes a physical field whose strength, or curvature from the mathematicians' point of view, is the measurable quantity like an electromagnetic field for example.

This subject now known as topological quantum field theory (TQFT) has been since then intensively treated in the community of both mathematicians and physicists and tools familiar to physicists such as functional integrals turned out to be a relevant approach. The definition and the classification of TQFTs are provided in particular in [Birmingham et al., 1991], a TQFT being defined as:

- a collection of fields (which are Grassmann graded) defined on a Riemannian manifold $(M^{(n)}, g)$,
- a nilpotent operator Q , which is odd with respect to the Grassmann grading,
- physical states defined to be Q -cohomology classes,
- an energy-momentum tensor which is Q -exact: $T_{\alpha\beta} = \{Q, V_{\alpha\beta}\}$ for some functional $V_{\alpha\beta}$ of the fields and the metric.

In most examples treated along the years, Q is the BRST operator, but such a statement is not mandatory. Those criteria imply in particular that the partition function Z defined as in equation (3) is a topological invariant of $M^{(n)}$. Hence, Z will be noted $Z(M^{(n)})$ where $M^{(n)}$ is the manifold over which the TQFT is defined. Still according to [Birmingham et al., 1991], TQFTs are classified according to two families. Schwarz type theories, in opposition to Witten type theories, are described by a local action which is independent of the metric as well as the correlators, which implies in particular that the energy-momentum tensor vanishes. As a consequence, for an observable $\mathcal{O} : \varphi \mapsto \mathcal{O}(\varphi)$, which is a functional over the space of fields \mathcal{H} , its expectation value, defined as:

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{N}} \int_{\mathcal{H}} \mathcal{D}\varphi \mathcal{O}(\varphi) e^{iS(\varphi)}, \quad (4)$$

is also a topological invariant. It should not be forgotten that functional integration is an ill-defined operation. It is a formal intermediary of computations. The results

of these computations are expected to be dominated by the contributions of flat connections, since flat connections are the classical solutions of the theory, that is, the solutions of EL equations. Hence, the partition function and the expectation value of observables are topological invariants in the sense that so is the result of their underlying computations.

From now on, this study will exclusively focus on Schwarz type TQFT. A theory which is, *stricto sensu*, a BF theory, was studied in 1978 by Schwarz and related with Ray-Singer torsion [Schwarz, 1978]. However, it is often considered that BF theory appeared for the first time in the work of G. Horowitz [Horowitz, 1989] who was investigating a class of TQFT that can be defined in any dimension.

Using the “dreibein” variables, the action of GR in dimension 3 can be written as ¹:

$$S_{\text{BF}_g}(A, B) = \frac{g}{4\pi} \int_{M^{(3)}} \text{Tr}(B \wedge F_A), \quad (5)$$

where $F_A = dA + A \wedge A$ is the curvature 2-form associated with a $\text{SU}(2)$ -connection A and B is a $\mathfrak{su}(2)$ -valued 1-form². The normalization $\frac{g}{4\pi}$ has been chosen this way for later convenience and differs from a lot of publications which prefer to absorb it completely by a redefinition of B . This action is invariant under the transformation:

$$\begin{cases} A \longrightarrow A^g = g^{-1}Ag + g^{-1}dg \Rightarrow F_A \longrightarrow F_{A^g} = g^{-1}F_Ag, \\ B \longrightarrow B^g = g^{-1}Bg, \end{cases} \quad (6)$$

with $g \in \text{SU}(2)$. It should be noted that actions of the same kind on a manifold $M^{(n)}$ of any dimension n can be built just taking B to be a $\mathfrak{su}(2)$ -valued $(n-2)$ -form.

The BF action is quite similar to CS action which is given by¹:

$$S_{\text{CS}_k}(A) = \frac{k}{4\pi} \int_{M^{(3)}} \text{Tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right), \quad (7)$$

where A is a $\text{SU}(2)$ -connection². The normalization here cannot be absorbed properly by a redefinition of A . It is actually part of the definition as explained further (up to a 2π factor¹). Under a gauge transformation, that is:

$$A \longrightarrow A^g = g^{-1}Ag + g^{-1}dg, \quad (8)$$

with $g \in \text{SU}(2)$, this action transforms as:

$$S_{\text{CS}_k}(A) \longrightarrow S_{\text{CS}_k}(A^g) = S_{\text{CS}_k}(A) + S_{\text{WZ}}(g), \quad (9)$$

¹ Several conventions of normalization of the action S are possible. If the normalization of S is $\frac{1}{4\pi}$, then S is invariant up to $2\pi n$ with $n \in \mathbb{Z}$ under gauge transformation. With this choice, the quantity strictly invariant under gauge transformation is e^{iS} . If the normalization of S is $\frac{1}{8\pi^2}$, then S is invariant up to $n \in \mathbb{Z}$ under gauge transformation. With this choice, the quantity strictly invariant under gauge transformation is $e^{2i\pi S}$.

²The theory can be defined as well with $\text{SU}(N)$ instead of $\text{SU}(2)$.

Introduction (English version)

where $S_{\text{WZ}}(g)$ is a so-called Wess-Zumino term and is actually an integer. Thus, CS action is well-defined on gauge classes of fields only in \mathbb{R}/\mathbb{Z} . Contrary to BF, this is a purely quantum action, in the sense that it cannot be regarded simply in \mathbb{R} as any classical action. One of the most remarkable uses of CS action is due to Witten who proved that the $SU(2)$ CS theory was related to the invariant of knots called Jones polynomial [Witten, 1989]. Of course, Witten never performed a functional integration which remains an ill-defined operation, but proved his result by indirect arguments related in particular to conformal field theory (CFT).

It is possible, thanks to a surgery procedure, to relate the computation in $M^{(3)}$ with a computation in S^3 which is completely similar to \mathbb{R}^3 . In \mathbb{R}^3 , the standard procedures of quantum field theory (QFT) like gauge fixing and perturbative expansion can be used, which has been done by E. Guadagnini et al. [Guadagnini et al., 1990]. Their perturbative result is compatible with Witten's result.

The expression (7) is not absolutely rigorous. Indeed, writing the integral this way implies that A is an object globally defined, which is not true in general. According to the theorem of stepwise extension of a section of a fiber bundle [Steenrod, 1999], since for any integer N , the Lie group $SU(N)$ is arcwise connected and $\pi_1(SU(N)) = \pi_2(SU(N)) = 0$ (which is not the case of $U(1)$), then any principal $SU(N)$ -bundle over a closed manifold of dimension 3 is trivializable. Hence, a representative that is globally defined can be found in the gauge class of any $SU(N)$ -connection. Therefore, in this formula, A should actually be this particular representative. It should be noted that fixing a globally defined representative is the same as focusing on the trivial $SU(N)$ -bundle over $M^{(3)}$.

Moreover, when considering a gauge theory like $SU(N)$ CS theory and claiming to perform functional integrals such as equation (3) implies summing over all the connections, or equivalently, all the principal bundles over the manifold considered $M^{(3)}$. However, the gauge invariance of the action induces an infinite redundancy. To deal with this issue, two approaches can be considered. Firstly, a gauge can be fixed, that is, a constraint that should guarantee that one and only one representative in each gauge class is to be considered, which is the usual approach in physics. Secondly, the gauge classes can be considered directly. If the principal fiber bundle associated with the theory is trivializable, then those two approaches are equivalent. This is the case when the gauge group is $SU(N)$ or when the manifold considered is \mathbb{R}^3 or S^3 . However, if the principal bundle is not trivializable, there is *a priori* no reason, from the physicists' point of view, to restrict to the connections of the trivial principal bundle. It should be pointed out, however, that introducing a matter field or equivalently considering a vector bundle on which the principal bundle of the theory acts modifies the properties of gauge invariance and actually fixes the principal bundle of the theory. This study will develop the second approach for the $U(1)$ CS and BF theories on a connected closed manifold of dimension 3 without any matter field. Until now, this approach, initiated in [Bauer et al., 2005], has

been developed since then by very few authors [Guadagnini and Thuillier, 2013], [Mathieu and Thuillier, 2016a]. The fact that the fiber bundles considered here are not trivializable is a crucial difference with the usual $SU(N)$ case due to the specific topological structure of $U(1)$.

In the 3-dimensional BF theory, a cosmological term can be incorporated in the action:

$$S_{\text{BF}_{g,\kappa}}(A, B) = \frac{g}{4\pi} \int_{M^{(3)}} \text{Tr} \left(B \wedge F_A + \frac{\kappa^2}{3} B \wedge B \wedge B \right), \quad (10)$$

as done by Horowitz in his derivation of 3 + 1-dimensional Einstein-Hilbert action [Horowitz, 1989].

If the normalization is absorbed by rescaling B in $\tilde{B} = \frac{g}{4\pi} B$ and κ redefined as $\tilde{\kappa} = \frac{4\pi\kappa}{g}$, then:

$$\tilde{S}_{\text{BF}_{g,\kappa}}(A, B) = S_{\text{BF}_{\tilde{\kappa}}}(A, \tilde{B}) = \int_{M^{(3)}} \text{Tr} \left(\tilde{B} \wedge F_A + \frac{\tilde{\kappa}^2}{3} \tilde{B} \wedge \tilde{B} \wedge \tilde{B} \right), \quad (11)$$

and $\tilde{\kappa}$ is nothing but the cosmological constant analogous in dimension 3 of the one defined by Einstein in dimension 3 + 1.

In 1998 A. Cattaneo et al. showed that the $SU(N)$ the partition functions of BF and CS theories could be related performing formal manipulations on functional integrals [Cattaneo et al., 1998]:

$$\left| Z_{\text{CS}_k}(M^{(3)}) \right|^2 = Z_{\text{BF}_{4k\kappa,\kappa}}(M^{(3)}). \quad (12)$$

Such expression of the result justifies the choice to express the BF theory with two parameters g and κ .

Beyond the scope of this study, and in order to complete the current genealogy of the BF theory in quantum gravity, L. Crane and D. Yetter studied what an analog of the TV invariant should be in dimension 4 keeping in mind the idea expressed by Ponzano and Regge and its regularization by the TV invariant using a quantum group [Crane and Yetter, 1993]. They wrote:

$$\int_{M^{(4)}} \text{Tr}(B \wedge F_A), \quad (13)$$

with B a 2-form $\mathfrak{su}(2)$ -valued. This is again a topological action. Unfortunately EL equations constrain the Riemann tensor which is too strong to be relevant in quantum gravity. Another attempt by J. Barrett and Crane in [Barrett and Crane, 1998] consisted in writing the action as:

$$\int_{M^{(4)}} e \wedge e \wedge F_A, \quad (14)$$

Introduction (English version)

introducing new degrees of freedom and constraining this time the Ricci tensor in EL equations. This should have been more adapted to a theory of quantum gravity but was actually wrong. A correction of this led to the EPRL model defined in [Engle et al., 2008], which is the most up-to-date model of LQG which has no UV divergence and controlled IR divergence, the limit with large spins being thus of course the GR.

Hence, understanding the regularization of the Ponzano-Regge model and its relation to the TV invariant and the $SU(2)$ BF theory turned out to be relevant as a step towards the definition of a quantum theory of gravitation. S. Mizoguchi and T. Tada showed in 1992 that the quantum deformation parameter q had to be related to the cosmological constant $\tilde{\kappa}$ [Mizoguchi and Tada, 1992]. C. Rovelli showed in 1993 the equivalence of the so-called loop representation in LQG and the combinatorics underlying the quantum gravity in the work of Ponzano, Regge, H. Ooguri, Turaev and Viro [Rovelli, 1993]. This question has been under consideration until recently for example with the works of K. Krasnov, D. Louapre and L. Freidel [Freidel and Krasnov, 2005, Freidel and Louapre, 2004, Freidel and Louapre, 2005] or the works of V. Bonzom and M. Smerlak [Bonzom and Smerlak, 2012]. Research on the CS and BF theories for themselves is also still active ([Cattaneo et al., 2017] for example).

So far, the state of the art can be summed up in the following diagram:

$$\begin{array}{ccc}
 |Z_{\text{CS}_k}|^2 & = & Z_{\text{BF}_{4k\kappa,\kappa}} \\
 \parallel & & \parallel \\
 |Z_{\text{RT}_{q(k)}}|^2 & = & Z_{\text{TV}_{q(k)}}
 \end{array}$$

The index $q(k)$ is the root of unity that appears in $\mathcal{U}_q(\mathfrak{sl}_2)$ and is built from the coupling constant of CS theory k . On the TV side, $q = q(\tilde{\kappa})$ but $\tilde{\kappa} = \frac{4\pi\kappa}{g} = \frac{\pi}{k}$ since $g = 4k\kappa$ to satisfy the equality of the partition functions. Hence $q = q(\tilde{\kappa}) = q(k)$.

The lower equality relates two perfectly defined quantities. This is a theorem, established by Reshetikhin, Turaev and Viro in the general framework of modular categories. On the contrary, the other equalities deal with ill-defined objects. The upper equality has been established by formal manipulations and the two vertical equalities have been proved by indirect arguments.

The aim of the present study is to investigate the abelian BF theory on a manifold $M^{(3)}$ of dimension 3 by computing the partition function and the expectation value of observables, starting from formal functional integrals. Several differences with usual studies on the subject have to be underlined. The abelian BF theory under study here is a TQFT, in the sense that the action considered is completely independent of any metric on $M^{(3)}$. It is even a cohomological TQFT in the sense that it will

be written, not as usual in terms of $U(1)$ -connections, but in terms of gauge classes of $U(1)$ -connections that will be proved to be Deligne-Beilinson (DB) cohomology classes. The structure of the space of DB cohomology classes will make it possible to extract the quantities of interest exactly from formal functional integrals, as done for the abelian CS theory [Guadagnini and Thuillier, 2013]. This approach is thus non-perturbative. In the present study, the computations will rely on DB cohomology and will not be restricted to gauge classes arising from the trivial $U(1)$ -bundle over $M^{(3)}$. Indeed, physicists interpret gauge theories in the following way: Observers perform a measurement of a physical quantity in their neighbourhood. If two observers compare their respective result, there is no reason for them to have the same one, but their results will be related by a gauge transformation. Hence, there is no reason for restricting to the configurations where the measurement is the same for each observer. As a consequence, in some sense, the partition functions and expectation value of observables will be considered to be specific sums over all the $U(1)$ -bundles over $M^{(3)}$.

Beyond the computations of the partition function and the expectation value of observables, which had never been treated this way before, this study aims at highlighting the relations between the abelian BF and CS theories as well as the relations between those theories and TQFT in the formalism of categories as described mostly by N. Reshetikhin, V. Turaev and O. Viro. Some subtleties have to be mentioned. First, to get correct relations between all those theories, the normalization of the RT and TV invariants will be taken to be S^3 and not $S^1 \times S^2$ contrary to the usual convention appearing in the mathematics literature. Then, the weaker hypothesis of finite semisimple spherical category regarding the RT theory has to be considered in order to obtain a relation between the abelian CS and RT theories. This weaker hypothesis is, in some sense, that of H. Murakami, T. Ohtsuki and M. Okada [Murakami et al., 1992] and is also evocated by Turaev [Turaev, 2010]. This is a fundamental difference with the non-abelian (mainly $SU(2)$) case. Finally, as already stated, the $SU(2)$ CS theory is related to the RT theory built on a quantum group. This cannot be directly extended to the abelian case since no quantum deformation of $U(1)$ is possible. This is another crucial difference with the $SU(2)$ case due to the specific topological structure of $U(1)$. As in the $SU(2)$ case, the point is actually to consider objects satisfying the domination axiom. In the cases considered, this will consist in an infinite set of objects with a periodic structure so that sums can be restricted to a period without losing any information carried by those objects. It will be contended that, in the abelian case, the correct objects to consider are representations of \mathbb{Z}_k . Before this study, such class of examples had never been exhibited, probably because of their seeming triviality. However, it will be shown further that they actually constitute a subtle class of examples with interesting links with abelian BF and CS theory.

The first part of this dissertation focuses on fundamental notions of TQFT from a

Introduction (English version)

mathematical approach. This approach is based on the theory of categories. The first chapter introduces the main definitions and abelian realizations are derived. The second chapter presents the abelian RT invariant founded on the abelian modular category derived in the first chapter, following the work of Reshetikhin and Turaev. The extension studied in particular by Murakami, Ohtsuki and Okada [Murakami et al., 1992] is also considered. Likewise, the RT invariant founded on the abelian Drinfeld center is presented, as well as associated new developments [Turaev and Virelizier, 2013]. The third chapter of this part presents the abelian TV invariant based on the abelian finite semisimple spherical category derived in the first chapter. Its definition is founded on the work of B. Balsam and A. Kirillov [Balsam and Kirillov, 2010] which generalizes the original construction on triangulations to any cellular decomposition. Within the abelian framework, several results relating the TV theory to the RT theory that have been established in the general case can thus be verified. The main original contribution of this part is the derivation of an abelian version of all those notions and the discussion about the subtleties that occur. Also, comparing the shape of abelian RT and TV invariants with abelian CS and BF partition function given in the next part led to introduce expectation value of observables in abelian RT and TV theories, which had never been done before strictly speaking.

The second part focuses on the abelian CS and BF theories from a physical approach. Those theories are defined by classical actions, that is, functionals over the space of classical fields. The quantities computed are extracted from functional integrals, but the aim of this study is neither to give a rigorous definition nor to perform any computation of such kind of formal objects. The first chapter is a review on DB cohomology whose classes will be precisely the gauge classes of fields. The structure of the space of DB cohomology classes and some properties are derived for further use, following [Mathieu, 2017]. In the second chapter, the abelian CS partition function and the expectation value of observables are defined and computed. Those results are related to the abelian RT theory presented in the previous part. In the third chapter, the abelian BF partition function and the expectation value of observables are defined and computed. Those results are related to the abelian TV theory presented in the previous part and with the results on abelian CS theory obtained in the previous chapter concerning the partition functions following [Mathieu and Thuillier, 2016a] and concerning the expectation value of observables following [Mathieu and Thuillier, 2016b]. It is often asserted that the main contributions to functional integrals come from the neighbourhood of the solutions of the classical EL equations which are the flat connections, that is, the connections A such that $F_A = 0$. Using DB cohomology implies that this assertion is correct and even proved by direct computations. The results established in the last chapter of this part, and in particular the links between the quantities defined and computed in this chapter and in the previous ones, constitute the main original production of this study.

Those two parts provide a complete and quite closed set of definitions and results concerning abelian RT, TV, CS and BF theories on closed manifolds of dimension 3. Those results are beyond the initial aim of this study.

The third part gathers new developments slightly out of the scope of the main theme of this study, without being totally unrelated. The first chapter introduces a completely new method founded on Heegaard diagrams to compute SU(2) CS invariant following [Guadagnini et al., 2017]. In the second chapter, the propagator of abelian CS theory in \mathbb{R}^{4+3} in an anisotropic metric is computed. Several limits are studied and geometric interpretations are proposed, completing the particular cases studied in [Gallot et al., 2018].

Introduction (version française)

L'un des principaux problèmes de la physique aujourd'hui consiste en la réconciliation de la relativité générale (GR) et de la mécanique quantique (QM). Une tentative d'atteindre ce but est proposée par la gravitation quantique à boules (LQG) dont le modèle le plus abouti a été défini relativement récemment, en 2008 [Engle et al., 2008]. Cependant, ce modèle est le résultat de travaux développés durant près de 50 ans, mélangeant des considérations à la fois mathématiques et physiques, la théorie BF se situant à l'intersection.

En 1961, T. Regge proposa d'exprimer la GR sans coordonnées en utilisant une décomposition cellulaire de l'espace-temps [Regge, 1961]. Un peu plus tard, Regge étudia avec G. Ponzano la limite des grands spins des coefficients de Racah [Ponzano and Regge, 1968]. Les coefficients de Racah sont, à un signe près, reliés aux symboles $6j$ définis par Wigner qui ont des propriétés de symétrie représentées sur la Figure 1. Sur ce schéma, les $j_{\alpha\beta}$ sont des indices de représentations de $SU(2)$ liées aux symboles $6j$ qui représentent donc des moments angulaires, mais ils peuvent être considérés comme les longueurs des arêtes du tétraèdre.

La relation entre tétraèdres et symboles $6j$ est bien plus profonde qu'un simple moyen mnémotechnique. En effet, dans leur travail, Ponzano et Regge ont montré qu'un symbole $6j$ est lié au volume de son tétraèdre associé par:

$$\left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\} = \frac{F}{\sqrt{12\pi V}}, \quad (1)$$

où V est le volume du tétraèdre et F est une fonction des angles diédraux du tétraèdre et des moments angulaires oscillant rapidement de sorte que la moyenne de F^2 est égale à $\frac{1}{2}$.

De plus, de façon analogue, les symboles $3nj$ sont associés à des polyèdres plus compliqués P_{3nj} . Ils peuvent aussi s'écrire comme fonctions des symboles $3mj$ avec $m < n$. Ces symboles $3mj$ sont eux-mêmes associés à des polyèdres P_{3mj} et la décomposition des symboles $3nj$ en fonction des symboles $3mj$ est reliée à la décomposition géométrique des P_{3nj} en P_{3mj} . Formalisant cette idée, Ponzano et Regge ont été conduits à considérer des sommes de symboles $6j$ représentant des tétraèdres élémentaires qui se recollent pour donner un polyèdre plus compliqué associé à un

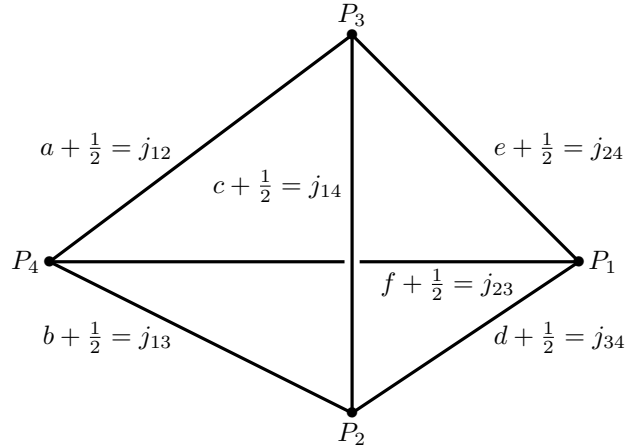


Figure 1: Symmetry of the $6j$ -symbols.

symbole $3nj$. L'unicité du symbole $3nj$ en question implique que ces sommes doivent être indépendantes de la décomposition du polyèdre P_{3nj} en tétraèdres élémentaires.

Pour se recoller le long d'une face, deux tétraèdres doivent porter les mêmes charges sur leurs arêtes. Faire la somme sur toutes les configurations autorisées doit donc produire une quantité indépendante de la triangulation du polyèdre P_{3nj} . Malheureusement, une telle somme sur les représentations de $SU(2)$ est infinie.

Cela étant, menant une étude formelle de la limite des grands spins, Ponzano et Regge reconnurent, grâce aux résultats obtenus par Regge quelques années auparavant, une discrétisation de la fonction de partition associée à la GR euclidienne en dimension 3. La fonction de partition est aussi mal définie puisqu'elle représente une somme sur l'espace de dimension infinie de toutes les décompositions cellulaires de l'espace-temps.

La GR euclidienne en dimension 3 est triviale en ce sens que, pour un espace vide, les équations d'Euler-Lagrange (EL) conduisent à un espace-temps plat. Néanmoins, c'est une première étape d'une tentative de quantification de la GR par quantification de l'espace-temps lui-même. Cette histoire a dormi pendant près de 20 ans avant de réémerger sous différentes formes.

En 1992, V. Turaev and O. Viro ont défini un invariant dit de somme d'états associé aux variétés de dimension 3, ou invariant de Turaev-Viro (TV), utilisant la même idée que Ponzano et Regge consistant à associer des représentations aux arêtes d'une triangulation de la variété considérée [Turaev and Viro, 1992]. La construction est basée sur les catégories modulaires, mais l'hypothèse plus faible de catégorie sphérique semisimple finie peut aussi être considérée [Barrett and Westbury, 1996]. Dans les deux cas, la contrainte dite axiome de domination permet de construire

l'invariant avec un ensemble fini d'objets. L'expression de l'invariant TV est alors formellement très similaire à la formule de Ponzano et Regge dans le cas d'une réalisation de la catégorie par des représentations de $\mathcal{U}_q(\mathfrak{sl}_2)$ (déformation quantique de l'algèbre enveloppante de \mathfrak{sl}_2 en une racine de l'unité q). Contrairement à la formule de Ponzano et Regge, l'invariant TV est bien défini, puisque l'axiome de domination impose à la somme d'être prise sur une période des représentations cycliques de $\mathcal{U}_q(\mathfrak{sl}_2)$. Il peut en ce sens être considéré comme une régularisation de la formule de Ponzano-Regge.

Les catégories modulaires sont aussi au cœur de la construction de l'invariant dit de Reshetikhin-Turaev (RT), dont il a été prouvé qu'il est lié à l'invariant TV construit sur la même catégorie modulaire \mathcal{C} par la relation :

$$\left| Z_{\text{RT}_{\mathcal{C}}} (M^{(3)}) \right|^2 = Z_{\text{TV}_{\mathcal{C}}} (M^{(3)}). \quad (2)$$

Il faut garder en tête que cette égalité n'est plus valable avec l'hypothèse plus faible selon laquelle \mathcal{C} ne serait que sphérique semisimple finie. L'invariant RT est construit à partir d'un nœud encadré associé à la variété considérée par une procédure de chirurgie. Il est intéressant de noter que, dans le cas de $\mathcal{U}_q(\mathfrak{sl}_2)$, il a été prouvé que cet invariant est équivalent au polynôme de Jones de ce nœud de chirurgie [Reshetikhin and Turaev, 1991], et par conséquent, d'après E. Witten, il est lié à la fonction de partition associée à une théorie de Chern-Simons (CS) $SU(2)$ comme discuté plus loin.

En 1978, A. S. Schwarz fut le premier à introduire l'idée que des invariants topologiques pouvaient être retrouvés avec la fonction de partition Z d'une théorie quantique des champs construite à partir d'une action classique S invariante de jauge [Schwarz, 1978], c'est-à-dire, avec une intégrale formelle pouvant s'écrire comme :

$$Z = \frac{1}{\mathcal{N}} \int_{\mathcal{H}} \mathcal{D}\varphi e^{iS(\varphi)}, \quad (3)$$

où le domaine d'intégration \mathcal{H} est l'espace des configurations φ de la théorie et \mathcal{D} est une mesure sur cet espace. Les mathématiciens affirment qu'une mesure non-triviale invariante par translation sur \mathcal{H} ne peut exister. J. Baez proposa néanmoins une alternative consistant à considérer des fonctionnelles linéaires sur des sous-espaces particuliers de \mathcal{H} [Baez, 1993]. Cependant, il n'existe aucune méthode pratique analogue au calcul de primitives permettant d'obtenir un résultat exact. Une telle fonction de partition est en réalité écrite par analogie avec la physique statistique. Un développement perturbatif et des manipulations formelles permettent de trouver les règles de Feynman sans passer par des contractions de Wick sur les opérateurs de champs qui vivent dans l'espace de Fock des états quantiques associés au système. La normalisation \mathcal{N} doit permettre l'extraction des quantités pertinentes en absorbant les divergences intrinsèques de l'intégrale, survenant par exemple à cause de l'invariance de jauge de l'action (mais pas seulement).

En 1982, Witten établit des relations entre théorie de Morse et QM supersymétrique [Witten, 1982], donnant un deuxième exemple de développement mathématique réalisé grâce à des outils issus de la physique. L'année d'après, S. Donaldson utilisa des considérations relatives aux théories de jauge afin de produire des résultats de topologie en dimension 4 [Donaldson, 1983]. Rassemblant en particulier les résultats de Witten et Donaldson, M. Atiyah montra en 1987 que l'étude de la variété de dimension infinie des connections plates du fibré trivial au dessus d'une variété $M^{(3)}$ de dimension 3 était particulièrement intéressante pour l'étude des propriétés topologiques de $M^{(3)}$ elle-même, de même que les propriétés topologiques des variétés qui bordent $M^{(3)}$ et des variétés qui sont bordées par $M^{(3)}$ [Atiyah, 1987].

Plus rigoureusement, dans la littérature mathématique, une connexion est une 1-forme globale sur un fibré principal. Son image par l'application tangente d'une section locale définit sur la variété de base des 1-formes locales à coefficients dans une algèbre de Lie. Dans les intersections de leurs domaines de définition respectifs, deux telles 1-formes locales se recollent par transformation de jauge. Ainsi, par abus de langage, une famille de telles 1-formes locales sera également appelée *connexion*. Une telle famille modélise un champ physique dont l'intensité, ou la courbure d'un point de vue mathématique, est la quantité mesurable, comme un champ électromagnétique par exemple.

Ce sujet désormais appelé théorie quantique des champs topologique (TQFT) a été étudié intensivement depuis lors, tant par les mathématiciens que par les physiciens et des outils familiers des physiciens tels que les intégrales fonctionnelles se sont avérés pertinents. La définition et la classification des TQFT sont notamment fournies dans [Birmingham et al., 1991], une TQFT étant définie par :

- une collection de champs (Grassmann-gradués) définis sur une variété riemannienne $(M^{(n)}, g)$,
- un opérateur nilpotent Q , impair relativement à la Grassmann-gradation,
- des états physiques définies comme étant des classes de Q -cohomologie,
- un tenseur énergie-impulsion Q -exact: $T_{\alpha\beta} = \{Q, V_{\alpha\beta}\}$, les $V_{\alpha\beta}$ étant des fonctionnelles des champs et de la métrique.

Dans la plupart des exemples traités au fil du temps, Q est l'opérateur BRST, mais il ne s'agit pas là d'une contrainte supplémentaire. Ces critères impliquent en particulier que la fonction de partition Z définie comme dans l'équation (3) est un invariant topologique de $M^{(n)}$. Ainsi, Z sera notée $Z(M^{(n)})$ où $M^{(n)}$ est la variété sur laquelle la TQFT est définie. Toujours selon [Birmingham et al., 1991], les TQFT sont classifiées en deux familles. Les théories de type Schwarz, par opposition aux théories de type Witten, sont décrites par une action locale indépendante de la métrique, de même que les corrélateurs, ce qui implique en particulier que le tenseur

énergie-impulsion est nul. Par conséquent, pour une observable $\mathcal{O} : \varphi \mapsto \mathcal{O}(\varphi)$, qui est une fonctionnelle sur l'espace des champs \mathcal{H} , sa valeur moyenne, définie par :

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{N}} \int_{\mathcal{H}} \mathcal{D}\varphi \mathcal{O}(\varphi) e^{iS(\varphi)}, \quad (4)$$

est également un invariant topologique. Il ne doit pas être oublié que l'intégration fonctionnelle est une opération mal définie. Il s'agit d'un intermédiaire formel de calculs. Une domination des résultats de ces calculs par les contributions des connexions plates est attendue étant donné que les connexions plates sont les solutions classiques de la théorie, c'est-à-dire, les solutions des équations EL. Ainsi, la fonction de partition et les valeurs moyennes d'observables sont des invariants topologiques en ce sens qu'il en est ainsi pour les résultats des calculs sous-jacents.

Ce manuscrit porte exclusivement sur des TQFT de type Schwarz. Une théorie qui est, *stricto sensu*, une théorie BF, a été étudiée en 1978 par Schwarz et reliée à la torsion de Ray-Singer [Schwarz, 1978]. Cependant, il est souvent considéré que la théorie BF est apparue pour la première fois dans les travaux de G. Horowitz [Horowitz, 1989] qui portaient sur une classe de TQFT pouvant être définies en n'importe quelle dimension.

Utilisant des "triades", l'action de la GR en dimension 3 peut s'écrire comme¹ :

$$S_{\text{BF}_g}(A, B) = \frac{g}{4\pi} \int_{M^{(3)}} \text{Tr}(B \wedge F_A), \quad (5)$$

où $F_A = dA + A \wedge A$ est la 2-forme de courbure associée à la connexion $SU(2)$ A et B est une 1-forme à valeurs dans $\mathfrak{su}(2)$ ². La normalisation $\frac{g}{4\pi}$ a été choisie de la sorte pour des raisons pratiques qui apparaîtront ultérieurement et diffère de nombreuses publications qui préfèrent l'absorber complètement en redéfinissant B . Cette action est invariante par transformation de jauge :

$$\begin{cases} A \longrightarrow A^g = g^{-1}Ag + g^{-1}dg \Rightarrow F_A \longrightarrow F_{A^g} = g^{-1}F_Ag, \\ B \longrightarrow B^g = g^{-1}Bg, \end{cases} \quad (6)$$

avec $g \in SU(2)$. Il est à noter que des actions du même type sur des variétés $M^{(n)}$ de dimension n quelconque peuvent être construites simplement en considérant B comme étant une $(n-2)$ -forme à valeurs dans $\mathfrak{su}(2)$.

¹ Plusieurs conventions de normalisation de l'action S sont possibles. Si la normalisation de S est $\frac{1}{4\pi}$, alors S est invariante à $2\pi n$ près avec $n \in \mathbb{Z}$ par transformation de jauge. Avec ce choix, la quantité invariante par transformation de jauge est e^{iS} strictement parlant. Si la normalisation de S est $\frac{1}{8\pi^2}$, alors S est invariante à $n \in \mathbb{Z}$ près par transformation de jauge. Avec ce choix, la quantité invariante par transformation de jauge est $e^{2i\pi S}$ strictement parlant.

²La théorie peut être définie aussi bien avec $SU(N)$ qu'avec $SU(2)$.

Introduction (version française)

L'action BF est relativement similaire à l'action CS donnée par¹:

$$S_{\text{CS}_k}(A) = \frac{k}{4\pi} \int_{M^{(3)}} \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \quad (7)$$

où A est une connexion $\text{SU}(2)$ ². La normalisation ici ne peut être absorbée par une redéfinition de A . Elle fait en réalité partie de la définition comme expliqué plus loin (à un facteur 2π près¹). Par transformation de jauge, c'est-à-dire :

$$A \longrightarrow A^g = g^{-1} A g + g^{-1} dg, \quad (8)$$

avec $g \in \text{SU}(2)$, cette action se transforme comme :

$$S_{\text{CS}_k}(A) \longrightarrow S_{\text{CS}_k}(A^g) = S_{\text{CS}_k}(A) + S_{\text{WZ}}(g), \quad (9)$$

où $S_{\text{WZ}}(g)$ est le terme dit de Wess-Zumino et s'avère être un entier. Ainsi, l'action CS n'est bien définie sur les classes de jauge de champs que dans \mathbb{R}/\mathbb{Z} . Contrairement à BF, cette action est purement quantique, en ce sens qu'elle ne peut être considérée simplement dans \mathbb{R} comme n'importe quelle action classique. L'une des applications les plus remarquables de l'action CS est due à Witten, qui prouva que la théorie CS $\text{SU}(2)$ était reliée à l'invariant de nœuds appelé polynôme de Jones [Witten, 1989]. Bien sûr, Witten n'a jamais effectué d'intégration fonctionnelle qui reste une opération mal définie, mais il prouva son résultat par des arguments indirects liés aux théories de champs conformes (CFT).

Il est possible, par une procédure de chirurgie, de relier le calcul dans $M^{(3)}$ à un calcul dans S^3 qui est complètement similaire à celui dans \mathbb{R}^3 . Dans \mathbb{R}^3 , les procédures standard de théorie quantique des champs (QFT) comme la fixation de jauge et les développements perturbatifs peuvent être utilisées, ce qui a été fait par E. Guadagnini et al. [Guadagnini et al., 1990]. Leur résultat est compatible avec le résultat de Witten.

L'expression (7) n'est pas tout à fait rigoureuse. En effet, écrire l'intégrale de cette manière suppose que A est un objet globalement défini, ce qui n'est pas nécessairement le cas en général. D'après le théorème d'extension progressive des sections d'un fibré [Steenrod, 1999], étant donné que pour tout entier N , le groupe de Lie $\text{SU}(N)$ est connexe par arc et $\pi_1(\text{SU}(N)) = \pi_2(\text{SU}(N)) = 0$ (ce qui n'est pas le cas pour $\text{U}(1)$), tout fibré principal $\text{SU}(N)$ au-dessus d'une variété fermée de dimension 3 est trivialisable. Ainsi, il est possible de trouver un représentant globalement défini dans la classe de jauge de n'importe quelle connexion $\text{SU}(N)$. Par conséquent, dans cette formule, A devrait être en fait ce représentant particulier. Il est à noter que fixer un représentant défini globalement revient à considérer le fibré $\text{SU}(N)$ trivial au-dessus de $M^{(3)}$.

De plus, considérer une théorie de jauge comme CS $\text{SU}(N)$ et prétendre calculer des intégrales fonctionnelles telles que l'expression (3) suppose de sommer sur

l'ensemble des connexions, ou, de façon équivalente, sur l'ensemble des fibrés principaux au-dessus de la variété $M^{(3)}$. Cependant, l'invariance de jauge de l'action induit une redondance infinie. Pour gérer ce problème, deux approches sont possibles. Premièrement, il est possible de fixer une jauge, c'est-à-dire, de poser une contrainte permettant de sélectionner un représentant et un seul dans chaque classe de jauge, ce qui est l'approche usuelle en physique. Deuxièmement, il est possible de considérer directement les classes de jauge. Si le fibré principal associé à la théorie est trivialisable, alors ces deux approches sont équivalentes. C'est le cas quand le groupe de jauge est $SU(N)$ ou quand la variété considérée est \mathbb{R}^3 ou S^3 . Cependant, si le fibré principal n'est pas trivialisable, il n'y a *a priori* aucune raison, d'un point de vue de physicien, de se restreindre aux connexions du fibré trivial. Il est à signaler cependant qu'introduire un champ de matière, ou, de façon équivalente, considérer un fibré vectoriel sur lequel le fibré principal agit, modifie les propriétés d'invariance de jauge et fixe en fait le fibré principal de la théorie. Ce travail présente la deuxième approche dans le cas des théories CS et BF $U(1)$ sur une variété fermée connexe de dimension 3 sans champ de matière. Jusqu'à présent, cette approche, initiée dans [Bauer et al., 2005], a été développée depuis lors par très peu d'auteurs [Guadagnini and Thuillier, 2013], [Mathieu and Thuillier, 2016a]. Le fait que les fibrés considérés ici sont non-trivialisables est une différence cruciale avec le cas $SU(N)$ usuel, différence due à la structure topologique spécifique de $U(1)$.

Pour la théorie BF en dimension 3, un terme cosmologique peut être incorporé à l'action :

$$S_{\text{BF}_{g,\kappa}}(A, B) = \frac{g}{4\pi} \int_{M^{(3)}} \text{Tr} \left(B \wedge F_A + \frac{\kappa^2}{3} B \wedge B \wedge B \right), \quad (10)$$

comme proposé par Horowitz dans son calcul de l'action d'Einstein-Hilbert en dimension $3 + 1$ [Horowitz, 1989].

Si la normalisation est absorbée en changeant B en $\tilde{B} = \frac{g}{4\pi} B$ et κ en $\tilde{\kappa} = \frac{4\pi\kappa}{g}$, alors :

$$S_{\text{BF}_{g,\kappa}}(A, B) = \tilde{S}_{\text{BF}_{\tilde{\kappa}}}(A, \tilde{B}) = \int_{M^{(3)}} \text{Tr} \left(\tilde{B} \wedge F_A + \frac{\tilde{\kappa}^2}{3} \tilde{B} \wedge \tilde{B} \wedge \tilde{B} \right), \quad (11)$$

et $\tilde{\kappa}$ n'est rien d'autre que la constante cosmologique analogue en dimension 3 de celle définie par Einstein en dimension $3 + 1$.

En 1998 A. Cattaneo et al. ont montré que les fonctions de partition des théories CS et BF $SU(N)$ pouvaient être reliées par des manipulations formelles sur les intégrales fonctionnelles [Cattaneo et al., 1998] :

$$\left| Z_{\text{CS}_k}(M^{(3)}) \right|^2 = Z_{\text{BF}_{4k\kappa,\kappa}}(M^{(3)}). \quad (12)$$

Cette expression du résultat justifie le choix d'exprimer la théorie BF avec deux paramètres g et κ .

Introduction (version française)

Au-delà du cadre de ce travail, et afin de compléter cette généalogie de la théorie BF en gravité quantique, L. Crane and D. Yetter étudièrent ce qui devrait être un analogue de l'invariant TV en dimension 4, gardant en tête l'idée exprimée par Ponzano et Regge et sa régularisation par l'invariant TV en utilisant un groupe quantique [Crane and Yetter, 1993]. Ils ont écrit :

$$\int_{M^{(4)}} \text{Tr}(B \wedge F_A), \quad (13)$$

avec B une 2-forme à valeurs dans $\mathfrak{su}(2)$. C'est encore une action topologique. Malheureusement, les équations EL contraignent le tenseur de Riemann, ce qui est trop fort pour être pertinent en gravité quantique. Une autre tentative par J. Barrett et Crane dans [Barrett and Crane, 1998] consista à écrire l'action comme :

$$\int_{M^{(4)}} e \wedge e \wedge F_A, \quad (14)$$

introduisant ainsi de nouveaux degrés de liberté et contraignant cette fois le tenseur de Ricci dans les équations EL. Ceci aurait été plus adapté à une théorie de la gravité quantique mais s'avère en fait faux. Une correction de ce modèle a mené néanmoins au modèle EPRL défini dans [Engle et al., 2008], qui est le modèle de LQG le plus abouti actuellement, qui n'a pas de divergences UV et dont les divergences IR sont contrôlées, la limite des grands spins étant bien sûr la GR.

Par conséquent, comprendre la régularisation du modèle de Ponzano-Regge et sa relation avec l'invariant TV et la théorie BF $SU(2)$ semble être une étape pertinente dans la recherche d'une théorie quantique de la gravitation. S. Mizoguchi et T. Tada ont montré en 1992 que le paramètre de déformation quantique q devait être relié à la constante cosmologique $\tilde{\kappa}$ [Mizoguchi and Tada, 1992]. C. Rovelli a montré en 1993 l'équivalence de la représentation dite en boucle en LQG et la combinatoire sous-jacente dans la gravité quantique présentée dans les travaux de Ponzano, Regge, H. Ooguri, Turaev and Viro [Rovelli, 1993]. Cette question a été considérée jusqu'à récemment par exemple avec les travaux de K. Krasnov, D. Louapre et L. Freidel [Freidel and Krasnov, 2005, Freidel and Louapre, 2004, Freidel and Louapre, 2005] ou les travaux de V. Bonzom et M. Smerlak [Bonzom and Smerlak, 2012]. La recherche sur les théories CS et BF pour elles-mêmes est également toujours active ([Cattaneo et al., 2017] par exemple).

Jusqu'à présent, l'étendue actuelle des connaissances peut être résumée sur le diagramme suivant :

$$\begin{array}{ccc} |Z_{CS_k}|^2 & = & Z_{BF_{4k\kappa, \kappa}} \\ \parallel & & \parallel \\ |Z_{RT_{q(k)}}|^2 & = & Z_{TV_{q(k)}} \end{array}$$

L'indice $q(k)$ est la racine de l'unité apparaissant dans $\mathcal{U}_q(\mathfrak{sl}_2)$ et est construite à partir de la constante de couplage k de la théorie CS. Du côté de TV, $q = q(\tilde{\kappa})$ mais $\tilde{\kappa} = \frac{4\pi\kappa}{g} = \frac{\pi}{k}$ puisque $g = 4k\kappa$ afin de satisfaire l'égalité des fonctions de partition. Par conséquent $q = q(\tilde{\kappa}) = q(k)$.

L'égalité du bas relie deux quantités parfaitement définies. Il s'agit d'un théorème établi par Reshetikhin, Turaev et Viro dans le cadre général des catégories modulaires. Au contraire, les autres égalités mettent en jeu des objets mal définis. L'égalité du haut a été établie par des manipulations formelles et les deux égalités verticales ont été prouvées par des arguments indirects.

Le but de ce travail est d'étudier la théorie BF abélienne sur une variété $M^{(3)}$ de dimension 3 en calculant la fonction de partition et les valeurs moyennes d'observables, en partant d'intégrales fonctionnelles formelles. Plusieurs différences avec les études habituelles du sujet doivent être soulignées. La théorie BF abélienne considérée ici est une TQFT, en ce sens que l'action considérée est complètement indépendante de toute métrique sur $M^{(3)}$. C'est même une TQFT cohomologique en ce sens qu'elle sera écrite, non pas, comme d'habitude, en termes de connexions $U(1)$, mais en termes de classes de jauge de connexions $U(1)$ qui correspondent en fait à des classes de cohomologie de Deligne-Beilinson (DB). La structure de l'espace des classes de cohomologie DB permettra d'extraire d'une manière exacte les quantités pertinentes des intégrales fonctionnelles, comme cela a été fait pour la théorie CS abélienne [Guadagnini and Thuillier, 2013]. Cette approche est donc non-perturbative. Dans ce travail, les calculs reposeront sur la cohomologie DB et ne se limiteront pas aux classes de jauge associées au fibré $U(1)$ trivial au-dessus de $M^{(3)}$. En effet, les physiciens interprètent les théories de jauge de la façon suivante : les observateurs font une mesure d'une quantité physique dans leur voisinage. Si deux observateurs comparent leurs résultats respectifs, il n'y a pas de raison pour qu'ils soient identiques, mais ils seront reliés par une transformation de jauge. Ainsi, il n'y a pas de raison de se restreindre aux configurations dans lesquelles les mesures de tous les observateurs sont identiques. Par conséquent, en un certain sens, les fonctions de partition et valeurs moyennes d'observables seront des sommes particulières sur tous les fibrés $U(1)$ au-dessus de $M^{(3)}$.

Au-delà des calculs de la fonction de partition et des valeurs moyennes d'observables qui n'avaient jamais été traités de cette façon auparavant, ce travail vise à mettre en lumière les relations entre les théories BF et CS abéliennes de même que les relations entre ces théories et les TQFT dans le formalisme des catégories telles que décrites notamment par N. Reshetikhin, V. Turaev et O. Viro. Quelques subtilités doivent être mentionnées. Tout d'abord, pour obtenir des relations correctes entre toutes ces théories, la normalisation des invariants RT et TV sera S^3 et non $S^1 \times S^2$ contrairement à la convention usuelle dans la littérature mathématique. Ensuite, l'hypothèse plus faible de catégorie sphérique semisimple finie doit être considérée pour la théorie RT afin d'obtenir une relation entre les théories CS et RT abéliennes.

Introduction (version française)

Cette hypothèse plus faible est, en un certain sens, celle de H. Murakami, T. Ohtsuki et M. Okada [Murakami et al., 1992] et est également évoquée par Turaev [Turaev, 2010]. Il s'agit d'une différence fondamentale avec le cas non-abélien (notamment $SU(2)$). Finalement, comme mentionné précédemment, la théorie CS $SU(2)$ est reliée à la théorie RT construite sur un groupe quantique. Cette idée ne peut être directement étendue au cas abélien étant donné qu'aucune déformation quantique de $U(1)$ n'est possible. C'est une autre différence cruciale avec le cas $SU(2)$ due à la structure topologique spécifique de $U(1)$. Comme dans le cas $SU(2)$, le problème est en fait de considérer des objets satisfaisant l'axiome de domination. Dans les cas considérés, cela consiste en un ensemble infini d'objets avec une structure périodique de sorte que les sommes peuvent être restreintes à une période sans perdre d'information portée par ces objets. Il sera montré que, dans le cas abélien, les objets corrects à considérer sont les représentations de \mathbb{Z}_k . Avant ce travail, une telle classe d'exemples n'avait jamais été exhibée, probablement à cause de leur trivialité apparente. Cependant, il sera montré plus loin qu'ils constituent en fait une classe subtile d'exemples ayant des liens intéressants avec les théories BF et CS abéliennes.

La première partie de ce manuscrit se concentre sur des notions fondamentales de TQFT d'un point de vue mathématique. Cette approche repose sur la théorie des catégories. Le premier chapitre introduit les définitions principales et des réalisations abéliennes sont présentées. Le deuxième chapitre présente l'invariant RT abélien fondé sur la catégorie modulaire abélienne présentée dans le premier chapitre, suivant le travail de Reshetikhin et Turaev. L'extension étudiée en particulier par Murakami, Ohtsuki et Okada [Murakami et al., 1992] est aussi considérée. De façon similaire, l'invariant RT fondé sur la version abélienne du centre de Drinfeld est présenté, ainsi que des développements récents associés [Turaev and Virelizier, 2013]. Le troisième chapitre de cette partie présente l'invariant TV abélien fondé sur la catégorie sphérique semisimple finie présentée dans le premier chapitre. Sa définition est fondée sur les travaux de B. Balsam et A. Kirillov [Balsam and Kirillov, 2010] qui généralisent la construction originale reposant sur les triangulations aux décompositions cellulaires quelconques. Dans le cadre abélien, certains résultats reliant la théorie TV et la théorie RT établis dans le cas général peuvent ainsi être vérifiés. La principale contribution originale de cette partie est la détermination d'une version abélienne de toutes ces notions et la discussion des subtilités qui apparaissent alors. Aussi, la comparaison entre la forme des invariants RT et TV abéliens et les fonctions de partition CS et BF données dans la partie suivante conduit à introduire des valeurs moyennes d'observables dans les théories RT et TV abéliennes, ce qui n'avait jamais été fait auparavant à proprement parler.

La deuxième partie se concentre sur les théories CS et BF abéliennes dans une approche physique. Ces théories sont définies par des actions classiques, c'est-à-dire, des fonctionnelles sur l'espace des champs classiques. Les quantités calculées sont extraites d'intégrales fonctionnelles, mais le but de ce travail n'est pas de donner une

définition rigoureuse ni d'effectuer des calculs de tels objets. Le premier chapitre présente la cohomologie DB dont les classes seront précisément les classes de jauge de champs. La structure de l'espace des classes de cohomologie DB et quelques propriétés sont détaillées en vue d'une utilisation ultérieure en suivant [Mathieu, 2017]. Dans le deuxième chapitre, la fonction de partition CS et les valeurs moyennes d'observables sont définies et calculées. Ces résultats sont reliés à la théorie abélienne RT présentée dans la partie précédente. Dans le troisième chapitre, la fonction de partition BF et les valeurs moyennes d'observables sont définies et calculées. Ces résultats sont reliés à la théorie abélienne TV présentée dans la partie précédente et aux résultats relatifs à la théorie CS abélienne obtenus dans le chapitre précédent concernant les fonctions de partition en suivant [Mathieu and Thuillier, 2016a] et concernant les valeurs moyennes d'observables suivant [Mathieu and Thuillier, 2016b]. Il est souvent affirmé que les contributions principales aux intégrales fonctionnelles proviennent d'un voisinage des solutions des équations EL qui sont les connexions plates, c'est-à-dire, les connexions A telles que $F_A = 0$. L'utilisation de la cohomologie DB implique que cette assertion est correcte, et même prouvée par un calcul direct. Les résultats établis dans le dernier chapitre de cette partie, en particulier les liens entre les quantités définies et calculées dans ce chapitre et dans les précédents, constituent la principale production originale de ce travail.

Ces deux parties fournissent un ensemble complet et relativement fermé de définitions et de résultats concernant les théories RT, TV, CS et BF sur des variétés de dimension 3. Ces résultats vont au-delà de l'objectif initial de ce travail.

La troisième partie rassemble des résultats nouveaux sortant légèrement du cadre central de ce travail, sans pour autant qu'ils soient complètement hors de propos. Le premier chapitre introduit une méthode complètement nouvelle fondée sur des diagrammes de Heegaard afin de calculer l'invariant CS $SU(2)$ suivant [Guadagnini et al., 2017]. Dans le deuxième chapitre, le propagateur de la théorie CS abélienne dans \mathbb{R}^{4+3} muni d'une métrique anisotrope est calculé. Plusieurs limites sont étudiées et des interprétations géométriques sont proposées, complétant ainsi les cas particuliers traités dans [Gallot et al., 2018].

Some preliminary remarks

In this study, if no opposite assumption is pointed out, a manifold $M^{(n)}$ will be assumed to be n -dimensional (most of the time $n = 3$), oriented, connected and closed, that is, compact without a boundary. For further convenience, such a manifold will always be supposed to be provided with a **good covering**, that is, a covering of contractible open sets, whose intersections are also contractible or empty. Such a good covering always exists. The results obtained will be independent of the good covering in question.

Still if no opposite assumption is pointed out, the homology groups H_p (respectively the cohomology groups H^p) will have integer coefficients and will be regarded as \mathbb{Z} -modules. As abelian groups, they will be decomposed in the direct sum of their free part F_p (respectively F^p) and their torsion part T_p (respectively T^p). To avoid any confusion later on, it is important to remind that the cohomology group $H^p(M^{(n)}, \mathbb{Z})$ of a manifold $M^{(n)}$ with integer coefficients is not the dual of the homology group $H_p(M^{(n)}, \mathbb{Z})$ of the same manifold $M^{(n)}$ with integer coefficients, that is:

$$H^p(M^{(n)}, \mathbb{Z}) \not\cong \text{Hom}(H_p(M^{(n)}, \mathbb{Z}), \mathbb{Z}). \quad (15)$$

However, the duality exists between the chains and cochains, the cycles and cocycles and the boundaries and coboundaries:

$$C^p(M^{(n)}, \mathbb{Z}) = \text{Hom}(C_p(M^{(n)}, \mathbb{Z}), \mathbb{Z}), \quad (16)$$

$$Z^p(M^{(n)}, \mathbb{Z}) = \text{Hom}(Z_p(M^{(n)}, \mathbb{Z}), \mathbb{Z}), \quad (17)$$

$$B^p(M^{(n)}, \mathbb{Z}) = \text{Hom}(B_p(M^{(n)}, \mathbb{Z}), \mathbb{Z}) \quad (18)$$

and:

$$H^p(M^{(n)}, \mathbb{Z}) = Z^p(M^{(n)}, \mathbb{Z}) / B^p(M^{(n)}, \mathbb{Z}). \quad (19)$$

Some preliminary remarks

Hence, this reminder just means that, in general:

$$\begin{aligned} \operatorname{Hom}(Z_p(M^{(n)}, \mathbb{Z}), \mathbb{Z}) / \operatorname{Hom}(B_p(M^{(n)}, \mathbb{Z}), \mathbb{Z}) \\ \not\cong \operatorname{Hom}(Z_p(M^{(n)}, \mathbb{Z}) / B_p(M^{(n)}, \mathbb{Z}), \mathbb{Z}). \end{aligned} \quad (20)$$

However, the universal coefficients theorem claims that, for any abelian group G [Mac Lane, 1994]:

$$H^p(M^{(n)}, G) \simeq \operatorname{Hom}(H_p(M^{(n)}, \mathbb{Z}), G) \oplus \operatorname{Ext}(H_{p-1}(M^{(n)}, \mathbb{Z}), G). \quad (21)$$

Several particular cases can then be considered. First, in general, if G is divisible, then $\operatorname{Ext}(H_{p-1}(M^{(n)}, \mathbb{Z}), G)$ is trivial and as a consequence:

$$H^p(M^{(n)}, G) \simeq \operatorname{Hom}(H_p(M^{(n)}, \mathbb{Z}), G). \quad (22)$$

A case of divisible group of particular interest for this dissertation is \mathbb{R}/\mathbb{Z} [Harvey et al., 2003]. Another particular case of interest for this dissertation is $p = 1$ and $G = \mathbb{Z}_k$. Indeed, since $M^{(n)}$ is supposed to be connected, then $H_0(M^{(n)}, \mathbb{Z}) = \mathbb{Z}$ and thus $\operatorname{Ext}(H_0(M^{(n)}, \mathbb{Z}), \mathbb{Z}_k)$ is trivial [Bott and Tu, 1982] and as a consequence:

$$H^1(M^{(n)}, \mathbb{Z}_k) \simeq \operatorname{Hom}(H_1(M^{(n)}, \mathbb{Z}), \mathbb{Z}_k). \quad (23)$$

The universal coefficients theorem provides other useful isomorphisms:

$$\begin{aligned} H_p(M^{(n)}, \mathbb{Z}) & \simeq & H^{n-p}(M^{(n)}, \mathbb{Z}) \\ \wr & & \wr \\ T^{p+1}(M^{(n)}, \mathbb{Z}) \oplus F^p(M^{(n)}, \mathbb{Z}) & \simeq & T^{n-p-1}(M^{(n)}, \mathbb{Z}) \oplus F_{n-p}(M^{(n)}, \mathbb{Z}). \\ \not\cong & & \not\cong \\ H^p(M^{(n)}, \mathbb{Z}) & \simeq & H_{n-p}(M^{(n)}, \mathbb{Z}) \end{aligned} \quad (24)$$

However, with real coefficients, that is, taking the tensor product of those groups by \mathbb{R} , the torsion is trivialized and as a consequence:

$$\begin{aligned} H_p(M^{(n)}, \mathbb{R}) & \simeq & H^{n-p}(M^{(n)}, \mathbb{R}) \\ \wr & & \wr \\ H^p(M^{(n)}, \mathbb{R}) & \simeq & H_{n-p}(M^{(n)}, \mathbb{R}) \end{aligned} . \quad (25)$$

More generally, for a given field \mathbb{K} , by the universal coefficients theorem:

$$H^p(M^{(n)}, \mathbb{K}) \simeq \operatorname{Hom}(H_p(M^{(n)}, \mathbb{K}), \mathbb{K}) \oplus \operatorname{Ext}(H_{p-1}(M^{(n)}, \mathbb{K}), \mathbb{K}), \quad (26)$$

and it happens that, in that case also, $\operatorname{Ext}(H_{p-1}(M^{(n)}, \mathbb{K}), \mathbb{K})$ is trivial so that:

$$H^p(M^{(n)}, \mathbb{K}) \simeq \operatorname{Hom}(H_p(M^{(n)}, \mathbb{K}), \mathbb{K}). \quad (27)$$

If, moreover, $H_p(M^{(n)}, \mathbb{K})$ is a vector space of finite dimension, then it is isomorphic to its dual and then:

$$H^p(M^{(n)}, \mathbb{K}) \simeq H_p(M^{(n)}, \mathbb{K}). \quad (28)$$

When no confusion is possible, $H_p(M^{(n)}, \mathbb{Z}) = T_p(M^{(n)}, \mathbb{Z}) \oplus F_p(M^{(n)}, \mathbb{Z})$ and $H^p(M^{(n)}, \mathbb{Z}) = T^p(M^{(n)}, \mathbb{Z}) \oplus F^p(M^{(n)}, \mathbb{Z})$ will be simply denoted $H_p(M^{(3)}) = T_p(M^{(n)}) \oplus F_p(M^{(n)})$ and $H^p(M^{(3)}) = T^p(M^{(n)}) \oplus F^p(M^{(n)})$ or even $H_p = T_p \oplus F_p$ and $H^p = T^p \oplus F^p$.

Otherwise, a link \mathcal{L} in a manifold of dimension 3 will always be provided with a framing for each component and supposed to have his components ordered. The framing of a component is an integer that can be interpreted in the following way: a component is actually not a 1-dimensional closed strand, but a closed strip twisted n times, with $n \in \mathbb{Z}$ being the so-called framing (the sign occuring according to a conventional sense of twist).

Finally, in this study, “abelian gauge theory” will be considered as a terminology equivalent to “U(1) gauge theory”.

Part I

A mathematical approach

Chapter 1

Notions on categories

Contents

Introduction (English version)	6
Introduction (version française)	8
1.1 The category $\mathbb{C}^{\mathbb{Z}_k}$ of representations of \mathbb{Z}_k	10
1.2 The category $\mathbb{C}^{\mathbb{Z}_k}$ as a monoidal category	11
1.3 The category $\mathbb{C}^{\mathbb{Z}_k}$ as a braided and twisted category . .	11
1.4 The category $\mathbb{C}^{\mathbb{Z}_k}$ as a ribbon and spherical category . .	13
1.5 The category $\mathbb{C}^{\mathbb{Z}_k}$ as a modular category?	14
1.6 Drinfeld center of $\mathbb{C}^{\mathbb{Z}_k}$	15

Introduction (English version)

Originally, the theory of categories was founded in the 40s mostly by Eilenberg and Mac Lane in the context of algebraic topology before spreading to the other branches of mathematics, as well as in mathematical physics and theoretical computer science. The interest in the theory of categories became bigger as it turned out to be a unifying framework. It was developed notably under the influence of works by Grothendieck in the 70s.

A **category** \mathcal{C} consists of:

- A collection of **objects** $\text{Ob}(\mathcal{C})$.
- For every two objects X and Y in \mathcal{C} , a set of **morphisms** or **arrows** $\text{Hom}_{\mathcal{C}}(X, Y)$.
- For any triple of objects X, Y and Z a composition law:

$$\begin{array}{ccc} \circ : \text{Hom}_{\mathcal{C}}(Y, Z) \times \text{Hom}_{\mathcal{C}}(X, Y) & \longrightarrow & \text{Hom}_{\mathcal{C}}(X, Z) \\ g, f & \longmapsto & g \circ f, \end{array}$$

subject to the following conditions:

1. The composition is associative: if $f \in \text{Hom}_{\mathcal{C}}(X, Y)$, $g \in \text{Hom}_{\mathcal{C}}(Y, Z)$ and $h \in \text{Hom}_{\mathcal{C}}(Z, T)$, then $h \circ (g \circ f) = (h \circ g) \circ f$.
2. For every object X in \mathcal{C} there is an identity morphism $\text{Id}_X : X \rightarrow X$ with the property that, for any morphism $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ and any morphism $g \in \text{Hom}_{\mathcal{C}}(Z, X)$:

$$f \circ \text{Id}_X = f \quad \text{and} \quad \text{Id}_X \circ g = g.$$

3. The sets $\text{Hom}_{\mathcal{C}}(X, Y)$ and $\text{Hom}_{\mathcal{C}}(X', Y')$ are disjoint unless $X = X'$ and $Y = Y'$.

As a group can be sent to another group, a category can be sent to another category. For this purpose, let \mathcal{C} and \mathcal{D} be categories. A **(covariant) functor** $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of:

- For every object X in \mathcal{C} an object $F(X)$ in \mathcal{D} .
- For every morphism $f : X \rightarrow Y$ in \mathcal{C} a morphism $F(f) : F(X) \rightarrow F(Y)$ in \mathcal{D} subject to the following conditions:
 1. For any pair of morphisms $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \text{Hom}_{\mathcal{C}}(Y, Z)$, $F(g) \circ F(f) = F(g \circ f)$.
 2. For any object X in \mathcal{C} , $F(\text{Id}_X) = \text{Id}_{F(X)}$.

The definitions of categories and functors are so general that they make it possible to relate different branches of mathematics and consider them in a unified systematic way. For instance, homology can be seen as a functor that relates a category of topological spaces and a category of algebraic objects. What mathematicians define as a TQFT is also a functor that transforms a category of topological space into a category of algebraic quantities. This is the particular case developed in this first part.

In this chapter, several notions of the theory of categories that are involved in the framework of TQFT from the mathematics point of view are defined. At the same time abelian realizations of the notions defined are presented and a few properties are derived. The following definitions come from [Turaev, 2010] but they have been refined more recently [Turaev and Virelizier, 2013]. Those refinements have no incidence on the present study. The lecture notes [Berglund, 2009] give a concise overview of the basics of the theory of categories and the website <https://ncatlab.org/> proposes a nice tree view of all the advanced notions presented in this chapter. Some materials can also be found in general perspective in [Mac Lane, 1998] and in the perspective of quantum groups in [Chari and Pressley, 1994].

Introduction (version française)

À l'origine, la théorie des catégories a été fondée dans les années 1940 par Eilenberg et Mac Lane dans le contexte de la topologie algébrique avant de toucher aux autres branches des mathématiques, ainsi qu'à la physique mathématique et à l'informatique théorique. L'intérêt pour la théorie des catégories est allé croissant à mesure qu'il s'est avéré être un cadre unificateur. Elle a été développée notamment sous l'influence des travaux de Grothendieck dans les années 1970.

Une **catégorie** \mathcal{C} consiste en :

- Une collection d'**objets** $\text{Ob}(\mathcal{C})$.
- Pour toute paire d'objets X et Y de \mathcal{C} , un ensemble de **morphismes** ou **flèches** $\text{Hom}_{\mathcal{C}}(X, Y)$.
- Pour tout triplet d'objets X, Y et Z , une loi de composition :

$$\begin{array}{ccc} \circ : \text{Hom}_{\mathcal{C}}(Y, Z) \times \text{Hom}_{\mathcal{C}}(X, Y) & \longrightarrow & \text{Hom}_{\mathcal{C}}(X, Z) \\ g, f & \longmapsto & g \circ f, \end{array}$$

soumise aux contraintes suivantes :

1. La composition est associative : si $f \in \text{Hom}_{\mathcal{C}}(X, Y)$, $g \in \text{Hom}_{\mathcal{C}}(Y, Z)$ et $h \in \text{Hom}_{\mathcal{C}}(Z, T)$, alors $h \circ (g \circ f) = (h \circ g) \circ f$.
2. Pour tout objet X de \mathcal{C} il existe un morphisme identité $\text{Id}_X : X \rightarrow X$ tel que, pour tout morphisme $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ et tout morphisme $g \in \text{Hom}_{\mathcal{C}}(Z, X)$:

$$f \circ \text{Id}_X = f \quad \text{et} \quad \text{Id}_X \circ g = g.$$

3. Les ensembles $\text{Hom}_{\mathcal{C}}(X, Y)$ et $\text{Hom}_{\mathcal{C}}(X', Y')$ sont disjoints à moins que $X = X'$ et $Y = Y'$.

Tout comme un groupe peut être envoyé dans un autre groupe, une catégorie peut être envoyée dans une autre catégorie. Pour cela, soit \mathcal{C} et \mathcal{D} deux catégories. Un **foncteur (covariant)** $F : \mathcal{C} \rightarrow \mathcal{D}$ consiste en la donnée :

- Pour tout objet X de \mathcal{C} , un objet $F(X)$ de \mathcal{D} .
 - Pour tout morphisme $f : X \rightarrow Y$ de \mathcal{C} , un morphisme $F(f) : F(X) \rightarrow F(Y)$ de \mathcal{D} soumis aux contraintes suivantes :
1. Pour toute paire de morphismes $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ et $g \in \text{Hom}_{\mathcal{C}}(Y, Z)$, $F(g) \circ F(f) = F(g \circ f)$.

2. Pour tout objet X de \mathcal{C} , $F(\text{Id}_X) = \text{Id}_{F(X)}$.

Les définitions de catégories et de foncteurs sont tellement générales qu'elles permettent de relier différentes branches des mathématiques et de les considérer d'une façon systématique et unifiée. Par exemple, l'homologie peut être vue comme un foncteur envoyant une catégorie d'espaces topologiques dans une catégorie d'objets algébriques. Ce que les mathématiciens définissent comme une TQFT est aussi un foncteur qui transforme une catégorie d'espaces topologiques en une catégorie de quantités algébriques. C'est le cas particulier qui est développé dans cette première partie.

Dans ce chapitre, plusieurs notions de théorie des catégories utiles dans le cadre des TQFT d'un point de vue mathématique sont définies. Des réalisations abéliennes de ces notions sont également définies et quelques propriétés sont présentées. Les définitions suivantes proviennent de [Turaev, 2010] mais celles-ci ont été raffinées récemment [Turaev and Virelizier, 2013]. Ces raffinements n'ont pas d'incidence sur ce travail. Les notes de cours [Berghlund, 2009] donnent un aperçu large et concis des bases de la théorie des catégories et le site web <https://ncatlab.org/> propose une belle arborescence de toutes les notions avancées présentées dans ce chapitre. Quelques éléments peuvent également être trouvés dans une perspective générale dans [Mac Lane, 1998] et dans la perspective des groupes quantiques dans [Chari and Pressley, 1994].

1.1 The category $\mathbb{C}^{\mathbb{Z}_k}$ of representations of \mathbb{Z}_k

The point of this section is to define a set of objects of a category that are representations related to $U(1)$. Those representations will have to verify some properties so that it will be possible later to consider only a finite set of those objects. This has been achieved by Reshetikhin and Turaev with $SU(2)$. They considered cyclic representations of a quantum deformation of $SU(2)$ at a root of unity. Unfortunately, the same procedure cannot be used for $U(1)$ which cannot be deformed since it is abelian. A way to solve the problem is to consider representations of a discrete and finite subgroup of $U(1)$ which is isomorphic to \mathbb{Z}_k for $k \in \mathbb{Z}$.

Like any group, the group \mathbb{Z}_k can be regarded as the set of morphisms of a category $\mathcal{C}_{\mathbb{Z}_k}$ that would have only one object and whose morphisms would all be invertible. This holds also for $Gl(\mathbb{C})$ with a category $\mathcal{C}_{Gl(\mathbb{C})}$. An irreducible representation $R : p \in \mathbb{Z}_k \mapsto R_p = e^{2i\pi \frac{p}{k}} \in Gl(\mathbb{C})$ of \mathbb{Z}_k over \mathbb{C} can thus be considered as a functor between $\mathcal{C}_{\mathbb{Z}_k}$ and $\mathcal{C}_{Gl(\mathbb{C})}$. The set of functors can be itself regarded as the set of objects of a category $\mathbb{C}^{\mathbb{Z}_k}$ whose morphisms are natural transformations, that is, in this case, the linear transformations $\eta_{p,q} : R_p \longrightarrow R_q$ such that:

$$R_q \circ \eta_{p,q} = \eta_{p,q} \circ R_p, \quad (1.1)$$

or, acting on a complex z :

$$R_q(\eta_{p,q}(z)) = \eta_{p,q}(R_p(z)), \quad (1.2)$$

which can be written explicitly:

$$e^{2i\pi \frac{q}{k}} \eta_{p,q}(z) = \eta_{p,q}(e^{2i\pi \frac{p}{k}} z) = e^{2i\pi \frac{p}{k}} \eta_{p,q}(z), \quad (1.3)$$

since for any $p \in \mathbb{Z}_k$, $R_p = e^{2i\pi \frac{p}{k}}$. Hence:

$$\eta_{p,q} = \delta_{p,q}^{[k]} z_{p,q}, \quad (1.4)$$

where $z_{p,q} \in \mathbb{C}$ and $\delta_{p,q}^{[k]} = 1$ if $p \equiv q [k]$ or $\delta_{p,q}^{[k]} = 0$ otherwise.

Strictly speaking, the index p of R_p is an element of \mathbb{Z}_k and in that case there is a finite number of objects R_p . It is also possible to consider that p is an element of \mathbb{Z} and in that case, there is an infinite number of objects R_p with a periodicity k . This is the case considered from now on. The family of objects $\{R_p, p \in \llbracket 0; k-1 \rrbracket\}$ is finite and contains all the information carried by the R_p , $p \in \mathbb{Z}$. This idea will be formalized later in particular with the axiom of domination.

To sum up, explicitly, the objects of the category $\mathbb{C}^{\mathbb{Z}_k}$ are the $R_p = e^{2i\pi \frac{p}{k}}$, $p \in \mathbb{Z}$ and its morphisms are the natural transformations $\eta_{p,q} = \delta_{p,q}^{[k]} z_{p,q} : R_p \longrightarrow R_q$, $p, q \in \mathbb{Z}$ ¹².

¹When there is no ambiguity, this integer index will be used to refer to objects or morphisms.

²The composition of morphisms is just the product in \mathbb{C} .

1.2 The category $\mathbb{C}^{\mathbb{Z}_k}$ as a monoidal category

The category $\mathbb{C}^{\mathbb{Z}_k}$ can be endowed with a tensor product $\otimes : \mathbb{C}^{\mathbb{Z}_k} \times \mathbb{C}^{\mathbb{Z}_k} \longrightarrow \mathbb{C}^{\mathbb{Z}_k}$ explicitly defined to be the usual product in \mathbb{C} :

$$R_p \otimes R_q = R_{p+q} \quad (1.5)$$

regarding the objects and:

$$\delta_{p,q}^{[k]} z_{p,q} \otimes \delta_{r,s}^{[k]} z_{r,s} = \delta_{p,q}^{[k]} \delta_{r,s}^{[k]} z_{p,q} z_{r,s} \quad (1.6)$$

regarding the morphisms.

Such a tensor product has several properties:

- It admits a neutral element:

- R_0 concerning the objects:

$$R_p \otimes R_0 = R_0 \otimes R_p = R_p, \quad (1.7)$$

- $\delta_{0,0}^{[k]} z_{0,0} = 1$ concerning the morphisms:

$$\delta_{p,q}^{[k]} z_{p,q} \otimes 1 = 1 \otimes \delta_{p,q}^{[k]} z_{p,q} = \delta_{p,q}^{[k]} z_{p,q}. \quad (1.8)$$

- It is associative:

- regarding the objects:

$$R_p \otimes (R_q \otimes R_r) = R_{p+q+r} = (R_p \otimes R_q) \otimes R_r, \quad (1.9)$$

- regarding the morphisms:

$$\delta_{p,q}^{[k]} z_{p,q} \otimes (\delta_{r,s}^{[k]} z_{r,s} \otimes \delta_{u,v}^{[k]} z_{u,v}) = (\delta_{p,q}^{[k]} z_{p,q} \otimes \delta_{r,s}^{[k]} z_{r,s}) \otimes \delta_{u,v}^{[k]} z_{u,v}. \quad (1.10)$$

This tensor product, together with this set of properties, endows $\mathbb{C}^{\mathbb{Z}_k}$ with the structure of **monoidal category** [Turaev, 2010]. It is commutative and thus encodes *a priori* less information than a non-commutative tensor product would.

1.3 The category $\mathbb{C}^{\mathbb{Z}_k}$ as a braided and twisted category

Among the morphisms of $\mathbb{C}^{\mathbb{Z}_k}$, two particular families of morphisms can be identified. A family of morphisms $c = \{c_{p,q} : R_p \otimes R_q \longrightarrow R_q \otimes R_p = R_p \otimes R_q, c_{p,q} = e^{2i\pi \frac{pq}{k}}\}$ is called **braiding** if it verifies:

$$c_{p,q \otimes r} = (\text{Id}_q \otimes c_{p,r}) \circ (c_{p,q} \otimes \text{Id}_r), \quad (1.11)$$

Chapter 1. Notions on categories

see the diagrams in Figure 1.1,

$$c_{p \otimes q, r} = (c_{p, r} \otimes \text{Id}_q) \circ (\text{Id}_p \otimes c_{q, r}), \quad (1.12)$$

see the diagrams in Figure 1.2,

$$\begin{aligned} & (\text{Id}_r \otimes c_{p, q}) \circ (c_{p, r} \otimes \text{Id}_q) \circ (\text{Id}_p \otimes c_{q, r}) \\ &= (c_{q, r} \otimes \text{Id}_p) \circ (\text{Id}_q \otimes c_{p, r}) \circ (c_{p, q} \otimes \text{Id}_r), \end{aligned} \quad (1.13)$$

(Yang-Baxter equation) see the diagrams in Figure 1.3. A family verifying this set of properties endows $\mathbb{C}^{\mathbb{Z}_k}$ with a structure of **braided category** [Turaev and Virelizier, 2013].

A family of morphisms $\theta = \{\theta_p : R_p \rightarrow R_p, \theta_p = e^{2i\pi \frac{p^2}{k}}\}$ is called **twist** if it verifies:

$$\theta_{p \otimes q} = c_{q, p} \circ c_{p, q} \circ (\theta_p \otimes \theta_q), \quad (1.14)$$

see the diagrams in Figure 1.4. A family verifying this set of properties endows $\mathbb{C}^{\mathbb{Z}_k}$ with a structure of **twisted category**.

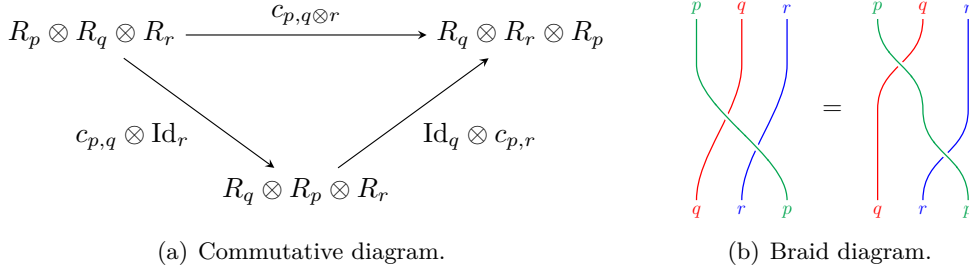


Figure 1.1: Diagrams associated with equation (1.11).

1.4 The category $\mathbb{C}^{\mathbb{Z}_k}$ as a ribbon and spherical category

So far, the information of a braiding has been encoded. However, the purpose here is to encode information on links. It is therefore necessary to encode the information of the closure of a braid. This is done by introducing a notion of **duality**³: for any

³More rigorously, the duality is given by two families of morphisms, the **evaluations** $\text{ev}_X : X \otimes X^* \rightarrow \mathbb{1}$ and the **coevaluations** $\text{coev}_X : \mathbb{1} \rightarrow X^* \otimes X$ for objects X . Consider two branches of same weight, one oriented downward, the other upward. The evaluation (respectively the coevaluation) related to that weight is the operators that closes the loop at the top (respectively at the bottom) of the diagram. In this abelian context, those morphisms are trivial since $R_p \otimes (R_p)^* = (R_p)^* \otimes R_p = \mathbb{1}$. See Figure 2.1.

1.4. The category $\mathbb{C}^{\mathbb{Z}_k}$ as a ribbon and spherical category

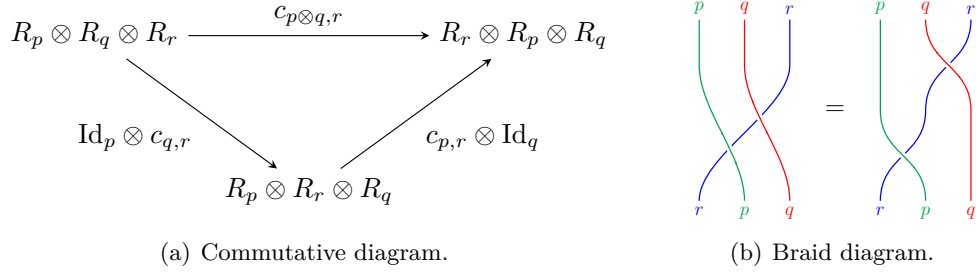


Figure 1.2: Diagrams associated with equation (1.12).

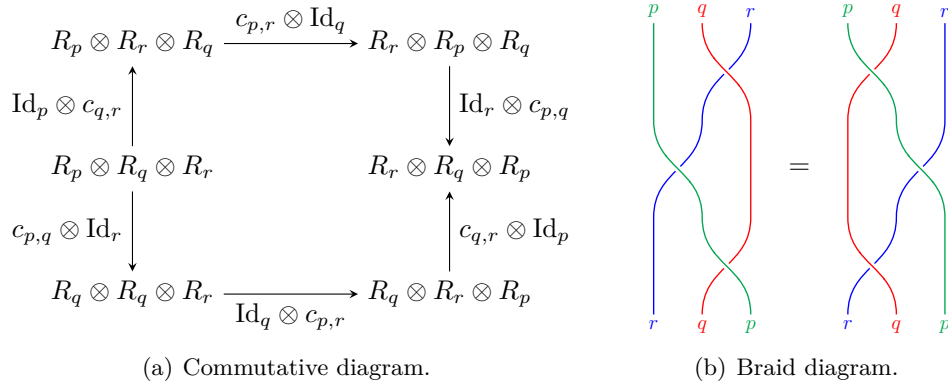


Figure 1.3: Diagrams associated with equation (1.13).

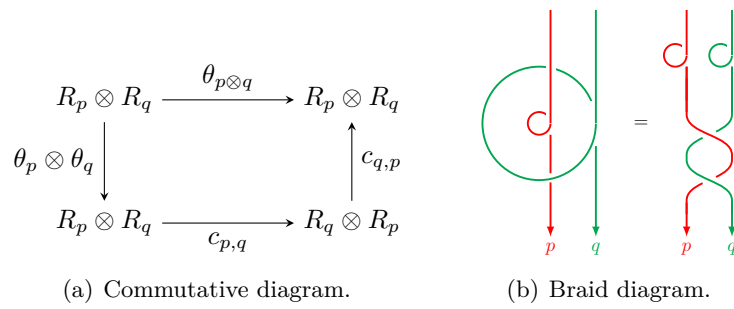


Figure 1.4: Diagrams associated with equation (1.14).

Chapter 1. Notions on categories

object R_p the dual $(R_p)^*$ of R_p is defined as:

$$(R_p)^* = R_{k-p}. \quad (1.15)$$

Further, $(R_p)^*$ will be denoted R_{p^*} or simply p^* if no confusion is possible. The duality makes it possible to materialize the idea of orientation of a branch on a braid diagram.

It should be noted that the twist verifies the so-called **compatibility** constraint expressed in that case as:

$$\theta_{p^*} = \theta_p, \quad (1.16)$$

where $\theta_{p^*} = e^{2i\pi \frac{(k-p)^2}{k}} : R_{p^*} \longrightarrow R_{p^*}$.

Duality, together with compatibility, turns $\mathbb{C}^{\mathbb{Z}_k}$ into a **ribbon category**. Interestingly, if $k = 2l$ then there are two self dual objects: R_0 and R_l .

In the specific example of $\mathbb{C}^{\mathbb{Z}_k}$ endowed with the structure of braided and twisted category provided together with a duality, given a morphism $f_p : R_p \longrightarrow R_p$, its **trace** is defined as:

$$\text{Tr}(f_p) = c_{p,p^*} \circ ((\theta_p \circ f_p) \otimes \text{Id}_p) = f_p, \quad (1.17)$$

and the **dimension** of an object R_p of $\mathbb{C}^{\mathbb{Z}_k}$ to be:

$$\dim(R_p) = \text{Tr}(\text{Id}_p) = 1. \quad (1.18)$$

With such a definition of the trace, $\mathbb{C}^{\mathbb{Z}_k}$ is a **spherical category** [Barrett and Westbury, 1996]. In a more general context, the difference between a left and a right trace should be made. However, a spherical category is the particular case where the left and right traces are the same.

1.5 The category $\mathbb{C}^{\mathbb{Z}_k}$ as a modular category?

Considering a period of objects of $\mathbb{C}^{\mathbb{Z}_k}$ with k odd, that is, $\{R_p, p \in \llbracket 0; k-1 \rrbracket\}$, then this family of objects verifies:

- $R_0 \in \{R_p, p \in \llbracket 0; k-1 \rrbracket\}$ (normalization axiom),
- if $R_q \in \{R_p, p \in \llbracket 0; k-1 \rrbracket\}$ then $(R_q)^* = R_{k-q} \in \{R_p, p \in \llbracket 0; k-1 \rrbracket\}$ (duality axiom),
- for any object R_q of $\mathbb{C}^{\mathbb{Z}_k}$, $\text{Id}_q = \delta_{q,q}^{[k]} = \delta_{q,q'}^{[k]} \delta_{q',q}^{[k]}$ with q' the representative of the class of q modulo k so that $R_{q'} \in \{R_p, p \in \llbracket 0; k-1 \rrbracket\}$ (axiom of domination),

1.5. The category $\mathbb{C}^{\mathbb{Z}_k}$ as a modular category?

- $(S_{p,q})_{0 \leq p, q \leq k-1} = (c_{p,q} \circ c_{q,p})_{0 \leq p, q \leq k-1} = \left(e^{4i\pi \frac{pq}{k}} \right)_{0 \leq p, q \leq k-1}$ is invertible⁴ (non-degeneracy axiom).

It is fundamental to note that the last property is verified if and only if k is odd. In that case and only in that case, $\mathbb{C}^{\mathbb{Z}_k}$ is a **modular category** [Turaev, 2010].

It shall be noted also that the domination axiom implies dealing with a finite set of representations. Hence, summing over that set does not cause any problem of convergence. This axiom is thus fundamental in the construction of the RT and TV invariants. The choice of considering $p \in \mathbb{Z}$ has been done to highlight this axiom which would have been rather trivial with $p \in \mathbb{Z}_k$.

If only the last property is not verified but the others are, that is, here, if k is even, then the category is said to be a **finite semisimple spherical** category [Barrett and Westbury, 1996] and it will be seen that this is sufficient to build the abelian TV invariant. The objects of the set $\{R_p, p \in \llbracket 0; k-1 \rrbracket\}$ are called **simple objects**.

For a finite semisimple spherical category, and in particular, for a modular category, the **global dimension** \mathcal{D}_k of $\mathbb{C}^{\mathbb{Z}_k}$ is defined such that:

$$\mathcal{D}_k^2 = \sum_{p=0}^{k-1} (\dim(R_p))^2 = k \tag{1.19}$$

In general, several choices are possible and the invariants that will be defined later on will depend on this choice. The convention adopted here consists in choosing the positive root:

$$\mathcal{D}_k = \sqrt{k} \tag{1.20}$$

In the quest of an abelian realization of a modular category $\mathbb{C}^{U(1)}$ could have been considered instead of $\mathbb{C}^{\mathbb{Z}_k}$ but it would have not been modular in any case, in particular because the domination axiom would have been violated. The modularity is a fundamental property that will be required when using an important theorem from Turaev.

The example of $\mathbb{C}^{\mathbb{Z}_k}$ had *a priori* never been derived following the construction of Turaev. This may be explained by its seeming triviality or the fact that no quantum deformation is possible in this case, whereas it was the key point of the examples using quantum groups. However, it appears here that some subtleties arise. Those subtleties are not trifling as it will be shown later on.

⁴This is actually nothing but a Vandermonde matrix:

$$\det(S_{p,q}) = \prod_{1 \leq p < q \leq k} \left(e^{\frac{4i\pi q}{k}} - e^{\frac{4i\pi p}{k}} \right)$$

1.6 Drinfeld center of $\mathbb{C}^{\mathbb{Z}_k}$

The **Drinfeld center** $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$ related to $\mathbb{C}^{\mathbb{Z}_k}$ consists in a category whose objects are couples (R_p, σ) where R_p is an object of $\mathbb{C}^{\mathbb{Z}_k}$ and σ is a collection of morphisms $\sigma_p : R_p \otimes R_q \longrightarrow R_q \otimes R_p$ such that:

$$\sigma_{q \otimes r} = \sigma_{q+r} = (\text{Id}_q \otimes \sigma_r) \circ (\sigma_q \otimes \text{Id}_r) = \sigma_q \sigma_r, \quad (1.21)$$

in this specific case of $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$ (the general construction can be found in [Turaev and Virelizier, 2013]).

Equation (1.21) implies that:

$$\sigma_q = \sigma_q^{(u)} = e^{2i\pi \frac{qu}{k}}, \quad (1.22)$$

with $u \in \mathbb{Z}_k$.

Let us underline that the set of objects of $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$ is thus isomorphic to $\mathbb{Z}_k \times \mathbb{Z}_k$ and that each $\sigma_q^{(u)}$ is a braiding for $\mathbb{C}^{\mathbb{Z}_k}$.

A morphism $f : (R_p, \sigma^{(u)}) \longrightarrow (R_q, \sigma^{(v)})$ of $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$ is a morphism $R_p \longrightarrow R_q$ such that:

$$(\text{Id}_r \otimes f) \circ \sigma_r^{(u)} = \sigma_r^{(v)} \circ (\text{Id}_r \otimes f), \quad (1.23)$$

which implies in this specific case of $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$ that:

$$f = \delta_{p,q}^{[k]} \delta_{u,v}^{[k]} z_{p,q} z_{u,v}. \quad (1.24)$$

The category $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$ endowed with the tensor product \otimes such that:

$$(R_p, \sigma^{(u)}) \otimes (R_q, \sigma^{(v)}) = (R_{p+q}, \sigma^{(u+v)}) \quad (1.25)$$

is a monoidal category with unit object $(R_0, \sigma^{(0)})$

Consider the morphism of $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$ given by:

$$c_{(p,u),(q,v)} = \sigma_p^{(v)} = e^{2i\pi \frac{pv}{k}}. \quad (1.26)$$

It verifies:

$$c_{(p,u),(q,v) \otimes (r,w)} = (\text{Id}_{(q,v)} \otimes c_{(p,u),(r,w)}) \circ (c_{(p,u),(q,v)} \otimes \text{Id}_{(r,w)}), \quad (1.27)$$

$$c_{(p,u) \otimes (q,v),(r,w)} = (c_{(p,u),(r,w)} \otimes \text{Id}_{(q,v)}) \circ (\text{Id}_{(p,u)} \otimes c_{(q,v),(r,w)}) \quad (1.28)$$

and:

$$\begin{aligned} & (\text{Id}_{(r,w)} \otimes c_{(p,u),q}) \circ (c_{(p,u),(r,w)} \otimes \text{Id}_{(q,v)}) \circ (\text{Id}_{(p,u)} \otimes c_{q,(r,w)}) \\ &= (c_{(q,v),(r,w)} \otimes \text{Id}_{(p,u)}) \circ (\text{Id}_{(q,v)} \otimes c_{(p,u),(r,w)}) \circ (c_{(p,u),q} \otimes \text{Id}_{(r,w)}). \end{aligned} \quad (1.29)$$

It is thus a braiding in $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$. It should be pointed out that:

$$c_{(p,u),(q,v)} \neq c_{(q,v),(p,u)}. \quad (1.30)$$

Consider also the morphism of $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$ given by:

$$\theta_{(p,u)} = \sigma_p^{(u)} = e^{2i\pi \frac{pu}{k}}. \quad (1.31)$$

It verifies:

$$\theta_{(p,u) \otimes (q,v)} = c_{(q,v),(p,u)} \circ c_{(p,u),(q,v)} \circ (\theta_{(p,u)} \otimes \theta_{(q,v)}). \quad (1.32)$$

It is therefore a twist in $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$

The Drinfeld center $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$ can also be endowed with a duality:

$$(R_p, \sigma^{(u)})^* = (R_{k-p}, \sigma^{(k-u)}), \quad (1.33)$$

verifying the (abelian) compatibility constraint:

$$\theta_{(p,u)}^* = \theta_{(p,u)}. \quad (1.34)$$

Hence, the category $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$ is a ribbon category.

Consider now the $k^2 \times k^2$ matrix S defined as:

$$S = (c_{(p,u),(q,v)} \circ c_{(q,v),(p,u)})_{0 \leq p,q,u,v \leq k-1} = (e^{2i\pi \frac{pv+qu}{k}})_{0 \leq p,q,u,v \leq k-1}, \quad (1.35)$$

then:

$$\det(S) = \pm \det(A \otimes A), \quad (1.36)$$

with:

$$A = (e^{2i\pi \frac{pq}{k}})_{0 \leq p,q \leq k-1}. \quad (1.37)$$

Finally:

$$\det(S) = \pm \left(\prod_{0 \leq u < v \leq k-1}^{k-1} e^{2i\pi \frac{v}{k}} - e^{2i\pi \frac{u}{k}} \right)^{2k} \neq 0, \quad (1.38)$$

which means that $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$ is modular for any k .

Also, for any object $(R_p, \sigma^{(u)})$:

$$\dim((R_p, \sigma^{(u)})) = 1, \quad (1.39)$$

and thus the global dimension $\mathcal{D}_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})}$ of $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$ is such that:

$$\mathcal{D}_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})}^2 = \sum_{p,u=0}^{k-1} (\dim((R_p, \sigma^{(u)})))^2 = k^2. \quad (1.40)$$

Chapter 1. Notions on categories

Therefore, according to the convention fixed previously:

$$\mathcal{D}_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})} = k = \mathcal{D}_k^2. \quad (1.41)$$

As a consequence, this construction makes it possible to extract a modular category from a finite semisimple spherical one. This fact will be exploited later on. Although the general construction and properties of the Drinfeld center of a category have been known for a few years, *a priori* no abelian realization had been exhibited.

Chapter 2

Abelian Reshetikhin-Turaev invariant

Contents

Introduction (English version)	20
Introduction (version française)	21
2.1 Generalities	22
2.2 Extension of abelian RT construction	25
2.3 Abelian RT invariant associated with $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$	27
2.4 Abelian expectation values of observables	28
2.5 Examples	30

Introduction (English version)

Any manifold $M^{(3)}$ of dimension 3 can be represented by a framed link \mathcal{L} in S^3 thanks to a so-called surgery procedure. This procedure will be developed in the main body of this chapter. The Reshetikhin-Turaev (RT) invariant [Reshetikhin and Turaev, 1990] relies on such a link. The graphical information carried by a diagram of \mathcal{L} is encoded by labeling the branches with objects of a modular category. The specific elements of the diagram like the braiding of branches will be encoded by morphisms acting on the objects associated with the concerned branches. The RT invariant is obtained by summing over the set of objects (which implies that this set has to be chosen so that the sum will converge). For this algebraic quantity to be a true invariant of $M^{(3)}$, it has to keep the same value if some manipulations on the diagram that produce a manifold homeomorphic to $M^{(3)}$ are performed. Those manipulations are called **Kirby moves**. As shown mostly by Reshetikhin and Turaev, the correct framework is the one provided by modular categories.

In this chapter, an abelian realization of the RT invariant founded on $\mathbb{C}^{\mathbb{Z}_k}$ is derived. As shown previously, the modularity imposes $k = 2l + 1$ but it will be shown that this hypothesis can be weakened and that $k = 4l$ can also lead to a RT-like invariant. This case has actually been presented by Murakami, Ohtsuki and Okada [Murakami et al., 1992]. An abelian realization of the RT invariant founded on $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$ will be derived for any k since $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$ is always modular. Finally, the expectation value of observables related to the abelian RT theory founded on $\mathbb{C}^{\mathbb{Z}_{4l}}$ and $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$ will be defined. Surprisingly, the modular case $\mathbb{C}^{\mathbb{Z}_{2l+1}}$ will not appear later.

Introduction (version française)

Toute variété $M^{(3)}$ de dimension 3 peut être représentée par un entrelac encadré \mathcal{L} dans S^3 grâce à une procédure dite de chirurgie. Cette procédure sera détaillée dans ce chapitre. L'invariant de Reshetikhin-Turaev (RT) [Reshetikhin and Turaev, 1990] se construit à partir d'un tel entrelac. L'information graphique contenue dans un diagramme de \mathcal{L} est encodée en associant aux branches du diagramme des objets d'une catégorie modulaire. Les éléments particuliers du diagramme tels que les croisements de branches seront encodés par des morphismes agissant sur les objets associés aux branches concernées. L'invariant RT est obtenu en sommant sur l'ensemble des objets (ce qui implique que cet ensemble doit être choisi de sorte à ce que la somme converge). Afin que cette quantité algébrique soit effectivement un invariant de $M^{(3)}$, elle doit conserver la même valeur si des manipulations du diagramme qui produisent une variété homéomorphe sont effectuées. Ces manipulations sont appelées **mouvements de Kirby**. Comme démontré principalement par Reshetikhin et Turaev, le cadre correct est celui fourni par les catégories modulaires.

Dans ce chapitre, une réalisation abélienne de l'invariant RT fondée sur $\mathbb{C}^{\mathbb{Z}_k}$ est présentée. Comme montré précédemment, la modularité impose que $k = 2l + 1$ mais il sera montré que cette hypothèse peut être affaiblie et que $k = 4l$ peut aussi mener à un invariant de type RT. Ce cas a en fait été présenté par Murakami, Ohtsuki et Okada [Murakami et al., 1992]. Une réalisation abélienne de l'invariant RT fondée sur $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$ sera présentée pour k quelconque puisque $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$ est toujours modulaire. Enfin, les valeurs moyennes d'observables liées aux théories RT abéliennes fondées sur $\mathbb{C}^{\mathbb{Z}_{4l}}$ et $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$ seront définies. De façon surprenante, le cas modulaire $\mathbb{C}^{\mathbb{Z}_{2l+1}}$ n'interviendra pas dans la suite.

2.1 Generalities

Surgery. Consider a knot \mathcal{K} in a manifold $X^{(3)}$ of dimension 3. Let $N(\mathcal{K})$ be a tubular neighbourhood of \mathcal{K} . Then $N(\mathcal{K})$ can be identified with the solid torus $D^2 \times S^1$ so that $\partial(X^{(3)} \setminus N(\mathcal{K})) = \partial N(\mathcal{K}) = \partial(D^2 \times S^1) = (\partial D^2) \times S^1 = T^2$. Taking away $N(\mathcal{K})$ from $X^{(3)}$ then gluing it back thanks to a homeomorphism $h : T_s^2 \rightarrow T_t^2$ generates a manifold $M^{(3)} = N(\mathcal{K}) \cup_h (X^{(3)} \setminus N(\mathcal{K}))$. The map h is completely determined by the image of the longitude l_s of the source torus T_s^2 (boundary of $N(\mathcal{K})$). This image can be decomposed on the target torus T_t^2 (boundary of $X^{(3)} \setminus N(\mathcal{K})$) along its longitude l_t and its meridian m_t . Necessarily:

$$h(l_s) = p \cdot m_t + q \cdot l_t, \quad (2.1)$$

with (p, q) a pair of integers that can be chosen coprime. Hence, the fraction p/q called the **framing** of \mathcal{K} completely determines h ¹. The manifold $M^{(3)}$ is said to be generated from $X^{(3)}$ by **rational surgery** along \mathcal{K} . A rational surgery is said to be an **integral surgery** when $q = 1$, the relevant information being therefore carried by p . Rational (respectively integral) surgery along a link \mathcal{L} is defined to be the rational (respectively integral) surgery along each link component. Then the following theorem holds:

Theorem ([Lickorish, 1962, Wallace, 1960]). Every closed orientable manifold $M^{(3)}$ of dimension 3 can be obtained from S^3 by an integral surgery on a link $\mathcal{L} \subset S^3$.

A more recent presentation of this result can be found in [Saveliev, 2012].

Definition of abelian RT invariant. Let $M^{(3)}$ be a manifold of dimension 3 generated from S^3 by integral surgery along a link $\mathcal{L} = \mathcal{L}_1 \cup \dots \cup \mathcal{L}_m$ of m components \mathcal{L}_i carrying a charge $p^i \in \mathbb{Z}_k$ so that $\mathbf{p} = (p^1 \dots p^m) \in \mathbb{Z}_k^m$. The linking matrix $\mathbf{L}_{\mathcal{L}} = (L_{ij})_{1 \leq i, j \leq m} = (\ellk_{S^3}(\mathcal{L}_i, \mathcal{L}_j))_{1 \leq i, j \leq m}$ has a signature $\sigma(\mathbf{L}_{\mathcal{L}})$, that is $\sigma(\mathbf{L}_{\mathcal{L}}) = m - n$ where m is the number of strictly positive eigenvalues and n the number of strictly negative eigenvalues.

The abelian RT invariant of $M^{(3)}$ is defined from a modular category. In the abelian context, that means that $k = 2l + 1$ as shown previously.

Hence, for $k = 2l + 1$ the abelian RT invariant of $M^{(3)}$ is given by:

$$Z_{\text{RT}_k}(M^{(3)}) = \Delta_k^{\sigma(\mathbf{L}_{\mathcal{L}})} \mathcal{D}_k^{-\sigma(\mathbf{L}_{\mathcal{L}}) - m} \sum_{\mathbf{p} \in \mathbb{Z}_k^m} \mathcal{F}_{\text{RT}_k}(\mathcal{L}, \mathbf{p}), \quad (2.2)$$

Δ_k being a normalization factor and $\mathcal{F}_{\text{RT}_k}$ representing the transformation of the graphical information into algebraic information of \mathcal{L} .

¹It should be pointed out that the pair $(1, 0)$ is allowed so that a framing written as $1/0$ is formally allowed.

More precisely:

$$\Delta_k = \sum_{p \in \mathbb{Z}_k} (\theta_p)^{-1} (\dim(R_p))^2 = \sum_{p \in \mathbb{Z}_k} e^{-2i\pi \frac{p^2}{k}}. \quad (2.3)$$

The value of this sum as a function of k is known to be [Loo Keng, 1983]:

$$\begin{cases} \Delta_k = k \text{ for } k = 1, \\ \Delta_k = \sqrt{k} - i\sqrt{k} \text{ for } k \equiv 0 [4], \\ \Delta_k = \sqrt{k} \text{ for } k \equiv 1 [4], \\ \Delta_k = 0 \text{ for } k \equiv 2 [4], \\ \Delta_k = -i\sqrt{k} \text{ for } k \equiv 3 [4]. \end{cases} \quad (2.4)$$

So for $k = 2l + 1$:

$$\begin{cases} \Delta_k = k \text{ for } k = 1, \\ \Delta_k = \sqrt{k} \text{ for } k \equiv 1 [4], \\ \Delta_k = -i\sqrt{k} \text{ for } k \equiv 3 [4]. \end{cases} \quad (2.5)$$

Such a derivation had *a priori* never been studied under this shape before [Mathieu and Thuillier, 2016a]. Clearly, the definition of the abelian RT invariant cannot be extended to the case where k is purely even (that is, when $k = 2(2l + 1)$). Indeed, if the signature $\sigma(\mathbf{L}_{\mathcal{L}})$ of $\mathbf{L}_{\mathcal{L}}$ is negative then $Z_{\text{RT}_k}(M^{(3)})$ is ill-defined. But, given an oriented manifold $M^{(3)}$ with linking matrix $\mathbf{L}_{\mathcal{L}}$, the manifold with reversed orientation $-M^{(3)}$ has a linking matrix $-\mathbf{L}_{\mathcal{L}}$. Hence, in the case $k = 2(2l + 1)$, $Z_{\text{RT}_k}(M^{(3)})$ is ill-defined or 0. Therefore, the hypothesis of modularity of $\mathbb{C}^{\mathbb{Z}_k}$ is sufficient to avoid that case. However, it is not necessary. Indeed, as treated further, the definition can be extended to the case where $k = 4l$. However, even if the construction of an abelian RT invariant is possible for $k = 4l$ some results that assume the modularity of the category do not extend to the case where $k = 4l$. Conversely, some results established for $k = 4l$ are not extensions of the modular case. In particular, it will be shown that the abelian RT theory for $k = 4l$ is related to the abelian CS theory with a coupling constant l whereas no abelian CS theory will be associated with the abelian RT theory for $k = 2l + 1$.

The operation $\mathcal{F}_{\text{RT}_k}$ is the transformation of the graphical information into algebraic information of \mathcal{L} reading, by convention, for example, the link diagram of the surgery link from the top to the bottom. The set of link diagrams can actually be regarded as objects of a category, and the various classes of movements of the strands of a diagram as morphisms of this category. Hence, the operation $\mathcal{F}_{\text{RT}_k}$ can be proved to be a covariant functor from this category of diagrams into the abelian realization of a modular category defined previously.

Chapter 2. Abelian Reshetikhin-Turaev invariant

Thanks to the properties of $\mathbb{C}^{\mathbb{Z}_k}$ imposed in the previous sections, it is possible to encode the graphical information as:

$$\begin{aligned}
\mathcal{F}_{\text{RT}_k}(X_{p,q}^+, (p, q)) &= c_{p,q}, & \mathcal{F}_{\text{RT}_k}(\varphi_p^+, p) &= \theta_p, \\
\mathcal{F}_{\text{RT}_k}(X_{p,q}^-, (p, q)) &= (c_{p,q})^{-1}, & \mathcal{F}_{\text{RT}_k}(\varphi_p^-, p) &= (\theta_p)^{-1}, \\
\mathcal{F}_{\text{RT}_k}(Y_{p,q}^+, (p, q)) &= (c_{q,k-p})^{-1}, & \mathcal{F}_{\text{RT}_k}(\cap_p^+, p) &= 1, \\
\mathcal{F}_{\text{RT}_k}(Y_{p,q}^-, (p, q)) &= c_{k-p,q}, & \mathcal{F}_{\text{RT}_k}(\cap_p^-, p) &= 1, \\
\mathcal{F}_{\text{RT}_k}(Z_{p,q}^+, (p, q)) &= (c_{k-q,p})^{-1}, & \mathcal{F}_{\text{RT}_k}(\cup_p^+, p) &= 1, \\
\mathcal{F}_{\text{RT}_k}(Z_{p,q}^-, (p, q)) &= c_{p,k-q}, & \mathcal{F}_{\text{RT}_k}(\cup_p^-, p) &= 1, \\
\mathcal{F}_{\text{RT}_k}(T_{p,q}^+, (p, q)) &= c_{k-p,k-q}, & \mathcal{F}_{\text{RT}_k}(\downarrow_p, p) &= 1, \\
\mathcal{F}_{\text{RT}_k}(T_{p,q}^-, (p, q)) &= (c_{k-q,k-p})^{-1}, & \mathcal{F}_{\text{RT}_k}(\uparrow_p, p) &= 1.
\end{aligned} \tag{2.6}$$

following the notations of Turaev [Turaev, 2010] reminded in Figure 2.1. The functor $\mathcal{F}_{\text{RT}_k}$ is covariant and built such that it preserves the tensor product, that is:

$$\mathcal{F}_{\text{RT}_k}(A \circ B) = \mathcal{F}_{\text{RT}_k}(A) \circ \mathcal{F}_{\text{RT}_k}(B) \tag{2.7}$$

and:

$$\mathcal{F}_{\text{RT}_k}(A \otimes B) = \mathcal{F}_{\text{RT}_k}(A) \otimes \mathcal{F}_{\text{RT}_k}(B). \tag{2.8}$$

Hence, the diagram of the surgery link can be cut into horizontal slices and its image by $\mathcal{F}_{\text{RT}_k}$ is the composition of the image of each slice.

Thus, it can be shown that:

$$\mathcal{F}_{\text{RT}_k}(\mathcal{L}, \mathbf{p}) = \prod_{1 \leq i < j \leq m} (c_{p^j, p^i} c_{p^i, p^j})^{L_{ij}} \prod_{i=1}^m (c_{p^i, p^i})^{L_{ii}}, \tag{2.9}$$

which can be actually rewritten as:

$$\mathcal{F}_{\text{RT}_k}(\mathcal{L}, \mathbf{p}) = \prod_{1 \leq i < j \leq m} (S_{p^i, p^j})^{L_{ij}} \prod_{i=1}^m (\theta_{p^i})^{L_{ii}} = (c_{1,1})_{i,j=1}^m \sum p^i L_{ij} p^j \tag{2.10}$$

so finally:

$$\mathcal{F}_{\text{RT}_k}(\mathcal{L}, \mathbf{p}) = e^{\frac{2i\pi}{k}(\mathbf{p}, \mathbf{L}\mathcal{L}\mathbf{p})}, \tag{2.11}$$

and:

$$Z_{\text{RT}_k}(M^{(3)}) = \Delta_k^{\sigma(\mathbf{L}\mathcal{L})} \sqrt{k}^{-\sigma(\mathbf{L}\mathcal{L})-m} \sum_{\mathbf{p} \in \mathbb{Z}_k^m} e^{\frac{2i\pi}{k}(\mathbf{p}, \mathbf{L}\mathcal{L}\mathbf{p})}. \tag{2.12}$$

2.2. Extension of abelian RT construction

It should be noted that contrary to the convention often taken in the mathematics community², the abelian RT invariant is normalized by S^3 (as it will be shown in the examples). With this normalization, it will be shown later on that:

$$\left| Z_{\text{RT}_{2l+1}}(M^{(3)}) \right|^2 = \left| H^1(M^{(3)}, \mathbb{Z}_{2l+1}) \right|. \quad (2.13)$$

From this last formula, it appears that the topological invariant obtained, that is, mainly $\left| H^1(M^{(3)}, \mathbb{Z}_{2l+1}) \right|$ (up to a phase), is not very accurate but not trivial at all. The determination of the phase is not obvious and can contain more than the information of the homology group, as shown in the examples given further.

Considering $Z_{\text{RT}_k}(M^{(3)})$ for several values of k should make it possible to get a better knowledge of the topology of $M^{(3)}$ than considering only one value of k . However, the accuracy of this knowledge is limited. Indeed, for example, the abelian RT theory cannot make the difference between S^3 and the Poincaré homology sphere, whatever k . It is also noteworthy that having the order of the homology group does not provide the group structure itself.

2.2 Extension of abelian RT construction

As explained in the previous section, the modularity of $\mathbb{C}^{\mathbb{Z}_k}$ is a sufficient condition for the existence of $Z_{\text{RT}_k}(M^{(3)})$. Indeed, in the non-modular case, some ill-defined cases would occur because $\Delta_k = 0$. However, this would happen only in the purely even case, that is, in the case where $k = 2(2l + 1)$. In the case where $k = 4l$ the definition (2.12) in itself makes sense and can be adapted in order to produce a true invariant of $M^{(3)}$, taking into account in particular the periodicity of the exponentials which is $2l$ in that case and not $4l$. Indeed:

$$((\mathbf{p} + 2l\mathbf{m}), \mathbf{L}_{\mathcal{L}}(\mathbf{p} + 2l\mathbf{m})) \equiv (\mathbf{p}, \mathbf{L}_{\mathcal{L}}\mathbf{p}) [4l]. \quad (2.14)$$

Hence:

$$\sum_{\mathbf{p} \in \mathbb{Z}_{4l}^m} \mathcal{F}_{\text{RT}_{4l}}(\mathcal{L}, \mathbf{p}) = 2^m \sum_{\mathbf{p} \in \mathbb{Z}_{2l}^m} \mathcal{F}_{\text{RT}_{4l}}(\mathcal{L}, \mathbf{p}). \quad (2.15)$$

The new definition is then:

$$\tilde{Z}_{\text{RT}_{4l}}(M^{(3)}) = \tilde{\Delta}_{4l}^{\sigma(\mathbf{L}_{\mathcal{L}})} \tilde{\mathcal{D}}_{4l}^{-\sigma(\mathbf{L}_{\mathcal{L}})-m} \sum_{\mathbf{p} \in \mathbb{Z}_{2l}^m} e^{\frac{2i\pi}{4l}(\mathbf{p}, \mathbf{L}_{\mathcal{L}}\mathbf{p})}, \quad (2.16)$$

with:

$$\tilde{\Delta}_{4l} = \sum_{p \in \mathbb{Z}_{2l}} e^{-2i\pi \frac{p^2}{4l}} = \frac{1}{2} \Delta_{4l} \quad (2.17)$$

²In the mathematics literature, in particular in [Turaev, 2010], the normalization is most of the time taken to be $S^1 \times S^2$. This detail does not change anything in this study.

Chapter 2. Abelian Reshetikhin-Turaev invariant

and:

$$\tilde{\mathcal{D}}_{4l} = \sqrt{2l} = \frac{\mathcal{D}_{4l}}{\sqrt{2}}. \quad (2.18)$$

Therefore:

$$\tilde{Z}_{\text{RT}_{4l}}(M^{(3)}) = \left(\frac{\tilde{\Delta}_{4l}}{|\tilde{\Delta}_{4l}|} \right)^{\sigma(\mathbf{L}_{\mathcal{L}})} |\tilde{\Delta}_{4l}|^{-m} \sum_{\mathbf{p} \in \mathbb{Z}_{2l}^m} e^{\frac{2i\pi}{4l}(\mathbf{p}, \mathbf{L}_{\mathcal{L}} \mathbf{p})}. \quad (2.19)$$

This quantity has been shown by Murakami, Ohtsuki and Okada to be an invariant of $M^{(3)}$. Its original construction is related to quasi-triangular Hopf algebras and \mathbb{Z}_{2l} -fusion rules [Murakami et al., 1992].

Furthermore, taking into account that:

$$\tilde{\Delta}_{4l} = \frac{1}{2} \Delta_{4l} = (\sqrt{l} - i\sqrt{l}), \quad (2.20)$$

then:

$$\frac{\tilde{\Delta}_{4l}}{|\tilde{\Delta}_{4l}|} = e^{-i\frac{\pi}{4}} \quad (2.21)$$

and $\tilde{Z}_{\text{RT}_{4l}}(M^{(3)})$ can be rewritten as:

$$\tilde{Z}_{\text{RT}_{4l}}(M^{(3)}) = \sqrt{2l}^{-m} e^{-i\frac{\pi}{4}\sigma(\mathbf{L}_{\mathcal{L}})} \sum_{\mathbf{p} \in \mathbb{Z}_{2l}^m} e^{\frac{2i\pi}{4l}(\mathbf{p}, \mathbf{L}_{\mathcal{L}} \mathbf{p})}, \quad (2.22)$$

which is another formulation widespread in the mathematics literature³. This new definition of $\tilde{Z}_{\text{RT}_{4l}}(M^{(3)})$ differs from the definition (2.12) taken with $k = 4l$ by a factor $\sqrt{2}^{m+\sigma(\mathbf{L}_{\mathcal{L}})}$ that might be irrelevant, but the choice to take the version of Murakami will appear particularly adapted further to use the reciprocity formula due to F. Deloup and V. Turaev [Deloup and Turaev, 2007] and to establish a relation with abelian CS theory.

Also, it can be noted that the modulus square of $\tilde{Z}_{\text{RT}_{4l}}(M^{(3)})$ can be expressed as [Murakami et al., 1992, Mathieu and Thuillier, 2016a]:

$$\left| \tilde{Z}_{\text{RT}_{4l}}(M^{(3)}) \right|^2 = \begin{cases} \left| H^1(M^{(3)}, \mathbb{Z}_{2l}) \right| & \text{if } \forall \alpha \in H^1(M^{(3)}, \mathbb{Z}_{2l}), \alpha \cup \alpha \cup \alpha = 0 \\ 0 & \text{otherwise.} \end{cases} \quad (2.23)$$

The piece of information contained in this extension is then similar to the piece of information contained in the modular case. However, it appears that there exists a degeneracy.

³The complex conjugation can also be found in the mathematics literature but would correspond to the same manifold with a reversed convention of orientation.

2.3. Abelian RT invariant associated with $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$

2.3 Abelian RT invariant associated with $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$

In section 1.6 the Drinfeld center $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$ has been shown to be a modular category for any k^4 . Hence, the abelian RT invariant $Z_{\text{RT}_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})}}(M^{(3)})$ associated with $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$ is:

$$\begin{aligned} Z_{\text{RT}_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})}}(M^{(3)}) &= \Delta_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})}^{\sigma(\mathbf{L}_{\mathcal{L}})} \mathcal{D}_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})}^{-\sigma(\mathbf{L}_{\mathcal{L}})-m} \sum_{(\mathbf{p}, \mathbf{u}) \in \mathbb{Z}_k^m \times \mathbb{Z}_k^m} \mathcal{F}_{\text{RT}_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})}}(\mathcal{L}, (\mathbf{p}, \mathbf{u})). \end{aligned} \quad (2.24)$$

Moreover, on the one hand:

$$\mathcal{D}_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})} = k \quad (2.25)$$

and on the other hand:

$$\Delta_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})} = \sum_{(p, u) \in \mathbb{Z}_k \times \mathbb{Z}_k} e^{2i\pi \frac{pu}{k}} = k = \mathcal{D}_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})}. \quad (2.26)$$

The operation $\mathcal{F}_{\text{RT}_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})}}(\mathcal{L}, (\mathbf{p}, \mathbf{u}))$ is given here by:

$$\mathcal{F}_{\text{RT}_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})}}(\mathcal{L}, (\mathbf{p}, \mathbf{u})) = e^{\frac{2i\pi}{k}(\mathbf{p}, \mathbf{L}_{\mathcal{L}} \mathbf{u})}, \quad (2.27)$$

so that finally:

$$Z_{\text{RT}_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})}}(M^{(3)}) = \frac{1}{k^m} \sum_{(\mathbf{p}, \mathbf{u}) \in \mathbb{Z}_k^m \times \mathbb{Z}_k^m} e^{\frac{2i\pi}{k}(\mathbf{p}, \mathbf{L}_{\mathcal{L}} \mathbf{u})}. \quad (2.28)$$

Otherwise:

$$Z_{\text{RT}_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})}}(M^{(3)}) = \sum_{\mathbf{u} \in \mathbb{Z}_k^m} \delta_{\mathbf{L}_{\mathcal{L}} \mathbf{u}, \mathbf{0}}^{[k]} = |\text{Ker } \mathbf{L}_{\mathcal{L}}^{[k]}|, \quad (2.29)$$

where $\mathbf{L}_{\mathcal{L}}^{[k]} : \mathbb{Z}_k^m \rightarrow \mathbb{Z}_k^m$ is the morphism canonically induced by $\mathbf{L}_{\mathcal{L}} : \mathbb{Z}^m \rightarrow \mathbb{Z}^m$. On the other hand, it is known that the sequence:

$$0 \rightarrow \text{Ker}(\mathbf{L}_{\mathcal{L}}) \rightarrow \mathbb{Z}^m \xrightarrow{\mathbf{L}_{\mathcal{L}}} \mathbb{Z}^m \rightarrow \text{Coker}(\mathbf{L}_{\mathcal{L}}) \rightarrow 0 \quad (2.30)$$

is exact, and hence, applying the functor $\text{Hom}(\cdot, \mathbb{Z}_k)$ the sequence:

$$0 \rightarrow \text{Ker}(\mathbf{L}_{\mathcal{L}}^{[k]}) \rightarrow (\mathbb{Z}_k)^m \xrightarrow{\mathbf{L}_{\mathcal{L}}^{[k]}} (\mathbb{Z}_k)^m \rightarrow \text{Coker}(\mathbf{L}_{\mathcal{L}}^{[k]}) \rightarrow 0 \quad (2.31)$$

⁴Of course it would be more correct to exclude the trivial case $k = 0$.

Chapter 2. Abelian Reshetikhin-Turaev invariant

is also exact since:

$$\text{Ker}(\mathbf{L}^{[k]}) \simeq \text{Hom}(\text{Coker}(\mathbf{L}^{[k]}), \mathbb{Z}_k). \quad (2.32)$$

Furthermore:

$$\text{Coker}(\mathbf{L}) \simeq H_1(M^{(3)}) \quad (2.33)$$

and:

$$H^1(M^{(3)}, \mathbb{Z}_k) \simeq \text{Hom}(H_1(M^{(3)}), \mathbb{Z}_k) \oplus \text{Ext}(H_0(M^{(3)}), \mathbb{Z}_k), \quad (2.34)$$

so:

$$H^1(M^{(3)}, \mathbb{Z}_k) = \text{Hom}(H_1(M^{(3)}), \mathbb{Z}_k). \quad (2.35)$$

As a conclusion:

$$Z_{\text{RT}_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k)}}}(M^{(3)}) = |H^1(M^{(3)}, \mathbb{Z}_k)|. \quad (2.36)$$

Here also, such a derivation had *a priori* never been exhibited before [Mathieu and Thuillier, 2017]. The invariant $Z_{\text{RT}_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k)}}}(M^{(3)})$ is defined for any k but, compared to the two previous cases studied, there is no phase carrying potentially more information. In this sense, $Z_{\text{RT}_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k)}}}(M^{(3)})$ is weaker than $Z_{\text{RT}_{2l+1}}(M^{(3)})$ and $\tilde{Z}_{\text{RT}_{4l}}(M^{(3)})$ when $\alpha \cup \alpha \cup \alpha = 0$ for any $\alpha \in H^1(M^{(3)}, \mathbb{Z}_{2l})$.

2.4 Abelian expectation values of observables

Consider a manifold $M^{(3)}$ obtained from S^3 by integral surgery along a link \mathcal{L} . Let γ be a framed loop with framing q in $M^{(3)}$ whose preimage in S^3 by the surgery procedure is $\tilde{\gamma}$. For the modular case $k = 2l + 1$ the quantity:

$$\langle \gamma \rangle_{\text{RT}_k, M^{(3)}} = \Delta_k^{\sigma(\mathbf{L}_{\mathcal{L}})} \sqrt{k}^{-\sigma(\mathbf{L}_{\mathcal{L}}) - m} \sum_{\mathbf{p} \in \mathbb{Z}_k^m} e^{\frac{2i\pi}{k}((\mathbf{p}, q), \mathbf{L}_{\mathcal{L} \cup \tilde{\gamma}}(\mathbf{p}, q))} \quad (2.37)$$

has been shown by Turaev to be a topological invariant of the pair $(M^{(3)}, \gamma)$ [Turaev, 2010]. This quantity will be called the **expectation value** of the observable γ according to the RT theory defined on $\mathbb{C}^{\mathbb{Z}_{2l+1}}$.

Such a result is *a priori* not relevant in the scope of this study since the abelian CS theory for a coupling $l \in \mathbb{Z}$ will be shown to be associated with the abelian RT theory for $k = 4l$ and the abelian BF theory for a coupling k to the Drinfeld center $\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})$ for any $k \in \mathbb{Z}$.

However, it has been shown also for $k = 4l$ that the following quantity is a topological invariant of the pair $(M^{(3)}, \gamma)$ [Deloup, 2001]:

$$\langle \gamma \rangle_{\text{RT}_{4l}, M^{(3)}} = \left(\frac{\tilde{\Delta}_{4l}}{|\tilde{\Delta}_{4l}|} \right)^{\sigma(\mathbf{L}_{\mathcal{L}})} |\tilde{\Delta}_{4l}|^{-m} \sum_{\mathbf{p} \in \mathbb{Z}_{2l}^m} e^{\frac{2i\pi}{4l}((\mathbf{p}, q), \mathbf{L}_{\mathcal{L} \cup \tilde{\gamma}}(\mathbf{p}, q))}. \quad (2.38)$$

2.4. Abelian expectation values of observables

that is:

$$\langle \gamma \rangle_{\text{RT}_{4l}, M^{(3)}} = e^{-i\frac{\pi}{4}\sigma(\mathbf{L}_{\mathcal{L}})} \sqrt{2l}^{-m} \sum_{\mathbf{p} \in \mathbb{Z}_{2l}^m} e^{\frac{2i\pi}{4l}((\mathbf{p}, q), \mathbf{L}_{\mathcal{L} \cup \tilde{\gamma}}(\mathbf{p}, q))}. \quad (2.39)$$

Hence, the previous terminology extends to the non-modular case $k = 4l$: this quantity will be called the **expectation value** of the observable γ according to the RT theory defined on $\mathbb{C}^{\mathbb{Z}_{4l}}$. The origin of this terminology will become clearer comparing the shape of (2.4) with the shape of expectation value of observables in the abelian CS theory presented in chapter 2 of the second part. Although, as mentioned before, the quantities (2.37) and (2.4) were known to be topological invariants of the pair $(M^{(3)}, \gamma)$, they had never been interpreted in terms of what physicists call expectation value of observables.

Another convenient definition of the expectation value that will be useful is:

$$\langle \langle \gamma \rangle \rangle_{\text{RT}_{4l}, M^{(3)}} = \frac{\langle \gamma \rangle_{\text{RT}_{4l}, M^{(3)}}}{\tilde{Z}_{\text{RT}_{4l}}(M^{(3)})}. \quad (2.40)$$

All those formulæ presented for a loop γ extend naturally to links.

In the case of the Drinfeld center, let γ_1 and γ_2 be a pair of loops with framing q and v respectively together with their respective preimage in S^3 by the surgery procedure $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$. Then:

$$\begin{aligned} \langle \gamma_1, \gamma_2 \rangle_{\text{RT}_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})}, M^{(3)}} &= \Delta_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})}^{\sigma(\mathbf{L}_{\mathcal{L}})} \mathcal{D}_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})}^{-\sigma(\mathbf{L}_{\mathcal{L}})-m} \\ &\quad \sum_{\mathbf{p} \in \mathbb{Z}_k^m} \sum_{\mathbf{u} \in \mathbb{Z}_k^m} e^{\frac{2i\pi}{k}((\mathbf{p}, q, 0), \mathbf{L}_{\mathcal{L} \cup \tilde{\gamma}_1 \cup \tilde{\gamma}_2}(\mathbf{u}, 0, v))} \\ &= \frac{1}{k^m} \sum_{\mathbf{p} \in \mathbb{Z}_k^m} \sum_{\mathbf{u} \in \mathbb{Z}_k^m} e^{\frac{2i\pi}{k}((\mathbf{p}, q, 0), \mathbf{L}_{\mathcal{L} \cup \tilde{\gamma}_1 \cup \tilde{\gamma}_2}(\mathbf{u}, 0, v))}, \end{aligned} \quad (2.41)$$

or:

$$\langle \gamma_1, \gamma_2 \rangle_{\text{RT}_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})}, M^{(3)}} = \frac{1}{k^m} \sum_{\mathbf{p} \in \mathbb{Z}_k^m} \sum_{\mathbf{u} \in \mathbb{Z}_k^m} e^{\frac{2i\pi}{k}((\mathbf{p}, q), \mathbf{L}_{\mathcal{C}_1, \mathcal{C}_2}(\mathbf{u}, v))}, \quad (2.42)$$

with $\mathbf{L}_{\mathcal{C}_1, \mathcal{C}_2} = \ell k_{S^3}(\mathcal{C}_{1i}, \mathcal{C}_{2j})$ and $\mathcal{C}_1 = \mathcal{L} \cup \tilde{\gamma}_1$ and $\mathcal{C}_2 = \mathcal{L} \cup \tilde{\gamma}_2$.

This formula presented for two loops γ_1 and γ_2 extends naturally to links.

2.5 Examples

Lens spaces. The so-called **lens spaces** $L(p, q)$ with (p, q) a pair of coprime integers provide here a convenient class of examples. Consider the sphere S^3 as a subset of \mathbb{C}^2 and the group action $(z_1, z_2) \longrightarrow \left(e^{\frac{2i\pi}{p}} \cdot z_1, e^{\frac{2i\pi q}{p}} \cdot z_2 \right)$. The lens space $L(p, q)$ is the quotient of S^3 by this group action.

It is known that $\pi_1(L(p, q)) = H_1(L(p, q)) = \mathbb{Z}_p$ and $\pi_i(L(p, q)) = \pi_i(S^3)$ for $i > 1$. Moreover, $L(p, q_1)$ and $L(p, q_2)$ are homeomorphic if and only if $q_2 = \pm q_1^{\pm 1}$ in \mathbb{Z}_p . This family of manifolds of dimension 3 is thereby interesting in the context of algebraic topology since it provides examples where the homotopy groups are not classifying.

The lens space $L(p, 1)$ can be obtained from S^3 by integral surgery along the unknot with a framing $-p$ [Saveliev, 2012] as represented in Figure 2.2.

With this representation, the linking matrix $\mathbf{L}_{\mathcal{L}}$ is simply:

$$\mathbf{L}_{\mathcal{L}} = -p \tag{2.43}$$

The specific case $p = 0$ and $q = 1$ corresponds to $L(0, 1) = S^1 \times S^2$ which is the only lens space to have no torsion part and a free part \mathbb{Z} in its homology. The signature is then $\sigma(\mathbf{L}_{\mathcal{L}}) = 0$ and:

$$Z_{\text{RT}_{2l+1}}(S^1 \times S^2) = \frac{1}{\sqrt{2l+1}} (2l+1) = \sqrt{2l+1} \tag{2.44}$$

and:

$$\tilde{Z}_{\text{RT}_{4l}}(S^1 \times S^2) = \frac{1}{\sqrt{2l}} (2l) = \sqrt{2l}. \tag{2.45}$$

If $p \geq 1$ then its signature is $\sigma(\mathbf{L}_{\mathcal{L}}) = -1$ and:

$$Z_{\text{RT}_{2l+1}}(L(p, 1)) = \frac{1}{\Delta_{2l+1}} \sum_{u \in \mathbb{Z}_{2l+1}} e^{-\frac{2i\pi}{2l+1} pu^2} \tag{2.46}$$

and:

$$\tilde{Z}_{\text{RT}_{4l}}(L(p, 1)) = \frac{e^{i\frac{\pi}{4}}}{\sqrt{2l}} \sum_{u \in \mathbb{Z}_{2l}} e^{-\frac{2i\pi}{4l} pu^2}. \tag{2.47}$$

A particular case arises when $p = 1$ which corresponds to $L(1, 1) = S^3$ by definition of a lens space and it happens that:

$$Z_{\text{RT}_{2l+1}}(L(1, 1)) = 1 \tag{2.48}$$

and:

$$\tilde{Z}_{\text{RT}_{4l}}(L(1, 1)) = 1. \tag{2.49}$$

The lens space $L(p = qr - 1, q)$ (q and r strictly positive) can be obtained from S^3 by integral surgery along the Hopf link with a framing $-r$ on the first component and a framing $-q$ on the second component [Saveliev, 2012] as represented in Figure 2.3. With this representation, the linking matrix $\mathbf{L}_{\mathcal{L}}$ is:

$$\mathbf{L}_{\mathcal{L}} = \begin{pmatrix} -r & 1 \\ 1 & -q \end{pmatrix} \quad (2.50)$$

then its signature is $\sigma(\mathbf{L}_{\mathcal{L}}) = -2$ and:

$$Z_{\text{RT}_{2l+1}}(L(p = qr - 1, q)) = \frac{1}{\Delta_{2l+1}^2} \sum_{u,v \in \mathbb{Z}_{2l+1}} e^{-\frac{2i\pi}{2l+1}(-qu^2 + 2uv - rv^2)} \quad (2.51)$$

and:

$$\tilde{Z}_{\text{RT}_{4l}}(L(p = qr - 1, q)) = \frac{i}{2l} \sum_{u,v \in \mathbb{Z}_{2l}} e^{-\frac{2i\pi}{4l}(-qu^2 + 2uv - rv^2)}. \quad (2.52)$$

A particular case arises when $r = 2$ and $q = 1$ which corresponds to $L(1, 1) = S^3$ by definition of a lens space and the same result as previously is found:

$$Z_{\text{RT}_{2l+1}}(L(1, 1)) = 1 \quad (2.53)$$

and:

$$\tilde{Z}_{\text{RT}_{4l}}(L(1, 1)) = 1. \quad (2.54)$$

More generally, $L(p, q)$ is obtained from S^3 by integral surgery along the link represented in Figure 2.4, the x_i are the element of the continued fraction decomposition of p/q :

$$\frac{p}{q} = x_1 - \frac{1}{x_2 - \frac{1}{\dots - \frac{1}{x_n}}}. \quad (2.55)$$

With this representation, the linking matrix $\mathbf{L}_{\mathcal{L}}$ is:

$$\mathbf{L}_{\mathcal{L}} = \begin{pmatrix} -x_1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & -x_2 & 1 & 0 & & & \vdots \\ 0 & 1 & -x_3 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & 0 & 1 & -x_{n-2} & 1 & 0 \\ \vdots & & & 0 & 1 & -x_{n-1} & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & -x_n \end{pmatrix} \quad (2.56)$$

Chapter 2. Abelian Reshetikhin-Turaev invariant

then its signature is $\sigma(\mathbf{L}_{\mathcal{L}}) = -n$ and:

$$Z_{\text{RT}_{2l+1}}(L(p, q)) = \frac{1}{\Delta_{2l+1}^n} \sum_{\mathbf{u} \in \mathbb{Z}_{2l+1}^n} e^{-\frac{2i\pi}{2l+1}(-x_1 u_1^2 - \dots - x_n u_n^2 + 2u_1 u_2 + \dots + 2u_{n-1} u_n)} \quad (2.57)$$

and:

$$\tilde{Z}_{\text{RT}_{4l}}(L(p, q)) = \frac{e^{i\frac{\pi}{4}n}}{\sqrt{2l}^n} \sum_{\mathbf{u} \in \mathbb{Z}_{2l}^n} e^{-\frac{2i\pi}{4l}(-x_1 u_1^2 - \dots - x_n u_n^2 + 2u_1 u_2 + \dots + 2u_{n-1} u_n)}. \quad (2.58)$$

For further use, it should be mentioned that, when $\mathbb{C}^{\mathbb{Z}_k}$ is modular, that is, when $k = 2l + 1$, then:

$$\left| Z_{\text{RT}_{2l+1}}(L(p, q)) \right|^2 = \gcd(2l + 1, p) \quad (2.59)$$

for any positive p (even $p = 0$ and $p = 1$). Taking into account equation (2.57), the phase of $Z_{\text{RT}_{2l+1}}(L(p, q))$ should depend on $k = 2l + 1$, p and q .

Poincaré homology sphere. Thanks in particular to the lens spaces, it is known that the fundamental group cannot classify the manifolds of dimension 3 up to homeomorphism. It is also known that the homology cannot classify them as well. For example, the Poincaré homology sphere is not homeomorphic to S^3 but has the same homology groups.

The Poincaré homology sphere is obtained from S^3 by integral surgery along the link \mathcal{L} represented in Figure 2.5. With this representation, the linking matrix $\mathbf{L}_{\mathcal{L}}$ is:

$$\mathbf{L}_{\mathcal{L}} = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix} \quad (2.60)$$

and its signature is $\sigma(\mathbf{L}_{\mathcal{L}}) = -8$. It can be shown that, as a consequence:

$$Z_{\text{RT}_{2l+1}}(M^{(3)}) = 1 \quad (2.61)$$

and:

$$\tilde{Z}_{\text{RT}_{4l}}(M^{(3)}) = 1. \quad (2.62)$$

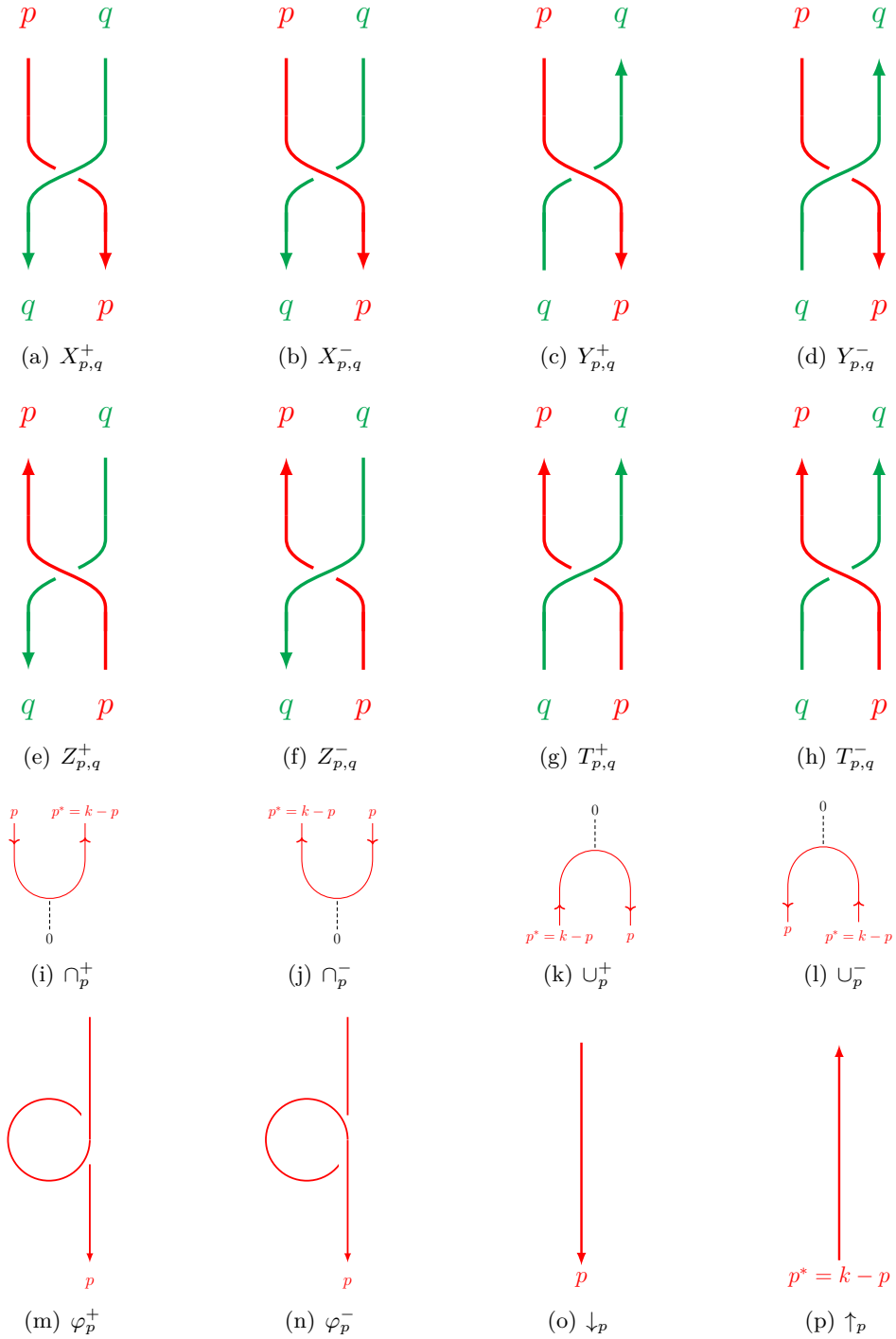


Figure 2.1: Notations associated with equations (2.6).

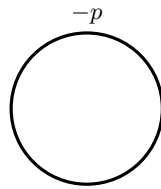


Figure 2.2: Surgery link of $L(p, 1)$.

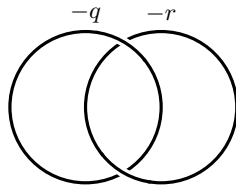


Figure 2.3: Surgery link of $L(p = qr - 1, q)$.

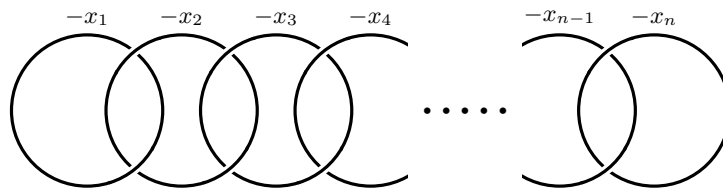


Figure 2.4: Surgery link of $L(p, q)$.

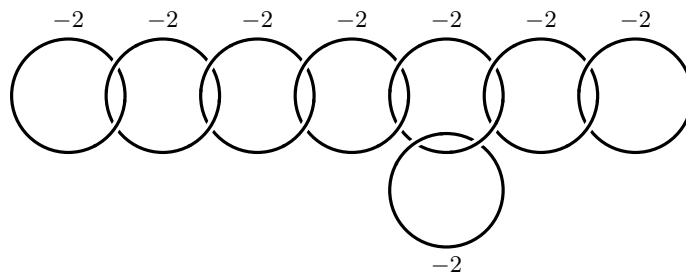


Figure 2.5: Surgery link of the Poincaré homology sphere.

Chapter 3

Abelian Turaev-Viro invariant

Contents

Introduction (English version)	36
Introduction (version française)	37
3.1 Generalities	38
3.2 Abelian expectation values of observables	43
3.3 Examples	44
3.4 Relations between abelian RT and TV theories	47

Introduction (English version)

The Turaev-Viro (TV) invariant originally relies on a triangulation of the manifold $M^{(3)}$ considered [Turaev and Viro, 1992]. However, the construction can be generalized to cellular decomposition of $M^{(3)}$ [Balsam and Kirillov, 2010]. The topological information is thus given by the set of the cells and how the cells are glued together. The edges will then be labeled by objects of a finite semisimple spherical category and morphisms will encode the gluing rule. The TV invariant is obtained by summing over the set of objects (which implies that this set has to be chosen so that the sum will converge). For this algebraic quantity to be a true invariant of $M^{(3)}$, it has to keep the same value if the cellular decomposition that represents $M^{(3)}$ is refined. As shown mostly by Turaev and Viro, then refined by Barrett and Westbury [Barrett and Westbury, 1996], the correct framework is the one provided by finite semisimple spherical categories which are less constrained than modular categories.

In this chapter, an abelian realization of the TV invariant founded on $\mathbb{C}^{\mathbb{Z}_k}$ is derived. Contrary to the RT case, there will be no obstruction on k for the invariant to be defined. This invariant will be related to the cardinal of the first homology group $H_1(M^{(3)})$ of the manifold $M^{(3)}$ considered. In the modular case, the theorem which claims that the modulus square of the RT invariant coincides with the TV invariant will be verified. Finally, observables related to the abelian TV theory and their expectation value will be introduced. They define an invariant of links in manifolds of dimension 3 that had *a priori* never been considered before [Mathieu and Thuillier, 2016b].

Introduction (version française)

L'invariant de Turaev-Viro (TV) se construit à l'origine à partir d'une triangulation de la variété $M^{(3)}$ considérée [Turaev and Viro, 1992]. Cependant, la construction peut être généralisée aux décompositions cellulaires de $M^{(3)}$ [Balsam and Kirillov, 2010]. L'information topologique est ainsi donnée par l'ensemble des cellules et la manière dont les cellules se recollent entre elles. Aux arêtes seront associés des objets d'une catégorie sphérique semisimple finie et des morphismes encoderont la règle de recollement. L'invariant TV est obtenu en sommant sur l'ensemble des objets (ce qui implique que cet ensemble doit être choisi de sorte à ce que la somme converge). Afin que cette quantité algébrique soit effectivement un invariant de $M^{(3)}$, elle doit conserver la même valeur si la décomposition cellulaire de $M^{(3)}$ est raffinée. Comme démontré principalement par Turaev et Viro, puis raffiné par Barrett et Westbury [Barrett and Westbury, 1996], le cadre correct est celui fourni par les catégories sphériques semisimples finies qui sont moins contraintes que les catégories modulaires.

Dans ce chapitre, une réalisation abélienne de l'invariant TV fondée sur $\mathbb{C}^{\mathbb{Z}_k}$ est présentée. Contrairement au cas RT, il n'y aura pas de contrainte sur k pour que l'invariant soit défini. Cet invariant sera relié au cardinal du premier groupe d'homologie $H_1(M^{(3)})$ de la variété $M^{(3)}$ considérée. Dans le cas modulaire, le théorème qui affirme que le module carré de l'invariant RT coïncide avec l'invariant TV sera vérifié. Enfin, des observables liées à la théorie TV abélienne et leurs valeurs moyennes seront introduites. Elles définissent des invariants d'entrelacs dans les variétés de dimension 3 qui n'ont *a priori* jamais été considérés auparavant [Mathieu and Thuillier, 2016b].

3.1 Generalities

Let $M^{(3)}$ be a manifold of dimension 3 provided with an oriented cellular decomposition $\mathcal{C} = (\mathcal{P}, \mathcal{F}, \mathcal{E}, \mathcal{V})$ where \mathcal{P} is the set of 3-cells (polyhedra), \mathcal{F} the set of 2-cells (faces), \mathcal{E} the set of 1-cells (edges) and \mathcal{V} the set of 0-cells (vertices). These sets are given by:

$$\begin{aligned} \mathcal{P} &= (P_\alpha)_{\alpha=1, \dots, P}, & \mathcal{F} &= (S_a)_{a=1, \dots, F}, \\ \mathcal{E} &= (e_i)_{i=1, \dots, E}, & \mathcal{V} &= (x_\mu)_{\mu=1, \dots, V}. \end{aligned} \quad (3.1)$$

As $M^{(3)}$ is closed, let $\mathcal{C}^* = (\mathcal{P}^*, \mathcal{F}^*, \mathcal{E}^*, \mathcal{V}^*)$ be a dual oriented decomposition of \mathcal{C} given by:

$$\begin{aligned} \mathcal{V}^* &= (x^\alpha)_{\alpha=1, \dots, P}, & \mathcal{E}^* &= (e^a)_{a=1, \dots, F}, \\ \mathcal{F}^* &= (S^i)_{i=1, \dots, E}, & \mathcal{P}^* &= (P^\mu)_{\mu=1, \dots, V}, \end{aligned} \quad (3.2)$$

in such a way that:

$$\begin{aligned} P_\alpha \odot x^\beta &= \delta_\alpha^\beta, & S_a \odot e^b &= \delta_a^b, \\ e_i \odot S^j &= \delta_i^j, & x_\mu \odot P^\nu &= \delta_\mu^\nu, \end{aligned} \quad (3.3)$$

with \odot denoting the intersection number in $M^{(3)}$. The decompositions \mathcal{C} and \mathcal{C}^* are naturally endowed with the structure of abelian graded groups.

Boundary operator. Let ∂ (respectively ∂^*) be boundary operator on \mathcal{C} (respectively \mathcal{C}^*) such that:

$$\left\{ \begin{array}{l} \partial P_\alpha = \partial_\alpha^a S_a, \\ \partial S_a = \partial_a^i e_i, \\ \partial e_i = \partial_i^\mu x_\mu, \\ \partial x_\mu = 0, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \partial^* P^\mu = \partial^{*\mu}_i S^i, \\ \partial^* S^i = \partial^{*i}_a e^a, \\ \partial^* e^a = \partial^{*a}_\alpha x^\alpha, \\ \partial^* x^\alpha = 0, \end{array} \right. \quad (3.4)$$

with:

$$\partial \circ \partial = 0 = \partial^* \circ \partial^*, \quad (3.5)$$

all matrix elements of ∂ and ∂^* being integers. By introducing the notations:

$$\left\{ \begin{array}{l} \partial_{(3)} = (\partial_\alpha^a)_{\substack{1 \leq a \leq F \\ 1 \leq \alpha \leq P}}, \\ \partial_{(2)} = (\partial_a^i)_{\substack{1 \leq i \leq E \\ 1 \leq a \leq F}}, \\ \partial_{(1)} = (\partial_i^\mu)_{\substack{1 \leq \mu \leq V \\ 1 \leq i \leq E}}, \\ \partial_{(0)} = \underbrace{(0, \dots, 0)}_V, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \partial_{(3)}^* = (\partial_{(1)})^\dagger, \\ \partial_{(2)}^* = (\partial_{(2)})^\dagger, \\ \partial_{(1)}^* = (\partial_{(3)})^\dagger, \\ \partial_{(0)}^* = \underbrace{(0, \dots, 0)}_P. \end{array} \right. \quad (3.6)$$

The boundary operator ∂ can be regarded as a matrix operator:

$$\partial = \left(\begin{array}{c|c|c|c} \partial_{(3)} & 0 & 0 & 0 \\ \hline 0 & \partial_{(2)} & 0 & 0 \\ \hline 0 & 0 & \partial_{(1)} & 0 \\ \hline 0 & 0 & 0 & \partial_{(0)} \end{array} \right). \quad (3.7)$$

The boundary operators ∂ and ∂^* turn \mathcal{C} and \mathcal{C}^* into differential groups thus yielding homology groups $H_\bullet(\mathcal{C})$ and $H_\bullet(\mathcal{C}^*)$. The decomposition \mathcal{C} will always be assumed to be **good**, which means that:

$$H_\bullet(\mathcal{C}) \simeq H_\bullet(M^{(3)}). \quad (3.8)$$

By construction the dual decomposition \mathcal{C}^* is good too.

Cochains and differentials. Relations (3.3) lead to the following correspondences:

$$\left\{ \begin{array}{l} P^\mu \rightarrow \hat{P}^\mu \in \text{Hom}(\mathcal{V}, \mathbb{Z}) \equiv C_{\mathcal{C}}^0 \quad | \quad \hat{P}^\mu(x_\nu) = \delta_\nu^\mu \\ S^i \rightarrow \hat{S}^i \in \text{Hom}(\mathcal{E}, \mathbb{Z}) \equiv C_{\mathcal{C}}^1 \quad | \quad \hat{S}^i(e_j) = \delta_j^i, \\ e^a \rightarrow \hat{e}^a \in \text{Hom}(\mathcal{F}, \mathbb{Z}) \equiv C_{\mathcal{C}}^2 \quad | \quad \hat{e}^a(S_b) = \delta_b^a, \\ x^\alpha \rightarrow \hat{x}^\alpha \in \text{Hom}(\mathcal{P}, \mathbb{Z}) \equiv C_{\mathcal{C}}^3 \quad | \quad \hat{x}^\alpha(P_\beta) = \delta_\beta^\alpha, \end{array} \right. \quad (3.9)$$

and once $(\mathcal{C}^*)^*$ has been canonically identified with \mathcal{C} the following additional correspondences can be done:

$$\left\{ \begin{array}{l} P_\alpha \rightarrow \hat{P}_\alpha \in \text{Hom}(\mathcal{V}^*, \mathbb{Z}) \equiv C_{\mathcal{C}^*}^0 \quad | \quad \hat{P}_\alpha(x^\beta) = \delta_\alpha^\beta, \\ S_a \rightarrow \hat{S}_a \in \text{Hom}(\mathcal{E}^*, \mathbb{Z}) \equiv C_{\mathcal{C}^*}^1 \quad | \quad \hat{S}_a(e^b) = \delta_a^b, \\ e_i \rightarrow \hat{e}_i \in \text{Hom}(\mathcal{F}^*, \mathbb{Z}) \equiv C_{\mathcal{C}^*}^2 \quad | \quad \hat{e}_i(S^j) = \delta_i^j, \\ x_\mu \rightarrow \hat{x}_\mu \in \text{Hom}(\mathcal{P}^*, \mathbb{Z}) \equiv C_{\mathcal{C}^*}^3 \quad | \quad \hat{x}_\mu(P^\nu) = \delta_\mu^\nu. \end{array} \right. \quad (3.10)$$

A cochain of \mathcal{C} and \mathcal{C}^* is then a linear combination of these fundamental cochains.

Let $C_{\mathcal{C}}^\bullet$ (respectively $C_{\mathcal{C}^*}^\bullet$) be the graded group of cochains of \mathcal{C} (respectively \mathcal{C}^*). Then $C_{\mathcal{C}}^\bullet$ (respectively $C_{\mathcal{C}^*}^\bullet$) is turned into a differential group by endowing it with the endomorphism $d : C_{\mathcal{C}}^\bullet \rightarrow C_{\mathcal{C}}^\bullet$ (respectively $d^* : C_{\mathcal{C}^*}^\bullet \rightarrow C_{\mathcal{C}^*}^\bullet$) defined by:

$$\begin{aligned} \forall \hat{u} \in C_{\mathcal{C}}^\bullet, \quad d \circ \hat{u} &= \hat{u} \circ \partial, \\ (\text{respectively } \forall \hat{v} \in C_{\mathcal{C}^*}^\bullet, \quad d^* \circ \hat{v} &= \hat{v} \circ \partial^*). \end{aligned} \quad (3.11)$$

This induces therefore the matrix relations:

$$\left\{ \begin{array}{l} d_{(0)} = (\partial_{(1)})^\dagger, \\ d_{(1)} = (\partial_{(2)})^\dagger, \\ d_{(2)} = (\partial_{(3)})^\dagger, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} (d_{(0)})^* = (d_{(2)})^\dagger = \partial_{(3)}, \\ (d_{(1)})^* = (d_{(1)})^\dagger = \partial_{(2)}, \\ (d_{(2)})^* = (d_{(0)})^\dagger = \partial_{(1)}. \end{array} \right. \quad (3.12)$$

Chapter 3. Abelian Turaev-Viro invariant

Since the decomposition \mathcal{C} is good the cohomology groups of $(C_{\mathcal{C}}^{\bullet}, d)$ and $(C_{\mathcal{C}^*}^{\bullet}, d^*)$ coincide with those of $M^{(3)}$.

Cap and cup. The symmetric non-degenerate pairings defined by:

$$\begin{cases} \langle \hat{P}_{\alpha}, \hat{x}^{\beta} \rangle \equiv \hat{P}_{\alpha}(x^{\beta}) = P_{\alpha} \odot x^{\beta} = \delta_{\alpha}^{\beta}, \\ \langle \hat{S}_a, \hat{e}^b \rangle \equiv \hat{S}_a(e^b) = S_a \odot e^b = \delta_a^b, \\ \langle \hat{e}_i, \hat{S}^j \rangle \equiv \hat{e}_i(S^j) = e_i \odot S^j = \delta_i^j, \\ \langle \hat{x}_{\mu}, \hat{P}^{\nu} \rangle \equiv \hat{x}_{\mu}(P^{\nu}) = x_{\mu} \odot P^{\nu} = \delta_{\mu}^{\nu} \end{cases} \quad (3.13)$$

yield the following cap products:

$$\begin{cases} M^{(3)} \cap \hat{P}_{\alpha} = P_{\alpha}, \\ M^{(3)} \cap \hat{S}_a = S_a, \\ M^{(3)} \cap \hat{e}_i = e_i, \\ M^{(3)} \cap \hat{x}_{\mu} = x_{\mu}, \end{cases} \quad \text{and} \quad \begin{cases} M^{(3)} \cap \hat{P}^{\mu} = P^{\mu}, \\ M^{(3)} \cap \hat{S}^i = S^i, \\ M^{(3)} \cap \hat{e}^a = e^a, \\ M^{(3)} \cap \hat{x}^{\alpha} = x^{\alpha}. \end{cases} \quad (3.14)$$

These relations are nothing but the Poincaré duality. For instance, for a 1-chain $c = c^i e_i$ then its Poincaré dual is just $\hat{c} = c^i \hat{e}_i \in C_{\mathcal{C}^*}^2$. It shall be noted that starting with a chain in \mathcal{C} implies ending with a cochain in \mathcal{C}^* .

The cup products associated with the previous cap products are:

$$\begin{cases} (\hat{P}_{\alpha} \cup \hat{x}^{\beta})(M^{(3)}) \equiv \hat{P}_{\alpha}(M^{(3)} \cap \hat{x}^{\beta}) = \hat{x}^{\beta}(M^{(3)} \cap \hat{P}_{\alpha}) = \hat{P}_{\alpha}(x^{\beta}) = \delta_{\alpha}^{\beta}, \\ (\hat{S}_a \cup \hat{e}^b)(M^{(3)}) \equiv \hat{S}_a(M^{(3)} \cap \hat{e}^b) = \hat{e}^b(M^{(3)} \cap \hat{S}_a) = \hat{S}_a(e^b) = \delta_a^b, \\ (\hat{e}_i \cup \hat{S}^j)(M^{(3)}) \equiv \hat{e}_i(M^{(3)} \cap \hat{S}^j) = \hat{S}^j(M^{(3)} \cap \hat{e}_i) = \hat{e}_i(S^j) = \delta_i^j, \\ (\hat{x}_{\mu} \cup \hat{P}^{\nu})(M^{(3)}) \equiv \hat{x}_{\mu}(M^{(3)} \cap \hat{P}^{\nu}) = \hat{P}^{\nu}(M^{(3)} \cap \hat{x}_{\mu}) = \hat{x}_{\mu}(P^{\nu}) = \delta_{\mu}^{\nu}. \end{cases} \quad (3.15)$$

Labelings and gaugings. The previous construction extends to \mathbb{Z}_k -valued cochains of \mathcal{C} and \mathcal{C}^* the differential groups of which are denoted $C_{\mathcal{C}}^{k,\bullet}$ and $C_{\mathcal{C}^*}^{k,\bullet}$. In the context of the TV theory elements of $C_{\mathcal{C}}^{k,1}$ (respectively $C_{\mathcal{C}^*}^{k,1}$) are called \mathbb{Z}_k **labelings** of \mathcal{C} (respectively \mathcal{C}^*) whereas elements of $C_{\mathcal{C}}^{k,0}$ (respectively $C_{\mathcal{C}^*}^{k,0}$) are called \mathbb{Z}_k **gaugings** of \mathcal{C} (respectively \mathcal{C}^*). By construction, the differential of a \mathbb{Z}_k gauging is a \mathbb{Z}_k labeling.

Consider $\hat{l} \in C_{\mathcal{C}}^{k,1}$ and $\hat{m} \in C_{\mathcal{C}^*}^{k,1}$ such that:

$$\hat{l} = l_i \hat{S}^i \quad \text{and} \quad \hat{m} = m^a \hat{S}_a, \quad (3.16)$$

with $l_i, m^a \in \mathbb{Z}_k$. The 2-cochains $d\hat{l} \in C_{\mathcal{C}}^{k,2}$ and $d^*\hat{m} \in C_{\mathcal{C}^*}^{k,2}$ are defined by:

$$d\hat{l} = l_i \partial_a^i \hat{e}^a = (d\hat{l})_a \hat{e}^a \quad \text{and} \quad d^*\hat{m} = m^a \partial_a^{*i} \hat{e}_i = (d^*\hat{m})^i \hat{e}_i. \quad (3.17)$$

It should be noted that $(d\hat{l})_a \in \mathbb{Z}_k$ since $\partial_a^i \in \mathbb{Z}$ and \mathbb{Z}_k is a \mathbb{Z} -module. Thanks to the ring structure of \mathbb{Z}_k the cup products can be extended to \mathbb{Z}_k -valued cochains. In particular:

$$(\hat{m} \cup d\hat{l})(M^{(3)}) = m^a (d\hat{l})_b (\hat{S}_a \cup \hat{e}^b)(M^{(3)}) = m^a (dl)_a. \quad (3.18)$$

As $C_C^{k,1} = \text{Hom}(\mathcal{E}, \mathbb{Z}_k) \simeq \mathbb{Z}_k^E$ and $C_{C^*}^{k,1} = \text{Hom}(\mathcal{E}^*, \mathbb{Z}_k) \simeq \mathbb{Z}_k^F$ there are canonical bijections:

$$\begin{aligned} \hat{l} = l_i \hat{S}^i \in C_C^{k,1} &\longrightarrow \mathbf{l} = (l_i)_{1 \leq i \leq E} \in \mathbb{Z}_k^E, \\ \hat{m} = m^a \hat{S}_a \in C_{C^*}^{k,1} &\longrightarrow \mathbf{m} = (m^a)_{1 \leq a \leq F} \in \mathbb{Z}_k^F, \end{aligned} \quad (3.19)$$

leading to:

$$\begin{aligned} d\hat{l} \in C_C^{k,2} &\longrightarrow d\mathbf{l} = ((dl)_a)_{1 \leq a \leq F} \in \mathbb{Z}_k^F, \\ d^* \hat{m} \in C_{C^*}^{k,2} &\longrightarrow d^* \mathbf{m} = ((d^* m)^i)_{1 \leq i \leq E} \in \mathbb{Z}_k^E. \end{aligned} \quad (3.20)$$

The Poincaré duality implies that a chain has the same components as its Poincaré dual regardless of the fact that these components are taken in \mathbb{Z} or \mathbb{Z}_k . For instance the Poincaré dual of $c = c^i e_i$ is the 2-cochain $\hat{c} = c^i \hat{e}_i$ since $M^{(3)} \cap (c^i \hat{e}_i) = c^i (M^{(3)} \cap \hat{e}_i) = c^i e_i$.

Using correspondences (3.19) and (3.20), equation (3.18) can be rewritten as:

$$(\hat{m} \cup d\hat{l})(M^{(3)}) = \mathbf{m} \cdot d\mathbf{l} = (d^* \hat{m} \cup \hat{l})(M^{(3)}), \quad (3.21)$$

where the \cdot denotes the euclidian scalar product.

Definition of abelian TV invariant. Let S^0 be an oriented face with bounding edges $(e_i^0)_{1 \leq i \leq n}$. Given a labeling \hat{l} of \mathcal{C} that induces a labeling $\hat{l}^0 = l_i^0 \hat{S}^{i,0}$ dualizing the edges $(e_i^0)_{1 \leq i \leq n}$, the **state space** of S^0 is defined as:

$$\mathcal{F}_{\text{TV}_k}(S^0, \hat{l}) = \text{Hom}(R_0, R_{l_1^0} \otimes \cdots \otimes R_{l_n^0}) = \text{Hom}(R_0, R_{\Sigma_{S^0}^{\hat{l}}}) = \delta_{\Sigma_{S^0,0}^{\hat{l}}}, \quad (3.22)$$

where the R_i are the representations associated with $i \in \mathbb{Z}_k$ according to the notations introduced in chapter 1 of this part and $\Sigma_{S^0}^{\hat{l}} = \sum_{i=1}^n l_i^0 = \sum_{i=1}^n \hat{l}(e_i^0)$, the edges e_i^0 being canonically oriented with respect to S^0 . Having S^0 running over the set of oriented faces of \mathcal{C} , the sums $\Sigma_{S^0}^{\hat{l}}$ generate a linear map:

$$\Sigma^{\hat{l}} : \mathcal{F} \rightarrow \mathbb{Z}_k. \quad (3.23)$$

The **total state space** of \mathcal{C} with labeling \hat{l} is defined as:

$$\mathcal{F}_{\text{TV}_k}(\mathcal{C}, l) = \bigotimes_{S \in \mathcal{F}} (\mathcal{F}_{\text{TV}_k}(S, \hat{l}) \otimes \mathcal{F}_{\text{TV}_k}(-S, \hat{l})) = \prod_{S \in \mathcal{F}} (\delta_{\Sigma_S^{\hat{l}},0} \delta_{\Sigma_{-S},0}), \quad (3.24)$$

Chapter 3. Abelian Turaev-Viro invariant

where S is arbitrarily oriented and $-S$ denotes S endowed with the opposite orientation. However, as $\text{Hom}(R_0, R_p)^* \simeq \text{Hom}(R_0, R_p^*)$ and $\delta_{k-p,0}^{[k]} = \delta_{p,0}^{[k]}$, then:

$$\mathcal{F}_{\text{TV}_k}(S, \hat{l})^* = \mathcal{F}_{\text{TV}_k}(-S, \hat{l}) = \delta_{k-\Sigma_S^{\hat{l}},0}^{[k]} = \delta_{\Sigma_S^{\hat{l}},0}^{[k]} = \mathcal{F}_{\text{TV}_k}(S, \hat{l}) \quad (3.25)$$

and hence the total state space of \mathcal{C} for the finite semisimple spherical category $\mathbb{C}^{\mathbb{Z}_k}$ takes the simpler form:

$$\mathcal{F}_{\text{TV}_k}(\mathcal{C}, \hat{l}) = \left(\prod_{S \in \mathcal{F}} \delta_{\Sigma_S^{\hat{l}},0} \right) = \prod_{S \in \mathcal{F}} \delta_{\text{dl}(S)}^{[k]}, \quad (3.26)$$

where $\Sigma_S^{\hat{l}}$ is computed by using any of the two possible orientations of S .

The abelian TV invariant of $M^{(3)}$ corresponding to the finite semisimple spherical category $\mathbb{C}^{\mathbb{Z}_k}$ (no specific constraint on k for this invariant) is defined as¹:

$$Z_{\text{TV}_k}(M^{(3)}) = \mathcal{D}_k^{-2(V-1)} \sum_{\hat{l} \in C_c^{k,1}} \mathcal{F}_{\text{TV}_k}(\mathcal{C}, \hat{l}), \quad (3.27)$$

that is:

$$Z_{\text{TV}_k}(M^{(3)}) = \frac{1}{k^{V-1}} \sum_{\hat{l} \in C_c^{k,1}} \prod_{S \in \mathcal{F}} \delta_{\text{dl}(S)}^{[k]}. \quad (3.28)$$

This formula can be exponentiated:

$$Z_{\text{TV}_k}(M^{(3)}) = \frac{1}{k^{V+F-1}} \sum_{\hat{l} \in C_c^{k,1}} \sum_{\hat{m} \in C_{c^*}^{k,1}} e^{\frac{2i\pi}{k}(\hat{m} \cup \text{dl})}(M^{(3)}) \quad (3.29)$$

and a convenient formulation is also:

$$Z_{\text{TV}_k}(M^{(3)}) = \frac{1}{k^{V+F-1}} \sum_{\hat{l} \in C_c^{k,1}} \sum_{\hat{m} \in C_{c^*}^{k,1}} e^{2i\pi k \left(\left(\frac{\hat{m}}{k} \right) \cup \text{d} \left(\frac{\hat{l}}{k} \right) \right)}(M^{(3)}). \quad (3.30)$$

Alternatively:

$$Z_{\text{TV}_k}(M^{(3)}) = \frac{1}{k^{V-1}} \sum_{\mathbf{l} \in \mathbb{Z}_k^E} \delta_{\mathbf{dl}}^{[k]} \quad (3.31)$$

and:

$$Z_{\text{TV}_k}(M^{(3)}) = \frac{1}{k^{V+F-1}} \sum_{\mathbf{l} \in \mathbb{Z}_k^E} \sum_{\mathbf{m} \in \mathbb{Z}_k^F} e^{\frac{2i\pi}{k} \mathbf{m} \cdot \mathbf{dl}}. \quad (3.32)$$

¹The TV invariant of $M^{(3)}$ is computed from a cellular decomposition \mathcal{C} of $M^{(3)}$ but it does not depend on the choice of this cellular decomposition.

3.2. Abelian expectation values of observables

It can be shown that [Mathieu and Thuillier, 2016a]:

$$Z_{\text{TV}_k}(M^{(3)}) = |H_1(M^{(3)}, \mathbb{Z}_k)| = k^{b_1} \prod_{i=1}^n \text{gcd}(p_i, k), \quad (3.33)$$

where b_1 is the first Betti number of $M^{(3)}$ and the p_i are the torsion indices of $M^{(3)}$, that is:

$$H_1(M^{(3)}, \mathbb{Z}) = F_1 \oplus T_1, \quad (3.34)$$

where $F_1 = \mathbb{Z}^{b_1}$ and $T_1 = \mathbb{Z}_{p_1} \oplus \dots \oplus \mathbb{Z}_{p_n}$.

It should be borne in mind that, as well as for the RT invariant, contrary to the convention often applied in the mathematics community, the abelian TV invariant is normalized here by S^{3^2} .

3.2 Abelian expectation values of observables

The expression (3.30) can be likened to a \mathbb{Z}_k gauge theory on a lattice. Hence for a physicist it is tempting to define observables and the expectation value for these observables.

The observables for such a theory would be simply the product of two Wilson loops for two \mathbb{Z}_k -connections, that is, the product of two \mathbb{Z}_k holonomies:

$$W_{M^{(3)}}\left(\frac{\hat{l}}{k}, \gamma_1\right) W_{M^{(3)}}\left(\frac{\hat{m}}{k}, \gamma_2\right) = e^{2i\pi\left(\frac{\hat{l}}{k} \cup \hat{\gamma}_1 + \frac{\hat{m}}{k} \cup \hat{\gamma}_2\right)(M^{(3)})}, \quad (3.35)$$

provided the loops γ_1 and γ_2 are built respectively with elements of \mathcal{E} and elements of \mathcal{E}^* . In what follows it will always be assumed that it is true. The expectation value of those observables would be:

$$\begin{aligned} & \left\langle W_{M^{(3)}}\left(\frac{\hat{l}}{k}, \gamma_1\right) W_{M^{(3)}}\left(\frac{\hat{m}}{k}, \gamma_2\right) \right\rangle_{\text{TV}_k, M^{(3)}} \\ &= \frac{1}{k^{F+V-1}} \sum_{\hat{l} \in C_C^{k,1}} \sum_{\hat{m} \in C_{C^*}^{k,1}} e^{2i\pi\left(k\left(\frac{\hat{m}}{k} \cup d\frac{\hat{l}}{k}\right) + \frac{\hat{l}}{k} \cup \hat{z}_1 + \frac{\hat{m}}{k} \cup \hat{z}_2\right)(M^{(3)})}, \end{aligned} \quad (3.36)$$

or alternatively:

$$\begin{aligned} & \left\langle W_{M^{(3)}}\left(\frac{\mathbf{1}}{k}, \gamma_1\right) W_{M^{(3)}}\left(\frac{\mathbf{m}}{k}, \gamma_2\right) \right\rangle_{\text{TV}_k, M^{(3)}} \\ &= \frac{1}{k^{V+F-1}} \sum_{\mathbf{l} \in \mathbb{Z}_k^E} \sum_{\mathbf{m} \in \mathbb{Z}_k^F} e^{2i\pi k\left(\frac{\mathbf{m}}{k} \cdot d\frac{\mathbf{1}}{k} + \frac{\mathbf{l}}{k} \cdot \mathbf{z}_1 + \frac{\mathbf{m}}{k} \cdot \mathbf{z}_2\right)}, \end{aligned} \quad (3.37)$$

²In the mathematics literature, the normalization is most of the time taken to be $S^1 \times S^2$. This detail does not change anything in this study.

Chapter 3. Abelian Turaev-Viro invariant

with \mathbf{z}_1 such that $\mathbf{l} \cdot \mathbf{z}_1 = \hat{l} \cup \hat{z}_1 (M^{(3)})$ and \mathbf{z}_2 such that $\mathbf{m} \cdot \mathbf{z}_2 = \hat{m} \cup \hat{z}_2 (M^{(3)})$.

This defines a topological invariant of the triplet $(M^{(3)}, \gamma_1, \gamma_2)$ that has *a priori* never been studied. However, it is important to note that γ_1 and γ_2 have to be contained in the cellular decomposition chosen to compute their expectation value although the final result does not depend on this choice.

Another convenient definition of the expectation value is:

$$\begin{aligned} & \left\langle \left\langle W_{M^{(3)}} \left(\frac{\mathbf{l}}{k}, \gamma_1 \right) W_{M^{(3)}} \left(\frac{\mathbf{m}}{k}, \gamma_2 \right) \right\rangle \right\rangle_{\text{TV}_k, M^{(3)}} \\ &= \frac{\langle W_{M^{(3)}} \left(\frac{\mathbf{l}}{k}, \gamma_1 \right) W_{M^{(3)}} \left(\frac{\mathbf{m}}{k}, \gamma_2 \right) \rangle_{\text{TV}_k, M^{(3)}}}{Z_{\text{TV}_k} (M^{(3)})}, \end{aligned} \quad (3.38)$$

which implies in particular that:

$$\left\langle \left\langle W_{M^{(3)}} \left(\frac{\mathbf{l}}{k}, 0 \right) W_{M^{(3)}} \left(\frac{\mathbf{m}}{k}, 0 \right) \right\rangle \right\rangle_{\text{TV}_k, M^{(3)}} = 1, \quad (3.39)$$

since:

$$\left\langle W_{M^{(3)}} \left(\frac{\mathbf{l}}{k}, 0 \right) W_{M^{(3)}} \left(\frac{\mathbf{m}}{k}, 0 \right) \right\rangle_{\text{TV}_k, M^{(3)}} = Z_{\text{TV}_k} (M^{(3)}). \quad (3.40)$$

All those formulæ presented for two loops γ_1 and γ_2 extend naturally to links.

3.3 Examples

The case of the lens spaces can be considered once again as examples. For this purpose, a cellular decomposition is required. The following construction provides a convenient one. This choice is arbitrary and will not affect the result.

Heegaard splittings. Consider two identical genus g handle bodies $H_{g,L}$ and $H_{g,R}$ glued together on their surface $\partial H_{g,L}$ and $\partial H_{g,R}$ via a homeomorphism $h : \partial H_{g,L} \rightarrow \partial H_{g,R}$. This construction called **Heegaard splitting** generates a closed manifold $M^{(3)}$ of dimension 3. Moreover:

Theorem. Any manifold of dimension 3 admits a Heegaard splitting.

A proof of this classical result is presented in [Saveliev, 2012]. It stems directly from the triangulability of the manifolds of dimension 3.

The relevant information lies on the gluing surface and is just the image of the meridians $\mu_{i,L}$ of $\partial H_{g,L}$ (which is contractible inside $H_{g,L}$) on $\partial H_{g,R}$ as a function of its meridians $\mu_{i,R}$ and longitudes $\lambda_{j,R}$. If $\partial H_{g,R}$ is cut and unfolded, keeping in mind

the identifications separated by the cutting, a planar representation of the splitting called a **Heegaard diagram** is obtained³.

Consider now a manifold $M^{(3)}$ of dimension 3 and a genus g Heegaard splitting $M^{(3)} = H_{g,L} \cup H_{g,R}$. As the interior of a n -cell of a cellular decomposition of a given manifold must be homeomorphic to an open n -ball, if $g = 0$ then $H_{0,L}$ and $H_{0,R}$ can be regarded as 3-cells of a cellular decomposition of $M^{(3)}$ since their interior are open 3-balls. However, $\partial H_{0,R} = \partial H_{0,L} = S^2$ cannot be regarded as a 2-cell since it has no boundary and is not an open 2-ball, that is, an open disk. It is necessary to cut it. Consider for example a loop γ on this S^2 . Then S^2 is cut into two halves whose interior can be regarded as 2-cells since they are homeomorphic to open disks. In its turn, γ cannot be regarded as a 1-cell since it has no boundary and is not homeomorphic to an open 1-ball, that is, a segment. Consider a point on γ . Then γ is cut and its interior is homeomorphic to a segment. If $g \neq 0$ then $H_{g,L}$ and $H_{g,R}$ cannot be regarded as 3-cells of a cellular decomposition of $M^{(3)}$ since their interior are not open 3-balls. It is necessary to cut them with their meridian disks. The procedure is then analogous to the case $g = 0$. What is interesting is that the boundary of the meridian disks is precisely what is represented on the Heegaard diagram. Hence, in some sense, a Heegaard diagram of $M^{(3)}$ provides a planar representation of a cellular decomposition of $M^{(3)}$.

Lens spaces. A Heegaard splitting of $L(p, q)$ is given by two solid tori with the homeomorphism h represented by the matrix:

$$[h] = \begin{pmatrix} q & -s \\ p & r \end{pmatrix} \quad (3.41)$$

with $\det([h]) = p \cdot s + q \cdot r = 1$ in the basis (μ_R, λ_R) as a function of the basis (μ_L, λ_L) , that is:

$$\begin{cases} h(\mu_L) = q \cdot \mu_R + p \cdot \lambda_R, \\ h(\lambda_L) = -s \cdot \mu_R + r \cdot \lambda_R. \end{cases} \quad (3.42)$$

A Heegaard diagram processing the right torus (for example) is obtained the following way indicated in Figure 3.1. The diagram given in Figure 3.2 is obtained for $L(p, q)$ considering on the right torus the image of the trace of the left meridian.

From this decomposition $Z_{\text{TV}_k}(L(p, q))$ can be computed in general as:

$$Z_{\text{TV}_k}(L(p, q)) = \text{gcd}(k, p), \quad (3.43)$$

with the specific case $L(0, 1) = S^1 \times S^2$:

$$Z_{\text{TV}_k}(L(0, 1)) = Z_{\text{TV}_k}(S^1 \times S^2) = k. \quad (3.44)$$

³It should be pointed out that all the curves are defined up to isotopy, thus continuous distortions of those curves can be performed for convenience.

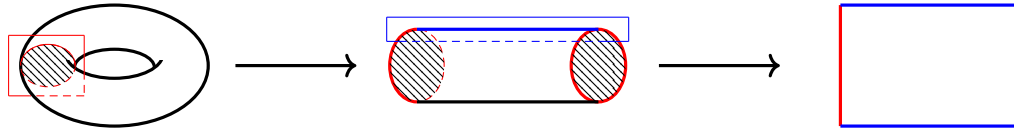


Figure 3.1: Cellular decomposition from a (genus 1) Heegaard splitting.

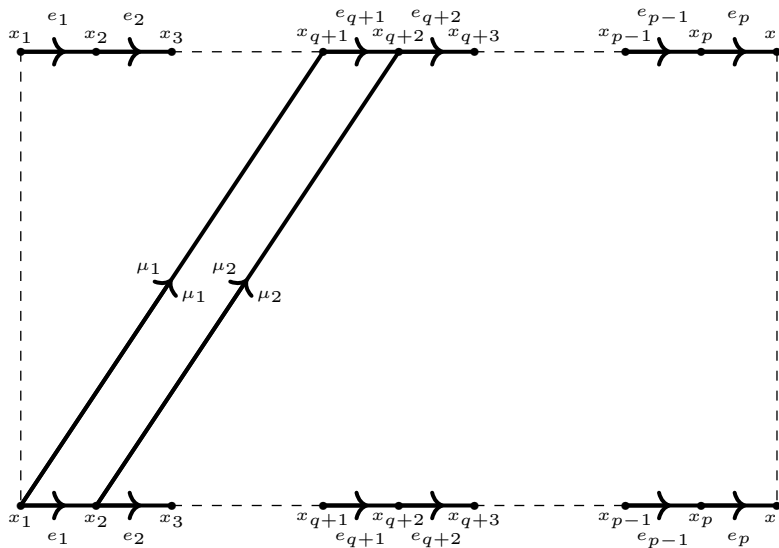


Figure 3.2: Cellular decomposition of $L(p, q)$ from a genus 1 Heegaard splitting.

Concerning the sphere S^3 :

$$Z_{\text{TV}_k}(S^3) = 1, \quad (3.45)$$

where a cellular decomposition is, for example, given in Figure 3.3.

Poincaré homology sphere. The Poincaré homology sphere is not homeomorphic to S^3 but has the same homology groups. Thus, its abelian TV invariant is expected to be 1. A Heegaard splitting can be considered in order to build a cellular decomposition. A Heegaard diagram of the Poincaré homology sphere will be anyway provided in chapter 1 of the last part of this study. However, to emphasize the fact that a Heegaard splitting is not the only way to build a cellular decomposition, another decomposition is used and represented in Figure 3.4. It is actually the template of a dodecahedron, each face being identified to its opposite after a rotation of an angle $\pi/5$.

3.4. Relations between abelian RT and TV theories

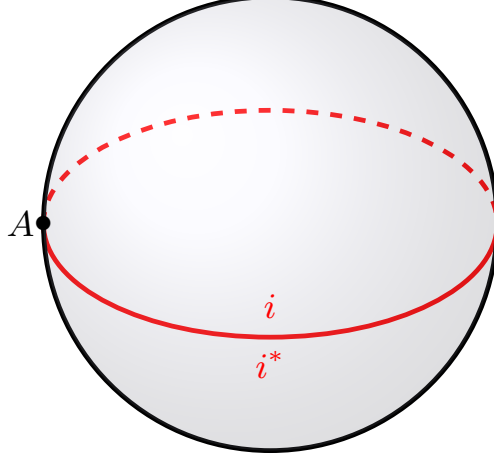


Figure 3.3: Cellular decomposition of S^3 from a genus 0 Heegaard splitting.

According to the rules defined in the previous sections, the abelian TV invariant of the Poincaré homology sphere can be expressed as:

$$Z_{\text{TV}_k}(M^{(3)}) = \sum_{\substack{a,b,c,d,e, \\ f,g,h,i,j=0}}^{k-1} \delta_{a+b+c+d+e}^{[k]} \delta_{a+j+d+h-i}^{[k]} \delta_{c+i+e-j-g}^{[k]} \delta_{b+g+d-i-f}^{[k]} \delta_{a+f+c+h-g}^{[k]} \delta_{h+j+f-b-e}^{[k]} \quad (3.46)$$

and finally:

$$Z_{\text{TV}_k}(M^{(3)}) = 1, \quad (3.47)$$

as expected.

3.4 Relations between abelian RT and TV theories

From the studies of Reshetikhin, Turaev and Viro it appears that:

Theorem ([Turaev, 2010]). If \mathcal{C} is a modular category, then:

$$\left| Z_{\text{RT}_{\mathcal{C}}}(M^{(3)}) \right|^2 = Z_{\text{TV}_{\mathcal{C}}}(M^{(3)}). \quad (3.48)$$

It shall be borne in mind that in the present case the theorem can be applied for $\mathcal{C} = \mathbb{C}^{\mathbb{Z}_k}$ with $k = 2l + 1$ which is the only case where $\mathbb{C}^{\mathbb{Z}_k}$ is modular. Indeed, for $k = 2l + 1$, the theorem is verified:

$$\left| Z_{\text{RT}_{2l+1}}(M^{(3)}) \right|^2 = \left| H^1(M^{(3)}, \mathbb{Z}_{2l+1}) \right| = Z_{\text{TV}_{2l+1}}(M^{(3)}). \quad (3.49)$$

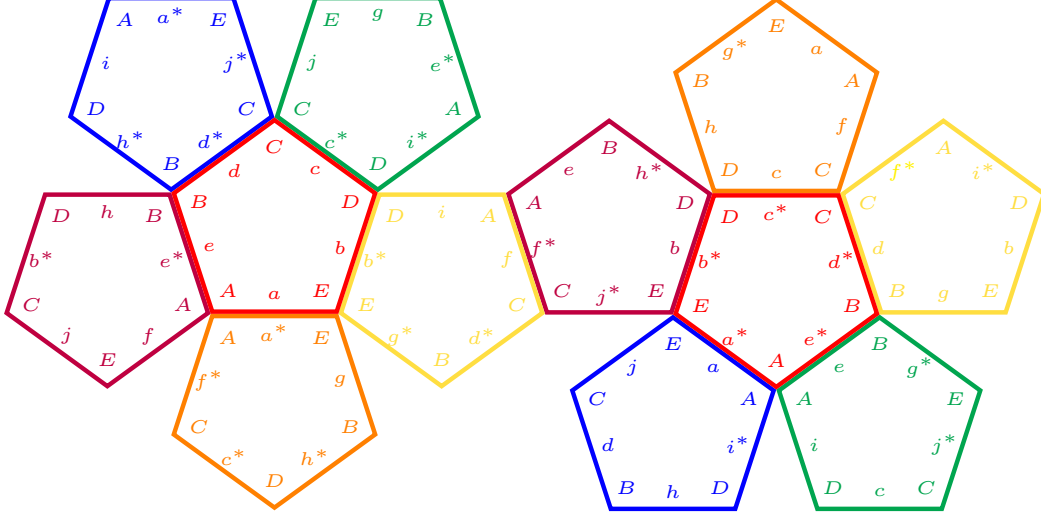


Figure 3.4: Cellular decomposition of the Poincaré homology sphere.

This is a first example of a relation between a quantity computed in S^3 (via a surgery link of $M^{(3)}$ in S^3) and a quantity computed in $M^{(3)}$ (via a cellular decomposition of $M^{(3)}$). It establishes a relation between a pair of lattices $(\mathbb{Z}_{2l+1}^E, \mathbb{Z}_{2l+1}^F)$ on which a non-symmetric operator d acts bilinearly and a lattice \mathbb{Z}_{2l+1}^m on which a symmetric operator $\mathbf{L}_{\mathcal{L}}$ acts quadratically. The lattices here have not the same dimension but the same periodicity. The importance of the modularity can be understood at this point: it ensures that $Z_{\text{RT}_{2l+1}}(M^{(3)})$ is well-defined and that no sub-periodicity appears.

Concerning the extension for $k = 4l$:

$$\left| \tilde{Z}_{\text{RT}_{4l}}(M^{(3)}) \right|^2 = \begin{cases} \left| H^1(M^{(3)}, \mathbb{Z}_{2l}) \right| & \text{if } \forall \alpha \in H^1(M^{(3)}, \mathbb{Z}_{2l}), \alpha \cup \alpha \cup \alpha = 0 \\ 0 & \text{otherwise} \end{cases} \quad (3.50)$$

and thus, if $\forall \alpha \in H^1(M^{(3)}, \mathbb{Z}_{2l}), \alpha \cup \alpha \cup \alpha = 0$:

$$\left| \tilde{Z}_{\text{RT}_{4l}}(M^{(3)}) \right|^2 = Z_{\text{TV}_{2l}}(M^{(3)}). \quad (3.51)$$

Hence, it shall be noted that this formula relates an abelian RT theory to an abelian TV theory with different k . This is because of the sub-periodicity that arises in the abelian RT theory when $k = 4l$.

It could be interesting also to verify if the formulæ (3.49) and (3.51) extend to

3.4. Relations between abelian RT and TV theories

the expectation value of observables as something like:

$$\left| \langle \langle \gamma \rangle \rangle_{\text{RT}_{2l+1}, M^{(3)}} \right|^2 = \left\langle \left\langle W_{M^{(3)}} \left(\frac{\mathbf{1}}{k}, \gamma \right) W_{M^{(3)}} \left(\frac{\mathbf{m}}{k}, \gamma \right) \right\rangle \right\rangle_{\text{TV}_{2l+1}, M^{(3)}} \quad (3.52)$$

and:

$$\left| \langle \langle \gamma \rangle \rangle_{\text{RT}_{4l}, M^{(3)}} \right|^2 = \left\langle \left\langle W_{M^{(3)}} \left(\frac{\mathbf{1}}{k}, \gamma \right) W_{M^{(3)}} \left(\frac{\mathbf{m}}{k}, \gamma \right) \right\rangle \right\rangle_{\text{TV}_{2l}, M^{(3)}} \quad (3.53)$$

up to some restrictions for this last case.

Another theorem due to V. Turaev and A. Virelizier claims:

Theorem ([Turaev and Virelizier, 2013]). If \mathcal{C} is a finite semisimple spherical category, then:

$$Z_{\text{RT}_{\mathcal{Z}(\mathcal{C})}}(M^{(3)}) = Z_{\text{TV}_{\mathcal{C}}}(M^{(3)}), \quad (3.54)$$

which is verified since:

$$Z_{\text{RT}_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})}}(M^{(3)}) = \left| H_1(M^{(3)}, \mathbb{Z}_k) \right| = Z_{\text{TV}_k}(M^{(3)}). \quad (3.55)$$

Strictly speaking, the hypotheses of Turaev and Virelizier are slightly different since their work refined again the vocabulary. However, for the present purpose, these approximations do not lead to any mistake. The formula (3.54) means actually that, given a category that makes it possible to build a TV invariant, it can be extracted from this category a new category that makes it possible to build a RT invariant. Those TV and RT invariants are then equal.

The previous result extends to the expectation value of observables:

$$\langle \gamma_1, \gamma_2 \rangle_{\text{RT}_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})}, M^{(3)}} = \left\langle W_{M^{(3)}} \left(\frac{\mathbf{1}}{k}, \gamma_1 \right) W_{M^{(3)}} \left(\frac{\mathbf{m}}{k}, \gamma_2 \right) \right\rangle_{\text{TV}_k, M^{(3)}}, \quad (3.56)$$

that is:

$$\begin{aligned} & \frac{1}{k^m} \sum_{\mathbf{p} \in \mathbb{Z}_k^m} \sum_{\mathbf{u} \in \mathbb{Z}_k^m} e^{\frac{2i\pi}{k}(\langle \mathbf{p}, \mathbf{q} \rangle, \mathbf{L}_{\mathcal{C}_1, \mathcal{C}_2}(\mathbf{u}, \mathbf{v}))} \\ &= \frac{1}{k^{V+F-1}} \sum_{\mathbf{l} \in \mathbb{Z}_k^E} \sum_{\mathbf{m} \in \mathbb{Z}_k^F} e^{\frac{2i\pi}{k}(\mathbf{m} \cdot \mathbf{d}\mathbf{l} + \mathbf{l} \cdot \mathbf{z}_1 + \mathbf{m} \cdot \mathbf{z}_2)}. \end{aligned} \quad (3.57)$$

This is the first reciprocity formula appearing in this study. The proof is not published but is provided by a straightforward adaptation of the proof of the reciprocity formula for abelian BF and TV theories [Mathieu and Thuillier, 2016b] by replacing the operator d by the linking matrix $\mathbf{L}_{\mathcal{L}}$. It follows that:

$$\langle \gamma_1, \gamma_2 \rangle_{\text{RT}_{\mathcal{Z}(\mathbb{C}^{\mathbb{Z}_k})}, M^{(3)}} = \frac{k^{b_1}}{p_1 \dots p_n} \langle W_{M^{(3)}}(A, \gamma_1) W_{M^{(3)}}(B, \gamma_2) \rangle_{\text{BF}_k, M^{(3)}}, \quad (3.58)$$

Chapter 3. Abelian Turaev-Viro invariant

(see chapter 3 of the next part) and the reciprocity formula for abelian BF and TV theories itself makes it possible to conclude. The main ingredients that are needed in the proof are the fact that:

$$\text{Coker } \mathbf{L}_{\mathcal{L}} \simeq \mathbb{Z}^m / \text{Im } \mathbf{L}_{\mathcal{L}} \simeq H_1(M^{(3)}) \quad (3.59)$$

and:

$$\text{Ker } \mathbf{L}_{\mathcal{L}} \simeq H_2(M^{(3)}). \quad (3.60)$$

As a consequence, because of equations (3.55) and (3.56):

$$\langle\langle \gamma_1, \gamma_2 \rangle\rangle_{\text{RT}_{\mathbb{Z}(\mathbb{C}^{\mathbb{Z}_k})}, M^{(3)}} = \left\langle\left\langle W_{M^{(3)}}\left(\frac{\mathbf{1}}{k}, \gamma_1\right) W_{M^{(3)}}\left(\frac{\mathbf{m}}{k}, \gamma_2\right) \right\rangle\right\rangle_{\text{TV}_{k, M^{(3)}}}. \quad (3.61)$$

To conclude, the abelian TV theory can be considered as weaker than the abelian RT theory, in the sense that the information contained in the phase of the abelian RT invariant is lost in the abelian TV invariant. However, in compensation, the abelian TV theory is defined for any k . It has also been shown that the abelian TV theory coincides actually with the abelian RT theory defined on the Drinfeld center of $\mathbb{C}^{\mathbb{Z}_k}$.

Part II

A physical approach

Chapter 1

Deligne-Beilinson cohomology

Contents

Introduction (English version)	56
Introduction (version française)	57
1.1 Definition of DB cohomology	58
1.2 Structure of the space of DB cohomology classes	64
1.3 Operations and duality on DB cohomology classes	66
1.4 Decomposition of DB cohomology classes	69

Introduction (English version)

In this chapter Deligne-Beilinson (DB) cohomology is introduced as in [Mathieu, 2017]. Consider the abelian holonomy (which will be the observable of the abelian CS and BF theories) along a loop γ :

$$h_\gamma(A) = e^{2i\pi \oint_\gamma A}, \quad (1.1)$$

where A is a $U(1)$ connection. This is the observable involved in the Aharonov-Bohm effect. The classical gauge transformation $A \rightarrow A + d\Lambda$ where Λ is a function leaves h_γ invariant. However, the invariance of h_γ is more general. Indeed, it turns out that the most general gauge transformation is $A \rightarrow A + \omega_{\mathbb{Z}}$ where $\omega_{\mathbb{Z}}$ is a closed 1-form with integral periods. In any manifold $M^{(3)}$ such that $H_1(M^{(3)}) = 0$, like S^3 for example, this transformation comes down to the previous classical one since, by Poincaré lemma, there exists a function Λ such that $\omega_{\mathbb{Z}} = d\Lambda$. If $H_1(M^{(3)}) \neq 0$, then this reduction happens only locally in contractible open sets that define a covering of $M^{(3)}$ (such a covering always exists). Hence, this quantum gauge transformation generalizes on any closed manifold of dimension 3 the classical gauge transformation that usually appears in S^3 .

In this chapter the equivalence classes according to this quantum gauge transformation are considered. These classes classify $U(1)$ -bundles over $M^{(3)}$ endowed with connections and their collection is the so-called first DB cohomology group of $M^{(3)}$. Some properties of DB classes are shown and to be used later on to extract relevant quantities from functional integrals.

Introduction (version française)

La cohomologie de Deligne-Beilinson (DB) est introduite dans ce chapitre comme dans [Mathieu, 2017]. L'holonomie abélienne (qui sera l'observable des théories CS et BF abéliennes) le long d'une boucle γ est définie par :

$$h_\gamma(A) = e^{2i\pi \oint_\gamma A}, \quad (1.1)$$

où A est une connexion $U(1)$. C'est l'observable qui est en jeu dans l'effet Aharonov-Bohm. La transformation de jauge classique $A \rightarrow A + d\Lambda$, où Λ est une fonction, laisse h_γ invariante. Cependant, l'invariance de h_γ est bien plus générale. En effet, la transformation de jauge la plus générale est $A \rightarrow A + \omega_{\mathbb{Z}}$ où $\omega_{\mathbb{Z}}$ est une 1-forme à périodes entières. Dans toute variété $M^{(3)}$ telle que $H_1(M^{(3)}) = 0$, comme S^3 par exemple, cette transformation se réduit à la transformation classique puisque, d'après le lemme de Poincaré, il existe une fonction Λ telle que $\omega_{\mathbb{Z}} = d\Lambda$. Si $H_1(M^{(3)}) \neq 0$, alors cette réduction ne survient que localement, sur les ouverts contractiles qui définissent un recouvrement de $M^{(3)}$ (un tel recouvrement existe toujours). Ainsi, cette transformation de jauge quantique généralise sur toute variété de dimension 3 la transformation de jauge classique qui apparaît usuellement dans S^3 .

Dans ce chapitre sont considérées les classes d'équivalence modulo cette transformation de jauge quantique. Ces classes classifient les fibrés $U(1)$ au-dessus de $M^{(3)}$ munis de connexions et leur collection définit le premier groupe de cohomologie DB de $M^{(3)}$. Quelques propriétés des classes de cohomologie DB sont montrées afin de servir plus loin à extraire les quantités pertinentes des intégrales fonctionnelles.

1.1 Definition of DB cohomology

In the following, $M^{(3)}$ is a closed and compact manifold of dimension 3 provided with a good covering, that is, some contractible open sets U_α covering $M^{(3)}$ and such that the multiple (but finite) intersections $U_{i_0 \dots i_n} = U_{i_0} \cap \dots \cap U_{i_n}$ are also contractible or empty. In particular the Poincaré lemma can be applied inside these open sets and intersections. Dimension 3 is not a necessary assumption but is the main framework of this dissertation.

In general, a U(1) gauge potential, or U(1)-connection, is defined only locally. Thus, the most general data to start with are local potentials $A_\alpha^{(1)}$, that is, local 1-forms with coefficients in the Lie algebra of U(1). The upper index (1) indicates the de Rham degree of $A_\alpha^{(1)}$ and the lower index α is a Čech index, meaning that $A_\alpha^{(1)}$ is defined in the open set U_α .

To define a global field, the gluing rule between $A_\alpha^{(1)}$ and $A_\beta^{(1)}$ in the intersection $U_{\alpha\beta}$ has to be given. This, by definition, is done thanks to a gluing relation:

$$A_\beta^{(1)} = A_\alpha^{(1)} + d^{(0)} \Lambda_{\alpha\beta}^{(0)} \text{ in } U_{\alpha\beta}, \quad (1.2)$$

the indices of $\Lambda_{\alpha\beta}^{(0)}$ being thus ordered. The antisymmetry of this relation in α and β implies that $d^{(0)} (\Lambda_{\alpha\beta}^{(0)} + \Lambda_{\beta\gamma}^{(0)} + \Lambda_{\gamma\alpha}^{(0)}) = 0$, making $\Lambda_{\alpha\beta}^{(0)} + \Lambda_{\beta\gamma}^{(0)} + \Lambda_{\gamma\alpha}^{(0)}$ a constant in $U_{\alpha\beta\gamma}$ which is an integer:

$$\Lambda_{\alpha\beta}^{(0)} + \Lambda_{\beta\gamma}^{(0)} + \Lambda_{\gamma\alpha}^{(0)} = n_{\mathbb{Z}\alpha\beta\gamma}^{(-1)} \in \mathbb{Z} \text{ in } U_{\alpha\beta\gamma}. \quad (1.3)$$

Indeed, considering $g_{\alpha\beta} = e^{2i\pi\Lambda_{\alpha\beta}}$, equation (1.3) becomes nothing but the cocycle condition for a U(1) fiber bundle, the $g_{\alpha\beta}$ being the transition functions. The subscript \mathbb{Z} has been put under $n_{\mathbb{Z}\alpha\beta\gamma}^{(-1)}$ in order to insist on the fact that the family $\left(n_{\mathbb{Z}\alpha\beta\gamma}^{(-1)} \right)$ has to be a family of integers. The de Rham degree of pure integers is considered here by convention to be (-1) . The symmetry in α , β and γ of this last relation implies that:

$$n_{\mathbb{Z}\alpha\beta\gamma}^{(-1)} - n_{\mathbb{Z}\alpha\beta\delta}^{(-1)} + n_{\mathbb{Z}\alpha\gamma\delta}^{(-1)} - n_{\mathbb{Z}\beta\gamma\delta}^{(-1)} = 0 \quad (1.4)$$

Thus, the case where the potential can be globally defined in the manifold under consideration generalizes to the case where it can only be defined locally by considering the collection $\left((A_\alpha^{(1)})^{(0)}, (\Lambda_{\alpha\beta}^{(0)})^{(1)}, \left(n_{\mathbb{Z}\alpha\beta\gamma}^{(-1)} \right)^{(2)} \right)$ constituted of the family of the local potentials $(A_\alpha^{(1)})^{(0)}$ whose elements $A_\alpha^{(1)}$ are defined in the open sets

1.1. Definition of DB cohomology

U_α , the family of functions $(\Lambda_{\alpha\beta}^{(0)})^{(1)}$ whose elements $\Lambda_{\alpha\beta}^{(0)}$ are defined in the intersections $U_{\alpha\beta}$ and ensure the gluing of the local potentials $A_\alpha^{(1)}$ and the family of integers $(n_{\mathbb{Z}\alpha\beta\gamma}^{(-1)})^{(2)}$ whose elements $n_{\mathbb{Z}\alpha\beta\gamma}^{(-1)}$ are defined in the intersections $U_{\alpha\beta\gamma}$ and constraint the functions $\Lambda_{\alpha\beta}^{(0)}$. Equations (1.2), (1.3) and (1.4) summarize the gluing of the potentials and the consistency of this operation. For each family, the interior upper index indicates the form or de Rham degree of its elements and the exterior upper index indicates the Čech degree of the family. The lower indices of the elements of each family indicate in which open sets or intersections those elements are defined.

Consider $\delta^{(p)}$ to be the Čech operator of cohomology transforming a Čech p -cochain, that is, a collection $(\omega_{\alpha_0\dots\alpha_p})^{(p)}$ of elements $\omega_{\alpha_0\dots\alpha_p}$ defined in the intersections $U_{\alpha_0\dots\alpha_p}$, into a Čech $(p+1)$ -cochain, that is, a collection $(\omega_{\alpha_0\dots\alpha_{p+1}})^{(p+1)}$ of elements $\omega_{\alpha_0\dots\alpha_{p+1}}$ defined in the intersections $U_{\alpha_0\dots\alpha_{p+1}}$. Then $\delta^{(p)}$ acts as:

$$\delta^{(p)} (\omega_{\alpha_0\dots\alpha_p})^{(p)} = (\omega_{\alpha_0\dots\alpha_p\alpha_{p+1}})^{(p+1)}, \quad (1.5)$$

with:

$$\omega_{\alpha_0\dots\alpha_p\alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0\dots\check{\alpha}_i\dots\alpha_{p+1}}, \quad (1.6)$$

where $\check{\alpha}_i$ means that α_i is omitted. It shall be underlined that, as a consequence, the indices are ordered and:

$$\omega_{\alpha_0\dots\alpha_i\alpha_{i+1}\dots\alpha_p} = -\omega_{\alpha_0\dots\alpha_{i+1}\alpha_i\dots\alpha_p}. \quad (1.7)$$

It can be shown that the operators $\delta^{(p)}$ are group morphisms and, according to their definition:

$$\delta^{(p)} \circ \delta^{(p-1)} = 0. \quad (1.8)$$

The elements of $\check{Z}^p = \text{Ker } \delta^{(p)}$ are called the Čech p -cocycles and the elements of $\check{B}^p = \text{Ker } \delta^{(p-1)}$ are called the Čech coboundaries. According to equation (1.8), \check{B}^p is a subgroup of \check{Z}^p . The Čech cohomology group $\check{H}^p(M^{(3)}, \mathbb{Z})$ of degree p of $M^{(3)}$ with coefficients in \mathbb{Z} is then:

$$\check{H}^p(M^{(3)}, \mathbb{Z}) = \check{Z}^p / \check{B}^p. \quad (1.9)$$

The elements of the collection of the families $\left((A_\alpha^{(1)})^{(0)}, (\Lambda_{\alpha\beta}^{(0)})^{(1)}, (n_{\mathbb{Z}\alpha\beta\gamma}^{(-1)})^{(2)} \right)$

Chapter 1. Deligne-Beilinson cohomology

are related by:

$$\begin{cases} A_\beta^{(1)} = A_\alpha^{(1)} + d\Lambda_{\alpha\beta}^{(0)} \text{ in } U_{\alpha\beta}, \\ \Lambda_{\alpha\beta}^{(0)} + \Lambda_{\beta\gamma}^{(0)} + \Lambda_{\gamma\alpha}^{(0)} = n_{\mathbb{Z}\alpha\beta\gamma}^{(-1)} \in \mathbb{Z} \text{ in } U_{\alpha\beta\gamma}, \\ n_{\mathbb{Z}\alpha\beta\gamma}^{(-1)} - n_{\mathbb{Z}\alpha\beta\delta}^{(-1)} + n_{\mathbb{Z}\alpha\gamma\delta}^{(-1)} - n_{\mathbb{Z}\beta\gamma\delta}^{(-1)} = 0, \end{cases} \quad (1.10)$$

which can be rewritten as:

$$\begin{cases} \delta^{(0)} \left(A_\alpha^{(1)} \right)^{(0)} - \left(d^{(0)} \Lambda_{\alpha\beta}^{(0)} \right)^{(1)} = 0 \text{ in the } U_{\alpha\beta}, \\ \delta^{(1)} \left(\Lambda_{\alpha\beta}^{(0)} \right)^{(1)} - \left(d^{(-1)} n_{\mathbb{Z}\alpha\beta\gamma}^{(-1)} \right)^{(2)} = 0 \text{ in the } U_{\alpha\beta\gamma}, \\ \delta^{(2)} \left(n_{\mathbb{Z}\alpha\beta\gamma}^{(-1)} \right)^{(2)} = 0, \end{cases} \quad (1.11)$$

where it is understood that $d^{(-1)}$ simply transforms an integer into the constant function equal to this integer and $d^{(p)}$ for $p \geq 0$ is the usual de Rham differential operator. In a more compact form, this reads:

$$D_{\text{DB}}^{[1]} A^{[1]} = 0, \quad (1.12)$$

with:

$$D_{\text{DB}}^{[1]} = \begin{pmatrix} \delta^{(0)} & -d^{(0)} & 0 \\ 0 & \delta^{(1)} & -d^{(-1)} \\ 0 & 0 & \delta^{(2)} \end{pmatrix} \quad (1.13)$$

and:

$$A^{[1]} = \begin{pmatrix} \left(A_\alpha^{(1)} \right)^{(0)} \\ \left(\Lambda_{\alpha\beta}^{(0)} \right)^{(1)} \\ \left(n_{\mathbb{Z}\alpha\beta\gamma}^{(-1)} \right)^{(2)} \end{pmatrix}. \quad (1.14)$$

A collection $A^{[1]}$ such that $D_{\text{DB}}^{[1]} A^{[1]} = 0$ is called a DB 1-cocycle. The set of DB 1-cocycles will be denoted $Z_{\text{DB}}^{[1]}$. By definition, $Z_{\text{DB}}^{[1]} = \text{Ker } D_{\text{DB}}^{[1]}$. The same reasoning starting from local p -forms makes it possible to define the DB p -cocycles $Z_{\text{DB}}^{[p]} = \text{Ker } D_{\text{DB}}^{[p]}$. It can be shown that $Z_{\text{DB}}^{[p]}$ can be endowed with a structure of \mathbb{Z} -module.

The particular case that will also be used further is the set of DB 3-cocycles $Z_{\text{DB}}^{[3]}$. A collection $\left(\left(V_\alpha^{(3)} \right)^{(0)}, \left(S_{\alpha\beta}^{(2)} \right)^{(1)}, \left(L_{\alpha\beta\gamma}^{(1)} \right)^{(2)}, \left(P_{\alpha\beta\gamma\delta}^{(0)} \right)^{(3)}, \left(n_{\mathbb{Z}\alpha\beta\gamma\delta\varepsilon}^{(-1)} \right)^{(4)} \right)$ is a

1.1. Definition of DB cohomology

DB 3-cocycle if the families of the collection verify:

$$\left\{ \begin{array}{l} \delta^{(0)} \left(V_\alpha^{(3)} \right)^{(0)} - \left(d^{(2)} S_{\alpha\beta}^{(2)} \right)^{(1)} = 0 \text{ in the } U_{\alpha\beta}, \\ \delta^{(1)} \left(S_{\alpha\beta}^{(2)} \right)^{(1)} - \left(d^{(1)} L_{\alpha\beta\gamma}^{(1)} \right)^{(2)} = 0 \text{ in the } U_{\alpha\beta\gamma}, \\ \delta^{(2)} \left(L_{\alpha\beta\gamma}^{(1)} \right)^{(2)} - \left(d^{(0)} P_{\alpha\beta\gamma\delta}^{(0)} \right)^{(3)} = 0 \text{ in the } U_{\alpha\beta\gamma\delta}, \\ \delta^{(3)} \left(P_{\alpha\beta\gamma\delta}^{(0)} \right)^{(3)} - \left(d^{(-1)} n_{\mathbb{Z}\alpha\beta\gamma\delta\varepsilon}^{(-1)} \right)^{(4)} = 0 \text{ in the } U_{\alpha\beta\gamma\delta\varepsilon}, \\ \delta^{(4)} \left(n_{\mathbb{Z}\alpha\beta\gamma\delta\varepsilon}^{(-1)} \right)^{(4)} = 0, \end{array} \right. \quad (1.15)$$

or in a more compact form:

$$D_{\text{DB}}^{[3]} V^{[3]} = 0, \quad (1.16)$$

with:

$$D_{\text{DB}}^{[3]} = \begin{pmatrix} \delta^{(0)} & -d^{(2)} & 0 & 0 & 0 \\ 0 & \delta^{(1)} & -d^{(1)} & 0 & 0 \\ 0 & 0 & \delta^{(2)} & -d^{(0)} & 0 \\ 0 & 0 & 0 & \delta^{(3)} & -d^{(-1)} \\ 0 & 0 & 0 & 0 & \delta^{(4)} \end{pmatrix} \quad (1.17)$$

and:

$$V^{[3]} = \begin{pmatrix} \left(V_\alpha^{(3)} \right)^{(0)} \\ \left(S_{\alpha\beta}^{(2)} \right)^{(1)} \\ \left(L_{\alpha\beta\gamma}^{(1)} \right)^{(2)} \\ \left(P_{\alpha\beta\gamma\delta}^{(0)} \right)^{(3)} \\ \left(n_{\mathbb{Z}\alpha\beta\gamma\delta\varepsilon}^{(-1)} \right)^{(4)} \end{pmatrix}. \quad (1.18)$$

It shall be reminded that the family $\left(V_\alpha^{(3)} \right)^{(0)}$ is a family of 3-forms $V_\alpha^{(3)}$ defined in the open sets U_α , the family $\left(S_{\alpha\beta}^{(2)} \right)^{(1)}$ is a family of 2-forms $S_{\alpha\beta}^{(2)}$ defined in the intersections $U_{\alpha\beta}$, the family $\left(L_{\alpha\beta\gamma}^{(1)} \right)^{(2)}$ is a family of 1-forms $L_{\alpha\beta\gamma}^{(1)}$ defined in the intersections $U_{\alpha\beta\gamma}$, the family $\left(P_{\alpha\beta\gamma\delta}^{(0)} \right)^{(3)}$ is a family of functions $P_{\alpha\beta\gamma\delta}^{(0)}$ defined in the intersections $U_{\alpha\beta\gamma\delta}$ and finally, the family $\left(n_{\mathbb{Z}\alpha\beta\gamma\delta\varepsilon}^{(-1)} \right)^{(4)}$ is a family of integers $n_{\mathbb{Z}\alpha\beta\gamma\delta\varepsilon}^{(-1)}$ defined in the intersections $U_{\alpha\beta\gamma\delta\varepsilon}$.

Now, perform a gauge transformation of the $A_\alpha^{(1)}$:

$$A_\alpha^{(1)} \longrightarrow A_\alpha^{(1)} + d^{(0)} q_\alpha^{(0)} \text{ in } U_\alpha, \quad (1.19)$$

Chapter 1. Deligne-Beilinson cohomology

where the family $(q_\alpha^{(0)})^{(0)}$ is a family of functions $q_\alpha^{(0)}$ defined in the U_α . This implies that the $\Lambda_{\alpha\beta}^{(0)}$ have to transform according to:

$$\Lambda_{\alpha\beta}^{(0)} \longrightarrow \Lambda_{\alpha\beta}^{(0)} + q_\beta^{(0)} - q_\alpha^{(0)} + m_{\mathbb{Z}\alpha\beta}^{(-1)} \text{ in } U_{\alpha\beta}, \quad (1.20)$$

where the family $(m_{\mathbb{Z}\alpha\beta}^{(-1)})^{(1)}$ consists in integers $m_{\mathbb{Z}\alpha\beta}^{(-1)}$ defined in the intersections $U_{\alpha\beta}$. Finally, the $n_{\mathbb{Z}\alpha\beta\gamma}^{(-1)}$ have to transform according to:

$$n_{\mathbb{Z}\alpha\beta\gamma}^{(-1)} \longrightarrow n_{\mathbb{Z}\alpha\beta\gamma}^{(-1)} + m_{\mathbb{Z}\beta\gamma}^{(-1)} - m_{\mathbb{Z}\alpha\gamma}^{(-1)} + m_{\mathbb{Z}\alpha\beta}^{(-1)}. \quad (1.21)$$

Hence, the collection $\left((q_\alpha^{(0)})^{(0)}, (m_{\mathbb{Z}\alpha\beta}^{(-1)})^{(1)} \right)$ together with the set of transformation rules:

$$\begin{cases} A_\alpha^{(1)} \longrightarrow A_\alpha^{(1)} + d^{(0)}q_\alpha^{(0)} \text{ in } U_\alpha, \\ \Lambda_{\alpha\beta}^{(0)} \longrightarrow \Lambda_{\alpha\beta}^{(0)} + q_\beta^{(0)} - q_\alpha^{(0)} + m_{\mathbb{Z}\alpha\beta}^{(-1)} \text{ in } U_{\alpha\beta}, \\ n_{\mathbb{Z}\alpha\beta\gamma}^{(-1)} \longrightarrow n_{\mathbb{Z}\alpha\beta\gamma}^{(-1)} + m_{\mathbb{Z}\beta\gamma}^{(-1)} - m_{\mathbb{Z}\alpha\gamma}^{(-1)} + m_{\mathbb{Z}\alpha\beta}^{(-1)} \text{ in } U_{\alpha\beta\gamma}, \end{cases} \quad (1.22)$$

generalize the idea of gauge transformation from the case where the potential is globally defined to the case where it is only locally defined. The set of rules (1.22) can be written as:

$$\begin{cases} (A_\alpha^{(1)})^{(0)} \longrightarrow (A_\alpha^{(1)})^{(0)} + (d^{(0)}q_\alpha^{(0)})^{(0)} \text{ in the } U_\alpha, \\ (\Lambda_{\alpha\beta}^{(0)})^{(1)} \longrightarrow (\Lambda_{\alpha\beta}^{(0)})^{(1)} + \delta^{(0)}(q_\alpha^{(0)})^{(0)} + (d^{(-1)}m_{\mathbb{Z}\alpha\beta}^{(-1)})^{(1)} \text{ in the } U_{\alpha\beta}, \\ (n_{\mathbb{Z}\alpha\beta\gamma}^{(-1)})^{(2)} \longrightarrow (n_{\mathbb{Z}\alpha\beta\gamma}^{(-1)})^{(2)} + \delta^{(1)}(m_{\mathbb{Z}\alpha\beta}^{(-1)})^{(1)} \text{ in the } U_{\alpha\beta\gamma}. \end{cases} \quad (1.23)$$

In a more compact form, this reads:

$$A^{[1]} \longrightarrow A^{[1]} + D_{\text{DB}}^{[0]}q^{[0]}, \quad (1.24)$$

with:

$$D_{\text{DB}}^{[0]} = \begin{pmatrix} d^{(0)} & 0 \\ \delta^{(0)} & d^{(-1)} \\ 0 & \delta^{(1)} \end{pmatrix} \quad (1.25)$$

and:

$$q^{[0]} = \begin{pmatrix} (q_\alpha^{(0)})^{(0)} \\ (m_{\mathbb{Z}\alpha\beta}^{(-1)})^{(1)} \end{pmatrix}. \quad (1.26)$$

1.1. Definition of DB cohomology

A collection that can be written as $D_{\text{DB}}^{[0]}q^{[0]}$ is called a DB 1-coboundary. The set of DB 1-coboundaries will be denoted $B_{\text{DB}}^{[1]}$. By definition, $B_{\text{DB}}^{[1]} = \text{Im } D_{\text{DB}}^{[0]}$. The same reasoning studying the transformation of local p -forms makes it possible to define the DB p -coboundary $B_{\text{DB}}^{[p]} = \text{Im } D_{\text{DB}}^{[p-1]}$. It can be shown that $B_{\text{DB}}^{[p]}$ can be endowed with a structure of \mathbb{Z} -module.

The particular case that will also be used further is $B_{\text{DB}}^{[p]}$. A DB 3-coboundary arises from the transformation rules of the families of a DB 3-cycle:

$$\left\{ \begin{array}{l} \left(V_{\alpha}^{(3)} \right)^{(0)} \longrightarrow \left(V_{\alpha}^{(3)} \right)^{(0)} + \left(d^{(2)} \Sigma_{\alpha}^{(2)} \right)^{(0)} \text{ in the } U_{\alpha}, \\ \left(S_{\alpha\beta}^{(2)} \right)^{(1)} \longrightarrow \left(S_{\alpha\beta}^{(2)} \right)^{(1)} + \delta^{(0)} \left(\Sigma_{\alpha}^{(2)} \right)^{(0)} + \left(d^{(1)} \Theta_{\alpha\beta}^{(1)} \right)^{(1)} \text{ in the } U_{\alpha\beta}, \\ \left(L_{\alpha\beta\gamma}^{(1)} \right)^{(2)} \longrightarrow \left(L_{\alpha\beta\gamma}^{(1)} \right)^{(2)} + \delta^{(1)} \left(\Theta_{\alpha\beta}^{(1)} \right)^{(1)} + \left(d^{(0)} \Pi_{\alpha\beta\gamma}^{(0)} \right)^{(2)} \text{ in the } U_{\alpha\beta\gamma}, \\ \left(P_{\alpha\beta\gamma\delta}^{(0)} \right)^{(3)} \longrightarrow \left(P_{\alpha\beta\gamma\delta}^{(0)} \right)^{(3)} + \delta^{(2)} \left(\Pi_{\alpha\beta\gamma}^{(0)} \right)^{(2)} + \left(d^{(-1)} m_{\mathbb{Z}\alpha\beta\gamma\delta}^{(-1)} \right)^{(3)} \text{ in the } U_{\alpha\beta\gamma\delta}, \\ \left(n_{\mathbb{Z}\alpha\beta\gamma\delta\varepsilon}^{(-1)} \right)^{(4)} \longrightarrow \left(n_{\mathbb{Z}\alpha\beta\gamma\delta\varepsilon}^{(-1)} \right)^{(4)} + \delta^{(3)} \left(m_{\mathbb{Z}\alpha\beta\gamma\delta}^{(-1)} \right)^{(3)} \text{ in the } U_{\alpha\beta\gamma\delta\varepsilon}, \end{array} \right. \quad (1.27)$$

or in a more compact form:

$$V^{[3]} \longrightarrow V^{[3]} + D_{\text{DB}}^{[2]} \Sigma^{[2]}, \quad (1.28)$$

with:

$$D_{\text{DB}}^{[2]} = \begin{pmatrix} d^{(2)} & 0 & 0 & 0 \\ \delta^{(0)} & d^{(1)} & 0 & 0 \\ 0 & \delta^{(1)} & d^{(0)} & 0 \\ 0 & 0 & \delta^{(2)} & d^{(-1)} \\ 0 & 0 & 0 & \delta^{(3)} \end{pmatrix} \quad (1.29)$$

and:

$$\Sigma^{[2]} = \begin{pmatrix} \left(\Sigma_{\alpha}^{(2)} \right)^{(0)} \\ \left(\Theta_{\alpha\beta}^{(1)} \right)^{(1)} \\ \left(\Pi_{\alpha\beta\gamma}^{(0)} \right)^{(2)} \\ \left(m_{\mathbb{Z}\alpha\beta\gamma\delta}^{(-1)} \right)^{(3)} \end{pmatrix}. \quad (1.30)$$

It shall be reminded that the family $\left(\Sigma_{\alpha}^{(2)} \right)^{(0)}$ is a family of 2-forms $\Sigma_{\alpha}^{(2)}$ defined in the open sets U_{α} , the family $\left(\Theta_{\alpha\beta}^{(1)} \right)^{(1)}$ is a family of 1-forms $\Theta_{\alpha\beta}^{(1)}$ defined in the intersections $U_{\alpha\beta}$, the family $\left(\Pi_{\alpha\beta\gamma}^{(0)} \right)^{(2)}$ is a family of functions $\Pi_{\alpha\beta\gamma}^{(0)}$ defined in the

Chapter 1. Deligne-Beilinson cohomology

intersections $U_{\alpha\beta\gamma}$ and finally, the family $\left(m_{\mathbb{Z}\alpha\beta\gamma\delta}^{(-1)}\right)^{(3)}$ is a family of integers $m_{\mathbb{Z}\alpha\beta\gamma\delta}^{(-1)}$ defined in the intersections $U_{\alpha\beta\gamma\delta}$.

It can be verified that the operators $D_{\text{DB}}^{[p]}$ are morphisms of modules and:

$$D_{\text{DB}}^{[p+1]} \circ D_{\text{DB}}^{[p]} = 0. \quad (1.31)$$

As a consequence, $B^{[p]}$ is a submodule of $Z^{[p]}$. The \mathbb{Z} -module arising from the quotient of the DB p -cocycles by the DB p -coboundaries is the DB cohomology group $H_{\text{DB}}^{[p]}(M^{(3)}, \mathbb{Z})$ of degree p . The degrees 1 and 3 will be the most relevant for the present purpose but higher degree generalizations are possible. Taking the previous construction into account, the elements of $H_{\text{DB}}^{[1]}(M^{(3)}, \mathbb{Z})$ are nothing but the gauge classes of U(1) gauge potentials over $M^{(3)}$, or equivalently the gauge classes of U(1)-connections over $M^{(3)}$.

What has been presented in this section is actually just a realization of DB cohomology from the Čech-de Rham bicomplex [Bott and Tu, 1982] by a truncation and imposing the pure Čech components to be integers. This makes it possible to access the torsion of $M^{(3)}$ contrary to the Čech-de Rham cohomology. The complete construction of DB cohomology from the Čech-de Rham bicomplex can be found in [Bauer et al., 2005].

It is important to remark that the degree of a DB class is considered here to be the de Rham degree of the first component of its representatives. This convention has been chosen because this first component is the most important from the point of view of physics (the local potentials for the degree 1 and the Lagrangian for the degree 3). However, this is not the usual convention in mathematics regarding differential cohomology [Bunke, 2013].

For convenience, from now on, the parenthesis around the elements of the families, as well as the de Rham, the Čech and the DB degrees will be omitted.

1.2 Structure of the space of DB cohomology classes

$H_{\text{DB}}^{[1]}$ is naturally endowed with a structure of \mathbb{Z} -module¹. It can be described in particular through two exact sequences. The first one is:

$$0 \longrightarrow \Omega^1/\Omega_{\mathbb{Z}}^1 \xrightarrow{\delta} H_{\text{DB}}^{[1]} \xrightarrow{i} \check{H}^2 \longrightarrow 0, \quad (1.32)$$

with:

$$\delta \left| \begin{array}{l} \Omega^1/\Omega_{\mathbb{Z}}^1 \longrightarrow H_{\text{DB}}^{[1]} \\ \omega \longmapsto (\delta\omega, 0, 0) \end{array} \right. \quad (1.33)$$

¹It must be borne in mind that, as a consequence, a DB class cannot be divided in general. However, a class that admits a representative $(\omega_{\alpha}, 0, 0)$ can be divided by any non-zero number a , producing a new class which admits a representative $(\frac{\omega_{\alpha}}{a}, 0, 0)$

1.2. Structure of the space of DB cohomology classes

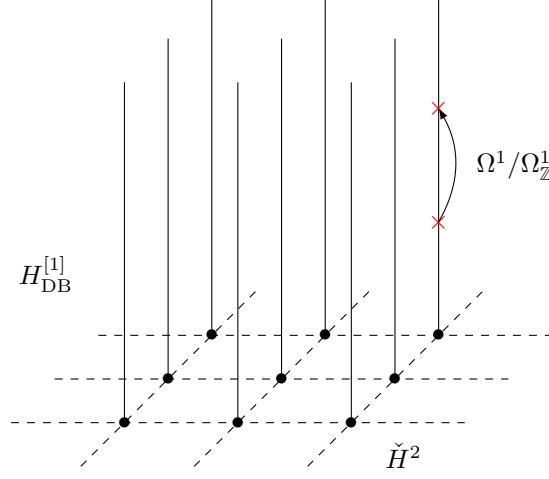


Figure 1.1: First representation of $H_{\text{DB}}^{[1]}$.

and:

$$i \left| \begin{array}{ccc} H_{\text{DB}}^{[1]} & \longrightarrow & \check{H}^2 \\ (A_\alpha, \Lambda_{\alpha\beta}, n_{\mathbb{Z}\alpha\beta\gamma}) & \longmapsto & n_{\mathbb{Z}\alpha\beta\gamma}, \end{array} \right. \quad (1.34)$$

where $\Omega^1/\Omega_{\mathbb{Z}}^1$ is the quotient of the 1-forms by the closed 1-forms with integral periods and \check{H}^2 is the space of Čech cohomology classes of the manifold $M^{(3)}$ considered. This is an abelian group, which can thus be decomposed as a direct sum of a free part $\check{F}^2 = \mathbb{Z}^{b^2}$ and a torsion part $\check{T}^2 = \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_n}$.

This exact sequence shows that the space of DB cohomology classes can be thought as a set of connected fibers over the discrete net constituted by \check{H}^2 and inside which it is possible to move thanks to the elements of $\Omega^1/\Omega_{\mathbb{Z}}^1$ (see Figure 1.1).

Another important remark that can be made at this step is that the gauge classes of flat connections are in bijection with the elements of $\check{H}^2 \simeq H_1$ [Guadagnini and Thuillier, 2014]. The correspondence between the gauge classes of flat connections and the elements of H_1 will be made explicit further.

The second exact sequence through which $H_{\text{DB}}^{[1]}$ can be represented is:

$$0 \longrightarrow \check{H}^1(M^{(3)}, \mathbb{R}/\mathbb{Z}) \xrightarrow{j} H_{\text{DB}}^{[1]} \xrightarrow{d} \Omega_{\mathbb{Z}}^2 \longrightarrow 0, \quad (1.35)$$

with:

$$j \left| \begin{array}{ccc} \check{H}^1(M^{(3)}, \mathbb{R}/\mathbb{Z}) & \longrightarrow & H_{\text{DB}}^{[1]} \\ m_{\alpha\beta} & \longmapsto & (0, dm_{\alpha\beta}, \delta m_{\alpha\beta}) \end{array} \right. \quad (1.36)$$

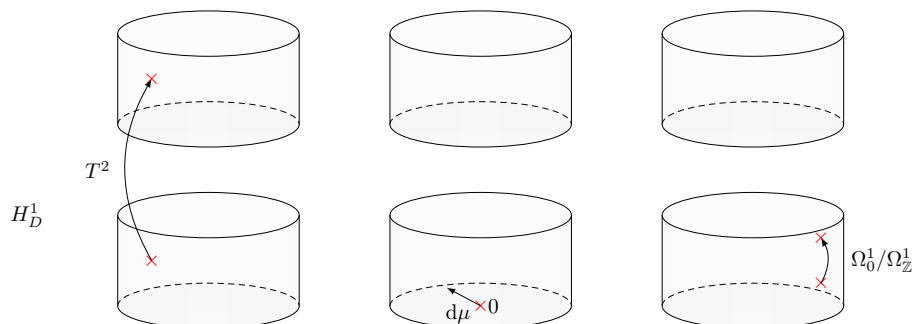


Figure 1.2: Second representation of $H_{\text{DB}}^{[1]}$.

and:

$$d \left| \begin{array}{ccc} H_{\text{DB}}^{[1]} & \longrightarrow & \Omega_{\mathbb{Z}}^2 \\ (A_\alpha, \Lambda_{\alpha\beta}, n_{\alpha\beta\gamma}) & \longmapsto & F \mid \delta F = dA_\alpha, \end{array} \right. \quad (1.37)$$

where $\check{H}^1(M^{(3)}, \mathbb{R}/\mathbb{Z})$ is the first Čech cohomology (multiplicative) group \mathbb{R}/\mathbb{Z} -valued and $\Omega_{\mathbb{Z}}^2$ is the set of closed 2-forms with integral periods. The classes of $\check{H}^1(M^{(3)}, \mathbb{R}/\mathbb{Z})$ can be regarded as the classes of equivalent $U(1)$ coordinates bundles, since they are actually the classes of equivalent transition functions of $U(1)$ coordinates bundles. The group $\check{H}^1(M^{(3)}, \mathbb{R}/\mathbb{Z})$ is actually isomorphic to the group $\check{H}^2(M^{(3)}, \mathbb{Z})$.

This exact sequence shows that the space of DB cohomology classes can be thought as clusters of cylinders, each cluster being indexed by an integer arising from \check{F}^2 . Moving from one cylinder to another of the same cluster can be performed thanks to an element of \check{T}^2 and inside a given cylinder it is possible to move radially thanks to exact 1-forms and along the axis thanks to an element of Ω^1/Ω_0^1 (see Figure 1.2).

Note that the gauge classes of flat connections are the elements of $\text{Ker } d$ in this second exact sequence, and thus, they are in bijection with the elements of $\check{H}^1(M^{(3)}, \mathbb{R}/\mathbb{Z})$.

Figure 1.3 shows how those two exact sequences carry the same information.

1.3 Operations and duality on DB cohomology classes

It is possible to define a product $\star : H_{\text{DB}}^{[p]} \times H_{\text{DB}}^{[q]} \longrightarrow H_{\text{DB}}^{[p+q+1]}$ that is graded-commutative, that is, if $\alpha \in H_{\text{DB}}^{[p]}$ and $\beta \in H_{\text{DB}}^{[q]}$, then $\alpha \star \beta = (-1)^{(p+1)(q+1)} \beta \star \alpha$, which means that, to be commutative, at least one of the classes should be of odd

1.3. Operations and duality on DB cohomology classes

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Omega^1/\Omega_{\mathbb{Z}}^1 & \longrightarrow & H_{\text{DB}}^{[1]} & \longrightarrow & \check{H}^2 \longrightarrow 0 \\
 & & \wr & & & & \wr \\
 & & \left(\Omega_0^1/\Omega_{\mathbb{Z}}^1 \right) \times \left(\Omega^1/\Omega_0^1 \right) & & & & \check{T}^2 \times \check{F}^2 \\
 & & \left(\Omega_0^1/\Omega_{\mathbb{Z}}^1 \right) \times \check{T}^2 & & & & \left(\Omega^1/\Omega_0^1 \right) \times \check{F}^2 \\
 & & \wr & & & & \wr \\
 0 & \longrightarrow & \check{H}^1(\mathbb{R}/\mathbb{Z}) & \longrightarrow & H_{\text{DB}}^{[1]} & \longrightarrow & \Omega_{\mathbb{Z}}^2 \longrightarrow 0
 \end{array}$$

Figure 1.3: Information carried by the two exact sequences.

degree.

Given a DB cohomology class $A \in H_{\text{DB}}^{[1]}$ with representative $(A_\alpha, \Lambda_{\alpha\beta}, n_{\alpha\beta\gamma})$ and a DB cohomology $B \in H_{\text{DB}}^{[1]}$ with representative $(B_\alpha, \Theta_{\alpha\beta}, m_{\alpha\beta\gamma})$, the product $A \star B \in H_{\text{DB}}^{[3]}$ is defined to be the class that admits:

$$(A_\alpha \wedge dB_\alpha, \Lambda_{\alpha\beta} dB_\beta, n_{\alpha\beta\gamma} B_\gamma, n_{\alpha\beta\gamma} \Theta_{\gamma\rho}, n_{\alpha\beta\gamma} m_{\gamma\rho\sigma}) \quad (1.38)$$

as representative.

Let z be a cycle in $M^{(3)}$, and pick in each intersection $U_{\alpha\beta}$ a point $z_{\alpha\beta}$ of z . Denote z_α the portion of z lying in U_α and connecting two points $z_{\alpha\beta}$. Such a decomposition of z is of course not unique. The integral of a DB cohomology class A with representative $(A_\alpha, \Lambda_{\alpha\beta}, n_{\alpha\beta\gamma})$ over the cycle z is defined by:

$$\oint_z A \stackrel{=}{\mathbb{Z}} \sum_\alpha \int_{z_\alpha} A_\alpha - \sum_{\alpha\beta} \int_{z_{\alpha\beta}} \Lambda_{\alpha\beta}, \quad (1.39)$$

where the $\stackrel{=}{\mathbb{Z}}$ means that the equality is satisfied in \mathbb{R}/\mathbb{Z} , that is, up to an integer. Changing the decomposition of z produces an integer and as a consequence does not change the result in \mathbb{R}/\mathbb{Z} . This integral is well defined on the DB classes

Chapter 1. Deligne-Beilinson cohomology

thanks to equations (1.19) and (1.20) in particular. The first term is nothing but the local usual holonomy in \mathbb{R}^3 . The following terms ensure the quantum gauge invariance, that is the invariance of the complex exponential of this integral under gauge transformation. In other words, they ensure that local expressions of the holonomy stick together. This integral is thus a generalized holonomy on $M^{(3)}$, which will be the observable of the CS and BF theories generalized on $M^{(3)}$. This definition ensures quantum gauge invariance, that is, invariance of the complex exponential of the holonomy under gauge transformations of the local fields A_α .

The integral over $M^{(3)}$ of $A \star B$ is defined as:

$$\begin{aligned} \int_{M^{(3)}} A \star B \stackrel{\mathbb{Z}}{=} & \sum_{\alpha} \int_{U_{\alpha}} A_{\alpha} \wedge dB_{\alpha} - \sum_{\alpha\beta} \int_{U_{\alpha\beta}} \Lambda_{\alpha\beta} dB_{\beta}, \\ & + \sum_{\alpha\beta\gamma} \int_{U_{\alpha\beta\gamma}} n_{\alpha\beta\gamma} B_{\gamma} - \sum_{\alpha\beta\gamma\rho} \int_{U_{\alpha\beta\gamma\rho}} n_{\alpha\beta\gamma} \Theta_{\gamma\rho}. \end{aligned} \quad (1.40)$$

This integral is also well defined on the DB classes. The first term is nothing but the the local BF classical action (or CS if $A = B$) in \mathbb{R}^3 . The following terms ensure the quantum gauge invariance. In other words, they ensure that local expressions of the action stick together. This integral provides thus a generalized action on $M^{(3)}$.

It should be noted that:

$$\left| \begin{array}{ll} Z_1 \times H_{\text{DB}}^{[1]} & \longrightarrow \mathbb{R}/\mathbb{Z} \\ (z, A) & \longmapsto \oint_z A \end{array} \right. \quad (1.41)$$

defines a bilinear pairing from the space Z_1 of 1-cycles and the space of DB cohomology classes (both considered as \mathbb{Z} -modules) in \mathbb{R}/\mathbb{Z} as well as:

$$\left| \begin{array}{ll} H_{\text{DB}}^{[1]} \times H_{\text{DB}}^{[1]} & \longrightarrow \mathbb{R}/\mathbb{Z} \\ (A, B) & \longmapsto \int_{M^{(3)}} A \star B. \end{array} \right. \quad (1.42)$$

Starting from that remark, and for later convenience, the Pontrjagin dual $(X)^{\#} = \text{Hom}(X, \mathbb{R}/\mathbb{Z})$ of a group X has to be considered. Regarding Hom as a functor, it can be shown that the following sequences are exact:

$$0 \longrightarrow (\check{H}^2)^{\#} \longrightarrow (H_{\text{DB}}^{[1]})^{\#} \longrightarrow (\Omega^1/\Omega_{\mathbb{Z}}^1)^{\#} \longrightarrow 0 \quad (1.43)$$

and:

$$0 \longrightarrow (\Omega_{\mathbb{Z}}^2)^{\#} \longrightarrow (H_{\text{DB}}^{[1]})^{\#} \longrightarrow (\check{H}^1(M^{(3)}, \mathbb{R}/\mathbb{Z}))^{\#} \longrightarrow 0. \quad (1.44)$$

Moreover, the information of the first two exact sequences is included in those two new ones, as shown in Figure 1.4. The Pontrjagin dual is a generalization to distributional objects. Finally, $Z_1 \subset (H_{\text{DB}}^{[1]})^{\#}$ in the sense of equation (1.41).

1.4. Decomposition of DB cohomology classes

Hence, given a cycle γ its associated class in $(H_{\text{DB}}^{[1]})^\#$ is defined to be the unique element η_γ such that:

$$\forall A \in H_{\text{DB}}^{[1]}, \int_\gamma A \stackrel{\mathbb{Z}}{=} \int_{M^{(3)}} A \star \eta_\gamma. \quad (1.45)$$

More details on this construction can be found in [Bauer et al., 2005].

Consider two such classes η_{γ_1} and η_{γ_2} in $(H_{\text{DB}}^{[1]})^\#$ associated with two cycles γ_1 and γ_2 . The so-called **zero regularization** consists in setting:

$$\int_{M^{(3)}} \eta_{\gamma_1} \star \eta_{\gamma_2} \stackrel{\mathbb{Z}}{=} 0. \quad (1.46)$$

Indeed, such a DB product comes down to an ill-defined product of distributions. A convention is thus required to be able to perform computations.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Omega^1/\Omega_{\mathbb{Z}}^1 & \longrightarrow & H_{\text{DB}}^{[1]} & \longrightarrow & \check{H}^2 \longrightarrow 0 \\
 & & \cap & & \cap & & \wr \\
 0 & \longrightarrow & (\Omega_{\mathbb{Z}}^2)^\# & \longrightarrow & (H_{\text{DB}}^{[1]})^\# & \longrightarrow & (\check{H}^1(\mathbb{R}/\mathbb{Z}))^\# \longrightarrow 0 \\
 & & & & & & \\
 0 & \longrightarrow & \check{H}^1(\mathbb{R}/\mathbb{Z}) & \longrightarrow & H_{\text{DB}}^{[1]} & \longrightarrow & \Omega_{\mathbb{Z}}^2 \longrightarrow 0 \\
 & & \wr & & \cap & & \cap \\
 0 & \longrightarrow & (\check{H}^2)^\# & \longrightarrow & (H_{\text{DB}}^{[1]})^\# & \longrightarrow & (\Omega^1/\Omega_{\mathbb{Z}}^1)^\# \longrightarrow 0
 \end{array}$$

Figure 1.4: Inclusion of the two exact sequences in their dual.

1.4 Decomposition of DB cohomology classes

The structure of DB cohomology classes is such that each class A can be decomposed as the sum of an origin indexed on the cohomology of $M^{(3)}$ (basis of the discrete

Chapter 1. Deligne-Beilinson cohomology

fiber bundle of DB cohomology classes) and a translation taken in $\Omega^1/\Omega_{\mathbb{Z}}^1$:

$$A = A_{\mathbf{a}}^0 + \omega, \quad \mathbf{a} \in \check{H}^2. \quad (1.47)$$

In this equality ω denotes the DB cohomology class that admits $(\omega_{\alpha}, 0, 0)$ as representative, ω_{α} being ω restricted to the open set U_{α} .

The result of functional integrals over the space of DB cohomology classes will not depend on the choice of the origins, but the complexity of the computations will. Thus, the goal is to find convenient origins with algebraic properties that will make it possible to perform computations easily.

Concerning the translations, $\Omega^1/\Omega_{\mathbb{Z}}^1$ can be included in a short exact sequence:

$$0 \longrightarrow \Omega_0^1/\Omega_{\mathbb{Z}}^1 \longrightarrow \Omega^1/\Omega_{\mathbb{Z}}^1 \longrightarrow \Omega^1/\Omega_0^1 \longrightarrow 0. \quad (1.48)$$

Hence, it can be decomposed (non canonically) as:

$$\Omega^1/\Omega_{\mathbb{Z}}^1 \simeq (\Omega^1/\Omega_0^1) \times (\Omega_0^1/\Omega_{\mathbb{Z}}^1), \quad (1.49)$$

where Ω_0^1 denotes the set of closed 1-forms. Furthermore:

$$\Omega_0^1/\Omega_{\mathbb{Z}}^1 \simeq (\mathbb{R}/\mathbb{Z})^{b_1}, \quad (1.50)$$

b_1 being the first Betti number. The elements $\omega_0 \in \Omega_0^1/\Omega_{\mathbb{Z}}^1$ will be called **zero modes**. This terminology has been chosen since it will appear in the next chapter that the functional measure considered is invariant under translation of those elements. With this decomposition:

$$\forall (\omega_0, \omega) \in (\Omega_0^1/\Omega_{\mathbb{Z}}^1) \times (\Omega^1/\Omega_{\mathbb{Z}}^1), \quad \int_{M^{(3)}} \omega \star \omega_0 \stackrel{\mathbb{Z}}{=} 0. \quad (1.51)$$

As shown further, the zero modes will consist in a residual invariance of the functional measure that will be used in path integrals. Moreover:

$$\omega_0 \in \Omega_0^1/\Omega_{\mathbb{Z}}^1 \simeq (\mathbb{R}/\mathbb{Z})^{b_1} \simeq \check{F}^1 \simeq F_2 = H_2, \quad (1.52)$$

\check{F}^1 being the free part of \check{H}^1 and F_2 being the free part of H_2 . The last equality occurs since H_2 has no torsion. This part has actually no volume, in the sense that the integral over this part is not infinite but simply 1. The zero modes are of capital importance. In the computation of the partition functions, they impose the free part to be 0 so that it does not couple to the elements $\omega_{\perp} \in \Omega^1/\Omega_0^1$. In the computation of the expectation value observables, they impose a decoupling between the free part and the torsion part of the links.

Consider generators z_a of the free part of the homology of $M^{(3)}$. Then, by Pontrjagin duality, consider the unique class η_{z_a} in $(H_{\text{DB}}^{[1]})^{\#}$ associated with z_a in

1.4. Decomposition of DB cohomology classes

the sense of equation (1.45). Thus, for a fiber over $\mathbf{m} = \sum_a m^a z_a \in F_1 \simeq \check{F}^2$ the origin of the fiber will be the element:

$$A_{\mathbf{m}}^0 = \sum_a m^a \eta_{z_a} \in \left(H_{\text{DB}}^{[1]} \right)^{\#}. \quad (1.53)$$

Thanks to the zero regularization introduced before in equation (1.46):

$$\int_{M^{(3)}} A_{\mathbf{m}}^0 \star A_{\mathbf{n}}^0 \stackrel{\mathbb{Z}}{=} 0. \quad (1.54)$$

Finally, decomposing $\omega_0 \in \Omega_0^1 / \Omega_{\mathbb{Z}}^1$ as $\omega_0 = \sum_b \theta_b \rho^b$ with $\oint_{z_a} \rho^b = \delta_a^b$, then:

$$\int_{M^{(3)}} A_{\mathbf{m}}^0 \star \omega_0 \stackrel{\mathbb{Z}}{=} \mathbf{m} \cdot \boldsymbol{\theta}. \quad (1.55)$$

Consider now a generator τ_a of the component \mathbb{Z}_{p_a} of the torsion part of the homology of $M^{(3)}$. This means that τ_a is the boundary of no surface, but $p_a \tau_a$ is. Consider now the element $\tilde{\eta}_{\tau_a} \in \left(H_{\text{DB}}^{[1]} \right)^{\#}$ which admits as representative²:

$$\left(0, \frac{m_{\alpha\beta}}{p_a}, n_{\alpha\beta\gamma} \right), \quad (1.56)$$

where $p_a n_{\alpha\beta\gamma} = m_{\beta\gamma} - m_{\alpha\gamma} + m_{\alpha\beta}$. Thus, for a fiber over $\boldsymbol{\kappa} = \sum_a \kappa^a \tau_a \in T_1 \simeq \check{T}^2$ the element:

$$A_{\boldsymbol{\kappa}}^0 = \sum_a \kappa^a \tilde{\eta}_{\tau_a} \in \left(H_{\text{DB}}^{[1]} \right)^{\#} \quad (1.57)$$

will be considered as origin. This choice has several advantages. Indeed, the definition of the product over DB cohomology classes (1.40) leads to:

$$\int_{M^{(3)}} A_{\boldsymbol{\kappa}_1}^0 \star A_{\boldsymbol{\kappa}_2}^0 \stackrel{\mathbb{Z}}{=} -Q(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2), \quad (1.58)$$

where Q is the so-called linking form, which is a quadratic form over the torsion of the cohomology. Also:

$$\int_{M^{(3)}} A_{\boldsymbol{\kappa}}^0 \star A_{\mathbf{m}}^0 \stackrel{\mathbb{Z}}{=} 0 \quad (1.59)$$

for any free origin $A_{\mathbf{m}}^0$ and:

$$\int_{M^{(3)}} A_{\boldsymbol{\kappa}}^0 \star \omega \stackrel{\mathbb{Z}}{=} 0 \quad (1.60)$$

²It should be pointed out that $\tilde{\eta}_{\tau_a}$ on the one hand and η_{τ_a} as defined by equation (1.45) on the other hand are **not** the same DB cohomology class.

Chapter 1. Deligne-Beilinson cohomology

for any translation ω .

The correspondence between the elements of H_1 and the gauge classes of flat connections can be presented at this step. With $z = \sum_a m^a z_a \in F_1$ is associated the DB class that admits $(\omega_0, 0, 0)$ as representative, with $\omega_0 = \sum_a m^a \rho^a \in \Omega_0^1/\Omega_{\mathbb{Z}}^1$ and $\rho^a \in \Omega_0^1/\Omega_{\mathbb{Z}}^1$ such that $\phi_{z_a} \rho^a = \delta_a^b$. With $\tau \in T_1$ is simply associated the origin A_τ^0 .

Now, a few remarks can be made about the zero regularization. This convention is also called **regularization by framing** for the following reason. Consider a trivial cycle γ and its associated class η_γ according to equation (1.45). Then by zero regularization:

$$\int_{M^{(3)}} \eta_\gamma \star \eta_\gamma \stackrel{\mathbb{Z}}{=} 0. \quad (1.61)$$

But:

$$\int_{M^{(3)}} \eta_\gamma \star \eta_\gamma \stackrel{\mathbb{Z}}{=} \int_{M^{(3)}} j_\gamma \wedge dj_\gamma, \quad (1.62)$$

j_γ being de Rham current associated with γ . Hence, this quantity can be interpreted as a self-linking as the linking between two trivial cycles γ_1 and γ_2 is defined to be:

$$\ellk_{M^{(3)}}(\gamma_1, \gamma_2) = \int_{M^{(3)}} j_{\gamma_1} \wedge dj_{\gamma_2}, \quad (1.63)$$

j_{γ_1} and j_{γ_2} being de Rham currents associated with γ_1 and γ_2 respectively. Consider γ not to be a cycle but a (closed) ribbon, that is, a 2-dimensional object. Then, this ribbon can be twisted. The self-linking can be defined to be this twist as the width of the ribbon goes to zero. According to this attempt to find a definition, the self-linking is ill-defined, but it is anyway an integer. Hence, in \mathbb{R}/\mathbb{Z} , it is zero.

An analogous consideration can be proposed for a pure torsion cycle τ . A representative of its associated class η_τ according to equation (1.45) is:

$$\left(j_\alpha^\tau, \frac{m_{\alpha\beta}}{p}, n_{\alpha\beta\gamma} \right), \quad (1.64)$$

j_α^τ being the de Rham current associated with τ restricted to the open set U_α . Hence, by definition of the DB product (1.40):

$$\int_{M^{(3)}} \eta_\tau \star \eta_\tau \stackrel{\mathbb{Z}}{=} \sum_\alpha \int_{U_\alpha} j_\alpha^\tau \wedge dj_\alpha^\tau - \frac{n_{\alpha\beta\gamma} \cup m_{\alpha\beta}}{p} \quad (1.65)$$

and as a consequence, zero regularization implies:

$$\sum_\alpha \int_{U_\alpha} j_\alpha^\tau \wedge dj_\alpha^\tau \stackrel{\mathbb{Z}}{=} \frac{n_{\alpha\beta\gamma} \cup m_{\alpha\beta}}{p}, \quad (1.66)$$

1.4. Decomposition of DB cohomology classes

that is:

$$\sum_{\alpha} \int_{U_{\alpha}} j_{\alpha}^{\tau} \wedge dj_{\alpha}^{\tau} \stackrel{\mathbb{Z}}{=} Q(\boldsymbol{\tau}, \boldsymbol{\tau}), \quad (1.67)$$

according to equation (1.58).

Taking those properties into account, it is remarkable that very few contributions remain in the computation of the integral of the product of two DB classes. Indeed, if $A = A_{\mathbf{m}_1}^0 + A_{\boldsymbol{\kappa}_1}^0 + \alpha_{\perp} + \alpha_0$ and $B = B_{\mathbf{m}_2}^0 + B_{\boldsymbol{\kappa}_2}^0 + \beta_{\perp} + \beta_0$, then:

$$\begin{aligned} \int_{M^{(3)}} A \star B \stackrel{\mathbb{Z}}{=} & \int_{M^{(3)}} A_{\mathbf{m}_1}^0 \star \beta_{\perp} + \int_{M^{(3)}} B_{\mathbf{m}_2}^0 \star \alpha_{\perp} \\ & + \mathbf{m}_1 \cdot \boldsymbol{\theta}_{\beta} + \mathbf{m}_2 \cdot \boldsymbol{\theta}_{\alpha} - Q(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) + \alpha_{\perp} \star \beta_{\perp} \end{aligned} \quad (1.68)$$

In the computation of the partition functions, the (functional) integral over the zero modes will ensure the complete decoupling between the remaining terms. Indeed, it will impose $\mathbf{m}_1 = \mathbf{m}_2 = 0$, so that only $-Q(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) + \int_{M^{(3)}} \alpha_{\perp} \star \beta_{\perp}$ will remain. The topological quantities of interest will be the contributions provided by $-Q(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2)$, in the sense that these contributions will be linked to the RT invariant in the case of the abelian CS partition function and to the TV invariant in the case of the abelian BF partition function. The contributions provided by $\int_{M^{(3)}} \alpha_{\perp} \star \beta_{\perp}$ will be eliminated by normalization. Hence, taking into account the correspondence established earlier, the gauge classes of flat connections is of fundamental importance for such a computation.

Chapter 2

Abelian Chern-Simons theory

Contents

Introduction (English version)	76
Introduction (version française)	79
2.1 Computation of Z_{CS_k}	82
2.2 Abelian expectation value of observables	84

Introduction (English version)

In the 70s, the works of J. Cheeger, S. S. Chern and J. Simons on secondary classes [Chern and Simons, 1974, Cheeger and Simons, 1985, Chern, 1979, Koszul, 1975] led them to introduce what is called now the Chern-Simons (CS) invariant, that can be written in the $SU(N)$ case as:

$$S_{\text{CS}}(A) = \frac{1}{8\pi^2} \int_{M^{(3)}} \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \quad (2.1)$$

where A is a $SU(N)$ -connection and $M^{(3)}$ is a closed manifold of dimension 3. Actually, to be written this way, A has to be an object which is globally defined, that is, which is a global 1-form with coefficients in the Lie algebra of $SU(N)$. Given a gauge class, finding such a representative is always possible since $SU(N)$ fiber bundles over a closed manifold of dimension 3 are all trivializable. The theory of secondary classes is out of purpose here, but it should be mentioned that $S_{\text{CS}}(A)$ is a primitive of the quadratic term in F_A that appears in the expansion of $\det(\text{Id} + \frac{iF_A}{2\pi})$ where $F_A = dA + \frac{1}{2}A \wedge A$.

Considered as an element of \mathbb{R}/\mathbb{Z} , $S_{\text{CS}}(A)$ does not depend on the (globally defined) representative of the gauge class of A , that is:

$$A \rightarrow A^g = g^{-1}Ag + g^{-1}dg \Rightarrow S_{\text{CS}}(A) \rightarrow S_{\text{CS}}(A^g) = S_{\text{CS}}(A) + n, \quad (2.2)$$

with $n \in \mathbb{Z}$. This integer n arises as:

$$n = -\frac{1}{24\pi^2} \int_{M^{(3)}} L_{\text{WZ}}(g), \quad (2.3)$$

where:

$$L_{\text{WZ}}(g) = \text{Tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg). \quad (2.4)$$

is a so-called Wess-Zumino (WZ) Lagrangian. The integer n actually labels the homotopy class of g .

Still in the 70s, physicists as R. Jackiw, C. Rebbi [Jackiw and Rebbi, 1976] or W. Marciano and H. Pagels [Marciano and Pagels, 1978] noticed the appearance of $S_{\text{CS}}(A)$ in the theory of instantons of Yang-Mills theory in S^3 . Indeed, if the configuration A is an instanton, then $S_{\text{CS}}(A)$ labels its homotopy class. The CS invariant and its properties have then been studied carefully as well as the WZ term. Considering those properties, it has been tried to use the CS invariant to introduce a mass term in gravity in dimension 3 [Deser et al., 1982].

Since $S_{\text{CS}}(A) \in \mathbb{R}/\mathbb{Z}$ then $S_{\text{CS}_k}(A) = kS_{\text{CS}}(A) \in \mathbb{R}/\mathbb{Z}$ with coupling constant $k \in \mathbb{Z}$ can be considered as a quantum action as well. In 1989, Witten showed that a QFT using such a quantum action is related to Jones polynomials [Witten, 1989].

Since then, various extensions of the $SU(N)$ CS theory have been deeply studied such as supersymmetric versions, in various fields of physics.

However, the abelian case, that is to say, replacing $SU(N)$ by $U(1)$, has often been considered as trivial and has rarely been treated carefully. Yet, this case is very different from the non-abelian one and, to keep the original ideas of secondary classes, its definition cannot be considered to be the simple abelianization of the $SU(N)$ case. Indeed, the determinant $\det(\text{Id} + \frac{iF_A}{2\pi})$ is just $1 + \frac{iF_A}{2\pi}$ in the abelian case and thus has no quadratic term, hence the obligation to define CS action carefully following another rule and keeping the idea of secondary classes. It should not be forgotten also that $U(1)$ fiber bundles over a closed manifold of dimension 3 are not trivializable which means that, in general, it is not possible to select an element globally defined in the gauge class of a given connection.

Consider first the abelian holonomy¹:

$$h_\gamma(A) = e^{2i\pi \oint_\gamma A}, \quad (2.5)$$

where the integral $\oint_\gamma A$ has to be understood regarding DB theory² presented in chapter 1 of this part, that is, A is actually a DB cohomology class. This cohomology class is associated with a curvature 2-form F with integral periods. The square of F is thus a 4-form, still with integral periods. Hence, consider a manifold $N^{(4)}$ of dimension 4 divided in two parts $N^{(4)} = N_L^{(4)} \cup N_R^{(4)}$ such that $\partial N_R^{(4)} = M^{(3)} = -\partial N_L^{(4)}$. Consider then a good covering $(U_\alpha)_{\alpha \in I}$ of $M^{(3)}$ and extend it to a good covering $(V_\alpha)_{\alpha \in I}$ of $N^{(4)}$ so that $\forall \alpha \in I, \partial V_{\alpha,R} = U_\alpha = -\partial V_{\alpha,L}$. Consider now local gauge potentials A_α in each U_α and assume that they admit a continuation in the $V_{\alpha,R}$ and $V_{\alpha,L}$. Then:

$$\begin{cases} \sum_{\alpha \in I} \int_{U_\alpha = \partial V_{\alpha,R}} A_\alpha \wedge dA_\alpha = \sum_{\alpha \in I} \int_{V_{\alpha,R}} F_\alpha \wedge F_\alpha, \\ \sum_{\alpha \in I} \int_{U_\alpha = \partial V_{\alpha,L}} A_\alpha \wedge dA_\alpha = \sum_{\alpha \in I} \int_{V_{\alpha,L}} F_\alpha \wedge F_\alpha. \end{cases} \quad (2.6)$$

Comparing the results obtained looking from the right side and looking from the left side, then:

$$\Delta = \sum_{\alpha \in I} \int_{V_{\alpha,R}} F_\alpha \wedge F_\alpha - \sum_{\alpha \in I} \int_{V_{\alpha,L}} F_\alpha \wedge F_\alpha = \int_{N^{(4)}} F \wedge F \in \mathbb{Z}. \quad (2.7)$$

¹This is actually the observable that arises in Aharonov-Bohm effect.

²Usually, $h_\gamma(\hat{A}) = e^{i \oint_\gamma \hat{A}}$. It must be borne in mind that because of the structure of the space of DB cohomology classes, a class cannot be divided by any number. This choice of normalization is not a simple division of the class A by 2π that would have no meaning. The factor 2π arises because of the definition of DB cohomology classes.

Chapter 2. Abelian Chern-Simons theory

Therefore, $\sum_{\alpha \in I} \int_{U_\alpha = bV_\alpha} A_\alpha \wedge dA_\alpha$ has to be considered in \mathbb{R}/\mathbb{Z} and this is the starting point of the construction of the relevant quantity studied here. Constraining this quantity to be gauge invariant (up to an integer) has been treated in chapter 1 of this part and finally, for any closed manifold $M^{(3)}$ of dimension 3, the correct quantity to consider is:

$$S_{\text{CS}}(A) = \int_{M^{(3)}} A \star A \in \mathbb{R}/\mathbb{Z}, \quad (2.8)$$

where A is a DB cohomology class associated with the gauge class of a $U(1)$ connection and the operation \star is the DB product³.

The purpose of the first section is to compute mainly the partition function of the abelian CS theory, that is:

$$Z_{\text{CS}_k}(M^{(3)}) = \frac{1}{\mathcal{N}_{\text{CS}_k}} \int_{\mathcal{H}} \mathcal{D}A e^{2i\pi S_{\text{CS}_k}(A)}, \quad (2.9)$$

where:

$$S_{\text{CS}_k}(A) = k S_{\text{CS}}(A) = k \int_{M^{(3)}} A \star A \quad (2.10)$$

and $k \in \mathbb{Z}$ since $\int_{M^{(3)}} A \star A \in \mathbb{R}/\mathbb{Z}$. It is remarkable that the local expression of $A \star A$ is nothing but $A_\alpha \wedge dA_\alpha$ which is the expression of the abelian CS Lagrangian in \mathbb{R}^3 . The space \mathcal{H} should be the space of abelian connections on fiber bundles over $M^{(3)}$, that is, as shown previously, the space $H_{\text{DB}}^{[1]}$. However, to use the origins defined in chapter 1 of this part, $(H_{\text{DB}}^{[1]})^\#$ or at least a subspace of $(H_{\text{DB}}^{[1]})^\#$ that contains $H_{\text{DB}}^{[1]}$ and Z_1 has to be considered. The measure \mathcal{D} is purely formal and a few properties will be postulated to be able to perform the computation of Z_{CS_k} , the normalization $\mathcal{N}_{\text{CS}_k}$ being here to cancel the intrinsic divergency of the functional integral. This will make it possible then to relate this result with the abelian RT invariant.

In the second section, the expectation value of observables are computed and related to that of the abelian RT theory. At this point the so-called surgery formula which relates computations performed in S^3 and computations performed directly in $M^{(3)}$ will be reminded.

This chapter gathers the results that can be found with proofs and details in [Guadagnini and Thuillier, 2013]. Those results are presented here with unified notations in order to propose through this study an overview as complete as possible.

³If the abelian CS theory is defined in S^3 , then A can be chosen to be an abelian connection globally defined and $S_{\text{CS}_k}(A) = k \int_{S^3} A \wedge dA = \frac{k}{4\pi^2} \int_{S^3} \tilde{A} \wedge d\tilde{A}$ (see previous footnote). This means that the abelian case and the non-abelian case have different normalizations. The abelian case cannot therefore be deduced by an abelianization of the non-abelian case, provided this abelian case is built following the same idea as the non-abelian one regarding the theory of secondary classes.

Introduction (version française)

Dans les années 1970, le travail de J. Cheeger, S. S. Chern et J. Simons sur les classes secondaires [Chern and Simons, 1974, Cheeger and Simons, 1985, Chern, 1979, Koszul, 1975] a amené ceux-ci à introduire ce qui est maintenant appelé l'invariant de Chern-Simons (CS), qui peut s'écrire, dans le cas $SU(N)$ comme :

$$S_{\text{CS}}(A) = \frac{1}{8\pi^2} \int_{M^{(3)}} \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \quad (2.1)$$

où A est une connexion $SU(N)$ et $M^{(3)}$ est une variété fermée de dimension 3. En fait, pour s'écrire de cette manière, A doit être un objet globalement défini, c'est-à-dire, une 1-forme globale à coefficients dans l'algèbre de Lie de $SU(N)$. Étant donnée une classe de jauge, trouver un tel représentant est toujours possible puisque les fibrés $SU(N)$ au-dessus des variétés fermées de dimension 3 sont tous trivialisables. La théorie des classes secondaires est hors de propos ici, mais il doit être mentionné que $S_{\text{CS}}(A)$ est une primitive du terme quadratique en F_A qui apparaît dans le développement de $\det \left(\text{Id} + \frac{iF_A}{2\pi} \right)$ où $F_A = dA + \frac{1}{2} A \wedge A$.

Considéré comme un élément de \mathbb{R}/\mathbb{Z} , $S_{\text{CS}}(A)$ ne dépend pas du représentant (globalement défini) de la classe de jauge de A , c'est-à-dire :

$$A \rightarrow A^g = g^{-1}Ag + g^{-1}dg \Rightarrow S_{\text{CS}}(A) \rightarrow S_{\text{CS}}(A^g) = S_{\text{CS}}(A) + n, \quad (2.2)$$

avec $n \in \mathbb{Z}$. Cet entier n survient comme :

$$n = -\frac{1}{24\pi^2} \int_{M^{(3)}} L_{\text{WZ}}(g), \quad (2.3)$$

où :

$$L_{\text{WZ}}(g) = \text{Tr} \left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg \right). \quad (2.4)$$

est le lagrangien dit de Wess-Zumino (WZ). L'entier n caractérise en fait la classe d'homotopie de g .

Toujours dans les années 1970, des physiciens tels que R. Jackiw, C. Rebbi [Jackiw and Rebbi, 1976] ou W. Marciano et H. Pagels [Marciano and Pagels, 1978] ont noté l'apparition de $S_{\text{CS}}(A)$ dans la théorie des instantons des théories de Yang-Mills dans S^3 . En effet, si la configuration A est un instanton, alors $S_{\text{CS}}(A)$ caractérise sa classe d'homotopie. L'invariant CS et ses propriétés ont alors été étudiées soigneusement de même que le terme WZ. Considérant ces propriétés, l'invariant CS a été utilisé pour tenter d'introduire un terme de masse dans la gravité en dimension 3 [Deser et al., 1982].

Étant donné que $S_{\text{CS}}(A) \in \mathbb{R}/\mathbb{Z}$, il s'en suit que $S_{\text{CS}_k}(A) = kS_{\text{CS}}(A) \in \mathbb{R}/\mathbb{Z}$, avec une constante de couplage $k \in \mathbb{Z}$, peut être également considérée comme action

Chapter 2. Abelian Chern-Simons theory

quantique. En 1989, Witten montra qu'une QFT utilisant une telle action quantique était liée aux polynômes de Jones [Witten, 1989].

Depuis lors, divers extensions de la théorie CS $SU(N)$ ont été étudiées en profondeur telles que des extensions supersymétriques, dans divers domaines de la physique.

Cependant, le cas abélien, c'est-à-dire, en remplaçant $SU(N)$ par $U(1)$, a souvent été considéré comme trivial et a rarement été traité soigneusement. Pourtant, ce cas est très différent du cas non-abélien et, afin de garder l'idée originale des classes secondaires, sa définition ne peut pas être considérée comme étant la simple abélianisation du cas $SU(N)$. En effet, le déterminant $\det\left(\text{Id} + \frac{iFA}{2\pi}\right)$ vaut simplement $1 + \frac{iFA}{2\pi}$ dans le cas abélien et par conséquent, il ne contient pas de terme quadratique, ce qui impose de définir l'action CS soigneusement en suivant une autre règle tout en gardant l'idée des classes secondaires. Il ne doit pas être oublié non plus que les fibrés $U(1)$ au-dessus d'une variété fermée de dimension 3 ne sont pas trivialisables, ce qui signifie qu'en général, il n'est pas possible de sélectionner un élément globalement défini dans la classe de jauge d'une connexion donnée.

Tout d'abord, soit l'holonomie abélienne¹ :

$$h_\gamma(A) = e^{2i\pi \oint_\gamma A}, \quad (2.5)$$

où l'intégrale $\oint_\gamma A$ doit être entendue au sens de la théorie DB² présentée au chapitre 1 de cette partie, c'est-à-dire, A est en fait une classe de cohomologie DB. Cette classe de cohomologie est associée à une 2-forme de courbure F à périodes entières. Le carré de F est donc une 4-forme, toujours à périodes entières. Soit $N^{(4)}$ une variété de dimension 4 découpée en deux morceaux $N^{(4)} = N_L^{(4)} \cup N_R^{(4)}$ tels que $\partial N_R^{(4)} = M^{(3)} = -\partial N_L^{(4)}$. Ensuite, soit $(U_\alpha)_{\alpha \in I}$ un bon recouvrement de $M^{(3)}$ et soit $(V_\alpha)_{\alpha \in I}$ une extension de ce bon recouvrement à un bon recouvrement de $N^{(4)}$ tel que $\forall \alpha \in I, \partial V_{\alpha,R} = U_\alpha = -\partial V_{\alpha,L}$. Soient maintenant A_α , des potentiels de jauge locaux dans chaque U_α qui admettent des prolongements à $V_{\alpha,R}$ et $V_{\alpha,L}$. Alors :

$$\begin{cases} \sum_{\alpha \in I} \int_{U_\alpha = \partial V_{\alpha,R}} A_\alpha \wedge dA_\alpha = \sum_{\alpha \in I} \int_{V_{\alpha,R}} F_\alpha \wedge F_\alpha, \\ \sum_{\alpha \in I} \int_{U_\alpha = \partial V_{\alpha,L}} A_\alpha \wedge dA_\alpha = \sum_{\alpha \in I} \int_{V_{\alpha,L}} F_\alpha \wedge F_\alpha. \end{cases} \quad (2.6)$$

En comparant les résultats obtenus en regardant du côté droit et du côté gauche,

¹Il s'agit en fait de l'observable intervenant dans l'effet Aharonov-Bohm.

²Habituellement, $h_\gamma(\tilde{A}) = e^{i \oint_\gamma \tilde{A}}$. Il doit être gardé en tête qu'à cause de la structure de l'espace des classes de cohomologie DB, une classe ne peut pas être divisée par un nombre quelconque. Ce choix de normalisation n'est pas une simple division de la classe de A par 2π qui n'aurait aucun sens. Le facteur 2π survient à cause de la définition des classes de cohomologie DB.

il vient :

$$\Delta = \sum_{\alpha \in I} \int_{V_{\alpha,R}} F_{\alpha} \wedge F_{\alpha} - \sum_{\alpha \in I} \int_{V_{\alpha,L}} F_{\alpha} \wedge F_{\alpha} = \int_{N^{(4)}} F \wedge F \in \mathbb{Z}. \quad (2.7)$$

Par conséquent, $\sum_{\alpha \in I} \int_{U_{\alpha}=bV_{\alpha}} A_{\alpha} \wedge dA_{\alpha}$ doit être considérée dans \mathbb{R}/\mathbb{Z} et c'est le point de départ de la construction des quantités pertinentes étudiées ici. Contraindre cette quantité à être invariante de jauge (à un entier près) a été traité au chapitre 1 de cette partie et finalement, pour toute variété fermée $M^{(3)}$ de dimension 3, la quantité correcte à considérer est :

$$S_{\text{CS}}(A) = \int_{M^{(3)}} A \star A \in \mathbb{R}/\mathbb{Z}, \quad (2.8)$$

où A est une classe de cohomologie DB associée à la classe de jauge d'une connexion $U(1)$ et l'opération \star est le produit DB³.

Le but de la première section est de calculer principalement la fonction de partition de la théorie CS abélienne, c'est-à-dire :

$$Z_{\text{CS}_k}(M^{(3)}) = \frac{1}{\mathcal{N}_{\text{CS}_k}} \int_{\mathcal{H}} \mathcal{D}A e^{2i\pi S_{\text{CS}_k}(A)}, \quad (2.9)$$

où :

$$S_{\text{CS}_k}(A) = k S_{\text{CS}}(A) = k \int_{M^{(3)}} A \star A \quad (2.10)$$

et $k \in \mathbb{Z}$ puisque $\int_{M^{(3)}} A \star A \in \mathbb{R}/\mathbb{Z}$. Il est remarquable que l'expression locale de $A \star A$ n'est rien d'autre que $A_{\alpha} \wedge dA_{\alpha}$ qui est l'expression du lagrangien CS abélien dans \mathbb{R}^3 . L'espace \mathcal{H} devrait être l'espace des connexions abéliennes sur les fibrés $U(1)$ au-dessus de $M^{(3)}$, c'est-à-dire, comme montré précédemment, l'espace $H_{\text{DB}}^{[1]}$. Cependant, pour utiliser les origines définies au chapitre 1 de cette partie, $(H_{\text{DB}}^{[1]})^{\#}$ ou du moins un sous-espace de $(H_{\text{DB}}^{[1]})^{\#}$ contenant $H_{\text{DB}}^{[1]}$ et Z_1 doit être considéré. La mesure \mathcal{D} est purement formelle et quelques propriétés seront postulées afin de pouvoir effectuer le calcul de Z_{CS_k} , la normalisation $\mathcal{N}_{\text{CS}_k}$ étant là afin d'absorber les divergences intrinsèques de l'intégrale fonctionnelle. Ceci permettra alors de relier ce résultat avec l'invariant RT abélien.

³Si la théorie CS abélienne est définie dans S^3 , alors A peut être choisi comme étant une connexion abélienne globalement définie et $S_{\text{CS}_k}(A) = k \int_{S^3} A \wedge dA = \frac{k}{4\pi^2} \int_{S^3} \tilde{A} \wedge d\tilde{A}$ (voir la note de bas de page précédente). Cela signifie que le cas abélien et le cas non-abélien ont une normalisation différente. Le cas abélien ne peut par conséquent pas se déduire d'une abélianisation du cas non-abélien, sous réserve de définir le cas abélien en suivant la même idée que le cas non-abélien du point de vue de la théorie des classes secondaires.

Chapter 2. Abelian Chern-Simons theory

Dans la deuxième section, les valeurs moyennes d'observables sont calculées et reliées à celles de la théorie RT abélienne. Là, la formule dite de chirurgie, qui relie des calculs dans S^3 à des calculs effectués directement sur la variété $M^{(3)}$ sera rappelée.

Ce chapitre rassemble des résultats qui peuvent être trouvés avec leur démonstration et plus de détails dans [Guadagnini and Thuillier, 2013]. Ces résultats sont présentés ici avec des notations unifiées afin de proposer à travers ce travail un panorama le plus complet possible.

2.1 Computation of Z_{CS_k}

As mentioned in the introduction, the purpose of this first section is first to compute the partition function of the abelian CS theory, that is:

$$Z_{\text{CS}_k}(M^{(3)}) = \frac{1}{\mathcal{N}_{\text{CS}_k}} \int_{\mathcal{H}} \mathcal{D}A e^{2i\pi S_{\text{CS}_k}(A)}, \quad (2.11)$$

where:

$$S_{\text{CS}_k}(A) = k \int_{M^{(3)}} A \star A \quad (2.12)$$

and $k \in \mathbb{Z}$ since $\int_{M^{(3)}} A \star A \in \mathbb{R}/\mathbb{Z}$.

It can be noticed that the partition function is a sum of complex exponentials of a quadratic term. This is formally reminiscent of the shape of the abelian RT invariant presented in chapter 2 of the first part.

The functional measure considered here is:

$$d\mu_{\text{CS}_k}(A) = \mathcal{D}A e^{2i\pi S_{\text{CS}_k}(A)}. \quad (2.13)$$

The main hypothesis made in this study is that it follows the so-called Cameron-Martin property, that is:

$$d\mu_{\text{CS}_k}(A + \alpha) = d\mu_{\text{CS}_k}(A) e^{4i\pi k \int_{M^{(3)}} A \star \alpha} e^{2i\pi S_{\text{CS}_k}(\alpha)} \quad (2.14)$$

for a fixed connections A and a translation α .

Consider now a closed surface Σ in $M^{(3)}$ which is not the boundary of any volume. Such kind of surface cannot exist in S^3 but appears as soon as the homology of $M^{(3)}$ has a non trivial free homology. By the Poincaré duality, Σ can be associated with a de Rham current j_Σ which defines canonically the representative:

$$(j_{\Sigma, \alpha}, 0, 0) \quad (2.15)$$

and is actually in the null class of $(H_D^{[1]})^\#$. However, the representative:

$$\left(\frac{j_{\Sigma, \alpha}}{2k}, 0, 0\right) \quad (2.16)$$

is not in the null class of $(H_D^{[1]})^\#$ anymore. By a misuse of language, its class is denoted $\frac{j_\Sigma}{2k}$ and thanks to the Cameron-Martin property:

$$d\mu_{\text{CS}_k}\left(A + m \frac{j_\Sigma}{2k}\right) = d\mu_{\text{CS}_k}(A) \quad (2.17)$$

Chapter 2. Abelian Chern-Simons theory

for any $m \in \mathbb{Z}$. This is the so-called **zero mode property** for the abelian CS theory. Since there is no homological torsion in degree 2, for any $\omega \in \Omega_{\mathbb{Z}}^1$, there is a surface Σ such that $(j_{\Sigma, \alpha}, 0, 0)$ is a representative in the same class as $(\omega_{\alpha}, 0, 0)$, where ω_{α} is the restrictions of ω to the open set U_{α} . Hence, the zero mode property is true for any $\omega \in \Omega_{\mathbb{Z}}^1$.

Using the algebraic properties given in chapter 1 of this part, the abelian CS partition function can be computed exactly.

As a convention, the normalization is chosen to be:

$$\mathcal{N}_{\text{CS}_k} = \int_{(\Omega^1/\Omega_{\mathbb{Z}}^1)^{\#}} \mathcal{D}\alpha e^{2i\pi S_{\text{CS}_k}(\alpha)}, \quad (2.18)$$

which is the same as:

$$\mathcal{N}_{\text{CS}_k} = \int_{(\Omega^1/\Omega_0^1)^{\#}} \mathcal{D}\alpha e^{2i\pi S_{\text{CS}_k}(\alpha)}, \quad (2.19)$$

which corresponds to the infinite volume associated with the trivial fiber of DB bundle associated with the manifold $M^{(3)}$ in which the abelian CS theory is defined with respect to the measure induced by the CS action considered.

This trivial fiber is the only one that constitutes the DB bundle when the manifold considered is S^3 and it arises *formally* for any manifold $M^{(3)}$ of dimension 3, in the sense that $\Omega^1/\Omega_{\mathbb{Z}}^1$ appears for any manifold $M^{(3)}$ of dimension 3. However, of course, $\Omega^1/\Omega_{\mathbb{Z}}^1$ depends on $M^{(3)}$, so, to that extent, the trivial fiber is not common to all closed manifolds of dimension 3. The normalization is performed with respect to S^3 only in the sense that, with this convention, $Z_{\text{CS}_k}(S^3) = 1$.

This choice of normalization is mainly justified by the fact that it leads to establishing a link with the abelian RT invariant as defined in chapter 2 of the previous part. Usually the normalization of the RT invariant is chosen to be related to $S^1 \times S^2$. However, if the normalization is chosen with respect to S^3 then the abelian RT invariant coincides with the abelian CS partition function as shown further.

This way:

$$Z_{\text{CS}_k}(M^{(3)}) = \sum_{\tau \in T^2} e^{-2i\pi k Q(\tau, \tau)}, \quad (2.20)$$

where Q is the so-called linking form introduced in chapter 1 of this part, in equation (1.58), which is a non-degenerate quadratic form over the torsion part T^2 of $H^2(M^{(3)})$. As a matter of fact, by the standard classification of abelian groups, $T^2 = \mathbb{Z}_{p_1} \oplus \dots \oplus \mathbb{Z}_{p_n}$ with $\forall i \in \llbracket 1; n-1 \rrbracket, p_i | p_{i+1}$. The p_i will be called **torsion indices** of $M^{(3)}$. Moreover, Q can be represented by a symmetric matrix $(Q_{ij})_{1 \leq i, j \leq n} = \left(\frac{q_{ij}}{p_i} \right)_{1 \leq i, j \leq n}$ with, by symmetry, $\frac{q_{ij}}{p_i} = \frac{q_{ji}}{p_j}$.

This computation shows that the abelian CS theory written in terms of gauge classes directly on $M^{(3)}$ make it possible to obtain a topological invariant of $M^{(3)}$

2.2. Abelian expectation value of observables

by an exact computation. The topological information probed is contained in Q which is slightly stronger than the torsion of the first homology group. Indeed, for example, for a lens space, $Q = \frac{q}{p}$ while $T_1(L(p, q)) = H_1(L(p, q)) = \mathbb{Z}_p$. Also, the abelian CS theory will never see the free part of the homology.

Finally, since the gauge classes of flat connections over $M^{(3)}$ are in bijection with the elements of $H_1(M^{(3)}, \mathbb{Z}) = \check{H}^2(M^{(3)}, \mathbb{Z})$, the expression (2.20) of the partition function shows that only the gauge classes of flat connections contribute. Hence, this computation validates the common guess claiming that the partition function should be dominated by the contributions of the flat connections. In this abelian case, the flat connections not only dominate, but they are the only contributions. In the non-abelian case, such contributions would be expected to be “dressed” by the contributions of non-flat connections.

2.2 Abelian expectation value of observables

In this second section, the expectation value of observables of the abelian CS theory are computed. Those observables are Wilson loops for one connection, that is, abelian holonomies:

$$W_{M^{(3)}}(A, \gamma) = e^{2i\pi \oint_\gamma A} \quad (2.21)$$

and their expectation value is given by:

$$\langle W_{M^{(3)}}(A, \gamma) \rangle_{\text{CS}_k, M^{(3)}} = \frac{1}{\mathcal{N}_{\text{CS}_k}(M^{(3)})} \int_{\mathcal{H}} d\mu_{\text{CS}_k}(A) e^{2i\pi \oint_\gamma A}. \quad (2.22)$$

Define also:

$$\langle\langle W_{M^{(3)}}(A, \gamma) \rangle\rangle_{\text{CS}_k, M^{(3)}} = \frac{\langle W_{M^{(3)}}(A, \gamma) \rangle_{\text{CS}_k}}{Z_{\text{CS}_k}(M^{(3)})}. \quad (2.23)$$

Consider $\gamma = \gamma^0 + \gamma^f + \gamma^\tau$ where the superscript 0 refers to the homologically trivial part of the loop, f to its non-trivial free part and τ to its non-trivial torsion part. Then:

$$\langle W_{M^{(3)}}(A, \gamma) \rangle_{\text{CS}_k, M^{(3)}} = \delta_{\mathbf{f}}^{[2k]} e^{-\frac{2i\pi}{4k} \ell_{k, M^{(3)}}(\gamma^0 + \gamma^\tau, \gamma^0 + \gamma^\tau)} \sum_{\kappa \in T_1} e^{-2i\pi(kQ(\kappa, \kappa) + Q(\kappa, \tau))} \quad (2.24)$$

and:

$$\langle\langle W_{M^{(3)}}(A, \gamma) \rangle\rangle_{\text{CS}_k, M^{(3)}} = \frac{1}{Z_{\text{CS}_k}} \delta_{\mathbf{f}}^{[2k]} e^{-\frac{2i\pi}{4k} \ell_{k, M^{(3)}}(\gamma^0 + \gamma^\tau, \gamma^0 + \gamma^\tau)} \sum_{\kappa \in T_1} e^{-2i\pi(kQ(\kappa, \kappa) + Q(\kappa, \tau))}, \quad (2.25)$$

Chapter 2. Abelian Chern-Simons theory

so that:

$$\langle W_{M^{(3)}}(A, 0) \rangle_{\text{CS}_k, M^{(3)}} = Z_{\text{CS}_k}(M^{(3)}) \quad (2.26)$$

and:

$$\langle \langle W_{M^{(3)}}(A, 0) \rangle \rangle_{\text{CS}_k, M^{(3)}} = 1. \quad (2.27)$$

The decoupling of the free part happens thanks to the properties of the zero modes presented in chapter 1 of this part.

This formula written with loops generalizes to links. It seems important to point out that the Kronecker symbols depend on the free part of the whole link. Hence, the whole link can have no free homology, but the components can independently have free homology, which might be counter-intuitive.

In their studies of Gauss sums, F. Deloup and V. Turaev showed a reciprocity formula [Deloup and Turaev, 2007] which can be written in particular, taking the results obtained previously into account⁴, as:

$$\langle \gamma \rangle_{\text{RT}_{4l}, M^{(3)}} = \sqrt{\frac{(2l)^{b_1}}{p_1 \dots p_n}} \langle W_{M^{(3)}}(A, \gamma) \rangle_{\text{CS}_l, M^{(3)}}, \quad (2.28)$$

according to the definition given by equation (2.4) in the first part, that is:

$$\begin{aligned} & e^{-i\frac{\pi}{4}\sigma(\mathbf{L}_\mathcal{L})} \sqrt{2l}^{-m} \sum_{\mathbf{p} \in \mathbb{Z}_{2l}^m} e^{\frac{2i\pi}{4l}((\mathbf{p}, q), \mathbf{L}_\mathcal{L} \cup \tilde{\gamma}(\mathbf{p}, q))} \\ &= \sqrt{\frac{(2l)^{b_1}}{p_1 \dots p_n}} \delta_{\mathbf{f}}^{[2l]} e^{-\frac{2i\pi}{4l} \ell_{k, M^{(3)}}(\gamma^0 + \gamma^\tau, \gamma^0 + \gamma^\tau)} \sum_{\kappa \in T_1} e^{-2i\pi(lQ(\kappa, \kappa) + Q(\kappa, \tau))}. \end{aligned} \quad (2.29)$$

In particular:

$$\tilde{Z}_{\text{RT}_{4l}}(M^{(3)}) = \sqrt{\frac{(2l)^{b_1}}{p_1 \dots p_n}} Z_{\text{CS}_l}(M^{(3)}) \quad (2.30)$$

for $\gamma = 0$. Therefore:

$$\langle \langle \gamma \rangle \rangle_{\text{RT}_{4l}, M^{(3)}} = \langle \langle W_{M^{(3)}}(A, \gamma) \rangle \rangle_{\text{CS}_l, M^{(3)}}. \quad (2.31)$$

This formula relates on the left-hand side a computation associated with $M^{(3)}$ but performed strictly speaking in S^3 , and on the right-hand side a computation performed directly in $M^{(3)}$.

⁴This formula was not first shown in a context of study of manifolds of dimension 3 but on generic quadratic forms on lattices having some specific properties. Hence, strictly speaking, the formula has to be read only in one sense, meaning that under some hypotheses on the objects of one side, the existence of the other side is guaranteed, but the reverse is not true in general. Also, it shall be noticed that the reciprocity formula corresponds in this context to the non-modular case $k = 4l$.

2.2. Abelian expectation value of observables

In the $SU(N)$ case, it is often said that the CS theory and the RT theory coincides. It is remarkable that there is always an abelian RT theory related to a given abelian CS theory whereas the converse is not true in general. In any case, an abelian CS theory is never associated with an abelian modular RT theory.

Moreover, for a given abelian CS theory, the correspondence with its associated abelian RT theory is not trivial at all, both sides of the reciprocity formula having completely different shapes. If it were possible to compute exactly the $SU(N)$ CS partition function and expectation value of observables as it is done here in the abelian case, then there should also be a “non-abelian reciprocity formula” relating an expression derived from a given $SU(N)$ CS theory with a coupling k and an expression derived from a modular category built with representations of $\mathcal{U}_q(\mathfrak{sl}_N)$, with $q = e^{\frac{2i\pi}{k}}$. It is interesting to notice that the reciprocity formula “quantizes” abelian CS theories, in the sense that it transforms an expression derived from a given abelian CS theory with a coupling l , thus related to $U(1)$, into an expression related to \mathbb{Z}_{4l} . This interpretation would still hold in the $SU(N)$ case.

Finally, the abelian CS **surgery function** of $M^{(3)}$ is defined as:

$$\widehat{W}_{S^3}(A, \mathcal{L}) = \sum_{\mathbf{p} \in \mathbb{Z}_{2l}^m} \prod_{i=1}^m e^{2i\pi p_i \oint_{\mathcal{L}_i} A}, \quad (2.32)$$

where $\mathcal{L} \subset S^3$ is a surgery link of $M^{(3)}$, so that:

$$\langle \widehat{W}_{S^3}(A, \mathcal{L}) \rangle_{CS_l, S^3} = \sum_{\mathbf{p} \in \mathbb{Z}_{2l}^m} e^{-\frac{2i\pi}{4l}(\mathbf{p}, \mathbf{L}\mathcal{L}\mathbf{p})} \quad (2.33)$$

and thus, from equation (2.22)⁵ obtained in chapter 2 of the first part and equation (2.30) given in the present chapter:

$$\langle \widehat{W}_{S^3}(A, \mathcal{L}) \rangle_{CS_l, S^3} = \sqrt{2l}^m e^{i\frac{\pi}{4}\sigma(\mathbf{L}\mathcal{L})} \sqrt{\frac{(2l)^{b_1}}{p_1 \dots p_n}} Z_{CS_l}(M^{(3)}). \quad (2.34)$$

Also:

$$\langle \widehat{W}_{S^3}(A, \mathcal{L}) W_{S^3}(A, \tilde{\gamma}) \rangle_{CS_l, S^3} = \sum_{\mathbf{p} \in \mathbb{Z}_{2l}^m} e^{-\frac{2i\pi}{4l}((\mathbf{p}, q), \mathbf{L}\mathcal{L} \cup_{\tilde{\gamma}}(\mathbf{p}, q))} \quad (2.35)$$

where $\tilde{\gamma} \subset S^3$ is the preimage of a loop $\gamma \subset M^{(3)}$ with a framing q by the surgery procedure. Moreover, from equation (2.4)⁵ obtained in chapter 2 of the first part and equation (2.28) given in the present chapter:

$$\langle \widehat{W}_{S^3}(A, \mathcal{L}) W_{S^3}(A, \tilde{\gamma}) \rangle_{CS_l, S^3} = \sqrt{2l}^m e^{i\frac{\pi}{4}\sigma(\mathbf{L}\mathcal{L})} \sqrt{\frac{(2l)^{b_1}}{p_1 \dots p_n}} \langle \gamma \rangle_{CS_l, M^{(3)}}. \quad (2.36)$$

⁵Strictly speaking, the correspondence is up to a complex conjugation, that is, actually, up to the orientation of the manifold.

Chapter 2. Abelian Chern-Simons theory

Finally:

$$\frac{\langle\langle \widehat{W}_{S^3}(A, \mathcal{L}) W_{S^3}(A, \tilde{\gamma}) \rangle\rangle_{\text{CS}_l, S^3}}{\langle\langle \widehat{W}_{S^3}(A, \mathcal{L}) \rangle\rangle_{\text{CS}_l, S^3}} = \langle\langle W_{M^{(3)}}(A, \gamma) \rangle\rangle_{\text{CS}_l, M^{(3)}} = \langle\langle \gamma \rangle\rangle_{\text{RT}_{4l}, M^{(3)}}. \quad (2.37)$$

This formula is called the **surgery formula** in reference to that of the non-abelian case [Reshetikhin and Turaev, 1991]. It extends naturally from loops to links. It is noteworthy that the abelian surgery formula has been proved here by a direct computation, whereas the non-abelian surgery formula is showed in [Guadagnini, 1993] by indirect arguments, similar to that of Witten for proving the correspondence between SU(2) CS theory and Jones polynomials. It uses in particular the formulation of QFT in terms of operators and Fock spaces and assumes the correspondence with the formulation of QFT in terms of functional integral. This correspondence is one of the tricky points of QFT since several quantities are completely ill-defined in both formulations.

Chapter 3

Abelian BF theory

Contents

Introduction (English version)	90
Introduction (version française)	91
3.1 Computation of Z_{BF_k}	92
3.2 Abelian expectation value of observables	94

Introduction (English version)

Mimicking the abelian CS theory, the abelian BF theory studied here is given by the action:

$$S_{\text{BF}_k}(A, B) = k \int_{M^{(3)}} A \star B, \quad (3.1)$$

where A and B are two DB classes associated with the gauge classes of two $U(1)$ -connections.

It should be underlined here that in the non-abelian case, B is a true 1-form (that is, globally defined) with coefficients in the Lie algebra of the gauge group. The choice made here is to treat B as if it were more specifically a connection in the non-abelian case so that it can be associated now with a DB class in the abelian case. This choice *a priori* arbitrary is the one that leads to obtaining a relation between the abelian BF theory and the abelian TV theory as shown further.

The purpose of the first section is to compute mainly the partition function of the abelian BF theory, that is:

$$Z_{\text{BF}_k}(M^{(3)}) = \frac{1}{\mathcal{N}_{\text{BF}_k}} \int_{\mathcal{H} \times \mathcal{H}} \mathcal{D}A \mathcal{D}B e^{2i\pi S_{\text{BF}_k}(A, B)}, \quad (3.2)$$

where:

$$S_{\text{BF}_k}(A, B) = k \int_{M^{(3)}} A \star B \quad (3.3)$$

and $k \in \mathbb{Z}$ since $\int_{M^{(3)}} A \star B \in \mathbb{R}/\mathbb{Z}$. The local expression of $A \star B$ is nothing but $A_\alpha \wedge dB_\alpha$ which is the expression of the abelian BF Lagrangian in \mathbb{R}^3 . The space \mathcal{H} is defined in the same way as in the previous chapter. The measure \mathcal{D} is still purely formal and again a few properties will be postulated to be able to perform the computation of Z_{BF_k} , the normalization $\mathcal{N}_{\text{BF}_k}$ being here to cancel the intrinsic divergency of the functional integral. This result will be related, on the one hand, to the abelian CS partition function and, on the other hand, to the abelian TV invariant.

In the second section, the expectation value of observables will be computed and related to that of the abelian TV theory. The abelian RT invariant related to the Drinfeld center of $\mathbb{C}^{\mathbb{Z}_k}$ will be used to obtain a surgery formula analogous to that obtained in the case of the abelian CS theory.

This chapter gathers the results that can be found with proofs and details in [Mathieu and Thuillier, 2016a], [Mathieu and Thuillier, 2016b] and [Mathieu and Thuillier, 2017], with unified notations. The computations related to the abelian BF theory have been obtained adapting the methods used for the abelian CS theory. They constitute the main contribution of this study together with the relations established with abelian RT, TV and CS theories.

Introduction (version française)

Calquant la théorie CS abélienne, la théorie BF abélienne étudiée ici est donnée par l'action :

$$S_{\text{BF}_k}(A, B) = k \int_{M^{(3)}} A \star B, \quad (3.1)$$

où A et B sont deux classes de cohomologie DB associées à la classe de jauge de deux connexions $U(1)$.

Il doit être souligné ici que dans le cas non-abélien, B est une vraie 1-forme (c'est-à-dire, globalement définie) avec des coefficients dans l'algèbre de Lie du groupe de jauge. Le choix fait ici est de traiter B comme si c'était plus spécifiquement une connexion dans le cas non-abélien, de sorte qu'elle peut être associée maintenant à une classe de cohomologie DB dans le cas abélien. Ce choix *a priori* arbitraire est celui qui permet d'obtenir une relation entre la théorie BF abélienne et la théorie TV abélienne comme il sera montré plus loin.

Le but de la première section est de calculer principalement la fonction de partition de la théorie BF abélienne, c'est-à-dire :

$$Z_{\text{BF}_k}(M^{(3)}) = \frac{1}{\mathcal{N}_{\text{BF}_k}} \int_{\mathcal{H} \times \mathcal{H}} \mathcal{D}A \mathcal{D}B e^{2i\pi S_{\text{BF}_k}(A, B)}, \quad (3.2)$$

où :

$$S_{\text{BF}_k}(A, B) = k \int_{M^{(3)}} A \star B \quad (3.3)$$

et $k \in \mathbb{Z}$ puisque $\int_{M^{(3)}} A \star B \in \mathbb{R}/\mathbb{Z}$. L'expression locale de $A \star B$ n'est rien d'autre que $A_\alpha \wedge dB_\alpha$ ce qui est l'expression du lagrangien BF dans \mathbb{R}^3 . L'espace \mathcal{H} est défini de la même façon qu'au chapitre précédent. La mesure \mathcal{D} est toujours purement formelle et, de nouveau, quelques propriétés seront postulées afin de pouvoir effectuer le calcul de Z_{BF_k} , la normalisation $\mathcal{N}_{\text{BF}_k}$ étant là afin d'absorber les divergences intrinsèques de l'intégrale fonctionnelle. Ce résultat sera relié, d'une part, à la fonction de partition CS abélienne et, d'autre part, à l'invariant TV abélien.

Dans la deuxième section, les valeurs moyennes d'observables sont calculées et reliées à celles de la théorie TV abélienne. L'invariant RT abélien lié au centre de Drinfeld de $\mathbb{C}^{\mathbb{Z}_k}$ sera utilisé afin d'obtenir une formule de chirurgie analogue à celle obtenue dans le cas de la théorie CS abélienne.

Ce chapitre rassemble des résultats qui peuvent être trouvés avec leur démonstration et plus de détails dans [Mathieu and Thuillier, 2016a], [Mathieu and Thuillier, 2016b] et [Mathieu and Thuillier, 2017], avec des notations unifiées. Les calculs liés à la théorie BF abélienne ont été obtenus en adaptant les méthodes utilisées pour la théorie CS abélienne. Ils constituent la principale contribution de ce travail de même que les relations établies avec les théories RT, TV et CS abéliennes.

3.1 Computation of Z_{BF_k}

As mentioned in the introduction, the purpose of this first section is first to compute the partition function of the abelian BF theory, that is:

$$Z_{\text{BF}_k}(M^{(3)}) = \frac{1}{\mathcal{N}_{\text{BF}_k}} \int_{\mathcal{H} \times \mathcal{H}} \mathcal{D}A \mathcal{D}B e^{2i\pi S_{\text{BF}_k}(A,B)}, \quad (3.4)$$

where:

$$S_{\text{BF}_k}(A, B) = k \int_{M^{(3)}} A \star B \quad (3.5)$$

and $k \in \mathbb{Z}$ since $\int_{M^{(3)}} A \star B \in \mathbb{R}/\mathbb{Z}$.

It can be noticed that the partition function is a sum of complex exponentials of a bilinear term. This is formally reminiscent of the shape of the abelian TV invariant presented in chapter 3 of the first part.

The functional measure considered here is:

$$d\mu_{\text{BF}_k}(A, B) = \mathcal{D}A \mathcal{D}B e^{2i\pi S_{\text{BF}_k}(A,B)}. \quad (3.6)$$

The main hypothesis made in this study is that it follows a ‘‘Cameron-Martin-like’’ property, that is:

$$\begin{aligned} d\mu_{\text{BF}_k}(A + \alpha, B + \beta) \\ = d\mu_{\text{BF}_k}(A, B) e^{2i\pi S_{\text{BF}_k}(A,\beta)} e^{2i\pi S_{\text{BF}_k}(\alpha,B)} e^{2i\pi S_{\text{BF}_k}(\alpha,\beta)} \end{aligned} \quad (3.7)$$

for fixed connections A and B and translations α and β .

Keeping the notations introduced for the abelian CS case, the so-called **zero mode property** becomes, for the abelian BF theory:

$$d\mu_{\text{BF}_k}\left(A + m \frac{j_{\Sigma_1}}{k}, B + n \frac{j_{\Sigma_2}}{k}\right) = d\mu_{\text{BF}_k}(A, B) \quad (3.8)$$

for any $n, m \in \mathbb{Z}$. It is similar to the one presented in the case of the abelian CS theory but here, the period is k instead of $2k$.

Using the algebraic properties given in chapter 1 of this part, the abelian BF partition function can be computed exactly.

As a convention, the normalization is chosen to be:

$$\mathcal{N}_{\text{BF}_k} = \int_{(\Omega^1/\Omega_{\mathbb{Z}}^1)^{\#} \times (\Omega^1/\Omega_{\mathbb{Z}}^1)^{\#}} e^{2i\pi S_{\text{BF}_k}(\alpha,\beta)} \mathcal{D}\alpha \mathcal{D}\beta, \quad (3.9)$$

which is the same as:

$$\mathcal{N}_{\text{BF}_k} = \int_{(\Omega^1/\Omega_0^1)^{\#} \times (\Omega^1/\Omega_0^1)^{\#}} e^{2i\pi S_{\text{BF}_k}(\alpha,\beta)} \mathcal{D}\alpha \mathcal{D}\beta, \quad (3.10)$$

3.1. Computation of Z_{BF_k}

which corresponds to the infinite volume associated with the trivial fiber of DB bundle associated with the manifold $M^{(3)}$ in which the abelian BF theory is defined with respect to the measure induced by the BF action considered.

It shall be reminded that this trivial fiber is the only one that constitutes the DB bundle when the manifold considered is S^3 and it arises *formally* for any manifold $M^{(3)}$ of dimension 3, in the sense that $\Omega^1/\Omega_{\mathbb{Z}}^1$ appears for any manifold $M^{(3)}$ of dimension 3. However, of course, $\Omega^1/\Omega_{\mathbb{Z}}^1$ depends on $M^{(3)}$, so, to that extent, the trivial fiber is not common to all closed manifolds of dimension 3. The normalization is performed with respect to S^3 only in the sense that, with this convention, $Z_{\text{BF}_k}(S^3) = 1$.

This choice of normalization is mainly justified by the fact that it leads to establishing a link with the abelian TV invariant as defined in chapter 3 of the previous part. Usually the normalization of the TV invariant is chosen to be related to $S^1 \times S^2$. However, if the normalization is chosen with respect to S^3 then the abelian TV invariant coincides with the abelian BF partition function as shown further.

The computation of the partition function finally gives:

$$Z_{\text{BF}_k}(M^{(3)}) = \sum_{\tau_1, \tau_2 \in T^2} e^{-2i\pi k Q(\tau_1, \tau_2)} = \prod_{i=1}^n \text{gcd}(p_i, k) p_i, \quad (3.11)$$

where Q is still the linking form over the torsion part T^2 of $H^2(M^{(3)})$ as introduced in the preliminaries. Here also, it is remarkable that the partition function contains a non-trivial topological piece of information related to the homological torsion of $M^{(3)}$ but nothing about the free homology.

The abelian BF case is very similar to the abelian CS case. The abelian BF action can be seen as an asymmetrization of the abelian CS action and the computation leads to a partition function which is the asymmetrization of the abelian CS partition function. Like in the abelian CS, it is remarkable that only the gauge classes of flat connections contribute to the partition function.

Furthermore, setting $T^2 = \mathbb{Z}_{p_1} \oplus \dots \mathbb{Z}_{p_n}$ and assuming:

- p_i is odd for $1 \leq i \leq \alpha$,
- p_i is purely even (divisible by 2 but not 4) for $\alpha + 1 \leq i \leq \alpha + \beta$,
- p_i is divisible by 4 for $\alpha + \beta + 1 \leq i \leq \alpha + \beta + \gamma = n$,

then:

$$\left| Z_{\text{CS}_k}(M^{(3)}) \right|^2 = 2^\gamma \left(\prod_{i=1}^n \text{gcd}(p_i, k) p_i \right) \delta_{0,\beta} = 2^\gamma Z_{\text{BF}_k}(M^{(3)}) \delta_{0,\beta}. \quad (3.12)$$

Chapter 3. Abelian BF theory

As a conclusion, in general:

$$\left| Z_{\text{CS}_k} (M^{(3)}) \right|^2 \neq Z_{\text{BF}_k} (M^{(3)}). \quad (3.13)$$

The equality occurs if and only if all the torsion indices of $M^{(3)}$ are odd.

The kronecker symbol $\delta_{0,\beta}$ in (3.12) is reminiscent of the condition:

$$\forall \alpha \in H^1 (M^{(3)}, \mathbb{Z}_{2k}), \alpha \cup \alpha \cup \alpha = 0 \quad (3.14)$$

in (2.23) of chapter 2 of the first part. Indeed:

$$\frac{1}{2k} \alpha \cup \alpha \cup \alpha (M^{(3)}) = kQ (\bar{\alpha}, \bar{\alpha}) \quad (3.15)$$

with $\bar{\alpha} \in T_1$ such that:

$$\forall \bar{u} \in T_1, Q (\bar{\alpha}, \bar{u}) = \frac{1}{2k} \alpha (\bar{u}). \quad (3.16)$$

Hence:

$$\alpha \cup \alpha \cup \alpha (M^{(3)}) = 0 \text{ in } \mathbb{Z}_{2k} \Leftrightarrow kQ (\bar{\alpha}, \bar{\alpha}) \in \mathbb{Z} \quad (3.17)$$

which can be showed to be equivalent to $\beta = 0$ [Mathieu and Thuillier, 2016a].

The point here is that the operator Q acts on the torsion of $M^{(3)}$ as a quadratic form for abelian CS theory and as a bilinear form for abelian BF theory. Turning Q acting as a quadratic form into Q bilinear sounds actually easy, just writing:

$$2(u, Qv) = ((u + v), Q(u + v)) - (u, Qu) - (v, Qv) \quad (3.18)$$

But the difficulty arises from the fact that 2 is not necessarily invertible on the torsion lattice and can make appear a sub-periodicity in the exponential. This phenomenon is quite similar to what happens with abelian RT and TV invariants.

This result is different from the $\text{SU}(2)$ case where it has been shown formally that:

$$\left| Z_{\text{CS}_k}^{\text{SU}(2)} (M^{(3)}) \right|^2 = Z_{\text{BF}_k}^{\text{SU}(2)} (M^{(3)}). \quad (3.19)$$

This difference is not really surprising in so far as the formal proof of the equality in the $\text{SU}(2)$ case relies on a variable change depending deeply on the cosmological constant κ of the theory [Cattaneo et al., 1998], whereas such term cannot exist here.

3.2 Abelian expectation value of observables

In this second section the expectation value of observables of the abelian BF theory are computed. Those observables are simply the product of two Wilson loops, that is, the product of two abelian holonomies:

$$W_{M^{(3)}}(A, \gamma_1) W_{M^{(3)}}(B, \gamma_2) = e^{2i\pi \oint_{\gamma_1} A} e^{2i\pi \oint_{\gamma_2} B} \quad (3.20)$$

and their expectation value is given by:

$$\begin{aligned} & \langle W_{M^{(3)}}(A, \gamma_1) W_{M^{(3)}}(B, \gamma_2) \rangle_{\text{BF}_k, M^{(3)}} \\ &= \frac{1}{\mathcal{N}_{\text{BF}_k}} \int_{\mathcal{H} \times \mathcal{H}} d\mu_{\text{BF}_k}(A, B) e^{2i\pi \oint_{\gamma_1} A} e^{2i\pi \oint_{\gamma_2} B}. \end{aligned} \quad (3.21)$$

Define also:

$$\begin{aligned} & \langle \langle W_{M^{(3)}}(A, \gamma_1) W_{M^{(3)}}(B, \gamma_2) \rangle \rangle_{\text{BF}_k, M^{(3)}} \\ &= \frac{\langle W_{M^{(3)}}(A, \gamma_1) W_{M^{(3)}}(B, \gamma_2) \rangle_{\text{BF}_k, M^{(3)}}}{Z_{\text{BF}_k}(M^{(3)})}. \end{aligned} \quad (3.22)$$

Consider $\gamma_1 = \gamma_1^0 + \gamma_1^f + \gamma_1^\tau$ and $\gamma_2 = \gamma_2^0 + \gamma_2^f + \gamma_2^\tau$ where the superscript 0 refers to the homologically trivial part of the loop, f to its non-trivial free part and τ to its non-trivial torsion part. Then:

$$\begin{aligned} & \langle W_{M^{(3)}}(A, \gamma_1) W_{M^{(3)}}(B, \gamma_2) \rangle_{\text{BF}_k, M^{(3)}} \\ &= \delta_{\mathbf{f}_1}^{[k]} \delta_{\mathbf{f}_2}^{[k]} e^{-\frac{2i\pi}{k} \ell_{k, M^{(3)}}(\gamma_1^0 + \gamma_1^\tau, \gamma_2^0 + \gamma_2^\tau)} \\ & \quad \sum_{\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2 \in T_1} e^{-2i\pi(kQ(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) + Q(\boldsymbol{\tau}_1, \boldsymbol{\kappa}_2) + Q(\boldsymbol{\kappa}_1, \boldsymbol{\tau}_2))} \end{aligned} \quad (3.23)$$

and:

$$\begin{aligned} & \langle \langle W_{M^{(3)}}(A, \gamma_1) W_{M^{(3)}}(B, \gamma_2) \rangle \rangle_{\text{BF}_k, M^{(3)}} \\ &= \frac{1}{Z_{\text{BF}_k}(M^{(3)})} \delta_{\mathbf{f}_1}^{[k]} \delta_{\mathbf{f}_2}^{[k]} e^{-\frac{2i\pi}{k} \ell_{k, M^{(3)}}(\gamma_1^0 + \gamma_1^\tau, \gamma_2^0 + \gamma_2^\tau)} \\ & \quad \sum_{\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2 \in T_1} e^{-2i\pi(kQ(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) + Q(\boldsymbol{\tau}_1, \boldsymbol{\kappa}_2) + Q(\boldsymbol{\kappa}_1, \boldsymbol{\tau}_2))}, \end{aligned} \quad (3.24)$$

so that:

$$\langle W_{M^{(3)}}(A, 0) W_{M^{(3)}}(B, 0) \rangle_{\text{BF}_k, M^{(3)}} = Z_{\text{BF}_k}(M^{(3)}) \quad (3.25)$$

and:

$$\langle \langle W_{M^{(3)}}(A, 0) W_{M^{(3)}}(B, 0) \rangle \rangle_{\text{BF}_k, M^{(3)}} = 1. \quad (3.26)$$

Chapter 3. Abelian BF theory

Here also of course, the decoupling of the free part happens thanks to the properties of the zero modes presented in chapter 1 of this part.

As the CS expectation value of observables, this formula written with loops generalizes to links. It shall be underlined that, for a given coupling k , the linking factor in the abelian BF case is $e^{-\frac{2i\pi}{k} \ell_{k_{M^{(3)}}}(\gamma_1^0 + \gamma_1^\tau, \gamma_2^0 + \gamma_2^\tau)}$ while it is $e^{-\frac{2i\pi}{4k} \ell_{k_{M^{(3)}}}(\gamma_1^0 + \gamma_1^\tau, \gamma_1^0 + \gamma_1^\tau)}$ in the abelian CS case. Hence, from this point of view, the abelian CS theory can be regarded as more accurate.

As for the abelian BF and CS partition functions, there is no relation in general between the abelian BF and CS expectation value of observables. Nevertheless, in the case where γ is homologically trivial and $M^{(3)}$ has no torsion, which implies that the quadratic form Q is trivial, then it is possible to write:

$$\begin{aligned} \langle \langle W_{M^{(3)}}(A, \gamma) \rangle \rangle_{\text{CS}_l, M^{(3)}} &= \langle \langle W_{M^{(3)}}(A, \gamma) W_{M^{(3)}}(B, \gamma) \rangle \rangle_{\text{BF}_{4l}, M^{(3)}} \\ &= e^{-\frac{2i\pi}{4l} \ell_{k_{M^{(3)}}}(\gamma, \gamma)} \end{aligned} \quad (3.27)$$

and:

$$Z_{\text{CS}_l}(M^{(3)}) = Z_{\text{BF}_{4l}}(M^{(3)}) = 1. \quad (3.28)$$

The expectation value of observables of the abelian BF theory turns out to be related to the one defined for the abelian TV thanks to a reciprocity formula analogous to the one which has been mentioned in the abelian RT and CS case:

$$\begin{aligned} \left\langle W_{M^{(3)}}\left(\frac{\mathbf{1}}{k}, \gamma_1\right) W_{M^{(3)}}\left(\frac{\mathbf{m}}{k}, \gamma_2\right) \right\rangle_{\text{TV}_k, M^{(3)}} \\ = \frac{k^{b_1}}{p_1 \dots p_n} \langle W_{M^{(3)}}(A, \gamma_1) W_{M^{(3)}}(B, \gamma_2) \rangle_{\text{BF}_k, M^{(3)}}, \end{aligned} \quad (3.29)$$

according to the definition given by equation (3.37) in the first part, that is:

$$\begin{aligned} \frac{1}{k^{V+F-1}} \sum_{\mathbf{l} \in \mathbb{Z}_k^E} \sum_{\mathbf{m} \in \mathbb{Z}_k^F} e^{\frac{2i\pi}{k} (\mathbf{m} \cdot \mathbf{d} + \mathbf{l} \cdot \mathbf{z}_1 + \mathbf{m} \cdot \mathbf{z}_2)} \\ = \frac{k^{b_1}}{p_1 \dots p_n} \delta_{\mathbf{f}_1}^{[k]} \delta_{\mathbf{f}_2}^{[k]} e^{-\frac{2i\pi}{k} \ell_{k_{M^{(3)}}}(\gamma_1^0 + \gamma_1^\tau, \gamma_2^0 + \gamma_2^\tau)} \sum_{\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2 \in T_1} e^{-2i\pi(kQ(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) + Q(\boldsymbol{\tau}_1, \boldsymbol{\kappa}_2) + Q(\boldsymbol{\kappa}_1, \boldsymbol{\tau}_2))}. \end{aligned} \quad (3.30)$$

or:

$$\begin{aligned} \frac{1}{k^{V+F-1}} \sum_{\mathbf{l} \in \mathbb{Z}_k^E} \sum_{\mathbf{m} \in \mathbb{Z}_k^F} e^{2i\pi k \left(\left(\frac{\mathbf{m}}{k}\right) \cdot \mathbf{d} + \left(\frac{\mathbf{l}}{k}\right) \cdot \mathbf{z}_1 + \left(\frac{\mathbf{m}}{k}\right) \cdot \mathbf{z}_2 \right)} \\ = \frac{k^{b_1}}{p_1 \dots p_n} \delta_{\mathbf{f}_1}^{[k]} \delta_{\mathbf{f}_2}^{[k]} e^{-\frac{2i\pi}{k} \ell_{k_{M^{(3)}}}(\gamma_1^0 + \gamma_1^\tau, \gamma_2^0 + \gamma_2^\tau)} \sum_{\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2 \in T_1} e^{-2i\pi(kQ(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) + Q(\boldsymbol{\tau}_1, \boldsymbol{\kappa}_2) + Q(\boldsymbol{\kappa}_1, \boldsymbol{\tau}_2))}. \end{aligned} \quad (3.31)$$

3.2. Abelian expectation value of observables

with a given cellular decomposition of $M^{(3)}$, assuming that γ_1 and γ_2 can be built from the edges of this cellular decomposition. If it is not the case, then it is always possible to refine the cellular decomposition so that this statement becomes true.

In particular:

$$Z_{\text{TV}_k}(M^{(3)}) = \frac{k^{b_1}}{p_1 \dots p_n} Z_{\text{BF}_k}(M^{(3)}) \quad (3.32)$$

for $\gamma_1 = 0$ and $\gamma_2 = 0$, which can be seen immediately from equation (3.33) of chapter 3 of the first part and equation (3.11) of this chapter. Therefore:

$$\begin{aligned} & \left\langle \left\langle W_{M^{(3)}} \left(\frac{\mathbf{1}}{k}, \gamma_1 \right) W_{M^{(3)}} \left(\frac{\mathbf{m}}{k}, \gamma_2 \right) \right\rangle \right\rangle_{\text{TV}_k, M^{(3)}} \\ &= \left\langle \left\langle W_{M^{(3)}}(A, \gamma_1) W_{M^{(3)}}(B, \gamma_2) \right\rangle \right\rangle_{\text{BF}_k, M^{(3)}}. \end{aligned} \quad (3.33)$$

This reciprocity formula, under the form (3.30), is very similar to that of Deloup and Turaev. It is remarkable that it relates a lattice of periodicity k to the torsion lattice, k passing, here also, from the numerator to the denominator in the complex exponential. Contrary to the reciprocity formula of Deloup and Turaev where the operators act quadratically, the operators here act bilinearly, introducing then an asymmetry. This asymmetry of the formula is even more visible due to the fact that d is even not a square matrix. Under the form (3.31), both sides are very similar. Indeed, the left-hand side can be considered as a BF theory with Z_k -connections over a lattice. The reciprocity formula can thus be regarded as a discretization of the abelian BF theory presented here which comes down also to a lattice on the right-hand side.

Furthermore, the computations on both sides of the equality are performed in $M^{(3)}$. Nevertheless, thanks to the results given in chapter 2 of the first part:

$$\langle \langle \gamma_1, \gamma_2 \rangle \rangle_{\text{RT}_{\mathbb{Z}(\mathbb{C}^{\mathbb{Z}_k})}, M^{(3)}} = \langle \langle W_{M^{(3)}}(A, \gamma_1) W_{M^{(3)}}(B, \gamma_2) \rangle \rangle_{\text{BF}_k, M^{(3)}}. \quad (3.34)$$

Hence, there is finally anyway a correspondence between a computation performed in $M^{(3)}$ and a computation performed in S^3 .

Moreover, the abelian BF **surgery function** of $M^{(3)}$ is defined as:

$$\widehat{W}_{S^3}(A, \mathcal{L}) \widehat{W}_{S^3}(B, \mathcal{L}) = \sum_{\mathbf{p} \in \mathbb{Z}_k^m} \prod_{i=1}^m e^{2i\pi p_i \phi_{\mathcal{L}_i}^A} \sum_{\mathbf{u} \in \mathbb{Z}_k^m} \prod_{i=1}^m e^{2i\pi u_i \phi_{\mathcal{L}_i}^B}, \quad (3.35)$$

where $\mathcal{L} \subset S^3$ is a surgery link of $M^{(3)}$, so that:

$$\langle \widehat{W}_{S^3}(A, \mathcal{L}) \widehat{W}_{S^3}(B, \mathcal{L}) \rangle_{\text{BF}_k, S^3} = \sum_{\mathbf{p} \in \mathbb{Z}_k^m} \sum_{\mathbf{u} \in \mathbb{Z}_k^m} e^{-\frac{2i\pi}{k}(\mathbf{p}, \mathbf{L} \mathbf{u})} \quad (3.36)$$

Chapter 3. Abelian BF theory

and thus, from the expressions obtained in chapters 2 and 3 of the first part, in particular equations (2.28) and (3.54):

$$Z_{\text{TV}_k}(M^{(3)}) = \frac{1}{k^m} \langle \widehat{W}_{S^3}(A, \mathcal{L}) \widehat{W}_{S^3}(B, \mathcal{L}) \rangle_{\text{BF}_k, S^3}. \quad (3.37)$$

Also:

$$\begin{aligned} & \langle \widehat{W}_{S^3}(A, \mathcal{L}) \widehat{W}_{S^3}(B, \mathcal{L}) W_{S^3}(A, \tilde{\gamma}_1) W_{S^3}(B, \tilde{\gamma}_2) \rangle_{\text{BF}_k, S^3} \\ &= \sum_{\mathbf{p} \in \mathbb{Z}_k^m} \sum_{\mathbf{u} \in \mathbb{Z}_k^m} e^{-\frac{2i\pi}{k}((\mathbf{p}, q, 0), \mathbf{L}_{\mathcal{L} \cup \tilde{\gamma}_1 \cup \tilde{\gamma}_2}(\mathbf{u}, 0, v))} \end{aligned} \quad (3.38)$$

where $\tilde{\gamma}_1 \subset S^3$ is the preimage of a loop $\gamma_1 \subset M^{(3)}$ with framing q and $\tilde{\gamma}_2 \subset S^3$ is the preimage of a loop $\gamma_2 \subset M^{(3)}$ with framing v by the surgery procedure. Moreover, from the expressions obtained in chapters 2 and 3 of the first part, in particular equations (2.41) and (3.56):

$$\begin{aligned} & \left\langle W_{M^{(3)}}\left(\frac{\mathbf{1}}{k}, \gamma_1\right) W_{M^{(3)}}\left(\frac{\mathbf{m}}{k}, \gamma_2\right) \right\rangle_{\text{TV}_k, M^{(3)}} \\ &= \frac{1}{k^m} \langle \widehat{W}(A, \mathcal{L}) \widehat{W}(B, \mathcal{L}) W_{M^{(3)}}(A, \tilde{\gamma}_1) W_{M^{(3)}}(B, \tilde{\gamma}_2) \rangle_{\text{BF}_k, S^3}. \end{aligned} \quad (3.39)$$

Finally the so-called **surgery formula** for BF theory can be written under three different forms:

$$\begin{aligned} & \langle \langle W_{M^{(3)}}(A, \gamma_1) W_{M^{(3)}}(B, \gamma_2) \rangle \rangle_{\text{BF}_k, M^{(3)}} \\ &= \frac{\langle \langle \widehat{W}_{S^3}(A, \mathcal{L}) \widehat{W}_{S^3}(B, \mathcal{L}) W_{S^3}(A, \tilde{\gamma}_1) W_{S^3}(B, \tilde{\gamma}_2) \rangle \rangle_{\text{BF}_k, S^3}}{\langle \langle \widehat{W}_{S^3}(A, \mathcal{L}) \widehat{W}_{S^3}(B, \mathcal{L}) \rangle \rangle_{\text{BF}_k, S^3}}, \end{aligned} \quad (3.40)$$

$$\begin{aligned} & \langle \langle W_{M^{(3)}}(A, \gamma_1) W_{M^{(3)}}(B, \gamma_2) \rangle \rangle_{\text{TV}_k, M^{(3)}} \\ &= \frac{\langle \langle \widehat{W}_{S^3}(A, \mathcal{L}) \widehat{W}_{S^3}(B, \mathcal{L}) W_{S^3}(A, \tilde{\gamma}_1) W_{S^3}(B, \tilde{\gamma}_2) \rangle \rangle_{\text{BF}_k, S^3}}{\langle \langle \widehat{W}_{S^3}(A, \mathcal{L}) \widehat{W}_{S^3}(B, \mathcal{L}) \rangle \rangle_{\text{BF}_k, S^3}}, \end{aligned} \quad (3.41)$$

$$\begin{aligned} & \langle \langle \gamma_1, \gamma_2 \rangle \rangle_{\text{RT}_{\mathbb{Z}(\mathbb{C}^{\mathbb{Z}_k)}, M^{(3)}}} \\ &= \frac{\langle \langle \widehat{W}_{S^3}(A, \mathcal{L}) \widehat{W}_{S^3}(B, \mathcal{L}) W_{S^3}(A, \tilde{\gamma}_1) W_{S^3}(B, \tilde{\gamma}_2) \rangle \rangle_{\text{BF}_k, S^3}}{\langle \langle \widehat{W}_{S^3}(A, \mathcal{L}) \widehat{W}_{S^3}(B, \mathcal{L}) \rangle \rangle_{\text{BF}_k, S^3}}, \end{aligned} \quad (3.42)$$

the last one being the less relevant one since it establishes a relation between two computations performed in S^3 strictly speaking. These formulæ extend naturally

3.2. Abelian expectation value of observables

form loops to links. They constitute the most important new results of this study. They sum up the relations that exist between the abelian BF, TV and RT theory and prove that, as well as the abelian CS theory, the abelian BF theory can be taken back in S^3 by a surgery procedure. It is important to insist on the fact that those results are not obtained in a perturbative way, nor by indirect arguments assuming correspondences between several different formulations of QFT, but only expressing BF theory in terms of gauge classes over $M^{(3)}$.

Part III

Additional matters

Chapter 1

$SU(N)$ Chern-Simons invariant of flat connections

Contents

Introduction (English version)	104
Introduction (version française)	106
1.1 General method	108
1.2 Example of lens spaces	116
1.3 Example of some genus 2 Heegaard splittings	119

Introduction (English version)

In this chapter some elements presented in [Guadagnini et al., 2017] on the $SU(N)$ CS invariant are recalled. It is defined as:

$$S_{\text{CS}}(A) = \frac{1}{8\pi^2} \int_{M^{(3)}} Q_{\text{CS}}(A), \quad (1.1)$$

Q_{CS} being the so-called CS 3-form defined as:

$$Q_{\text{CS}} = \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \quad (1.2)$$

where A is a $SU(N)$ -connection and $M^{(3)}$ is a closed manifold of dimension 3¹. Actually, to be written this way, A has to be an object which is globally defined, that is, which is a global 1-form with coefficients in the Lie algebra of $SU(N)$. Given a gauge class, finding such a representative is always possible since $SU(N)$ fiber bundles over a closed manifold of dimension 3 are all trivializable.

As said in the introduction of chapter 2 of the second part, the CS invariant, and mostly its $SU(N)$ version, has been considered in various branches of physics such as particle physics, quantum gravity, condensed matter physics [Fröhlich et al., 1994] and mathematical physics.

The most famous application in mathematical physics is probably its relation to Jones polynomials. The idea of Witten's proof is basically the following. Consider in a closed manifold $M^{(3)}$ of dimension 3 and a link \mathcal{L} whose components are labeled by elements of $SU(N)$. Cut $M^{(3)}$ along a sphere S^2 in a neighbourhood of a crossing of two strands of \mathcal{L} . Two volumes are obtained and in particular one which is a ball B^3 of dimension 3 whose boundary S^2 is punctured four times in total by the two strands of \mathcal{L} which cross each other inside B^3 . Those four points are pairwise associated inside B^3 by the two strands connecting them. Witten showed that the state space associated with the punctured S^2 is 2-dimensional. The present configuration of strands inside B^3 is a vector of this state space, as well as any other configuration of strands connecting the same points inside B^3 . There are three such configurations in number. Since the state space is 2-dimensional, those three different configurations are linearly dependent. The state space is also the space of conformal blocks for the four-point functions on S^2 punctured by the two strands of \mathcal{L} . The linear dependence between the three configurations can then be written explicitly. Witten discovered that it corresponds to the so-called skein relation which is a fundamental property that arises in the construction of Jones polynomials.

This proof could indicate that there is actually a CFT underlying the CS theory, but this is actually not the case. Indeed, a CS theory has a vanishing stress-energy

¹It should be pointed out that the $SU(N)$ CS quantum action S_{CS_k} with coupling k is just given by $S_{\text{CS}_k} = kS_{\text{CS}}$.

tensor, which should not happen in a CFT. The link between CFT and the CS theory relies on the common symmetries of those theories. Indeed, on the one hand, in CFT the monodromies of the correlation functions provide representations of the braid group and the $SU(N)$ -representations of the braid group are classified by the quantum groups $\mathcal{U}_q(\mathfrak{su}(N))$. On the other hand, the $SU(N)$ CS theory is equivalent to the RT theory founded on the same quantum groups.

However, the purpose of this chapter is to prove that it is possible anyway to reduce some computations of the CS invariant S_{CS} for flat $SU(N)$ -connections to 2-dimensional terms. The method presented here relies on Heegaard splittings as introduced in chapter 3 of the first part. The CS invariant of a flat connection is a relevant quantity to compute since flat connections are the solutions of EL equations and are thus expected to provide the main contributions to the functional integrals in the quantum case.

Although some methods have been proposed to compute the CS invariant of a flat connection, for example in [Kirk and Klassen, 1990], this quantity is quite tricky to compute and remains unknown in general for a given manifold. This chapter presents some preliminary results of an ongoing work. In particular the main new idea is that the CS invariant of a flat connection can always be reduced to a surface term which splits into a sum of two surface terms, one being an intersection term that can be computed completely graphically, the other coming from a WZ term whose graphical signification remains mysterious but which has been investigated on several classes of examples.

Introduction (version française)

Dans ce chapitre, des éléments présentés dans [Guadagnini et al., 2017] relatifs à l'invariant CS $SU(N)$ sont rappelés. Il est défini comme :

$$S_{\text{CS}}(A) = \frac{1}{8\pi^2} \int_{M^{(3)}} Q_{\text{CS}}(A), \quad (1.1)$$

Q_{CS} étant la 3-forme dite CS définie comme :

$$Q_{\text{CS}} = \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \quad (1.2)$$

où A est une connexion $SU(N)$ et $M^{(3)}$ est une variété fermée de dimension 3¹. En fait, pour être écrite de cette façon, A doit être un objet globalement défini, c'est-à-dire, être une 1-forme globale à coefficients dans l'algèbre de Lie de $SU(N)$. Étant donné une classe de jauge, trouver un tel représentant est toujours possible puisque les fibrés $SU(N)$ au-dessus d'une variété fermée de dimension 3 sont tous trivialisables.

Comme précisé dans l'introduction du chapitre 2 de la deuxième partie, l'invariant CS, et essentiellement sa version $SU(N)$, a été considéré dans diverses branches de la physique, comme la physique des particules, la gravité quantique, la physique de la matière condensée et la physique mathématique.

L'application en physique mathématique la plus célèbre est probablement sa relation avec les polynômes de Jones. L'idée de la preuve de Witten peut être résumée de la façon suivante. Le point de départ est une variété fermée $M^{(3)}$ de dimension 3 et un entrelac \mathcal{L} dont les composantes portent chacune un élément de $SU(N)$. La variété $M^{(3)}$ est ensuite découpée selon une sphère S^2 contenant un voisinage d'un croisement de deux brins de \mathcal{L} . Deux volumes sont alors obtenus et en particulier un qui est une boule B^3 de dimension 3 dont le bord S^2 est percé quatre fois au total par les deux brins de \mathcal{L} qui se croisent à l'intérieur de B^3 . Ces quatre points sont associés deux par deux dans B^3 par les deux brins qui les relient. Witten a montré que l'espace des états associés à la sphère percée S^2 est de dimension 2. La configuration considérée des brins dans B^3 est un vecteur de cet espace des états, de même que toute autre configuration de brins reliant les mêmes points dans B^3 . Ces configurations sont au nombre de trois. Puisque l'espace des états est de dimension 2, ces trois configurations différentes sont linéairement dépendantes. L'espace des états est aussi l'espace des blocs conformes de la fonction quatre points de S^2 percée par les deux brins de \mathcal{L} . La relation de dépendance linéaire entre les trois configurations peut être ensuite écrite explicitement. Witten a découvert qu'elle

¹L'action quantique CS $SU(N)$ notée S_{CS_k} , k étant une constante de couplage, est simplement définie par $S_{\text{CS}_k} = kS_{\text{CS}}$.

correspond à la relation dite d'échevaux qui est une propriété fondamentale qui intervient dans la construction des polynômes de Jones.

Cette preuve pourrait indiquer qu'il existe en fait une CFT sous-jacente dans la théorie CS, mais ce n'est pas le cas. En effet, le tenseur énergie-impulsion d'une théorie CS est nul, ce qui ne peut se produire dans une CFT. Le lien entre CFT et théorie CS repose sur les symétries communes de ces théories. En effet, d'une part, en CFT, les monodromies des fonctions de corrélation fournissent des représentations du groupe de tresse et les représentations $SU(N)$ du groupe de tresse sont classifiées par les groupes quantiques $\mathcal{U}_q(\mathfrak{su}(N))$. D'autre part, la théorie CS $SU(N)$ est équivalente à la théorie RT fondée sur ces mêmes groupes quantiques.

Cependant, l'objectif de ce chapitre est de montrer qu'il est tout de même possible de réduire certains calculs de l'invariant CS S_{CS} associé aux connexions plates $SU(N)$ à des termes de dimension 2. La méthode présentée ici repose sur les scindements de Heegaard introduits au chapitre 3 de la première partie. L'invariant CS associé à une connexion plate est une quantité pertinente à calculer étant donné que les connexions plates sont les solutions des équations EL et sont donc supposées fournir la principale contribution aux intégrales fonctionnelles dans le cas quantique.

Bien que des méthodes aient été proposées afin de calculer l'invariant CS d'une connexion plate, par exemple dans [Kirk and Klassen, 1990], cette quantité est assez compliquée à calculer et demeure inconnue en général pour une variété donnée. Ce chapitre présente quelques résultats préliminaires d'un travail en cours. En particulier, la principale nouvelle idée est que l'invariant CS d'une connexion plate peut toujours se réduire à la somme de deux termes de surface, l'un étant un terme d'intersection qui peut être calculé entièrement graphiquement, l'autre étant issu d'un terme WZ dont la signification graphique demeure mystérieuse mais qui a été étudié sur plusieurs classes d'exemples.

1.1 General method

Reduction of the $SU(N)$ CS invariant for a flat connection. It should not be forgotten that any closed manifold $M^{(3)}$ of dimension 3 can be decomposed as the gluing of two identical genus g handlebodies H_L and H_R along their boundary Σ which is thus a Riemann surface of genus g according to a gluing rule h . The relevant information of this so-called Heegaard splitting is provided by the image of the trace of the meridian discs of H_L on $\partial H_R = \Sigma$ (by convention here, but the opposite convention is of course possible), see Figure 1.1. This information can thus be represented on a diagram called Heegaard diagram. A planar representation will be adopted according to Figure 1.2.

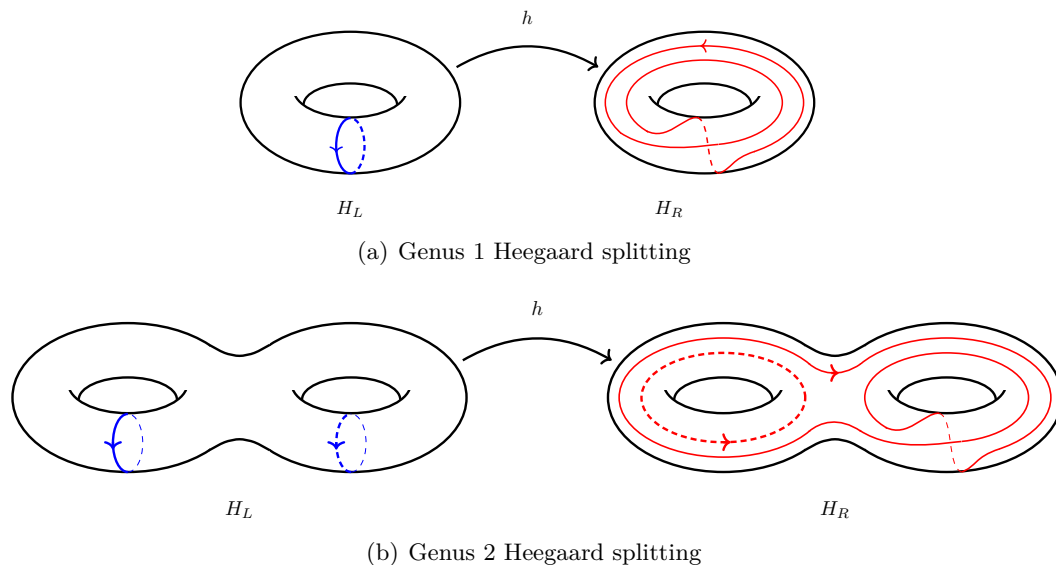


Figure 1.1: Heegaard splitting.

It should be pointed out that the meridian discs dualize the generators $(\gamma_j)_{1 \leq j \leq g}$ of the fundamental group of H_L , and furthermore, the following theorem holds:

Theorem. There exists a bijection between the gauge classes of flat $SU(N)$ -connections over $M^{(3)}$ and the conjugation classes of $SU(N)$ -representations of $\pi_1(M^{(3)})$.

A proof of this classical result can be found in [Morita, 2001]. A fundamental object arising is the $SU(N)$ holonomy of a $SU(N)$ -connection A along a loop γ , which is given by:

$$h_\gamma(A) = \mathcal{P}e^{i\oint_\gamma A} \in SU(N), \quad (1.3)$$

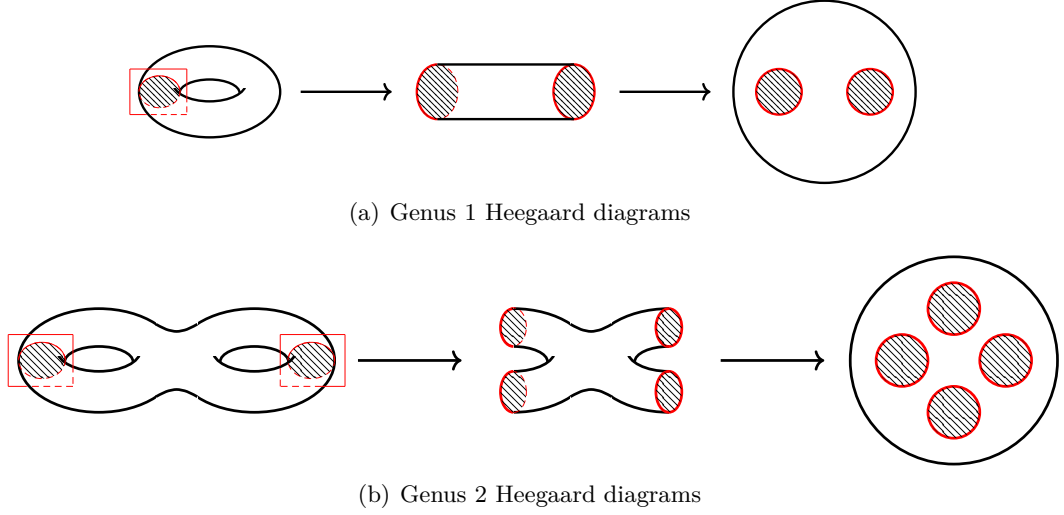


Figure 1.2: Convention of representation of Heegaard diagrams.

The previous theorem just states that the pairing given by the holomy between gauge classes of flat connections and conjugation classes of $SU(N)$ -representations of loops in the fundamental group is non-degenerate.

Consider a flat connection ω in H_L and for all j in $\llbracket 1; g \rrbracket$, a generator γ_j of $\pi_1(H)$. Then, for all j in $\llbracket 1; g \rrbracket$:

$$h_{\gamma_j}(\omega) = e^{i\sigma_{\gamma_j}}, \quad (1.4)$$

where $\sigma_{\gamma_j} \in \mathfrak{su}(N)$. Let x_j be an arbitrary point on γ_j . Consider now:

$$A = i \sum_{j=1}^g \sigma_{\gamma_j} \delta_{x_j}(s_j) ds_j = i \sum_{j=1}^g \sigma_{\gamma_j} d(H_{x_j}(s_j)), \quad (1.5)$$

with s_j the natural parameter of γ_j and H_{x_j} the Heaviside distribution defined between $s_j = 0$ and $s_j = l(\gamma_j)$ (length of γ_j) switching at x_j . Of course, in this context, the differential operator d has to be understood in the sense of distributions.

By construction, A is a connection which is exact, thus:

$$dA = 0. \quad (1.6)$$

Also, still by construction:

$$A \wedge A = 0. \quad (1.7)$$

This implies that:

$$F_A = dA + A \wedge A = 0. \quad (1.8)$$

Chapter 1. $SU(N)$ Chern-Simons invariant of flat connections

Hence, a flat connection that has the same holonomy along γ_j as ω is built. Moreover, it has the remarkable property to be localized on a disc D_j in H_L that intersects once with γ_j that is, in a transverse position towards γ_j . Therefore the holonomy can be considered to be the intersection between γ_j and D_j with a charge σ_{γ_j} provided an orientation for D_j has been set. This orientation defines -thanks to the “right-hand rule”- an orientation of its boundary which is represented on the Heegaard diagram. All these orientations have to be consistent.

In the following, by a misuse of language, the confusion will be made between a localized connection and its associated charge:

$$“ A = i \sum_{j=1}^g \sigma_{\gamma_j} ”, \quad (1.9)$$

The connection can also be localized on thickened discs of thickness ε_j by the same arguments:

$$A = i \sum_{j=1}^g \sigma_{\gamma_j} \theta'_j(s_j) ds_j = i \sum_{j=1}^g \sigma_{\gamma_j} d(\theta_j(s_j)), \quad (1.10)$$

where θ_j is a smooth function such that $\theta_j(0) = 0$ and $\theta_j(\varepsilon_j) = 1$. Such semi-localized connections are flat and abelian. They are related to ω by a gauge transformation, as the completely localized one and for the same reason : it has the same holonomy along each γ_i as ω .

The image on $\Sigma = \partial H_R$ of the trace of meridional discs of H_L is precisely what is represented on a Heegaard diagram associated with a splitting of a manifold $M^{(3)} = H_L \cup_h H_R$. The curves appearing on such diagram can thus be regarded as the trace of a flat connection on the Riemann surface $\Sigma = \partial H_R$.

In the following, a flat connection represented by a collection of g discs in a handlebody of genus g will be noted with a capital latin letter.

Let ω be a flat $SU(N)$ -valued 1-form over $M^{(3)}$ such that:

$$\begin{cases} \omega|_{H_R} = A_R, \\ \omega|_{H_L} = \omega_L. \end{cases} \quad (1.11)$$

Then:

$$A_R^\Sigma = h_* \omega_L^\Sigma, \quad (1.12)$$

on $\partial H_R = \Sigma_R$ and there exists $g : H_L \rightarrow SU(N)$ such that:

$$\omega_L = g^{-1} A_L g + g^{-1} dg. \quad (1.13)$$

Hence, the gluing rule is:

$$A_R^\Sigma = V^{-1} h_* A_L^\Sigma V + V^{-1} dV, \quad (1.14)$$

with:

$$V = h_*g^\Sigma, \quad (1.15)$$

on $\partial H_R = \Sigma_R$.

Equation (1.7) shows that A_L and A_R are on top of that abelian. Hence:

$$Q_{CS}(A_R) = Q_{CS}(A_L) = 0, \quad (1.16)$$

since the cubic term is 0 and the quadratic term is exact on H_L and H_R which are manifolds with a boundary.

Hence, taking this fact into account:

$$\begin{aligned} Q_{CS}(\omega_L) &= d\text{Tr}(g^{-1}A_Lg \wedge (g^{-1}dg)) \\ &\quad - \frac{1}{3}\text{Tr}((g^{-1}dg) \wedge (g^{-1}dg) \wedge (g^{-1}dg)) \end{aligned} \quad (1.17)$$

and finally:

$$\begin{aligned} S_{CS}(\omega) &= \text{deg}(h) \left\{ \frac{1}{8\pi^2} \oint_{\Sigma} \text{Tr}(V^{-1}h_*A_L^\Sigma V \wedge V^{-1}dV) \right. \\ &\quad \left. - \frac{1}{24\pi^2} \iiint_{H_R} \text{Tr}(U^{-1}dU \wedge U^{-1}dU \wedge U^{-1}dU) \right\}, \end{aligned} \quad (1.18)$$

where $U : H_R \rightarrow \text{SU}(N)$ is a continuation of $V : \partial H_R^\Sigma \rightarrow \text{SU}(N)$ inside H_R .

Let us write this equality as:

$$S_{CS}(\omega) = I - S_{WZ}, \quad (1.19)$$

where I is the surface term and S_{WZ} the volume integral of the Wess-Zumino term.

Computation of the surface term. Using the gluing rule, the commutativity in the trace, the anti-commutativity of the wedge product and the fact that:

$$d(V^{-1}dV) = (dV^{-1})dV = -V^{-1}dV \wedge V^{-1}dV, \quad (1.20)$$

the surface term can be written in several equivalent ways:

$$I = \text{deg}(h) \left\{ \frac{1}{8\pi^2} \oint_{\Sigma} \text{Tr}(V^{-1}h_*A_L^\Sigma V \wedge V^{-1}dV) \right\} \quad (1.21)$$

$$= \text{deg}(h) \left\{ \frac{1}{8\pi^2} \oint_{\Sigma} \text{Tr}(A_R^\Sigma \wedge V^{-1}dV) \right\} \quad (1.22)$$

$$= \text{deg}(h) \left\{ \frac{1}{8\pi^2} \oint_{\Sigma} \text{Tr}(V^{-1}h_*A_L^\Sigma V \wedge A_R^\Sigma) \right\} \quad (1.23)$$

$$I = \text{deg}(h) \left\{ \frac{1}{8\pi^2} \oint_{\Sigma} \text{Tr}(h_*A_L^\Sigma \wedge dVV^{-1}) \right\}. \quad (1.24)$$

Chapter 1. $SU(N)$ Chern-Simons invariant of flat connections

It should be pointed out that if the representation of $\pi_1(M^{(3)})$ is abelian, then:

$$S_{\text{CS}}(\omega) = \deg(h) \left\{ \frac{1}{8\pi^2} \oint_{\Sigma} \text{Tr}(h_* A_L^{\Sigma} \wedge A_R^{\Sigma}) \right\} \quad (1.25)$$

and the WZ term is 0.

To compute the surface term V has to be determined. For this purpose, the main equation used is the gluing rule:

$$A_R^{\Sigma} = V^{-1} h_* A_L^{\Sigma} V + V^{-1} dV, \quad (1.26)$$

which just says that A_L and A_R are related by a gauge transformation at the gluing surface.

As for the Heegaard diagram, $A_L = A_R = 0$ inside the domains delimited by the traces of $h_* A_L$ and/or A_R on $\partial H_R = \Sigma$ and the gluing rule gives thus:

$$V^{-1} dV = 0. \quad (1.27)$$

Therefore, V is a constant map inside those domains.

The trace on $\partial H_R = \Sigma$ of each disc D_j which represents A_R^{Σ} carries a charge associated with sigma according to equation (1.5). Given two domains of $\partial H_R = \Sigma$ separated by the trace of a disc D_j which represents A_R^{Σ} , the discontinuity of V between the two domains is provided by the gluing rule:

$$A_R^{\Sigma} = V^{-1} dV, \quad (1.28)$$

since $h_* A_L^{\Sigma} = 0$, that is, the height of the step between the value of V in each domain is precisely the charge of A_R^{Σ} .

Consider an oriented path γ on Σ starting at x_0 and finishing at x_1 . Then:

$$\tau_{\gamma}(A_R^{\Sigma}) = V^{-1}(x_0) \tau_{\gamma}(h_* A_L^{\Sigma}) V(x_1), \quad (1.29)$$

where $\tau_{\gamma}(A_R^{\Sigma})$ is the parallel transport of A_R^{Σ} along γ . This implies that:

$$V(x_1) = \tau_{\gamma}^{-1}(h_* A_L^{\Sigma}) V(x_0) \tau_{\gamma}(A_R^{\Sigma}). \quad (1.30)$$

If $V(x_0) = \mathbb{1}$ then:

$$\forall x \in \Sigma, V(x) = \tau_{\gamma}^{-1}(h_* A_L^{\Sigma}) \tau_{\gamma}(A_R^{\Sigma}), \quad (1.31)$$

so V can be computed from parallel transports of A_R^{Σ} and $h_* A_L^{\Sigma}$.

Moreover, A_R^{Σ} induces in the universal covering a flat connection \tilde{A}_R^{Σ} which is gauge equivalent to the null connection :

$$\tilde{A}_R^{\Sigma} = \tilde{V}^{-1} d\tilde{V}, \quad (1.32)$$

with $\tilde{V} : \tilde{\Sigma} \rightarrow \text{SU}(N)$. Consider two points x_0 and x_1 on Σ on both sides of the trace of A_R^Σ and a path γ that joins them. They induce respectively two points \tilde{x}_0 and \tilde{x}_1 and a path $\tilde{\gamma}$ on $\tilde{\Sigma}$. Since \tilde{A}_R^Σ is gauge equivalent to the null connection, then:

$$\tilde{\tau}_{\tilde{\gamma}}(\tilde{A}_R^\Sigma) = \tilde{V}^{-1}(\tilde{x}_0) \tilde{\tau}_{\tilde{\gamma}}(0) \tilde{V}(\tilde{x}_1) = \tilde{V}^{-1}(\tilde{x}_0) \tilde{V}(\tilde{x}_1) \quad (1.33)$$

and this relation is locally preserved on Σ :

$$\tau_\gamma(A_R^\Sigma) = V^{-1}(x_0) V(x_1). \quad (1.34)$$

Hence, the height of the step can be computed from the evaluation of V inside the two domains separated by the trace of A_R^Σ .

This way, the holonomy along a trivial generator of $\pi_1(H_R)$ is $\mathbb{1}$ (up to a sign depending on the orientation of the curves) and its computation according to the previous rule gives a constraint that appears in the presentation of $\pi_1(M^{(3)})$.

As an example, consider the Heegaard splitting of the lens space $L(5, 3)$ shown in Figure 1.3 (keeping in mind that, according to Figure 1.2, the genus 1 handlebody of the splitting is recovered identifying the two black discs and twisting them so that the equal numbers coincide). According to the “right hand rule” for the orientation:

$$a^5 = \mathbb{1}, \quad (1.35)$$

which is actually the presentation of $\pi_1(L(5, 3)) = \mathbb{Z}_5$. There are as many relations as handles in the splitting but some might be trivial or redundant.

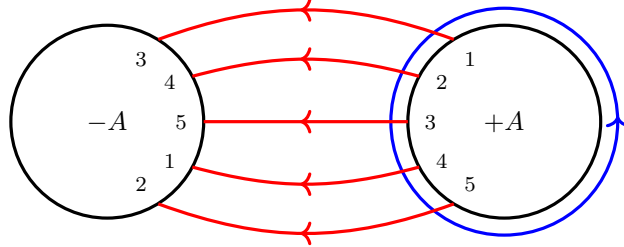


Figure 1.3: Heegaard diagram associated with $L(5, 3)$.

A way to see that is the following : a trivial loop in a manifold $M^{(3)}$ induces a trivial loop in $\tilde{M}^{(3)}$, but a non-trivial loop in $M^{(3)}$ induces an open path in $\tilde{M}^{(3)}$

By a misuse of language, the confusion will be made between the charge of the connection and its holonomy² and equations (1.28) and (1.34) will be identified as:

$$“ A_R^\Sigma = V_R V_L^{-1} ”, \quad (1.36)$$

²It must be borne in mind that this misuse of language will be adopted also to label the Heegaard diagrams, where the traces of A_R^Σ and A_L^Σ will be labeled with their holonomy.

Chapter 1. $SU(N)$ Chern-Simons invariant of flat connections

with the indices L and R referring to the right side and the left side of the trace of A_R^Σ according to its orientation. Keeping this convention, when two domains are separated by the trace of $h_*A_L^\Sigma$, then the gluing rule gives:

$$h_*A_L^\Sigma = -(\mathrm{d}V)V^{-1} = (V^{-1})^{-1}\mathrm{d}(V^{-1}), \quad (1.37)$$

so:

$$\text{“ } h_*A_L^\Sigma = V_R^{-1}V_L \text{ ”}. \quad (1.38)$$

The quantity:

$$-V^{-1}h_*A_L^\Sigma V = A_R^\Sigma \quad (1.39)$$

is also needed to compute the surface term. It is given by:

$$\text{“ } -V^{-1}h_*A_L^\Sigma V = V_R V_L^{-1} \text{ ”}. \quad (1.40)$$

As a conclusion, if the value of V is set inside a domain, thanks to the continuity at the handles and rule (1.36), it is possible to determine the value of V in every domain and then to compute the charge of the trace of $h_*A_L^\Sigma$ thanks to rule (1.38) as well as $V^{-1}h_*A_L^\Sigma V$ thanks to rule (1.39).

To complete the diagram, an element V_0 can be placed in any domain, then propagated thanks to the continuity at the handles and equation (1.36). The label of each domain depends thus on this V_0 . However, in the examples treated below, equation (1.38), to be true, imposes a constraint on this element V_0 which limits the possibilities of the labels. In the case of lens spaces treated as a first family of examples, there is actually only one possibility. In the case of the second family of examples, the labels are defined up to a global sign.

Once the diagram is completed with all the charges on each line, the surface term is nothing but an intersection term. Contributions arise specifically at intersections between traces of $h_*A_L^\Sigma$ and A_R^Σ and are just the product of the charge of A_R^Σ and the charge of $V^{-1}h_*A_L^\Sigma V$ in this order or the product of the charge of $-V^{-1}h_*A_L^\Sigma V$ and the charge of A_R^Σ in this order. It must be borne in mind that the charge appearing on the completed diagrams on the trace of $V^{-1}h_*A_L^\Sigma V$ is not the charge of A_L^Σ but the charge of $-V^{-1}h_*A_L^\Sigma V$ computed with rule (1.39) since this is the quantity that appears in the computation of the surface term as given by equations (1.21)-(1.24). The charge $V^{-1}h_*A_L^\Sigma V$ has no reason for being continuous along the trace of $h_*A_L^\Sigma$ and there are in fact some discontinuities in the examples treated. What is remarkable is the fact that although $V^{-1}h_*A_L^\Sigma V$ is discontinuous on both sides of A_R^Σ , the trace of the product of $V^{-1}h_*A_L^\Sigma V$ by A_R^Σ is continuous.

Such a way of computing with localized connections can be proved to be consistent with the smooth approach by just studying limits in the sense of the topology of weak convergence. On the side of localized connections, the main issue is the appearance of ill-defined distributional products. However, in the present case, because

the connections are flat, the square of a distributional connection can be regarded as its differential up to a sign.

Computation of the WZ term. Concerning the WZ term, the following study is restricted to the SU(2) case and a semi-localized expression of $V^{-1} dV$ ³ is used. Indeed, the WZ term involves a distributional cube with *a priori* no canonical way to define it contrary to the distributional square as mentioned above. Thickening the trace of A_R^Σ and $h_* A_L^\Sigma$, the points of intersection of those lines become by thickening intersections of strips and, thanks to equation (1.31), considering a system of axis (x, y) , x being directed along the trace of A_R^Σ and y being directed along the trace of A_L^Σ (or conversely), V can be written as⁴:

$$V(x, y) = a(x) b(y), \quad (x, y) \in [0; 1]^2 \quad (1.41)$$

and the functions a and b are determined by the value of V at the corners of the domain defined by the intersection of the strips which represent the thickening of A_R^Σ and $h_* A_L^\Sigma$. Actually, written this way, the whole function V is completely determined by its value at only three corners. Its value at the last corner immediately comes out of the triviality of the holonomy around the intersection domain.

To compute the S_{WZ} term a volume continuation of such a V has to be found. Since $V(x, y) \in \text{SU}(2)$ for any x and y , then:

$$V(x, y) = e^{i\xi(x,y)\sigma(x,y)} = \cos(\xi(x, y)) + i \sin(\xi(x, y)) \sigma(x, y), \quad (1.42)$$

with:

$$\sigma(x, y) = n_1(x, y) \sigma_1 + n_2(x, y) \sigma_2 + n_3(x, y) \sigma_3, \quad (1.43)$$

σ_1, σ_2 and σ_3 being the Pauli matrices. Then, the continuation:

$$U(x, y, t) = e^{it\xi(x,y)\sigma(x,y)} = \cos(t\xi(x, y)) + i \sin(t\xi(x, y)) \sigma(x, y) \quad (1.44)$$

seems relevant⁵ since the integration over t can be easily performed:

$$S_{WZ} = \frac{1}{4\pi^2} \oint_{\Sigma} (\xi - \cos(\xi) \sin(\xi)) n_2^3 d\left(\frac{n_1}{n_2}\right) \wedge d\left(\frac{n_3}{n_2}\right). \quad (1.45)$$

As a conclusion, the CS invariant, which is actually a volume term, can be transformed into surface terms, related to the gluing surface of a Heegaard splitting of $M^{(3)}$. It is noteworthy that this is the symmetry of the Heegaard splitting construction that makes it possible to get this result, the localization of the connections

³A map V can be used to compute I and another one to compute S_{WZ} if they are gauge-related since a gauge transformation produces only an integer on S_{WZ} and the global result is taken in \mathbb{R}/\mathbb{Z}

⁴Considering the thickness to be normalized to 1.

⁵Any other continuation would lead to the same S_{WZ} up to an integer.

being impossible for example in a surgery construction. Heegaard splittings are known in the world of mathematical physics [Kohno, 1992] but had never been used the present way until now. The method presented in [Kirk and Klassen, 1990] is completely different and relies on a surgery construction.

It should be pointed out that S_{WZ} computed with thickened connections is nothing but the volume of the body of dimension 3 whose boundary is defined by V ⁶. Different choices of V would lead to the same value of S_{WZ} in \mathbb{R}/\mathbb{Z} . This value in \mathbb{R}/\mathbb{Z} depends only on the family of elements of $SU(2)$ that label the domains where V is constant, which are simply isolated points when the connections are completely thickened. Hence, conversely, this statement is true also when connections tend to be completely localized. In some sense, S_{WZ} computed with thickened connections defines the value of S_{WZ} with localized connections in \mathbb{R}/\mathbb{Z} . However, finding a universal combinatorial formula on the values of V might lead to a new geometrical interpretation of S_{WZ} related to dimension 2.

Lastly, taking into account what has just been said, in the case of completely thickened connections, the map V sends the surface Σ in $SU(2)$. Thus the geometry of $V(\Sigma)$ could be expected to have the same genus as Σ . In the example investigated, this is not exactly the case, in the sense that all the handles degenerate in dimension 1 objects making the volume bounded by $V(\Sigma)$ homeomorphic to the 3-ball B^3 (volume such that $\partial B^3 = S^2$).

1.2 Example of lens spaces

Flat connections and Heegaard diagram. In this section the general method presented in section 1.1 is applied to compute the $SU(N)$ CS invariant of a lens space. A lens space $L(p, q)$ is considered here. Its fundamental group $\pi_1(L(p, q))$ is \mathbb{Z}_p and there are $\lfloor \frac{p}{2} \rfloor$ conjugacy classes of $SU(N)$ -representations of $\pi_1(L(p, q))$, $\lfloor \frac{p}{2} \rfloor$ being the integer part of $\frac{p}{2}$. Each class admits a representative that can be written:

$$\rho(\gamma) = e^{2i\pi n \frac{1}{p} \sigma}, \quad (1.46)$$

where n is an integer in $\llbracket 0; \lfloor \frac{p}{2} \rfloor \rrbracket$ ⁷, γ is a generator of $\pi_1(L(p, q))$ and $\sigma = \sigma_\gamma$ is an element of $SU(N)$. Since the fundamental group is generated by only one generator,

⁶Indeed, at each intersection of strips that represent A_R^Σ and $h_* A_L^\Sigma$, V is a map from $[0; \varepsilon_x] \times [0; \varepsilon_y]$ to $SU(2)$. The continuity of V in this smooth case guarantees that each intersection domain defines a face of a closed body in $SU(2)$.

⁷It could be expected that $n \in \llbracket 0; p-1 \rrbracket$ but there would be then two representatives for each class. Indeed, if $\rho(\gamma) = e^{2i\pi n \frac{1}{p} \sigma}$ with $n \in \llbracket \lfloor \frac{p}{2} \rfloor + 1; p-1 \rrbracket$, then, there exists an element $V \in SU(N)$ such that $V\rho(\gamma)V^{-1} = e^{2i\pi n' \frac{1}{p} \sigma}$ with $n' \in \llbracket 0; \lfloor \frac{p}{2} \rfloor \rrbracket$.

1.2. Example of lens spaces

there is no specific constraint on σ , which can be chosen, for example, to be:

$$\sigma = \begin{pmatrix} 1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ \vdots & & \ddots & & & & & & & \vdots \\ \vdots & & & \ddots & & & & & & \vdots \\ \vdots & & & & \ddots & & & & & \vdots \\ \vdots & & & & & \ddots & & & & \vdots \\ \vdots & & & & & & \ddots & & & \vdots \\ \vdots & & & & & & & \ddots & & \vdots \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 & -(N-1) \end{pmatrix} \quad (1.47)$$

Thanks to the theorem introduced in the previous section, the class of conjugation of this representation is associated with the gauge class of a flat connection, among which a localized representative A_R , and so in particular A_R^Σ , carries a charge:

$$A_R^\Sigma = 2i\pi n \frac{1}{p} \sigma. \quad (1.48)$$

A Heegaard diagram associated with $L(p, q)$ is given in Figure 1.4. Filling out this diagram thanks to rules (1.36) and (1.38):

$$h_* A_L^\Sigma = 2i\pi n \frac{q}{p} \sigma. \quad (1.49)$$

Computation of the surface term. Since this case is abelian, the CS invariant coincides with the surface term, which has a simpler expression than the general one:

$$S_{\text{CS}} = \frac{1}{8\pi^2} \oint_{\Sigma} \text{Tr}(h_* A_L^\Sigma \wedge A_R^\Sigma). \quad (1.50)$$

This integral has p contributions arising at the p intersections of the trace of $h_* A_L^\Sigma$ and A_R^Σ , each contribution being:

$$\text{Tr}(h_* A_L^\Sigma \wedge A_R^\Sigma) = \left(2i\pi n \frac{q}{p}\right) \left(2i\pi n \frac{1}{p}\right) \text{Tr}(\sigma^2), \quad (1.51)$$

that is:

$$\text{Tr}(h_* A_L^\Sigma \wedge A_R^\Sigma) = -8\pi^2 n^2 \frac{q}{p^2} \frac{N(N-1)}{2} \quad (1.52)$$

and thus:

$$S_{\text{CS}} = \frac{1}{8\pi^2} \times p \times \left(-8\pi^2 n^2 \frac{q}{p^2} \frac{N(N-1)}{2}\right) = -n^2 \frac{q}{p} \frac{N(N-1)}{2}, \quad (1.53)$$

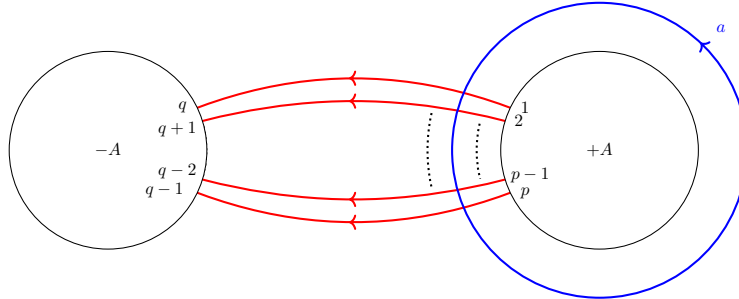


Figure 1.4: Heegaard diagram associated with $L(p, q)$.

since, as already mentioned, there is no WZ term in the abelian case.

In particular, in the $SU(2)$ case:

$$S_{\text{CS}} \underset{\mathbb{Z}}{=} -n^2 \frac{q}{p}, \quad (1.54)$$

keeping in mind that $n \in \llbracket 0; \lfloor \frac{p}{2} \rfloor \rrbracket$. It is important to notice that the $SU(2)$ case looks like the $U(1)$ case but does not produce exactly the same CS invariant. Indeed, in the abelian partition given in equation (2.20) of chapter 1 of the second part, for a lens space $L(p, q)$:

$$S_{\text{CS}} \underset{\mathbb{Z}}{=} Q(\tau_A, \tau_A) \underset{\mathbb{Z}}{=} -n^2 \frac{q}{p}, \quad (1.55)$$

with $\tau_A \in T^2 = \mathbb{Z}_p$ and $n \in \llbracket 0; p-1 \rrbracket$ associated with τ_A by the universal coefficients theorem⁸.

In the $SU(2)$ case, although the representations of $\pi_1(L(p, q))$ are abelian, n is not the index of the cohomology class of a given generator, but the index of the conjugacy class of its representation of $\pi_1(L(p, q))$. Hence, n is in $\llbracket 0; \lfloor \frac{p}{2} \rfloor \rrbracket$ and not in $\llbracket 0; p-1 \rrbracket$.

⁸It shall be reminded at this step that, according to part II, only the flat connections contribute to the abelian CS and BF partition functions whereas in the non-abelian case, some fluctuations around this contribution are expected.

1.3 Example of some genus 2 Heegaard splittings

Flat connections and Heegaard diagram. In this section the general method presented in section 1.1 is used to compute the $SU(2)$ CS invariant of a family of closed manifolds $M^{(3)}$ of dimension 3 whose general Heegaard diagram is given in Figure 1.5. This family of manifolds includes some homology spheres, that are manifolds having the same homology as the sphere but which are not homeomorphic to the sphere (they show, as a consequence, that the homology is not classifying). The non-abelian case is expected to probe a quantity related to the homotopy.

Since a loop around a handle is homotopically trivial inside H_R , the relations generating the fundamental group can be read turning around the handles (the discs on the diagrams). Hence the fundamental group⁹ of such a family of manifolds is:

$$\pi_1(M^{(3)}) = \langle \gamma_a, \gamma_b \mid \gamma_a^{n_a} = \gamma_b^{n_b} = (\gamma_a \gamma_b)^{n_{ab}} \rangle. \quad (1.56)$$

In the presentation (1.56) the element $(\gamma_a \gamma_b)^{n_{ab}}$ turns out to be central. Since the center of $SU(2)$ is $\pm \mathbf{1}$, there are two possibilities to build a $SU(2)$ -representation ρ of $\pi_1(M^{(3)})$. The choice made here is¹⁰:

$$a^{n_a} = b^{n_b} = (ab)^{n_{ab}} = -\mathbf{1}, \quad (1.57)$$

with:

$$\begin{cases} \rho(\gamma_a) = a = e^{i \frac{(2\beta+1)\pi}{n_a} \sigma_3} = \cos\left(\frac{(2\alpha+1)\pi}{n_a}\right) + i \sin\left(\frac{(2\alpha+1)\pi}{n_a}\right) \sigma_3 \\ \rho(\gamma_b) = b = e^{i \frac{(2\beta+1)\pi}{n_b} \sigma_r} = \cos\left(\frac{(2\beta+1)\pi}{n_b}\right) + i \sin\left(\frac{(2\beta+1)\pi}{n_b}\right) \sigma_r \\ (ab) = e^{i \frac{(2\gamma+1)\pi}{n_{ab}} \sigma_{ab}} = \cos\left(\frac{(2\gamma+1)\pi}{n_{ab}}\right) + i \sin\left(\frac{(2\gamma+1)\pi}{n_{ab}}\right) \sigma_{ab}, \end{cases} \quad (1.58)$$

where:

$$\sigma_r = \sqrt{1-r^2} \sigma_1 + r \sigma_3, \quad (1.59)$$

σ_1, σ_2 and σ_3 being still Pauli matrices. Furthermore, $\alpha \in \llbracket 0; \lceil \frac{n_a-1}{2} \rceil \rrbracket$, $\beta \in \llbracket 0; \lceil \frac{n_b-1}{2} \rceil \rrbracket$ and $\gamma \in \llbracket 0; \lceil \frac{n_{ab}-1}{2} \rceil \rrbracket$. Indeed, according to [Kirk and Klassen, 1990], contrary to the lens spaces, there is no redundancy here.

Computing the product ab and identifying the real part with $\cos\left(\frac{(2\gamma+1)\pi}{n_{ab}}\right)$:

$$r = \frac{\cos\left(\frac{(2\alpha+1)\pi}{n_a}\right) \cos\left(\frac{(2\beta+1)\pi}{n_b}\right) - \cos\left(\frac{(2\gamma+1)\pi}{n_{ab}}\right)}{\sin\left(\frac{(2\alpha+1)\pi}{n_a}\right) \sin\left(\frac{(2\beta+1)\pi}{n_b}\right)} \quad (1.60)$$

⁹The general fundamental group of a manifold $M^{(3)}$ of dimension 3 is not classifying, which means that it does not contain the whole information on the topology of $M^{(3)}$, but the Heegaard diagram does.

¹⁰It can be verified that the opposite choice actually leads to the trivial representation.

Chapter 1. $SU(N)$ Chern-Simons invariant of flat connections

and identifying the imaginary part with $\sin\left(\frac{(2\gamma+1)\pi}{n_{ab}}\right)$:

$$\begin{aligned}
 \sigma_{ab} = & \frac{\cos\left(\frac{(2\alpha+1)\pi}{n_a}\right) \sin\left(\frac{(2\beta+1)\pi}{n_b}\right)}{\sin\left(\frac{(2\gamma+1)\pi}{n_{ab}}\right)} \sqrt{1-r^2} \sigma_1 \\
 & - \frac{\sin\left(\frac{(2\alpha+1)\pi}{n_a}\right) \sin\left(\frac{(2\beta+1)\pi}{n_b}\right)}{\sin\left(\frac{(2\gamma+1)\pi}{n_{ab}}\right)} \sqrt{1-r^2} \sigma_2 \\
 & + \frac{\sin\left(\frac{(2\alpha+1)\pi}{n_a}\right) \cos\left(\frac{(2\beta+1)\pi}{n_b}\right) + r \cos\left(\frac{(2\alpha+1)\pi}{n_a}\right) \sin\left(\frac{(2\beta+1)\pi}{n_b}\right)}{\sin\left(\frac{(2\gamma+1)\pi}{n_{ab}}\right)} \sqrt{1-r^2} \sigma_3.
 \end{aligned} \tag{1.61}$$

Set from now on:

$$\begin{cases} \theta_a = \frac{(2\alpha+1)\pi}{n_a}, \\ \theta_b = \frac{(2\beta+1)\pi}{n_b}, \\ \theta_{ab} = \frac{(2\gamma+1)\pi}{n_{ab}}. \end{cases} \tag{1.62}$$

Rules (1.36) and (1.39) given in section 1.1 of this chapter can then be used to fill out a diagram with the piece of information that is needed for the calculations. The result is given in Figure 1.6 with focuses around the handles in Figures 1.7 and 1.8.

1.3. Example of some genus 2 Heegaard splittings

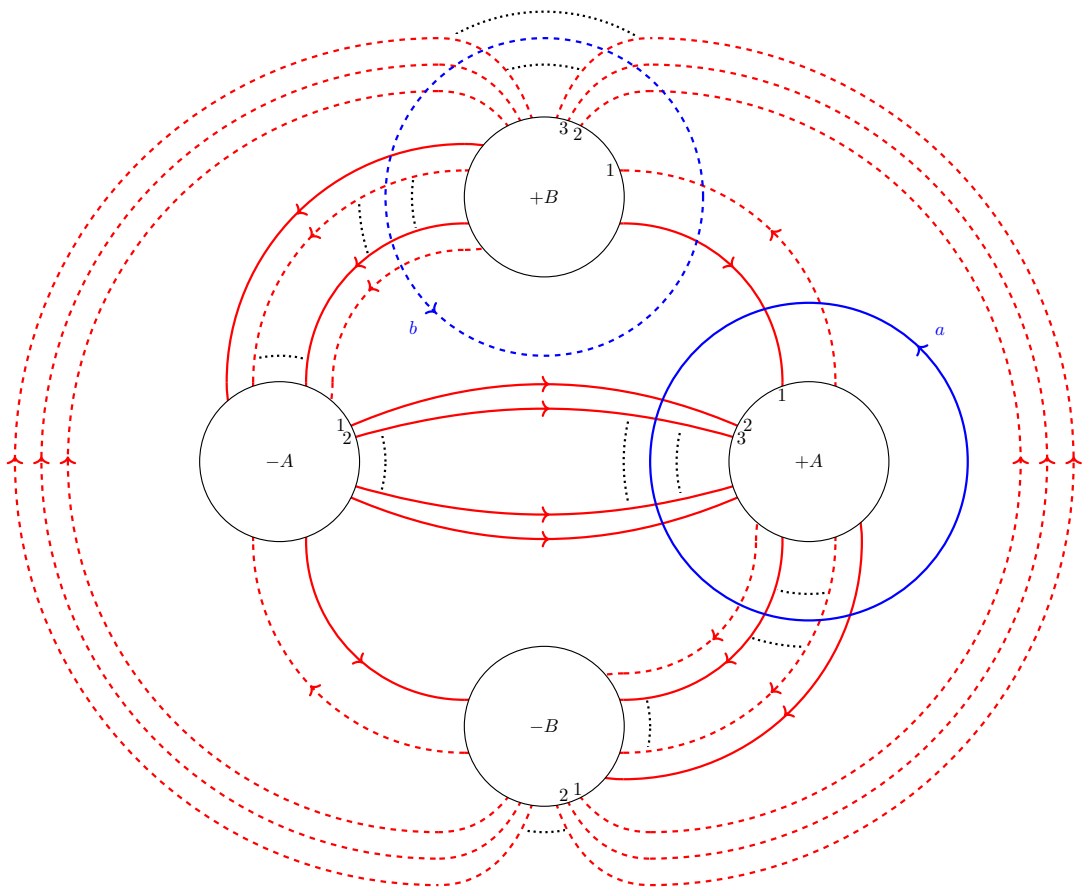


Figure 1.5: Heegaard diagram associated with $M^{(3)}$.

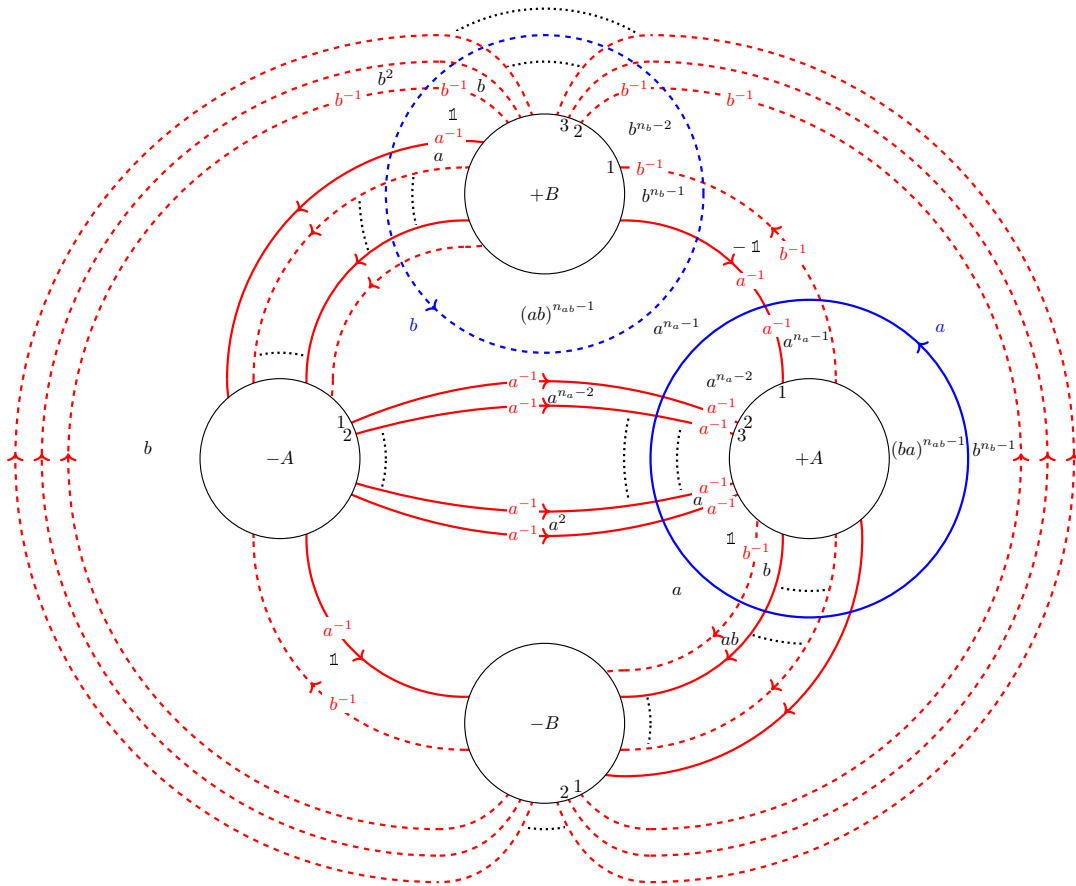


Figure 1.6: Heegaard diagram associated with $M^{(3)}$ with charges.

1.3. Example of some genus 2 Heegaard splittings

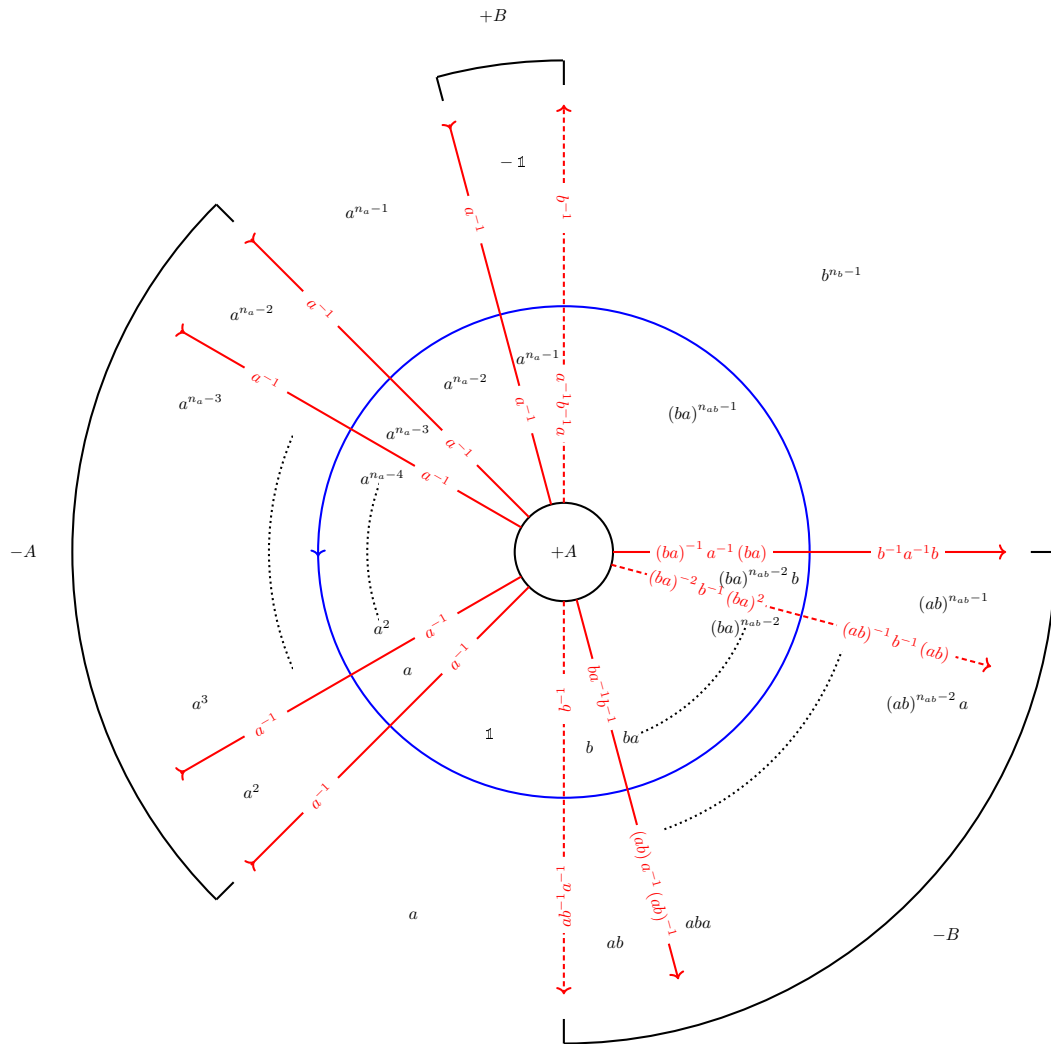


Figure 1.7: Focus on the handle $+A$.

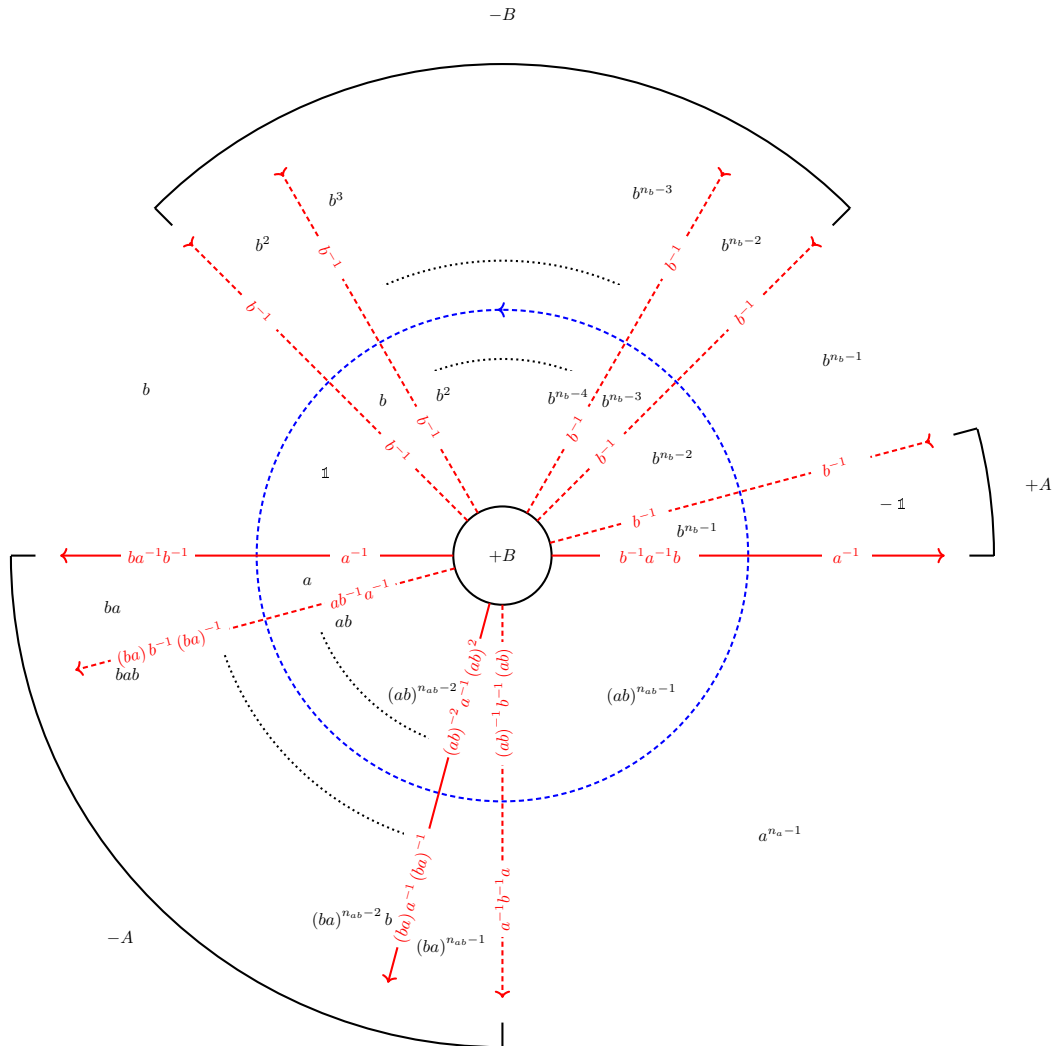


Figure 1.8: Focus on the handle $+B$.

1.3. Example of some genus 2 Heegaard splittings

Computation of the surface term. Computing the surface term I:

$$\begin{aligned} \text{I} = \frac{1}{8\pi^2} \left\{ & -\theta_a^2 \left[(n_{ab} - 2) - 2(n_a - 1) + \frac{n_{ab}}{2} \text{Tr}(\sigma_3 \sigma_{ab} \sigma_3 \sigma_{ab}) \right] \right. \\ & - \theta_b^2 \left[(n_{ab} - 2) - 2r^2(n_b - 1) + \frac{n_{ab}}{2} \text{Tr}(\sigma_r \sigma_{ba} \sigma_r \sigma_{ba}) \right] \\ & \left. - \theta_a \theta_b \left[2rn_{ab} + \frac{n_{ab}}{2} \text{Tr}(\sigma_3 \sigma_{ba} \sigma_r \sigma_{ba}) + \frac{n_{ab}}{2} \text{Tr}(\sigma_r \sigma_{ab} \sigma_3 \sigma_{ab}) \right] \right\}, \end{aligned} \quad (1.63)$$

with:

$$\left\{ \begin{aligned} \text{Tr}(\sigma_3 \sigma_{ab} \sigma_3 \sigma_{ab}) &= \frac{1}{\sin^2(\theta_a) \sin^2(\theta_{ab})} \\ &\quad \left(-8 \cos(\theta_a) \cos(\theta_b) \cos(\theta_{ab}) \right. \\ &\quad \left. + 2 \cos^2(\theta_a) (1 + \cos^2(\theta_{ab})) \right. \\ &\quad \left. + 4 \cos^2(\theta_b) - 2 \sin^2(\theta_{ab}) \right), \\ \text{Tr}(\sigma_r \sigma_{ba} \sigma_r \sigma_{ba}) &= \frac{1}{\sin^2(\theta_b) \sin^2(\theta_{ab})} \\ &\quad \left(-8 \cos(\theta_a) \cos(\theta_b) \cos(\theta_{ab}) \right. \\ &\quad \left. + 2 \cos^2(\theta_b) (1 + \cos^2(\theta_{ab})) \right. \\ &\quad \left. + 4 \cos^2(\theta_a) - 2 \sin^2(\theta_{ab}) \right), \\ \text{Tr}(\sigma_r \sigma_{ab} \sigma_3 \sigma_{ab}) &= \frac{1}{\sin(\theta_a) \sin(\theta_b) \sin^2(\theta_{ab})} \\ &\quad \left(6 \cos(\theta_a) \cos(\theta_b) \cos^2(\theta_{ab}) \right. \\ &\quad \left. - 4 \cos(\theta_{ab}) (\cos^2(\theta_a) + \cos^2(\theta_b)) \right. \\ &\quad \left. + 2 \cos(\theta_a) \cos(\theta_b) + 2 \cos(\theta_{ab}) \right. \\ &\quad \left. - 2 \cos^3(\theta_{ab}) \right), \\ \text{Tr}(\sigma_3 \sigma_{ba} \sigma_r \sigma_{ba}) &= \text{Tr}(\sigma_r \sigma_{ab} \sigma_3 \sigma_{ab}). \end{aligned} \right. \quad (1.64)$$

The case $n_{ab} = 2$ has been completely solved analytically. The associated Heegaard diagram is given in Figure 1.9. The diagram filled out with charges on each line is given in Figure 1.10. After computation of the intersection terms:

$$\text{I} = \frac{1}{4} \left(n_a \left(\frac{\theta_a}{\pi} \right)^2 + n_b \left(\frac{\theta_b}{\pi} \right)^2 - 2A^2 \right), \quad (1.65)$$

with:

$$A = \left(\frac{\cos(\theta_a)}{\sin(\theta_b)} \left(\frac{\theta_a}{\pi} \right) + \frac{\cos(\theta_b)}{\sin(\theta_a)} \left(\frac{\theta_b}{\pi} \right) \right), \quad (1.66)$$

that is:

$$I = \frac{1}{4} \left(\frac{(2\alpha + 1)^2}{n_a} + \frac{(2\beta + 1)^2}{n_b} - 2A^2 \right). \quad (1.67)$$

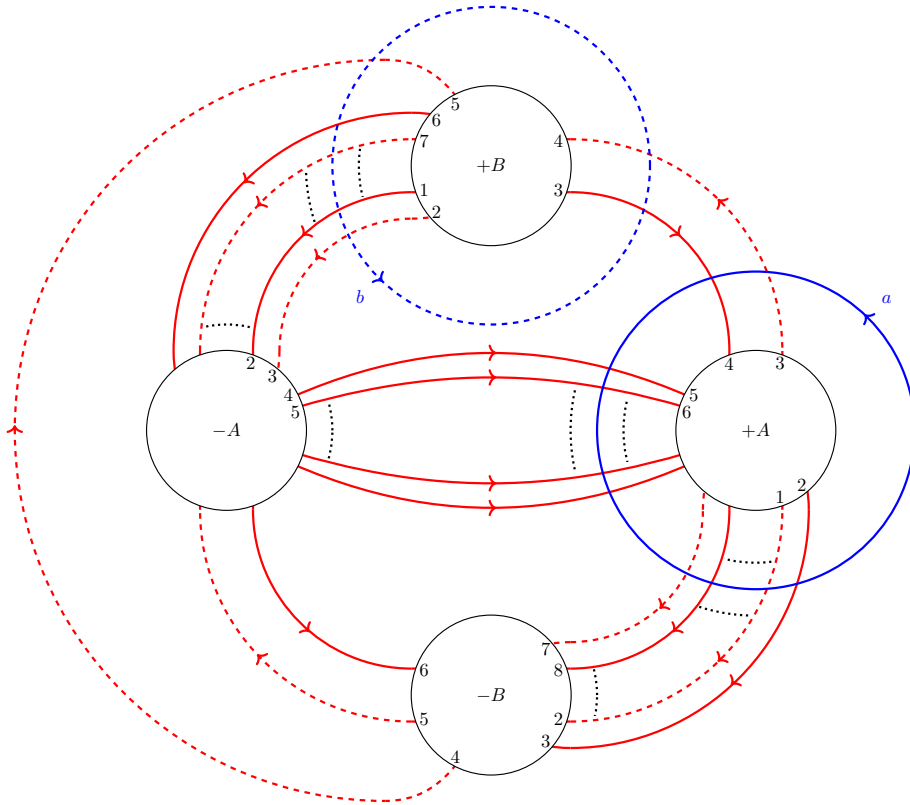


Figure 1.9: Particular case $n_{ab} = 2$.

1.3. Example of some genus 2 Heegaard splittings

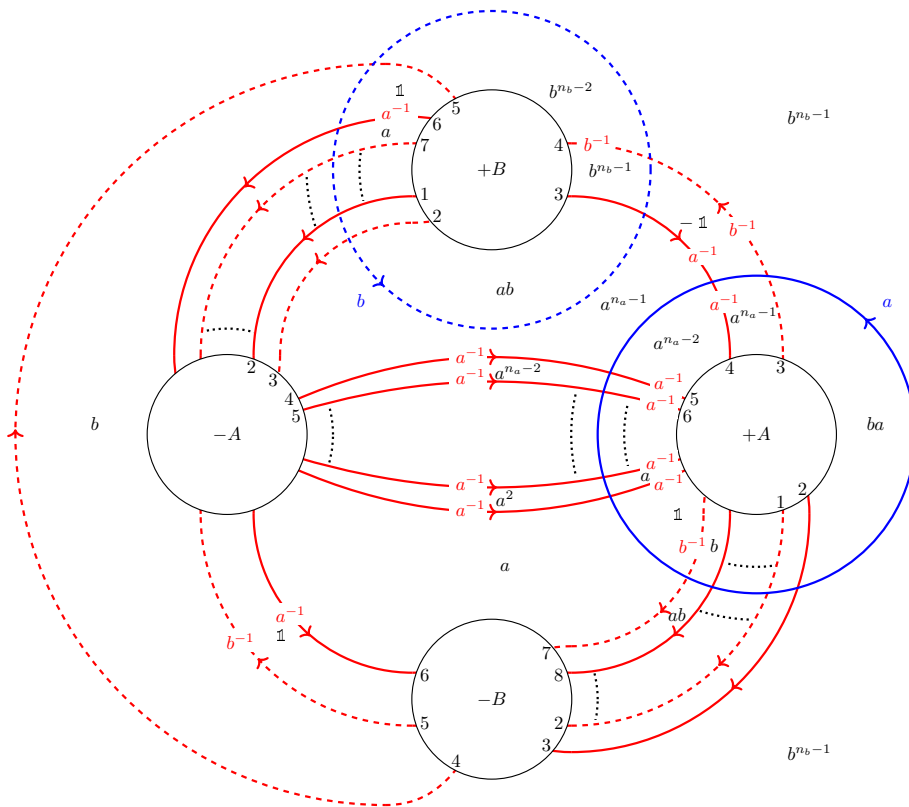


Figure 1.10: Particular case $n_{ab} = 2$ with charges.

Chapter 1. $SU(N)$ Chern-Simons invariant of flat connections

Computation of the WZ term. The WZ term is quite tricky to compute and for the moment only the particular case $n_{ab} = 2$ has been completely solved analytically. Equation (1.45) of section 1.1 of this chapter can be rewritten in this particular case as:

$$\begin{aligned} & \frac{1}{4\pi^2} \oint_{\Sigma} (\xi - \cos(\xi) \sin(\xi)) n_2^3 d\left(\frac{n_1}{n_2}\right) \wedge d\left(\frac{n_3}{n_2}\right) \\ &= \oint_{\Sigma} \partial_x \partial_y \left\{ \frac{1}{4\pi^2} \left(\frac{1}{2} \xi^2(x, y) - r(\theta_a x)(\theta_b y) \right) \right\}. \end{aligned} \quad (1.68)$$

Thus, the contribution on each domain $(x, y) \in [0; 1]^2$ is:

$$\frac{1}{4\pi^2} \left(\frac{1}{2} \xi^2(0, 0) + \frac{1}{2} \xi^2(1, 1) - \frac{1}{2} \xi^2(1, 0) - \frac{1}{2} \xi^2(0, 1) - r\theta_a \theta_b \right). \quad (1.69)$$

Summing over all the domains, finally:

$$S_{\text{WZ}} = \frac{1}{4} \left(\frac{1}{2} - 2A^2 \right), \quad (1.70)$$

so that:

$$S_{\text{CS}} = \frac{1}{4} \left(\frac{(2\alpha + 1)^2}{n_a} + \frac{(2\beta + 1)^2}{n_b} - \frac{1}{2} \right). \quad (1.71)$$

Moreover, a group $G = \langle \gamma_a, \gamma_b \mid \gamma_a^{n_a} = \gamma_b^{n_b} = (\gamma_a \gamma_b)^{n_{ab}} = e \rangle$ is called a **Von Dyck group**, e being the neutral element. The Von Dyck groups are classified according to three different types:

- spherical type if $\frac{1}{n_a} + \frac{1}{n_b} + \frac{1}{n_{ab}} > 1$ (all finite),
- Euclidean type if $\frac{1}{n_a} + \frac{1}{n_b} + \frac{1}{n_{ab}} = 1$ (all infinite),
- hyperbolic type if $\frac{1}{n_a} + \frac{1}{n_b} + \frac{1}{n_{ab}} < 1$ (all infinite).

There are actually finitely many Von Dyck groups of spherical type which are:

$$(n_a, n_b, n_{ab}) = \begin{cases} (3, 3, 2), \\ (4, 3, 2), \\ (5, 3, 2), \\ (n, 2, 2). \end{cases} \quad (1.72)$$

All those spherical type Von Dyck groups are finite and of order $|G|$ such that:

$$\frac{1}{|G|} = \frac{1}{2} \left(\frac{1}{n_a} + \frac{1}{n_b} - \frac{1}{n_{ab}} \right), \quad (1.73)$$

1.3. Example of some genus 2 Heegaard splittings

with $n_{ab} = 2$.

Assume now $\pi_1(M^{(3)}) = \langle \gamma_a, \gamma_b \mid \gamma_a^{n_a} = \gamma_b^{n_b} = (\gamma_a \gamma_b)^{n_{ab}} \rangle$ with $\frac{1}{n_a} + \frac{1}{n_b} + \frac{1}{n_{ab}} > 1$. Consider also $\rho((\gamma_a \gamma_b)^{n_{ab}}) = -\mathbf{1}$. Then, the elements of $\rho(\pi_1(M^{(3)}))$ are the elements of the corresponding Von Dyck group up to a sign.

Hence, when $\frac{1}{n_a} + \frac{1}{n_b} + \frac{1}{n_{ab}} > 1$:

$$\frac{1}{|\rho(\pi_1(M^{(3)}))|} = \frac{1}{2|G|} = \frac{1}{4} \left(\frac{1}{n_a} + \frac{1}{n_b} - \frac{1}{n_{ab}} \right) \quad (1.74)$$

and thus, for $\alpha = \beta = 0$:

$$S_{\text{CS}} = \frac{1}{|\rho(\pi_1(M^{(3)}))|}. \quad (1.75)$$

This result is still true in \mathbb{R}/\mathbb{Z} for the trivial representation.

It can be stated in particular that the Poincaré homology sphere corresponds to $(n_a, n_b, n_{ab}) = (5, 3, 2)$. It is known that its fundamental group is of order 120. Therefore, in that case, the representation ρ is faithful.

The general case where n_{ab} is not specified has been investigated but not solved analytically. Numerical computations seem to indicate that the result for $n_{ab} = 2$ can be generalized to:

$$S_{\text{CS}} = \frac{1}{4} \left(\frac{(2\alpha + 1)^2}{n_a} + \frac{(2\beta + 1)^2}{n_b} - \frac{(2\gamma + 1)^2}{n_{ab}} \right). \quad (1.76)$$

Comparison with [Kirk and Klassen, 1990]. The Poincaré homology sphere admits a Heegaard diagram given in Figure 1.11. The diagram filled out with charges on each line is given in Figure 1.12. On the one hand, this is a particular case of the family of diagrams considered in this section, whose fundamental group parameters are:

$$\begin{cases} n_a = 5, \\ n_b = 3, \\ n_{ab} = 2. \end{cases} \quad (1.77)$$

On the other hand, the Poincaré homology sphere is a particular case of so-called **Seifert manifolds** over S^2 with three exceptional fibers, described by:

$$\{0, (o_1, 0), (5, 1), (3, 1), (2, -1)\} \quad (1.78)$$

according to Seifert's notations.

Hence, it seems reasonable to assume that the whole family of diagrams considered is associated with such kind of manifolds, described more generally by:

$$\{0, (o_1, 0), (n_a, 1), (n_b, 1), (n_{ab}, -1)\} \quad (1.79)$$

Chapter 1. $SU(N)$ Chern-Simons invariant of flat connections

and whose fundamental group admits the presentation:

$$\begin{aligned} \pi_1(M^{(3)}) \\ = \langle x_a, x_b, x_{ab} \mid hx_i = x_ih, x_a^{n_a}h = x_b^{n_b}h = x_{ab}^{n_{ab}}h^{-1} = 1, x_ax_bx_{ab} = 1 \rangle, \end{aligned} \quad (1.80)$$

which coincides in fact with the fundamental group read on the Heegaard diagrams presented before, noticing that the presentation implies:

$$x_{ab} = (x_ax_b)^{-1}. \quad (1.81)$$

As a consequence the previous equivalent notations will be used from now on.

It turns out that such a manifold is a homology sphere if and only if n_a , n_b and n_{ab} are pairwise relatively prime and:

$$\frac{1}{n_a} + \frac{1}{n_b} - \frac{1}{n_{ab}} = \frac{1}{n_an_bn_{ab}}. \quad (1.82)$$

In this case, P. Kirk and E. Klassen found that:

$$S_{\text{CS}}^{\text{KK}} = \frac{n_an_bn_{ab}}{4} \left(\frac{(2\alpha+1)}{n_a} + \frac{(2\beta+1)}{n_b} + \frac{(2\gamma+1)}{n_{ab}} \right)^2. \quad (1.83)$$

It can be shown that, actually, in \mathbb{R}/\mathbb{Z} :

$$S_{\text{CS}}^{\text{KK}} \stackrel{\mathbb{Z}}{=} \frac{1}{4} \left(\frac{(2\alpha+1)^2}{n_a} + \frac{(2\beta+1)^2}{n_b} - \frac{(2\gamma+1)^2}{n_{ab}} \right), \quad (1.84)$$

which is consistent with the result (1.71) computed before and the expression (1.76) conjectured after numerical computations.

1.3. Example of some genus 2 Heegaard splittings

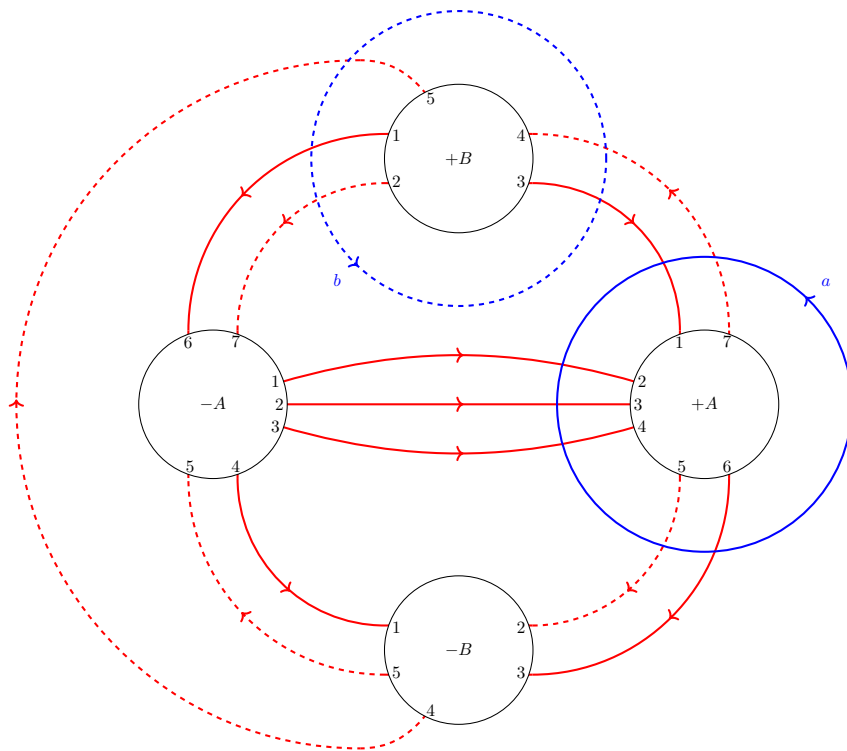


Figure 1.11: Particular case of the Poincaré homology sphere.

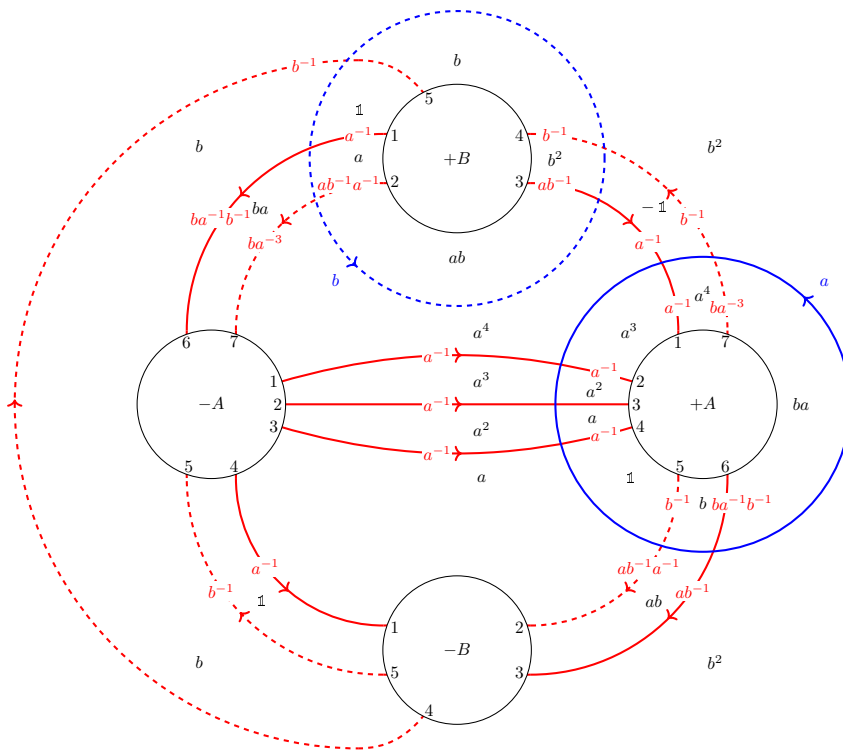


Figure 1.12: Particular case of the Poincaré homology sphere with charges.

Chapter 2

Geometric interpretation of gauge fixing in \mathbb{R}^{4l+3}

Contents

Introduction (English version)	134
Introduction (version française)	136
2.1 Conventions	138
2.2 Correlators	140
2.3 Limits	146
2.4 Geometric interpretation	148

Introduction (English version)

The abelian CS action is considered in \mathbb{R}^{4l+3} in this chapter:

$$S_{\text{CS}_k} = \frac{k}{4\pi^2} \int_{\mathbb{R}^{4l+3}} \tilde{A}^{(2l+1)} \wedge d\tilde{A}^{(2l+1)}. \quad (2.1)$$

It should not be forgotten that this expression is not the abelianization of the SU(2) CS action (see the normalization).

Set $A = \frac{\tilde{A}}{2\pi}$ (to be compared with the choice made in chapter 1 of the first part). Then:

$$S_{\text{CS}_k} = k \int_{\mathbb{R}^{4l+3}} A^{(2l+1)} \wedge dA^{(2l+1)}, \quad (2.2)$$

which is the action considered further.

It should be noted that the connections, which are objects of degree 1 in the 3-dimensional case, are replaced here by objects $A^{(2l+1)}$ of degree $2l+1$ in the $4l+3$ -dimensional case. Since \mathbb{R}^{4l+3} is contractible, $A^{(2l+1)} \wedge dA^{(2l+1)}$ is an object of degree $4l+3$ which is defined globally.

For using standard methods of computation of QFT, this action should be written under the form:

$$S_{\text{CS}_k} = k (A^{(2l+1)}, \mathcal{D}A^{(2l+1)}), \quad (2.3)$$

where \mathcal{D} is an invertible operator. This is not possible for the moment. Under this form, the theory has too many degrees of freedom and \mathcal{D} is not invertible. Some degrees of freedom have to be fixed. For this purpose, a gauge has to be fixed and this so-called **gauge fixing procedure** has to be implemented in the action by introducing auxiliary fields $B^{(i)}$ analogous to Lagrange multipliers. This procedure therefore puts some constraints on the fields of the theory and the action becomes:

$$S_{\text{CS}_k} = k (\mathcal{A}, \mathcal{D}\mathcal{A}), \quad (2.4)$$

with $\mathcal{A} = (A^{(2l+1)}, B^{(2l)}, \dots, B^{(0)})$.

Then, consider two $2l+1$ -cycles $\gamma_1^{(2l+1)}$ and $\gamma_2^{(2l+1)}$ and their associated de Rham current $j_1^{(2l+2)}$ and $j_2^{(2l+2)}$, that is, distributional objects such that:

$$\int_{\gamma_i^{(2l+1)}} A^{(2l+1)} = \int_{\mathbb{R}^{4l+3}} A^{(2l+1)} \wedge j_i^{(2l+2)}. \quad (2.5)$$

Suppose γ_i is carrying a charge q_i . Then:

$$q_1 \int_{\gamma_1^{(2l+1)}} A^{(2l+1)} + q_2 \int_{\gamma_2^{(2l+1)}} A^{(2l+1)} = (\mathcal{A}, \mathcal{J}), \quad (2.6)$$

where $\mathcal{J} = (q_1 j_1^{(2l+2)} + q_2 j_2^{(2l+2)}, 0, \dots, 0)$. In the formalism of path integral, the expectation value $\langle \gamma_1^{(2l+1)}, \gamma_2^{(2l+1)} \rangle$ is given by:

$$\langle \gamma_1^{(2l+1)}, \gamma_2^{(2l+1)} \rangle = \frac{1}{\mathcal{N}} \int_{\mathcal{H}} \mathcal{D}A^{(2l+1)} e^{2i\pi k \int_{\mathbb{R}^{4l+3}} A^{(2l+1)} \wedge dA^{(2l+1)}} e^{2i\pi \oint_{\gamma_1^{(2l+1)}} A^{(2l+1)}} e^{2i\pi \oint_{\gamma_2^{(2l+1)}} A^{(2l+1)}}. \quad (2.7)$$

Taking into account the degeneracy of the theory and using a gauge fixing procedure, the expectation value $\langle \gamma_1^{(2l+1)}, \gamma_2^{(2l+1)} \rangle$ is actually given by:

$$\langle \gamma_1, \gamma_2 \rangle = \frac{1}{\mathcal{N}_{\text{CS}_k}} \int_{\mathcal{H}} \mathcal{D}A e^{2i\pi k((\mathcal{A}, \mathcal{D}\mathcal{A}) + (\mathcal{A}, \mathcal{J}))}. \quad (2.8)$$

Now perform a shift:

$$\mathcal{A} \longrightarrow \mathcal{A} - \frac{1}{2k} \mathcal{D}^{-1} \mathcal{J}. \quad (2.9)$$

Thanks to the self-adjointness of \mathcal{D} and admitting that the formal measure \mathcal{D} is invariant under translation:

$$\langle \gamma_1, \gamma_2 \rangle = \frac{e^{-\frac{2i\pi}{4k}(\mathcal{J}, \mathcal{D}^{-1} \mathcal{J})}}{\mathcal{N}_{\text{CS}_k}} \int_{\mathcal{H}} \mathcal{D}A e^{2i\pi k(\mathcal{A}, \mathcal{D}\mathcal{A})}. \quad (2.10)$$

Hence, the quantity of interest is actually $(\mathcal{J}, \mathcal{D}^{-1} \mathcal{J})$ and the technical difficulty still consists in inverting the suitable operator \mathcal{D} . It should be noted otherwise that the $\gamma_i^{(2l+1)}$ are not exactly cycles but framed cycles¹ and even more precisely with ambient isotopy classes of framed cycles.

The point of this chapter is to interpret geometrically a set of gauge fixing procedures. The propagator of the abelian CS theory in \mathbb{R}^{4l+3} is computed with respect to a distortion of the covariant gauge in Euclidean metric following two methods. The first method is the usual one. The second consists in absorbing the distortion of the gauge in the metric so as to consider the usual covariant gauge with an anisotropic metric. The results are then interpreted in terms of linking number.

The study presented here is a generalization of [Gallot et al., 2018].

¹It should not be forgotten that the framing of a cycle can be interpreted in the following way: a cycle is not exactly a loop but a closed strip which is twisted n times, $n \in \mathbb{Z}$.

Introduction (version française)

L'action CS abélienne est considérée dans \mathbb{R}^{4l+3} dans ce chapitre :

$$S_{\text{CS}_k} = \frac{k}{4\pi^2} \int_{\mathbb{R}^{4l+3}} \tilde{A}^{(2l+1)} \wedge d\tilde{A}^{(2l+1)}. \quad (2.1)$$

Il ne doit pas être oublié que cette expression n'est pas l'abélianisation de l'action CS SU(2) (voir la normalisation).

Soit $A = \frac{\tilde{A}}{2\pi}$ (à comparer avec le choix fait dans le chapitre 1 de la première partie). Alors :

$$S_{\text{CS}_k} = k \int_{\mathbb{R}^{4l+3}} A^{(2l+1)} \wedge dA^{(2l+1)}, \quad (2.2)$$

qui est l'action considérée plus loin.

Il doit être noté que les connexions, qui sont des objets de degré 1 dans le cas de dimension 3, sont remplacées ici par des objets $A^{(2l+1)}$ de degré $2l+1$ dans le cas de dimension $4l+3$. Puisque \mathbb{R}^{4l+3} est contractile, $A^{(2l+1)} \wedge dA^{(2l+1)}$ est un objet de degré $4l+3$ qui est défini globalement.

Pour utiliser les méthodes standard de calcul de QFT, cette action doit être écrite sous la forme :

$$S_{\text{CS}_k} = k (A^{(2l+1)}, \mathcal{D}A^{(2l+1)}), \quad (2.3)$$

où \mathcal{D} est un opérateur inversible. Ce n'est pour l'instant pas possible. Sous cette forme, la théorie a trop de degrés de liberté et \mathcal{D} n'est pas inversible. Des degrés de liberté doivent être fixés. Pour cela, une jauge doit être fixée et cette procédure dite de **fixation de jauge** doit être implémentée dans l'action en utilisant des champs auxiliaires $B^{(i)}$ analogues à des multiplicateurs de Lagrange. Cette procédure contraint par conséquent les champs de la théorie et l'action devient :

$$S_{\text{CS}_k} = k (\mathcal{A}, \mathcal{D}\mathcal{A}), \quad (2.4)$$

avec $\mathcal{A} = (A^{(2l+1)}, B^{(2l)}, \dots, B^{(0)})$.

Ensuite, des $2l+1$ -cycles $\gamma_1^{(2l+1)}$ et $\gamma_2^{(2l+1)}$ sont considérés ainsi que leurs courants de de Rham associés $j_1^{(2l+2)}$ et $j_2^{(2l+2)}$, c'est-à-dire, des objets distributionnels tels que :

$$\int_{\gamma_i^{(2l+1)}} A^{(2l+1)} = \int_{\mathbb{R}^{4l+3}} A^{(2l+1)} \wedge j_i^{(2l+2)}. \quad (2.5)$$

À γ_i est associée une charge q_i . Alors :

$$q_1 \int_{\gamma_1^{(2l+1)}} A^{(2l+1)} + q_2 \int_{\gamma_2^{(2l+1)}} A^{(2l+1)} = (\mathcal{A}, \mathcal{J}), \quad (2.6)$$

où $\mathcal{J} = (q_1 j_1^{(2l+2)} + q_2 j_2^{(2l+2)}, 0, \dots, 0)$. Dans le formalisme des intégrales fonctionnelles, la valeur moyenne d'observables $\langle \gamma_1^{(2l+1)}, \gamma_2^{(2l+1)} \rangle$ est donnée par :

$$\langle \gamma_1^{(2l+1)}, \gamma_2^{(2l+1)} \rangle = \frac{1}{\mathcal{N}} \int_{\mathcal{H}} \mathcal{D}A^{(2l+1)} e^{2i\pi k \int_{\mathbb{R}^{4l+3}} A^{(2l+1)} \wedge dA^{(2l+1)}} e^{2i\pi \oint_{\gamma_1^{(2l+1)}} A^{(2l+1)}} e^{2i\pi \oint_{\gamma_2^{(2l+1)}} A^{(2l+1)}}. \quad (2.7)$$

Tenant compte de la dégénérescence de la théorie et utilisant une procédure de fixation de jauge, la valeur moyenne $\langle \gamma_1^{(2l+1)}, \gamma_2^{(2l+1)} \rangle$ est en fait donnée par :

$$\langle \gamma_1, \gamma_2 \rangle = \frac{1}{\mathcal{N}_{\text{CS}_k}} \int_{\mathcal{H}} \mathcal{D}A e^{2i\pi k((\mathcal{A}, \mathcal{D}\mathcal{A}) + (\mathcal{A}, \mathcal{J}))}. \quad (2.8)$$

Un changement de variable est ensuite effectué :

$$\mathcal{A} \longrightarrow \mathcal{A} - \frac{1}{2k} \mathcal{D}^{-1} \mathcal{J}. \quad (2.9)$$

Grâce au fait que \mathcal{D} est auto-adjoint, et supposant que la mesure formelle \mathcal{D} est invariante par translation :

$$\langle \gamma_1, \gamma_2 \rangle = \frac{e^{-\frac{2i\pi}{4k}(\mathcal{J}, \mathcal{D}^{-1} \mathcal{J})}}{\mathcal{N}_{\text{CS}_k}} \int_{\mathcal{H}} \mathcal{D}A e^{2i\pi k(\mathcal{A}, \mathcal{D}\mathcal{A})}. \quad (2.10)$$

Ainsi, la quantité intéressante est en fait $(\mathcal{J}, \mathcal{D}^{-1} \mathcal{J})$ et la difficulté technique consiste toujours à inverser l'opérateur \mathcal{D} approprié. Il doit être noté par ailleurs que les $\gamma_i^{(2l+1)}$ ne sont pas exactement des cycles mais des cycles encadrés¹ et même plus précisément des classes d'isotopie ambiante de cycles encadrés.

L'objectif de ce chapitre est d'interpréter géométriquement un ensemble de procédures de fixation de jauge. Le propagateur de la théorie CS abélienne dans \mathbb{R}^{4l+3} est calculé pour une déformation de la jauge covariante en métrique euclidienne suivant deux méthodes. La première méthode est la méthode usuelle. La deuxième consiste à absorber la déformation de la jauge dans la métrique de sorte à considérer la jauge covariante habituelle dans une métrique anisotrope. Les résultats sont alors interprétés en termes de nombre d'enlacements.

Le travail présenté ici est une généralisation de [Gallot et al., 2018].

¹Il ne doit pas être oublié que le cadre d'un cycle peut être interprété de la façon suivante : un cycle n'est pas tout à fait une boucle mais une bande fermée vrillée n fois, $n \in \mathbb{Z}$.

2.1 Conventions

Fields. The vectors are noted here by lower case letters $n = n^\mu \partial_\mu$ and capital or Greek letters are used for the forms $\omega^{(r)} = \omega_{\mu_1 \dots \mu_r} \psi^{\mu_1} \wedge \dots \wedge \psi^{\mu_r}$, with $(\psi^{\mu_1} \wedge \dots \wedge \psi^{\mu_r})_{\mu_1, \dots, \mu_r \in \llbracket 1; 4l+3 \rrbracket}$ a basis of the subspace of the r -forms of the exterior algebra $\bigwedge^{4l+3}(\mathbb{R}^{4l+3})$. Hence, a lowercase letter can be transformed into a capital one via the action of a metric g on \mathbb{R}^{4l+3} as $N_\mu^g = g_{\mu\nu} n^\nu$ for a given vector n . When g is simply the Euclidean metric, the superscript g will be omitted. It should be noted that although the notations of differential geometry are adopted here, the covariance involved in those computations is just that of $\text{Gl}(\mathbb{R}^{4l+3})$.

Consider the so-called Levi-Civita symbol which is the totally antisymmetric tensor defined by:

$$\varepsilon_{\mu_1 \dots \mu_{4l+3}} = \begin{cases} +1 & \text{if } \varepsilon((\mu_1 \dots \mu_{4l+3})) = +1, \\ 0 & \text{if } \exists i, j \in \llbracket 1; 4l+3 \rrbracket \mid \mu_i = \mu_j, \\ -1 & \text{if } \varepsilon((\mu_1 \dots \mu_{4l+3})) = -1, \end{cases} \quad (2.11)$$

where $\varepsilon((\mu_1 \dots \mu_{4l+3}))$ is the signature of the permutation $(\mu_1 \dots \mu_{4l+3})$.

Let $\omega^{(r)} = \frac{1}{r!} \omega_{\mu_1 \dots \mu_r} \psi^{\mu_1} \wedge \dots \wedge \psi^{\mu_r}$ be a r -form. The Hodge dualization $*_g$ with respect to a metric g of ω is defined by:

$$*_g \omega^{(r)} = \frac{\sqrt{|g|}}{r! (4l+3-r)!} g^{\mu_1 \nu_1} \dots g^{\nu_r \mu_r} \varepsilon_{\nu_1 \dots \nu_{4l+3}} \omega_{\mu_1 \dots \mu_r} \psi^{\nu_{r+1}} \wedge \dots \wedge \psi^{\nu_{4l+3}}, \quad (2.12)$$

with $|g| = \det(g)$.

It should be noted that since $4l+3$ is odd and the metrics considered are riemannian:

$$*_g *_g \omega^{(r)} = \omega^{(r)}. \quad (2.13)$$

Let $\eta^{(q)} = \frac{1}{q!} \eta_{\mu_1 \dots \mu_q} \psi^{\mu_1} \wedge \dots \wedge \psi^{\mu_q}$, then the exterior product $\omega^{(r)} \wedge \eta^{(q)}$ is:

$$\omega^{(r)} \wedge \eta^{(q)} = \frac{1}{r! q!} \omega_{\mu_1 \dots \mu_r} \eta_{\mu_{r+1} \dots \mu_{r+q}} \psi^{\mu_1} \wedge \dots \wedge \psi^{\mu_{r+q}}. \quad (2.14)$$

Let m and n be a pair of vectors. The scalar product of m and n with respect to a metric g will be denoted $m \cdot_g n$ and the norm of m with respect to g will be denoted $|m|_g$. Here also, the index g will be omitted when g is the Euclidean metric.

The correspondence between the Lagrangian form \mathcal{L} and the Lagrangian density L is then $\mathcal{L} = *_g L$ so that $\int \mathcal{L} = \int L \sqrt{|g|} d^n x$ according to equation (2.12).

Fourier transform with respect to a constant anisotropic metric. In the following, the fields considered are elements of the tensor product between smooth functions and the exterior algebra $\mathcal{C}^\infty(\mathbb{R}_x^{4l+3}) \otimes \wedge^*(\mathbb{R}^{4l+3})$. The Fourier transform is defined so as to send the set of fields in a copy of itself $\mathcal{C}^\infty(\mathbb{R}_k^{4l+3}) \otimes \wedge^*(\mathbb{R}^{4l+3})$. In particular this Fourier transform preserves form degrees.

The Fourier transform of $\omega^{(r)} = \frac{1}{r!} \omega_{\mu_1 \dots \mu_r}(x) \psi_x^{\mu_1} \wedge \dots \wedge \psi_x^{\mu_r}$ with respect to the metric g is defined by:

$$\text{FT}_g[\omega^{(r)}] = \frac{1}{r!} \text{FT}_g[\omega_{\mu_1 \dots \mu_r}](k) \psi_k^{\mu_1} \wedge \dots \wedge \psi_k^{\mu_r}, \quad (2.15)$$

where:

$$\text{FT}_g[\omega_{\mu_1 \dots \mu_r}](k) = \int_{\mathbb{R}^{4l+3}} \omega_{\mu_1 \dots \mu_r}(x) e^{-ik^\mu g_{\mu\nu} x^\nu} d^{4l+3}x. \quad (2.16)$$

The definition of a Fourier transform with respect to the metric g relies heavily on the fact that this metric is constant albeit anisotropic .

The inverse Fourier transform of $\phi^{(r)} = \frac{1}{r!} \phi_{\mu_1 \dots \mu_r}(k) \psi_k^{\mu_1} \wedge \dots \wedge \psi_k^{\mu_r}$ with respect to the metric g is defined by:

$$\text{FT}_g^{-1}[\phi^{(r)}] = \frac{1}{r!} \text{FT}_g^{-1}[\phi_{\mu_1 \dots \mu_r}](x) \psi_x^{\mu_1} \wedge \dots \wedge \psi_x^{\mu_r}, \quad (2.17)$$

where:

$$\text{FT}_g^{-1}[\phi_{\mu_1 \dots \mu_r}](x) = \frac{1}{(2\pi)^{4l+3}} \int_{\mathbb{R}^{4l+3}} \phi_{\mu_1 \dots \mu_r}(k) e^{ik^\mu g_{\mu\nu} x^\nu} |g| d^{4l+3}k. \quad (2.18)$$

With these definitions:

$$\text{FT}_g[\partial_\mu] = -iK_\mu^g, \quad (2.19)$$

where $K_\mu^g = g_{\mu\nu} k^\nu$ according to the convention set before.

Convolution is defined as usual:

$$u \odot v(x) = \int_{\mathbb{R}^{4l+3}} u(y) v(x-y) dy, \quad (2.20)$$

so that FT_g turns convolutions into pointwise products:

$$\text{FT}_g[u \odot v] = \text{FT}_g[u] \cdot \text{FT}_g[v]. \quad (2.21)$$

Here also, the index g will be omitted when g is the Euclidean metric.

2.2 Correlators

In the Euclidean metric. Consider the gauge fixing function:

$$\mathcal{F}(A) = \alpha_1 \partial_1 A_1 + \dots + \alpha_{4l+3} \partial_{4l+3} A_{4l+3} \quad (2.22)$$

in Euclidean metric, or:

$$\mathcal{F}(A) = g^{\mu\nu} \partial_\mu A_\nu, \quad (2.23)$$

with $g^{\mu\nu}$ the inverse of the anisotropic metric:

$$g_{\mu\nu} = \begin{pmatrix} \frac{1}{\alpha_1} & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{\alpha_2} & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & \frac{1}{\alpha_{4l+2}} & 0 \\ 0 & 0 & \dots & 0 & \frac{1}{\alpha_{4l+3}} \end{pmatrix}. \quad (2.24)$$

Rewrite \mathcal{F} as:

$$\mathcal{F}(A) = \sum_{i=1}^{4l+3} \alpha_i (n_i^\nu \partial_\nu) (n_i^\mu A_\mu), \quad (2.25)$$

where the n_i form the canonical basis of \mathbb{R}_x^{4l+3} and the α_i are real numbers that will be considered to vary in $]0; 1]$. This constraint is implemented by adding the Lagrange multiplier term $B \mathcal{F}(A)$ where B is an auxiliary field, to the CS Lagrangian.

In order to easily generalize to $4l + 3$ dimensions, it is convenient to translate the Lagrange constraint into geometric terms using the exterior derivative, wedge product and Hodge dualization:

$$\mathcal{L}_{\text{GF}} = B \sum_{i=1}^{4l+3} \alpha_i N_i \wedge * d * \wedge N_i * A, \quad (2.26)$$

where the α_i are defined as above. Expression (2.26) shall be understood as follows: the Hodge dualization $*$, wedge product and exterior derivative successively act from right to left on the whole forms on their respective right sides. For example,

$$B \wedge N_i \wedge * d * \wedge N_i * A \Leftrightarrow B \wedge \langle N_i \wedge \{ * [d (* (N_i \wedge * A))] \} \rangle. \quad (2.27)$$

In $4l + 3$ dimensions $A = A^{(2l+1)}$ is a $(2l + 1)$ -form, thus $B = B^{(2l)}$ is a $2l$ -form and the above gauge-fixing constraint is incomplete as a large residual gauge invariance with respect to $B^{(2l)}$ is left. The latter can be fixed in its turn using a similar

2.2. Correlators

procedure with a $(2l - 1)$ -form of Lagrange multipliers $B^{(2l-1)}$ etc. in a cascading way. The complete gauge fixing Lagrangian then reads:

$$\begin{aligned}
\mathcal{L}_{\text{GF}} = & B^{(2l)} \wedge \sum_{i=1}^{4l+3} \alpha_i N_i \wedge {}^* d {}^* \wedge N_i {}^* A^{(2l+1)} \\
& + B^{(2l-1)} \wedge \sum_{i=1}^{4l+3} \alpha_i N_i \wedge {}^* d {}^* \wedge N_i {}^* B^{(2l)} \\
& + \dots \\
& + B^{(0)} \wedge \sum_{i=1}^{4l+3} \alpha_i N_i \wedge {}^* d {}^* \wedge N_i {}^* B^{(1)}. \tag{2.28}
\end{aligned}$$

Focus now on the computation of this two-point autocorrelator of A in the interpolating gauge considered. For this purpose, define $\mathcal{A} = (A^{(2l+1)}, B^{(2l)}, B^{(2l-1)}, \dots, B^{(0)})$. The complete action can be written as a scalar product:

$$\frac{1}{2} (\mathcal{A}, \mathcal{D}\mathcal{A}) = \int_{\mathbb{R}^{4l+3}} \mathcal{L}_{\text{CS}} + \mathcal{L}_{\text{GF}}. \tag{2.29}$$

In this equation, \mathcal{D} is the $(2l + 2) \times (2l + 2)$ matrix differential operator:

$$\left(\begin{array}{cccc}
({}^* d)^{(2l+1, 2l+1)} & -\Psi^{(2l+1, 2l)} & 0 & 0 \\
\Xi^{(2l, 2l+1)} & 0 & \Psi^{(2l, 2l-1)} & 0 \\
0 & \Xi^{(2l-1, 2l)} & 0 & -\Psi^{(2l-1, 2l-2)} \\
0 & 0 & \Xi^{(2l-2, 2l-1)} & 0 \\
& & & \ddots \\
& & & 0 & -\Psi^{(1, 0)} \\
& & & \Xi^{(0, 1)} & 0
\end{array} \right), \tag{2.30}$$

where:

$$\Xi = \sum_{i=1}^{4l+3} \alpha_i {}^* N_i \wedge {}^* d {}^* \wedge N_i {}^* \tag{2.31}$$

and:

$$\Psi = \sum_{i=1}^{4l+3} \alpha_i N_i \wedge {}^* d {}^* \wedge N_i. \tag{2.32}$$

The right superscript indicates the degree of the form on which the operator acts, the left superscript indicates the degree of the resulting form. It should be pointed out that $\Xi \neq \pm {}^* \Psi {}^*$ since the degree on which each operates is not the same.

2.2. Correlators

Let us focus on the component $\text{FT}[\mathcal{P}]_{1,1}$ since it is the only one that matters in the computation of the link invariant. Finally:

$$\text{FT}[\mathcal{P}]_{1,1} = -i^* \frac{Q}{k \cdot q} \wedge, \quad (2.43)$$

where $k = \sum_{i=1}^{4l+3} k_i$, taking the convention $k_i = (0, \dots, 0, k_i, 0, \dots, 0)$. It must be borne in mind that this equation describes actually an action on $2l + 1$ -forms: the Hodge dualization does not act on Q but on $Q \wedge \omega^{(2l+1)}$.

Rewrite $\text{FT}[\mathcal{P}]_{1,1}$ in terms of components:

$$\text{FT}[\mathcal{P}]_{\mu_1 \dots \mu_{2l+1} \nu_1 \dots \nu_{2l+1}} = -\frac{i}{(2l+1)!} \varepsilon_{\mu_1 \dots \mu_{2l+1} \nu_1 \dots \nu_{2l+1} \rho} \frac{\alpha_\rho k^\rho}{\sum_{i=1}^{4l+3} \alpha_i k_i^2}, \quad (2.44)$$

where the matrix indices are omitted for convenience.

It should be noted that the normalization $\frac{1}{(2l+1)!}$ comes from the Hodge dualization $*$ applied to a $2l + 2$ -form.

The computation of the inverse Fourier transform leads to:

$$\begin{aligned} & \mathcal{P}_{\mu_1 \dots \mu_{2l+1} \nu_1 \dots \nu_{2l+1}}(x) \\ &= \frac{1}{(2l+1)!} \frac{\Gamma\left(\frac{4l+3}{2}\right)}{2\pi^{\frac{4l+3}{2}}} \frac{1}{\sqrt{\alpha_1 \dots \alpha_{4l+3}}} \varepsilon_{\mu_1 \dots \mu_{2l+1} \nu_1 \dots \nu_{2l+1} \rho} \frac{x^\rho}{|x|_g^{4l+3}}. \end{aligned} \quad (2.45)$$

This two-point autocorrelator is reminiscent of the well-known Gauss linking formula according to the anisotropic metric g .

In the anisotropic metric. The gauge fixing function used here is:

$$\mathcal{F}(A) = g^{\mu\nu} \partial_\mu A_\nu, \quad (2.46)$$

with $g^{\mu\nu}$ the inverse of the anisotropic metric:

$$g_{\mu\nu} = \begin{pmatrix} \frac{1}{\alpha_1} & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{\alpha_2} & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & \frac{1}{\alpha_{4l+2}} & 0 \\ 0 & 0 & \dots & 0 & \frac{1}{\alpha_{4l+3}} \end{pmatrix}. \quad (2.47)$$

with:

$$\text{FT}_g \left[(\text{*}_g \text{d})^{(2l+1, 2l+1)} \right] = -i \text{*}_g K^g \wedge, \quad (2.53)$$

$$\text{FT}_g \left[\Phi_g^{(r, r+1)} \right] = -i \text{*}_g K^g \wedge \text{*}_g, \quad (2.54)$$

$$\text{FT}_g \left[\text{d}^{(r+1, r)} \right] = -i K^g \wedge. \quad (2.55)$$

The aim is to find a right-inverse operator from the point of view of the convolution in the direct space:

$$\mathcal{D}_g \odot \mathcal{P}_g(x) = \delta(x) \quad (2.56)$$

thus satisfying in the Fourier space:

$$\text{FT}_g [\mathcal{D}_g] \text{FT}_g [\mathcal{P}_g] = \text{Id}, \quad (2.57)$$

with:

$$\text{FT}_g [\mathcal{P}_g] = \left(\text{FT}_g \left[\mathcal{P}_g^{(2l+2-i, 2l+2-j)} \right]_{i,j} \right)_{1 \leq i, j \leq 2l+2}. \quad (2.58)$$

From now on, the superscripts will be omitted since they are completely determined by the subscripts.

Finally:

$$\text{FT}_g [\mathcal{P}_g]_{1,1} = -i \text{*}_g \frac{K^g}{k \cdot_g k} \wedge, \quad (2.59)$$

which is perfectly analogous to the previous result:

$$\text{FT} [\mathcal{P}]_{1,1} = -i \text{*} \frac{Q}{k \cdot q} \wedge, \quad (2.60)$$

up to a dissymmetry between q and k and with \cdot_g and *_g instead of \cdot and * .

Rewrite $\text{FT}_g [\mathcal{P}_g]_{1,1}$ in terms of components:

$$\text{FT}_g [\mathcal{P}_g]_{\mu_1 \dots \mu_{2l+1} \nu_1 \dots \nu_{2l+1}} = -\frac{i \sqrt{|g|}}{(2l+1)!} \varepsilon^{\mu_1 \dots \mu_{2l+1} \nu_1 \dots \nu_{2l+1} \rho} \frac{k^\rho}{\sum_{i=1}^{4l+3} \frac{1}{\alpha_i} k_i^2}, \quad (2.61)$$

where the matrix indices are omitted for convenience.

The computation of the inverse Fourier transform leads to:

$$\begin{aligned} & (\mathcal{P}_g)_{\mu_1 \dots \mu_{2l+1} \nu_1 \dots \nu_{2l+1}}(x) \\ &= \frac{1}{(2l+1)!} \frac{\Gamma\left(\frac{4l+3}{2}\right)}{2\pi^{\frac{4l+3}{2}}} \frac{1}{\alpha_1 \dots \alpha_{4l+3}} \varepsilon^{\mu_1 \dots \mu_{2l+1} \nu_1 \dots \nu_{2l+1} \rho} \frac{x^\rho}{|x|_g^{4l+3}}, \end{aligned} \quad (2.62)$$

which is not exactly the same result as the \mathcal{P} found in the previous part because of the normalization factor coming from the Hodge dualization in $\text{FT} [\mathcal{P}]$. However,

Chapter 2. Geometric interpretation of gauge fixing in \mathbb{R}^{4l+3}

the normalization factor is absorbed in the computations of the building blocks of the expectation value of observables:

$$(\mathcal{J}_1, \mathcal{P}\mathcal{J}_2) = (\mathcal{J}_1, \mathcal{P}_g\mathcal{J}_2)_g \quad (2.63)$$

by definition, so this operation contains a Hodge dualization and $\text{FT}_g[\mathcal{P}_g]$ too and in section 2.1 it has been shown that the double Hodge dualization is the identity. Both approaches are thus equivalent for the computation of observables.

To sum up, the Euclidean metric $\delta_{\mu\nu}$ has been changed into the constant anisotropic metric $g_{\mu\nu}$. Instead, the active $\text{Gl}(\mathbb{R}^{4l+3})$ transformation of \mathbb{R}^{4l+3} which transforms (via pullback or pushforward) the Euclidean metric into the anisotropic one could have been considered. However, it appeared that this approach would have been more cumbersome since it would have been necessary to take gauge potential transformations into account.

2.3 Limits

The limit cases investigated here can be generically written as:

$$\begin{cases} \alpha_1, \dots, \alpha_k \longrightarrow 0, \\ \alpha_{k+1}, \dots, \alpha_{4l+3} \longrightarrow 1, \end{cases} \quad (2.64)$$

for $k \in \llbracket 1; 4l+2 \rrbracket$. This operation makes the metric singular with k singularities. Geometrically, this amounts to flatten k directions of the space.

Since:

$$(\mathcal{I}_1, \mathcal{P}\mathcal{I}_2) = (\mathcal{I}_1, \mathcal{P}_g\mathcal{I}_2)_g, \quad (2.65)$$

consider \mathcal{P} which has the simplest expression:

$$\begin{aligned} & \mathcal{P}_{\mu_1 \dots \mu_{2l+1} \nu_1 \dots \nu_{2l+1}}(x) \\ &= \frac{1}{(2l+1)!} \frac{\Gamma\left(\frac{4l+3}{2}\right)}{2\pi^{\frac{4l+3}{2}}} \frac{1}{\sqrt{\alpha_1 \dots \alpha_{4l+3}}} \varepsilon_{\mu_1 \dots \mu_{2l+1} \nu_1 \dots \nu_{2l+1} \rho} \frac{x^\rho}{|x|_g^{4l+3}}. \end{aligned} \quad (2.66)$$

Since the area of the unit $4l+2$ -dimensional sphere S^{4l+2} is $|S^{4l+2}| = \frac{2\pi^{\frac{4l+3}{2}}}{\Gamma\left(\frac{4l+3}{2}\right)}$, then:

$$\begin{aligned} & \mathcal{P}_{\mu_1 \dots \mu_{2l+1} \nu_1 \dots \nu_{2l+1}}(x) \\ &= \frac{1}{(2l+1)!} \frac{1}{|S^{4l+2}|} \frac{1}{\sqrt{\alpha_1 \dots \alpha_{4l+3}}} \varepsilon_{\mu_1 \dots \mu_{2l+1} \nu_1 \dots \nu_{2l+1} \rho} \frac{x^\rho}{|x|_g^{4l+3}}. \end{aligned} \quad (2.67)$$

The limit investigated can be written:

$$\begin{aligned} & \mathcal{P}_{\mu_1 \dots \mu_{2l+1} \nu_1 \dots \nu_{2l+1}}(x) \\ &= \frac{1}{(2l+1)!} \frac{|S^{k-1}|}{|S^{4l+2}|} I_k \varepsilon_{\mu_1 \dots \mu_{2l+1} \nu_1 \dots \nu_{2l+1} \rho} \frac{x_{\perp}^{\rho}}{|x_{\perp}|^{4l+3-k}} \delta^{(k)}(x_{\parallel}), \end{aligned} \quad (2.68)$$

with:

$$x_{\parallel} = (x^1, \dots, x^k, 0 \dots 0), \quad (2.69)$$

and:

$$x_{\perp} = (0, \dots, 0, x^{k+1}, \dots, x^{4l+3}), \quad (2.70)$$

so that:

$$x = x_{\parallel} + x_{\perp} \quad (2.71)$$

and:

$$I_k = \int_0^{+\infty} dr \frac{r^{k-1}}{(1+r^2)^{\frac{4l+3}{2}}}. \quad (2.72)$$

For k even, then:

$$I_k = \frac{(k-2)(k-4)\dots 2}{(4l+1)(4l-1)\dots(4l-(k-5))(4l-(k-3))} \quad (2.73)$$

and for k odd, then:

$$I_k = \frac{(k-2)(k-4)\dots 1}{(4l+1)(4l-1)\dots(4l-(k-4))} \frac{2^{4l-(k-2)} \left(2l - \frac{k-3}{2}\right)! \left(2l - \frac{k-1}{2}\right)!}{(4l-(k-3))!}. \quad (2.74)$$

The explicit formula of \mathcal{P} can be given for some specific cases:

- $k = 1$ would correspond to a generalization of the Coulomb gauge and:

$$\begin{aligned} & \mathcal{P}_{\mu_1 \dots \mu_{2l+1} \nu_1 \dots \nu_{2l+1}}(x) \\ &= \frac{1}{(2l+1)} \frac{1}{2\pi^{2l+1}} \varepsilon_{\mu_1 \dots \mu_{2l+1} \nu_1 \dots \nu_{2l+1} \rho} \frac{x_{\perp}^{\rho}}{|x_{\perp}|^{4l+2}} \delta^{(1)}(x_{\parallel}), \end{aligned} \quad (2.75)$$

- $k = 4l + 2$ would correspond to a generalization of an axial-like gauge² and:

$$\begin{aligned} & \mathcal{P}_{\mu_1 \dots \mu_{2l+1} \nu_1 \dots \nu_{2l+1}}(x) \\ &= \frac{1}{2} \frac{1}{(2l+1)!} \varepsilon_{\mu_1 \dots \mu_{2l+1} \nu_1 \dots \nu_{2l+1} \rho} \frac{x_{\perp}^{\rho}}{|x_{\perp}|} \delta^{(4l+2)}(x_{\parallel}). \end{aligned} \quad (2.76)$$

²In the sense that an axial gauge $n^{\mu} A_{\mu} = 0$ for a given vector n would imply *a fortiori* $n^{\nu} \partial_{\nu} (n^{\mu} A_{\mu}) = 0$.

In the previous computation two particular cases were eliminated:

- $k = 0$ is purely covariant gauge, so there is no integral to compute and:

$$\begin{aligned} & \mathcal{P}_{\mu_1 \dots \mu_{2l+1} \nu_1 \dots \nu_{2l+1}}(x) \\ &= \frac{1}{(2l+1)!} \frac{\Gamma\left(\frac{4l+3}{2}\right)}{2\pi^{\frac{4l+3}{2}}} \varepsilon_{\mu_1 \dots \mu_{2l+1} \nu_1 \dots \nu_{2l+1} \rho} \frac{x^\rho}{|x|^{4l+3}}, \end{aligned} \quad (2.77)$$

- $k = 4l + 3$ has no meaning from the geometric point of view, and it can be noticed through the fact that $\frac{r^{k-1}}{(1+r^2)^{\frac{4l+3}{2}}}$ is not integrable on \mathbb{R}_+ .

The covariant, Coulomb and axial-like gauges were investigated in [Gallot et al., 2018] while the formula (2.68) is a generalization that is not published. It provides a general expression of the propagator of abelian CS theory in \mathbb{R}^{4l+3} for the complete family of derivative gauges (2.22). It is noteworthy that the Coulomb and axial-like gauges provide examples of the two different situations that arise through the computation of I_k in the general case.

2.4 Geometric interpretation

The linking formula has a universal expression:

$$(\mathcal{I}_1, \mathcal{P}\mathcal{I}_2) = (\mathcal{I}_1, \mathcal{P}_g \mathcal{I}_2)_g = \frac{1}{|S^{4l+2}|} \oint_{\gamma_1} \oint_{\gamma_2} \frac{[e_g(x, y), dx, dy]_g}{|x - y|_g^{4l+2}}, \quad (2.78)$$

where \mathcal{I}_i is the de Rham current associated with γ_i ,

$$e_g(x, y) = \frac{x - y}{|x - y|_g} \quad (2.79)$$

and:

$$[a, b, c]_g = \frac{\sqrt{|g|}}{(2l+1)!2} \varepsilon_{\rho\mu_1 \dots \mu_{2l+1} \nu_1 \dots \nu_{2l+1}} a^\rho b^{\mu_1 \dots \mu_{2l+1}} c^{\nu_1 \dots \nu_{2l+1}}. \quad (2.80)$$

Universal means here that the right-hand side of equation (2.78) does not depend on the choice between the two methods presented in the previous sections, that is, computation in Euclidean metric with anisotropic gauge fixing or in anisotropic metric with covariant gauge.

Given two cycles γ_1 and γ_2 , the Gauss map $(x, y) \mapsto e_g(x, y)$, $(x, y) \in \gamma_1 \times \gamma_2$ is a mapping from a product of two manifolds, each being homeomorphic to S^{2l+1} , in a manifold homeomorphic to S^{4l+2} . The linking formula represents the degree of this map, that is, the number of times the manifold homeomorphic to S^{4l+2} is covered.

2.4. Geometric interpretation

This manifold can be seen to be an ellipsoid embedded in \mathbb{R}^{4l+3} endowed with the Euclidean metric or to be exactly S^{4l+3} in \mathbb{R}^{4l+3} endowed with the anisotropic metric g . The product $\gamma_1 \times \gamma_2$ has to be homeomorphic to $S^{2l+1} \times S^{2l+1}$ for the degree map to be well-defined. Thus, depending on the limit chosen, some classes of degenerated cases arise. However, since the linking formula is invariant under ambient isotopy, a representative in the ambient isotopy class can be chosen for the linking number to be well-defined.

To illustrate this, let us consider the case of \mathbb{R}^3 . Keeping the notations of the previous section, $k = 0$ is the covariant gauge, $k = 1$ is the Coulomb gauge and $k = 2$ is an axial-like gauge. The typical non-pathological configuration probed by the covariant gauge is represented in Figure 2.1(a) while an example of a degenerate configuration is presented in Figure 2.1(b). Similarly for Coulomb gauge, those configurations are shown in Figures 2.2(a) and 2.2(b) and for axial-like gauge in Figures 2.3(a) and 2.3(b).

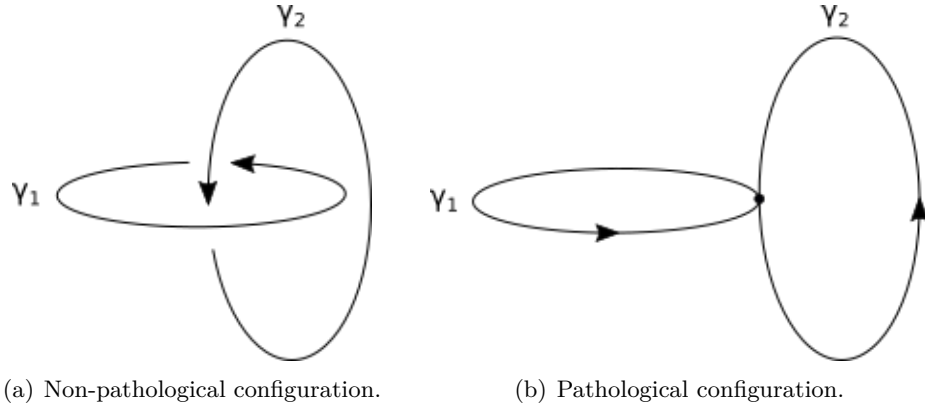


Figure 2.1: The case of covariant gauge.

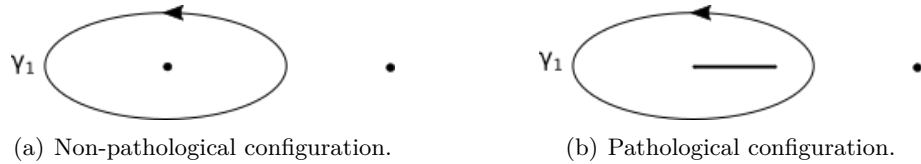


Figure 2.2: The case of covariant gauge.

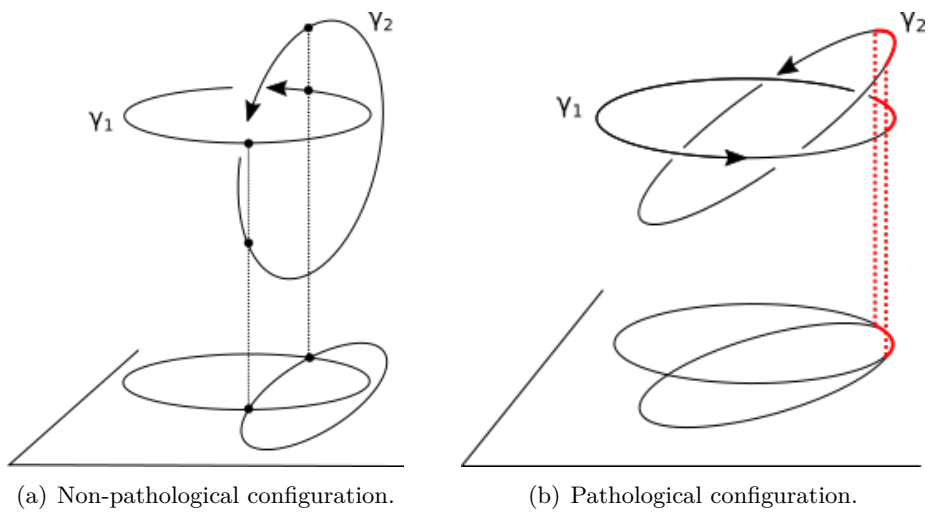


Figure 2.3: The case of covariant gauge.

Conclusion (English version)

This study presented an abelian modular category. It produced the corresponding abelian RT and TV invariants related together, in accordance with the theorem of Reshetikhin, Turaev and Viro. A weaker version of the abelian RT invariant has been studied. However, it is not related to the abelian TV invariant derived before since its construction relies on a finite semisimple spherical category and not a modular category anymore.

The abelian CS and BF theories have been defined on any closed manifold $M^{(3)}$ of dimension 3. Those abelian theories are not written as usual in terms of connections but in terms of gauge classes of connections which are nothing but DB cohomology classes. By introducing DB cohomology in formal functional integrals, it was then possible to switch from a purely formal object to finite integrals which finally produce topological invariants. The partition functions of the abelian CS and BF theories turned out then to be equal respectively to the abelian RT and TV invariants derived previously, up to a factor depending on the homology of $M^{(3)}$. Those equalities establish a non-trivial correspondence between topological invariants probed by categories on the one hand, and by connections on the other hand. It is noteworthy that the connections probe the topology of the manifold $M^{(3)}$ itself whereas the categories probe the topology of a graph related to $M^{(3)}$, sometimes located in $M^{(3)}$ (a triangulation), sometimes located in S^3 (a surgery link).

Moreover, it is natural to introduce the expectation value of observables in QFT from physicists' point of view, whereas it is not in the perspective of mathematicians. Categories are not really adapted for that. Through the identification made in this study by direct computations between those two different perspectives, it is now possible to identify invariants of links in the manifold $M^{(3)}$ under consideration with the expectation value of observables where categories are concerned and show correspondences with the usual expectation value of observables, where connections are concerned.

Such kind of results have been known for several years in the non-abelian case, mostly $SU(2)$. Through this study, it appeared that subtleties arise in the $U(1)$ case

Conclusion (English version)

which is fundamentally different from the $SU(N)$ case¹. In particular, regarding the partition functions, in general:

$$\left| Z_{CS_\kappa} (M^{(3)}) \right|^2 \neq Z_{BF_\kappa} (M^{(3)}). \quad (1)$$

The equality can occur depending on the topology of $M^{(3)}$. This fact could be expected since the formal equality established in the non-abelian case is strongly based on the cosmological constant κ that cannot appear in the abelian case. Moreover, surprisingly, the abelian CS theory is not related to the modular abelian RT theory but its extension, contrary to the non-abelian case. The *ad hoc* regularization by quantum groups, appearing in particular in LQG, arises here naturally through the reciprocity formulæ. Those formulæ can be understood in some sense as a proof of the surgery formulæ.

It has been shown all along this study that the abelian CS and BF theories, through the quadratic form Q over the cohomological torsion, have something to do with linking, intersection and crossings... The extensions of this study presented in part III show that this is still true for abelian CS theory in \mathbb{R}^{4l+3} , using customary physicists' methods of computation. Some pieces of those ideas also seem to appear in the computation of the $SU(N)$ CS invariant. This potentially opens new ways for understanding the non-abelian case.

It is however important to keep in mind that what was called CS and BF actions in the present study is very specific: the fields of the theories have been considered to be DB cohomology classes, that is, $U(1)$ gauge classes, whose representatives are local forms defined in open sets, which stick together inside the intersections of those open sets by means of gauge transformations. The DB cohomology is the natural tool with which computations on such kind of actions can be performed. Sometimes, in the mathematics literature, the abelian CS and BF actions are defined globally. This is the same as considering only the fiber over the null class in DB affine bundle and it is actually the choice of normalization taken here for partition function. What is remarkable is that this way of defining the abelian CS and BF theories starts from the usual physical idea of local observers and in addition, it is related to the abelian RT and TV theory.

The formalism of functional integrals and the theory of categories produce two different approaches of TQFT that appeared in this study to be strongly related in the particular cases of the abelian CS and BF theories. However, the aim of this study is not to define properly the formalism of functional integral nor to propose a systematical way of computation. What has been done is just an extraction of relevant topological quantities from purely formal functional integrals.

The formalism of functional integrals is not limited to those two theories since it can be used for any theory that can be expressed in terms of an action functional,

¹It should not be forgotten that the $SU(N)$ groups are simply connected whereas $U(1)$ is not.

although the DB cohomology non-perturbative approach remains very specific to the present study. In general, only standard perturbative methods provide some results. The theory of categories provides also a framework to study other QFT like CFT [Fuchs et al., 2002] reducing the problems to the computation of quantities related to graphs.

This study presents a quite closed set of results related to abelian RT, TV, CS and BF theories in dimension 3. However, it raises several open questions.

First, since the DB product maps $H_{\text{DB}}^{[p]} \times H_{\text{DB}}^{[q]}$ to $H_{\text{DB}}^{[p+q+1]}$, it is possible to consider integrals of polynomial functions of DB cohomology classes over closed manifolds $M^{(n)}$ of any dimension n as actions. For example, the abelian CS theory defined here in terms of DB cohomology classes in dimension 3 has already been extended to closed manifolds $M^{(4l+3)}$ of dimension $4l + 3$. It has similar properties as that presented in this study and produces analogous invariants of manifolds and links in those manifolds [Gallot et al., 2013]. Hence, an action that consists in an integral of a polynomial function of some DB cohomology classes should also lead to some topological invariants, and since the reciprocity formulæ presented in this study do not depend on the dimension, those invariants might be related to invariants built from some generalizations of a surgery procedure in dimension n . Some relations might also exist with generalizations of abelian RT and TV theories.

Furthermore, in the so-called Batalin-Vilkovisky formalism, abelian BF theory has been proved to be in relation with the Reidemeister torsion of the manifold under consideration [Cattaneo et al., 2017]. The comparison of this approach and the approach discussed in this study could lead to a new understanding of this topic.

Ultimately, the abelian CS theory plays a role in the quantum Hall effect and it appeared relevant to study tensor powers of $U(1)$ as a gauge group [Fräbendorf, 2018]. As a consequence, the theory is not characterized by a unique coupling constant anymore, but by a matrix of coupling constants. For such a theory, there might be an extension of DB cohomology approach that makes it possible to get exact results.

Conclusion (version française)

Ce travail a présenté une catégorie modulaire abélienne. Il a produit les invariants RT et TV abéliens correspondants reliés entre eux, en accord avec le théorème de Reshetikhin, Turaev et Viro. Une version plus faible de l'invariant RT abélien a été étudiée. Cependant, elle n'est pas reliée à l'invariant TV considéré précédemment puisque sa construction ne repose plus sur une catégorie modulaire mais sur une catégorie sphérique semisimple finie.

Les théories CS et BF abéliennes ont été définies sur toute variété fermée $M^{(3)}$ de dimension 3. Ces théories abéliennes ne sont pas écrites comme d'habitude en termes de connexions mais en termes de classes de jauge de connexions qui ne sont rien d'autre que des classes de cohomologie DB. L'introduction de la cohomologie DB dans des intégrales fonctionnelles formelles a permis de transformer des objets purement formels en des intégrales finies qui produisent finalement des invariants topologiques. Les fonctions de partition des théories CS et BF abéliennes se sont avérées respectivement égales aux invariants RT et TV abéliens présentés précédemment, à un facteur près dépendant de l'homologie de $M^{(3)}$. Ces égalités établissent une correspondance non-triviale entre les invariants topologiques sondés par les catégories d'une part et par les connexions d'autre part. Il est important de noter que les connexions sondent la topologie de la variété $M^{(3)}$ elle-même alors que les catégories sondent la topologie d'un graphe relié à $M^{(3)}$, parfois localisé dans $M^{(3)}$ (une triangulation), parfois localisé dans S^3 (un entrelac de chirurgie).

De plus, il est naturel d'introduire des valeurs moyennes d'observables en QFT d'un point de vue de physicien, alors que ça ne l'est pas d'un point de vue de mathématicien. Les catégories ne sont pas vraiment adaptées pour cela. A travers l'identification faite dans ce travail par des calculs directs entre ces deux différentes perspectives, il est maintenant possible d'identifier des invariants d'entrelacs dans la variété $M^{(3)}$ considérée avec des valeurs moyennes d'observables en termes de catégories et de montrer des correspondances avec les valeurs moyennes d'observables usuelles en termes de connexions.

Des résultats de ce type sont connus depuis plusieurs années dans le cas non-abélien, principalement SU(2). A travers ce travail, il s'avère que des subtilités

Conclusion (version française)

surviennent dans le cas $U(1)$ qui est fondamentalement différent du cas $SU(N)$ ¹. En particulier, concernant les fonctions de partition, en général :

$$\left| Z_{CS_\kappa} (M^{(3)}) \right|^2 \neq Z_{BF_\kappa} (M^{(3)}) . \quad (1)$$

L'égalité peut survenir en fonction de la topologie de $M^{(3)}$. Ce fait pouvait être attendu puisque l'égalité formelle établie dans le cas non-abélien repose fortement sur la constante cosmologique κ qui ne peut apparaître dans le cas abélien. De plus, de façon surprenante, la théorie CS abélienne n'est pas reliée à la théorie RT abélienne modulaire, mais à son extension, contrairement au cas non-abélien. La régularisation *ad hoc* par des groupes quantiques, apparaissant en particulier en LQG, survient ici naturellement à travers les formules de réciprocité. Ces formules peuvent être comprises d'une certaine façon comme une preuve des formules de chirurgie.

Il a été montré tout au long de ce travail que les théories CS et BF abéliennes, à travers la forme quadratique Q sur la torsion cohomologique, ont quelque chose à voir avec de l'enlacement, de l'intersection, des croisements... Les extensions de ce travail présentées en partie III montrent que c'est toujours vrai pour la théorie CS abélienne dans \mathbb{R}^{4l+3} , en utilisant les méthodes standard de calcul des physiciens. Des éléments de telles idées ont aussi l'air d'apparaître dans le calcul de l'invariant CS $SU(N)$. Ceci ouvre de nouvelles perspectives potentielles pour la compréhension du cas non-abélien.

Il est cependant important de garder en tête que ce qui a été appelé actions CS et BF dans ce travail est très spécifique : les champs de la théorie ont été en fait pris comme étant des classes de cohomologie DB, c'est-à-dire, des classes de jauge de connexions $U(1)$, dont les représentants sont des formes locales définies dans des ouverts, qui se recollent dans les intersections de ces ouverts par transformations de jauge. La cohomologie DB est l'outil naturel avec lequel les calculs de ce genre d'actions peuvent être effectués. Parfois, dans la littérature, les actions CS et BF abéliennes sont définies globalement. Cela revient à considérer seulement la fibre au-dessus de la classe nulle dans le fibré affine représentant les classes de cohomologie DB, ce qui est le choix de normalisation pris ici pour la fonction de partition. Il est remarquable, que cette façon de définir les théories CS et BF abéliennes parte de l'idée physique usuelle d'observateurs locaux et s'avère en plus reliée aux théories RT et TV abéliennes.

Le formalisme des intégrales fonctionnelles et la théorie des catégories constituent deux approches différentes des TQFT qui se sont avérées dans ce travail fortement liées dans les cas particuliers des théories CS et BF abéliennes. Cependant, le but de ce travail n'est pas de définir proprement l'intégrale fonctionnelle ni de proposer

¹Il doit être gardé en tête que les groupes $SU(N)$ sont simplement connexes alors que $U(1)$ ne l'est pas.

une méthode systématique de calcul. Ce qui a été fait n'est qu'une extraction des quantités topologiques pertinentes d'intégrales fonctionnelles purement formelles.

Le formalisme des intégrales fonctionnelles n'est pas limité à ces deux théories puisqu'il peut être utilisé pour n'importe quelle théorie qui peut s'exprimer en termes d'action, bien que l'approche non-perturbative par la cohomologie DB reste très spécifique à ce travail. En général, seules les méthodes perturbatives standard fournissent des résultats. La théorie des catégories fournit aussi un cadre pour étudier d'autres QFT comme les CFT [Fuchs et al., 2002] en réduisant les problèmes à des calculs de quantités liées à des graphes.

Ce travail présente un ensemble relativement fermé de résultats reliant les théories RT, TV, CS et BF abéliennes en dimension 3. Cela étant, il soulève un certain nombre de questions ouvertes.

Premièrement, puisque le produit DB envoie $H_{\text{DB}}^{[p]} \times H_{\text{DB}}^{[q]}$ dans $H_{\text{DB}}^{[p+q+1]}$, il est possible de considérer des intégrales de polynômes en des classes de cohomologie DB sur des variétés fermées $M^{(n)}$ de dimension n quelconque comme actions. Par exemple, la théorie CS abélienne définie ici en termes de classes de cohomologie DB en dimension 3 a déjà été étendue aux variétés fermées de dimension $4l + 3$. Ses propriétés sont similaires à celles présentées dans ce travail et elle produit de façon analogue des invariants des variétés et des entrelacs dans ces variétés [Gallot et al., 2013]. Ainsi, une action qui consiste en une intégrale d'un polynôme en des classes de cohomologie DB devrait aussi conduire à des invariants topologiques, et puisque les formules de réciprocity présentées dans ce travail ne dépendent pas de la dimension, ces invariants pourraient être reliés à des invariants construits à partir de généralisations d'une procédure de chirurgie en dimension n . Des relations pourraient également exister avec des généralisations des théories RT et TV abéliennes.

Ensuite, dans le formalisme dit de Batalin-Vilkovisky, il a été prouvé que la théorie BF abélienne est en relation avec la torsion de Reidemeister de la variété considérée [Cattaneo et al., 2017]. La comparaison entre cette approche et l'approche discutée dans ce travail pourrait conduire à une nouvelle compréhension de ce sujet.

Enfin, la théorie CS abélienne joue un rôle dans l'effet Hall quantique et il s'avère pertinent d'étudier des puissances tensorielles de $U(1)$ comme groupe de jauge [Fräßdorf, 2018]. Par conséquent, la théorie n'est plus caractérisée par une unique constante de couplage mais par une matrice de constantes de couplage. Pour une telle théorie, il pourrait y avoir une extension de l'approche par la cohomologie DB qui permette d'obtenir des résultats exacts.

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Abstract

This dissertation considers the abelian BF theory on a closed manifold of dimension 3. It is formulated in terms of gauge classes which turn out to be Deligne-Beilinson cohomology classes. This formulation offers the possibility to extract the mathematically relevant quantities from formal functional integrals. The partition function and the expectation value of observables are therefore computed in this sense. They are proved to be connected, on the one hand, with their analogue obtained in the same manner from the abelian Chern-Simons theory, and, on the other hand, with their analogue defined in the framework of the abelian Reshetikhin-Turaev and Turaev-Viro theories. Two extensions of this study are also discussed. Firstly, a graphical approach is proposed to compute the $SU(N)$ classical Chern-Simons invariant. Secondly, a geometric interpretation of the gauge fixing procedure is presented for the abelian Chern-Simons theory in \mathbb{R}^{4l+3} .

Keywords: abelian BF theory, abelian Chern-Simons theory, Deligne-Beilinson cohomology, abelian Turaev-Viro invariant, abelian Reshetikhin-Turaev invariant, modular category, $SU(N)$ classical Chern-Simons invariant, Heegaard diagram, interpolating gauge.

Résumé

Cette thèse porte sur la théorie BF abélienne sur une variété fermée de dimension 3. Elle est formulée en termes de classes de jauge qui sont en fait des classes de cohomologie de Deligne-Beilinson. Cette formulation offre la possibilité d'extraire les quantités mathématiquement pertinentes d'intégrales fonctionnelles formelles. La fonction de partition et les valeurs moyennes d'observables sont ainsi calculées en ce sens. Elles s'avèrent reliées, premièrement, aux quantités analogues obtenues de la même manière avec la théorie abélienne de Chern-Simons, et, deuxièmement, avec les quantités analogues définies dans le cadre des théories abéliennes de Reshetikhin-Turaev et de Turaev-Viro. Deux extensions de ce travail sont discutées. Premièrement, une approche graphique est proposée afin de calculer l'invariant classique $SU(N)$ de Chern-Simons. Deuxièmement, une interprétation géométrique de la procédure de fixation de jauge est présentée pour la théorie de Chern-Simons abélienne dans \mathbb{R}^{4l+3} .

Mots-clefs: théorie BF abélienne, théorie de Chern-Simons abélienne, cohomologie de Deligne-Beilinson, invariant de Reshetikhin-Turaev abélien, invariant de Turaev-Viro abélien, catégorie modulaire, invariant classique $SU(N)$ de Chern-Simons, diagramme de Heegaard, jauge interpolante.