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Méthodes d'analyse fonctionnelle pour des systèmes de dimension infinie issus de la dynamique de populations

Nicolas Hegoburu

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Nicolas Hegoburu. Méthodes d'analyse fonctionnelle pour des systèmes de dimension infinie issus de la dynamique de populations. Equations aux dérivées partielles [math.AP]. Université de Bordeaux, 2019. Français. NNT : 2019BORD0061 . tel-02133006

HAL Id: tel-02133006

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THÈSE

PRÉSENTÉE À

L'UNIVERSITÉ DE BORDEAUX

ÉCOLE DOCTORALE DE MATHÉMATIQUES ET
D'INFORMATIQUE

par **Nicolas HGOBURU**

POUR OBTENIR LE GRADE DE

DOCTEUR

SPÉCIALITÉ — MATHÉMATIQUES

**Méthodes d'analyse fonctionnelle pour des systèmes de
dimension infinie issus de la dynamique de populations**

Date de soutenance : Le 7 Mai 2019

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– Financement I.D.E.X. –

Résumé Cette thèse étudie les propriétés de contrôle par migration d'équations aux dérivées partielles modélisant la dynamique d'une population structurée en âge. Les équations de populations considérées seront essentiellement celles décrites par Lotka et McKendrick, en tenant compte ou non de la diffusion spatiale des individus, ainsi que leur pendant non-linéaire décrit par les équations de Gurtin et MacCamy.

La première partie étudie les propriétés de contrôlabilité interne des équations linéaires de Lotka et McKendrick (sans diffusion), lorsque le contrôle n'agit que pour les jeunes individus formant la population. La contrôlabilité à zéro ainsi que la contrôlabilité vers les solutions stationnaires du système considéré est démontrée, en utilisant les propriétés du semi-groupe associé à l'opérateur de population originellement étudié par Song (contrôleur supposé responsable de la radicalisation de la politique de l'enfant unique en Chine). En outre, la conservation au cours du temps de la positivité de la densité de population contrôlée est étudiée.

Les deux parties suivantes établissent respectivement des propriétés de contrôle à zéro et de contrôle en temps optimal pour l'équation de Lotka et McKendrick, lorsque le déplacement spatial des individus est considéré (ici, le contrôle agit pour tous les âges mais seulement dans une certaine zone du milieu considéré). Les méthodes employées relèvent d'une adaptation de celles originellement développées pour le contrôle d'équations paraboliques, notamment la méthode de Lebeau et Robbiano (pour l'étude du contrôle à zéro de l'équation de la chaleur), ainsi que leur généralisation développée par Wang pour l'étude du contrôle en temps optimal de l'équation de la chaleur.

Une dernière partie étudie les propriétés de contrôlabilité des équations non-linéaires de Gurtin et MacCamy (sans diffusion), lorsque le contrôle est voué à n'agir que pour une certaine tranche d'âge d'individus. L'utilisation de principes de comparaison en dynamique de populations permet notamment d'obtenir le contrôle à zéro des équations considérées.

Mots-clés Dynamique de populations, structure en âge, contrôlabilité, positivité

Title Functional analysis methods for infinite dimensional systems coming from population dynamics

Abstract This work is devoted to study the controllability properties of some infinite dimensional systems modeling an age structured population dynamics. The considered equations are essentially those described by Lotka and McKendrick, with or without spatial diffusion, and their nonlinear versions described by the Gurtin and MacCamy equations.

The first part of this thesis aims to study the controllability properties of the linear Lotka and McKendrick system (without diffusion), in the case when the control acts for the very young individuals. The null controllability and the controllability towards the stationary solutions of the considered system are established, using a semigroup approach. In addition, the nonnegativity of the controlled population dynamics is studied.

The next two parts are respectively devoted to establish a null controllability result and a time optimal control result for the Lotka McKendrick equation with spatial diffusion (here, the control acts for every ages but only on a subdomain of the considered spatial domain). The methods employed are those originally devoted to study the internal controllability properties of the heat equation.

A last part studies the controllability properties of the Gurtin and MacCamy nonlinear equations (without diffusion), when the control acts only in an arbitrary age range. In this case, the use of comparison principles in age structured population dynamics ensures the null controllability of the considered equations.

Keywords Population dynamics, age structuring, controllability, nonnegativity

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CHAPITRE I

Introduction Générale

Le contexte général dans lequel s'inscrit cette thèse est celui de l'étude des propriétés de contrôle par *migration* d'équations aux dérivées partielles modélisant la dynamique d'une population structurée en âge, en tenant compte ou non de la diffusion spatiale des individus et des éventuelles non-linéarités associées aux équations.

Une section préliminaire de ce chapitre introductif vise à présenter - de manière relativement heuristique et informelle - un historique non exhaustif des principaux modèles continus de populations (en mettant l'accent sur ceux structurés en âge), dont certains ont conduit à une politique de contrôle effectif d'une population. En particulier, nous exposerons l'influence des équations linéaires de Sharpe, Lotka & McKendrick quant à la décision du gouvernement Chinois d'imposer la politique de l'enfant unique dès les années 1980, et ce via un contrôle des *naissances*.

Le contrôle par migration étudié dans cette thèse étant mathématiquement associé au contrôle additif d'équations aux dérivées partielles, nous développerons dans une prochaine section les notions mathématiques correspondantes. Nous présenterons ensuite les modèles de dynamique de populations structurés en âge étudiés dans ce manuscrit, et nous dresserons un panorama des résultats de contrôlabilité associés existants (notamment, ceux obtenus pendant la durée de la thèse). Une dernière sous-section de cette introduction précisera l'organisation globale du reste du manuscrit.

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I.1 EXEMPLE DE CONTRÔLE MATHÉMATIQUE DE POPULATIONS : LE SPECTRE DE MALTHUS DANS LA CHINE DES ANNÉES 1980

Cette section se veut être relativement informelle : elle vise à dresser, de Malthus à Gurtin et MacCamy, l'essentiel des principaux modèles continus d'une dynamique de population isolée et leurs éventuelles conséquences politiques d'un point de vue *contrôle*. L'existence et l'unicité des solutions aux équations présentées dans l'ensemble de cette section sera discutée en Section I.3. Les objectifs généraux de la thèse seront présentés en dernière sous-section de ce chapitre introductif.

§ I.1.1. Quelques principaux modèles continus de dynamique de population Le pessimisme de Malthus (1798)

Il est largement considéré que c'est au crédit de Thomas. R. Malthus que l'on accorde la paternité de l'un des premiers modèles mathématiques de croissance de population, qui publie (anonymement, dans sa première version) son célèbre *Essai sur le principe de population* [Mal98] en 1798¹. La pierre angulaire de son modèle de dynamique de population se résume en ces présentes phrases :

"I say, that the power of population is indefinitely greater than the power in the earth to produce subsistence for man. Population, when unchecked, increases in a geometrical ratio. Subsistence increases only in an arithmetical ratio." [Mal98, Chap. I]

Selon ce postulat², le dit « pouvoir des populations » serait tel qu'il induit une croissance *géométrique* du nombre d'individus (2, 4, 8, 16, 32, 64, ...), alors que les moyens de subsistance n'augmenteraient que de manière *arithmétique* (1, 2, 3, 4, 5, 6, ...). Malthus estime que la population mondiale double toutes les 25 années - fatalement, la population

1. En réalité, Leonhard Euler [Eul67] publia seulement deux décennies avant Malthus (en 1767) un modèle discret de croissance de populations. Voir Iannelli et Milner [IM17] pour un historique relativement plus exhaustif que celui présenté dans cette introduction.

2. Malthus ne vérifie pas rigoureusement cette théorie de rapport de croissances, prenant comme argument d'autorité l'évidence de ses propos.

tendra rapidement à s'accroître au-delà des moyens de subsistance, engendrant selon lui pléthores de conséquences plutôt dévastatrices (guerres, famines, épidémies, ...) :

“The power of population is so superior to the power in the earth to produce subsistence for man, that premature death must in some shape or other visit the human race. The vices of mankind are active and able ministers of depopulation. They are the precursors in the great army of destructions ; and often finish the dreadful work themselves. But should they fail in this war of extermination, sickly seasons, epidemics, pestilence, and plague, advance in terrific array, and sweep of their thousands and ten thousands. Should success be still incomplete, gigantic inevitable familes stalks in the rear, and with one mighty blow levels the population with the food of the world.” [Mal98, Chap. VII]

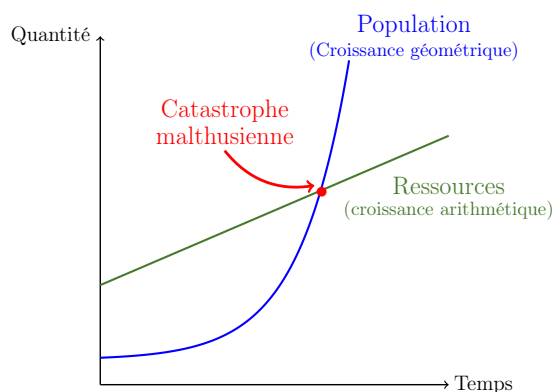


FIGURE I.1 – Représentation heuristique du modèle de Malthus. Lorsque la quantité de population (augmentation géométrique) dépasse celle des ressources (augmentation arithmétique), la catastrophe malthusienne s’enclenche.

L’appréhension d’une telle catastrophe démographique associée à une préconisation de la limitation du nombre de naissances porte désormais le nom de *malthusianisme*. Notons que, comme le souligne l’anthropologue C. Meilassoux [Mei91], cette peur d’une croissance excessive au delà des moyens de subsistances constitue un certain paralogisme vis-à-vis de la pensée de Malthus : en réalité, comment une population pourrait-elle continuer à croître exponentiellement en ayant épuisé les ressources nécessaires à son développement ? Malthus le reconnaît d’ailleurs lui-même : « Je sais bien, que les millions d’habitants en excès dont j’ai parlé n’existeront jamais³ ».

Quand bien même, Malthus impute - implicitement, de manière plus ou moins ambiguë et donc libre à interprétation - une hypothétique croissance explosive du nombre d’individus au-delà des moyens de subsistance aux supposés vices des classes inférieures qui, dépourvus des vertus de prévoyance et de chasteté, se reproduiraient de manière déraisonnée. Cet excès de population se doit donc d’être maîtrisé et *contrôlé* par les classes

3. Citation tirée de la traduction française.

supérieures, plus raisonnables. Malthus suggère ainsi aux populations les plus pauvres de ne procréer que s'ils en ont les moyens, et de retarder l'âge des mariages le cas échéant. En réalité, la haute bourgeoisie anglaise coupa court aux aides des chômeurs et plaça les pauvres dans des maisons de travail dont les conditions de labeur furent similaires à celles des camps nazis (c.f. [Mei91] qui cite lui-même Engels [Eng45]). C'est, selon l'auteur de ce présent manuscrit, l'un des premiers exemples de *contrôle de populations* inspiré de notions mathématiques (croissance géométrique versus croissance arithmétique). Cette doctrine malthusienne aura pour plus récente conséquence la radicalisation et le gèlement de la politique de l'enfant unique en Chine post-maoïste dans les années 1980, décrite en sous-section I.1.2.

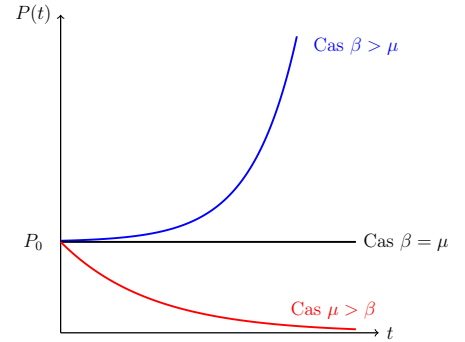
Mathématiquement parlant, le pendant moderne du modèle de Malthus en termes d'équations différentielles peut se traduire de la manière classique suivante : en notant $\beta \geq 0$ le taux de fertilité et $\mu \geq 0$ le taux de mortalité des individus, le nombre total d'individus $P(t)$ dépendant du temps $t \geq 0$ est solution de l'équation différentielle

$$\begin{cases} \dot{P}(t) = (\beta - \mu)P(t) & (t \geq 0), \\ P(0) = P_0, \end{cases} \quad (\text{I.1.1})$$

dont la solution est donnée par

$$P(t) = P_0 e^{(\beta - \mu)t} \quad (t \geq 0),$$

où P_0 désigne le nombre initial d'individus.



Aussi simpliste soit-il, ce modèle de croissance de populations met en évidence un point fondamental commun à l'ensemble des modèles de populations : la dynamique d'une population est essentiellement dictée par le taux de fertilité et de mortalité des individus formant la dite population. La balance entre taux de fertilité et taux de mortalité régit la croissance d'une population - dans le modèle ci-dessus, cette croissance n'est pas freinée dès lors que le taux de fertilité est strictement supérieur au taux de mortalité, *i.e.* dans le cas où $\beta > \mu$. Ici, une stratégie de contrôle des naissances d'une population peut consister à chercher à moduler la valeur du taux de fertilité β , afin d'obtenir (si possible) un profil de population souhaité en un temps fixé. C'est précisément ce type de stratégie qui fut employée dans les modèles de prédiction de population par la Chine dans les années 1980, en utilisant non pas le modèle (I.1.1) inspiré par Malthus, mais les équations de Sharpe-Lotka-McKendrick que nous décrivons peu après et qui seront étudiées plus en détail dans ce manuscrit d'un point de vue *contrôle par migration*.

Notons que la population humaine n'a pas encore subi dans sa globalité de « catastrophe malthusienne » en raison de la révolution industrielle du XIX^e siècle (que n'avait pas pu présager Malthus). Notons aussi que, bien que le modèle (I.1.1) ne constitue évidemment pas la quintessence des modèles de populations, il permet néanmoins de décrire

l'évolution de la croissance d'une population dans la mesure où celle-ci n'est pas freinée ou influée par des limitations en son sein.

Le modèle logistique de Verhulst (1838)

En 1838, Pierre-François Verhulst répond à Malthus (décédé quatre ans plus tôt) en proposant un modèle *logistique* de dynamique de population dans une note intitulée *Notice sur la loi que la population suit dans son accroissement* [Ver38]. A l'instar du modèle de Malthus, il suppose qu'une population sans limitation de ressources croît de manière exponentielle mais que la croissance de la population est freinée par sa propre dynamique et par la limitation des ressources du milieu :

« On sait que le célèbre *Malthus* a établi comme principe que la population humaine *tend* à croître en progression géométrique, de manière à se doubler après une certaine période, par exemple, tous les vingt-cinq ans. Cette proposition est incontestable, si l'on fait abstraction de la difficulté toujours croissante de se procurer des subsistances lorsque la population a acquis un certain degré d'agglomération, ou des ressources que la population puise dans son accroissement, même lorsque la société est encore naissante, telles qu'une plus grande division du travail, l'existence d'un gouvernement régulier et de moyens de défense qui assurent la tranquillité publique, etc. » [Ver38, p. 113]

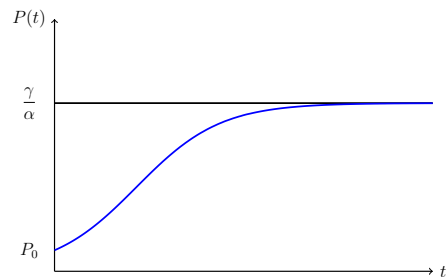
Dans sa note, Verhulst suppose la résistance à la croissance d'une population proportionnelle au carré de la vitesse avec laquelle la population tend à croître (à l'instar d'un mobile en chute libre traversant un milieu résistant - cette intuition fut d'ailleurs fournie à Verhulst par le physicien Quetelet). En notant $\alpha \geq 0$ le taux de croissance de la population et α/γ la capacité d'accueil du milieu (où $\gamma > 0$), Verhulst modélise le nombre total d'individus $P(t)$ dépendant du temps via l'équation différentielle suivante :

$$\begin{cases} \dot{P}(t) = \alpha P(t) - \gamma P(t)^2, \\ P(0) = P_0 \end{cases} \quad (\text{I.1.2})$$

dont la solution est donnée par

$$P(t) = \frac{\frac{\gamma}{\alpha} P_0}{P_0 + e^{-\alpha t} (\frac{\gamma}{\alpha} - P_0)} \quad (t \geq 0),$$

où P_0 désigne le nombre initial d'individus.



Ainsi, toute solution de (I.1.2) tend vers la capacité d'accueil du milieu (dès lors que $P_0 > 0$). La croissance exponentielle relative au terme $\alpha P(t)$ est ainsi freinée par la dynamique propre à la population, modélisée par le terme $-\gamma P(t)^2$. Verhulst compare les données réelles de croissance des populations Françaises et Belges de l'époque avec la solution donnée via son modèle, et juge les résultats concordants. La présence de nonlinéarités du type logistique (I.1.2) dans les modèles de croissance de population se retrouvera notamment dans les modèles structurés en âge établis par Gurtin et MacCamy des années 1970.

Le modèle de Sharpe, Lotka & McKendrick : vers une structure en âge (1911 - 1926)

Au début du XX^e siècle, F.R.Sharpe et A.Lotka [SL11] (en 1911) et A.G.McKendrick [M'K25] (en 1926) proposent de modéliser la dynamique d'une population via une *structure en âge* (i.e. en tenant compte de la pyramide des âges des individus pour décrire leur évolution). Leur modèle est basé sur les deux faits suivants (plutôt incontestables!) :

- ♣ l'âge des individus formant une population défile à la même vitesse que le temps et « la translation est irréversible » [M'K25, p. 122],
- ♣ les individus sont susceptibles de décéder à tout âge, et la population se renouvelle via reproduction des individus.

Plus précisément, Sharpe, Lotka et McKendrick décrivent le comportement d'une population isolée via sa densité $p(t, a)$, dépendante du temps $t \geq 0$ et de l'âge $a \in [0, a_+]$ (où a_+ désigne l'âge maximal des individus, supposé fini ici), au sens où si $a_1 < a_2$ sont deux âges appartenant à l'intervalle $[0, a_+]$, l'intégrale

$$\int_{a_1}^{a_2} p(t, a) da$$

compte le nombre d'individus, au temps t , dont l'âge est compris dans l'intervalle $[a_1, a_2]$.

En notant $\mu(a)$ le taux de mortalité des individus d'âge $a \in [0, a_+]$ au temps $t \geq 0$ (au sens où $\mu(a)$ donne le nombre d'individus de la population qui meurent durant l'intervalle d'âge infinitésimal $[a, a + da]$), la densité de population $p(t, a)$ est en partie décrite par l'équation ci-dessous :

$$\underbrace{\frac{\partial p}{\partial t}(t, a) + \frac{\partial p}{\partial a}(t, a)}_{\text{Transport en âge à vitesse constante}} + \underbrace{\mu(a)p(t, a)}_{\text{Mortalité}} = 0, \quad t > 0, \quad a \in [0, a_+]. \quad (\text{I.1.3})$$

En outre, notons $\beta(a)$ le taux de fertilité pour l'âge $a \in [0, a_+]$, au sens où $\beta(a)$ donne le nombre de nouveaux-nés conçus par des individus d'âge compris dans l'intervalle d'âge infinitésimal $[a, a + da]$. En particulier, l'intégrale

$$B(t) = \int_0^{a_+} \beta(a)p(t, a) da$$

donne le nombre total de nouveaux-nés dans l'intervalle de temps infinitésimal $[t, t + dt]$, de sorte que l'on obtienne l'équation de renouvellement suivante :

$$\underbrace{p(t, 0) = B(t) = \int_0^{a_+} \beta(a)p(t, a) da}_{\text{Equation de renouvellement}}, \quad t > 0. \quad (\text{I.1.4})$$

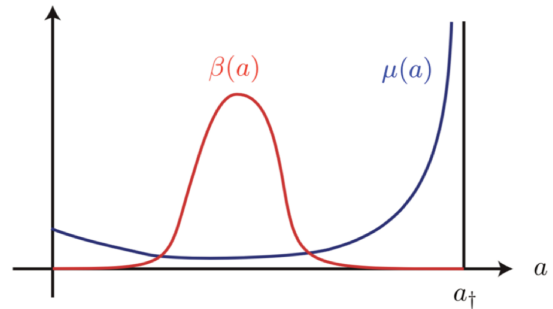
La donnée des équations (I.1.3) et (I.1.4) conduit au système suivant :

$$\begin{cases} \frac{\partial p}{\partial t}(t, a) + \frac{\partial p}{\partial a}(t, a) + \mu(a)p(t, a) = 0, & t > 0, a \in (0, a_{\dagger}), \\ p(t, 0) = \int_0^{a_{\dagger}} \beta(a)p(t, a) da, & t > 0, \\ p(0, a) = p_0(a), & a \in (0, a_{\dagger}), \end{cases} \quad (\text{I.1.5})$$

où p_0 désigne la densité initiale de population. Les équations décrites par le système (I.1.5) portent communément le nom d'*équations de Sharpe-Lotka-McKendrick* (ou, plus brièvement, de Lotka-McKendrick).

Les hypothèses mathématiques classiques portant sur la densité initiale de population et les fonctions β et μ sont les suivantes :

- (A1) $\beta \in L^\infty(0, a_{\dagger})$, $\beta(a) \geq 0$ pour presque tout $a \in (0, a_{\dagger})$,
- (A2) $\mu \in L^1[0, a^*]$ pour tout $a^* \in (0, a_{\dagger})$, $\mu(a) \geq 0$ pour presque tout $a \in (0, a_{\dagger})$,
- (A3) $\int_0^{a_{\dagger}} \mu(a) da = +\infty$,



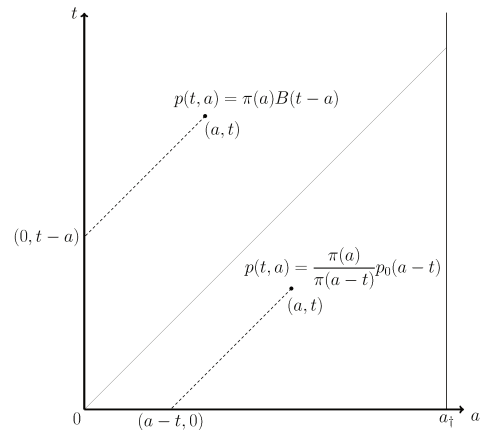
Une hypothèse supplémentaire « naturelle » peut consister à supposer que la fonction β est à support compact dans l'intervalle $]0, a_{\dagger}[$ (ce qui implique que les individus ne peuvent pas se reproduire avant et après un certain âge). Néanmoins, il existe des cas où cela n'est pas satisfait (voir par exemple [GGQ⁺12, eq. (20)] pour un modèle de division cellulaire).

Notons $\pi(a) := \exp\left(-\int_0^a \mu(s) ds\right)$ la probabilité de survivre jusqu'à un âge $a \in [0, a_{\dagger}]$ de sorte que $\pi(0) = 1$, et $\pi(a_{\dagger}) = 0$ en vertu de l'hypothèse (A3) portant sur le taux de mortalité μ . En intégrant formellement le long des caractéristiques, la solution p de (I.1.5) est donnée par

$$p(t, a) = \begin{cases} \frac{\pi(a)}{\pi(a-t)} p_0(a-t) & \text{si } t \leq a, \\ \pi(a) B(t-a) & \text{si } t > a, \end{cases}$$

où

$$B(t) = \int_0^{a_{\dagger}} \beta(a)p(t, a) da \text{ pour } t > 0.$$



La fonction B est l'unique solution continue de l'équation de Volterra de seconde

espèce suivante :

$$B(t) = \int_0^{\min(t, a_+)} K(t-s)B(s) ds + \int_{\min(t, a_+)}^{a_+} \beta(a) \frac{\pi(a)}{\pi(a-t)} p_0(a-t) da \quad (t > 0),$$

où K désigne la *fonction de maternité* définie par $K(a) = \pi(a)\beta(a)$ pour presque tout $a \in (0, a_+)$ (voir, par exemple, Iannelli [Ian95a] ou Webb [Web85]).

A l'instar du modèle de Malthus (I.1.1), le comportement asymptotique des solutions de (I.1.5) dépend de la balance entre les taux de fertilité et de mortalité. En notant $\mathcal{R}_0 := \int_0^{a_+} \beta(a)\pi(a) da$ le *taux de reproduction* de la population, pour toute densité initiale de population p_0 « non-triviale » (au sens où la population ne s'éteint pas naturellement en temps fini) :

- si $\mathcal{R}_0 > 1$, alors la population croît exponentiellement,
- si $\mathcal{R}_0 = 1$, alors la population tend vers un état stationnaire,
- si $\mathcal{R}_0 < 1$, alors la population décroît vers zéro.

Plus précisément, il existe un réel α^* (appelé *paramètre Malthusien*) et une fonction b_0 de sorte que la solution p de (I.1.5) s'écrive sous la forme

$$p(t, a) = e^{\alpha^*(t-a)} b_0(t-a) \pi(a) \quad \text{si } t > a,$$

où la fonction b_0 (dépendante de la densité initiale p_0) tend vers une constante en l'infini (voir, par exemple, [Ian95b, p. 21 and 22] ou Anița [Ani00, p. 54]). Le taux de reproduction \mathcal{R} et le paramètre malthusien α^* sont reliés comme suit :

- $\mathcal{R}_0 > 1$ si et seulement si $\alpha^* > 0$,
- $\mathcal{R}_0 = 1$ si et seulement si $\alpha^* = 0$,
- $\mathcal{R}_0 < 1$ si et seulement si $\alpha^* < 0$.

A l'instar du modèle purement malthusien (I.1.1), la *dynamique de la population* décrite par les équations (I.1.5) est ainsi essentiellement régie par son taux de reproduction \mathcal{R}_0 . Le cas particulier où $\mathcal{R}_0 = 1$ correspond au cas où la population est « idéale », au sens où l'équilibre entre taux de croissance et taux de mortalité est parfaitement maintenu au cours du temps. C'est en partie cet équilibre mathématique qui fut visé dans la politique de l'enfant unique en Chine, en cherchant à moduler efficacement le taux de fertilité β des équations (I.1.5).

Un raffinement du modèle. En 1972, H. Langhaar [Lan72] note que le modèle original de Lotka et McKendrick (I.1.5) ne prend pas en compte les éventuelles variations des taux de fertilité et de mortalité suivant les générations, les fonctions β et μ du système (I.1.5) ne dépendant pas du temps. En outre, une densité de population quelconque ne saurait être réellement totalement isolée et peut être sujette à des variations externes ou internes (notamment dues aux effets de migration). À cet effet, Langhaar complète le

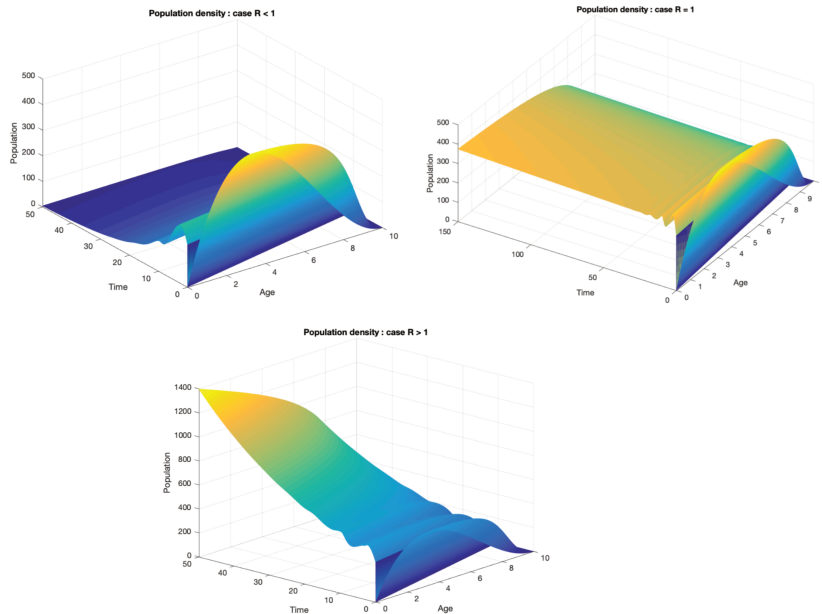


FIGURE I.2 – Allure de la solution de l'équation de Sharpe, Lotka et McKendrick (I.1.5) suivant les cas $\mathcal{R}_0 < 1$ (extinction de la population), $\mathcal{R}_0 = 1$ (stabilisation) et $\mathcal{R}_0 > 1$ (croissance exponentielle malthusienne).

modèle (I.1.5) comme suit :

$$\begin{cases} \frac{\partial p}{\partial t}(t, a) + \frac{\partial p}{\partial a}(t, a) + \mu(t, a)p(t, a) = \underbrace{g(t, a)}_{\text{Migration}}, & t > 0, a \in (0, a_{\dagger}), \\ p(t, 0) = \int_0^{a_{\dagger}} \beta(t, a)p(t, a) da, & t > 0, \\ p(0, a) = p_0(a), & a \in (0, a_{\dagger}). \end{cases} \quad (\text{I.1.6})$$

où les fonctions de fertilité et de mortalité dépendent ici du temps, et où l'application g représente les effets d'une éventuelle migration considérée.

Une structure en âge non linéaire - Gurtin & MacCamy (1974).

Un pendant *non-linéaire* des équations de Sharpe, Lotka et McKendrick fut premièrement étudié par Gurtin et MacCamy [GM74] afin de modéliser - à l'instar du modèle logistique de Verhulst - la compétition entre individus pour accéder aux ressources, ainsi que l'influence du nombre total d'individus sur les taux de fertilité et de mortalité associés

à la population. La densité p en devient décrite par les équations :

$$\begin{cases} \frac{\partial p}{\partial t}(t, a) + \frac{\partial p}{\partial a}(t, a) + \mu(t, a, P(t))p(t, a) = g(t, a), & t > 0, a \in (0, a_+), \\ p(t, 0) = \int_0^{a_+} \beta(t, a, P(t))p(t, a) da, & t > 0, \\ p(0, a) = p_0(a), & a \in (0, a_+), \\ P(t) = \int_0^{a_+} p(t, a) da, & t > 0, \end{cases} \quad (\text{I.1.7})$$

où $P(t) = \int_0^{a_+} p(t, a) da$ représente le nombre total d'individus au temps t et où les taux de fertilité et de mortalité β et μ dépendent de cette dernière quantité (la fonction g représente encore ici les effets de migration). Une hypothèse « naturelle » portant sur les taux de mortalité et de fertilité consiste à supposer que, à t et a fixés, les fonctions $\mu(t, a, \cdot)$ et $\beta(t, a, \cdot)$ sont respectivement croissantes et décroissantes : plus le nombre d'individus augmente, plus la compétition est rude et le développement de la population induit une baisse du taux de reproduction (cette dernière hypothèse est susceptible de surprendre, mais est largement vérifiée par les récents pays en développement [C⁺72]).

Une structure en âge avec diffusion - Gurtin & MacCamy (1973).

Dans une courte lettre adressé aux éditeurs du Journal de Biologie Théorique [Gur73], Gurtin et MacCamy proposent de modéliser le déplacement des individus via un processus de diffusion, originellement posé dans un espace euclidien (tout entier) de dimension finie quelconque. Le système structuré en âge considéré en devient ainsi de nature mixte : un terme hyperbolique modélisant le transport en âge des individus au cours du temps est mêlé à un terme parabolique associé au déplacement des individus dans l'espace considéré.

Lorsque les individus sont restreints à se déplacer dans une certaine zone bornée de l'espace, la structure en âge proposée par Lotka & McKendrick se décrit comme suit. Soit Ω un ouvert connexe borné non-vide dans \mathbb{R}^n , où $n \geq 1$ (biologiquement parlant, on a $n \leq 3$). Considérons qu'une densité de population isolée $p(t, a, x)$ dépende du temps t , de l'âge a des individus et de leur position spatiale $x \in \Omega$, au sens où l'intégrale

$$\int_{a_1}^{a_2} \int_{\omega} p(t, a, x) dx da$$

compte le nombre d'individus, au temps t , dont l'âge est compris entre a_1 et a_2 résidant dans l'ensemble $\omega \subset \Omega$. En supposant que la population reste confinée dans le milieu Ω ,

sa dynamique peut se décrire via les équations

$$\left\{ \begin{array}{l}
 \underbrace{\frac{\partial p}{\partial t}(t, a, x) + \frac{\partial p}{\partial a}(t, a, x)}_{\text{Transport en âge}} + \underbrace{\mu(a)p(t, a, x)}_{\text{Mortalité}} - \underbrace{\Delta p(t, a, x)}_{\substack{\text{Déplacement des} \\ \text{individus}}} = \underbrace{g(t, a, x)}_{\text{Migration}}, \quad t > 0, a \in (0, a_+), x \in \Omega, \\
 \underbrace{\frac{\partial p}{\partial \nu}(t, a, x)}_{\text{Population isolée}} = 0, \quad t > 0, a \in (0, a_+), x \in \partial\Omega, \\
 \underbrace{p(t, 0, x) = \int_0^{a_+} \beta(a)p(t, a, x) da}_{\text{Renouvellement de la population}}, \quad t > 0, x \in \Omega, \\
 \underbrace{p(0, a, x) = p_0(a, x)}_{\text{Densité initiale d'individus}}, \quad a \in (0, a_+), x \in \Omega,
 \end{array} \right. \quad (\text{I.1.8})$$

où $\Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$ est le laplacien par rapport à la variable d'espace et où l'équation $\frac{\partial}{\partial \nu} = 0$

sur $\partial\Omega$ désigne les conditions au bord de Neumann (qui peuvent être remplacées par des conditions d'autres types). Un « pendant non-linéaire » plus complet du système (I.1.8) s'écrit

$$\left\{ \begin{array}{l}
 \frac{\partial p}{\partial t}(t, a, x) + \frac{\partial p}{\partial a}(t, a, x) + \mu(a, t, x, P(t, x))p(t, a, x) - \Delta p(t, a, x) \\
 = g(t, a, x), \quad t > 0, a \in (0, a_+), x \in \Omega, \\
 \frac{\partial p}{\partial \nu}(t, a, x) = 0, \quad t > 0, a \in (0, a_+), x \in \partial\Omega, \\
 p(t, 0, x) = \int_0^{a_+} \beta(t, a, x, P(t, x))p(t, a, x) da, \quad t > 0, x \in \Omega, \\
 p(0, a, x) = p_0(a, x), \quad a \in (0, a_+), x \in \Omega, \\
 P(t, x) = \int_0^{a_+} p(t, a, x) da, \quad t > 0, x \in \Omega,
 \end{array} \right. \quad (\text{I.1.9})$$

où les fonctions β et μ dépendent ici de l'ensemble des variables considérées (âge, temps et espace) ainsi que du nombre total d'individus, et où la fonction g représente les effets de migration.

§ I.1.2. La politique de l'enfant unique en Chine (dès 1979)

Cette sous-section vise à présenter l'influence des équations de Lotka et McKendrick (I.1.5) et de la pensée malthusienne dans la mise en place de la politique de l'enfant unique en Chine. Les prochaines lignes s'inspirent conjointement de deux articles de Susan Greenhalgh [Gre03], [Gre05], anthropologue et spécialiste des politiques démographiques,

de l'ouvrage des mathématiciens Song et Yu [SY88] ainsi que d'un mémoire de Pascal Rocha da Silva [RdS06].

Les Trois Campagnes : avant la politique de l'enfant unique [RdS06] A l'aube de l'instauration de la République Populaire de Chine (en 1949), le Communisme chinois ne se souciait guère des éventuels problèmes de surpopulation, favorisant d'ailleurs les familles nombreuses via l'application de positions pro-natalistes. Mao Zedong encourageait lui même les femmes à procréer le plus possible, décernant le titre de « Maman d'honneur » à celles qui parviennent à avoir plus de dix enfants. En outre, les visions communistes et malthusiennes sont peu compatibles dans la Chine socialiste, où Malthus représente une certaine mentalité bourgeoise. Cependant, la réalité rattrapa l'idéologie Communiste entre 1953 et 1954, où la croissance de population très rapide rend nettement insuffisantes l'ensemble des récoltes agricoles. Le Parti en devient favorable à la pratique de la contraception et au contrôle des naissances.

En 1956 a lieu la Première Campagne (de contrôle des naissances), qui reposera essentiellement sur une propagande pour les mariages tardifs. Mi-1957, Mao Zedong change d'avis et provoque la fin du mouvement, sa brièveté ne parvient donc pas à diminuer notablement la fécondité. Une Seconde Campagne débute en 1962, où l'application repose encore sur une propagande en faveur de mariages tardifs (de plus, un trop grand nombre d'enfants par couple est susceptible de s'accompagner d'une absence de primes à la naissance). La nécessité de cette Campagne ne faisant finalement pas l'unanimité au sein du Parti, celle-ci ne fut appliquée que quelques années et son incidence sur la fécondité en sera quasi-nulle. Une Troisième Campagne nettement plus coercitive prend forme dès 1973 où, dorénavant, la limitation du nombre d'enfants n'est pas seulement suggérée : « un ce n'est pas peu, deux c'est bien et trois c'est trop » (slogan de la Troisième Campagne). L'exécution de cette politique repose sous de multiples formes de pression psychologiques (propagandes de dissuasion, dénonciations, harcèlement, ...), économiques et physiques (avortement et stérilisation forcés, espacement des naissances via dispositifs intra-utérins, ...) qui assurent sa « réussite » : la fécondité chute de 5,8 à 2,7 enfants entre les années 1970 et 1979.

La politique de l'enfant unique (wan-xi-shao) de 1979 [RdS06] [Gre03] [Gre05]. Après la mort de Mao en 1976, un consensus émerge au sein du Parti sur la nécessité et l'urgence de contrôler la croissance, toujours gargantuesque, de la population chinoise. Cette croissance malthusienne constitue un obstacle économique majeur en vue de la réforme des « Quatre Modernisations », qui vise à concentrer l'essentiel des ressources au sein du pays. Entre 1977 et 1978, selon Susan Greenhalgh [Gre05], Deng Xiaoping (successeur de Mao) semble encourager l'élite scientifique à réfléchir sur les nombreux problèmes du pays, dont celui de la croissance exponentielle de la population. C'est dans ce contexte qu'intervient Song Jian, mathématicien originellement spécialiste du contrôle optimal du guidage des missiles.

En 1978, lors d'un Congrès à Helsinki (the Seventh Triennial World Congress of the International Federation of Automatic Control, IFAC, 12-16 June), Song prend connais-

sance des travaux du *Club de Rome* qui présentent des modèles de croissance de population, l'influence pseudo dramatique de cette croissance quant à la destruction de l'environnement et les possibilités de contrôle de la dynamique. Song comprend alors directement que sa discipline de prédilection originelle - le contrôle des trajectoires de missiles - pouvait s'adapter de manière efficace à la prédiction et au contrôle de la croissance démographique chinoise. De retour en Chine, Song constitue une petite équipe constituée de Yu Jingyuan (ingénieur), Li Guangyuan (informaticien) et Tian Xueyuan (économiste) pour mettre au point leur modèle de prédiction.

Le modèle de dynamique de population chinoise [SY88] Le modèle retenu par le groupe de Song afin de modéliser (en vue de la contrôler) la densité de population chinoise est celui décrit par les équations linéaires (sans diffusion) de Lotka & McKendrick (I.1.6), lorsque les fonctions de mortalité et de fertilité dépendent du temps. Plus précisément, Song modélise la dynamique de la population Chinoise au cours du temps via sa densité p sujette aux équations

$$\begin{cases} \frac{\partial p}{\partial t}(t, a) + \frac{\partial p}{\partial a}(t, a) + \mu(t, a)p(t, a) = 0, & t > 0, a \in (0, a_{\dagger}), \\ p(t, 0) = \int_0^{a_{\dagger}} f(t, a)k(t, a)p(t, a) da, & t > 0, \\ p(0, a) = p_0(a), & a \in (0, a_{\dagger}), \end{cases} \quad (\text{I.1.10})$$

où les paramètres intervenant dans (I.1.10) sont décrits ci-dessous :

- a_{\dagger} est l'âge maximal de la population chinoise,
- $k(t, a)$ représente la proportion de femmes d'âge a à l'instant t ,
- $f(t, a)$ représente le nombre de nouveaux-nés provenant de femelles d'âge a à l'instant t ,
- $\mu(t, a)$ représente le taux de mortalité à l'instant t des individus d'âge a ,
- p_0 est la densité initiale de Chinois.

Notons que la migration est négligée dans les équations (I.1.10), car ce n'est pas - selon Song et Yu - une variable de contrôle satisfaisante⁴, en vue notamment de l'argument suivant : "[...] no populous contry or nation can expect to reduce its population by emigration. Take China for example. No country in the world can recieve even one-tenth of the Chinese population, thus reducing the population of China by 100 million. Even the addition of 10 million people, still less 100 million people, would have a big impact on a medium size country, and might bring difficulties to its economy" [SY88, p. 26]. Aussi, hors de question d'envoyer la population chinoise sur la face cachée de la Lune car il y fait un peu froid : "Should any man-made structures in outer space go wrong, this will lead to the extinction of all kinds of life within them. Therefore, if people must open up new living space, they should go to the deserts, oceans, highlands or the North and South Pole areas but not to outer space" [SY88, p. 4].

4. C'est pourtant l'objet d'étude de toute cette présente thèse!

De la même façon, Song remarque ingénument que la variable de temps des équations (I.1.13) ne saurait être contrôlée : “[...] in short, the passage of time cannot be slowed down, neither can it be turned back ; this is a natural law that no one can change. In our study of population development, we can only reluctantly regard time as an independant variable free from human will.” [SY88, p. 24] En outre, Song remarque qu’augmenter la mortalité μ du système (I.1.13) en vue d’obtenir le profil de population souhaité n’est peut être pas la meilleure méthode⁵, éthiquement parlant. La variable de contrôle considérée par le groupe de Song est le nombre moyen d’enfants donnés par une génération au cours du temps, défini par

$$\beta(t) := \int_0^{a_{\dagger}} f(t, a) da, \quad (t > 0). \quad (\text{I.1.11})$$

En notant

$$h(t, a) := \frac{f(t, a)}{\int_0^{a_{\dagger}} f(t, a) da} \quad (t > 0, a \in (0, a_{\dagger})), \quad (\text{I.1.12})$$

le système (I.1.10) se réécrit de la façon suivante :

$$\begin{cases} \frac{\partial p}{\partial t}(t, a) + \frac{\partial p}{\partial a}(t, a) + \mu(t, a)p(t, a) = 0, & t > 0, a \in (0, a_{\dagger}), \\ p(t, 0) = \beta(t) \int_0^{a_{\dagger}} h(t, a)k(t, a)p(t, a) da, & t > 0, \\ p(0, a) = p_0(a), & a \in (0, a_{\dagger}). \end{cases} \quad (\text{I.1.13})$$

La méthodologie optée par le groupe de Song en vue de prédire la densité de population future via modulation de la fonction β consiste en les deux points suivants :

- Les fonctions μ , h et k sont mesurées de manière ponctuelle entre les années 1973 et 1978, interpolées continûment (via notamment l’utilisation de statistiques à l’instar de la loi du χ^2 pour la fonction h - voir [SY88, p. 70 et p.199]), puis extrapolées pour les années postérieures à 1979.
- à fonction β fixée, la simulation de la solution p du système (I.1.13) donne une approximation de la densité de population p à devenir.

La Figure I.3 recense les résultats des simulations obtenues par l’équipe de Song pour différentes valeurs constantes au cours du temps de la fonction β . Le nombre total d’individus au cours du temps P donnée par $P(t) = \int_0^{a_{\dagger}} p(t, a) da$, où p est la solution de (I.1.13) est esquissé entre les années 1980 et 2080 pour différents nombre moyens d’enfants par générations. A la lumière des courbes dressés en Figure I.3, le résultat est sans appel : seules des faibles valeurs du nombre moyen d’enfants par couple ($\beta = 1$ et $\beta =$

5. En réalité, le contrôle de la fonction μ des équations (I.1.13) correspond quantitativement au contrôle d’une certaine fraction de la densité de population, et peut correspondre qualitativement à une mise en quarantaine des individus contrôlés. C’est un contrôle du type “harvest” (« récolte »), largement étudié dans la littérature d’un point de vue contrôle optimal (voir, par exemple, [Ani00, Chapitre 3] ainsi que la collection de références en page 104 de [Ani00]) et dans le Chapitre V de ce manuscrit d’un point de vue contrôle à zéro.

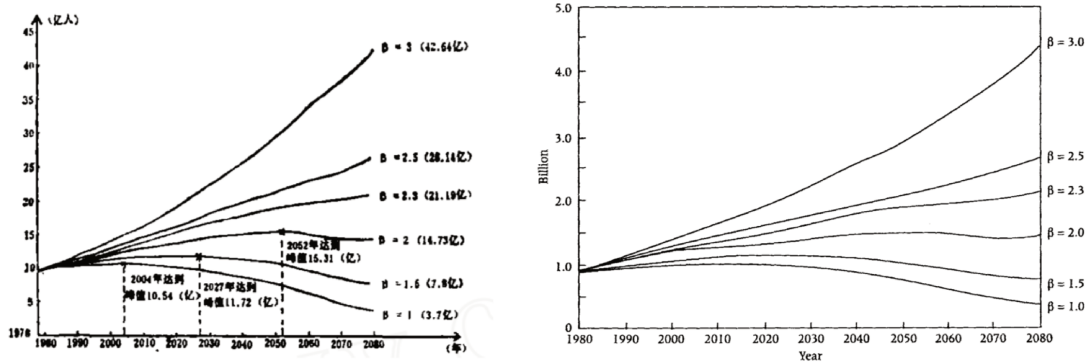


FIGURE I.3 – Evolution du nombre total d’individus chinois en fonction du nombre moyen d’enfants donné par femme. La figure de gauche est extraite du rapport de Song (version chinoise originale), celle de droite en est une reproduction fournie par S. Greenhalgh [Gre03] (voir aussi [SY88, p. 213]).

1,5) permettent de réduire progressivement le nombre total d’individus Chinois. Selon Susan Greenhalgh [Gre05], d’après plusieurs interviews d’anciens gouvernants Chinois (notamment l’ancien premier Ministre), ceux-ci furent éblouis par les mathématiques, frappés par ces projections et convaincus, comme Song, qu’une politique de l’enfant unique n’était pas une mais la solution au problème démographique. Le rapport de Song fut validé par les plus hautes instances du Parti et la politique de l’enfant unique fut mise en place dans la foulée via une lettre ouverte, dont une première version (trop longue et trop académique) fut d’ailleurs rédigée par Song lui-même. L’histoire ne dit pas si Deng Xiapoing fut réellement frappé par les conclusions de la Figure I.3, ou si cette figure fut utilisée comme outil de propagande afin de garantir la subsistance économique de la population chinoise.

Contrôle optimal vers une population « idéale ». Une politique de contrôle des naissances ne pouvant raisonnablement pas être une solution pérenne (durable indéfiniment), Song et Yu définissent dans [SY88, Chap. 7], en vue de l’atteindre, le concept subjectif de « population idéale » dont l’essence repose essentiellement sur les points suivants :

- la population doit avoir un taux de croissance nul, *i.e.* le taux de reproduction \mathcal{R}_0 associé aux équations (I.1.13) doit être égal à 1 pour obtenir un équilibre parfait entre naissances et morts des individus,
- le ratio entre individus travailleurs et non-travailleurs (“the social dependency index”) ainsi que le ratio entre l’âge moyen des individus et l’espérance de vie moyenne des individus (“the ageing index”) doivent être supérieurement bornés,
- chaque individu doit pouvoir subvenir à ses besoins naturels (eau, oxygène, ...) et économiques tout au long de son existence.

Le but de Song est de moduler efficacement, au cours du temps, la valeur du taux

de fertilité β afin d'obtenir en 100 ans (donc en 2080, la politique radicale de contrôle des naissances prenant effet dès 1980) une telle population idéale, et ensuite de ne plus contrôler la population (de la laisser évoluer librement). Par des arguments hautement non scientifiques (via notamment l'introduction et l'utilisation de la notion de « degrés de possibilité et de satisfaction »), Song construit une population idéale définie par sa densité en âge notée p^* , dont le nombre d'individus associés se situe entre 650 et 700 millions (sous-estimant au passage la capacité porteuse de la Chine, réévaluée par la suite).

Comment atteindre cette population idéale à moindre coût ? Il s'agit de résoudre le problème de contrôle optimal suivant :

$$\min_{\beta(\cdot)} \left\{ \int_0^{100} \int_0^{a+} [p(t, a) - p^*(a)]^2 dt da \right\},$$

où p désigne la solution de (I.1.13) associée à la fonction β . Song résout ce problème en obtenant des conditions nécessaires d'optimalité (voir [SY88, Chap. 8]) et dresse le programme de fertilité le plus « humainement acceptable » en Figure I.4. Selon cette

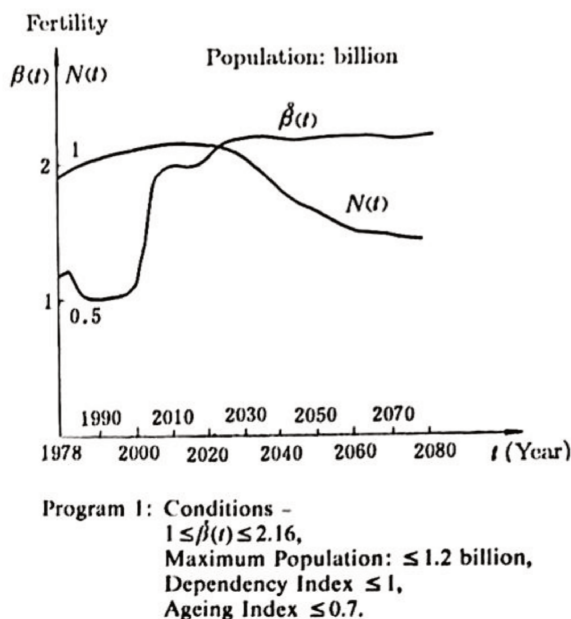


FIGURE I.4 – Fertilité optimale pour atteindre la densité de population optimale. La valeur des ratios de la population idéale est totalement arbitraire. Tiré de [SY88, p. 274].

figure, afin d'obtenir le profil de population idéal en 100 ans à partir de 1980, la fertilité doit effectivement rapidement chuter à un enfant par couple jusqu'aux environs de l'année 2005 avant de pouvoir se redresser progressivement vers le taux de fertilité permettant de rendre la solution de (I.1.13) stationnaire.

Critique et résultats [Gre05], [Coh90]. L'absence de synergie entre sciences dures et sciences sociales lors de la mise en place de cette politique conduit un bilan bien évidemment socialement difficile (plus de 20 million de stérilisation forcées, plus de 15 millions d'avortements, ...). La mortalité infantile - essentiellement des jeunes filles - fut particulièrement élevée, amenant un sex-ratio à la naissance aux environs de 1,1 en faveur des garçons. Notons que ces statistiques sont d'ailleurs biaisées par la présence de millions d'enfants non-déclarés (« enfants noirs »), sans accès à la santé ou à l'éducation.

Une review scientifique du livre de Song et Yu établie par J. E. Cohen [Coh90] résume l'ensemble des critiques communément appliquées au travail du groupe de Song. Quant au choix du modèle de dynamique de population lui-même (I.1.10), Cohen déplore l'absence de non-linéarités et de considération spatiale (les ruraux se reproduisant nettement plus que les citadins). En outre, il est souligné que le concept de population idéale établi par Song est purement subjectif et que les valeurs de certaines contraintes ne sont absolument pas justifiées. Cela est aussi remarqué par de nombreux démographes, attestant de manière tranchée que les résultats du groupe de Song « ne présentent aucun intérêt scientifique » (voir [RdS06, p. 25] qui cite Y. Blayo, spécialiste en démographie). Le groupe de Song ne modélise en rien la possibilité de changement de comportement sociaux, modélisant la trajectoire de la population telle un missile calibré par son taux de fertilité.

Notons qu'une alternative à la politique de l'enfant unique fut proposée notamment par S. Greenhalgh, mais sous-estimée par le groupe de Song. Cette alternative proposait de limiter le nombre de naissances à deux enfants (« stop-at-two »), mais en définissant un âge minimal de conception du premier enfant (entre 25 et 27 ans) et en limitant d'au moins quatre années les âges de naissance des deux enfants (voir [Coh90, p. 497] ou [RdS06, p. 27]). Selon les projections, cela aurait pu conduire à une diminution notable du nombre d'individus via une politique de contrôle objectivement plus humaine que celle que la Chine a connue.

§ I.1.3. Contrôle par migration : objectif général de la thèse

Si l'objectif du groupe de Song fut de contrôler les équations linéaires de Lotka et McKendrick par un contrôle des naissances, celui de cette thèse est de contrôler une densité de population par un contrôle des migrations. Autrement dit, dans tout ce qui suit, la variable de contrôle ne sera plus la fonction de fertilité β , mais la fonction g de chacune des équations linéaires (I.1.6), des équations non-linéaires (I.1.7) et des équations avec diffusion (I.1.8). En général, le contrôle g sera voué à n'agir que pour une certaine tranche d'âge d'individus, et additionnellement dans une certaine zone du milieu considéré pour les équations modélisant de plus le déplacement des individus.

En tenant compte de telles restrictions, les questions générales associées à cette thèse sont les suivantes :

- Etant donné une densité initiale de population p_0 et une densité de population cible p_1 , est-il possible de trouver un taux de migration g conduisant la densité de

population de p_0 à p_1 ?

- Si oui, est-il possible de moduler le taux de migration g afin de préserver la positivité de la trajectoire contrôlée (ce qui est biologiquement parlant indispensable pour une population d'individus) ?
- Si oui, combien de temps le processus de contrôle peut-il prendre ? Peut-on décrire le contrôle associé au temps minimal de contrôle (sous réserve d'existence) ?
- Si oui, peut-on décrire théoriquement ou numériquement un contrôle optimal satisfaisant un critère prescrit à l'avance (à l'instar du contrôle à zéro de norme optimale dans le cas où $p_1 \equiv 0$) ?

Les notions associées aux questions ci-dessus sont classiques en théorie du contrôle de systèmes de dimension infinie. Nous décrivons ces notions dans la section suivante.

I.2 QUELQUES NOTIONS DE CONTRÔLABILITÉ D'ÉQUATIONS AUX DÉRIVÉES PARTIELLES

Les notions de contrôlabilité et d'observabilité développées dans cette présente section sont largement discutées dans Tucsnak et Weiss [TW09], ou dans Coron [Cor07] (pour ne citer qu'eux). Concernant les notions de semi-groupe d'opérateurs et de générateur infinitésimal, nous renvoyons le lecteur intéressé à [TW09, Chap. 2]. Chacune des notions abordées sera illustrée par l'étude des propriétés de contrôlabilité de l'équation de la chaleur (certaines équations de dynamique de populations étudiées dans ce manuscrit étant en partie de nature parabolique lorsque le déplacement des individus est considéré, c.f. les Chapitres III et IV).

§ I.2.1. Contrôlabilité et observabilité

Contrôlabilité Soient $(X, \|\cdot\|_X)$ et $(U, \|\cdot\|_U)$ deux espaces de Hilbert, identifiés à leur duaux. Soit $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0} \subset \mathcal{L}(X)$ un semi-groupe fortement continu d'opérateurs sur X dont le générateur infinitésimal non-borné est noté $A : \mathcal{D}(A) \rightarrow X$, et soit $B \in \mathcal{L}(U, X)$ un opérateur borné⁶. Considérons le problème de Cauchy abstrait suivant :

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), & t \geq 0, \\ z(0) = z_0, \end{cases} \quad (\text{I.2.1})$$

où $z_0 \in X$ et $u \in L^2([0, +\infty); U)$. Les espaces X et U sont respectivement communément appelés *espace d'état* (dans lequel les conditions initiales z_0 du système dynamique

6. En général, l'opérateur B n'est pas nécessairement borné (par exemple, pour l'étude du contrôle frontière). Dans ce manuscrit, le contrôle est essentiellement additif et l'opérateur de restriction associé restera continu de U dans X .

(I.2.1) appartiennent) et *espace de contrôle* (dans lequel les valeurs du contrôle $u(t)$ appartiennent, pour tout $t \geq 0$). En dimension infinie, trois principales notions distinctes de contrôlabilité doivent être définies.

Définition – I.2.1 (Contrôlabilité). *Soit $\tau > 0$.*

- ♣ *Le système (I.2.1) est **exactement contrôlable en temps τ** si, pour toute donnée initiale $z_0 \in X$ et pour toute cible $z_1 \in X$, il existe un contrôle $u \in L^2([0, \tau]; U)$ de sorte que la solution z de (I.2.1) vérifie $z(\tau) = z_1$ dans X .*
- ♣ *Le système (I.2.1) est **contrôlable à zéro en temps τ** si, pour toute donnée initiale $z_0 \in X$ il existe un contrôle $u \in L^2([0, \tau]; U)$ de sorte que la solution z de (I.2.1) vérifie $z(\tau) = 0$ dans X .*
- ♣ *Le système (I.2.1) est **approximativement contrôlable en temps τ** si, pour tout $\varepsilon > 0$, pour toute donnée initiale $z_0 \in X$ et pour toute cible $z_1 \in X$, il existe un contrôle $u \in L^2([0, \tau]; U)$ de sorte que la solution z de (I.2.1) vérifie $\|z(\tau) - z_1\|_X < \varepsilon$.*

Notons que l'unique solution $z \in C([0, +\infty); X)$ du système (I.2.1) peut s'exprimer sous la forme

$$z(t, \cdot) = \mathbb{T}_t z_0 + \Phi_t u \quad (t \geq 0, \quad u \in L^2([0, \infty); U)), \quad (\text{I.2.2})$$

où pour tout $t \geq 0$, $\Phi_t \in \mathcal{L}(L^2([0, \infty); U); X)$ désigne l'opérateur défini par

$$\Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma) d\sigma \quad (t \geq 0, \quad u \in L^2([0, \infty); U)). \quad (\text{I.2.3})$$

Les propriétés de contrôlabilités en temps $\tau > 0$ du système (I.2.1) sont ainsi équivalentes aux propriétés suivantes, relatives aux images des opérateurs \mathbb{T}_τ et Φ_τ :

Proposition – I.2.1. *Soit $\tau > 0$. Le système (I.2.1) est :*

- ♣ *exactement contrôlable en temps τ si et seulement si $\text{Im } \Phi_\tau = X$,*
- ♣ *contrôlable à zéro en temps τ si et seulement si $\text{Im } \mathbb{T}_\tau \subset \text{Im } \Phi_\tau$,*
- ♣ *approximativement contrôlable en temps τ si et seulement si $\overline{\text{Im } \Phi_\tau} = X$.*

Etant donné $\tau > 0$, l'image de l'opérateur Φ_τ défini par (I.2.3) est appelé **espace atteignable au temps τ** . En vertu de la Proposition I.2.1, la description (souvent délicate) de l'espace atteignable peut permettre d'obtenir certains résultats de contrôlabilité.

Il est clair que la contrôlabilité exacte en un temps $\tau > 0$ implique la contrôlabilité à zéro ainsi que la contrôlabilité approchée au même temps τ , les implications réciproques étant fausses en général (hormis le cas où les espaces X et U sont de dimension finie).

Observabilité. Les trois notions de contrôlabilité présentées dans la Définition (I.2.1) possèdent chacune une formulation duale qui leur est équivalente, en vertu de la Proposition suivante :

Proposition I.2.1. *Soit $\tau > 0$.*

- ♣ *Le système (I.2.1) est exactement contrôlable en temps τ si et seulement si il existe une constante $C_\tau > 0$ de sorte que*

$$C_\tau^2 \int_0^\tau \|B^* \mathbb{T}_t^* \psi\|_U^2 dt \geq \|\psi\|_X^2 \quad \text{pour tout } \psi \in X. \quad (\text{I.2.4})$$

- ♣ *Le système (I.2.1) est contrôlable à zéro en temps τ si et seulement si il existe une constante $C_\tau > 0$ de sorte que*

$$C_\tau^2 \int_0^\tau \|B^* \mathbb{T}_t^* \psi\|_U^2 dt \geq \|\mathbb{T}_\tau^* \psi\|_X^2 \quad \text{pour tout } \psi \in X. \quad (\text{I.2.5})$$

- ♣ *Le système (I.2.1) est approximativement contrôlable en temps τ si et seulement si la propriété de continuation unique suivante est vérifiée : pour tout $\psi \in X$,*

$$(\text{pour tout } t \in [0, \tau], B^* \mathbb{T}_t^* \psi = 0) \implies (\psi = 0 \text{ dans } X).$$

Les inégalités (I.2.4) et (I.2.5) portent le nom d'*inégalités d'observabilité*, au sens où l'observation de la quantité

$$\int_0^\tau \|B^* \mathbb{T}_t^* \psi\|_U^2 dt$$

permet d'obtenir une estimation de $\|\psi\|_X^2$ dans le cadre de la contrôlabilité exacte, ou bien une estimation de $\|\mathbb{T}_\tau^* \psi\|_X^2$ dans le cadre de la contrôlabilité à zéro. Cela revient à estimer l'état initial ou bien l'état final du *système adjoint* :

$$\begin{cases} \dot{z}(t) = A^* z(t), & t \geq 0, \\ z(0) = z_0, \end{cases} \quad (\text{I.2.6})$$

via l'observation de la quantité $\int_0^\tau \|B^* z(t, \cdot)\|_U^2 dt$, où z est la solution de (I.2.6) :

Proposition I.2.2. *Soit $\tau > 0$.*

- ♣ *Le système (I.2.1) est exactement contrôlable en temps τ si et seulement si il existe une constante $C_\tau > 0$ de sorte que pour toute solution z de (I.2.6),*

$$\|z_0\|_X^2 \leq C_\tau^2 \int_0^\tau \|B^* z(t, \cdot)\|_U^2 dt. \quad (\text{I.2.7})$$

- ♣ *Le système (I.2.1) est contrôlable à zéro en temps τ si et seulement si il existe une constante $C_\tau > 0$ de sorte que pour toute solution z de (I.2.6),*

$$\|z(\tau)\|_X^2 \leq C_\tau^2 \int_0^\tau \|B^* z(t, \cdot)\|_U^2 dt. \quad (\text{I.2.8})$$

- ♣ *Le système (I.2.1) est approximativement contrôlable en temps τ si et seulement si,*

pour toute solution z de (I.2.6),

$$(pour\ tout\ t \in [0, \tau],\ B^*z(t, \cdot) = 0) \implies (z_0 = 0\ dans\ X).$$

Remarque – I.2.1. Notons que la contrôlabilité approché en temps τ peut se déduire de la contrôlabilité à zéro en temps τ dans le cas où le système (I.2.6) possède la propriété d'unicité rétrograde suivante :

$$si\ z(\tau) = 0,\ alors\ z_0 = 0, \tag{I.2.9}$$

propriété notamment vérifiée pour certaines équations paraboliques.

Exemple : contrôle interne de l'équation de la chaleur. Soit $\Omega \subset \mathbb{R}^n$ un ouvert non-vide, connexe et borné de \mathbb{R}^n (où $n \geq 1$), et notons $X = U = L^2(\Omega)$. Supposons que le bord de Ω soit de classe C^2 . Soit ω (la zone de contrôle) un ouvert relativement compact non-vide dans Ω , et considérons l'équation de la chaleur contrôlée :

$$\begin{cases} \frac{\partial z}{\partial t}(t, x) = \Delta z(t, x) + \chi_\omega(x)u(t, x), & t \geq 0, x \in \Omega, \\ z(t, x) = 0, & t > 0, x \in \partial\Omega, \\ z(0, x) = z_0(x), & x \in \Omega, \end{cases} \tag{I.2.10}$$

où χ_ω désigne la fonction caractéristique de l'ensemble ω et u est le contrôle. L'équation (I.2.10) représente un processus de diffusion contrôlé dans un sous-domaine ω de l'ensemble Ω . Soit \mathcal{A} l'opérateur infinitésimal défini par

$$\begin{aligned} \mathcal{D}(\mathcal{A}) &= H^2(\Omega) \cap H_0^1(\Omega), \\ \mathcal{A}\varphi &= \Delta\varphi \quad (\varphi \in \mathcal{D}(\mathcal{A})), \end{aligned}$$

où $\Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$ est l'opérateur laplacien. L'opérateur \mathcal{A} génère un semi-groupe d'opérateurs fortement continu sur X , noté $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$. Notons $B \in \mathcal{L}(X)$ l'opérateur de restriction défini par

$$Bu = \chi_\omega u \quad (u \in X),$$

de sorte que le système (I.2.10) s'écrive sous la forme

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), & t \geq 0, \\ z(0) = z_0. \end{cases} \tag{I.2.11}$$

Les différentes notions de contrôlabilité associées à l'équation (I.2.10) sont présentes ci-après.

- ♣ Le système (I.2.10) n'est pas exactement contrôlable. En effet, étant donné $z_0 \in X$, $\tau > 0$ et $u \in L^2((0, \tau); X)$, l'effet dit régularisant de l'équation de la chaleur

implique que la solution z de (I.2.10) satisfait $z(t, \cdot) \in \mathcal{C}^\infty(\Omega)$ pour tout $t > 0$, de sorte que n'importe quelle fonction $z_1 \in X \setminus \mathcal{C}^\infty(\Omega)$ ne saurait être atteinte, quelque soit le temps de contrôle $\tau > 0$.

- ♣ Le système (I.2.10) est contrôlable à zéro en tout temps $\tau > 0$. Ce résultat fut obtenu indépendamment par Fursikov et Imanuvilov [FI96] et par Lebeau et Robbiano [LR95]. Le point commun de ces deux démonstrations réside dans le fait que les auteurs démontrent l'observabilité du système adjoint de l'équation (I.2.10) via l'utilisation d'inégalités de Carleman - dans leur version globale dans [FI96] et dans leur version locale dans [LR95], conséquences d'analyses microlocales. Nous renvoyons le lecteur au résumé de Lebeau et Rousseau [LRL12] pour une présentation pédagogique des inégalités de Carleman et de leur application pour le contrôle à zéro de l'équation de la chaleur.
- ♣ Le système (I.2.10) est approximativement contrôlable en tout temps $\tau > 0$. En effet, ce système est contrôlable à zéro et possède la propriété d'unicité rétrograde (I.2.9) mentionnée en Remarque I.2.1.

§ I.2.2. A propos des contrôles optimaux

Lorsqu'un système est contrôlable d'un état initial $z_0 \in X$ vers un état final $z_1 \in X$, le contrôle associé n'est pas nécessairement unique. Parmi cette classe de contrôles transportant z_0 vers z_1 , certains sont dits optimaux au sens où ils permettent de satisfaire un critère de minimisation.

Contrôle à zéro de norme optimale. Supposons qu'une condition initiale $z_0 \in X$ soit contrôlable à zéro en temps $\tau > 0$, au sens donné par la Définition I.2.1. L'ensemble des contrôles à zéro en temps τ pour l'état initial z_0 , noté $\text{Adm}(z_0, \tau)$ et défini par

$$\text{Adm}(z_0, \tau) := \{u \in L^2((0, \tau); U) \mid \Phi_\tau u = -\mathbb{T}_\tau z_0\},$$

est ainsi un ensemble non-vidé, convexe et fermé. La projection de la fonction nulle dans $L^2((0, \tau); U)$ sur ce convexe fermé non-vidé fournit le *contrôle à zéro de norme optimale*, noté u_* , qui satisfait donc

$$u_* = \text{Argmin}\{\|u\|_{L^2((0, \tau); U)} \mid u \in \text{Adm}(z_0, \tau)\}.$$

Un tel contrôle à zéro de norme optimale s'appelle aussi contrôle *HUM*, en vertu de la méthode du même nom (Hilbert Uniqueness Method) introduite par Jacques-Louis Lions [Lio68] afin d'approcher les contrôles à zéro de norme optimale.

Obtenir une formulation explicite du contrôle HUM est une tâche délicate. Afin de pallier à ce problème, la *méthode HUM pénalisée* permet par exemple d'obtenir une approximation du contrôle à zéro de norme optimale recherché. Etant donné $\varepsilon > 0$, définissons

la fonctionnelle F_ε sur l'espace des contrôles $L^2((0, \tau); U)$ par

$$F_\varepsilon(u) := \|u\|_{L^2((0, \tau); U)}^2 + \frac{1}{\varepsilon} \|z(\tau, \cdot)\|_X^2 \quad (u \in L^2((0, \tau); U)), \quad (\text{I.2.12})$$

où z désigne la solution de (I.2.1) associée au contrôle u . Pour $\varepsilon > 0$ fixé, la fonctionnelle F_ε est continue, coercive et strictement convexe sur $L^2((0, \tau); U)$: elle possède ainsi un unique minimiseur, noté ici u_ε . Ce minimiseur constitue une approximation du contrôle HUM, au sens où on a convergence forte de u_ε vers u_* dans $L^2((0, \tau); U)$ lorsque ε tend vers 0^+ (voir, par exemple, Boyer [Boy13, Théorème 1.7.]). En pratique, une discrétisation du système (I.2.1) et de la fonctionnelle associée (I.2.12) peut permettre d'obtenir, en un certain sens, une approximation du contrôle à zéro de norme optimale recherché (voir, par exemple, [Boy13] concernant l'utilisation de discrétisations pour la recherche du contrôle HUM associée au contrôle de l'équation de la chaleur).

Contrôle en temps optimal. Dans un problème du type contrôle en *temps optimal*, l'objectif général est d'emmener une condition initiale d'un système contrôlé vers un état cible le plus rapidement possible (en minimisant le temps nécessaire pour transporter z_0 vers z_1), en tenant compte de contraintes spécifiques à la fonction de contrôle (et éventuellement sur la solution du système contrôlé, ce qui ne sera pas considéré dans ce manuscrit). Plus précisément, étant donné un ensemble convexe, compact, borné et non-vide $K \subset U$ (l'ensemble des contraintes associées à la fonction de contrôle), l'ensemble des contrôles admissibles est défini ici par $\mathcal{U}_{ad} := L^\infty([0, \infty); K)$. Étant donné $z_0 \in X$, l'ensemble $\mathcal{R}(z_0, \mathcal{U}_{ad})$ des états atteignables partant de z_0 à l'aide de contrôles admissibles est défini par

$$\mathcal{R}(z_0, \mathcal{U}_{ad}) := \{z(\tau, \cdot) \mid \tau > 0 \text{ et } z \text{ est la solution de (I.2.1) avec } u \in \mathcal{U}_{ad}\}.$$

Pour $z_0 \in X$ et $z_1 \in \mathcal{R}(z_0, \mathcal{U}_{ad})$, le problème de contrôle en temps optimal pour le système (I.2.1) consiste à déterminer (sous réserve d'existence) un contrôle $u^* \in \mathcal{U}_{ad}$ de sorte que la solution associée z^* de (I.2.1) vérifie

$$z^*(\tau^*(z_0, z_1)) = z_1,$$

où $\tau^*(z_0, z_1)$ est le temps minimal nécessaire pour transporter l'état initial z_0 vers la cible z_1 à l'aide de contrôles admissibles,

$$\tau^*(z_0, z_1) := \min_{u \in \mathcal{U}_{ad}} \{\tau \mid z(\tau, \cdot) = z_1\}. \quad (\text{I.2.13})$$

Sous certaines hypothèses, il est possible de démontrer que le contrôle optimal possède la propriété de *bang-bang*, au sens où le contrôle optimal satisfait $u(t, \cdot) \in \partial K$ pour presque tout $t \in [0, \tau^*(z_0, z_1)]$. Ce principe est par exemple vérifié en dimension finie (voir, par exemple, la découverte de ce résultat dans l'article originel de Bellman, Glicksberg et Gross [BGG56]). En dimension infinie, la propriété de bang-bang peut être obtenue en démontrant la contrôlabilité L^∞ des équations (I.2.1) en tout temps $\tau > 0$, lorsque le

contrôle est additionnellement supporté en temps dans un ensemble de Lebesgue quelconque de mesure strictement positive dans l'intervalle $[0, \tau]$ (voir par exemple Micu, Roventa et Tucsnak [MRT12] - Proposition 2.6.). A titre d'exemple, concernant l'équation de la chaleur, cette méthode fut récemment employée par Wang [Wan08], Apraiz et al. [AEWZ14] dans le cadre du contrôle interne et par Micu, Roventa et Tucsnak [MRT12] dans celui du contrôle frontière. Ces démonstrations reposent sur une adaptation et une généralisation de la méthode de Lebeau et Robbiano [LR95].

I.3 CONTRÔLE PAR MIGRATION D'ÉQUATIONS STRUCTURÉES EN ÂGE : ÉTAT DE L'ART

Cette section a pour ambition de donner une vue panoramique des résultats de contrôlabilité par migration des équations de Lotka & McKendrick. Les deux premières sous-sections dressent respectivement les résultats existants lorsque la diffusion spatiale des individus n'est pas prise en compte (où l'on contrôle ainsi un système du type hyperbolique), ou lorsque cette diffusion est considérée dans les équations (contrôle d'un système mixte du type hyperbolique et parabolique).

§ I.3.1. Le cas sans diffusion

Considérons une densité de population p décrite par les équations de Lotka & McKendrick :

$$\begin{cases} \frac{\partial p}{\partial t}(t, a) + \frac{\partial p}{\partial a}(t, a) + \mu(a)p(t, a) = 0, & t > 0, a \in (0, a_+), \\ p(t, 0) = \int_0^{a_+} \beta(a)p(t, a) da, & t > 0, \\ p(0, a) = p_0(a), & a \in (0, a_+), \end{cases} \quad (\text{I.3.1})$$

où a_+ désigne l'âge maximal des individus, $p_0 \in L^2[0, a_+]$ désigne la densité initiale de population et les fonctions β et μ désignent respectivement les taux de fertilité et de mortalité des individus. Rappelons les hypothèses « classiques » portant sur les fonctions β et μ , déjà mentionnées dans la Section I.1 :

- (A1) $\beta \in L^\infty(0, a_+)$, $\beta(a) \geq 0$ pour presque tout $a \in (0, a_+)$,
- (A2) $\mu \in L^1[0, a^*]$ pour tout $a^* \in (0, a_+)$, $\mu(a) \geq 0$ pour presque tout $a \in (0, a_+)$,
- (A3) $\int_0^{a_+} \mu(a) da = +\infty$.

De la même façon, rappelons qu'en vertu des hypothèses ci-dessus, la fonction π définie par

$$\pi(a) := \exp\left(-\int_0^a \mu(s) ds\right) \quad (a \in (0, a_+) \text{ p.p.}), \quad (\text{I.3.2})$$

décrit la probabilité pour un individu de survivre jusqu'à un âge $a \in (0, a_+)$.

Dans toute la suite, étant donné $\tau > 0$, nous noterons $Q_\tau := (0, \tau) \times (0, a_+)$.

A propos de l'opérateur de population sans diffusion. Considérons l'opérateur de population \mathcal{A}_0 défini par

$$\begin{aligned} \mathcal{D}(\mathcal{A}_0) &= \left\{ \varphi \in L^2[0, a_+] \mid \varphi \text{ est localement absolument continue sur } [0, a_+), \right. \\ &\quad \left. \varphi(0) = \int_0^{a_+} \beta(a)\varphi(a) da; -\frac{d\varphi}{da} - \mu\varphi \in L^2[0, a_+] \right\}, \\ \mathcal{A}_0\varphi &= -\frac{d\varphi}{da} - \mu\varphi \quad (\varphi \in \mathcal{D}(\mathcal{A}_0)). \end{aligned} \tag{I.3.3}$$

L'opérateur \mathcal{A}_0 défini ci-dessus est le générateur infinitésimal d'un semi groupe, noté $\mathbb{T}^{\mathcal{A}_0} = (\mathbb{T}_t^{\mathcal{A}_0})_{t \geq 0}$ dans cette introduction. La démonstration de ce résultat peut se trouver dans Song et al. [SYW⁺82, Théorème 4]⁷ via l'utilisation du théorème de Lumer-Phillips (voir, par exemple, Tucsnak et Weiss [TW09, p. 103]), Kappel and Zhang [KZ93, Théorème 2.1], Webb [Web84] ou bien dans le livre de Song et Yu [SY88, Section 4.4].

L'adjoint \mathcal{A}_0^* de l'opérateur \mathcal{A}_0 défini par (I.3.3) est donné par

$$\begin{aligned} \mathcal{D}(\mathcal{A}_0^*) &= \left\{ \psi \in L^2[0, a_+] \mid \psi \text{ est localement absolument continue sur } [0, a_+), \right. \\ &\quad \left. \lim_{a \rightarrow a_+^-} \psi(a) = 0; \frac{d\psi}{da} - \mu\psi \in L^2[0, a_+] \right\}, \\ \mathcal{A}_0^*\psi &= \frac{d\psi}{da} - \mu\psi + \beta\psi(0) \quad (\psi \in \mathcal{D}(\mathcal{A}_0^*)). \end{aligned} \tag{I.3.4}$$

La formulation rigoureuse de cet adjoint et de son domaine fut démontrée récemment par Maity [Mai18] (voir aussi [MTZ18a]), en vue d'étudier les propriétés de contrôlabilité des équations de Lotka & McKendrick avec diffusion.

Contrôle par migration d'une tranche d'âge : problématiques. Supposons que l'on contrôle une densité de population via migration (ce qui se traduit mathématiquement par un contrôle additif) d'une certaine tranche d'âge d'individus. Autrement dit, la fonction de contrôle u possède un support en âge dans un certain intervalle $[a_1, a_2]$, où

7. En réalité, la démonstration présentée dans l'article de Song [SYW⁺82] datant de 1982 est fautive en l'état. Il manque dans l'article l'hypothèse de surjectivité (de $\mathcal{D}(\mathcal{A}_0)$ dans $L^2[0, a_+]$) de l'opérateur $\mathcal{A}_0 - \lambda \text{Id}_{L^2[0, a_+]}$ pour λ assez grand, afin d'appliquer correctement le théorème de Lumer-Phillips. Néanmoins, cette surjectivité se vérifie de manière relativement simple.

$0 \leq a_1 < a_2 \leq a_+$, et le système contrôlé s'écrit

$$\begin{cases} \frac{\partial p}{\partial t}(t, a) + \frac{\partial p}{\partial a}(t, a) + \mu(a)p(t, a) = \chi_{[a_1, a_2]}(a)u(t, a), & t > 0, a \in (0, a_+), \\ p(t, 0) = \int_0^{a_+} \beta(a)p(t, a) da, & t > 0, \\ p(0, a) = p_0(a), & a \in (0, a_+), \end{cases} \quad (\text{I.3.5})$$

où $\chi_{[a_1, a_2]}$ (la couleur bleue fait référence à celle de la Figure I.5) désigne la fonction caractéristique de l'intervalle $[a_1, a_2]$. Les principales questions de contrôlabilité étudiées dans la littérature et dans ce manuscrit associées au système (I.3.5) sont les suivantes :

- Etant donné un temps $\tau > 0$, peut-on exterminer la population en temps τ en agissant seulement sur les individus d'âge compris entre a_1 et a_2 - autrement dit, le système (I.3.5) est-il contrôlable à zéro ?
- Peut-on conduire la densité de population p vers un état stationnaire du système (I.3.5) en temps τ , *i.e.* peut-on transporter la densité de population p vers une solution \tilde{p} de

$$\begin{cases} \frac{\partial \tilde{p}}{\partial a}(a) + \mu(a)\tilde{p}(a) = \chi_{[a_1, a_2]}w(a) & (a \in (0, a_+)), \\ \tilde{p}(0) = \int_0^{a_+} \beta(a)\tilde{p}(a) da, \end{cases} \quad (\text{I.3.6})$$

où $w \in L^2[0, a_+]$?

- Peut-on conserver la positivité de la trajectoire contrôlée durant le temps de contrôle ?

En notant $X = L^2[0, a_+]$ et $U = L^2[0, a_+]$, notons que la solution p de (I.3.5) s'écrit sous la forme

$$p(t, \cdot) = \mathbb{T}_t^{A_0} p_0 + \Phi_t u \quad (t \geq 0, u \in L^2([0, \infty); X)),$$

où l'opérateur de restriction $B \in \mathcal{L}(X)$ est défini par

$$Bu = \chi_{[a_1, a_2]}u \quad (u \in X),$$

et où

$$\Phi_t u = \int_0^t \mathbb{T}_{t-\sigma}^{A_0} B u(\sigma) d\sigma \quad (t \geq 0, u \in L^2([0, \infty); X)).$$

Autour du contrôle à zéro

Les présentes lignes visent à donner un état de l'art concernant le contrôle à zéro des équations (I.3.5), suivant certaines hypothèses portant sur la tranche d'âge $[a_1, a_2]$ du contrôle et le support de la fonction de fertilité β .

Le cas $a_1 = 0$: contrôle des jeunes individus. Intéressons nous premièrement au cas (originellement étudié dans l'article de Barbu, Iannelli et Martcheva [BIM01]) du contrôle des jeunes individus, *i.e.* le cas où $a_1 = 0$ dans le système (I.3.5). Dans ce cas, nous noterons plus simplement la tranche d'âge de contrôle comme étant $[0, a_0]$ afin de rester fidèle aux notations de [BIM01] et des articles qui le suivent. Le système contrôlé s'écrit ainsi ici sous la forme

$$\begin{cases} \frac{\partial p}{\partial t}(t, a) + \frac{\partial p}{\partial t}(t, a) + \mu(a)p(t, a) = \chi_{[0, a_0]}(a)u(t, a), & t > 0, a \in (0, a_1), \\ p(t, 0) = \int_0^{a_1} \beta(a)p(t, a) da, & t > 0, \\ p(0, a) = p_0(a), & a \in (0, a_1), \end{cases} \quad (\text{I.3.7})$$

où $a_0 \in (0, a_1]$. Le point de départ de cette thèse fut de tenter de répondre aux questions ouvertes posées dans l'article de Barbu, Martcheva & Iannelli [BIM01] qui suivent l'énoncé de leur résultat principal :

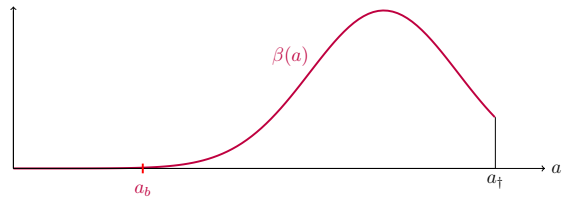
Théorème – I.3.1 (Barbu, Martcheva & Iannelli, 2001, [BIM01]). *Soit $\delta \in (0, a_0]$ et $\tau \in [a_1, a_1 + \delta)$. Alors, pour toute densité initiale $p_0 \in L^2[0, a_1]$, il existe $u \in L^2([0, \tau]; L^2[0, a_1])$ de sorte que la solution p de (I.3.7) vérifie*

$$p(\tau, a) = 0, \quad \text{pour presque tout } a \in [\delta, a_1]. \quad (\text{I.3.8})$$

La démonstration du théorème I.3.1 se base sur l'observabilité du système adjoint de (I.3.1). Notons que la conclusion de ce théorème ne permet pas d'examiner la population dans sa totalité, puisqu'étant donné $\delta > 0$ le processus de contrôle ne parvient pas à éradiquer les individus d'âge compris entre 0 et δ . En outre, la positivité de la densité de population contrôlée n'est pas garantie. Les questions posées dans [BIM01] furent ainsi les suivantes : peut-on prendre $\delta = 0$ dans l'énoncé du théorème, et conserver la positivité de la densité de population contrôlée lorsque la densité initiale p_0 est positive ?

Une réponse positive à ces deux questions fut premièrement donnée dans Hegoburu, Magal et Tucsnak [HMT18], dans le cas où les jeunes individus ne peuvent pas se reproduire (hypothèse vérifiée par une large classe d'espèces). Plus précisément, sous l'hypothèse ($\mathcal{N}ull_\beta$) suivante :

Il existe $a_b > 0$ de sorte que

$$\beta(a) = 0 \quad (a \in (0, a_b) \text{ p.p.}) \quad (\mathcal{N}ull_\beta)$$


la conclusion du Théorème I.3.1 est valable pour $\delta = 0$ et la positivité de la densité de population peut être conservée :

Théorème – I.3.2 (Hegoburu, Magal & Tucsnak, 2018, [HMT18]). *Supposons que l'hypothèse ($\mathcal{N}ull_\beta$) soit vérifiée pour un certain $a_b > 0$. Alors, le système (I.3.7) est contrôlable à zéro en tout temps $\tau > a_1 - a_0$. De plus, si la densité initiale de population p_0*

est positive, alors il existe des contrôles à zéro préservant la positivité de la densité de population contrôlée.

La démonstration heuristique associée au Théorème I.3.2 est relativement simple : la condition $(\mathcal{N}ull_\beta)$ permet d'avoir « un temps d'avance » pour contrôler les individus avant qu'ils ne se reproduisent. Afin d'exterminer la population, il suffit de tuer continuellement les individus d'âge compris entre 0 et a_0 et d'attendre suffisamment longtemps pour réduire la population totale à néant. Pour préserver la positivité de la solution, il faut prendre garde à ne pas créer d'individus « négatifs », en tuant tout juste le nombre d'individus vivants.

Mathématiquement parlant, la démonstration du contrôle à zéro du Théorème I.3.2 repose dans [HMT18] sur l'obtention de l'inclusion : $\text{Im } \mathbb{T}_\tau \subset \text{Im } \Phi_\tau$, et ce pour $\tau > a_\dagger - a_0$. Cette inclusion est démontrée en deux temps :

- une première étape consiste à construire un ensemble \mathcal{R} inclus dans l'image de Φ_τ via l'utilisation de contrôles constants le long des caractéristiques (*i.e.* qui suivent les cohortes de populations). La construction s'effectue de manière récursive en vertu de l'hypothèse $(\mathcal{N}ull_\beta)$.
- Une seconde étape montre l'inclusion $\text{Im } \mathbb{T}_\tau^{A_0} \subset \mathcal{R}$, conséquence de l'inégalité de Hardy (voir, par exemple, [HLP88]).

Le Théorème I.3.2 a été récemment amélioré par Anița et Hegoburu, au sens où l'hypothèse $(\mathcal{N}ull_\beta)$ n'est pas nécessaire pour pouvoir contrôler à zéro le système (I.3.7) :

Théorème – I.3.3 (Anița & Hegoburu, 2018, [AH18]). *Le système (I.3.7) est contrôlable à zéro en tout temps $\tau > a_\dagger - a_0$. De plus, si la densité initiale de population p_0 est positive, alors il existe des contrôles à zéro préservant la positivité de la densité de population contrôlée.*

Ici, l'absence de l'hypothèse $(\mathcal{N}ull_\beta)$ induit une absence de temps d'avance pour contrôler les individus. La démonstration du Théorème I.3.3 en devient un peu plus technique que celle du Théorème I.3.2. Elle repose sur la construction de contrôles *feedback* et d'un découpage minutieux de la zone de contrôle (voir [AH18] ou le Chapitre V de ce manuscrit). Notons qu'en l'absence d'hypothèses du type $(\mathcal{N}ull_\beta)$ (ce qui est par exemple le cas pour une division frénétique du type division cellulaire), il est essentiel de pouvoir contrôler les très jeunes individus pour espérer exterminer la population.

Le cas $a_1 > 0$: contrôle d'une tranche d'âge quelconque. Le contrôle d'une tranche d'âge quelconque fut premièrement étudié par D. Maity [Mai18] en vue d'obtenir les théorèmes correspondants dans le cadre d'une dynamique de population avec diffusion [MTZ18a]. Sous l'hypothèse $(\mathcal{N}ull_\beta)$, dans le cas où il y a une marge pour contrôler les individus avant qu'ils ne se reproduisent (*i.e.* dans le cas où $a_1 < a_b$), l'auteur montre que le système (I.3.5) est contrôlable à zéro en temps assez grand :

Théorème – I.3.4 (Maity, 2018, [Mai18]). *Supposons que l'hypothèse $(\mathcal{N}ull_\beta)$ soit vérifiée pour un certain $a_b > 0$, avec $a_1 < a_b$. Alors, pour tout $\tau > a_1 + \max(a_1, a_\dagger - a_2)$, le système (I.3.5) est contrôlable à zéro.*

Le théorème ci-dessus est démontré via une inégalité d'observabilité portant sur le système adjoint de (I.3.1). Avec les hypothèses du théorème I.3.4, l'auteur démontre que pour tout temps $\tau > a_1 + \max(a_1, a_\dagger - a_2)$, il existe une constante C_τ de sorte que la solution q du système adjoint

$$\begin{cases} \frac{\partial q}{\partial t}(t, a) - \frac{\partial q}{\partial a}(t, a) + \mu(a)q(t, a) - \beta(a)q(t, 0) = 0, & t > 0, a \in (0, a_\dagger), \\ q(t, a_\dagger) = 0, & t > 0, \\ q(0, a) = q_0(a), & a \in (0, a_\dagger). \end{cases}$$

vérifie

$$\int_0^{a_\dagger} q^2(\tau, a) da \leq C_\tau^2 \int_0^\tau \int_{a_1}^{a_2} q^2(t, a) da dt,$$

et ce pour tout $q_0 \in \mathcal{D}(\mathcal{A}_0^*)$. Le théorème I.3.4 est le premier à démontrer le contrôle à zéro (total) d'une dynamique de populations à l'aide d'inégalités d'observabilité. Cependant, une telle démonstration ne permet pas conclure quant à une éventuelle positivité de la trajectoire contrôlée (notons que ce n'était de toute manière pas le but de l'article [Mai18]). La positivité de la trajectoire contrôlée est cependant plausible, et ce grâce au théorème suivant :

Théorème – I.3.5 (Anița & Hegoburu, 2018, [AH18]). *Supposons que l'hypothèse ($\mathcal{N}_{\text{null}_\beta}$) soit vérifiée pour un certain $a_b > 0$, avec $a_1 < a_b$. Alors, pour tout $\tau > a_\dagger - (a_2 - a_1)$, le système (I.3.5) est contrôlable à zéro. De plus, si la densité initiale de population p_0 est positive, alors il existe des contrôles à zéro préservant la positivité de la densité de population contrôlée.*

Le théorème ci-dessus raffine légèrement le temps de contrôle nécessaire pour exermier une densité de population (par rapport au temps de contrôle énoncé dans le Théorème I.3.4), avec préservation de la positivité de la densité de population contrôlée. A l'instar du Théorème I.3.3, la démonstration du Théorème I.3.5 repose sur la construction de contrôles de type feedback.

La Figure I.5 résume les principaux résultats de contrôle à zéro pour l'équation (I.3.5) obtenus depuis l'article originel de Barbu, Iannelli et Martcheva [BIM01].

Contrôle vers les populations « idéales » (stationnaires)

Dans le cas du contrôle des jeunes individus (*i.e.* dans le cas où $a_1 = 0$), la contrôlabilité du système (I.3.7) en temps assez long vers une population stationnaire solution de (I.3.6) fut partiellement démontrée dans Barbu et al. [BIM01, Théorème 1], et démontrée dans Hegoburu, Magal et Tucsnaç [HMT18, Théorème 1.1]. Le résultat le plus complet concernant la contrôlabilité d'une tranche d'âge quelconque vers une population stationnaire est le suivant :

Théorème – I.3.6 (Anița & Hegoburu, 2018, [AH18]). *Supposons que l'une des deux conditions suivantes soit vérifiée :*

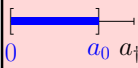
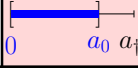
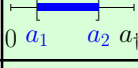
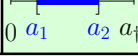
	Tranche d'âge du support de contrôle	Les jeunes individus se reproduisent	Contrôle à zéro?	Temps optimal de contrôle à zéro
Barbu, Martcheva & Iannelli (2001) [BIM01]		OUI	NON	×
Hegoburu, Magal & Tucsnaik (2018) [HMT18]		NON	OUI	OUI ($\tau > a_+ - a_0$)
Maity (2018) [Mai18]		NON	OUI	NON ($\tau > a_1$ + $\max(a_1, a_+ - a_2)$)
Anița & Hegoburu (2018) [AH18]		OUI (cas $a_1 = 0$)	OUI	OUI ($\tau > a_+ - a_2 + a_1$)

FIGURE I.5 – Sans considération de la diffusion spatiale, état de l'art du contrôle à zéro par migration d'une certaine tranche d'âge des équations de Sharpe, Lotka et McKendrick décrites par (I.3.5). Les cases vertes décrivent des hypothèses « optimales ».

Condition 1 : l'hypothèse $(\mathcal{N}ull_\beta)$ est vérifiée pour un certain $a_b > 0$, avec $a_1 < a_b$ et $\tau > a_+ - a_2 + a_1$,

Condition 2 : $a_1 = 0$ et $\tau > a_+ - a_2$.

Supposons que w et \tilde{p} soient deux fonctions positives presque partout dans $L^2(0, a_+)$, où \tilde{p} est une solution de (I.3.6). Alors, pour tout $p_0 \in L^2(0, a_+)$ avec $p_0(a) \geq 0$ pour presque tout $a \in (0, a_+)$, il existe $u \in L^2(Q_\tau)$ de sorte que la solution p de (I.3.6) soit positive dans $(0, \tau) \times (0, a_+)$ et vérifie :

$$p(\tau, a) = \tilde{p}(a) \quad \text{pour presque tout } a \in (0, a_+).$$

Contrôle d'équations non-linéaires

Le contrôle par migration d'une tranche d'âge quelconque des équations structurées en âge non-linéaires est décrit par le système suivant :

$$\left\{ \begin{array}{l} \frac{\partial p}{\partial t}(t, a) + \frac{\partial p}{\partial a}(t, a) + \mu(t, a, P(t))p(t, a) = \chi_{[a_1, a_2]}(a)u(t, a) \quad (t, a) \in Q_\tau, \\ p(t, 0) = \int_0^{a_+} \beta(t, a, P(t))p(t, a) da, \quad t \in (0, \tau), \\ p(0, a) = p_0(a), \quad a \in (0, a_+), \\ P(t) = \int_0^{a_+} p(t, a) da, \quad t \in (0, \tau), \end{array} \right. \quad (\text{I.3.9})$$

dont les hypothèses classiques associées sont les suivantes :

- (H1) pour tout $s \geq 0$, les $\beta(\cdot, \cdot, s)$ et $\mu(\cdot, \cdot, s)$ appartiennent à $L^\infty(Q_\tau)$ et $L^1_{loc}([0, \tau] \times [0, a_+))$ respectivement,
- (H2) les fonctions β et μ sont localement Lipschitziennes par rapport à la troisième variable,
- (H3) pour presque tout $(t, a, s) \in Q_\tau \times (0, +\infty)$, on a

$$\beta(t, a, s) \geq 0 \quad \text{et} \quad \mu(t, a, s) \geq 0,$$

- (H4) pour presque tout $(t, a) \in Q_\tau$, la fonction $\beta(t, a, \cdot)$ est décroissante,
- (H5) pour presque tout $(t, a) \in Q_\tau$, la fonction $\mu(t, a, \cdot)$ est croissante,
- (H6) $p_0 \in L^2(0, a_+)$, $p_0(a) \geq 0$ pour presque tout $a \in (0, a_+)$,
- (H7) le contrôle u appartient à $L^2(Q_\tau)$.

En présence des hypothèses décrites ci-dessus, l'existence et l'unicité d'une solution globale à l'équation (I.3.9) est démontrée dans [Ani00, Chapitre 2]. Le contrôle à zéro des équations (I.3.9) fut partiellement obtenu par Ainseba et Iannelli [AI⁺03] lorsqu'il est possible de contrôler les jeunes individus :

Théorème – I.3.7 (Ainseba & Iannelli, 2003, [AI⁺03]). *Supposons que $a_1 = 0$, et soit $\delta \in (0, a_2)$. Alors, pour tout $\tau \in (a_+, a_+ + \delta)$ et pour toute densité initiale de population $p_0 \in L^2(0, a_+)$ telle que $\frac{p_0}{\pi} \in L^2(0, a_+)$, il existe $u \in L^2((0, \tau) \times (0, a_+))$ de sorte que la solution p de (I.3.9) vérifie*

$$p(\tau, a) = 0 \quad \text{pour presque tout } a \in [\delta, a_+].$$

Dans l'article [AI⁺03], la démonstration du théorème (I.3.7) repose sur les deux points suivants :

- le résultat est d'abord démontré dans le cas linéaire, *i.e.* dans le cas où les fonctions β et μ ne dépendent pas de la troisième variable,
- le passage du cadre linéaire au cadre non linéaire s'effectue grâce au théorème du point fixe de Kakutani pour les fonctions multivaluées, en utilisant l'inégalité d'observabilité obtenue dans Barbu, Iannelli et Martcheva [BIM01].

Ce théorème fut récemment amélioré par Anița & Hegoburu [AH18] en suivant la même philosophie de comparaison avec le cas linéaire :

Théorème – I.3.8 (Anița & Hegoburu, 2018, [AH18]). *Supposons que l'une des deux conditions suivantes soit vérifiée :*

Condition 1 : *l'hypothèse $(\mathcal{N}ull_\beta)$ est vérifiée pour un certain $a_b > 0$, avec $a_1 < a_b$ et $\tau > a_+ - a_2 + a_1$,*

Condition 2 : *$a_1 = 0$ et $\tau > a_+ - a_2$.*

Alors, pour tout $p_0 \in L^2(0, a_+)$ avec $p_0(a) \geq 0$ pour presque tout $a \in (0, a_+)$, il existe $u \in L^2(Q_\tau)$ de sorte que la solution p de (I.3.9) soit positive dans $(0, \tau) \times (0, a_+)$ et vérifie :

$$p(\tau, a) = 0 \quad \text{pour presque tout } a \in (0, a_+).$$

La démonstration repose sur des principes de comparaison en dynamique de population - essentiellement, pour démontrer le contrôle à zéro du système (I.3.9), on montre qu'il est possible de construire une solution positive de (I.3.9) plus petite qu'une solution positive contrôlée à zéro du système linéaire (I.3.5).

§ I.3.2. Le cas diffusif

Soit $\Omega \subset \mathbb{R}^n$, $n \geq 1$ un domaine connexe borné non-vidé, et considérons une densité de population p (dépendante du temps, de l'âge et de l'espace) structurée en âge se déplaçant dans le milieu Ω . En notant p_0 la densité initiale de population, le contrôle de la densité p par migration d'une certaine tranche d'âge $[a_1, a_2]$ d'individus dans un ouvert non-vidé $\omega \subset\subset \Omega$ est décrit par les équations :

$$\begin{cases} \frac{\partial p}{\partial t}(t, a, x) + \frac{\partial p}{\partial a}(t, a, x) + \mu(a)p(t, a, x) - \Delta p(t, a, x) \\ \quad = \chi_\omega(x)\chi_{[a_1, a_2]}(a)u(t, a, x), & t > 0, a \in (0, a_\dagger), x \in \Omega, \\ \frac{\partial p}{\partial \nu}(t, a, x) = 0, & t > 0, a \in (0, a_\dagger), x \in \partial\Omega, \\ p(t, 0, x) = \int_0^{a_\dagger} \beta(a)p(t, a, x) da, & t > 0, x \in \Omega, \\ p(0, a, x) = p_0(a, x), & a \in (0, a_\dagger), x \in \Omega. \end{cases} \quad (\text{I.3.10})$$

Ici, l'opérateur Δ désigne le laplacien par rapport à la variable d'espace, les fonctions β et μ désignent respectivement les fonctions de fertilité et de mortalité et $\frac{\partial}{\partial \nu}$ désigne l'opérateur de dérivation par rapport à la normale extérieure sortante de $\partial\Omega$. Les hypothèses classiques portant sur les fonctions β et μ sont les mêmes que dans le cadre sans diffusion (cf. la sous-section précédente).

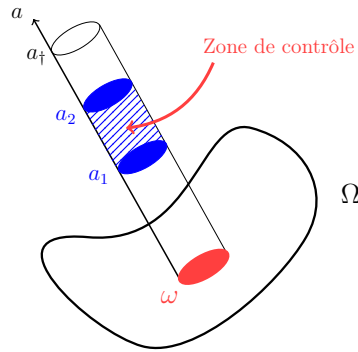


FIGURE I.6 – Représentation de la zone de contrôle associée aux équations de Lotka McKendrick avec diffusion (I.3.10). Le contrôle s'effectue dans une certaine zone du milieu considéré et se restreint à une tranche d'âge d'individus.

A propos de l'opérateur de population avec diffusion. Notons $H := L^2((0, a_+) \times \Omega)$ et considérons l'opérateur de population sans diffusion $\mathcal{A}_1 : \mathcal{D}(\mathcal{A}_1) \rightarrow H$ défini par

$$\begin{aligned} \mathcal{D}(\mathcal{A}_1) &= \left\{ \varphi \in H \mid \varphi(\cdot, x) \text{ est localement absolument continue sur } [0, a_+), \right. \\ &\quad \left. \varphi(0, x) = \int_0^{a_+} \beta(a) \varphi(a, x) da \text{ pour presque tout } x \in \Omega, \frac{\partial \varphi}{\partial a} + \mu \varphi \in H \right\}, \\ \mathcal{A}_1 \varphi &= -\frac{\partial \varphi}{\partial a} - \mu \varphi, \end{aligned}$$

et l'opérateur de diffusion $\mathcal{A}_2 : \mathcal{D}(\mathcal{A}_2) \rightarrow H$ défini par

$$\mathcal{D}(\mathcal{A}_2) = \left\{ \varphi \in H \mid \frac{\partial \varphi}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}, \quad \mathcal{A}_2 \varphi = \Delta \varphi.$$

L'opérateur de population $\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow H$ est défini par

$$\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}_1) \cap \mathcal{D}(\mathcal{A}_2), \quad \mathcal{A} \varphi = \mathcal{A}_1 \varphi + \mathcal{A}_2 \varphi.$$

L'opérateur \mathcal{A} génère un semi-groupe d'opérateurs (voir par exemple Huyer [Huy94, Théorème 2.8]), noté $\mathbb{T}^{\mathcal{A}} = (\mathbb{T}_t^{\mathcal{A}})_{t \geq 0}$ dans cette introduction. L'adjoint \mathcal{A}^* de l'opérateur \mathcal{A} est défini par

$$\begin{aligned} \mathcal{D}(\mathcal{A}^*) &= \left\{ \varphi \in H \mid \varphi(\cdot, x) \text{ est localement absolument continue sur } [0, a_+), \varphi \in L^2(0, a_+; H^2(\Omega)), \right. \\ &\quad \left. \lim_{a \rightarrow a_+^-} \varphi(a, x) = 0 \text{ pour presque tout } x \in \Omega, \frac{\partial \varphi}{\partial \nu} = 0 \text{ sur } \partial\Omega, \frac{\partial \varphi}{\partial a} - \mu \varphi + \Delta \varphi \in H \right\}, \\ \mathcal{A}^* \varphi &= \frac{\partial \varphi}{\partial a} - \mu \varphi + \Delta \varphi + \beta \psi(0, x). \end{aligned} \tag{I.3.11}$$

La formulation rigoureuse de l'adjoint de \mathcal{A} et de son domaine peut se trouver dans Maity, Tucsnak et Zuazua [MTZ18a, Proposition 2.3].

Autour du contrôle à zéro

L'étude du contrôle à zéro des équations (I.3.10) fut amorcée par B. Ainseba [Ain02] - [Ain12], dans la cas où le contrôle agit pour tous les âges.

Théorème – I.3.9 (Ainseba, 2002, [Ain02] et [Ain12]). *Supposons que $a_1 = 0$ et $a_2 = a_+$, et qu'il existe $a_m \in (0, a_+)$ de sorte que $\beta(a) = 0$ pour presque tout $a \in [a_m, a_+]$. De plus, soit $\delta \in (0, a_m)$. Alors, pour tout $\tau > a_m$ et pour toute densité initiale de population $p_0 \in L^2((0, a_+) \times \Omega)$, il existe $u \in L^2((0, \tau); L^2((0, a_+) \times \Omega))$ de sorte que la solution p de (I.3.10) vérifie*

$$p(\tau, a, x) = 0 \quad \text{pour presque tout } (a, x) \in [\delta, a_m] \times \Omega.$$

La démonstration du Théorème (I.3.9) est basée sur l'observation du système adjoint de (I.1.8) via l'utilisation d'inégalités de Carleman globales, proches de celles utili-

sées par Fursikov et Imanuvilov [FI96] dans leur démonstration du contrôle à zéro pour l'équation de la chaleur. Notons qu'en réalité, les fonctions β et μ de l'article [Ain02] sont susceptibles de dépendre du temps et de l'espace (la démonstration des propriétés de contrôlabilité des équations de dynamique de populations en devient nettement plus délicate notamment à cause de la dépendance en temps, cf. les commentaires en fin de l'article [MTZ18a]).

Lorsque le contrôle est actif pour tous les âges, le contrôle à zéro des équations (I.3.10) fut récemment démontré par Hegoburu et Tucsnak [HT17] :

Théorème – I.3.10 (Hegoburu & Tucsnak, 2017, [HT17]). *Lorsque $a_1 = 0$ et $a_2 = a_\dagger$, le système (I.3.10) est contrôlable à zéro en tout temps $\tau > 0$.*

Le théorème I.3.10 ne nécessite aucune hypothèse supplémentaire sur la fonction de fertilité β . Sa démonstration est basée sur la résolution d'un problème de moments et sur une adaptation de la méthode développée par Lebeau et Robbiano [LR95], originellement utilisée pour démontrer le contrôle à zéro de l'équation de la chaleur. Il est par ailleurs naturel d'imaginer qu'une trajectoire contrôlée à zéro ne saurait rester positive dans l'ensemble Ω tout entier si l'on n'agit que dans un sous-domaine $\omega \subset \Omega$: cela est démontré dans Anița et Hegoburu [AH18, Proposition 7.1] via l'utilisation de principes de comparaison.

Les deux théorèmes présentés ci-dessus furent très récemment améliorés par Maity, Tucsnak et Zuazua [MTZ18a] dans le cadre du contrôle d'une tranche d'âge quelconque :

Théorème – I.3.11 (Maity, Tucsnak & Zuazua, 2018, [MTZ18a]). *Supposons que l'hypothèse $(\mathcal{N}_{\text{null}_\beta})$ soit vérifiée pour un certain $a_b > 0$, avec $a_1 < a_b$. Alors, pour tout $\tau > a_\dagger - a_2 + a_1$, le système (I.3.10) est contrôlable à zéro.*

La démonstration de ce théorème réside dans l'observabilité du système adjoint, en mêlant l'observabilité du système sans diffusion obtenue avec la méthodologie de l'article de Maity [Mai18] et l'observabilité de l'équation de la chaleur obtenue par Fursikov et Imanuvilov [FI96]. Notons que les théorèmes I.3.10 et I.3.11 n'utilisent pas à proprement parler d'inégalités de Carleman, mais leur « produit » via la stratégie de Lebeau Robbiano dans [HT17] et l'utilisation directe de l'inégalité d'observabilité pour l'équation de la chaleur avec contrôle interne dans [MTZ18a].

Contrôle vers les populations « idéales » (stationnaires)

Par population « idéale » associée aux équations (I.3.10), on entend ici (en suivant la définition donnée dans [AA01], [AA04] et [MTZ18a]) toute solution positive stationnaire $p_s \in L^\infty((0, a_\dagger) \times \Omega)$ du système

$$\begin{cases} \frac{\partial p_s}{\partial a}(a, x) + \mu(a)p_s(a, x) - \Delta p_s(a, x) = \chi_{[a_1, a_2]} v_s(a, x), & a \in (0, a_\dagger), x \in \Omega, \\ \frac{\partial p_s}{\partial \nu}(a, x) = 0, & a \in (0, a_\dagger), x \in \partial\Omega, \\ p_s(0, x) = \int_0^{a_\dagger} \beta(a)p_s(a, x) da, & x \in \Omega, \end{cases} \quad (\text{I.3.12})$$

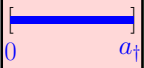
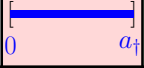
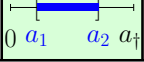
	Tranche d'âge du support de contrôle	Les jeunes individus se reproduisent	Contrôle à zéro?	Temps optimal de contrôle à zéro
Ainseba (2002) [Ain02]		OUI (mais pas les très âgés)	NON	×
Hegoburu & Tucsnaк (2017) [HT17]		OUI	OUI	OUI (évident !)
Maity, Tucsnaк & Zuazua (2018) [MTZ18a]		NON	OUI	OUI ($\tau > a_+ - a_2 + a_1$)

FIGURE I.7 – Avec considération de la diffusion spatiale, état de l’art du contrôle à zéro par migration d’une certaine tranche d’âge des équations de Sharpe, Lotka et McKendrick décrites par (I.3.10). Les cases vertes décrivent des hypothèses « optimales ».

où $v_s \in L^\infty((0, a_+) \times \Omega)$ est un contrôle interne positif (indépendant du temps). La question de contrôlabilité vers une solution (I.3.12) est la suivante : peut-on conduire une densité de population initiale p_0 vers une solution p_s de (I.3.12) en préservant la positivité de la population contrôlée ?

Cette question fut étudiée par Ainseba et Anița dans [AA01] et [AA04], dans les cas respectifs où le contrôle est actif pour tous les âges (dans [AA01]) et pour les jeunes individus (dans [AA04]). Les résultats dans ces deux articles indiquent que la contrôlabilité vers une solution de (I.3.12) en préservant la positivité de la solution est assurée, en temps suffisamment grand (dépendant du support en âge du contrôle) dès lors que la densité initiale de population est assez proche de la population cible. Ces deux résultats sont démontrés via des inégalités de Carleman globales, en dimension d’espace plus petite que 3 et furent généralisés par Maity, Tucsnaк et Zuazua [MTZ18a] dans le cadre du contrôle d’une tranche d’âge quelconque :

Théorème – I.3.12 (Maity, Tucsnaк & Zuazua, 2018, [MTZ18a]). *Supposons que l’hypothèse $(\mathcal{N}_{\text{null}_\beta})$ soit vérifiée pour un certain $a_b > 0$, avec $a_1 < a_b$. Soient $p_{s,I}$ et $p_{s,F}$ deux états d’équilibres positifs de l’équation (I.3.10). Supposons qu’il existe $a_* \in (0, a_+)$ et $\delta > 0$ de sorte que*

$$p_{s,I}(a, x), p_{s,F}(a, x) \geq \delta \quad \text{pour presque tout } (a, x) \in [0, a_*] \times \bar{\Omega}.$$

Alors, il existe un temps $\tau > 0$ et un contrôle $u \in L^\infty((0, \tau) \times (0, a_+) \times \Omega)$ de sorte que la solution p de (I.3.10) avec

$$p_0(a, x) = p_{s,I}(a, x) \quad ((a, x) \in (0, a_+) \times \Omega \text{ p.p}),$$

vérifie

$$p(\tau, a, x) = p_{s,F}(a, x) \quad ((a, x) \in (0, a_{\dagger}) \times \Omega \text{ p.p.}),$$

et

$$p(t, a, x) \geq 0 \quad ((t, a, x) \in (0, \tau) \times (0, a_{\dagger}) \times \Omega \text{ p.p.}).$$

Notons que le temps de contrôle τ dans le théorème ci-dessus dépend de la distance entre les deux densités de population $p_{s,I}$ et $p_{s,F}$ - plus cette distance est grande, plus le temps de contrôle l'est. La démonstration de ce théorème repose sur des inégalités d'observabilité dans L^1 afin d'obtenir par dualité les résultats de contrôle correspondants dans L^∞ , avec une procédure du type « step by step » afin de modifier pas-à-pas, légèrement, la population contrôlée pour conserver sa positivité et obtenir la densité de population souhaitée en temps assez long.

Le cadre non-linéaire

Afin de conclure cette section, mentionnons que le contrôle des équations (I.3.10) dans le cadre non-linéaire fut étudié par Ainseba et Iannelli [AI⁺03], lorsqu'il est possible d'influer l'ensemble des individus d'une certaine zone du milieu considéré. Plus précisément, le système étudié s'écrit

$$\left\{ \begin{array}{l} \partial_t p(t, a, x) + \partial_a p(t, a, x) + \mu(a, t, x, P(t, x))p(t, a, x) - \Delta p(t, a, x) \\ \quad = \chi_\omega(x)u(t, a, x), \quad t > 0, a \in (0, a_{\dagger}), x \in \Omega, \\ \frac{\partial p}{\partial \nu}(t, a, x) = 0, \quad t > 0, a \in (0, a_{\dagger}), x \in \partial\Omega, \\ p(t, 0, x) = \int_0^{a_{\dagger}} \beta(t, a, x, P(t, x))p(t, a, x) da, \quad t > 0, x \in \Omega, \\ p(0, a, x) = p_0(a, x), \quad a \in (0, a_{\dagger}), x \in \Omega, \\ P(t, x) = \int_0^{a_{\dagger}} p(t, a, x) da, \quad t > 0, x \in \Omega. \end{array} \right. \quad (\text{I.3.13})$$

Concernant les hypothèses portant sur les fonctions β et μ qui garantissent l'existence et l'unicité d'une solution (I.3.13), nous renvoyons à [Ani00, Section 4.2] ou [AI⁺03]. La fonction μ se décompose ici comme étant

$$\mu(a, t, x, \xi) = \mu_0(a) + \mathcal{M}(a, t, x, \xi),$$

où μ_0 vérifie les hypothèses (A2) - (A3) du début de cette section. En notant

$$\pi_0(a) := \exp\left(-\int_0^a \mu_0(s) ds\right) \quad (a \in (0, a_{\dagger}) \text{ p.p.}),$$

le contrôle à zéro des équations obtenu dans [AI⁺03] est quasi-total :

Théorème – I.3.13 (Ainseba & Iannelli, 2003, [AI⁺03]). *Soit $\delta \in (0, a_{\dagger})$. Alors, pour tout $\tau > 0$ et pour toute densité de population initiale $p_0 \in L^2((0, \tau) \times (0, a_{\dagger}) \times \Omega)$ telle*

que $\frac{p_0}{\pi_0} \in L^2((0, \tau) \times (0, a_+) \times \Omega)$, il existe $u \in L^2((0, \tau) \times (0, a_+) \times \Omega)$ de sorte que la solution p de (I.3.13) vérifie

$$p(\tau, a, x) = 0 \quad \text{pour presque tout } (a, x) \in [\delta, a_+] \times \Omega.$$

Basé sur des inégalités de Carlemann globales et le théorème du point fixe de Kakutani, ce résultat est pour l'heure le plus complet concernant le contrôle à zéro des équations non linéaires (I.3.13).

§ I.3.3. Organisation du manuscrit

Outre cette présente introduction, ce manuscrit est constitué de quatre chapitres qui recouvrent de manière chronologique la production scientifique obtenue pendant la durée de la thèse.

- Le chapitre II reprend dans sa grande majorité l'article [HMT18] qui étudie les propriétés de contrôlabilité du système (I.3.7) sous l'hypothèse ($\mathcal{N}ull_\beta$). Le point central du papier consiste à étudier finement l'espace atteignable afin d'en déduire les résultats de contrôlabilité associés. En outre, une nouvelle section (par rapport à l'article originel), dédiée à la recherche des contrôles HUM, complète l'article [HMT18] rédigé avec P. Magal et M. Tucsnak.
- Le chapitre III dresse la démonstration du contrôle à zéro des équations (I.3.10) avec diffusion, lorsque le contrôle est actif pour tous les âges. Ce résultat est démontré dans l'article [HT17], rédigé en collaboration avec M. Tucsnak
- Le chapitre IV étudie les propriétés de contrôle en temps optimal associées au système (I.3.10), lorsque le contrôle est actif pour tous les âges. Ce chapitre reprend la méthodologie du précédent, afin de démontrer un principe de type « bang-bang » pour les contrôles en temps optimaux, lorsque ceux-ci sont voués à appartenir en tout temps dans une boule centrée en 0 et de rayon $M > 0$ dans $L^\infty((0, a_+) \times \Omega)$.
- Enfin, le chapitre V étudie les propriétés de contrôlabilité du système non linéaire de Gurtin et MacCamy (sans diffusion) (I.3.9), et généralise notamment les résultats du chapitre II concernant le contrôle à zéro des équations (I.3.5). Les résultats sont obtenus dans l'article [AH18] en collaboration avec S. Anița.

Contrôlabilité avec contraintes de positivité des équations de Lotka-McKendrick via migration des jeunes individus

Ce chapitre reprend dans sa majeure partie l'article accepté et paru dans *SIAM Journal of Control and Optimization (SICON)* [HMT18]. Par rapport à l'article originel, une section a été rajoutée (la sixième) dans le but d'approcher numériquement les contrôles à zéro de norme optimale (contrôle HUM) dans le cadre des équations de dynamique de populations.

	Tranche d'âge du support de contrôle	Les jeunes individus se reproduisent	Contrôle à zéro ?	Temps optimal de contrôle à zéro
Barbu, Martcheva & Iannelli (2001) [BIM01]		OUI	NON	×
Hegoburu, Magal & Tunesak (2018) [HMT18]		NON	OUI	OUI ($\tau > a_1 - a_0$)
Maiti (2018) [Mait18]		NON	OUI	NON ($\tau > a_2 - a_1$)
Anifa & Hegoburu (2018) [AH18]		OUI (avec $a_1 = 0$)	OUI	OUI ($\tau > a_2 - a_1 + a_0$)

Sommaire

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II.3	Some controllability results	51
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The purpose of this work is the study of the controllability properties of an infinite dimensional dynamical system issued of population dynamics. More precisely, we revisit from a control theoretical view point the linear Lotka-McKendrick model. We aim, in particular, answer to some open questions raised in Barbu, Iannelli and Martcheva [BIM01]. More precisely, let $p(t, a)$ be the distribution of individuals of age $a \geq 0$ at time

$t \geq 0$. Let a_{\dagger} be the life expectancy of an individual and τ be a positive constant. Let $\beta(a) \geq 0$ be the natural fertility-rate and $\mu(a) \geq 0$ the natural death-rate of individuals of age a . The system we consider, already studied from a control theoretic view point in [BIM01], is described by the equations

$$\begin{cases} \frac{\partial p}{\partial t}(t, a) + \frac{\partial p}{\partial a}(t, a) + \mu(a)p(t, a) \\ \quad = m(a)u(t, a), & (t, a) \in Q_{\tau}, \\ p(t, 0) = \int_0^{a_{\dagger}} \beta(a)p(t, a) da, & t \in (0, \tau), \\ p(0, a) = p_0(a), & a \in (0, a_{\dagger}), \end{cases} \quad (\text{II.0.1})$$

where u is a control function, $m = \mathbb{1}_{[0, a_0]}$ is the characteristic function of the interval $(0, a_0)$ (where $0 < a_0 < a_{\dagger}$), p_0 is the initial population density and $Q_{\tau} = (0, \tau) \times (0, a_{\dagger})$.

In [BIM01] the main result asserts controllability of the above system to any quasi steady state by a distributed control, except for a small interval of ages near zero. The main open questions raised in the above mentioned reference are the possibility of controlling the whole range of ages and designing controls which preserve the positivity of the state trajectories. We give a positive answer to both questions above, in the particular case when the age of individuals able to reproduce is bounded away from zero.

To state our main results, we first recall the standard assumptions, used in particular in [BIM01], on the functions μ and β :

(H1) $\beta \in L^{\infty}(0, a_{\dagger})$, $\beta(a) \geq 0$ for almost every $a \in (0, a_{\dagger})$,

(H2) $\mu \in L^1[0, a^*]$ for every $a^* \in (0, a_{\dagger})$, $\mu(a) \geq 0$ for almost every $a \in (0, a_{\dagger})$,

(H3) $\int_0^{a_{\dagger}} \mu(a) da = +\infty$.

The controllability result [BIM01] asserts that if $\delta \in (0, a_0]$ and if \tilde{p} is a time-independant function which satisfies the equations

$$\begin{cases} \frac{\partial \tilde{p}}{\partial a}(a) + \mu(a)\tilde{p}(a) = m(a)w(a) & (a \in (0, a_{\dagger})), \\ \tilde{p}(0) = \int_0^{a_{\dagger}} \beta(a)\tilde{p}(a) da, \end{cases} \quad (\text{II.0.2})$$

for some function $w \in L^2[0, a_{\dagger}]$, then for every $p_0 \in L^2[0, a_{\dagger}]$ and every time

$$a_{\dagger} \leq \tau < a_{\dagger} + \delta, \quad (\text{II.0.3})$$

there exists $u \in L^2([0, \tau]; L^2[0, a_{\dagger}])$ such that

$$p(\tau, a) = \tilde{p}(a) \quad (a \in [\delta, a_{\dagger}]).$$

In other words, the above result from [BIM01] asserts that we can control in a time τ satisfying (II.0.3) the population of individuals of age at least equal to some $\delta > 0$. Our

main result asserts that this restriction is not necessary, provided that we assume that the age of individuals able to reproduce is bounded away from zero. More precisely, we have

Theorem II.0.1. *With the above notation and assumptions, suppose that there exists $a_b \in (0, a_+)$ such that*

$$\beta(a) = 0 \quad (a \in (0, a_b) \text{ a.e.}). \quad (\text{II.0.4})$$

Then for every $\tau > a_+ - a_0$, for every $p_0 \in L^2([0, a_+])$ with $p_0(a) \geq 0$ for almost every $a \in (0, a_+)$, and for any positive function \tilde{p} satisfying (II.0.2), there exists $u \in L^2([0, \tau]; L^2[0, a_+])$ such that the solution p of (II.0.1) satisfies

$$p(\tau, a) = \tilde{p}(a) \quad (a \in (0, a_+) \text{ a.e.}),$$

and

$$p(t, a) \geq 0 \text{ a.e. } (t, a) \in Q_\tau.$$

For an overview on age structured population dynamics models we refer to Webb [Web85], Iannelli [Ian95b], Kunisch et al [KSW85] and references therein. For related work on size structured population dynamics we refer to Ackleh and Ito [AI97], Kappel and Ito [IKP91]

The literature devoted to the associated control problems is less abundant but several important results and methods are available. For optimal control problems (namely applied to human population) we refer to Song and Yu [SY88]. The null-controllability of the age-dependant population dynamics (II.0.1) with spatial dependance in the particular case when the control acts for all ages a (the case corresponding to $a_0 = a_+$) was investigated by S. Anița (see [Ani00], p 148). The case when the control acts in a spatial subdomain ω and only for small age classes was investigated by B. Ainseba and S. Anița [AA04], for initial data p_0 in a neighborhood of the target \tilde{p} . Related approximate and exact controllability issues have also been studied in Ainseba [Ain02], Ainseba and Langlais [AL00], Ainseba and Iannelli [AI12], Traore [Tra06], Kavian and Traore [KT11]. Using a direct approach, the approximate controllability by birth or boundary control is studied in Yu, Guo and Zhu [YGZ87].

The remaining part of this work is organized as follows. In Section II.1 we recall some basic results on the Lotka-McKendrick semigroup. Section II.2 is devoted to the study of the reachable space of (II.0.1). In Section II.3 we investigate some controllability properties of the system (II.0.1). In particular, we give the proof of Theorem II.0.1. In Section II.4 we discuss a singular perturbation problem. More precisely we show that if the age interval on which our control is active shrinks to $\{0\}$ the system (including controls and state trajectories) converges to a direct birth controlled system (which can be seen as an “impulse” control). In Section II.5 we show how our results apply to a nonlinear system arising in an infection-age model. In Section II.6 we investigate the norm optimal controls driving an initial population density p_0 to zero, theoretically and numerically. We end up with an appendix, showing that the considered system is still

null controllable in an L^1 setting.

II.1 SOME BACKGROUND ON THE LOTKA-MCKENDRICK SEMIGROUP

In this section we recall, with no claim of originality, the formulation of equations (II.0.1) using semigroup theory.

Denote $X = L^2[0, a_+]$ and consider the operator $A : \mathcal{D}(A) \rightarrow X$ defined by

$$\begin{aligned} \mathcal{D}(A) &= \left\{ \varphi \in L^2[0, a_+] \mid \varphi(0) = \int_0^{a_+} \beta(a)\varphi(a) da; -\frac{d\varphi}{da} - \mu\varphi \in L^2[0, a_+] \right\}, \\ A\varphi &= -\frac{d\varphi}{da} - \mu\varphi \quad (\varphi \in \mathcal{D}(A)). \end{aligned} \tag{II.1.1}$$

It is well known (see, for instance, Song et al. [SYW⁺82] or Kappel and Zhang [KZ93]) that A generates a C^0 semigroup of linear operators in X which we denote by $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$. This allows to define the concept of (mild) solution of (II.0.1) in the following standard way: we say that p is a mild solution of (II.0.1) if

$$p(t, \cdot) = \mathbb{T}_t p_0 + \Phi_t u \quad (t \geq 0, \quad u \in L^2([0, \infty); X)), \tag{II.1.2}$$

where the control operator $B \in \mathcal{L}(X)$ is defined by

$$Bu = mu \quad (u \in X),$$

and where

$$\Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma) d\sigma \quad (t \geq 0, \quad u \in L^2([0, \infty); X)). \tag{II.1.3}$$

It is well known (see, for instance, [Ian95b] or [Web85]) that the semigroup \mathbb{T} satisfies

$$(\mathbb{T}_t f)(a) = \begin{cases} \frac{\pi(a)}{\pi(a-t)} f(a-t) & \text{if } t \leq a, \\ \pi(a) b(t-a) & \text{if } t > a, \end{cases} \tag{II.1.4}$$

where $\pi(a) = e^{-\int_0^a \mu(\sigma) d\sigma}$ is the probability of survival of an individual from age 0 to a and $b(t) = (\mathbb{T}_t f)(0) = \int_0^{a_+} \beta(a)(\mathbb{T}_t f)(a) da$ is the total birth rate function. Moreover, according, for instance, to [Ian95b, p.12], b is the unique continuous function satisfying

the integral Volterra equation of the second kind

$$b(t) = \int_0^{\min(t, a_+)} \beta(t-s)\pi(t-s)b(s) ds + \int_{\min(t, a_+)}^{a_+} \beta(a) \frac{\pi(a)}{\pi(a-t)} f(a-t) da. \quad (\text{II.1.5})$$

The state trajectory p of the controlled system (II.0.1) is given (see, for instance, [BIM01]) by

$$p(t, a) := \begin{cases} \frac{\pi(a)}{\pi(a-t)} p_0(a-t) + c(t, a), & t \leq a, \\ \pi(a) \tilde{b}(t-a) + c(t, a), & a < t, \end{cases} \quad (\text{II.1.6})$$

where

$$c(t, a) := \begin{cases} \int_{a-t}^a \frac{\pi(a)}{\pi(s)} m(s) u(s - (a-t), s) ds, & t \leq a, \\ \int_0^a \frac{\pi(a)}{\pi(s)} m(s) u(s + (t-a), s) ds, & a < t, \end{cases} \quad (\text{II.1.7})$$

where $\tilde{b}(t) = \int_0^{a_+} \beta(a)p(t, a) da$ is the flux of newly born individuals for the controlled system. It is easy to check that \tilde{b} is the unique continuous function satisfying the Volterra integral equation

$$\begin{aligned} \tilde{b}(t) = & \int_0^{\min(t, a_+)} \beta(t-s) \left[\pi(t-s) \tilde{b}(s) + c(t, t-s) \right] ds \\ & + \int_{\min(t, a_+)}^{a_+} \beta(a) \left[\frac{\pi(a)}{\pi(a-t)} p_0(a-t) + c(t, a) \right] da. \end{aligned} \quad (\text{II.1.8})$$

Remark II.1.1. *The control u constructed in the following section will be of the form*

$$u(t, a) = v(t-a) \quad (a \in [0, a_+], t \in [0, \tau]),$$

where $v : [-a_+, \tau] \rightarrow \mathbb{R}$. In this case (II.1.6) becomes

$$p(t, a) = \begin{cases} \frac{\pi(a)}{\pi(a-t)} p_0(a-t) + v(t-a) \int_{a-t}^a \frac{\pi(a)}{\pi(s)} m(s) ds, & a \geq t, \\ \pi(a) \tilde{b}(t-a) + v(t-a) \int_0^a \frac{\pi(a)}{\pi(s)} m(s) ds, & a < t. \end{cases} \quad (\text{II.1.9})$$

Remark II.1.2. *If the state space $X = L^2[0, a_+]$ is substituted by $X = L^1[0, a_+]$, the*

operator $A : \mathcal{D}(A) \rightarrow X$ defined by

$$\mathcal{D}(A) = \left\{ \varphi \in L^1[0, a_+] \mid \varphi(0) = \int_0^{a_+} \beta(a)\varphi(a) da; -\frac{d\varphi}{da} - \mu\varphi \in L^1[0, a_+] \right\},$$

$$A\varphi = -\frac{d\varphi}{da} - \mu\varphi \quad (\varphi \in \mathcal{D}(A)), \quad (\text{II.1.10})$$

generates a C^0 semigroup of linear operators in X denoted by $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$ (see, for instance, Banks and Kappel [BK89]). This allows to define the concept of (mild) solution of (II.0.1) in the same way as above : we say that p is a mild solution of (II.0.1) if

$$p(t, \cdot) = \mathbb{T}_t p_0 + \Phi_t u \quad (t \geq 0, \quad u \in L^1([0, \infty); X)), \quad (\text{II.1.11})$$

where the control operator $B \in \mathcal{L}(X)$ is defined by

$$Bu = mu \quad (u \in X),$$

and where

$$\Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma) d\sigma \quad (t \geq 0, \quad u \in L^1([0, \infty); X)). \quad (\text{II.1.12})$$

The semigroup \mathbb{T} satisfies the same formulas (II.1.4) as in the L^2 setting.

II.2 STUDY OF THE REACHABLE SPACE

In this section, we study the reachable space of (II.0.1) by means of controls $u \in L^2([0, \tau]; L^2[0, a_+])$. In other words, we study the space $\text{Ran } \Phi_\tau$, for every $\tau > 0$, where the operator Φ_τ has been defined in (II.1.3). In order to obtain controllability results, we focus on the case $\tau > a_+ - a_0$ since in the opposite case, as shown below, we have neither approximate nor null controllability. For the sake of completeness, we recall that the property of *approximate controllability* in some time $\tau > 0$ means that $\text{Ran } \Phi_\tau$ is dense in X and that the *null controllability* in time τ means that $\text{Ran } \Phi_\tau \supset \text{Ran } \mathbb{T}_\tau$ (we refer, for instance, to [TW09] - Chapter 11 for a detailed description of these concepts).

Proposition II.2.1. *Suppose that $\tau < a_+ - a_0$. Then the system (II.0.1) is neither approximately controllable, nor null controllable in time τ .*

Proof. Let $\tau \in]0, a_+ - a_0[$. Then for every $u \in L^2([0, \tau]; L^2[0, a_+])$ and for almost every $a \in (a_0 + \tau, a_+)$, we have from (II.1.6) that

$$(\Phi_\tau u)(a) = \int_{a-\tau}^a \frac{\pi(a)}{\pi(s)} m(s) u(s + (\tau - a), s) ds = 0, \quad (\text{II.2.1})$$

since $m(s) = 0$ if $s \geq a - \tau > a_0$. This clearly implies that the system (II.0.1) is not

approximately controllable in time τ .

On the other hand, (II.1.4) implies that there exists $p_0 \in X$ such that $(\mathbb{T}_\tau p_0)(a) \neq 0$ on $[a_0 + \tau, a_\dagger]$. Consequently, $\text{Ran } \Phi_\tau \not\subset \text{Ran } \mathbb{T}_\tau$, so that (II.0.1) is not null controllable in time τ . \square

The main result in this section is:

Theorem II.2.2. *Under the assumptions of Theorem II.0.1, define*

$$\mathcal{R} := \left\{ \tilde{p} \in X \mid a \mapsto \frac{\tilde{p}(a)}{\pi(a)} \in X, \quad a \mapsto \frac{\frac{\tilde{p}(a)}{\pi(a)} - \int_0^{a_\dagger - a} \beta(x) \frac{\pi(x)}{\pi(a+x)} \tilde{p}(a+x) dx}{\int_0^a \frac{m(x)}{\pi(x)} dx} \in X \right\}. \quad (\text{II.2.2})$$

Then for every $\tau > a_\dagger - a_0$ we have $\mathcal{R} \subset \text{Ran } \Phi_\tau$.

Proof. We can assume, without loss of generality, that $a_0 \leq a_b$ (otherwise we simply control the system for smaller ages). Let $\tau \in (a_\dagger - a_0, a_\dagger)$.

We define the sets $\{P_\tau^j\}_{j \geq -1}$ by

$$P_\tau^{-1} = \{(t, a) \in (0, +\infty)^2 : \tau - a_\dagger \leq t - a \leq 0\}$$

and for $j \geq 0$

$$P_\tau^j = \{(t, a) \in (0, +\infty)^2 : ja_b < t - a \leq (j+1)a_b\}.$$

Given $\tilde{p} \in \mathcal{R}$, we look for $v \in L^2[-a_\dagger, \tau]$ such that setting $u(t, a) = v(t - a)$ for every $t \in [0, \tau]$ and $a \in [0, a_\dagger]$ we have

$$\Phi_\tau u(a) = \tilde{p}(a) \quad (a \in [0, a_\dagger] \text{ a.e.}) \quad (\text{II.2.3})$$

Case $t - a < \tau - a_\dagger$: We first set $v(s) = 0$ for $s \in [-a_\dagger, \tau - a_\dagger]$ so that, according to (II.1.9) we have

$$(\Phi_t u)(a) = 0 \quad (t - a < \tau - a_\dagger). \quad (\text{II.2.4})$$

Case $\tau - a_\dagger \leq t - a \leq 0$: By using again (II.1.9), it follows that

$$(\Phi_\tau u)(a) = v(\tau - a) \int_{a-\tau}^a \frac{\pi(a)}{\pi(s)} m(s) ds \quad (a \in [\tau, a_\dagger] \text{ a.e.}).$$

Consequently, (II.2.3) is satisfied for $a \in [\tau, a_\dagger]$ iff

$$v(s) = \frac{\frac{\tilde{p}(\tau-s)}{\pi(\tau-s)}}{\int_{-s}^{\tau-s} \frac{m(x)}{\pi(x)} dx} \quad (s \in [\tau - a_\dagger, 0] \text{ a.e.}). \quad (\text{II.2.5})$$

Note that, since $\tau > a_\dagger - a_0$, $s \in [\tau - a_\dagger, 0]$ implies that $-s < a_0$ therefore for every

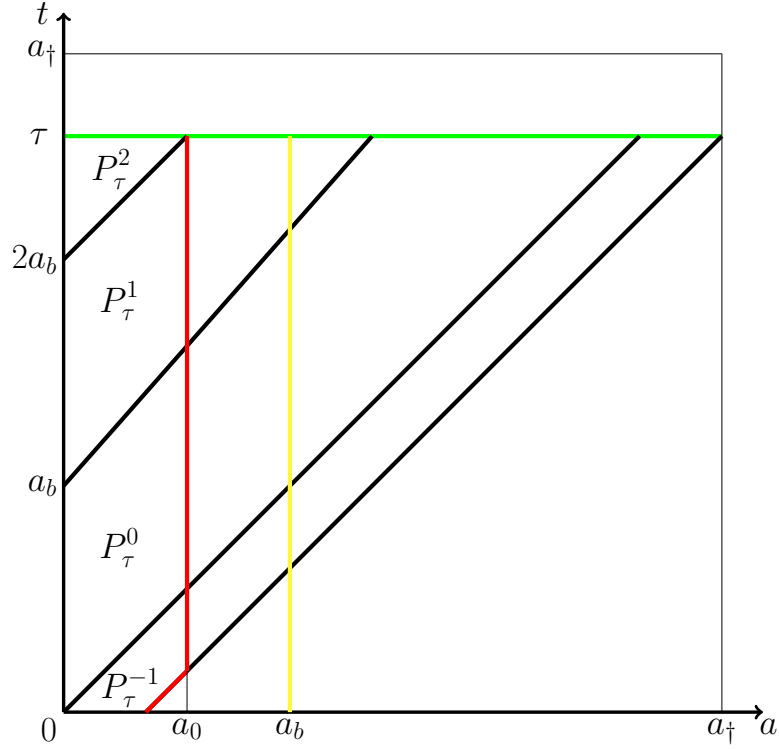


Figure II.1 – In this figure the control possibly acts in the subset of $Q_\tau = [0, \tau] \times [0, a_\dagger]$ delimited by the line $a = 0$ and the red curve. The domain Q_τ is decomposed into subregions $P_\tau^{-1}, P_\tau^0, P_\tau^1, P_\tau^2, \dots$ which are delimited respectively by the characteristics lines $t - a = \tau - a_\dagger, t - a = 0, t - a = a_b, t - a = 2a_b, \dots$. For each subregion P_τ^j (for $j \geq 0$) the values of $\tilde{b}(t) = \int_0^{a_\dagger} \beta(a)p(t, a) da$ for $t \in [ja_b, (j+1)a_b]$ depends only on the control in the previous subregions $P_\tau^{-1}, \dots, P_\tau^{j-1}$. This will allow us, using an inductive method, to derive an explicit formula of the control functions in terms of the targeted distribution of population.

$s \in [\tau - a_\dagger, 0]$ we have

$$\int_{-s}^{\tau-s} \frac{m(x)}{\pi(x)} dx > 0.$$

Moreover by assumption we have $a \mapsto \frac{\tilde{p}(a)}{\pi(a)} \in L^2[0, a_\dagger]$ hence

$$v \in L^2[\tau - a_\dagger, 0].$$

Case $0 < t - a < a_b$: According to (II.1.9) and (II.2.5), for almost every $(t, a) \in P_\tau^{-1}$

we have

$$\begin{aligned} (\Phi_t u)(a) &= v(t-a) \int_{a-t}^a \frac{\pi(a)}{\pi(s)} m(s) ds \\ &= \pi(a) \frac{\int_{a-t}^a \frac{m(s)}{\pi(s)} ds}{\int_{a-t}^{\tau+(a-t)} \frac{m(s)}{\pi(s)} ds} \frac{\tilde{p}(\tau+(a-t))}{\pi(\tau+(a-t))}. \end{aligned} \quad (\text{II.2.6})$$

To determine v on $[0, a_b]$ we note that from (II.1.9) it follows that

$$(\Phi_\tau u)(a) = \pi(a) \tilde{b}(\tau-a) + v(\tau-a) \int_0^a \frac{\pi(a)}{\pi(s)} m(s) ds \quad (a \in [\tau - a_b, \tau] \text{ a.e.}), \quad (\text{II.2.7})$$

where

$$\tilde{b}(t) = \int_{a_b}^{a_+} \beta(a) (\Phi_t u)(a) da \quad (t \in [0, a_b]). \quad (\text{II.2.8})$$

To determine \tilde{b} , after some simple calculations and using (II.2.4), (II.2.6) and (II.2.8) it follows that

$$\tilde{b}(t) = \int_{a_b}^{t+a_+-\tau} \beta(a) \pi(a) \frac{\tilde{p}(\tau+(a-t))}{\pi(\tau+(a-t))} da \quad (t \in [0, a_b]). \quad (\text{II.2.9})$$

Inserting the above formula (II.2.9) in (II.2.7) it follows that

$$(\Phi_\tau u)(a) = \pi(a) \int_{a_b}^{a_+-a} \beta(x) \pi(x) \frac{\tilde{p}(a+x)}{\pi(a+x)} dx + v(\tau-a) \int_0^a \frac{\pi(a)}{\pi(s)} m(s) ds \quad (a \in [\tau - a_b, \tau] \text{ a.e.}). \quad (\text{II.2.10})$$

Thus, (II.2.3) is satisfied for $a \in [\tau - a_b, \tau]$ iff

$$v(s) = \frac{\tilde{p}(\tau-s) - \pi(\tau-s) \chi(s)}{\pi(\tau-s) \int_0^{\tau-s} \frac{m(x)}{\pi(x)} dx} \quad (s \in [0, a_b] \text{ a.e.}), \quad (\text{II.2.11})$$

where

$$\chi(s) := \int_{a_b}^{s+a_+-\tau} \beta(x) \pi(x) \frac{\tilde{p}(\tau-(s-x))}{\pi(\tau-(s-x))} dx \quad (s \in [0, \tau]). \quad (\text{II.2.12})$$

Since $\tilde{p} \in L^2[0, a_+]$, we clearly have that $v \in L^2[0, a_b]$.

Case $ja_b < t-a < (j-1)a_b$ for some integer $j \geq 1$: The above calculations suggest

as possible control driving the zero initial state to $u(t, a) = v(t - a)$ where

$$v(s) = \begin{cases} 0 & \text{for } s \in [-a_{\dagger}, \tau - a_{\dagger}], \\ \frac{\tilde{p}(\tau-s)}{\pi(\tau-s)} & \text{for } s \in [\tau - a_{\dagger}, 0], \\ \frac{\int_{-s}^{\tau-s} \frac{m(x)}{\pi(x)} dx}{\tilde{p}(\tau-s) - \pi(\tau-s)\chi(s)} & \text{for } s \in [0, \tau]. \\ \frac{\pi(\tau-s) \int_0^{\tau-s} \frac{m(x)}{\pi(x)} dx}{\pi(\tau-s)} & \text{for } s \in [0, \tau]. \end{cases} \quad (\text{II.2.13})$$

where $\chi(s)$ is defined by (II.2.12).

To prove this fact we first note that v defined above is in $L^2[-a_{\dagger}, \tau]$ due to the fact that $\tilde{p} \in \mathcal{R}$. Moreover, we have already seen that (II.2.13) gives

$$(\Phi_{\tau}u)(a) = \tilde{p}(a) \quad (a \in [\tau - a_b, a_{\dagger}] \text{ a.e.}).$$

We next show by induction over $j \geq 0$ the following properties which we denote by (\mathcal{P}_j) :

- $\tilde{b}(t) = \chi(t)$ for every $t \in [ja_b, (j+1)a_b]$,
- $(\Phi_{\tau}u)(a) = \tilde{p}(a)$ for a.e. $a \in (\tau - (j+1)a_b, \tau - ja_b)$.

We have seen above that (\mathcal{P}_0) is true.

Let $j \geq 0$ be an integer fixed, and suppose that (\mathcal{P}_i) holds for every $i \in \{0, \dots, j\}$. Then, for every $i \in \{0, \dots, j\}$ and for almost every $(t, a) \in P_{\tau}^i$, from (II.1.9) and (II.2.13) we have

$$(\Phi_t u)(a) = \pi(a)\chi(t-a) \left[1 - \frac{\int_0^a \frac{m(s)}{\pi(s)} ds}{\int_0^{\tau-(t-a)} \frac{m(s)}{\pi(s)} ds} \right] + \pi(a) \frac{\int_0^a \frac{m(x)}{\pi(x)} dx}{\int_0^{\tau-(t-a)} \frac{m(x)}{\pi(x)} dx} \frac{\tilde{p}(\tau - (t-a))}{\pi(\tau - (t-a))}. \quad (\text{II.2.14})$$

Combining (II.2.4), (II.2.6) and the above formula (II.2.14), we are in position to compute $\tilde{b}(t)$ for $t \in [(j+1)a_b, (j+2)a_b]$. Indeed, since $\tilde{b}(t) = \int_{a_b}^{a_{\dagger}} \beta(a)(\Phi_t u)(a) da$ for every $t \in [(j+1)a_b, (j+2)a_b]$, we can use (II.2.4) to obtain that

$$\tilde{b}(t) = \int_{a_b}^t \beta(a)(\Phi_t u)(a) da + \int_t^{t+a_{\dagger}-\tau} \beta(a)(\Phi_t u)(a) da \quad (t \in [(j+1)a_b, (j+2)a_b]). \quad (\text{II.2.15})$$

Firstly, according to (II.2.14), the first integral in the above formula (II.2.15) writes

$$\int_{a_b}^t \beta(a)(\Phi_t u)(a) da = \int_{a_b}^t \beta(a) \left(\pi(a)\chi(t-a) \left[1 - \frac{\int_0^a \frac{m(s)}{\pi(s)} ds}{\int_0^{\tau-(t-a)} \frac{m(s)}{\pi(s)} ds} \right] + \pi(a) \left[\frac{\int_0^a \frac{m(x)}{\pi(x)} dx}{\int_0^{\tau-(t-a)} \frac{m(x)}{\pi(x)} dx} \frac{\tilde{p}(\tau-(t-a))}{\pi(\tau-(t-a))} \right] \right) da.$$

We note that the term contained in the first bracket of the above formula involves only values of a which are larger than a_0 , so it vanishes. Moreover the term in the second bracket of the last formula is clearly equal to one. Consequently,

$$\int_{a_b}^t \beta(a)(\Phi_t u)(a) da = \int_{a_b}^t \beta(a)\pi(a) \frac{\tilde{p}(\tau+(a-t))}{\pi(\tau+(a-t))} da \quad (t \in [(j+1)a_b, (j+2)a_b]). \quad (\text{II.2.16})$$

Moreover, by using (II.2.6) it is not difficult to check that for every $t \in [(j+1)a_b, (j+2)a_b]$,

$$\int_t^{t+a_1-\tau} \beta(a)(\Phi_t u)(a) da = \int_t^{t+a_1-\tau} \beta(a)\pi(a) \frac{\tilde{p}(\tau+(a-t))}{\pi(\tau+(a-t))} da. \quad (\text{II.2.17})$$

Using (II.2.15), (II.2.16) and (II.2.17) we have

$$\tilde{b}(t) = \int_{a_b}^{t+a_1-\tau} \beta(a)\pi(a) \frac{\tilde{p}(\tau+(a-t))}{\pi(\tau+(a-t))} da = \chi(t) \quad (t \in [(j+1)a_b, (j+2)a_b]). \quad (\text{II.2.18})$$

Consequently, if $(j+1)a_b < t-a \leq (j+2)a_b$, we obtain from (II.1.9), (II.2.13) and (II.2.18) that

$$(\Phi_t u)(a) = \pi(a)\chi(t-a) \left[1 - \frac{\int_0^a \frac{m(s)}{\pi(s)} ds}{\int_0^{\tau-(t-a)} \frac{m(s)}{\pi(s)} ds} \right] + \pi(a) \frac{\int_0^a \frac{m(x)}{\pi(x)} dx}{\int_0^{\tau-(t-a)} \frac{m(x)}{\pi(x)} dx} \frac{\tilde{p}(\tau-(t-a))}{\pi(\tau-(t-a))}.$$

The above formula implies that $(\Phi_\tau u)(a) = \tilde{p}(a)$ for almost every $a \in (\tau-(j+2)a_b, \tau-(j+1)a_b)$, which ends the induction proof. \square

Remark II.2.3. Given $\tau \in]a_+ - a_0, a_+[$ and $\tilde{p} \in \mathcal{R}$, the proof of Theorem II.2.2 gives an explicit formula for a control $u \in L^2([0, \tau]; L^2[0, a_+])$ satisfying $\Phi_\tau u = \tilde{p}$. More precisely, for almost every $(t, a) \in Q_\tau$ we have

$$u(t, a) = \begin{cases} 0 & \text{if } t - a < \tau - a_{\dagger}, \\ \frac{\tilde{p}(\tau + (a - t))}{\pi(\tau + (a - t))} & \text{if } \tau - a_{\dagger} \leq t - a \leq 0, \\ \frac{\int_{a-t}^{\tau+(a-t)} \frac{m(x)}{\pi(x)} dx}{\tilde{p}(\tau - (t - a)) - \pi(\tau - (t - a))\chi(t - a)} & \text{if } 0 < t - a \leq \tau, \\ \pi(\tau - (t - a)) \int_0^{\tau-(t-a)} \frac{m(x)}{\pi(x)} dx & \end{cases}$$

where

$$\chi(s) = \int_{a_b}^{s+a_{\dagger}-\tau} \beta(x)\pi(x) \frac{\tilde{p}(\tau - (s - x))}{\pi(\tau - (s - x))} dx \quad (s \in [0, \tau]).$$

Moreover, for almost every $(t, a) \in Q_{\tau}$, we have

$$(\Phi_t u)(a) = \begin{cases} 0 & \text{if } t - a < \tau - a_{\dagger}, \\ \pi(a) \frac{\int_{a-t}^a \frac{m(s)}{\pi(s)} ds}{\int_{a-t}^{\tau+(a-t)} \frac{m(s)}{\pi(s)} ds} \frac{\tilde{p}(\tau + (a - t))}{\pi(\tau + (a - t))} & \text{if } \tau - a_{\dagger} \leq t - a \leq 0, \\ \pi(a)\chi(t - a) \left[1 - \frac{\int_0^a \frac{m(s)}{\pi(s)} ds}{\int_0^{\tau-(t-a)} \frac{m(s)}{\pi(s)} ds} \right] & \\ + \pi(a) \frac{\int_0^a \frac{m(x)}{\pi(x)} dx}{\int_0^{\tau-(t-a)} \frac{m(x)}{\pi(x)} dx} \frac{\tilde{p}(\tau - (t - a))}{\pi(\tau - (t - a))} & \text{if } 0 < t - a \leq \tau. \end{cases}$$

Remark II.2.4. For every $\varphi \in \mathcal{D}(A)$ with $\frac{\varphi}{\pi} \in X$, we have $\varphi \in \mathcal{R}$. Indeed, for every $\varphi \in \mathcal{D}(A)$, there exists $f \in X$ such that $\varphi' + \mu\varphi = f$, so that assuming the condition $\frac{\varphi}{\pi} \in X$, for almost every $a \in (0, a_{\dagger})$ we have

$$\begin{aligned} \frac{\varphi(a)}{\pi(a)} &= \varphi(0) + \int_0^a \frac{\varphi'(s)\pi(s) + \mu(s)\pi(s)\varphi(s)}{\pi^2(s)} ds \\ &= \int_0^{a_{\dagger}} \beta(a)\varphi(a) da + \int_0^a \frac{f(s)}{\pi(s)} ds. \end{aligned} \quad (\text{II.2.19})$$

Denoting by $I := \int_0^{a_{\dagger}} \beta(a)\varphi(a) da$ and $i(a) = \int_0^{a_{\dagger}-a} \beta(x) \frac{\pi(x)}{\pi(a+x)} \varphi(a+x) dx$, it follows

from the above formula (II.2.19) that we have

$$\frac{\varphi(a)}{\pi(a)} - \int_0^{a_+ - a} \beta(x) \frac{\pi(x)}{\pi(a+x)} \varphi(a+x) dx = \int_0^a \frac{f(s)}{\pi(s)} ds + I - i(a).$$

Let us show that the map

$$a \mapsto \frac{\int_0^a \frac{f(s)}{\pi(s)} ds + I - i(a)}{a}, \quad (\text{II.2.20})$$

is in $L^2[0, a_+]$. By Hardy's inequality (see [HLP88, p 240]), the map $a \mapsto \frac{\int_0^a \frac{f(s)}{\pi(s)} ds}{a}$ lies in $L^2[0, a_+]$. For the second term at the numerator in (II.2.20) we note that for almost every $a \in (0, a_+)$ we have

$$\frac{d}{da}(I - i(a)) = -\beta(a_+ - a)\pi(a_+ - a) \left(\varphi(0) + \int_0^{a_+} \frac{f(s)}{\pi(s)} ds \right) - \int_0^{a_+ - a} \beta(x)\pi(x) \frac{f(a+x)}{\pi(a+x)} dx,$$

so that the map $a \mapsto \frac{d}{da}(I - i(a))$ is in $L^2[0, a_+]$. It follows that the map $F : a \mapsto I - i(a)$ satisfies $F \in H^1[0, a_+]$ with $F(0) = 0$, so that from Hardy's inequality (see [HLP88, p 240]) the map $a \mapsto \frac{F(a)}{a}$ is in $L^2[0, a_+]$. Consequently, from (II.2.20) we have $\varphi \in \mathcal{R}$.

II.3 SOME CONTROLLABILITY RESULTS

In this section we investigate some controllability properties of the system (II.0.1).

We first note that the following null controllability result holds.

Proposition II.3.1. *For every $\tau > a_+ - a_0$, we have $\text{Ran } \mathbb{T}_\tau \subset \mathcal{R}$, where \mathcal{R} has been defined in (II.2.2). Consequently, if β satisfies (II.0.4) then the system (II.0.1) is null controllable in any time $\tau > a_+ - a_0$.*

Proof. Without loss of generality, we can assume that $a_+ - a_0 < \tau < a_+$. Indeed, for $\tau \geq a_+$, the result follows using the semigroup property and replacing p_0 by $\mathbb{T}_\varepsilon p_0$ for some convenient $\varepsilon > 0$. Let $\tilde{p} \in \text{Ran } \mathbb{T}_\tau$. Then from (II.1.4) there exists $p_0 \in X$ such that

$$\tilde{p}(a) = \begin{cases} \frac{\pi(a)}{\pi(a-\tau)} p_0(a-\tau) & \text{if } \tau \leq a, \\ \pi(a)b(\tau-a) & \text{if } \tau > a, \end{cases} \quad (a \in [0, a_+] \text{ a.e.}), \quad (\text{II.3.1})$$

with $b(t) = \int_0^{a_+} \beta(a)(\mathbb{T}_t p_0)(a) da$. Consequently, for almost every $a \in (0, \tau)$ we have

$\frac{\tilde{p}(a)}{\pi(a)} = b(\tau - a)$ and for almost every $a \in (\tau, a_+)$ we have $\frac{\tilde{p}(a)}{\pi(a)} = \frac{p_0(a - \tau)}{\pi(a - \tau)}$. Hence, the function $a \mapsto \frac{\tilde{p}}{\pi}$ is in $L^2[0, a_+]$. From that it is easy to check that

$$a \mapsto \frac{\frac{\tilde{p}(a)}{\pi(a)} - \int_0^{a_+ - a} \beta(x) \frac{\pi(x)}{\pi(a+x)} \tilde{p}(a+x) dx}{\int_0^a \frac{m(x)}{\pi(x)} dx}, \quad (\text{II.3.2})$$

is in $L^2[\varepsilon, a_+]$ for every $\varepsilon > 0$. To show that $\tilde{p} \in \mathcal{R}$, we still have to check that the map defined in (II.3.2) is in $L^2[0, \varepsilon]$ for every $\varepsilon > 0$ small enough. To this aim, applying first (II.1.4) we note that, for almost every $a \in (0, \varepsilon)$,

$$\begin{aligned} \frac{\tilde{p}(a)}{\pi(a)} - \int_0^{a_+ - a} \beta(x) \frac{\pi(x)}{\pi(a+x)} \tilde{p}(a+x) dx \\ &= b(\tau - a) - \int_0^{a_+ - a} \beta(x) \frac{\pi(x)}{\pi(a+x)} (\mathbb{T}_\tau p_0)(a+x) dx \\ &= b(\tau - a) - \int_0^{\tau - a} \beta(x) \frac{\pi(x)}{\pi(a+x)} (\mathbb{T}_\tau p_0)(a+x) dx - \int_{\tau - a}^{a_+ - a} \beta(x) \frac{\pi(x)}{\pi(a+x)} (\mathbb{T}_\tau p_0)(a+x) dx \\ &= b(\tau - a) - \int_0^{\tau - a} \beta(x) \pi(x) b(\tau - (a+x)) dx - \int_{\tau - a}^{a_+ - a} \beta(x) \pi(x) \frac{p_0(a+x-\tau)}{\pi(a+x-\tau)} dx. \end{aligned}$$

Moreover, using (II.1.5) it follows that for every $a \in (0, \varepsilon)$ we have

$$b(\tau - a) = \int_0^{\tau - a} \beta(x) \pi(x) b(\tau - (a+x)) dx + \int_{\tau - a}^{a_+} \beta(x) \pi(x) \frac{p_0(a+x-\tau)}{\pi(a+x-\tau)} dx.$$

We can thus combine the last two formulas to obtain that for almost every $a \in (0, \varepsilon)$ we have

$$\frac{\tilde{p}(a)}{\pi(a)} - \int_0^{a_+ - a} \beta(x) \frac{\pi(x)}{\pi(a+x)} \tilde{p}(a+x) dx = \int_{a_+ - a}^{a_+} \beta(x) \pi(x) \frac{p_0(a+x-\tau)}{\pi(a+x-\tau)} dx.$$

Since $\beta \in L^\infty[0, a_+]$ and π is a decreasing function, it follows that for almost every $a \in (0, \varepsilon)$,

$$\left| \frac{\tilde{p}(a)}{\pi(a)} - \int_0^{a_+ - a} \beta(x) \frac{\pi(x)}{\pi(a+x)} \tilde{p}(a+x) dx \right| \leq \|\beta\|_{L^\infty[0, a_+]} \int_{a_+ - \tau}^{a_+ - \tau + a} |p_0(s)| ds. \quad (\text{II.3.3})$$

On the other hand, Hardy's inequality (see, for instance, [HLP88, p. 240]) ensures that

$$a \mapsto \frac{\int_0^a |p_0(s + (a_+ - \tau))| ds}{a} \in L^2[0, \varepsilon],$$

hence $\tilde{p} \in \mathcal{R}$.

To obtain the null controllability assertion it suffices to combine the inclusions $\text{Ran } \mathbb{T}_\tau \subset \mathcal{R}$ which we have just proved and $\mathcal{R} \subset \text{Ran } \Phi_\tau$ which was proved in Theorem II.2.2. \square

The result below and the following one answer to an open question raised in [BIM01] and it essentially asserts that, with an extra assumption, the system can be “positively” steered from any positive initial state in X to any positive state in \mathcal{R} by means of a L^2 control. More precisely, we have,

Theorem II.3.2. *Let the assumptions of Theorem II.0.1 be satisfied. Let $p_0 \in X$ and $\tilde{p} \in \mathcal{R}$. Then for every $\tau > a_\dagger - a_0$, there exists a control $u \in L^2([0, \tau]; L^2[0, a_\dagger])$ such that the controlled solution p of (II.0.1) satisfies*

$$p(\tau, a) = \tilde{p}(a) \quad (a \in (0, a_\dagger) \text{ a.e.}).$$

Moreover, if $p_0(a) \geq 0$ and $\tilde{p}(a) \geq 0$ for almost every $a \in (0, a_\dagger)$, then the control can be chosen such that the controlled state trajectory remains positive, i.e., such that for every $t \in [0, \tau]$ we have

$$p(t, a) \geq 0 \quad (a \in [0, a_\dagger] \text{ a.e.}).$$

Proof. We give the proof in the case $\tau \in (a_\dagger - a_0, a_\dagger)$.

We set $z = p - \tilde{p}$. The aim is to find a control function $u \in L^2([0, \tau]; L^2[0, a_\dagger])$ such as $z(\tau) = 0$ and $p(t, a) \geq 0$ for almost every $(t, a) \in Q_\tau$. To achieve this goal we set

$$z(t) = \mathbb{T}_t p_0 + \Phi_t u - \tilde{p} \quad (t \geq 0).$$

Since from Theorem II.2.2 and Proposition II.3.1 we have $\tilde{p} - \mathbb{T}_\tau p_0 \in \mathcal{R}$ we can apply the construction in Theorem II.2.2 to obtain a control $u \in L^2([0, \tau]; L^2[0, a_\dagger])$ such as

$$\Phi_\tau u = \tilde{p} - \mathbb{T}_\tau p_0.$$

The corresponding state trajectory will satisfy, in particular, the condition $p(\tau, \cdot) = \tilde{p}$.

To prove the positivity of the state trajectory constructed above we first assume that p_0 and \tilde{p} are continuous on $[0, a_\dagger]$. Then the the control u described in Remark II.2.3 is of the form $u(t, a) = v(t - a)$, with v continuous on each one of the intervals $[-a_\dagger, \tau - a_\dagger]$, $[\tau - a_\dagger, 0]$ and $[0, \tau]$. Moreover, the corresponding state trajectory p , also described in Remark II.2.3, is continuous on each one of the domains Q_τ^1 , Q_τ^2 and Q_τ^3 , where

$$Q_\tau^1 = \{(t, a) \in Q_\tau \mid -a_\dagger \leq t - a < \tau - a_\dagger\},$$

$$Q_\tau^2 = \{(t, a) \in Q_\tau \mid \tau - a_\dagger \leq t - a \leq 0\},$$

$$Q_\tau^3 = \{(t, a) \in Q_\tau \mid 0 < t - a \leq \tau\}.$$

Since the initial condition p_0 is supposed positive and the control u vanishes on Q_τ^1 , the solution p of the controlled system (II.0.1) is positive on Q_τ^1 . Indeed, from (II.1.9) we

have

$$p(t, a) = \frac{\pi(a)}{\pi(a-t)} p_0(a-t) \quad ((t, a) \in Q_\tau^1). \quad (\text{II.3.4})$$

Moreover, if $(t, a) \in Q_\tau^2$ with $a \geq a_0$, using (II.1.9) we have

$$p(t, a) = \frac{\pi(a)}{\pi(a-t)} p_0(a-t) + v(t-a) \int_{a-t}^{a_0} \frac{\pi(a)}{\pi(s)} ds \quad ((t, a) \in Q_\tau^2, a \geq a_0). \quad (\text{II.3.5})$$

It follows from the above formula that for every $(t, a) \in Q_\tau^2$ with $a \geq a_0$ we have

$$\begin{aligned} \tilde{p}(\tau - (t-a)) &= p(\tau, \tau - (t-a)) = \frac{\pi(\tau - (t-a))}{\pi(a-t)} p_0(a-t) \\ &+ v(t-a) \int_{a-t}^{a_0} \frac{\pi(\tau - (t-a))}{\pi(s)} ds \quad ((t, a) \in Q_\tau^2, a \geq a_0). \end{aligned} \quad (\text{II.3.6})$$

By combining (II.3.5) and (II.3.6) we get

$$p(t, a) = \frac{\pi(a)}{\pi(\tau - (t-a))} \tilde{p}(\tau - (t-a)) \quad ((t, a) \in Q_\tau^2, a \geq a_0). \quad (\text{II.3.7})$$

Since the target \tilde{p} is supposed positive, it follows from (II.3.7) that the solution p of the controlled system (II.0.1) is positive on the subdomain $\{(t, a) \in Q_\tau^2, a \geq a_0\}$.

If $(t, a) \in Q_\tau^3$ with $a \geq a_0$, using (II.1.9) we have

$$p(t, a) = \pi(a) \tilde{b}(t-a) + v(t-a) \int_0^{a_0} \frac{\pi(a)}{\pi(s)} ds \quad ((t, a) \in Q_\tau^3, a \geq a_0). \quad (\text{II.3.8})$$

It follows from the above formula that for every $(t, a) \in Q_\tau^3$ with $a \geq a_0$ we have

$$\tilde{p}(\tau - (t-a)) = \pi(\tau - (t-a)) \tilde{b}(t-a) + v(t-a) \int_0^{a_0} \frac{\pi(\tau - (t-a))}{\pi(s)} ds \quad ((t, a) \in Q_\tau^3, a \geq a_0), \quad (\text{II.3.9})$$

so that using (II.3.8) and (II.3.9) we get

$$p(t, a) = \frac{\pi(a)}{\pi(\tau - (t-a))} \tilde{p}(\tau - (t-a)) \quad ((t, a) \in Q_\tau^3, a \geq a_0). \quad (\text{II.3.10})$$

Since the target \tilde{p} is supposed positive, it follows from (II.3.10) that the solution p of the controlled system (II.0.1) is positive on the subdomain $\{(t, a) \in Q_\tau^3, a \geq a_0\}$. In particular, using (II.3.7) and (II.3.10), the solution p is positive along the red curve represented in Figure 1.

It remains to show the positivity of the controlled solution p into the subregions $\{(t, a) \in Q_\tau^2, a \leq a_0\}$ and $\{(t, a) \in Q_\tau^3, a \leq a_0\}$. To this aim, note first that if

$(t, a) \in Q_\tau^2$ with $a \leq a_0$, then from (II.1.9) we have

$$p(t, a) = \pi(a) \left(\frac{p_0(a-t)}{\pi(a-t)} + v(t-a) \int_{a-t}^a \frac{m(s)}{\pi(s)} ds \right) \quad ((t, a) \in Q_\tau^2, a \leq a_0). \quad (\text{II.3.11})$$

Let us show that the density $p(t, a)$ is positive along each characteristic line $t - a = c$, with $c \in [\tau - a_\dagger, 0]$ and $a \leq a_0$. Note first that if $c \in [\tau - a_\dagger, 0]$, then for every $(t, a) \in Q_\tau^2$ with $a \leq a_0$ and $t - a = c$, from (II.3.11) we have

$$p(a+c, a) = \pi(a) \left(\frac{p_0(-c)}{\pi(-c)} + v(c) \int_{-c}^a \frac{m(s)}{\pi(s)} ds \right). \quad (\text{II.3.12})$$

If $v(c) \geq 0$, then from the above formula (II.3.12), the density $p(t, a)$ is clearly positive along the characteristic line $t - a = c$. If $v(c) < 0$, the control acts like a retrieval of individuals so that along the characteristic line $t - a = c$, the population density decreases and must achieve its minimum value on the red vertical line represented in Figure 1, minimum which is positive due to (II.3.7) and the fact that the solution p is continuous on Q_τ^2 . In fact, with c fixed in $[\tau - a_\dagger, 0]$, in the case where $v(c) < 0$, the quantity

$$\frac{p_0(-c)}{\pi(-c)} + v(c) \int_{-c}^a \frac{m(s)}{\pi(s)} ds$$

is a decreasing expression of a which attains its positive minimum for $a = a_0$. It follows that $p(t, a) \geq 0$ for almost every $(t, a) \in Q_\tau^2$ with $a \leq a_0$.

Similar arguments can be used to show the positivity of the solution p on $\{(t, a) \in Q_\tau^3, a \leq a_0\}$. Indeed, if $(t, a) \in Q_\tau^3$ with $a \leq a_0$, then from (II.1.9) we have

$$p(t, a) = \pi(a) \left(\tilde{b}(t-a) + v(t-a) \int_0^a \frac{m(s)}{\pi(s)} ds \right) \quad ((t, a) \in Q_\tau^3, a \leq a_0), \quad (\text{II.3.13})$$

with

$$\tilde{b}(t) = \int_{a_b}^{a_\dagger} \beta(a) p(t, a) da \quad (t \in [0, \tau]). \quad (\text{II.3.14})$$

From (II.3.4), (II.3.7), (II.3.10) and (II.3.14) it follows that we have $\tilde{b}(t) \geq 0$ for every $t \in [0, \tau]$, since the density $p(t, a)$ is positive for $a \geq a_b \geq a_0$. Distinguish again the cases $v(t-a) \geq 0$ and $v(t-a) < 0$ in (II.3.13) and using the positivity and the continuity of p along the red and green lines represented in Figure 1 provides the positivity of the controlled solution p on the subdomain $\{(t, a) \in Q_\tau^3, a \leq a_0\}$.

The case where $p_0 \in X$ and $\tilde{p} \in \mathcal{R}$ are not necessarily continuous can be deduced from a density argument. Indeed, it suffices to note that from Remark II.2.3 it follows that if $(p_{0,n})_{n \geq 1} \subset X$ and $(\tilde{p}_n)_{n \geq 1} \subset \mathcal{R}$ are such that $p_{0,n} \rightarrow p_0$ and $\tilde{p}_n \rightarrow \tilde{p}$ then the corresponding state trajectories satisfy $p_n \rightarrow p$ in $L^2(Q_\tau)$. \square

Remark II.3.3. *The control and the state trajectory constructed in the above proof can be explicitly written in terms of p_0 and \tilde{p} . Indeed, applying the formulas in Remark II.2.3,*

with $\tilde{p} - \mathbb{T}_\tau p_0$ instead of \tilde{p} and after some calculations, it can be checked that, for almost every $(t, a) \in Q_\tau$ we have

$$u(t, a) = \begin{cases} 0 & \text{if } t - a < \tau - a_\dagger, \\ \frac{\tilde{p}(\tau + (a - t)) - p_0(a - t)}{\pi(\tau + (a - t)) - \pi(a - t)} & \text{if } \tau - a_\dagger \leq t - a \leq 0, \\ \frac{\int_{a-t}^{\tau+(a-t)} \frac{m(x)}{\pi(x)} dx}{\tilde{p}(\tau - (t - a)) - \pi(\tau - (t - a))\eta(t - a)} & \text{if } 0 < t - a \leq \tau, \\ \frac{\pi(\tau - (t - a)) \int_0^{\tau-(t-a)} \frac{m(x)}{\pi(x)} dx}{\tilde{p}(\tau - (t - a)) - \pi(\tau - (t - a))\eta(t - a)} & \text{if } 0 < t - a \leq \tau, \end{cases}$$

where

$$\eta(s) = \int_{a_b}^{s+a_\dagger-\tau} \beta(x)\pi(x) \frac{\tilde{p}(\tau - (s - x))}{\pi(\tau - (s - x))} dx + \int_{s+a_\dagger-\tau}^{a_\dagger} \beta(x)\pi(x) \frac{p_0(x - s)}{\pi(x - s)} dx \quad (s \in [0, \tau]).$$

Moreover, for almost every $(t, a) \in Q_\tau$, we have

$$p(t, a) = \begin{cases} \frac{\pi(a)}{\pi(a - t)} p_0(a - t) & \text{if } t - a < \tau - a_\dagger, \\ \frac{\pi(a)}{\pi(a - t)} p_0(a - t) \left[1 - \frac{\int_{a-t}^a \frac{m(x)}{\pi(x)} dx}{\int_{a-t}^{a+(\tau-t)} \frac{m(x)}{\pi(x)} dx} \right] & \text{if } \tau - a_\dagger \leq t - a \leq 0, \\ + \pi(a) \frac{\int_{a-t}^a \frac{m(s)}{\pi(s)} ds}{\int_{a-t}^{\tau+(a-t)} \frac{m(s)}{\pi(s)} ds} \frac{\tilde{p}(\tau + (a - t))}{\pi(\tau + (a - t))} & \text{if } \tau - a_\dagger \leq t - a \leq 0, \\ \pi(a)\eta(t - a) \left[1 - \frac{\int_0^a \frac{m(s)}{\pi(s)} ds}{\int_0^{\tau-(t-a)} \frac{m(s)}{\pi(s)} ds} \right] & \text{if } 0 < t - a \leq \tau, \\ + \pi(a) \frac{\int_0^a \frac{m(x)}{\pi(x)} dx}{\int_0^{\tau-(t-a)} \frac{m(x)}{\pi(x)} dx} \frac{\tilde{p}(\tau - (t - a))}{\pi(\tau - (t - a))} & \text{if } 0 < t - a \leq \tau. \end{cases}$$

We show below that the result in Theorem II.3.2 applies for target states which are stationary solutions of (II.0.1), which is the case investigated in [BIM01].

Proof of Theorem II.0.1. According to Theorem II.3.2, we only have to show that any solution \tilde{p} of (II.0.2) satisfies $\tilde{p} \in \mathcal{R}$.

From [BIM01] it follows that any solution \tilde{p} of (II.0.2) is given by

$$\tilde{p}(a) = C\pi(a) + \int_0^a \frac{\pi(a)}{\pi(\sigma)} m(\sigma)w(\sigma) d\sigma \quad (a \in (0, a_+)), \quad (\text{II.3.15})$$

where C is determined through

$$(1 - R)C = \int_0^{a_+} \beta(x)\pi(x) \int_0^x \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma dx, \quad (\text{II.3.16})$$

and R is the reproductive number of the population given by

$$R = \int_0^{a_+} \beta(a)\pi(a) da.$$

From (II.3.15) it follows that for almost every $a \in (0, a_+)$ we have

$$\frac{\tilde{p}(a)}{\pi(a)} = C + \int_0^a \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma,$$

so that $\frac{\tilde{p}}{\pi} \in L^2[0, a_+]$. Moreover, for almost every $a \in (0, a_+)$ we have from (II.3.15) that

$$\begin{aligned} & \frac{\tilde{p}(a)}{\pi(a)} - \int_0^{a_+-a} \beta(x) \frac{\pi(x)}{\pi(a+x)} \tilde{p}(a+x) dx \\ &= C + \int_0^a \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma - \int_0^{a_+-a} \beta(x)\pi(x) \left[C + \int_0^{a+x} \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma \right] dx \\ &= \int_0^a \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma + C \left(1 - \int_0^{a_+-a} \beta(x)\pi(x) dx \right) - \int_0^{a_+-a} \beta(x)\pi(x) \int_0^{a+x} \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma dx. \end{aligned} \quad (\text{II.3.17})$$

Define $\mathcal{G} := \int_0^{a_+} \beta(x)\pi(x) \int_0^x \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma dx$ such that from (II.3.16) we have $(1 -$

$R)C = \mathcal{G}$. Define $r(a) := \int_0^{a_+-a} \beta(x)\pi(x) dx$ and $g(a) := \int_0^{a_+-a} \beta(x)\pi(x) \int_0^{a+x} \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma dx$.

Then (II.3.17) yields that for almost every $a \in (0, a_+)$, we have

$$\begin{aligned} & \frac{\tilde{p}(a)}{\pi(a)} - \int_0^{a_+-a} \beta(x) \frac{\pi(x)}{\pi(a+x)} \tilde{p}(a+x) dx \\ &= \int_0^a \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma + C(1 - r(a)) - g(a) \\ &= \int_0^a \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma + C(R - r(a)) + (\mathcal{G} - g(a)). \end{aligned} \quad (\text{II.3.18})$$

Let us show that the map

$$a \mapsto \frac{\int_0^a \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma + C(R - r(a)) + (\mathcal{G} - g(a))}{a}, \quad (\text{II.3.19})$$

is in $L^2[0, a_\dagger]$. By Hardy's inequality (see [HLP88, p 240]), the map

$$a \mapsto \frac{\int_0^a \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma}{a}$$

lies in $L^2[0, a_\dagger]$. For the second term at the numerator in (II.3.19) we note that for almost every $a \in (0, a_\dagger)$ we have

$$\begin{aligned} |R - r(a)| &= \int_{a_\dagger - a}^{a_\dagger} \beta(x) \pi(x) dx \\ &\leq a \|\beta\|_{L^\infty[0, a_\dagger]}, \end{aligned}$$

so that the map $a \mapsto \frac{R - r(a)}{a}$ is in $L^2[0, a_\dagger]$.

We still have to tackle the third term at the numerator in (II.3.19). To this aim we note that for almost every $a \in (0, a_\dagger)$ we have

$$\frac{d}{da}(\mathcal{G} - g(a)) = -\beta(a_\dagger - a) \pi(a_\dagger - a) \int_0^{a_\dagger} \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma - \int_0^{a_\dagger - a} \beta(x) \pi(x) \frac{m(a+x)}{\pi(a+x)} w(a+x) dx,$$

so that the map $a \mapsto \frac{d}{da}(\mathcal{G} - g(a))$ is in $L^2[0, a_\dagger]$. It follows that the map $F : a \mapsto \mathcal{G} - g(a)$ satisfies $F \in H^1[0, a_\dagger]$ with $F(0) = 0$, so that from Hardy's inequality (see [HLP88, p 240]) the map $a \mapsto \frac{F(a)}{a}$ is in $L^2[0, a_\dagger]$. Consequently, from (II.3.19) we have $\tilde{p} \in \mathcal{R}$. \square

Remark II.3.4. *The method employed above to prove Theorem II.0.1 cannot be adapted to the case in which the state space $X = L^2[0, a_\dagger]$ is substituted by $X = L^1[0, a_\dagger]$. Indeed, if w and \tilde{p} are in L^1 the fact that \tilde{p} is in \mathcal{R} no longer holds, in general. Indeed, the map*

$$a \mapsto \frac{\int_0^a \frac{m(\sigma)}{\pi(\sigma)} w(\sigma) d\sigma}{a}$$

does not necessarily belong to $L^1[0, a_\dagger]$ for any $w \in L^1[0, a_\dagger]$ (see Hardy's inequality [HLP88]). However, the inclusion $\text{Ran } \mathbb{T}_\tau \subset \text{Ran } \Phi_\tau$ remains true for every $\tau > a_\dagger - a_0$ - see Appendix A.

II.4 A SINGULAR PERTURBATION PROBLEM

In this section we study the behavior of the null controls constructed in Section II.3, together with the behavior of the corresponding state trajectories, when the age interval where the control is active shrinks to the singleton $\{0\}$. Not surprisingly, we show that the null controls converge to a null control for a system in which the input appears in an additive manner in the formula giving the birth rate. The control in this limit system can be seen as an “impulse” (in the age domain) type control or, simpler, as a direct control of the birth rate. Similar singular perturbation problems for the wave or for the Schrödinger equations have been studied in Fabre [FP94, Fab92], Fabre and Puel [FP93] and Joly [Jol06].

To be more precise, we consider the control system (II.0.1) for $a_0 = \varepsilon \in (0, a_\dagger)$. According to Proposition II.3.1, given $\varepsilon > 0$, for every $p_0 \in X = L^2[0, a_\dagger]$ there exists $u^\varepsilon \in L^2([0, a_\dagger]; X)$ such that the solution of

$$\begin{cases} \dot{p}^\varepsilon(t, a) + \frac{\partial p^\varepsilon}{\partial a}(t, a) + \mu(a)p^\varepsilon(t, a) = \mathbb{1}_{[0, \varepsilon]}(a)u^\varepsilon(t, a), & (t, a) \in (0, a_\dagger)^2, \\ p^\varepsilon(t, 0) = \int_0^{a_\dagger} \beta(a)p^\varepsilon(t, a) da, & t \in (0, a_\dagger), \\ p^\varepsilon(0, a) = p_0(a), & a \in (0, a_\dagger), \end{cases} \quad (\text{II.4.1})$$

satisfies

$$p^\varepsilon(a_\dagger, a) = 0 \quad (a \in (0, a_\dagger)). \quad (\text{II.4.2})$$

Moreover, by a slight variation of the calculations used to derive the formula in Remark II.3.3, it is easy to check that the control u^ε defined by

$$u^\varepsilon(t, a) = \begin{cases} -\frac{\int_{t-a}^{a_\dagger} \beta(x)\pi(x) \frac{p_0(x-(t-a))}{\pi(x-(t-a))} dx}{\int_0^{a_\dagger-(t-a)} \frac{\mathbb{1}_{[0, \varepsilon]}(x)}{\pi(x)} dx} & ((t, a) \in (0, a_\dagger)^2, t-a \geq 0), \\ 0 & ((t, a) \in (0, a_\dagger)^2, t-a < 0), \end{cases} \quad (\text{II.4.3})$$

satisfies the above conditions. With u_ε defined by (II.4.3), the controlled density p^ε solution of (II.4.1) is given by

$$p^\varepsilon(t, a) = \begin{cases} \frac{\pi(a)}{\pi(a-t)} p_0(a-t) & \text{if } t-a < 0, \\ \pi(a)\eta(t-a) \left[1 - \frac{\int_0^a \frac{\mathbb{1}_{[0, \varepsilon]}(x)}{\pi(x)} dx}{\int_0^{a_\dagger-(t-a)} \frac{\mathbb{1}_{[0, \varepsilon]}(x)}{\pi(x)} dx} \right] & \text{if } 0 < t-a \leq a_\dagger, \end{cases} \quad (\text{II.4.4})$$

where

$$\eta(s) = \int_s^{a_\dagger} \beta(x)\pi(x)\frac{p_0(x-s)}{\pi(x-s)} dx \quad (s \in [0, a_\dagger]).$$

We also consider the following boundary controlled system, already studied in [BIM01]:

$$\begin{cases} \dot{p}(t, a) + \frac{\partial p}{\partial a}(t, a) + \mu(a)p(t, a) = 0, & (t, a) \in (0, a_\dagger)^2, \\ p(t, 0) = \int_0^{a_\dagger} \beta(a)p(t, a) da + v(t), & t \in (0, a_\dagger), \\ p(0, a) = p_0(a), & a \in (0, a_\dagger), \end{cases} \quad (\text{II.4.5})$$

where $v \in L^2([0, a_\dagger])$. Using a recursive argument as in the proof of Theorem II.2.2, it can be easily shown that taking

$$v(t) := - \int_t^{a_\dagger} \beta(a)\frac{\pi(a)}{\pi(a-t)}p_0(a-t) da \quad (t \in [0, a_\dagger]), \quad (\text{II.4.6})$$

the solution p of (II.4.5) satisfies

$$p(a_\dagger, a) = 0 \quad (a \in (0, a_\dagger)). \quad (\text{II.4.7})$$

More precisely, with v defined by (II.4.6), the controlled density p solution of (II.4.5) is given by

$$p(t, a) = \begin{cases} \frac{\pi(a)}{\pi(a-t)}p_0(a-t) & \text{if } t \leq a, \\ 0 & \text{if } t > a. \end{cases} \quad (\text{II.4.8})$$

The main result of this section is:

Theorem II.4.1. *Given $p_0 \in L^2([0, a_\dagger])$, consider the family of distributed controls $(u^\varepsilon)_{\varepsilon \in (0, a_\dagger)}$ in $L^2([0, a_\dagger]; L^2([0, a_\dagger]))$ and the boundary control v in $L^2[0, a_\dagger]$ respectively defined by (II.4.3) and (II.4.6). Then we have*

1. $\lim_{\varepsilon \rightarrow 0^+} u^\varepsilon \mathbb{1}_{[0, \varepsilon]} = v\delta_0$ weakly in $L^2([0, a_\dagger], \mathcal{D}(A^*)')$, where δ_0 stands for the Dirac mass concentrated at the origin;

2.

$$\lim_{\varepsilon \rightarrow 0^+} \|p^\varepsilon - p\|_{C([0, a_\dagger]; L^2([0, a_\dagger]))} = 0,$$

where p is the solution of (II.4.5).

Proof. Let $\varepsilon \in (0, a_\dagger/2)$. It is easy to check that every function in $\mathcal{D}(A^*)$ belongs to $H^1(0, a^*)$ for every $a^* \in (0, a_\dagger)$. In particular, for every $\varphi \in L^2([0, a_\dagger], \mathcal{D}(A^*))$ and every $t \in [0, a_\dagger]$, we have $\varphi(t, \cdot) \in L^\infty(0, a_\dagger/2)$.

We introduce the following notations:

$$f(t, a) := \begin{cases} - \int_{t-a}^{a_{\dagger}} \beta(x) \pi(x) \frac{p_0(x - (t - a))}{\pi(x - (t - a))} dx & ((t, a) \in (0, a_{\dagger})^2, t - a \geq 0), \\ 0 & ((t, a) \in (0, a_{\dagger})^2, t - a < 0), \end{cases} \quad (\text{II.4.9})$$

and

$$k^{\varepsilon}(t, a) := \int_0^{a_{\dagger} - (t - a)} \frac{\mathbb{1}_{[0, \varepsilon]}(x)}{\pi(x)} dx \quad ((t, a) \in (0, a_{\dagger})^2), \quad (\text{II.4.10})$$

so that from (II.4.3), (II.4.9) and (II.4.10) we have

$$u^{\varepsilon}(t, a) = \frac{f(t, a)}{k^{\varepsilon}(t, a)} \quad ((t, a) \in (0, a_{\dagger})^2).$$

Let $\varphi \in L^2([0, a_{\dagger}], \mathcal{D}(A^*))$, and denote by

$$g_{\varepsilon}(t) := \int_0^{a_{\dagger}} \mathbb{1}_{[0, \varepsilon]}(a) u^{\varepsilon}(t, a) \varphi(t, a) da = \int_0^{\varepsilon} \frac{f(t, a)}{k^{\varepsilon}(t, a)} \varphi(t, a) da \quad (t \in [0, a_{\dagger}]). \quad (\text{II.4.11})$$

The aim is to show that $\lim_{\varepsilon \rightarrow 0} g_{\varepsilon} = v\varphi(\cdot, 0)$ in $L^2[0, a_{\dagger}]$. To this aim, we use the Lebesgue's Dominated Convergence Theorem. Firstly, let us show that for every $t \in [0, a_{\dagger}]$ we have the following pointwise convergence:

$$\lim_{\varepsilon \rightarrow 0^+} g_{\varepsilon}(t) = v(t)\varphi(t, 0) \quad (t \in [0, a_{\dagger}]). \quad (\text{II.4.12})$$

For every $t \in [0, a_{\dagger})$, take $\varepsilon < a_{\dagger} - t$ small enough so that $\varepsilon < a_{\dagger} - (t - a)$ for every $a \in (0, \varepsilon)$. It follows from (II.4.10) that we have

$$k^{\varepsilon}(t, a) = \int_0^{\varepsilon} \frac{dx}{\pi(x)} \quad (a \in (0, \varepsilon)), \quad (\text{II.4.13})$$

so that from (II.4.11) and (II.4.13) we have

$$g_{\varepsilon}(t) = \frac{1}{\int_0^{\varepsilon} \frac{1}{\pi(x)} dx} \int_0^{\varepsilon} f(t, a) \varphi(t, a) da,$$

and, using the last formula and the fact that $f(t, 0) = v(t)$, we have

$$\lim_{\varepsilon \rightarrow 0} g_{\varepsilon}(t) = v(t)\varphi(t, 0) \quad (t \in [0, a_{\dagger}]). \quad (\text{II.4.14})$$

For $t = a_{\dagger}$, it follows from (II.4.10) that we have

$$k^{\varepsilon}(a_{\dagger}, a) = \int_0^a \frac{dx}{\pi(x)} \quad (a \in (0, \varepsilon)), \quad (\text{II.4.15})$$

so that from (II.4.11) and (II.4.15) we have

$$g_\varepsilon(a_\dagger) = \int_0^\varepsilon \frac{f(a_\dagger, a)}{\int_0^a \frac{1}{\pi(x)} dx} \varphi(a_\dagger, a) da,$$

and, using the above formula, together with the fact that $v(a_\dagger) = 0$, we obtain that

$$\lim_{\varepsilon \rightarrow 0} g_\varepsilon(a_\dagger) = v(a_\dagger) \varphi(a_\dagger, 0), \quad (\text{II.4.16})$$

Consequently, it follows from (II.4.14) and (II.4.16) that the pointwise convergence (II.4.12) holds for every $t \in [0, a_\dagger]$.

Secondly, let us find a function $h \in L^2[0, a_\dagger]$ such that

$$|g_\varepsilon(t)| \leq h(t) \quad (\varepsilon \in (0, a_\dagger/2), t \in [0, a_\dagger]). \quad (\text{II.4.17})$$

It is easy to check that for every $t \in [0, a_\dagger - \varepsilon]$ we have

$$k^\varepsilon(t, a) = \int_0^\varepsilon \frac{dx}{\pi(x)} \quad \text{if } a \in (0, \varepsilon), \quad (\text{II.4.18})$$

and for every $t \in [a_\dagger - \varepsilon, a_\dagger]$ we have

$$k^\varepsilon(t, a) = \begin{cases} \int_0^{a_\dagger - (t-a)} \frac{dx}{\pi(x)} & \text{if } a \in (0, \varepsilon + t - a_\dagger), \\ \int_0^\varepsilon \frac{dx}{\pi(x)} & \text{if } a \in (\varepsilon + t - a_\dagger, \varepsilon). \end{cases} \quad (\text{II.4.19})$$

It follows from (II.4.11), (II.4.18) and (II.4.19) that we have

$$g_\varepsilon(t) = \begin{cases} \frac{1}{\int_0^\varepsilon \frac{1}{\pi(x)} dx} \int_0^\varepsilon f(t, a) \varphi(t, a) da, & (t \in [0, a_\dagger - \varepsilon]), \\ \int_0^{\varepsilon + t - a_\dagger} \frac{f(t, a) \varphi(t, a)}{\int_0^{a_\dagger - (t-a)} \frac{1}{\pi(x)} dx} da + \frac{\int_{\varepsilon + t - a_\dagger}^\varepsilon f(t, a) \varphi(t, a) da}{\int_0^\varepsilon \frac{1}{\pi(x)} dx}, & (t \in [a_\dagger - \varepsilon, a_\dagger]). \end{cases} \quad (\text{II.4.20})$$

For every $t \in [0, a_\dagger - \varepsilon]$, since $f \in L^\infty([0, a_\dagger]^2)$ and $\varphi(t, \cdot) \in L^\infty([0, a_\dagger/2])$, from (II.4.20) we get

$$|g_\varepsilon(t)| \leq \|f\|_{L^\infty([0, a_\dagger]^2)} \|\varphi(t, \cdot)\|_{L^\infty([0, a_\dagger/2])}. \quad (\text{II.4.21})$$

For every $t \in [a_{\dagger} - \varepsilon, a_{\dagger}]$, from (II.4.20) we have

$$|g_{\varepsilon}(t)| \leq \int_0^{\varepsilon} \frac{f(t, a)\varphi(t, a)}{a_{\dagger} - (t - a)} da + \frac{1}{\varepsilon} \int_0^{\varepsilon} f(t, a)\varphi(t, a) da. \quad (\text{II.4.22})$$

Since $p_0 \in L^2[0, a_{\dagger}]$, Cauchy-Schwarz inequality gives

$$|f(t, a)| \leq K \|p_0\|_{L^2(0, a_{\dagger})} \sqrt{a_{\dagger} - (t - a)}, \quad (\text{II.4.23})$$

for some constant $K \geq 0$. It follows from (II.4.22) and (II.4.23) that we have

$$|g_{\varepsilon}(t)| \leq K \|p_0\|_{L^2(0, a_{\dagger})} \|\varphi(t, \cdot)\|_{L^{\infty}([0, a_{\dagger}/2])} \int_0^{a_{\dagger}/2} \frac{da}{\sqrt{a_{\dagger} - (t - a)}} + \|f\|_{L^{\infty}([0, a_{\dagger}]^2)} \|\varphi(t, \cdot)\|_{L^{\infty}([0, a_{\dagger}/2])}, \quad (\text{II.4.24})$$

and the following majoration holds:

$$|g_{\varepsilon}(t)| \leq 2K \|p_0\|_{L^2(0, a_{\dagger})} \|\varphi(t, \cdot)\|_{L^{\infty}([0, a_{\dagger}/2])} \sqrt{3a_{\dagger}/2} + \|f\|_{L^{\infty}([0, a_{\dagger}]^2)} \|\varphi(t, \cdot)\|_{L^{\infty}([0, a_{\dagger}/2])}. \quad (\text{II.4.25})$$

Define, for every $t \in [0, a_{\dagger}]$,

$$h(t) := 2K \|p_0\|_{L^2(0, a_{\dagger})} \|\varphi(t, \cdot)\|_{L^{\infty}([0, a_{\dagger}/2])} \sqrt{3a_{\dagger}/2} + \|f\|_{L^{\infty}([0, a_{\dagger}]^2)} \|\varphi(t, \cdot)\|_{L^{\infty}([0, a_{\dagger}/2])}. \quad (\text{II.4.26})$$

It follows from (II.4.21) and (II.4.25) that for every $\varepsilon \in (0, a_{\dagger}/2)$ and every $t \in [0, a_{\dagger}]$, we have

$$|g_{\varepsilon}(t)| \leq h(t) \quad (\varepsilon \in (0, a_{\dagger}/2), t \in [0, a_{\dagger}]), \quad (\text{II.4.27})$$

with $h \in L^2([0, a_{\dagger}])$. The pointwise convergence (II.4.12) and the above domination (II.4.27) gives

$$\lim_{\varepsilon \rightarrow 0} g_{\varepsilon} = v\varphi(\cdot, 0) \text{ in } L^2[0, a_{\dagger}], \quad (\text{II.4.28})$$

so that we have

$$\lim_{\varepsilon \rightarrow 0^+} u^{\varepsilon} \mathbb{1}_{[0, \varepsilon]} = v\delta_0 \text{ weakly in } L^2([0, a_{\dagger}], \mathcal{D}(A^*)'). \quad (\text{II.4.29})$$

Concerning the convergence of the state trajectories, it follows from (II.4.4) and (II.4.8) that for every $t \in [0, a_{\dagger}]$,

$$\begin{aligned} \int_0^{a_{\dagger}} |p^{\varepsilon}(t, a) - p(t, a)|^2 da &\leq \|\eta\|_{L^{\infty}[0, a_{\dagger}]}^2 \int_0^{\varepsilon} \left(1 - \frac{\int_0^a \frac{\mathbb{1}_{[0, \varepsilon]}(x)}{\pi(x)} dx}{\int_0^{a_{\dagger} - (t - a)} \frac{\mathbb{1}_{[0, \varepsilon]}(x)}{\pi(x)} dx} \right)^2 da \\ &\leq \|\eta\|_{L^{\infty}[0, a_{\dagger}]}^2 \varepsilon. \end{aligned} \quad (\text{II.4.30})$$

The convergence stated in Theorem (II.4.1) easily follows from (II.4.30). □

II.5 CONTROL OF AN INFECTION-AGE MODEL

The methodology developed in the previous sections is essentially applicable to linear systems. However, the fact that controls are explicitly determined by calculations along characteristics, allows, in some particular, situations to tackle nonlinear systems. The aim of this section is thus to provide an example of application of our methods for a nonlinear controlled infection-age model which writes:

$$\begin{cases} \frac{dS}{dt}(t) = -S(t) \int_0^{a_{\dagger}} \beta(a)i(t, a) da, \\ \frac{\partial i}{\partial t}(t, a) + \frac{\partial i}{\partial a}(t, a) = -\mu(a)i(t, a) - m(a)u(t, a), \\ i(t, 0) = S(t) \int_0^{a_{\dagger}} \beta(a)i(t, a) da, \end{cases} \quad (\text{II.5.1})$$

with the initial condition

$$S(0) = S_0 \geq 0 \text{ and } i(0, \cdot) = i_0 \in L^2_+[0, a_{\dagger}].$$

In the model (II.5.1), the population is decomposed into the class (S) of susceptible individuals and the class (I) of infected individuals. The number of individuals in the class (S) at time t is $S(t)$. The age of infection $a \in [0, a_{\dagger}]$ is the time since the infection began (where a_{\dagger} denotes the maximum age of the epidemic), and $i(t, a)$ is the density of infected individuals with respect to the age infection. That is to say that for two given age values $a_1, a_2 : 0 \leq a_1 < a_2 \leq a_{\dagger}$ the number of infected individuals with age of infection a between a_1 and a_2 is

$$\int_{a_1}^{a_2} i(t, a) da.$$

The function $\beta(a)$ is the force of infection at the age of infection a . We assume that there exists $a_b \in (0, a_{\dagger})$ such that

$$\beta(a) = 0 \quad (a \in (0, a_b) \text{ a.e.}), \quad (\text{II.5.2})$$

that is to say we interpret infection age to an exposed period (infected, but not yet infectious) from $a = 0$ to $a = a_b$ and an infectious period from $a = a_b$ to $a = a_{\dagger}$. The quantity

$$\int_0^{a_{\dagger}} \beta(a)i(t, a) da$$

is the number of infectious individuals within the subpopulation (I). Finally, $\mu(a)$ is the exit (or (and) mortality or (and) recovery) rate of infected individuals with an age of infection $a \geq 0$. As a consequence the quantity

$$\pi(a) := \exp\left(-\int_0^a \mu(\sigma) d\sigma\right)$$

is the probability for an individual to stay in the class (I) after a period of time $a \geq 0$.

System (II.5.1) can be seen as a semilinear Cauchy problem in an appropriate Banach space, in which the closed linear operator is not densely defined (see, for instance, Magal and Ruan [MR07, MR09, MR17]). In the above references, the existence and the uniqueness of solutions, for a given L^2 input function u , is proved by using a step by step procedure which preserves the positivity of solutions. The global in time existence and uniqueness follows since no finite time blowup occurs. Here, we aim to find a control function u such as the density of infected individuals in the age-infection model (II.5.1) becomes null in finite time. We suppose that $m = \mathbb{1}_{[0, a_0]}$ is the characteristic function of the interval $(0, a_0)$, with $a_0 \leq a_b$: this means that we are able to control infected individuals that are still not infectious. The below proposition show that by controlling the infected individuals (i.e. by using quarantine) before their infectiousness period we can eradicate the epidemic in finite time.

Proposition II.5.1. *Let assumption (II.5.2) be satisfied. Let $i_0 \in L^2[0, a_\dagger]$ with $i_0(a) \geq 0$ for almost every $a \in (0, a_\dagger)$ and let $S_0 \geq 0$. Then for every $\tau > a_\dagger - a_0$, there exists a control $u \in L^2([0, \tau]; L^2[0, a_\dagger])$ such that the controlled density of infected individuals i given by (II.5.1) satisfies*

$$i(\tau, a) = 0 \quad (a \in (0, a_\dagger) \text{ a.e.}).$$

Moreover the controlled density of infected individuals remains positive that is to say

$$i(t, a) \geq 0 \text{ a.e. } (t, a) \in Q_\tau.$$

By a slight variation of the calculations in Section II.1 we can check the solution of (II.5.1) satisfies:

$$S(t) = S_0 \exp\left(-\int_0^t \int_0^{a_\dagger} \beta(a) i(s, a) da ds\right), \quad (\text{II.5.3})$$

and

$$i(t, a) := \begin{cases} \frac{\pi(a)}{\pi(a-t)} i_0(a-t) + c(t, a), & t \leq a, \\ \pi(a) \tilde{b}(t-a) + c(t, a), & a < t, \end{cases} \quad (\text{II.5.4})$$

where

$$c(t, a) := \begin{cases} \int_{a-t}^a \frac{\pi(a)}{\pi(s)} m(s) u(s - (a - t), s) ds, & t \leq a, \\ \int_0^a \frac{\pi(a)}{\pi(s)} m(s) u(s + (t - a), s) ds, & a < t, \end{cases} \quad (\text{II.5.5})$$

and $t \mapsto \tilde{b}(t)$ is the unique continuous function satisfying the Volterra integral equation

$$\tilde{b}(t) = S(t) \left\{ \int_0^{\min(t, a_\dagger)} \beta(a) \left[\pi(a) \tilde{b}(t - a) + c(t, a) \right] da + \int_{\min(t, a_\dagger)}^{a_\dagger} \beta(a) \left[\frac{\pi(a)}{\pi(a - t)} i_0(a - t) + c(t, a) \right] da \right\}. \quad (\text{II.5.6})$$

Proof of Proposition II.5.1. The remark suggesting the approach below is that, assuming that S is a given function, the construction of the control in Theorem II.3.2 and, more precisely, in Remark II.3.3, adapts to the controllability of the system formed by the last two equations in (II.5.1). This motivates the below construction of the control function for the system (II.5.1).

As previously, we focus on the case where $\tau \in]a_\dagger - a_0, a_\dagger[$.

Case $t - a < 0$: We first set

$$u(t, a) = \begin{cases} 0 & \text{if } t - a < \tau - a_\dagger, \\ \frac{i_0(a - t)}{\pi(a - t)} & \text{if } \tau - a_\dagger \leq t - a \leq 0, \\ \int_{a-t}^{\tau+(a-t)} \frac{m(x)}{\pi(x)} dx & \end{cases} \quad (\text{II.5.7})$$

so that from (II.5.4) and (II.5.7) we have

$$i(t, a) = \begin{cases} \frac{\pi(a)}{\pi(a - t)} i_0(a - t) & \text{if } t - a < \tau - a_\dagger, \\ \frac{\pi(a)}{\pi(a - t)} i_0(a - t) \left[1 - \frac{\int_{a-t}^a \frac{m(x)}{\pi(x)} dx}{\int_{a-t}^{a+(\tau-t)} \frac{m(x)}{\pi(x)} dx} \right] & \text{if } \tau - a_\dagger \leq t - a \leq 0. \end{cases} \quad (\text{II.5.8})$$

The above formula (II.5.8) clearly implies that

$$i(\tau, a) = 0 \quad (a \in (\tau, a_\dagger) \text{ a.e.}). \quad (\text{II.5.9})$$

Case $0 < t - a < a_b$: using (II.5.8), we are able to compute $S(t)$ for every $t \in [0, a_b]$.

Indeed, after an easy calculation using (II.5.8) it follows that for every $t \in [0, a_b]$,

$$\int_0^{a_\dagger} \beta(a)i(t, a) da = \int_{a_b}^{a_\dagger} \beta(a)i(t, a) da = \int_{t+a_\dagger-\tau}^{a_\dagger} \beta(a) \frac{\pi(a)}{\pi(a-t)} i_0(a-t) da. \quad (\text{II.5.10})$$

In the sequel, we set

$$f(t) := \int_{t+a_\dagger-\tau}^{a_\dagger} \beta(a) \frac{\pi(a)}{\pi(a-t)} i_0(a-t) da \quad (t \in [0, \tau]), \quad (\text{II.5.11})$$

so that using (II.5.3), (II.5.10) and (II.5.11) we have

$$S(t) = S_0 e^{-\int_0^t f(s) ds} \quad (t \in [0, a_b]). \quad (\text{II.5.12})$$

Then for every $t \in [0, a_b]$, using (II.5.2), (II.5.6), (II.5.10) and (II.5.12) we have

$$\tilde{b}(t) = S(t) \int_{a_b}^{a_\dagger} \beta(a)i(t, a) da = S_0 f(t) e^{-\int_0^t f(s) ds} \quad (t \in [0, a_b]). \quad (\text{II.5.13})$$

Denoting by

$$\chi(t) := S_0 f(t) e^{-\int_0^t f(s) ds} \quad (t \in [0, \tau]), \quad (\text{II.5.14})$$

it follows from (II.5.13) and (II.5.14) that we have $\tilde{b}(t) = \chi(t)$ for every $t \in [0, a_b]$. Then we set for almost every $(t, a) \in Q_\tau$ with $0 < t - a \leq a_b$:

$$u(t, a) = \frac{\chi(t-a)}{\int_0^{\tau-(t-a)} \frac{m(x)}{\pi(x)} dx}, \quad (\text{II.5.15})$$

so that from (II.5.4) we have

$$i(t, a) = \pi(a) \chi(t-a) \left[1 - \frac{\int_0^a \frac{m(s)}{\pi(s)} ds}{\int_0^{\tau-(t-a)} \frac{m(s)}{\pi(s)} ds} \right] \text{ if } 0 < t - a \leq a_b. \quad (\text{II.5.16})$$

It follows from (II.5.16) that we have

$$i(\tau, a) = 0 \quad (a \in (\tau - a_b, \tau) \text{ a.e.}). \quad (\text{II.5.17})$$

Case $ja_b < t - a < (j-1)a_b$ for some integer $j \geq 1$: The above calculations suggest as possible control driving the initial state to $u(t, a)$ where

$$u(t, a) = \begin{cases} 0 & \text{if } t - a < \tau - a_{\dagger}, \\ \frac{i_0(a-t)}{\pi(a-t)} & \text{if } \tau - a_{\dagger} \leq t - a \leq 0, \\ \frac{\int_{a-t}^{\tau+(a-t)} \frac{m(x)}{\pi(x)} dx}{\int_{a-t}^{\tau+(a-t)} \frac{m(x)}{\pi(x)} dx} & \text{if } \tau - a_{\dagger} \leq t - a \leq 0, \\ \frac{\chi(t-a)}{\int_0^{\tau-(t-a)} \frac{m(x)}{\pi(x)} dx} & \text{if } 0 < t - a \leq \tau, \\ \frac{\int_0^{\tau-(t-a)} \frac{m(x)}{\pi(x)} dx}{\int_0^{\tau-(t-a)} \frac{m(x)}{\pi(x)} dx} & \text{if } 0 < t - a \leq \tau, \end{cases} \quad (\text{II.5.18})$$

where χ is defined by (II.5.14). With u defined by (II.5.18), it can be checked using a recursive argument as in the proof of Theorem II.2.2 that we have

$$S(t) = S_0 e^{-\int_0^t f(s) ds} \quad (t \in [ja_b, (j+1)a_b]), \quad (\text{II.5.19})$$

which implies that

$$\tilde{b}(t) = \chi(t) \quad (t \in [ja_b, (j+1)a_b]), \quad (\text{II.5.20})$$

so that using (II.5.4), (II.5.18) and (II.5.20) we get

$$i(t, a) = \pi(a)\chi(t-a) \left[1 - \frac{\int_0^a \frac{m(s)}{\pi(s)} ds}{\int_0^{\tau-(t-a)} \frac{m(s)}{\pi(s)} ds} \right] \text{ if } ja_b < t - a \leq (j+1)a_b, \quad (\text{II.5.21})$$

and it follows from the above formula (II.5.21) that we have

$$i(\tau, a) = 0 \quad (a \in [\tau - (j+1)a_b, \tau - ja_b] \text{ a.e.}). \quad (\text{II.5.22})$$

Finally, with u defined by (II.5.18) we have

$$i(t, a) = \begin{cases} \frac{\pi(a)}{\pi(a-t)} i_0(a-t) & \text{if } t - a < \tau - a_{\dagger}, \\ \frac{\pi(a)}{\pi(a-t)} i_0(a-t) \left[1 - \frac{\int_{a-t}^a \frac{m(x)}{\pi(x)} dx}{\int_{a-t}^{a+(\tau-t)} \frac{m(x)}{\pi(x)} dx} \right] & \text{if } \tau - a_{\dagger} \leq t - a \leq 0, \\ \pi(a)\chi(t-a) \left[1 - \frac{\int_0^a \frac{m(s)}{\pi(s)} ds}{\int_0^{\tau-(t-a)} \frac{m(s)}{\pi(s)} ds} \right] & \text{if } 0 < t - a \leq \tau, \end{cases}$$

where we recall that we have defined for every $t \in [0, \tau]$:

$$\chi(t) := S_0 f(t) e^{-\int_0^t f(s) ds},$$

so that the positivity of the controlled density i is clear. Moreover, it can be checked that we have $u \in L^2([0, \tau]; L^2[0, a_+])$ thanks to Hardy's inequality [HLP88]. \square

II.6 ABOUT THE POSITIVITY OF THE POPULATION ASSOCIATED TO THE HUM CONTROL

This section is devoted to investigate the norm optimal control driving an initial population density p_0 to zero in time τ . Let us recall that the null controllability of equations (II.0.1) in any time $\tau > a_+ - a_0$ has been proved in Section II.2. The main goal of this present Section is to investigate and build null controls in time τ with minimal L^2 -norm, and to study the positivity of the corresponding controlled population solution to (II.0.1).

For every $p_0 \in L^2[0, a_+]$ and every $\tau > 0$, let us introduce the (possibly empty) closed convex set constituted by the null control functions from p_0 in time τ :

$$\text{Adm}(p_0, \tau) := \{u \in L^2(Q_\tau) \mid \Phi_\tau u = -\mathbb{T}_\tau p_0\}.$$

In the case when the set $\text{Adm}(p_0, \tau)$ is not empty (for instance, in the case when $\tau > a_+ - a_0$ for every $p_0 \in L^2(0, a_+)$), we introduce the HUM control u_* as defined by

$$u_* = \text{Argmin}\{\|u\|_{L^2(Q_\tau)} \mid u \in \text{Adm}(p_0, \tau)\},$$

which has the minimal L^2 -norm among all the null controls in time τ .

This section is divided in two main parts.

- A first subsection investigates the case when $\tau = a_+ - a_0$ (the "critical null control time"), with an additional assumption on the initial population density p_0 which ensures that the set $\text{Adm}(p_0, \tau)$ is not empty. In this case, we shall give an explicit expression for the HUM control and we shall check that the associated population remains nonnegative when the initial population density is nonnegative.
- A second subsection investigates the case when $\tau > a_+ - a_0$ (large control times). In these cases, an explicit expression of the norm optimal control may not be obtained, due to the complexity of the constraints inherent to the set $\text{Adm}(p_0, \tau)$. A numerical approach is developed here, using a penalised HUM method as explained in Boyer [Boy13]. The numerical experiments suggest that the nonnegativity of the state trajectory may not be preserved in the case when $\tau > a_+ - a_0$.

§ II.6.1. An explicit norm optimal control for the critical null control time

In this subection, we prove that an explicit HUM control may be obtained in the case when $\tau = a_{\dagger} - a_0$, with an additional assumption on the initial populaiton density p_0 which ensures that the set $\text{Adm}(p_0, \tau)$ is not empty.

Proposition II.6.1. *With the above notations and assumptions, assume that there exists $a_b > 0$ such that (II.0.4) is satisfied, and let $\tau = a_{\dagger} - a_0$ (with $a_0 \leq a_b$). Let $p_0 \in L^2[0, a_{\dagger}]$ be such that :*

$$\int_{\tau}^{a_{\dagger}} \frac{p_0^2(a - \tau)}{\int_{a-\tau}^a \frac{m(x)}{\pi^2(x)} dx} da < +\infty. \quad (\text{II.6.1})$$

Then, the set $\text{Adm}(p_0, \tau)$ is not empty. Moreover, assuming that p_0 satisfies the above condition (II.6.1) with $p_0(a) \geq 0$ for almost every $a \in (0, a_{\dagger})$, the controlled population associated to the HUM control satisfies

$$p(t, a) \geq 0 \quad \text{a.e. } (t, a) \in Q_{\tau}.$$

Proof. Assume that there exists $a_b > 0$ such that (II.0.4) is satisfied, and let $\tau = a_{\dagger} - a_0$ with $a_0 \leq a_b$. Let $p_0 \in L^2[0, a_{\dagger}]$ be such that the condition (II.6.1) is satisfied. For almost every $(t, a) \in Q_{\tau}$, let us define

$$u_*(t, a) := \begin{cases} 0 & \text{if } t - a < \tau - a_{\dagger}, \\ -m(a) \frac{\frac{p_0(a-t)}{\pi(a-t)}}{\pi(a) \int_{a-t}^{\tau+(a-t)} \frac{m(x)}{\pi^2(x)} dx} & \text{if } \tau - a_{\dagger} \leq t - a \leq 0, \\ -m(a) \frac{\frac{\eta(t-a)}{\pi(a)}}{\pi(a) \int_0^{\tau-(t-a)} \frac{m(x)}{\pi^2(x)} dx} & \text{if } 0 < t - a \leq \tau, \end{cases} \quad (\text{II.6.2})$$

where

$$\eta(s) = \int_{s+a_{\dagger}-\tau}^{a_{\dagger}} \beta(x) \pi(x) \frac{p_0(x-s)}{\pi(x-s)} dx \quad (s \in [0, \tau]).$$

First, let us show that the condition (II.6.1) ensures that the function u^* defined as above satisfies $u^* \in L^2(Q_{\tau})$. To this aim, we shall compute the $L^2(Q_{\tau})$ - norm of u^* by integrating along the characteristic lines of the rectangle

$$\Delta = \{(t, a) \in Q_{\tau} \mid 0 \leq a \leq a_0\},$$

using the following formulation of the scalar product in $L^2(\Delta)$:

$$(u, v)_{L^2(\Delta)} = \int_{\tau}^{a_{\dagger}} \int_{a-\tau}^a m(s)u(s - (a - \tau), s)v(s - (a - \tau), s) ds da + \int_0^{\tau} \int_0^a m(s)u(s + (\tau - a), s)v(s + (\tau - a), s) ds da, \quad (\text{II.6.3})$$

for every $u, v \in L^2(\Delta)$. Using (II.6.2) and (II.6.3) we get that

$$\|u_*\|_{L^2(Q_{\tau})}^2 = \int_{\tau}^{a_{\dagger}} \int_{a-\tau}^a m(s) \left(\frac{\frac{p_0(a-\tau)}{\pi(a-\tau)}}{\pi(s) \int_{a-\tau}^a \frac{m(x)}{\pi^2(x)} dx} \right)^2 ds da + \int_0^{\tau} \int_0^a m(s) \left(\frac{\eta(\tau-a)}{\pi(s) \int_0^a \frac{m(x)}{\pi^2(x)} dx} \right)^2 ds da,$$

so that we have

$$\|u_*\|_{L^2(Q_{\tau})}^2 = \int_{\tau}^{a_{\dagger}} \frac{\frac{p_0^2(a-\tau)}{\pi^2(a-\tau)}}{\int_{a-\tau}^a \frac{m(x)}{\pi^2(x)} dx} da + \int_0^{\tau} \frac{\eta^2(\tau-a)}{\int_0^a \frac{m(x)}{\pi^2(x)} dx} da. \quad (\text{II.6.4})$$

Both integrals in the right side of (II.6.4) are convergent, respectively due to the condition (II.6.1) and Hardy's inequality [HLP88].

Moreover, it may be shown by induction that the solution p^* to (II.0.1) associated to the control function u^* satisfies, for almost every $(t, a) \in Q_{\tau}$,

$$p_*(t, a) = \begin{cases} \frac{\pi(a)}{\pi(a-t)} p_0(a-t) & \text{if } t-a < \tau - a_{\dagger}, \\ \frac{\pi(a)}{\pi(a-t)} p_0(a-t) \left[1 - \frac{\int_{a-t}^a \frac{m(x)}{\pi^2(x)} dx}{\int_{a-t}^{\tau+a-t} \frac{m(x)}{\pi^2(x)} dx} \right] & \text{if } \tau - a_{\dagger} \leq t-a \leq 0, \\ \pi(a)\eta(t-a) \left[1 - \frac{\int_0^a \frac{m(s)}{\pi^2(s)} ds}{\int_0^{\tau-(t-a)} \frac{m(s)}{\pi^2(s)} ds} \right] & \text{if } 0 < t-a \leq \tau, \end{cases} \quad (\text{II.6.5})$$

and it follows that $p^*(\tau, \cdot) = 0$ in $L^2[0, a_{\dagger}]$, so that we have $u^* \in \text{Adm}(p_0, \tau)$.

Now, assume that $u \in \text{Adm}(p_0, \tau)$. The corresponding solution p to (II.0.1) satisfies

$p(\tau, a) = 0$ for almost every $a \in (0, a_\dagger)$. Using (II.1.6) and (II.1.7), it follows that for almost every $a \in (\tau, a_\dagger)$ we have

$$\int_{a-\tau}^a \frac{m(s)}{\pi(s)} u(s - (a - \tau), s) ds = -\frac{p_0(a - \tau)}{\pi(a - \tau)} \quad (a \in (\tau, a_\dagger) \text{ a.e.}). \quad (\text{II.6.6})$$

Moreover, since we have $p(\tau, a) = 0$ for almost every $a \in (a_0, a_\dagger)$, by integrating along the characteristic lines we get that for almost every $(t, a) \in Q_\tau$ with $a \geq a_0$ and $a - t \geq a_\dagger - \tau$ we have

$$p(t, a) = 0 \quad ((t, a) \in Q_\tau, a \geq a_0, a - t \geq a_\dagger - \tau). \quad (\text{II.6.7})$$

From the above equality, and using that $a_0 \leq a_b$, it follows that the total birth rate \tilde{b} satisfies, for almost every $t \in (0, \tau)$,

$$\tilde{b}(t) = \int_0^{a_\dagger} \beta(a) p(t, a) da = \int_{t+a_\dagger-\tau}^{a_\dagger} \beta(a) \pi(a) \frac{p_0(a-t)}{\pi(a-t)} da = \eta(t) \quad (t \in (0, \tau)), \quad (\text{II.6.8})$$

so that using the fact that $p(\tau, a) = 0$ for almost every $a \in (0, \tau)$ together with (II.1.6), (II.1.7) and (II.6.8) we get that for almost every $a \in (0, \tau)$ we have

$$\int_0^a \frac{m(s)}{\pi(s)} u(s + (\tau - a), s) ds = -\eta(\tau - a) \quad (a \in (0, \tau) \text{ a.e.}). \quad (\text{II.6.9})$$

In other words, a function $u \in L^2(Q_\tau)$ belongs to $\text{Adm}(p_0, \tau)$ if, and only if, both

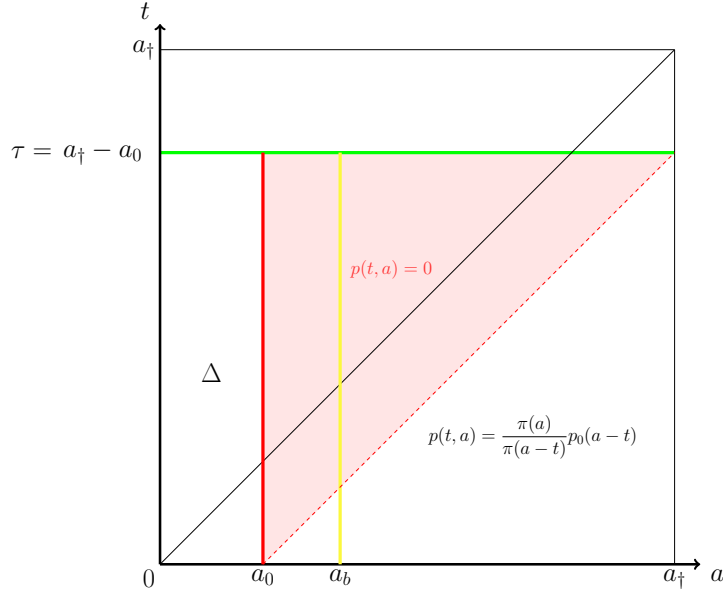


Figure II.2 – Assuming that $u \in \text{Adm}(p_0, \tau)$, the associated population density p has to be null on the whole filled red domain and the total birth rate $\tilde{b}(t)$, for every $t \in (0, \tau)$, has an explicit expression.

conditions (II.6.6) and (II.6.9) are satisfied.

Let $u \in \text{Adm}(p_0, \tau)$. Using (II.6.2) and (II.6.3) we have

$$\begin{aligned} (u, u_*)_{L^2(\Delta)} = & - \int_{\tau}^{a_{\dagger}} \left(\frac{p_0(a-\tau)}{\pi(a-\tau)} \right) \int_{a-\tau}^a \frac{m(s)}{\pi^2(x)} u(s-(a-\tau), s) ds da \\ & - \int_0^{\tau} \left(\frac{\eta(\tau-a)}{\int_0^a \frac{m(x)}{\pi^2(x)} dx} \right) \int_0^a \frac{m(s)}{\pi(s)} u(s+(\tau-a), s) ds da. \end{aligned} \quad (\text{II.6.10})$$

Moreover, since $u \in \text{Adm}(p_0, \tau)$, both condition (II.6.6) and (II.6.9) are satisfied, so that from (II.6.10) we have

$$(u, u_*)_{L^2(\Delta)} = \int_{\tau}^{a_{\dagger}} \frac{p_0^2(a-\tau)}{\int_{a-\tau}^a \frac{m(x)}{\pi^2(x)} dx} da + \int_0^{\tau} \frac{\eta^2(\tau-a)}{\int_0^a \frac{m(x)}{\pi^2(x)} dx} da. \quad (\text{II.6.11})$$

It follows from (II.6.4) and (II.6.11) that for every $u \in \text{Adm}(p_0, \tau)$ we have

$$\|u_*\|_{L^2(\Delta)}^2 = (u, u_*)_{L^2(\Delta)}, \quad (\text{II.6.12})$$

and the Cauchy-Schwarz inequality gives

$$\|u_*\|_{L^2(\Delta)} \leq \|u\|_{L^2(\Delta)}, \quad (\text{II.6.13})$$

for every $u \in \text{Adm}(p_0, \tau)$. The above inequality ensures that the control function defined by (II.6.2) is the HUM control associated to the initial population p_0 in time $\tau = a_{\dagger} - a_0$. Moreover, the controlled population p^* associated to the HUM control u^* is given by formulas (II.6.5), so that the nonnegativity of the initial population density p_0 ensures the nonnegativity of the population density p_* . \square

Remark II.6.2. Given $\tau = a_{\dagger} - a_0$, the above Proposition II.6.1 ensures that an initial population density p_0 may be driven to zero by means of controls $u \in L^2((0, \tau); L^2(0, a_{\dagger}))$ in the case when p_0 satisfies (II.6.1), which means that p_0 has to be “sufficiently null” in the neighborhood of a_0 . Indeed, since $\tau = a_{\dagger} - a_0$, the condition (II.6.1) writes as

$$\int_{a_{\dagger}-a_0}^{a_{\dagger}} \frac{p_0^2(a-(a_{\dagger}-a_0))}{\int_{a-(a_{\dagger}-a_0)}^a \frac{m(x)}{\pi^2(x)} dx} da = \int_0^{a_0} \frac{p_0^2(s)}{\int_s^{s+(a_{\dagger}-a_0)} \frac{\mathbb{1}_{[0, a_0]}(x)}{\pi^2(x)} dx} ds < +\infty,$$

where the integral in the right side of the above equality may be singular in the neighborhood of $s = a_0$.

In fact, this condition (II.6.1) is necessary to get the existence of a control $u \in L^2((0, \tau); L^2(0, a_\dagger))$ such that the solution p to (II.0.1) satisfies $p(\tau, \cdot) = 0$ in $L^2(0, a_\dagger)$. Indeed, let $p_0 \in L^2(0, a_\dagger)$ and let $u \in L^2((0, \tau); L^2(0, a_\dagger))$ be an associate null control function. From (II.6.6) and the Cauchy-Schwarz inequality, it follows that for almost every $a \in (\tau, a_\dagger)$ we have

$$\int_{\tau}^{a_\dagger} \frac{\frac{p_0^2(a-\tau)}{\pi(a-\tau)}}{\int_{a-\tau}^a \frac{m(x)}{\pi^2(x)} dx} \leq \int_{a-\tau}^a m(s)u^2(s-(a-\tau), s) ds \quad (a \in (\tau, a_\dagger) \text{ a.e.}),$$

and integrating the above inequality on (τ, a_\dagger) ensures that

$$\begin{aligned} \int_{\tau}^{a_\dagger} \frac{\frac{p_0^2(a-\tau)}{\pi(a-\tau)}}{\int_{a-\tau}^a \frac{m(x)}{\pi^2(x)} dx} da &\leq \int_{\tau}^{a_\dagger} \int_{a-\tau}^a m(s)u^2(s-(a-\tau), s) ds da \\ &\leq \|u_*\|_{L^2(Q_\tau)}^2, \end{aligned}$$

so that the condition (II.6.1) is satisfied since $u \in L^2((0, \tau); L^2(0, a_\dagger))$.

§ II.6.2. The case $\tau > a_\dagger - a_0$: a numerical toy model using a penalised HUM approach

In this subsection, we shall assume that $\tau \geq a_\dagger - a_0$. Note that, in this case, the strategy developed in the previous subsection to find an explicit HUM control may fail due to the "complexity" of the set $\text{Adm}(p_0, \tau)$. For instance, in the case when $\tau \in (a_\dagger - a_0, a_\dagger]$, a control u belongs to $\text{Adm}(p_0, \tau)$ if and only if both conditions are satisfied:

$$\begin{aligned} \int_{a_\dagger - a}^{a_0 + \tau - a} \beta(x)\pi(x) \int_{a+x-\tau}^{a_0} \frac{m(s)}{\pi(s)} u(s + \tau - a - x, s) ds dx \\ + \int_0^a \frac{m(s)}{\pi(s)} u(s + (\tau - a), s) ds = -\eta(\tau - a) \quad (a \in (0, \tau) \text{ a.e.}), \end{aligned} \tag{II.6.14}$$

and

$$\int_{a-\tau}^a \frac{m(s)}{\pi(s)} u(s - (a - \tau), s) ds = -\frac{p_0(a - \tau)}{\pi(a - \tau)} \quad (a \in (\tau, a_\dagger) \text{ a.e.}).$$

Such conditions may not be included in the expression of an L^2 -scalar product on Q_τ , since it may be difficult - or even impossible - to tackle with the first integral of (II.6.14) which comes from the renewal law. Working with the conditions arising from $\text{Adm}(p_0, \tau)$ may be even worse for largest null control times, *i.e.* for $\tau > a_\dagger$.

Another stragety is developed in the present subsection : given $p_0 \in L^2(0, a_\dagger)$ and

$\tau > a_{\dagger} - a_0$, we aim to build numerically the norm optimal control u_{\star} driving p_0 to zero in time τ using a discrete version of the *penalised HUM method*. More precisely, given $p_0 \in L^2(0, a_{\dagger})$ and $\tau > 0$, we define the quadratic functional F_{ε} on $L^2((0, \tau); L^2(0, a_{\dagger}))$ by

$$F_{\varepsilon}(u) = \|u\|_{L^2(Q_{\tau})}^2 + \frac{1}{\varepsilon} \|p(\tau, \cdot)\|_{L^2[0, a_{\dagger}]}^2 \quad (u \in L^2((0, \tau); L^2(0, a_{\dagger}))), \quad (\text{II.6.15})$$

where p denotes the solution of (II.0.1) associated to the control function u . This functional has been introduced by Lions [Lio68] and studied in a (semi-)discrete setting, for instance, in Lions and Glowinski [GL95], [GLH08] and Boyer [Boy13].

Given $\varepsilon > 0$, the above continuous functional F_{ε} is strictly convex and coercive on $L^2((0, \tau); L^2(0, a_{\dagger}))$. In the following, we denote by u_{ε} the unique minimiser of F_{ε} . The link between u_{ε} and the HUM control is stated in the following Proposition (see, for instance, [Boy13, Theorem 1.7.] for a detailed proof):

Proposition II.6.3. *Assume that $p_0 \in L^2(0, a_{\dagger})$ and $\tau > 0$ are such that the set $\text{Adm}(p_0, \tau)$ is not empty. Then the unique minimizer u_{ε} of the functional F_{ε} defined by (II.6.15) satisfies,*

$$u_{\varepsilon} \xrightarrow{\varepsilon \rightarrow 0^+} u_{\star}, \quad \text{strongly in } L^2((0, \tau); L^2(0, a_{\dagger})),$$

where u_{\star} denotes the norm optimal null control driving the initial population density p_0 to zero.

Given $\varepsilon > 0$, finding the minimiser u_{ε} of the continuous functional F_{ε} seems to be a difficult task, even for the “simple” transport equation (*i.e.* in the case when $\beta \equiv \mu \equiv 0$). We shall follow the methodology developed in Boyer [Boy13] and discretize equations (II.0.1) in order to approach, in some sense, the minimiser u_{ε} for ε small enough using here a simple procedure.

First, let us introduce a discrete scheme associated to the system (II.0.1) without control function. Let $M \in \mathbb{N}^*$. Define $h := \frac{a_{\dagger}}{M}$ the age-step discretization and set $N := \left\lfloor \frac{\tau}{h} \right\rfloor$. For every $j \in \llbracket 0, M \rrbracket$ and every $n \in \llbracket 0, N \rrbracket$, let us denote by

$$a_j = jh \quad (j \in \llbracket 0, M \rrbracket) \quad \text{and} \quad t^n = nh \quad (n \in \llbracket 0, N \rrbracket), \quad (\text{II.6.16})$$

such that the sequences $\{a_j\}_{j=0}^M$ and $\{t^n\}_{n=0}^N$ are respectively age and time meshes of $[0, a_{\dagger}]$ and $[0, \tau]$, with the same step $h > 0$. For any $j \in \llbracket 0, M \rrbracket$, we set $\beta_j := \beta(a_j)$ and $\mu_j := \mu(a_j)$. Assume that, for every $(j, n) \in \llbracket 0, M \rrbracket \times \llbracket 0, N \rrbracket$, p_j^n is an approximation of

$p(t^n, a_j)$ given by the mortality implicit Euler scheme:

$$\begin{cases} \frac{p_{j+1}^{n+1} - p_j^n}{h} + \mu_{j+1} p_{j+1}^{n+1} = 0 & \text{if } \begin{cases} 0 \leq n \leq N-1, \\ 0 \leq j \leq M-1, \end{cases} \\ p_0^{n+1} = h \sum_{i=0}^M \beta_i p_i^n & \text{if } 0 \leq n \leq N-1, \\ p_j^0 = p_0(a_j) & \text{if } 0 \leq j \leq M. \end{cases} \quad (\text{II.6.17})$$

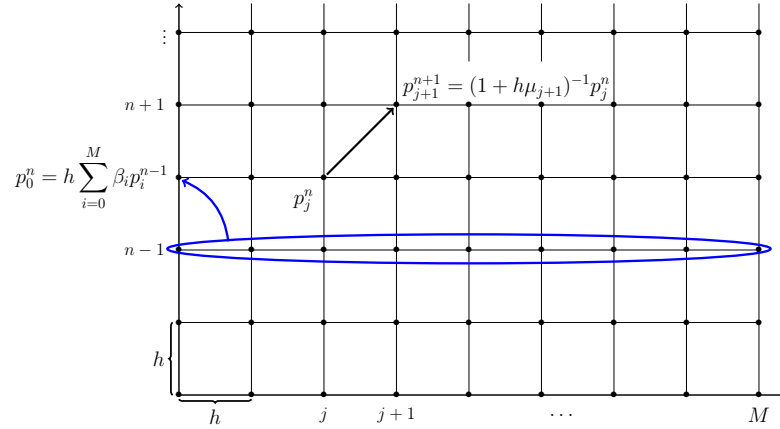


Figure II.3 – The age-time numerical grid.

Such a scheme (II.6.17) is studied in a convergence view point, for instance, in Iannelli and Milner [IM17, Chapter 3] in the case when the function β has compact support on $(0, a_\dagger)$. We associate to (II.6.17) the *controlled discrete scheme*:

$$\begin{cases} \frac{p_{j+1}^{n+1} - p_j^n}{h} + \mu_{j+1} p_{j+1}^{n+1} = m_j u_j^n & \text{if } \begin{cases} 0 \leq n \leq N-1, \\ 0 \leq j \leq M-1, \end{cases} \\ p_0^{n+1} = h \sum_{i=0}^M \beta_i p_i^n & \text{if } 0 \leq n \leq N-1, \\ p_j^0 = p_0(a_j) & \text{if } 0 \leq j \leq M, \end{cases} \quad (\text{II.6.18})$$

where $m_j := m(a_j) = \mathbb{1}_{[0, a_0]}(a_j)$ for every $j \in \llbracket 0, M \rrbracket$ and $\{u_j^n\}_{j=0, \dots, M}^{n=0, \dots, N}$ is the discrete control function. Note that, for every $n \in \llbracket 0, N \rrbracket$, denoting by

$$\mathcal{P}^n := \begin{pmatrix} p_0^n \\ p_1^n \\ \vdots \\ p_M^n \end{pmatrix} \quad \text{and} \quad \mathcal{U}^n := \begin{pmatrix} u_0^n \\ u_1^n \\ \vdots \\ u_M^n \end{pmatrix} \quad (n \in \llbracket 0, N \rrbracket),$$

we have

$$\mathcal{P}^{n+1} = \mathcal{A}_h \mathcal{P}^n + \mathcal{B}_h \mathcal{U}^n \quad (n \in \llbracket 0, N-1 \rrbracket),$$

where $\mathcal{A}_h \in \mathcal{M}_{M+1}(\mathbb{R})$ (the so-called *Leslie matrix*) and $\mathcal{B}_h \in \mathcal{M}_{M+1}(\mathbb{R})$ are respectively defined by

$$\mathcal{A}_h = \begin{bmatrix} h\beta_0 & h\beta_1 & \dots & \dots & h\beta_M \\ (1+h\mu_1)^{-1} & 0 & \dots & \dots & 0 \\ 0 & (1+h\mu_2)^{-1} & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & (1+h\mu_M)^{-1} & 0 \end{bmatrix},$$

that is to say

$$\begin{cases} \mathcal{A}_h[1, j] = h\beta_{j-1} & \text{if } 1 \leq j \leq M, \\ \mathcal{A}_h[i+1, i] = (1+h\mu_i)^{-1} & \text{if } 1 \leq i \leq M, \\ \mathcal{A}_h[i, j] = 0 & \text{otherwise.} \end{cases} \quad (\text{II.6.19})$$

and

$$\mathcal{B}_h = \begin{bmatrix} 0 & \dots & \dots & 0 \\ hm_0(1+h\mu_1)^{-1} & 0 & \dots & 0 \\ 0 & hm_1(1+h\mu_2)^{-1} & & (0) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & hm_{M-1}(1+h\mu_M)^{-1} & 0 \end{bmatrix},$$

i.e.

$$\begin{cases} \mathcal{B}_h[i+1, i] = hm_{i-1}(1+h\mu_i)^{-1} & \text{if } 1 \leq i \leq M, \\ \mathcal{B}_h[i, j] = 0 & \text{otherwise.} \end{cases} \quad (\text{II.6.20})$$

Let us introduce some notations. Given $\varepsilon > 0$ and $M \in \mathbb{N}^*$, we denote by $L_h^2(Q_\tau)$ the space $\mathbb{R}^{(M+1) \times (N+1)}$ endowed with the discrete norm

$$\llbracket u \rrbracket_{L_h^2(Q_\tau)} := \left(h^2 \sum_{j=0}^M \sum_{n=0}^N |u_j^n|^2 \right)^{\frac{1}{2}} \quad (u = \{u_j^n\} \in L_h^2(Q_\tau)), \quad (\text{II.6.21})$$

where we recall that h is defined by $h = \frac{a_\dagger}{M}$. We associate to (II.6.15) and (II.6.18) the following discrete functional $F_{\varepsilon, h}$ defined on $L_h^2(Q_\tau)$ by

$$F_{\varepsilon, h}(u) = \llbracket u \rrbracket_{L_h^2(Q_\tau)}^2 + \frac{h}{\varepsilon} \sum_{j=0}^M |p_j^N|^2 \quad (u = \{u_j^n\} \in L_h^2(Q_\tau)), \quad (\text{II.6.22})$$

where $p = \{p_j^n\}$ denotes the solution p to (II.6.18) associated to $u = \{u_j^n\}$. We denote

by u_ε^h the unique minimiser of the above functional $F_{\varepsilon,h}$. Moreover, we denote by $u_\star^h \in L_h^2(Q_\tau)$ a discretization of the HUM control u_\star as defined by

$$(u_\star^h)_j^n = u_\star(t^n, a_j) \quad (j \in \llbracket 0, M \rrbracket, n \in \llbracket 0, N \rrbracket), \quad (\text{II.6.23})$$

where $\{a_j\}_{j=0}^M$ and $\{t^n\}_{n=0}^N$ are defined by (II.6.16).

The main question arising here is the following : can we describe the behaviour of the minimiser u_ε^h of the function $F_{\varepsilon,h}$ when h and ε goes to zero ? More precisely, do we have the convergence of u_ε^h towards the HUM control u_\star , in the sense that

$$\lim_{(\varepsilon,h) \rightarrow (0,0)} \llbracket u_\varepsilon^h - u_\star^h \rrbracket_{L_h^2(Q_\tau)} = 0 ? \quad (\text{II.6.24})$$

About the convergence of the discrete controlled scheme. We aim to observe numerically the convergence of u_ε^h towards u_\star in a special case when the HUM control u_\star can be computed explicitly, *i.e.* in the case when $\tau = a_\dagger - a_0$ with p_0 satisfying the condition (II.6.1) (see the previous subsection II.6.1). To this aim, we set the parameters as defined by the following Table II.1 and we compute numerically the minimiser u_ε^h of $F_{\varepsilon,h}$ using MATLAB, for some decreasing values of h and ε . The collection of the norm differences $\llbracket u_\varepsilon^h - u_\star^h \rrbracket_{L_h^2(Q_\tau)}$ are dressed in Table II.2. The discretisation of the HUM control (II.6.2) with parameters of Table (II.1) is pictured in Figure II.4 for $M = 115$, and the simulated minimiser u_ε^h of $F_{\varepsilon,h}$ is represented in Figure II.5 for $M = 115$ and $\varepsilon = 10^{-9}$.

Remark II.6.4. *In this example, we may observe numerically the convergence of u_ε^h towards the explicit HUM control u_\star (see Table II.2 and Figures II.4 and II.5.) We emphasize that this convergence is also observed for any other set of parameters where the HUM control can be computed explicitly, but a precise proof of the theoretical convergence has not been obtained in this manuscript. This theoretical strong convergence seems true due to numerical experiments, and a rigorous proof of this phenomenon may be an interesting and quite challenging task.*

a_\dagger	a_0	τ	$\beta(a)$	$\mu(a)$	$p_0(a)$
10	3	7	$\mathbb{1}_{[3,9]}(a)$	$(a_\dagger - a)^{-1}$	$10 a_0 - a $

Table II.1 – Parameters for the convergence observation.

About the positivity of the state trajectory associated to the HUM control. It has been already proved, in the previous subsection, that the HUM control preserves the nonnegativity of a nonnegative initial population density in the case when $\tau = a_\dagger - a_0$. This phenomenon seems not to be guaranteed for largest control times - assuming that the convergence of u_ε^h towards u_\star is true.

$\varepsilon \setminus M$	25	55	75	95	115
10^{-1}	11,0360	4.4455	3.2806	2.6740	2,2571
10^{-3}	13,0033	5,2918	3,8329	3.0345	2.5197
10^{-5}	8,2756	1,6903	0,9159	2.3977	2,1073
10^{-7}	8,2741	1,6875	0,9036	0,5616	0.3825
10^{-9}	8,2741	1,6874	0,9036	0,5616	0.3825

Table II.2 – Estimation of the norm $\|u_\varepsilon^h - u_\star^h\|_{L_h^2(Q_\tau)}$, where u_\star^h is a discretization of the HUM control as defined by (II.6.2) and u_ε^h is the minimiser of the discrete functional cost $F_{\varepsilon,h}$ defined by (II.6.22).

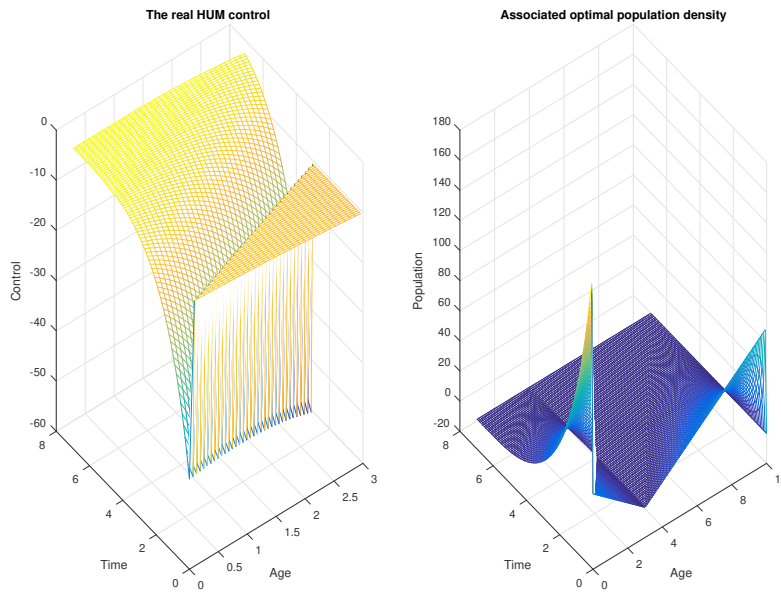


Figure II.4 – Shape of the discretization of the “real HUM control” given by (II.6.2), and its associated population density given by (II.6.5) with $M = 115$.

In Table II.3 are defined some set of parameters that may not preserve the nonnegativity of the population associated to the simulated HUM control - see Figure II.6. Once again, a precise proof of this phenomenon type has not been obtained here.

a_\dagger	a_0	τ	$\beta(a)$	$\mu(a)$	$p_0(a)$
10	3	15	$\mathbb{1}_{[4,8]}(a) \times \tanh(a)$	$(a_\dagger - a)^{-1}$	$a(a_\dagger - a)$

Table II.3 – Parameters for the nonpositivity observation.

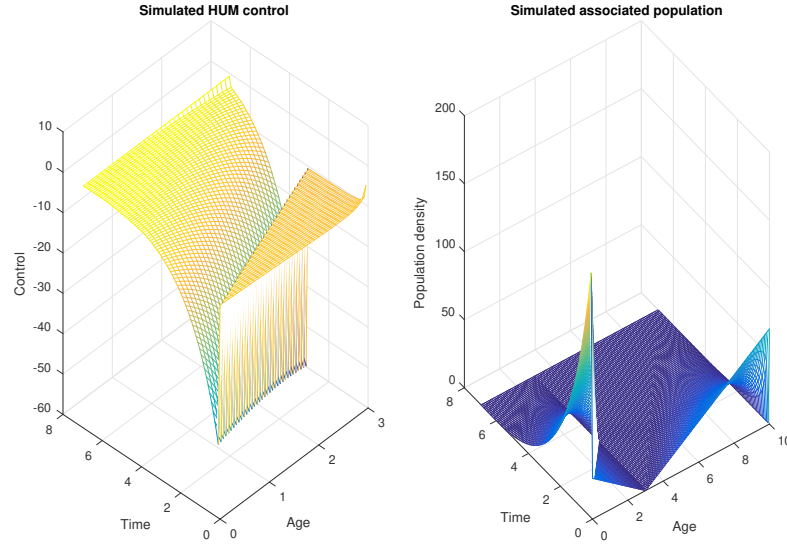


Figure II.5 – Shape of the simulated HUM control with $M = 115$ and $\varepsilon = 10^{-9}$ for the set of parameters given by II.1.

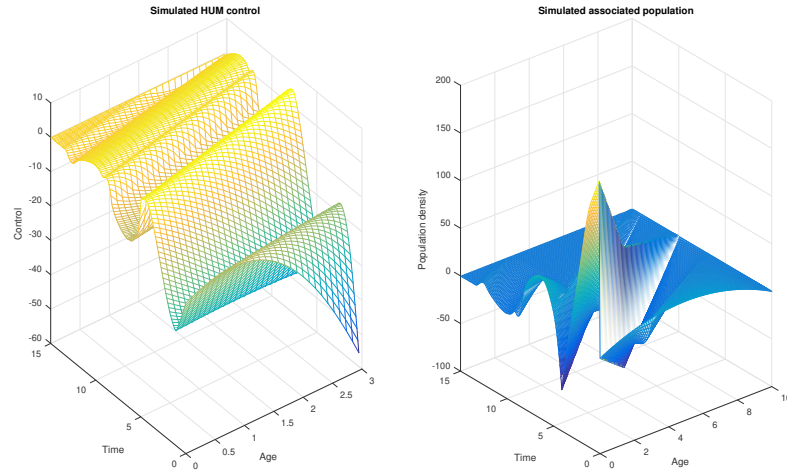


Figure II.6 – Numerical observation of the lack of positivity of the population associated to the simulated HUM control for large time controls (here, $\tau > a_+$) with $M = 115$, $\varepsilon = 10^{-9}$ and the set of parameters given by II.3.

II.7 NULL-CONTROLLABILITY IN $L^1[0, a_+]$.

In this Appendix, we investigate the null controllability of system (II.0.1) when the state space X is chosen to be $L^1[0, a_+]$ instead of $L^2[0, a_+]$.

Proposition II.7.1. *For $\tau > 0$ we denote by $(\mathbb{T}_\tau)_{\tau>0}$ and $(\Phi_\tau)_{\tau>0}$ the semigroup acting on $L^1[0, a_+]$ and the control to state maps, defined on $L^1[0, \tau]$ with values in $L^1[0, a_+]$, defined according to Remark (II.1.2). Then for every $\tau > a_+ - a_0$ we have $\text{Ran } \mathbb{T}_\tau \subset$*

Ran Φ_τ , i.e., the system (II.0.1), with state space $X = L^1[0, a_\dagger]$ and input space $U = L^1[0, \tau]$ is null-controllable in any time $\tau > a_\dagger - a_0$.

Proof. We can assume, without loss of generality, that $a_0 \leq a_b$. Let $\tau \in (a_\dagger - a_0, a_\dagger)$. Given $p_0 \in X$, we look for a control $u \in L^1([0, \tau]; L^1[0, a_\dagger])$ such that the solution p of (II.0.1) satisfies $p(\tau, \cdot) = 0$. To this aim, we use the control constructed in Remark II.2.3 (with $\tilde{p} = 0$) with a slight modification (see Figure 2 for an illustration of the modification). Let $\delta \in (a_\dagger - a_0, \tau)$. We set

$$u(t, a) = \begin{cases} 0 & \text{if } t - a < \delta - a_\dagger, \\ -\frac{\frac{p_0(a-t)}{\pi(a-t)}}{\int_{a-t}^{\tau+(a-t)} \frac{m(x)}{\pi(x)} dx} & \text{if } \delta - a_\dagger \leq t - a \leq 0, \\ -\frac{\eta(t-a)}{\int_0^{\tau-(t-a)} \frac{m(x)}{\pi(x)} dx} & \text{if } 0 < t - a \leq \delta, \\ 0 & \text{if } \delta < t - a \leq \tau, \end{cases} \quad (\text{II.7.1})$$

where

$$\eta(s) = \int_{s+a_\dagger-\delta}^{a_\dagger} \beta(x)\pi(x) \frac{p_0(x-s)}{\pi(x-s)} dx \quad (s \in [0, \delta]).$$

With u defined by (II.7.1), it can be checked using a recursive argument as in the proof of Theorem II.2.2 that we have

$$p(t, a) = \begin{cases} \frac{\pi(a)}{\pi(a-t)} p_0(a-t) & \text{if } t - a < \delta - a_\dagger, \\ \frac{\pi(a)}{\pi(a-t)} p_0(a-t) \left[1 - \frac{\int_{a-t}^a \frac{m(x)}{\pi(x)} dx}{\int_{a-t}^{a+(\tau-t)} \frac{m(x)}{\pi(x)} dx} \right] & \text{if } \delta - a_\dagger \leq t - a \leq 0, \\ \pi(a)\eta(t-a) \left[1 - \frac{\int_0^a \frac{m(s)}{\pi(s)} ds}{\int_0^{\tau-(t-a)} \frac{m(s)}{\pi(s)} ds} \right] & \text{if } 0 < t - a \leq \delta. \end{cases} \quad (\text{II.7.2})$$

It follows from the above formula (II.7.2) that we have

$$p(\tau, a) = 0 \quad (a \in [a_\dagger - \delta, a_\dagger] \text{ a.e.}). \quad (\text{II.7.3})$$

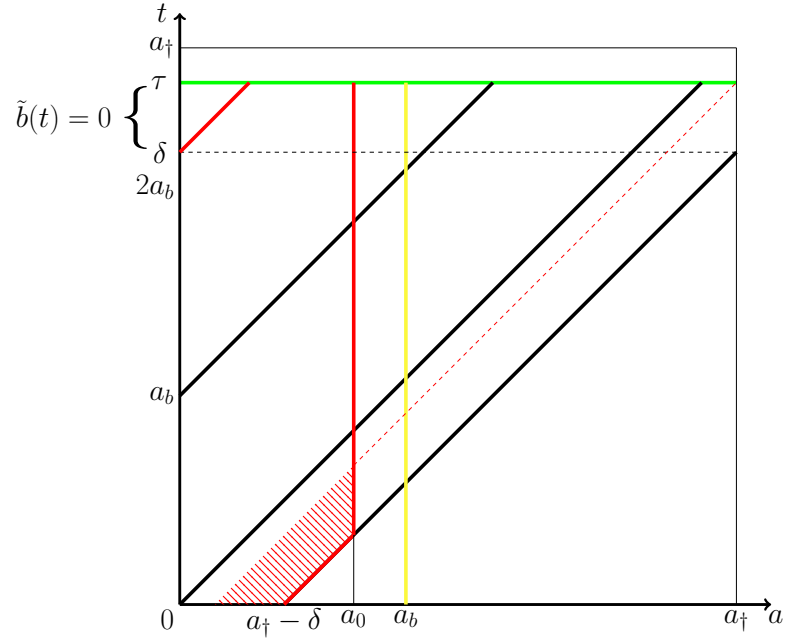


Figure II.7 – In this Figure, the control acts in the subset of $Q_{\tau} = [0, \tau] \times [0, a_{\dagger}]$ delimited by the line $a = a_0$ and the red curve. The hatched region corresponds to the slight modification of the control region (compared to the control pictured in Figure 1). Due to this modification, the value of $\tilde{b}(t) = \int_0^{a_{\dagger}} \beta(a)p(t, a) da$ is null for every $t \in [\delta, a_{\dagger}]$, and we can choose $u(t, a) = 0$ for $\delta \leq t - a \leq a_{\dagger}$ to avoid the singularity in the neighborhood of the point $(\tau, 0)$.

If $t \in [\delta, a_{\dagger}]$, it follows from (II.7.2) that we have

$$\tilde{b}(t) = \int_{a_b}^{a_{\dagger}} \beta(a)p(t, a) da = 0,$$

so that from (II.1.9) and (II.7.1) we get

$$p(t, a) = 0 \quad (\delta < t - a \leq \tau). \quad (\text{II.7.4})$$

It follows from (II.7.2) and (II.7.4) that the controlled density of population p is given

by




$$p(t, a) = \begin{cases} \frac{\pi(a)}{\pi(a-t)} p_0(a-t) & \text{if } t-a < \delta - a_{\dagger}, \\ \frac{\pi(a)}{\pi(a-t)} p_0(a-t) \left[1 - \frac{\int_{a-t}^a \frac{m(x)}{\pi(x)} dx}{\int_{a-t}^{a+(\tau-t)} \frac{m(x)}{\pi(x)} dx} \right] & \text{if } \delta - a_{\dagger} \leq t-a \leq 0, \\ \pi(a) \eta(t-a) \left[1 - \frac{\int_0^a \frac{m(s)}{\pi(s)} ds}{\int_0^{\tau-(t-a)} \frac{m(s)}{\pi(s)} ds} \right] & \text{if } 0 < t-a \leq \delta, \\ 0 & \text{if } \delta < t-a \leq \tau, \end{cases}$$

so that we have $p(\tau, \cdot) = 0$ and the control u defined by (II.7.1) is in $L^1([0, \tau]; L^1[0, a_{\dagger}])$. \square

CHAPITRE III

Contrôle à zéro de l'équation de Lotka-McKendrick avec diffusion

Ce chapitre reprend la démonstration de la contrôlabilité à zéro des équations de Lotka et McKendrick avec diffusion de l'article [HT17], accepté et paru dans *Mathematical Control and Related Fields (MCRF)*. L'essence de la preuve réside dans la résolution d'un problème de moments multidimensionnel à l'aide de l'inégalité de Lebeau et Robbiano [LR95], originellement dédiée à la résolution du problème de contrôle à zéro pour le contrôle interne de l'équation de la chaleur.

	Tranche d'âge du support de contrôle	Les jeunes individus se reproduisent	Contrôle à zéro ?	Temps optimal de contrôle à zéro
Ainscha (2002) [Ain02]		OUI (mais pas les $\theta_1, \theta_2, \dots$)	NON	×
Hegoburu, Tucsnak (2017) [HT17]		OUI	OUI	OUI (evident !)
Mindy, Tucsnak & Zoussou (2018) [MTZ18a]		NON	OUI	OUI ($\theta_1, \theta_2, \dots, \theta_{n-1}, \theta_n$)

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III.1 INTRODUCTION

We consider a linear controlled age-structured population model with spatial diffusion described by the following system:

$$\begin{cases} \begin{cases} \partial_t p(t, a, x) + \partial_a p(t, a, x) + \mu(a)p(t, a, x) - \Delta p(t, a, x) \\ \quad = \chi_\omega(x)u(t, a, x), & t > 0, a \in (0, a_\dagger), x \in \Omega, \\ \frac{\partial p}{\partial \nu}(t, a, x) = 0, & t > 0, a \in (0, a_\dagger), x \in \partial\Omega, \\ p(t, 0, x) = \int_0^{a_\dagger} \beta(a)p(t, a, x) da, & t > 0, x \in \Omega, \\ p(0, a, x) = p_0(a, x), & a \in (0, a_\dagger), x \in \Omega. \end{cases} \end{cases} \quad (\text{III.1.1})$$

In the above equations:

- $\Omega \subset \mathbb{R}^n$, $n \geq 1$, denotes a smooth connected bounded domain and Δ is the laplacian with respect to the variable x ;
- $\frac{\partial}{\partial \nu}$ denotes the derivation operator in the direction of the unit outer normal to $\partial\Omega$. We thus have homogeneous Neumann boundary conditions, thus the considered population is isolated from the exterior of Ω ;
- $p(t, a, x)$ denotes the distribution density of the population at time t , of age a at spatial position $x \in \Omega$;
- p_0 denotes the initial population distribution;
- $a_\dagger \in (0, +\infty)$ is the maximal life expectancy;
- $\beta(a)$ and $\mu(a)$ are positive functions denoting respectively the birth and death rates, which are supposed to be independent of t and x ;
- $\omega \subset \Omega$ is a nonempty open subset of Ω and χ_ω denotes the characteristic function of ω .

We make the following classical assumptions on β and μ :

$$(H1) \quad \beta \in L^\infty(0, a_\dagger), \beta(a) \geq 0 \text{ for almost every } a \in (0, a_\dagger),$$

$$(H2) \quad \mu \in L^1[0, a^*] \text{ for every } a^* \in (0, a_\dagger), \mu(a) \geq 0 \text{ for almost every } a \in (0, a_\dagger),$$

$$(H3) \quad \int_0^{a_\dagger} \mu(a) da = +\infty.$$

We also introduce the function

$$\pi(a) := \exp\left(-\int_0^a \mu(s) ds\right), \quad (\text{III.1.2})$$

which is the probability of survival of an individual from age 0 to a .

Our main result is

Theorem III.1.1. *With the above notations and assumptions, for every $\tau > 0$ and for every $p_0 \in L^2((0, a_+) \times \Omega)$, there exists $u \in L^\infty((0, \tau); L^2((0, a_+) \times \omega))$ such that the solution p of III.1.1 satisfies*

$$p(\tau, a, x) = 0 \quad (a \in (0, a_+), x \in \Omega \text{ a.e.}) \quad (\text{III.1.3})$$

The null-controllability of the system modelling age-dependant population dynamics is by now well understood in the case in which diffusion is neglected (see Barbu, Iannelli and Martcheva [BIM01] and Hegoburu, Magal, Tucsnaak [HMT18]). In the case when spatial diffusion is taken into account, namely for III.1.1, the particular case when the control acts in the whole space (the case corresponding to $\omega = \Omega$) was investigated by S. Anița (see [Ani00], p 148). The case when the control acts in a spatial subdomain ω was firstly studied by B. Ainseba [Ain02], where the author proves the null controllability of the above system III.1.1, except for a small interval of ages near zero. The case when the control acts in a spatial subdomain ω and also only for small age classes was investigated by B. Ainseba and S. Anița [AA04], for initial data p_0 in a neighborhood of the target \tilde{p} . Related approximate and exact controllability issues have also been studied in Ainseba and Langlais [AL00], Ainseba and Iannelli [AI12], Traore [Tra06], Kavian and Traore [KT11].

We also mention that the controllability of related degenerated parabolic systems (namely of Kolmogorov type), not containing the renewal term, has been studied in Beauchard [Bea14], Beauchard and Pravda-Starov [BPS18].

The main novelty (with respect to the literature quoted above) brought in Theorem III.1.1 is in our case u can be chosen such that III.1.3 holds for every $a \in (0, a_+)$ instead of (δ, a_+) , with $\delta > 0$, as it has been done, for instance, in [Ain02]. We can thus find a control driving to zero the whole population, without excluding very young individuals. Moreover, we do not assume that the birth rate vanishes for small ages.

The remaining part of this work is organized as follows. In Section III.2 we recall some basic results on the Lotka-McKendrick semigroup, with or without spatial diffusion, and we state a null controllability result associated to system III.1.1 without spatial diffusion. Section III.3 is devoted to study the null controllability of low frequencies for the solution of system III.1.1. We prove the main result in Section III.4, by using a version of the Lebeau-Robbiano strategy.

Notation: In all what follows, C will denote a generic constant, depending only of the coefficients in III.1.1, on Ω and ω , whose value may change from line to line.

III.2 SOME BACKGROUND ON THE LOTKA-MCKENDRICK SEMIGROUP

In this section we recall, with no claim of originality, some existing results on the population semigroup for the linear age-structured model without and with spatial diffu-

sion. In particular, we recall the structure of the spectrum of the semigroups generators, together with some controllability results concerning the free diffusion case.

§ III.2.1. The free diffusion case

In this paragraph we remind some results on the diffusion free case, which is described by the so-called McKendrick-Von Foster model. With one exception, we do not give proofs and we refer, for instance, to Song et al. [SYW⁺82] or Inaba [Ina17] for a detailed presentation of these issues.

The considered system is:

$$\begin{cases} \partial_t p(t, a) + \partial_a p(t, a) = -\mu(a)p(t, a), & t > 0, a \in (0, a_\dagger), \\ p(t, 0) = \int_0^{a_\dagger} \beta(a)p(t, a) da, & t > 0, \\ p(0, a) = p_0(a), & a \in (0, a_\dagger), \end{cases} \quad (\text{III.2.1})$$

where μ and β satisfy the assumptions in Theorem III.1.1.

The above system is described by the operator A_0 defined by

$$\begin{aligned} \mathcal{D}(A_0) &= \left\{ \varphi \in L^2[0, a_\dagger] \mid \varphi(0) = \int_0^{a_\dagger} \beta(a)\varphi(a) da; -\frac{d\varphi}{da} - \mu\varphi \in L^2[0, a_\dagger] \right\}, \\ A_0\varphi &= -\frac{d\varphi}{da} - \mu\varphi \quad (\varphi \in \mathcal{D}(A_0)). \end{aligned} \quad (\text{III.2.2})$$

Theorem III.2.1. *The operator A_0 defined by III.2.2 has compact resolvent and its spectrum is constituted of a countable (infinite) set of isolated eigenvalues with finite algebraic multiplicity. The eigenvalues $(\lambda_n^0)_{n \geq 1}$ of A_0 (counted without multiplicity) are the solutions of the characteristic equation*

$$F(\lambda) := \int_0^{a_\dagger} \beta(a)e^{-\lambda a} \pi(a) da = 1. \quad (\text{III.2.3})$$

The eigenvalues $(\lambda_n^0)_{n \geq 1}$ are of geometric multiplicity one, the eigenspace associated to λ_n^0 being the one-dimensional subspace of $L^2(0, a_\dagger)$ generated by the function

$$\varphi_n^0(a) = e^{-\lambda_n^0 a} \pi(a) = e^{-\lambda_n^0 a - \int_0^a \mu(s) ds}.$$

Finally, every vertical strip of the complex plane $\alpha_1 \leq \text{Re}(z) \leq \alpha_2$, $\alpha_1, \alpha_2 \in \mathbb{R}$, contains a finite number of eigenvalues of A_0 .

Theorem III.2.2. *The operator A_0 defined by III.2.2 has a unique real eigenvalue λ_1^0 . Moreover, we have the following properties :*

1. λ_1^0 is of algebraic multiplicity one;

2. $\lambda_1^0 > 0$ (resp. $\lambda_1^0 < 0$) if and only if $F(0) > 1$ (resp. $F(0) < 1$);
3. λ_1^0 is a real dominant eigenvalue:

$$\lambda_1^0 > \operatorname{Re}(\lambda_n^0), \quad \forall n \geq 2. \quad (\text{III.2.4})$$

It is well known (see, for instance, Song et al. [SYW⁺82] or Kappel and Zhang [KZ93]) that A generates a C^0 semigroup of linear operators in $L^2[0, a_+]$ which we denote by $\mathbb{T}^{A_0} = (\mathbb{T}_t^{A_0})_{t \geq 0}$. We also have the following useful result (see, for instance, [Ina17, p 23]):

Proposition III.2.3. *The semigroup \mathbb{T}^{A_0} generated on $L^2([0, a_+])$ by A_0 is compact for $t \geq a_+$.*

According to Zabczyk [Zab76, Section 2]), this implies in particular that

$$\omega_a(A_0) = \omega_0(A_0),$$

where $\omega_a(A_0) := \lim_{t \rightarrow +\infty} t^{-1} \ln \|\mathbb{T}_t^{A_0}\|_{L^2(0, a_+)}$ denotes the growth bound of the semigroup $\mathbb{T}_t^{A_0}$ and $\omega_0(A_0) := \sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A_0)\}$ the spectral bound of A_0 . It is worth noticing that the above condition ensures that the exponential stability of \mathbb{T}^{A_0} is equivalent to the condition $\omega_0(A_0) < 0$. According to Theorem III.2.1 and III.2.2, it follows that the exponential stability of \mathbb{T}^{A_0} is equivalent to the condition $\lambda_1^0 < 0$, where $\lambda_1^0 < 0$ is the unique real solution to the characteristic equation defined by III.2.3.

The free diffusion control problem associated to system III.1.1 writes as

$$\begin{cases} \partial_t p(t, a) + \partial_a p(t, a) + \mu(a)p(t, a) = v(t, a), & t > 0, a \in (0, a_+), \\ p(t, 0) = \int_0^{a_+} \beta(a)p(t, a) da, & t > 0, \\ p(0, a) = p_0(a), & a \in (0, a_+), \end{cases} \quad (\text{III.2.5})$$

where v is the control function. Let us state a null controllability result for system III.2.5:

Proposition III.2.4. *Under the assumptions of Theorem III.1.1, for every $\tau > 0$, there exists $v \in L^\infty((0, \tau); L^2(0, a_+))$ such that the solution p of III.2.5 satisfies*

$$p(\tau, a) = 0 \quad (a \in (0, a_+) \text{ a.e.}).$$

Moreover, we have

$$\|v\|_{L^\infty((0, \tau); L^2(0, a_+))} \leq \frac{\sup_{\sigma \in [0, \tau]} \|\mathbb{T}_\sigma^{A_0}\|_{L^2(0, a_+)}}{\tau} \|p_0\|_{L^2(0, a_+)}. \quad (\text{III.2.6})$$

Proof. Let $\tau > 0$. For almost every $(t, a) \in (0, \tau) \times (0, a_+)$, we set

$$v(t, a) = -\frac{1}{\tau} (\mathbb{T}_t^{A_0} p_0)(a) \quad (t \in (0, \tau), a \in (0, a_+)). \quad (\text{III.2.7})$$

It is easy to check that the (mild) solution p of III.2.5, given by

$$p(t, a) = \mathbb{T}_t^{A_0} p_0(a) + \int_0^t \mathbb{T}_{t-\sigma}^{A_0} v(\sigma)(a) \, d\sigma, \quad (\text{III.2.8})$$

with v defined in III.2.7, satisfies $p(\tau, \cdot) = 0$. Moreover, the cost stated in inequality III.2.6 follows from the definition of the control v given by III.2.7. \square

§ III.2.2. The population dynamics with diffusion

The existence of a semigroup on $L^2((0, a_+) \times \Omega)$ describing the linear age-structured population model with diffusion coefficient and age dependent birth and death rates, with homogeneous Neumann boundary conditions has been proved in Huyer [Huy94] (see also [GC94] for the case of homogeneous Dirichlet boundary conditions).

More precisely, let $X := L^2((0, a_+) \times \Omega)$ and define the following unbounded operator A on X :

$$\begin{aligned} \mathcal{D}(A) &= \{ \varphi \in X \mid \varphi \in C([0, a_+]; L^2(\Omega)) \cap L^2([0, a_+], H^1(\Omega)), \\ \varphi(0, x) &= \int_0^{a_+} \beta(a) \varphi(a, x) \, da; \quad -\frac{d\varphi}{da} - \mu\varphi + \Delta\varphi \in X \}, \\ A\varphi &= -\frac{d\varphi}{da} - \mu\varphi + \Delta\varphi \quad (\varphi \in \mathcal{D}(A)). \end{aligned} \quad (\text{III.2.9})$$

The generator A of the population semigroup can be seen as the sum of a population operator without diffusion $-d/da - \mu I$ and a spatial diffusion term Δ . It turns out that spectral properties of A can be easily obtained from those of these two operators.

Theorem III.2.5. *Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ be the increasing sequence of eigenvalues of $-\Delta$ with Neumann boundary conditions and let $(\varphi_n)_{n \geq 0}$ be a corresponding orthonormal basis of $L^2(\Omega)$. Let $(\lambda_n^0)_{n \geq 1}$ and $(\varphi_n^0)_{n \geq 1}$ be respectively the sequence of eigenvalues and eigenfunctions of the free diffusion operator A_0 defined by III.2.2 (see Theorem III.2.1). Then the following assertions hold:*

1. *The eigenvalues of A are given by*

$$\sigma(A) = \{ \lambda_i^0 - \lambda_j \mid i \in \mathbb{N}^*, j \in \mathbb{N} \}.$$

2. *A has a dominant eigenvalue:*

$$\lambda_1^0 > \operatorname{Re}(\lambda), \quad \forall \lambda \in \sigma(A), \lambda \neq \lambda_1^0.$$

3. *The eigenspace associated to an eigenvalue λ of A is given by*

$$\operatorname{Span}\{ \varphi_i^0(a) \varphi_j(x) = e^{-\lambda_i^0 a} \pi(a) \varphi_j(x) \mid \lambda_i^0 - \lambda_j = \lambda \}.$$

Since the operator A generates a \mathcal{C}^0 semigroup of linear operators in X which we

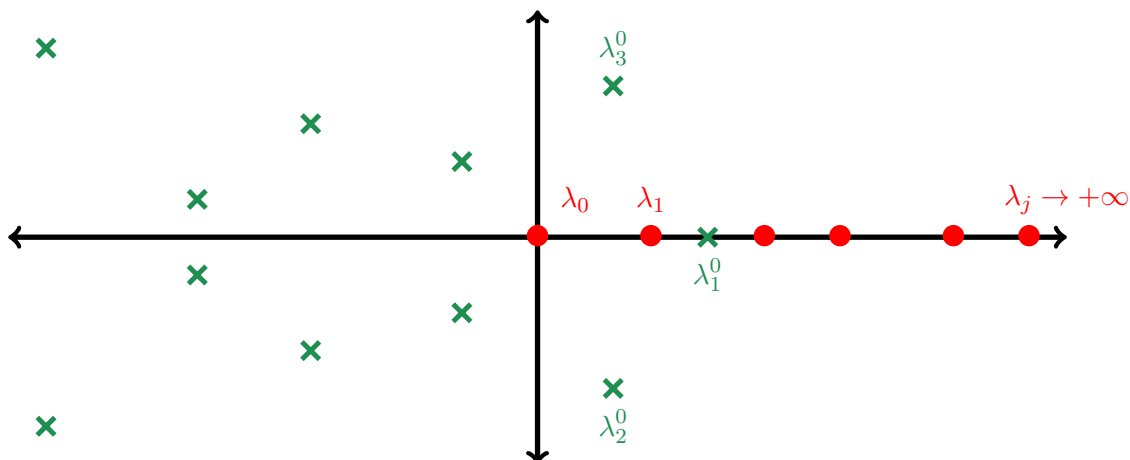


Figure III.1 – The spectrum of the free diffusion operator A_0 (green crosses) and of $-\Delta$ (red circles)

denote by $\mathbb{T}^A = (\mathbb{T}_t^A)_{t \geq 0}$, this allows to define the concept of (mild) solution of III.1.1 in the following standard way: we say that p is a mild solution of III.1.1 if

$$p(t, \cdot) = \mathbb{T}_t^A p_0 + \Phi_t u \quad (t \geq 0, \quad u \in L^2([0, \infty); X)), \quad (\text{III.2.10})$$

where the control operator $B \in \mathcal{L}(X)$ is defined by

$$Bu = \chi_\omega u \quad (u \in X),$$

and where

$$\Phi_t u = \int_0^t \mathbb{T}_{t-\sigma}^A B u(\sigma) d\sigma \quad (t \geq 0, \quad u \in L^2([0, \infty); X)). \quad (\text{III.2.11})$$

It is worth noticing, for instance by using a spectral decomposition, that the semi-group \mathbb{T}^A is exponentially stable if $\lambda_1^0 < 0$, where we recall that λ_1^0 denotes the unique real solution to the characteristic equation defined by III.2.3.

Remark III.2.6. In order to prove the null controllability of system III.1.1, we may assume, without loss of generality, that the so called reproductive number satisfies

$$\int_0^{a^\dagger} \beta(a) \pi(a) da < 1, \quad (\text{III.2.12})$$

which implies that both semigroups \mathbb{T}^{A_0} and \mathbb{T}^A are exponentially stable - see the above results. Indeed, in the case where $\int_0^{a^\dagger} \beta(a) \pi(a) da \geq 1$, we may consider the auxiliary

system

$$\begin{cases} \begin{cases} \partial_t z(t, a, x) + \partial_a z(t, a, x) + \tilde{\mu}(a)z(t, a, x) - \Delta z(t, a, x) \\ \quad = \chi_\omega(x)v(t, a, x), & t > 0, a \in (0, a_\dagger), x \in \Omega, \\ \frac{\partial z}{\partial \nu}(t, a, x) = 0, & t > 0, a \in (0, a_\dagger), x \in \partial\Omega, \\ z(t, 0, x) = \int_0^{a_\dagger} \beta(a)z(t, a, x) da, & t > 0, x \in \Omega, \\ z(0, a, x) = p_0(a, x), & a \in (0, a_\dagger), x \in \Omega, \end{cases} \end{cases} \quad (\text{III.2.13})$$

with $\tilde{\mu}(a) := \mu(a) + \lambda$, where $\lambda \geq 0$ is large enough to have

$$\int_0^{a_\dagger} \beta(a) e^{-\int_0^a \tilde{\mu}(s) ds} da < 1.$$

Suppose that the above system III.2.13 is null controllable with control function v . Then, system III.1.1 is null controllable with control function $u = e^{\lambda t}v$, which has the same regularity as v .

From now on, without loss of generality (see Remark III.2.6), we assume that both semigroups \mathbb{T}^{A_0} and \mathbb{T}^A are exponentially stable, which implies that there exists a constant $C \geq 0$ and a constant $M \geq 0$ such that for every $t \geq 0$, we have

$$\|\mathbb{T}_t^{A_0}\|_{\mathcal{L}(L^2[0, a_\dagger])} \leq C \quad \text{and} \quad \|\mathbb{T}_t^A\|_{\mathcal{L}(X)} \leq M \quad (t \geq 0). \quad (\text{III.2.14})$$

III.3 LOW FREQUENCY CONTROL

In this section, we prove that the projection of the state trajectory of III.1.1 on an infinite subspace of X (defined using the eigenfunctions of the Neumann Laplacian) can be steered to zero in any time and we estimate the norm of the associated control. More precisely, let $\{\varphi_j\}_{j \geq 0}$ be an orthonormal basis in $L^2(\Omega)$ formed of eigenvectors of the Neumann Laplacian and let $(\lambda_j)_{j \geq 0}$ be the corresponding non decreasing sequence of eigenvalues. In other words $(\varphi_j)_{j \geq 0}$ is an orthonormal basis in $L^2(\Omega)$ such that for every $j \geq 0$ we have

$$\begin{cases} -\Delta \varphi_j = \lambda_j \varphi_j & \text{in } \Omega, \\ \frac{\partial \varphi_j}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \quad (\text{III.3.1})$$

In the sequel, for any $\mu > 0$, we denote by

$$\begin{aligned} \mathcal{N}(\mu) &:= \text{Card}\{k : \lambda_k \leq \mu\}, \\ E_\mu &= \text{Span}\{\phi_k : \lambda_k \leq \mu\}, \end{aligned} \quad (\text{III.3.2})$$

and $\Pi_{E_\mu} : L^2(\Omega) \rightarrow L^2(\Omega)$ the orthogonal projection from $L^2(\Omega)$ onto E_μ . The main result of this section is:

Proposition III.3.1. *Let $\mu > 0$, and let $T > 0$. There exists $u_\mu \in L^\infty((0, T); X)$ such that the solution p of III.1.1 satisfies*

$$\Pi_{E_\mu} p(T, a, \cdot) = 0 \quad (a \in (0, a_\dagger) \text{ a.e.})$$

Moreover, we have the following estimate:

$$\|u_\mu\|_{L^\infty((0, T); X)} \leq \frac{C e^{C\sqrt{\mu}}}{T} \|p_0\|_X.$$

The main ingredient of the proof is an inequality involving the eigenfunctions of the Neumann Laplacian. This inequality, obtained in Jerison and Lebeau [JL99], can also be obtained by combining results and methods from Lebeau and Robbiano [LR95], [LR97] (see also Lü [L13]).

Theorem III.3.2. *For any non-empty open subset ω of Ω , there exists $C > 0$ such that for any $\mu > 0$, for any sequence $(a_j)_{j \geq 0} \subset \mathbb{R}$, we have*

$$\sum_{j: \lambda_j \leq \mu} |a_j|^2 \leq C e^{C\sqrt{\mu}} \int_\omega \left| \sum_{j: \lambda_j \leq \mu} a_j \varphi_j(x) \right|^2 dx. \quad (\text{III.3.3})$$

We will also use the following classical lemma, whose proof can be found, for instance, in [TT11] - Section 2.

Lemma III.3.3. *Suppose that W, Y, Z are Hilbert spaces, $\mathcal{F} \in \mathcal{L}(W, Z)$ and that $\mathcal{G} \in \mathcal{L}(Y, Z)$. Denote by A^* the adjoint of an operator A . Then the following statements are equivalent:*

(i) *there exists $c > 0$ such that*

$$\|\mathcal{F}^* z\|_W \leq c \|\mathcal{G}^* z\|_Y \quad (z \in Z);$$

(ii) *there exists $\mathcal{H} \in \mathcal{L}(W, Y)$ such that $\mathcal{G}\mathcal{H} = \mathcal{F}$ and $\|\mathcal{H}\|_{\mathcal{L}(W, Y)} \leq c$.*

Proof of Proposition III.3.1. Note that the solution p of III.1.1 writes

$$p(t, a, \cdot) = \sum_{j=0}^{+\infty} p^j(t, a) \varphi_j \quad \text{in } L^2(\Omega), \text{ a.e. in } (0, T) \times (0, a_\dagger),$$

where p^j denotes the solution of

$$\begin{cases} \partial_t p^j(t, a) + \partial_a p^j(t, a) + (\mu(a) + \lambda_j) p^j(t, a) = \int_{\omega} u(t, a, x) \varphi_j(x) dx, \\ p^j(t, 0) = \int_0^{a_{\dagger}} \beta(a) p^j(t, a) da, \\ p^j(0, a) = p_0^j(a), \end{cases} \quad (\text{III.3.4})$$

and where

$$p_0(a, \cdot) = \sum_{j=0}^{+\infty} p_0^j(a) \varphi_j \quad \text{in } L^2(\Omega), \quad \text{a.e. } a \in (0, a_{\dagger}).$$

The aim is to solve the following moment problem: find $u \in L^\infty((0, T); X)$ such that for every $j \in [0, \mathcal{N}(\mu)]$, we have

$$\int_{\omega} u_{\mu}(t, a, x) \varphi_j(x) dx = v_j(t, a) \quad ((t, a) \in (0, T) \times (0, a_{\dagger}) \quad \text{a.e.}), \quad (\text{III.3.5})$$

where v_j denotes a null control associated to the system

$$\begin{cases} \partial_t p^j(t, a) + \partial_a p^j(t, a) + (\mu(a) + \lambda_j) p^j(t, a) = v_j(t, a), & t > 0, a \in (0, a_{\dagger}), \\ p^j(t, 0) = \int_0^{a_{\dagger}} \beta(a) p^j(t, a) da, & t > 0, \\ p^j(0, a) = p_0^j(a), & a \in (0, a_{\dagger}). \end{cases} \quad (\text{III.3.6})$$

Recall from Proposition III.2.4 and assumptions stated in Remark III.2.6 that for every $j \geq 0$, there exists $v_j \in L^\infty((0, T); L^2(0, a_{\dagger}))$ so that the solution p^j of the above system III.3.6 satisfies $p^j(T, a) = 0$ for almost every $a \in (0, a_{\dagger})$, with $\|v_j\|_{L^\infty((0, T); L^2(0, a_{\dagger}))} \leq \frac{C}{T} \|p_0^j\|_{L^2(0, a_{\dagger})}$, where C is a constant independent of λ_j (since we can choose $v_j(t, a) = -\frac{e^{-t\lambda_j}}{T} (\mathbb{T}_t^{A_0} p_0^j)(a)$).

Let $\mu > 0$. Define the map $\mathcal{G} : L^2(\omega) \rightarrow \mathbb{R}^{\mathcal{N}(\mu)+1}$ by

$$\mathcal{G}u := \left(\int_{\omega} u(x) \varphi_j(x) dx \right)_{0 \leq j \leq \mathcal{N}(\mu)}.$$

It is easy to check that for every $w = (w_j)_{0 \leq j \leq \mathcal{N}(\mu)} \in \mathbb{R}^{\mathcal{N}(\mu)+1}$, we have

$$\mathcal{G}^* w = \sum_{j=0}^{\mathcal{N}(\mu)} w_j \varphi_j. \quad (\text{III.3.7})$$

Inequality III.3.3 ensures that for every $w \in \mathbb{R}^{\mathcal{N}(\mu)+1}$, we have

$$\|w\|_{\mathbb{R}^{\mathcal{N}(\mu)+1}} \leq C e^{C\sqrt{\mu}} \|\mathcal{G}^* w\|_{L^2(\omega)}. \quad (\text{III.3.8})$$

By Lemma III.3.3 with $W = Z = \mathbb{R}^{\mathcal{N}(\mu)+1}$, $Y = L^2(\omega)$ and $\mathcal{F} = \text{Id}_{\mathcal{L}(\mathbb{R}^{\mathcal{N}(\mu)+1})}$, it follows that there exists $\mathcal{H} \in \mathcal{L}(\mathbb{R}^{\mathcal{N}(\mu)+1}, L^2(\omega))$ such that $\mathcal{G}\mathcal{H} = \text{Id}_{\mathcal{L}(\mathbb{R}^{\mathcal{N}(\mu)+1})}$. Moreover, $\|\mathcal{H}\|_{\mathcal{L}(\mathbb{R}^{\mathcal{N}(\mu)+1})} \leq C e^{C\sqrt{\mu}}$ so that for every $w \in \mathbb{R}^{\mathcal{N}(\mu)+1}$, there exists $u := \mathcal{H}(w) \in L^2(\omega)$ such that $\mathcal{G}u = w$ and $\|u\|_{L^2(\omega)} \leq C e^{C\sqrt{\mu}} \|w\|_{\mathbb{R}^{\mathcal{N}(\mu)+1}}$.

Let $(t, a) \in (0, T) \times (0, a_+)$. Setting $w := (v_j(t, a))_{0 \leq j \leq \mathcal{N}(\mu)}$ where v_j is the null control defined by Proposition III.2.4 (with $\mu(a)$ replaced by $(\mu(a) + \lambda_j)$), it follows that there exists $u_\mu(t, a) \in L^2(\omega)$ such that $\mathcal{G}u_\mu(t, a) = w$, *i.e.*

$$\int_{\omega} u_\mu(t, a, x) \varphi_j(x) dx = v_j(t, a) \quad (j \in [0, \mathcal{N}(\mu)], (t, a) \in (0, T) \times (0, a_+) \text{ a.e.}), \quad (\text{III.3.9})$$

with

$$\|u_\mu(t, a, \cdot)\|_{L^2(\omega)} \leq C e^{C\sqrt{\mu}} \|w\|_{\mathbb{R}^{\mathcal{N}(\mu)+1}}. \quad (\text{III.3.10})$$

From the above inequality III.3.9, it follows that

$$\|u_\mu\|_{L^\infty((0, T); X)}^2 \leq C e^{C\sqrt{\mu}} \sum_{j=0}^{\mathcal{N}(\mu)} \|v_j\|_{L^\infty((0, T); L^2(0, a_+))}^2, \quad (\text{III.3.11})$$

where $\|v_j\|_{L^\infty((0, T); L^2(0, a_+))}^2 \leq \frac{C}{T^2} \|p_0^j\|_{L^2(0, a_+)}^2$ by Proposition III.2.4, so that from the above inequality we have

$$\|u_\mu\|_{L^\infty((0, T); X)}^2 \leq \frac{C e^{C\sqrt{\mu}}}{T^2} (\mathcal{N}(\mu) + 1) \|p_0\|_X^2.$$

By Weyl's formula (see, for instance, Netrusov and Safarov [NS05] for a reminder), there exists a constant $K > 0$ such that $\mathcal{N}(\mu) \leq K\mu^{\frac{n}{2}}$, so that from the above formula we have

$$\|u_\mu\|_{L^\infty((0, T); X)} \leq \frac{C e^{C\sqrt{\mu}}}{T} \|p_0\|_X,$$

for some constant $C \geq 0$. □

III.4 PROOF OF THE MAIN RESULT

In this section we prove Theorem III.1.1 by applying a version of the Lebeau-Robbiano strategy, initially proposed for the heat equation. A distinctive feature of the version of this methodology we propose here is that the projected systems are infinite dimensional,

being described by equations similar to the Lotka-McKendrick system without spatial diffusion. To apply this strategy, we need these systems to be null controllable in arbitrarily small time so that our method is limited to the case of a control which is active for all ages.

Recall from Proposition III.3.1 that, given a time $T > 0$ and a frequency $\mu > 0$, there exists $u_\mu \in L^\infty((0, T); X)$ such that the solution p of III.1.1 belongs to the orthogonal of E_μ at time T , for every $a \in (0, a_+)$. The control cost behaves like $e^{C\sqrt{\mu}}$, and can be compensated by the natural dissipation of the solution stated in the following proposition.

Proposition III.4.1. *Let $\mu > 0$. Suppose that $\Pi_{E_\mu} p_0(a, \cdot) = 0$ for almost every $a \in (0, a_+)$. Then there exists a constant $M \geq 0$ such that for every $t > 0$, the solution p of III.1.1 with $u \equiv 0$ satisfies*

$$\|p(t, \cdot, \cdot)\|_X \leq M e^{-\mu t} \|p_0\|_X \quad (t \geq 0).$$

Proof. Suppose that $\Pi_{E_\mu} p_0(a) = 0$ for almost every $a \in (0, a_+)$. With $u \equiv 0$, the solution p of III.1.1 satisfies

$$p(t, a, \cdot) = \sum_{j: \lambda_j > \mu} p^j(t, a) \varphi_j \quad \text{in } L^2(\Omega), \quad \text{a.e. in } (0, \tau) \times (0, a_+),$$

where

$$\begin{cases} \partial_t p^j(t, a) + \partial_a p^j(t, a) + (\mu(a) + \lambda_j) p^j(t, a) = 0 & t > 0, \quad a \in (0, a_+), \\ p^j(t, 0) = \int_0^{a_+} \beta(a) p^j(t, a) da, & t > 0, \\ p^j(0, a) = p_0^j(a), & a \in (0, a_+). \end{cases} \quad (\text{III.4.1})$$

Let $\lambda_j > \mu$. It is easy to check that the solution p^j of III.4.1 satisfies $p_j(t, a) = e^{-t\lambda_j} \mathbb{T}_t^{A_0}(p_0^j)(a)$, so that using III.2.14 we have

$$\begin{aligned} \|p^j(t, \cdot)\|_{L^2(0, a_+)} &\leq M e^{-\lambda_j t} \|p_0^j\|_{L^2(0, a_+)} \\ &\leq M e^{-\mu t} \|p_0^j\|_{L^2(0, a_+)}, \end{aligned} \quad (\text{III.4.2})$$

since $\lambda_j > \mu$. Using

$$\|p(t, a, \cdot)\|_{L^2(\Omega)}^2 = \sum_{j: \lambda_j > \mu} |p^j(t, a)|^2,$$

it follows that we have

$$\begin{aligned} \|p(t, \cdot, \cdot)\|_{L^2((0, a_+) \times \Omega)}^2 &= \sum_{j: \lambda_j > \mu} \|p^j(t, \cdot)\|_{L^2(0, a_+)}^2 \\ &\leq M^2 e^{-2\mu t} \sum_{j: \lambda_j > \mu} \|p_0^j\|_{L^2(0, a_+)}^2 \\ &\leq M^2 e^{-2\mu t} \|p_0\|_X^2, \end{aligned} \quad (\text{III.4.3})$$

so that the estimation of Proposition III.4.1 holds. \square

The following corollary is the key point of the strategy of Lebeau and Robbiano, originally developed for the null controllability of the heat equation [LR95]. Given a time $T > 0$, we construct a control in two steps in the time interval $(0, T)$: we control the first frequencies in the time interval $(0, T/2)$ with the cost obtained in Proposition III.3.1, and we let the system to evolve freely in the time interval $(T/2, T)$ in order to dissipate the energy possibly transferred to the high frequencies (see Proposition III.4.1).

Given Proposition III.3.1 and Proposition III.4.1, the following Corollary III.4.2 and the proof of Theorem III.1.1 can be obtained as a consequence of Theorem 2.1 from [BPS18]. For the sake of completeness, we provide detailed proofs.

Corollary III.4.2. *For every $\mu > 0$ and every $T \in (0, \tau)$, there exists $u_\mu \in L^\infty((0, T); X)$ such that*

$$\|u_\mu\|_{L^\infty((0, T); X)} \leq \frac{Ce^{C\sqrt{\mu}}}{T} \|p_0\|_X,$$

and

$$\|p(T, \cdot, \cdot)\|_X \leq Ce^{C\sqrt{\mu} - \frac{T\mu}{2}} \|p_0\|_X, \quad (\text{III.4.4})$$

where p is the solution of III.1.1 with control function u_μ .

Proof. Let $T \in (0, \tau)$. We first use Proposition III.3.1 on the time interval $(0, T/2)$. This gives us a control function $v_\mu \in L^\infty((0, T/2); X)$ such that the solution p of III.1.1 with control function v_μ satisfies

$$\Pi_{E_\mu} p(T/2, a) = 0 \quad (a \in (0, a_\dagger) \text{ a.e.}), \quad (\text{III.4.5})$$

and

$$\|v_\mu\|_{L^\infty((0, T/2); X)} \leq \frac{Ce^{C\sqrt{\mu}}}{T} \|p_0\|_X. \quad (\text{III.4.6})$$

In the second half interval $(T/2, T)$, we choose a null control in order to take advantage of the natural dissipation given by Proposition III.4.1. More precisely, for almost every $t \in (0, T)$, we set

$$u_\mu(t) = \begin{cases} v_\mu(t) & \text{if } t \in (0, T/2), \\ 0 & \text{if } t \in (T/2, T). \end{cases} \quad (\text{III.4.7})$$

Thanks to III.4.6 and III.4.7, it is clear that we have

$$\|u_\mu\|_{L^\infty((0, T); X)} \leq \frac{Ce^{C\sqrt{\mu}}}{T} \|p_0\|_X, \quad (\text{III.4.8})$$

and since $u_\mu = 0$ on the time interval $(T/2, T)$, formula III.2.10 gives

$$p(T) = \mathbb{T}_{T/2}^A p(T/2), \quad (\text{III.4.9})$$

so that using Proposition III.4.1 together with III.4.5 and III.4.9, it follows that we have

$$\|p(T, \cdot, \cdot)\|_X \leq M e^{-\frac{T\mu}{2}} \|p(T/2, \cdot, \cdot)\|_X. \quad (\text{III.4.10})$$

Moreover, it follows from formula III.2.10 and III.4.7 that we have

$$p(T/2) = \mathbb{T}_{T/2}^A p_0 + \int_0^{T/2} \mathbb{T}_{T/2-\sigma}^A B v_\mu(\sigma) d\sigma,$$

and it follows from the assumption III.2.14 and Cauchy-Schwarz inequality that we have

$$\begin{aligned} \|p(T/2, \cdot, \cdot)\|_X &\leq M \|p_0\|_X + M \sqrt{T/2} \|v_\mu\|_{L^2((0, T/2); X)} \\ &\leq M \|p_0\|_X + \frac{M}{2} T \|v_\mu\|_{L^\infty((0, T/2); X)} \end{aligned} \quad (\text{III.4.11})$$

so that using III.4.6, III.4.10 and III.4.11 we have

$$\|p(T, \cdot, \cdot)\|_X \leq C e^{C\sqrt{\mu} - \frac{T\mu}{2}} \|p_0\|_X, \quad (\text{III.4.12})$$

for some constant $C \geq 0$. \square

We then use a time-splitting procedure, as described in [LR95], to get the null controllability of system III.1.1.

Proof of theorem III.1.1. Let $\tau > 0$. Consider the sequences

$$T_j = \frac{\tau}{2^j} \quad \text{and} \quad \mu_j = \beta(2^j)^2 \quad (j \geq 1),$$

where $\beta > 0$ is determined later. Denote by $\tau_0 = 0$ and $\tau_j = \sum_{k=1}^j T_k$, for every $j \geq 1$,

so that we have $(0, \tau) = \bigcup_{j \geq 0} (\tau_j, \tau_{j+1})$. Following the strategy of Lebeau and Robbiano [LR95], we will define a control by induction on each interval (τ_j, τ_{j+1}) which drives the initial state to zero in time τ .

Firstly, during the time interval $(0, \tau_1) = (0, T_1)$, we apply a control u_1 as given by Corollary III.4.2 with $\mu = \mu_1$, so that we have

$$\|u_1\|_{L^\infty((0, \tau_1); X)} \leq \frac{C e^{C\sqrt{\mu_1}}}{T_1} \|p_0\|_X,$$

and

$$\|p(\tau_1, \cdot, \cdot)\|_X \leq C e^{C\sqrt{\mu_1} - \frac{T_1 \mu_1}{2}} \|p_0\|_X.$$

Given $j \geq 1$, during the time interval (τ_{j-1}, τ_j) , we apply by induction a control function denoted by u_j as given by Corollary III.4.2 with $\mu = \mu_j$ so that we have

$$\|u_j\|_{L^\infty((\tau_{j-1}, \tau_j); X)} \leq \frac{C e^{C\sqrt{\mu_j}}}{T_j} \|p(\tau_{j-1}, \cdot, \cdot)\|_X, \quad (\text{III.4.13})$$

and

$$\|p(\tau_j, \cdot, \cdot)\|_X \leq C e^{C\sqrt{\mu_j} - \frac{T_j \mu_j}{2}} \|p(\tau_{j-1}, \cdot, \cdot)\|_X. \quad (\text{III.4.14})$$

From III.4.14, it follows that for every $j \geq 1$ we have

$$\|p(T_j, \cdot, \cdot)\|_X \leq C^j e^{C \sum_{k=1}^j \sqrt{\mu_k} - \frac{1}{2} \sum_{k=1}^j T_k \mu_k} \|p_0\|_X, \quad (\text{III.4.15})$$

with

$$C \sum_{k=1}^j \sqrt{\mu_k} - \frac{1}{2} \sum_{k=1}^j T_k \mu_k = 2(2^j - 1)(C\sqrt{\beta} - \frac{\beta}{2}\tau), \quad (\text{III.4.16})$$

by construction of the sequences $(T_j)_{j \geq 1}$ and $(\mu_j)_{j \geq 1}$. Then we choose $\beta > 0$ large enough so that

$$\tilde{\beta} := \frac{\beta}{2}\tau - C\sqrt{\beta} > 0, \quad (\text{III.4.17})$$

so that from III.4.15, III.4.16 and III.4.17 we have

$$\|p(\tau_j, \cdot, \cdot)\|_X \leq K C^j e^{-\tilde{\beta} 2^{j+1}} \|p_0\|_X, \quad (\text{III.4.18})$$

for some constant $K \geq 0$.

Going back to III.4.13 and using the estimate given by III.4.18, it follows that we have

$$\begin{aligned} \|u_j\|_{L^\infty((\tau_{j-1}, \tau_j); X)} &\leq \frac{K C}{\tau} e^{C\sqrt{\beta} 2^j} 2^{-j} C^{j-1} e^{-2\tilde{\beta}(2^{j-1}-1)} \|p_0\|_X \\ &= \frac{\tilde{K}}{\tau} C^j 2^{-j} e^{2^j(C\sqrt{\beta}-\tilde{\beta})} \|p_0\|_X, \end{aligned} \quad (\text{III.4.19})$$

with $\tilde{K} = K e^{2\tilde{\beta}}$. We choose $\beta > 0$ large enough such that

$$\bar{\beta} := \tilde{\beta} - C\sqrt{\beta} = \frac{\beta}{2}\tau - 2C\sqrt{\beta} > 0, \quad (\text{III.4.20})$$

so that from III.4.19 and III.4.20 we have

$$\|u_j\|_{L^\infty((\tau_{j-1}, \tau_j); X)} \leq \frac{\tilde{K}}{\sqrt{\tau}} C^j 2^{-j} e^{-\bar{\beta} 2^j} \|p_0\|_X. \quad (\text{III.4.21})$$

The above estimate III.4.21 ensures that

$$\sup_{j \geq 1} \|u_j\|_{L^\infty((\tau_{j-1}, \tau_j); X)} < +\infty, \quad (\text{III.4.22})$$

so that defining the control u by

$$u = \sum_{j=1}^{+\infty} u_j \mathbb{1}_{(\tau_{j-1}, \tau_j)} \tag{III.4.23}$$

gives a control function in $L^\infty((0, \tau); X)$. The corresponding trajectory p is continuous in time with values in X and satisfies

$$\|p(\tau_j, \cdot, \cdot)\|_X \xrightarrow{j \rightarrow \infty} 0, \tag{III.4.24}$$

thanks to estimation III.4.18. This implies that $p(\tau, \cdot, \cdot) = 0$, since $\tau_j \rightarrow \tau$ as $j \rightarrow +\infty$.

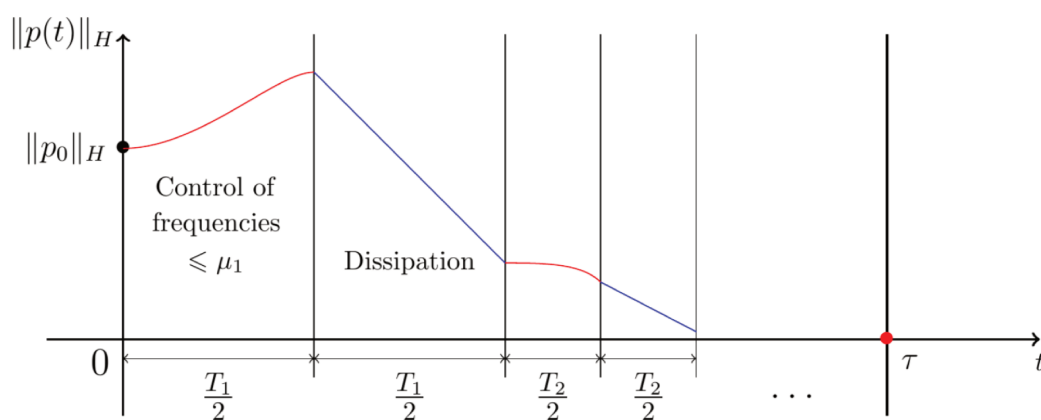


Figure III.2 – Adaptation of the Lebeau-Robbiano strategy : evolution shape of the norm of the controlled solution.

□

CHAPITRE IV

Un principe de bang-bang pour l'équation de Lotka-McKendrick avec diffusion

Ce chapitre reprend la méthodologie du chapitre précédent afin de démontrer le contrôle à zéro des équations de Lotka et McKendrick avec diffusion, avec des contrôles $u \in L^\infty((0, \tau) \times (0, a_\dagger) \times \omega)$ supportés en temps dans un ensemble de Lebesgue quelconque de mesure positive dans $(0, \tau)$. Le but est d'obtenir les résultats associés de contrôle en temps optimal et un principe de type "bang-bang" à l'aide d'une méthode développée par Wang [Wan08], qui généralise celle de Lebeau et Robbiano [LR95].

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IV.1 INTRODUCTION

We consider a linear controlled age-structured population model with spatial diffusion described by the following system:

$$\begin{cases} \begin{cases} \partial_t p(t, a, x) + \partial_a p(t, a, x) + \mu(a)p(t, a, x) - \Delta p(t, a, x) \\ \quad = \chi_\omega(x)u(t, a, x), & t > 0, a \in (0, a_\dagger), x \in \Omega, \\ \frac{\partial p}{\partial \nu}(t, a, x) = 0, & t > 0, a \in (0, a_\dagger), x \in \partial\Omega, \\ p(t, 0, x) = \int_0^{a_\dagger} \beta(a)p(t, a, x) da, & t > 0, x \in \Omega, \\ p(0, a, x) = p_0(a, x), & a \in (0, a_\dagger), x \in \Omega. \end{cases} \end{cases} \quad (\text{IV.1.1})$$

In the above equations:

- $\Omega \subset \mathbb{R}^N$, $N \geq 1$, denotes a smooth connected bounded domain and Δ is the laplacian with respect to the variable x ;
- $\frac{\partial}{\partial \nu}$ denotes the derivation operator in the direction of the unit outer normal to $\partial\Omega$. We thus have homogeneous Neumann boundary conditions, thus the considered population is isolated from the exterior of Ω ;
- $p(t, a, x)$ denotes the distribution density of the population at time t , of age a at spatial position $x \in \Omega$;
- p_0 denotes the initial population distribution;
- $a_\dagger \in (0, +\infty)$ is the maximal life expectancy;
- $\beta(a)$ and $\mu(a)$ are positive functions denoting respectively the birth and death rates, which are supposed to be independent of t and x ;
- $\omega \subset \Omega$ is a nonempty open subset of Ω and χ_ω denotes the characteristic function of ω .

We make the following classical assumptions on β and μ :

- (H1) $\beta \in L^\infty(0, a_\dagger)$, $\beta(a) \geq 0$ for almost every $a \in (0, a_\dagger)$,
- (H2) $\mu \in L^1[0, a^*]$ for every $a^* \in (0, a_\dagger)$, $\mu(a) \geq 0$ for almost every $a \in (0, a_\dagger)$,
- (H3) $\int_0^{a_\dagger} \mu(a) da = +\infty$.

In a recent work, Hegoburu and Tucsnak [HT17] proved that the above system (IV.1.1) is null controllable in any time $\tau > 0$, in the sense that for any $p_0 \in L^2((0, a_\dagger) \times \Omega)$, there exists a control function $u \in L^\infty((0, \tau); L^2((0, a_\dagger) \times \omega))$ such that the corresponding solution p of (IV.1.1) satisfies

$$p(\tau, a, x) = 0 \quad ((a, x) \in (0, a_\dagger) \times \Omega \text{ a.e.}).$$

Our aim is to study the associated time optimal control problem, in an L^∞ setting. More precisely, given $M > 0$, we define the set of admissible controls by

$$\mathcal{U}_{ad} := \{u \in L^\infty([0, \infty) \times (0, a_+) \times \omega) \mid |u(t, a, x)| \leq M \text{ a.e. in } [0, \infty) \times (0, a_+) \times \omega\}.$$

Given $p_0 \in L^\infty((0, a_+); L^2(\Omega))$, we define the set of reachable states from p_0 as

$$\mathcal{R}(p_0, \mathcal{U}_{ad}) := \{p(\tau, \cdot, \cdot) \mid \tau > 0 \text{ and } p \text{ is the solution of (IV.1.1) with } u \in \mathcal{U}_{ad}\}.$$

For $p_0 \in L^\infty((0, a_+); L^2(\Omega))$ and $p_1 \in \mathcal{R}(p_0, \mathcal{U}_{ad})$, the *time optimal control problem for system (IV.1.1)* consists in determining an input $u^* \in \mathcal{U}_{ad}$ such that the corresponding solution p^* of (IV.1.1) satisfies

$$p^*(\tau^*(p_0, p_1)) = p_1,$$

where $\tau^*(p_0, p_1)$ is the minimal time needed to steer the initial data p_0 towards the target population p_1 with controls in \mathcal{U}_{ad} ,

$$\tau^*(p_0, p_1) = \min_{u \in \mathcal{U}_{ad}} \{\tau \mid p(\tau, \cdot, \cdot) = p_1\}. \quad (\text{IV.1.2})$$

The main result in this work asserts that the solution is bang-bang and unique. More precisely, we have

Theorem IV.1.1. *With the above notations and assumptions, for any $p_0 \in L^\infty((0, a_+); L^2(\Omega))$ and any $p_1 \in \mathcal{R}(p_0, \mathcal{U}_{ad})$, there exists a unique solution u^* of the time optimal control problem (IV.1.2). This solution u^* has the bang-bang property:*

$$|u(t, a, x)| = M \text{ a.e. in } [0, \infty) \times (0, a_+) \times \omega. \quad (\text{IV.1.3})$$

It is known (see, for instance, Micu, Roventa and Tucsnak [MRT12] - Proposition 2.6.) that the existence and uniqueness of the time optimal control problem (IV.1.2) which has the bang-bang property (IV.1.3) may be induced by the L^∞ null-controllability of the system (IV.1.1) in any arbitrary time $\tau > 0$ over any subset E of positive measure in $[0, \tau]$. Hence, in order to prove the above Theorem IV.1.1, we need to derive from [HT17] the following L^∞ null controllability result for the Lotka-McKendrick equation:

Theorem IV.1.2. *With the above notations and assumptions, let τ be a positive constant and let E be a subset of the interval $[0, \tau]$ with positive measure. Then the system (IV.1.1) is L^∞ null-controllable in time $\tau > 0$ over E , in the sense that for every $p_0 \in L^\infty((0, a_+); L^2(\Omega))$, there exists a control $u \in L^\infty((0, \tau) \times (0, a_+) \times \omega)$ such that the*

solution p of the following controlled population equation:

$$\begin{cases} \partial_t p(t, a, x) + \partial_a p(t, a, x) + \mu(a)p(t, a, x) - \Delta p(t, a, x) \\ \quad = \chi_E(t)\chi_\omega(x)u(t, a, x), & t \in (0, \tau), a \in (0, a_\dagger), x \in \Omega, \\ \frac{\partial p}{\partial \nu}(t, a, x) = 0, & t \in (0, \tau), a \in (0, a_\dagger), x \in \partial\Omega, \\ p(t, 0, x) = \int_0^{a_\dagger} \beta(a)p(t, a, x) da, & t \in (0, \tau), x \in \Omega, \\ p(0, a, x) = p_0(a, x), & a \in (0, a_\dagger), x \in \Omega, \end{cases} \quad (\text{IV.1.4})$$

satisfies $p(\tau, a, x) = 0$ for almost every $(a, x) \in (0, a_\dagger) \times \Omega$. Moreover, we have

$$\|u\|_{L^\infty((0, \tau) \times (0, a_\dagger) \times \omega)} \leq L \|p_0\|_{L^\infty((0, a_\dagger); L^2(\Omega))}, \quad (\text{IV.1.5})$$

where L is a positive constant independent of p_0 .

Remark IV.1.3. In this work, the initial condition p_0 is restricted to belong to $L^\infty((0, a_\dagger); L^2(\Omega))$ in order to get a corresponding L^∞ null controllability result - more precisely, in the case when p_0 does not belong to $L^\infty((0, a_\dagger); L^2(\Omega))$, there may not exist a control function in $L^\infty((0, \tau) \times (0, a_\dagger) \times \omega)$ driving p_0 to zero in any arbitrary small time τ .

The null-controllability of the the system modelling age-dependant population dynamics is by now well understood in the case in which diffusion is neglected (see Barbu, Ianneli and Martcheva [BIM01], Hegoburu, Magal, Tucsnak [HMT18], Maity [Mai18], Anița and Hegoburu [AH18]). In the case when spatial diffusion is taken into account, namely for (IV.1.1), the particular case when the control acts in the whole space (the case corresponding to $\omega = \Omega$) was investigated by S. Anița (see [Ani00], p 148). The case when the control acts in a spatial subdomain ω was firstly studied by B. Ainseba [Ain02], where the author proves the null controllability of the above system (IV.1.1), except for a small interval of ages near zero. The case when the control acts in a spatial subdomain ω and also only for small age classes was investigated by B. Ainseba and S. Anița [AA04], for initial data p_0 in a neighborhood of the target \tilde{p} . As already mentioned, Hegoburu and Tucsnak proved the null controllability of system (IV.1.1), using an adaptation of the Lebeau Robbiano strategy originally developed for the null-controllability of the heat equation. This result has been recently improved by Maity, Tucsnak and Zuazua [MTZ18b], assuming that the young individuals are not able to reproduce before some age $a_b > 0$, where the control function u in system (IV.1.1) has support in some interval of ages $[a_1, a_2]$, where $0 \leq a_1 < a_2 \leq a_\dagger$. In [MTZ18b] the authors proved the null controllability result with this additional age restriction, provided that the control time τ is large enough, and the age a_1 is smaller than a_b . Related approximate and exact controllability issues have also been studied in Ainseba and Langlais [AL00], Ainseba and Ianneli [AI⁺03], Traore [Tra06], Kavian and Traore [KT11].

The time optimal control problems for age-structured populations dynamics without diffusion has been extensively studied in the past decades, essentially in the case when the

control acts as an harvesting rate (see, for instance, Brokate [Bro85], Barbu and Iannelli [BI99]). In this case, the bang-bang structure of the optimal harvesting has been obtained in several papers (see, for instance, Medhin [Med92], Anița [Ani98, Ani00] Anița et al [PIKA98], Hritonenko and Yatsenko [HY07], Hritonenko et al [HYGX09] and references therein). The literature devoted to the time optimal additive control problems for age-structured populations dynamics with spatial diffusion (namely for system (IV.1.1)) is less abundant, but several important results and methods are available for the heat equation (see, for instance, Apraiz, Escauriaza, Wang, Zhang [AEWZ14], Wang [Wan08], Micu, Roventa, Tucsnak [MRT12], Wang and Zhang [WZ17], Apraiz and Escauriaza [AE13] and references therein). Here, we shall use the strategy developed by Wang [Wan08] (which is, roughly speaking, a generalization of the Lebeau Robianno strategy) with the methodology developed in Hegoburu and Tucsnak [HT17] (based on the Lebeau Robianno strategy) in order to prove the L^∞ null controllability of system (IV.1.4) in any time $\tau > 0$ over any subset $E \subset (0, \tau)$ of positive measure. As already claimed, this L^∞ null controllability result implies the bang-bang principle stated in Theorem IV.1.1.

The remaining part of this work is organized as follows. In Section IV.2 we introduce some basic results on the Lotka-McKendrick semigroup without spatial diffusion, corresponding to the L^2 and the L^∞ settings, and we state an L^∞ null controllability result associated to system (IV.1.4) without spatial diffusion. In Section IV.3 we recall and introduce some results corresponding to the Lotka-McKendrick semigroup with spatial diffusion, associated to the L^2 and the L^∞ settings. Section IV.4 is devoted to study the L^∞ null controllability of low frequencies for the solution of system (IV.1.4). We prove the main result in Section IV.5, by using a strategy developed by Wang [Wan08], originally intended to study the time optimal internal controls for the heat equation.

Notation: In all what follows, C will denote a generic constant, depending only of the coefficients in (IV.1.1), on Ω and ω , whose value may change from line to line.

IV.2 SOME BACKGROUND ON THE LOTKA-MCKENDRICK SEMIGROUP WITHOUT DIFFUSION

This section is devoted to recall some existing results on the population semigroup for the linear age-structured model without spatial diffusion relatively to the classical L^2 setting, and to introduce the corresponding results relatively to the L^∞ setting. In particular, we recall the structure of the spectrum of the semigroup generator and we shall state a null controllability result, in the L^∞ setting, concerning the diffusion free case.

§ IV.2.1. The free diffusion semigroup in $L^2(0, a_+)$

In this paragraph we remind some results on the diffusion free case, which is described by the so-called McKendrick-Von Foster model. We do not give proofs and we refer, for instance, to Song et al. [SYW⁺82] or Inaba [Ina17] for a detailed presentation of these issues.

The considered system is:

$$\begin{cases} \partial_t p(t, a) + \partial_a p(t, a) = -\mu(a)p(t, a), & t > 0, a \in (0, a_+), \\ p(t, 0) = \int_0^{a_+} \beta(a)p(t, a) da, & t > 0, \\ p(0, a) = p_0(a), & a \in (0, a_+), \end{cases}$$

where μ and β satisfy the assumptions in Theorem IV.1.2.

The above system is described by the operator A_0 defined by

$$\begin{aligned} \mathcal{D}(A_0) &= \left\{ \varphi \in L^2[0, a_+] \mid \varphi(0) = \int_0^{a_+} \beta(a)\varphi(a) da; -\frac{d\varphi}{da} - \mu\varphi \in L^2[0, a_+] \right\}, \\ A_0\varphi &= -\frac{d\varphi}{da} - \mu\varphi \quad (\varphi \in \mathcal{D}(A_0)). \end{aligned} \tag{IV.2.1}$$

Theorem IV.2.1. *The operator A_0 defined by (IV.2.1) has compact resolvent and its spectrum is constituted of a countable (infinite) set of isolated eigenvalues with finite algebraic multiplicity. The eigenvalues $(\lambda_n^0)_{n \geq 1}$ of A_0 (counted without multiplicity) are the solutions of the characteristic equation*

$$F(\lambda) := \int_0^{a_+} \beta(a)e^{-\lambda a} \pi(a) da = 1. \tag{IV.2.2}$$

The eigenvalues $(\lambda_n^0)_{n \geq 1}$ are of geometric multiplicity one, the eigenspace associated to λ_n^0 being the one-dimensional subspace of $L^2(0, a_+)$ generated by the function

$$\varphi_n^0(a) = e^{-\lambda_n^0 a} \pi(a) = e^{-\lambda_n^0 a - \int_0^a \mu(s) ds}.$$

Finally, every vertical strip of the complex plane $\alpha_1 \leq \operatorname{Re}(z) \leq \alpha_2$, $\alpha_1, \alpha_2 \in \mathbb{R}$, contains a finite number of eigenvalues of A_0 .

Theorem IV.2.2. *The operator A_0 defined by (IV.2.1) has a unique real eigenvalue λ_1^0 . Moreover, we have the following properties :*

1. λ_1^0 is of algebraic multiplicity one;
2. $\lambda_1^0 > 0$ (resp. $\lambda_1^0 < 0$) if and only if $F(0) > 1$ (resp. $F(0) < 1$);
3. λ_1^0 is a real dominant eigenvalue:

$$\lambda_1^0 > \operatorname{Re}(\lambda_n^0), \quad \forall n \geq 2.$$

It is well known (see, for instance, Song et al. [SYW⁺82] or Kappel and Zhang [KZ93]) that A generates a C^0 semigroup of linear operators in $L^2[0, a_+]$ which we denote by $\mathbb{T}^{A_0} = (\mathbb{T}_t^{A_0})_{t \geq 0}$. We also have the following useful result (see, for instance, [Ina17, p 23]):

Proposition IV.2.3. *The semigroup \mathbb{T}^{A_0} generated on $L^2([0, a_+])$ by A_0 is compact for $t \geq a_+$.*

According to Zabczyk [Zab76, Section 2]), this implies in particular that

$$\omega_a(A_0) = \omega_0(A_0),$$

where $\omega_a(A_0) := \lim_{t \rightarrow +\infty} t^{-1} \ln \|\mathbb{T}_t^{A_0}\|_{L^2(0, a_+)}$ denotes the growth bound of the semigroup $\mathbb{T}_t^{A_0}$ and $\omega_0(A_0) := \sup\{\operatorname{Re} \lambda \mid \lambda \in \sigma(A_0)\}$ the spectral bound of A_0 . It is worth noticing that the above condition ensures that the exponential stability of \mathbb{T}^{A_0} is equivalent to the condition $\omega_0(A_0) < 0$. According to Theorem IV.2.1 and IV.2.2, it follows that the exponential stability of \mathbb{T}^{A_0} is equivalent to the condition $\lambda_1^0 < 0$, where $\lambda_1^0 < 0$ is the unique real solution to the characteristic equation defined by (IV.2.2).

The following subsection is intended to show that the condition $\lambda_1^0 < 0$ is also sufficient to get a stability type result in the L^∞ setting.

§ IV.2.2. About the diffusion free semigroup in $L^\infty(0, a_+)$

This subsection is devoted to discuss some properties of the Lotka-McKendrick semigroup \mathbb{T}^{A_0} without diffusion, in the case when the initial inputs are restricted to belong to the state space $L^\infty(0, a_+)$. More precisely, we shall introduce an L^∞ exponential stability type result, and state an L^∞ null-controllability result associated to system (IV.1.4) in the diffusion free case.

The following Lemma IV.2.4 states that the condition $\lambda_1^0 < 0$ (where $\lambda_1^0 < 0$ is the unique real solution to the characteristic equation defined by (IV.2.2)) is sufficient to get an L^∞ exponential stability type result:

Lemma IV.2.4. *Let λ_1^0 be the unique real solution to the characteristic equation defined by (IV.2.2). There exists a constant $C \geq 0$ such that for every $p_0 \in L^\infty(0, a_+)$, we have*

$$|\mathbb{T}_t^{A_0} p_0(a)| \leq \begin{cases} |p_0(a-t)| & \text{if } a \geq t, \\ C e^{\lambda_1^0(t-a)} \|p_0\|_{L^1(0, a_+)} & \text{if } t > a. \end{cases} \quad (\text{IV.2.3})$$

Proof. Let $p_0 \in L^\infty(0, a_+)$. It is well known (see, for instance, Iannelli [Ian95b] or Webb [Web85]) that the semigroup \mathbb{T}^{A_0} satisfies

$$(\mathbb{T}_t^{A_0} p_0)(a) = \begin{cases} \frac{\pi(a)}{\pi(a-t)} p_0(a-t) & \text{if } t \leq a, \\ \pi(a) b(t-a) & \text{if } t > a, \end{cases} \quad (\text{IV.2.4})$$

where $\pi(a) = e^{-\int_0^a \mu(\sigma) d\sigma}$ is the probability of survival of an individual from age 0 to a and $b(t) = (\mathbb{T}_t^{A_0} p_0)(0) = \int_0^{a_\dagger} \beta(a)(\mathbb{T}_t^{A_0} p_0)(a) da$ is the total birth rate function.

If $a \geq t$, it is clear that we have $|\mathbb{T}_t^{A_0} p_0(a)| \leq |p_0(a-t)|$ due to (IV.2.4), so that the first estimate of (IV.2.3) holds.

Let $s > 0$. It is shown in Iannelli [Ian95b, p. 21 and 22] (see also Anița [Ani00, p. 54]) that for every $s > 0$ we may write the total birth rate $b(s)$ in the following form:

$$b(s) = e^{\lambda_1^0 s} (b_0 + e^{-\lambda_1^0 s} F(s) + \Omega_0(s)) \quad (s > 0), \quad (\text{IV.2.5})$$

where b_0 is a nonnegative constant satisfying

$$0 \leq b_0 \leq M_0 \|p_0\|_{L^1(0, a_\dagger)}, \quad (\text{IV.2.6})$$

the function $\Omega_0 \in L^\infty(0, +\infty)$ satisfies, for every $s > 0$,

$$|\Omega_0(s)| \leq M_0 \|p_0\|_{L^1(0, a_\dagger)}, \quad (\text{IV.2.7})$$

and the map $F \in L^\infty(0, +\infty)$ is defined by

$$F(s) := \begin{cases} \int_s^{a_\dagger} \beta(x) \frac{\pi(x)}{\pi(x-s)} p_0(x-s) dx & \text{if } s \in [0, a_\dagger], \\ 0 & \text{if } s > a_\dagger, \end{cases} \quad (\text{IV.2.8})$$

where the constant M_0 given in the inequalities (IV.2.6) and (IV.2.7) do not depend on p_0 . It follows from (IV.2.5), (IV.2.6), (IV.2.7) and (IV.2.8) that for every $s > 0$ we have

$$|b(s)| \leq e^{\lambda_1^0 s} (M_0 \|p_0\|_{L^1(0, a_\dagger)} + (\sup_{\sigma \in [0, a_\dagger]} e^{-\lambda_1^0 \sigma} \cdot \|\beta\|_{L^\infty(0, a_\dagger)} \|p_0\|_{L^1(0, a_\dagger)} + M_0 \|p_0\|_{L^1(0, a_\dagger)}) \quad (s > 0),$$

so that the second estimate of (IV.2.3) follows from the above inequality, setting

$$C := 3 \max \left(M_0 ; \left(\sup_{\sigma \in [0, a_\dagger]} e^{-\lambda_1^0 \sigma} \right) \cdot \|\beta\|_{L^\infty(0, a_\dagger)} \right).$$

□

Recall that the main motivation of this paper is to show that the system (IV.1.4) is null controllable in any time $\tau > 0$ with controls $u \in L^\infty((0, \tau) \times (0, a_\dagger) \times \omega)$ which are additionally supported in time in an arbitrary subset $E \subset (0, \tau)$ of positive measure. To this aim, it will be needed - in the following Section IV.4 - to get a corresponding null controllability result in the diffusion free case. More precisely, the diffusion free control

problem associated to system (IV.1.4) writes as

$$\begin{cases} \partial_t p(t, a) + \partial_a p(t, a) + \mu(a)p(t, a) = v(t, a)\chi_E(t) & t \in (0, \tau), a \in (0, a_\dagger), \\ p(t, 0) = \int_0^{a_\dagger} \beta(a)p(t, a) da, & t \in (0, \tau), \\ p(0, a) = p_0(a), & a \in (0, a_\dagger), \end{cases} \quad (\text{IV.2.9})$$

where v denotes the control function and E is a subset of $[0, \tau]$ with positive measure. Let us state an L^∞ null-controllability result for the above system (IV.2.9):

Proposition IV.2.5. *Under the above assumptions, let $\tau > 0$ and E be a subset of $[0, \tau]$ with positive measure, i.e. $m(E) > 0$. Then for every $p_0 \in L^\infty(0, a_\dagger)$, there exists $v \in L^\infty((0, \tau) \times (0, a_\dagger))$ such that the solution p of (IV.2.9) satisfies*

$$p(\tau, a) = 0 \quad (a \in (0, a_\dagger) \text{ a.e.}).$$

Moreover, for almost every $(t, a) \in (0, \tau) \times (0, a_\dagger)$, we have

$$|v(t, a)| \leq \frac{1}{m(E)} |\mathbb{T}_t^{A_0} p_0(a)| \quad ((t, a) \in (0, \tau) \times (0, a_\dagger) \text{ a.e.}). \quad (\text{IV.2.10})$$

Proof. Let $\tau > 0$. For almost every $(t, a) \in (0, \tau) \times (0, a_\dagger)$, we set

$$v(t, a) = -\frac{1}{m(E)} (\mathbb{T}_t^{A_0} p_0)(a) \quad (t \in (0, \tau), a \in (0, a_\dagger)). \quad (\text{IV.2.11})$$

With v defined as above, the (mild) solution p of (IV.2.9) satisfies

$$\begin{aligned} p(t, a) &= \mathbb{T}_t^{A_0} p_0(a) + \int_0^t (\mathbb{T}_{t-\sigma}^{A_0} \chi_E(\sigma) v(\sigma))(a) d\sigma \\ &= \mathbb{T}_t^{A_0} p_0(a) - \frac{1}{m(E)} \int_0^t \chi_E(\sigma) \mathbb{T}_{t-\sigma}^{A_0} (\mathbb{T}_\sigma^{A_0} p_0)(a) d\sigma \\ &= \mathbb{T}_t^{A_0} p_0(a) - \frac{1}{m(E)} \int_0^t \chi_E(\sigma) \mathbb{T}_t^{A_0} p_0(a) d\sigma \\ &= \mathbb{T}_t^{A_0} p_0(a) \left(1 - \frac{m(E \cap (0, t))}{m(E)} \right), \end{aligned}$$

so that we have $p(\tau, \cdot) = 0$ since E is a subset of $(0, \tau)$. The estimate (IV.2.10) is then a direct consequence of (IV.2.11). \square

IV.3 THE POPULATION DYNAMICS WITH DIFFUSION

This section is devoted to recall and introduce some results concerning the Lotka-McKendrick semigroup with spatial diffusion, in both L^2 and L^∞ settings. More precisely, we recall the structure of the spectrum of the semigroup generator and we give a stability result in both L^2 and L^∞ settings, conditionally to the sign of λ_1^0 (where λ_1^0 denotes the unique real solution to (IV.2.2)).

§ IV.3.1. The Lotka-McKendrick semigroup with diffusion in $L^2((0, a_+) \times \Omega)$

The existence of a semigroup on $L^2((0, a_+) \times \Omega)$ describing the linear age-structured population model with diffusion coefficient and age dependent birth and death rates, with homogeneous Neumann boundary conditions has been proved in Huyer [Huy94, Theorem 2.8] (see also Guo and Chan [GC94] for the case of homogeneous Dirichlet boundary conditions).

More precisely, let $H := L^2((0, a_+) \times \Omega)$ and let us consider the diffusion free population operator $A_1 : \mathcal{D}(A_1) \rightarrow H$ defined by

$$\begin{aligned} \mathcal{D}(A_1) &= \left\{ \varphi \in H \mid \varphi(\cdot, x) \text{ is locally absolutely continuous on } [0, a_+), \right. \\ &\quad \left. \varphi(0, x) = \int_0^{a_+} \beta(a) \varphi(a, x) da \text{ for a.e. } x \in \Omega, \frac{\partial \varphi}{\partial a} + \mu \varphi \in H \right\}, \\ A_1 \varphi &= -\frac{\partial \varphi}{\partial a} - \mu \varphi, \end{aligned}$$

and the diffusion operator $A_2 : \mathcal{D}(A_2) \rightarrow H$ defined by

$$\mathcal{D}(A_2) = \left\{ \varphi \in H \mid \frac{\partial \varphi}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}, \quad A_2 \varphi = \Delta \varphi.$$

The population operator with diffusion $A : \mathcal{D}(A) \rightarrow H$ is defined by

$$\mathcal{D}(A) = \mathcal{D}(A_1) \cap \mathcal{D}(A_2), \quad A\varphi = A_1\varphi + A_2\varphi.$$

The generator A of the population semigroup can be seen as the sum of a population operator without diffusion $-d/da - \mu I$ and a spatial diffusion term Δ . It turns out that spectral properties of A can be easily obtained from those of these two operators.

Theorem IV.3.1. *Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ be the increasing sequence of eigenvalues of $-\Delta$ with Neumann boundary conditions and let $(\varphi_n)_{n \geq 0}$ be a corresponding orthonormal basis of $L^2(\Omega)$. Let $(\lambda_n^0)_{n \geq 1}$ and $(\varphi_n^0)_{n \geq 1}$ be respectively the sequence of eigenvalues and eigenfunctions of the free diffusion operator A_0 defined by (IV.2.1) (see Theorem IV.2.1). Then the following assertions hold:*

1. *The eigenvalues of A are given by*

$$\sigma(A) = \{ \lambda_i^0 - \lambda_j \mid i \in \mathbb{N}^*, j \in \mathbb{N} \}.$$

2. A has a dominant eigenvalue:

$$\lambda_1^0 > \operatorname{Re}(\lambda), \quad \forall \lambda \in \sigma(A), \lambda \neq \lambda_1^0.$$

3. The eigenspace associated to an eigenvalue λ of A is given by

$$\operatorname{Span}\{\varphi_i^0(a)\varphi_j(x) = e^{-\lambda_i^0 a}\pi(a)\varphi_j(x) \mid \lambda_i^0 - \lambda_j = \lambda\}.$$

Since the operator A generates a \mathcal{C}^0 semigroup of linear operators in H which we denote by $\mathbb{T}^A = (\mathbb{T}_t^A)_{t \geq 0}$, this allows to define the concept of (mild) solution of (IV.1.4) in the following standard way: we say that p is a mild solution of (IV.1.4) if

$$p(t, \cdot) = \mathbb{T}_t^A p_0 + \Phi_{t,E} u \quad (t \geq 0, \quad u \in L^2([0, \infty); H)), \quad (\text{IV.3.1})$$

where the control operator $B \in \mathcal{L}(H)$ is defined by

$$Bu = \chi_\omega u \quad (u \in H),$$

and where

$$\Phi_{t,E} u = \int_0^t \mathbb{T}_{t-\sigma}^A B \chi_E(\sigma) u(\sigma) \, d\sigma \quad (t \geq 0, \quad u \in L^2([0, \infty); H)). \quad (\text{IV.3.2})$$

It is worth noticing, for instance by using a spectral decomposition, that the semigroup \mathbb{T}^A is exponentially stable if $\lambda_1^0 < 0$, where we recall that λ_1^0 denotes the unique real solution to the characteristic equation defined by (IV.2.2).

Remark IV.3.2. *In order to prove the null controllability of system (IV.1.4), we may assume, without loss of generality, that the so called reproductive number satisfies*

$$\int_0^{a_\dagger} \beta(a)\pi(a) \, da < 1,$$

which implies that the unique real solution λ_1^0 to the characteristic equation defined by (IV.2.2) satisfies $\lambda_1^0 < 0$. Indeed, in the case when $\int_0^{a_\dagger} \beta(a)\pi(a) \, da \geq 1$, we may consider the auxiliary system

$$\left\{ \begin{array}{l} \partial_t z(t, a, x) + \partial_a z(t, a, x) + \tilde{\mu}(a)z(t, a, x) - \Delta z(t, a, x) \\ \quad = \chi_\omega(x)\chi_E(t)v(t, a, x), \quad t > 0, \quad a \in (0, a_\dagger), \quad x \in \Omega, \\ \frac{\partial z}{\partial \nu}(t, a, x) = 0, \quad t > 0, \quad a \in (0, a_\dagger), \quad x \in \partial\Omega, \\ z(t, 0, x) = \int_0^{a_\dagger} \beta(a)z(t, a, x) \, da, \quad t > 0, \quad x \in \Omega, \\ z(0, a, x) = p_0(a, x), \quad a \in (0, a_\dagger), \quad x \in \Omega, \end{array} \right. \quad (\text{IV.3.3})$$

with $\tilde{\mu}(a) := \mu(a) + \lambda$, where $\lambda \geq 0$ is large enough to have

$$\int_0^{a_+} \beta(a) e^{-\int_0^a \tilde{\mu}(s) ds} da < 1.$$

Suppose that the above system (IV.3.3) is null controllable with control function v . Then, system (IV.1.4) is null controllable with control function $u = e^{\lambda t} v$, which has the same regularity as v .

From now on, without loss of generality (see the above Remark IV.3.2), we assume that the unique real solution λ_1^0 to the characteristic equation defined by (IV.2.2) satisfies the following assumption:

(Stable) : the unique real solution λ_1^0 to the characteristic equation (IV.2.2) satisfies $\lambda_1^0 < 0$.

Recall that the above assumption gives the stability of the semigroup \mathbb{T}^A in the space $H = L^\infty((0, a_+); L^2(\Omega))$. The aim of the following subsection is to prove that the above assumption **(Stable)** also induces some stability type results in the subspace $L^\infty((0, a_+); L^2(\Omega))$ of H .

§ IV.3.2. Stability results in $L^\infty((0, a_+); L^2(\Omega))$

In order to derive from the above subsection some stability results in the L^∞ setting, let us recall from Theorem IV.3.1 that $\{\varphi_j\}_{j \geq 0}$ denotes an orthonormal basis in $L^2(\Omega)$ formed of eigenvectors of the Neumann Laplacian and $(\lambda_j)_{j \geq 0}$ is the corresponding non decreasing sequence of eigenvalues. In other words $(\varphi_j)_{j \geq 0}$ is an orthonormal basis in $L^2(\Omega)$ such that for every $j \geq 0$ we have

$$\begin{cases} -\Delta \varphi_j = \lambda_j \varphi_j & \text{in } \Omega, \\ \frac{\partial \varphi_j}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

The following Lemma IV.3.3 states that the condition $\lambda_1^0 < 0$ (where λ_1^0 is the unique real solution the characteristic equation defined by (IV.2.2)) ensures some stability results in the space $L^\infty((0, a_+); L^2(\Omega))$. More precisely, we have

Lemma IV.3.3. *Under the assumption **(Stable)**, there exists a constant $C \geq 0$ such that for every $p_0 \in L^\infty((0, a_+); L^2(\Omega))$, for every $(t, a) \in (0, +\infty) \times (0, a_+)$ we have*

$$\sum_{j=0}^{+\infty} (\mathbb{T}_t^{A_0} p_0^j(a))^2 \leq C \|p_0\|_{L^\infty((0, a_+); L^2(\Omega))}^2 \quad (t \geq 0, a \in (0, a_+) \text{ a.e.}), \quad (\text{IV.3.4})$$

and for every $t \geq 0$ we have

$$\|\mathbb{T}_t^A p_0\|_{L^\infty((0, a_+); L^2(\Omega))}^2 \leq C \|p_0\|_{L^\infty((0, a_+); L^2(\Omega))}^2 \quad (t \geq 0), \quad (\text{IV.3.5})$$

where $p_0^j(a) := (p_0(a, \cdot), \varphi_j)_{L^2(\Omega)}$ for almost every $a \in (0, a_+)$ and every integer $j \geq 0$.

Proof. Let $p_0 \in L^\infty((0, a_+) \times \Omega)$. For every $j \geq 0$ and for almost every $a \in (0, a_+)$, denote by $p_0^j(a) = (p_0(a, \cdot), \varphi_j)_{L^2(\Omega)}$. Note that, for almost every $a \in (0, a_+)$ we have

$$\sum_{j=0}^{+\infty} (p_0^j(a))^2 = \int_{\Omega} p_0(a, x)^2 dx, \quad (\text{IV.3.6})$$

as a consequence of Parseval's formula.

Let $a \geq t$. Using (IV.2.3) together with (IV.3.6), we have

$$\sum_{j=0}^{+\infty} (\mathbb{T}_t^{A_0} p_0^j(a))^2 \leq \sum_{j=0}^{+\infty} (p_0^j(a-t))^2 = \int_{\Omega} p_0(a-t, x)^2 dx \leq \|p_0\|_{L^\infty((0, a_+); L^2(\Omega))}^2. \quad (\text{IV.3.7})$$

If $t \geq a$, since $\lambda_1^0 \leq 0$, from (IV.2.3) we have

$$\begin{aligned} \sum_{j=0}^{+\infty} (\mathbb{T}_t^{A_0} p_0^j(a))^2 &\leq C \sum_{j=0}^{+\infty} \|p_0^j\|_{L^1(0, a_+)}^2 \\ &\leq C \sum_{j=0}^{+\infty} \|p_0^j\|_{L^2(0, a_+)}^2 \\ &= C \|p_0\|_{L^2((0, a_+); L^2(\Omega))}^2 \\ &\leq C \|p_0\|_{L^\infty((0, a_+); L^2(\Omega))}^2, \end{aligned} \quad (\text{IV.3.8})$$

so that the estimation (IV.3.4) follows from (IV.3.7) and (IV.3.8).

In order to get the estimation (IV.3.5), notice that for almost every $(t, a) \in (0, +\infty) \times (0, a_+)$ we have

$$\mathbb{T}_t^A p_0(a, \cdot) = \sum_{j=0}^{+\infty} p^j(t, a) \varphi_j \quad \text{in } L^2(\Omega), \quad \text{a.e. in } (0, +\infty) \times (0, a_+),$$

where

$$\begin{cases} \partial_t p^j(t, a) + \partial_a p^j(t, a) + (\mu(a) + \lambda_j) p^j(t, a) = 0 & t > 0, \quad a \in (0, a_+), \\ p^j(t, 0) = \int_0^{a_+} \beta(a) p^j(t, a) da, & t > 0, \\ p^j(0, a) = p_0^j(a), & a \in (0, a_+). \end{cases} \quad (\text{IV.3.9})$$

It is easy to check that the solution p^j of (IV.3.9) satisfies $p^j(t, a) = e^{-t\lambda_j} \mathbb{T}_t^{A_0}(p_0^j)(a)$, so

that for almost every $(t, a) \in (0, +\infty) \times (0, a_\dagger)$ we have

$$\|\mathbb{T}_t^A p_0(a, \cdot)\|_{L^2(\Omega)}^2 = \sum_{j=0}^{+\infty} |p^j(t, a)|^2 = \sum_{j=0}^{+\infty} e^{-2t\lambda_j} (\mathbb{T}_t^{A_0} p_0^j(a))^2 \leq \sum_{j=0}^{+\infty} (\mathbb{T}_t^{A_0} p_0^j(a))^2,$$

and the estimation (IV.3.5) follows from (IV.3.4) and the above inequality. \square

IV.4 LOW FREQUENCY CONTROL IN $L^\infty((0, a_\dagger); L^2(\Omega))$

In this section, we prove that the projection of the state trajectory of (IV.1.4) on an infinite subspace of $H = L^2((0, a_\dagger); L^2(\Omega))$ (defined using the eigenfunctions of the Neumann Laplacian) can be steered to zero in any time and we estimate the norm of the associated control.

In the sequel, for any $\mu > 0$, we denote by

$$\mathcal{N}(\mu) := \text{Card}\{k : \lambda_k \leq \mu\},$$

$$E_\mu := \text{Span}\{\varphi_k : \lambda_k \leq \mu\},$$

and $\Pi_{E_\mu} : L^2(\Omega) \rightarrow L^2(\Omega)$ the orthogonal projection from $L^2(\Omega)$ onto E_μ . The main result of this section is:

Proposition IV.4.1. *Under the assumption (Stable), let $\mu > 0$ and let $T > 0$. Let E be a subset of positive measure in the interval $[0, T]$. There exists $u_\mu \in L^\infty((0, T) \times (0, a_\dagger) \times \omega)$ such that the solution p of (IV.1.4) satisfies*

$$\Pi_{E_\mu} p(T, a, \cdot) = 0 \quad (a \in (0, a_\dagger) \text{ a.e.}).$$

Moreover, we have the following estimate:

$$\|u_\mu\|_{L^\infty((0, T) \times (0, a_\dagger) \times \omega)}^2 \leq \frac{C_1 e^{C_2 \sqrt{\mu}}}{(m(E))^2} \|p_0\|_{L^\infty((0, a_\dagger); L^2(\Omega))}^2,$$

where C_1 and C_2 are two positive constants.

The main ingredient of the proof is an inequality involving the eigenfunctions of the Neumann Laplacian:

Theorem IV.4.2. *For any non-empty open subset ω of Ω , there exists two positive constants $C_1, C_2 \geq 0$ such that for any $\mu > 0$, for any sequence $(a_j)_{j \geq 0} \subset \mathbb{R}$, we have*

$$\left(\sum_{j: \lambda_j \leq \mu} |a_j|^2 \right)^{1/2} \leq C_1 e^{C_2 \sqrt{\mu}} \int_\omega \left| \sum_{j: \lambda_j \leq \mu} a_j \varphi_j(x) \right| dx. \quad (\text{IV.4.1})$$

The above inequality may be obtained by combining results and methods from Lebeau and Robbiano [LR95], [LR97] (see also Lü [L13]), and analyticity arguments due to Apraiz et al. [AEWZ14, Proof of Theorem 5]. Besides the above inequality, we will use a classical duality argument, following the methodology in Micu, Roventa and Tucsnak [MRT12] - roughly speaking, the L^1 observability induces the L^∞ controllability.

Proof of Proposition IV.4.1. Note that the solution p of (IV.1.4) writes

$$p(t, a, \cdot) = \sum_{j=0}^{+\infty} p^j(t, a) \varphi_j \quad \text{in } L^2(\Omega), \quad \text{a.e. in } (0, T) \times (0, a_\dagger),$$

where

$$\begin{cases} \partial_t p^j(t, a) + \partial_a p^j(t, a) + (\mu(a) + \lambda_j) p^j(t, a) = \chi_E(t) \int_{\omega} u(t, a, x) \varphi_j(x) dx, & t \in (0, T), \quad a \in (0, a_\dagger), \\ p^j(t, 0) = \int_0^{a_\dagger} \beta(a) p^j(t, a) da, & t \in (0, T), \\ p^j(0, a) = p_0^j(a), & a \in (0, a_\dagger), \end{cases}$$

and where

$$p_0(a, \cdot) = \sum_{j=0}^{+\infty} p_0^j(a) \varphi_j \quad \text{in } L^2(\Omega), \quad \text{a.e. } a \in (0, a_\dagger).$$

The aim is to solve the following moment problem: find $u \in L^\infty((0, T) \times (0, a_\dagger) \times \omega)$ such that for every $j \in [0, \mathcal{N}(\mu)]$, we have

$$\int_{\omega} u_\mu(t, a, x) \varphi_j(x) dx = v_j(t, a) \quad ((t, a) \in (0, T) \times (0, a_\dagger) \quad \text{a.e.}),$$

where v_j denotes a null control associated to the system

$$\begin{cases} \partial_t p^j(t, a) + \partial_a p^j(t, a) + (\mu(a) + \lambda_j) p^j(t, a) = \chi_E(t) v_j(t, a), & t \in (0, T), \quad a \in (0, a_\dagger), \\ p^j(t, 0) = \int_0^{a_\dagger} \beta(a) p^j(t, a) da, & t \in (0, T), \\ p^j(0, a) = p_0^j(a), & a \in (0, a_\dagger). \end{cases} \quad (\text{IV.4.2})$$

Recall from Proposition IV.2.5 that for every $j \geq 0$, there exists $v_j \in L^\infty((0, T) \times (0, a_\dagger))$ such that the corresponding solution p^j of the above system (IV.4.2) satisfies $p^j(T, a) = 0$ for almost every $a \in (0, a_\dagger)$, with $|v_j(t, a)| \leq \frac{1}{m(E)} |(\mathbb{T}_t^{A_0} p_0^j)(a)|$ (since we can choose $v_j(t, a) = -\frac{e^{-t\lambda_j}}{m(E)} (\mathbb{T}_t^{A_0} p_0^j)(a)$).

Let $\mu > 0$. Define the map $\mathcal{G} : L^2(\omega) \rightarrow \mathbb{R}^{\mathcal{N}(\mu)+1}$ by

$$\mathcal{G}u := \left(\int_{\omega} u(x) \varphi_j(x) \, dx \right)_{0 \leq j \leq \mathcal{N}(\mu)}.$$

It is easy to check that for every $w = (w_j)_{0 \leq j \leq \mathcal{N}(\mu)} \in \mathbb{R}^{\mathcal{N}(\mu)+1}$, we have

$$\mathcal{G}^*w = \sum_{j=0}^{\mathcal{N}(\mu)} w_j \varphi_j.$$

Inequality (IV.4.1) ensures that for every $w \in \mathbb{R}^{\mathcal{N}(\mu)+1}$, we have

$$\|w\|_{\mathbb{R}^{\mathcal{N}(\mu)+1}} \leq C_1 e^{C_2 \sqrt{\mu}} \|\mathcal{G}^*w\|_{L^1(\omega)}. \quad (\text{IV.4.3})$$

Define the mapping $\mathcal{K} : E_\mu \rightarrow \mathbb{R}^{\mathcal{N}(\mu)+1}$ by the formula $\mathcal{K}(\mathcal{G}^*w) = w$ for all $w \in \mathbb{R}^{\mathcal{N}(\mu)+1}$. Using (IV.4.3), it follows that the mapping \mathcal{K} is well defined and that

$$\|\mathcal{K}z\|_{\mathbb{R}^{\mathcal{N}(\mu)+1}} \leq C_1 e^{C_2 \sqrt{\mu}} \|z\|_{L^1(\omega)} \quad (z \in E_\mu). \quad (\text{IV.4.4})$$

For every $i \in \{0, \dots, \mathcal{N}(\mu)\}$, define the linear mapping $\mathcal{K}_i : E_\mu \rightarrow \mathbb{R}$ by the formula $\mathcal{K}_i(z) = (\mathcal{K}(z))_i$ for all $z \in E_\mu$, where $(\mathcal{K}(z))_i$ denotes the i -th component of the vector $\mathcal{K}(z)$. It follows from (IV.4.4) that, for every $i \in \{0, \dots, \mathcal{N}(\mu)\}$ and for every $z \in E_\mu$ we have

$$|\mathcal{K}_i(z)| \leq C_1 e^{C_2 \sqrt{\mu}} \|z\|_{L^1(\omega)} \quad (i \in \{0, \dots, \mathcal{N}(\mu)\}, z \in E_\mu),$$

so that from the Hahn-Banach theorem we can extend each linear functional \mathcal{K}_i to a bounded linear functional $\tilde{\mathcal{K}}_i$ on $L^1(\omega)$ such that

$$|\tilde{\mathcal{K}}_i(z)| \leq C_1 e^{C_2 \sqrt{\mu}} \|z\|_{L^1(\omega)} \quad (i \in \{0, \dots, \mathcal{N}(\mu)\}, z \in L^1(\omega)). \quad (\text{IV.4.5})$$

Now, let us define the mapping $\tilde{\mathcal{K}} : L^1(\omega) \rightarrow \mathbb{R}^{\mathcal{N}(\mu)+1}$ by the formula $\tilde{\mathcal{K}}(z) := (\tilde{\mathcal{K}}_i(z))_{0 \leq i \leq \mathcal{N}(\mu)}$ for all $z \in L^1(\omega)$, so that the mapping $\tilde{\mathcal{K}}$ is a bounded linear extension of the mapping \mathcal{K} on $L^1(\omega)$. Notice that, using (IV.4.5), for every $z \in L^1(\omega)$ we have

$$\|\tilde{\mathcal{K}}z\|_{\mathbb{R}^{\mathcal{N}(\mu)+1}} \leq C_1 e^{C_2 \sqrt{\mu}} (\mathcal{N}(\mu) + 1)^{\frac{1}{2}} \|z\|_{L^1(\omega)} \quad (z \in L^1(\omega)).$$

By Weyl's formula (see, for instance, Netrusov and Safarov [NS05] for a reminder), there exists a constant $K > 0$ such that $\mathcal{N}(\mu) \leq K \mu^{\frac{N}{2}}$, so that we may infer from the above estimation that

$$\|\tilde{\mathcal{K}}z\|_{\mathbb{R}^{\mathcal{N}(\mu)+1}} \leq C_1 e^{C_2 \sqrt{\mu}} \|z\|_{L^1(\omega)} \quad (z \in L^1(\omega)), \quad (\text{IV.4.6})$$

for some other constants $C_1, C_2 \geq 0$.

Note that for every $w \in \mathbb{R}^{\mathcal{N}(\mu)+1}$, since $\mathcal{G}^*w \in E_\mu$ we have

$$\tilde{\mathcal{K}}(\mathcal{G}^*w) = \tilde{\mathcal{K}}|_{E_\mu}(\mathcal{G}^*w) = \mathcal{K}(\mathcal{G}^*w) = w,$$

so that $\tilde{\mathcal{K}}\mathcal{G}^* = \text{Id}_{\mathcal{L}(\mathbb{R}^{\mathcal{N}(\mu)+1})}$ and since $\tilde{\mathcal{K}}$ and \mathcal{G} are bounded operators, it follows that we have

$$\mathcal{G} \left(\tilde{\mathcal{K}} \right)^* = \text{Id}_{\mathcal{L}(\mathbb{R}^{\mathcal{N}(\mu)+1})}. \quad (\text{IV.4.7})$$

Noting that the range of $\left(\tilde{\mathcal{K}} \right)^*$ is included in $(L^1(\omega))' = L^\infty(\omega)$ (from the Riesz representation theorem), it follows from the above equality (IV.4.7) that for every $w \in \mathbb{R}^{\mathcal{N}(\mu)+1}$, there exists $u := \left(\tilde{\mathcal{K}} \right)^* (w) \in L^\infty(\omega)$ such that $\mathcal{G}u = w$, with

$$\begin{aligned} \|u\|_{L^\infty(\omega)} &= \left\| \left(\tilde{\mathcal{K}} \right)^* (w) \right\|_{L^\infty(\omega)} \\ &\leq \left\| \left(\tilde{\mathcal{K}} \right)^* \right\|_{\mathcal{L}(\mathbb{R}^{\mathcal{N}(\mu)+1}, L^\infty(\omega))} \|w\|_{\mathbb{R}^{\mathcal{N}(\mu)+1}} \\ &= \left\| \tilde{\mathcal{K}} \right\|_{\mathcal{L}(L^1(\omega), \mathbb{R}^{\mathcal{N}(\mu)+1})} \|w\|_{\mathbb{R}^{\mathcal{N}(\mu)+1}} \\ &\leq C_1 e^{C_2 \sqrt{\mu}} \|w\|_{\mathbb{R}^{\mathcal{N}(\mu)+1}}. \end{aligned}$$

In the above inequalities, we have used that $\left\| \left(\tilde{\mathcal{K}} \right)^* \right\|_{\mathcal{L}(\mathbb{R}^{\mathcal{N}(\mu)+1}, L^\infty(\omega))} = \left\| \tilde{\mathcal{K}} \right\|_{\mathcal{L}(L^1(\omega), \mathbb{R}^{\mathcal{N}(\mu)+1})}$ together with the estimation $\left\| \tilde{\mathcal{K}} \right\|_{\mathcal{L}(L^1(\omega), \mathbb{R}^{\mathcal{N}(\mu)+1})} \leq C_1 e^{C_2 \sqrt{\mu}}$ which follows from (IV.4.6).

Let $(t, a) \in (0, T) \times (0, a_\dagger)$. Setting $w(t, a) := (v_j(t, a))_{0 \leq j \leq \mathcal{N}(\mu)}$ where v_j is the null control defined by Proposition IV.2.5 (with $\mu(a)$ replaced by $(\mu(a) + \lambda_j)$), it follows that there exists $u_\mu(t, a, \cdot) \in L^\infty(\omega)$ such that $\mathcal{G}u_\mu(t, a) = w(t, a)$, *i.e.*

$$\int_\omega u_\mu(t, a, x) \varphi_j(x) \, dx = v_j(t, a) \quad (j \in [0, \mathcal{N}(\mu)]), \quad (t, a) \in (0, T) \times (0, a_\dagger) \text{ a.e.},$$

with

$$\|u_\mu(t, a, \cdot)\|_{L^\infty(\omega)} \leq C_1 e^{C_2 \sqrt{\mu}} \|w(t, a)\|_{\mathbb{R}^{\mathcal{N}(\mu)+1}}. \quad (\text{IV.4.8})$$

From the above inequality (IV.4.8), it follows that for almost every $(t, a) \in (0, T) \times (0, a_\dagger)$, we have

$$\|u_\mu(t, a, \cdot)\|_{L^\infty(\omega)} \leq C_1 e^{C_2 \sqrt{\mu}} \left(\sum_{j=0}^{\mathcal{N}(\mu)} |v_j(t, a)|^2 \right)^{1/2},$$

where $|v_j(t, a)|^2 \leq \frac{(\mathbb{T}_t^{A_0} p_0^j(a))^2}{(m(E))^2}$ by Proposition IV.2.5, so that using (IV.3.4) together

with the above inequality we get that

$$\|u_\mu(t, a, \cdot)\|_{L^\infty(\omega)}^2 \leq C \times \frac{C_1^2 e^{2C_2\sqrt{\mu}}}{(m(E))^2} \|p_0\|_{L^\infty((0, a_\dagger); L^2(\Omega))}^2 \quad ((t, a) \in (0, T) \times (0, a_\dagger) \text{ a.e.}),$$

and we deduce from the above inequality (recalling the constant $C \times C_1^2$ by C_1 and recalling $2C_2$ by C_2) that we have

$$\|u_\mu\|_{L^\infty((0, T) \times (0, a_\dagger) \times \omega)}^2 \leq \frac{C_1 e^{C_2\sqrt{\mu}}}{(m(E))^2} \|p_0\|_{L^\infty((0, a_\dagger); L^2(\Omega))}^2.$$

□

IV.5 PROOF OF THE MAIN RESULT

In this section we prove Theorem IV.1.2 by using a slight adaptation of the strategy developed by Wang [Wan08], initially proposed to study the time optimal internal control problem for the heat equation.

First, recall from Proposition IV.4.1 that, given a time $T > 0$, a subset $E \subset (0, T)$ of positive measure and a cutting frequency $\mu > 0$, there exists $u_\mu \in L^\infty((0, T) \times (0, a_\dagger) \times \omega)$ such that the solution p of (IV.1.4) belongs to the orthogonal of E_μ at time T , for every $a \in (0, a_\dagger)$. The control cost behaves like $e^{C\sqrt{\mu}}$, and may be compensated by the natural dissipation of the solution - under assumption **(Stable)** - stated in the following Proposition.

Proposition IV.5.1. *Under the assumption **(Stable)**, let $\mu > 0$ and suppose that $\Pi_{E_\mu} p_0(a, \cdot) = 0$ for almost every $a \in (0, a_\dagger)$. Then there exists a constant $C \geq 0$ such that for every $t > 0$, the solution p of (IV.1.4) with $u \equiv 0$ satisfies*

$$\|p(t, \cdot, \cdot)\|_{L^\infty((0, a_\dagger); L^2(\Omega))} \leq C e^{-\mu t} \|p_0\|_{L^\infty((0, a_\dagger); L^2(\Omega))} \quad (t \geq 0).$$

Proof. Suppose that $\Pi_{E_\mu} p_0(a) = 0$ for almost every $a \in (0, a_\dagger)$. With $u \equiv 0$, the solution p of (IV.1.4) satisfies

$$p(t, a, \cdot) = \sum_{j: \lambda_j > \mu} p^j(t, a) \varphi_j \text{ in } L^2(\Omega), \text{ a.e. in } (0, \tau) \times (0, a_\dagger),$$

where

$$\begin{cases} \partial_t p^j(t, a) + \partial_a p^j(t, a) + (\mu(a) + \lambda_j) p^j(t, a) = 0 & t > 0, a \in (0, a_\dagger), \\ p^j(t, 0) = \int_0^{a_\dagger} \beta(a) p^j(t, a) da, & t > 0, \\ p^j(0, a) = p_0^j(a), & a \in (0, a_\dagger). \end{cases} \quad (\text{IV.5.1})$$

Let $\lambda_j > \mu$. It is easy to check that the solution p^j of (IV.5.1) satisfies $p^j(t, a) = e^{-t\lambda_j} \mathbb{T}_t^{A_0}(p_0^j)(a)$, so that for almost every $(t, a) \in (0, \tau) \times (0, a_\dagger)$ we have

$$\begin{aligned} \|p(t, a, \cdot)\|_{L^2(\Omega)}^2 &= \sum_{j: \lambda_j > \mu} |p^j(t, a)|^2 \\ &\leq e^{-2\mu t} \sum_{j=0}^{+\infty} (\mathbb{T}_t^{A_0} p_0^j(a))^2, \end{aligned}$$

and using inequality (IV.3.4) we deduce from the above inequality that we have

$$\|p(t, a, \cdot)\|_{L^2(\Omega)}^2 \leq C e^{-2\mu t} \|p_0\|_{L^\infty((0, a_\dagger); L^2(\Omega))}^2,$$

so that the estimation of Proposition IV.5.1 holds. \square

In order to prove Theorem IV.1.2, we will need the following known and useful result from the measure theory, whose proof may be found in Lions [Lio68, p. 275].

Lemma IV.5.2. *Let $T > 0$ and E be a Lebesgue measurable set with positive measure in $[0, T]$. For almost every $\tilde{t} \in E$, there exists a sequence of numbers $\{t_i\}_{i=1}^\infty$ in the interval $[0, T]$ such that*

$$t_1 < t_2 < \dots < t_i < t_{i+1} < \dots < \tilde{t}, \quad t_i \rightarrow \tilde{t} \text{ as } i \rightarrow \infty, \quad (\text{IV.5.2})$$

$$m(E \cap [t_i, t_{i+1}]) \geq \rho(t_{i+1} - t_i), \quad i = 1, 2, \dots, \quad (\text{IV.5.3})$$

$$\frac{t_{i+1} - t_i}{t_{i+2} - t_{i+1}} \leq C_0, \quad i = 1, 2, \dots, \quad (\text{IV.5.4})$$

where $\rho \leq 1$ and C_0 are two positive constants which are independant on i .

We now have all the ingredients to prove Theorem IV.1.2, following the ideas developed in Wang [Wan08].

Proof of theorem (IV.1.2). With no claim of originality, we borrow some ideas from [Wan08] (see also Lü [Lü10]). Without loss of generality, we may assume that the assumption **(Stable)** is satisfied (see Remark IV.3.2). We may also assume that $C_1 \geq 1$, where C_1 is the positive constant given in (IV.4.1) (resp. in (IV.5.3)). Let $C \geq 1$ be a fixed constant such that (IV.3.5) and (IV.5.1) hold. By Lemma IV.5.2, we can take a number $\tilde{t} \in E$ with $\tilde{t} < T$ and a sequence $\{t_N\}_{N=1}^\infty$ in the interval $(0, T)$ such that (IV.5.2) - (IV.5.4) hold for some positive number $\rho \leq 1$ and C_0 , and

$$\tilde{t} - t_1 \leq \min\left(1, \frac{1}{|\omega|}\right).$$

Let us consider the following equation :

$$\left\{ \begin{array}{l} \partial_t \tilde{p}(t, a, x) + \partial_a \tilde{p}(t, a, x) + \mu(a) \tilde{p}(t, a, x) - \Delta \tilde{p}(t, a, x) \\ \quad = \chi_\omega(x) \chi_E(t) \tilde{u}(t, a, x), \quad t \in [t_1, \tilde{t}], a \in (0, a_\dagger), x \in \Omega, \\ \frac{\partial \tilde{p}}{\partial \nu}(t, a, x) = 0, \quad t \in [t_1, \tilde{t}], a \in (0, a_\dagger), x \in \partial\Omega, \\ \tilde{p}(t, 0, x) = \int_0^{a_\dagger} \beta(a) \tilde{p}(t, a, x) da, \quad t \in [t_1, \tilde{t}], x \in \Omega, \\ \tilde{p}(t_1, a, x) = \tilde{p}_0(a, x), \quad a \in (0, a_\dagger), x \in \Omega. \end{array} \right. \quad (\text{IV.5.5})$$

We shall first prove that for each $\tilde{p}_0 \in L^\infty((0, a_\dagger); L^2(\Omega))$, there exists a control \tilde{u} in the space $L^\infty((t_1, \tilde{t}) \times (0, a_\dagger) \times \omega)$ with the estimate $\|\tilde{u}\|_{L^\infty((t_1, \tilde{t}) \times (0, a_\dagger) \times \omega)} \leq L \|\tilde{p}_0\|_{L^\infty((0, a_\dagger); L^2(\Omega))}$ for some positive constant L independant of \tilde{p}_0 , such that the solution \tilde{p} to (IV.5.5) vanishes at time \tilde{t} , *i.e.* $\tilde{p}(\tilde{t}, a, x) = 0$ for almost every $(a, x) \in (0, a_\dagger) \times \Omega$.

Set $I_N := [t_{2N-1}, t_{2N}]$, $J_N := [t_{2N}, t_{2N+1}]$ for $N = 1, 2, \dots$. Then we have

$$[t_1, \tilde{t}] = \bigcup_{N=1}^{\infty} (I_N \cup J_N).$$

Notice that for each $N \geq 1$, we have $m(E \cap I_N) > 0$ thanks to (IV.5.3).

Now, on the interval $I_1 \equiv [t_1, t_2]$, we consider the following controlled equation:

$$\left\{ \begin{array}{l} \partial_t p_1(t, a, x) + \partial_a p_1(t, a, x) + \mu(a) p_1(t, a, x) - \Delta p_1(t, a, x) \\ \quad = \chi_\omega(x) \chi_E(t) u_1(t, a, x), \quad t \in [t_1, t_2], a \in (0, a_\dagger), x \in \Omega, \\ \frac{\partial p_1}{\partial \nu}(t, a, x) = 0, \quad t \in [t_1, t_2], a \in (0, a_\dagger), x \in \partial\Omega, \\ p_1(t, 0, x) = \int_0^{a_\dagger} \beta(a) p_1(t, a, x) da, \quad t \in [t_1, t_2], x \in \Omega, \\ p_1(t_1, a, x) = \tilde{p}_0(a, x), \quad a \in (0, a_\dagger), x \in \Omega. \end{array} \right. \quad (\text{IV.5.6})$$

By Proposition IV.4.1, for any $r_1 > 0$, there exists a control u_1 in the space $L^\infty((t_1, t_2) \times (0, a_\dagger) \times \omega)$ with the estimate:

$$\|u_1\|_{L^\infty((t_1, t_2) \times (0, a_\dagger) \times \omega)}^2 \leq \frac{C_1 e^{C_2 \sqrt{r_1}}}{(m(E \cap [t_1, t_2]))^2} \|\tilde{p}_0\|_{L^\infty((0, a_\dagger); L^2(\Omega))}^2, \quad (\text{IV.5.7})$$

such that $\Pi_{E_{r_1}} p_1(t_2, a, \cdot) = 0$ for almost every $a \in (0, a_\dagger)$. Then, using (IV.5.3) with (IV.5.7) we have

$$\begin{aligned} \|u_1\|_{L^\infty((t_1, t_2) \times (0, a_\dagger) \times \omega)}^2 &\leq \frac{C_1 e^{C_2 \sqrt{r_1}}}{\rho^2 (t_2 - t_1)^2} \|\tilde{p}_0\|_{L^\infty((0, a_\dagger); L^2(\Omega))}^2 \\ &= \frac{C_1}{\rho^2 (t_2 - t_1)^2} \alpha_1 \|\tilde{p}_0\|_{L^\infty((0, a_\dagger); L^2(\Omega))}^2, \end{aligned} \quad (\text{IV.5.8})$$

where $\alpha_1 := e^{C_2\sqrt{r_1}}$. Moreover, from (IV.3.1), (IV.3.2) and (IV.3.5) we get that the solution p_1 of (IV.5.6) satisfies

$$\begin{aligned}
 \|p_1(t_2, \cdot, \cdot)\|_{L^\infty((0, a_\dagger); L^2(\Omega))} &\leq C \|\tilde{p}_0\|_{L^\infty((0, a_\dagger); L^2(\Omega))} + C \int_{t_1}^{t_2} \|u_1(\sigma)\|_{L^\infty((0, a_\dagger); L^2(\omega))} d\sigma \\
 &\leq C \|\tilde{p}_0\|_{L^\infty((0, a_\dagger); L^2(\Omega))} + C(t_2 - t_1)|\omega| \|u_1\|_{L^\infty((t_1, t_2) \times (0, a_\dagger) \times \omega)} \\
 &\leq C(\|\tilde{p}_0\|_{L^\infty((0, a_\dagger); L^2(\Omega))} + \|u_1\|_{L^\infty((t_1, t_2) \times (0, a_\dagger) \times \omega)}) \\
 &\leq 2C \left(\frac{C_1}{\rho^2(t_2 - t_1)^2} \alpha_1 \right)^{\frac{1}{2}} \|\tilde{p}_0\|_{L^\infty((0, a_\dagger); L^2(\Omega))}.
 \end{aligned} \tag{IV.5.9}$$

Here, we have used the fact that $(t_2 - t_1) \leq \min\left(1, \frac{1}{|\omega|}\right)$, $\rho \leq 1$ and $C_1 \geq 1$, together with (IV.5.8).

On the interval $J_1 \equiv [t_2, t_3]$, we consider the following population equation without control:

$$\begin{cases}
 \partial_t q_1(t, a, x) + \partial_a q_1(t, a, x) + \mu(a)q_1(t, a, x) - \Delta q_1(t, a, x) = 0, & t \in [t_2, t_3], a \in (0, a_\dagger), x \in \Omega, \\
 \frac{\partial q_1}{\partial \nu}(t, a, x) = 0, & t \in [t_2, t_3], a \in (0, a_\dagger), x \in \partial\Omega, \\
 q_1(t, 0, x) = \int_0^{a_\dagger} \beta(a)q_1(t, a, x) da, & t \in [t_2, t_3], x \in \Omega, \\
 q_1(t_2, a, x) = p_1(t_2, a, x), & a \in (0, a_\dagger), x \in \Omega.
 \end{cases}$$

Since $\Pi_{E_{r_1}} p_1(t_2, a, \cdot) = 0$ for almost every $a \in (0, a_\dagger)$, from Proposition IV.5.1 we have

$$\|q_1(t_3, \cdot, \cdot)\|_{L^\infty((0, a_\dagger); L^2(\Omega))}^2 \leq C^2 \exp(-2r_1(t_3 - t_2)) \cdot \|p_1(t_2, \cdot, \cdot)\|_{L^\infty((0, a_\dagger); L^2(\Omega))}^2, \tag{IV.5.10}$$

and using (IV.5.9) with (IV.5.10) we get that

$$\|q_1(t_3, \cdot, \cdot)\|_{L^\infty((0, a_\dagger); L^2(\Omega))}^2 \leq 4C^4 \times \frac{C_1}{\rho^2(t_2 - t_1)^2} \alpha_1 \cdot \exp(-2r_1(t_3 - t_2)) \cdot \|\tilde{p}_0\|_{L^\infty((0, a_\dagger); L^2(\Omega))}^2. \tag{IV.5.11}$$

On the interval $I_2 \equiv [t_3, t_4]$, we consider the controlled population equation as follows:

$$\begin{cases}
 \partial_t p_2(t, a, x) + \partial_a p_2(t, a, x) + \mu(a)p_2(t, a, x) - \Delta p_2(t, a, x) \\
 \quad = \chi_\omega(x)\chi_E(t)u_2(t, a, x), & t \in [t_3, t_4], a \in (0, a_\dagger), x \in \Omega, \\
 \frac{\partial p_2}{\partial \nu}(t, a, x) = 0, & t \in [t_3, t_4], a \in (0, a_\dagger), x \in \partial\Omega, \\
 p_2(t, 0, x) = \int_0^{a_\dagger} \beta(a)p_2(t, a, x) da, & t \in [t_3, t_4], x \in \Omega, \\
 p_2(t_3, a, x) = q_1(t_3, a, x), & a \in (0, a_\dagger), x \in \Omega.
 \end{cases} \tag{IV.5.12}$$

Then by using Proposition IV.4.1, for any $r_2 > 0$, there exists a control u_2 in the space

$L^\infty((t_3, t_4) \times (0, a_\dagger) \times \omega)$ with the estimate:

$$\|u_2\|_{L^\infty((t_3, t_4) \times (0, a_\dagger) \times \omega)}^2 \leq \frac{C_1 e^{C_2 \sqrt{r_2}}}{(m(E \cap [t_3, t_4]))^2} \|q_1(t_3, \cdot, \cdot)\|_{L^\infty((0, a_\dagger); L^2(\Omega))}^2, \quad (\text{IV.5.13})$$

such that $\Pi_{E_{r_2}} p_2(t_4, a, \cdot) = 0$ for almost every $a \in (0, a_\dagger)$. By using (IV.5.4), (IV.5.11) and (IV.5.13) we have

$$\|u_2\|_{L^\infty((t_3, t_4) \times (0, a_\dagger) \times \omega)}^2 \leq 4C^4 \left(\frac{C_1}{\rho^2(t_2 - t_1)^2} \right)^2 C_0^4 \cdot \alpha_1 \cdot \alpha_2 \cdot \|\tilde{p}_0\|_{L^\infty((0, a_\dagger); L^2(\Omega))}^2, \quad (\text{IV.5.14})$$

where $\alpha_2 := \exp(C_2 \sqrt{r_2}) \exp(-2r_1(t_3 - t_2))$. Given (IV.3.1), (IV.3.2) and (IV.3.5), we may infer from (IV.5.14) that the solution p_2 of (IV.5.12) satisfies

$$\|p_2(t_4, \cdot, \cdot)\|_{L^\infty((0, a_\dagger); L^2(\Omega))}^2 \leq 4^2 C^6 \left(\frac{C_1 e^{C_2 \sqrt{r_2}}}{\rho^2(t_2 - t_1)^2} \right)^2 C_0^4 \cdot \alpha_1 \cdot \alpha_2 \cdot \|\tilde{p}_0\|_{L^\infty((0, a_\dagger); L^2(\Omega))}^2. \quad (\text{IV.5.15})$$

On the interval $J_2 \equiv [t_4, t_5]$, we consider the following controlled population without control:

$$\begin{cases} \partial_t q_2(t, a, x) + \partial_a q_2(t, a, x) + \mu(a) q_2(t, a, x) - \Delta q_2(t, a, x) = 0, & t \in [t_4, t_5], a \in (0, a_\dagger), x \in \Omega, \\ \frac{\partial q_2}{\partial \nu}(t, a, x) = 0, & t \in [t_4, t_5], a \in (0, a_\dagger), x \in \partial\Omega, \\ q_2(t, 0, x) = \int_0^{a_\dagger} \beta(a) q_2(t, a, x) da, & t \in [t_4, t_5], x \in \Omega, \\ q_2(t_4, a, x) = p_2(t_4, a, x), & a \in (0, a_\dagger), x \in \Omega. \end{cases}$$

Since $\Pi_{E_{r_2}} p_2(t_4, a, \cdot) = 0$ for almost every $a \in (0, a_\dagger)$, from Proposition (IV.5.1) we have

$$\|q_2(t_5, \cdot, \cdot)\|_{L^\infty((0, a_\dagger); L^2(\Omega))}^2 \leq C^2 \exp(-2r_2(t_5 - t_4)) \cdot \|p_2(t_4, \cdot, \cdot)\|_{L^\infty((0, a_\dagger); L^2(\Omega))}^2, \quad (\text{IV.5.16})$$

and using (IV.5.15) with (IV.5.16) we get that

$$\|q_2(t_5, \cdot, \cdot)\|_{L^\infty((0, a_\dagger); L^2(\Omega))}^2 \leq 4^2 C^8 \left(\frac{C_1}{\rho^2(t_2 - t_1)^2} \right)^2 C_0^4 \cdot \alpha_1 \cdot \alpha_2 \cdot \exp(-2r_2(t_5 - t_4)) \cdot \|\tilde{p}_0\|_{L^\infty((0, a_\dagger); L^2(\Omega))}^2. \quad (\text{IV.5.17})$$

On the interval $I_3 \equiv [t_5, t_6]$, we consider the controlled population equation as follows:

$$\begin{cases} \partial_t p_3(t, a, x) + \partial_a p_3(t, a, x) + \mu(a)p_3(t, a, x) - \Delta p_3(t, a, x) \\ \quad = \chi_\omega(x)\chi_E(t)u_3(t, a, x), & t \in [t_5, t_6], a \in (0, a_\dagger), x \in \Omega, \\ \frac{\partial p_3}{\partial \nu}(t, a, x) = 0, & t \in [t_5, t_6], a \in (0, a_\dagger), x \in \partial\Omega, \\ p_3(t, 0, x) = \int_0^{a_\dagger} \beta(a)p_3(t, a, x) da, & t \in [t_5, t_6], x \in \Omega, \\ p_3(t_5, a, x) = q_2(t_5, a, x), & a \in (0, a_\dagger), x \in \Omega. \end{cases}$$

Then by using Proposition IV.4.1, for any $r_3 > 0$, there exists a control u_3 in the space $L^\infty((t_5, t_6) \times (0, a_\dagger) \times \omega)$ with the estimate:

$$\|u_3\|_{L^\infty((t_5, t_6) \times (0, a_\dagger) \times \omega)}^2 \leq \frac{C_1 e^{C_2 \sqrt{r_3}}}{(m(E \cap [t_5, t_6]))^2} \|q_2(t_5, \cdot, \cdot)\|_{L^\infty((0, a_\dagger); L^2(\Omega))}^2, \quad (\text{IV.5.18})$$

such that $\Pi_{E_{r_3}} p_3(t_6, a, \cdot) = 0$ for almost every $a \in (0, a_\dagger)$. By using (IV.5.4), (IV.5.17) and (IV.5.18) we have

$$\|u_3\|_{L^\infty((t_5, t_6) \times (0, a_\dagger) \times \omega)}^2 \leq 4^2 C^8 \left(\frac{C_1}{\rho^2(t_2 - t_1)^2} \right)^3 C_0^4 C_0^{4 \cdot 2} \cdot \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \|\tilde{p}_0\|_{L^\infty((0, a_\dagger); L^2(\Omega))}^2,$$

where $\alpha_3 := \exp(C_2 \sqrt{r_3}) \exp(-2r_2(t_3 - t_2)C_0^{-2})$.

Generally, on the interval I_N , we consider the following controlled population equation:

$$\begin{cases} \partial_t p_N(t, a, x) + \partial_a p_N(t, a, x) + \mu(a)p_N(t, a, x) \\ \quad - \Delta p_N(t, a, x) = \chi_\omega(x)\chi_E(t)u_N(t, a, x), & t \in [t_{2N-1}, t_{2N}], a \in (0, a_\dagger), x \in \Omega, \\ \frac{\partial p_N}{\partial \nu}(t, a, x) = 0, & t \in [t_{2N-1}, t_{2N}], a \in (0, a_\dagger), x \in \partial\Omega, \\ p_N(t, 0, x) = \int_0^{a_\dagger} \beta(a)p_N(t, a, x) da, & t \in [t_{2N-1}, t_{2N}], x \in \Omega, \\ p_N(t_{2N-1}, a, x) = q_{N-1}(t_{2N-1}, a, x), & a \in (0, a_\dagger), x \in \Omega. \end{cases} \quad (\text{IV.5.19})$$

On the interval J_N , we consider the following uncontrolled population equation:

$$\begin{cases} \partial_t q_N(t, a, x) + \partial_a q_N(t, a, x) + \mu(a)q_N(t, a, x) \\ \quad - \Delta q_N(t, a, x) = 0, & t \in [t_{2N}, t_{2N+1}], a \in (0, a_\dagger), x \in \Omega, \\ \frac{\partial q_N}{\partial \nu}(t, a, x) = 0, & t \in [t_{2N}, t_{2N+1}], a \in (0, a_\dagger), x \in \partial\Omega, \\ q_N(t, 0, x) = \int_0^{a_\dagger} \beta(a)q_N(t, a, x) da, & t \in [t_{2N}, t_{2N+1}], x \in \Omega, \\ q_N(t_{2N}, a, x) = p_N(t_{2N}, a, x), & a \in (0, a_\dagger), x \in \Omega. \end{cases}$$

It may be shown by induction that, for each $r_N > 0$, there exists a control $u_N \in L^\infty(I_N \times (0, a_\dagger) \times \omega)$ satisfying:

$$\begin{aligned} & \|u_N\|_{L^\infty(I_N \times (0, a_\dagger) \times \omega)}^2 \\ & \leq (4C^4)^{N-1} \left(\frac{C_1}{\rho^2(t_2 - t_1)^2} \right)^N C_0^4 C_0^{4 \times 2} \dots C_0^{4(N-1)} \alpha_1 \alpha_2 \dots \alpha_N \|\tilde{p}_0\|_{L^\infty((0, a_\dagger); L^2(\Omega))}^2, \end{aligned} \quad (\text{IV.5.20})$$

where

$$\alpha_N := \begin{cases} \exp(C_2 \sqrt{r_1}), & N = 1, \\ \exp(C_2 \sqrt{r_N}) \exp(-2r_{N_1}(t_3 - t_2) C_0^{-2(N-2)}), & N \geq 2, \end{cases}$$

such that $\Pi_{E_{r_N}} p_N(t_{2N}, a, \cdot) = 0$ for almost every $a \in (0, a_\dagger)$. It is easy to check that for each $N \geq 1$ we have

$$\|u_N\|_{L^\infty(I_N \times (0, a_\dagger) \times \omega)} \leq (\tilde{C})^{N(N-1)} \alpha_1 \dots \alpha_N \cdot \|\tilde{p}_0\|_{L^\infty((0, a_\dagger); L^2(\Omega))},$$

where

$$\tilde{C} := \frac{4C^4 C_1}{\rho^2(t_2 - t_1)^2} \cdot C_0^2.$$

Now, for every $N \geq 1$ we set

$$r_N := \left[\frac{2}{(t_3 - t_2)} \tilde{C}^{N-1} \right]^4 \quad (N \geq 1).$$

With this choice of the sequence $\{r_N\}_{N \geq 1}$ given by the above formula, it is shown in [Wan08] that the sequence $\{(\tilde{C})^{N(N-1)} \alpha_1 \dots \alpha_N\}_{N \geq 1}$ is bounded by some nonnegative constant L , so that from (IV.5.20) it follows that for every $N \geq 1$ we have

$$\|u_N\|_{L^\infty(I_N \times (0, a_\dagger) \times \omega)} \leq L \|\tilde{p}_0\|_{L^\infty((0, a_\dagger); L^2(\Omega))} \quad (N \geq 1). \quad (\text{IV.5.21})$$

We now construct a control \tilde{u} by setting

$$\tilde{u}(t, a, x) = \begin{cases} u_N(t, a, x), & t \in I_N, a \in (0, a_\dagger), N \geq 1, \\ 0, & t \in J_N, a \in (0, a_\dagger), N \geq 1, \end{cases} \quad (\text{IV.5.22})$$

so that from (IV.5.21) it is clear that $\tilde{u} \in L^\infty((t_1, \tilde{t}) \times (0, a_\dagger) \times \omega)$ with the estimate

$$\|\tilde{u}\|_{L^\infty((t_1, \tilde{t}) \times (0, a_\dagger) \times \omega)} \leq L \|\tilde{p}_0\|_{L^\infty((0, a_\dagger); L^2(\Omega))}. \quad (\text{IV.5.23})$$

Denote by \tilde{p} the solution of (IV.5.5) corresponding to the control \tilde{u} defined by (IV.5.22). Then on any interval I_N , we have $\tilde{p}(t, \cdot, \cdot) = p_N(t, \cdot, \cdot)$, where p_N is the solution of (IV.5.19). Since we have $\Pi_{E_{r_N}} p_N(t_{2N}, a, \cdot) = 0$ for almost every $a \in (0, a_\dagger)$ and for every $N \geq 1$, using the fact that $r_1 < r_2 < \dots < r_N < \dots$, by making use of (IV.5.22) we get

that

$$\Pi_{E_{r_N}} p_N(t_{2M}, a, \cdot) = 0 \text{ for all } M \geq N. \quad (\text{IV.5.24})$$

On the other hand, since $t_{2M} \rightarrow \tilde{t}$ as $M \rightarrow \infty$, we obtain that

$$\tilde{p}(t_{2M}, \cdot, \cdot) \rightarrow \tilde{p}(\tilde{t}, \cdot, \cdot) \text{ strongly in } L^2((0, a_+) \times \Omega) \text{ as } M \rightarrow \infty.$$

This, together with (IV.5.24), implies that $\Pi_{E_{r_N}} p_N(\tilde{t}, a, \cdot) = 0$ for almost every $a \in (0, a_+)$ and for every $N \geq 1$. Since $r_N \rightarrow \infty$ when $N \rightarrow \infty$, it holds that $\tilde{p}(\tilde{t}, \cdot, \cdot) = 0$. Thus, we have proved that for each $\tilde{p}_0 \in L^\infty((0, a_+); L^2(\Omega))$, there exists a control $\tilde{u} \in L^\infty((t_1, \tilde{t}) \times (0, a_+) \times \omega)$ with the estimate $\|\tilde{u}\|_{L^\infty((t_1, \tilde{t}) \times (0, a_+) \times \omega)} \leq L \|\tilde{p}_0\|_{L^\infty((0, a_+); L^2(\Omega))}$ such that the solution \tilde{p} to (IV.5.5) reaches zero value at time \tilde{t} , namely, $\tilde{p}(\tilde{t}, \cdot, \cdot) = 0$.

Now, we take \tilde{p}_0 to be $\mathbb{T}_{t_1}^A p_0$ and we construct a control u by setting, for almost every $(t, a, x) \in (0, \tau) \times (0, a_+) \times \Omega$,

$$u(t, a, x) = \begin{cases} 0 & \text{in } (0, t_1) \times (0, a_+) \times \Omega, \\ \tilde{u}(t, a, x) & \text{in } (t_1, \tilde{t}) \times (0, a_+) \times \Omega, \\ 0 & \text{in } (\tilde{t}, \tau) \times (0, a_+) \times \Omega. \end{cases} \quad (\text{IV.5.25})$$

It is clear that this control u is in the space $L^\infty((0, \tau) \times (0, a_+) \times \Omega)$ and that the corresponding solution p of (IV.1.4) satisfies $p(\tau, \cdot, \cdot) = 0$. The norm estimation (IV.1.5) in Theorem IV.1.2 easily follows from (IV.3.5), (IV.5.23) and (IV.5.25). \square

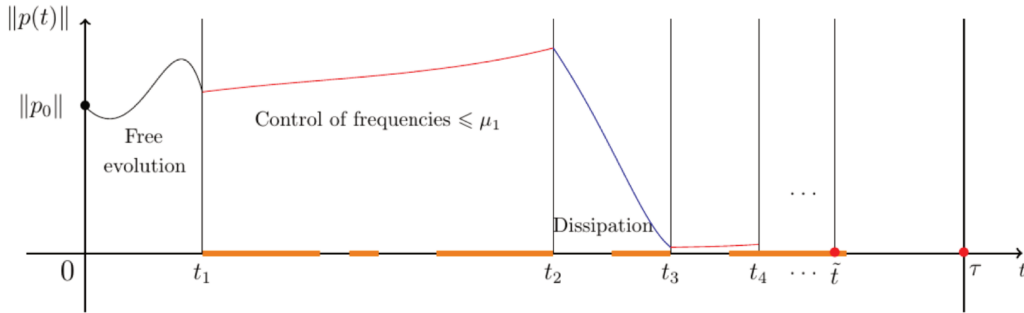


Figure IV.1 – Evolution shape of the norm of the controlled solution. The orange domain represents the Lebesgue measurable control time zone $E \subset (0, \tau)$.

Principes de comparaison pour l'étude du contrôle des équations de Gurtin et MacCamy

Ce chapitre reprend l'article rédigé en collaboration avec S. Anița [AH18], accepté et paru dans *Mathematics of Controls, Signals and Systems (MCSS)*. Le contrôle à zéro des équations non linéaires structurées en âge décrites par Gurtin et MacCamy [GM74] y est démontré. Le présent chapitre généralise les résultats de contrôle à zéro et de contrôlabilité vers les trajectoires stationnaires obtenus dans le Chapitre II.

	Tranche d'âge du support de contrôle	Les jeunes individus se reproduisent	Contrôle à zéro ?	Temps optimal de contrôle à zéro
Barbu, Martcheva & Iannelli (2001) [BM01]		OUI	NON	×
Hegoburu, Magal & Tarsnak (2018) [HMT18]		NON	OUI	OUI ($\tau > a_1 - a_2$)
Maity (2018) [Mai18]		NON	OUI	NON ($\tau > a_1 - a_2 - a_3$)
Anița & Hegoburu (2018) [AH18]		OUI (cas $\alpha_1 = 0$)	OUI	OUI ($\tau > a_1 - a_2 + a_3$)

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V.1 INTRODUCTION

We shall continue and extend the investigation in Hegoburu, Magal and TucsnaK [HMT18] concerning the null controllability of the age-dependent population dynamics. More precisely, let $p(t, a)$ be the distribution of individuals of age a at time t of a biological population. Let a_{\dagger} be the maximal age of an individual and τ be a positive constant. Consider that the population dynamics is described by the following nonlinear system, firstly proposed (without control) by Gurtin and MacCamy [GM74]:

$$\begin{cases} \frac{\partial p}{\partial t}(t, a) + \frac{\partial p}{\partial a}(t, a) + \mu(t, a, P(t))p(t, a) = \chi_{[a_1, a_2]}(a)u(t, a), & (t, a) \in Q_{\tau} \\ p(t, 0) = \int_0^{a_{\dagger}} \beta(t, a, P(t))p(t, a) da, & t \in (0, \tau) \\ p(0, a) = p_0(a), & a \in (0, a_{\dagger}) \\ P(t) = \int_0^{a_{\dagger}} p(t, a) da, & t \in (0, \tau), \end{cases} \quad (\text{V.1.1})$$

where u is a control function, $\chi_{[a_1, a_2]}$ is the characteristic function of the interval $[a_1, a_2]$ (where $0 \leq a_1 < a_2 \leq a_{\dagger}$), p_0 is the initial population density and $Q_{\tau} = (0, \tau) \times (0, a_{\dagger})$. β and μ are the fertility rate and the mortality rate, respectively. In the following, the nonlocal boundary condition

$$p(t, 0) = \int_0^{a_{\dagger}} \beta(t, a, P(t))p(t, a) da, \quad t \in (0, \tau),$$

will be referred as the *renewal law*.

To state our main results, we first recall the standard assumptions, used for instance in Anița [Ani00, p. 30], on the functions μ , β , p_0 and u :

- (H1) for every $s \geq 0$, the functions $\beta(\cdot, \cdot, s)$ and $\mu(\cdot, \cdot, s)$ belong to $L^{\infty}(Q_{\tau})$ and $L^1_{loc}([0, \tau] \times [0, a_{\dagger}))$ respectively,
- (H2) the functions β and μ are locally Lipschitz functions with respect to the third variable, *i.e.* for any $M > 0$, there exists $L(M) > 0$ such that for almost every $(t, a) \in Q_{\tau}$ and for every $s_1, s_2 \in [0, M]$, we have

$$\begin{aligned} |\beta(t, a, s_1) - \beta(t, a, s_2)| &\leq L(M) \cdot |s_1 - s_2|, \\ |\mu(t, a, s_1) - \mu(t, a, s_2)| &\leq L(M) \cdot |s_1 - s_2|, \end{aligned}$$

- (H3) for almost every $(t, a, s) \in Q_{\tau} \times (0, +\infty)$, we have

$$\beta(t, a, s) \geq 0 \quad \text{and} \quad \mu(t, a, s) \geq 0,$$

- (H4) for almost every $(t, a) \in Q_{\tau}$, the function $\beta(t, a, \cdot)$ is nonincreasing,
- (H5) for almost every $(t, a) \in Q_{\tau}$, the function $\mu(t, a, \cdot)$ is nondecreasing,
- (H6) $p_0 \in L^2(0, a_{\dagger})$, $p_0(a) \geq 0$ a.e. $a \in (0, a_{\dagger})$,

(H7) the control function u belongs to $L^2(Q_\tau)$.

By a solution to (V.1.1), we mean a function $p \in L^\infty(0, \tau; L^2(0, a_\dagger))$, absolutely continuous along almost every characteristic line of equation $t - a = c$ (with $c \in (-a_\dagger, \tau)$), which satisfies

$$\begin{cases} Dp(t, a) + \mu(t, a, P(t))p(t, a) = \chi_{[a_1, a_2]}(a)u(t, a) & \text{a.e. } (t, a) \in Q_\tau \\ \lim_{\varepsilon \rightarrow 0^+} p(t + \varepsilon, \varepsilon) = \int_0^{a_\dagger} \beta(t, a, P(t))p(t, a) da & \text{a.e. } t \in (0, \tau) \\ \lim_{\varepsilon \rightarrow 0^+} p(\varepsilon, a + \varepsilon) = p_0(a) & \text{a.e. } a \in (0, a_\dagger), \end{cases}$$

where $P(t) = \int_0^{a_\dagger} p(t, a) da$ for almost every $t \in (0, \tau)$. Here

$$Dp(t, a) = \lim_{\varepsilon \rightarrow 0} \frac{p(t + \varepsilon, a + \varepsilon) - p(t, a)}{\varepsilon}$$

is the directional derivative of p at (t, a) for direction $(1, 1)$. The definition of the solution implies that there is a trace of p on any curve of equation $t = c$, with $c \in [0, \tau]$. With the above assumptions system (V.1.1) has at most one nonnegative solution, and if in addition $u(t, a) \geq 0$ a.e. $(t, a) \in Q_\tau$, the existence and uniqueness of a nonnegative solution to (V.1.1) can be proved via the Banach fixed point theorem (see, for instance, [Ani00, Chapter 2]).

Notice that actually the control acts only for $a \in [a_1, a_2]$.

For almost every $(t, a) \in Q_\tau$, denote by $\beta_+(t, a) := \beta(t, a, 0)$ so that, due to assumption (H4), for almost every $(t, a, s) \in Q_\tau \times (0, +\infty)$ we have

$$\beta(t, a, s) \leq \beta(t, a, 0) = \beta_+(t, a) \quad \text{a.e. } (t, a, s) \in Q_\tau \times (0, +\infty). \quad (\text{V.1.2})$$

Our first result asserts that, assuming that the age of individuals able to reproduce is bounded away from zero, given a_1 small enough and τ large enough, we are able to bring the solution p of (V.1.1) to zero by means of a control function $u \in L^2(Q_\tau)$, preserving the nonnegativity of the state trajectory. More precisely, we have

Theorem V.1.1. *With the above notations and assumptions, suppose that there exists $a_b \in (0, a_\dagger)$ such that for almost every $t \in (0, \tau)$, we have*

$$\beta_+(t, a) = 0 \quad \text{a.e. } a \in (0, a_b), \quad (\text{V.1.3})$$

where the function β_+ is defined in (V.1.2).

Assume that we have $a_1 < a_b$. If $\tau > a_\dagger - a_2 + a_1$, then for every $p_0 \in L^2(0, a_\dagger)$ with $p_0(a) \geq 0$ a.e. $a \in (0, a_\dagger)$, there exists $u \in L^2(Q_\tau)$ and a corresponding unique nonnegative solution p of (V.1.1) such that

$$p(\tau, a) = 0 \quad \text{a.e. } a \in (0, a_\dagger).$$

The following Theorem V.1.2 states that the previous result is sharp, in the sense that if a_1 (respectively τ) is bigger than a_b (respectively smaller than $a_\dagger - a_2 + a_1$), the system (V.1.1) may not be null controllable by means of controls $u \in L^2(Q_\tau)$ while preserving the nonnegativity of the controlled state trajectory. More precisely, we have

Theorem V.1.2. *With the above notations and assumptions,*

- (i) *Assume that there exists $a_b \in (0, a_\dagger)$ and $a_m \in (0, a_\dagger]$ with $a_b < a_m$ such that, for almost every $t \in (0, \tau)$ we have*

$$\beta(t, a, s) > 0 \quad \text{a.e. } (a, s) \in (a_b, a_m) \times (0, +\infty).$$

If $a_1 > a_b$, then for every $\tau > 0$, there exists $p_0 \in L^2(0, a_\dagger)$ with $p_0(a) \geq 0$ a.e. $a \in (0, a_\dagger)$ such that there is no control u and a corresponding nonnegative solution p to (V.1.1) satisfying

$$p(\tau, a) = 0 \quad \text{a.e. } a \in (0, a_\dagger).$$

- (ii) *If $\tau < a_\dagger - a_2 + a_1$, there exists a function β satisfying (H1) - (H4) and an initial population density $p_0 \in L^2(0, a_\dagger)$ with $p_0(a) \geq 0$ a.e. $a \in (0, a_\dagger)$ such that there is no control u and a corresponding nonnegative solution p to (V.1.1) satisfying*

$$p(\tau, a) = 0 \quad \text{a.e. } a \in (0, a_\dagger).$$

Our third result asserts that condition (V.1.3) is not necessary to get the null controllability of system (V.1.1), provided that we are able to control the very young individuals. More precisely, provided that $a_1 = 0$ and $a_2 > 0$, for any function β satisfying (H1) - (H4), we are able to bring the solution p of (V.1.1) to zero in sufficiently large time τ by means of a control $u \in L^2(Q_\tau)$, preserving the nonnegativity of the state trajectory:

Theorem V.1.3. *With the above notations and assumptions,*

- (i) *Assume that $a_1 = 0$ and $a_2 > 0$. If $\tau > a_\dagger - a_2$, then for every $p_0 \in L^2(0, a_\dagger)$ with $p_0(a) \geq 0$ a.e. $a \in (0, a_\dagger)$, there exists $u \in L^2(Q_\tau)$ and a unique nonnegative solution p of (V.1.1) such that*

$$p(\tau, a) = 0 \quad \text{a.e. } a \in (0, a_\dagger).$$

- (ii) *Assume that there exists $a_m \in (0, a_\dagger]$ such that, for almost every $t \in (0, \tau)$ we have*

$$\beta(t, a, s) > 0 \quad \text{a.e. } (a, s) \in (0, a_m) \times (0, +\infty).$$

If $a_1 > 0$, then for every $\tau > 0$, there exists $p_0 \in L^2(0, a_\dagger)$ with $p_0(a) \geq 0$ a.e. $a \in (0, a_\dagger)$ such that there is no control u and a corresponding nonnegative solution p to (V.1.1) satisfying

$$p(\tau, a) = 0 \quad \text{a.e. } a \in (0, a_\dagger).$$

(iii) If $\tau < a_{\dagger} - a_2$, there exists an initial population density $p_0 \in L^2(0, a_{\dagger})$ with $p_0(a) \geq 0$ a.e. $a \in (0, a_{\dagger})$ such that there is no control u and a corresponding nonnegative solution p to (V.1.1) satisfying

$$p(\tau, a) = 0 \quad \text{a.e. } a \in (0, a_{\dagger}).$$

For an overview on age-structured population dynamics models we refer, for instance, to Webb [Web85], Iannelli [Ian95b], Kunisch et al [KSW85] and the references therein.

In the case where both functions β and μ depend only on the age variable, some null controllability results of the age-dependent population dynamics model (without diffusion) were first obtained by Barbu, Iannelli and Martcheva [BIM01]. Assuming that the control is supported in the age interval $(0, a_0)$, for some $a_0 < a_{\dagger}$, the authors proved that the controlled population may be steered to any steady state of (V.1.1), except for a small interval of ages near zero. Recently, Hegoburu, Magal and Tucsnaak [HMT18] proved that this restriction is not necessary, provided individuals do not reproduce at the age close to zero. They also proved there exists controls which preserves the nonnegativity of the state trajectory. More recently, Maity [Mai18] proved that null controllability can be achieved by controls supported in any subinterval $[a_1, a_2]$ of $[0, a_{\dagger}]$, provided we control before the individuals start to reproduce. In this case, the author proved that the system is null controllable in sufficiently large time, *i.e.* when $\tau > a_1 + \max\{a_1, a_{\dagger} - a_2\}$ by means of observability inequalities.

The null controllability of systems modelling nonlinear age-structured population dynamics has been studied by Ainseba and Iannelli [AI⁺03], in the case when $a_1 = 0$ and $a_2 \in (0, a_{\dagger})$, using as a main ingredient the Kakutani fixed-point theorem. In [AI⁺03] the main result asserts controllability of the system (V.1.1) to zero in time $\tau > a_{\dagger}$ by a distributed control, except for a small interval of ages near zero. In our present paper, we get an exact null controllability result for the system (V.1.1). Moreover, the minimal time needed to steer the population to zero (*i.e.* $\tau > a_{\dagger} - a_2 + a_1$) seems to be sharp. Instead of using observability results we use here as a main ingredient the comparison principle for linear age-structured population dynamics. In case of null controllability we provide a feedback control with a very simple structure and such that (V.1.1) admits a unique nonnegative solution p satisfying $p(\tau, a) = 0$ a.e. $a \in (0, a_{\dagger})$.

As already mentionned, in the present work we use comparison results for age-structured population dynamics and some feedback controls with a simple structure (as a harvesting term) in order to prove some null-controllability results. Actually, the use of such harvesting term in order to get the null-controllability in an abstract space is presented in Section V.2. Sections V.3, V.4 and V.5 are devoted to prove respectively Theorems V.1.1, V.1.2 and V.1.3, firstly in the case where the functions β and μ do not depend on the third variable (namely, in the *linear setting*), and to deduce the results of the above mentionned Theorems in the more general nonlinear setting by using the results of the linear case and the comparison principle for age-structured population dynamics. In Section V.6 we give an application to the results obtained in the linear case: we show how to steer a population to another one in the linear setting, preserving the

nonnegativity of the population. In Section V.7, we show how the comparison principle for age-structured population dynamics may be used to prove the lack of the null controllability for the linear Lotka-McKendrick equation with spatial diffusion when the control acts in a subset of the habitat.

V.2 A NULL-CONTROLLABILITY RESULT IN AN ABSTRACT SPACE

Consider the following controlled system:

$$\begin{cases} y'(t) - Ay(t) = u(t), & t \in (0, \tau) \\ y(0) = y_0, \end{cases} \quad (\text{V.2.1})$$

where A is the generator of a \mathcal{C}_0 -semigroup $(e^{tA})_{t \geq 0}$ of linear and bounded operators in the real Banach space X and τ is a positive constant. We remind that by a mild solution to (V.2.1), we mean the function $y \in \mathcal{C}([0, \tau]; X)$ given by

$$y(t) = e^{tA}y_0 + \int_0^t e^{(t-s)A}u(s) ds, \quad \text{for every } t \in [0, \tau].$$

If $y_0 \in X$ and $u \in L^2(0, \tau; X)$, then (V.2.1) has a unique mild solution.

The controllability properties of (V.2.1) have been investigated by Tucsnak and Weiss [TW09] via the observability. Here we use a different approach in order to get the null-controllability: we use a feedback control. Actually, if

$$u(t) := -\frac{1}{\tau - t}y(t), \quad t \in (0, \tau), \quad (\text{V.2.2})$$

then (V.2.1) becomes

$$\begin{cases} y'(t) - Ay(t) = -\frac{1}{\tau - t}y(t), & t \in (0, \tau) \\ y(0) = y_0, \end{cases} \quad (\text{V.2.3})$$

and by a mild solution to (V.2.3) we mean a function $y \in \mathcal{C}([0, \tau]; X)$ such that

$$y(t) = e^{tA}y_0 - \int_0^t e^{(t-s)A} \left(\frac{1}{\tau - s}y(s) \right) ds, \quad \text{for every } t \in [0, \tau].$$

By Gronwall-Bellman's inequality we may conclude that there exists at most one mild solution to (V.2.3). Let us prove that actually, the function y defined by

$$y(t) := \frac{\tau - t}{\tau}e^{tA}y_0 \quad t \in [0, \tau], \quad (\text{V.2.4})$$

is a mild solution to (V.2.3) and that u given by (V.2.2) belongs to $L^2(0, \tau; X)$.

Indeed, y given by (V.2.4) belongs to $\mathcal{C}([0, \tau]; X)$ and for any $t \in [0, \tau]$ we have

$$\begin{aligned} & e^{tA}y_0 - \int_0^t \frac{1}{\tau-s} e^{(t-s)A}y(s) \, ds \\ &= e^{tA}y_0 - \frac{1}{\tau} \int_0^t \frac{1}{\tau-s} (\tau-s) e^{(t-s)A} e^{sA} y_0 \, ds \\ &= e^{tA}y_0 - \frac{t}{\tau} e^{tA}y_0 = \frac{\tau-t}{\tau} e^{tA}y_0 = y(t). \end{aligned}$$

On the other hand, the control u given by (V.2.2) satisfies

$$u(t) = -\frac{1}{\tau-t} y(t) = -\frac{1}{\tau} e^{tA}y_0, \quad t \in [0, \tau],$$

which indeed belongs to $C([0, \tau]; X)$ and consequently to $L^2(0, \tau; X)$.

Notice that, assuming that the functions β and μ depend only on the age variable, the formulation of equations (V.1.1) may be considered using semigroup theory. Indeed, denote by $X = L^2(0, a_+)$ and consider the operator $A : \mathcal{D}(A) \rightarrow X$ defined by

$$\begin{aligned} \mathcal{D}(A) &= \left\{ \varphi \in L^2(0, a_+); \varphi(0) = \int_0^{a_+} \beta(a)\varphi(a) \, da, \quad -\frac{d\varphi}{da} - \mu\varphi \in L^2(0, a_+) \right\}, \\ A\varphi &= -\frac{d\varphi}{da} - \mu\varphi, \quad \varphi \in \mathcal{D}(A). \end{aligned}$$

It is well known (see, for instance, Song et al. [SYW⁺82] or Kappel and Zhang [KZ93]) that A generates a \mathcal{C}_0 -semigroup of linear and bounded operators in X which we denote by $(e^{tA})_{t \geq 0}$. If the control function u in (V.1.1) acts in the whole age range $(0, a_+)$ (i.e. if $a_1 = 0$ and $a_2 = a_+$) and is defined by the feedback form

$$u(t) := -\frac{1}{\tau-t} p(t), \quad t \in [0, \tau], \tag{V.2.5}$$

where p is the corresponding solution to (V.1.1), we deduce that p satisfies

$$\begin{cases} \frac{\partial p}{\partial t}(t, a) + \frac{\partial p}{\partial a}(t, a) + \left(\mu(a) + \frac{1}{\tau-t} \right) p(t, a) = 0, & (t, a) \in Q_\tau \\ p(t, 0) = \int_0^{a_+} \beta(a)p(t, a) \, da, & t \in (0, \tau) \\ p(0, a) = p_0(a), & a \in (0, a_+), \end{cases} \tag{V.2.6}$$

and so that for every $t \in [0, \tau]$ we get from (V.2.4) that $p(t) = \frac{\tau-t}{\tau} e^{tA}p_0$ in X for any $t \in [0, \tau]$, and $p(\tau) = 0$ in X . It can be seen in (V.2.6) that, u given in such a feedback form (V.2.5), it behaves like an additional mortality (as a harvesting rate) for the age-structured population dynamics. In Sections V.3 and V.5 we shall develop this

idea in the more general case when the functions β and μ depend on both time and age variable, and the control u acts only in a subinterval of $[0, a_+]$.

V.3 A POSITIVE NULL CONTROLLABILITY RESULT WHEN THE YOUNG INDIVIDUALS DO NOT REPRODUCE

This section is devoted to prove Theorem V.1.1, firstly in the case where the functions β and μ do not depend on the third variable (see Proposition V.3.2). The proof of Theorem V.1.1 in the more general nonlinear case will be deduced from the result of Proposition V.3.2 and the comparison principle for age-structured population dynamics.

More precisely, we first assume that the population dynamics is described by the following linear system:

$$\begin{cases} \frac{\partial p}{\partial t}(t, a) + \frac{\partial p}{\partial a}(t, a) + \mu_0(t, a)p(t, a) = \chi_{[a_1, a_2]}(a)u(t, a), & (t, a) \in Q_\tau \\ p(t, 0) = \int_0^{a_+} \beta_0(t, a)p(t, a) da, & t \in (0, \tau) \\ p(0, a) = p_0(a), & a \in (0, a_+), \end{cases} \quad (\text{V.3.1})$$

where τ is a positive constant and $Q_\tau = (0, \tau) \times (0, a_+)$. For the sake of completeness, let us recall the classical assumptions on the functions μ_0, β_0, p_0 and u relatively to the linear case:

- (L1) the functions β_0 and μ_0 belong to $L^\infty(Q_\tau)$ and $L^1_{loc}([0, \tau] \times [0, a_+))$ respectively,
- (L2) for almost every $(t, a) \in Q_\tau$, we have

$$\beta_0(t, a) \geq 0 \quad \text{and} \quad \mu_0(t, a) \geq 0,$$

- (L3) $p_0 \in L^2(0, a_+)$, $p_0(a) \geq 0$ a.e. $a \in (0, a_+)$,

- (L4) the control function u belongs to $L^2(Q_\tau)$.

Note that, using the definition of a solution to (V.3.1), we can obtain, using an integration along the characteristic lines (see, for instance, [Ani00, p. 16]) that for almost every $(t, a) \in Q_\tau$ with $a \geq t$, we have that the solution p to (V.3.1) satisfies

$$\begin{aligned} p(t, a) = & \exp \left\{ - \int_0^t \mu_0(s, a - t + s) ds \right\} p_0(a - t) \\ & + \int_0^t \exp \left\{ - \int_s^t \mu_0(\sigma, a - t + \sigma) d\sigma \right\} \chi_{[a_1, a_2]}(a - t + s) u(s, a - t + s) ds, \end{aligned} \quad (\text{V.3.2})$$

and for almost every $(t, a) \in Q_\tau$ with $t > a$, we have

$$p(t, a) = \exp \left\{ - \int_0^a \mu_0(t - a + s, s) ds \right\} B(t - a) \\ + \int_0^a \exp \left\{ - \int_s^a \mu_0(t - a + \sigma, \sigma) d\sigma \right\} \chi_{[a_1, a_2]}(s) u(t - a + s, s) ds,$$

where

$$B(t) = \int_0^{a_\dagger} \beta_0(t, a) p(t, a) da \quad \text{a.e. } t \in (0, \tau).$$

Notice that, as a consequence of the Banach fixed-point theorem, we get that $B \in L^\infty(0, \tau)$ (see, for instance, [Ani00]).

In the following, we shall make several uses of the *comparison principle for age-structured population dynamics* (see [Ani00, Theorem 2.1.2]) which we recall in the following Theorem V.3.1:

Theorem V.3.1. *With the above notations and assumptions, assume that $\tau > 0$. Let p be the solution of (V.3.1). We get*

- (i) *if $u(t, a) \geq 0$ a.e. in Q_τ , then $p(t, a) \geq 0$ a.e. in Q_τ ;*
- (ii) *if $\beta_{0i}, \mu_{0i}, p_{0i}, u_i$ satisfy (L1) - (L4) ($i = \overline{1, 2}$) and*

$$\begin{aligned} \beta_{01}(t, a) &\geq \beta_{02}(t, a), & \mu_{01}(t, a) &\leq \mu_{02}(t, a) & \text{a.e. in } Q_\tau, \\ p_{01}(a) &\geq p_{02}(a), & & & \text{a.e. in } (0, a_\dagger), \\ u_1(t, a) &\geq u_2(t, a), & & & \text{a.e. in } Q_\tau, \end{aligned}$$

then $p^1(t, a) \geq p^2(t, a)$ a.e. in Q_τ , where p^i is the solution of (V.3.1), corresponding to $\beta_0 := \beta_{0i}, \mu_0 := \mu_{0i}, p_0 := p_{0i}, u := u_i, i \in \overline{1, 2}$.

In this section, we will first prove the following Proposition V.3.2 which is, roughly speaking, the “linear version” of Theorem V.1.1:

Proposition V.3.2. *With the above notations and assumptions, suppose that there exists $a_b \in (0, a_\dagger)$ such that for almost every $t \in (0, \tau)$, we have*

$$\beta_0(t, a) = 0 \quad \text{a.e. } a \in (0, a_b). \quad (\text{V.3.3})$$

Assume that we have $a_1 < a_b$. If $\tau > a_\dagger - a_2 + a_1$, then for every $p_0 \in L^2(0, a_\dagger)$ with $p_0(a) \geq 0$ a.e. $a \in (0, a_\dagger)$, there exists $u \in L^2(Q_\tau)$ and a corresponding unique nonnegative solution p of (V.3.1) such that

$$p(\tau, a) = 0 \quad \text{a.e. } a \in (0, a_\dagger).$$

In this whole section, we suppose that there exists $a_b > 0$ such that (V.3.3) is satisfied, with $a_1 < a_b$ and $\tau > a_\dagger - a_2 + a_1$.

We may divide our proof in two cases, since the shapes of the controls functions driving the initial population to zero in time τ may be slightly different in the two following cases, depending on the order relationship of a_2 and a_b :

Case 1 : $\tau > a_{\dagger} - a_2 + a_1$ and $a_1 < a_2 \leq a_b$,

Case 2 : $\tau > a_{\dagger} - a_2 + a_1$ and $a_1 < a_b < a_2$.

The two following subsections are devoted to prove Proposition V.3.2 in the two above mentioned cases.

§ V.3.1. **The first case :** $\tau > a_{\dagger} - a_2 + a_1$ and $a_1 < a_2 \leq a_b$

This subsection is devoted to prove Proposition V.3.2, in the case where $\tau > a_{\dagger} - a_2 + a_1$ and $a_1 < a_2 \leq a_b$.

Proof of Proposition V.3.2 with $a_2 \leq a_b$. Suppose that $\tau > a_{\dagger} - a_2 + a_1$ and $a_1 < a_2 \leq a_b$. Without loss of generality, we may assume that $\tau < a_{\dagger}$. Let us prove that there exists a feedback control u and a corresponding unique nonnegative solution p of (V.3.1) such that

$$p(\tau, a) = 0 \quad \text{a.e. } a \in (0, a_{\dagger}).$$

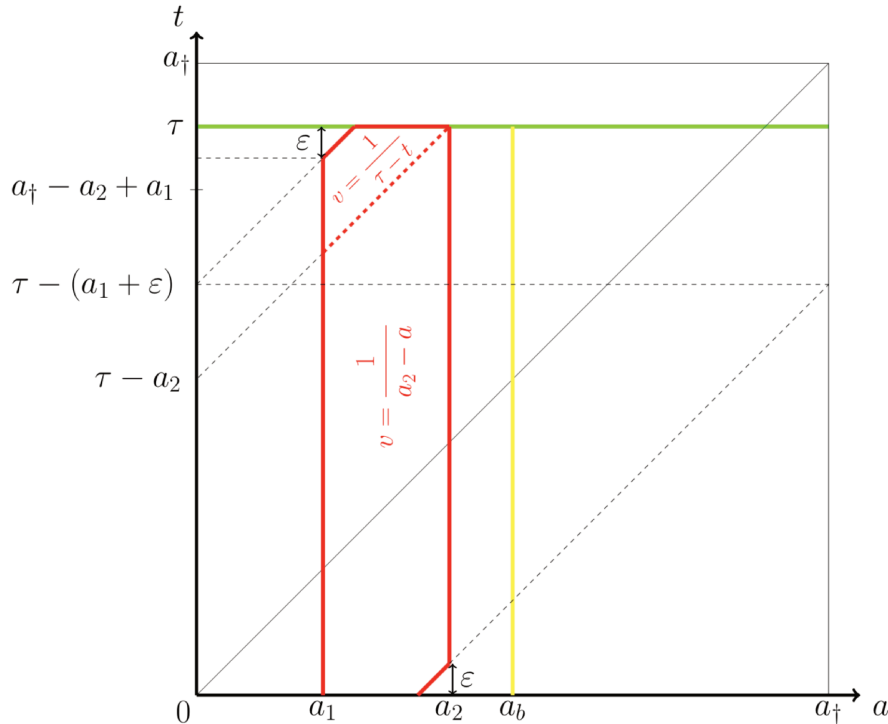


Figure V.1 – The structure of the control v when $\tau > a_{\dagger} - a_2 + a_1$ and $a_1 < a_2 \leq a_b$.

Denote by $\varepsilon := \frac{1}{2}(\tau - (a_+ - a_2 + a_1))$. For almost every $(t, a) \in Q_\tau$, let us consider the feedback control $u(t, a) := -v(t, a)p(t, a)$, where v is a control itself and acts as a harvesting rate, and is given by

$$v(t, a) := \begin{cases} \frac{1}{a_2 - a} & \text{if } \begin{cases} \varepsilon - a_2 \leq t - a < \tau - a_2, \\ a \in [a_1, a_2], \end{cases} \\ \frac{1}{\tau - t} & \text{if } \begin{cases} \tau - a_2 \leq t - a \leq \tau - (a_1 + \varepsilon), \\ a \in [a_1, a_2], \end{cases} \\ 0 & \text{otherwise.} \end{cases} \quad (\text{V.3.4})$$

The structure of v is given in Figure V.1. The solution to (V.3.1) with $u = -vp$ exists, is unique and nonnegative (see Anița [Ani00]), and via the comparison principle for age-dependent population dynamics (see Theorem V.3.1 - (ii)) we get that

$$0 \leq p(t, a) \leq \tilde{p}(t, a) \quad \text{a.e. } (t, a) \in Q_\tau, \quad (\text{V.3.5})$$

where \tilde{p} is the solution to

$$\begin{cases} \frac{\partial \tilde{p}}{\partial t}(t, a) + \frac{\partial \tilde{p}}{\partial a}(t, a) + v(t, a)\tilde{p}(t, a) = 0, & (t, a) \in Q_\tau \\ \tilde{p}(t, 0) = \int_{a_b}^{a_+} \beta_0(t, a)\tilde{p}(t, a) da, & t \in (0, \tau) \\ \tilde{p}(0, a) = p_0(a), & a \in (0, a_+). \end{cases} \quad (\text{V.3.6})$$

Integrating along the characteristic lines, we get that the solution \tilde{p} of (V.3.6) satisfies

$$\tilde{p}(t, a) = \begin{cases} p_0(a - t) \exp\left(-\int_{a-t}^a \frac{ds}{a_2 - s}\right) & \text{if } \begin{cases} \varepsilon - a_2 \leq t - a \leq -a_1, \\ a \in [a_1, a_2], \end{cases} \\ p_0(a - t) \exp\left(-\int_{a_1}^a \frac{ds}{a_2 - s}\right) & \text{if } \begin{cases} -a_1 \leq t - a \leq 0, \\ a \in [a_1, a_2], \end{cases} \\ \tilde{p}(t - a, 0) \exp\left(-\int_{a_1}^a \frac{ds}{a_2 - s}\right) & \text{if } \begin{cases} 0 < t - a < \tau - a_2, \\ a \in [a_1, a_2], \end{cases} \\ \tilde{p}(t - a, 0) \exp\left(-\int_{t-a+a_1}^t \frac{ds}{\tau - s}\right) & \text{if } \begin{cases} \tau - a_2 \leq t - a \leq \tau - (a_1 + \varepsilon), \\ a \in [a_1, a_2], \end{cases} \end{cases}$$

so that we have

$$\tilde{p}(t, a) = \begin{cases} \frac{a_2 - a}{a_2 - (a - t)} p_0(a - t) & \text{if } \begin{cases} \varepsilon - a_2 \leq t - a \leq -a_1, \\ a \in [a_1, a_2], \end{cases} \\ \frac{a_2 - a}{a_2 - a_1} p_0(a - t) & \text{if } \begin{cases} -a_1 \leq t - a \leq 0, \\ a \in [a_1, a_2], \end{cases} \\ \frac{a_2 - a}{a_2 - a_1} \tilde{p}(t - a, 0) & \text{if } \begin{cases} 0 < t - a < \tau - a_2, \\ a \in [a_1, a_2], \end{cases} \\ \frac{\tau - t}{\tau - (t - a + a_1)} \tilde{p}(t - a, 0) & \text{if } \begin{cases} \tau - a_2 \leq t - a \leq \tau - (a_1 + \varepsilon), \\ a \in [a_1, a_2]. \end{cases} \end{cases} \quad (\text{V.3.7})$$

We may conclude from the above formula that we have

$$\begin{cases} \tilde{p}(t, a_2) = 0 & \text{if } t \in (\varepsilon, \tau), \\ \tilde{p}(\tau, a) = 0 & \text{if } a \in (a_1 + \varepsilon, a_2). \end{cases} \quad (\text{V.3.8})$$

Integrating on the characteristic lines the equation satisfied by \tilde{p} , we get from (V.3.8) that

$$\tilde{p}(t, a) = 0 \quad \text{if } (t, a) \in [\tau - (a_1 + \varepsilon), \tau] \times [a_2, a_+]. \quad (\text{V.3.9})$$

Since $a_b \geq a_2$, we obviously get from the above equation that

$$\tilde{p}(t, a) = 0 \quad \text{if } (t, a) \in [\tau - (a_1 + \varepsilon), \tau] \times [a_b, a_+].$$

Since \tilde{p} satisfies the renewal law, we may infer that

$$\tilde{p}(t, 0) = 0 \quad \text{if } t \in [\tau - (a_1 + \varepsilon), \tau],$$

and integrating along the characteristic lines we get that

$$\tilde{p}(\tau, a) = 0 \quad \text{if } a \in [0, a_1 + \varepsilon]. \quad (\text{V.3.10})$$

By (V.3.8), (V.3.9) and (V.3.10) we may conclude that

$$\tilde{p}(\tau, a) = 0 \quad \text{a.e. } a \in (0, a_+).$$

It remains to prove that the control $u = -vp$ belongs to $L^2(Q_\tau)$. For almost every

$(t, a) \in Q_\tau$, from (V.3.4) and (V.3.7) we have

$$v(t, a)\tilde{p}(t, a) = \begin{cases} \frac{p_0(a-t)}{a_2 - (a-t)} & \text{if } \begin{cases} \varepsilon - a_2 \leq t - a \leq -a_1, \\ a \in [a_1, a_2], \end{cases} \\ \frac{p_0(a-t)}{a_2 - a_1} & \text{if } \begin{cases} -a_1 \leq t - a \leq 0, \\ a \in [a_1, a_2], \end{cases} \\ \frac{\tilde{p}(t-a, 0)}{a_2 - a_1} & \text{if } \begin{cases} 0 < t - a < \tau - a_2, \\ a \in [a_1, a_2], \end{cases} \\ \frac{\tilde{p}(t-a, 0)}{\tau - (t-a+a_1)} & \text{if } \begin{cases} \tau - a_2 \leq t - a \leq \tau - (a_1 + \varepsilon), \\ a \in [a_1, a_2]. \end{cases} \\ 0 & \text{otherwise,} \end{cases}$$

and this implies that

$$|v(t, a)\tilde{p}(t, a)| \leq \begin{cases} \frac{|p_0(a-t)|}{\varepsilon} & \text{if } \begin{cases} \varepsilon - a_2 \leq t - a \leq -a_1, \\ a \in [a_1, a_2], \end{cases} \\ \frac{|p_0(a-t)|}{a_2 - a_1} & \text{if } \begin{cases} -a_1 \leq t - a \leq 0, \\ a \in [a_1, a_2], \end{cases} \\ \frac{|\tilde{p}(t-a, 0)|}{a_2 - a_1} & \text{if } \begin{cases} 0 < t - a < \tau - a_2, \\ a \in [a_1, a_2], \end{cases} \\ \frac{|\tilde{p}(t-a, 0)|}{\varepsilon} & \text{if } \begin{cases} \tau - a_2 \leq t - a \leq \tau - (a_1 + \varepsilon), \\ a \in [a_1, a_2]. \end{cases} \\ 0 & \text{otherwise.} \end{cases} \quad (\text{V.3.11})$$

Since $\tilde{p} \in L^2(Q_\tau)$, we may infer from the renewal law that we have $\tilde{p}(\cdot, 0) \in L^2(0, \tau)$, so that from (V.3.11) we have $v\tilde{p} \in L^2(Q_\tau)$. Given (V.3.5), it follows that we have $vp \in L^2(Q_\tau)$, which concludes the proof of Proposition V.3.2 in the case where $a_2 \leq a_b$. \square

§ V.3.2. The second case : $\tau > a_\dagger - a_2 + a_1$ and $a_1 < a_b < a_2$

This subsection is devoted to prove Proposition V.3.2, in the case where $\tau > a_\dagger - a_2 + a_1$ and $a_1 < a_b < a_2$.

Proof of Proposition V.3.2 with $a_b < a_2$. Suppose that $\tau > a_\dagger - a_2 + a_1$ and $a_1 < a_b < a_2$. Without loss of generality, we may assume that $\tau < a_\dagger$. We may also assume, without loss of generality, that we have $a_b \in (a_1, \tau)$: indeed, in the case when there exists $a_b > 0$ satisfying (V.3.3) such that $a_b \geq \tau$, we may consider \tilde{a}_b such that $\tilde{a}_b \in (a_1, \tau)$ and replace a_b by \tilde{a}_b , noting that \tilde{a}_b also satisfies (V.3.3) since $\tilde{a}_b < a_b$.

For any $\varepsilon > 0$, denote by $\theta(\varepsilon) := \tau - (a_\dagger - a_2 + a_1) - \varepsilon$. Let us introduce $\varepsilon > 0$ small enough such that we have $a_1 + \varepsilon < a_b$ and $\theta(\varepsilon) > 0$ (this is possible, since $a_1 < a_b$ and

$\tau > a_{\dagger} - a_2 + a_1$).

Let us prove that there exists a feedback control u and a corresponding unique non-negative solution p of (V.3.1) satisfying

$$p(\tau, a) = 0 \quad \text{a.e. } a \in (0, a_{\dagger}).$$

For almost every $(t, a) \in Q_{\tau}$, let us consider the feedback control $u(t, a) := -v(t, a)p(t, a)$, where

$$v(t, a) := \begin{cases} \frac{1}{a_2 - a} & \text{if } \begin{cases} \theta(\varepsilon) - a_2 \leq t - a < \tau - a_1 - a_2 - \varepsilon, \\ a \in [a_1, a_2], \end{cases} \\ \frac{1}{\tau - (a_1 + \varepsilon) - t} & \text{if } \begin{cases} \tau - a_1 - a_2 - \varepsilon \leq t - a < \tau - a_1 - a_b - \varepsilon, \\ a \in [a_1, a_2], \end{cases} \\ \frac{1}{a_b - a} & \text{if } \begin{cases} \tau - a_1 - a_b - \varepsilon \leq t - a < \tau - a_b, \\ a \in [a_1, a_b], \end{cases} \\ \frac{1}{\tau - t} & \text{if } \begin{cases} \tau - a_b \leq t - a < \tau - (a_1 + \varepsilon), \\ a \in [a_1, a_b], \end{cases} \\ 0 & \text{otherwise.} \end{cases} \quad (\text{V.3.12})$$

The structure of v is given in Figure V.2. The solution to (V.3.1) with $u = -vp$ exists, is unique and nonnegative (see Anița [Ani00]), and via the comparison principle for age-dependent population dynamics (see Theorem V.3.1 - (ii)) we get that

$$0 \leq p(t, a) \leq \tilde{p}(t, a) \quad \text{a.e. } (t, a) \in Q_{\tau}, \quad (\text{V.3.13})$$

where \tilde{p} is the solution to

$$\begin{cases} \frac{\partial \tilde{p}}{\partial t}(t, a) + \frac{\partial \tilde{p}}{\partial a}(t, a) + v(t, a)\tilde{p}(t, a) = 0, & (t, a) \in Q_{\tau} \\ \tilde{p}(t, 0) = \int_{a_b}^{a_{\dagger}} \beta_0(t, a)\tilde{p}(t, a) da, & t \in (0, \tau) \\ \tilde{p}(0, a) = p_0(a), & a \in (0, a_{\dagger}). \end{cases} \quad (\text{V.3.14})$$

We may discuss three subcases, depending on the order relationship of $\tau - a_1 - a_b - \varepsilon$, $\tau - a_1 - a_2 - \varepsilon$ and 0 (since the expression of the solution \tilde{p} of (V.3.14) may be different in each subcase). The first subcase (see below) is shown in Figure V.2.

Subcase 1 : assume that $\tau - a_1 - a_b - \varepsilon \leq 0$. Integrating along the characteristic

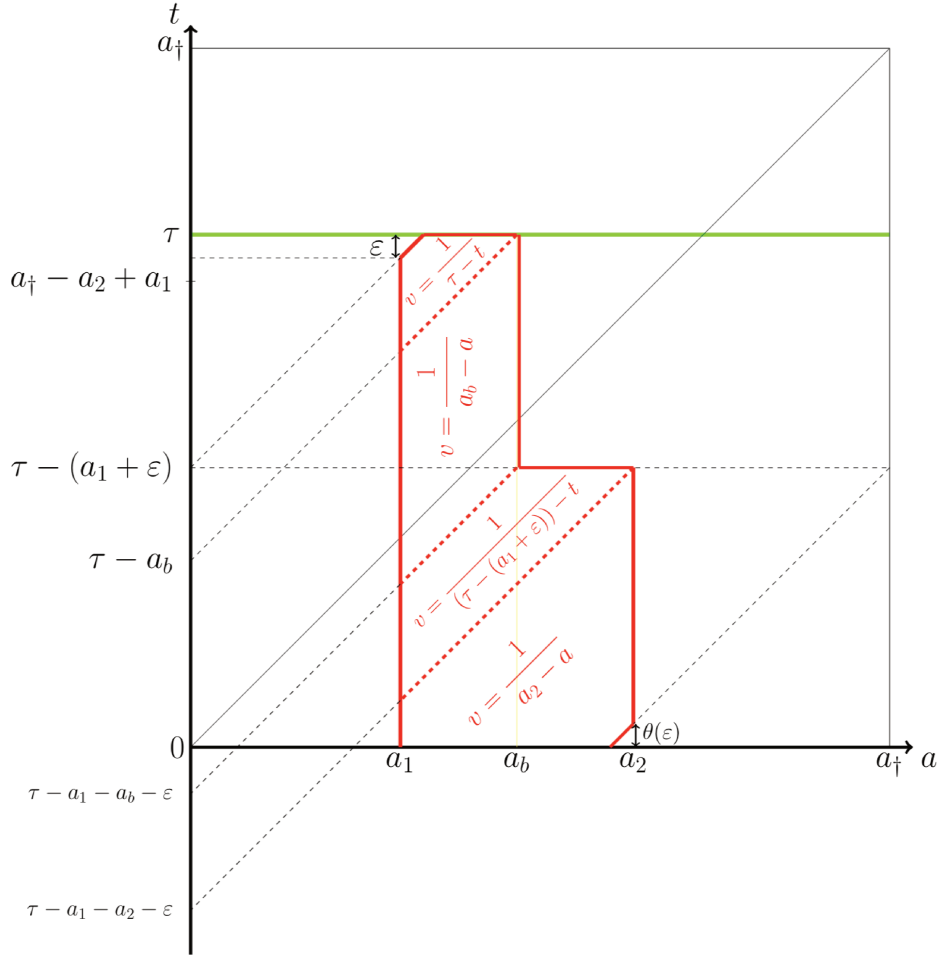


Figure V.2 – The structure of the control v when $\tau > a_{\dagger} - a_2 + a_1$ and $a_1 < a_b < a_2$.

lines, we get that the solution \tilde{p} of (V.3.14) satisfies

$$\tilde{p} = \begin{cases} p_0(a-t) \exp\left(-\int_{\max(a-t, a_1)}^a \frac{ds}{a_2-s}\right) & \text{if } \begin{cases} \theta(\varepsilon) - a_2 \leq t-a < \tau - a_1 - a_2 - \varepsilon, \\ a \in [a_1, a_2], \end{cases} \\ p_0(a-t) \exp\left(-\int_{\max(t-a+a_1, 0)}^t \frac{ds}{\tau - (a_1 + \varepsilon) - s}\right) & \text{if } \begin{cases} \tau - a_1 - a_2 - \varepsilon \leq t-a < \tau - a_1 - a_b - \varepsilon, \\ a \in [a_1, a_2], \end{cases} \\ p_0(a-t) \exp\left(-\int_{\max(a-t, a_1)}^a \frac{ds}{a_b-s}\right) & \text{if } \begin{cases} \tau - a_1 - a_b - \varepsilon \leq t-a \leq 0, \\ a \in [a_1, a_b], \end{cases} \\ \tilde{p}(t-a, 0) \exp\left(-\int_{a_1}^a \frac{ds}{a_b-s}\right) & \text{if } \begin{cases} 0 < t-a \leq \tau - a_b, \\ a \in [a_1, a_b], \end{cases} \\ \tilde{p}(t-a, 0) \exp\left(-\int_{t-a+a_1}^t \frac{ds}{\tau-s}\right) & \text{if } \begin{cases} \tau - a_b < t-a \leq \tau - (a_1 + \varepsilon), \\ a \in [a_1, a_b], \end{cases} \end{cases}$$

and so that we have

$$\tilde{p} = \begin{cases} \frac{a_2 - a}{a_2 - \max(a - t, a_1)} p_0(a - t) & \text{if } \begin{cases} \theta(\varepsilon) - a_2 \leq t - a < \tau - a_1 - a_2 - \varepsilon, \\ a \in [a_1, a_2], \end{cases} \\ \frac{\tau - (a_1 + \varepsilon) - t}{\tau - (a_1 + \varepsilon) - \max(t - a + a_1, 0)} p_0(a - t) & \text{if } \begin{cases} \tau - a_1 - a_2 - \varepsilon \leq t - a < \tau - a_1 - a_b - \varepsilon, \\ a \in [a_1, a_2], \end{cases} \\ \frac{a_b - a}{a_b - \max(a - t, a_1)} p_0(a - t) & \text{if } \begin{cases} \tau - a_1 - a_b - \varepsilon \leq t - a \leq 0, \\ a \in [a_1, a_b], \end{cases} \\ \frac{a_b - a}{a_b - a_1} \tilde{p}(t - a, 0) & \text{if } \begin{cases} 0 < t - a \leq \tau - a_b, \\ a \in [a_1, a_b], \end{cases} \\ \frac{\tau - t}{\tau - (t - a + a_1)} \tilde{p}(t - a, 0) & \text{if } \begin{cases} \tau - a_b < t - a \leq \tau - (a_1 + \varepsilon), \\ a \in [a_1, a_b]. \end{cases} \end{cases} \quad (\text{V.3.15})$$

Subcase 2 : assume that $\tau - a_1 - a_2 - \varepsilon \leq 0 < \tau - a_1 - a_b - \varepsilon$. Integrating along the characteristic lines, we get that the solution \tilde{p} of (V.3.14) satisfies

$$\tilde{p} = \begin{cases} p_0(a - t) \exp\left(-\int_{\max(a-t, a_1)}^a \frac{ds}{a_2 - s}\right) & \text{if } \begin{cases} \theta(\varepsilon) - a_2 \leq t - a < \tau - a_1 - a_2 - \varepsilon, \\ a \in [a_1, a_2], \end{cases} \\ p_0(a - t) \exp\left(-\int_{\max(t-a+a_1, 0)}^t \frac{ds}{\tau - (a_1 + \varepsilon) - s}\right) & \text{if } \begin{cases} \tau - a_1 - a_2 - \varepsilon \leq t - a \leq 0, \\ a \in [a_1, a_2], \end{cases} \\ \tilde{p}(t - a, 0) \exp\left(-\int_{t-a+a_1}^t \frac{ds}{\tau - (a_1 + \varepsilon) - s}\right) & \text{if } \begin{cases} 0 \leq t - a \leq \tau - a_1 - a_b - \varepsilon, \\ a \in [a_1, a_2], \end{cases} \\ \tilde{p}(t - a, 0) \exp\left(-\int_{a_1}^a \frac{ds}{a_b - s}\right) & \text{if } \begin{cases} \tau - a_1 - a_b - \varepsilon < t - a \leq \tau - a_b, \\ a \in [a_1, a_b], \end{cases} \\ \tilde{p}(t - a, 0) \exp\left(-\int_{t-a+a_1}^t \frac{ds}{\tau - s}\right) & \text{if } \begin{cases} \tau - a_b < t - a \leq \tau - (a_1 + \varepsilon), \\ a \in [a_1, a_b], \end{cases} \end{cases}$$

and so that we have

$$\tilde{p}(t, a) = \begin{cases} \frac{a_2 - a}{a_2 - \max(a - t, a_1)} p_0(a - t) & \text{if } \begin{cases} \theta(\varepsilon) - a_2 \leq t - a < \tau - a_1 - a_2 - \varepsilon, \\ a \in [a_1, a_2], \end{cases} \\ \frac{\tau - (a_1 + \varepsilon) - t}{\tau - (a_1 + \varepsilon) - \max(t - a + a_1, 0)} p_0(a - t) & \text{if } \begin{cases} \tau - a_1 - a_2 - \varepsilon \leq t - a \leq 0, \\ a \in [a_1, a_2], \end{cases} \\ \frac{\tau - (a_1 + \varepsilon) - t}{\tau - (a_1 + \varepsilon) - (t - a + a_1)} \tilde{p}(t - a, 0) & \text{if } \begin{cases} 0 < t - a \leq \tau - a_1 - a_b - \varepsilon, \\ a \in [a_1, a_2], \end{cases} \\ \frac{a_b - a}{a_b - a_1} \tilde{p}(t - a, 0) & \text{if } \begin{cases} \tau - a_1 - a_b - \varepsilon < t - a \leq \tau - a_b, \\ a \in [a_1, a_b], \end{cases} \\ \frac{\tau - t}{\tau - (t - a + a_1)} \tilde{p}(t - a, 0) & \text{if } \begin{cases} \tau - a_b < t - a \leq \tau - (a_1 + \varepsilon), \\ a \in [a_1, a_b]. \end{cases} \end{cases} \quad (\text{V.3.16})$$

Subcase 3 : assume that $\tau - a_1 - a_2 - \varepsilon > 0$. Integrating along the characteristic lines, we get that the solution \tilde{p} of (V.3.14) satisfies

$$\tilde{p} = \begin{cases} p_0(a - t) \exp\left(-\int_{\max(a-t, a_1)}^a \frac{ds}{a_2 - s}\right) & \text{if } \begin{cases} \theta(\varepsilon) - a_2 \leq t - a \leq 0, \\ a \in [a_1, a_2], \end{cases} \\ \tilde{p}(t - a, 0) \exp\left(-\int_{a_1}^a \frac{ds}{a_2 - s}\right) & \text{if } \begin{cases} 0 < t - a \leq \tau - a_1 - a_2 - \varepsilon, \\ a \in [a_1, a_2], \end{cases} \\ \tilde{p}(t - a, 0) \exp\left(-\int_{t-a+a_1}^t \frac{ds}{\tau - (a_1 + \varepsilon) - s}\right) & \text{if } \begin{cases} \tau - a_1 - a_2 - \varepsilon < t - a \leq \tau - a_1 - a_b - \varepsilon, \\ a \in [a_1, a_2], \end{cases} \\ \tilde{p}(t - a, 0) \exp\left(-\int_{a_1}^a \frac{ds}{a_b - s}\right) & \text{if } \begin{cases} \tau - a_1 - a_b - \varepsilon < t - a \leq \tau - a_b, \\ a \in [a_1, a_b], \end{cases} \\ \tilde{p}(t - a, 0) \exp\left(-\int_{t-a+a_1}^t \frac{ds}{\tau - s}\right) & \text{if } \begin{cases} \tau - a_b < t - a \leq \tau - (a_1 + \varepsilon), \\ a \in [a_1, a_b], \end{cases} \end{cases}$$

and so that we have

$$\tilde{p}(t, a) = \begin{cases} \frac{a_2 - a}{a_2 - \max(a - t, a_1)} p_0(a - t) & \text{if } \begin{cases} \theta(\varepsilon) - a_2 \leq t - a \leq 0, \\ a \in [a_1, a_2], \end{cases} \\ \frac{a_2 - a}{a_2 - a_1} \tilde{p}(t - a, 0) & \text{if } \begin{cases} 0 < t - a \leq \tau - a_1 - a_2 - \varepsilon, \\ a \in [a_1, a_2], \end{cases} \\ \frac{\tau - (a_1 + \varepsilon) - t}{\tau - (a_1 + \varepsilon) - (t - a + a_1)} \tilde{p}(t - a, 0) & \text{if } \begin{cases} \tau - a_1 - a_2 - \varepsilon < t - a \leq \tau - a_1 - a_b - \varepsilon, \\ a \in [a_1, a_2], \end{cases} \\ \frac{a_b - a}{a_b - a_1} \tilde{p}(t - a, 0) & \text{if } \begin{cases} \tau - a_1 - a_b - \varepsilon < t - a \leq \tau - a_b, \\ a \in [a_1, a_b], \end{cases} \\ \frac{\tau - t}{\tau - (t - a + a_1)} \tilde{p}(t - a, 0) & \text{if } \begin{cases} \tau - a_b < t - a \leq \tau - (a_1 + \varepsilon), \\ a \in [a_1, a_b]. \end{cases} \end{cases} \quad (\text{V.3.17})$$

In the three subcases, we may conclude that we have

$$\begin{cases} \tilde{p}(t, a_2) = 0 & \text{if } t \in (\theta(\varepsilon), \tau - (a_1 + \varepsilon)), \\ \tilde{p}(\tau - (a_1 + \varepsilon), a) = 0 & \text{if } a \in (a_b, a_2), \\ \tilde{p}(t, a_b) = 0 & \text{if } t \in (\tau - (a_1 + \varepsilon), \tau), \\ \tilde{p}(\tau, a) = 0 & \text{if } a \in (a_1 + \varepsilon, a_b). \end{cases} \quad (\text{V.3.18})$$

Integrating on the characteristic lines the equation satisfied by \tilde{p} , we get that

$$\tilde{p}(t, a) = 0 \quad \text{if } (t, a) \in [\tau - (a_1 + \varepsilon), \tau] \times [a_b, a_+]. \quad (\text{V.3.19})$$

Since \tilde{p} satisfies the renewal law, we may infer that

$$\tilde{p}(t, 0) = 0 \quad \text{if } t \in [\tau - (a_1 + \varepsilon), \tau],$$

and integrating along the characteristic lines we get that

$$\tilde{p}(\tau, a) = 0 \quad \text{if } a \in [0, a_1 + \varepsilon]. \quad (\text{V.3.20})$$

By (V.3.18), (V.3.19) and (V.3.20) we get that

$$\tilde{p}(\tau, a) = 0 \quad \text{a.e. } a \in (0, a_+).$$

It remains to prove that, for each subcase, the control $u = -vp$ belongs to $L^2(Q_\tau)$. In order to avoid repetitive arguments, we prove that $u \in L^2(Q_\tau)$ only in the first subcase. In this subcase, for almost every $(t, a) \in Q_\tau$, from (V.3.12) and the corresponding expression

of \tilde{p} we have that

$$v(t, a)\tilde{p}(t, a) = \begin{cases} \frac{p_0(a-t)}{a_2 - \max(a-t, a_1)} & \text{if } \begin{cases} \theta(\varepsilon) - a_2 \leq t-a < \tau - a_1 - a_2 - \varepsilon, \\ a \in [a_1, a_2], \end{cases} \\ \frac{p_0(a-t)}{\tau - (a_1 + \varepsilon) - \max(t-a+a_1, 0)} & \text{if } \begin{cases} \tau - a_1 - a_2 - \varepsilon \leq t-a \leq \tau - a_1 - a_b - \varepsilon, \\ a \in [a_1, a_2], \end{cases} \\ \frac{p_0(a-t)}{a_b - \max(a-t, a_1)} & \text{if } \begin{cases} \tau - a_1 - a_b - \varepsilon < t-a \leq 0, \\ a \in [a_1, a_b], \end{cases} \\ \frac{\tilde{p}(t-a, 0)}{a_b - a_1} & \text{if } \begin{cases} 0 < t-a \leq \tau - a_b, \\ a \in [a_1, a_b], \end{cases} \\ \frac{\tilde{p}(t-a, 0)}{\tau - (t-a+a_1)} & \text{if } \begin{cases} \tau - a_b < t-a \leq \tau - (a_1 + \varepsilon), \\ a \in [a_1, a_b], \end{cases} \\ 0 & \text{otherwise,} \end{cases}$$

and it follows that we have

$$|v(t, a)\tilde{p}(t, a)| \leq \begin{cases} \frac{|p_0(a-t)|}{\min(\theta(\varepsilon), a_2 - a_1)} & \text{if } \begin{cases} \theta(\varepsilon) - a_2 \leq t-a < \tau - a_1 - a_2 - \varepsilon, \\ a \in [a_1, a_2], \end{cases} \\ \frac{|p_0(a-t)|}{\min(\tau - (a_1 + \varepsilon), a_b - a_1)} & \text{if } \begin{cases} \tau - a_1 - a_2 - \varepsilon \leq t-a \leq \tau - a_1 - a_b - \varepsilon, \\ a \in [a_1, a_2], \end{cases} \\ \frac{|p_0(a-t)|}{\min(\tau - (a_1 + \varepsilon), a_b - a_1)} & \text{if } \begin{cases} \tau - a_1 - a_b - \varepsilon < t-a \leq 0 \\ a \in [a_1, a_b], \end{cases} \\ \frac{|\tilde{p}(t-a, 0)|}{a_b - a_1} & \text{if } \begin{cases} 0 < t-a \leq \tau - a_b, \\ a \in [a_1, a_b], \end{cases} \\ \frac{|\tilde{p}(t-a, 0)|}{\varepsilon} & \text{if } \begin{cases} \tau - a_b < t-a \leq \tau - (a_1 + \varepsilon), \\ a \in [a_1, a_b], \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

Since $\tilde{p} \in L^2(Q_\tau)$, we may infer from the renewal law that we have $\tilde{p}(\cdot, 0) \in L^2(0, \tau)$, so that we have $v\tilde{p} \in L^2(Q_\tau)$. Given (V.3.13), it follows that we have $vp \in L^2(Q_\tau)$, which concludes the proof of Proposition V.3.2 in the case where $a_2 > a_b$. \square

§ V.3.3. Proof of Theorem V.1.1 : from the linear to the nonlinear case

This subsection is devoted to prove Theorem V.1.1, using the result of Proposition V.3.2 and the comparison principle for age-structured population dynamics (see Theorem V.3.1).

Proof of Theorem V.1.1. Assume that there exists $a_b > 0$ such that (V.1.3) holds. Sup-

pose that $a_1 < a_b$, and let $\tau > a_{\dagger} - a_2 + a_1$. Suppose, first, that $a_2 \leq a_b$. Denote by v the function defined by (V.3.4). The solution p to (V.1.1) with $u := -vp$ exists, is unique, and via the comparison result for age-dependent population dynamics, we get that

$$0 \leq p(t, a) \leq \bar{p}(t, a) \quad \text{a.e. } (t, a) \in Q_{\tau}, \quad (\text{V.3.21})$$

where \bar{p} is solution to the linear system:

$$\begin{cases} \frac{\partial \bar{p}}{\partial t}(t, a) + \frac{\partial \bar{p}}{\partial a}(t, a) \\ \quad + (\mu(t, a, 0) + v(t, a)\chi_{[a_1, a_2]}(a))\bar{p}(t, a) = 0, & (t, a) \in Q_{\tau} \\ \bar{p}(t, 0) = \int_0^{a_{\dagger}} \beta_+(t, a)\bar{p}(t, a) da, & t \in (0, \tau) \\ \bar{p}(0, a) = p_0(a), & a \in (0, a_{\dagger}). \end{cases} \quad (\text{V.3.22})$$

From the proof of Proposition V.3.2, we get that the solution \bar{p} of (V.3.22) satisfies $\bar{p}(\tau, \cdot) = 0$, with $v\bar{p} \in L^2(Q_{\tau})$, so that using (V.3.21) the solution p to (V.1.1) with $u = -vp$ satisfies $p(\tau, \cdot) = 0$ with $u \in L^2(Q_{\tau})$.

The case where $a_b < a_2$ is similar, denoting by v the function defined by (V.3.12). This concludes the proof of Theorem V.1.1. \square

V.4 NEGATIVE NULL CONTROLLABILITY RESULTS WHEN THE YOUNG INDIVIDUALS DO NOT REPRODUCE

This section is devoted to prove that the results obtained in Theorem V.1.1 are sharp, in the sense that if we do not control before the individuals start to reproduce or if the control time horizon τ is too small, then system (V.1.1) may not be null controllable. Following the methodology of the previous Section V.1, we first prove Theorem V.1.2 in the linear setting (see Proposition V.4.1) and we shall deduce the results of Theorem V.1.2 from those of Proposition V.4.1 and the comparison principle for age-structured population dynamics (see Theorem V.3.1).

More precisely, let us first consider the linear controlled system (V.3.1) subject to assumptions (L1) - (L4) stated in the beginning of Section V.3. We first prove the following Proposition V.4.1 which is the “linear version” of Theorem V.1.2:

Proposition V.4.1. *With the above notations and assumptions,*

- (i) *Assume that there exists $a_b \in (0, a_{\dagger})$ and $a_m \in (0, a_{\dagger}]$ with $a_b < a_m$ such that, for almost every $t \in (0, \tau)$ we have*

$$\beta_0(t, a) > 0 \quad \text{a.e. } a \in (a_b, a_m). \quad (\text{V.4.1})$$

If $a_1 > a_b$, then for every $\tau > 0$, there exists $p_0 \in L^2(0, a_{\dagger})$ with $p_0(a) \geq 0$ a.e.

$a \in (0, a_+)$ such that there is no control u and a corresponding nonnegative solution p to (V.3.1) satisfying

$$p(\tau, a) = 0 \quad \text{a.e. } a \in (0, a_+).$$

(ii) If $\tau < a_+ - a_2 + a_1$, then there exists a function β_0 satisfying (L1) - (L2) and an initial population density $p_0 \in L^2(0, a_+)$ with $p_0(a) \geq 0$ a.e. $a \in (0, a_+)$ such that there is no control u and a corresponding nonnegative solution p to (V.3.1) satisfying

$$p(\tau, a) = 0 \quad \text{a.e. } a \in (0, a_+).$$

Proof of Proposition V.4.1 - (i). Suppose that there exists $a_b \in (0, a_+)$ and $a_m \in (0, a_+]$ with $a_b < a_m$ such that (V.4.1) holds. Assume, first, that $a_1 = a_b$ and let $\tau > 0$. Since $a_1 > 0$, note that there exists $m \in \mathbb{N}$ and $r \in [0, a_1)$ such that $\tau = ma_1 + r$. We may suppose, without loss of generality, that $m \geq 2$. Let $p_0 \in L^2(0, a_+)$ such that

$$p_0(a) > 0 \quad \text{a.e. } a \in (0, a_1 - r). \quad (\text{V.4.2})$$

Suppose that there exists a control function $u \in L^2(Q_\tau)$ and a corresponding nonnegative solution p of (V.3.1) such that

$$p(\tau, a) = 0 \quad \text{a.e. } a \in (0, a_+).$$

Since $p(\tau, a) = 0$ for almost every $a \in (0, a_1)$, integrating along the characteristic lines we have that

$$p(t, 0) = 0 \quad \text{a.e. } t \in (\tau - a_1, \tau). \quad (\text{V.4.3})$$

Since p satisfies the renewal law, for almost every $t \in (\tau - a_1, \tau)$ we have that

$$p(t, 0) = \int_0^{a_+} \beta_0(t, a)p(t, a) da \geq \int_{a_b}^{a_m} \beta_0(t, a)p(t, a) da. \quad (\text{V.4.4})$$

Since $p(t, a) \geq 0$ for almost every $(t, a) \in Q_\tau$, using (V.4.3) and (V.4.4) we have, for almost every $t \in (\tau - a_1, \tau)$,

$$\int_{a_b}^{a_m} \beta_0(t, a)p(t, a) da = 0. \quad (\text{V.4.5})$$

Using the assumptions on the function β_0 and the fact that $p(t, a) \geq 0$ for almost every $(t, a) \in Q_\tau$, together with the above equality (V.4.5) we have that

$$p(t, a) = 0 \quad \text{a.e. } (t, a) \in (\tau - a_1, \tau) \times (a_b, a_m).$$

Integrating along the characteristic lines we have that

$$p(t, 0) = 0 \quad \text{a.e. } t \in (\tau - 2a_1, \tau - a_1).$$

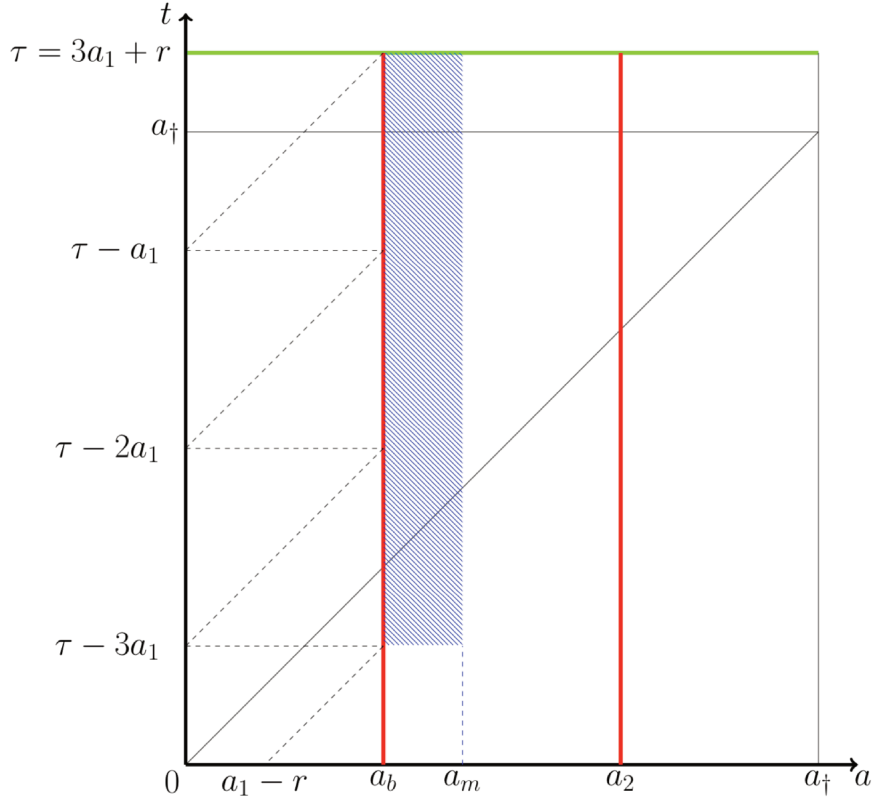


Figure V.3 – In this figure, we have $\tau = 3a_1 + r$ with $r \in [0, a_1)$. If we suppose that the population vanishes in time τ , we may deduce that the population vanishes in the whole blue dashed domain, so that the initial population has to be null in the age interval $(0, a_1 - r)$.

Using an induction argument, we may infer that

$$p(t, 0) = 0 \quad \text{a.e. } t \in (\tau - (m - 1)a_1, \tau - ma_1).$$

Recall that $\tau - ma_1 = r$. Using again the renewal law, it follows from the above equality that

$$p(t, a) = 0 \quad \text{a.e. } (t, a) \in (\tau - (m - 1)a_1, r) \times (a_b, a_m),$$

and integrating along the characteristic lines we have that

$$p(0, a) = 0 \quad \text{a.e. } a \in (0, a_1 - r),$$

which is a contradiction with (V.4.2) (see Figure V.3).

Since it is not possible to bring the population to zero in any time τ in the case where $a_1 = a_b$, we may infer that it is not possible to bring the population to zero in the case where $a_1 > a_b$, so that Proposition V.4.1 - (i) is proved. \square

Proof of Proposition V.4.1 - (ii). Let $\tau < a_{\dagger} - a_2 + a_1$. Suppose that there exists a control function $u \in L^2(Q_{\tau})$ and a corresponding nonnegative solution p of (V.3.1) such that

$$p(\tau, a) = 0 \quad \text{a.e. } a \in (0, a_{\dagger}).$$

We may divide this proof in two cases.

Case 1 : suppose that $\tau < a_{\dagger} - a_2$. Then, for almost every $(t, a) \in Q_{\tau}$ with $a_2 \leq a - t \leq a_{\dagger} - \tau$, we have that

$$\int_0^t \exp \left\{ - \int_s^t \mu_0(\sigma, a - t + \sigma) d\sigma \right\} \chi_{[a_1, a_2]}(a - t + s) u(s, a - t + s) ds = 0,$$

so that from (V.3.2), for almost every $(t, a) \in Q_{\tau}$ with $a_2 \leq a - t \leq a_{\dagger} - \tau$ we have that

$$p(t, a) = \exp \left\{ - \int_0^t \mu_0(s, a - t + s) ds \right\} p_0(a - t),$$

and it follows from the above equality that we have that

$$p(\tau, a) = \exp \left\{ - \int_0^{\tau} \mu_0(s, a - \tau + s) ds \right\} p_0(a - \tau) \quad \text{a.e. } a \in (\tau + a_2, a_{\dagger}),$$

so that from the above equation the condition $p(\tau, \cdot) = 0$ cannot be fulfilled in the case where $p_0(a) \neq 0$ for almost every $a \in (a_2, a_{\dagger} - \tau)$.

Case 2 : suppose that $\tau \geq a_{\dagger} - a_2$. Since $p(\tau, a) = 0$ for almost every $a \in (0, a_1)$, integrating along the characteristic lines we have that

$$p(t, 0) = 0 \quad \text{a.e. } t \in (\tau - a_1, \tau). \quad (\text{V.4.6})$$

Note that, since $\tau < a_{\dagger} - a_2 + a_1$, we have $\tau - a_1 < a_{\dagger} - a_2$ so that from (V.4.6) and using the fact that $a_{\dagger} - a_2 \leq \tau$, we have, in particular,

$$p(t, 0) = 0 \quad \text{a.e. } t \in (\tau - a_1, a_{\dagger} - a_2).$$

Since p satisfies the renewal law, for almost every $t \in (\tau - a_1, a_{\dagger} - a_2)$ we have that

$$p(t, 0) = \int_0^{a_{\dagger}} \beta_0(t, a) p(t, a) da = 0. \quad (\text{V.4.7})$$

Since $\beta_0(t, a) \geq 0$ and $p(t, a) \geq 0$ for almost every $(t, a) \in (\tau - a_1, a_{\dagger} - a_2) \times (0, a_{\dagger})$, we deduce from the above equality (V.4.7) that we have

$$\beta_0(t, a) p(t, a) = 0 \quad \text{a.e. } (t, a) \in (\tau - a_1, a_{\dagger} - a_2) \times (0, a_{\dagger}). \quad (\text{V.4.8})$$

Define the triangle

$$\mathcal{T} := \{(t, a) \in Q_{\tau}, a_2 \leq a - t \leq a_{\dagger} - (\tau - a_1), a \geq \tau - a_1 + a_2, t \geq \tau - a_1\}. \quad (\text{V.4.9})$$

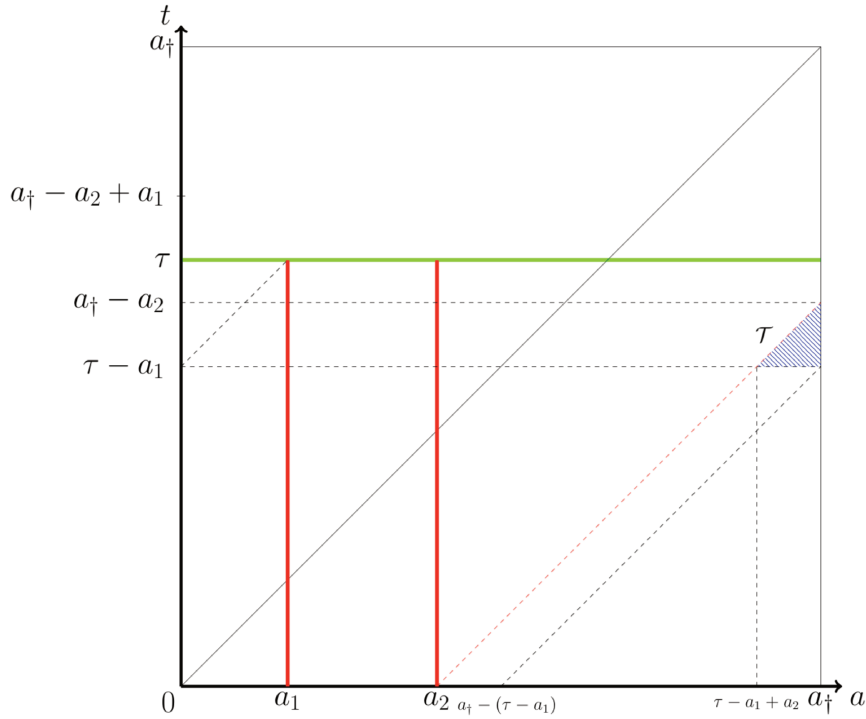


Figure V.4 – In this figure, we have $\tau < a_\dagger - a_2 + a_1$ and $\tau \geq a_\dagger - a_2$. If we suppose that the population vanishes in time τ , because of the renewal law, we may deduce that the population or the function β has to be null within the blue dashed triangle \mathcal{T} (region which is not influenced by the control). This is not the case in general : we may choose an initial condition which is strictly positive in the age interval $(a_2, a_\dagger - (\tau - a_1))$ and a strictly positive function β in the blue dashed triangle.

See Figure V.4 for a picture of the triangle position. From (V.4.8) and (V.4.9), for almost every $(t, a) \in \mathcal{T}$ we have

$$\beta_0(t, a)p(t, a) = 0. \quad (\text{V.4.10})$$

Moreover, for almost every $(t, a) \in \mathcal{T}$ we have that

$$\int_0^t \exp \left\{ - \int_s^t \mu_0(\sigma, a - t + \sigma) d\sigma \right\} \chi_{[a_1, a_2]}(a - t + s) u(s, a - t + s) ds = 0,$$

so that from (V.3.2), for almost every $(t, a) \in \mathcal{T}$ we have that

$$p(t, a) = \exp \left\{ - \int_0^t \mu_0(s, a - t + s) ds \right\} p_0(a - t),$$

and using the above equality, the condition (V.4.10) may be rewritten as

$$\beta_0(t, a) \exp \left\{ - \int_0^t \mu_0(s, a - t + s) ds \right\} p_0(a - t) = 0 \quad \text{a.e. } (t, a) \in \mathcal{T}.$$

This condition may not be fulfilled in general: take, for instance, $\beta_0(t, a) = 1$ for almost every $(t, a) \in \mathcal{T}$ and $p_0(a) > 0$ for almost every $a \in (a_2, a_+ - (\tau - a_1))$. This concludes the proof of Proposition V.4.1 - (ii). \square

We now prove Theorem V.1.2 in the more general nonlinear case, using again the comparison principle and the results of Proposition V.4.1 relatively to the linear setting.

Proof of Theorem V.1.2 - (i). Let $\tau > 0$ and $a_1 > 0$. Assume by contradiction that there exists a control u and a corresponding unique nonnegative solution p of (V.1.1) such that $p(\tau, a) = 0$ a.e. $a \in (0, a_+)$. We have that $P \in L^\infty(0, \tau)$, where $P(t) = \int_0^{a_+} p(t, a) da$. Denote by $M = \|P\|_{L^\infty(0, \tau)}$ and by

$$\beta_-(t, a) := \beta(t, a, M), \quad \mu_+(t, a) := \mu(t, a, 0), \quad \text{a.e. } (t, a) \in Q_\tau.$$

By the comparison principle for age-structured population dynamics we get that

$$0 \leq \tilde{p}(t, a) \leq p(t, a), \quad (t, a) \in Q_\tau^1$$

(along almost every characteristic line), where $Q_\tau^1 = (0, \tau) \times (0, \bar{a}_1)$, $\bar{a}_1 = \min\{a_1, a_m\}$, and \tilde{p} is the solution to the linear system

$$\begin{cases} \frac{\partial \tilde{p}}{\partial t}(t, a) + \frac{\partial \tilde{p}}{\partial a}(t, a) + \mu_+(t, a)\tilde{p}(t, a) = 0, & (t, a) \in Q_\tau^1 \\ \tilde{p}(t, 0) = \int_0^{\bar{a}_1} \beta_-(t, a)\tilde{p}(t, a) da, & t \in (0, \tau) \\ \tilde{p}(0, a) = p_0(a), & a \in (0, \bar{a}_1). \end{cases}$$

More precisely, we have used the comparison principle (see Theorem V.3.1 - (ii)) for the functions β_1, β_2, μ_1 and μ_2 defined on Q_τ by

$$\beta_1(t, a) := \beta_-(t, a), \quad \beta_2(t, a) := \begin{cases} \beta_-(t, a) & \text{if } (t, a) \in Q_\tau^1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu_1(t, a) := \mu_+(t, a), \quad \mu_2(t, a) := \mu(t, a, P(t)),$$

noting that we have

$$\beta_1(t, a) \geq \beta_2(t, a) \quad \text{and} \quad \mu_1(t, a) \leq \mu_2(t, a) \quad \text{a.e. in } Q_\tau.$$

Since $\beta_-(t, a) > 0$ a.e. $(t, a) \in (0, \tau) \times (a_b, \bar{a}_1)$, we may conclude that for p_0 satisfying

$$p_0(a) > 0 \quad \text{a.e. } a \in (0, \bar{a}_1),$$

we get that $\tilde{p}(t, a) > 0$ along almost any characteristic line. Since we have $\tilde{p}(\tau, a) > 0$ a.e. $a \in (0, \bar{a}_1)$, we get a contradiction with the fact that $p(\tau, a) = 0$ a.e. $a \in (0, \bar{a}_1)$. \square

Proof of Theorem V.1.2 - (ii). Let us take $\beta := \beta_0$, where β_0 is chosen as at the end of the proof of Proposition V.4.1 - (ii), then arguing again by contradiction and using the comparison principle for age-structured population dynamics we get the conclusion as in Theorem V.1.2 - (i). \square

V.5 CONTROLLABILITY RESULTS WITH GENERAL ASSUMPTIONS ON THE FERTILITY RATE

This section is devoted to prove Theorem V.1.3, when no extra assumption on the function β is assumed (in particular, we may have $\beta > 0$ in some interval of ages near zero). To this aim, we follow the methodology of the two previous sections and we first prove Theorem V.1.3 in the linear setting. The results of Theorem V.1.3 will be deduced from those of Proposition V.5.1 and by the comparison principle.

Let us first consider the linear controlled system (V.3.1) subject to assumptions **(L1)** - **(L4)** stated in Section V.3. We will prove the following Proposition V.5.1 which is the “linear version” of Theorem V.1.3:

Proposition V.5.1. *With the above notations and assumptions,*

- (i) *Assume that $a_1 = 0$ and $a_2 > 0$. If $\tau > a_{\dagger} - a_2$, then for every $p_0 \in L^2(0, a_{\dagger})$ with $p_0(a) \geq 0$ a.e. $a \in (0, a_{\dagger})$, there exists $u \in L^2(Q_{\tau})$ and a corresponding unique nonnegative solution p of (V.3.1) such that*

$$p(\tau, a) = 0 \quad \text{a.e. } a \in (0, a_{\dagger}).$$

- (ii) *Assume that there exists $a_m \in (0, a_{\dagger}]$ such that, for almost every $t \in (0, \tau)$ we have*

$$\beta_0(t, a) > 0 \quad \text{a.e. } a \in (0, a_m).$$

If $a_1 > 0$, then for every $\tau > 0$, there exists $p_0 \in L^2(0, a_{\dagger})$ with $p_0(a) \geq 0$ a.e. $a \in (0, a_{\dagger})$ such that there is no control u and a corresponding nonnegative solution p to (V.3.1) satisfying

$$p(\tau, a) = 0 \quad \text{a.e. } a \in (0, a_{\dagger}).$$

- (iii) *If $\tau < a_{\dagger} - a_2$, there exists an initial population density $p_0 \in L^2(0, a_{\dagger})$ with $p_0(a) \geq 0$ a.e. $a \in (0, a_{\dagger})$ such that there is no control u and a corresponding nonnegative*

solution p to (V.3.1) satisfying

$$p(\tau, a) = 0 \quad \text{a.e. } a \in (0, a_{\dagger}).$$

Proof of Proposition V.5.1 - (i). Suppose that $\tau > a_{\dagger} - a_2$. Without loss of generality, we may assume that $\tau < a_{\dagger}$. Let us prove that there exists a feedback control u and a corresponding unique nonnegative solution p of (V.3.1) such that

$$p(\tau, a) = 0 \quad \text{a.e. } a \in (0, a_{\dagger}).$$

Denote by $\varepsilon := \frac{1}{2}(\tau - (a_{\dagger} - a_2))$. For almost every $(t, a) \in Q_{\tau}$, let us consider the feedback control $u(t, a) := -v(t, a)p(t, a)$, where

$$v(t, a) := \begin{cases} \frac{1}{a_2 - a} & \text{if } \begin{cases} \varepsilon - a_2 \leq t - a < a_{\dagger} - 2a_2 + \varepsilon, \\ a \in [0, a_2], \end{cases} \\ \frac{1}{a_{\dagger} - a_2 + \varepsilon - t} & \text{if } \begin{cases} a_{\dagger} - 2a_2 + \varepsilon \leq t - a < a_{\dagger} - 2a_2 + 2\varepsilon, \\ a \in [0, a_2], \end{cases} \\ 0 & \text{if } \begin{cases} a_{\dagger} - 2a_2 + 2\varepsilon \leq t - a < a_{\dagger} - a_2 + \varepsilon, \\ t \in [a_{\dagger} - 2a_2 + 2\varepsilon, a_{\dagger} - a_2 + \varepsilon], \\ a \in [0, a_2 - \varepsilon], \end{cases} \\ \frac{1}{\tau - t} & \text{if } \begin{cases} a_{\dagger} - a_2 + \varepsilon \leq t \leq \tau, \\ a \in [0, a_2], \end{cases} \\ 0 & \text{otherwise.} \end{cases} \quad (\text{V.5.1})$$

The structure of v is given in Figure V.5. The solution to (V.3.1) with $u = -vp$ exists, is unique and nonnegative (see Anița [Ani00]), and via the comparison principle for age-dependent population dynamics (see Theorem V.3.1 - (ii)) we get that

$$0 \leq p(t, a) \leq \tilde{p}(t, a) \quad \text{a.e. } (t, a) \in Q_{\tau}, \quad (\text{V.5.2})$$

where \tilde{p} is the solution to

$$\begin{cases} \frac{\partial \tilde{p}}{\partial t}(t, a) + \frac{\partial \tilde{p}}{\partial a}(t, a) + v(t, a)\tilde{p}(t, a) = 0, & (t, a) \in Q_{\tau} \\ \tilde{p}(t, 0) = M \int_0^{a_{\dagger}} \tilde{p}(t, a) da, & t \in (0, \tau) \\ \tilde{p}(0, a) = p_0(a), & a \in (0, a_{\dagger}), \end{cases} \quad (\text{V.5.3})$$

where $M := \|\beta_0\|_{L^{\infty}(Q_{\tau})}$.

There may be again three subcases to discuss, depending on the order relationship of $a_{\dagger} - 2a_2 + \varepsilon$, $a_{\dagger} - 2a_2 + 2\varepsilon$ and 0 (since the expression of the solution \tilde{p} of (V.5.3) may be different in each subcase). Here, we do the proof only in the case when $a_{\dagger} - 2a_2 + \varepsilon > 0$ (namely, the case when a_2 is sufficiently small) which is pictured in Figure V.5. In

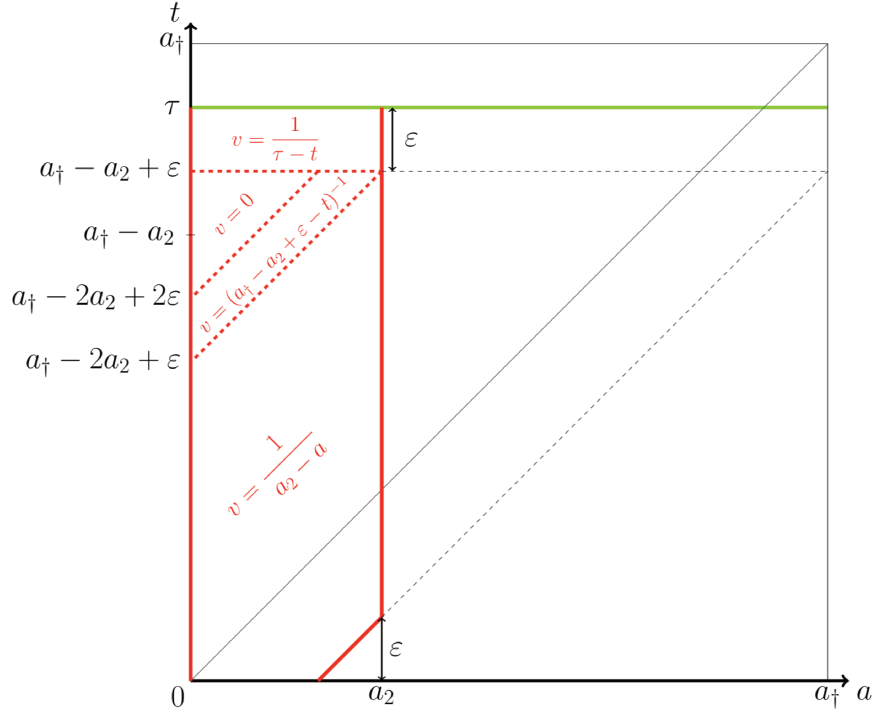


Figure V.5 – The structure of the control v when $a_1 = 0$ and $\tau > a_+ - a_2$.

this case, integrating along the characteristic lines, we get that the solution \tilde{p} of (V.5.3) satisfies

$$\tilde{p}(t, a) = p_0(a - t) \exp\left(-\int_{a-t}^a \frac{ds}{a_2 - s}\right)$$

if $\varepsilon - a_2 \leq t - a \leq 0$, $a \in [0, a_2]$,

$$\tilde{p}(t, a) = \tilde{p}(t - a, 0) \exp\left(-\int_0^a \frac{ds}{a_2 - s}\right)$$

if $0 < t - a \leq a_+ - 2a_2 + \varepsilon$, $a \in [0, a_2]$,

$$\tilde{p}(t, a) = \tilde{p}(t - a, 0) \exp\left(-\int_{t-a}^t \frac{ds}{a_+ - a_2 + \varepsilon - s}\right)$$

if $a_+ - 2a_2 + \varepsilon < t - a \leq a_+ - 2a_2 + 2\varepsilon$, $a \in [0, a_2]$,

$$\tilde{p}(t, a) = \tilde{p}(t - a, 0)$$

if $a_+ - 2a_2 + 2\varepsilon < t - a \leq a_+ - a_2 + \varepsilon$, $t \in [a_+ - 2a_2 + 2\varepsilon, a_+ - a_2 + \varepsilon]$, $a \in [0, a_2 - \varepsilon]$, so

that we have

$$\tilde{p}(t, a) = \begin{cases} \frac{a_2 - a}{a_2 - (a - t)} p_0(a - t) & \text{if } \begin{cases} \varepsilon - a_2 \leq t - a \leq 0, \\ a \in [0, a_2], \end{cases} \\ \frac{a_2 - a}{a_2} \tilde{p}(t - a, 0) & \text{if } \begin{cases} 0 < t - a \leq a_+ - 2a_2 + \varepsilon, \\ a \in [0, a_2], \end{cases} \\ \frac{a_+ - a_2 + \varepsilon - t}{a_+ - a_2 + \varepsilon - (t - a)} \tilde{p}(t - a, 0) & \text{if } \begin{cases} a_+ - 2a_2 + \varepsilon < t - a \leq a_+ - 2a_2 + 2\varepsilon, \\ a \in [0, a_2], \end{cases} \\ \tilde{p}(t - a, 0) & \text{if } \begin{cases} a_+ - 2a_2 + 2\varepsilon < t - a \leq a_+ - a_2 + \varepsilon, \\ t \in [a_+ - 2a_2 + 2\varepsilon, a_+ - a_2 + \varepsilon], \\ a \in [0, a_2 - \varepsilon]. \end{cases} \end{cases} \quad (\text{V.5.4})$$

We may conclude from the above formulas that we have

$$\begin{cases} \tilde{p}(t, a_2) = 0 & \text{if } t \in (\varepsilon, a_+ - a_2 + \varepsilon), \\ \tilde{p}(a_+ - a_2 + \varepsilon, a) = 0 & \text{if } a \in (a_2 - \varepsilon, a_2). \end{cases} \quad (\text{V.5.5})$$

Integrating on the characteristic lines the equation satisfied by \tilde{p} , we get from (V.5.5) that

$$\tilde{p}(t, a) = 0 \quad \text{if } (t, a) \in [a_+ - a_2 + \varepsilon, \tau] \times [a_2, a_+]. \quad (\text{V.5.6})$$

By (V.5.6) we get that

$$-\frac{1}{\tau - t} \chi_{[0, a_2]}(a) \tilde{p}(t, a) = -\frac{1}{\tau - t} \tilde{p}(t, a)$$

a.e. $(t, a) \in [a_+ - a_2 + \varepsilon, \tau] \times (0, a_+)$, and so we may infer that \tilde{p} is the unique nonnegative solution to

$$\begin{cases} \frac{\partial \tilde{p}}{\partial t}(t, a) + \frac{\partial \tilde{p}}{\partial a}(t, a) + \frac{1}{\tau - t} \tilde{p}(t, a) = 0, & (t, a) \in (a_+ - a_2 + \varepsilon, \tau) \times (0, a_+) \\ \tilde{p}(t, 0) = M \int_0^{a_+} \tilde{p}(t, a) da, & t \in (a_+ - a_2 + \varepsilon, \tau) \\ \tilde{p}(a_+ - a_2 + \varepsilon, a) = \tilde{p}(a_+ - a_2 + \varepsilon, a), & a \in (0, a_+). \end{cases}$$

We may conclude that

$$\tilde{p}(t, a) = h(t)g(t, a)$$

a.e. $(t, a) \in [a_+ - a_2 + \varepsilon, \tau] \times (0, a_+)$ (\tilde{p} is separable), where g is the solution to

$$\begin{cases} \frac{\partial g}{\partial t}(t, a) + \frac{\partial g}{\partial a}(t, a) = 0, & (t, a) \in (a_+ - a_2 + \varepsilon, \tau) \times (0, a_+) \\ g(t, 0) = M \int_0^{a_+} g(t, a) da, & t \in (a_+ - a_2 + \varepsilon, \tau) \\ g(a_+ - a_2 + \varepsilon, a) = \tilde{p}(a_+ - a_2 + \varepsilon, a), & a \in (0, a_+), \end{cases}$$

and h is the solution to

$$\begin{cases} h'(t) + \frac{1}{\tau - t}h(t) = 0, & t \in (a_+ - a_2 + \varepsilon, \tau) \\ h(a_+ - a_2 + \varepsilon) = 1. \end{cases}$$

Hence for every $t \in [a_+ - a_2 + \varepsilon, \tau]$ we have

$$h(t) = \frac{1}{\tau - (a_+ - a_2 + \varepsilon)}(\tau - t). \quad (\text{V.5.7})$$

By (V.5.7) we get that $\tilde{p}(\tau, a) = 0$ a.e. $a \in (0, a_+)$ and consequently

$$p(\tau, a) = 0$$

a.e. $a \in (0, a_+)$.

It remains to prove that the control $u = -vp$ belongs to $L^2(Q_\tau)$. For almost every $(t, a) \in (0, a_+ - a_2 + \varepsilon) \times (0, a_2)$, from (V.5.1) and (V.5.4) we have that

$$|v(t, a)\tilde{p}(t, a)| = \begin{cases} \frac{p_0(a-t)}{a_2 - (a-t)} & \text{if } \begin{cases} \varepsilon - a_2 \leq t - a \leq 0, \\ a \in [0, a_2], \end{cases} \\ \frac{\tilde{p}(t-a, 0)}{a_2} & \text{if } \begin{cases} 0 < t - a \leq a_+ - 2a_2 + \varepsilon, \\ a \in [0, a_2], \end{cases} \\ \frac{\tilde{p}(t-a, 0)}{a_+ - a_2 + \varepsilon - (t-a)} & \text{if } \begin{cases} a_+ - 2a_2 + \varepsilon < t - a \leq a_+ - 2a_2 + 2\varepsilon, \\ a \in [0, a_2], \end{cases} \\ 0 & \text{if } \begin{cases} a_+ - 2a_2 + 2\varepsilon < t - a \leq a_+ - a_2 + \varepsilon, \\ t \in [a_+ - 2a_2 + 2\varepsilon, a_+ - a_2 + \varepsilon], \\ a \in [0, a_2 - \varepsilon], \end{cases} \end{cases} \quad (\text{V.5.8})$$

and it follows from (V.5.8) that we have

$$|v(t, a)\tilde{p}(t, a)| \leq \begin{cases} \frac{|p_0(a-t)|}{\varepsilon} & \text{if } \begin{cases} \varepsilon - a_2 \leq t - a \leq 0, \\ a \in [0, a_2], \end{cases} \\ \frac{|\tilde{p}(t-a, 0)|}{a_2} & \text{if } \begin{cases} 0 < t - a \leq a_{\dagger} - 2a_2 + \varepsilon, \\ a \in [0, a_2], \end{cases} \\ \frac{|\tilde{p}(t-a, 0)|}{a_2 - \varepsilon} & \text{if } \begin{cases} a_{\dagger} - 2a_2 + \varepsilon < t - a \leq a_{\dagger} - 2a_2 + 2\varepsilon, \\ a \in [0, a_2], \end{cases} \\ 0 & \text{if } \begin{cases} a_{\dagger} - 2a_2 + 2\varepsilon < t - a \leq a_{\dagger} - a_2 + \varepsilon, \\ t \in [a_{\dagger} - 2a_2 + 2\varepsilon, a_{\dagger} - a_2 + \varepsilon], \\ a \in [0, a_2 - \varepsilon]. \end{cases} \end{cases} \quad (\text{V.5.9})$$

Since $\tilde{p} \in L^2((0, a_{\dagger} - a_2 + \varepsilon) \times (0, a_{\dagger}))$, we may infer from the renewal law that we have $\tilde{p}(\cdot, 0) \in L^2(0, a_{\dagger} - a_2 + \varepsilon)$, so that from (V.5.9) we have $v\tilde{p} \in L^2((0, a_{\dagger} - a_2 + \varepsilon) \times (0, a_{\dagger}))$. Given (V.5.2), it follows that we have $vp \in L^2((0, a_{\dagger} - a_2 + \varepsilon) \times (0, a_{\dagger}))$.

On the other hand, for almost every $(t, a) \in (a_{\dagger} - a_2 + \varepsilon) \times (0, a_2)$ we have

$$|v(t, a)p(t, a)| \leq |v(t, a)\tilde{p}(t, a)| = \frac{1}{\tau - t}h(t)g(t, a) = \frac{1}{\tau - (a_{\dagger} - a_2 + \varepsilon)}g(t, a),$$

and so that $vp \in L^2((a_{\dagger} - a_2 + \varepsilon, \tau) \times (0, a_2))$, which ends the proof of Proposition V.5.1 - (i). \square

Remark V.5.2. Note that actually, in the above proof, the control u defined by $u(t) = -\frac{1}{\tau - t}\tilde{p}(t)$ for every $t \in (a_{\dagger} - a_2 + \varepsilon, \tau)$ is the feedback null control as defined by (V.2.5) (see Section V.2) associated to the Lotka-McKendrick semigroup where the functions β and μ are defined by $\beta(a) := M$ and $\mu(a) := 0$ for almost every $a \in (0, a_2)$.

Proof of Proposition V.5.1 - (ii) and (iii). The proof of Proposition V.5.1 - (ii) follows the proof of Proposition V.4.1 - (i), taking $a_b = 0$ and $a_1 > 0$, so we omit it. Moreover, the proof of Proposition V.5.1 - (iii) is the same as the proof of Proposition V.4.1 - (ii), Case 1. \square

Note that the proof of Theorem V.1.3 follow closely the proofs of Theorem V.1.1 and V.1.2 (see Sections V.3 and V.4), so we omit it.

V.6 AN APPLICATION OF THE LINEAR CONTROLLABILITY RESULTS : FROM A POPULATION TO ANOTHER ONE

In this section, we give an application of Proposition V.3.2 and Proposition V.5.1 to a linear population control problem. The aim of this section is, roughly speaking, to

steer an initial population density to another population density in sufficiently large time, dealing with age restriction. More precisely, let τ be a positive constant and assume that \tilde{p} is solution to the system:

$$\begin{cases} \frac{\partial \tilde{p}}{\partial t}(t, a) + \frac{\partial \tilde{p}}{\partial a}(t, a) + \mu_0(t, a)\tilde{p}(t, a) = \chi_{[a_1, a_2]}(a)w(t, a), & (t, a) \in Q_\tau \\ \tilde{p}(t, 0) = \int_0^{a_\dagger} \beta_0(t, a)\tilde{p}(t, a) da, & t \in (0, \tau) \\ \tilde{p}(0, a) = \tilde{p}_0(a), & a \in (0, a_\dagger), \end{cases} \quad (\text{V.6.1})$$

where the functions μ_0 and β_0 are assumed to follow **(L1)** - **(L4)** (see Section V.3), $w \in L^2(Q_\tau)$ is a nonnegative function and $\tilde{p}_0 \in L^2(0, a_\dagger)$ is a nonnegative initial population density. We aim to find a control function $u \in L^2(Q_\tau)$ such that the corresponding unique solution p to (V.3.1) coincides with the solution \tilde{p} to (V.6.1) in time τ , while preserving the nonnegativity of p . Such a problem has been studied, for instance, in the pioneering work [BIM01] and more recently in [HMT18], assuming that the functions μ_0 , β_0 and w do not depend on the time, the control function u is supported in the age interval $(0, a_0)$ for some $a_0 < a_\dagger$ and \tilde{p} is a steady state of (V.3.1). The main novelty we bring here is that we may be able to steer the solution p of (V.3.1) to the solution \tilde{p} of (V.6.1) by means of a feedback control function u , without considering the above mentioned restrictions. More precisely, we have

Proposition V.6.1. *With the above notations and assumptions, assume that at least one of the two following conditions is fulfilled:*

Condition 1: *there exists $a_b \in (0, a_\dagger)$ such that (V.3.3) is satisfied, with $a_1 < a_b$ and $\tau > a_\dagger - a_2 + a_1$,*

Condition 2: *$a_1 = 0$ and $\tau > a_\dagger - a_2$.*

Suppose that w and \tilde{p} are both nonnegative functions in $L^2(Q_\tau)$, where \tilde{p} is the solution of (V.6.1). Then for every $p_0 \in L^2(0, a_\dagger)$ with $p_0(a) \geq 0$ a.e. $a \in (0, a_\dagger)$, there exists $u \in L^2(Q_\tau)$ and a unique nonnegative solution p of (V.3.1) such that

$$p(\tau, a) = \tilde{p}(\tau, a) \quad \text{a.e. } a \in (0, a_\dagger).$$

Proof. Suppose, first, that the Condition 1 of Proposition V.6.1 is fulfilled. For almost every $(t, a) \in Q_\tau$, we set

$$u(t, a) := w(t, a) - v(t, a)(p(t, a) - \tilde{p}(t, a)) \quad \text{a.e. } (t, a) \in Q_\tau, \quad (\text{V.6.2})$$

where v is defined by (V.3.4) (resp. by (V.3.12)) in the case where $a_2 \leq a_b$ (resp. in the case where $a_b < a_2$).

Denoting by $h := p - \tilde{p}$, using (V.3.1), (V.6.1) together with the above definition of

the control function u given by (V.6.2), we get that h is solution of the following system:

$$\begin{cases} \frac{\partial h}{\partial t}(t, a) + \frac{\partial h}{\partial a}(t, a) \\ \quad + \mu_0(t, a)h(t, a) = -\chi_{[a_1, a_2]}(a)v(t, a)h(t, a), & (t, a) \in Q_\tau \\ h(t, 0) = \int_0^{a_\dagger} \beta_0(t, a)h(t, a) da, & t \in (0, \tau) \\ h(0, a) = h_0(a), & a \in (0, a_\dagger), \end{cases} \quad (\text{V.6.3})$$

where $h_0(a) := p_0(a) - \tilde{p}_0(a)$ for almost every $a \in (0, a_\dagger)$. From the proof of Proposition V.3.2, we get that the solution h of (V.6.3) satisfies $h(\tau, \cdot) = 0$, with $vh \in L^2(Q_\tau)$. It follows that we have $p(\tau, \cdot) = \tilde{p}(\tau, \cdot)$ and $u = w - vh \in L^2(Q_\tau)$.

It remains to show that the unique solution p to (V.3.1) corresponding to u defined by (V.6.2) is nonnegative. Since u is defined by (V.6.2), we get that p is solution to

$$\begin{cases} \frac{\partial p}{\partial t}(t, a) + \frac{\partial p}{\partial a}(t, a) \\ \quad + \tilde{\mu}(t, a)p(t, a) = \chi_{[a_1, a_2]}(a)(w(t, a) + v(t, a)\tilde{p}(t, a)), & (t, a) \in Q_\tau \\ p(t, 0) = \int_0^{a_\dagger} \beta_0(t, a)p(t, a) da, & t \in (0, \tau) \\ p(0, a) = p_0(a), & a \in (0, a_\dagger), \end{cases} \quad (\text{V.6.4})$$

where $\tilde{\mu}(t, a) := \mu_0(t, a) + \chi_{[a_1, a_2]}v(t, a)$ for almost every $(t, a) \in Q_\tau$. Noting that, for almost every $(t, a) \in Q_\tau$ we have

$$\chi_{[a_1, a_2]}(a)(w(t, a) + v(t, a)\tilde{p}(t, a)) \geq 0 \quad \text{a.e. } (t, a) \in Q_\tau,$$

it follows from the comparison principle for linear age-structured population dynamics (see, for instance, [Ani00]) that the solution p of (V.6.4) satisfies $p(t, a) \geq 0$ for almost every $(t, a) \in Q_\tau$.

In the case where the Condition 2 is fulfilled, we may consider v as defined by (V.5.1) and follow the above proof to get the desired result. \square

V.7 LACK OF NULL CONTROLLABILITY FOR THE LOTKA-MCKENDRICK EQUATION WITH SPATIAL DIFFUSION AND POSITIVITY CONSTRAINTS

In this section, we consider a linear controlled age-structured population model with spatial diffusion described by the following system:

$$\left\{ \begin{array}{ll} \frac{\partial p}{\partial t}(t, a, x) + \frac{\partial p}{\partial a}(t, a, x) + \mu(t, a)p(t, a, x) - \Delta p(t, a, x) \\ \quad = \chi_\omega(x)u(t, a, x), & (t, a) \in Q_\tau, x \in \Omega \\ \frac{\partial p}{\partial \nu}(t, a, x) = 0, & (t, a) \in Q_\tau, x \in \partial\Omega \\ p(t, 0, x) = \int_0^{a_\dagger} \beta(t, a)p(t, a, x) da, & t \in (0, \tau), x \in \Omega \\ p(0, a, x) = p_0(a, x), & a \in (0, a_\dagger), x \in \Omega. \end{array} \right. \quad (\text{V.7.1})$$

In the above equations:

- $\Omega \subset \mathbb{R}^N$, $N \geq 1$, denotes a smooth connected bounded domain and Δ is the laplacian with respect to the variable x ;
- $\frac{\partial}{\partial \nu}$ denotes the derivation operator in the direction of the unit outer normal to $\partial\Omega$. We thus have homogeneous Neumann boundary conditions, thus the considered population is isolated from the exterior of Ω ;
- $p(t, a, x)$ denotes the distribution density of the population at time t , of age a at spatial position $x \in \Omega$;
- p_0 denotes the initial population distribution;
- $a_\dagger \in (0, +\infty)$ is the maximal age of an individual;
- β and μ are nonnegative functions denoting respectively the birth and death rates;
- $\omega \subset\subset \Omega$ is a nonempty open subset of Ω and χ_ω denotes the characteristic function of ω .

Let τ be a positive constant. We make the following classical assumptions on β and μ :

(D1) $\beta \in L^\infty(Q_\tau)$, $\beta(t, a) \geq 0$ for almost every $(t, a) \in Q_\tau$,

(D2) $\mu \in L^1_{loc}([0, \tau] \times [0, a_\dagger))$, $\mu(t, a) \geq 0$ almost every $(t, a) \in Q_\tau$.

From a controllability view point, system (V.7.1) has been extensively studied in the past decades. The particular case when the control acts in the whole space (the case corresponding to $\omega = \Omega$) was investigated by Anița (see [Ani00], p 148). The case when the control acts in a spatial subdomain ω was firstly studied by Ainseba [Ain02], where the author proves the null controllability of the above system (V.7.1), except for a small interval of ages near zero. The case when the control acts in a spatial subdomain ω and also only for small age classes was investigated by Ainseba and Anița [AA04], for initial data p_0 in a neighborhood of the target \tilde{p} . Related approximate and exact controllability

issues have also been studied in Ainseba and Langlais [AL00], Ainseba and Iannelli, Traore [Tra06], Kavian and Traore [KT11].

In a recent work, Hegoburu and Tucsnak [HT17] proved that, in the case where both functions β and μ do not depend on the time variable, the above system (V.7.1) is null controllable in any time $\tau > 0$, in the sense that for any $p_0 \in L^2((0, a_+) \times \Omega)$, there exists a control function $u \in L^\infty(0, \tau; L^2((0, a_+) \times \Omega))$ such that the corresponding solution p of (V.7.1) satisfies

$$p(\tau, a, x) = 0 \quad \text{a.e. } (a, x) \in (0, a_+) \times \Omega.$$

This result has been recently improved by Maity, Tucsnak and Zuazua [MTZ18b], assuming that the young individuals are not able to reproduce before some age $a_b > 0$, where the control function u in system (V.7.1) has support in some interval of ages $[a_1, a_2]$, where $0 \leq a_1 < a_2 \leq a_+$. In [MTZ18b] the authors proved the null controllability result with this additional age restriction, provided that the control time τ is large enough, and the age a_1 is smaller than a_b .

The aim of this section is to prove that, in general, the solution p of the controlled system may not be positive in the whole time interval $[0, \tau]$. More precisely, we have:

Proposition V.7.1. *Let $\tau > 0$. With the above notations and assumptions, suppose that for almost every $t \in [0, \tau]$, we have*

$$\beta(t, \cdot) \text{ is positive on a subset of positive measure of } (0, a_+), \quad (\text{V.7.2})$$

and denote by

$$E_t := \{a \in (0, a_+); \beta(t, a) \neq 0\} \quad \text{and} \quad E_t^+ := \sup E_t. \quad (\text{V.7.3})$$

Suppose that

$$m := \inf \{E_t^+, t \in [0, \tau]\} > 0. \quad (\text{V.7.4})$$

Then there exists $p_0 \in L^2((0, a_+) \times \Omega)$ with $p_0(a, x) \geq 0$ for almost every $(a, x) \in (0, a_+) \times \Omega$ such that there does not exist a control function $u \in L^2((0, \tau); L^2((0, a_+) \times \Omega))$ such that

$$p(\tau, a, x) = 0 \quad \text{a.e. } (a, x) \in (0, a_+) \times \Omega,$$

together with

$$p(t, a, x) \geq 0 \quad \text{a.e. } (t, a, x) \in Q_\tau \times \Omega.$$

In other words, for some well chosen nonnegative initial condition $p_0 \in L^2((0, a_+) \times \Omega)$, it is not possible to eradicate the whole population in time τ while keeping the nonnegativity of the state trajectory.

Proof. Let $\tau > 0$, and assume that (V.7.2) and (V.7.4) are satisfied. Denote by ω^c the complementary of ω in Ω . Let $h_0 \in L^2(\Omega)$ be a nonnegative function such that for almost every $x \in \omega^c$ we have

$$h_0(x) > 0 \quad \text{a.e. } x \in \omega^c. \quad (\text{V.7.5})$$

For almost every $(a, x) \in (0, a_+) \times \Omega$, we set

$$p_0(a, x) := h_0(x). \quad (\text{V.7.6})$$

It is clear that the initial condition p_0 defined by (V.7.6) belongs to $L^2((0, a_+) \times \Omega)$ and is a nonnegative function. Let $\tau > 0$. Suppose that there exists a control function $u \in L^2((0, \tau); L^2((0, a_+) \times \Omega))$ such that the two following conditions are fulfilled:

$$p(\tau, a, x) = 0 \quad \text{a.e. } (a, x) \in (0, a_+) \times \Omega,$$

and

$$p(t, a, x) \geq 0 \quad \text{a.e. } (t, a, x) \in Q_\tau \times \Omega,$$

where p is the solution of (V.7.1).

In the subdomain $Q_\tau \times \omega^c$, given (V.7.1) and (V.7.6), the function p satisfies

$$\begin{cases} \frac{\partial p}{\partial t}(t, a, x) + \frac{\partial p}{\partial a}(t, a, x) \\ \quad + \mu(t, a)p(t, a, x) - \Delta p(t, a, x) = 0, & (t, a) \in Q_\tau, x \in \omega^c \\ \frac{\partial p}{\partial \nu}(t, a, x) = 0, & (t, a) \in Q_\tau, x \in \partial\Omega \\ p(t, 0, x) = \int_0^{a_+} \beta(t, a)p(t, a, x) da, & t \in (0, \tau), x \in \omega^c \\ p(0, a, x) = h_0(x), & a \in (0, a_+), x \in \omega^c. \end{cases}$$

By the comparison principle, for almost every $(t, a, x) \in Q_\tau \times \omega^c$, we have that

$$p(t, a, x) \geq \tilde{p}(t, a, x) \geq 0, \quad (\text{V.7.7})$$

where the function \tilde{p} satisfies the following equation:

$$\begin{cases} \frac{\partial \tilde{p}}{\partial t}(t, a, x) + \frac{\partial \tilde{p}}{\partial a}(t, a, x) \\ \quad + \mu(t, a)\tilde{p}(t, a, x) - \Delta \tilde{p}(t, a, x) = 0, & (t, a) \in Q_\tau, x \in \omega^c \\ \tilde{p}(t, a, x) = 0, & (t, a) \in Q_\tau, x \in \partial\omega \\ \frac{\partial \tilde{p}}{\partial \nu}(t, a, x) = 0, & (t, a) \in Q_\tau, x \in \partial\Omega \\ \tilde{p}(t, 0, x) = \int_0^{a_+} \beta(t, a)\tilde{p}(t, a, x) da, & t \in (0, \tau), x \in \omega^c \\ \tilde{p}(0, a, x) = h_0(x), & a \in (0, a_+), x \in \omega^c. \end{cases} \quad (\text{V.7.8})$$

Note that the above system (V.7.8) is separable, in the sense that the solution \tilde{p} to (V.7.8) writes as

$$\tilde{p}(t, a, x) = g(t, a)h(t, x) \quad \text{a.e. } (t, a, x) \in Q_\tau \times \omega^c, \quad (\text{V.7.9})$$

where the functions $g \in C([0, \tau] \times [0, a_\dagger] \setminus S)$ ($S = \{(t, a) \in [0, \tau] \times [0, a_\dagger]; t = a\}$) and $h \in C([0, \tau]; L^2(\Omega))$ are respectively solutions to

$$\begin{cases} \frac{\partial g}{\partial t}(t, a) + \frac{\partial g}{\partial a}(t, a) + \mu(t, a)g(t, a) = 0, & (t, a) \in Q_\tau \\ g(t, 0) = \int_0^{a_\dagger} \beta(t, a)g(t, a) da, & t \in (0, \tau) \\ g(0, a) = 1, & a \in (0, a_\dagger), \end{cases} \quad (\text{V.7.10})$$

and

$$\begin{cases} \frac{\partial h}{\partial t}(t, x) - \Delta h(t, x) = 0, & t \in (0, \tau), x \in \omega^c \\ h(t, x) = 0, & t \in (0, \tau), x \in \partial\omega \\ \frac{\partial h}{\partial \nu}(t, x) = 0, & t \in (0, \tau), x \in \partial\Omega \\ h(0, x) = h_0(x), & x \in \omega^c. \end{cases}$$

Given assumptions (V.7.2) and (V.7.4), let us prove that, for every $\tau > 0$, we have that $g(\tau, a) > 0$ on a subset of positive measure of $(0, a_\dagger)$. To this aim, integrating along the characteristics the equation satisfied by g , for almost every $(t, a) \in Q_\tau$ we have

$$g(t, a) = \begin{cases} \exp\left(-\int_0^t \mu(s, a-t+s) ds\right), & a \geq t, \\ \exp\left(-\int_0^a \mu(t-a+s, s) ds\right) B(t-a), & a < t, \end{cases} \quad (\text{V.7.11})$$

where

$$B(t) = \int_0^{a_\dagger} \beta(t, a)g(t, a) da \quad \text{a.e. } t \in (0, \tau).$$

Suppose that there exists $\tau > 0$ such that for almost every $a \in (0, a_\dagger)$ we have $g(t, a) = 0$, where g satisfies (V.7.10). Given (V.7.11), necessarily, we have $\tau \geq a_\dagger$. Assuming that $g(\tau, a) = 0$ for almost every $a \in (0, a_\dagger)$ for some $\tau \geq a_\dagger$, from (V.7.11) we have

$$B(s) = 0 \quad \text{a.e. } s \in (\tau - a_\dagger, \tau),$$

so that, in particular, we have

$$B(\tau - a_\dagger) = \int_{E_{\tau-a_\dagger}} \beta(\tau - a_\dagger, a)g(\tau - a_\dagger, a) da = 0, \quad (\text{V.7.12})$$

where $E_{\tau-a_\dagger}$ is defined by (V.7.3). Since $\beta(\tau - a_\dagger, a) > 0$ for almost every $a \in E_{\tau-a_\dagger}$ and p is assumed to be positive, from (V.7.12) we get that for almost every $a \in E_{\tau-a_\dagger}$, we have

$$g(\tau - a_\dagger, a) = 0 \quad \text{a.e. } a \in E_{\tau-a_\dagger}. \quad (\text{V.7.13})$$

Denote by $E^{(1)} := E_{\tau-a_{\dagger}}^+$, where $E_{\tau-a_{\dagger}}^+$ is defined by (V.7.3). From (V.7.13), we have

$$g(\tau - a_{\dagger}, E^{(1)}) = 0. \quad (\text{V.7.14})$$

If $E^{(1)} \geq \tau - a_{\dagger}$, the above condition (V.7.14) gives a contradiction with (V.7.11). Otherwise, we have $E^{(1)} < \tau - a_{\dagger}$ and integrating along the characteristics, we have

$$g(\tau - a_{\dagger} - E^{(1)}, 0) = 0.$$

Denote by $E^{(2)} = E_{\tau-a_{\dagger}-E^{(1)}}^+$. Following the previous arguments, we may deduce from the renewal law that we have

$$g(\tau - a_{\dagger} - E^{(1)}, E^{(2)}) = 0. \quad (\text{V.7.15})$$

If $E^{(2)} \geq \tau - a_{\dagger} - E^{(1)}$, the above condition (V.7.15) gives a contradiction with (V.7.11). Otherwise, we have $E^{(2)} < \tau - a_{\dagger} - E^{(1)}$ and integrating along the characteristics, we have

$$g(\tau - a_{\dagger} - E^{(1)} - E^{(2)}, 0) = 0.$$

Define, recursively, the sequence $(E^{(n)})_{n \geq 1}$ by $E^{(1)} := E_{\tau-a_{\dagger}}^+$ and $E^{(n+1)} := E_{\tau-a_{\dagger}-E^{(1)}-\dots-E^{(n)}}^+$. Following the above procedure, we may obtain by induction that for every $n \geq 1$ such that $\tau - a_{\dagger} - E^{(1)} - \dots - E^{(n)} \geq 0$, we have

$$g(\tau - a_{\dagger} - E^{(1)} - \dots - E^{(n)}, E^{(n+1)}) = 0.$$

Recall that there exists $m > 0$ such that, for almost every $t \in [0, \tau]$ we have $E_t^+ \geq m > 0$. Thus, denote by l the first integer such that

$$\tau - a_{\dagger} - E^{(1)} - \dots - E^{(l)} - E^{(l+1)} \leq 0.$$

Then, the condition

$$g(\tau - a_{\dagger} - E^{(1)} - \dots - E^{(l)}, E^{(l+1)}) = 0,$$

gives a contradiction with (V.7.11), since $E^{(l+1)} \geq \tau - a_{\dagger} - E^{(1)} - \dots - E^{(l)}$. It follows that $g(\tau, \cdot)$ cannot be the null function. By (V.7.9) we have that $h(\tau, x) = 0$ for almost every $x \in \omega^c$. From the backward uniqueness for the parabolic equations, we have $h_0(x) = 0$ for almost every $x \in \omega^c$. This is a contradiction with (V.7.5). \square

Remark V.7.2. *Using the comparison results for age-structured population dynamics with spatial diffusion and feedback controls of harvesting type it is possible to obtain null-controllability results when the control function u acts in the whole habitat Ω , for any $t \in (0, \tau)$ but only for some age subinterval.*

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