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Asymptotic study of covariance operator of fractional processes : analytic approach with applications

Dmytro Marushkevych

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Dmytro Marushkevych. Asymptotic study of covariance operator of fractional processes: analytic approach with applications. General Mathematics [math.GM]. Le Mans Université, 2019. English. NNT : 2019LEMA1010 . tel-02150298

HAL Id: tel-02150298

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THESE DE DOCTORAT DE

LE MANS UNIVERSITE
COMUE UNIVERSITE BRETAGNE LOIRE

ECOLE DOCTORALE N° 601
*Mathématiques et Sciences et Technologies
de l'Information et de la Communication*
Spécialité : *Mathématiques*

Par

Dmytro MARUSHKEVYCH

**« Asymptotic study of covariance operator of fractional processes:
analytic approach with applications »**

Thèse présentée et soutenue à Le Mans, le 22 mai 2019
Unité de recherche : Laboratoire Manceau de Mathématiques
Thèse N° : 2019LEMA1010

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ACKNOWLEDGEMENT

First and foremost I want to thank my advisors Marina KLEPTSYNA, Yuliya MISHURA and Alexandre POPIER. I appreciate all their contributions of time, ideas and energy to make my Ph.D. experience productive and stimulating. Their doors were always open whenever I ran into a trouble spot or had a question about my research or writing. I would also like to thank Pavel CHIGANSKY for all the work we have done together, his insightful comments and willing to help.

Further, I would like to express my gratitude to David NUALART and Mark PODOLSKIJ, the referees of this thesis, for carefully reading this manuscript and for giving such constructive comments. I would like to thank my thesis committee members for all of their guidance through this process; your discussion and feedback have been absolutely invaluable.

I would also like to thank the whole team of LMM. The members of the Probability and Statistics group have contributed immensely to my personal and professional time at Le Mans University. The group has been a source of friendships as well as good advice and collaboration.

My time at Le Mans University was made enjoyable in large part due to the many friends that became a part of my life. I thank my fellow Ph.D. students and friends for all the fun we have had in the last four years.

Finally, I must express my very profound gratitude to my parents, who raised me with a love of science and supported me in all my pursuits. They have been providing me with unfailing support and continuous encouragement throughout my years of study and through the process of researching and writing this thesis. This accomplishment would not have been possible without them. Thank you.

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INTRODUCTION

The purpose of this thesis is to develop new methodology of the asymptotic spectral analysis of covariance operators of several fractional processes and to apply it in a number of related problems of probability and statistics.

This thesis is organized as follows. Introduction contains brief description of obtained results and their applications, as well as the main ideas of proofs. In Chapter 1 we develop methodology for the spectral analysis of the fractional type covariance operator and present the results of illustrative numerical experiments. In Chapter 2 we consider the covariance eigenproblem for Gaussian bridges and introduce an alternative approach to its solution, based on the spectral asymptotics of base process. Finally, in Chapter 3 we consider three representative applications of the developed theory, namely, filtering problem for fractional Gaussian signals in white noise, large deviation problem for the mixed fractional Ornstein-Uhlenbeck process and L_2 -small ball probabilities for fractional Gaussian processes.

Motivations and summary

Any centered Gaussian process X_t is determined by its covariance function $K(s, t) = \mathbb{E}X_s X_t$. This function defines the corresponding *covariance* integral operator

$$(K\varphi)(t) = \int_0^1 K(s, t)\varphi(s) ds, \quad t \in [0, 1].$$

This operator is associated to the process in a natural way, as on $L_2([0, 1])$ it can also be defined by the relation

$$\langle \varphi, K\psi \rangle = \text{cov}(\langle \varphi, X \rangle, \langle \psi, X \rangle) \quad \forall \varphi, \psi \in L_2,$$

where $\langle \cdot, \cdot \rangle$ is the L_2 -scalar product. It is thus not surprising that the behaviour of its eigenfunctions and eigenvalues has various applications in the theory of stochastic processes: the Karhunen-Loève series expansion, equivalence and orthogonality of

Gaussian measures, exact asymptotics of the small ball probabilities, sampling from heavy tailed distributions, non-central limit theorems, optimal linear filtering, large deviations problem, statistical estimation problem etc.

This motivates the study of the eigenproblem for the operator K :

$$K\varphi = \lambda\varphi, \quad \varphi \neq 0.$$

Under mild conditions on the covariance function, being a compact non-negative self-adjoint operator, K admits a countable family of solutions (λ_n, φ_n) , where the eigenvalues λ_n are real and converge to zero, when put in the decreasing order. The corresponding eigenfunctions φ_n form a complete orthonormal basis in $L^2([0, 1])$.

Eigenproblems rarely admit closed form solutions and one notable exception is the standard Brownian motion, for which the covariance eigenproblem can be easily solved by reduction to the Sturm–Liouville problem for a simple differential operator. Similar approach also works for a number of related processes, such as the Brownian bridge, the Ornstein-Uhlenbeck process, the integrated Brownian motion, etc.

Many natural, social and economical phenomena (e.g. telecommunication connections, asset prices, turbulence) exhibit "long memory" property, an effect, that cannot be conveniently described by means of the standard Brownian motion with its independent increments (see, e.g., [46]). A.N. Kolmogorov [27] was the first to consider a continuous Gaussian process with stationary increments and with the following self-similarity property:

$$\text{Law}(X(ct); t \geq 0) = \text{Law}(c^H X(t); t \geq 0), \quad c > 0$$

for some constant H . It turns out that such process with zero mean has very specific covariance function:

$$K(s, t) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |s - t|^{2H}),$$

where $H \in (0, 1)$ is the Hurst exponent. The process defined by this covariance function, is called a *fractional Brownian motion*.

Following publication of the influential survey paper [35], the fractional Brownian motion (fBm), along with the closely related processes, such as the fractional Ornstein–Uhlenbeck process, integrated fBm, the fBm bridge, became important building blocks

in models with long range dependence, which cannot be easily captured by their standard Brownian counterparts.

Despite the fact that the fractional Gaussian processes have been extensively studied ever since, their spectral structure remained elusive. No closed form expressions for the eigenvalues or eigenfunctions of the fBm or any other related process have been found so far, and, perhaps, not without a reason, as commented in [36]. Nonetheless, many valuable results can be deduced from sufficiently accurate asymptotic approximations, which raises the main question to be addressed in this thesis:

What can we tell about the spectral asymptotics of the fractional Gaussian processes?

The exact first order asymptotics of the fBm eigenvalues was found in [7, 8] and, by different methods, in [40] and [33]. These results remained state of the art on the subject until the recent paper [12], where the authors obtained a much more informative asymptotic description of both the eigenvalues and the eigenfunctions of the fBm. The main results of Chapter 1 in the present thesis extend the technique from [12] to a larger family of fractional processes, which includes the fBm as a particular case. This generalization reveals a whole new level of complexity in the spectral structure of such processes and poses a number of challenging technical problems.

Specifically, exact asymptotics of both eigenvalues and eigenfunctions is derived in this thesis for the following processes:

- the fractional Ornstein-Uhlenbeck process;
- the integrated fractional Brownian motion;
- the mixed fractional Brownian motion.

We obtain asymptotic approximation for the eigenvalues, exact up to the second order term, and find the exact asymptotic expression for the leading term of the eigenfunctions with respect to the uniform norm. These results reveal new interesting effects, such as separation of various parameters in the asymptotic formulas, variety of dependence patterns on the value of the Hurst exponent, different orders of the residual terms, etc.

Section 1 also contains results of numerical experiments, which show that the actual accuracy of our approximation is surprisingly good, already, for relatively small values of n .

Chapter 2 of this thesis is concerned with spectral approximations of bridges, derived from Gaussian processes by conditioning them to start and terminate at given points. In principle, we could use the analytic approach, described in Chapter 1, however, somewhat surprisingly, it does not produce explicit results. In this regard we were interested in the following question:

Can the exact asymptotics of the eigenvalues and the eigenfunctions for a Gaussian bridge be deduced from that of its "base" process?

From the perspective of the general perturbation theory, the covariance operator of the bridge can be viewed as a certain perturbation of the covariance operator of the base process. It follows then that their first order spectral asymptotics should coincide, but any finer details of the spectra may differ and be highly dependent on specificities of the perturbation. As shown in this thesis, in the case of bridges, it is possible to derive the spectral asymptotics, accurate almost to the same order, as asymptotics of the base process. The suggested technique is not specific to a particular process and is based on the structure of perturbation that is inherent to bridges. The case of the fractional Brownian bridge is considered in detail.

We conclude this thesis by presenting three applications:

- Optimal linear filtering problem for the fractional Ornstein-Uhlenbeck process.

We consider the problem of estimation error computing for the fractional Ornstein-Uhlenbeck process observed in white noise in large time and high signal-to-noise regimes. The latter is derived using the spectral approximations from Chapter 1, while the former is obtained by extending the analytic approach to solution of integral equations of the second kind.

- Large deviations problem for the mixed fractional Ornstein-Uhlenbeck process.

Large deviations principle is established for the maximum likelihood estimator of the drift parameter of the mixed fractional Ornstein-Uhlenbeck process. This estimator was previously shown to be consistent and asymptotically normal in the large sample limit and the large deviations analysis is a natural further step.

We use a variation of the Gartner-Ellis theorem, whose main ingredient is an appropriately scaled Laplace transform. The main difficulty here is to compute its precise asymptotics, which derive from the spectral approximations from Chapter 1.

- Small L_2 - ball probabilities problem for fractional processes.

Our purpose was to find of the probabilities of fractional processes to be confined to the L_2 -ball of a vanishing radius. The solution to this problem can be formulated in terms of the eigenvalues of the corresponding covariance operator. However, to obtain closed form results for a concrete process one has to know their exact asymptotics. We solve the small ball probability problem for the fractional processes, using the obtained exact spectral asymptotics.

Exact spectral asymptotics for fractional processes

Covariance eigenproblem for the fractional Brownian motion

The eigenproblem for a centered random process $X = (X_t, t \in [0, 1])$ with covariance function $K(s, t) = \mathbb{E}X_s X_t$ and the corresponding covariance operator

$$(Kf)(t) = \int_0^1 K(s, t)f(s)ds, \quad t \in [0, 1].$$

consists of finding all non-trivial pairs (λ, φ) satisfying the equation

$$K\varphi = \lambda\varphi.$$

For a self adjoint positive definite operator K , compact in $L^2(0, 1)$, this problem is well known to have countably many solutions (λ_n, φ_n) , $n \in \mathbb{N}$, where eigenvalues λ_n are real and nonnegative and converge to zero, when put in the decreasing order, and the corresponding eigenfunctions φ_n form a complete orthonormal basis in $L^2(0, 1)$.

Nevertheless, the eigenvalues and eigenfunctions are notoriously hard to find explicitly. One notable exception is the standard Brownian motion $B = (B_t, t \in [0, 1])$ for which

$$\lambda_n = \frac{1}{(n - \frac{1}{2})^2 \pi^2} \quad \text{and} \quad \varphi_n(t) = \sqrt{2} \sin \left((n - \frac{1}{2})\pi t \right), \quad n = 1, 2, \dots \quad .$$

The exact expressions for the eigenfunctions and eigenvalues of the fractional Brownian motion remain unknown, but the following recent result [12] sheds much light on their asymptotic structure:

Theorem 0.1.

1. For $H \in (0, 1)$ the eigenvalues are given by the formula

$$\lambda_n = \sin(\pi H) \Gamma(2H + 1) \nu_n^{-2H-1} \quad n = 1, 2, \dots$$

where the sequence ν_n satisfies

$$\nu_n = \left(n - \frac{1}{2}\right)\pi - \frac{1}{2} \frac{(H - \frac{1}{2})^2}{H + \frac{1}{2}} \pi + O(n^{-1}) \quad \text{as } n \rightarrow \infty,$$

2. The corresponding unit norm eigenfunctions admit the approximation

$$\varphi_n(t) = \sqrt{2} \sin(\nu_n t + \pi \eta_H) + \int_0^\infty f_0(u) e^{-t \nu_n u} du + (-1)^n \int_0^\infty f_1(u) e^{-(1-t) \nu_n u} du + n^{-1} r_n(t),$$

where $f_0(\cdot)$ and $f_1(\cdot)$ are explicit functions defined in [12] and

$$\eta_H := \frac{H - \frac{3}{2}}{4} \frac{H - \frac{1}{2}}{H + \frac{1}{2}}$$

and the residual $r_n(t)$ is bounded uniformly over $n \in \mathbb{N}$ and $t \in [0, 1]$.

3. The values of the eigenfunctions at $t = 1$ satisfy

$$\varphi_n(1) = (-1)^n \sqrt{2H + 1} (1 + O(n^{-1}))$$

The proof uses analytic properties of the Laplace transform of the eigenfunctions. The main idea is to reduce the eigenproblem to a certain integro-algebraic system of equations, that can be solved asymptotically. In the thesis we develop this approach for a larger family of fractional processes, as described below.

Processes under consideration

Fractional Ornstein–Uhlenbeck process. In the fractional setting, the Ornstein–Uhlenbeck process can be defined in a number of nonequivalent ways [11]. We will be concerned with the solution of the Langevin equation, driven by the fBm:

$$X_t = X_0 + \beta \int_0^t X_s ds + B_t^H,$$

where $\beta \in \mathbb{R}$ is the drift parameter and $X_0 \sim N(0, \sigma^2)$ is the initial condition, independent of B^H . The covariance function of this process is given by the formula (see, e.g., [46]):

$$K_\beta(s, t) = \int_0^t e^{\beta(t-v)} \frac{d}{dv} \int_0^s H |v - u|^{2H-1} \text{sign}(v - u) e^{\beta(s-u)} du dv.$$

The fractional Ornstein–Uhlenbeck process of this type inherits long-range dependence property from the fractional Brownian motion, see [11].

Integrated fractional Brownian motion. The first integral of the fractional Brownian motion $X_t = \int_0^t B_s^H ds$ is the centered process with covariance function

$$K(s, t) = \int_0^t \int_0^s \frac{1}{2} (u^{2H} + v^{2H} - |v - u|^{2H}) du dv.$$

It is a Gaussian process with a.s. continuous first derivative. It occurs in the ruin problem in finance, storage theory, information traffic theory, etc.

Mixed fractional Brownian motion. Let $B = (B_t)$ be a standard Brownian motion and $B^H = (B_t^H)$ be an independent fBm with the Hurst exponent $H \in (0, 1)$. The *mixed* fractional Brownian motion is the process

$$\tilde{B}_t = B_t + B_t^H, \quad t \in [0, T].$$

The covariance function of \tilde{B} is given by

$$\tilde{K}(s, t) = s \wedge t + \frac{1}{2} (s^{2H} + t^{2H} - |t - s|^{2H}), \quad s, t \in [0, 1].$$

The interest in the mixed fBm was triggered by the paper [10]. Further results and related problems were comprehensively considered in [37]. Finally, recent results con-

cerning the canonical representation of the mixed fBm were obtained in [9].

Summary of the main results

Fractional Ornstein–Uhlenbeck process. The covariance function of the fractional Ornstein–Uhlenbeck process satisfies the scaling property

$$K_\beta(sT, tT) = T^{2H} K_{\beta T}(s, t), \quad s, t \in [0, 1] \quad T > 0$$

and, consequently, asymptotic solution to the eigenproblem on an arbitrary interval $[0, T]$ can be obtained from that on the unit interval.

Theorem 0.2.

1. For any $H \in (0, 1)$ the eigenvalues of covariance operator of the fractional Ornstein–Uhlenbeck process satisfy

$$\lambda_n = \sin(\pi H) \Gamma(2H + 1) \frac{\nu_n^{1-2H}}{\nu_n^2 + \beta^2}, \quad n = 1, 2, \dots,$$

where ν_n is the sequence with asymptotics

$$\nu_n = \left(n - \frac{1}{2}\right)\pi - \frac{1}{2} \frac{(H - \frac{1}{2})^2}{H + \frac{1}{2}} \pi + O(n^{-1}) \quad \text{as } n \rightarrow \infty.$$

2. The corresponding unit norm eigenfunctions admit the approximation

$$\varphi_n(x) = \sqrt{2} \sin(\nu_n x + \pi \eta_H) - \frac{\sqrt{2H+1}}{\pi} \int_0^\infty \rho_0(u) \left(e^{-x\nu_n u} \frac{u - \ell_H}{\sqrt{1 + \ell_H^2}} + (-1)^n e^{-(1-x)\nu_n u} \right) du + n^{-1} r_n(x),$$

where the residual $r_n(x)$ is bounded by a constant, depending only on H and β , $\rho_0(u)$ is an explicit function and the constants

$$\eta_H = \frac{H - \frac{3}{2}}{4} \frac{H - \frac{1}{2}}{H + \frac{1}{2}} \quad \text{and} \quad \ell_H = \tan\left(\frac{\pi}{2} \frac{H - \frac{1}{2}}{H + \frac{1}{2}}\right).$$

3. The eigenfunctions satisfy

$$\varphi_n(1) = (-1)^n \sqrt{2H+1} (1 + O(n^{-1})) \quad \text{and} \quad \int_0^1 \varphi_n(x) dx = -\sqrt{\frac{2H+1}{1+\ell_H^2}} \nu_n^{-1}.$$

This result exhibits a curious separation in dependence of the spectral asymptotics on its parameters. The drift parameter β enters eigenvalues formula only in the dominator, while all of its other ingredients depend solely on the Hurst parameter H . Moreover, the eigenfunctions do not depend on β in the first approximation.

Integrated fractional Brownian motion. Covariance function of the integrated fractional Brownian motion satisfies the scaling property

$$K(sT, tT) = T^{2H+2} K(s, t), \quad s, t \in [0, 1] \quad T > 0,$$

and, consequently, without loss of generality the corresponding covariance eigenproblem can be considered on the unit interval.

Theorem 0.3.

1. *The eigenvalues of covariance operator of the integrated fractional Brownian motion satisfy*

$$\lambda_n = \sin(\pi H) \Gamma(2H+1) \nu_n^{-2H-3} \quad n = 1, 2, \dots$$

where

$$\nu_n = \pi \left(n - \frac{1}{2} \right) + \frac{1-2H}{4} \pi + \frac{H - \frac{1}{2}}{H + \frac{3}{2}} \pi + O(n^{-1}), \quad n \rightarrow \infty.$$

2. *The corresponding unit norm eigenfunctions admit the approximation*

$$\varphi_n(x) = \varphi_n^{(1)}(x) + \varphi_n^{(2)}(x) + \varphi_n^{(3)}(x) + n^{-1} r_n(x)$$

where residual $r_n(x)$ is bounded uniformly in both $n \in \mathbb{N}$ and $x \in [0, 1]$ and

a. *the oscillatory term is given by*

$$\varphi_n^{(1)}(x) = \sqrt{2} \cos \left(\nu_n x + \frac{2H+1}{8} \pi - \frac{H - \frac{1}{2}}{H + \frac{3}{2}} \pi \right);$$

b. the polynomial boundary layer term is given by

$$\varphi_n^{(2)}(x) = -\frac{\sqrt{2H+3}}{\pi} \int_0^\infty \rho_0(t) \left(Q_0(t)e^{-t\nu_n x} - (-1)^n Q_1(t)e^{-t\nu_n(1-x)} \right) dt,$$

where function $\rho_0(t)$ and polynomials $Q_0(t)$ and $Q_1(t)$ are given by explicit expressions;

c. the exponential boundary layer term vanishes for $H \leq \frac{1}{2}$ and otherwise is given by

$$\varphi_n^{(3)}(x) = C_0 e^{-c\nu_n x} \cos\left(s\nu_n x + \varkappa_0\right) + C_1 e^{-c\nu_n(1-x)} \cos\left(s\nu_n(1-x) + \varkappa_1\right)$$

where amplitudes C_0 and C_1 and phases \varkappa_0 and \varkappa_1 are explicit constants and

$$c := \cos\left(\frac{\pi}{2} \frac{H - \frac{1}{2}}{H + \frac{3}{2}}\right) > 0 \quad \text{and} \quad s := \sin\left(\frac{\pi}{2} \frac{H - \frac{1}{2}}{H + \frac{3}{2}}\right) > 0.$$

3. The eigenfunctions satisfy

$$\varphi_n(1) = (-1)^n \sqrt{2H+3} \left(1 + O(n^{-1})\right)$$

$$\int_0^1 \varphi_n(x) dx = \nu_n^{-1} \sqrt{2H+3} C_H \left(1 + O(n^{-1})\right)$$

with explicit constant C_H , depending only on H .

In this case the eigenfunctions also have the oscillatory and boundary layer terms. The effect of the boundary layer is asymptotically negligible in the interior of the interval, and it pushes the eigenfunctions to zero at $x = 0$ and to $\pm\sqrt{2H+3}$ at $x = 1$. However, in this case it exhibits multiscale behaviour: for $H > \frac{1}{2}$, the two components $\varphi_n^{(2)}(x)$ and $\varphi_n^{(3)}(x)$ vanish at different, polynomial and exponential, rates respectively as $n \rightarrow \infty$.

Mixed fractional Brownian motion. Asymptotic formulas of the eigenvalues of the mixed fractional Brownian motion separate contributions of the standard and fractional components. In particular, it follows that the first order asymptotics of the eigenvalues and eigenfunctions coincides with that of the Brownian component for $H > \frac{1}{2}$ and of the fractional Brownian component for $H < \frac{1}{2}$.

Theorem 0.4. 1. *The ordered sequence of eigenvalues of covariance operator of the mixed fractional Brownian motion is given by*

$$\lambda_n := \frac{1}{\nu_n^2} + \frac{\sin(\pi H)\Gamma(2H+1)}{\nu_n^{2H+1}}, \quad n = 1, 2, \dots$$

where

$$\nu_n = \left(n - \frac{1}{2}\right)\pi - \frac{1}{2} \frac{(H - \frac{1}{2})^2}{H + \frac{1}{2}} \pi \mathbf{1}_{\{H < \frac{1}{2}\}} + O(n^{-|2H-1|}), \quad n \rightarrow \infty.$$

2. *The corresponding unit norm eigenfunctions admit the approximation*

$$\begin{aligned} \varphi_n(x) = & \sqrt{2} \sin(\nu_n x) \mathbf{1}_{\{H > \frac{1}{2}\}} + \sqrt{2} \sin\left(\nu_n x + \pi \eta_H\right) \mathbf{1}_{\{H < \frac{1}{2}\}} - \\ & - \frac{\sqrt{2H+1}}{\pi} \int_0^\infty \rho_0(u) \left(e^{-x\nu_n u} \frac{u - \ell_H}{\sqrt{1 + \ell_H^2}} + (-1)^n e^{-(1-x)\nu_n u} \right) du \mathbf{1}_{\{H < \frac{1}{2}\}} + \frac{r_n(x)}{n^{|2H-1|}}, \end{aligned}$$

where the residual $r_n(x)$ is bounded by a constant, depending only on H , $\rho_0(u)$ is an explicit function and the constants

$$\eta_H = \frac{H - \frac{3}{2}}{4} \frac{H - \frac{1}{2}}{H + \frac{1}{2}} \quad \text{and} \quad \ell_H = \tan\left(\frac{\pi}{2} \frac{H - \frac{1}{2}}{H + \frac{1}{2}}\right).$$

Description of the analytic approach

The covariance eigenproblems for processes related to the standard Brownian motion are usually solved by reduction to ordinary Sturm–Liouville boundary value problems for differential operators. For the fractional processes such approach is not applicable and we consider a different route, based on analytic properties of the Laplace transform

$$\widehat{\varphi}(z) = \int_0^1 \varphi(x) e^{-zx} dx, \quad z \in \mathbb{C}.$$

The proof is inspired by [12] and the approach from [54] to asymptotic approximation of the eigenvalues for weakly singular integral operators. Given the particular structure of the covariance eigenproblem, an expression for $\widehat{\varphi}(z)$ with singularities can be derived. Since $\widehat{\varphi}(z)$ must be an entire function, removal of these singularities gives

an alternative characterization for the eigenvalues and eigenfunctions in the form of integro-algebraic system of equations.

In principle, this approach can be applied to operators with the difference kernels satisfying

$$K(|x - y|) = \int_0^\infty \kappa(t) e^{-t|x-y|} dt,$$

with some function $\kappa(t)$, and their compositions with the integration operator. In principle, the developed methodology works for kernels with logarithmic singularities (see [44]). While the implementation is very specific in each particular situation, in general, the proof includes the steps, described below.

The Laplace transform. For covariance eigenproblems considered in this paper, it is possible to find an expression for the Laplace transform of eigenfunctions in the following typical form:

$$\hat{\varphi}(z) = P(z) - Q(z) \frac{\Phi_0(z) + e^{-z}\Phi_1(-z)}{\Lambda(z)}$$

where $P(z)$ and $Q(z)$ are polynomials of a finite degree, $\Phi_0(z)$ and $\Phi_1(z)$ are certain functions and $\Lambda(z)$ is given by an explicit formula. Such representation is tailored to the particular structure of the operator and is constructed on the case to case basis. The function $\Lambda(z)$ usually has a finite number of zeros $z_1(\lambda), \dots, z_k(\lambda)$ and a discontinuity along the real line and $\Phi_0(z)$ and $\Phi_1(z)$ are sectionally holomorphic on the complex plane, cut along the real axis.

Removal of the singularities. Since the Laplace transform is a priori an entire function, all singularities must be removable. Removal of the poles gives the algebraic conditions

$$\Phi_0(z_j(\lambda)) + e^{-z_j(\lambda)}\Phi_1(-z_j(\lambda)) = 0, \quad j = 1, \dots, k.$$

and removing the discontinuity along the real line imposes the following boundary conditions:

$$\lim_{z \rightarrow t^+} \frac{\Phi_0(z) + e^{-z}\Phi_1(-z)}{\Lambda(z)} = \lim_{z \rightarrow t^-} \frac{\Phi_0(z) + e^{-z}\Phi_1(-z)}{\Lambda(z)}, \quad t \in \mathbb{R}$$

Thus, the covariance eigenproblem reduces to an equivalent problem of finding a pair of functions $\Phi_0(z)$ and $\Phi_1(z)$, sectionally holomorphic on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$, satisfying the following conditions:

- limits $\Phi_0^\pm(t) = \lim_{z \rightarrow t^\pm} \Phi_0(z)$ and $\Phi_1^\pm(t) = \lim_{z \rightarrow t^\pm} \Phi_1(z)$ comply with the boundary conditions;
- the behaviour $\Phi_0(z)$ and $\Phi_1(z)$ at infinity matches the a priori growth estimates, determined by polynomials $P(z)$ and $Q(z)$;
- the values of $\Phi_0(z)$ and $\Phi_1(z)$ satisfy the algebraic constraints imposed by the zeros of $\Lambda(z)$.

Equivalent formulation of the eigenproblem. The conditions for $\Phi_0(z)$ and $\Phi_1(z)$ can be rewritten as an integro-algebraic system of equations as follows.

Let $\theta(t)$ be the argument of the limit $\Lambda^+(t) := \lim_{z \rightarrow t^+} \Lambda(z)$. The homogenous Riemann boundary value problem is solved to find a nonvanishing function $X(z)$, which is sectionally holomorphic on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$ and satisfies

$$\frac{X^+(t)}{X^-(t)} = e^{2i\theta(t)}, \quad t > 0.$$

The solution of this problem is not unique and is fixed to match the particular properties of $\theta(t)$ and a priori estimates on $\Phi_0(z)$ and $\Phi_1(z)$ in each particular case. The boundary conditions can be written in the decoupled form

$$\begin{aligned} S^+(t) - S^-(t) &= 2ih(t)e^{-t}S(-t) \\ D^+(t) - D^-(t) &= -2ih(t)e^{-t}D(-t), \quad t > 0 \end{aligned}$$

where we defined

$$S(z) := \frac{\Phi_0(z) + \Phi_1(z)}{2X(z)} \quad \text{and} \quad D(z) := \frac{\Phi_0(z) - \Phi_1(z)}{2X(z)}$$

and the real-valued function

$$h(t) := e^{i\theta(t)} \sin \theta(t) \frac{X(-t)}{X^+(t)}.$$

Applying the Sokhotski-Plemelj formula, we can rewrite the boundary conditions as

$$S(z) = \frac{1}{\pi} \int_0^\infty \frac{h_0(t/\nu)e^{-t}}{t-z} S(-t) dt + P_S(z)$$

$$D(z) = -\frac{1}{\pi} \int_0^\infty \frac{h_0(t/\nu)e^{-t}}{t-z} D(-t) dt + P_D(z)$$

where $P_S(z)$ and $P_D(z)$ are polynomials that match the a priori growth of $S(z)$ and $D(z)$ at infinity: $S(z) - P_S(z) \rightarrow 0$ and $D(z) - P_D(z) \rightarrow 0$ as $z \rightarrow \infty$. Moreover, by construction, both functions $S - P_S$ and $D - P_D$ are square integrable and, consequently, the integral equations have unique solutions with such property, since the operator on the right side is a contraction on $L_2(0, \infty)$.

Therefore, non-trivial solutions (S, D) to the integro-algebraic system are at one-to-one correspondence with the solutions (λ, φ) to the eigenproblem and thus we obtain an equivalent characterization of the eigenvalues and the eigenfunctions: any non-trivial solution of the eigenproblem determines functions $S(z)$ and $D(z)$ which solve the integro-algebraic system and vice versa.

Inversion of the Laplace transform and asymptotic analysis. Enumerating solutions of the integro-algebraic system in a certain convenient way, it is possible to find asymptotic approximation for their algebraic part as $\lambda_n \rightarrow 0$. Plugging it back into the system and using suitable estimates for the norm of the operator in the integral equations, the eigenfunctions asymptotics is extracted by inverting the Laplace transform:

$$\varphi(x) = \frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{-Ri}^{Ri} \hat{\varphi}(z) e^{zx} dz, \quad t \in \mathbb{R}.$$

Enumeration alignment. The enumeration chosen in the previous step may differ from the "natural" enumeration, which puts the eigenvalues into decreasing order. Usually it does not affect the leading term asymptotics, but may change the second order term. Alignment of the two enumerations is done through a calibration procedure, based on continuity of the spectrum.

Covariance eigenproblem for Gaussian bridges

Gaussian bridges

For a centered Gaussian *base* process $X = (X_t, t \in [0, 1])$ with the starting point $X_0 = 0$, the corresponding zero to zero bridge $\tilde{X} = (\tilde{X}_t, t \in [0, 1])$ is obtained by "restricting" the trajectories to terminate at zero:

$$\tilde{X}_t = X_t - \frac{K(t, 1)}{K(1, 1)} X_1, \quad t \in [0, 1]$$

Therefore, \tilde{X} is a centered Gaussian process with the covariance function

$$\tilde{K}(s, t) = K(s, t) - \frac{K(s, 1)K(t, 1)}{K(1, 1)}.$$

Aside of being interesting mathematical objects, Gaussian bridges have many important applications, such as statistical hypothesis testing, exact sampling of diffusions, etc.

We are interested in the solution of the covariance eigenproblem $\tilde{K}\tilde{\varphi} = \tilde{\lambda}\tilde{\varphi}$. In principle, we could use the analytic approach, described in the previous section, directly; however, somewhat surprisingly, it does not produce explicit results. We suggest an alternative approach to the solution of this problem.

The covariance function of the Gaussian bridge is a linear transformation of the covariance function of its base process. This implies similarity between the solutions of the covariance eigenproblems for the bridge and the *base* process and suggests that the spectra of the two processes must be closely related.

To illustrate, consider the Brownian motion with $K(s, t) = t \wedge s$ and the Brownian bridge with covariance function $\tilde{K}(s, t) = s \wedge t - st$. Then the eigenelements of the *base* process are given by:

$$\lambda_n = \frac{1}{\left((n - \frac{1}{2})\pi\right)^2} \quad \text{and} \quad \varphi_n(t) = \sqrt{2} \sin\left(\left(n - \frac{1}{2}\right)\pi t\right)$$

and solution to the eigenproblem for the bridge is

$$\tilde{\lambda}_n = \frac{1}{(\pi n)^2} \quad \text{and} \quad \tilde{\varphi}_n(t) = \sqrt{2} \sin(\pi n t).$$

The formulas for eigenvalues and eigenfunctions have the identical form, but there is a frequency shift of $\pi/2$. As mentioned above, in general case the precise formulas for the eigenvalues and eigenfunctions of K are rarely known. Nevertheless, exact asymptotic approximations, such as those we obtained in this thesis, are often available. We show how the the spectral asymptotics for a bridge can be deduced from that of the base process.

Connection between solutions of the covariance eigenproblem for bridge and base process

The general relation between the spectra of a bridge and its base process can be explained by perturbation theory [25], which implies that the eigenvalues of K and \widetilde{K} agree in the leading asymptotic term, as it happens for the standard Brownian motion and the Brownian bridge. This is vividly demonstrated in the paper [43], where the kernels of the following form are considered:

$$\widetilde{K}_Q(s, t) = K(s, t) + Q\psi(s)\psi(t).$$

Here Q is a scalar real valued parameter and ψ is a function in the range of K . It turns out that for any Q greater than a certain critical value Q^* , the spectrum of \widetilde{K}_Q coincides with that of K in the first two asymptotic terms. For $Q = Q^*$ the spectra depart in the second term. The deviation is quantified in [43], when ψ is an image of an $L^2(0, 1)$ function, under the action of K . This corresponding approach is not applicable in the case of fractional Gaussian bridges, since this condition does not hold.

We develop a different technique, using the particular structure of the perturbation inherent to bridges. Observe that the eigenproblem $\widetilde{K}\tilde{\varphi} = \tilde{\lambda}\tilde{\varphi}$ can be written in terms of the covariance operator of the base process

$$\int_0^1 K(s, t)\tilde{\varphi}(s)ds - K(1, t) \int_0^1 K(1, s)\tilde{\varphi}(s)ds = \tilde{\lambda}\tilde{\varphi}(t), \quad t \in [0, 1],$$

where, without loss of generality, X is assumed to be normalized so that $K(1, 1) = 1$. Let (φ_n, λ_n) be the eigenelements of the base process. Since the eigenfunctions (φ_n) of K form a complete orthonormal basis in $L^2([0, 1])$, for any $\lambda \neq \lambda_n$ we have the expansion

$$\tilde{\varphi}(t) = c \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k - \tilde{\lambda}} \tilde{\varphi}_k(1) \varphi_k(t) \quad \text{with} \quad c := \int_0^1 K(1, s) \tilde{\varphi}(s) ds.$$

The above-mentioned equation implies that $\tilde{\varphi}(1) = 0$. Noting that $c \neq 0$ whenever $\tilde{\lambda} \neq \lambda_n$, we obtain the following equation for the eigenvalues of the bridge:

$$\sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k - \tilde{\lambda}} \tilde{\varphi}_k(1)^2 = 0.$$

Note that its roots are not determined solely by the eigenvalues of the base process, but also require some information on its eigenfunctions. In Chapter 2 we show how this equation can be used to construct asymptotic approximation for the solutions of the bridge eigenproblem, given the exact asymptotics of the Karhunen–Loève expansion for the corresponding base process. As an example, we consider the covariance eigenproblem for the fractional Brownian bridge, but the technique is applicable to other Gaussian bridges.

Exact spectral asymptotic for the fractional Brownian bridge

As mentioned before, the eigenvalues of the fBm are given by the formula

$$\lambda_n = \frac{\sin(\pi H) \Gamma(2H + 1)}{\nu_n^{2H+1}}, \quad n = 1, 2, \dots$$

where $\nu_n = \pi n + \pi \gamma_H + O(n^{-1})$ as $n \rightarrow \infty$ and

$$\gamma_H = -\frac{1}{2} - \frac{1}{2} \frac{(H - \frac{1}{2})^2}{H + \frac{1}{2}}.$$

The corresponding unit norm eigenfunctions admit the approximation

$$\begin{aligned} \varphi_n(x) = & \sqrt{2} \sin\left(\nu_n x + \pi \eta_H\right) - \\ & - \frac{\sqrt{2H+1}}{\pi} \int_0^\infty \rho_0(u) \left(e^{-x\nu_n u} \frac{u - \ell_H}{\sqrt{1 + \ell_H^2}} + (-1)^n e^{-(1-x)\nu_n u} \right) du + n^{-1} r_n(x), \end{aligned}$$

where the residual $r_n(x)$ is bounded by a constant, depending only on H , $\rho_0(u)$ is an explicit function and the constants

$$\eta_H = \frac{H - \frac{3}{2}}{4} \frac{H - \frac{1}{2}}{H + \frac{1}{2}} \quad \text{and} \quad \ell_H = \tan\left(\frac{\pi}{2} \frac{H - \frac{1}{2}}{H + \frac{1}{2}}\right).$$

Moreover, the eigenfunctions satisfy

$$\varphi_n(1) = (-1)^n \sqrt{2H+1} (1 + O(n^{-1})).$$

Let us introduce a new variable $\tilde{\mu}$ such that

$$\tilde{\lambda} = \frac{\sin(\pi H) \Gamma(2H+1)}{(\pi \tilde{\mu})^{2H+1}}.$$

Then $\tilde{\mu}$ solves the following equation:

$$g(\tilde{\mu}) := \sum_{k=1}^{\infty} \frac{\varphi_k(1)^2}{\mu_k^{2H+1} - \tilde{\mu}^{2H+1}} = 0,$$

where $\mu_k := \nu_k/\pi$. Since the asymptotics of the eigenvalues of the base process is known, we may first consider the *perturbed* equation

$$g^a(\mu) := \sum_{k=1}^{\infty} \frac{2H+1}{(k + \gamma_H)^{2H+1} - \tilde{\mu}^{2H+1}} = 0,$$

where the eigenvalues and eigenfunctions of the base process are replaced with their asymptotic approximations. We give a characterization of roots of perturbed equation and show that they are asymptotically close to roots of original equation on a suitable scale, that gives the asymptotics of the eigenvalues. The approximation for the eigenfunctions are obtained by plugging these asymptotics into the formula:

$$\tilde{\varphi}_n(t) = c_n \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k - \tilde{\lambda}_n} \varphi_k(1) \varphi_k(t) = -c_n \tilde{\mu}_n^{2H+1} \sum_{k=1}^{\infty} \frac{1}{(\nu_k/\pi)^{2H+1} - \tilde{\mu}_n^{2H+1}} \varphi_k(1) \varphi_k(t)$$

and normalizing. Combining these results, we prove the following theorem:

Theorem 0.5. 1. *For the fractional Brownian bridge with $H \in (0, 1)$, the ordered sequence of the eigenvalues satisfies*

$$\tilde{\lambda}_n = \frac{\sin(\pi H) \Gamma(2H+1)}{\tilde{\nu}_n^{2H+1}}, \quad n = 1, 2, \dots$$

where $\tilde{\nu}_n = \pi n + \pi \tilde{\gamma}_H + O(n^{-1} \log n)$ as $n \rightarrow \infty$ and $\tilde{\gamma}_H := \gamma_H + \frac{H}{H+\frac{1}{2}}$.

2. The corresponding eigenfunctions admit the approximation

$$\begin{aligned} \tilde{\varphi}_n(t) = & \sqrt{2} \sin(\tilde{\nu}_n t + \pi\eta_H) + \int_0^\infty f_0(u) e^{-\tilde{\nu}_n t u} du + \\ & (-1)^n \int_0^\infty e^{-\tilde{\nu}_n(1-t)u} \left(\sin(\pi(\tilde{\gamma}_H - \gamma_H)) \tilde{f}_1(u) du + \cos(\pi(\tilde{\gamma}_H - \gamma_H)) f_1(u) \right) du + \\ & (-1)^n \sin(\pi(\tilde{\gamma}_H - \gamma_H)) \int_0^\infty \tilde{g}_1(\tilde{\nu}_n(1-t)u) f_1(u) du + \frac{\log n}{n} \tilde{r}_n(t) \end{aligned}$$

where \tilde{f}_1 and \tilde{g}_1 are functions, defined in the closed forms by (2.38) and (2.39), and the residual $\tilde{r}_n(t)$ is bounded, uniformly over $n \in \mathbb{N}$ and $t \in [0, 1]$.

The eigenvalues of the fractional Brownian motion and its bridge differ by a constant shift in the second order asymptotic term

$$\pi\tilde{\gamma}_H - \pi\gamma_H = \frac{\pi H}{H + \frac{1}{2}},$$

which reduces to the familiar constant $\pi/2$ for $H = \frac{1}{2}$ in the standard Brownian case. On the other hand the residuals in ν_n and $\tilde{\nu}_n$ differ by the $\log n$ factor, which may well be an artifact of the approach.

The eigenfunctions of the bridge inherit the oscillatory term from the corresponding term of the base process, however, with a frequency shift.

A more complicated modification occurs in the integral terms, which are responsible for the boundary layer: their contribution is asymptotically negligible away from the endpoints of the interval, but is persistent near the boundary. For the base process, these terms force the eigenfunctions to vanish at $t = 0$ and approach the alternating values at $t = 1$; for the bridge, they push the eigenfunctions to zero at both endpoints. Consequently, the change is more significant near 1 than near the origin. It can also be seen that the boundary layer vanishes for $H = \frac{1}{2}$ and the leading asymptotic term reduces to the familiar formula for the standard Brownian motion.

Applications of exact spectral asymptotics

Estimation of fractional signals in white noise

The linear filtering problem. Filtering is concerned with estimation of signals from noisy observations. The standard setup in continuous time, known in engineering literature as the additive white Gaussian noise model, consists of the *signal* process X_t , whose trajectory is to be estimated given the trajectory of the observation process of the form

$$Y_t = \mu \int_0^t X_s ds + \sqrt{\varepsilon} B_t, \quad t \in [0, T],$$

where μ and $\varepsilon > 0$ are real constants and B_t is the Brownian motion independent of X_t .

If X is a centered Gaussian process with covariance function $K(s, t) = \mathbb{E}X_s X_t$, then the optimal in the mean squared sense estimator of X_t is the orthogonal projection on the closed linear subspace of random variables generated by increments of Y up to time T and is given by

$$\widehat{X}_t = \frac{1}{\mu} \int_0^T h(s, t) dY_s,$$

where the kernel $h(s, t)$ solves the Wiener-Hopf integral equation

$$\varepsilon h(s, t) + \int_0^T \mu^2 K(r, s) h(r, t) dr = \mu^2 K(s, t), \quad 0 \leq s \leq t \leq T.$$

The corresponding minimal mean square error satisfies

$$P_\varepsilon(t) = \mathbb{E}(X_t - \widehat{X}_t)^2 = K(t, t) - \int_0^T h(r, t) K(r, t) dr = \frac{\varepsilon}{\mu^2} h(t, t),$$

and the important engineering question is how it scales with the noise intensity and what is its behaviour in large time limit.

High signal-to-noise asymptotics for fractional processes The integral equation for $h(s, t)$ is rarely solvable in a closed form. One important exception is the Markov case, when X is the Ornstein-Uhlenbeck process generated by the stochastic differential equation

$$dX_t = \beta X_t dt + dW_t, \quad t > 0,$$

driven by an independent Brownian motion $W = (W_t)_{t \in \mathbb{R}_+}$. In this case, the integral equation can be solved explicitly by reduction to the Riccati o.d.e. (see, e.g., Theorem 12.10 in [32]) and, $P_\varepsilon(t)$ can be computed in a closed form. A calculation gives the following *high signal-to-noise* asymptotics

$$P_\varepsilon(t) \simeq \sqrt{\varepsilon/\mu^2} \begin{cases} \frac{1}{2} & t \in (0, T) \\ 1 & t = T \end{cases}, \quad \text{as } \varepsilon \rightarrow 0,$$

We aim at generalize this result to fractional setting, that cannot be approached along the same lines in view of the lack of the Markov property. We prove the following result, using the exact spectral asymptotics of the fractional Ornstein-Uhlenbeck process:

Proposition 1. *The minimal mean squared error in the estimation problem of the fractional Ornstein–Uhlenbeck process in the additive Gaussian white noise satisfies*

$$P_\varepsilon(t) \simeq (\varepsilon/\mu^2)^{\frac{2H}{1+2H}} \frac{\left(\sin(\pi H)\Gamma(2H+1)\right)^{\frac{1}{1+2H}}}{\sin \frac{\pi}{2H+1}} \begin{cases} \frac{1}{2H+1} & t \in (0, T) \\ 1 & t = T \end{cases} \quad \text{as } \varepsilon \rightarrow 0.$$

Note that the *smoothing* estimator \widehat{X}_t with $t < T$ outperforms the *filtering* estimator \widehat{X}_T by factor $2H+1$ in the limit.

Large time asymptotics for fractional signals. Another important objective is to compute the steady state limit of optimal linear filtering error $P_\infty = \lim_{t \rightarrow \infty} P_t$, if it exists. Without loss of generality let $\varepsilon = 1$.

Again, when X is the Ornstein–Uhlenbeck process, the optimal estimator \widehat{X}_t and the minimal error P_t solve the famous Kalman-Bucy differential equations

$$\begin{aligned} d\widehat{X}_t &= \beta \widehat{X}_t dt + \mu P_t (dY_t - \mu \widehat{X}_t dt) \\ \dot{P}_t &= 2\beta P_t + 1 - \mu^2 P_t^2 \end{aligned}$$

and the steady state error $P_\infty = \lim_{t \rightarrow \infty} P_t$ exists for $\mu \neq 0$ and is found by setting derivative in the Riccati o.d.e to zero and solving the obtained quadratic equation:

$$P_\infty = \frac{\beta + \sqrt{\beta^2 + \mu^2}}{\mu^2}.$$

In absence of the Markov structure, the large time asymptotic analysis of the Wiener-Hopf equation is a nontrivial matter and, to the best of our knowledge, no general theory exists to this end. Nevertheless, quite remarkably, using the analytic approach, we were able to obtain a fairly explicit formula for its steady state limit error:

Theorem 0.6. *For the fractional Ornstein–Uhlenbeck process X_t with Hurst parameter $H \in (0, 1) \setminus \{\frac{1}{2}\}$ and $\mu \neq 0$*

$$P_\infty(H) = \lim_{t \rightarrow \infty} \mathbb{E}(X_t - \widehat{X}_t)^2 = \frac{1}{\mu^2} (b_0 + \beta) + \frac{2}{\mu^2} \operatorname{Re}\{z_0\} \mathbf{1}_{\{H > \frac{1}{2}\}}$$

where

$$b_0 = \frac{\operatorname{sign}(\frac{1}{2} - H)}{\pi} \int_0^\infty \operatorname{arccot} \left(-|\tan(\pi H)| - \frac{2}{\mu^2} \frac{(\beta^2 - t^2)t^{2H-1}}{\Gamma(2H + 1)|\sin(2\pi H)|} \right) dt,$$

and, for $H > \frac{1}{2}$, z_0 is the unique root of equation

$$z^2 - \beta^2 - \mu^2 \Gamma(2H + 1) \sin(\pi H) (z/i)^{1-2H} = 0$$

in the first quadrant of the complex plane.

The expression for $P_\infty(H)$ has right and left limits at $H = \frac{1}{2}$, which coincide with the classic formula for the standard Ornstein–Uhlenbeck process.

Large deviations for drift parameter estimator of mixed fractional Ornstein-Uhlenbeck process

Maximum likelihood estimation for mixed fractional Ornstein-Uhlenbeck process. Consider the Ornstein-Uhlenbeck process driven by the mixed fractional Brownian motion:

$$dX_t = -\vartheta X_t dt + d\tilde{B}_t, \quad t \in [0, T], \quad T > 0$$

where the initial state $X_0 = 0$ and the drift parameter ϑ is strictly positive. The process \tilde{B}_t is the mixed fractional Brownian motion

$$\tilde{B}_t = B_t + B_t^H, \quad t \in [0, T]$$

where $B = (B_t)$ is a Brownian motion and $B^H = (B_t^H)$ is an independent fractional Brownian motion with the Hurst exponent $H \in (0, 1]$.

Let us briefly recall the canonical innovation representation constructed in [9]. Let $g(s, t)$ be the solution of the following integro-differential equation

$$g(s, t) + \frac{d}{ds} \int_0^t g(r, t) H |s - r|^{2H-1} \text{sign}(s - r) dr = 1, \quad 0 < s < t \leq T.$$

Then the process $M_t = \int_0^t g(s, t) d\tilde{B}_s$ is a Gaussian martingale on $[0, T]$ with quadratic variation $\langle M \rangle_t = \int_0^t g(s, t) ds$. Moreover, the natural filtration of the martingale M coincides with the natural filtration of the mixed fractional Brownian motion \tilde{B} .

In [9] authors also show that the following processes can be properly defined:

$$Q_t = \frac{d}{d\langle M \rangle_t} \int_0^t g(s, t) X_s ds \quad \text{and} \quad Z_t = \int_0^t g(s, t) dX_s.$$

The process Z is a semimartingale with the decomposition $Z_t = -\vartheta \int_0^t Q_s d\langle M \rangle_s + M_t$ fundamental for X , i.e. the natural filtrations (\mathcal{X}_t) and (\mathcal{Z}_t) of X and Z respectively coincide.

Specific structure of the process Q allows to determine the likelihood function for the mixed fractional Ornstein-Uhlenbeck process:

$$L_T(\vartheta, X) = \frac{d\mu^X}{d\mu^{\tilde{B}}}(X) = \exp \left(-\vartheta \int_0^T Q_t dZ_t - \frac{1}{2} \vartheta^2 \int_0^T Q_t^2 d\langle M \rangle_t \right),$$

where μ^X and $\mu^{\tilde{B}}$ are the probability measures induced by processes X and \tilde{B} respectively. Thus, the score function for the mixed fractional Ornstein-Uhlenbeck process, i.e. the derivative of the log-likelihood function on the interval $[0, T]$ is given by

$$\Sigma_T(\theta) = - \int_0^T Q_t dZ_t - \vartheta \int_0^T Q_t^2 d\langle M \rangle_t$$

Equating this to zero, yields a formula for the maximum likelihood estimator for the drift parameter ϑ , which, by Theorem 2.9 in [12], is consistent as $T \rightarrow \infty$ and asymptotically normal.

Theorem 0.7. *The maximum likelihood estimator of ϑ is given by*

$$\hat{\vartheta}_T(X) = -\frac{\int_0^T Q_t dZ_t}{\int_0^T Q_t^2 d\langle M \rangle_t}.$$

For $\vartheta > 0$ this estimator is asymptotically normal at the usual parametric rate

$$\sqrt{T} \left(\hat{\vartheta}_T(X) - \vartheta \right) \xrightarrow[T \rightarrow \infty]{d} N(0, 2\vartheta).$$

In this thesis we will complement this result by the Large Deviations Principle for the estimation error.

Large Deviations Principle. The Large Deviations Principle (LDP) characterizes the limiting behavior of a family of random variables (or corresponding probability measures) in terms of a rate function.

A rate function I is a lower semicontinuous function $I : \mathbb{R} \rightarrow [0, +\infty]$, such that for all $\alpha \in [0, +\infty)$ the level sets $\{x : I(x) \leq \alpha\}$ are closed subsets of \mathbb{R} . Moreover, I is a good rate function if its level sets are compact.

A family of real random variables $(Y_T)_{T>0}$ satisfies the LDP with a rate function $I : \mathbb{R} \rightarrow [0, +\infty]$, if for any Borel set $\Gamma \subset \mathbb{R}$,

$$-\inf_{x \in \Gamma^\circ} I(x) \leq \liminf_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{P}(Y_T \in \Gamma) \leq \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{P}(Y_T \in \Gamma) \leq -\inf_{x \in \bar{\Gamma}} I(x)$$

where Γ° and $\bar{\Gamma}$ denote the closure and the interior of Γ correspondingly.

In order to prove the LDP for the drift parameter estimator of the mixed fractional Ornstein-Uhlenbeck process we will use the normalized cumulant generating function

$$\mathcal{L}_T(a, b) = \frac{1}{T} \log \mathbb{E} \left[\exp \left(a \int_0^T Q_t dZ_t + b \int_0^T Q_t^2 d\langle M \rangle_t \right) \right].$$

In particular, the following implication of the Gärtner-Ellis theorem (Theorem 2.3.6 in [14]) applies:

Lemma 0.1. *Let $(\hat{\vartheta}_T)_{T>0}$ be a family of drift parameter maximum likelihood estimators for the mixed fractional Ornstein-Uhlenbeck process and let $\mathcal{L}_T(a, b)$ be defined as above. For each fixed value of $x \in \mathbb{R}$, let Δ_x denote the set of $a \in \mathbb{R}$ for which $\lim_{T \rightarrow \infty} \mathcal{L}_T(a, -xa)$ exists and is finite. If Δ_x is not empty for each value of x , then $(\hat{\vartheta}_T)_{T>0}$ satisfies the LDP with the good rate function*

$$I(x) = - \inf_{a \in \Delta_x} \lim_{T \rightarrow \infty} \mathcal{L}_T(a, -xa).$$

Main results. In order to establish the LDP for $\hat{\vartheta}_T$ it is necessary to calculate the following limit

$$\mathcal{L}(a, b) = \lim_{T \rightarrow \infty} \mathcal{L}_T(a, b)$$

and determine the set of $(a, b) \in \mathbb{R}^2$ for which this limit is finite. For arbitrary $\varphi \in \mathbb{R}$ we can apply a usual change of measures and consider new probability \mathbb{P}_φ defined by the local density

$$\Lambda_\varphi(T) = \frac{d\mathbb{P}_\varphi}{d\mathbb{P}} = \exp \left((\varphi + \vartheta) \int_0^T Q_t dM_t - \frac{(\varphi + \vartheta)^2}{2} \int_0^T Q_t^2 d\langle M \rangle_t \right).$$

and rewrite $\mathcal{L}_T(a, b)$ as

$$\begin{aligned} \mathcal{L}_T(a, b) &= \frac{1}{T} \log \mathbb{E} \left[\exp(\mathcal{Z}_T(a, b)) \right] = \frac{1}{T} \log \mathbb{E}_\varphi \left[\exp(\mathcal{Z}_T(a, b)) \Lambda_\varphi(T)^{-1} \right] = \\ &= \frac{1}{T} \log \mathbb{E}_\varphi \exp \left((a - \varphi - \vartheta) \int_0^T Q_t dZ_t + \frac{1}{2} (2b - \vartheta^2 + \varphi^2) \int_0^T Q_t^2 d\langle M \rangle_t \right). \end{aligned}$$

Since φ is an arbitrary real number, we can choose $\varphi = a - \vartheta$ and denote $\mu = -\frac{1}{2}(2b - \vartheta^2 + (a - \vartheta)^2)$:

$$\mathcal{L}_T(a, b) = \frac{1}{T} \log \mathbb{E}_\varphi \exp \left(-\mu \int_0^T Q_t^2 d\langle M \rangle_t \right).$$

By the Girsanov theorem, under \mathbb{P}_φ , the process

$$M_t - (\varphi + \vartheta) \int_0^t Q_s d\langle M \rangle_s = Z_t - \varphi \int_0^t Q_s d\langle M \rangle_s$$

has the same distribution as M under \mathbb{P} . Consequently, under \mathbb{P}_φ , the process $X_t -$

$\varphi \int_0^t X_s ds$ is the mixed fractional Brownian motion and in order to calculate the limit of $\mathcal{L}_T(a, b)$ as $T \rightarrow \infty$ we prove the following proposition:

Proposition 2. *For the mixed fractional Ornstein-Uhlenbeck process X with the drift parameter ϑ the following limit holds*

$$\mathcal{K}_T(\mu) = \frac{1}{T} \log \mathbb{E} \exp \left(-\mu \int_0^T Q_t^2 d\langle M \rangle_t \right) \rightarrow \frac{\vartheta}{2} - \sqrt{\frac{\vartheta^2}{4} + \frac{\mu}{2}}, \quad T \rightarrow \infty$$

for all $\mu > -\frac{\vartheta^2}{2}$.

Then the straightforward calculations give the following result:

Theorem 0.8. *The maximum likelihood estimator of the drift parameter of the mixed fractional Ornstein-Uhlenbeck process satisfies the LDP with the good rate function*

$$I(x) = \begin{cases} -\frac{(x + \vartheta)^2}{4x}, & \text{if } x < -\frac{\vartheta}{3} \\ 2x + \vartheta, & \text{if } x \geq -\frac{\vartheta}{3}. \end{cases}$$

The rate function $I(x)$ does not depend on the parameter H . Hence, $\hat{\vartheta}_T$ satisfies the same LDP as the standard Ornstein-Uhlenbeck process [17] and the fractional Ornstein-Uhlenbeck process [1].

Small L^2 -ball probabilities for fractional processes

Small L^2 -ball probabilities problem. For a given process $X = (X_t, t \in [0, 1])$ and a norm $\|\cdot\|$, the problem of small ball probabilities is to find the asymptotics of

$$\mathbb{P}(\|X\| \leq \varepsilon) \quad \text{as } \varepsilon \rightarrow 0.$$

This problem has been studied extensively in the past and was found to have deep connections to various topics in probability theory. The case of Gaussian processes and L^2 -norm is the simplest, in which asymptotics of small ball probabilities is determined by eigenvalues of the covariance operator of X . The principal difficulty with this formula is that it requires an accurate asymptotic approximation of the sequence of eigenvalues, which is typically hard to find in concrete cases.

For the standard Brownian motion, the exact asymptotics is long known:

$$\mathbb{P}(\|B\|_2 \leq \varepsilon) = \frac{4}{\sqrt{\pi}} \varepsilon \exp\left(-\frac{1}{8}\varepsilon^{-2}\right) (1 + o(1)), \quad \varepsilon \rightarrow 0.$$

For the fractional Brownian motion, the rough logarithmic asymptotics of small ball probabilities was derived in [7, 8] and, by different methods, in [40] and [33], and was recently improved in [12] up to

$$\mathbb{P}(\|B^H\| \leq \varepsilon) \sim \varepsilon^{\gamma_0(H)} \exp\left(-\beta_0(H)\varepsilon^{-\frac{1}{H}}\right)$$

with explicit constant

$$\beta_0(H) := \frac{H}{(2H+1)^{\frac{2H+1}{2H}}} \left(\frac{\sin(\pi H)\Gamma(2H+1)}{\left(\sin\frac{\pi}{2H+1}\right)^{2H+1}} \right)^{\frac{1}{2H}}$$

and the exact power

$$\gamma_0(H) = \frac{1}{2H} \left(H^2 - H + \frac{5}{4} \right).$$

This refinement follows from asymptotic formula for the eigenvalues of the fractional Brownian motion. Our objective is to consider the small ball probabilities problem for the fractional processes.

Main results. The exact spectral asymptotics lead to the following results:

Proposition 3. *For the fractional Ornstein-Uhlenbeck process X_{OU}^H with the Hurst exponent $H \in (0, 1)$*

$$\mathbb{P}(\|X_{OU}^H\|_2 \leq \varepsilon) \sim \varepsilon^{\gamma_0(H)} \exp\left(-\beta_0(H)\varepsilon^{-\frac{1}{H}}\right), \quad \varepsilon \rightarrow 0$$

We should notice that the small L^2 -ball probabilities for the fractional Ornstein-Uhlenbeck process do not depend on the drift parameter β and, consequently coincide with those of the fractional Brownian motion.

In case of the integrated fractional Brownian motion we have the following small L^2 -ball probabilities:

Proposition 4. *For all $H \in (0, 1)$*

$$\mathbb{P}\left(\int_0^1 \left(\int_0^t B_s^H ds\right)^2 dt \leq \varepsilon^2\right) \sim \varepsilon^{\gamma_1(H)} \exp\left(-\beta_1(H)\varepsilon^{-\frac{1}{H+1}}\right), \quad \varepsilon \rightarrow 0$$

where $\beta_1(H)$ is an explicit constant depending only on H and

$$\gamma_1(H) = \frac{1}{2H+2} \left(H^2 - H + \frac{5}{4}\right).$$

The case of the mixed fractional Brownian motion was considered in [40] (see also [41], [34]), where logarithmic asymptotics was shown to be inherited either from the standard or fractional parts, depending on the value of H :

$$\log \mathbb{P}(\|\tilde{B}\|_2 \leq \varepsilon) \sim \begin{cases} \log \mathbb{P}(\|B\| \leq \varepsilon) & H > \frac{1}{2} \\ \log \mathbb{P}(\|B^H\| \leq \varepsilon) & H < \frac{1}{2} \end{cases}$$

Using the exact spectral asymptotics of the mixed fractional Brownian motion, we were able to show that the exact asymptotics of the mixed process is more intricate than could have been expected in view of the above formula and discovers new certain special values of H , further referred to as *thresholds*:

Theorem 0.9. 1. For $H \in (\frac{1}{2}, 1)$ let h_k and g_k be the real sequences, defined by formulas (3.89) evaluated at the constants in (3.85). Then for any $r > 0$ the equation

$$y/y_0 + \sum_{k=1}^{\lfloor \frac{1}{2} \frac{1}{2H-1} \rfloor} h_k r^{k(2H-1)} y^{k(2H-1)+1} = 1$$

with $y_0 = 2\sqrt{2}$ has unique positive root $y(r)$, which admits expansion into series

$$y(r) = y_0 + \sum_{j=1}^{\infty} y_j r^{j(2H-1)},$$

convergent for all r small enough. Let ξ_j and $\eta_{k,j}$ be coefficients of the power expansions

$$y(r)^{-2} = \sum_{j=0}^{\infty} \xi_j r^{j(2H-1)} \quad \text{and} \quad y(r)^{k(2H-1)-1} = \sum_{j=0}^{\infty} \eta_{k,j} r^{j(2H-1)}$$

and define

$$\beta_\ell := \frac{1}{\sqrt{2}} \eta_{0,\ell} - \sum_{j=0}^{\ell-1} g_{\ell-j} \eta_{\ell-j,j} - \xi_\ell.$$

Then

$$\mathbb{P}(\|\tilde{B}\|_2 \leq \varepsilon) \sim \varepsilon \exp\left(-\frac{1}{8}\varepsilon^{-2} - \sum_{\ell=1}^{\lfloor \frac{1}{2H-1} \rfloor} \beta_\ell \varepsilon^{2\ell(2H-1)-2}\right), \quad \varepsilon \rightarrow 0.$$

2. For $H \in (0, \frac{1}{2})$ let h_k and g_k be the real sequences, defined by formulas (3.89), evaluated at the constants (3.91). Then for any $r > 0$ the equation

$$y/y_0 + \sum_{k=1}^{\lfloor \frac{1}{2} \frac{1}{1-2H} \rfloor} h_k r^k y^{k \frac{1-2H}{2H} + 1} = 1$$

with

$$y_0 = (2H + 1) \left(\frac{2^{2H} \left(\sin \frac{\pi}{2H+1} \right)^{2H+1}}{\sin(\pi H) \Gamma(2H + 1)} \right)^{\frac{1}{2H+1}}$$

has unique positive root $y(r)$, which admits expansion into series

$$y(r) = y_0 + \sum_{k=1}^{\infty} y_k r^{k \frac{1-2H}{2H}}$$

convergent for all r small enough. Let ξ_j and $\eta_{k,j}$ be coefficients of the power expansions

$$y(r)^{-\frac{2H+1}{2H}} = \sum_{j=0}^{\infty} \xi_j r^{j \frac{1-2H}{2H}} \quad \text{and} \quad y(r)^{\frac{k(1-2H)-1}{2H}} = \sum_{j=0}^{\infty} \eta_{k,j} r^{j \frac{1-2H}{2H}}$$

and define

$$\beta_\ell(H) := \frac{2H + 1}{y_0} \eta_{0,\ell} - \sum_{j=0}^{\ell-1} g_{\ell-j} \eta_{\ell-j,j} - \xi_\ell.$$

Then

$$\mathbb{P}(\|\tilde{B}\|_2 \leq \varepsilon) \sim \varepsilon^{\gamma_0(H)} \exp\left(-\beta_0(H) \varepsilon^{-\frac{1}{H}} - \sum_{\ell=1}^{\lfloor \frac{1}{1-2H} \rfloor} \beta_\ell(H) \varepsilon^{\frac{\ell(1-2H)-1}{H}}\right), \quad \varepsilon \rightarrow 0$$

where $\gamma(H)$ and $\beta_0(H)$ were defined in previous paragraph.

It should be noticed that while general closed form formula for constants β_ℓ would be

cumbersome to find, they can be easily computed for any given value of H , at least numerically.

Remark 1. The fractional Brownian motion B^H is known to change some of its properties abruptly at certain special values of H . The most obvious example is $H = \frac{1}{2}$, at which it coincides with the standard Brownian motion. Moreover, the increments of the fractional Brownian motion exhibit long range dependence if and only if $H > \frac{1}{2}$. Other thresholds are less apparent and some of them become visible when the fractional Brownian motion is mixed with other processes, in particular, with the standard Brownian motion.

The mixed fractional Brownian motion $\tilde{B}_t = B_t + B_t^H$ is a semimartingale, measure equivalent to B , if and only if $H > \frac{3}{4}$, that was shown in [10] and follows from Shepp's general criteria for equivalence of such mixtures [48]. The counterpart threshold $H = \frac{1}{4}$ was discovered in [58], where it was shown that \tilde{B} is equivalent to B^H if and only if $H < \frac{1}{4}$.

Another threshold $H = \frac{2}{3}$ emerges in statistical applications, based on the canonical innovation representation, and in the optimal linear filtering problem. The worst steady state filtering error of B^H given the white noise observations is attained at this threshold.

A close look to the formula for the small ball probabilities for the mixed fractional Brownian motion reveals that additional terms join the sum in the exponent at the following values of H :

$$\frac{1}{4}, \frac{1}{3}, \frac{3}{8}, \dots, \frac{1}{2}, \dots, \frac{5}{8}, \frac{2}{3}, \frac{3}{4}$$

where we emphasized the already known thresholds, mentioned above. Note that the classical threshold $H = \frac{1}{2}$ here is the accumulation point of all the others.

EXACT SPECTRAL ASYMPTOTICS FOR FRACTIONAL PROCESSES

1.1 Introduction

1.1.1 Covariance eigenproblem

Covariance operator of a centered stochastic process $X = (X_t, t \in [0, 1])$ with covariance kernel $K(s, t) = \mathbb{E}X_s X_t$ is the self-adjoint integral operator

$$f \mapsto (Kf)(t) = \int_0^1 K(s, t)f(s)ds.$$

The associated eigenproblem consists of finding all pairs (λ, φ) satisfying the equation

$$K\varphi = \lambda\varphi. \tag{1.1}$$

For square integrable kernels, this problem is well known to have countably many solutions (λ_n, φ_n) , $n \in \mathbb{N}$, where eigenvalues λ_n are real and nonnegative and converge to zero, when put in the decreasing order, and the corresponding eigenfunctions φ_n form a complete orthonormal basis in $L^2(0, 1)$.

This fact has numerous applications in stochastic processes: the Karhunen-Loève series expansion [3], equivalence and orthogonality of Gaussian measures [48], asymptotics of the small ball probabilities [31], sampling from heavy tailed distributions [55], non-central limit theorems [30] are only a few problems to mention. However the eigenvalues and eigenfunctions are notoriously hard to find explicitly.

Even though spectral decomposition is one of the earliest and most useful tools in the theory of stochastic processes, eigenproblems are rarely tractable and only a few are known to have reasonably explicit solutions. One general solution technique is reduction to linear differential equations. It is applicable whenever the covariance

operator can be identified with the Green function of a differential operator. In this case the two share the same eigenstructure, which is typically more accessible for the latter through the generalized Sturm–Liouville theory (see [39], [42, 43]).

The simplest example is the Brownian motion with $K(s, t) = s \wedge t$, for which (1.1) readily reduces to the boundary value problem for the simple o.d.e

$$\lambda\varphi''(t) + \varphi(t) = 0 \tag{1.2}$$

subject to $\varphi(0) = 0$ and $\varphi'(1) = 0$. The explicit solution yields familiar formulas:

$$\lambda_n = \frac{1}{\nu_n^2} \quad \text{and} \quad \varphi_n(t) = \sqrt{2} \sin \nu_n t, \tag{1.3}$$

where $\nu_n = \pi n - \pi/2$. Essentially the same method works for a whole class of processes, which derive from the Brownian motion through linear transformations, including the integrated Brownian motion, the Brownian bridge, the Ornstein–Uhlenbeck process and others. However, this approach does not apply to covariance operators with a more complicated structure, including fractional processes.

1.1.2 Exact spectral asymptotics for fractional Brownian motion

The *fractional* Brownian motion (f.B.m.) is the centered Gaussian process $B^H = (B_t^H, t \in [0, 1])$ with covariance function

$$K(s, t) = \frac{1}{2} \left(s^{2H} + t^{2H} - |s - t|^{2H} \right),$$

where $H \in (0, 1)$ is its Hurst exponent and $H = \frac{1}{2}$ corresponds to the standard Brownian motion. The f.B.m. is the only H -self similar Gaussian process with stationary increments.

Unlike the standard Brownian motion, the f.B.m. is neither a semimartingale nor a Markov process for any value of $H \neq \frac{1}{2}$. For $H > \frac{1}{2}$ its increments are positively correlated and have long range dependence

$$\sum_{n=1}^{\infty} \mathbb{E} B_1^H (B_n^H - B_{n-1}^H) = \infty.$$

that makes the f.B.m. useful in a variety of applications, see e.g. [46].

In particular, the diversity of properties makes the fBm useful in modeling (see

e.g. [4]), various aspects of the related theory and applications can be found in [16], [6], [37], [45], [52].

Despite the fact that the f.B.m. has been extensively studied since its introduction in [35], its eigenstructure remained elusive for quite a while, see [36]. The exact first order asymptotics of the eigenvalues was obtained only a decade and a half ago in [7,8] (see also [33], [40]) and this result remained state of the art until recently, when a more detailed asymptotic picture was revealed in [12]:

Theorem 1.1.

1. For $H \in (0, 1)$ the eigenvalues are given by the formula

$$\lambda_n = \sin(\pi H) \Gamma(2H + 1) \nu_n^{-2H-1} \quad n = 1, 2, \dots \quad (1.4)$$

where the sequence ν_n satisfies

$$\nu_n = \left(n - \frac{1}{2}\right)\pi - \frac{1}{2} \frac{(H - \frac{1}{2})^2}{H + \frac{1}{2}} \pi + O(n^{-1}) \quad \text{as } n \rightarrow \infty. \quad (1.5)$$

2. The corresponding unit norm eigenfunctions admit the approximation

$$\varphi_n(x) = \sqrt{2} \sin(\nu_n x + \pi \eta_H) - \frac{\sqrt{2H+1}}{\pi} \int_0^\infty \rho_0(u) \left(e^{-x\nu_n u} \frac{u - \ell_H}{\sqrt{1 + \ell_H^2}} + (-1)^n e^{-(1-x)\nu_n u} \right) du + n^{-1} r_n(x), \quad (1.6)$$

where the residual $r_n(x)$ is bounded by a constant, depending only on H , and $\rho_0(u)$ is an explicit function (see (1.84) below) and the constants

$$\eta_H = \frac{H - \frac{3}{2}}{4} \frac{H - \frac{1}{2}}{H + \frac{1}{2}} \quad \text{and} \quad \ell_H = \tan\left(\frac{\pi}{2} \frac{H - \frac{1}{2}}{H + \frac{1}{2}}\right).$$

3. The eigenfunctions satisfy

$$\varphi_n(1) = (-1)^n \sqrt{2H+1} (1 + O(n^{-1})) \quad \text{and} \quad \int_0^1 \varphi_n(x) dx = -\sqrt{\frac{2H+1}{1 + \ell_H^2}} \nu_n^{-1} \quad (1.7)$$

Formulas (1.4)-(1.5) furnish asymptotic approximation of the eigenvalues, accurate up to the second order with an estimate for residual. Asymptotics (1.6) reveals that

the eigenfunctions comprise of the oscillatory term, similar to that in (1.3), and the boundary layer, whose contribution persists only near the endpoints of the interval. The boundary layer vanishes in the standard Brownian case $H = \frac{1}{2}$. Expressions (1.7) give the exact asymptotics for two particular linear functionals of the eigenfunctions, which turn out to be useful in applications discussed in [12]. It is possible to derive explicit formulas for other functionals of interest, either directly, using approximation (1.6), or through intermediate steps of the proof.

Theorem 1.1 is proved in [12] by a technique, based on analytic properties of the Laplace transform. It reduces the eigenproblem to a certain integro-algebraic system of equations, which turns out to be more amenable to asymptotic analysis. It would be natural to expect that, just as in the standard Brownian case, the same method applies to processes, which derive from the f.B.m by means of linear transformations. While ultimately this is indeed the case, such extension is far from being straightforward and its implementation faces a whole new level of complexity.

In this chapter we consider the following processes, related to the fractional Brownian motion:

- the Ornstein-Uhlenbeck type process, generated by linear stochastic equation driven by fractional noise;
- the integrated fractional Brownian motion;
- the mixture of fractional Brownian motion with an independent standard Brownian motion.

1.1.3 Frequently used notations

For numerical sequences a_n and b_n , we write $a_n \propto b_n$ if $a_n = Cb_n$ with a constant $C \neq 0$ and $a_n \sim b_n$ and $a_n \simeq b_n$ if $a_n \propto b_n(1 + o(1))$ and $a_n = b_n(1 + o(1))$ respectively as $n \rightarrow \infty$. Similarly, $f(x) \propto g(x)$ stands for the equality $f(x) = Cg(x)$ with a constant $C \neq 0$, etc.

Our main reference for tools from complex analysis is the text [18], where the particular form of the Riemann boundary value problem used below is detailed in §43. Another reference on the subject is the classic book [38]. Unless stated otherwise, standard principle branches of the multivalued complex functions will be used. We will frequently work with functions, sectionally holomorphic on the complex plane cut along

the real line. For such a function Ψ , we denote by $\Psi^+(t)$ and $\Psi^-(t)$ the limits of $\Psi(z)$ as z approaches the point t on the real line from above and below respectively:

$$\Psi^\pm(t) := \lim_{z \rightarrow t^\pm} \Psi(z).$$

We will frequently use the Sokhotski–Plemelj formula, which asserts that the Cauchy-type integral

$$\Phi(z) := \frac{1}{2\pi i} \int_0^\infty \frac{\phi(\tau)}{\tau - z} d\tau$$

of Hölder continuous function ϕ on $\mathbb{R}_{>0}$, possibly with integrable singularities at the origin and infinity, defines a function, analytic on the cut plane $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$, whose limits across the real axis satisfy

$$\Phi^\pm(t) = \frac{1}{2\pi i} \int_0^\infty \frac{\phi(\tau)}{\tau - z} d\tau \pm \frac{1}{2} \phi(t), \quad t \in \mathbb{R}_{>0},$$

where the dash integral stands for the Cauchy principle value. In particular, $\Phi(z)$ is a solution of the Riemann boundary value problem $\Phi^+(t) - \Phi^-(t) = \phi(t)$, $t \in \mathbb{R}_{>0}$, which vanishes at infinity. This fact is the main building block in all boundary problems to be encountered in this paper.

1.2 Main results

1.2.1 Fractional Ornstein-Uhlenbeck process

In stochastic analysis the Ornstein–Uhlenbeck process $X = (X_t, t \in [0, 1])$ can be derived from the standard Brownian motion $B = (B_t, t \in [0, 1])$ as the solution of the Langevin equation

$$X_t = X_0 + \beta \int_0^t X_s ds + B_t, \quad (1.8)$$

where $\beta \in \mathbb{R}$ is the drift parameter and $X_0 \sim N(0, \sigma^2)$ is the initial condition, independent of B . A non-trivial initial condition introduces a rank one perturbation to the covariance operator, which is inessential for our purposes (see [43]), and hereafter we will set $X_0 = 0$ for simplicity. In this case, the covariance function of X is given by the formula

$$K(s, t) = \int_0^{t \wedge s} e^{\beta(t-\tau)} e^{\beta(s-\tau)} d\tau.$$

Eigenproblem (1.1) with this kernel reduces to a boundary value problem similar to (1.2), whose explicit solution yields

$$\lambda_n = \frac{1}{\nu_n^2 + \beta^2} \quad \text{and} \quad \varphi_n(t) \propto \sqrt{2} \sin \nu_n t, \quad (1.9)$$

where $\nu_n = \pi n - \pi/2 + O(n^{-1})$ is the increasing sequence of positive roots of equation

$$\nu/\beta = \tan \nu.$$

In the fractional setting, the Ornstein–Uhlenbeck process can be defined in a number of nonequivalent ways [11], and here we consider solution of the Langevin equation (1.8), driven by the f.B.m. The covariance function is given in this case by the formula (see, e.g., [46]):

$$K_\beta(s, t) = \int_0^t e^{\beta(t-v)} \frac{d}{dv} \int_0^s H |v - u|^{2H-1} \text{sign}(v - u) e^{\beta(s-u)} dudv. \quad (1.10)$$

Note that this kernel satisfies the scaling property

$$K_\beta(sT, tT) = T^{2H} K_{\beta T}(s, t), \quad s, t \in [0, 1] \quad T > 0 \quad (1.11)$$

and therefore asymptotic approximation of solutions to the eigenproblem on an arbitrary interval $[0, T]$ can be obtained from that on the unit interval. For $H > \frac{1}{2}$ integration and derivative in (1.10) are interchangeable and it simplifies to

$$K_\beta(s, t) = \int_0^t \int_0^s e^{\beta(t-v)} e^{\beta(s-u)} H(2H - 1) |v - u|^{2H-2} dudv. \quad (1.12)$$

The fractional Ornstein–Uhlenbeck process of this type inherits long-range dependence property from the f.B.m., see [11].

The following result generalizes (1.9) beyond the standard Brownian case $H = \frac{1}{2}$:

Theorem 1.2. *For any $H \in (0, 1)$ the eigenvalues of covariance operator (1.10) satisfy*

$$\lambda_n = \sin(\pi H) \Gamma(2H + 1) \frac{\nu_n^{1-2H}}{\nu_n^2 + \beta^2}, \quad n = 1, 2, \dots \quad (1.13)$$

where ν_n is the sequence with asymptotics (1.5). The unit norm eigenfunctions φ_n admit approximation (1.6) and satisfy (1.7).

This result exhibits a curious dependence separation of the spectral asymptotics on its parameters. The drift parameter β enters eigenvalues formula (1.13) only in the dominator, while all of its other ingredients depend solely on the Hurst parameter H . This includes ν_n , which turns out to be the same as for the f.B.m., at least up to the second asymptotic term. Moreover, the eigenfunctions do not depend on β in the first approximation.

1.2.2 Integrated fractional Brownian motion

The first integral of the Brownian motion $X_t = \int_0^t B_s ds$ is the centered process with covariance function

$$K(s, t) = \int_0^s \int_0^t (u \wedge v) dudv.$$

For covariance operator with this kernel, eigenproblem (1.1) also reduces to a simple differential equation, whose explicit solution yields

$$\lambda_n = \frac{1}{\nu_n^4}, \quad n = 1, 2, \dots$$

with $\nu_n = \pi n - \pi/2 + O(e^{-n})$, being the increasing sequence of positive roots of equation

$$\cos \nu \cosh \nu + 1 = 0.$$

The corresponding normalized eigenfunctions satisfy

$$\varphi_n(t) \propto \frac{\cos \nu_n + \cosh \nu_n}{\sin \nu_n + \sinh \nu_n} (\sinh \nu_n t - \sin \nu_n t) - (\cosh \nu_n t - \cos \nu_n t),$$

where proportionality symbol \propto stands for equality up to a multiplicative factor asymptotic to 1, which normalizes φ_n to unit $L^2(0, 1)$ norm.

The eigenstructure of integrated processes was comprehensively studied in [19], [39]. However none of the approaches suggested so far applies to the integrated f.B.m. $X_t := \int_0^t B_s^H ds$. Covariance function of this process

$$K(s, t) = \int_0^t \int_0^s \frac{1}{2} (u^{2H} + v^{2H} - |v - u|^{2H}) dudv, \quad (1.14)$$

satisfies the scaling property

$$K(sT, tT) = T^{2H+2}K(s, t), \quad s, t \in [0, 1], \quad T > 0,$$

and hence no generality is lost if eigenproblem (1.1) is considered on the unit interval. The following result details the corresponding asymptotic spectral structure:

Theorem 1.3.

1. The eigenvalues of covariance operator with kernel (1.14) satisfy

$$\lambda_n = \sin(\pi H)\Gamma(2H + 1)\nu_n^{-2H-3} \quad n = 1, 2, \dots \quad (1.15)$$

where

$$\nu_n = \pi\left(n - \frac{1}{2}\right) + \frac{1-2H}{4}\pi + \frac{H - \frac{1}{2}}{H + \frac{3}{2}}\pi + O(n^{-1}), \quad n \rightarrow \infty. \quad (1.16)$$

2. The corresponding unit norm eigenfunctions admit the approximation

$$\varphi_n(x) = \varphi_n^{(1)}(x) + \varphi_n^{(2)}(x) + \varphi_n^{(3)}(x) + n^{-1}r_n(x)$$

where residual $r_n(x)$ is bounded uniformly in both $n \in \mathbb{N}$ and $x \in [0, 1]$ and

a. the oscillatory term is given by

$$\varphi_n^{(1)}(x) = \sqrt{2} \cos\left(\nu_n x + \frac{2H+1}{8}\pi - \frac{H - \frac{1}{2}}{H + \frac{3}{2}}\pi\right);$$

b. the polynomial boundary layer term is given by

$$\varphi_n^{(2)}(x) = -\frac{\sqrt{2H+3}}{\pi} \int_0^\infty \rho_0(t) \left(Q_0(t)e^{-t\nu_n x} - (-1)^n Q_1(t)e^{-t\nu_n(1-x)}\right) dt,$$

where function $\rho_0(t)$ and polynomials $Q_0(t)$ and $Q_1(t)$ are given by explicit expressions (1.132)-(1.133) for $H < \frac{1}{2}$ and (1.181)-(1.182) for $H > \frac{1}{2}$;

c. the exponential boundary layer term vanishes for $H \leq \frac{1}{2}$ and otherwise is given by

$$\varphi_n^{(3)}(x) = C_0 e^{-c\nu_n x} \cos\left(s\nu_n x + \varkappa_0\right) + C_1 e^{-c\nu_n(1-x)} \cos\left(s\nu_n(1-x) + \varkappa_1\right)$$

where amplitudes C_0 and C_1 and phases \varkappa_0 and \varkappa_1 are explicit constants and

$$c := \cos \frac{\pi}{2} \frac{H - \frac{1}{2}}{H + \frac{3}{2}} > 0 \quad \text{and} \quad s := \sin \frac{\pi}{2} \frac{H - \frac{1}{2}}{H + \frac{3}{2}} > 0.$$

3. The eigenfunctions satisfy

$$\varphi_n(1) = (-1)^n \sqrt{2H + 3} (1 + O(n^{-1}))$$

and

$$\int_0^1 \varphi_n(x) dx = \nu_n^{-1} \sqrt{2H + 3} C_H (1 + O(n^{-1}))$$

with explicit constant C_H , given by (1.134) for $H < \frac{1}{2}$ and (1.183) for $H > \frac{1}{2}$.

As for the f.B.m the eigenfunctions in this case also comprise of the oscillatory and boundary layer terms. The effect of the boundary layer is asymptotically negligible in the interior of the interval, and it pushes the eigenfunctions to zero at $x = 0$ and to $\pm\sqrt{2H + 3}$ at $x = 1$. However, unlike before, it exhibits multiscale behavior: for $H > \frac{1}{2}$, the two components $\varphi_n^{(2)}(x)$ and $\varphi_n^{(3)}(x)$ vanish at different, polynomial and exponential, rates respectively as $n \rightarrow \infty$.

1.2.3 Mixed fractional Brownian motion

The mixed fractional Brownian motion is the process

$$\tilde{B}_t = B_t + B_t^H, \quad t \in [0, T] \tag{1.17}$$

where $B = (B_t)$ is a Brownian motion and $B^H = (B_t^H)$ is an independent fractional Brownian motion. The covariance function of \tilde{B} is given by

$$\tilde{K}(s, t) = s \wedge t + \frac{1}{2} \left(s^{2H} + t^{2H} - |t - s|^{2H} \right), \quad s, t \in [0, 1]. \tag{1.18}$$

Theorem 1.4. 1. *The ordered sequence of eigenvalues of covariance operator of the mixed fBm (1.17) is given by*

$$\lambda_n := \frac{1}{\nu_n^2} + \frac{\sin(\pi H)\Gamma(2H+1)}{\nu_n^{2H+1}}, \quad n = 1, 2, \dots \quad (1.19)$$

where

$$\nu_n = \left(n - \frac{1}{2}\right)\pi - \frac{1}{2} \frac{(H - \frac{1}{2})^2}{H + \frac{1}{2}} \pi \mathbf{1}_{\{H < \frac{1}{2}\}} + O\left(n^{-|2H-1|}\right), \quad n \rightarrow \infty. \quad (1.20)$$

2. *The corresponding unit norm eigenfunctions admit the approximation*

$$\begin{aligned} \varphi_n(x) = & \sqrt{2} \sin(\nu_n x) \mathbf{1}_{\{H > \frac{1}{2}\}} + \sqrt{2} \sin\left(\nu_n x + \pi \eta_H\right) \mathbf{1}_{\{H < \frac{1}{2}\}} - \\ & - \frac{\sqrt{2H+1}}{\pi} \int_0^\infty \rho_0(u) \left(e^{-x\nu_n u} \frac{u - \ell_H}{\sqrt{1 + \ell_H^2}} + (-1)^n e^{-(1-x)\nu_n u} \right) du \mathbf{1}_{\{H < \frac{1}{2}\}} + \frac{r_n(x)}{n^{|2H-1|}}, \end{aligned} \quad (1.21)$$

where the residual $r_n(x)$ is bounded by a constant, depending only on H , and $\rho_0(u)$ is an explicit function and the constants

$$\eta_H = \frac{H - \frac{3}{2}}{4} \frac{H - \frac{1}{2}}{H + \frac{1}{2}} \quad \text{and} \quad \ell_H = \tan\left(\frac{\pi}{2} \frac{H - \frac{1}{2}}{H + \frac{1}{2}}\right).$$

Remark 2.

(a) Eigenvalue formulas (1.19)-(1.20) for the mixed process separate contributions of the standard and fractional components. In particular, it follows that the first order asymptotics of the eigenvalues coincides with that of the Brownian component for $H > \frac{1}{2}$ and of the fractional Brownian component for $H < \frac{1}{2}$.

(b) The proof of Theorem 1.4 suggests that the order of the residual in (1.20) is sharp, being strictly slower than in (1.5).

(c) Eigenfunctions formula (1.21) also separates contributions of the standard and fractional components. More precisely, the main terms of the asymptotics of the eigenfunctions coincides with those of the Brownian motion for $H > \frac{1}{2}$ and of the fractional Brownian motion for $H < \frac{1}{2}$.

1.3 Proofs

1.3.1 Preliminaries

The proofs follow the framework, set up in [12], which, in turn, is inspired by approach in [54]. The implementation of this program depends heavily on specificities of the kernel. Complexity of the problem is largely determined by the *structural* function $\Lambda(z)$, see (1.23) below, and particularly by the number of its zeros and scalability with respect to λ .

The processes considered in this chapter differ in both aspects from the simpler situation studied in [12] and pose entirely new challenges, which is the main focus of these results on the technical side.

Let us now briefly describe the main ideas behind the proofs. The basic object is the Laplace transform

$$\widehat{\varphi}(z) = \int_0^1 \varphi(x) e^{-zx} dx, \quad z \in \mathbb{C}, \quad (1.22)$$

where $\varphi(x)$ is a solution to (1.1). Using underlying fractional structure of the kernels under consideration, it is possible to derive an expression for $\widehat{\varphi}(z)$, which contains singularities. A typical expression has the form (see e.g. (1.27) and (1.88) below):

$$\widehat{\varphi}(z) = P(z) - Q(z) \frac{\Phi_0(z) + e^{-z}\Phi_1(-z)}{\Lambda(z)} \quad (1.23)$$

where $P(z)$ and $Q(z)$ are polynomials of a finite degree, $\Phi_0(z)$ and $\Phi_1(z)$ are certain functions and $\Lambda(z)$ is given by an explicit formula, such as e.g. (1.28).

Function $\Lambda(z)$ usually has a finite number of zeros $z_1(\lambda), \dots, z_k(\lambda)$ and a discontinuity along the real line and $\Phi_0(z)$ and $\Phi_1(z)$ are sectionally holomorphic on the complex plane, cut along the real axis. Since the Laplace transform is a priori an entire function, all singularities must be removable. Therefore the numerator in (1.23) must compensate the discontinuity and the zeros of $\Lambda(z)$, thus imposing on $\Phi_0(z)$ and $\Phi_1(z)$ two structural conditions:

$$\lim_{z \rightarrow t^+} \frac{\Phi_0(z) + e^{-z}\Phi_1(-z)}{\Lambda(z)} = \lim_{z \rightarrow t^-} \frac{\Phi_0(z) + e^{-z}\Phi_1(-z)}{\Lambda(z)}, \quad t \in \mathbb{R} \quad (1.24)$$

and

$$\Phi_0(z_j(\lambda)) + e^{-z_j(\lambda)}\Phi_1(-z_j(\lambda)) = 0, \quad j = 1, \dots, k. \quad (1.25)$$

Using various symmetries of the problem, it is possible to rewrite (1.24) as a more explicit boundary condition for the limits of $\Phi_0(z)$ and $\Phi_1(z)$ across the positive real semiaxis (see e.g. (1.56) below). Since the eigenfunction is recovered by computing inverse Laplace transform of (1.23), the original eigenproblem reduces to finding sectionally holomorphic functions $\Phi_0(z)$ and $\Phi_1(z)$ satisfying the following conditions:

- i. limits $\Phi_0^\pm(t) = \lim_{z \rightarrow t^\pm} \Phi_0(z)$ and $\Phi_1^\pm(t) = \lim_{z \rightarrow t^\pm} \Phi_1(z)$ comply with (1.24)
- ii. the behaviour $\Phi_0(z)$ and $\Phi_1(z)$ at infinity matches the a priori growth estimates, determined by polynomials $P(z)$ and $Q(z)$
- iii. the values of $\Phi_0(z)$ and $\Phi_1(z)$ satisfy the constraints imposed by (1.25)

Finding sectionally holomorphic functions subject to conditions such as (i)-(ii) is known in complex analysis as the Riemann boundary value problem. Remarkably, in our case, its unique solution can be expressed in terms of certain integral equations on the real semiaxis (see, e.g., (1.69) below). Along with (iii) this furnishes an integro-algebraic system of equations, which gives an equivalent characterisation for the eigenvalues and eigenfunctions: any solution of this system corresponds to a solution of the eigenproblem and vice versa. Despite its seeming complexity, this system turns out to be more amenable to asymptotic analysis, which is where the approximations of Theorems 1.2, 1.3 and 1.4 ultimately come from.

On the technical level, the case of the fractional Ornstein–Uhlenbeck process turns out to be less involved than the integrated f.B.m. and the mixed f.B.m. Hence we start with the proof of Theorem 1.2 in the next subsection.

1.3.2 Proof for fractional Ornstein–Uhlenbeck process

The case $H > \frac{1}{2}$

It will be convenient to define $\alpha := 2 - 2H \in (0, 1)$, so that (1.12) reads

$$K_\beta(s, t) = \int_0^t \int_0^s e^{\beta(t-v)} e^{\beta(s-u)} c_\alpha |u - v|^{-\alpha} du dv$$

where $c_\alpha = (1 - \frac{\alpha}{2})(1 - \alpha)$. The eigenproblem (1.1) takes the form

$$\int_0^1 \left(\int_0^x \int_0^y e^{\beta(x-u)} e^{\beta(y-v)} c_\alpha |u - v|^{-\alpha} dv du \right) \varphi(y) dy = \lambda \varphi(x), \quad x \in [0, 1]. \quad (1.26)$$

The Laplace transform. The starting point of the proof is an expression for the Laplace transform, derived on the basis of (1.26):

Lemma 1.1. *Let (λ, φ) be a solution of (1.26), then the Laplace transform (1.22) satisfies*

$$\widehat{\varphi}(z) = \widehat{\varphi}(-\beta) - \frac{z + \beta}{\Lambda(z)} \left(\Phi_0(z) + e^{-z} \Phi_1(-z) \right), \quad (1.27)$$

where

$$\Lambda(z) = \frac{\Gamma(\alpha)\lambda}{c_\alpha} (z^2 - \beta^2) + \int_0^\infty \frac{2t^\alpha}{t^2 - z^2} dt \quad (1.28)$$

and functions $\Phi_0(z)$ and $\Phi_1(z)$ are sectionally holomorphic on the cut plane $\mathbb{C} \setminus \mathbb{R}_{>0}$.

Proof. Taking derivative of both sides of (1.26) we get

$$\int_0^1 \left(\int_0^y e^{-\beta v} c_\alpha |x - v|^{-\alpha} dv \right) e^{\beta y} \varphi(y) dy + \beta \lambda \varphi(x) = \lambda \varphi'(x), \quad x \in [0, 1]. \quad (1.29)$$

Integrating by parts, the first term can be rewritten as

$$\int_0^1 \left(\int_0^y e^{-\beta v} c_\alpha |x - v|^{-\alpha} dv \right) e^{\beta y} \varphi(y) dy = \int_0^1 c_\alpha |x - y|^{-\alpha} \psi(y) dy,$$

where we defined

$$\psi(x) := e^{-\beta x} \int_x^1 e^{\beta r} \varphi(r) dr. \quad (1.30)$$

By definition $\psi(x)$ satisfies

$$\psi'(x) + \beta \psi(x) = -\varphi(x), \quad (1.31)$$

and therefore (1.29) is equivalent to the generalized eigenproblem:

$$\begin{aligned} \int_0^1 c_\alpha |x-y|^{-\alpha} \psi(y) dy &= \lambda \left(\beta^2 \psi(x) - \psi''(x) \right), \quad x \in [0, 1] \\ \psi(1) &= 0, \quad \psi'(0) + \beta \psi(0) = 0 \end{aligned} \quad (1.32)$$

Plugging the identity

$$|x-y|^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t|x-y|} dt, \quad \alpha \in (0, 1), \quad (1.33)$$

into (1.32) gives

$$\int_0^\infty t^{\alpha-1} u(x, t) dt = \frac{\Gamma(\alpha)\lambda}{c_\alpha} \left(\beta^2 \psi(x) - \psi''(x) \right), \quad (1.34)$$

where we defined

$$u(x, t) := \int_0^1 \psi(y) e^{-t|x-y|} dy. \quad (1.35)$$

On the other hand, integrating twice by parts gives

$$\begin{aligned} \widehat{\psi}''(z) &= \int_0^1 \psi''(x) e^{-zx} dx = \psi'(1)e^{-z} - \psi'(0) + z\psi(1)e^{-z} - z\psi(0) + z^2 \widehat{\psi}(z) = \\ &= \psi'(1)e^{-z} + (\beta - z)\psi(0) + z^2 \widehat{\psi}(z), \end{aligned} \quad (1.36)$$

where the last equality holds due to the boundary conditions in (1.32). Taking the Laplace transform of (1.34) and plugging (1.36) we get

$$\int_0^\infty t^{\alpha-1} \widehat{u}(z, t) dt = \frac{\Gamma(\alpha)\lambda}{c_\alpha} \left((\beta^2 - z^2) \widehat{\psi}(z) - \psi'(1)e^{-z} - (\beta - z)\psi(0) \right). \quad (1.37)$$

A different expression for $\widehat{u}(z, t)$ can be obtained, using definition (1.35) and taking two derivatives

$$u''(x, t) = t^2 u(x, t) - 2t\psi(x), \quad (1.38)$$

subject to the boundary conditions

$$\begin{aligned} u'(0, t) &= tu(0, t) \\ u'(1, t) &= -tu(1, t). \end{aligned} \quad (1.39)$$

Integrating twice by parts gives

$$\begin{aligned} \hat{u}''(z, t) &= \int_0^1 u''(x, t)e^{-zx} dx = u'(1, t)e^{-z} - u'(0, t) + z\hat{u}'(z, t) = \\ &= (z - t)e^{-z}u(1, t) - (z + t)u(0, t) + z^2\hat{u}(z, t), \end{aligned} \quad (1.40)$$

where we used (1.39). Taking the Laplace transform of (1.38) and plugging (1.40) we get

$$\hat{u}(z, t) = \frac{1}{z - t}u(0, t) - \frac{1}{z + t}u(1, t)e^{-z} - \frac{2t}{z^2 - t^2}\hat{\psi}(z). \quad (1.41)$$

Now combining (1.37) and (1.41) and rearranging we obtain

$$\hat{\psi}(z) = \frac{1}{\Lambda(z)} \left(\Phi_0(z) + e^{-z}\Phi_1(-z) \right)$$

where $\Lambda(z)$ is defined in (1.28) and

$$\begin{aligned} \Phi_0(z) &:= -\frac{\Gamma(\alpha)\lambda}{c_\alpha}(\beta - z)\psi(0) + \int_0^\infty \frac{t^{\alpha-1}}{t - z}u(0, t)dt \\ \Phi_1(z) &:= -\frac{\Gamma(\alpha)\lambda}{c_\alpha}\psi'(1) + \int_0^\infty \frac{t^{\alpha-1}}{t - z}u(1, t)dt \end{aligned} \quad (1.42)$$

Since $\psi(1) = 0$, we have

$$\hat{\psi}'(z) = -\psi(0) + z\hat{\psi}(z)$$

and, on the other hand, by (1.31)

$$\hat{\psi}'(z) + \beta\hat{\psi}(z) = -\hat{\varphi}(z).$$

Combining these two expressions gives

$$\hat{\varphi}(z) = \psi(0) - (z + \beta)\hat{\psi}(z)$$

and, in turn, the expression claimed in (1.27).

□

The following lemma details the structure of $\Lambda(z)$ and summarizes some of its properties, to be used later in the proofs:

Lemma 1.2.

a) $\Lambda(z)$ admits the expression

$$\Lambda(z) = \frac{\Gamma(\alpha)\lambda}{c_\alpha}(z^2 - \beta^2) + z^{\alpha-1} \frac{\pi}{\cos \frac{\pi}{2}\alpha} \begin{cases} e^{\frac{1-\alpha}{2}\pi i} & \arg(z) \in (0, \pi) \\ e^{-\frac{1-\alpha}{2}\pi i} & \arg(z) \in (-\pi, 0) \end{cases} \quad (1.43)$$

and has two zeros at $\pm z_0 = \pm i\nu$ with $\nu > 0$ satisfying the equation

$$\lambda = \frac{c_\alpha}{\Gamma(\alpha)} \frac{\pi}{\cos \frac{\pi}{2}\alpha} \frac{\nu^{\alpha-1}}{\beta^2 + \nu^2}. \quad (1.44)$$

b) The limits $\Lambda^\pm(t) = \lim_{z \rightarrow t^\pm} \Lambda(z)$ across the real line are given by

$$\Lambda^\pm(t) = \frac{\Gamma(\alpha)\lambda}{c_\alpha}(t^2 - \beta^2) + |t|^{\alpha-1} \frac{\pi}{\cos \frac{\pi}{2}\alpha} \begin{cases} e^{\pm \frac{1-\alpha}{2}\pi i} & t > 0 \\ e^{\mp \frac{1-\alpha}{2}\pi i} & t < 0 \end{cases}$$

and satisfy the symmetries

$$\Lambda^+(t) = \overline{\Lambda^-(t)} \quad (1.45)$$

$$\frac{\Lambda^+(t)}{\Lambda^-(t)} = \frac{\Lambda^-(-t)}{\Lambda^+(-t)} \quad (1.46)$$

$$|\Lambda^+(t)| = |\Lambda^+(-t)| \quad (1.47)$$

c) The argument $\theta(t) := \arg\{\Lambda^+(t)\} \in (-\pi, \pi]$ is an odd function

$$\theta(t) = \arctan \frac{\sin \frac{1-\alpha}{2}\pi}{\frac{(t/\nu)^2 - (\beta/\nu)^2}{1 + (\beta/\nu)^2} (t/\nu)^{1-\alpha} + \cos \frac{1-\alpha}{2}\pi}, \quad t > 0 \quad (1.48)$$

continuous on $(0, \infty)$ with $\theta(0+) := \frac{1-\alpha}{2}\pi > 0$ and $\theta(\infty) := \lim_{t \rightarrow \infty} \theta(t) = 0$. For all ν large enough, the scaled function $\theta(u; \nu) := \theta(u\nu)$ satisfies the bound

$$\left| \theta(u; \nu) - \theta_0(u) \right| \leq g(u)(\beta/\nu)^2 \quad (1.49)$$

where $g(u)$ does not depend on ν , is continuous on $[0, \infty)$, $g(u) \sim u^{1-\alpha}$ as $u \rightarrow 0$ and $g(u) \sim u^{\alpha-3}$ as $u \rightarrow \infty$ and

$$\theta_0(u) := \lim_{\nu \rightarrow \infty} \theta(u\nu) = \arctan \frac{\sin \frac{1-\alpha}{2}\pi}{u^{3-\alpha} + \cos \frac{1-\alpha}{2}\pi}. \quad (1.50)$$

For any fixed $\beta \in \mathbb{R}$,

$$b_\alpha(\beta, \nu) := \frac{1}{\pi} \int_0^\infty \theta(u; \nu) du \xrightarrow{\nu \rightarrow \infty} \frac{1}{\pi} \int_0^\infty \theta_0(u) du = \tan\left(\frac{1-\alpha}{3-\alpha} \frac{\pi}{2}\right) =: b_\alpha \quad (1.51)$$

and with a constant C

$$|b_\alpha(\beta, \nu) - b_\alpha| \leq C(\beta/\nu)^2. \quad (1.52)$$

Proof.

a) Standard contour integration gives the identity

$$\int_0^\infty \frac{t^\alpha}{t^2 - z^2} dt = z^{\alpha-1} \frac{1}{2} \frac{\pi}{\cos \frac{\pi}{2}\alpha} \begin{cases} e^{\frac{1-\alpha}{2}\pi i} & \arg(z) \in (0, \pi) \\ e^{-\frac{1-\alpha}{2}\pi i} & \arg(z) \in (-\pi, 0) \end{cases} \quad (1.53)$$

and in turn the expression in (1.43).

To find the roots of $\Lambda(z)$ in the upper half plane, let $z = \nu e^{i\omega}$ with $\nu > 0$ and $\omega \in (0, \pi)$ so that the equation $\Lambda(z) = 0$ becomes

$$\kappa(\nu^2 e^{2i\omega} - \beta^2) + \nu^{\alpha-1} e^{i(\omega - \frac{\pi}{2})(\alpha-1)} = 0,$$

where we set $\kappa := \frac{\Gamma(\alpha)\lambda \cos \frac{\pi}{2}\alpha}{c_\alpha \pi}$ for brevity. This implies

$$\kappa \nu^2 \sin 2\omega + \nu^{\alpha-1} \sin(\omega - \frac{\pi}{2})(\alpha - 1) = 0.$$

For both $\omega \in (0, \frac{\pi}{2})$ and $\omega \in (\frac{\pi}{2}, \pi)$, the sines are either positive or negative simultaneously and hence the equality cannot hold. Therefore the only root in the upper half plane is $i\nu$ with ν solving the equation (1.44). By definition (1.28), all zeros of $\Lambda(z)$ must be conjugates and the only other root in the lower half plane is $-i\nu$.

b) All the claims are checked by direct calculations.

c) The function $f(u) := (u^2 - (\beta/\nu)^2)u^{1-\alpha}$, $u \in \mathbb{R}_+$ vanishes at $u = 0$ and $u = \beta/\nu$ and has a unique $\min_{u \geq 0} f(u) = -(\beta/\nu)^{3-\alpha} \frac{2}{3-\alpha} \left(\frac{1-\alpha}{3-\alpha}\right)^{\frac{1-\alpha}{2}}$. Therefore for a fixed β and all ν large enough the denominator in (1.48) is bounded away from zero uniformly in u . The rest of the claims readily follows by direct inspection (the value of β_α is computed in [12]).

□

Removal of singularities. Since the Laplace transform is a priori an entire function, both poles and the discontinuity on the real line in (1.27) must be removable. Therefore the functions $\Phi_0(z)$ and $\Phi_1(z)$ must satisfy

$$\Phi_0(\pm z_0) + e^{\mp z_0} \Phi_1(\mp z_0) = 0 \quad (1.54)$$

and

$$\lim_{z \rightarrow t^+} \frac{1}{\Lambda(z)} (e^{-z} \Phi_1(-z) + \Phi_0(z)) = \lim_{z \rightarrow t^-} \frac{1}{\Lambda(z)} (e^{-z} \Phi_1(-z) + \Phi_0(z)).$$

The latter condition gives

$$\begin{aligned} \frac{1}{\Lambda^+(t)} (\Phi_0^+(t) + e^{-t} \Phi_1(-t)) &= \frac{1}{\Lambda^-(t)} (\Phi_0^-(t) + e^{-t} \Phi_1(-t)), \quad t > 0 \\ \frac{1}{\Lambda^+(t)} (\Phi_0(t) + e^{-t} \Phi_1^-(t)) &= \frac{1}{\Lambda^-(t)} (\Phi_0(t) + e^{-t} \Phi_1^+(t)), \quad t < 0 \end{aligned}$$

which, in view of (1.46), can be rewritten as

$$\begin{aligned} \Phi_0^+(t) - \frac{\Lambda^+(t)}{\Lambda^-(t)} \Phi_0^-(t) &= e^{-t} \Phi_1(-t) \left(\frac{\Lambda^+(t)}{\Lambda^-(t)} - 1 \right) \\ \Phi_1^+(t) - \frac{\Lambda^+(t)}{\Lambda^-(t)} \Phi_1^-(t) &= e^{-t} \Phi_0(-t) \left(\frac{\Lambda^+(t)}{\Lambda^-(t)} - 1 \right) \end{aligned} \quad (1.55)$$

Further, since $\Lambda^+(t)$ and $\Lambda^-(t)$ are complex conjugates, $\Lambda^+(t)/\Lambda^-(t) = e^{2i\theta(t)}$ and

$$\frac{\Lambda^+(t)}{\Lambda^-(t)} - 1 = e^{2i\theta(t)} - 1 = 2ie^{i\theta(t)} \sin \theta(t).$$

Therefore (1.55) takes the form

$$\begin{aligned}\Phi_0^+(t) - e^{2i\theta(t)}\Phi_0^-(t) &= 2ie^{-t}e^{i\theta(t)}\sin\theta(t)\Phi_1(-t) \\ \Phi_1^+(t) - e^{2i\theta(t)}\Phi_1^-(t) &= 2ie^{-t}e^{i\theta(t)}\sin\theta(t)\Phi_0(-t)\end{aligned}\quad t > 0. \quad (1.56)$$

By definition (1.35) both $tu(0, t)$ and $tu(1, t)$ are bounded and hence the functions in (1.42) satisfy the following a priori estimates:

$$\Phi_1(z) \sim z^{\alpha-1} \quad \text{and} \quad \Phi_0(z) \sim z^{\alpha-1} \quad \text{as } z \rightarrow 0 \quad (1.57)$$

and

$$\begin{aligned}\Phi_0(z) &= 2c_2(\beta - z) + O(z^{-1}) \\ \Phi_1(z) &= 2c_1 + O(z^{-1})\end{aligned}\quad \text{as } z \rightarrow \infty \quad (1.58)$$

where we defined

$$c_1 = -\frac{1}{2} \frac{\Gamma(\alpha)\lambda}{c_\alpha} \psi'(1) \quad \text{and} \quad c_2 = -\frac{1}{2} \frac{\Gamma(\alpha)\lambda}{c_\alpha} \psi(0). \quad (1.59)$$

An equivalent formulation of the eigenproblem. The Laplace transform of any φ , which satisfies (1.26), is given by the formula (1.27), in which sectionally holomorphic functions $\Phi_0(z)$ and $\Phi_1(z)$ have the growth rates (1.58) and (1.57) and satisfy the boundary conditions (1.56) and the constraint (1.54). In this subsection we will establish one-to-one correspondence between such functions and square integrable solutions of a certain system of integral equations. This is done by the solution technique of the Riemann boundary value problems.

Let us start with the homogeneous Riemann problem of finding a function $X(z)$, sectionally holomorphic on the cut plane $\mathbb{C} \setminus \mathbb{R}_{>0}$ and satisfying the boundary condition

$$X^+(t) - e^{2i\theta(t)}X^-(t) = 0, \quad t \in \mathbb{R}_{>0}. \quad (1.60)$$

All such functions can be found by the Sokhotski–Plemelj formula

$$X(z) = z^k X_c(z) = z^k \exp\left(\frac{1}{\pi} \int_0^\infty \frac{\theta(t)}{t-z} dt\right), \quad z \in \mathbb{C} \setminus \mathbb{R}_{>0}, \quad (1.61)$$

where k is an arbitrary integer to be fixed later. Canonical part $X_c(z)$ satisfies

$$X_c(z) = 1 - z^{-1}\nu b_\alpha(\beta, \nu) + O(z^{-2}) \quad \text{as } z \rightarrow \infty, \quad (1.62)$$

where $b_\alpha(\beta, \nu)$ is defined in (1.51), and

$$X_c(z) \sim z^{\frac{\alpha-1}{2}} \quad \text{as } z \rightarrow 0. \quad (1.63)$$

Now define

$$\begin{aligned} S(z) &:= \frac{\Phi_0(z) + \Phi_1(z)}{2X(z)} \\ D(z) &:= \frac{\Phi_0(z) - \Phi_1(z)}{2X(z)} \end{aligned} \quad (1.64)$$

Combining (1.56) and (1.60) shows that these function satisfy the boundary conditions

$$\begin{aligned} S^+(t) - S^-(t) &= 2ih(t)e^{-t}S(-t) \\ D^+(t) - D^-(t) &= -2ih(t)e^{-t}D(-t) \end{aligned} \quad t > 0, \quad (1.65)$$

where

$$h(t) := e^{i\theta(t)} \sin \theta(t) \frac{X(-t)}{X^+(t)}. \quad (1.66)$$

As in [12] (see eq. (5.37)) this function satisfies the formula

$$h(t) = \exp\left(-\frac{1}{\pi} \int_0^\infty \theta'(s) \log \left| \frac{t+s}{t-s} \right| ds\right) \sin \theta(t) \quad (1.67)$$

and therefore is real valued, Holder continuous on $\mathbb{R}_{>0}$ with

$$h(0) := \sin \theta(0+) = \sin \frac{1-\alpha}{2} \pi.$$

Applying the Sokhotski–Plemelj formula to (1.65), we obtain the following representation

$$\begin{aligned} S(z) &= \frac{1}{\pi} \int_0^\infty \frac{h(t)e^{-t}}{t-z} S(-t) dt + P_S(z) \\ D(z) &= -\frac{1}{\pi} \int_0^\infty \frac{h(t)e^{-t}}{t-z} D(-t) dt + P_D(z) \end{aligned} \quad (1.68)$$

where $P_S(z)$ and $P_D(z)$ are polynomials to be chosen to match the a priori growth of $S(z)$ and $D(z)$ as $z \rightarrow \infty$. Note that (1.68) requires that $S(-t)$ and $D(-t)$ are integrable near the origin. In view of the a priori estimates (1.57) and (1.63), the choice of k in (1.61) is limited by condition $k < \frac{\alpha+1}{2}$. In fact, in what follows we will also need $S(-t)$ and $D(-t)$ to be *square* integrable, which implies further restriction $k < \frac{\alpha}{2}$. A convenient choice is $k = 0$, which we will fix hereafter, that is, set $X(z) := X_c(z)$.

Since for any numbers a, b and c

$$\frac{az + b + O(z^{-1})}{1 - cz^{-1} + O(z^{-2})} = a(z + c) + b + O(z^{-1}) \quad \text{as } z \rightarrow \infty$$

the a priori estimates (1.58) and (1.62) imply

$$\begin{aligned} S(z) &= c_2 \left(-z + \beta - \nu b_\alpha(\beta, \nu) \right) + c_1 + O(z^{-1}) \\ D(z) &= c_2 \left(-z + \beta - \nu b_\alpha(\beta, \nu) \right) - c_1 + O(z^{-1}) \end{aligned}$$

This determines the choice of the polynomials in (1.68)

$$\begin{aligned} P_S(z) &:= c_2(-z + \beta - \nu b_\alpha(\beta, \nu)) + c_1 \\ P_D(z) &:= c_2(-z + \beta - \nu b_\alpha(\beta, \nu)) - c_1, \end{aligned}$$

where constants c_1 and c_2 are defined in (1.59). If we now set $z := -t$ with $t > 0$ in (1.68), the integral equations for $S(-t)$ and $D(-t)$ are obtained:

$$\begin{aligned} S(-t) &= \frac{1}{\pi} \int_0^\infty \frac{h(s)e^{-s}}{s+t} S(-s) ds + c_2(t + \beta - \nu b_\alpha(\beta, \nu)) + c_1 \\ D(-t) &= -\frac{1}{\pi} \int_0^\infty \frac{h(s)e^{-s}}{s+t} D(-s) ds + c_2(t + \beta - \nu b_\alpha(\beta, \nu)) - c_1 \end{aligned}$$

Consider now the integral equations

$$p_j^\pm(t) = \pm \frac{1}{\pi} \int_0^\infty \frac{h_\beta(s; \nu) e^{-\nu s}}{s+t} p_j^\pm(s) ds + t^j, \quad j \in \{0, 1\} \quad (1.69)$$

where $h_\beta(u; \nu) := h(u\nu)$, $u > 0$. Below we will argue that, for all sufficiently large ν , these equations have unique solutions, such that $p_0^\pm(t) - 1$ and $p_1^\pm(t) - t$ belong to $L^2(0, \infty)$. Let us extend the domain of $p_j^\pm(z)$ to the cut plane by replacing t with $z \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ in the

right hand side of (1.69).

Since $S(-t)$ and $D(-t)$ are a priori square integrable near the origin, by linearity

$$\begin{aligned} S(z\nu) &= c_2\nu p_1^+(-z) + \left(c_2(\beta - \nu b_\alpha(\beta, \nu)) + c_1 \right) p_0^+(-z) \\ D(z\nu) &= c_2\nu p_1^-(-z) + \left(c_2(\beta - \nu b_\alpha(\beta, \nu)) - c_1 \right) p_0^-(-z) \end{aligned}$$

Consequently, if we let

$$\begin{aligned} a_\pm(z) &:= p_0^\pm(z) \pm p_0^\mp(z) \\ b_\pm(z) &:= p_1^\pm(z) \pm p_1^\mp(z) \end{aligned}$$

by definition (1.64)

$$\begin{aligned} \Phi_0(z\nu) &= c_2\nu X(z\nu) \left(b_+(-z) + (\beta/\nu - b_\alpha(\beta, \nu)) a_+(-z) \right) + c_1 X(z\nu) a_-(-z) \\ \Phi_1(z\nu) &= c_2\nu X(z\nu) \left(b_-(-z) + (\beta/\nu - b_\alpha(\beta, \nu)) a_-(-z) \right) + c_1 X(z\nu) a_+(-z) \end{aligned} \quad (1.70)$$

Setting $X_\beta(z; \nu) := X(z\nu)$ and plugging (1.54), we therefore obtain

$$c_2\nu\xi + c_1\eta = 0 \quad (1.71)$$

where

$$\begin{aligned} \xi &:= e^{i\nu/2} X_\beta(i; \nu) \left(b_+(-i) + (\beta/\nu - b_\alpha(\beta, \nu)) a_+(-i) \right) + \\ &\quad e^{-i\nu/2} X_\beta(-i; \nu) \left(b_-(i) + (\beta/\nu - b_\alpha(\beta, \nu)) a_-(i) \right) \\ \eta &:= e^{i\nu/2} X_\beta(i; \nu) a_-(-i) + e^{-i\nu/2} X_\beta(-i; \nu) a_+(i) \end{aligned} \quad (1.72)$$

Since c_1 and c_2 are real, nontrivial solutions to (1.71) are possible if and only if

$$\text{Im}\{\xi\bar{\eta}\} = 0, \quad (1.73)$$

in which case $c_1 = -c_2\nu\xi/\eta$. Plugging this into (1.70) we get

$$\begin{aligned}\Phi_0(z)/c_2\nu &= X(z)\left(b_+(-z/\nu) + (\beta/\nu - b_\alpha(\beta, \nu))a_+(-z/\nu)\right) - \frac{\xi}{\eta}X(z)a_-(-z/\nu) \\ \Phi_1(z)/c_2\nu &= X(z)\left(b_-(-z/\nu) + (\beta/\nu - b_\alpha(\beta, \nu))a_-(-z/\nu)\right) - \frac{\xi}{\eta}X(z)a_+(-z/\nu)\end{aligned}\quad (1.74)$$

To recap, we arrive at the following equivalent formulation of eigenproblem (1.26):

Lemma 1.3. *Let (p_0^\pm, p_1^\pm, ν) with $\nu > 0$ be a solution of the system, which consists of the integral equations (1.69) and the algebraic equation (1.73). Let φ be defined by the Laplace transform, given by the formula (1.27), where $\Phi_0(z)$ and $\Phi_1(z)$ are given by (1.74) and let λ be defined by (1.44). Then the pair (λ, φ) solves the eigenproblem (1.26). Conversely, any solution (λ, φ) of (1.26) defines a solution to the above integro-algebraic system.*

The following lemma determines the precise asymptotics of $X_\beta(i; \nu)$ as $\nu \rightarrow \infty$, used in the calculations to follow:

Lemma 1.4.

$$\arg\{X_\beta(i; \nu)\} = \frac{1-\alpha}{8}\pi + O(\nu^{-2}) \quad \text{and} \quad |X_\beta(i; \nu)| = \sqrt{\frac{3-\alpha}{2}} + O(\nu^{-2}), \quad \nu \rightarrow \infty.$$

Proof. The claimed constants are the argument and the absolute value of (see (1.50))

$$X_0(i) := \lim_{\nu \rightarrow \infty} X_\beta(i; \nu) = \exp\left(\frac{1}{\pi} \int_0^\infty \frac{\theta_0(u)}{u-i} du\right),$$

computed in Lemma 5.5 [12]. The estimates of the residuals follow from (1.49). \square

Properties of the integro-algebraic system. Solvability of the system, introduced in Lemma 1.3 relies on the contracting properties of the operator

$$(Af)(t) := \frac{1}{\pi} \int_0^\infty \frac{h_\beta(s; \nu)e^{-\nu s}}{s+t} f(s) ds$$

where $h_\beta(u; \nu) := h(u\nu)$ (see (1.67)):

$$h_\beta(u; \nu) = \exp\left(-\frac{1}{\pi} \int_0^\infty \theta'(v; \nu) \log\left|\frac{u+v}{u-v}\right| dv\right) \sin \theta(u; \nu). \quad (1.75)$$

Lemma 1.5. *The operator A is a contraction on $L^2(0, \infty)$ for all ν large enough. More precisely, for any $\alpha_0 \in (0, 1]$ there exists an $\varepsilon > 0$ and a constant $\nu' > 0$, such that $\|A\| \leq 1 - \varepsilon$ for all $\nu \geq \nu'$ and all $\alpha \in [\alpha_0, 1]$.*

Proof. A direct calculation shows that for all ν large enough and all $\alpha \in [\alpha_0, 1]$, the exponent in (1.75) is bounded by a continuous function $f(u)$, which does not depend neither on α nor on ν and converges to 1 as $u \rightarrow 0$ and $u \rightarrow \infty$. Consequently

$$h_\beta(u, \nu) \leq f(u) \sin \theta(u; \nu) \leq \|f\|_\infty.$$

Further, since $\theta_0(0+) = \frac{1-\alpha}{2}\pi$, the estimate (1.49) implies that on a neighbourhood of the origin we have

$$\sin \theta(u; \nu) \leq \frac{1}{2} + \frac{1}{2} \sin \frac{1-\alpha_0}{2}\pi =: 1 - 3\varepsilon$$

for all sufficiently large ν . Since $f(0) = 1$, this also guarantees that $h_\beta(u, \nu) < 1 - 2\varepsilon$ on some neighbourhood of the origin, for all large ν . But then, since $\sup_{\nu > 0} \|h_\beta\|_\infty \leq \|f\|_\infty$, a constant ν' can be chosen so that $h_\beta(u, \nu)e^{-\nu u} < 1 - \varepsilon$ for all $u > 0$ and all $\nu \geq \nu'$. The claim now follows by the same calculation as in Lemma 5.6 in [12]. □

The following estimates are the key to asymptotic analysis of the integro-algebraic system:

Lemma 1.6. *For any $\alpha_0 \in (0, 1]$ there exist constants ν' and C , such that for all $\nu \geq \nu'$ and all $\alpha \in [\alpha_0, 1]$*

$$\begin{aligned} |a_-(\pm i)| &\leq C\nu^{-1}, & |a_+(\pm i) - 2| &\leq C\nu^{-1} \\ |b_-(\pm i)| &\leq C\nu^{-2}, & |b_+(\pm i) \mp 2i| &\leq C\nu^{-2} \end{aligned}$$

and for all $\tau > 0$

$$\begin{aligned} |a_-(\tau)| &\leq C\nu^{-1}\tau^{-1}, & |a_+(\tau) - 2| &\leq C\nu^{-1}\tau^{-1} \\ |b_-(\tau)| &\leq C\nu^{-2}\tau^{-1}, & |b_+(\tau) \mp 2\tau| &\leq C\nu^{-2}\tau^{-1} \end{aligned}$$

Proof. As shown in the proof of the previous lemma, $h_\beta(u; \nu)$ is bounded by a constant, which depends only on α_0 , for all sufficiently large ν . With this estimate, the bounds are obtained as in Lemma 5.7 in [12]. □

Inversion of the Laplace transform. The following lemma derives an expression for the eigenfunctions in terms of solutions to the integro-algebraic system of equations introduced in Lemma 1.3:

Lemma 1.7. *Let (Φ_0, Φ_1, ν) satisfy the integro-algebraic system from Lemma 1.3. Then the corresponding eigenfunction φ , given by the Laplace transform (1.27), satisfies*

$$\begin{aligned} \varphi(x) = & -\nu^{3-\alpha} \frac{\cos \frac{\pi}{2} \alpha}{\pi} 2\operatorname{Re} \left\{ e^{i\nu x} \Phi_0(i\nu) \frac{1 - i(\beta/\nu)}{(\beta/\nu)^2 + 1 - \alpha + 1} \right\} + \\ & \nu^{3-\alpha} \frac{\cos \frac{\pi}{2} \alpha}{\pi} \frac{1}{\pi} \int_0^\infty \frac{\sin \theta_\beta(u; \nu)}{\gamma_\beta(u; \nu)} \left(e^{-(1-x)u\nu} \left(u + \frac{\beta}{\nu} \right) \Phi_1(-u\nu) - e^{-u\nu x} \left(u - \frac{\beta}{\nu} \right) \Phi_0(-u\nu) \right) du, \end{aligned} \quad (1.76)$$

where $\gamma_\beta(u; \nu)$ is defined in (1.80) below. Moreover,

$$\begin{aligned} \int_0^1 e^{\beta x} \varphi(x) dx &= -\nu^{3-\alpha} \frac{\cos \frac{\pi}{2} \alpha}{\pi} 2c_2 \left(1 + (\beta/\nu)^2 \right) \\ \varphi(1) &= -\nu^{3-\alpha} \frac{\cos \frac{\pi}{2} \alpha}{\pi} 2c_2 \nu \frac{\xi}{\eta} \left(1 + (\beta/\nu)^2 \right) \end{aligned} \quad (1.77)$$

Proof. Since $\widehat{\varphi}(z)$ is an entire function, the inversion of the Laplace transform (1.27) can be carried out by integration on the imaginary axis:

$$\begin{aligned} \varphi(x) &= -\frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{-iR}^{iR} \left(\frac{z + \beta}{\Lambda(z)} \Phi_0(z) + \frac{z + \beta}{\Lambda(z)} e^{-z} \Phi_1(-z) - \psi(0) \right) e^{zx} dz = \\ &= -\frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{-iR}^{iR} (f_0(z) + f_1(z)) dz \end{aligned} \quad (1.78)$$

where we defined

$$f_0(z) = e^{zx} \left((z + \beta) \frac{\Phi_0(z)}{\Lambda(z)} - \psi(0) \right) \quad \text{and} \quad f_1(z) = e^{(x-1)z} (z + \beta) \frac{\Phi_1(-z)}{\Lambda(z)}.$$

Computing the contour integral, we get

$$\int_{-i\infty}^{i\infty} (f_1(z) + f_0(z)) dz = 2\pi i \left(\text{Res}(f_0, z_0) + \text{Res}(f_0, -z_0) \right) + \int_0^\infty (f_1^+(t) - f_1^-(t)) dt + \int_0^\infty (f_0^-(-t) - f_0^+(-t)) dt. \quad (1.79)$$

Using the symmetries (1.45) and (1.47) and the definition of $\theta(t)$, we have

$$\begin{aligned} f_1^+(t) - f_1^-(t) &= -e^{(x-1)t}(t + \beta)\Phi_1(-t) \frac{2i \sin \theta(t)}{\gamma(t)} \\ + f_0^-(-t) - f_0^+(-t) &= -e^{-tx}(-t + \beta)\Phi_0(-t) \frac{2i \sin \theta(t)}{\gamma(t)} \end{aligned}$$

where $\gamma(t) = |\Lambda^+(t)|$, and hence

$$\begin{aligned} \varphi(x) &= -\text{Res}(f_0, z_0) - \text{Res}(f_0, -z_0) + \\ &+ \frac{1}{\pi} \int_0^\infty \frac{\sin \theta(t)}{\gamma(t)} \left(e^{-(1-x)t}(t + \beta)\Phi_1(-t) - e^{-tx}(t - \beta)\Phi_0(-t) \right) dt. \end{aligned}$$

The residues can be readily computed:

$$\begin{aligned} \text{Res}(f_0, z_0) &= e^{i\nu x}(i\nu + \beta) \frac{\Phi_0(i\nu)}{\Lambda'(i\nu)} = e^{i\nu x} \Phi_0(i\nu) \frac{\cos \frac{\pi}{2} \alpha}{\pi} \nu^{3-\alpha} \frac{1 - i(\beta/\nu)}{\frac{2}{(\beta/\nu)^2+1} - \alpha + 1} \\ \text{Res}(f_0, -z_0) &= e^{-i\nu x}(-i\nu + \beta) \frac{\Phi_0(-i\nu)}{\Lambda'(-i\nu)} = e^{-i\nu x} \Phi_0(-i\nu) \frac{\cos \frac{\pi}{2} \alpha}{\pi} \nu^{3-\alpha} \frac{1 + i(\beta/\nu)}{\frac{2}{(\beta/\nu)^2+1} - \alpha + 1} \end{aligned}$$

and therefore

$$\text{Res}(f_0, z_0) + \text{Res}(f_0, -z_0) = 2\nu^{3-\alpha} \frac{\cos \frac{\pi}{2} \alpha}{\pi} \text{Re} \left\{ e^{i\nu x} \Phi_0(i\nu) \frac{1 - i(\beta/\nu)}{\frac{2}{(\beta/\nu)^2+1} - \alpha + 1} \right\}.$$

Plugging this back we get (1.76), where

$$\gamma_\beta(u; \nu) = \nu^{1-\alpha} \frac{\cos \frac{\pi}{2} \alpha}{\pi} |\Lambda^+(u\nu)| = \left| \frac{u^2 - (\beta/\nu)^2}{(\beta/\nu)^2 + 1} + u^{\alpha-1} e^{\frac{1-\alpha}{2}\pi i} \right|. \quad (1.80)$$

Equations (1.77) are obtained in view of (1.59), (1.44) and (1.71). □

Asymptotic analysis. Lemma 1.3 reduces the eigenproblem (1.26) to integro–algebraic system of equations. The following lemma derives exact asymptotics for the algebraic part of its solution:

Lemma 1.8. *The integro-algebraic system of Lemma 1.3 has countably many solutions, which can be enumerated so that*

$$\nu_n = \pi \left(n + \frac{1}{2} \right) - \frac{1-\alpha}{4} \pi + \frac{1-\alpha}{3-\alpha} \frac{\pi}{2} + n^{-1} r_n(\alpha), \quad n \rightarrow \infty \quad (1.81)$$

where the residual $r_n(\alpha)$ is bounded, uniformly in $n \in \mathbb{N}$ and $\alpha \in [\alpha_0, 1]$ for any $\alpha_0 \in (0, 1]$.

Proof. The arguments parallel those of Lemma 5.9 in [12]. Plugging the estimates from Lemma 1.6 and Lemma 1.4 and the estimate (1.52) into the definition (1.72), we can write

$$\xi \bar{\eta} = 4 \frac{3-\alpha}{2} \sqrt{1+b_\alpha^2} \exp \left\{ i \left(\nu + \frac{1-\alpha}{4} \pi - \pi + \arg \{i + b_\alpha\} \right) \right\} (1 + R(\nu)),$$

where $R(\nu)$ is a function satisfying the bound $|R(\nu)| \leq C_1 \nu^{-1}$ with a constant C_1 , depending only on α_0 . Hence the equation (1.73) reads

$$\nu + \frac{1-\alpha}{4} \pi - \pi + \arg \{i + b_\alpha\} - \pi n + \arctan \frac{\operatorname{Im}\{R(\nu)\}}{1 + \operatorname{Re}\{R(\nu)\}} = 0, \quad n \in \mathbb{Z}. \quad (1.82)$$

This enumerates all possible solutions of the integro-algebraic system from Lemma 1.3. Obviously, ν is positive for all n greater than some integer. Note that at this point it is not yet clear whether there is a solution for any such n .

Existence of the unique solution can be argued for all n large enough as follows. Recall that by Lemma 1.5 the integral operator in the right hand side of equations (1.69) is contracting in $L_2(0, 1)$ for all ν large enough. A direct calculation also shows that $|R'(\nu)| \leq C_2 \nu^{-1}$ with a constant C_2 . Hence for any sufficiently large n , the system consisting of the integral equations (1.69) and the algebraic equation (1.82), has the

unique solution, given by the fixed point iterations. The asymptotics (1.81) now follows from (1.82), since $\arg\{i + b_\alpha\} = \frac{\pi}{2} - \arcsin \frac{b_\alpha}{\sqrt{1+b_\alpha^2}} = \frac{\pi}{2} - \frac{1-\alpha}{3-\alpha} \frac{\pi}{2}$.

□

The corresponding asymptotic approximation of the eigenfunctions is obtained using Lemma 1.7:

Lemma 1.9. *Under the enumeration, introduced by Lemma 1.8, the eigenfunctions admit the approximation:*

$$\begin{aligned} \varphi_n(x) = & \sqrt{2} \cos \left(\nu_n x + \frac{1-\alpha}{8} \pi + \frac{\pi}{2} - \frac{1-\alpha}{3-\alpha} \frac{\pi}{2} \right) + \\ & + \frac{\sqrt{3-\alpha}}{\pi} \int_0^\infty \rho_0(u) \left(-e^{-u\nu_n x} \frac{u-b_\alpha}{\sqrt{1+b_\alpha^2}} - (-1)^n e^{-(1-x)u\nu_n} \right) du + n^{-1} r_n(x), \end{aligned} \quad (1.83)$$

where the residual $r_n(x)$ is uniformly bounded in both $n \in \mathbb{N}$ and $x \in [0, 1]$ and

$$\rho_0(u) = \frac{\sin \theta_0(u)}{\gamma_0(u)} X_0(-u). \quad (1.84)$$

Moreover,

$$\varphi_n(1) \propto -(-1)^n \sqrt{3-\alpha} (1 + O(n^{-1})) \quad \text{and} \quad \int_0^1 e^{\beta x} \varphi_n(x) dx \propto -\sqrt{\frac{3-\alpha}{1+b_\alpha^2}} \nu_n^{-1} \quad (1.85)$$

and

$$\int_0^1 \varphi_n(x) dx \propto -\sqrt{\frac{3-\alpha}{1+b_\alpha^2}} \nu_n^{-1}. \quad (1.86)$$

Proof. Let $\gamma_0(u) := \left| u + u^{\alpha-2} e^{\frac{1-\alpha}{2} \pi i} \right|$, then by (1.80)

$$\left| \gamma_\beta(u; \nu) - u\gamma_0(u) \right| \leq 2(\beta/\nu)^2 (u^2 + 1).$$

Along with (1.49), formula (1.76) gives:

$$\varphi_n(x) \propto -\frac{2}{3-\alpha} \operatorname{Re}\left\{e^{i\nu_n x} \Phi_0(i\nu_n)\right\} + \frac{1}{\pi} \int_0^\infty \frac{\sin \theta_0(u)}{\gamma_0(u)} \left(e^{-(1-x)u\nu_n} \Phi_1(-u\nu_n) - e^{-u\nu_n x} \Phi_0(-u\nu_n)\right) du + n^{-1} r_n(x)$$

where the residual $r_n(x)$ is bounded in both $n \in \mathbb{N}$ and $x \in [0, 1]$. Now plugging the estimates from Lemma 1.6 and Lemma 1.4 and (1.52) into the expressions (1.74) and normalizing to the unit $L^2(0, 1)$ norm, as in (5.52) in [12], gives (1.83). Formulas (1.77) give (1.85) after the same normalizing.

Asymptotics (1.86) is obtained by integrating (1.76): a direct calculation shows that

$$\int_0^1 \varphi_n(x) dx = C\nu_n^{-1} \left(1 + O(\nu_n^{-1})\right) \quad n \rightarrow \infty,$$

where $C\nu_n^{-1}$ coincides with the integral of the expression in (1.83) without the residual. Since this expression does not depend on β , the constant factor C must coincide with that obtained for $\beta = 0$. In other words, the sequence of integrals $\int_0^1 \varphi_n(x) dx$ for the fractional Ornstein–Uhlenbeck process and the f.B.m. has the exactly same leading order asymptotics. The exact constant in (1.86) can therefore be taken from (5.53) in [12].

□

Enumeration alignment. The enumeration introduced in Lemma 1.8 may not coincide with the *natural* enumeration, which puts the eigenvalues into decreasing order. Note that when the expression (1.81) is plugged into (1.44) the emerging sequence of λ_n 's in our enumeration is strictly decreasing; hence starting from some index it can differ from the natural enumeration only by a finite shift. To identify this shift we can use the calibration procedure, based on continuity of the spectrum with respect to α and the known asymptotics (1.9) for the standard Ornstein–Uhlenbeck process, corresponding to $\alpha = 1$. Consequently, the formulas (1.81) and (1.83)-(1.85) should be shifted by one: replacing n with $n - 1$ and α with $2 - 2H$ the expressions claimed in Theorem 1.2 are obtained.

The case $H < \frac{1}{2}$

In this case the covariance function has the form (1.10) and the eigenproblem reads:

$$\int_0^1 \left(\int_0^x e^{\beta(x-u)} \frac{d}{du} \int_0^y e^{\beta(y-v)} C_\alpha |u-v|^{1-\alpha} \text{sign}(u-v) dv du \right) \varphi(y) dy = \lambda \varphi(x),$$

where $C_\alpha := 1 - \frac{\alpha}{2}$. Taking the derivative of both sides gives

$$\int_0^1 \left(\frac{d}{dx} \int_0^y e^{\beta(y-v)} C_\alpha |x-v|^{1-\alpha} \text{sign}(x-v) dv \right) \varphi(y) dy + \beta \lambda \varphi(x) = \lambda \varphi'(x).$$

This can be rewritten as

$$-\frac{d}{dx} \int_0^1 \frac{d}{dy} \left(\int_y^1 e^{\beta r} \varphi(r) dr \right) \left(\int_0^y e^{-\beta v} C_\alpha |x-v|^{1-\alpha} \text{sign}(x-v) dv \right) dy + \beta \lambda \varphi(x) = \lambda \varphi'(x),$$

and integrating by parts we get

$$\frac{d}{dx} \int_0^1 C_\alpha |x-y|^{1-\alpha} \text{sign}(x-y) \psi(y) dy + \beta \lambda \varphi(x) = \lambda \varphi'(x),$$

where $\psi(x)$ is defined as in (1.30). Plugging in the identity (1.31) we obtain the generalized eigenproblem (cf. (1.32)):

$$\begin{aligned} \frac{d}{dx} \int_0^1 C_\alpha |x-y|^{1-\alpha} \text{sign}(x-y) \psi(y) dy &= \lambda \left(\beta^2 \psi(x) - \psi''(x) \right), \quad x \in [0, 1] \\ \psi(1) &= 0, \quad \psi'(0) + \beta \psi(0) = 0 \end{aligned}$$

From here on, the proof proceeds as in the case $H > \frac{1}{2}$.

1.3.3 Proof for integrated fractional Brownian motion

For the integrated f.B.m. with covariance function (1.14), eigenproblem (1.1) reads

$$\int_0^1 \left(\int_0^y \int_0^x K(u, v) du dv \right) \varphi(y) dy = \lambda \varphi(x) \quad x \in [0, 1], \quad (1.87)$$

where $K(u, v) := \frac{1}{2}(u^{2-\alpha} + v^{2-\alpha} - |v - u|^{2-\alpha})$ and $\alpha := 2 - 2H \in (0, 2)$.

The case $H < \frac{1}{2}$

The Laplace transform. Our starting point is again a suitable expression for the Laplace transform:

Lemma 1.10. *Let (λ, φ) be a solution of (1.87), then the Laplace transform of φ satisfies*

$$\widehat{\varphi}(z) = \widehat{\varphi}(0) + z \frac{d}{dz} \widehat{\varphi}(z) \Big|_{z=0} - \frac{e^{-z} \Phi_1(-z) + \Phi_0(z)}{\Lambda(z)} \quad (1.88)$$

where functions $\Phi_0(z)$ and $\Phi_1(z)$ are sectionally holomorphic on the cut plane $\mathbb{C} \setminus \mathbb{R}_{>0}$ and

$$\Lambda(z) := \frac{\lambda \Gamma(\alpha - 1)}{1 - \frac{\alpha}{2}} z^2 - \int_0^\infty \frac{2t^{\alpha-2}}{z^2 - t^2}. \quad (1.89)$$

Proof. Taking derivative of (1.87) and changing the order of integration we get

$$\int_0^1 K(x, y) \int_y^1 \varphi(u) du dy = \lambda \varphi'(x), \quad x \in [0, 1]. \quad (1.90)$$

Define $\psi(x) = \int_x^1 \int_y^1 \varphi(u) du dy$, then integration by parts gives

$$\begin{aligned} \int_0^1 K(x, y) \int_y^1 \varphi(u) du dy &= - \int_0^1 K(x, y) \psi'(y) dy = \\ &= \int_0^1 \left(1 - \frac{\alpha}{2}\right) (y^{1-\alpha} + \text{sign}(x - y) |x - y|^{1-\alpha}) \psi(y) dy. \end{aligned}$$

Since $\varphi(x) = \psi''(x)$ equation (1.90) reads

$$\left(1 - \frac{\alpha}{2}\right) \int_0^1 \left(y^{1-\alpha} + \text{sign}(x - y) |x - y|^{1-\alpha}\right) \psi(y) dy = \lambda \psi^{(3)}(x), \quad x \in [0, 1],$$

and, taking another derivative we arrive at the generalized eigenproblem

$$\begin{aligned} \left(1 - \frac{\alpha}{2}\right) \frac{d}{dx} \int_0^1 \text{sign}(x-y) |x-y|^{1-\alpha} \psi(y) dy &= \lambda \psi^{(4)}(x), \quad x \in [0, 1] \\ \psi(1) = 0, \quad \psi'(1) = 0, \quad \psi''(0) = 0, \quad \psi^{(3)}(0) = 0. \end{aligned} \quad (1.91)$$

Using the identity (1.33), with α replaced by $\alpha - 1 \in (0, 1)$, the expression in the left hand side can be rewritten in the form:

$$\begin{aligned} &\frac{d}{dx} \int_0^1 \text{sign}(x-y) |x-y|^{1-\alpha} \psi(y) dy = \\ &= \frac{1}{\Gamma(\alpha-1)} \frac{d}{dx} \int_0^\infty t^{\alpha-2} \left(\int_0^1 \text{sign}(x-y) e^{-t|x-y|} \psi(y) dy \right) dt \end{aligned}$$

If we now define

$$\begin{aligned} u(x, t) &:= \int_0^1 \text{sign}(x-y) e^{-t|x-y|} \psi(y) dy \\ u_0(x) &:= \int_0^\infty t^{\alpha-2} u(x, t) dt \end{aligned} \quad (1.92)$$

equation (1.91) becomes

$$u'_0(x) = \frac{\lambda \Gamma(\alpha-1)}{1 - \frac{\alpha}{2}} \psi^{(4)}(x), \quad x \in [0, 1]. \quad (1.93)$$

Taking the Laplace transform and plugging the boundary conditions from (1.91) we get

$$\begin{aligned} \widehat{u}'_0(z) &= \frac{\lambda \Gamma(\alpha-1)}{1 - \frac{\alpha}{2}} \int_0^1 e^{-zx} \psi^{(4)}(x) dx = \\ &= \frac{\lambda \Gamma(\alpha-1)}{1 - \frac{\alpha}{2}} \left(e^{-z} \psi^{(3)}(1) + z e^{-z} \psi''(1) - z^2 \psi'(0) - z^3 \psi(0) + z^4 \widehat{\psi}(z) \right). \end{aligned} \quad (1.94)$$

Another expression for $\widehat{u}'_0(z)$ can be obtained, using definitions (1.92) directly. Taking two derivatives of the expression for $u(x, t)$ gives the equation

$$u''(x, t) = 2\psi'(x) + t^2 u(x, t) \quad (1.95)$$

with the boundary conditions

$$\begin{aligned} u'(0, t) &= tu(0, t) + 2\psi(0) \\ u'(1, t) &= -tu(1, t). \end{aligned}$$

Integrating by parts twice and plugging these conditions yields

$$\begin{aligned} \widehat{u}''(z, t) &= \int_0^1 u''(x, t)e^{-zx} dx = u'(1, t)e^{-z} - u'(0, t) + zu(1, t)e^{-z} - zu(0, t) + z^2\widehat{u}(z, t) = \\ &= (z - t)u(1, t)e^{-z} - (z + t)u(0, t) - 2\psi(0) + z^2\widehat{u}(z, t). \end{aligned}$$

Combining this with the Laplace transform of (1.95) gives

$$\widehat{u}(z, t) = \frac{2z}{z^2 - t^2}\widehat{\psi}(z) - \frac{1}{z + t}u(1, t)e^{-z} + \frac{1}{z - t}u(0, t).$$

Multiplying by $t^{\alpha-2}$ and integrating we obtain

$$\widehat{u}_0(z) = \int_0^\infty \frac{t^{\alpha-2}}{z - t}u(0, t)dt - e^{-z} \int_0^\infty \frac{t^{\alpha-2}}{z + t}u(1, t)dt + z\widehat{\psi}(z) \int_0^\infty \frac{2t^{2-\alpha}}{z^2 - t^2}dt,$$

and, since $\widehat{u}'_0(z) = u_0(1)e^{-z} - u_0(0) + z\widehat{u}_0(z)$,

$$\widehat{u}'_0(z) = \int_0^\infty \frac{t^{\alpha-1}}{z - t}u(0, t)dt + e^{-z} \int_0^\infty \frac{t^{\alpha-1}}{z + t}u(1, t)dt + z^2\widehat{\psi}(z) \int_0^\infty \frac{2t^{\alpha-2}}{z^2 - t^2}dt.$$

Combining this with (1.94) and rearranging we get

$$z^2\widehat{\psi}(z) = -\frac{e^{-z}\Phi_1(-z) + \Phi_0(z)}{\Lambda(z)},$$

where $\Lambda(z)$ is given in (1.89) and

$$\begin{aligned} \Phi_0(z) &:= -\frac{\lambda\Gamma(\alpha - 1)}{1 - \frac{\alpha}{2}}\psi'(0)z^2 - \frac{\lambda\Gamma(\alpha - 1)}{1 - \frac{\alpha}{2}}\psi(0)z^3 + \int_0^\infty \frac{t^{\alpha-1}}{t - z}u(0, t)dt \\ \Phi_1(z) &:= \frac{\lambda\Gamma(\alpha - 1)}{1 - \frac{\alpha}{2}}\psi^{(3)}(1) - \frac{\lambda\Gamma(\alpha - 1)}{1 - \frac{\alpha}{2}}\psi''(1)z - \int_0^\infty \frac{t^{\alpha-1}}{t - z}u(1, t)dt. \end{aligned} \tag{1.96}$$

The expression (1.88) follows since $\widehat{\varphi}(z) = \widehat{\psi}''(z) = -\psi'(0) - \psi(0)z + z^2\widehat{\psi}(z)$.

□

The next lemma details the structure of $\Lambda(z)$:

Lemma 1.11.

a) Function $\Lambda(z)$ admits the expression

$$\Lambda(z) = \frac{\lambda\Gamma(\alpha)}{|c_\alpha|}z^2 - \frac{\pi}{|\cos\frac{\pi}{2}\alpha|}z^{\alpha-3} \begin{cases} e^{\frac{1-\alpha}{2}\pi i} & \arg(z) \in (0, \pi) \\ e^{-\frac{1-\alpha}{2}\pi i} & \arg(z) \in (-\pi, 0) \end{cases} \quad (1.97)$$

where $c_\alpha = (1 - \frac{\alpha}{2})(1 - \alpha)$. It has two zeros at $\pm z_0 = \pm\nu i$ with $\nu \in \mathbb{R}_{>0}$ given by

$$\nu^{\alpha-5} = \frac{\lambda\Gamma(\alpha)}{|c_\alpha|} \frac{|\cos\frac{\pi}{2}\alpha|}{\pi}. \quad (1.98)$$

b) The limits of $\Lambda(z)$ across the real axis are given by

$$\Lambda^\pm(t) = \frac{\lambda\Gamma(\alpha)}{|c_\alpha|}t^2 \mp |t|^{\alpha-3} \frac{\pi}{|\cos\frac{\pi}{2}\alpha|} \begin{cases} e^{\frac{1-\alpha}{2}\pi i} & t > 0 \\ e^{-\frac{1-\alpha}{2}\pi i} & t < 0 \end{cases}$$

and satisfy symmetries (1.45)-(1.47).

c) The argument $\theta(t) := \arg\{\Lambda^+(t)\} \in (-\pi, \pi]$ is an odd function $\theta(-t) = -\theta(t)$

$$\theta(t) = \arctan \frac{-\sin\frac{1-\alpha}{2}\pi}{(t/\nu)^{5-\alpha} - \cos\frac{1-\alpha}{2}\pi}, \quad t > 0$$

decreasing continuously from $\theta(0+) := \frac{3-\alpha}{2}\pi$ to 0 as $t \rightarrow \infty$. Function $\theta_0(u) := \theta(u\nu)$ satisfies

$$\begin{aligned} b_0 &:= \frac{1}{\pi} \int_0^\infty \theta_0(t) dt = \frac{\sin\left(\frac{\pi}{2} \frac{3-\alpha}{5-\alpha}\right)}{\sin\frac{\pi}{5-\alpha}} \\ b_1 &:= \frac{1}{\pi} \int_0^\infty t\theta_0(t) dt = \frac{1}{2} \\ b_2 &:= \frac{1}{\pi} \int_0^\infty t^2\theta_0(t) dt = \frac{1}{3} \frac{\sin\left(\frac{3\pi}{2} \frac{3-\alpha}{5-\alpha}\right)}{\sin\frac{3\pi}{5-\alpha}} \end{aligned} \quad (1.99)$$

and consequently

$$\frac{1}{\pi} \int_0^\infty \frac{\theta(s)}{s-z} ds = -\frac{\nu b_0}{z} - \frac{\nu^2 b_1}{z^2} - \frac{\nu^3 b_2}{z^3} + O(z^{\alpha-5}), \quad z \rightarrow \infty, \quad z \in \mathbb{C} \setminus \mathbb{R}_{>0}. \quad (1.100)$$

Proof. The proof is similar to that of Lemma 1.2: identity (1.53) with α replaced by $\alpha - 2$ gives expression (1.97) and, in turn, the formulas in b) and c). By a change of variables the integrals in (1.99) reduce to

$$\frac{1}{\pi} \int_0^\infty u^k \theta_0(u) du = \frac{\left| \cos \frac{\pi}{2} \alpha \right|^{\frac{k+1}{5-\alpha}}}{\pi(k+1)} \int_0^\infty \frac{s^{\frac{k+1}{5-\alpha}}}{1 + \left(\cot \frac{3-\alpha}{2} \pi + s \right)^2} ds$$

and the formulas claimed in (1.99) are obtained by appropriate contour integration. Asymptotics (1.100) holds by virtue of the elementary formula

$$\frac{1}{s-z} = -\frac{1}{z} - \frac{s}{z^2} - \frac{s^2}{z^3} + \frac{s^3}{z^3} \frac{1}{s-z},$$

since $\theta(t) \sim t^{\alpha-5}$ as $t \rightarrow \infty$ and

$$\int_0^\infty t^k \theta(t) dt = \nu^{k+1} \int_0^\infty u^k \theta_0(u) du, \quad k = 0, 1, 2.$$

□

Removal of singularities. As for the fractional Ornstein-Uhlenbeck process, removal of singularities in (1.88) imposes the boundary condition

$$\begin{aligned} \Phi_0^+(t) - e^{2i\theta(t)} \Phi_0^-(t) &= 2ie^{-t} e^{i\theta(t)} \sin \theta(t) \Phi_1(-t) \\ \Phi_1^+(t) - e^{2i\theta(t)} \Phi_1^-(t) &= 2ie^{-t} e^{i\theta(t)} \sin \theta(t) \Phi_0(-t) \end{aligned} \quad t > 0.$$

and

$$\Phi_0(\pm z_0) + e^{\mp z_0} \Phi_1(\mp z_0) = 0. \quad (1.101)$$

It follows from (1.96) that

$$\begin{aligned} \Phi_0(z) &= 2k_2 z^2 + 2k_3 z^3 + O(z^{\alpha-2}) \\ \Phi_1(z) &= 2k_0 + 2k_1 z + O(z^{\alpha-2}) \end{aligned} \quad \text{as } z \rightarrow \infty, \quad (1.102)$$

where we defined the constants

$$\begin{aligned} k_0 &= \frac{1}{2} \frac{\lambda \Gamma(\alpha)}{|c_\alpha|} \psi^{(3)}(1), & k_1 &= -\frac{1}{2} \frac{\lambda \Gamma(\alpha)}{|c_\alpha|} \psi''(1) \\ k_2 &= -\frac{1}{2} \frac{\lambda \Gamma(\alpha)}{|c_\alpha|} \psi'(0), & k_3 &= -\frac{1}{2} \frac{\lambda \Gamma(\alpha)}{|c_\alpha|} \psi(0). \end{aligned} \quad (1.103)$$

Integrating (1.93) we get $2k_0 = u_0(1) - u_0(0)$ and hence (1.96) reads

$$\begin{aligned} \Phi_0(z) &= u_0(0) + 2k_2 z^2 + 2k_3 z^3 + z \int_0^\infty \frac{t^{\alpha-2}}{t-z} u(0, t) dt \\ \Phi_1(z) &= -u_0(0) + 2k_1 z - z \int_0^\infty \frac{t^{\alpha-2}}{t-z} u(1, t) dt \end{aligned}$$

Consequently,

$$\Phi_0(z) = u_0(0) + O(z^{\alpha-1}) \quad \text{and} \quad \Phi_1(z) = -u_0(0) + O(z^{\alpha-1}) \quad \text{as } z \rightarrow 0. \quad (1.104)$$

An equivalent formulation of the eigenproblem. The suitable solution of the homogeneous Riemann boundary value problem

$$X^+(t) - e^{2i\theta(t)} X^-(t) = 0, \quad t \in \mathbb{R}_{>0}.$$

in this case has the form

$$X(z) = z X_c(z) = z \exp\left(\frac{1}{\pi} \int_0^\infty \frac{\theta(t)}{t-z} dt\right), \quad z \in \mathbb{C} \setminus \mathbb{R}_{>0}. \quad (1.105)$$

Factor z in front of the exponential is fixed here to guarantee that functions

$$S(z) := \frac{\Phi_0(z) + \Phi_1(z)}{2X(z)} \quad \text{and} \quad D(z) := \frac{\Phi_0(z) - \Phi_1(z)}{2X(z)} \quad (1.106)$$

are square integrable at the origin, when restricted to the negative real semiaxis. This is indeed the case in view of a priori estimates (1.104) and asymptotics (1.107) of $X(z)$ at the origin, derived along with other useful properties in the following lemma:

Lemma 1.12. *Function $X(z)$ defined in (1.105) satisfies*

$$X(z) \sim z^{\frac{\alpha-1}{2}} \quad \text{as } z \rightarrow 0 \quad (1.107)$$

and

$$X(z) \simeq z - \nu b_0 + \left(\frac{1}{2}b_0^2 - b_1\right)\frac{\nu^2}{z} - \left(\frac{1}{6}b_0^3 - b_0b_1 + b_2\right)\frac{\nu^3}{z^2} \quad \text{as } z \rightarrow \infty, \quad (1.108)$$

where $b_{j,\alpha}$ are the constants defined in (1.99). Moreover, $X_0(z) := X_c(\nu z)$ satisfies

$$\arg\{X_0(i)\} = \frac{3-\alpha}{8}\pi \quad \text{and} \quad |X_0(i)| = \sqrt{\frac{5-\alpha}{2}}. \quad (1.109)$$

Proof. The growth estimate (1.107) holds since

$$X_c(z) = \exp\left(\frac{1}{\pi} \int_0^\infty \frac{\theta(t)}{t-z} dt\right) \sim z^{-\theta(0+)/\pi} \quad \text{as } z \rightarrow 0$$

and $\theta(0+) = \frac{3-\alpha}{2}\pi$ (see Lemma 1.11 (c)). The asymptotics at infinity is obtained from (1.100) and the Taylor expansion of exponential. Formulas (1.109) follow from the identities

$$\arg\{X_c(i\nu)\} = \frac{\theta_0(0+)}{4} \quad \text{and} \quad |X_c(i\nu)|^2 = \frac{|c_\alpha|}{\lambda\Gamma(\alpha)} \lim_{z \rightarrow z_0} \frac{z^4 \Lambda(z)}{z^2 - z_0^2}$$

proved in Lemma 5.5 [12].

□

Due to a priori estimates (1.102) and (1.108), functions $S(z)$ and $D(z)$ satisfy equations (c.f. (1.68))

$$\begin{aligned} S(z) &= \frac{1}{\pi} \int_0^\infty \frac{h(t)e^{-t}}{t-z} S(-t) dt + P_S(z) \\ D(z) &= -\frac{1}{\pi} \int_0^\infty \frac{h(t)e^{-t}}{t-z} D(-t) dt + P_D(z) \end{aligned}$$

with $h(t)$ defined as in (1.66) and polynomials, whose degrees do not exceed 2

$$P_S(z) = l_0 + l_1 z + l_2 z^2 \quad \text{and} \quad P_D(z) = m_0 + m_1 z + m_2 z^2.$$

By definition (1.106) and estimate (1.102)

$$S(z)X(z) = \frac{1}{2} \left(\Phi_0(z) + \Phi_1(z) \right) = k_0 + k_1 z + k_2 z^2 + k_3 z^3 + o(1), \quad z \rightarrow \infty. \quad (1.110)$$

On the other hand, (1.68) implies

$$S(z)X(z) = \left(P_S(z) + k_S z^{-1} + o(z^{-1}) \right) X(z), \quad z \rightarrow \infty, \quad (1.111)$$

where we defined

$$k_S := \lim_{z \rightarrow \infty} \frac{z}{\pi} \int_0^\infty \frac{h(t)e^{-t}}{t-z} S(-t) dt. \quad (1.112)$$

Plugging expansion (1.108) into (1.111) and matching the powers with (1.110), we obtain the relations

$$\begin{aligned} l_2 &= k_3 \\ l_1 &= k_2 + \nu k_3 b_0 \\ l_0 &= k_1 + \nu k_2 b_0 + \nu^2 k_3 \sigma_1 \\ k_S &= k_0 + \nu k_1 b_0 + \nu^2 k_2 \sigma_1 + \nu^3 k_3 \sigma_2, \end{aligned} \quad (1.113)$$

where we defined

$$\sigma_1 = \frac{1}{2} b_0^2 + b_1 \quad \text{and} \quad \sigma_2 = \frac{1}{6} b_0^3 + b_0 b_1 + b_2.$$

Analogous calculations for $D(z)X(z)$ give

$$\begin{aligned} m_2 &= k_3 \\ m_1 &= k_2 + \nu k_3 b_0 \\ m_0 &= -k_1 + k_2 \nu b_0 + \nu^2 k_3 \sigma_1 \\ k_D &= -k_0 - k_1 \nu b_0 + k_2 \nu^2 \sigma_1 + k_3 \nu^3 \sigma_2, \end{aligned} \quad (1.114)$$

with the constant

$$k_D := - \lim_{z \rightarrow \infty} \frac{z}{\pi} \int_0^\infty \frac{h(t)e^{-t}}{t-z} D(-t) dt. \quad (1.115)$$

Consider now the integral equations

$$p_j^\pm(t) = \pm \frac{1}{\pi} \int_0^\infty \frac{h_0(s)e^{-\nu s}}{s+t} p_j^\pm(s) ds + t^j, \quad t > 0, \quad j \in \{0, 1, 2\}, \quad (1.116)$$

where $h_0(s) := h(s\nu)$ with $h(s)$ being defined as in (1.66). As in Lemma 1.5 the operator in the right hand side is contracting on $L^2(0, \infty)$ for all ν large enough. Consequently these equations have unique solutions, such that functions $p_j^\pm(t) - t^j$ belong to $L^2(0, \infty)$. Since $S(-t)$ and $D(-t)$ are a priori square integrable at the origin, by linearity

$$\begin{aligned} S(z\nu) &= l_0 p_0^+(-z) - l_1 \nu p_1^+(-z) + l_2 \nu^2 p_2^+(-z) = \\ &= \left(k_1 + \nu k_2 b_0 + \nu^2 k_3 \sigma_1 \right) p_0^+(-z) - \left(\nu k_2 + \nu^2 k_3 b_0 \right) p_1^+(-z) + \nu^2 k_3 p_2^+(-z) \end{aligned} \quad (1.117)$$

and

$$\begin{aligned} D(z\nu) &= m_0 p_0^-(-z) - m_1 \nu p_1^-(-z) + m_2 \nu^2 p_2^-(-z) = \\ &= \left(-k_1 + \nu k_2 b_0 + \nu^2 k_3 \sigma_1 \right) p_0^-(-z) - \left(\nu k_2 + \nu^2 k_3 b_0 \right) p_1^-(-z) + \nu^2 k_3 p_2^-(-z) \end{aligned} \quad (1.118)$$

where we substituted (1.113)-(1.114) and extended the domain of $p_j^\pm(z)$ to the cut plane by replacing t with $z \in \mathbb{C} \setminus \mathbb{R}_{<0}$ in (1.116).

Combining the definitions of $S(z)$ and $D(z)$ with (1.117) and (1.118) we get

$$\begin{aligned} \Phi_0(\nu z)/X(\nu z) &= k_1 \xi_1(-z) + \nu k_2 \xi_2(-z) + \nu^2 k_3 \xi_3(-z) \\ \Phi_1(\nu z)/X(\nu z) &= k_1 \eta_1(-z) + \nu k_2 \eta_2(-z) + \nu^2 k_3 \eta_3(-z) \end{aligned} \quad (1.119)$$

where $a_j^\pm(z) := p_j^+(z) - p_j^-(z)$ and

$$\begin{aligned} \xi_1(z) &:= a_0^-(z) & \eta_1(z) &:= a_0^+(z) \\ \xi_2(z) &:= b_0 a_0^+(z) - a_1^+(z) & \eta_2(z) &:= b_0 a_0^-(z) - a_1^-(z) \\ \xi_3(z) &:= \sigma_1 a_0^+(z) - b_0 a_1^+(z) + a_2^+(z) & \eta_3(z) &:= \sigma_1 a_0^-(z) - b_0 a_1^-(z) + a_2^-(z). \end{aligned} \quad (1.120)$$

Now plugging these expressions into (1.101) we obtain

$$k_1 \gamma_1 + \nu k_2 \gamma_2 + \nu^2 k_3 \gamma_3 = 0 \quad (1.121)$$

with

$$\gamma_j = \eta_j(i) + e^{\phi_{\nu} i} \xi_j(-i) \quad \text{and} \quad \phi_{\nu} = \nu + 2 \arg\{X(i\nu)\}. \quad (1.122)$$

Since k_j 's are real and γ_j 's have nontrivial imaginary parts, (1.121) furnishes two equations with real coefficients. An additional third equation can be obtained as follows. By definitions (1.112) and (1.115), we have

$$\begin{aligned} k_S &= -\nu l_0 c_{S,0} + \nu^2 l_1 c_{S,1} - l_2 \nu^3 c_{S,2} \\ c_D &= -\nu m_0 c_{D,0} + \nu^2 m_1 c_{D,1} - \nu^3 m_2 c_{D,2} \end{aligned} \quad (1.123)$$

where

$$\begin{aligned} c_{S,j} &:= -\lim_{z \rightarrow \infty} \frac{z}{\pi} \int_0^{\infty} \frac{h_0(t) e^{-\nu t}}{t-z} p_j^+(t) dt = \frac{1}{\pi} \int_0^{\infty} h_0(t) e^{-\nu t} p_j^+(t) dt \\ c_{D,j} &:= \lim_{z \rightarrow \infty} \frac{z}{\pi} \int_0^{\infty} \frac{h_0(t) e^{-\nu t}}{t-z} p_j^-(t) dt = -\frac{1}{\pi} \int_0^{\infty} h_0(t) e^{-\nu t} p_j^-(t) dt. \end{aligned}$$

Plugging (1.123) into the equations (1.113) and (1.114) we obtain

$$\begin{aligned} k_0 + \nu k_1 (b_0 + c_{S,0}) + \nu^2 k_2 (\sigma_1 + b_0 c_{S,0} - c_{S,1}) + \nu^3 k_3 (\sigma_2 + \sigma_1 c_{S,0} - b_0 c_{S,1} + c_{S,2}) &= 0 \\ k_0 + \nu k_1 (b_0 + c_{D,0}) + \nu^2 k_2 (-\sigma_1 - b_0 c_{D,0} + c_{D,1}) + \nu^3 k_3 (-\sigma_2 - \sigma_1 c_{D,0} + b_0 c_{D,1} - c_{D,2}) &= 0 \end{aligned}$$

which upon subtraction yield

$$\begin{aligned} &k_1 (c_{S,0} - c_{D,0}) + \nu k_2 (2\sigma_1 + b_0 (c_{S,0} + c_{D,0}) - (c_{S,1} + c_{D,1})) + \\ &+ \nu^2 k_3 (2\sigma_2 + \sigma_1 (c_{S,0} + c_{D,0}) - b_0 (c_{S,1} + c_{D,1}) + (c_{S,2} + c_{D,2})) = 0 \end{aligned} \quad (1.124)$$

Thus (1.121) and (1.124) form a system of three linear equations for $k_1, \nu k_2, \nu^2 k_3$, whose coefficients are real valued and are functions of ν . Letting $M(\nu)$ be the matrix of coefficients, this system admits a nontrivial solution if and only if ν satisfies the algebraic equation

$$\det\{M(\nu)\} = 0. \quad (1.125)$$

In summary, we arrive at the following equivalent formulation of the eigenproblem:

Lemma 1.13. Let $(p_0^\pm, p_1^\pm, p_2^\pm, \nu)$ with $\nu > 0$ be a solution of the system, which consists of the integral equations (1.116) and the algebraic equations (1.125). Let φ be defined by the Laplace transform, given by the formula (1.88), where $\Phi_0(z)$ and $\Phi_1(z)$ are given by (1.119) and let λ be defined by (1.98). Then the pair (λ, φ) solves the eigenproblem (1.87). Conversely, any solution (λ, φ) of (1.87) defines a solution to the above integro-algebraic system.

Properties of the integro-algebraic system. As mentioned above, equations (1.116) have unique solutions, such that $p_j^\pm(t) - t^j$ belong to $L^2(0, \infty)$. Asymptotic analysis of the integro-algebraic system of Lemma 1.13 is based on the following estimates:

Lemma 1.14. For any $\alpha_0 \in (1, 2)$ there exist constants ν' and C , such that for all $\nu \geq \nu'$, $\alpha \in [1, \alpha_0]$ the following estimates hold:

$$\begin{aligned} |p_j^\pm(i) - i^j| &\leq C\nu^{-(j+1)} \\ |p_j^\pm(\tau) - \tau^j| &\leq C\nu^{-(j+1)}\tau^{-1}, \quad \tau > 0. \\ |c_{S,j}| \vee |c_{D,j}| &\leq C\nu^{-(j+1)} \end{aligned}$$

Inversion of the Laplace transform. The eigenfunctions are recovered from the solution of the integro-algebraic system by inversion of the Laplace transform:

Lemma 1.15. Let (Φ_0, Φ_1, ν) satisfy the integro-algebraic system introduced in Lemma 1.13, then the pair (λ, φ) with λ defined by the formula (1.98) and the function

$$\begin{aligned} \varphi(x) = &-\frac{1}{\nu} \frac{|c_\alpha|}{\lambda\Gamma(\alpha)} \frac{2}{5-\alpha} \operatorname{Re} \left\{ e^{i\nu x} \Phi_0(i\nu) \frac{1}{i} \right\} + \\ &+ \frac{1}{\nu} \frac{|c_\alpha|}{\lambda\Gamma(\alpha)} \frac{1}{\pi} \int_0^\infty \frac{\sin \theta_0(t)}{\gamma_0(t)} \left(e^{-t\nu(1-x)} \Phi_1(-t\nu) + e^{-t\nu x} \Phi_0(-t\nu) \right) dt \end{aligned} \quad (1.126)$$

where $\gamma_0(u) = \left| u^2 - u^{\alpha-3} e^{\frac{1-\alpha}{2}\pi i} \right|$, solves the eigenproblem (1.87) with $\alpha \in (1, 2)$.

Proof. As in the proof of Lemma 1.7 eigenfunction $\varphi(x)$ satisfies (1.78), this time with

$$f_1(z) := e^{z(x-1)} \frac{\Phi_1(-z)}{\Lambda(z)} \quad \text{and} \quad f_0(z) := e^{zx} \left(\psi'(0) + \psi(0)z + \frac{\Phi_0(z)}{\Lambda(z)} \right).$$

Equation (1.79) holds with

$$f_1^+(t) - f_1^-(t) = -e^{-t(1-x)}\Phi_1(-t)\frac{2i \sin \theta(t)}{\gamma(t)}$$

$$f_0^-(t) - f_0^+(t) = -e^{-tx}\Phi_0(-t)\frac{2i \sin \theta(t)}{\gamma(t)}$$

and $\gamma(t) = |\Lambda^+(t)|$. The residues are computed, using the explicit expression for $\Lambda(z)$ from Lemma 1.11 (a):

$$\text{Res}(f_0, z_0) = e^{i\nu x}\Phi_0(i\nu)\frac{|c_\alpha|}{\lambda\Gamma(\alpha)}\frac{1}{\nu i}\frac{1}{5-\alpha}$$

$$\text{Res}(f_0, -z_0) = e^{-i\nu x}\Phi_0(-i\nu)\frac{|c_\alpha|}{\lambda\Gamma(\alpha)}\frac{1}{-\nu i}\frac{1}{5-\alpha}$$

and hence

$$\text{Res}(f_0, z_0) + \text{Res}(f_0, -z_0) = \frac{2}{\nu}\frac{1}{5-\alpha}\frac{|c_\alpha|}{\lambda\Gamma(\alpha)}\text{Re}\left\{e^{i\nu x}\Phi_0(i\nu)\frac{1}{i}\right\}.$$

Assembling all parts together we obtain formula (1.126). □

Asymptotic analysis. The following lemma determines asymptotics of the algebraic part of solutions to the system from Lemma 1.13:

Lemma 1.16. *The integro-algebraic system, introduced in Lemma 1.13, has countably many solutions, which can be enumerated so that*

$$\nu_n = \pi(n-1) - \frac{3-\alpha}{4}\pi + \arctan \Delta_\alpha + r_n(\alpha)n^{-1} \quad \text{as } n \rightarrow \infty \quad (1.127)$$

where

$$\Delta_\alpha = \frac{\frac{1}{3}b_0^3 - b_2}{\frac{1}{4} + \frac{1}{2}b_0^2 + b_2b_0 - \frac{1}{12}b_0^4} \quad (1.128)$$

and the residual $r_n(\alpha)$ is bounded uniformly in n and $\alpha \in [1, \alpha_0]$ for any $\alpha_0 \in (1, 2)$.

Proof. Definition (1.120) and the estimates from Lemma 1.14 imply

$$\begin{aligned}
\xi_1(i) &\simeq 0 & \eta_1(i) &\simeq 2 \\
\xi_2(i) &\simeq 2b_0 - 2i & \eta_2(i) &\simeq 0 \\
\xi_3(i) &\simeq \sigma_1 2 - b_0 2i - 2 & \eta_3(i) &\simeq 0
\end{aligned}$$

where \simeq stands for equality up to $O(\nu^{-1})$ residual, uniform with respect to $\alpha \in [1, \alpha_0]$. Hence γ_j 's from (1.122) satisfy

$$\begin{aligned}
\gamma_1 &\simeq 2 \\
\gamma_2 &\simeq 2e^{\phi_\nu i}(b_0 + i) \\
\gamma_3 &\simeq 2e^{\phi_\nu i}(\sigma_1 - 1 + b_0 i)
\end{aligned}$$

where ϕ_ν is defined in (1.122) and, in view of (1.124) and (1.121) the matrix in (1.125) satisfies

$$M(\nu) \simeq 2 \begin{pmatrix} 1 & b_0 \cos \phi_\nu - \sin \phi_\nu & (\sigma_1 - 1) \cos \phi_\nu - b_0 \sin \phi_\nu \\ 0 & \cos \phi_\nu + b_0 \sin \phi_\nu & b_0 \cos \phi_\nu + (\sigma_1 - 1) \sin \phi_\nu \\ 0 & \sigma_1 & \sigma_2 \end{pmatrix} \quad (1.129)$$

Consequently

$$\det\{M(\nu)\} \simeq (\cos \phi_\nu + b_0 \sin \phi_\nu) \sigma_2 - (b_0 \cos \phi_\nu + (\sigma_1 - 1) \sin \phi_\nu) \sigma_1.$$

Hence the root of (1.125) satisfies

$$\begin{aligned}
\tan \phi_\nu &\simeq \frac{\sigma_2 - b_0 \sigma_1}{(\sigma_1 - 1) \sigma_1 - b_0 \sigma_2} = \frac{-\frac{1}{3} b_0^3 + b_2}{\frac{1}{12} b_0^4 + b_1^2 - \frac{1}{2} b_0^2 - b_1 - b_2 b_0} = \\
&= \frac{\frac{1}{3} b_0^3 - b_2}{\frac{1}{4} + \frac{1}{2} b_0^2 + b_2 b_0 - \frac{1}{12} b_0^4} =: \Delta_\alpha.
\end{aligned}$$

A direct calculation shows that the residual in this equality is differentiable with respect to ν and its derivative is less than 1 in magnitude for all ν large enough. Hence for all sufficiently large integer n the integro-algebraic system has the unique solution,

obtained through fixed-point iterations, and its algebraic part ν_n satisfies

$$\nu_n = \pi n - 2 \arg\{iX_0(i)\} + \arctan \Delta_\alpha + r_n n^{-1} \quad n \rightarrow \infty$$

where r_n is a sequence, uniformly bounded in n and $\alpha \in [1, \alpha_0]$. Asymptotics (1.127) now follows from (1.109). \square

The following lemma derives asymptotic approximation for the eigenfunctions:

Lemma 1.17. *Under the enumeration, introduced by Lemma 1.16, the eigenfunctions admit the approximation:*

$$\varphi_n(x) = \sqrt{2} \cos\left(\nu_n x + \frac{3-\alpha}{8}\pi - \arctan \Delta_\alpha\right) - \quad (1.130)$$

$$-\frac{\sqrt{5-\alpha}}{\pi} \int_0^\infty \rho_0(t) \left(Q_0(t)e^{-t\nu_n x} + (-1)^n Q_1(t)e^{-t\nu_n(1-x)}\right) dt + r_n(x)n^{-1} \quad (1.131)$$

where the residual $r_n(x)$ is uniformly bounded in both $n \in \mathbb{N}$ and $x \in [0, 1]$ and

$$\rho_0(t) = \frac{\sin \theta_0(t)}{\gamma_0(t)} t \exp\left(\frac{1}{\pi} \int_0^\infty \frac{\theta_0(s)}{s+t} ds\right) \quad (1.132)$$

and

$$Q_0(t) := \frac{\Delta_\alpha}{\sqrt{\Delta_\alpha^2 + 1}} \frac{\sigma_1}{\sigma_2 - b_0 \sigma_1} \left(\frac{\sigma_2}{\sigma_1} b_0 - \sigma_1 + \left(b_0 - \frac{\sigma_2}{\sigma_1}\right)t - t^2\right) \quad (1.133)$$

$$Q_1(t) := 1$$

Moreover,

$$\begin{aligned} \varphi_n(1) &= -(-1)^n \sqrt{5-\alpha} \left(1 + O(n^{-1})\right) \\ \int_0^1 \varphi_n(x) dx &= \nu_n^{-1} \sqrt{5-\alpha} \frac{\Delta_\alpha}{\sqrt{\Delta_\alpha^2 + 1}} \frac{\sigma_2}{\sigma_2 - b_0 \sigma_1} \left(1 + O(n^{-1})\right). \end{aligned} \quad (1.134)$$

Proof. The asymptotic structure of matrix $M(\nu)$ in (1.129) implies the following relations between the coefficients k_1 , $k_2 \nu_n$ and $k_3 \nu_n^2$:

$$\begin{aligned} k_1 &\simeq -\left(b_0 \cos \phi_{\nu_n} - \sin \phi_{\nu_n}\right) k_2 \nu_n - \left((\sigma_1 - 1) \cos \phi_{\nu_n} - b_0 \sin \phi_{\nu_n}\right) k_3 \nu_n^2 \\ k_2 \nu_n &\simeq -\frac{\sigma_2}{\sigma_1} k_3 \nu_n^2 \end{aligned} \quad (1.135)$$

Plugging these expressions and the estimates from Lemma 1.14 into (1.119) gives

$$\Phi_0(i\nu_n) \simeq 2X(i\nu_n) \left((b_0 + i)k_2\nu_n + (\sigma_1 - 1 + b_0i)k_3\nu_n^2 \right) = 2i\nu_n X_c(i\nu_n) \zeta k_3\nu_n^2 \quad (1.136)$$

where we used the equality $X(i\nu_n) = i\nu_n X_c(i\nu_n)$ and defined

$$\zeta := -b_0 \frac{\sigma_2}{\sigma_1} + \sigma_1 - 1 + i \left(b_0 - \frac{\sigma_2}{\sigma_1} \right).$$

The argument and the absolute value of this constant are given by

$$\arg\{\zeta\} = -\arctan \Delta_\alpha \quad \text{and} \quad |\zeta|^2 = \left(\frac{1}{\Delta_\alpha^2} + 1 \right) \frac{(\sigma_2 - b_0\sigma_1)^2}{\sigma_1^2}.$$

Similarly we have

$$\begin{aligned} \Phi_0(-\nu_n t) &\simeq -2\nu_n t X_c(-t\nu_n) \left(-\frac{\sigma_2}{\sigma_1}(b_0 - t) + (\sigma_1 - b_0 t + t^2) \right) k_3\nu_n^2 \\ \Phi_1(-\nu_n t) &\simeq -2\nu_n t X_c(-t\nu_n) k_1 \end{aligned} \quad (1.137)$$

where the residuals are bounded uniformly with respect to t by Lemma 1.14. Combining the equations in (1.135) and using the definition of Δ_α , we also have

$$\begin{aligned} k_1 &\simeq \left((b_0 \cos \phi_{\nu_n} - \sin \phi_{\nu_n}) \frac{\sigma_2}{\sigma_1} - ((\sigma_1 - 1) \cos \phi_{\nu_n} - b_0 \sin \phi_{\nu_n}) \right) k_3\nu_n^2 = \\ &= -(-1)^n \frac{1}{\sqrt{1 + \Delta_\alpha^2}} \left((b_0 - \Delta_\alpha) \frac{\sigma_2}{\sigma_1} - ((\sigma_1 - 1) - b_0 \Delta_\alpha) \right) k_3\nu_n^2 = -(-1)^n |\zeta| k_3\nu_n^2. \end{aligned}$$

Expression (1.130) is now obtained by plugging (1.136)-(1.137) and (1.109) into (1.126) and normalizing by the factor

$$C_n := -2\nu_n^2 k_3 \frac{|c_\alpha|}{\lambda \Gamma(\alpha)} \frac{|\zeta|}{\sqrt{5 - \alpha}}.$$

Asymptotic formulas (1.134) follow by normalizing expressions (1.103) by the same factor. \square

Enumeration alignment. The enumeration, introduced in Lemma 1.16, may differ from the *natural* enumeration, which puts all the eigenvalues into increasing order, only by a constant shift. This shift can be identified by the calibration procedure, based on continuity of the spectrum. Since for the integrated Brownian motion, corresponding to $\alpha = 1$, the sequence ν_n is asymptotic to $\pi(n - \frac{1}{2})$, the asymptotics in (1.127) should be shifted by π and the formulas in (2) and (3) of Theorem 1.3 are obtained by the corresponding adjustment of the expressions from Lemma 1.17.

The case $H > \frac{1}{2}$

The Laplace transform. For $\alpha \in (0, 1)$, corresponding to $H \in (\frac{1}{2}, 1)$, the proof is done completely differently, since as we will see below, structural function $\Lambda(z)$ in this case has more roots than before.

Lemma 1.18. *Let (λ, φ) be a solution of (1.87), then the Laplace transform of φ satisfies*

$$\widehat{\varphi}(z) = \widehat{\varphi}(0) + z \frac{d}{dz} \widehat{\varphi}(z) \Big|_{z=0} - \frac{e^{-z} \Phi_1(-z) + \Phi_0(z)}{\Lambda(z)} \quad (1.138)$$

where functions $\Phi_0(z)$ and $\Phi_1(z)$ are sectionally holomorphic on the cut plane $\mathbb{C} \setminus \mathbb{R}_{>0}$ and

$$\Lambda(z) := \frac{\lambda \Gamma(\alpha)}{c_\alpha} z^2 - \frac{1}{z^2} \int_0^\infty \frac{2t^\alpha}{t^2 - z^2} dt. \quad (1.139)$$

Proof. As in the case $H < \frac{1}{2}$, the function $\psi(x) = \int_x^1 \int_y^1 \varphi(u) du dy$, satisfies (1.91). Setting $c_\alpha = (1 - \frac{\alpha}{2})(1 - \alpha)$ and interchanging derivative and integration we arrive at the following generalized eigenproblem:

$$\begin{aligned} c_\alpha \int_0^1 |x - y|^{-\alpha} \psi(y) dy &= \lambda \psi^{(4)}(x), \quad x \in [0, 1] \\ \psi(1) = 0, \quad \psi'(1) = 0, \quad \psi''(0) = 0, \quad \psi^{(3)}(0) = 0. \end{aligned} \quad (1.140)$$

Define

$$u(x, t) = \int_0^1 e^{-t|x-y|} \psi(y) dy \quad \text{and} \quad u_0(x) = \int_0^\infty t^{\alpha-1} u(x, t) dt,$$

then, plugging the identity (1.33) into (1.140) we get

$$\frac{c_\alpha}{\Gamma(\alpha)} u_0(x) = \lambda \psi^{(4)}(x), \quad x \in [0, 1]$$

and therefore, in view of the boundary conditions in (1.140),

$$\begin{aligned}\hat{u}_0(z) &= \frac{\lambda\Gamma(\alpha)}{c_\alpha} \int_0^1 e^{-zx} \psi^{(4)}(x) dx = \\ &= \frac{\lambda\Gamma(\alpha)}{c_\alpha} \left(e^{-z} \psi^{(3)}(1) + z e^{-z} \psi''(1) - z^2 \psi'(0) - z^3 \psi(0) + z^4 \hat{\psi}(z) \right).\end{aligned}\tag{1.141}$$

An additional relation between $\hat{u}_0(z)$ and $\hat{\varphi}(z)$ is obtained as in the proof Lemma 1.1 (c.f. (1.41)):

$$\hat{u}_0(z) = \int_0^\infty \frac{t^{\alpha-1}}{z-t} u(0,t) dt - e^{-z} \int_0^\infty \frac{t^{\alpha-1}}{z+t} u(1,t) dt - \hat{\psi}(z) \int_0^\infty \frac{2t^\alpha}{z^2-t^2} dt.$$

Combining this with (1.141) and rearranging we obtain the expression

$$z^2 \hat{\psi}(z) = -\frac{e^{-z} \Phi_1(-z) + \Phi_0(z)}{\Lambda(z)},$$

where $\Lambda(z)$ is defined in (1.139) and

$$\begin{aligned}\Phi_0(z) &:= -\frac{\lambda\Gamma(\alpha)}{c_\alpha} \psi'(0) z^2 - \frac{\lambda\Gamma(\alpha)}{c_\alpha} \psi(0) z^3 + \int_0^\infty \frac{t^{\alpha-1}}{t-z} u(0,t) dt \\ \Phi_1(z) &:= \frac{\lambda\Gamma(\alpha)}{c_\alpha} \psi^{(3)}(1) - \frac{\lambda\Gamma(\alpha)}{c_\alpha} \psi''(1) z + \int_0^\infty \frac{t^{\alpha-1}}{t-z} u(1,t) dt.\end{aligned}\tag{1.142}$$

Formula (1.138) follows since

$$\hat{\varphi}(z) = \hat{\psi}''(z) = -\psi'(0) - \psi(0)z + z^2 \hat{\psi}(z).$$

□

Next lemma details the structure of $\Lambda(z)$:

Lemma 1.19.

a) $\Lambda(z)$ admits the expression

$$\Lambda(z) = \frac{\lambda\Gamma(\alpha)}{c_\alpha} z^2 - \frac{\pi}{\cos \frac{\pi}{2} \alpha} z^{\alpha-3} \begin{cases} e^{\frac{1-\alpha}{2} \pi i} & \arg(z) \in (0, \pi) \\ e^{-\frac{1-\alpha}{2} \pi i} & \arg(z) \in (-\pi, 0) \end{cases}\tag{1.143}$$

and has six zeros

$$\pm z_0 = \pm i\nu, \quad \pm z_+ = \pm \nu e^{\frac{\pi}{2} \frac{1-\alpha}{5-\alpha} i}, \quad \pm z_- = \pm \nu e^{\frac{\pi}{2} \frac{9-\alpha}{5-\alpha} i}$$

where ν is given by

$$\nu^{\alpha-5} = \frac{\lambda \Gamma(\alpha) \cos \frac{\pi}{2} \alpha}{c_\alpha \pi}. \quad (1.144)$$

b) The limits of $\Lambda(z)$ across the real line are given by

$$\Lambda^\pm(t) = \frac{\lambda \Gamma(\alpha)}{c_\alpha} t^2 \mp |t|^{\alpha-3} \frac{\pi}{\cos \frac{\pi}{2} \alpha} \begin{cases} e^{\frac{1+\alpha}{2} \pi i} & t > 0 \\ e^{-\frac{1+\alpha}{2} \pi i} & t < 0 \end{cases} \quad (1.145)$$

and satisfy the symmetries (1.45)-(1.47).

c) The argument $\theta(t) := \arg\{\Lambda^+(t)\} \in (-\pi, \pi]$ is an odd function $\theta(t) = -\theta(-t)$,

$$\theta(t) = \arctan \frac{-\sin \frac{1-\alpha}{2} \pi}{(t/\nu)^{5-\alpha} - \cos \frac{1-\alpha}{2} \pi}, \quad t > 0$$

increasing continuously from $\theta(0+) = -\frac{1+\alpha}{2} \pi$ to 0 as $t \rightarrow \infty$. The rescaled function $\theta_0(u) := \theta(u\nu)$ satisfies

$$b_{k,\alpha} := \frac{1}{\pi} \int_0^\infty u^k \theta_0(u) du = -\frac{1}{k+1} \frac{\sin\left((k+1)\frac{1+\alpha}{2}\frac{\pi}{5-\alpha}\right)}{\sin\left((k+1)\frac{\pi}{5-\alpha}\right)} \quad k = 0, 1, 2 \quad (1.146)$$

and the following asymptotics holds:

$$\frac{1}{\pi} \int_0^\infty \frac{\theta(s)}{s-z} ds = -\frac{\nu b_0}{z} - \frac{\nu^2 b_1}{z^2} - \frac{\nu^3 b_2}{z^3} + O(z^{-4}), \quad z \rightarrow \infty, \quad z \in \mathbb{C} \setminus \mathbb{R}_{>0}.$$

Proof.

a) Formula (1.143) follows from definition (1.139) and identity (1.53). Note that conjugate of any zero of $\Lambda(z)$ is also a zero, hence it is enough to locate zeros only in the upper half plane. To this end let $z = \nu e^{i\omega}$ with $\nu > 0$ and $\omega \in (0, \pi)$, then equating (1.143) to zero gives

$$\frac{\lambda \Gamma(\alpha) \cos \frac{\pi}{2} \alpha}{c_\alpha \pi} \nu^{5-\alpha} = \exp\left(\frac{1-\alpha}{2} \pi i + (\alpha-5)\omega i\right).$$

Obviously, any solution must satisfy (1.144) and

$$\frac{1-\alpha}{2}\pi + (\alpha-5)\omega = 2\pi k \quad \text{for some } k \in \mathbb{Z}.$$

The only values of k , for which

$$\omega = \frac{\pi}{2} \frac{1-\alpha-4k}{5-\alpha} \in (0, \pi),$$

are 0, -1 and -2 , corresponding to three zeros in the upper half plane:

$$z_+ = \nu e^{\frac{\pi}{2} \frac{1-\alpha}{5-\alpha} i}, \quad z_0 = \nu i, \quad z_- = \nu e^{\frac{\pi}{2} \frac{9-\alpha}{5-\alpha} i}.$$

The zeros in the lower half plane are the conjugates

$$\bar{z}_0 = -z_0, \quad \bar{z}_+ = -z_-, \quad \bar{z}_- = -z_+.$$

b) All the formulas are obtained from (1.143) by direct calculations.

c) The expression for $\theta(t)$ follows from (1.145). The integrals in (1.146) reduce to

$$\frac{1}{\pi} \int_0^\infty u^k \theta_0(u) du = -\frac{(\cos(\frac{\pi}{2}\alpha))^{\frac{k+1}{5-\alpha}}}{\pi(k+1)} \int_0^\infty \frac{s^{\frac{k+1}{5-\alpha}}}{1 + (\cot(\frac{\pi}{2}(1+\alpha)) + s)^2} ds$$

by a change of variable and the claimed formulas are obtained by appropriate contour integration.

□

Removal of singularities. Since the Laplace transform is an entire function, removal of poles in (1.138) gives

$$\begin{aligned} \Phi_0(\pm z_0) + e^{\mp z_0} \Phi_1(\mp z_0) &= 0 \\ \Phi_0(\pm z_+) + e^{\mp z_+} \Phi_1(\mp z_+) &= 0 \\ \Phi_0(\pm z_-) + e^{\mp z_-} \Phi_1(\mp z_-) &= 0 \end{aligned} \tag{1.147}$$

and removal of discontinuity on the real line yields the boundary conditions, c.f. (1.56):

$$\begin{aligned} \Phi_0^+(t) - e^{2i\theta(t)} \Phi_0^-(t) &= 2ie^{-t} e^{i\theta(t)} \sin \theta(t) \Phi_1(-t) \\ \Phi_1^+(t) - e^{2i\theta(t)} \Phi_1^-(t) &= 2ie^{-t} e^{i\theta(t)} \sin \theta(t) \Phi_0(-t) \end{aligned} \quad t > 0.$$

Since $tu(0, t)$ and $tu(1, t)$ are bounded functions, it follows from (1.142) that

$$\begin{aligned}\Phi_0(z) &= 2k_5z^2 + 2k_6z^3 + O(z^{-1}) \\ \Phi_1(z) &= 2k_3 + 2k_4z + O(z^{-1})\end{aligned}\quad \text{as } z \rightarrow \infty \quad (1.148)$$

where we defined

$$\begin{aligned}k_3 &= \frac{\lambda\Gamma(\alpha)}{2c_\alpha}\psi^{(3)}(1), & k_4 &= -\frac{\lambda\Gamma(\alpha)}{2c_\alpha}\psi''(1) \\ k_5 &= -\frac{\lambda\Gamma(\alpha)}{2c_\alpha}\psi'(0), & k_6 &= -\frac{\lambda\Gamma(\alpha)}{2c_\alpha}\psi(0)\end{aligned}\quad (1.149)$$

Also we have

$$\Phi_0(z) \sim z^{\alpha-1} \quad \text{and} \quad \Phi_1(z) \sim z^{\alpha-1} \quad \text{as } z \rightarrow 0. \quad (1.150)$$

An equivalent formulation of the eigenproblem. The appropriate solution of the homogeneous Riemann boundary value problem

$$X^+(t) - e^{2i\theta(t)}X^-(t) = 0, \quad t \in \mathbb{R}_{>0},$$

in this case is given by the Sokhotski–Plemelj formula

$$X(z) = \frac{1}{z}X_c(z) = \frac{1}{z} \exp\left(\frac{1}{\pi} \int_0^\infty \frac{\theta(t)}{t-z} dt\right), \quad z \in \mathbb{C} \setminus \mathbb{R}_{>0}. \quad (1.151)$$

Factor $1/z$ in front of the exponential is chosen to guarantee square integrability of the functions

$$\begin{aligned}S(z) &:= \frac{\Phi_0(z) + \Phi_1(z)}{2X(z)} \\ D(z) &:= \frac{\Phi_0(z) - \Phi_1(z)}{2X(z)}\end{aligned}\quad (1.152)$$

at the origin, in view of estimates (1.150) and (1.153) below:

Lemma 1.20. *The function defined in (1.151) satisfies*

$$X(z) \sim z^{\frac{\alpha-1}{2}} \quad \text{as } z \rightarrow 0 \quad (1.153)$$

and

$$X(z) = \frac{1}{z} - b_0 \frac{\nu}{z^2} + \left(\frac{1}{2}b_0^2 - b_1\right) \frac{\nu^2}{z^3} - \left(\frac{1}{6}b_0^3 - b_0b_1 + b_2\right) \frac{\nu^3}{z^4} + O(z^{-5}), \quad z \rightarrow \infty. \quad (1.154)$$

Moreover, $X_0(z) := X_c(\nu z)$ satisfies

$$\arg\{X_0(i)\} = -\frac{1+\alpha}{8}\pi \quad \text{and} \quad |X_0(i)| = \sqrt{\frac{5-\alpha}{8}} \frac{1}{\cos \frac{\pi}{2} \frac{1-\alpha}{5-\alpha}}. \quad (1.155)$$

Proof. Estimate (1.153) is valid, since $X_c(z) \sim z^{-\theta(0+)/\pi}$ with $\theta(0+) = -\frac{1+\alpha}{2}\pi$ (see Lemma 1.19 (c)). The asymptotics at infinity is obtained by (1.146) and the Taylor expansion of exponential. The formulas in (1.155) follow from the identities

$$\arg\{X_0(i)\} = \frac{\theta_0(0+)}{4} \quad \text{and} \quad |X_0(i)|^2 = \frac{c_\alpha}{\lambda\Gamma(\alpha)} \lim_{z \rightarrow z_0} \frac{z^4 \Lambda(z)}{(z^2 - z_0^2)(z^2 - z_+^2)(z^2 - z_-^2)}.$$

□

Being integrable at the origin, functions $S(z)$ and $D(z)$ satisfy, c.f. (1.68):

$$S(z) = \frac{1}{\pi} \int_0^\infty \frac{h(t)e^{-t}}{t-z} S(-t) dt + P_S(z)$$

$$D(z) = -\frac{1}{\pi} \int_0^\infty \frac{h(t)e^{-t}}{t-z} D(-t) dt + P_D(z)$$

where polynomials are chosen to match a priori growth of $S(z)$ and $D(z)$ at infinity, determined by (1.148) and (1.154):

$$P_S(z) = k_1 + l_1 z + l_2 z^2 + l_3 z^3 + l_4 z^4$$

$$P_D(z) = k_2 + m_1 z + m_2 z^2 + m_3 z^3 + m_4 z^4$$

Here k_1 and k_2 are arbitrary and the rest of the constants are related to the previously introduced quantities through matching the powers in (1.152):

$$\begin{aligned} l_4 &= k_6 & m_4 &= k_6 \\ l_3 &= k_5 + b_0 \nu k_6 & m_3 &= k_5 + b_0 \nu k_6 \\ l_2 &= k_4 + b_0 \nu k_5 + \nu^2 \sigma_1 k_6 & m_2 &= -k_4 + b_0 \nu k_5 + \nu^2 \sigma_1 k_6 \\ l_1 &= k_3 + b_0 \nu k_4 + \nu^2 \sigma_1 k_5 + \nu^3 \sigma_2 k_6 & m_1 &= -k_3 - b_0 \nu k_4 + \nu^2 \sigma_1 k_5 + \nu^3 \sigma_2 k_6 \end{aligned}$$

where we defined

$$\sigma_1 = \frac{1}{2}b_0^2 + b_1 \quad \text{and} \quad \sigma_2 = \frac{1}{6}b_0^3 + b_0b_1 + b_2.$$

Consider now the integral equations

$$p_j^\pm(t) = \pm \frac{1}{\pi} \int_0^\infty \frac{h_0(s)e^{-\nu s}}{s+t} p_j^\pm(s) ds + t^j, \quad t > 0, \quad j \in \{0, 1, 2, 3, 4\}, \quad (1.156)$$

where $h_0(s) := h(s\nu)$ with $h(s)$ being defined as in (1.66). As in Lemma 1.5, the integral operator in the right hand side is a contraction on $L^2(0, \infty)$. Consequently equations (1.156) have unique solutions, such that functions $p_j^\pm(t) - t^j$ belong to $L^2(0, \infty)$. Since $S(-t)$ and $D(-t)$ are square integrable at the origin, by linearity we have

$$\begin{aligned} S(z\nu) &= k_1 p_0^+(-z) - k_3 \nu p_1^+(-z) + \\ &\quad + k_4 \nu^2 \left(-b_0 p_1^+(-z) + p_2^+(-z) \right) + \\ &\quad + k_5 \nu^3 \left(-\sigma_1 p_1^+(-z) + b_0 p_2^+(-z) - p_3^+(-z) \right) + \\ &\quad + k_6 \nu^4 \left(-\sigma_2 p_1^+(-z) + \sigma_1 p_2^+(-z) - b_0 p_3^+(-z) + p_4^+(-z) \right) \end{aligned} \quad (1.157)$$

and

$$\begin{aligned} D(z\nu) &= k_2 p_0^-(-z) + k_3 \nu p_1^-(-z) + \\ &\quad + k_4 \nu^2 \left(b_0 p_1^-(-z) - p_2^-(-z) \right) + \\ &\quad + k_5 \nu^3 \left(-\sigma_1 p_1^-(-z) + b_0 p_2^-(-z) - p_3^-(-z) \right) + \\ &\quad + k_6 \nu^4 \left(-\sigma_2 p_1^-(-z) + \sigma_1 p_2^-(-z) - b_0 p_3^-(-z) + p_4^-(-z) \right) \end{aligned} \quad (1.158)$$

where the domain of $p_j^\pm(z)$ is extended to the cut plane by replacing t with $z \in \mathbb{C} \setminus \mathbb{R}_{<0}$ in (1.156).

Now plugging (1.157) and (1.158) into definition (1.152) and letting $a_j^\pm(z) := p_j^\pm(z) \pm$

$p_j^-(z)$ we obtain

$$\frac{\Phi_0(z\nu)}{X(z\nu)} = k_1\xi_1(-z) + k_2\xi_2(-z) + k_3\nu\xi_3(-z) + k_4\nu^2\xi_4(-z) + k_5\nu^3\xi_5(-z) + k_6\nu^4\xi_6(-z) \quad (1.159)$$

$$\frac{\Phi_1(z\nu)}{X(z\nu)} = k_1\eta_1(-z) + k_2\eta_2(-z) + k_3\nu\eta_3(-z) + k_4\nu^2\eta_4(-z) + k_5\nu^3\eta_5(-z) + k_6\nu^4\eta_6(-z) \quad (1.160)$$

where

$$\begin{aligned} \xi_1(z) &:= p_0^+(z) \\ \xi_2(z) &:= p_0^-(z) \\ \xi_3(z) &:= -a_1^-(z) \\ \xi_4(z) &:= -b_0a_1^-(z) + a_2^-(z) \\ \xi_5(z) &:= -\sigma_1a_1^+(z) + b_0a_2^+(z) - a_3^+(z) \\ \xi_6(z) &:= -\sigma_2a_1^+(z) + \sigma_1a_2^+(z) - b_0a_3^+(z) + a_4^+(z) \end{aligned} \quad (1.161)$$

and

$$\begin{aligned} \eta_1(z) &:= p_0^+(z) \\ \eta_2(z) &:= -p_0^-(z) \\ \eta_3(z) &:= -a_1^+(z) \\ \eta_4(z) &:= -b_0a_1^+(z) + a_2^+(z) \\ \eta_5(z) &:= -\sigma_1a_1^-(z) + b_0a_2^-(z) - a_3^-(z) \\ \eta_6(z) &:= -\sigma_2a_1^-(z) + \sigma_1a_2^-(z) - b_0a_3^-(z) + a_4^-(z). \end{aligned} \quad (1.162)$$

In terms of the objects, introduced above, conditions (1.147) take the form of the system of linear equations

$$\begin{aligned} k_1\gamma_{1,1} + k_2\gamma_{1,2} + k_3\nu\gamma_{1,3} + k_4\nu^2\gamma_{1,4} + k_5\nu^3\gamma_{1,5} + k_6\nu^4\gamma_{1,6} &= 0 \\ k_1\gamma_{2,1} + k_2\gamma_{2,2} + k_3\nu\gamma_{2,3} + k_4\nu^2\gamma_{2,4} + k_5\nu^3\gamma_{2,5} + k_6\nu^4\gamma_{2,6} &= 0 \\ k_1\gamma_{3,1} + k_2\gamma_{3,2} + k_3\nu\gamma_{3,3} + k_4\nu^2\gamma_{3,4} + k_5\nu^3\gamma_{3,5} + k_6\nu^4\gamma_{3,6} &= 0 \end{aligned} \quad (1.163)$$

where we defined

$$\begin{aligned}
 \gamma_{1,j} &:= \xi_j(-i) + e^{-i\nu} \frac{X(-\nu i)}{X(\nu i)} \eta_j(i) \\
 \gamma_{2,j} &:= \xi_j(-e^{\phi i}) + e^{-\nu e^{\phi i}} \frac{X(-\nu e^{\phi i})}{X(\nu e^{\phi i})} \eta_j(e^{\phi i}) \\
 \gamma_{3,j} &:= \eta_j(-e^{-\phi i}) + e^{-\nu e^{-\phi i}} \frac{X(-\nu e^{-\phi i})}{X(\nu e^{-\nu \phi i})} \xi_j(e^{-\phi i})
 \end{aligned} \tag{1.164}$$

and $\phi = \frac{\pi}{2} \frac{1-\alpha}{5-\alpha}$. Since k_1, k_2, \dots, k_6 and ν are real, the system (1.163) comprise of six equations with real coefficients. Let $M(\nu)$ denote the matrix of these coefficients:

$$\begin{aligned}
 M_{1,j}(\nu) &= \operatorname{Re}\{\gamma_{1,j}\}, & M_{2,j}(\nu) &= \operatorname{Im}\{\gamma_{1,j}\} \\
 M_{3,j}(\nu) &= \operatorname{Re}\{\gamma_{2,j}\}, & M_{4,j}(\nu) &= \operatorname{Im}\{\gamma_{2,j}\} \\
 M_{5,j}(\nu) &= \operatorname{Re}\{\gamma_{3,j}\}, & M_{6,j}(\nu) &= \operatorname{Im}\{\gamma_{3,j}\}
 \end{aligned} \tag{1.165}$$

Non-trivial solutions are possible if and only if

$$\det\{M(\nu)\} = 0. \tag{1.166}$$

and thus we arrive at the following equivalent formulation for the eigenproblem:

Lemma 1.21. *Let $(p_0^\pm, \dots, p_4^\pm, \nu)$ with $\nu > 0$ be a solution of the system, which consists of the integral equations (1.156) and the algebraic equations (1.166). Let φ be defined by the Laplace transform, given by the formula (1.138), where $\Phi_0(z)$ and $\Phi_1(z)$ are given by (1.159)-(1.160) and let λ be defined by (1.144). Then the pair (λ, φ) solves eigenproblem (1.87). Conversely, any solution (λ, φ) of (1.87) defines a solution to the above integro-algebraic system.*

Properties of the integro-algebraic system. As mentioned above, equations (1.156) have unique solutions, such that $p_j^\pm(t) - t^j$ belong to $L^2(0, \infty)$. The following lemma derives several estimates useful in asymptotic analysis of the integro-algebraic system of Lemma 1.21:

Lemma 1.22. *For any $\alpha_0 \in (0, 1)$ there exist constants ν' and C , such that for all $\nu \geq \nu'$*

and $\alpha \in [\alpha_0, 1]$

$$\left| p_j^\pm(z/\nu) - (z/\nu)^j \right| \leq C\nu^{-(j+1)}, \quad 0 \leq j \leq 4, \quad z \in \{\pm z_0, \pm z_+, \pm z_-\} \quad (1.167)$$

and for all $\tau > 0$

$$\left| p_j^\pm(\tau) - \tau^j \right| \leq C\nu^{-(j+1)}\tau^{-1}, \quad 0 \leq j \leq 4. \quad (1.168)$$

Proof. The proof is analogous to Lemma 5.7 in [12]. \square

Inversion of the Laplace transform. The eigenfunctions are recovered by inverting the Laplace transform (1.138):

Lemma 1.23. *Let (Φ_0, Φ_1, ν) satisfy the integro-algebraic system introduced in Lemma 1.21, then the function*

$$\varphi(x) = \frac{1}{\nu} \frac{c_\alpha}{\lambda \Gamma(\alpha)} \frac{2}{5 - \alpha} \operatorname{Re} \left\{ e^{i\nu x} \Phi_0(i\nu) i \right\} + \quad (1.169a)$$

$$+ \frac{1}{\nu} \frac{c_\alpha}{\lambda \Gamma(\alpha)} \frac{1}{\pi} \int_0^\infty \frac{\sin \theta_0(u)}{\gamma_0(u)} \left(e^{-u\nu(1-x)} \Phi_1(-u\nu) + e^{-u\nu x} \Phi_0(-u\nu) \right) du + \quad (1.169b)$$

$$+ \frac{1}{\nu} \frac{c_\alpha}{\lambda \Gamma(\alpha)} \frac{2}{5 - \alpha} \operatorname{Re} \left\{ e^{-\nu e^{\phi i} x} \Phi_0(-\nu e^{\phi i}) e^{-\phi i} - e^{-\nu e^{\phi i} (1-x)} \Phi_1(-\nu e^{\phi i}) e^{-\phi i} \right\} \quad (1.169c)$$

with $\gamma_0(u) := \left| u^2 - u^{\alpha-3} e^{\frac{1-\alpha}{2}\pi i} \right|$ and $\phi = \frac{\pi}{2} \frac{1-\alpha}{5-\alpha}$, solves eigenproblem (1.87).

Proof. With the singularities being removed, the expression (1.138) is an entire function and therefore the inversion can be carried out on the imaginary axis:

$$\begin{aligned} \varphi(x) &= -\frac{1}{2\pi i} \int_{-iR}^{iR} \left(\psi'(0) + \psi(0)z + \frac{\Phi_0(z)}{\Lambda(z)} + \frac{e^{-z} \Phi_1(-z)}{\Lambda(z)} \right) e^{zx} dz = \\ &= -\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left(f_1(z) + f_0(z) \right) dz, \end{aligned}$$

where

$$f_1(z) := e^{z(x-1)} \frac{\Phi_1(-z)}{\Lambda(z)} \quad \text{and} \quad f_0(z) := e^{zx} \left(\psi'(0) + \psi(0)z + \frac{\Phi_0(z)}{\Lambda(z)} \right).$$

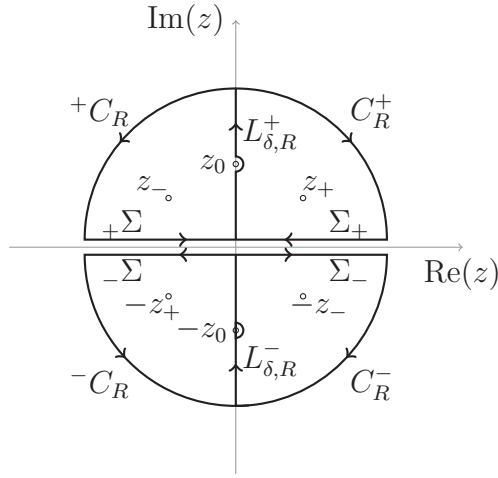


Figure 1.1 – Integration contour, used for the Laplace transform inversion: the outer circular arcs are of radius R and the half circles around the poles have radius δ

Integrating $f_1(z)$ and $f_0(z)$ over the contours on Figure 1.1 in the right and left half-planes respectively gives

$$\int_{L_{\delta,R}^+} f_1(z)dz + \int_{C_R^+} f_1(z)dz + \int_{\Sigma_+} f_1(z)dz = -2\pi i \text{Res}(f_1, z_+)$$

$$\int_{L_{\delta,R}^-} f_1(z)dz + \int_{\Sigma_-} f_1(z)dz + \int_{C_R^-} f_1(z)dz = -2\pi i \text{Res}(f_1, -z_-)$$

and

$$\int_{L_{\delta,R}^+} f_0(z)dz + \int_{C_R^+} f_0(z)dz + \int_{\Sigma_+} f_0(z)dz = 2\pi i \text{Res}(f_0, z_0) + 2\pi i \text{Res}(f_0, z_-)$$

$$\int_{L_{\delta,R}^-} f_0(z)dz + \int_{\Sigma_-} f_0(z)dz + \int_{C_R^-} f_0(z)dz = 2\pi i \text{Res}(f_0, -z_0) + 2\pi i \text{Res}(f_0, -z_+)$$

Taking $\delta \rightarrow 0$ and $R \rightarrow \infty$ and applying Jordan's lemma we get

$$\int_0^{i\infty} f_1(z)dz - \int_0^{\infty} f_1^+(t)dt = -2\pi i \text{Res}(f_1, z_+)$$

$$\int_{-i\infty}^0 f_1(z)dz + \int_0^{\infty} f_1^-(t)dt = -2\pi i \text{Res}(f_1, -z_-)$$

and

$$\int_0^{i\infty} f_0(z)dz + \int_{-\infty}^0 f_0^+(t)dt = 2\pi i \operatorname{Res}(f_0, z_0) + 2\pi i \operatorname{Res}(f_0, z_-)$$

$$\int_{-i\infty}^0 f_0(z)dz - \int_{-\infty}^0 f_0^-(t)dt = 2\pi i \operatorname{Res}(f_0, -z_0) + 2\pi i \operatorname{Res}(f_0, -z_+)$$

Summing up all these equations yields

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} (f_1(z) + f_2(z))dz = \frac{1}{2\pi i} \int_0^\infty (f_1^+(t) - f_1^-(t))dt + \frac{1}{2\pi i} \int_0^\infty (f_0^-(-t) - f_0^+(-t))dt +$$

$$+ \operatorname{Res}(f_0, z_0) + \operatorname{Res}(f_0, -z_0) + \operatorname{Res}(f_0, z_-) + \operatorname{Res}(f_0, -z_+) - \operatorname{Res}(f_1, z_+) - \operatorname{Res}(f_1, -z_-).$$

Further by symmetries (1.45)-(1.47)

$$f_1^+(t) - f_1^-(t) = -e^{-t(1-x)} \Phi_1(-t) \frac{2i \sin \theta(t)}{\gamma(t)}$$

$$f_0^-(-t) - f_0^+(-t) = -e^{-tx} \Phi_0(-t) \frac{2i \sin \theta(t)}{\gamma(t)}, \quad t > 0$$

where we defined $\gamma(t) = |\Lambda^+(t)|$. The residues can be computed using expression (1.143):

$$\operatorname{Res}(f_0, z_0) = e^{i\nu x} \Phi_0(i\nu) \frac{c_\alpha}{\lambda \Gamma(\alpha)} \frac{1}{\nu i} \frac{1}{5 - \alpha}$$

$$\operatorname{Res}(f_0, -z_0) = e^{-i\nu x} \Phi_0(-i\nu) \frac{c_\alpha}{\lambda \Gamma(\alpha)} \frac{1}{-\nu i} \frac{1}{5 - \alpha}$$

$$\operatorname{Res}(f_0, z_-) = e^{-\nu e^{-\phi i} x} \Phi_0(-\nu e^{-\phi i}) \frac{c_\alpha}{\lambda \Gamma(\alpha)} \frac{1}{-\nu e^{-\phi i}} \frac{1}{5 - \alpha}$$

$$\operatorname{Res}(f_0, -z_+) = e^{-\nu e^{\phi i} x} \Phi_0(-\nu e^{\phi i}) \frac{c_\alpha}{\lambda \Gamma(\alpha)} \frac{1}{-\nu e^{\phi i}} \frac{1}{5 - \alpha}$$

$$\operatorname{Res}(f_1, z_+) = e^{\nu e^{\phi i} (x-1)} \Phi_1(-\nu e^{\phi i}) \frac{c_\alpha}{\lambda \Gamma(\alpha)} \frac{1}{\nu e^{\phi i}} \frac{1}{5 - \alpha}$$

$$\operatorname{Res}(f_1, -z_-) = e^{\nu e^{-\phi i} (x-1)} \Phi_1(-\nu e^{-\phi i}) \frac{c_\alpha}{\lambda \Gamma(\alpha)} \frac{1}{\nu e^{-\phi i}} \frac{1}{5 - \alpha}$$

and hence

$$\begin{aligned} & \text{Res}(f_0, z_0) + \text{Res}(f_0, -z_0) + \text{Res}(f_0, z_-) + \text{Res}(f_0, -z_+) - \text{Res}(f_1, z_+) - \text{Res}(f_1, -z_-) = \\ &= \frac{2}{\nu} \frac{1}{5 - \alpha} \frac{c_\alpha}{\lambda \Gamma(\alpha)} \text{Re} \left\{ e^{i\nu x} \Phi_0(i\nu) \frac{1}{i} - e^{-\nu e^{\phi i} x} \Phi_0(-\nu e^{\phi i}) \frac{1}{e^{\phi i}} + e^{\nu e^{\phi i} (x-1)} \Phi_1(-\nu e^{\phi i}) \frac{1}{e^{\phi i}} \right\}. \end{aligned}$$

Assembling all parts together we obtain the expression in (1.169a)-(1.169c). □

Asymptotic analysis. The following lemma determines the asymptotics of algebraic part of the solutions to (1.166) under a particular enumeration:

Lemma 1.24. *The integro-algebraic system from Lemma 1.21 has countably many solutions, which can be enumerated so that*

$$\nu_n = \pi n + \pi + \frac{1 + \alpha}{4} \pi + \arctan \Delta_\alpha + \frac{r_n(\alpha)}{n}, \quad n \rightarrow \infty \quad (1.170)$$

where Δ_α is given by (1.174) below and the residual $r_n(\alpha)$ is bounded uniformly in n and $\alpha \in [\alpha_0, 1]$ for any $\alpha_0 \in (0, 1)$.

Proof. Formula (1.170) is obtained by asymptotic analysis of the equation (1.166), using the estimates of Lemma 1.22. Let us first find the leading asymptotics of $\xi_j(i)$'s and $\eta_j(i)$'s, defined in (1.161)-(1.162):

$$\begin{aligned} \xi_1(i) &\simeq 1 & \eta_1(i) &\simeq 1 \\ \xi_2(i) &\simeq 1 & \eta_2(i) &\simeq -1 \\ \xi_3(i) &\simeq 0 & \eta_3(i) &\simeq -2i \\ \xi_4(i) &\simeq 0 & \eta_4(i) &\simeq -2ib_0 - 2 \\ \xi_5(i) &\simeq 2i(1 - \sigma_1) - 2b_0 & \eta_5(i) &\simeq 0 \\ \xi_6(i) &\simeq 2i(b_0 - \sigma_2) + 2(1 - \sigma_1) & \eta_6(i) &\simeq 0 \end{aligned} \quad (1.171)$$

where \simeq stands for equality up to $O(\nu^{-1})$ term, uniform over $\alpha \in [\alpha_0, 1]$. Further by

(1.164)

$$\begin{aligned}
\gamma_{1,1} &\simeq 1 + c_\nu - is_\nu \\
\gamma_{1,2} &\simeq 1 - c_\nu + is_\nu \\
\gamma_{1,3} &\simeq -2c_\nu i - 2s_\nu \\
\gamma_{1,4} &\simeq -2(c_\nu + s_\nu b_0) + i2(s_\nu - c_\nu b_0) \\
\gamma_{1,5} &\simeq -2i(1 - \sigma_1) - 2b_0 \\
\gamma_{1,6} &\simeq -2i(b_0 - \sigma_2) + 2(1 - \sigma_1)
\end{aligned}$$

where we used the notations

$$c_\nu = \cos(\nu + 2 \arg \{X(\nu i)\}) \quad \text{and} \quad s_\nu = \sin(\nu + 2 \arg \{X(\nu i)\}).$$

Similarly, by (1.161)-(1.162) and (1.167)

$$\begin{aligned}
\xi_1(-e^{\phi i}) &\simeq 1 & \eta_1(-e^{-\phi i}) &\simeq 1 \\
\xi_2(-e^{\phi i}) &\simeq 1 & \eta_2(-e^{-\phi i}) &\simeq -1 \\
\xi_3(-e^{\phi i}) &\simeq 0 & \eta_3(-e^{-\phi i}) &\simeq 2e^{-\phi i} \\
\xi_4(-e^{\phi i}) &\simeq 0 & \eta_4(-e^{-\phi i}) &\simeq 2b_0 e^{-\phi i} + 2e^{-2\phi i} \\
\xi_5(-e^{\phi i}) &= 2\sigma_1 e^{\phi i} + 2b_0 e^{2\phi i} + 2e^{3\phi i} & \eta_5(-e^{-\phi i}) &\simeq 0 \\
\xi_6(-e^{\phi i}) &= 2\sigma_2 e^{\phi i} + 2\sigma_1 e^{2\phi i} + 2b_0 e^{3\phi i} + 2e^{4\phi i} & \eta_6(-e^{-\phi i}) &\simeq 0.
\end{aligned}$$

Since $\phi \in (0, \frac{\pi}{2})$, we have

$$\gamma_{2,j} \simeq \xi_j(-e^{\phi i}) \quad \text{and} \quad \gamma_{3,j} \simeq \eta_j(-e^{-\phi i}),$$

and, collecting all parts together, the leading term asymptotics of (1.165) reads

$$M(\nu) \simeq \begin{pmatrix} 1 + c_\nu & 1 - c_\nu & -2s_\nu & -2(c_\nu + s_\nu b_0) & -2b_0 & 2(1 - \sigma_1) \\ -s_\nu & s_\nu & -2c_\nu & 2(s_\nu - c_\nu b_0) & 2(\sigma_1 - 1) & 2(\sigma_2 - b_0) \\ 1 & 1 & 0 & 0 & 2A_1 & 2B_1 \\ 0 & 0 & 0 & 0 & 2A_2 & 2B_2 \\ 1 & -1 & 2c^{(1)} & 2(b_0 c^{(1)} + c^{(2)}) & 0 & 0 \\ 0 & 0 & -2s^{(1)} & -2(b_0 s^{(1)} + s^{(2)}) & 0 & 0 \end{pmatrix} \quad (1.172)$$

where we defined

$$\begin{aligned} A_1 &= \sigma_1 c^{(1)} + b_0 c^{(2)} + c^{(3)} & B_1 &= \sigma_2 c^{(1)} + \sigma_1 c^{(2)} + b_0 c^{(3)} + c^{(4)} \\ A_2 &= \sigma_1 s^{(1)} + b_0 s^{(2)} + s^{(3)} & B_2 &= \sigma_2 s^{(1)} + \sigma_1 s^{(2)} + b_0 s^{(3)} + s^{(4)} \end{aligned}$$

and

$$c^{(j)} = \cos(j\phi) \quad \text{and} \quad s^{(j)} = \sin(j\phi).$$

A straightforward calculation yields the following asymptotic expression for the determinant of this matrix:

$$\det\{M(\nu)\} \simeq -c_\nu s^{(2)} (A_1 B_2 - A_2 B_1) + c_\nu s^{(2)} (A_2(\sigma_1 - 1) - B_2 b_0) + s_\nu s^{(2)} (A_2(\sigma_2 - b_0) - B_2(\sigma_1 - 1))$$

and therefore the equation $\det\{M(\nu)\} = 0$ becomes

$$\tan\left(\nu + 2 \arg\{X(\nu i)\}\right) = \Delta_\alpha + R(\nu) \tag{1.173}$$

with

$$\Delta_\alpha = \frac{A_1 B_2 - A_2 B_1 - A_2(\sigma_1 - 1) + B_2 b_0}{A_2(\sigma_2 - b_0) - B_2(\sigma_1 - 1)}. \tag{1.174}$$

Here $|R(\nu)| \vee |R'(\nu)| \leq C\nu^{-1}$ for all ν large enough with some constant C , depending only on α_0 . Hence for any sufficiently large integer n , fixed point iterations produce the unique solution to the integro-algebraic system, with ν_n satisfying asymptotics (1.170), where we used the expression (1.155).

□

The next lemma derives asymptotic expression for the eigenfunctions:

Lemma 1.25. *Under the enumeration, introduced by Lemma 1.24, the unit norm eigen-*

functions admit the approximation:

$$\varphi_n(x) = \sqrt{2} \cos\left(\nu_n x + \frac{3-\alpha}{8}\pi - \arctan \Delta_\alpha\right) - \quad (1.175a)$$

$$- \frac{\sqrt{5-\alpha}}{\pi} \int_0^\infty \rho_0(u) \left(Q_0(u) e^{-u\nu_n x} - (-1)^n Q_1(u) e^{-u\nu_n(1-x)} \right) du + \quad (1.175b)$$

$$+ C_0 e^{-c\nu_n x} \cos\left(s\nu_n x + \varkappa_0\right) + C_1 e^{-c\nu_n(1-x)} \cos\left(s\nu_n(1-x) + \varkappa_1\right) + n^{-1} r_n(x) \quad (1.175c)$$

with residual $r_n(x)$, bounded uniformly in both $n \in \mathbb{N}$ and $x \in [0, 1]$. Here function $\rho_0(u)$ is defined in (1.181), polynomials $Q_0(u)$ and $Q_1(u)$ are given in (1.182),

$$c := \cos \frac{\pi}{2} \frac{1-\alpha}{5-\alpha} \quad \text{and} \quad s := \sin \frac{\pi}{2} \frac{1-\alpha}{5-\alpha},$$

and the amplitudes C_0 and C_1 and phases \varkappa_0 and \varkappa_1 are constants, which depend only on α . Moreover, the eigenfunctions satisfy

$$\varphi_n(1) = (-1)^n \sqrt{5-\alpha} \left(1 + O(n^{-1})\right) \quad (1.176)$$

and

$$\int_0^1 \varphi_n(x) dx = \nu_n^{-1} \sqrt{5-\alpha} C \left(1 + O(n^{-1})\right) \quad (1.177)$$

with explicit constant C , defined in (1.183).

Proof. All the formulas are derived from Lemma 1.23, expressions (1.159)-(1.160) and the relations between coefficients k_j 's, which follow from the equations defined by matrix (1.172). More precisely, equation corresponding to the fourth row of (1.172) implies

$$k_5 \simeq -\frac{B_2}{A_2} k_6 \nu_n,$$

and hence the third row gives

$$k_1 + k_2 \simeq 2 \left(A_1 \frac{B_2}{A_2} - B_1 \right) k_6 \nu_n^4.$$

The sixth row implies that

$$k_3 \simeq -\left(b_0 + \frac{s^{(2)}}{s^{(1)}}\right)k_4\nu_n,$$

and hence by the fifth row

$$k_1 - k_2 \simeq 2\left(\frac{s^{(2)}}{s^{(1)}}c^{(1)} - c^{(2)}\right)k_4\nu_n^2.$$

Plugging these expressions into the second row we obtain

$$\left(-s_{\nu_n}\frac{s^{(2)}}{s^{(1)}}c^{(1)} + s_{\nu_n}c^{(2)} + \frac{s^{(2)}}{s^{(1)}}c_{\nu_n} + s_{\nu_n}\right)k_4 + \left(-(\sigma_1 - 1)\frac{B_2}{A_2} + (\sigma_2 - b_0)\right)k_6\nu_n^2 \simeq 0.$$

Equation (1.173) implies $c_{\nu_n} \simeq (-1)^n/\sqrt{1 + \Delta_\alpha^2}$ and using trigonometric identities

$$s^{(2)} = 2s^{(1)}c^{(1)} \quad \text{and} \quad c^{(2)} = 2(c^{(1)})^2 - 1$$

we obtain

$$k_4 \simeq -(-1)^n \frac{1}{2c^{(1)}} \sqrt{A_3^2 + B_3^2} k_6 \nu_n^2 \quad (1.178)$$

where A_3 and B_3 denote the expressions in the numerator and denominator of (1.174).

The first row does not impose any further constraint on the coefficients since $\det\{M(\nu_n)\} = 0$.

Let us first consider the oscillatory term (1.169a). Plugging asymptotics (1.171) and the above relations between the coefficients into (1.159) gives

$$\frac{\Phi_0(i\nu_n)}{X(i\nu_n)} \simeq k_1 + k_2 - k_5\nu_n^3(2i(1-\sigma_1) + 2b_0) + k_6\nu_n^4(-2i(b_0 - \sigma_2) + 2(1-\sigma_1)) \simeq 2(A_3 + iB_3)k_6\nu_n^4.$$

Hence expression in (1.169a) satisfies

$$\begin{aligned} \frac{1}{\nu_n} \frac{c_\alpha}{\lambda\Gamma(\alpha)} \frac{2}{5-\alpha} \operatorname{Re}\left\{e^{i\nu_n x} \Phi_0(i\nu_n) i\right\} &\simeq k_6 \nu_n^3 \frac{c_\alpha}{\lambda\Gamma(\alpha)} \frac{4}{5-\alpha} \operatorname{Re}\left\{e^{i\nu_n x} (A_3 + iB_3) X(i\nu_n) i\right\} \simeq \\ &\simeq k_6 \nu_n^2 \frac{c_\alpha}{\lambda\Gamma(\alpha)} \sqrt{\frac{2}{5-\alpha}} \sqrt{A_3^2 + B_3^2} \frac{1}{c^{(1)}} \cos\left(\nu_n x + \arctan \frac{B_3}{A_3} - \frac{1+\alpha}{8}\pi\right) \end{aligned} \quad (1.179)$$

where we used the formulas from (1.155). Since $A_3/B_3 = \Delta_\alpha$ and $\arctan x + \arctan 1/x = \pi/2$, we have

$$\arctan \frac{B_3}{A_3} - \frac{1+\alpha}{8}\pi = \frac{3-\alpha}{8}\pi - \arctan \Delta_\alpha.$$

Thus normalizing (1.179) by the constant factor

$$C_n := k_6 \nu_n^2 \frac{c_\alpha}{\lambda \Gamma(\alpha)} \sqrt{\frac{1}{5-\alpha}} \sqrt{A_3^2 + B_3^2} \frac{1}{c^{(1)}} \quad (1.180)$$

we obtain the oscillatory term claimed in (1.175a).

The approximation for (1.169b) is obtained similarly: plugging the estimates from (1.168) into (1.159) yields

$$\begin{aligned} \frac{\Phi_0(-u\nu_n)}{X(-u\nu_n)} &\simeq k_1 + k_2 - k_5 \nu_n^3 (\sigma_1 2u - b_0 2u^2 + 2u^3) - k_6 \nu_n^4 (\sigma_2 2u - \sigma_1 2u^2 + b_0 2u^3 - 2u^4) \simeq \\ &\simeq 2 \left(A_1 \frac{B_2}{A_2} - B_1 + \left(\frac{B_2}{A_2} \sigma_1 - \sigma_2 \right) u + \left(\sigma_1 - \frac{B_2}{A_2} b_0 \right) u^2 + \left(\frac{B_2}{A_2} - b_0 \right) u^3 + u^4 \right) k_6 \nu_n^4 \end{aligned}$$

and

$$\begin{aligned} \frac{\Phi_1(-u\nu_n)}{X(-u\nu_n)} &\simeq k_1 - k_2 - k_3 \nu_n 2u + k_4 \nu_n^2 (-b_0 2u + 2u^2) \simeq \\ &\simeq -(-1)^n \left(\frac{s^{(2)}}{s^{(1)}} c^{(1)} - c^{(2)} + \frac{s^{(2)}}{s^{(1)}} u + u^2 \right) \frac{1}{c^{(1)}} \sqrt{A_3^2 + B_3^2} k_6 \nu_n^4. \end{aligned}$$

Hence the integral term (1.169b) contributes

$$\begin{aligned} &\frac{1}{\nu_n} \frac{c_\alpha}{\lambda \Gamma(\alpha)} \frac{1}{\pi} \int_0^\infty \frac{\sin \theta_0(u)}{\gamma_0(u)} \left(e^{-\nu_n(1-x)} \Phi_1(-u\nu_n) + e^{-\nu_n x} \Phi_0(-u\nu_n) \right) du \simeq \\ &\simeq -\frac{1}{\nu_n^2} \frac{c_\alpha}{\lambda \Gamma(\alpha)} \frac{1}{\pi} \int_0^\infty \frac{\sin \theta_0(u)}{\gamma_0(u)} X_0(-u) \frac{1}{u} \left(\frac{\Phi_0(-u\nu_n)}{X(-u\nu_n)} e^{-\nu_n x} + \frac{\Phi_1(-u\nu_n)}{X(-u\nu_n)} e^{-\nu_n(1-x)} \right) du \end{aligned}$$

which after normalizing by factor (1.180) becomes (1.175b) with

$$\rho_0(u) := \frac{\sin \theta_0(u)}{\gamma_0(u)} \frac{1}{u} X_0(-u) \quad (1.181)$$

and

$$\begin{aligned}
 Q_0(u) &= \frac{2c^{(1)}}{\sqrt{A_3^2 + B_3^2}} \left(A_1 \frac{B_2}{A_2} - B_1 + \left(\frac{B_2}{A_2} \sigma_1 - \sigma_2 \right) u + \left(\sigma_1 - \frac{B_2}{A_2} b_0 \right) u^2 + \left(\frac{B_2}{A_2} - b_0 \right) u^3 + u^4 \right) \\
 Q_1(u) &= \frac{s^{(2)}}{s^{(1)}} c^{(1)} - c^{(2)} + \frac{s^{(2)}}{s^{(1)}} u + u^2
 \end{aligned} \tag{1.182}$$

The last term (1.175c) is deduced from (1.169c) along the same lines. In principle, closed form formulas can be obtained for the amplitudes C_0 and C_1 and the phases \varkappa_0 and \varkappa_1 ; the emerging expressions are cumbersome and will be omitted.

Finally, by (1.149) and (1.178)

$$\varphi_n(1) = \psi_n''(1) = -k_4 \frac{2c_\alpha}{\lambda_n \Gamma(\alpha)} \simeq (-1)^n \frac{1}{2c^{(1)}} \sqrt{A_3^2 + B_3^2} \frac{2c_\alpha}{\lambda_n \Gamma(\alpha)} k_6 \nu_n^2$$

and after normalizing by factor (1.180) we get (1.176). Similarly,

$$\int_0^1 \varphi_n(x) dx = -\psi'(0) = k_5 \frac{2c_\alpha}{\lambda_n \Gamma(\alpha)} \simeq -\frac{B_2}{A_2} \frac{2c_\alpha}{\lambda_n \Gamma(\alpha)} k_6 \nu_n$$

and after normalising by (1.180) we get (1.177) with the constant

$$C := -\frac{B_2}{A_2} \frac{2c^{(1)}}{\sqrt{A_3^2 + B_3^2}}. \tag{1.183}$$

□

The seemingly different expressions for Δ_α , obtained for $\alpha \in (0, 1)$ and $\alpha \in (1, 2)$, in fact, coincide:

Lemma 1.26. *The expressions for Δ_α in (1.128) and (1.174) are equal for all $\alpha \in (0, 2) \setminus \{1\}$. Moreover $\arctan \Delta_\alpha = \frac{1-\alpha}{5-\alpha} \pi$.*

Proof. The claimed equality follows if we show that

$$\left(\frac{1}{3} d_0^3 - d_2 \right) \left(A_2 (\sigma_2 - b_0) - B_2 (\sigma_1 - 1) \right) = \tag{1.184}$$

$$= \left(\frac{1}{4} + \frac{1}{2} d_0^2 + d_2 d_0 - \frac{1}{12} d_0^4 \right) \left(A_1 B_2 - A_2 B_1 - A_2 (\sigma_1 - 1) + B_2 b_0 \right). \tag{1.185}$$

Define the quantities:

$$c_k := \cos k \frac{\pi}{2} \frac{1-\alpha}{5-\alpha} \quad \text{and} \quad s_k := \sin k \frac{\pi}{2} \frac{1-\alpha}{5-\alpha}, \quad k = 1, 2, 3, 4$$

and

$$C_k := \cos k \frac{\pi}{2} \frac{2}{5-\alpha} \quad \text{and} \quad S_k := \sin k \frac{\pi}{2} \frac{2}{5-\alpha}, \quad k = 1, 2, 3, 4.$$

Then for $k = 0, 1, 2$ we have

$$b_k = \frac{1}{k+1} \frac{\sin(k+1) \frac{\pi}{2} \left(\frac{1-\alpha}{5-\alpha} - \frac{2}{5-\alpha} \right)}{\sin(k+1) \frac{\pi}{2} \frac{2}{5-\alpha}} = \frac{1}{k+1} \frac{1}{S_{k+1}} \left(s_{k+1} C_{k+1} - S_{k+1} c_{k+1} \right)$$

and

$$d_0 = \frac{s_1 C_1 + c_1 S_1}{S_1} \quad \text{and} \quad d_2 = \frac{1}{3} \frac{s_3 C_3 + c_3 S_3}{S_3}.$$

After the multiplication of (1.184) by $(S_1 S_2 S_3)^6$, the left and right hand sides, which we now denote by E_ℓ and E_r turn into polynomials with respect to c_k 's, s_k 's, C_k 's and S_k 's. By the basic trigonometry c_k 's and s_k 's can be expressed in terms of c_1 and s_1 :

$$\begin{aligned} s_2 &= 2s_1 c_1 & c_2 &= 2c_1^2 - 1 \\ s_3 &= -4s_1^3 + 3s_1 & c_3 &= 4c_1^3 - 3c_1 \\ c_4 &= 2c_1 c_3 - c_2 & s_4 &= 2c_1 s_3 - s_2 \end{aligned}$$

The same is true for C_k 's and S_k 's. Plugging these identities, computer aided symbolic computation produces the following formula:

$$\begin{aligned} E_\ell - E_r &= -2 \cos x (\cos y)^4 \left((\cos x)^2 - 1 \right)^2 \left((\cos y)^2 - 1 \right)^6 \left(4 (\cos y)^2 - 1 \right)^5 \times \\ &\times \left(\sin(2y) + \sin(2x + 2y) - 2 \sin(2x + 4y) + \sin(2x + 6y) + \sin(4x + 6y) - \sin(4x + 8y) \right) \end{aligned} \quad (1.186)$$

where

$$x = \frac{\pi}{2} \frac{1-\alpha}{5-\alpha} \quad \text{and} \quad y = \frac{\pi}{2} \frac{2}{5-\alpha}.$$

Note that

$$\begin{aligned}
 \sin(2y) &= \sin\left(\pi \frac{2}{5-\alpha}\right) \\
 \sin(2x+2y) &= \sin\left(\pi \frac{3-\alpha}{5-\alpha}\right) \\
 \sin(2x+4y) &= \sin\left(\pi \frac{5-\alpha}{5-\alpha}\right) = 0 \\
 \sin(2x+6y) &= \sin\left(\pi \frac{7-\alpha}{5-\alpha}\right) = -\sin\left(\pi \frac{2}{5-\alpha}\right) \\
 \sin(4x+6y) &= \sin\left(2\pi \frac{1-\alpha}{5-\alpha} + 3\pi \frac{2}{5-\alpha}\right) = \sin\left(2\pi \frac{4-\alpha}{5-\alpha}\right) \\
 \sin(4x+8y) &= \sin\left(2\pi \frac{1-\alpha}{5-\alpha} + 2\pi \frac{4}{5-\alpha}\right) = \sin\left(2\pi \frac{5-\alpha}{5-\alpha}\right) = 0.
 \end{aligned}$$

and hence the expression in the last brackets in (1.186) becomes

$$\begin{aligned}
 &\sin\left(\pi \frac{2}{5-\alpha}\right) + \sin\left(\pi \frac{3-\alpha}{5-\alpha}\right) - \sin\left(\pi \frac{2}{5-\alpha}\right) + \sin\left(2\pi \frac{4-\alpha}{5-\alpha}\right) = \\
 &= 2 \sin\left(\frac{\pi}{2} \frac{3-\alpha}{5-\alpha} + \pi \frac{4-\alpha}{5-\alpha}\right) \cos\left(\frac{\pi}{2} \frac{3-\alpha}{5-\alpha} - \pi \frac{4-\alpha}{5-\alpha}\right) = 2 \sin\left(\frac{\pi}{2} \frac{11-3\alpha}{5-\alpha}\right) \cos\left(\frac{\pi}{2} \frac{5-\alpha}{5-\alpha}\right) = 0.
 \end{aligned}$$

The expression for Δ_α can be further simplified by virtue of (1.99). Note that

$$b_0 = \frac{\sin\left(\frac{\pi}{2} \frac{3-\alpha}{5-\alpha}\right)}{\sin\frac{\pi}{5-\alpha}} = \frac{\sin\left(\frac{\pi}{2} \frac{3-\alpha}{5-\alpha}\right)}{\sin\left(\frac{\pi}{2} - \frac{\pi}{2} \frac{3-\alpha}{5-\alpha}\right)} = \tan\left(\frac{\pi}{2} \frac{3-\alpha}{5-\alpha}\right) =: \tan\beta$$

and

$$b_2 = \frac{1}{3} \frac{\sin\left(\frac{3\pi}{2} \frac{3-\alpha}{5-\alpha}\right)}{\sin\frac{3\pi}{5-\alpha}} = \frac{1}{3} \frac{\sin\left(\frac{3\pi}{2} \frac{3-\alpha}{5-\alpha}\right)}{\sin\left(\frac{3\pi}{2} - \frac{3\pi}{2} \frac{3-\alpha}{5-\alpha}\right)} = -\frac{1}{3} \tan(3\beta) = -\frac{b_0}{3} \frac{3-b_0^2}{1-3b_0^2}$$

where the last equality holds by the triple-angle tangent formula. Hence

$$\frac{1}{3} b_0^3 - b_2 = b_0(1-b_0^4) \frac{1}{1-3b_0^2}$$

and, in turn, the denominator of Δ_α reads

$$\frac{1}{4} + \frac{1}{2} b_0^2 + b_2 b_0 - \frac{1}{12} b_0^4 = \frac{1}{4} \frac{1}{1-3b_0^2} (1+b_0^2)(b_0^4 - 6b_0^2 + 1).$$

Thus we obtain

$$\Delta_\alpha = \frac{4b_0(1-b_0^2)}{b_0^4-6b_0^2+1} = \frac{4\sin\beta\cos\beta(\cos^2\beta-\sin^2\beta)}{\sin^4\beta-6\sin^2\beta\cos^2\beta+\cos^4\beta} = \frac{\sin 4\beta}{1-8\sin^2\beta\cos^2\beta} = \tan(4\beta),$$

and $\arctan \Delta_\alpha = 4\beta - \pi = \frac{1-\alpha}{5-\alpha}\pi$.

□

Enumeration alignment. Similarly to the case $H < \frac{1}{2}$, the alignment between the enumeration, introduced in Lemma 1.24, and the enumeration which puts the eigenvalues in decreasing order, is achieved by shifting expression (1.170) for ν_n by -2π .

1.3.4 Proof for mixed fractional Brownian motion

The case $H > \frac{1}{2}$

For these values of H the parameter $\alpha := 2 - 2H$ takes values in $(0, 1)$ and

$$K(s, t) = s \wedge t + c_\alpha \int_0^t \int_0^s |u - v|^{-\alpha} dudv \quad (1.187)$$

with $\alpha := 2 - 2H \in (0, 1)$ and $c_\alpha := (1 - \frac{\alpha}{2})(1 - \alpha)$. The eigenproblem in this case reads:

$$\int_0^1 (s \wedge t) \varphi(s) ds + \int_0^1 c_\alpha \int_0^t \int_0^s |u - v|^{-\alpha} dudv \varphi(s) ds = \lambda \varphi(t), \quad t \in [0, 1]. \quad (1.188)$$

The Laplace transform. The first step is to derive a useful expression for the Laplace transform, based on the specific structure of the kernel (1.187):

Lemma 1.27. For $\alpha \in (0, 1)$,

$$\widehat{\varphi}(z) - \widehat{\varphi}(0) = -\frac{1}{\Lambda(z)} \left(\Phi_0(z) + e^{-z} \Phi_1(-z) \right), \quad (1.189)$$

where the functions $\Phi_0(z)$ and $\Phi_1(z)$, defined in (1.197) below, are sectionally holomorphic on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$ and

$$\Lambda(z) = \frac{\lambda \Gamma(\alpha)}{c_\alpha} z + \frac{\Gamma(\alpha)}{c_\alpha} \frac{1}{z} + \frac{1}{z} \int_0^\infty \frac{2t^\alpha}{t^2 - z^2} dt. \quad (1.190)$$

Proof. Taking derivative of (1.188) gives

$$\int_t^1 \varphi(s) ds + c_\alpha \int_0^1 \int_0^s |u-t|^{-\alpha} du \varphi(s) ds = \lambda \varphi'(t).$$

Integrating by parts and setting $\psi(x) := \int_x^1 \varphi(s) ds$, we arrive at the following generalized eigenproblem:

$$\begin{aligned} c_\alpha \int_0^1 |y-x|^{-\alpha} \psi(y) dy &= -\lambda \psi''(x) - \psi(x), \quad x \in [0, 1] \\ \psi(1) &= 0, \quad \psi'(0) = 0. \end{aligned} \tag{1.191}$$

Plugging the identity

$$|y-x|^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t|x-y|} dt \tag{1.192}$$

this equation becomes

$$\frac{c_\alpha}{\Gamma(\alpha)} u_0(x) = -\lambda \psi''(x) - \psi(x) \tag{1.193}$$

where we defined

$$u(x, t) := \int_0^1 e^{-t|x-y|} \psi(y) dy \quad \text{and} \quad u_0(x) := \int_0^\infty t^{\alpha-1} u(x, t) dt.$$

Differentiating $u(x, t)$ twice with respect to x yields the equation

$$u''(x, t) = t^2 u(x, t) - 2t \psi(x) \tag{1.194}$$

subject to the boundary conditions

$$\begin{aligned} u'(0, t) &= tu(0, t) \\ u'(1, t) &= -tu(1, t). \end{aligned} \tag{1.195}$$

The Laplace transform of the second derivative $u''(x, t)$ satisfies

$$\begin{aligned} \hat{u}''(z, t) &= \int_0^1 e^{-zx} u''(x, t) dx = e^{-z} \left(u'(1, t) + zu(1, t) \right) - \left(u'(0, t) + zu(0, t) \right) + z^2 \hat{u}(z, t) = \\ &= e^{-z} (z-t)u(1, t) - (z+t)u(0, t) + z^2 \hat{u}(z, t) \end{aligned}$$

where we integrated twice by parts and used the boundary conditions (1.195). Combining this with the expression for $\hat{u}''(z, t)$ which follows from (1.194) we obtain

$$(z^2 - t^2)\hat{u}(z, t) = u(0, t)(z + t) - e^{-z}u(1, t)(z - t) - 2t\hat{\psi}(z).$$

Consequently for all $z \in \mathbb{C} \setminus \mathbb{R}$

$$\begin{aligned} \hat{u}_0(z) &= \int_0^\infty t^{\alpha-1}u(z, t)dt = \\ &= \int_0^\infty \frac{t^{\alpha-1}}{z-t}u(0, t)dt - e^{-z} \int_0^\infty \frac{t^{\alpha-1}}{z+t}u(1, t)dt - \hat{\psi}(z) \int_0^\infty \frac{2t^\alpha}{z^2-t^2}dt. \end{aligned} \quad (1.196)$$

On the other hand by (1.193) and the boundary conditions in (1.191),

$$\frac{c_\alpha}{\Gamma(\alpha)}\hat{u}_0(z) = -\lambda\hat{\psi}''(z) - \hat{\psi}(z) = -\lambda\psi'(1)e^{-z} + \lambda z\psi(0) - (\lambda z^2 + 1)\hat{\psi}(z).$$

Combining these two expressions gives

$$z\hat{\psi}(z) = \frac{\Phi_0(z) + e^{-z}\Phi_1(-z)}{\Lambda(z)}$$

with $\Lambda(z)$ defined in (1.190) and

$$\begin{aligned} \Phi_0(z) &:= \frac{\lambda\Gamma(\alpha)}{c_\alpha}\psi(0)z + \int_0^\infty \frac{t^{\alpha-1}}{t-z}u(0, t)dt \\ \Phi_1(z) &:= -\frac{\lambda\Gamma(\alpha)}{c_\alpha}\psi'(1) + \int_0^\infty \frac{t^{\alpha-1}}{t-z}u(1, t)dt \end{aligned} \quad (1.197)$$

The claimed formula follows since $\hat{\varphi}(z) = \hat{\varphi}(0) - z\hat{\psi}(z)$. □

The structure of the problem is largely defined by the function $\Lambda(z)$, whose properties are summarized in the following lemma:

Lemma 1.28.

a) *The function defined in (1.190) admits closed form expression*

$$\Lambda(z) = \frac{\lambda\Gamma(\alpha)}{c_\alpha}z + \frac{\Gamma(\alpha)}{c_\alpha}\frac{1}{z} + z^{\alpha-2}\frac{\pi}{\cos\frac{\pi}{2}\alpha} \begin{cases} e^{\frac{1-\alpha}{2}\pi i} & \arg(z) \in (0, \pi) \\ e^{-\frac{1-\alpha}{2}\pi i} & \arg(z) \in (-\pi, 0) \end{cases} \quad (1.198)$$

It vanishes only at simple zeros $\pm z_0 := \pm i\nu$ with $\nu \in \mathbb{R}_{>0}$ being the unique root of equation

$$\lambda = \nu^{-2} + \kappa_\alpha \nu^{\alpha-3} \quad (1.199)$$

where $\kappa_\alpha := \frac{c_\alpha}{\Gamma(\alpha)} \frac{\pi}{\cos \frac{\pi}{2}\alpha}$.

b) $\Lambda(z)$ is analytic on the cut plane $\mathbb{C} \setminus \mathbb{R}$ and its limits across the real line

$$\Lambda^\pm(t) := \lim_{z \rightarrow t^\pm} \Lambda(z) = \frac{\lambda \Gamma(\alpha)}{c_\alpha} t + \frac{\Gamma(\alpha)}{c_\alpha} \frac{1}{t} \pm |t|^{\alpha-2} \frac{\pi}{\cos \frac{\pi}{2}\alpha} \begin{cases} e^{\frac{1-\alpha}{2}\pi i} & t > 0 \\ e^{\frac{1+\alpha}{2}\pi i} & t < 0 \end{cases}$$

satisfy the symmetries (1.45)-(1.47).

c) The argument $\theta(t) := \arg\{\Lambda^+(t)\} \in (-\pi, \pi]$ satisfies $\theta(-t) = \pi - \theta(t)$ and is given by

$$\theta(t) = \arctan \frac{\sin \frac{1-\alpha}{2}\pi}{\kappa_\alpha^{-1} \nu^{1-\alpha} \left((t/\nu)^{3-\alpha} + (t/\nu)^{1-\alpha} \right) + (t/\nu)^{3-\alpha} + \cos \frac{1-\alpha}{2}\pi}, \quad t > 0 \quad (1.200)$$

It decreases continuously from $\theta(0+) := \lim_{t \rightarrow 0} \theta(t) = \frac{1-\alpha}{2}\pi$ to zero as $t \rightarrow \infty$.

Proof. The explicit formula (1.198) follows from the identity

$$\int_0^\infty \frac{t^\alpha}{t^2 - z^2} dt = \frac{1}{2} z^{\alpha-1} \frac{\pi}{\cos \frac{\pi}{2}\alpha} \begin{cases} e^{\frac{1-\alpha}{2}\pi i} & \arg(z) \in (0, \pi) \\ e^{-\frac{1-\alpha}{2}\pi i} & \arg(z) \in (-\pi, 0) \end{cases} \quad (1.201)$$

Since $\overline{\Lambda(z)} = \Lambda(\bar{z})$ all zeros of $\Lambda(z)$, which are not purely real, appear in conjugate pairs. Hence it suffices to find the zeros in the upper half plane. To this end, let $z = \nu e^{i\phi}$ with $\nu > 0$ and $\phi \in (0, \pi)$. Then equation $\Lambda(z) = 0$ is equivalent to

$$\lambda \kappa_\alpha^{-1} \nu^2 e^{i2\phi} + \kappa_\alpha^{-1} + \nu^{\alpha-1} e^{(\frac{\pi}{2}-\phi)(1-\alpha)i} = 0. \quad (1.202)$$

Equating the imaginary part to zero yields

$$\lambda \kappa_\alpha^{-1} \nu^2 \sin(2\phi) + \nu^{\alpha-1} \sin\left(\left(\frac{\pi}{2} - \phi\right)(1-\alpha)\right) = 0$$

Note that for $\phi \in (0, \pi) \setminus \{\frac{\pi}{2}\}$ and $\alpha \in (0, 1)$, the sines have the same signs and hence $\phi = \frac{\pi}{2}$ is the only possibility. For this angle equation (1.202) reduces to (1.199), which

has unique root for any $\lambda > 0$, since the function in the right hand side is decreasing from $+\infty$ to 0 as ν increases. All other claims follow by direct calculations. \square

Removal of singularities. Since the Laplace transform $\widehat{\varphi}(z)$ is a priori an entire function, all singularities in (1.189) must be removable. Removing discontinuity amounts to equating limits from below and from above the real line of the right hand side of (1.189) and yields the condition

$$\begin{aligned}\Phi_0^+(t) - e^{2\theta(t)i}\Phi_0^-(t) &= 2ie^{-t}e^{i\theta(t)}\sin\theta(t)\Phi_1(-t) \\ \Phi_1^+(t) - e^{2\theta(t)i}\Phi_1^-(t) &= 2ie^{-t}e^{i\theta(t)}\sin\theta(t)\Phi_0(-t)\end{aligned}\quad t > 0. \quad (1.203)$$

Removal of poles in (1.189) gives

$$e^{-z_0}\Phi_1(-z_0) + \Phi_0(z_0) = 0. \quad (1.204)$$

The expressions in (1.197) determine asymptotic growth of $\Phi_0(z)$ and $\Phi_1(z)$:

$$\Phi_0(z) = O(z^{\alpha-1}) \quad \text{and} \quad \Phi_1(z) = O(z^{\alpha-1}) \quad \text{as } z \rightarrow 0 \quad (1.205)$$

and

$$\Phi_0(z) = -2c_2z + O(z^{-1}) \quad \text{and} \quad \Phi_1(z) = 2c_1 + O(z^{-1}) \quad \text{as } z \rightarrow \infty \quad (1.206)$$

where we defined constants

$$c_1 := -\frac{1}{2} \frac{\lambda\Gamma(\alpha)}{c_\alpha} \psi'(1) \quad \text{and} \quad c_2 := -\frac{1}{2} \frac{\lambda\Gamma(\alpha)}{c_\alpha} \psi(0).$$

These estimates hold since both $tu(j, t)$ and $u(j, t)$, $j \in \{0, 1\}$ are uniformly bounded.

Equivalent formulation of the eigenproblem. The preceding calculations show that any solution of eigenproblem (1.188) can be used to construct a pair of functions $\Phi_0(z)$ and $\Phi_1(z)$, which are sectionally holomorphic on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$ and satisfy the growth estimates (1.206) and (1.205), the boundary conditions (1.203) and algebraic constraints (1.204). Now we show that all such pairs can be characterized uniquely as solutions to certain integro-algebraic system of equations.

The key element in construction of this system is the solution to the homogeneous Riemann problem related to (1.203):

Lemma 1.29. *Any sectionally holomorphic and nonvanishing function on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$, satisfying the boundary condition*

$$\frac{X^+(t)}{X^-(t)} = e^{2i\theta(t)}, \quad t \in \mathbb{R}_{>0} \quad (1.207)$$

is given by $X(z) = z^m X_c(z)$ for some integer m and

$$X_c(z) = \exp\left(\frac{1}{\pi} \int_0^\infty \frac{\theta(\tau)}{\tau - z} d\tau\right). \quad (1.208)$$

The canonical part $X_c(z)$ satisfies

$$X_c(z) = 1 - \nu b_\alpha(\nu) \frac{1}{z} + O(z^{-2}) \quad \text{as } z \rightarrow \infty, \quad (1.209)$$

with $b_\alpha(\nu) := \frac{1}{\pi} \int_0^\infty \theta(\nu\tau) d\tau$ and

$$X_c(z) = O\left(z^{\frac{\alpha-1}{2}}\right) \quad \text{as } z \rightarrow 0. \quad (1.210)$$

Proof. The general expression for $X(z)$ is obtained by the Sokhotski-Plemelj formula. The growth estimate (1.209) is obtained from powers expansion of the exponent and

$$\frac{1}{\pi} \int_0^\infty \frac{\theta(\tau)}{\tau - z} d\tau = -\nu b_\alpha(\nu) \frac{1}{z} + O(z^{-2}) \quad \text{as } z \rightarrow \infty.$$

Estimate (1.210) holds since $X_c(z) = O\left(z^{-\theta(0+)/\pi}\right) = O\left(z^{\frac{\alpha-1}{2}}\right)$ as $z \rightarrow 0$. □

Since $X(z)$ does not vanish on the cut plane, the functions

$$S(z) := \frac{\Phi_0(z) + \Phi_1(z)}{2X(z)} \quad \text{and} \quad D(z) := \frac{\Phi_0(z) - \Phi_1(z)}{2X(z)} \quad (1.211)$$

are sectionally holomorphic and by (1.203), satisfy decoupled boundary conditions

$$\begin{aligned} S^+(t) - S^-(t) &= 2ih(t)e^{-t}S(-t) \\ D^+(t) - D^-(t) &= -2ih(t)e^{-t}D(-t) \end{aligned} \quad (1.212)$$

where we defined

$$h(t) := e^{i\theta(t)} \sin \theta(t) \frac{X(-t)}{X^+(t)}.$$

Clearly, function $h(t)$ does not depend on the choice of the integer m . Applying the Sokhotski-Plemelj formula to the Cauchy integral in the definition of $X_c(z)$ and integrating by parts reveals that $h(t)$ is real valued, bounded and admits the expression

$$h(t) = \exp \left(-\frac{1}{\pi} \int_0^\infty \theta'(s) \log \left| \frac{t+s}{t-s} \right| ds \right) \sin \theta(t).$$

Therefore, in view of estimates (1.205) and (1.210), the functions in the right hand side of (1.212) are integrable on $\mathbb{R}_{\geq 0}$ for all $m \leq 0$. Moreover, $S(-t)$ and $D(-t)$ are square integrable on $\mathbb{R}_{>0}$. Hereafter we fix $m = 0$ for convenience, that is, set $X(z) := X_c(z)$.

For this choice the unique functions satisfying (1.212) are found again by the Sokhotski-Plemelj formula:

$$\begin{aligned} S(z) &= \frac{1}{\pi} \int_0^\infty \frac{h(t)e^{-t}}{t-z} S(-t) dt + P_S(z) \\ D(z) &= -\frac{1}{\pi} \int_0^\infty \frac{h(t)e^{-t}}{t-z} D(-t) dt + P_D(z) \end{aligned} \quad z \in \mathbb{C} \setminus \mathbb{R}_{\geq 0} \quad (1.213)$$

where $P_S(z)$ and $P_D(z)$ are polynomials, which must fit the a priori growth of $S(z)$ and $D(z)$ at infinity. In view of estimates (1.206) and (1.209)

$$\begin{aligned} P_S(z) &= -c_2(z + \nu b_\alpha(\nu)) + c_1 \\ P_D(z) &= -c_2(z + \nu b_\alpha(\nu)) - c_1 \end{aligned}$$

If we now set $z := -\nu t$ with $t > 0$ in (1.213) and change the integration variable, we obtain a pair of integral equations for $S(-\nu t)$ and $D(-\nu t)$:

$$\begin{aligned} S(-\nu t) &= \frac{1}{\pi} \int_0^\infty \frac{h(\nu\tau)e^{-\nu\tau}}{\tau+t} S(-\nu\tau) d\tau + \nu c_2(t - b_\alpha(\nu)) + c_1 \\ D(-\nu t) &= -\frac{1}{\pi} \int_0^\infty \frac{h(\nu\tau)e^{-\nu\tau}}{\tau+t} D(-\nu\tau) d\tau + \nu c_2(t - b_\alpha(\nu)) - c_1 \end{aligned}, \quad t > 0.$$

To solve them, consider the auxiliary integral equations

$$\begin{aligned} p_{\pm}(t) &= \pm \frac{1}{\pi} \int_0^{\infty} \frac{h(\nu\tau)e^{-\nu\tau}}{\tau+t} p_{\pm}(\tau) d\tau + 1 \\ q_{\pm}(t) &= \pm \frac{1}{\pi} \int_0^{\infty} \frac{h(\nu\tau)e^{-\nu\tau}}{\tau+t} q_{\pm}(\tau) d\tau + t \end{aligned}, \quad t > 0. \quad (1.214)$$

For all ν large enough, the integral operator on the right is a contraction on $L^2(0, \infty)$ and therefore equations (1.214) have unique solutions such that the functions $p^{\pm}(t) - 1$ and $q^{\pm}(t) - t$ belong to $L^2(0, \infty)$. We extend the definition of $p_{\pm}(\cdot)$ and $q_{\pm}(\cdot)$ to the cut plane by:

$$\begin{aligned} p_{\pm}(z) &:= \pm \frac{1}{\pi} \int_0^{\infty} \frac{h(\nu s)e^{-\nu s}}{s+z} p_{\pm}(s) ds + 1 \\ q_{\pm}(z) &:= \pm \frac{1}{\pi} \int_0^{\infty} \frac{h(\nu s)e^{-\nu s}}{s+z} q_{\pm}(s) d\tau + z \end{aligned}, \quad z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}.$$

Then, by linearity, for $z \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$

$$\begin{aligned} S(z) &= c_2\nu(q_+(-z/\nu) - b_{\alpha}(\nu)p_+(-z/\nu)) + c_1p_+(-z/\nu) \\ D(z) &= c_2\nu(q_-(-z/\nu) - b_{\alpha}(\nu)p_-(-z/\nu)) - c_1p_-(-z/\nu) \end{aligned}$$

and, plugging these expressions into definitions (1.211), we get

$$\begin{aligned} \Phi_0(z) &= c_2\nu X(z)(b_+(-z/\nu) - b_{\alpha}(\nu)a_+(-z/\nu)) + c_1X(z)a_-(-z/\nu) \\ \Phi_1(z) &= c_2\nu X(z)(b_-(-z/\nu) - b_{\alpha}(\nu)a_-(-z/\nu)) + c_1X(z)a_+(-z/\nu) \end{aligned} \quad (1.215)$$

where we defined

$$\begin{aligned} a_{\pm}(z) &:= p_+(z) \pm p_-(z) \\ b_{\pm}(z) &:= q_+(z) \pm q_-(z) \end{aligned} \quad (1.216)$$

Consequently the algebraic constraint (1.204) takes the form

$$c_2\nu\xi(\nu) + c_1\eta(\nu) = 0 \quad (1.217)$$

with

$$\begin{aligned}\xi(\nu) &:= e^{i\nu/2}X(\nu i)\left(b_+(-i) - b_\alpha(\nu)a_+(-i)\right) + e^{-i\nu/2}X(-\nu i)\left(b_-(i) - b_\alpha(\nu)a_-(i)\right) \\ \eta(\nu) &:= e^{i\nu/2}X(\nu i)a_-(-i) + e^{-i\nu/2}X(-\nu i)a_+(i).\end{aligned}\tag{1.218}$$

Since c_1 and c_2 are real, equation (1.217) has a nontrivial solution if and only if

$$\operatorname{Im}\{\xi(\nu)\overline{\eta(\nu)}\} = 0.\tag{1.219}$$

To recap, we obtain the following result:

Lemma 1.30. *Let (p_\pm, q_\pm, ν) with $\nu > 0$ be a solution of the integro-algebraic system, which consists of equations (1.214) and (1.219) and let $\Phi_0(z)$ and $\Phi_1(z)$ be defined by (1.215). Then the pair (φ, ν) where $\varphi(t)$ is defined by the Laplace transform (1.189) and λ is given by (1.199) solves the eigenproblem (1.188). Conversely, given a solution (φ, λ) of the eigenproblem (1.188), we can construct a solution to the above integro-algebraic system.*

Asymptotic analysis. The equivalent characterization of the eigenproblem in Lemma 1.30 can be used to derive an approximation for the eigenvalues, exact up to the second order term. To this end, we need the following estimates:

Lemma 1.31. *For any $\alpha_0 \in (0, 1)$, there are constants C_0 and ν_0 such that the following bounds hold for all $\nu \geq \nu_0$ and $\alpha \in [\alpha_0, 1]$:*

a) *Constant $b_\alpha(\nu)$ introduced in Lemma 1.29 satisfies*

$$b_\alpha(\nu) \leq (1 - \alpha)C_0\nu^{\alpha-1}$$

b) *Function $X(z) = X_c(z)$ from (1.208) satisfies*

$$\left|\arg\{X(\nu i)\}\right| \leq (1 - \alpha)C_0\nu^{\alpha-1} \quad \text{and} \quad \left||X(\nu i)| - 1\right| \leq (1 - \alpha)C_0\nu^{\alpha-1}$$

c) Functions $a_{\pm}(z)$ and $b_{\pm}(z)$ defined in (1.216) satisfy

$$\begin{aligned} |a_{-}(\pm i)| &\leq C_0\nu^{-1}, & |a_{+}(\pm i) - 2| &\leq C_0\nu^{-1} \\ |b_{-}(\pm i)| &\leq C_0\nu^{-2}, & |b_{+}(\pm i) \mp 2i| &\leq C_0\nu^{-2} \end{aligned}$$

Proof. a) The bound is obtained using expression (1.200):

$$b_{\alpha}(\nu) = \frac{1}{\pi} \int_0^{\infty} \theta(\nu\tau) d\tau \leq \frac{\sin \frac{1-\alpha}{2}}{\kappa_{\alpha}^{-1}\nu^{1-\alpha}} \int_0^{\infty} \frac{1}{(\tau^{3-\alpha} + \tau^{1-\alpha})} d\tau \leq (1-\alpha)C_0\nu^{\alpha-1}$$

where the last inequality holds since $\min_{0 \leq \alpha \leq 1} \kappa_{\alpha}^{-1} > 0$ and the integral is bounded uniformly over $\alpha \in [\alpha_0, 1]$ for all $\alpha_0 \in (0, 1)$.

b) Define $\Gamma(z) := \frac{1}{\pi} \int_0^{\infty} \frac{\theta(\tau)}{\tau-z} d\tau$, then

$$\begin{aligned} \left| \arg\{X(\nu i)\} \right| &= \left| \text{Im}\{\Gamma(\nu i)\} \right| = \frac{1}{\pi} \int_0^{\infty} \frac{\theta(\tau)}{\tau^2 + \nu^2} \nu d\tau = \frac{1}{\pi} \int_0^{\infty} \frac{\theta(\nu s)}{s^2 + 1} ds \leq \\ &\leq \frac{\sin \frac{1-\alpha}{2}\pi}{\kappa_{\alpha}^{-1}\nu^{1-\alpha}} \int_0^{\infty} \frac{1}{s^2 + 1} \frac{1}{(s^{3-\alpha} + s^{1-\alpha})} ds \leq (1-\alpha)C_0\nu^{\alpha-1}. \end{aligned}$$

Similarly

$$\text{Re}\{\Gamma(\nu i)\} = \frac{1}{\pi} \int_0^{\infty} \frac{\tau}{\tau^2 + \nu^2} \theta(\tau) d\tau \leq (1-\alpha)C_0\nu^{\alpha-1}$$

and hence for all ν large enough, $\left| |X(\nu i)| - 1 \right| = \left| e^{\text{Re}\{\Gamma(i\nu)\}} - 1 \right| \leq (1-\alpha)C_0\nu^{\alpha-1}$ as claimed.

c) The proof is similar to Lemma 5.6 and Lemma 5.7 in [12]. □

The following lemma gives asymptotic approximation for the algebraic part of the integro-algebraic system from Lemma 1.30:

Lemma 1.32. *The integro-algebraic system (1.214) and (1.219) has countably many solutions, which can be enumerated so that*

$$\nu_n = \pi n - \frac{\pi}{2} + g(\nu_n) + n^{-1}r_n(\alpha), \quad n \in \mathbb{Z} \tag{1.220}$$

where the residual $r_n(\alpha)$ is bounded uniformly over $\alpha \in [\alpha_0, 1]$ for any $\alpha_0 \in (0, 1)$ and

$$g(\nu) := -2 \arg\{X(\nu i)\} + \frac{\pi}{2} - \arg\{i + b_\alpha(\nu)\}.$$

Proof. Plugging the estimates of Lemma 1.31 into expressions from (1.218) we obtain

$$\xi(\nu)\overline{\eta(\nu)} = -4e^{\nu i} X(\nu i)^2 (i + b_\alpha(\nu)) (1 + R_1(\nu)) = -4e^{\nu i + \frac{\pi}{2}i} (1 + R_2(\nu))$$

where $|R_1(\nu)| \leq C\nu^{-1}$ and $|R_2(\nu)| \leq C\nu^{\alpha-1}$ with a constant C which depends only on α_0 . Hence equation (1.219) with $\nu > 0$ reads

$$\nu + \frac{\pi}{2} + \arctan \frac{\operatorname{Im}\{R_2(\nu)\}}{1 + \operatorname{Re}\{R_2(\nu)\}} = \pi n, \quad n \in \mathbb{Z}.$$

A tedious but otherwise straightforward calculation reveals that $|R_2'(\nu)| \leq C\nu^{\alpha-1}$ as well. Hence for each n large enough, say greater than some integer n_0 , the unique solution to the integro-algebraic system is obtained by fixed-point iterations. The algebraic component ν_n of this solution satisfies

$$\nu_n + 2 \arg\{X(\nu_n i)\} + \arg\{i + b_\alpha(\nu_n)\} + \arctan \frac{\operatorname{Im}\{R_1(\nu_n)\}}{1 + \operatorname{Re}\{R_1(\nu_n)\}} = \pi n \quad (1.221)$$

which yields the claimed formula since $|R_1(\nu)| \leq C\nu^{-1}$. □

Inversion of the Laplace transform. The eigenfunctions are recovered by inverting the Laplace transform (1.189):

Lemma 1.33. *Under the enumeration, introduced by Lemma 1.8, the eigenfunctions admit the approximation:*

$$\varphi_n(x) = \sqrt{2} \sin(\nu_n x) + n^{\alpha-1} r_n(x)$$

where the residual $r_n(x)$ is uniformly bounded in both $n \in \mathbb{N}$ and $x \in [0, 1]$

The proof is analogical to the one for Lemmas 1.7 and 1.9 and uses the asymptotics from Lemma 1.31.

Enumeration alignment. The particular enumeration $\nu_n(\alpha)$ fixed in Lemma 1.32 does not necessarily coincide with the enumeration $\tilde{\nu}_n(\alpha)$, which puts all the solutions in increasing order. Since $\nu_n(\alpha)$ is strictly increasing for all positive n large enough and ceases to be positive for all sufficiently negative n 's, the two enumerations may differ only by a shift $k(\alpha) \in \mathbb{Z}$ in the index, possibly dependent on α :

$$\tilde{\nu}_n(\alpha) = \nu_{n+k(\alpha)}(\alpha).$$

We will argue that this shift is constant with respect to $\alpha \in (0, 1]$ and therefore it can be identified using the exact formula for the eigenvalues, available in the case $\alpha = 1$, corresponding to a multiple of the standard Brownian motion:

$$\tilde{\nu}_n(1) = \pi n - \frac{\pi}{2}, \quad n = 1, 2, \dots \quad (1.222)$$

To this end, fix an $\alpha_0 \in (0, 1)$, then by (1.221)

$$|k(\alpha)| \leq |\tilde{\nu}_n(1) - \tilde{\nu}_n(\alpha)| + 3\pi, \quad \alpha \in [\alpha_0, 1].$$

By continuity of the spectrum, the right hand side is continuous with respect to α and therefore bounded by a constant which depends only on α_0 . Subtracting again (1.222) from (1.220) we get

$$|k(\alpha)| \leq \left| \tilde{\nu}_n(\alpha) - \tilde{\nu}_n(1) \right| + \left| g(\nu_{n+k(\alpha)}) \right| + \frac{1}{n+k(\alpha)} \left| r_{n+k(\alpha)}(\alpha) \right|. \quad (1.223)$$

Since $k(\alpha)$ and $r_n(\alpha)$ are uniformly bounded on $[\alpha_0, 1]$, we can choose n large enough so that the last term does not exceed an $\varepsilon > 0$ for all $\alpha \in [\alpha_0, 1]$. The estimates of Lemma 1.31 imply that $g(\nu) \leq (1 - \alpha)C_0\nu^{\alpha-1}$ and hence the second term in (1.223) can be made less than ε by further increasing n if necessary. Finally, by continuity of the spectrum the first term does not exceed ε for all α near 1. Therefore being a signed integer $k(\alpha)$ must equal zero on an open vicinity of $\alpha = 1$. Similarly, $k(\alpha)$ is shown to have no jumps on an open vicinity of any point $\alpha \in (\alpha_0, 1)$. By arbitrariness of α_0 it follows that $k(\alpha) = 0$ for all $\alpha \in (0, 1]$ and the formula claimed in Theorem 1.4 is obtained for $H > \frac{1}{2}$ by plugging (1.220) into (1.199) and replacing α with $2 - 2H$.

The case $H < \frac{1}{2}$

For these values of H , the kernel with $\alpha \in (1, 2)$ can no longer be written in the form (1.187) and some of the calculations are done differently from the previous case.

The Laplace transform. The following lemma is the analog of Lemma 1.27:

Lemma 1.34. For $\alpha \in (1, 2)$,

$$\widehat{\varphi}(z) - \widehat{\varphi}(0) = -\frac{1}{\Lambda(z)} \left(\Phi_0(z) + e^{-z} \Phi_1(-z) \right), \quad (1.224)$$

where the functions $\Phi_0(z)$ and $\Phi_1(z)$, defined in (1.227) below, are sectionally holomorphic on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$ and

$$\Lambda(z) = \frac{\Gamma(\alpha)}{|c_\alpha|} \frac{1}{z} + \lambda \frac{\Gamma(\alpha)}{|c_\alpha|} z - z \int_0^\infty \frac{2t^{\alpha-2}}{t^2 - z^2} dt \quad (1.225)$$

Proof. The eigenproblem now takes the form

$$\int_0^1 (y \wedge x) \psi'(y) dy + \int_0^1 \frac{1}{2} \left(y^{2-\alpha} + x^{2-\alpha} - |x-y|^{2-\alpha} \right) \psi'(y) dy = \lambda \psi'(x), \quad x \in [0, 1],$$

where $\alpha = 2 - 2H \in (1, 2)$ and we defined $\psi(x) = \int_x^1 \varphi(y) dy$. Integrating by parts under the second integral and taking two derivatives gives

$$\psi(x) + \left(1 - \frac{\alpha}{2}\right) \frac{d}{dx} \int_0^1 \text{sign}(x-y) |x-y|^{1-\alpha} \psi(y) dy = -\lambda \psi''(x), \quad x \in [0, 1],$$

Plugging identity (1.192) with α replaced by $\alpha - 1$, we get

$$u'_0(x) = -\frac{\Gamma(\alpha)}{|c_\alpha|} \left(\psi(x) + \lambda \psi''(x) \right), \quad (1.226)$$

where we defined

$$u(x, t) := \int_0^1 \text{sign}(x-y) e^{-t|x-y|} \psi(y) dy \quad \text{and} \quad u_0(x) := \int_0^\infty t^{\alpha-2} u(x, t) dt.$$

Calculations as in the proof of Lemma 1.27 give:

$$\widehat{u}'_0(z) = e^{-z} \int_0^\infty \frac{t^{\alpha-1}}{z+t} u(1,t) dt + \int_0^\infty \frac{t^{\alpha-1}}{z-t} u(0,t) dt + z(\widehat{\psi}'(z) + \psi(0)) \int_0^\infty \frac{2t^{\alpha-2}}{z^2-t^2} dt,$$

whereas by (1.226) we have

$$\widehat{u}'_0(z) = -\frac{\Gamma(\alpha)}{|c_\alpha|} (\widehat{\psi}(z) + \lambda \widehat{\psi}''(z)) = -\frac{\Gamma(\alpha)}{|c_\alpha|} \left(\frac{1}{z} (\widehat{\psi}'(z) + \psi(0)) + \lambda e^{-z} \psi'(1) + \lambda z \widehat{\psi}'(z) \right).$$

Combining the two expressions for $\widehat{u}'_0(z)$ we obtain (1.224) with

$$\begin{aligned} \Phi_0(z) &= \lambda \frac{\Gamma(\alpha)}{|c_\alpha|} \psi(0) z + \int_0^\infty \frac{t^{\alpha-1}}{t-z} u(0,t) dt \\ \Phi_1(z) &= -\lambda \frac{\Gamma(\alpha)}{|c_\alpha|} \psi'(1) - \int_0^\infty \frac{t^{\alpha-1}}{t-z} u(1,t) dt \end{aligned} \quad (1.227)$$

and $\Lambda(z)$ defined in (1.225). □

The following lemma summarizes the relevant properties of $\Lambda(z)$:

Lemma 1.35. *a) The function defined in (1.225) admits the following closed form expression*

$$\Lambda(z) = \lambda \frac{\Gamma(\alpha)}{|c_\alpha|} z + \frac{\Gamma(\alpha)}{|c_\alpha|} \frac{1}{z} + z^{\alpha-2} \frac{\pi}{|\cos \frac{\pi}{2} \alpha|} \begin{cases} e^{\frac{1-\alpha}{2} \pi i} & \arg(z) \in (0, \pi) \\ e^{-\frac{1-\alpha}{2} \pi i} & \arg(z) \in (-\pi, 0) \end{cases} \quad (1.228)$$

It vanishes only at zeros $\pm z_0 = \pm i\nu$ where $\nu > 0$ is the only solution of equation (1.199).

b) $\Lambda(z)$ is analytic on the cut plane $\mathbb{C} \setminus \mathbb{R}$ and its limits across the real line

$$\Lambda^\pm(t) = \lambda \frac{\Gamma(\alpha)}{|c_\alpha|} t + \frac{\Gamma(\alpha)}{|c_\alpha|} \frac{1}{t} \pm |t|^{\alpha-2} \frac{\pi}{|\cos \frac{\pi}{2} \alpha|} \begin{cases} e^{\frac{1\mp\alpha}{2} \pi i}, & t > 0 \\ e^{\frac{1\pm\alpha}{2} \pi i}, & t < 0 \end{cases}$$

satisfy the symmetries (1.45)-(1.47).

c) The argument $\theta(t) := \arg\{\Lambda^+(t)\} \in (-\pi, \pi]$ satisfies $\theta(-t) = -\pi - \theta(t)$ and is given by

$$\theta(t) = \arctan \frac{(t/\nu)^{\alpha-3} \sin \frac{1-\alpha}{2} \pi}{\kappa_\alpha^{-1} \nu^{1-\alpha} (1 + (t/\nu)^{-2}) + 1 + (t/\nu)^{\alpha-3} \cos \frac{1-\alpha}{2} \pi}, \quad t > 0.$$

It is a continuous negative function with $\theta(0+) = \lim_{t \rightarrow 0+} \theta(t) = 0$ and $\lim_{t \rightarrow \infty} \theta(t) = 0$. Moreover, for any $\alpha_0 \in (0, 1)$, there is a constant C_0 such that

$$\left| \frac{1}{\pi} \int_0^\infty \theta(\nu t) dt - b_\alpha \right| \leq C_0 \nu^{1-\alpha} \quad (1.229)$$

where

$$b_\alpha = \frac{\sin\left(\frac{\pi}{3-\alpha} \frac{1-\alpha}{2}\right)}{\sin \frac{\pi}{3-\alpha}} = \tan\left(\frac{1-\alpha}{3-\alpha} \frac{\pi}{2}\right).$$

Proof. The explicit formula (1.228) is obtained by means of identity (1.201) with $\alpha \in (0, 1)$ replaced by $\alpha - 2 \in (-1, 0)$. Since $\overline{\Lambda(z)} = \Lambda(\bar{z})$ all zeros of $\Lambda(z)$ which are not purely real appear in conjugate pairs. Hence it suffices to find the zeros in the upper half plane. To this end, let $z = \nu e^{i\phi}$ with $\nu > 0$ and $\phi \in (0, \pi)$. Then equation $\Lambda(z) = 0$ is equivalent to

$$\lambda \kappa_\alpha^{-1} \nu^2 e^{2\phi i} + \kappa_\alpha^{-1} + \nu^{\alpha-1} e^{i(\phi - \frac{\pi}{2})(\alpha-1)} = 0 \quad (1.230)$$

where $\kappa_\alpha := \frac{|c_\alpha|}{\Gamma(\alpha)} \frac{\pi}{|\cos \frac{\pi}{2}\alpha|} = \frac{c_\alpha}{\Gamma(\alpha)} \frac{\pi}{\cos \frac{\pi}{2}\alpha}$. Equating the imaginary and real parts to zero gives

$$\begin{aligned} \lambda \kappa_\alpha^{-1} \nu^2 \sin(2\phi) + \nu^{\alpha-1} \sin\left(\left(\phi - \frac{\pi}{2}\right)(\alpha - 1)\right) &= 0 \\ \lambda \kappa_\alpha^{-1} \nu^2 \cos(2\phi) + \nu^{\alpha-1} \cos\left(\left(\phi - \frac{\pi}{2}\right)(\alpha - 1)\right) + \kappa_\alpha^{-1} &= 0. \end{aligned}$$

Plugging the first equation into the second, we obtain

$$g(\phi) + \kappa_\alpha^{-1} \nu^{1-\alpha} = 0, \quad \phi \in (0, \pi) \setminus \left\{\frac{\pi}{2}\right\} \quad (1.231)$$

where

$$g(\phi) := -\sin\left(\left(\phi - \frac{\pi}{2}\right)(\alpha - 1)\right) \cot(2\phi) + \cos\left(\left(\phi - \frac{\pi}{2}\right)(\alpha - 1)\right).$$

This function is continuous on $(0, \pi) \setminus \left\{\frac{\pi}{2}\right\}$ and diverges to $+\infty$ as $\phi \rightarrow 0$ or $\phi \rightarrow \pi$. Its zeros must also solve the equation

$$\tan\left(\left(\phi - \frac{\pi}{2}\right)(\alpha - 1)\right) = \tan(2\phi)$$

which is easily seen to have no roots on $(0, \pi) \setminus \left\{\frac{\pi}{2}\right\}$. Therefore $g(\phi) > 0$ in (1.231) and equation (1.230) does not have solutions with $\phi \neq \frac{\pi}{2}$ in the upper half plain. For $\phi = \frac{\pi}{2}$

(1.230) admits a pair of purely imaginary zeros and reduces to (1.199). The limits in (b) and the expression for $\theta(t)$ in (c) are derived from (1.228) by a direct calculation. The estimate (1.229) holds since

$$b_\alpha = \frac{1}{\pi} \int_0^\infty \arctan \frac{\tau^{\alpha-3} \sin \frac{1-\alpha}{2} \pi}{1 + \tau^{\alpha-3} \cos \frac{1-\alpha}{2} \pi} d\tau$$

and therefore

$$\left| \frac{1}{\pi} \int_0^\infty \theta(\nu\tau) d\tau - b_\alpha \right| \leq \nu^{1-\alpha} \kappa_\alpha^{-1} \sin \frac{1-\alpha}{2} \pi \int_0^\infty \frac{(1 + \tau^{-2}) \tau^{\alpha-3}}{1 + \tau^{2\alpha-6}} d\tau \quad (1.232)$$

where we used the identity $\arctan x - \arctan y = \arctan(x - y)/(1 + xy)$.

□

Removal of singularities. The removal of singularities gives the same conditions (1.203) and (1.204) as before. Since $tu(0, t)$ and $tu(1, t)$ are uniformly bounded, cf. (1.206)

$$\Phi_0(z) = -2c_2 z + O(z^{\alpha-2}) \quad \text{and} \quad \Phi_1(z) = 2c_1 + O(z^{\alpha-2}) \quad \text{as } z \rightarrow \infty$$

where we defined

$$c_1 = -\frac{1}{2} \lambda \frac{\Gamma(\alpha)}{|c_\alpha|} \psi'(1) \quad \text{and} \quad c_2 = -\frac{1}{2} \lambda \frac{\Gamma(\alpha)}{|c_\alpha|} \psi(0)$$

and, cf. (1.205),

$$\Phi_0(z) = O(1) \quad \text{and} \quad \Phi_1(z) = O(1) \quad z \rightarrow 0.$$

An equivalent formulation of the eigenproblem. As in the previous case, the starting point is to find a suitable solution to the homogeneous Riemann boundary value problem (cf. Lemma 1.29):

Lemma 1.36. *Any sectionally holomorphic and nonvanishing function on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$, satisfying the boundary condition (1.207) is given by $X(z) = z^m X_c(z)$ for some integer m and $X_c(z)$ defined in (1.208). The canonical part $X_c(z)$ satisfies*

$$X_c(z) = 1 - \nu b_\alpha(\nu) \frac{1}{z} + O(z^{\alpha-3}) \quad \text{as } z \rightarrow \infty,$$

with $b_\alpha(\nu) := \frac{1}{\pi} \int_0^\infty \theta(\nu\tau) d\tau$ and

$$X_c(z) = O(1), \quad \text{as } z \rightarrow 0.$$

Moreover for any $\alpha_0 \in (1, 2)$, there is a constant C_0 such that

$$\left| \arg\{X_c(\nu i)\} - \frac{1-\alpha}{8}\pi \right| \leq C_0\nu^{1-\alpha} \quad \text{and} \quad \left| |X_c(\nu i)| - \sqrt{\frac{3-\alpha}{2}} \right| \leq C_0\nu^{1-\alpha} \quad (1.233)$$

Proof. Define

$$\theta_0(\tau) := \arctan \frac{\tau^{\alpha-3} \sin \frac{1-\alpha}{2}\pi}{1 + \tau^{\alpha-3} \cos \frac{1-\alpha}{2}\pi}, \quad \tau > 0$$

and

$$X_0(i) := \exp \left(\frac{1}{\pi} \int_0^\infty \frac{\theta_0(\tau)}{\tau - i} d\tau \right).$$

It is shown in Lemma 5.5 [12] that

$$\arg\{X_0(i)\} = \frac{1-\alpha}{8}\pi \quad \text{and} \quad |X_0(i)| = \sqrt{\frac{3-\alpha}{2}}.$$

The estimates in (1.233) are obtained by bounding $|\theta(\nu\tau) - \theta_0(\tau)|$ as in (1.232). \square

All the calculations and the assertions of Lemma 1.30 continues to hold.

Asymptotic analysis. In this case estimates (a) and (b) of Lemma 1.31 are replaced with (1.229) and (1.233) and the bounds in (c) continue to hold. Hence the assertion of Lemma 1.32 remains intact. The formulae claimed in Theorem 1.4 for $H < \frac{1}{2}$ are derived by the calibration procedure, using estimates (1.229) and (1.233).

Inversion of the Laplace transform. The eigenfunctions are recovered by inverting the Laplace transform (1.224). The following lemma is analogue to Lemma 1.9 and follows from the asymptotics of Lemma 1.36:

Lemma 1.37. *The eigenfunctions admit the approximation:*

$$\begin{aligned} \varphi_n(x) = & \sqrt{2} \cos \left(\nu_n x + \frac{1-\alpha}{8} \pi + \frac{\pi}{2} - \frac{1-\alpha}{3-\alpha} \frac{\pi}{2} \right) + \\ & + \frac{\sqrt{3-\alpha}}{\pi} \int_0^\infty \rho_0(u) \left(-e^{-u\nu_n x} \frac{u-b_\alpha}{\sqrt{1+b_\alpha^2}} - (-1)^n e^{-(1-x)u\nu_n} \right) du + n^{1-\alpha} r_n(x), \end{aligned}$$

where the residual $r_n(x)$ is uniformly bounded in both $n \in \mathbb{N}$ and $x \in [0, 1]$ and $\rho_0(u)$ is given by (1.84).

1.4 Numerical experiments

1.4.1 Preliminaries

Theorems 1.2, 1.3 and 1.4 provide an approximation for the eigenvalues and eigenfunctions, which is asymptotically exact and therefore more accurate for smaller eigenvalues. By keeping track of all constants in the proofs, it is possible to obtain rough estimates for the residual terms in the suggested formulas; however, they are likely to be quite conservative. Numerical experiments, presented in this section, indicate that the actual accuracy of our approximation is surprisingly good, already, for relatively small values of n .

To solve our eigenproblems numerically we will use the quadrature method, which replaces integration with a numerical approximation. Since the kernels under consideration are smooth, it will suffice to work with uniform nodes and weights, in which case the eigenproblem (1.1) is replaced with the system of linear equations:

$$\sum_{i=0}^L K\left(\frac{i}{L}, \frac{j}{L}\right) \varphi_i \frac{1}{L} = \lambda \varphi_j, \quad j = 0, \dots, L.$$

The eigenvalues $\widehat{\lambda}_{n,L}$ and the corresponding eigenvectors $\widehat{\varphi}_{n,L}$, $n = 0, \dots, L$ for this problem approximate the solutions to the original eigenproblem of interest in the sense:

$$\lim_{L \rightarrow \infty} \widehat{\lambda}_{n,L} = \lambda_n \quad \text{and} \quad \lim_{L \rightarrow \infty} \widehat{\varphi}_{n,L}([xL]) = \varphi_n(x).$$

The convergence can be quantified, under appropriate assumptions on kernel K (see, e.g., [47]).

We will compare the numerical solutions to the approximations $\tilde{\lambda}_n$ and $\tilde{\varphi}_n$ of the eigenvalues and eigenfunctions, obtained by truncating the residual terms in our asymptotic formulas.

As a test case we consider the Brownian motion for which the exact formula for the eigenvalues is well known:

$$\lambda_n = \frac{1}{(n - \frac{1}{2})^2 \pi^2} \quad \text{and} \quad \varphi_n(t) = \sqrt{2} \sin\left((n - \frac{1}{2})\pi t\right), \quad n = 1, 2, \dots$$

Since λ_n rapidly decrease, it will be more convenient to compare the empirical frequency $\hat{\nu}_{n,L} = \hat{\lambda}_{n,L}^{-1/2}$ with the corresponding theoretical frequencies $\nu_n = (n - \frac{1}{2})\pi$. The corresponding results are presented in Table 1.1 and illustrated on the plots of Figure 1.2.

L	1000	3000	5000	8000	10000
$\hat{\lambda}_{50,L} \times 10^5$	4.1268	4.1342	4.1348	4.1350	4.1351
$\hat{\nu}_{50,L}$	155.6656	155.5262	155.5151	155.5113	155.5104
$ \hat{\lambda}_{50,L}/\lambda_{50} - 1 \times 100\%$	0.2013	0.0224	0.0081	0.0031	0.0020
$ \hat{\nu}_{50,L}/\nu_{50} - 1 \times 100\%$	0.1008	0.0112	0.00403	0.00157	0.00101

Table 1.1 – Numerical approximations versus theoretical values for different L .

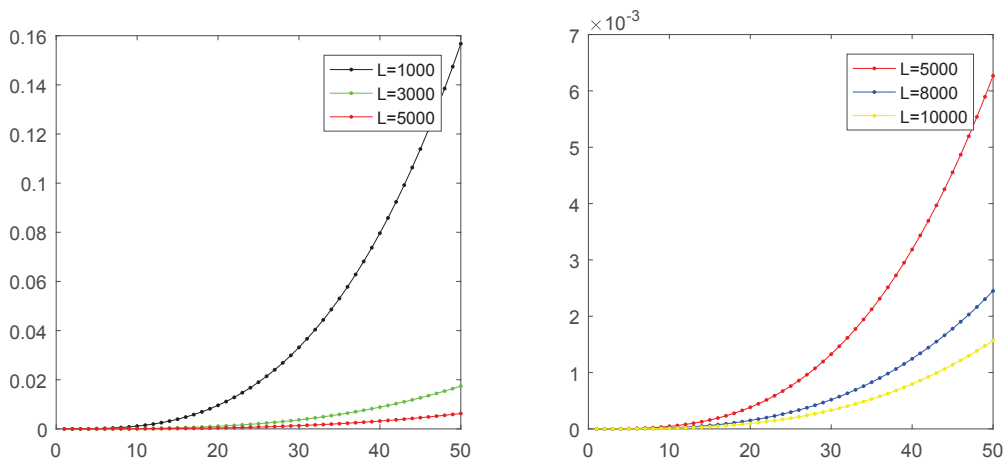


Figure 1.2 – Approximation error $\hat{\nu}_{n,L} - \nu_n$ for $n \in \{1, \dots, 50\}$ and different L .

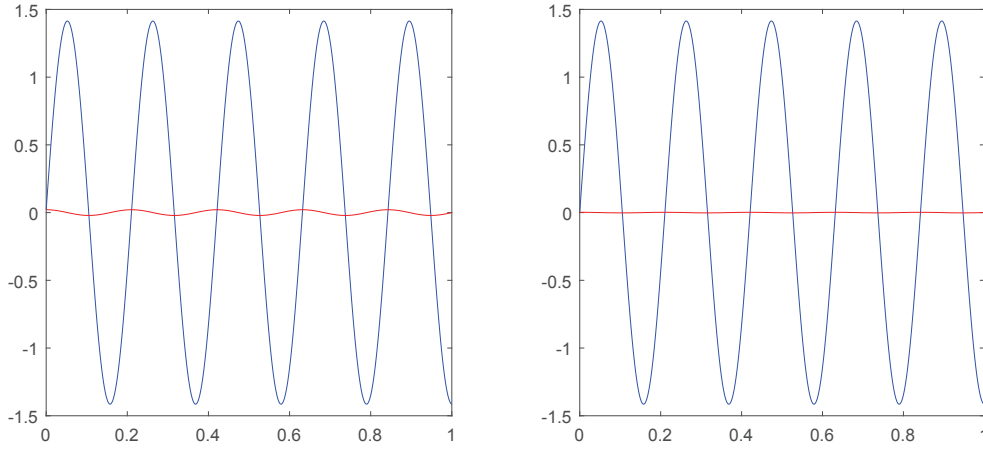


Figure 1.3 – Numerical estimate $\hat{\varphi}_{10,L}$ of the eigenfunction (blue) versus the approximation error (red) for $L = 10^3$ (left) and $L = 10^4$ (right).

In practical terms, L is chosen, so that its further increase does not improve the accuracy beyond the desired decimal digit. Consequently, for the following calculations we are going to take $L = 10^4$ since the corresponding values of relative error are quiet stable.

1.4.2 Fractional Ornstein-Uhlenbeck process.

A calculation shows that for the fractional Ornstein–Uhlenbeck process with the Hurst parameter H and the drift parameter β the kernel in (1.12) can be rewritten in the form

$$\begin{aligned}
 K(s, t) = & \frac{c_\alpha}{2\beta} e^{\beta(t+s)} \left(\Phi(t, \alpha, \beta) + \Phi(s, \alpha, \beta) \right) \\
 & - \frac{c_\alpha}{2\beta} e^{\beta(s-t)} \left(\Phi(t, \alpha, -\beta) - \Phi(t-s, \alpha, -\beta) \right) \\
 & - \frac{c_\alpha}{2\beta} e^{\beta(t-s)} \left(\Phi(t-s, \alpha, \beta) + \Phi(s, \alpha, -\beta) \right), \quad t > s
 \end{aligned}$$

where $\alpha = 2 - 2H$, $c_\alpha = (1 - \frac{\alpha}{2})(1 - \alpha)$ and function

$$\Phi(t, \alpha, \beta) = \int_0^t e^{-\beta x} x^{-\alpha} dx$$

can be simplified through a simple scaling to the incomplete Gamma function, routinely

available in numerical packages.

The empirical frequency $\hat{\nu}_{n,L}$ is obtained by solving the equation

$$\hat{\lambda}_{n,L} = \sin(\pi H)\Gamma(2H + 1) \frac{\nu^{1-2H}}{\nu^2 + \beta^2} \quad (1.234)$$

and compared to the corresponding theoretical approximation $\tilde{\nu}_n$, given by (1.5) after truncating the residual term. Transcendental equation (1.234) must also be solved numerically, but its root can be located to within *any* desired precision and hence this computation does not introduce any significant error.

Our study cases are the fractional Ornstein-Uhlenbeck processes with $H = \frac{3}{4}$ and $\beta = -1$ and $\beta = -1$. The obtained numbers are summarized in Tables 1.2 and 1.3 and illustrated graphically on the plots of Figure 1.4, which gives an idea about the actual magnitude of the $O(n^{-1})$ residual in the approximation (1.5).

n	1	2	3	4	5	6	7	8	9	10
$\hat{\lambda}_{n,L} \times 10^3$	182.46	17.62	5.348	2.3210	1.2519	0.7574	0.5005	0.3495	0.2560	0.1937
$\hat{\nu}_{n,L}$	1.7133	4.8245	7.8551	11.003	14.104	17.255	20.373	23.523	26.648	29.79
$ \hat{\lambda}_{n,L}/\tilde{\lambda}_n - 1 \times 100\%$	23.5	9.3	2.5	2.0	0.8	0.8	0.4	0.4	0.3	0.3
$ \hat{\nu}_{n,L}/\tilde{\nu}_n - 1 \times 100\%$	14.8	4.1	1.1	0.8	0.4	0.4	0.2	0.2	0.1	0.1

Table 1.2 – Numerical versus asymptotic approximation: obtained for $L = 10^4$, $H = \frac{3}{4}$ and $\beta = -1$.

n	1	2	3	4	5	6	7	8	9	10
$\hat{\lambda}_{n,L} \times 10^3$	860.13	22.12	5.731	2.4139	1.2797	0.7693	0.5059	0.3525	0.2577	0.1947
$\hat{\nu}_{n,L}$	0.6215	4.3901	7.6384	10.8314	13.9804	17.1479	20.2866	23.445	26.582	29.73
$ \hat{\lambda}_{n,L}/\tilde{\lambda}_n - 1 \times 100\%$	260.6	13.9	4.5	1.9	1.4	0.8	0.7	0.4	0.4	0.3
$ \hat{\nu}_{n,L}/\tilde{\nu}_n - 1 \times 100\%$	58.4	5.3	1.8	0.8	0.6	0.3	0.3	0.2	0.2	0.1

Table 1.3 – Numerical versus asymptotic approximation: obtained for $L = 10^4$, $H = \frac{3}{4}$ and $\beta = 1$.

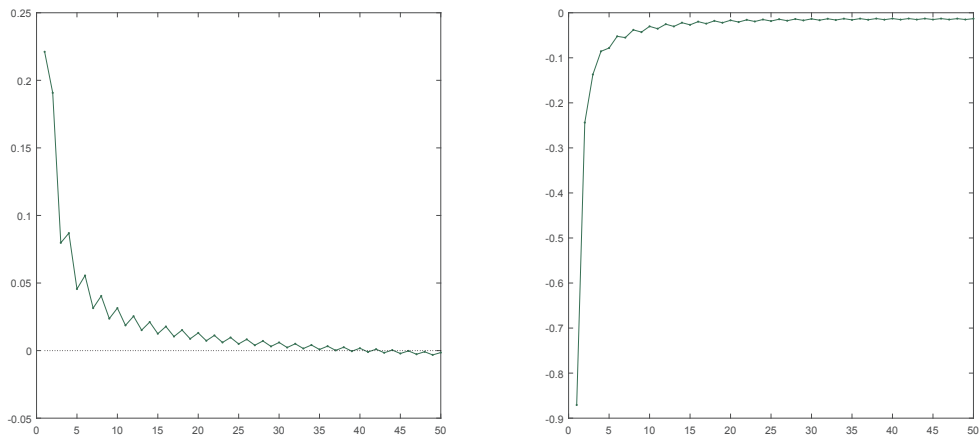


Figure 1.4 – Approximation error $\hat{\nu}_{n,L} - \tilde{\nu}_n$ for $L = 10^4$, $H = \frac{3}{4}$ and $\beta = -1$ (left) and $\beta = 1$ (right).

The plots of Figures 1.5 and 1.6 illustrate the numerical estimate $\hat{\varphi}_{n,L}$, along with the error of approximating the eigenfunction only by the oscillatory term from (1.6). Up to an $O(n^{-1})$ residual, this error coincides with the boundary layer in (1.6), which pushes $\varphi_n(0)$ to zero and $\varphi_n(1)$ to $\sqrt{2H + 1}$.

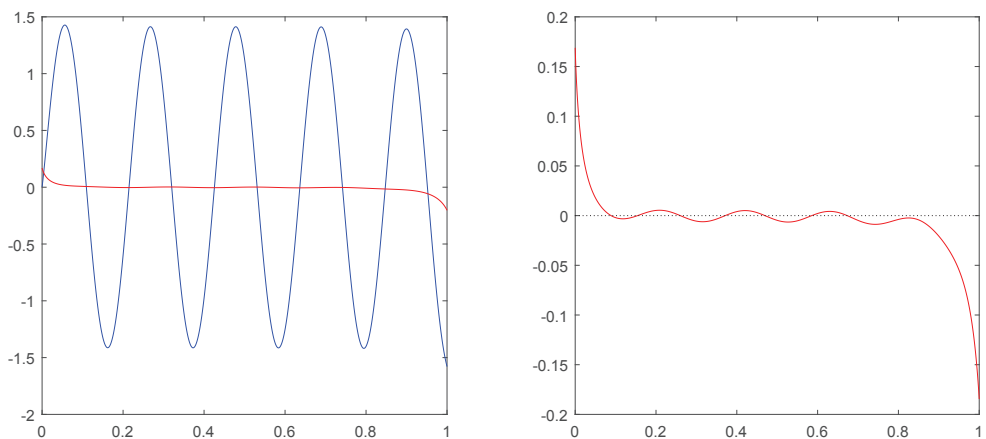


Figure 1.5 – Numerical estimate $\hat{\varphi}_{10,L}$ of the eigenfunction (blue) versus the approximation error (red) for $L = 10^3$, $H = \frac{3}{4}$ and $\beta = -1$.

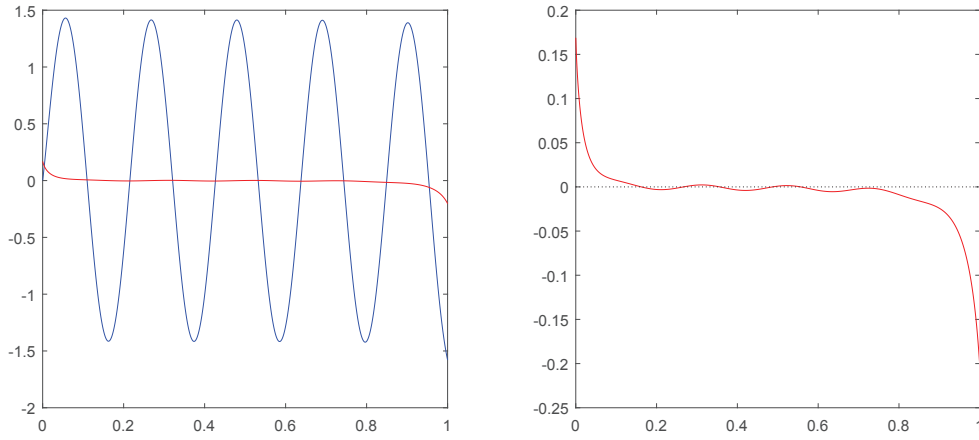


Figure 1.6 – Numerical estimate $\widehat{\varphi}_{10,L}$ of the eigenfunction (blue) versus the approximation error (red) for $L = 10^3$, $H = \frac{3}{4}$ and $\beta = 1$.

1.4.3 Integrated fractional Brownian motion.

For the integrated fractional Brownian motion with the Hurst parameter H the covariance function (1.14) can be rewritten in the following form:

$$K(s, t) = \frac{1}{4H + 2} \left(ts^{2H+1} + st^{2H+1} - \frac{1}{2H + 2} (t^{2H+2} + s^{2H+2} - |t - s|^{2H+2}) \right). \quad (1.235)$$

The empirical frequency $\widehat{\nu}_{n,L}$ is obtained from (1.15) and compared to the corresponding theoretical approximation $\widetilde{\nu}_n$, given by (1.16) after truncating the residual term.

We consider the integrated fractional Brownian motions with $H = \frac{3}{4}$ and $H = 14$ and summarize the obtained numbers in Tables 1.4 and 1.5. The corresponding illustrations are presented on the plots of Figures 1.7-1.9.

n	1	2	3	4	5	6	7	8	9	10
$\widehat{\lambda}_{n,L} \times 10^3$	62.4679	0.9151	0.0917	0.0197	0.0064	0.0026	0.0012	0.0006	0.0004	0.0002
$\widehat{\nu}_{n,L}$	1.8267	4.6693	7.7846	10.9542	14.0811	17.236	20.368	23.518	26.65	29.8
$ \widehat{\lambda}_{n,L}/\widetilde{\lambda}_n - 1 \times 100\%$	55.3	0.06	1.5	0.09	0.4	0.013	0.2	0.004	0.12	0.011
$ \widehat{\nu}_{n,L}/\widetilde{\nu}_n - 1 \times 100\%$	19.6	0.012	0.33	0.02	0.088	0.003	0.043	0.002	0.027	0.001

Table 1.4 – Numerical versus asymptotic approximation: obtained for $L = 10^4$ and $H = \frac{3}{4}$.

n	1	2	3	4	5	6	7	8	9	10
$\widehat{\lambda}_{n,L} \times 10^3$	110.63	2.91	0.48	0.144	0.06	0.0296	0.0165	0.0101	0.0064	0.0043
$\widehat{\nu}_{n,L}$	1.6413	4.6390	7.7783	10.9412	14.0682	17.224	20.354	23.51	26.64	29.8
$ \widehat{\lambda}_{n,L}/\widetilde{\lambda}_n - 1 \times 100\%$	24.5	1.31	0.883	0.0553	0.32	0.0252	0.165	0.0077	0.104	0.0012
$ \widehat{\nu}_{n,L}/\widetilde{\nu}_n - 1 \times 100\%$	8.36	0.372	0.251	0.016	0.091	0.0072	0.0471	0.0022	0.0298	0.0003

Table 1.5 – Numerical versus asymptotic approximation: obtained for $L = 10^4$ and $H = \frac{1}{4}$.

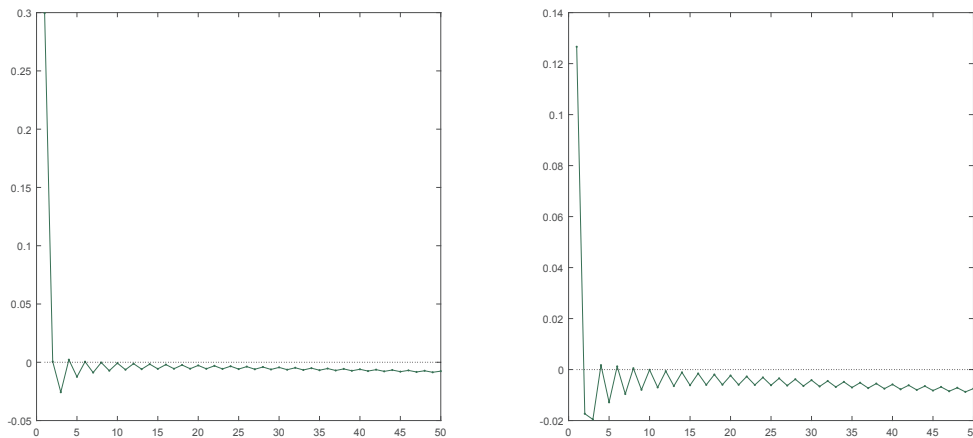


Figure 1.7 – Approximation error $\widehat{\nu}_{n,L} - \widetilde{\nu}_n$ for $L = 10^4$, $H = \frac{3}{4}$ (left) and $H = \frac{1}{4}$ (right).

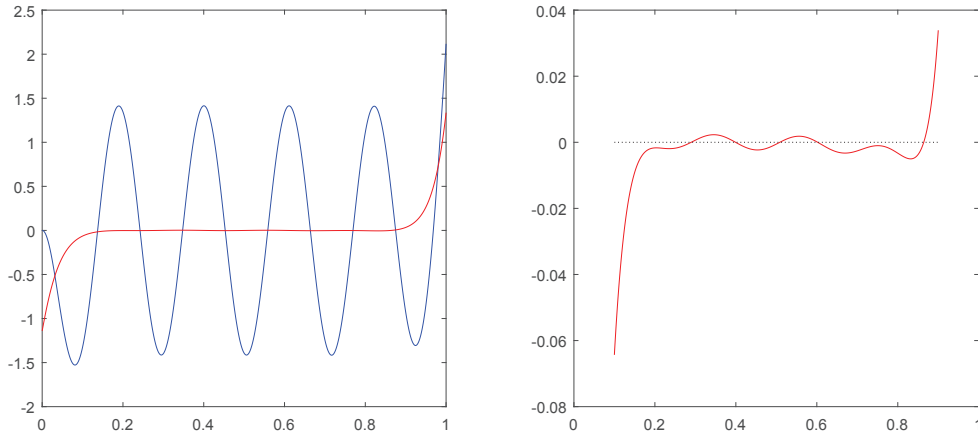


Figure 1.8 – Numerical estimate $\widehat{\varphi}_{10,L}$ of the eigenfunction (blue) versus the approximation error (red) for $L = 10^3$ and $H = \frac{3}{4}$.

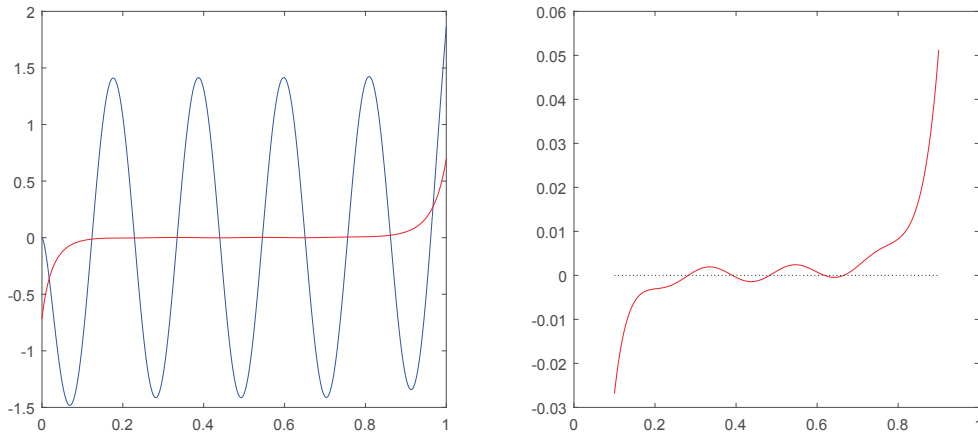


Figure 1.9 – Numerical estimate $\widehat{\varphi}_{10,L}$ of the eigenfunction (blue) versus the approximation error (red) for $L = 10^3$ and $H = \frac{1}{4}$.

1.4.4 Mixed fractional Brownian motion.

For the mixed fractional Brownian motion the empirical frequency $\widehat{\nu}_{n,L}$ is obtained by solving the equation

$$\lambda_n := \frac{1}{\nu_n^2} + \frac{\sin(\pi H)\Gamma(2H + 1)}{\nu_n^{2H+1}}, \quad n = 1, 2, \dots$$

numerically and compared to the corresponding theoretical approximation \tilde{v}_n , given by (1.20) after truncating the residual term.

We investigate the mixed fractional Brownian motions with $H = \frac{3}{4}$ and $\frac{1}{4}$. The obtained results are presented in Tables 1.6 and 1.7 and illustrated graphically on the plots of Figures 1.10-1.12.

n	1	2	3	4	5	6	7	8	9	10
$\hat{\lambda}_{n,L} \times 10^3$	771.65	66.13	21.91	10.695	6.289	4.124	2.9067	2.1559	1.6614	1.3184
$\hat{v}_{n,L}$	1.5121	4.6592	7.8098	10.957	14.1	17.245	20.39	23.5	26.7	29.8
$ \hat{\lambda}_{n,L}/\tilde{\lambda}_n - 1 \times 100\%$	8.8	2.5	1.2	0.74	0.55	0.4	0.34	0.27	0.235	0.197
$ \hat{v}_{n,L}/\tilde{v}_n - 1 \times 100\%$	3.7	1.1	0.56	0.35	0.26	0.19	0.16	0.129	0.113	0.095

Table 1.6 – Numerical versus asymptotic approximation: obtained for $L = 10^4$ and $H = \frac{3}{4}$.

n	1	2	3	4	5	6	7	8	9	10
$\hat{\lambda}_{n,L} \times 10^3$	845.8	110.5	46.04	25.95	17.07	12.23	9.29	7.35	5.99	5.01
$\hat{v}_{n,L}$	1.4391	4.6059	7.7155	10.869	13.99	17.1451	20.28	23.42	26.6	29.7
$ \hat{\lambda}_{n,L}/\tilde{\lambda}_n - 1 \times 100\%$	0.09	0.91	0.17	0.06	0.12	0.03	0.1	0.05	0.08	0.04
$ \hat{v}_{n,L}/\tilde{v}_n - 1 \times 100\%$	0.05	0.53	0.1	0.04	0.08	0.02	0.06	0.03	0.05	0.03

Table 1.7 – Numerical versus asymptotic approximation: obtained for $L = 10^4$ and $H = \frac{1}{4}$.

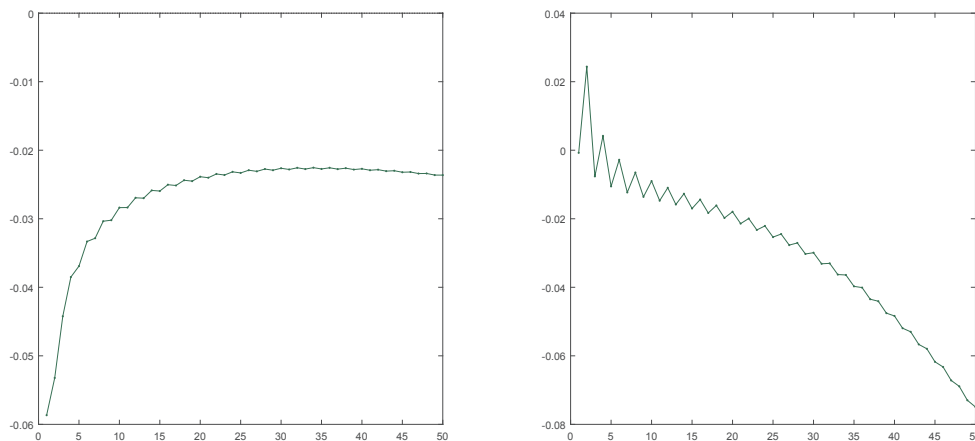


Figure 1.10 – Approximation error $\hat{v}_{n,L} - \tilde{v}_n$ for $L = 10^4$, $H = \frac{3}{4}$ (left) and $H = \frac{1}{4}$ (right).

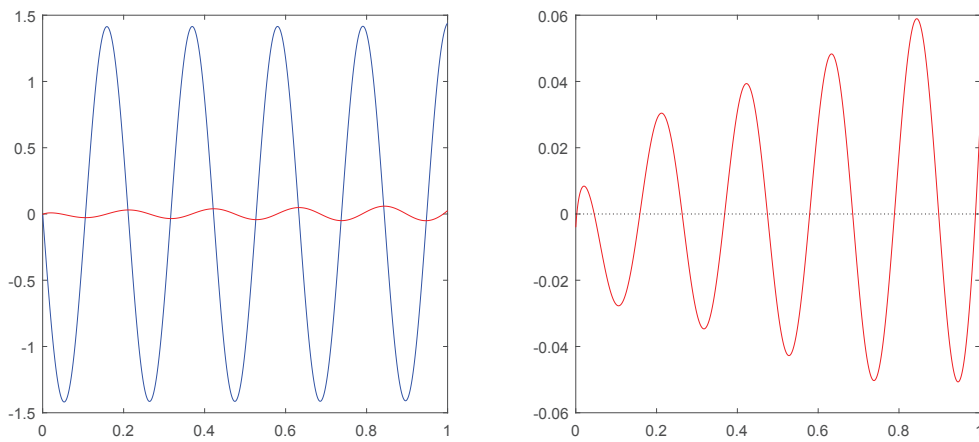


Figure 1.11 – Numerical estimate $\hat{\varphi}_{10,L}$ of the eigenfunction (blue) versus the approximation error (red) for $L = 10^3$ and $H = \frac{3}{4}$.

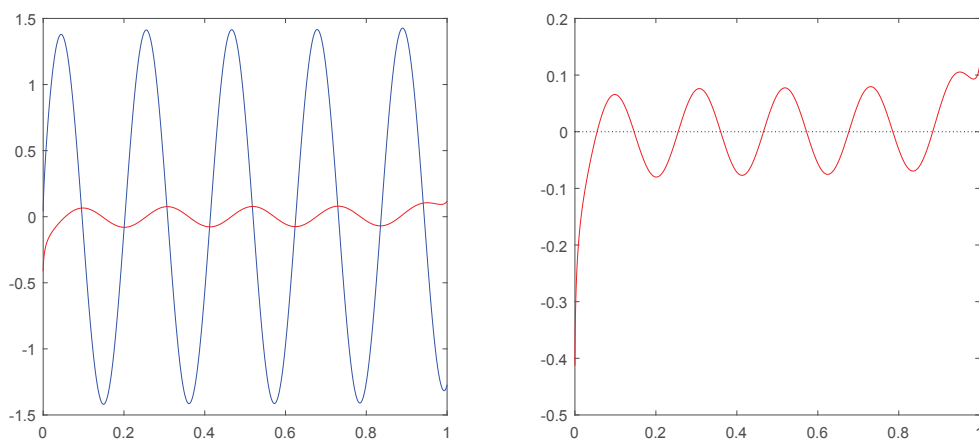


Figure 1.12 – Numerical estimate $\hat{\varphi}_{10,L}$ of the eigenfunction (blue) versus the approximation error (red) for $L = 10^3$ and $H = \frac{1}{4}$.

COVARIANCE EIGENPROBLEM FOR GAUSSIAN BRIDGES

2.1 Introduction

2.1.1 Gaussian bridges

In this chapter we consider the eigenproblem for Gaussian bridges. For a *base* process $X = (X_t, t \in [0, 1])$, the corresponding bridge $\tilde{X} = (\tilde{X}_t, t \in [0, 1])$ is obtained by "restricting" the trajectories to start and terminate at the given points.

For Gaussian processes, such restriction amounts to the usual conditioning. Hence if X is a centered Gaussian *base* process with the starting point $X_0 = 0$ and covariance function $K(s, t)$, the corresponding zero to zero bridge is the centered Gaussian process

$$\tilde{X}_t = X_t - \frac{K(t, 1)}{K(1, 1)} X_1, \quad t \in [0, 1]$$

with the covariance function

$$\tilde{K}(s, t) = K(s, t) - \frac{K(s, 1)K(t, 1)}{K(1, 1)}. \quad (2.1)$$

Various aspects of general Gaussian bridges are discussed in [20], [49]. Aside of being interesting mathematical objects, they are important ingredients in applications, such as statistical hypothesis testing [29], exact sampling of diffusions [5], etc.

2.1.2 Covariance eigenproblem for bridges

The covariance operator of the bridge with kernel (2.1) is a rank one perturbation of the covariance operator of its base process. This implies similarity between spectral

decomposition of the bridge and the *base* process and suggests that the spectra of the two processes must be closely related in general.

To illustrate, consider the Brownian motion with $K(s, t) = t \wedge s$ and the Brownian bridge with covariance function $\widetilde{K}(s, t) = s \wedge t - st$. Then explicit solution to the eigenproblem (1.1) for the *base* process is given by:

$$\lambda_n = \frac{1}{\left((n - \frac{1}{2})\pi\right)^2} \quad \text{and} \quad \varphi_n(t) = \sqrt{2} \sin\left(n - \frac{1}{2}\right)\pi t \quad (2.2)$$

and solution to the eigenproblem for bridge is

$$\tilde{\lambda}_n = \frac{1}{(\pi n)^2} \quad \text{and} \quad \tilde{\varphi}_n(t) = \sqrt{2} \sin \pi n t. \quad (2.3)$$

Thus, the formulas for eigenvalues and eigenfunctions have the identical form, but there is a shift in frequencies for $\pi/2$. This is indeed the case and one can find an exact expression for the Fredholm determinant of \widetilde{K} in terms of the Fredholm determinant of K even for more general finite rank perturbations (see, e.g., [50], Ch.II, 4.6 in [24]).

As mentioned before, the precise formulas for the eigenvalues and eigenfunctions of K are rarely known; however, the exact asymptotic approximations can be more tractable. This raises the following question:

Can the exact asymptotics of the eigenvalues and the eigenfunctions for the bridge be deduced from that of the base process ?

A rough answer to this question is given by the general perturbation theory [25], which implies that the eigenvalues of K and \widetilde{K} agree in the leading asymptotic term, as it happens for (2.2) and (2.3) (see, e.g., the proof of Lemma 2 in [7]). This is vividly demonstrated in the paper [43], where the kernels of the following form are considered:

$$\widetilde{K}_Q(s, t) = K(s, t) + Q\psi(s)\psi(t). \quad (2.4)$$

Here Q is a scalar real valued parameter and ψ is a function in the range of K . It turns out that for any Q greater than a certain critical value Q^* , the spectrum of \widetilde{K}_Q coincides with that of K in the first two asymptotic terms. For $Q = Q^*$ the spectra depart in the second term. The deviation is quantified in [43], when ψ is an image of an $L^2(0, 1)$ function, under the action of K .

The bridge process under consideration corresponds precisely to the critical case, but with $\psi(x) = K(1, x)$ being an image of the distribution $\delta(t - 1)$, rather than of a square integrable function; hence the approach of [43] is not directly applicable here.

In this chapter we will develop a different route towards answering the above question, using the particular structure of the perturbation inherent to bridges. Observe that the eigenproblem $\widetilde{K}\widetilde{\varphi} = \widetilde{\lambda}\widetilde{\varphi}$ can be written in terms of the covariance operator of the base process

$$\int_0^1 K(s, t)\widetilde{\varphi}(s)ds - K(1, t)\int_0^1 K(1, s)\widetilde{\varphi}(s)ds = \widetilde{\lambda}\widetilde{\varphi}(t), \quad t \in [0, 1], \quad (2.5)$$

where, without loss of generality, X is assumed to be normalized so that $K(1, 1) = 1$. Taking scalar product with the eigenfunction φ_n of K gives

$$\langle \widetilde{\varphi}, \varphi_n \rangle = c \frac{\lambda_n}{\lambda_n - \widetilde{\lambda}} \varphi_n(1), \quad \widetilde{\lambda} \notin \{\lambda_1, \lambda_2, \dots\} \quad (2.6)$$

where $c := \int_0^1 K(1, s)\widetilde{\varphi}(s)ds$. If $K(s, t)$ is continuous, by Mercer's theorem

$$K(s, t) = \sum_{n \geq 1} \lambda_n \varphi_n(t) \varphi_n(s)$$

where the convergence is absolute and uniform, and hence, in view of (2.6),

$$c = \int_0^1 K(1, s)\widetilde{\varphi}(s)ds = \sum_{n \geq 1} \lambda_n \varphi_n(1) \langle \widetilde{\varphi}, \varphi_n \rangle = c \sum_{n \geq 1} \frac{\lambda_n^2}{\lambda_n - \widetilde{\lambda}} \varphi_n(1)^2$$

Since $\sum_{n \geq 1} \lambda_n \varphi_n(1)^2 = K(1, 1) = 1$ and $c \neq 0$ whenever $\widetilde{\lambda} \notin \{\lambda_1, \lambda_2, \dots\}$, we obtain the following transcendental equation for the eigenvalues of the bridge

$$\sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k - \widetilde{\lambda}} \varphi_k(1)^2 = 0, \quad (2.7)$$

and the corresponding expression for its eigenfunctions:

$$\widetilde{\varphi}(t) = c \sum_{n \geq 1} \frac{\lambda_n}{\lambda_n - \widetilde{\lambda}} \varphi_n(1) \varphi_n(t), \quad t \in [0, 1]. \quad (2.8)$$

Note that the roots of (2.7) are not determined solely by the eigenvalues of the base process, but also require some information on its eigenfunctions.

The objective of this chapter is to show how equations (2.8) and (2.7) can be used to

construct asymptotic approximation for the solutions to the bridge eigenproblem (2.5), given the exact asymptotics of the Karhunen–Loève expansion for the corresponding base process.

2.2 Exact spectral asymptotics for fractional Brownian bridge

For definiteness we will work with a particular process, though the same approach applies whenever similar spectral approximation for the base process is available (as, e.g., for the processes considered in Chapter 1).

Our study case will be the fractional Brownian motion (fBm), that is, the centered Gaussian process $B^H = (B_t^H, t \in [0, 1])$ with covariance function

$$K(s, t) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}), \quad s, t \in [0, 1],$$

where $H \in (0, 1]$ is the Hurst exponent.

Smaller eigenvalues and corresponding eigenfunctions can be approximated using the following asymptotics (see [7, 8], [40], [33] for earlier results):

Theorem 2.1 (Theorem 2.1 in [12]).

1. For the fractional Brownian motion with $H \in (0, 1)$, the ordered sequence of the eigenvalues satisfies

$$\lambda_n = \frac{\sin(\pi H)\Gamma(2H + 1)}{\nu_n^{2H+1}} \quad n = 1, 2, \dots \quad (2.9)$$

where $\nu_n = \pi n + \pi\gamma_H + O(n^{-1})$ as $n \rightarrow \infty$ and

$$\gamma_H = -\frac{1}{2} - \frac{1}{2} \frac{(H - \frac{1}{2})^2}{H + \frac{1}{2}}. \quad (2.10)$$

2. The corresponding normalized eigenfunctions admit the approximation

$$\varphi_n(t) = \sqrt{2} \sin(\nu_n t + \pi\eta_H) + \int_0^\infty f_0(u) e^{-t\nu_n u} du + (-1)^n \int_0^\infty f_1(u) e^{-(1-t)\nu_n u} du + n^{-1} r_n(t), \quad (2.11)$$

where the functions f_0 and f_1 are defined in [12] in closed forms and

$$\eta_H = \frac{H - \frac{3}{2}}{4} \frac{H - \frac{1}{2}}{H + \frac{1}{2}}. \quad (2.12)$$

The residuals $r_n(t)$ in (2.11) are bounded uniformly over $n \in \mathbb{N}$ and $t \in [0, 1]$. Moreover, the values of the eigenfunctions at $t = 1$ are asymptotic to constants:

$$\varphi_n(1) = (-1)^n \sqrt{2H + 1} (1 + O(n^{-1})). \quad (2.13)$$

In principle, the spectral approximation technique developed in Chapter 1 is applicable to the fractional Brownian bridge directly. However, somewhat surprisingly, it does not produce results quite as explicit as those of Theorem 2.1. In this chapter an alternative approach, based on the the equations (2.8) and (2.7), is suggested.

We will show how the exact spectral asymptotics of the bridge follows from that of the base process. Specifically, we will prove the following result:

Theorem 2.2.

1. For the fractional Brownian bridge with $H \in (0, 1)$, the ordered sequence of the eigenvalues satisfies

$$\tilde{\lambda}_n = \frac{\sin(\pi H) \Gamma(2H + 1)}{\tilde{\nu}_n^{2H+1}} \quad n = 1, 2, \dots \quad (2.14)$$

where $\tilde{\nu}_n = \pi n + \pi \tilde{\gamma}_H + O(n^{-1} \log n)$ as $n \rightarrow \infty$ and $\tilde{\gamma}_H := \gamma_H + \frac{H}{H+\frac{1}{2}}$.

2. The corresponding eigenfunctions admit the approximation

$$\begin{aligned} \tilde{\varphi}_n(t) = & \sqrt{2} \sin(\tilde{\nu}_n t + \pi \eta_H) + \int_0^\infty f_0(u) e^{-\tilde{\nu}_n t u} du + \\ & (-1)^n \int_0^\infty e^{-\tilde{\nu}_n(1-t)u} \left(\sin(\pi(\tilde{\gamma}_H - \gamma_H)) \tilde{f}_1(u) du + \cos \pi(\tilde{\gamma}_H - \gamma_H) f_1(u) \right) du + \\ & (-1)^n \sin \pi(\tilde{\gamma}_H - \gamma_H) \int_0^\infty \tilde{g}_1(\tilde{\nu}_n(1-t)u) f_1(u) du + n^{-1} \log n \tilde{r}_n(t) \end{aligned} \quad (2.15)$$

where \tilde{f}_1 and \tilde{g}_1 are functions, defined in the closed forms by (2.38) and (2.39) below, and the residual $\tilde{r}_n(t)$ is bounded, uniformly over $n \in \mathbb{N}$ and $t \in [0, 1]$.

Remark 3.

a. The eigenvalues of the fBm and its bridge differ by a constant shift in the second order asymptotic term

$$\pi\tilde{\gamma}_H - \pi\gamma_H = \frac{\pi H}{H + \frac{1}{2}}$$

which reduces to the familiar constant $\pi/2$ for $H = \frac{1}{2}$, the standard Brownian case. The residuals in ν_n and $\tilde{\nu}_n$ differ by the $\log n$ factor, which may well be an artifact of the approach.

b. The eigenfunctions of the bridge inherit the oscillatory term in (2.15) from the corresponding term of the base process (2.11), however, with a frequency shift.

A more complicated modification occurs in the integral terms, which are responsible for the boundary layer: their contribution is asymptotically negligible away from the endpoints of the interval, but is persistent near the boundary. For the base process, these terms force the eigenfunctions to vanish at $t = 0$ and approach the alternating values (2.13) at $t = 1$; for the bridge, they push the eigenfunctions to zero at both endpoints.

Consequently the change is more significant near 1 than near the origin. Tracking back the definitions of all the functions involved, it can be seen that the boundary layer vanishes for $H = \frac{1}{2}$ and the leading asymptotic term in (2.15) reduces to the familiar formula (2.3) for the standard Brownian motion.

c. The approach, developed in this chapter, applies to the operator with the kernel (cf. (2.1) and (2.4)):

$$\tilde{K}_Q(s, t) = K(s, t) + QK(s, 1)K(t, 1),$$

where $Q \geq -1/K(1, 1)$ is a fixed parameter. In agreement with the results in [43], it can be seen that the second order asymptotic terms of the eigenvalues of \tilde{K}_Q and K coincide when $Q > -1/K(1, 1)$ and depart at the critical value $Q^* = -1/K(1, 1)$, corresponding to the bridge process. The deviation formula differs from the one, derived in [43] for the square integrable case.

2.3 Proofs

2.3.1 Preliminaries

Before giving the full proof, it is insightful to consider the special case $H = \frac{1}{2}$, corresponding to the standard Brownian motion. Let us see how the formulas (2.3) can be derived from (2.2), using the expansions (2.8) and (2.7).

To this end, it will be convenient to change the variables to

$$\mu_k := \frac{1}{\pi\sqrt{\lambda_k}} = k - \frac{1}{2} \quad \text{and} \quad \tilde{\mu} = \frac{1}{\pi\sqrt{\tilde{\lambda}}}$$

so that in view of (2.2) the equation (2.7) becomes

$$g(\tilde{\mu}) := \sum_{k=1}^{\infty} \frac{1}{(k - \frac{1}{2})^2 - \tilde{\mu}^2} = 0. \quad (2.16)$$

The explicit formula for this series is well known:

$$g(\tilde{\mu}) = \frac{\pi}{2\tilde{\mu}} \tan(\pi\tilde{\mu}) \quad (2.17)$$

and can be obtained by means of the residue calculus. It will be instructive to recall the calculation: define the function

$$f(z) = \frac{\operatorname{ctg} \pi(z + \frac{1}{2})}{z^2 - \tilde{\mu}^2}, \quad z \in \mathbb{Z}$$

which is analytic, except for the simple poles at $z_{\pm} = \pm\tilde{\mu}$ and $z_k = k - \frac{1}{2}$. Integrating $f(z)$ over a circular contour of radius R and taking the limit $R \rightarrow \infty$ we find that

$$\operatorname{Res}\{f; z_+\} + \operatorname{Res}\{f; z_-\} + \sum_{k \in \mathbb{Z}} \operatorname{Res}\{f; z_k\} = 0.$$

Here the residues are

$$\begin{aligned} \operatorname{Res}\{f; z_+\} &= \operatorname{Res}\{f; z_-\} = -\frac{1}{2\tilde{\mu}} \tan(\pi\tilde{\mu}) \\ \operatorname{Res}\{f; z_k\} &= \frac{1}{\pi} \frac{1}{(k - \frac{1}{2})^2 - \tilde{\mu}^2}. \end{aligned}$$

Since the sequence $(k - \frac{1}{2})^2 - \tilde{\mu}^2$, $k \in \mathbb{Z}$ is symmetric around $\frac{1}{2}$, the expression (2.17) is obtained and the equation (2.16) produces the roots $\tilde{\mu}_n = \pi n$, $n = 1, 2, \dots$, confirming the formula for the eigenvalues in (2.3).

The corresponding eigenfunctions can be found using (2.8):

$$\tilde{\varphi}_n(t) = 2c_n \tilde{\mu}_n^2 \sum_{k=1}^{\infty} \frac{(-1)^k \sin \mu_k \pi t}{\tilde{\mu}_n^2 - \mu_k^2} = 2c_n n^2 \sum_{k=1}^{\infty} \frac{(-1)^k \sin(k - \frac{1}{2})\pi t}{n^2 - (k - \frac{1}{2})^2}.$$

Using similar residue calculus, the series can be computed exactly:

$$\tilde{\varphi}_n(t) = -c_n n \pi (-1)^n \sin \pi n t,$$

which agrees with the formula in (2.3), after normalizing to the unit norm.

The more general case $H \in (0, 1)$ is different in two aspects:

1. The function $g(\tilde{\mu})$ for $H \neq \frac{1}{2}$ involves a power function with non-integer exponent (see (2.18) below) and hence, in addition to the poles, has a discontinuity across the branch cut. Consequently the Cauchy theorem cannot be applied as before and a different contour is to be chosen.

A natural choice is the boundary of half disk, which lies in the right half plane, but such integration produces an additional integral term along the imaginary axis. Asymptotic analysis shows that its contribution is non-negligible on the relevant scale for all values of H but $\frac{1}{2}$; thus it is "invisible" in the case of standard Brownian motion.

2. The exact formulas for the eigenvalues and eigenfunctions for $H \neq \frac{1}{2}$ are unavailable beyond their precise asymptotics as in (2.9).

It is reasonable to consider first the perturbed version of the equation (2.7), in which λ_k and φ_k are replaced with the corresponding asymptotic approximations from Theorem 2.1. This gives the main terms in the eigenvalues formula (2.14).

It remains then to show that the roots of the perturbed and the exact equations get close asymptotically on the suitable scale. Once the asymptotics of $\tilde{\lambda}_n$ becomes available, it can be plugged into (2.8), along with the expressions for λ_k and $\varphi_k(t)$, to construct the approximations for the bridge eigenfunctions.

2.3.2 The eigenvalues

Let us change the variable to $\tilde{\mu}$ such that

$$\tilde{\lambda} = \frac{\sin(\pi H)\Gamma(2H+1)}{(\pi\tilde{\mu})^{2H+1}},$$

in which case equation (2.7) becomes

$$g(\tilde{\mu}) := \sum_{k=1}^{\infty} \frac{\varphi_k(1)^2}{\mu_k^{2H+1} - \tilde{\mu}^{2H+1}} = 0, \quad (2.18)$$

where $\mu_k := \nu_k/\pi$. Observe that $g(\cdot)$ is continuous and increasing on $\mathbb{R}_+ \setminus \{\mu_k, k \in \mathbb{N}\}$ and

$$\lim_{\tilde{\mu} \searrow \mu_k} g(\tilde{\mu}) = -\infty \quad \text{and} \quad \lim_{\tilde{\mu} \nearrow \mu_k} g(\tilde{\mu}) = +\infty, \quad k \in \mathbb{N}.$$

Consequently it has a unique root $\tilde{\mu}_n$ at each one of the intervals (μ_n, μ_{n+1}) .

In view of the asymptotics (2.9)-(2.11), it makes sense to consider first the *perturbed* equation

$$g^a(\mu) := \sum_{k=1}^{\infty} \frac{2H+1}{(k+\gamma_H)^{2H+1} - \tilde{\mu}^{2H+1}} = 0, \quad (2.19)$$

where the eigenvalues and eigenfunctions of the base process are replaced with their asymptotic approximations.

The second step is to argue that the roots of the exact and perturbed equations (2.18) and (2.19) close on an appropriate scale, asymptotically as $n \rightarrow \infty$. This is done in Lemmas 2.1 and 2.2 below, which imply assertion (1) of Theorem 2.2.

Lemma 2.1. *The unique root $\tilde{\mu}_n^a \in (n, n+1)$ of equation (2.19) satisfies*

$$\pi\tilde{\mu}_n^a = \pi n + \frac{1-2H}{4}\pi + 2 \arcsin \frac{\ell_H}{\sqrt{1+\ell_H^2}} + O(n^{-1}),$$

where

$$\ell_H = \tan\left(\frac{\pi H - \frac{1}{2}}{2H + \frac{1}{2}}\right). \quad (2.20)$$

Proof. A more convenient expression can be found for $g^a(\tilde{\mu})$ using residue calculus. To this end note that the principal branch of the function

$$f(z) := \frac{\operatorname{ctg}(\pi(z - \gamma_H))}{z^{2H+1} - \tilde{\mu}^{2H+1}}, \quad z \in \mathbb{C} \setminus \mathbb{R}_-$$

is analytic on the right half plane, except for simple poles at $z_0 := \tilde{\mu}$ and $z_k := k + \gamma_H$, $k = 1, 2, \dots$. Note that $z_k > 0$ since $\gamma_H \in (-\frac{3}{4}, -\frac{1}{2})$ for $H \in (0, 1)$. Integrating this function over the boundary of the half disk of radius $R \in \mathbb{N}$ in the right half plane gives

$$\int_{iR}^{-iR} f(z)dz + \int_{C_R} f(z)dz = 2\pi i \operatorname{Res}\{f, z_0\} + 2\pi i \sum_{k=1}^R \operatorname{Res}\{f, z_k\}$$

where C_R denotes the half circle arc. Since $\operatorname{ctg}(\cdot)$ is bounded on C_R , by Jordan's lemma the integral over C_R vanishes as $R \rightarrow \infty$ and we obtain

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} f(it)dt = -\operatorname{Res}\{f, z_0\} - \sum_{k=1}^{\infty} \operatorname{Res}\{f, z_k\}. \quad (2.21)$$

Computing the residues

$$\begin{aligned} \operatorname{Res}\{f, z_0\} &= \operatorname{ctg}(\pi(\tilde{\mu} - \gamma_H)) \lim_{z \rightarrow \tilde{\mu}} \frac{z - \tilde{\mu}}{z^{2H+1} - \tilde{\mu}^{2H+1}} = \operatorname{ctg}(\pi(\tilde{\mu} - \gamma_H)) \frac{\tilde{\mu}^{-2H}}{2H+1} \\ \operatorname{Res}\{f, z_k\} &= \frac{1}{(k + \gamma_H)^{2H+1} - \tilde{\mu}^{2H+1}} \lim_{z \rightarrow 0} z \frac{\cos(\pi z)}{\sin(\pi z)} = \frac{1}{\pi} \frac{1}{(k + \gamma_H)^{2H+1} - \tilde{\mu}^{2H+1}} \end{aligned}$$

and plugging these expressions into (2.21) gives the explicit formula

$$\begin{aligned} g^a(\tilde{\mu}) &= -\frac{2H+1}{2} \int_{-\infty}^{\infty} f(it)dt - \frac{\pi}{\tilde{\mu}^{2H}} \operatorname{ctg}(\pi(\tilde{\mu} - \gamma_H)) = \\ &= -\frac{2H+1}{\tilde{\mu}^{2H}} \operatorname{Re} \left\{ \int_0^{\infty} \frac{\operatorname{ctg}(\pi(i\tau\tilde{\mu} - \gamma_H))}{(i\tau)^{2H+1} - 1} d\tau \right\} - \frac{\pi}{\tilde{\mu}^{2H}} \operatorname{ctg}(\pi(\tilde{\mu} - \gamma_H)). \end{aligned}$$

Hence the equation (2.19) becomes

$$\operatorname{ctg}(\pi(\tilde{\mu} - \gamma_H)) = -\frac{2H+1}{\pi} \operatorname{Re} \left\{ \int_0^{\infty} \frac{\operatorname{ctg}(\pi(i\tau\tilde{\mu} - \gamma_H))}{(i\tau)^{2H+1} - 1} d\tau \right\}. \quad (2.22)$$

Let $\tilde{\mu}_n^a$ be the unique root of (2.19) in the interval $(n, n + 1)$, then

$$\begin{aligned} \int_0^\infty \frac{\operatorname{ctg}(\pi(i\tau\tilde{\mu}_n^a - \gamma_H))}{(i\tau)^{2H+1} - 1} d\tau &= \frac{1}{i} \int_0^\infty \frac{1 + e^{-2\pi(\tau\tilde{\mu}_n^a + i\gamma_H)}}{1 - e^{-2\pi(\tau\tilde{\mu}_n^a + i\gamma_H)}} \frac{1}{(i\tau)^{2H+1} - 1} d\tau = \\ &= \frac{1}{i} \int_0^\infty \frac{1}{(i\tau)^{2H+1} - 1} d\tau + R_n \end{aligned} \quad (2.23)$$

with the residual satisfying

$$\begin{aligned} \tilde{\mu}_n^a |R_n| &\leq \int_0^\infty \left| \frac{2e^{-2\pi t}}{1 - e^{-2\pi t - 2\pi i\gamma_H}} \frac{1}{(it/\tilde{\mu}_n^a)^{2H+1} - 1} \right| dt \leq \\ &\leq \int_0^\infty \frac{2e^{-2\pi t}}{|(it/\tilde{\mu}_n^a)^{2H+1} - 1|} dt \xrightarrow{n \rightarrow \infty} \frac{1}{\pi} \end{aligned} \quad (2.24)$$

where the second inequality holds since $\cos(2\pi\gamma_H) \leq 0$.

The real part of the integral on the right hand side of (2.23) can be computed explicitly. We will give the details for $H > \frac{1}{2}$, leaving out similar calculation for $H < \frac{1}{2}$:

$$\begin{aligned} \operatorname{Re} \left\{ \frac{1}{i} \int_0^\infty \frac{1}{(i\tau)^{2H+1} - 1} d\tau \right\} &= \int_0^\infty \frac{-\tau^{2H+1} \sin \pi(H + \frac{1}{2})}{(\tau^{2H+1} - \cos \pi(H + \frac{1}{2}))^2 + \sin^2 \pi(H + \frac{1}{2})} d\tau = \\ &= \int_0^\infty \frac{\tau^{2H+1} \sin \pi(H - \frac{1}{2})}{(\tau^{2H+1} + \cos \pi(H - \frac{1}{2}))^2 + \sin^2 \pi(H - \frac{1}{2})} d\tau = \\ &= \left(\sin \pi(H - \frac{1}{2}) \right)^{\frac{1}{2H+1}} \frac{1}{2H+1} \int_0^\infty \frac{t^{\frac{1}{2H+1}}}{(t + \operatorname{ctg} \pi(H - \frac{1}{2}))^2 + 1} dt. \end{aligned}$$

The latter integral can be evaluated by integrating the appropriate branch of

$$f(z) := \frac{z^{\frac{1}{2H+1}}}{(z+c)^2 + 1} \quad \text{with} \quad c := \operatorname{ctg} \pi(H - \frac{1}{2}) > 0$$

on the circular contour, cut along the positive real semi-axis. By Jordan's lemma

$$\frac{1}{2\pi i} \int_0^\infty (f^+(t) - f^-(t)) dt = \operatorname{Res}\{f; z_+\} + \operatorname{Res}\{f; z_-\} \quad (2.25)$$

where $z_\pm = -c \pm i$ are the poles of f . The residues are given by

$$\begin{aligned} \operatorname{Res}\{f; z_+\} + \operatorname{Res}\{f; z_-\} &= \frac{z_+^{\frac{1}{2H+1}}}{z_+ - z_-} + \frac{z_-^{\frac{1}{2H+1}}}{z_- - z_+} = \\ &= \frac{(-c+i)^{\frac{1}{2H+1}}}{2i} + \frac{(-c-i)^{\frac{1}{2H+1}}}{-2i} = -(c^2+1)^{\frac{1}{2}} \frac{1}{2H+1} e^{\frac{1}{H+1/2} \frac{\pi}{2} i} \sin \frac{\pi}{2} \frac{H-1/2}{H+1/2} \end{aligned}$$

where we used the formula $\arctan c^{-1} = \pi(H - \frac{1}{2})$. Therefore, by (2.25)

$$\begin{aligned} \int_0^\infty \frac{t^{\frac{1}{2H+1}}}{(t+c)^2+1} dt &= -\frac{2\pi i}{1 - e^{\pi i \frac{1}{H+1/2}}} (c^2+1)^{\frac{1}{2}} \frac{1}{2H+1} e^{\frac{1}{H+1/2} \frac{\pi}{2} i} \sin \frac{\pi}{2} \frac{H-1/2}{H+1/2} = \\ &= \pi (c^2+1)^{\frac{1}{2}} \frac{1}{2H+1} \frac{\sin \frac{\pi}{2} \frac{H-1/2}{H+1/2}}{\sin \frac{\pi}{2} \frac{1}{H+1/2}} \end{aligned}$$

and consequently

$$\operatorname{Re} \left\{ \frac{1}{i} \int_0^\infty \frac{1}{(i\tau)^{2H+1} - 1} d\tau \right\} = \frac{\pi}{2H+1} \frac{\sin \frac{\pi}{2} \frac{H-1/2}{H+1/2}}{\sin \frac{\pi}{2} \frac{1}{H+1/2}} = \frac{\pi}{2H+1} \ell_H, \quad (2.26)$$

where ℓ_H is the constant defined in (2.20).

Plugging (2.24) and (2.26) into (2.23) and (2.22) and recalling that $\tilde{\mu}_n^a \in (n, n+1)$ and $\gamma_H < 0$, we obtain the claimed asymptotics:

$$\begin{aligned} \pi \tilde{\mu}_n^a &= \pi(n+1) + \pi\gamma_H + \operatorname{arctg} \left(-\ell_H + O(n^{-1}) \right) \\ &= \pi(n+1) + \pi\gamma_H + \arcsin \frac{\ell_H}{\sqrt{1+\ell_H^2}} - \frac{\pi}{2} + O(n^{-1}) = \\ &= \pi n + \frac{1-2H}{4} \pi + 2 \arcsin \frac{\ell_H}{\sqrt{1+\ell_H^2}} + O(n^{-1}). \end{aligned}$$

□

The next step is to show that the roots of (2.19) and (2.18) are close on a suitable scale:

Lemma 2.2. *The unique root $\tilde{\mu}_n \in (n, n+1)$ of equation (2.18) satisfies*

$$\tilde{\mu}_n - \tilde{\mu}_n^a = O(n^{-1} \log n).$$

Proof. Suppose f is a differentiable function with $\frac{d}{dx} f(x) \geq c > 0$ for some constant c ,

whose root x_0 belongs to an open interval I . Then any strictly increasing continuous function h , satisfying $\sup_{x \in I} |f(x) - h(x)| \leq c(b - a)$, has its root y_0 in I and

$$|y_0 - x_0| \leq c^{-1} \sup_{x \in I} |f(x) - h(x)|. \quad (2.27)$$

We will apply this elementary bound to $f := g^a$ and $h := g$ on the interval I_n with the endpoints at $n + \tilde{\gamma}_H \pm \delta$, where $\delta := (\tilde{\gamma}_H - \gamma_H)/2 > 0$. Recall that by Lemma 2.1, the unique root $\tilde{\mu}_n^a \in (n, n + 1)$ of g^a belongs to I_n for all sufficiently large n . The function g^a is differentiable on $\mathbb{R}_+ \setminus \{k + \gamma_H : k \in \mathbb{N}\}$ and

$$\begin{aligned} \inf_{\tilde{\mu} \in I_n} \frac{d}{d\tilde{\mu}} g^a(\tilde{\mu}) &= \inf_{\tilde{\mu} \in I_n} \sum_{k=1}^{\infty} \frac{(2H+1)^2 \tilde{\mu}^{2H}}{\left((k + \gamma_H)^{2H+1} - \tilde{\mu}^{2H+1}\right)^2} \geq \\ &\geq \inf_{\tilde{\mu} \in I_n} \frac{\tilde{\mu}^{2H}}{\left((n + \gamma_H)^{2H+1} - \tilde{\mu}^{2H+1}\right)^2} \geq \\ &\geq \frac{(n + \tilde{\gamma}_H - \delta)^{2H}}{\left((n + \gamma_H)^{2H+1} - (n + \tilde{\gamma}_H - \delta)^{2H+1}\right)^2} > cn^{-2H} \end{aligned} \quad (2.28)$$

with a constant $c > 0$.

Next we will estimate the oscillation of $g^a(\tilde{\mu}) - g(\tilde{\mu})$ on I_n :

$$\begin{aligned} g^a(\tilde{\mu}) - g(\tilde{\mu}) &= \sum_{k=1}^{\infty} \frac{(2H+1) - \varphi_k(1)^2}{(k + \gamma_H)^{2H+1} - \tilde{\mu}^{2H+1}} + \\ &+ \sum_{k=1}^{\infty} \varphi_k(1)^2 \frac{\mu_k^{2H+1} - (k + \gamma_H)^{2H+1}}{\left((k + \gamma_H)^{2H+1} - \tilde{\mu}^{2H+1}\right) \left(\mu_k^{2H+1} - \tilde{\mu}^{2H+1}\right)} =: D_1(\tilde{\mu}) + D_2(\tilde{\mu}). \end{aligned} \quad (2.29)$$

In view of (2.13),

$$\begin{aligned} \sup_{\tilde{\mu} \in I_n} |D_1(\tilde{\mu})| &\lesssim \sup_{\tilde{\mu} \in I_n} \sum_{k=1}^{\infty} \frac{1/k}{\left|(k + \gamma_H)^{2H+1} - \tilde{\mu}^{2H+1}\right|} \lesssim \\ &\lesssim \sum_{k=1}^n \frac{1/k}{(n + \tilde{\gamma}_H - \delta)^{2H+1} - (k + \gamma_H)^{2H+1}} + \\ &+ \sum_{k=n+1}^{\infty} \frac{1/k}{(k + \gamma_H)^{2H+1} - (n + \tilde{\gamma}_H + \delta)^{2H+1}} =: A_n + B_n, \end{aligned}$$

where $x \lesssim y$ means $x \leq Cy$ for some constant C . The first sum on the right satisfies

$$A_n \leq \int_1^n \frac{1/x}{(n + \tilde{\gamma}_H - \delta)^{2H+1} - (x + \gamma_H)^{2H+1}} dx + \frac{1/n}{(n + \tilde{\gamma}_H - \delta)^{2H+1} - (n + \gamma_H)^{2H+1}} =: A_n^{(1)} + A_n^{(2)}$$

where, obviously, $A_n^{(2)} \lesssim n^{-2H-1}$ and, for all n large enough

$$\begin{aligned} A_n^{(1)} &= n^{-2H-1} \int_{1/n}^1 \frac{1/y}{(1 + (\tilde{\gamma}_H - \delta)/n)^{2H+1} - (y + \gamma_H/n)^{2H+1}} dy \leq \\ &\leq n^{-2H-1} \int_{1/n}^{1/2} \frac{2}{y} dy + n^{-2H-1} \int_{1/2}^1 \frac{2}{1 - (y + \gamma_H/n)^{2H+1}} dy \lesssim n^{-2H-1} \log n. \end{aligned}$$

Similar estimate holds for B_n and therefore

$$\sup_{\tilde{\mu} \in I_n} |D_1(\tilde{\mu})| \lesssim O(n^{-2H-1} \log n). \quad (2.30)$$

Further, let k_0 be such that $|\mu_k - (k + \gamma_H)| \leq \delta/2$ for all $k \geq k_0$, then

$$\begin{aligned} \sup_{\tilde{\mu} \in I_n} |D_2(\tilde{\mu})| &\lesssim \sup_{\tilde{\mu} \in I_n} \sum_{k=1}^{\infty} \frac{k^{2H-1}}{\left| \left((k + \gamma_H)^{2H+1} - \tilde{\mu}^{2H+1} \right) \left(\mu_k^{2H+1} - \tilde{\mu}^{2H+1} \right) \right|} \lesssim \\ &\lesssim n^{-4H-2} + \sum_{k=k_0}^n \frac{k^{2H-1}}{\left((n + \tilde{\gamma}_H - \delta)^{2H+1} - (k + \gamma_H + \delta/2)^{2H+1} \right)^2} + \\ &+ \sum_{k=n+1}^{\infty} \frac{k^{2H-1}}{\left((k + \gamma_H - \delta/2)^{2H+1} - (n + \tilde{\gamma}_H + \delta)^{2H+1} \right)^2}. \end{aligned}$$

Denote the last two terms by R_n and Q_n respectively. Then

$$\begin{aligned} R_n &\leq \int_{k_0}^n \frac{x^{2H}/(x-1)}{\left((n + \tilde{\gamma}_H - \delta)^{2H+1} - (x + \gamma_H + \delta/2)^{2H+1} \right)^2} dx + \\ &\frac{n^{2H-1}}{\left((n + \tilde{\gamma}_H - \delta)^{2H+1} - (n + \gamma_H + \delta/2)^{2H+1} \right)^2} := R_n^{(1)} + R_n^{(2)}. \end{aligned}$$

Here $R_n^{(2)} \lesssim n^{-2H-1}$ and

$$\begin{aligned}
R_n^{(1)} &= n^{-2H-2} \int_{k_0/n}^1 \frac{y^{2H}/(y-1/n)}{\left((1+(\tilde{\gamma}_H-\delta)/n)^{2H+1} - (y+(\gamma_H+\delta/2)/n)^{2H+1}\right)^2} dy \\
&\lesssim n^{-2H-2} \int_0^1 \frac{y^{2H-1}}{\left(1 - (y+\tilde{\gamma}_H/n)^{2H+1}\right)^2} dy \lesssim n^{-2H-1}.
\end{aligned}$$

A similar bound holds for Q_n and therefore

$$\sup_{\tilde{\mu} \in I_n} |D_2(\tilde{\mu})| \lesssim n^{-2H-1}.$$

By (2.30) the second sum in (2.29) is asymptotically negligible, that is

$$\sup_{\tilde{\mu} \in I_n} |g^a(\tilde{\mu}) - g(\tilde{\mu})| \lesssim n^{-2H-1} \log n.$$

Plugging this estimate and (2.28) into (2.27) gives the claimed asymptotics. □

2.3.3 The eigenfunctions

The approximation (2.15) is obtained by plugging the asymptotics (2.9), (2.11) and (2.14) into the formula (2.8):

$$\tilde{\varphi}_n(t) = c_n \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k - \tilde{\lambda}_n} \varphi_k(1) \varphi_k(t) = -c_n \tilde{\mu}_n^{2H+1} \sum_{k=1}^{\infty} \frac{1}{\mu_k^{2H+1} - \tilde{\mu}_n^{2H+1}} \varphi_k(1) \varphi_k(t)$$

where we set $\mu_n := \nu_n/\pi$ and $\tilde{\mu}_n := \tilde{\nu}_n/\pi$ as in Lemma 2.1.

As before, we will first replace the exact values by their leading asymptotic terms and then will argue that the error, thus introduced, is negligible on the suitable scale. To this end, define (c.f. (2.11)):

$$\tilde{\varphi}_n^{1,a}(t) = -c_n \tilde{\mu}_n^{2H+1} \sqrt{2H+1} \sqrt{2} \sum_{k=1}^{\infty} \frac{(-1)^k \sin(\pi(k + \gamma_H)t + \pi\eta_H)}{(k + \gamma_H)^{2H+1} - \tilde{\mu}_n^{2H+1}} \quad (2.31)$$

$$\tilde{\varphi}_n^{2,a}(t) = -c_n \tilde{\mu}_n^{2H+1} \sqrt{2H+1} \int_0^{\infty} f_0(u) \left(\sum_{k=1}^{\infty} \frac{(-1)^k e^{-(k+\gamma_H)\pi tu}}{(k + \gamma_H)^{2H+1} - \tilde{\mu}_n^{2H+1}} \right) du \quad (2.32)$$

$$\tilde{\varphi}_n^{3,a}(t) = -c_n \tilde{\mu}_n^{2H+1} \sqrt{2H+1} \int_0^{\infty} f_1(u) \left(\sum_{k=1}^{\infty} \frac{e^{-(k+\gamma_H)\pi(1-t)u}}{(k + \gamma_H)^{2H+1} - \tilde{\mu}_n^{2H+1}} \right) du \quad (2.33)$$

where η_H is the constant defined in (2.12).

Lemma 2.3. *The function $\tilde{\varphi}_n^a(t) = \tilde{\varphi}_n^{1,a}(t) + \tilde{\varphi}_n^{2,a}(t) + \tilde{\varphi}_n^{3,a}(t)$ satisfies*

$$\begin{aligned} \tilde{\varphi}_n^a(t) / \|\tilde{\varphi}_n^a\| &= \sqrt{2} \sin(\tilde{\nu}_n t + \pi\eta_H) + \int_0^{\infty} f_0(u) e^{-\tilde{\nu}_n tu} du + \\ &+ (-1)^n \int_0^{\infty} e^{-\tilde{\nu}_n(1-t)u} \left(\sin(\pi(\tilde{\gamma}_H - \gamma_H)) \tilde{f}_1(u) du + \cos \pi(\tilde{\gamma}_H - \gamma_H) f_1(u) \right) du + \\ &+ (-1)^n \sin \pi(\tilde{\gamma}_H - \gamma_H) \int_0^{\infty} \tilde{g}_1(\tilde{\nu}_n(1-t)u) f_1(u) du + n^{-1} \tilde{r}_n(t), \end{aligned}$$

where \tilde{f}_1 and \tilde{g}_1 are explicit functions, defined in (2.38) and (2.39) below, and the residual $\tilde{r}_n(t)$ is bounded uniformly over $n \in \mathbb{N}$ and $t \in [0, 1]$.

Proof. The claimed approximation is obtained by finding the leading term asymptotics of the functions in (2.31)-(2.33) and normalizing their sum by a suitable common factor.

1) *Asymptotics of (2.31).* For fixed $t \in [0, 1]$ and $\tilde{\mu} > 0$, consider the series

$$h(\tilde{\mu}) := \sum_{k=1}^{\infty} \frac{(-1)^k \sin(\pi(k + \gamma_H)t + \pi\eta_H)}{(k + \gamma_H)^{2H+1} - \tilde{\mu}^{2H+1}}.$$

A closed form formula for this expression can be found by means of residue calculus as in Lemma 2.1. To this end, consider the principal branch of the function

$$f(z) := \frac{\sin(\pi(zt + \eta_H))}{z^{2H+1} - \tilde{\mu}^{2H+1}} \frac{1}{\sin(\pi(z - \gamma_H))} \quad z \in \mathbb{C} \setminus \mathbb{R}_- \quad (2.34)$$

which is analytic on the right half plane, except for the simple poles at $z_0 := \tilde{\mu}$ and $z_k = k + \gamma_H$, $k \in \mathbb{N}$. Integrating $f(z)$ over the half disc boundary in the right half plane we get

$$\int_{iR}^{-iR} f(z)dz + \int_{C_R} f(z)dz = 2\pi i \operatorname{Res}\{f; z_0\} + 2\pi i \sum_{k=1}^R \operatorname{Res}\{f; z_k\}$$

where C_R stands for the half circle arc with radius $R \in \mathbb{N}$. The ratio of sines in (2.34) is bounded for any $t \in [0, 1]$ and therefore, applying Jordan's lemma, we get

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} f(it)dt = -\operatorname{Res}\{f; z_0\} - \sum_{k=1}^{\infty} \operatorname{Res}\{f; z_k\}, \quad (2.35)$$

where the residues are

$$\begin{aligned} \operatorname{Res}\{f; z_0\} &= \frac{\sin(\pi(\tilde{\mu}t + \eta_H))}{\sin(\pi(\tilde{\mu} - \gamma_H))} \frac{1}{2H+1} \frac{1}{\tilde{\mu}^{2H}} \\ \operatorname{Res}\{f; z_k\} &= \frac{\sin(\pi(k + \gamma_H)t + \eta_H)}{(k + \gamma_H)^{2H+1} - \tilde{\mu}^{2H+1}} \frac{(-1)^k}{\pi}. \end{aligned}$$

Plugging these expressions into (2.35), we get

$$\begin{aligned} h(\tilde{\mu}) &= -\frac{\pi}{2H+1} \frac{1}{\tilde{\mu}^{2H}} \frac{\sin(\pi(\tilde{\mu}t + \eta_H))}{\sin(\pi(\tilde{\mu} - \gamma_H))} \\ &\quad - \frac{1}{\tilde{\mu}^{2H}} \operatorname{Re} \left\{ \int_0^{\infty} \frac{1}{(iu)^{2H+1} - 1} \frac{\sin(\pi(iu\tilde{\mu}t + \eta_H))}{\sin(\pi(iu\tilde{\mu} - \gamma_H))} du \right\}. \end{aligned}$$

The second term simplifies to

$$\begin{aligned} &\operatorname{Re} \left\{ \int_0^{\infty} \frac{1}{(iu)^{2H+1} - 1} \frac{\sin(\pi(iu\tilde{\mu}t + \eta_H))}{\sin(\pi(iu\tilde{\mu} - \gamma_H))} du \right\} = \\ &= \int_0^{\infty} e^{-\pi u\tilde{\mu}(1-t)} \operatorname{Re} \left\{ \frac{e^{-i\pi(\eta_H + \gamma_H)}}{(iu)^{2H+1} - 1} \right\} du + R(t, \tilde{\mu}) \end{aligned}$$

with the residual satisfying

$$\tilde{\mu} |R(t, \tilde{\mu})| \leq \tilde{\mu} \int_0^{\infty} \frac{2e^{-\pi u\tilde{\mu}}}{|(iu)^{2H+1} - 1|} du = \int_0^{\infty} \frac{2e^{-\pi s}}{|(is/\tilde{\mu})^{2H+1} - 1|} ds \xrightarrow{\tilde{\mu} \rightarrow \infty} \frac{2}{\pi}$$

Plugging these expressions back gives

$$\begin{aligned} \tilde{\varphi}_n^{1,a}(t) &\simeq \sqrt{2} \sin(\pi(\tilde{\mu}_n t + \eta_H)) + \\ &+ (-1)^n \sin(\pi(\tilde{\gamma}_H - \gamma_H)) \int_0^\infty e^{-\pi \tilde{\mu}_n (1-t)u} \tilde{f}_1(u) du + n^{-1} \tilde{r}_n^{(1)}(t) \end{aligned} \quad (2.36)$$

where $x \simeq y$ means $x \simeq Cy$ with a constant C and we normalized by the factor

$$\tilde{c}_n := c_n \tilde{\mu}_n \frac{\pi}{\sqrt{2H+1}} \frac{(-1)^n}{\sin \pi(\tilde{\gamma}_H - \gamma_H)}. \quad (2.37)$$

It can be seen (as in the calculation, concluding section 5.1.6. in [12]), that the norm of the integral term in (2.36) is of order $O(n^{-1})$ and hence the norm of $\tilde{\varphi}_n^{1,a}$ is asymptotic to 1 as $n \rightarrow \infty$. The residual $\tilde{r}_n^{(1)}(t)$ is uniformly bounded over $n \in \mathbb{N}$ and $t \in [0, 1]$ and the function \tilde{f}_1 is given by the formula

$$\tilde{f}_1(u) := \frac{2H+1}{\pi} \sqrt{2} \operatorname{Re} \left\{ \frac{e^{-i\pi(\eta_H + \gamma_H)}}{(iu)^{2H+1} - 1} \right\}. \quad (2.38)$$

2) *Asymptotics of (2.32).* A closed form expression for the series

$$h(\tilde{\mu}) := \sum_{k=1}^{\infty} \frac{(-1)^k}{(k + \gamma_H)^{2H+1} - \tilde{\mu}^{2H+1}} e^{-(k+\gamma_H)\pi t u}$$

can be found by integrating the principal branch of the function

$$f(z) := \frac{e^{-z\pi t u}}{z^{2H+1} - \tilde{\mu}^{2H+1}} \frac{1}{\sin(\pi(z - \gamma_H))}, \quad z \in \mathbb{C} \setminus \mathbb{R}_-$$

over the half disk boundary in the right half plane. As before,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} f(it) dt = -\operatorname{Res}\{f; z_0\} - \sum_{k=1}^{\infty} \operatorname{Res}\{f; z_k\},$$

with the same poles as defined above. The residues are given by

$$\begin{aligned} \operatorname{Res}\{f; z_0\} &= \frac{1}{2H+1} \frac{1}{\tilde{\mu}^{2H}} \frac{e^{-\tilde{\mu}\pi t u}}{\sin(\pi(\tilde{\mu} - \gamma_H))} \\ \operatorname{Res}\{f; z_k\} &= \frac{1}{\pi} \frac{(-1)^k}{(k + \gamma_H)^{2H+1} - \tilde{\mu}^{2H+1}} e^{-(k+\gamma_H)\pi t u} \end{aligned}$$

and therefore

$$h(\tilde{\mu}) = -\frac{\pi}{2H+1} \frac{1}{\tilde{\mu}^{2H}} \frac{e^{-\tilde{\mu}\pi t u}}{\sin(\pi(\tilde{\mu} - \gamma_H))} - \frac{1}{\tilde{\mu}^{2H}} \operatorname{Re} \left\{ \int_0^\infty \frac{e^{-is\tilde{\mu}\pi t u}}{(is)^{2H+1} - 1} \frac{1}{\sin(\pi(is\tilde{\mu} - \gamma_H))} ds \right\}.$$

The integral term satisfies

$$\left| \int_0^\infty \frac{e^{-is\tilde{\mu}\pi t u}}{(is)^{2H+1} - 1} \frac{1}{\sin(\pi(is\tilde{\mu} - \gamma_H))} ds \right| \leq \frac{1}{\tilde{\mu}} \int_0^\infty \frac{2e^{-\pi s}}{|(is/\tilde{\mu})^{2H+1} - 1|} ds$$

and hence, normalizing by the constant (2.37), we get

$$\tilde{\varphi}_n^{2,a}(t) \simeq \int_0^\infty f_0(u) e^{-\tilde{\mu}_n \pi t u} du + n^{-1} \tilde{r}_n^{(2)}(t)$$

with a uniformly bounded residual $\tilde{r}_n^{(2)}$.

3) *Asymptotics of (2.33)*. An explicit formula for the series

$$h(\tilde{\mu}) := \sum_{k=1}^{\infty} \frac{e^{-(k+\gamma_H)\pi(1-t)u}}{(k+\gamma_H)^{2H+1} - \tilde{\mu}^{2H+1}}$$

is obtained by integrating the principal branch of the function

$$f(z) := \frac{e^{-z\pi(1-t)u}}{z^{2H+1} - \tilde{\mu}^{2H+1}} \operatorname{ctg}(\pi(z - \gamma_H)) \quad z \in \mathbb{C} \setminus \mathbb{R}_-$$

over the same contour as above:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} f(it) dt = -\operatorname{Res}\{f; z_0\} - \sum_{k=1}^{\infty} \operatorname{Res}\{f; z_k\}.$$

The residues are

$$\begin{aligned} \operatorname{Res}\{f; z_0\} &= \frac{1}{2H+1} \frac{1}{\tilde{\mu}^{2H}} e^{-\tilde{\mu}\pi(1-t)u} \operatorname{ctg}(\pi(\tilde{\mu} - \gamma_H)) \\ \operatorname{Res}\{f; z_k\} &= \frac{1}{\pi} \frac{e^{-(k+\gamma_H)\pi(1-t)u}}{(k+\gamma_H)^{2H+1} - \tilde{\mu}^{2H+1}} \end{aligned}$$

and therefore

$$h(\tilde{\mu}) = -\frac{\pi}{2H+1} \frac{1}{\tilde{\mu}^{2H}} e^{-\tilde{\mu}\pi(1-t)u} \operatorname{ctg}(\pi(\tilde{\mu} - \gamma_H)) - \\ - \frac{1}{\tilde{\mu}^{2H}} \operatorname{Im} \left\{ \int_0^\infty \frac{e^{-is\tilde{\mu}\pi(1-t)u}}{(is)^{2H+1} - 1} ds \right\} - \frac{1}{\tilde{\mu}^{2H}} R(\tilde{\mu}),$$

where the function $R(\tilde{\mu})$ is bounded

$$|R(\tilde{\mu})| \leq \frac{1}{\tilde{\mu}} \int_0^\infty \frac{2e^{-2\pi s}}{|(is/\tilde{\mu})^{2H+1} - 1|} ds.$$

Plugging these into (2.33) and normalizing by (2.37) gives

$$\tilde{\varphi}_n^{3,a}(t) \simeq (-1)^n \sin(\pi(\tilde{\gamma}_H - \gamma_H)) \int_0^\infty f_1(u) \tilde{g}_1(\tilde{\mu}_n \pi(1-t)u) du + \\ + (-1)^n \cos \pi(\tilde{\gamma}_H - \gamma_H) \int_0^\infty f_1(u) e^{-\tilde{\mu}_n \pi(1-t)u} du + n^{-1} \tilde{r}_n^{(3)}(t)$$

where we defined

$$\tilde{g}_1(x) = \frac{2H+1}{\pi} \operatorname{Im} \left\{ \int_0^\infty \frac{e^{-isx}}{(is)^{2H+1} - 1} ds \right\}. \quad (2.39)$$

□

Finally, it is left to check that the eigenfunctions of the bridge are asymptotic to the expressions found in Lemma 2.3:

Lemma 2.4. For any $H \in (0, 1)$,

$$\left| \frac{\tilde{\varphi}_n(t)}{\|\tilde{\varphi}_n\|} - \frac{\tilde{\varphi}_n^a(t)}{\|\tilde{\varphi}_n^a\|} \right| \leq C n^{-1} \log n, \quad t \in [0, 1]$$

for some constant C .

Proof. Denote by $\varphi_n^a(t)$ the leading asymptotic term in the eigenfunctions approximation (2.11) for the base process, satisfying

$$|\varphi_n^a(t) - \varphi_n(t)| = |r_n(t)| n^{-1} \leq C n^{-1}$$

with a constant C . Then after normalizing by the factor (2.37)

$$\begin{aligned}
|\tilde{\varphi}_n(t) - \tilde{\varphi}_n^a(t)|/c_n &\leq \tilde{\mu}_n^{2H} \sum_{k=1}^{\infty} \left| \frac{\varphi_k(1)\varphi_k(t)}{\mu_k^{2H+1} - \tilde{\mu}_n^{2H+1}} - \frac{\varphi_k^a(1)\varphi_k^a(t)}{(k + \gamma_H)^{2H+1} - \tilde{\mu}_n^{2H+1}} \right| \lesssim \\
&\lesssim \tilde{\mu}_n^{2H} \sum_{k=1}^{\infty} \frac{(k + \gamma_H)^{2H-1}}{\left| (\mu_k^{2H+1} - \tilde{\mu}_n^{2H+1}) \left((k + \gamma_H)^{2H+1} - \tilde{\mu}_n^{2H+1} \right) \right|} + \\
&+ \tilde{\mu}_n^{2H} \sum_{k=1}^{\infty} \frac{1/k}{\left| \mu_k^{2H+1} - \tilde{\mu}_n^{2H+1} \right|} + \tilde{\mu}_n^{2H} \sum_{k=1}^{\infty} \frac{1/k^2}{\left| \mu_k^{2H+1} - \tilde{\mu}_n^{2H+1} \right|} \lesssim n^{-1} \log n
\end{aligned}$$

where all the estimates are obtained as in Lemma 2.2. The claim follows since $\|\tilde{\varphi}_n^a/c_n\| = 1 + O(n^{-1})$. \square

APPLICATIONS OF EXACT SPECTRAL ASYMPTOTICS

3.1 Estimation of fractional Gaussian signals in white noise

3.1.1 Signal estimation in white noise

In this section we revisit the classical problem of signal estimation in white noise. Consider the process

$$Y_t = \mu \int_0^t X_s ds + \sqrt{\varepsilon} B_t, \quad t \in [0, T] \quad (3.1)$$

where μ and $\varepsilon > 0$ are real constants, B_t is the Brownian motion and X_t is an independent *signal* process, whose trajectory is to be estimated given the observed trajectory of $Y = (Y_t, t \in [0, T])$. Here μ is interpreted as the *channel gain* and the formal derivative of B_t is viewed as the white noise disturbance, whose intensity is controlled by ε .

The optimal in the mean squared sense estimator of X_t given the observation path is the conditional expectation $\widehat{X}_t = \mathbb{E}(X_t | \mathcal{F}_T^Y)$, where $\mathcal{F}_T^Y = \sigma\{Y_t, t \in [0, T]\}$. If X is a centred Gaussian process with covariance function $K(s, t) = \mathbb{E}X_s X_t$, this estimator is given by suitably defined stochastic integral

$$\widehat{X}_t = \frac{1}{\mu} \int_0^T h(s, t) dY_s,$$

where kernel $h(s, t)$ solves the integral equation

$$\varepsilon h(s, t) + \int_0^T \mu^2 K(r, s) h(r, t) dr = \mu^2 K(s, t), \quad 0 \leq s \leq t \leq T. \quad (3.2)$$

The corresponding minimal mean square error $P_\varepsilon(t) = \mathbb{E}(X_t - \widehat{X}_t)^2$ satisfies

$$P_\varepsilon(t) = K(t, t) - \int_0^T h(r, t)K(r, t)dr = \frac{\varepsilon}{\mu^2}h(t, t),$$

and the important engineering questions are how it scales with the noise intensity and what is its behaviour in large time.

3.1.2 High signal-to-noise asymptotics for fractional processes

In the Gauss-Markov case, when X is the Ornstein–Uhlenbeck process driven by the standard Brownian motion, integral equation (3.2) can be solved explicitly by reduction to the Riccati o.d.e. (see, e.g., Theorem 12.10 in [32]) and, $P_\varepsilon(t)$ can be computed in a closed form. A calculation gives the following *high signal-to-noise* asymptotics

$$P_\varepsilon(t) \simeq \sqrt{\varepsilon/\mu^2} \begin{cases} \frac{1}{2} & t \in (0, T) \\ 1 & t = T \end{cases} \quad \text{as } \varepsilon \rightarrow 0,$$

where $f(\varepsilon) \simeq g(\varepsilon)$ stands for $f(\varepsilon) = g(\varepsilon)(1 + o(1))$ as $\varepsilon \rightarrow 0$. Note that the *smoothing* estimator \widehat{X}_t with $t < T$ outperforms the *filtering* estimator \widehat{X}_T by factor 2 in the limit.

Generalizing this result to fractional setting, when X is the fractional Ornstein - Uhlenbeck process (1.8), cannot be easily approached along the same lines due to lack of the Markov property. The more tractable alternative is to use spectral approximations from Theorem 1.2, which give the following result:

Proposition 5. *The minimal mean squared error in the estimation problem of the fractional Ornstein–Uhlenbeck process (1.8) given observations (3.1) satisfies*

$$P_\varepsilon(t) \simeq (\varepsilon/\mu^2)^{\frac{2H}{1+2H}} \frac{\left(\sin(\pi H)\Gamma(2H + 1)\right)^{\frac{1}{1+2H}}}{\sin \frac{\pi}{2H+1}} \begin{cases} \frac{1}{2H+1} & t \in (0, T) \\ 1 & t = T \end{cases} \quad \text{as } \varepsilon \rightarrow 0. \quad (3.3)$$

Remark 4. The scaling rate $\varepsilon^{\frac{2H}{1+2H}}$ of the estimation error here coincides with the optimal minimax rate in the nonparametric estimation problem of deterministic H -Hölder signals in the white noise model, see [53].

Proof. By scaling property (1.11), equation (3.2) reads

$$\varepsilon h_\varepsilon(u, v) + \int_0^1 \mu^2 T^{2H+1} K_{\beta T}(r, u) h_\varepsilon(r, v) dr = \mu^2 T^{2H} K_{\beta T}(u, v), \quad 0 \leq u \leq v \leq 1 \quad (3.4)$$

where $h_\varepsilon(u, v) := h(uT, vT)$, and

$$P_\varepsilon(u) = \frac{\varepsilon}{\mu^2} h_\varepsilon(u, u), \quad u \in (0, 1).$$

Expanding the solution of (3.4) into series of eigenfunctions of $K_{\beta T}$ we get

$$h_\varepsilon(u, v) = \sum_{n=1}^{\infty} \frac{\mu^2 T^{2H}}{\varepsilon \lambda_n^{-1} + \mu^2 T^{2H+1}} \varphi_n(u) \varphi_n(v), \quad 0 \leq u \leq v \leq 1,$$

where λ_n are the eigenvalues of $K_{\beta T}$. This series is absolutely convergent for any $\varepsilon > 0$ and its value diverges to $+\infty$ as $\varepsilon \rightarrow 0$. Its first order term asymptotics does not change if the eigenvalues and eigenfunctions are replaced with their first order approximations from Theorem 1.2. Let $C := \sin(\pi H) \Gamma(2H + 1)$, then

$$\begin{aligned} P_\varepsilon(1) &= \frac{\varepsilon}{\mu^2} h_\varepsilon(1, 1) = \sum_{n=1}^{\infty} \frac{\varepsilon T^{2H}}{\varepsilon \lambda_n^{-1} + \mu^2 T^{2H+1}} \varphi_n^2(1) \simeq \sum_{n=1}^{\infty} \frac{\varepsilon T^{2H}}{\frac{\varepsilon}{C} \frac{(\pi n)^2 + (\beta T)^2}{(\pi n)^{1-2H}} + \mu^2 T^{2H+1}} (2H + 1) \simeq \\ &\simeq (2H + 1) \int_1^\infty \frac{\varepsilon T^{2H}}{\frac{\varepsilon}{C} \left((\pi x)^{2H+1} + (\beta T)^2 (\pi x)^{2H-1} \right) + \mu^2 T^{2H+1}} dx \simeq \\ &\simeq \frac{\varepsilon}{\mu^2 T} \left(\frac{\varepsilon}{C} \frac{1}{T^{2H+1} \mu^2} \right)^{-\frac{1}{2H+1}} \frac{2H + 1}{\pi} \int_0^\infty \frac{1}{y^{2H+1} + 1} dy = \\ &= \frac{\varepsilon}{\mu^2 T} \left(\frac{\varepsilon}{C} \frac{1}{T^{2H+1} \mu^2} \right)^{-\frac{1}{2H+1}} \frac{1}{\sin \frac{\pi}{2H+1}} = (\varepsilon / \mu^2)^{\frac{2H}{2H+1}} \frac{C^{\frac{1}{2H+1}}}{\sin \frac{\pi}{2H+1}} \end{aligned}$$

which gives the expression for the filtering error at $t = T$ in (3.3).

Let us now calculate the limit value of $P_\varepsilon(u)$ for $u \in (0, 1)$. To this end, note that the contribution of the boundary layer term in (1.6) is of order $O(n^{-1})$ and hence

$$\varphi_n(u) = \sqrt{2} \sin(\nu_n u + \phi_H) + O(n^{-1}), \quad \text{as } n \rightarrow \infty,$$

where $\phi_H := \frac{2H-1}{8}\pi - \arcsin \frac{\ell_H}{\sqrt{1+\ell_H^2}}$. Hence

$$\begin{aligned} P_\varepsilon(u) &= \frac{\varepsilon}{\mu^2} h_\varepsilon(u, u) = \sum_{n=1}^{\infty} \frac{\varepsilon T^{2H}}{\varepsilon \lambda_n^{-1} + \mu^2 T^{2H+1}} \varphi_n^2(u) = \\ &= \sum_{n=1}^{\infty} \frac{\varepsilon T^{2H}}{\varepsilon \lambda_n^{-1} + \mu^2 T^{2H+1}} - \sum_{n=1}^{\infty} \frac{\varepsilon T^{2H}}{\varepsilon \lambda_n^{-1} + \mu^2 T^{2H+1}} \cos(2\nu_n u + 2\phi_H) := I_1(\varepsilon) + I_2(\varepsilon). \end{aligned}$$

The first term here differs from the previous case only by constant factor $2H + 1$ and hence the expression for smoothing error at $t \in (0, T)$ in (3.3) is obtained, once we show that the second term vanishes as $\varepsilon \rightarrow 0$. Define $S_0 = 0$ and

$$S_n = \sum_{k=1}^n \cos(2\nu_k u + 2\phi_H), \quad n \geq 1,$$

which is a bounded sequence for $u \in (0, 1)$. Then

$$I_2(\varepsilon) = \sum_{n=1}^{\infty} \frac{\varepsilon T^{2H}}{\varepsilon \lambda_n^{-1} + \mu^2 T^{2H+1}} (S_n - S_{n-1}) = \varepsilon^2 T^{2H} \sum_{n=1}^{\infty} S_n \frac{\lambda_{n+1}^{-1} - \lambda_n^{-1}}{(\varepsilon \lambda_{n+1}^{-1} + \mu^2 T^{2H+1})(\varepsilon \lambda_n^{-1} + \mu^2 T^{2H+1})}.$$

In view of (1.13), for all n large enough

$$|\lambda_{n+1}^{-1} - \lambda_n^{-1}| \leq C_1 n^{2H} \quad \text{and} \quad \lambda_n^{-1} \geq C_2 n^{2H+1}$$

with positive constants C_1 and C_2 . Since S_n is bounded, for some constant C_3

$$|I_2(\varepsilon)| \leq C_3 \varepsilon^2 \sum_{n=1}^{\infty} \frac{n^{2H}}{(\varepsilon n^{2H+1} + 1)^2} \simeq C_3 \varepsilon \int_0^{\infty} \frac{y^{2H}}{(y^{2H+1} + 1)^2} dy \xrightarrow{\varepsilon \rightarrow 0} 0.$$

□

Remark 5. Using the exact approximations from Theorems 1.2, 1.3 and 1.4 one can obtain the similar results for other fractional Gaussian processes.

3.1.3 Large time asymptotics for fractional signals

Another important objective is to compute the steady state limit of optimal linear filtering error $P_\infty = \lim_{t \rightarrow \infty} P_t$, if it exists. Without loss of generality let $\varepsilon = 1$ in (3.1) and let

$X = (X_t, t \in [0, T])$ be the fractional OU process (of the first kind) started from zero, that is, the solution of the Langevin equation

$$X_t = \beta \int_0^t X_s ds + B_t^H, \quad t \in [0, T]$$

subject to $X_0 = 0$. Here $\beta \in \mathbb{R}$ is a fixed constant and $B^H = (B_t^H, t \in [0, T])$ is the fractional Brownian motion with the Hurst parameter $H \in (0, 1]$.

Since X is a Gaussian process with covariance function $K_\beta(s, t)$ given by (1.10), a standard calculation yields the formula

$$\widehat{X}_t = \mu \int_0^t g(s, t) dY_s,$$

where $g(s, t)$ is the solution of the Wiener-Hopf integral equation

$$g(s, t) + \mu^2 \int_0^t g(r, t) R(r, s) dr = R(s, t), \quad 0 \leq s \leq t \leq T \quad (3.5)$$

and the estimation error is given by

$$P_t = g(t, t).$$

Unfortunately, in order to find the steady state limit we can no longer use the rescaling property (1.11) since βT is not bounded, when $T \rightarrow \infty$. Nevertheless, we can still prove the following result.

Theorem 3.1. (1) *The steady state filtering error for the fractional Ornstein-Uhlenbeck process with Hurst parameter $H > \frac{1}{2}$ is given by*

$$P_\infty = \lim_{T \rightarrow \infty} P_T = \frac{\beta + \delta_H + 2 \operatorname{Re} z_0}{\mu^2},$$

where $z_0 = \rho e^{i\phi}$ is the unique solution of (3.14) such that $\rho > 0$ and $\phi \in (0, \frac{\pi}{2})$ and the constant

$$\delta_H = -\frac{1}{\pi} \int_0^\infty \operatorname{arccot} \left(\frac{1}{\pi} \frac{\Gamma(2-2H)}{H(2H-1)} \frac{1}{\mu^2} (t^2 - \beta^2) t^{2H-1} - \cot \frac{2H-1}{2} \pi \right) dt.$$

(2) *The steady state filtering error for the fractional Ornstein-Uhlenbeck process with Hurst parameter $H < \frac{1}{2}$ is given by*

$$P_\infty = \lim_{T \rightarrow \infty} P_T = \frac{\beta + \delta_H}{\mu^2},$$

and the constant

$$\delta_H = \frac{1}{\pi} \int_0^\infty \operatorname{arccot} \left(\frac{1}{\pi} \frac{\Gamma(2-2H)}{H(1-2H)} \frac{1}{\mu^2} (t^2 - \beta^2) t^{2H-1} - \cot \frac{1-2H}{2} \pi \right) dt.$$

Proof for $H > \frac{1}{2}$

Define $\alpha := 2 - 2H \in (0, 1)$, then

$$K_\beta(s, t) = \int_0^t \int_0^s e^{\beta(t-v)} e^{\beta(s-u)} c_\alpha |u - v|^{-\alpha} du dv$$

with $c_\alpha = (1 - \frac{\alpha}{2})(1 - \alpha)$ and

$$\frac{\partial}{\partial s} K_\beta(s, t) = \beta K_\beta(s, t) + c_\alpha \int_0^t e^{\beta(t-v)} |u - v|^{-\alpha} dv.$$

Differentiating (3.5) we get

$$\frac{\partial}{\partial s} g(s, t) - \beta g(s, t) + \mu^2 c_\alpha \int_0^t g(r, t) \int_0^r e^{\beta(r-v)} |s - v|^{-\alpha} dv dr = f(s, t), \quad (3.6)$$

where

$$f(s, t) = c_\alpha \int_0^t e^{\beta(t-v)} |s - v|^{-\alpha} dv.$$

For the fixed value of $T > 0$ define the new function

$$\Upsilon_T(s) = e^{-\beta s} \int_s^T g(r, T) e^{\beta r} dr.$$

Denote $f_T(s) = f(s, T)$. Then according to (3.6), $\Upsilon_T(s)$ is a solution of the following equation

$$-\Upsilon_T''(s) + \beta^2 \Upsilon_T(s) + \mu^2 c_\alpha \int_0^T |s - r|^{-\alpha} \Upsilon_T(r) dr = f_T(s),$$

subject to the boundary condition

$$\Upsilon_T(T) = 0 \quad \text{and} \quad \beta \Upsilon_T(0) + \Upsilon_T'(0) = 0.$$

In order to simplify the equation and get rid of f_T on the right side we introduce a new function

$$\Psi_T(t) = \Upsilon_T(t) - \frac{1}{\mu^2} e^{\beta(T-t)}, \quad t \in [0, T]$$

that satisfies the equation

$$-\Psi_T''(t) + \beta^2 \Psi_T(t) + \mu^2 c_\alpha \int_0^T |t-r|^{-\alpha} \Psi_T(r) dr = 0 \quad (3.7)$$

and the boundary condition

$$\Psi_T(T) = -\frac{1}{\mu^2} \quad \text{and} \quad \beta \Psi_T(0) + \Psi_T'(0) = 0. \quad (3.8)$$

Our purpose is to find

$$P_\infty = \lim_{T \rightarrow \infty} g(T, T) = -\lim_{T \rightarrow \infty} \Upsilon_T'(T) = \frac{\beta}{\mu^2} - \lim_{T \rightarrow \infty} \Psi_T'(T).$$

The Laplace transform. Once again, let us start with deriving a suitable expression for the Laplace transform of Ψ_T :

Lemma 3.1. *The Laplace transform of Ψ_T satisfies*

$$\widehat{\Psi}_T(z) = \frac{\Phi_{0,T}(z) + e^{-zT} \Phi_{1,T}(-z)}{\Lambda(z)}, \quad (3.9)$$

where

$$\Lambda(z) = z^2 - \beta^2 - \frac{\mu^2 c_\alpha}{\Gamma(\alpha)} \int_0^\infty \frac{2t^\alpha}{t^2 - z^2} dt$$

and the functions $\Phi_{0,T}(z)$ and $\Phi_{1,T}(z)$ are sectionally holomorphic on the cut plane $\mathbb{C} \setminus \mathbb{R}_{>0}$.

Proof. Recall that

$$\frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t|x-y|} dt = |x-y|^{-\alpha}, \quad \alpha \in (0, 1)$$

and define

$$u_T(x, t) = \int_0^T \Psi_T(y) e^{-t|x-y|} dy$$

$$v_T(x) = \int_0^\infty t^{\alpha-1} u_T(x, t) dt.$$

Then applying Laplace transform to (3.7) we get

$$-\widehat{\Psi}_T''(z) + \beta^2 \widehat{\Psi}_T(z) + \frac{\mu^2 c_\alpha}{\Gamma(\alpha)} \widehat{v}_T(z) = 0. \quad (3.10)$$

Using the boundary conditions (3.8), the following expression is obtained for the Laplace transform of Ψ_T'' :

$$\widehat{\Psi}_T''(z) = A_T(\beta - z) + e^{-zT} \left(B_T - \frac{1}{\mu^2} z \right) + z^2 \widehat{\Psi}_T(z)$$

with $A_T = \Psi_T(0)$ and $B_T = \Psi_T'(T)$. An additional relation can be deduced as follows. The function $u_T(x, t)$ satisfies the equation

$$u_T''(x, t) = t^2 u_T(x, t) - 2t \Psi_T(x), \quad (3.11)$$

subject to the boundary conditions

$$u_T'(0, t) = t u_T(0, t) \quad \text{and} \quad u_T'(T, t) = -t u_T(T, t).$$

Consequently, the Laplace transform of $u''(x, t)$ satisfies

$$\widehat{u}_T''(z, t) = e^{-zT} (z - t) u_T(T, t) - (z + t) u_T(0, t) + z^2 \widehat{u}_T(z, t). \quad (3.12)$$

Combining (3.11) and (3.12) we obtain

$$\widehat{v}_T(z) = -\widehat{\Psi}_T(z) \int_0^\infty \frac{2t^\alpha}{z^2 - t^2} dt + \int_0^\infty \frac{t^{\alpha-1}}{z - t} u_T(0, t) dt - e^{-zT} \int_0^\infty \frac{t^{\alpha-1}}{z + t} u_T(T, t) dt.$$

Plugging this expression into (3.10) and rearranging gives (3.9) with

$$\begin{aligned}\Lambda(z) &= z^2 - \beta^2 - \frac{\mu^2 c_\alpha}{\Gamma(\alpha)} \int_0^\infty \frac{2t^\alpha}{t^2 - z^2} dt \\ \Phi_{0,T}(z) &= A_T(z - \beta) - \frac{\mu^2 c_\alpha}{\Gamma(\alpha)} \int_0^\infty \frac{t^{\alpha-1}}{t - z} u_T(0, t) dt \\ \Phi_{1,T}(z) &= -\frac{1}{\mu^2} z - B_T - \frac{\mu^2 c_\alpha}{\Gamma(\alpha)} \int_0^\infty \frac{t^{\alpha-1}}{t - z} u_T(T, t) dt\end{aligned}\quad (3.13)$$

□

Lemma 3.2. a) The function $\Lambda(z)$ is given by the formula

$$\Lambda(z) = z^2 - \beta^2 - \frac{\mu^2 c_\alpha}{\Gamma(\alpha)} \frac{\pi}{\cos \frac{\pi}{2} \alpha} z^{\alpha-1} \begin{cases} e^{\frac{1-\alpha}{2} \pi i} & \arg(z) \in (0, \pi) \\ e^{-\frac{1-\alpha}{2} \pi i} & \arg(z) \in (-\pi, 0) \end{cases}.$$

This expression vanishes only at $\pm z_0 = \pm \rho e^{i\varphi}$ and $\pm \bar{z}_0$, where $\rho = \rho(\alpha, \mu)$ and $\varphi = \varphi(\alpha, \mu)$ are the unique solution of the equation

$$\rho^2 e^{2i\varphi} - \beta^2 - \frac{\mu^2 c_\alpha}{\Gamma(\alpha)} \frac{\pi}{\cos \frac{\pi}{2} \alpha} \rho^{\alpha-1} e^{((\alpha-1)\varphi + \frac{1-\alpha}{2} \pi)i} = 0 \quad (3.14)$$

depending only on α and μ and such that $\rho > 0$ and $\varphi \in (0, \frac{1-\alpha}{3-\alpha} \pi)$.

b) $\Lambda(z)$ has a discontinuity along the real axis with the limits

$$\Lambda^\pm(t) = t^2 - \beta^2 - \frac{\mu^2 c_\alpha}{\Gamma(\alpha)} \frac{\pi}{\cos \frac{\pi}{2} \alpha} |t|^{\alpha-1} \frac{\pi}{\cos \frac{\pi}{2} \alpha} \begin{cases} e^{\pm \frac{1-\alpha}{2} \pi i} & t > 0 \\ e^{\mp \frac{1-\alpha}{2} \pi i} & t < 0 \end{cases}$$

and the symmetries (1.45)-(1.47) hold.

Proof. Follows from the formula

$$\int_0^\infty \frac{t^\alpha}{t^2 - z^2} dt = z^{\alpha-1} \frac{1}{2} \frac{\pi}{\cos \frac{\pi}{2} \alpha} \begin{cases} e^{\frac{1-\alpha}{2} \pi i} & \arg(z) \in (0, \pi) \\ e^{-\frac{1-\alpha}{2} \pi i} & \arg(z) \in (-\pi, 0) \end{cases}$$

□

For fixed values of α and β let us now introduce the function

$$\theta(t) := \arg\{\Lambda^+(t)\} = -\operatorname{arccot} \left(\frac{1}{\pi} \frac{\Gamma(\alpha)}{\mu^2 c_\alpha} (t^2 - \beta^2) t^{\alpha-1} - \cot \frac{1-\alpha}{2} \pi \right).$$

This function is not monotonic, but is increasing eventually and

$$\theta(0+) = -\frac{1+\alpha}{2}\pi \quad \text{and} \quad \theta(t) \sim -t^{\alpha-3}, \quad t \rightarrow \infty.$$

Removal of singularities. Since the Laplace transform is a priori an entire function, removal of the singularities gives the following conditions

$$\begin{aligned} \Phi_{0,T}(\pm z_0) + e^{\mp z_0 T} \Phi_{1,T}(\mp z_0) &= 0 \\ \Phi_{0,T}(\pm \bar{z}_0) + e^{\mp \bar{z}_0 T} \Phi_{1,T}(\mp \bar{z}_0) &= 0 \end{aligned} \quad (3.15)$$

and

$$\begin{aligned} \Phi_{0,T}^+(t) - e^{2i\theta(t)} \Phi_{0,T}^-(t) &= 2ie^{i\theta(t)} \sin \theta(t) e^{-tT} \Phi_{1,T}(-t) \\ \Phi_{1,T}^+(t) - e^{2i\theta(t)} \Phi_{1,T}^-(t) &= 2ie^{i\theta(t)} \sin \theta(t) e^{-tT} \Phi_{0,T}(-t) \end{aligned}, \quad t > 0. \quad (3.16)$$

Since $tu_T(0, t)$ and $tu_T(1, t)$ are bounded, it follows from (3.13) that

$$\Phi_{0,T}(z) \sim A_T(z - \beta) \quad \text{and} \quad \Phi_{1,T}(z) \sim -\frac{1}{\mu^2}z - B_T \quad \text{as } z \rightarrow \infty$$

and

$$|\Phi_{0,T}(z)| \sim |z|^{\alpha-1} \quad \text{and} \quad |\Phi_{1,T}(z)| \sim |z|^{\alpha-1} \quad \text{as } |z| \rightarrow 0.$$

An equivalent formulation of the eigenproblem. Consider the homogeneous Hilbert problem of finding a function $X(z)$, sectionally holomorphic on the cut plane $\mathbb{C} \setminus \mathbb{R}_{>0}$ and satisfying the boundary condition

$$X^+(t) - e^{2i\theta(t)} X^-(t) = 0, \quad t \in \mathbb{R}_{>0}.$$

One of the appropriate solutions obtained by the Plemelj formula is

$$X(z) = \frac{1}{z} \exp \left(\frac{1}{\pi} \int_0^\infty \frac{\theta(t)}{t-z} dt \right), \quad z \in \mathbb{C} \setminus \mathbb{R}_{>0}.$$

that satisfies

$$|X(z)| \sim |z|^{\frac{\alpha-1}{2}}, \quad z \rightarrow 0 \quad \text{and} \quad X(z) \sim \frac{1}{z} - \frac{b_0}{z^2} + O\left(\frac{1}{z^3}\right), \quad z \rightarrow \infty \quad (3.17)$$

where

$$b_0 = \frac{1}{\pi} \int_0^\infty \theta(u) du.$$

The functions

$$S_T(z) := \frac{\Phi_{0,T}(z) + \Phi_{1,T}(z)}{2X(z)} \quad \text{and} \quad D_T(z) := \frac{\Phi_{0,T}(z) - \Phi_{1,T}(z)}{2X(z)}$$

satisfy the decoupled boundary conditions:

$$\begin{aligned} S_T^+(t) - S_T^-(t) &= 2ih(t)e^{-tT}S_T(-t) \\ D_T^+(t) - D_T^-(t) &= -2ih(t)e^{-tT}D_T(-t) \end{aligned}$$

where we defined a real valued, Holder continuous on $\mathbb{R}_{>0}$ function

$$h(t) := e^{i\theta(t)} \sin \theta(t) \frac{X(-t)}{X^+(t)} = \exp\left(-\frac{1}{\pi} \int_0^\infty \theta'(s) \log \left| \frac{t+s}{t-s} \right| ds\right) \sin \theta(t).$$

with $h(0) := \sin \theta(0+) = -\sin \frac{1+\alpha}{2}\pi$. By the Plemelj formula

$$\begin{aligned} S_T(z) &= \frac{1}{\pi} \int_0^\infty \frac{h(t)e^{-tT}}{t-z} S_T(-t) dt + P_T^S(z) \\ D_T(z) &= -\frac{1}{\pi} \int_0^\infty \frac{h(t)e^{-tT}}{t-z} D_T(-t) dt + P_T^D(z), \end{aligned}$$

where $P_T^S(z)$ and $P_T^D(z)$ are arbitrary polynomials. Since (3.17), the growth of $\Phi_{0,T}(z)$ and $\Phi_{1,T}(z)$ in (3.13) implies

$$\begin{aligned} P_T^S(z) &:= \frac{1}{2} \left(A_T - \frac{1}{\mu^2} \right) z^2 + \frac{1}{2} \left((b_0 - \beta) A_T - B_T - \frac{b_0}{\mu^2} \right) z + k_T^S \\ P_T^D(z) &:= \frac{1}{2} \left(A_T + \frac{1}{\mu^2} \right) z^2 + \frac{1}{2} \left((b_0 - \beta) A_T + B_T + \frac{b_0}{\mu^2} \right) z + k_T^D, \end{aligned}$$

where k_T^S and k_T^D are some constants. Setting $z := -t$ with $t \in \mathbb{R}_{>0}$, the following integral equations are obtained:

$$\begin{aligned} S_T(-t) &= \frac{1}{\pi} \int_0^\infty \frac{h(s)e^{-sT}}{s+t} S_T(-s) ds + P_T^S(-t) \\ D_T(-t) &= -\frac{1}{\pi} \int_0^\infty \frac{h(s)e^{-sT}}{s+t} D_T(-s) ds + P_T^D(-t). \end{aligned}$$

For $j \in \{0, 1, 2\}$ let $p_{j,T}(t)$ and $q_{j,T}(t)$ be the solutions of the integral equations

$$\begin{aligned} p_{j,T}(t) &= \frac{1}{\pi} \int_0^\infty \frac{h(s)e^{-sT}}{s+t} p_{j,T}(s) ds + t^j \\ q_{j,T}(t) &= -\frac{1}{\pi} \int_0^\infty \frac{h(s)e^{-sT}}{s+t} q_{j,T}(s) ds + t^j \end{aligned} \tag{3.18}$$

then by linearity

$$\begin{aligned} S_T(z) &= \frac{1}{2} \left(A_T - \frac{1}{\mu^2} \right) p_{2,T}(-z) - \frac{1}{2} \left((b_0 - \beta) A_T - B_T - \frac{b_0}{\mu^2} \right) p_{1,T}(-z) + k_T^S p_{0,T}(-z) \\ D_T(z) &= \frac{1}{2} \left(A_T + \frac{1}{\mu^2} \right) q_{2,T}(-z) - \frac{1}{2} \left((b_0 - \beta) A_T + B_T + \frac{b_0}{\mu^2} \right) q_{1,T}(-z) + k_T^D q_{0,T}(-z), \end{aligned}$$

where the definition of all the functions on the cut plane is extended in the obvious way. Consequently, if we let

$$a_{j,T}(z) := p_{j,T}(z) + q_{j,T}(z) \quad \text{and} \quad b_{j,T}(z) := p_{j,T}(z) - q_{j,T}(z)$$

we get

$$\begin{aligned} \Phi_{0,T}(z) &= \left(\frac{1}{2} A_T a_{2,T}(-z) - \frac{1}{2\mu^2} b_{2,T}(-z) - \frac{1}{2} (b_0 - \beta) A_T a_{1,T}(-z) + \right. \\ &\quad \left. + \frac{1}{2} \left(B_T + \frac{b_0}{\mu^2} \right) b_{1,T}(-z) + k_T^S p_{0,T}(-z) + k_T^D q_{0,T}(-z) \right) X(z) \\ \Phi_{1,T}(z) &= \left(\frac{1}{2} A_T b_{2,T}(-z) - \frac{1}{2\mu^2} a_{2,T}(-z) - \frac{1}{2} (b_0 - \beta) A_T b_{1,T}(-z) + \right. \\ &\quad \left. + \frac{1}{2} \left(B_T + \frac{b_0}{\mu^2} \right) a_{1,T}(-z) + k_T^S p_{0,T}(-z) - k_T^D q_{0,T}(-z) \right) X(z). \end{aligned}$$

Applying this expressions to (3.15) we get

$$\begin{aligned} & \frac{1}{2}A_T\xi_{2,T}(z_0) - \frac{1}{2\mu^2}\eta_{2,T}(z_0) - \frac{1}{2}(b_0 - \beta)A_T\xi_{1,T}(z_0) + \\ & + \frac{1}{2}(B_T + \frac{b_0}{\mu^2})\eta_{1,T}(z_0) + k_T^S\zeta_T^S(z_0) + k_T^D\zeta_T^D(z_0) = 0, \end{aligned} \quad (3.19)$$

where

$$\begin{aligned} \xi_{j,T}(z) &= a_{j,T}(z) \frac{X(-z)}{X(z)} e^{-zT} + b_{j,T}(-z), \quad j \in \{1, 2\} \\ \eta_{j,T}(z) &= b_{j,T}(z) \frac{X(-z)}{X(z)} e^{-zT} + a_{j,T}(-z), \quad j \in \{1, 2\} \\ \zeta_T^S(z) &= p_{0,T}(z) \frac{X(-z)}{X(z)} e^{-zT} + p_{0,T}(-z) \\ \zeta_T^D(z) &= q_{0,T}(z) \frac{X(-z)}{X(z)} e^{-zT} - q_{0,T}(-z) \end{aligned}$$

Properties of the integro-algebraic system. Solvability of the integro-algebraic system, constructed in the previous section, relies on the contracting properties of the operator

$$(Af)(t) := \frac{1}{\pi} \int_0^\infty \frac{h(s)e^{-sT}}{s+t} f(s) ds.$$

Lemma 3.3. *The operator A contracts in $L^2(0, \infty)$ for all T large enough. More precisely, for any $\alpha_0 \in (0, 1]$ there exists an $\varepsilon > 0$ and a constant $T' > 0$, such that $\|A\| \leq 1 - \varepsilon$ for all $T \geq T'$ and all $\alpha \in [\alpha_0, 1]$.*

Lemma 3.4. *For any $\alpha_0 \in (0, 1]$ there exist constants T' and C , such that for all $T \geq T'$ and all $\alpha \in [\alpha_0, 1]$*

$$\begin{aligned} |a_{j,T}(\pm z_0) - 2(\pm z_0)^j| &\leq CT^{-1-j}, \quad |b_{j,T}(\pm z_0)| \leq CT^{-1-j}, \quad j \in \{1, 2\} \\ |p_{0,T}(\pm z_0) - 1| &\leq CT^{-1}, \quad |q_{0,T}(\pm z_0) - 1| \leq CT^{-1} \end{aligned}$$

Asymptotic analysis. Taking the limit as $T \rightarrow \infty$ in (3.19) and applying the estimates from Lemma 3.4 we get:

$$-\frac{z_0^2}{\mu^2} - \left(\lim_{T \rightarrow \infty} B_T + \frac{b_0}{\mu^2} \right) z_0 + \lim_{T \rightarrow \infty} k_T^S - \lim_{T \rightarrow \infty} k_T^D = 0$$

and consequently taking the imaginary part we get

$$\lim_{T \rightarrow \infty} B_T = -\frac{b_0}{\mu^2} - \frac{2\operatorname{Re}z_0}{\mu^2}$$

and

$$P_\infty = \frac{\beta}{\mu^2} - \lim_{T \rightarrow \infty} \Psi'_T(T) = \frac{\beta + b_0 + 2\operatorname{Re}z_0}{\mu^2}.$$

Proof for $H < \frac{1}{2}$

Define $\alpha := 2 - 2H \in (1, 2)$, then

$$K_\beta(s, t) = \int_0^s e^{\beta(s-v)} \frac{d}{dv} \int_0^t \left(1 - \frac{\alpha}{2}\right) |v - u|^{1-\alpha} \operatorname{sign}(v - u) e^{\beta(t-u)} dudv$$

and

$$\frac{\partial}{\partial s} K_\beta(s, t) = \beta K_\beta(s, t) + \frac{d}{ds} \int_0^t \left(1 - \frac{\alpha}{2}\right) |s - u|^{1-\alpha} \operatorname{sign}(s - u) e^{\beta(t-u)} du.$$

Differentiating (3.5) we get

$$\begin{aligned} \frac{\partial}{\partial s} g(s, t) - \beta g(s, t) + \mu^2 \int_0^t g(r, t) \frac{d}{ds} \int_0^r \left(1 - \frac{\alpha}{2}\right) e^{\beta(r-u)} |s - u|^{1-\alpha} \operatorname{sign}(s - u) dudr = \\ = \frac{d}{ds} \int_0^t \left(1 - \frac{\alpha}{2}\right) e^{\beta(t-u)} |s - u|^{1-\alpha} \operatorname{sign}(s - u) du. \end{aligned}$$

For the fixed value of $T > 0$ define the new function

$$\Upsilon_T(s) = e^{-\beta s} \int_s^T g(r, T) e^{\beta r} dr.$$

Then from integrating by parts follows the following equation

$$\begin{aligned} -\Upsilon_T''(s) + \beta^2 \Upsilon_T(s) + \mu^2 \frac{d}{ds} \int_0^T \left(1 - \frac{\alpha}{2}\right) |s - r|^{1-\alpha} \operatorname{sign}(s - r) \Upsilon_T(r) dr = \\ = e^{\beta T} \frac{d}{ds} \int_0^T \left(1 - \frac{\alpha}{2}\right) |s - r|^{1-\alpha} \operatorname{sign}(s - r) e^{-\beta r} dr. \end{aligned}$$

subject to the boundary condition

$$\Upsilon_T(T) = 0 \quad \text{and} \quad \beta\Upsilon_T(0) + \Upsilon_T'(0) = 0.$$

We introduce a new function

$$\Psi_T(t) = \Upsilon_T(t) - \frac{1}{\mu^2} e^{\beta(T-t)}, \quad t \in [0, T]$$

that satisfies the equation

$$-\Psi_T''(t) + \beta^2\Psi_T(t) + \mu^2 \frac{d}{dt} \int_0^T \left(1 - \frac{\alpha}{2}\right) |t-r|^{1-\alpha} \text{sign}(t-r) \Psi_T(r) dr = 0 \quad (3.20)$$

and the boundary conditions

$$\Psi_T(T) = -\frac{1}{\mu^2} \quad \text{and} \quad \beta\Psi_T(0) + \Psi_T'(0) = 0.$$

Our objective is to find

$$P_\infty = \lim_{T \rightarrow \infty} g(T, T) = -\lim_{T \rightarrow \infty} \Upsilon_T'(T) = \frac{\beta}{\mu^2} - \lim_{T \rightarrow \infty} \Psi_T'(T).$$

The Laplace transform. Let us proceed with deriving a suitable expression for the Laplace transform of Ψ_T :

Lemma 3.5. *The Laplace transform of Ψ_T satisfies*

$$\widehat{\Psi}_T(z) = \frac{\Phi_{0,T}(z) + e^{-zT}\Phi_{1,T}(-z)}{\Lambda(z)}, \quad (3.21)$$

where

$$\Lambda(z) = z^2 - \beta^2 - \frac{\mu^2 |c_\alpha|}{\Gamma(\alpha)} z^2 \int_0^\infty \frac{2t^{\alpha-2}}{z^2 - t^2} dt$$

and the functions $\Phi_{0,T}(z)$ and $\Phi_{1,T}(z)$ are sectionally holomorphic on the cut plane $\mathbb{C} \setminus \mathbb{R}_{>0}$.

Proof. Recall that

$$\frac{d}{dx} \int_0^T |x-y|^{1-\alpha} \text{sign}(x-y) \Psi_T(y) dy = \frac{1}{\Gamma(\alpha-1)} \frac{d}{dx} \int_0^\infty t^{\alpha-2} \int_0^T e^{-t|x-y|} \text{sign}(x-y) \Psi_T(y) dy dt.$$

and define

$$u_T(x, t) = \int_0^T \text{sign}(x-y) e^{-t|x-y|} \Psi_T(y) dy$$

$$v_T(x) = \int_0^\infty t^{\alpha-2} u_T(x, t) dt.$$

Then applying Laplace transform we get

$$-\widehat{\Psi}_T''(z) + \beta^2 \widehat{\Psi}_T(z) + \frac{\mu^2(1 - \frac{\alpha}{2})}{\Gamma(\alpha-1)} \widehat{v}_T'(z) = 0. \quad (3.22)$$

Using the boundary conditions for Ψ_T , the following expression is obtained:

$$\widehat{\Psi}_T''(z) = A_T(\beta - z) + e^{-zT} (B_T - \frac{1}{\mu^2} z) + z^2 \widehat{\Psi}_T(z)$$

with $A_T = \Psi_T(0)$ and $B_T = \Psi_T'(T)$.

An additional relation can be deduced as follows. The function $u_T(x, t)$ satisfies the equation

$$u_T''(x, t) = 2\Psi_T'(x) + t^2 u_T(x, t),$$

subject to the boundary conditions

$$u_T'(0, t) = 2\Psi_T(0) + t u_T(0, t) \quad \text{and} \quad u_T'(T, t) = 2\Psi_T(0) - t u_T(T, t).$$

Consequently, the Laplace transform of $u''(x, t)$ satisfies

$$\widehat{u}_T''(z, t) = -2\Psi_T(0) + 2\Psi_T(T) e^{-zT} + e^{-zT} (z-t) u_T(T, t) - (z+t) u_T(0, t) + z^2 \widehat{u}_T(z, t).$$

Combining these expressions we obtain

$$\widehat{v}'_T(z) = z^2 \widehat{\Psi}_T(z) \int_0^\infty \frac{2t^{\alpha-2}}{z^2 - t^2} dt + \int_0^\infty \frac{t^{\alpha-1}}{z - t} u_T(0, t) dt + e^{-zT} \int_0^\infty \frac{t^{\alpha-1}}{z + t} u_T(T, t) dt.$$

Plugging this expression into (3.22) and rearranging gives (3.21) with

$$\begin{aligned} \Lambda(z) &= z^2 - \beta^2 - \frac{\mu^2 |c_\alpha|}{\Gamma(\alpha)} z^2 \int_0^\infty \frac{2t^{\alpha-2}}{z^2 - t^2} dt \\ \Phi_{0,T}(z) &= A_T(z - \beta) - \frac{\mu^2 |c_\alpha|}{\Gamma(\alpha)} \int_0^\infty \frac{t^{\alpha-1}}{t - z} u_T(0, t) dt \\ \Phi_{1,T}(z) &= -\frac{1}{\mu^2} z - B_T + \frac{\mu^2 |c_\alpha|}{\Gamma(\alpha)} \int_0^\infty \frac{t^{\alpha-1}}{t - z} u_T(T, t) dt \end{aligned} \quad (3.23)$$

□

Lemma 3.6. a) The function $\Lambda(z)$ is given by the formula

$$\Lambda(z) = z^2 - \beta^2 - \frac{\mu^2 |c_\alpha|}{\Gamma(\alpha)} \frac{\pi}{|\cos \frac{\pi}{2} \alpha|} z^{\alpha-1} \begin{cases} e^{\frac{1-\alpha}{2} \pi i} & \arg(z) \in (0, \pi) \\ e^{-\frac{1-\alpha}{2} \pi i} & \arg(z) \in (-\pi, 0) \end{cases}.$$

and does not have zeros in the complex plane.

b) $\Lambda(z)$ has a discontinuity along the real axis with the limits

$$\Lambda^\pm(t) = t^2 - \beta^2 - \frac{\mu^2 |c_\alpha|}{\Gamma(\alpha)} \frac{\pi}{|\cos \frac{\pi}{2} \alpha|} |t|^{\alpha-1} \frac{\pi}{\cos \frac{\pi}{2} \alpha} \begin{cases} e^{\pm \frac{1-\alpha}{2} \pi i} & t > 0 \\ e^{\mp \frac{1-\alpha}{2} \pi i} & t < 0 \end{cases}$$

and the symmetries (1.45)-(1.47) hold.

Proof. Follows from the formula

$$\int_0^\infty \frac{t^{\alpha-2}}{t^2 - z^2} dt = z^{\alpha-3} \frac{1}{2} \frac{\pi}{|\cos \frac{\pi}{2} \alpha|} \begin{cases} e^{\frac{3-\alpha}{2} \pi i} & \arg(z) \in (0, \pi) \\ e^{-\frac{3-\alpha}{2} \pi i} & \arg(z) \in (-\pi, 0) \end{cases}.$$

□

For fixed values of α and β lets consider function

$$\theta(t) := \arg\{\Lambda^+(t)\} = \operatorname{arccot} \left(\frac{1}{\pi} \frac{\Gamma(\alpha)}{\mu^2 |c_\alpha|} (t^2 - \beta^2) t^{1-\alpha} - \cot \frac{\alpha - 1}{2} \pi \right).$$

This function is decreasing and

$$\theta(0+) = \pi \quad \text{and} \quad \theta(t) \sim -t^{\alpha-3}, \quad t \rightarrow \infty.$$

Removal of singularities. Analogously to the case $h > \frac{1}{2}$ the removal of the singularities of $\Lambda(z)$ gives

$$\begin{aligned} \Phi_{0,T}^+(t) - e^{2i\theta(t)}\Phi_{0,T}^-(t) &= 2ie^{i\theta(t)} \sin \theta(t)e^{-tT}\Phi_{1,T}(-t) \\ \Phi_{1,T}^+(t) - e^{2i\theta(t)}\Phi_{1,T}^-(t) &= 2ie^{i\theta(t)} \sin \theta(t)e^{-tT}\Phi_{0,T}(-t), \end{aligned} \quad t > 0. \quad (3.24)$$

Moreover, it follows from (3.23) that

$$\Phi_{0,T}(z) \sim A_T(z - \beta) \quad \text{and} \quad \Phi_{1,T}(z) \sim -\frac{1}{\mu^2}z - B_T \quad \text{as } z \rightarrow \infty$$

and

$$|\Phi_{0,T}(z)| \sim \text{const} \quad \text{and} \quad |\Phi_{1,T}(z)| \sim \text{const} \quad \text{as } |z| \rightarrow 0.$$

An equivalent formulation of the eigenproblem. Consider the solution of the homogeneous Hilbert problem obtained by Plemelj formula

$$X(z) = z \exp\left(\frac{1}{\pi} \int_0^\infty \frac{\theta(t)}{t-z} dt\right), \quad z \in \mathbb{C} \setminus \mathbb{R}_{>0}.$$

that satisfies

$$|X(z)| \sim \text{const}, \quad z \rightarrow 0 \quad \text{and} \quad X(z) \sim z - b_0 + O\left(\frac{1}{z}\right), \quad z \rightarrow \infty \quad (3.25)$$

where

$$b_0 = \frac{1}{\pi} \int_0^\infty \theta(u) du.$$

The functions

$$S_T(z) := \frac{\Phi_{0,T}(z) + \Phi_{1,T}(z)}{2X(z)} \quad \text{and} \quad D_T(z) := \frac{\Phi_{0,T}(z) - \Phi_{1,T}(z)}{2X(z)}$$

satisfy the decoupled boundary conditions:

$$\begin{aligned} S_T^+(t) - S_T^-(t) &= 2ih(t)e^{-tT} S_T(-t) \\ D_T^+(t) - D_T^-(t) &= -2ih(t)e^{-tT} D_T(-t), \end{aligned}$$

where we defined a real valued, Holder continuous function

$$h(t) := e^{i\theta(t)} \sin \theta(t) \frac{X(-t)}{X^+(t)}.$$

By the Plemelj formula and using the growth rates of $\Phi_{0,T}(z)$ and $\Phi_{1,T}(z)$ as $z \rightarrow \infty$ we get

$$\begin{aligned} S_T(z) &= \frac{1}{\pi} \int_0^\infty \frac{h(t)e^{-tT}}{t-z} S_T(-t) dt + \frac{1}{2} \left(A_T - \frac{1}{\mu^2} \right) \\ D_T(z) &= -\frac{1}{\pi} \int_0^\infty \frac{h(t)e^{-tT}}{t-z} D_T(-t) dt + \frac{1}{2} \left(A_T + \frac{1}{\mu^2} \right). \end{aligned}$$

Let $p_T(t)$ and $q_T(t)$ be the solutions of the integral equations

$$\begin{aligned} p_T(t) &= \frac{1}{\pi} \int_0^\infty \frac{h(s)e^{-sT}}{s+t} p_T(s) ds + 1 \\ q_T(t) &= -\frac{1}{\pi} \int_0^\infty \frac{h(s)e^{-sT}}{s+t} q_T(s) ds + 1, \end{aligned}$$

extended to the cut plane in the obvious way. Then by linearity

$$S_T(z) = \frac{1}{2} \left(A_T - \frac{1}{\mu^2} \right) p_T(-z) \quad \text{and} \quad D_T(z) = \frac{1}{2} \left(A_T + \frac{1}{\mu^2} \right) q_T(-z).$$

Consequently, if we let

$$a_T(z) := p_T(z) + q_T(z) \quad \text{and} \quad b_T(z) := p_T(z) - q_T(z)$$

we get

$$\begin{aligned} \Phi_{0,T}(z) &= \frac{1}{2} X(z) \left(A_T a_T(-z) - \frac{1}{\mu^2} b_T(-z) \right) \\ \Phi_{1,T}(z) &= \frac{1}{2} X(z) \left(A_T b_T(-z) - \frac{1}{\mu^2} a_T(-z) \right). \end{aligned} \tag{3.26}$$

Properties of the integro-algebraic system. Further we are going to use the contracting properties of the operator

$$(Af)(t) := \frac{1}{\pi} \int_0^\infty \frac{h(s)e^{-sT}}{s+t} f(s) ds.$$

Lemma 3.7. *The operator A contracts in $L^2(0, \infty)$ for all T large enough. More precisely, for any $\alpha_0 \in (1, 2)$ there exists an $\varepsilon > 0$ and a constant $T' > 0$, such that $\|A\| \leq 1 - \varepsilon$ for all $T \geq T'$ and all $\alpha \in (1, \alpha_0)$.*

Lemma 3.8. *For any $\alpha_0 \in (1, 2)$ there exist constants T' and C , such that for all $T \geq T'$ and all $\alpha \in (1, \alpha_0)$*

$$\left| a_T(-\tau) - 2 - \frac{\delta_T^+ + \delta_T^-}{\tau} \right| + \left| b_T(-\tau) - \frac{\delta_T^+ - \delta_T^-}{\tau} \right| \leq \frac{C}{T\tau^2}, \quad \tau > 0,$$

where

$$\delta_T^+ = \frac{1}{\pi} \int_0^\infty h(s)e^{-sT} p_T(s) ds \quad \text{and} \quad \delta_T^- = -\frac{1}{\pi} \int_0^\infty h(s)e^{-sT} q_T(s) ds$$

Moreover, the following inequality holds

$$|\delta_T^+| + |\delta_T^-| \leq \frac{C}{T}.$$

Asymptotic analysis. Using the asymptotics of $\Phi_{0,T}$ and $\Phi_{1,T}$, the asymptotics of a_T and b_T from Lemma 3.8 and the representation (3.26) we get:

$$\begin{aligned} A_T(\beta - b_0 - \frac{\delta_T^+ + \delta_T^-}{2}) &= -\frac{1}{2\mu^2}(\delta_T^+ - \delta_T^-) \\ B_T &= \frac{1}{\mu^2} \left(-b_0 - \frac{\delta_T^+ + \delta_T^-}{2} + \frac{\delta_T^+ - \delta_T^-}{2} A_T \right). \end{aligned}$$

and consequently, since $|\delta_T^+| + |\delta_T^-| \rightarrow 0$ as $T \rightarrow \infty$, we get

$$\lim_{T \rightarrow \infty} B_T = -\frac{b_0}{\mu^2}$$

and

$$P_\infty = \frac{\beta}{\mu^2} - \lim_{T \rightarrow \infty} \Psi'_T(T) = \frac{\beta + b_0}{\mu^2}.$$

Remark 6. Using the method described in this chapter similar results can be obtained for other fractional processes.

3.2 Large deviations for drift parameter estimator of mixed fractional Ornstein-Uhlenbeck process

3.2.1 Maximum likelihood estimation for mixed fractional Ornstein-Uhlenbeck process

Our purpose is to establish large deviations for the maximum likelihood estimator of drift parameter of the Ornstein-Uhlenbeck process driven by mixed fractional Brownian motion:

$$dX_t = -\vartheta X_t dt + d\tilde{B}_t, \quad t \in [0, T], \quad T > 0 \quad (3.27)$$

where the initial state $X_0 = 0$ and the drift parameter ϑ is strictly positive. The process \tilde{B}_t is a mixed fractional Brownian motion

$$\tilde{B}_t = B_t + B_t^H, \quad t \in [0, T] \quad (3.28)$$

where $B = (B_t)$ is a Brownian motion and $B^H = (B_t^H)$ is an independent fractional Brownian motion with the Hurst exponent $H \in (0, 1]$.

The interest to mixed fractional Brownian motion was triggered by P.Cheridito's paper [10]. Further, mixed fractional Brownian motion and related models were comprehensively considered by Y.Mishura in [37]. Finally, results of recent works of M.Kleptsyna and P.Chigansky [9] and [12] concerning the new canonical representation of mixed fractional Brownian motion present the great value for the purposes of this paper.

The main contribution of paper [9] is a novel approach to the analysis of mixed fractional Brownian motion based on the filtering theory of Gaussian processes. The core of this method is a new canonical representation of \tilde{B} .

Actually there is integral transformation which changes the mixed fractional Brownian motion to martingale. In particular (see [9]), let $g(s, t)$ be the solution of the following integro-differential equation

$$g(s, t) + \frac{d}{ds} \int_0^t g(r, t) H |s - r|^{2H-1} \text{sign}(s - r) dr = 1, \quad 0 < s < t \leq T. \quad (3.29)$$

Then the following process

$$M_t = \int_0^t g(s, t) d\tilde{B}_s, \quad t \in [0, T] \quad (3.30)$$

is a Gaussian martingale with quadratic variation

$$\langle M \rangle_t = \int_0^t g(s, t) ds, \quad t \in [0, T]. \quad (3.31)$$

Moreover, the natural filtration of the martingale M coincides with the natural filtration of the mixed fractional Brownian motion \tilde{B} . Further to what has just been mentioned concerning the mixed fractional Brownian motion, an auxiliary semimartingale, appropriate for the purposes of statistical analysis, can be also associated to the corresponding Ornstein-Uhlenbeck process X defined by (3.27). In particular, for the martingale M defined by (3.30), the sample paths of the process X are smooth enough in such sense that the following process is well-defined

$$Q_t = \frac{d}{d\langle M \rangle_t} \int_0^t g(s, t) X_s ds. \quad (3.32)$$

We also define the process $Z = (Z_t, t \in [0, T])$ by

$$Z_t = \int_0^t g(s, t) dX_s. \quad (3.33)$$

One of the most important results of [9] is that the process Z is the fundamental semimartingale associated to X in the following sense:

Theorem 3.2. *Let $g(s, t)$ be the solution of equation (3.29) and process Z be defined by (3.33). Then the following assertions hold:*

(1) Z is a semimartingale with the decomposition

$$Z_t = -\vartheta \int_0^t Q_s d\langle M \rangle_s + M_t \quad (3.34)$$

where M_t is the martingale defined by (3.30).

(2) X admits the representation

$$X_t = \int_0^t \hat{g}(s, t) dZ_s \quad (3.35)$$

where

$$\widehat{g}(s, t) = 1 - \frac{d}{d\langle M \rangle_s} \int_0^t g(r, s) dr. \quad (3.36)$$

(3) The natural filtrations (\mathcal{X}_t) and (\mathcal{Z}_t) of X and Z respectively coincide.

In addition, it was shown by Kleptsyna and Chigansky [12] that process Q admits the following representation

$$Q_t = \int_0^t \psi(s, t) dZ_s = \frac{1}{2} \psi(t, t) Z_t + \frac{1}{2} \int_0^t \psi(s, s) dZ_s, \quad t \in [0, T] \quad (3.37)$$

with

$$\psi(s, t) = \frac{1}{2} \left(\frac{dt}{d\langle M \rangle_t} + \frac{ds}{d\langle M \rangle_s} \right).$$

Specific structure of the process Q allows to determine the likelihood function for (3.27), which according to Corollary 2.9 in [9] equals

$$L_T(\vartheta, X) = \frac{d\mu^X}{d\mu^{\widetilde{B}}}(X) = \exp \left(-\vartheta \int_0^T Q_t dZ_t - \frac{1}{2} \vartheta^2 \int_0^T Q_t^2 d\langle M \rangle_t \right),$$

where μ^X and $\mu^{\widetilde{B}}$ are the probability measures induced by processes X and \widetilde{B} respectively. Thus, the score function for (3.27), i.e. the derivative of the log-likelihood function from observations over the interval $[0, T]$ is given by

$$\Sigma_T(\theta) = - \int_0^T Q_t dZ_t - \vartheta \int_0^T Q_t^2 d\langle M \rangle_t$$

which allows to determine the maximum likelihood estimator for the drift parameter ϑ . Moreover, according to theorem 2.9 in [12], which is also presented below, maximum likelihood estimator is asymptotically normal.

Theorem 3.3. *Let $g(s, t)$ be the solution of equation (3.29) and processes Q and Z defined by (3.32) and (3.33) respectively. The maximum likelihood estimator of ϑ is given by*

$$\widehat{\vartheta}_T(X) = - \frac{\int_0^T Q_t dZ_t}{\int_0^T Q_t^2 d\langle M \rangle_t}. \quad (3.38)$$

Since $\vartheta > 0$ this estimator is asymptotically normal at the usual rate

$$\sqrt{T} \left(\hat{\vartheta}_T(X) - \vartheta \right) \xrightarrow[T \rightarrow \infty]{d} N(0, 2\vartheta). \quad (3.39)$$

We develop this result by proving the large deviation principle for the maximum-likelihood estimator (3.38)

3.2.2 Large deviation principle

The large deviations principle characterizes the limiting behavior of a family of random variables (or corresponding probability measures) in terms of a rate function.

A rate function I is a lower semicontinuous function $I : \mathbb{R} \rightarrow [0, +\infty]$, such that for all $\alpha \in [0, +\infty)$ the level sets $\{x : I(x) \leq \alpha\}$ are closed subsets of \mathbb{R} . Moreover, I is a good rate function if its level sets are compact.

We shall say that a family of real random variables $(Z_T)_{T>0}$ satisfies the large deviation principle with a rate function $I : \mathbb{R} \rightarrow [0, +\infty]$, if for any Borel set $\Gamma \subset \mathbb{R}$,

$$-\inf_{x \in \Gamma^\circ} I(x) \leq \liminf_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{P}(Z_T \in \Gamma) \leq \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{P}(Z_T \in \Gamma) \leq -\inf_{x \in \bar{\Gamma}} I(x)$$

where Γ° and $\bar{\Gamma}$ denote the closure and the interior of Γ correspondingly. It should be noted that a family of random variables can have at most one rate function associated with its large deviation principle (for the proof we refer reader to the book by Dembo and Zeitouni [14]). Moreover, it is obvious that if $(Z_T)_{T>0}$ satisfies the large deviation principle and Borel set $\Gamma \subset \mathbb{R}$ is such that

$$\inf_{x \in \Gamma^\circ} I(x) = \inf_{x \in \bar{\Gamma}} I(x)$$

then

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{P}(Z_T \in \Gamma) = -\inf_{x \in \Gamma} I(x).$$

We shall prove the large deviation principle for a family of maximum likelihood estimators (3.38) via a similar approach as the one of [2] and [1] for Ornstein-Uhlenbeck process and fractional Ornstein-Uhlenbeck process respectively.

In order to prove the large deviations principle for the drift parameter estimator of mixed fractional Ornstein-Uhlenbeck process (3.27) the main tool is normalized cumulant generating function of arbitrary linear combination of $\int_0^T Q_t dZ_t$ and $\int_0^T Q_t^2 d\langle M \rangle_t$

$$\mathcal{L}_T(a, b) = \frac{1}{T} \log \mathbb{E} \left[\exp (\mathcal{Z}_T(a, b)) \right] \quad (3.40)$$

where for any $(a, b) \in \mathbb{R}^2$

$$\mathcal{Z}_T(a, b) = a \int_0^T Q_t dZ_t + b \int_0^T Q_t^2 d\langle M \rangle_t. \quad (3.41)$$

It should be noted that for some $(a, b) \in \mathbb{R}^2$ mathematical expectation in (3.40) might be infinite. In fact, in order to establish a large deviation principle for $\hat{\vartheta}_T$ it is enough to find the limit of $\mathcal{L}_T(a, b)$ as $T \rightarrow \infty$ and apply the following lemma, which is a consequence of Gärtner-Ellis theorem (Theorem 2.3.6 in [14]).

Lemma 3.9. *For a family of real random variables $(Z_T)_{T>0}$ let the function $\mathcal{L}_T(a, b)$ be defined by (3.40) and, for each fixed value of x , Δ_x denotes the set of $a \in \mathbb{R}$ for which $\lim_{T \rightarrow \infty} \mathcal{L}_T(a, -xa)$ exists and is finite. If Δ_x is not empty for each value of x then $(Z_T)_{T>0}$ satisfies large deviation principle with a good rate function*

$$I(x) = - \inf_{a \in \Delta_x} \lim_{T \rightarrow \infty} \mathcal{L}_T(a, -xa). \quad (3.42)$$

3.2.3 Main results

Theorem 3.4. *The maximum likelihood estimator $\hat{\vartheta}_T$ defined by (3.38) satisfies the large deviation principle with the good rate function*

$$I(x) = \begin{cases} -\frac{(x + \vartheta)^2}{4x}, & \text{if } x < -\frac{\vartheta}{3} \\ 2x + \vartheta, & \text{if } x \geq -\frac{\vartheta}{3}. \end{cases} \quad (3.43)$$

Proof. As it was mentioned in previous subsection, in order to establish the large deviation principle for $\hat{\vartheta}_T$ and determine the corresponding good rate function it is necessary to find the following limit

$$\mathcal{L}(a, b) = \lim_{T \rightarrow \infty} \mathcal{L}_T(a, b) \quad (3.44)$$

and determine the set of $(a, b) \in \mathbb{R}^2$ for which this limit is finite. For arbitrary $\varphi \in \mathbb{R}$ denote the Doleans exponential of $(\varphi + \vartheta) \int_0^t Q_s dM_s$ as

$$\Lambda_\varphi(t) = \exp \left((\varphi + \vartheta) \int_0^t Q_s dM_s - \frac{(\varphi + \vartheta)^2}{2} \int_0^t Q_s^2 d\langle M \rangle_s \right).$$

Given $\left(\frac{1}{\sqrt{\psi(t,t)}} Q_t \right)_{t \geq 0}$ is a Gaussian process, whose mean and variance functions are bounded on $[0, T]$, Λ_φ satisfies conditions of Girsanov's theorem in accordance with Example 3 of paragraph 2 of Section 6 in [32]. Thus, we can apply a usual change of measures and consider new probability \mathbb{P}_φ defined by the local density

$$\frac{d\mathbb{P}_\varphi}{d\mathbb{P}} = \Lambda_\varphi(T) = \exp \left((\varphi + \vartheta) \int_0^T Q_t dM_t - \frac{(\varphi + \vartheta)^2}{2} \int_0^T Q_t^2 d\langle M \rangle_t \right).$$

Now we observe that due to (3.34) $\Lambda_\varphi(T)$ can be rewritten in terms of the fundamental semimartingal

$$\begin{aligned} \Lambda_\varphi(T) &= \exp \left((\varphi + \vartheta) \int_0^T Q_t dZ_t + (\varphi + \vartheta)\vartheta \int_0^T Q_t^2 d\langle M \rangle_t - \frac{(\varphi + \vartheta)^2}{2} \int_0^T Q_t^2 d\langle M \rangle_t \right) = \\ &= \exp \left((\varphi + \vartheta) \int_0^T Q_t dZ_t - \frac{\varphi^2 - \vartheta^2}{2} \int_0^T Q_t^2 d\langle M \rangle_t \right). \end{aligned}$$

Consequently we can rewrite $\mathcal{L}_T(a, b)$ as

$$\begin{aligned} \mathcal{L}_T(a, b) &= \frac{1}{T} \log \mathbb{E} \left[\exp(\mathcal{Z}_T(a, b)) \right] = \frac{1}{T} \log \mathbb{E}_\varphi \left[\exp(\mathcal{Z}_T(a, b)) \Lambda_\varphi(T)^{-1} \right] = \\ &= \frac{1}{T} \log \mathbb{E}_\varphi \exp \left((a - \varphi - \vartheta) \int_0^T Q_t dZ_t + \frac{1}{2}(2b - \vartheta^2 + \varphi^2) \int_0^T Q_t^2 d\langle M \rangle_t \right). \end{aligned}$$

Given φ is an arbitrary real number, we can choose $\varphi = a - \vartheta$. Then

$$\mathcal{L}_T(a, b) = \frac{1}{T} \log \mathbb{E}_\varphi \exp \left(\frac{1}{2}(2b - \vartheta^2 + (a - \vartheta)^2) \int_0^T Q_t^2 d\langle M \rangle_t \right)$$

or denoting $\mu = -\frac{1}{2}(2b - \vartheta^2 + (a - \vartheta)^2)$

$$\mathcal{L}_T(a, b) = \frac{1}{T} \log \mathbb{E}_\varphi \exp \left(-\mu \int_0^T Q_t^2 d\langle M \rangle_t \right). \quad (3.45)$$

As it was mentioned before, mathematical expectation above can be infinite for some combinations of μ and φ . Our purpose is to determine the set of $(\varphi, \mu) \in \mathbb{R}^2$, for which this mathematical expectation and limit (3.44) are finite. According to Girsanov theorem, under \mathbb{P}_φ process

$$M_t - (\varphi + \vartheta) \int_0^t Q_s d\langle M \rangle_s = Z_t - \varphi \int_0^t Q_s d\langle M \rangle_s \quad (3.46)$$

has the same distribution as M under \mathbb{P} . Consequently, applying inverse integral transformation (3.35) to (3.46), we get that under \mathbb{P}_φ process $X_t - \varphi \int_0^t X_s ds$ is the mixed fractional Brownian motion.

Thus, under new probability measure \mathbb{P}_φ process X is the mixed fractional Ornstein-Uhlenbeck process with the drift parameter $-\varphi$. Consequently, in order to find a limit of (3.45) as $T \rightarrow \infty$ we can apply Lemma 3.10, which is presented further, and obtain the following result

$$\mathcal{L}(a, b) = -\frac{\varphi}{2} - \sqrt{\frac{\varphi^2}{4} + \frac{\mu}{2}} = -\frac{1}{2} (a - \vartheta + \sqrt{\vartheta^2 - 2b})$$

and convergence (3.44) holds for $\varphi < 0$ and $\mu > -\frac{\varphi^2}{2}$, which give $\vartheta - a > 0$ and $\vartheta^2 - 2b > 0$ respectively. For $x \in \mathbb{R}$ denote the function

$$L_x(a) = \mathcal{L}(a, -xa) = -\frac{1}{2} (a - \vartheta + \sqrt{\vartheta^2 + 2xa})$$

which is defined on the set

$$\Delta_x = \{a \in \mathbb{R} | \vartheta - a > 0 \text{ and } \vartheta^2 + 2xa > 0\}.$$

Then, according to (3.42) the rate function $I(x)$ for the maximum likelihood estimator $\hat{\vartheta}_T$ can be found as

$$I(x) = - \inf_{a \in \Delta_x} L_x(a).$$

Consequently, straightforward calculations of the infimum above finish the proof of the theorem. □

Remark 7. One can observe that rate function $I(x)$ does not depend on the parameter H . Hence, $\hat{\vartheta}_T$ shares the same large deviation principles that the ones established

by Florens-Landais and Pham [17] for standard Ornstein-Uhlenbeck process and by Bercu, Coutin and Savy [1] for fractional Ornstein-Uhlenbeck process.

3.2.4 Auxiliary results

One can observe that the following lemma plays key role in the proof of Theorem 3.4.

Lemma 3.10. *For the mixed fractional Ornstein-Uhlenbeck process X with the drift parameter ϑ the following limit holds*

$$\mathcal{K}_T(\mu) = \frac{1}{T} \log \mathbb{E} \exp \left(-\mu \int_0^T Q_t^2 d\langle M \rangle_t \right) \rightarrow \frac{\vartheta}{2} - \sqrt{\frac{\vartheta^2}{4} + \frac{\mu}{2}}, \quad T \rightarrow \infty \quad (3.47)$$

for all $\mu > -\frac{\vartheta^2}{2}$.

Proof. We prove the lemma using a similar approach as the one of [12]. Denote $V_t = \int_0^t \psi(s, s) dZ_s$. Then according to (3.37) we can rewrite

$$\begin{aligned} dZ_t &= -\vartheta Q_t d\langle M \rangle_t + dM_t = -\frac{\vartheta}{2} \psi(t, t) Z_t d\langle M \rangle_t - \frac{\vartheta}{2} V_t d\langle M \rangle_t + dM_t = \\ &= -\frac{\vartheta}{2} Z_t dt - \frac{\vartheta}{2} V_t \frac{1}{\psi(t, t)} dt + \frac{1}{\sqrt{\psi(t, t)}} dW_t \end{aligned}$$

where W_t is a Brownian motion. Consequently we get

$$dV_t = \psi(t, t) dZ_t = -\frac{\vartheta}{2} \psi(t, t) Z_t dt - \frac{\vartheta}{2} V_t dt + \sqrt{\psi(t, t)} dW_t.$$

Gaussian vector $\zeta_t = (Z_t, V_t)^T$ is a solution of the linear system of Ito stochastic differential equations

$$d\zeta_t = -\frac{\vartheta}{2} A(t) \zeta_t dt + b(t) dW_t$$

where

$$A(t) = \begin{pmatrix} 1 & \frac{1}{\psi(t,t)} \\ \psi(t,t) & 1 \end{pmatrix} \quad \text{and} \quad b(t) = \begin{pmatrix} 1 \\ \sqrt{\psi(t,t)} \\ \sqrt{\psi(t,t)} \end{pmatrix}.$$

Moreover, $\mathcal{K}_T(\mu)$ in (3.47) can be rewritten as

$$\begin{aligned} \mathcal{K}_T(\mu) &= \frac{1}{T} \log \mathbb{E} \exp \left(-\mu \int_0^T Q_t^2 d\langle M \rangle_t \right) = \frac{1}{T} \log \mathbb{E} \exp \left(-\frac{\mu}{4} \int_0^T (\psi(t,t) Z_t + V_t)^2 d\langle M \rangle_t \right) = \\ &= \frac{1}{T} \log \mathbb{E} \exp \left(-\frac{\mu}{4} \int_0^T \left(\sqrt{\psi(t,t)} Z_t + \frac{1}{\sqrt{\psi(t,t)}} V_t \right)^2 dt \right) = \frac{1}{T} \log \mathbb{E} \exp \left(-\frac{\mu}{4} \int_0^T \zeta_t^T R(t) \zeta_t dt \right) \end{aligned}$$

where

$$R(t) = \begin{pmatrix} \psi(t,t) & 1 \\ 1 & \frac{1}{\psi(t,t)} \end{pmatrix}.$$

By the Cameron-Martin type formula from Section 4.1 of [26]

$$\mathcal{K}_T(\mu) = -\frac{\mu}{4T} \int_0^T \text{tr}(\Gamma(t)R(t)) dt$$

where $\Gamma(t)$ is the solution of the equation

$$\dot{\Gamma}(t) = -\frac{\vartheta}{2} A(t)\Gamma(t) - \frac{\vartheta}{2} \Gamma(t)A^T(t) - \frac{\mu}{2} \Gamma(t)R(t)\Gamma(t) + B(t) \quad (3.48)$$

with $B(t) = b(t)b^T(t)$ and initial condition $\Gamma(0) = 0$.

We shall search solution of the equation (3.48) as the ratio $\Gamma(t) = \Psi_1^{-1}(t)\Psi_2(t)$, where $\Psi_1(t)$ and $\Psi_2(t)$ are the solutions of the following system of equation

$$\begin{aligned} \dot{\Psi}_1(t) &= \frac{\vartheta}{2} \Psi_1(t)A(t) + \frac{\mu}{2} \Psi_2(t)R(t) \\ \dot{\Psi}_2(t) &= \Psi_1(t)B(t) - \frac{\vartheta}{2} \Psi_2(t)A^T(t) \end{aligned} \quad (3.49)$$

with initial conditions $\Psi_1(0) = I$ and $\Psi_2(0) = 0$. From the first equation of (3.49) we get

$$\Psi_1^{-1}(t)\dot{\Psi}_1(t) = \frac{\vartheta}{2}A(t) + \frac{\mu}{2}\Gamma(t)R(t)$$

and since $\text{tr } A(t) = 2$ it follows that

$$\frac{\mu}{2}\text{tr}(\Gamma(t)R(t)) = \text{tr}(\Psi_1^{-1}(t)\dot{\Psi}_1(t)) - \vartheta.$$

Since $\dot{\Psi}_1(t) = \Psi_1(t)(\Psi_1^{-1}(t)\dot{\Psi}_1(t))$ by Liouville's formula

$$-\frac{\mu}{4T} \int_0^T \text{tr}(\Gamma(t)R(t)) dt = -\frac{1}{2T} \int_0^T \text{tr}(\Psi_1^{-1}(t)\dot{\Psi}_1(t)) dt + \frac{\vartheta}{2} = -\frac{1}{2T} \log \det \Psi_1(T) + \frac{\vartheta}{2}.$$

In order to calculate $\lim_{T \rightarrow \infty} \frac{1}{T} \log \det \Psi_1(T)$ define matrix

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and note that $A^T(t) = JA(t)J$, $R(t) = JA(t)$ and $B(t) = A(t)J$. If we set $\tilde{\Psi}_2(t) = \Psi_2(t)J$ from (3.49) we obtain the following system of equations

$$\begin{aligned} \dot{\Psi}_1(t) &= \frac{\vartheta}{2}\Psi_1(t)A(t) + \frac{\mu}{2}\Psi_2(t)A(t) \\ \dot{\tilde{\Psi}}_2(t) &= \Psi_1(t)A(t) - \frac{\vartheta}{2}\tilde{\Psi}_2(t)A(t) \end{aligned} \tag{3.50}$$

with initial condition $\Psi_1(0) = I$ and $\tilde{\Psi}_2(0) = 0$. When $\frac{\vartheta^2}{2} + \mu > 0$ the coefficients matrix of system (3.50)

$$\begin{pmatrix} \frac{\vartheta}{2} & \frac{\mu}{2} \\ 1 & -\frac{\vartheta}{2} \end{pmatrix}$$

has two real eigenvalues $\pm\lambda$ with $\lambda = \sqrt{\frac{\vartheta^2}{4} + \frac{\mu}{2}}$ and eigenvectors

$$v^\pm = \begin{pmatrix} \frac{\vartheta}{2} \pm \lambda \\ 1 \end{pmatrix}.$$

Denote $a^\pm = \frac{\vartheta}{2} \pm \lambda = \frac{\vartheta}{2} \pm \sqrt{\frac{\vartheta^2}{4} + \frac{\mu}{2}}$. Diagonalizing system (3.50) we get

$$\Psi_1(t) = a^+ \Upsilon_1(t) + a^- \Upsilon_2(t)$$

where $\Upsilon_1(t)$ and $\Upsilon_2(t)$ are the solutions of the following equations

$$\begin{aligned} \dot{\Upsilon}_1(t) &= \lambda \Upsilon_1(t) A(t) \\ \dot{\Upsilon}_2(t) &= -\lambda \Upsilon_2(t) A(t) \end{aligned} \quad (3.51)$$

with initial condition $\Upsilon_1(0) = \frac{1}{2\lambda} I$ and $\Upsilon_2(0) = -\frac{1}{2\lambda} I$. Denote matrix $M(T) = \Upsilon_2^{-1}(T) \Upsilon_1(T)$, which is the solution of the following equation

$$\dot{M}(t) = \lambda (A(t)M(t) + M(t)A(t)) \quad (3.52)$$

subject to initial condition $M(0) = -I$. Then

$$\begin{aligned} \frac{1}{T} \log \det \Psi_1(T) &= \frac{1}{T} \log \det (a^+ \Upsilon_1(T) + a^- \Upsilon_2(T)) = \\ &= \frac{1}{T} \log \det (a^- \Upsilon_2(T)) + \frac{1}{T} \log \det \left(I + \frac{a^+}{a^-} M(T) \right) = \\ &= \frac{1}{T} \log \det (a^- \Upsilon_2(T)) + \frac{1}{T} \log \left(1 + \left(\frac{a^+}{a^-} \right)^2 \det M(T) + \frac{a^+}{a^-} \text{tr} M(T) \right) = \\ &= \frac{1}{T} \log \det (a^- \Upsilon_2(T)) + \frac{1}{T} \log \left(1 + \left(\frac{a^+}{a^-} \right)^2 \det M(T) \right) + \frac{1}{T} \log \left(1 + \frac{\frac{a^+}{a^-} \text{tr} M(T)}{1 + \left(\frac{a^+}{a^-} \right)^2 \det M(T)} \right). \end{aligned}$$

Applying Liouville's formula to (3.51) we get

$$\begin{aligned} &\frac{1}{T} \log \det (a^- \Upsilon_2(t)) + \frac{1}{T} \log \left(1 + \left(\frac{a^+}{a^-} \right)^2 \det M(T) \right) = \\ &= \frac{1}{T} \log \left(\left(\frac{a^-}{2\lambda} \right)^2 \exp(-2\lambda T) \right) + \frac{1}{T} \log \left(1 + \left(\frac{a^+}{a^-} \right)^2 \exp(4\lambda T) \right) \rightarrow 2\lambda, \quad T \rightarrow \infty. \end{aligned}$$

Thus, in order to prove that limit (3.47) holds we should show that

$$\frac{1}{T} \log \left(1 + \frac{\frac{a^+}{a^-} \operatorname{tr} M(T)}{1 + \left(\frac{a^+}{a^-}\right)^2 \exp(4\lambda T)} \right) \rightarrow 0, \quad T \rightarrow \infty. \quad (3.53)$$

Given (3.52), by Theorem 3 in [56]

$$\left| \operatorname{tr} M(T) \right| \leq 2\sqrt{2} \exp(2\lambda T).$$

Thus limit (3.53) holds, that finishes proof of the lemma. □

3.3 Small L^2 -ball probabilities for fractional processes

3.3.1 Small L^2 -ball probabilities for Gaussian processes

For a given process $X = (X_t, t \in [0, 1])$ and a norm $\|\cdot\|$, the problem of small ball probabilities is to find the asymptotics of

$$\mathbb{P}(\|X\| \leq \varepsilon) \quad \text{as } \varepsilon \rightarrow 0. \quad (3.54)$$

This problem has been studied extensively in the past and was found to have deep connections to various topics in probability theory, see [31].

The case of Gaussian processes and L^2 -norm is the simplest, in which asymptotics of (3.54) is determined by eigenvalues of the covariance operator of X . More precisely, for the L^2 -norm

$$\|X\|_2^2 = \int_0^1 X(t)^2 dt = \sum_{n=1}^{\infty} \lambda_n \xi_n^2,$$

where ξ_n are i.i.d. $N(0, 1)$ random variables, and the complete solution in this case was found in [51]. The principal difficulty with this approach is that it requires an accurate asymptotic approximation of the sequence of eigenvalues, which is typically hard to find in concrete cases.

Computation of the precise asymptotics for particular processes is possible if a suf-

ficiently detailed asymptotic behavior of the eigenvalues is known. Roughly speaking, the exact first order asymptotic term of λ_n suffices for the *logarithmic* asymptotics of $\log \mathbb{P}(\|X\| \leq \varepsilon)$ and the exact second term gives the asymptotics of $\mathbb{P}(\|X\| \leq \varepsilon)$ itself, usually precise up to the so called *distortion* constant

$$C_d = \prod_{n=1}^{\infty} \left(\frac{\widehat{\lambda}_n}{\lambda_n} \right)^{\frac{1}{2}} \quad (3.55)$$

where $\widehat{\lambda}_n$ is the sequence obtained from λ_n , by leaving out all but the first two asymptotic terms. This constant can be found only in a few cases (see, e.g., [42]) and is typically left to numerical approximation.

For the standard Brownian motion, the exact asymptotics is long known:

$$\mathbb{P}(\|B\|_2 \leq \varepsilon) = \frac{4}{\sqrt{\pi}} \varepsilon \exp\left(-\frac{1}{8}\varepsilon^{-2}\right) (1 + o(1)), \quad \varepsilon \rightarrow 0.$$

For the fBm, the rough logarithmic asymptotics was derived in [7, 8] and, by different methods, in [33] and [40]:

$$\log \mathbb{P}(\|B^H\|_2 \leq \varepsilon) \sim -\beta_0(H) \varepsilon^{-\frac{1}{H}} \quad (3.56)$$

with explicit constant

$$\beta_0(H) := \frac{H}{(2H+1)^{\frac{2H+1}{2H}}} \left(\frac{\sin(\pi H) \Gamma(2H+1)}{\left(\sin \frac{\pi}{2H+1}\right)^{2H+1}} \right)^{\frac{1}{2H}} \quad (3.57)$$

Asymptotics (3.56) was recently improved in [12] up to

$$\mathbb{P}(\|B^H\| \leq \varepsilon) \sim \varepsilon^{\gamma_0(H)} \exp\left(-\beta_0(H) \varepsilon^{-\frac{1}{H}}\right)$$

with the exact power

$$\gamma_0(H) = \frac{1}{2H} \left(H^2 - H + \frac{5}{4} \right). \quad (3.58)$$

3.3.2 Small L^2 -ball probabilities for fractional Ornstein-Uhlenbeck process and integrated fractional Brownian motion

The asymptotics from Theorems 1.2 and 1.3 can be plugged into the general result of Theorem 6.2 in [39] leading to the following results:

Proposition 6. *For the fractional Ornstein-Uhlenbeck process X_{OU}^H with the Hurst exponent $H \in (0, 1)$*

$$\mathbb{P}(\|X_{OU}^H\|_2 \leq \varepsilon) \simeq \frac{C_d(H)C(H)}{(\sin(\pi H)\Gamma(2H+1))^{\frac{\gamma_0(H)}{2}}} \varepsilon^{\gamma_0(H)} \exp\left(-\beta_0(H)\varepsilon^{-\frac{1}{H}}\right), \quad \varepsilon \rightarrow 0$$

where $C_d(H)$ is the distortion constant, defined in (3.55), $C(H)$ is an explicit constant (given by (6.6) in [39]) and $\gamma_0(H)$ and $\beta_0(H)$ are given by (3.57) and (3.58) respectively.

We should notice that the small L^2 -ball probabilities for the fractional Ornstein-Uhlenbeck process do not depend on the drift parameter β and, consequently coincide with those of the fractional Brownian motion.

In case of the integrated fractional Brownian motion we have the following small L^2 -ball probabilities:

Proposition 7. *For all $H \in (0, 1)$*

$$\mathbb{P}\left(\int_0^1 \left(\int_0^t B_s^H ds\right)^2 dt \leq \varepsilon^2\right) \simeq \frac{C_d(H)C(H)}{(\sin(\pi H)\Gamma(2H+1))^{\frac{\gamma_1(H)}{2}}} \varepsilon^{\gamma_1(H)} \exp\left(-\beta_1(H)\varepsilon^{-\frac{1}{H+1}}\right), \quad \varepsilon \rightarrow 0$$

where $C_d(H)$ is the distortion constant, defined in (3.55), $C(H)$ is an explicit constant (given by (6.6) in [39]),

$$\beta_1(H) := \frac{H+1}{(2H+3)^{\frac{2H+3}{2H+2}}} \left(\frac{\sin(\pi H)\Gamma(2H+1)}{\left(\sin \frac{\pi}{2H+3}\right)^{2H+3}} \right)^{\frac{1}{2H+2}}$$

and

$$\gamma_1(H) = \frac{1}{2H+2} \left(H^2 - H + \frac{5}{4} \right).$$

3.3.3 Mixed fractional Brownian motion and its thresholds

The fractional Brownian motion B^H is known to change some of its properties abruptly at certain special values of H , referred to as *thresholds* in this section. The most obvious example is $H = \frac{1}{2}$, at which it coincides with the standard Brownian motion. A simple calculation shows that

$$\mathbb{E}(B_1^H - B_0^H)(B_{n+1}^H - B_n^H) \sim H(2H - 1)n^{2H-2} \quad \text{as } n \rightarrow \infty,$$

and hence increments of the fBm exhibit long range dependence in the sense

$$\sum_{n=1}^{\infty} \mathbb{E}B_1^H(B_{n+1}^H - B_n^H) = \infty,$$

if and only if $H > \frac{1}{2}$.

Other thresholds are less apparent and some of them become visible when the fBm is mixed with other processes. A particularly curious object in this regard is the mixture with an independent standard Brownian motion B :

$$\tilde{B}_t = B_t + B_t^H, \quad t \in [0, T]. \quad (3.59)$$

Measure equivalence: $H = \frac{3}{4}$ and $H = \frac{1}{4}$

The mixed fBm (3.59) was first studied in [10], where it was shown to be a semimartingale, measure equivalent to B , if and only if $H > \frac{3}{4}$. This can be deduced from Shepp's general criteria for equivalence of such mixtures, [48], which requires that

$$\int_0^T \int_0^T \left(\frac{\partial}{\partial t} \frac{\partial}{\partial s} \mathbb{E}B_t^H B_s^H \right)^2 ds dt < \infty.$$

Plugging expression for the covariance function of fractional Brownian motion, we get $2(2H - 2) > -1$ which implies $H > \frac{3}{4}$. Threshold $H = \frac{3}{4}$ also arises in other contexts, such as e.g. limit theorems of self-intersection local times [22].

The counterpart threshold $H = \frac{1}{4}$ was discovered in [58], which considers general linear combination

$$X_t = \sum_{k=1}^n a_k B_t^{H_k}, \quad t \in [0, T]$$

of independent fBm's B^{H_k} with distinct Hurst exponents $H_1 < \dots < H_k$ and constant

mixture coefficients $a_k \neq 0$. It is shown that X is measure equivalent to $a_1 B^{H_1}$ if and only if $H_2 - H_1 > \frac{1}{4}$. In particular, this implies that \tilde{B} in (3.59) is equivalent to B^H if and only if $H < \frac{1}{4}$.

Canonical innovation and filtering: $H = \frac{2}{3}$

Another threshold emerges in statistical applications, based on the canonical innovation representation of (3.59) from [9]:

$$\tilde{B}_t = \int_0^t \hat{g}(s, t) dM_s, \quad t \geq 0$$

with the martingale $M_t = \mathbb{E}(B_t | \mathcal{F}_t^{\tilde{B}})$ and a certain deterministic kernel function $\hat{g}(s, t)$. This martingale is generated by stochastic integral with respect to \tilde{B}

$$M_t = \int_0^t g(s, t) d\tilde{B}_s, \quad t > 0$$

where $g(s, t)$ is the unique solution of the integro-differential equation

$$g(s, t) + \frac{\partial}{\partial s} \int_0^t g(r, t) \frac{\partial}{\partial r} K(r, s) dr = 1, \quad 1 \leq s \leq t. \quad (3.60)$$

For $H > \frac{1}{2}$ the derivative and integration in (3.60) are interchangeable and it simplifies to

$$g(s, t) + \int_0^t g(r, t) c_H |r - s|^{2H-2} dr = 1, \quad 1 \leq s \leq t, \quad (3.61)$$

with the constant $c_H = H(2H - 1)$.

The large sample limit accuracy of statistical procedures in models involving mixed fBm is governed by the asymptotic behaviour of $\{g(s, t) : s \leq t\}$ as $t \rightarrow \infty$, which can be approached by reduction to a singular perturbation problem, [13]. If we define small parameter $\varepsilon := t^{1-2H}$ equation (3.61) becomes

$$\varepsilon g_\varepsilon(x) + \int_0^1 g_\varepsilon(y) c_H |x - y|^{2H-2} dy = 1, \quad 0 \leq x \leq 1 \quad (3.62)$$

where $g_\varepsilon(x) := t^{2H-1} g(xt, t)$. As $\varepsilon \rightarrow 0$ its solution $g_\varepsilon(x)$ converges to the solution $g_0(x)$ of the limit equation obtained by setting $\varepsilon := 0$ in (3.62). How fast is this convergence?

The following answer is given in [12] in terms of the L^2 -norm:

$$\|g_\varepsilon - g_0\|_2 \sim \begin{cases} \varepsilon^{\frac{1-H}{2H-1}} & H \in (\frac{2}{3}, 1) \\ \varepsilon \sqrt{\log \varepsilon^{-1}} & H = \frac{2}{3} \\ \varepsilon & H \in (\frac{1}{2}, \frac{2}{3}) \end{cases}$$

This asymptotics reveals threshold $H = \frac{2}{3}$, at which the convergence rate starts to slow down. The same threshold arises in the filtering problem of B_t^H given white noise observations

$$Y_t = \int_0^t B_s^H ds + B_t, \quad t \geq 0.$$

The following expression is found in [12] for the steady state filtering error:

$$V_\infty(H) := \lim_{t \rightarrow \infty} \text{Var}(B_t^H | \mathcal{F}_t^Y) = \frac{(\sin(\pi H) \Gamma(2H + 1))^{\frac{1}{2H+1}}}{\sin \frac{\pi}{2H+1}}.$$

The worst, maximal error is attained at $H = \frac{2}{3}$.

Small L^2 -ball probabilities: $H = \frac{1}{2}(1 \pm \frac{1}{k}), k \in \mathbb{N}$

In this section we discuss yet another context, namely the small L^2 -ball probabilities problem, in which these and many more thresholds make their appearance. We will show that as $\varepsilon \rightarrow 0$

$$\mathbb{P}(\|\tilde{B}\|_2 \leq \varepsilon) \sim \varepsilon^\gamma \exp\left(-\sum_{k=0}^{\lfloor \frac{1}{|2H-1|} \rfloor} \beta_k \varepsilon^{\frac{1}{H \wedge \frac{1}{2}}(k|2H-1|-1)}\right)$$

where $f(\varepsilon) \sim g(\varepsilon)$ stands for the equality $f(\varepsilon) = Cg(\varepsilon)(1 + o(1))$ for some positive constant C and γ and $\beta_k, k = 0, 1, 2, \dots$ are explicitly defined functions of H (see Theorem 3.5 below). A close look at this formula reveals that additional terms join the sum in the exponent at the following values of H :

$$\frac{1}{4}, \frac{1}{3}, \frac{3}{8}, \dots, \frac{1}{2}, \dots, \frac{5}{8}, \frac{2}{3}, \frac{3}{4}$$

where we emphasized the already known thresholds, mentioned above. Note that the

classical threshold $H = \frac{1}{2}$ here is the accumulation point of all the others.

3.3.4 Small L^2 -ball probabilities for mixed fractional Brownian motion

Main results

The case of mixed fBm was considered in [40] (see also [41], [34]), where logarithmic asymptotics was shown to be inherited either from the standard or fractional parts, depending on the value of H :

$$\log \mathbb{P}(\|\tilde{B}\|_2 \leq \varepsilon) \sim \begin{cases} \log \mathbb{P}(\|B\| \leq \varepsilon) & H > \frac{1}{2} \\ \log \mathbb{P}(\|B^H\| \leq \varepsilon) & H < \frac{1}{2} \end{cases} \quad (3.63)$$

Our main result is the following theorem, which shows that the exact asymptotics of the mixed process is more intricate than could have been expected in view of (3.63):

Theorem 3.5.

1. For $H \in (\frac{1}{2}, 1)$ let h_k and g_k be the real sequences, defined by formulas (3.89) evaluated at the constants in (3.85). Then for any $r > 0$ the equation

$$y/y_0 + \sum_{k=1}^{\lfloor \frac{1}{2} \frac{1}{2H-1} \rfloor} h_k r^{k(2H-1)} y^{k(2H-1)+1} = 1 \quad (3.64)$$

with $y_0 = 2\sqrt{2}$ has unique positive root $y(r)$, which admits expansion into series

$$y(r) = y_0 + \sum_{j=1}^{\infty} y_j r^{j(2H-1)}, \quad (3.65)$$

convergent for all r small enough. Let ξ_j and $\eta_{k,j}$ be coefficients of the power expansions

$$y(r)^{-2} = \sum_{j=0}^{\infty} \xi_j r^{j(2H-1)} \quad \text{and} \quad y(r)^{k(2H-1)-1} = \sum_{j=0}^{\infty} \eta_{k,j} r^{j(2H-1)} \quad (3.66)$$

and define

$$\beta_\ell := \frac{1}{\sqrt{2}} \eta_{0,\ell} - \sum_{j=0}^{\ell-1} g_{\ell-j} \eta_{\ell-j,j} - \xi_\ell.$$

Then

$$\mathbb{P}(\|\tilde{B}\|_2 \leq \varepsilon) \sim \varepsilon \exp\left(-\frac{1}{8}\varepsilon^{-2} - \sum_{\ell=1}^{\lfloor \frac{1}{2H-1} \rfloor} \beta_\ell \varepsilon^{2\ell(2H-1)-2}\right), \quad \varepsilon \rightarrow 0. \quad (3.67)$$

2. For $H \in (0, \frac{1}{2})$ let h_k and g_k be the real sequences, defined by formulas (3.89), evaluated at the constants (3.91). Then for any $r > 0$ the equation

$$y/y_0 + \sum_{k=1}^{\lfloor \frac{1}{2} \frac{1}{1-2H} \rfloor} h_k r^{k \frac{1-2H}{2H}} y^{k \frac{1-2H}{2H} + 1} = 1$$

with

$$y_0 = (2H + 1) \left(\frac{2^{2H} \left(\sin \frac{\pi}{2H+1} \right)^{2H+1}}{\sin(\pi H) \Gamma(2H + 1)} \right)^{\frac{1}{2H+1}}$$

has unique positive root $y(r)$, which admits expansion into series

$$y(r) = y_0 + \sum_{k=1}^{\infty} y_k r^{k \frac{1-2H}{2H}}$$

convergent for all r small enough. Let ξ_j and $\eta_{k,j}$ be coefficients of the power expansions

$$y(r)^{-\frac{2H+1}{2H}} = \sum_{j=0}^{\infty} \xi_j r^{j \frac{1-2H}{2H}} \quad \text{and} \quad y(r)^{\frac{k(1-2H)-1}{2H}} = \sum_{j=0}^{\infty} \eta_{k,j} r^{j \frac{1-2H}{2H}}$$

and define

$$\beta_\ell(H) := \frac{2H + 1}{y_0} \eta_{0,\ell} - \sum_{j=0}^{\ell-1} g_{\ell-j} \eta_{\ell-j,j} - \xi_\ell.$$

Then

$$\mathbb{P}(\|\tilde{B}\|_2 \leq \varepsilon) \sim \varepsilon^{\gamma_0(H)} \exp\left(-\beta_0(H) \varepsilon^{-\frac{1}{H}} - \sum_{\ell=1}^{\lfloor \frac{1}{1-2H} \rfloor} \beta_\ell(H) \varepsilon^{\frac{\ell(1-2H)-1}{H}}\right), \quad \varepsilon \rightarrow 0 \quad (3.68)$$

where $\gamma_0(H)$ and $\beta_0(H)$ are given by (3.58) and (3.57).

While general closed form formula for constants β_ℓ in (3.67) and (3.68) would be

cumbersome to find, they can be easily computed for any given value of H , at least numerically. The following example, demonstrates the algorithm given by Theorem 3.5.

Example 3.6.

Case $H \in [\frac{3}{4}, 1)$. For the values of H in this range, $\lfloor \frac{1}{2H-1} \rfloor = 1$ and the sum in (3.67) contains only one term:

$$\mathbb{P}(\|\tilde{B}\|_2 \leq \varepsilon) \sim C(H)\varepsilon \exp\left(-\frac{1}{8}\varepsilon^{-2} + \beta_1\varepsilon^{4H-4}\right), \quad \varepsilon \rightarrow 0,$$

Thus we need to calculate only $\beta_1 = \xi_1 - \frac{1}{\sqrt{2}}\eta_{0,1} + g_1\eta_{1,0}$.

Since $\lfloor \frac{1}{2H-1} \rfloor = 0$ equation (3.64) reduces to $y = y_0$. Comparing $y(r)^{-2} = y_0^{-2}$ to (3.66) gives $\xi_1 = 0$. Similarly, comparing $y(r)^{-1} = y_0^{-1}$ with $y(r)^{-1} = \eta_{0,0} + \eta_{0,1}r^{2H-1} + O(r^{4H-2})$ gives $\eta_{0,1} = 0$. Finally $\eta_{1,0} = y_0^{2H-2}$ and hence $\beta_1 = g_1y_0^{2H-2}$. After simplification, formula (3.89) yields $g_1 = -2^{-H-1}\Gamma(2H+1)$ and we obtain

$$\beta_1(H) = -2^{2H-4}\Gamma(2H+1).$$

Case $H \in [\frac{2}{3}, \frac{3}{4})$. In this case $\lfloor \frac{1}{2H-1} \rfloor = 2$ and the sum in (3.67) contains two terms:

$$\mathbb{P}(\|\tilde{B}\|_2 \leq \varepsilon) \sim C(H)\varepsilon \exp\left(-\frac{1}{8}\varepsilon^{-2} + \beta_1\varepsilon^{4H-4} + \beta_2\varepsilon^{8H-6}\right), \quad \varepsilon \rightarrow 0,$$

with coefficients

$$\begin{aligned} \beta_1 &= \xi_1 - \frac{1}{\sqrt{2}}\eta_{0,1} + g_1\eta_{1,0} \\ \beta_2 &= \xi_2 - \frac{1}{\sqrt{2}}\eta_{0,2} + g_2\eta_{2,0} + g_1\eta_{1,1}. \end{aligned} \tag{3.69}$$

To find ξ_1 and ξ_2 , note that

$$\begin{aligned} y(r)^{-2} &= \left(y_0 + \sum_{j=1}^{\infty} y_j r^{j(2H-1)}\right)^{-2} = y_0^{-2} - 2y_0^{-3} \sum_{j=1}^{\infty} y_j r^{j(2H-1)} + 3y_0^{-4} \left(\sum_{j=1}^{\infty} y_j r^{j(2H-1)}\right)^2 + \dots = \\ &= y_0^{-2} - 2y_0^{-3}y_1 r^{2H-1} + (3y_0^{-4}y_1^2 - 2y_0^{-3}y_2) r^{2(2H-1)} + O(r^{3(2H-1)}) \end{aligned}$$

which yields

$$\xi_1 = -2y_0^{-3}y_1 \quad \text{and} \quad \xi_2 = y_0^{-3}(3y_0^{-1}y_1^2 - 2y_2).$$

By (3.66) with $k = 0$, we have

$$y(r)^{-1} = \left(y_0 + \sum_{j=1}^{\infty} y_j r^{j(2H-1)} \right)^{-1} = y_0^{-1} - y_0^{-2} y_1 r^{2H-1} + O(r^{2(2H-1)})$$

and hence

$$\eta_{0,0} = y_0^{-1} \quad \text{and} \quad \eta_{0,1} = -y_0^{-2} y_1.$$

Similarly, for $k = 1$

$$y(r)^{2H-2} = \left(y_0 + \sum_{j=1}^{\infty} y_j r^{j(2H-1)} \right)^{2H-2} = y_0^{2H-2} - (2-2H)y_0^{2H-3} y_1 r^{2H-1} + O(r^{2(2H-1)})$$

which gives

$$\eta_{1,0} = y_0^{2H-2} \quad \text{and} \quad \eta_{1,1} = -(2H-2)y_0^{2H-3} y_1.$$

Finally, (3.66) with $k = 2$ yields $\eta_{2,0} = y_0^{4H-3}$. Plugging all these values into (3.69) we get

$$\begin{aligned} \beta_1 &= -2y_0^{-3} y_1 + \frac{1}{\sqrt{2}} y_0^{-2} y_1 + g_1 y_0^{2H-2} \\ \beta_2 &= y_0^{-3} (3y_0^{-1} y_1^2 - 2y_2) - \frac{1}{\sqrt{2}} y_0^{4H-3} + g_2 y_0^{4H-3} - g_1 (2-2H) y_0^{2H-3} y_1, \end{aligned}$$

where g_1 and g_2 are found using (3.89). It is left to find y_1 and y_2 . For $H \in [\frac{2}{3}, \frac{3}{4})$ we have $\lfloor \frac{1}{2} \frac{1}{2H-1} \rfloor = 1$ and equation (3.64) reads

$$y/y_0 + h_1 r^{2H-1} y^{2H} = 1.$$

Plugging expansion (3.65) we get

$$\sum_{j=1}^{\infty} \frac{y_j}{y_0} r^{j(2H-1)} + h_1 r^{2H-1} \left(y_0 + \sum_{j=1}^{\infty} y_j r^{j(2H-1)} \right)^{2H} = 0$$

where h_1 is defined in (3.89). Comparing coefficients of powers r^{2H-1} and r^{4H-2} we obtain

$$y_1 = -h_1 y_0^{2H+1} \quad \text{and} \quad y_2 = -2H h_1 y_0^{2H} y_1.$$

■

Proof

Our starting point is the theory of small deviations developed in [15]. It addresses the problem of calculating exact asymptotics of the probabilities

$$\mathbb{P}\left(\sum_{j=1}^{\infty} \phi(j)Z_j \leq r\right) \quad \text{as } r \rightarrow 0, \quad (3.70)$$

where Z_j 's are i.i.d. nonnegative random variables and $\phi(j)$ is a summable sequence of positive numbers. Squared L^2 -norm of a Gaussian process can be written as such series with $Z_j \sim \chi_1^2$ and $\phi(j) := \lambda_j$, where λ_j 's are the eigenvalues of its covariance operator. In what follows $\phi(t)$, $t \in \mathbb{R}_+$ stands for the function, obtained by replacing the integer index in $\phi(j)$ with a real positive variable t .

The main ingredients in the asymptotic analysis of (3.70) are functions, defined in terms of the Laplace transform $f(s) := \mathbb{E}e^{-sZ_1} = (1 - 2s)^{-\frac{1}{2}}$, $s \in (-\infty, \frac{1}{2})$ of the χ_1^2 -distribution:

$$\begin{aligned} I_0(u) &:= \int_1^{\infty} \log f(u\phi(t))dt = -\frac{1}{2} \int_1^{\infty} \log(1 + 2u\phi(t))dt \\ I_1(u) &:= \int_1^{\infty} u\phi(t)(\log f)'(u\phi(t))dt = - \int_1^{\infty} \frac{u\phi(t)}{1 + 2u\phi(t)}dt \\ I_2(u) &:= \int_1^{\infty} (u\phi(t))^2(\log f)''(u\phi(t))dt = 2 \int_1^{\infty} \left(\frac{u\phi(t)}{1 + 2u\phi(t)}\right)^2 dt \end{aligned} \quad (3.71)$$

We will apply Corollary 3.2 from [15], which takes the following form in our case:

Corollary 3.7. *The L^2 -ball probabilities satisfy*

$$\mathbb{P}(\|\tilde{B}\|_2^2 \leq r) \sim \left(\sqrt{u(r)}I_2(u(r))\right)^{-\frac{1}{2}} \exp\left(I_0(u(r)) + u(r)r\right) \quad \text{as } r \rightarrow 0, \quad (3.72)$$

where $u(r)$ is any function satisfying

$$\lim_{r \rightarrow 0} \frac{I_1(u(r)) + u(r)r}{\sqrt{I_2(u(r))}} = 0. \quad (3.73)$$

Asymptotic expansion of $I_j(u)$'s. A preliminary step towards application of Corollary 3.7 is to derive the exact asymptotics of the functions from (3.71) as $u \rightarrow \infty$ for

$$\phi(t) = \sum_{j=1}^k c_j t^{-d_j}. \quad (3.74)$$

This weight function with $k = 1$ has been considered in [15] and with $k = 2$ in [39]. In our case, $k = 3$ and c_j and $d_1 < d_2 < d_3$ are positive constants, whose values depend on $\alpha := 2 - 2H \in (0, 2) \setminus \{1\}$. In particular, $\lim_{t \rightarrow \infty} \frac{d}{dt} \log \phi(t) = 0$ holds, as required in [15]. It will be convenient to use constants

$$a_j := c_j/c_1 \quad \text{and} \quad \delta_j := d_j - d_1$$

and to define the new variable v by the formula

$$2uc_1 = v^{-d_1},$$

which converges to zero as $u \rightarrow \infty$. Obviously $a_1 = 1$ and $\delta_1 = 0$, and for the specific values of constants d_j 's needed below, we also have $\delta_3 = 1$.

Asymptotic expansion of $I_0(u)$. Integrating by parts we get

$$\begin{aligned} I_0(u) &= -\frac{1}{2} \int_1^\infty \log(1 + 2u\phi(t)) dt = \frac{1}{2} \log(1 + 2u\phi(1)) + \int_1^\infty \frac{ut\phi'(t)}{1 + 2u\phi(t)} dt = \\ &= \frac{1}{2} \log \left(1 + 2u \sum_{j=1}^k c_j \right) - \sum_{i=1}^k c_i d_i \int_1^\infty \frac{ut^{-d_i}}{1 + 2u \sum_{j=1}^k c_j t^{-d_j}} dt. \end{aligned}$$

Changing the integration variable and using the above notations, this can be written as

$$I_0(u) = \frac{1}{2} \log \left(1 + v^{-d_1} \sum_{j=1}^k a_j \right) - \frac{1}{2} \sum_{i=1}^k a_i d_i v^{\delta_i - 1} \int_v^\infty \frac{\tau^{-\delta_i}}{\tau^{d_1} + 1 + p(v/\tau)} d\tau, \quad (3.75)$$

where we defined

$$p(s) := a_2 s^{\delta_2} + a_3 s^{\delta_3}.$$

Let us find the exact asymptotics of each integral as $v \rightarrow 0$. The first one gives

$$\begin{aligned} v^{\delta_1-1} \int_v^\infty \frac{\tau^{-\delta_1}}{\tau^{d_1+1} + p(v/\tau)} d\tau &= v^{-1} \int_v^\infty \frac{1}{\tau^{d_1+1} + p(v/\tau)} d\tau = \\ &= v^{-1} \int_v^\infty \frac{1}{\tau^{d_1+1}} d\tau - v^{-1} \int_v^\infty \frac{p(v/\tau)}{(\tau^{d_1+1} + p(v/\tau))(\tau^{d_1+1})} d\tau = \\ &= v^{-1} \int_v^\infty \frac{1}{\tau^{d_1+1}} d\tau - J_{0,1}(v) - J_{0,2}(v) \end{aligned}$$

where we defined

$$J_{0,1}(v) := v^{\delta_2-1} a_2 \int_v^\infty \frac{\tau^{-\delta_2}}{(\tau^{d_1+1} + p(v/\tau))(\tau^{d_1+1})} d\tau$$

and

$$J_{0,2}(v) := v^{\delta_3-1} a_3 \int_v^\infty \frac{\tau^{-\delta_3}}{(\tau^{d_1+1} + p(v/\tau))(\tau^{d_1+1})} d\tau.$$

The latter term with $\delta_3 = 1$ satisfies

$$\begin{aligned} J_{0,2}(v) &= a_3 \int_v^\infty \frac{\tau^{-1}}{(\tau^{d_1+1} + p(v/\tau))(\tau^{d_1+1})} d\tau = \\ &= a_3 \int_v^\infty \frac{\tau^{-1}}{(\tau^{d_1+1})^2} d\tau + O(1) = -a_3 \log v + O(1), \end{aligned}$$

and similarly

$$J_{0,1}(v) = v^{\delta_2-1} a_2 \int_0^\infty \frac{\tau^{-\delta_2}}{(\tau^{d_1+1})^2} d\tau - v^{2\delta_2-1} a_2 \int_v^\infty \frac{\tau^{-2\delta_2}}{(\tau^{d_1+1})^2} \frac{a_2}{\tau^{d_1+1} + p(v/\tau)} d\tau + O(1).$$

If $2\delta_2 - 1 > 0$ then the second term is of order $O(1)$, otherwise we can proceed similarly to obtain the expansion

$$J_{0,1}(v) = - \sum_{k=1}^{m_\alpha} (-a_2)^k \chi_{1,k} v^{k\delta_2-1}$$

where m_α is the largest integer such that $m_\alpha \delta_2 - 1 < 0$ and (formula 3.241.4 in [21])

$$\chi_{1,k} := \int_0^\infty \frac{\tau^{-k\delta_2}}{(\tau^{d_1} + 1)^{k+1}} d\tau = \frac{1}{d_1} \frac{\Gamma\left(\frac{1-k\delta_2}{d_1}\right) \Gamma\left(k+1 - \frac{1-k\delta_2}{d_1}\right)}{\Gamma(k+1)}, \quad k = 0, \dots, m_\alpha \quad (3.76)$$

Plugging all the estimates back we obtain

$$v^{\delta_1-1} \int_v^\infty \frac{\tau^{-\delta_1}}{\tau^{d_1} + 1 + p(v/\tau)} d\tau = a_3 \log v + \sum_{k=0}^{m_\alpha} (-a_2)^k \chi_{1,k} v^{k\delta_2-1} + O(1), \quad \text{as } v \rightarrow 0. \quad (3.77)$$

Further, the second integral in the sum in (3.75) reads

$$v^{\delta_2-1} \int_v^\infty \frac{\tau^{-\delta_2}}{\tau^{d_1} + 1 + p(v/\tau)} d\tau = v^{\delta_2-1} \int_v^\infty \frac{\tau^{-\delta_2}}{\tau^{d_1} + 1} d\tau - J_{1,1}(v) - J_{1,2}(v).$$

Here

$$J_{1,2}(v) := v^{\delta_2+\delta_3-1} a_3 \int_v^\infty \frac{\tau^{-\delta_2-\delta_3}}{(\tau^{d_1} + 1 + p(v/\tau))(\tau^{d_1} + 1)} d\tau \leq v^{\delta_2+\delta_3-1} a_3 \int_v^\infty \frac{\tau^{-\delta_2-\delta_3}}{(\tau^{d_1} + 1)^2} d\tau = O(1)$$

since $\delta_2 + \delta_3 > 1$. The second term

$$J_{1,1}(v) := v^{2\delta_2-1} a_2 \int_v^\infty \frac{\tau^{-2\delta_2}}{(\tau^{d_1} + 1 + p(v/\tau))(\tau^{d_1} + 1)} d\tau$$

is of order $O(1)$, if $2\delta_2 > 1$. Otherwise,

$$J_{1,1}(v) = v^{2\delta_2-1} a_2 \int_0^\infty \frac{\tau^{-2\delta_2}}{(\tau^{d_1} + 1)^2} d\tau - v^{3\delta_2-1} a_2^2 \int_v^\infty \frac{\tau^{-3\delta_2}}{(\tau^{d_1} + 1)^2} \frac{1}{\tau^{d_1} + 1 + p(v/\tau)} d\tau + O(1)$$

where the second term is of order $O(1)$, if $3\delta_2 > 1$ and so on. Thus we obtain asymptotics

$$v^{\delta_2-1} \int_v^\infty \frac{\tau^{-\delta_2}}{\tau^{d_1} + 1 + p(v/\tau)} d\tau = \sum_{k=1}^{m_\alpha} (-a_2)^{k-1} \chi_{0,k} v^{k\delta_2-1} + O(1), \quad \text{as } v \rightarrow 0 \quad (3.78)$$

with

$$\chi_{0,k} := \int_0^\infty \frac{\tau^{-k\delta_2}}{(\tau^{d_1} + 1)^k} d\tau = \frac{1}{d_1} \frac{\Gamma\left(\frac{1-k\delta_2}{d_1}\right)\Gamma\left(k - \frac{1-k\delta_2}{d_1}\right)}{\Gamma(k)}, \quad k = 1, \dots, m_\alpha. \quad (3.79)$$

Finally, since $\delta_3 = 1$, the last summand in (3.75) contributes

$$v^{\delta_3-1} \int_v^\infty \frac{\tau^{-\delta_3}}{\tau^{d_1} + 1 + p(v/\tau)} d\tau = \int_v^\infty \frac{\tau^{-1}}{\tau^{d_1} + 1} d\tau - J_{3,1}(v) - J_{3,2}(v) = -\log v + O(1) \quad (3.80)$$

where we used the bounds

$$J_{3,1}(v) := v^{\delta_2+\delta_3-1} \int_v^\infty \frac{a_2 \tau^{-\delta_3-\delta_2}}{(\tau^{d_1} + 1 + p(v/\tau))(\tau^{d_1} + 1)} d\tau = O(1)$$

and

$$J_{3,2}(v) := v^{2\delta_3-1} \int_v^\infty \frac{a_3 \tau^{-2\delta_3}}{(\tau^{d_1} + 1 + p(v/\tau))(\tau^{d_1} + 1)} d\tau = O(1).$$

Plugging (3.77), (3.78) and (3.80) into (3.75) we obtain

$$\begin{aligned} I_0(u) &= -\frac{1}{2}d_1\left(1 + a_1a_3 - a_3\frac{d_3}{d_1}\right)\log v - \frac{1}{2}a_1d_1\chi_{1,0}v^{-1} + \\ &+ \frac{1}{2}\sum_{k=1}^{m_\alpha}(-a_2)^k\left(d_2\chi_{0,k} - a_1d_1\chi_{1,k}\right)v^{k\delta_2-1} + O(1). \end{aligned} \quad (3.81)$$

Asymptotic expansion of $I_1(u)$. Using the asymptotic formulas, already derived above, we have

$$\begin{aligned} I_1(u) &:= -\sum_{i=1}^k c_i \int_1^\infty \frac{ut^{-d_i}}{1 + 2u \sum_{j=1}^k c_j t^{-d_j}} dt = -\frac{1}{2} \sum_{i=1}^k a_i v^{\delta_i-1} \int_v^\infty \frac{\tau^{-\delta_i}}{\tau^{d_1} + 1 + p(v/\tau)} d\tau = \\ &= -\frac{1}{2}a_1\chi_{1,0}v^{-1} + \frac{1}{2}\sum_{k=1}^{m_\alpha}(-a_2)^k\left(\chi_{0,k} - a_1\chi_{1,k}\right)v^{k\delta_2-1} + O(1), \end{aligned} \quad (3.82)$$

where the logarithmic term vanishes since $a_1 = 1$.

Asymptotic expansion of $I_2(u)$. We will need only the leading asymptotic term of $I_2(u)$. To this end, we have

$$\begin{aligned} I_2(u) &= 2 \int_1^\infty \left(\frac{u \sum_{j=1}^k c_j t^{-d_j}}{1 + 2u \sum_{j=1}^k c_j t^{-d_j}} \right)^2 dt = \\ &= \frac{1}{2} v^{-1} \int_v^\infty \left(\frac{1 + p(v/\tau)}{\tau^{d_1} + 1 + p(v/\tau)} \right)^2 d\tau = \frac{1}{2} \chi_{3,1} v^{-1} (1 + o(1)) \end{aligned} \quad (3.83)$$

where we defined

$$\chi_{3,1} := \int_0^\infty \frac{1}{(\tau^{d_1} + 1)^2} d\tau = \frac{1}{d_1} \frac{\Gamma(\frac{1}{d_1}) \Gamma(2 - \frac{1}{d_1})}{\Gamma(2)}. \quad (3.84)$$

Case $H \in (\frac{1}{2}, 1)$. It will be convenient to work with parameter $\alpha := 2 - 2H \in (0, 2)$. By Theorem 1.4, eigenvalues of the covariance operator of \tilde{B} satisfy

$$\lambda(n) = \nu_n^{-2} + \kappa_\alpha \nu_n^{\alpha-3}$$

where κ_α denotes the constant from (1.19) expressed in terms of α (see (1.199)) and, for $\alpha \in (0, 1)$, $\nu_n = (n - \frac{1}{2})\pi + O(n^{\alpha-1})$ as $n \rightarrow \infty$. Taylor expansion yields

$$\begin{aligned} \nu_n^{-2} &= \frac{1}{\pi^2} n^{-2} + \frac{1}{\pi^2} n^{-3} + O(n^{\alpha-4}) \\ \nu_n^{\alpha-3} &= \frac{1}{\pi^{3-\alpha}} n^{\alpha-3} + \frac{3-\alpha}{2} \frac{1}{\pi^{3-\alpha}} n^{\alpha-4} + O(n^{2\alpha-5}) \end{aligned}$$

and consequently

$$\lambda(n) = \frac{1}{\pi^2} n^{-2} + \frac{\kappa_\alpha}{\pi^{3-\alpha}} n^{\alpha-3} + \frac{1}{\pi^2} n^{-3} + O(n^{\alpha-4}) =: \phi(n) + O(n^{\alpha-4}).$$

By Li's comparison theorem (Theorem 2 in [57])

$$\mathbb{P}\left(\sum_{n=1}^\infty \lambda(n) Z_n \leq \varepsilon^2\right) \sim \left(\prod_{n=1}^\infty \frac{\phi(n)}{\lambda(n)}\right)^{1/2} \mathbb{P}\left(\sum_{n=1}^\infty \phi(n) Z_n \leq \varepsilon^2\right), \quad \text{as } \varepsilon \rightarrow 0,$$

if $\sum_{n=1}^\infty |1 - \lambda(n)/\phi(n)| < \infty$, which holds in our case. Hence the desired asymptotics coincides with

$$\mathbb{P}\left(\sum_{n=1}^\infty \phi(n) Z_n \leq \varepsilon^2\right) \quad \text{as } \varepsilon \rightarrow 0,$$

up to a multiplicative constant. Function $\phi(t)$ has the form (3.74) with $k = 3$ and

$$\begin{aligned} c_1 &= \frac{1}{\pi^2}, & d_1 &= 2 \\ c_2 &= \frac{\kappa_\alpha}{\pi^{3-\alpha}}, & d_2 &= 3 - \alpha \\ c_3 &= \frac{1}{\pi^2}, & d_3 &= 3 \end{aligned} \quad (3.85)$$

Plugging these values into the asymptotic expansions (3.81), (3.82) and (3.83) gives

$$I_0(u) = \frac{1}{4} \log u - \frac{1}{\sqrt{2}} u^{\frac{1}{2}} + \sum_{k=1}^{m_\alpha} g_k u^{\frac{1-k(1-\alpha)}{2}} + O(1) \quad (3.86)$$

$$I_1(u) = -\frac{1}{2\sqrt{2}} u^{\frac{1}{2}} - \sum_{k=1}^{m_\alpha} h_k u^{\frac{1-k(1-\alpha)}{2}} + O(1) \quad (3.87)$$

$$I_2(u) = \sqrt{\frac{c_1}{2}} \chi_{3,1} u^{\frac{1}{2}} (1 + o(1)) \quad (3.88)$$

where we defined sequences

$$\begin{aligned} h_k &= \frac{1}{2} \left(-\frac{c_2}{c_1}\right)^k (\chi_{1,k} - \chi_{0,k}) (2c_1)^{\frac{1-k\delta_2}{d_1}} \\ g_k &= \frac{1}{2} \left(-\frac{c_2}{c_1}\right)^k (d_2 \chi_{0,k} - d_1 \chi_{1,k}) (2c_1)^{\frac{1-k\delta_2}{d_1}} \end{aligned} \quad (3.89)$$

and constants $\chi_{i,k}$ are defined in (3.76), (3.79) and (3.84).

Application of Corollary 3.7 requires finding a function $u(r)$ which satisfies condition (3.73). To this end consider the equation (c.f. (3.87))

$$\frac{1}{2\sqrt{2}} u^{\frac{1}{2}} + \sum_{k=1}^{\left\lfloor \frac{\frac{1}{2}}{1-\alpha} \right\rfloor} h_k u^{\frac{1-k(1-\alpha)}{2}} = ur, \quad (3.90)$$

with respect to $u > 0$. If we divide both sides by u , the left hand side becomes a monotonous function decreasing to zero as $u \rightarrow \infty$ and therefore this equation has unique positive solution $u(r)$, which grows to $+\infty$ as $r \rightarrow 0$. By the choice of the upper limit in the sum in (3.90), the power of $u(r)$ in the numerator of (3.73) is strictly less than $\frac{1}{4}$ and hence (3.73) holds in view of (3.88).

If we now let $u = (ry)^{-2}$ equation (3.90) reads

$$\frac{1}{2\sqrt{2}}y + \sum_{k=1}^{\lfloor \frac{1}{2} \frac{1}{1-\alpha} \rfloor} h_k r^{k(1-\alpha)} y^{k(1-\alpha)+1} = 1, \quad y > 0.$$

The function $r \mapsto y(r)$ is analytic in a vicinity of zero and can be expanded into series of the small parameter $r^{1-\alpha}$

$$y(r) = y_0 + \sum_{j=1}^{\infty} y_j r^{j(1-\alpha)}.$$

Let ξ_j and $\eta_{k,j}$ be the coefficients of the expansions

$$y(r)^{-2} = \sum_{j=0}^{\infty} \xi_j r^{j(1-\alpha)} \quad \text{and} \quad y(r)^{k(1-\alpha)-1} = \sum_{j=0}^{\infty} \eta_{k,j} r^{j(1-\alpha)}.$$

Note that both are expressible in terms of y_j 's. Plugging these expansions into (3.72) gives

$$\begin{aligned} \mathbb{P}(\|\tilde{B}\|_2^2 \leq r) &\sim r^{\frac{1}{2}} \exp\left(-\frac{1}{\sqrt{2}} \sum_{j=0}^{\infty} \eta_{0,j} r^{j(1-\alpha)-1} + \sum_{k=1}^{m_\alpha} g_k \sum_{j=0}^{\infty} \eta_{k,j} r^{(j+k)(1-\alpha)-1} + \sum_{j=0}^{\infty} \xi_j r^{j(1-\alpha)-1}\right) \sim \\ &\sim r^{\frac{1}{2}} \exp\left(-\frac{1}{8} r^{-1} + \sum_{\ell=1}^{m_\alpha} \left(\xi_\ell - \frac{1}{\sqrt{2}} \eta_{0,\ell}\right) r^{\ell(1-\alpha)-1} + \sum_{\ell=1}^{m_\alpha} \left(\sum_{j=0}^{\ell-1} g_{\ell-j} \eta_{\ell-j,j}\right) r^{\ell(1-\alpha)-1}\right) \end{aligned}$$

with $m_\alpha := \lfloor \frac{1}{1-\alpha} \rfloor$. The formula claimed in 1. of Theorem 3.5 is obtained after substitution of $r := \varepsilon^2$ and $\alpha = 2 - 2H$.

The case $H \in (0, \frac{1}{2})$. By Theorem 1.4 the eigenvalues satisfy the same formula

$$\lambda(n) = \nu_n^{-2} + \kappa_\alpha \nu_n^{\alpha-3}$$

but for $\alpha \in (1, 2)$, with $\nu_n = \pi n - \frac{\pi}{2} q_\alpha + O(n^{1-\alpha})$ as $n \rightarrow \infty$, where

$$q_\alpha = 1 - \frac{1-2H}{2} - \frac{2}{\pi} \arcsin \frac{\ell_H}{\sqrt{1+\ell_H^2}}.$$

By the Taylor expansion

$$\begin{aligned}\nu_n^{-2} &= \frac{1}{\pi^2}n^{-2} + \frac{q_\alpha}{\pi^2}n^{-3} + O(n^{-\alpha-2}) \\ \nu_n^{\alpha-3} &= \frac{1}{\pi^{3-\alpha}}n^{\alpha-3} + \frac{3-\alpha}{2} \frac{q_\alpha}{\pi^{3-\alpha}}n^{\alpha-4} + O(n^{-3})\end{aligned}$$

and therefore

$$\lambda(n) = \frac{\kappa_\alpha}{\pi^{3-\alpha}}n^{\alpha-3} + \frac{1}{\pi^2}n^{-2} + \kappa_\alpha \frac{3-\alpha}{2} \frac{q_\alpha}{\pi^{3-\alpha}}n^{\alpha-4} + O(n^{-3}) := \phi(n) + O(n^{-3}).$$

As in the previous case, omitting the residual $O(n^{-3})$ term alters the exact asymptotics of small ball probabilities only by a multiplicative constant. The weight function $\phi(t)$ has the form (3.74) with

$$\begin{aligned}c_1 &= \frac{\kappa_\alpha}{\pi^{3-\alpha}}, & d_1 &= 3 - \alpha \\ c_2 &= \frac{1}{\pi^2}, & d_2 &= 2 \\ c_3 &= \frac{\kappa_\alpha}{\pi^{3-\alpha}} \frac{3-\alpha}{2} q_\alpha, & d_3 &= 4 - \alpha\end{aligned}\tag{3.91}$$

and the asymptotic expansions (3.81), (3.82) and (3.83) read

$$I_0(u) = \frac{1}{2} \left(1 - \frac{1}{2}q_\alpha\right) \log u - \frac{3-\alpha}{2} \chi_{1,0}(2c_1)^{\frac{1}{3-\alpha}} u^{\frac{1}{3-\alpha}} + \sum_{k=1}^{m_\alpha} g_k u^{\frac{1-k(\alpha-1)}{3-\alpha}} + O(1)\tag{3.92}$$

$$I_1(u) = -\frac{1}{2} \chi_{1,0}(2c_1)^{\frac{1}{3-\alpha}} u^{\frac{1}{3-\alpha}} - \sum_{k=1}^{m_\alpha} h_k u^{\frac{1-k(\alpha-1)}{3-\alpha}} + O(1)\tag{3.93}$$

$$I_2(u) = \frac{1}{2} \chi_{3,1}(2c_1 u)^{\frac{1}{3-\alpha}} \left(1 + o(1)\right)\tag{3.94}$$

where sequences h_k and g_k are defined by the same formulas as in (3.89), evaluated at constants (3.91).

To find a suitable function $u(r)$ satisfying condition (3.73), consider equation (cf. (3.93))

$$\frac{1}{2} \chi_{1,0}(2c_1)^{\frac{1}{3-\alpha}} u^{\frac{1}{3-\alpha}} + \sum_{k=1}^{\lfloor \frac{1}{2} \frac{1}{\alpha-1} \rfloor} h_k u^{\frac{1-k(\alpha-1)}{3-\alpha}} = ur.\tag{3.95}$$

As in the previous case, it has the unique solution $u(r)$ for any $r > 0$ and it increases to $+\infty$ as $r \rightarrow 0$. By the choice of upper limit in the sum in (3.95), the power of $u(r)$ in

the numerator of (3.73) does not exceed $\frac{1}{2} \frac{1}{3-\alpha}$ and hence (3.73) holds in view of (3.94). Define new variable y by the relation $u = (ry)^{-\frac{3-\alpha}{2-\alpha}}$, then it solves equation

$$y/y_0 + \sum_{k=1}^{\lfloor \frac{1}{2} \frac{1}{\alpha-1} \rfloor} h_k r^k \frac{\alpha-1}{2-\alpha} y^{1+k \frac{\alpha-1}{2-\alpha}} = 1,$$

where $1/y_0 = \frac{1}{2} \chi_{1,0}(2c_1)^{\frac{1}{3-\alpha}}$. The function $r \mapsto y(r)$ is analytic in the vicinity of $r = 0$ and admits expansion into powers of the small parameter $r^{\frac{\alpha-1}{2-\alpha}}$:

$$y(r) = y_0 + \sum_{k=1}^{\infty} y_k r^{k \frac{\alpha-1}{2-\alpha}}$$

Plugging (3.92) and (3.94) into (3.72) yields

$$\begin{aligned} \mathbb{P}(\|\tilde{B}\|_2^2 \leq r) &\sim u(r)^{-\frac{1}{4} - \frac{1}{2} \frac{1}{3-\alpha} + \frac{1}{2}(1 - \frac{1}{2} q_\alpha)} \exp\left(-\frac{3-\alpha}{y_0} u(r)^{\frac{1}{3-\alpha}} + \sum_{k=1}^{m_\alpha} g_k u(r)^{\frac{1-k(\alpha-1)}{3-\alpha}} + u(r)r\right) \sim \\ &\sim r^{\gamma_\alpha} \exp\left(-\frac{3-\alpha}{y_0} (ry)^{-\frac{1}{2-\alpha}} + \sum_{k=1}^{m_\alpha} g_k (ry)^{-\frac{1-k(\alpha-1)2}{2-\alpha}} + (ry)^{-\frac{3-\alpha}{2-\alpha}} r\right) \sim \\ &\sim r^{\gamma_\alpha} \exp\left(-\frac{3-\alpha}{y_0} \sum_{j=0}^{\infty} \eta_{0,j} r^{\frac{j(\alpha-1)-1}{2-\alpha}} + \sum_{k=1}^{m_\alpha} \sum_{j=0}^{\infty} g_k \eta_{k,j} r^{\frac{(j+k)(\alpha-1)-1}{2-\alpha}} + \sum_{j=0}^{\infty} \xi_j r^{\frac{j(\alpha-1)-1}{2-\alpha}}\right) \sim \\ &\sim r^{\gamma_\alpha} \exp\left(-\beta_0 r^{-\frac{1}{2-\alpha}} - \sum_{\ell=1}^{m_\alpha} \left(\frac{3-\alpha}{y_0} \eta_{0,\ell} - \sum_{j=0}^{\ell-1} g_{\ell-j} \eta_{\ell-j,j} - \xi_\ell\right) r^{\frac{\ell(\alpha-1)-1}{2-\alpha}}\right) \end{aligned}$$

where we defined

$$\gamma_\alpha := -\frac{3-\alpha}{2-\alpha} \left(-\frac{1}{4} - \frac{1}{2} \frac{1}{3-\alpha} + \frac{1}{2} \left(1 - \frac{1}{2} q_\alpha\right)\right) \quad \text{and} \quad \beta_0 := \frac{3-\alpha}{y_0} \eta_{0,0} - \xi_0,$$

and ξ_j and $\eta_{k,j}$ are coefficients in the expansions

$$y^{-\frac{1-k(\alpha-1)}{2-\alpha}} = \sum_{j=0}^{\infty} \eta_{k,j} r^{j \frac{\alpha-1}{2-\alpha}} \quad \text{and} \quad y^{-\frac{3-\alpha}{2-\alpha}} = \sum_{j=0}^{\infty} \xi_j r^{j \frac{\alpha-1}{2-\alpha}}.$$

Replacing α with $2-2H$ and $r := \varepsilon^2$, we obtain the formulas claimed in 2. of Theorem 3.5 after simplification.

BIBLIOGRAPHY

- [1] B.Bercu, L.Coutin and N.Savy. *Sharp large deviations for the fractional Ornstein-Uhlenbeck process*. Theory Probab. Appl., 55(4), 575-610, 2010.
- [2] B. Bercu, A. Rouault. *Sharp large deviations for the Ornstein-Uhlenbeck process*. Theory Probab. Appl., 46(1):1-19, 2002.
- [3] A. Berlinet, C. Thomas-Agnan. *Reproducing kernel Hilbert spaces in probability and statistics*, Kluwer Academic Publishers, Boston, MA, 2004, with a preface by Persi Diaconis.
- [4] C. Berzin, A. Latour, J. R. León. *Inference on the Hurst parameter and the variance of diffusions driven by fractional Brownian motion*. Vol. 216 of Lecture Notes in Statistics, Springer, Cham, 2014, with a foreword by Aline Bonami.
- [5] A. Beskos and G. O. Roberts. *Exact simulation of diffusions*. Ann. Appl. Probab., 15(4):2422–2444, 2005.
- [6] F. Biagini, Y. Hu, B. Øksendal, T. Zhang. *Stochastic calculus for fractional Brownian motion and applications*. Probability and its Applications (New York), Springer-Verlag London Ltd., London, 2008.
- [7] J. C. Bronski. *Asymptotics of Karhunen-Loeve eigenvalues and tight constants for probability distributions of passive scalar transport*. Comm. Math. Phys., 238(3):563–582, 2003.
- [8] J. C. Bronski. *Small ball constants and tight eigenvalue asymptotics for fractional Brownian motions*. J. Theoret. Probab., 16(1):87–100, 2003.
- [9] C.Cai, P.Chigansky and M.Kleptsyna. *Mixed gaussian processes: a filtering approach*. Ann. Probab., Volume 44, Number 4 (2016), 3032-3075.
- [10] P. Cheridito. *Mixed fractional Brownian motion*. Bernoulli, 7(6):913–934, 2001.

BIBLIOGRAPHY

- [11] P. Cheridito, H. Kawaguchi, and M. Maejima. *Fractional Ornstein-Uhlenbeck processes*. Electron. J. Probab., 8:no. 3, 14, 2003.
- [12] P. Chigansky and M. Kleptsyna. *Exact asymptotics in eigenproblems for fractional Brownian covariance operators*. Stochastic Process. Appl., 128(6):2007–2059, 2018.
- [13] P. Chigansky and M. Kleptsyna. *Statistical analysis of the mixed fractional Ornstein–Uhlenbeck process*. Teor. Veroyatn. Primen., 63(3):500–519, 2018.
- [14] A. Dembo and O. Zeitouni. *Large deviations techniques and applications*. Volume 38 of Applications of mathematics (New York). Springer-Verlag, New York, second edition, 1998.
- [15] T. Dunker, M. A. Lifshits, and W. Linde. *Small deviation probabilities of sums of independent random variables*. High dimensional probability (Oberwolfach, 1996), volume 43 of Progr. Probab., pages 59–74. Birkhäuser, Basel, 1998.
- [16] P. Embrechts, M. Maejima. *Selfsimilar processes*. Princeton Series in Applied Mathematics, Princeton University Press, Princeton, NJ, 2002.
- [17] D. Florence-Landais and H. Pham. *Large deviations in estimation of an Ornstein-Uhlenbeck model*. J. Appl. Probab., 36(1):60-77, 1999.
- [18] F. D. Gakhov. *Boundary value problems*. Dover Publications, Inc., New York, 1990. Translated from the Russian, Reprint of the 1966 translation.
- [19] F. Gao, J. Hannig, and F. Torcaso. *Integrated Brownian motions and exact L_2 -small balls*. Ann. Probab., 31(3):1320–1337, 2003.
- [20] D. Gasbarra, T. Sottinen, and E. Valkeila. *Gaussian bridges*. Stochastic analysis and applications, volume 2 of Abel Symp., pages 361–382. Springer, Berlin, 2007.
- [21] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Elsevier/Academic Press, Amsterdam, seventh edition, 2007. Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, With one CD-ROM (Windows, Macintosh and UNIX).
- [22] Y. Hu and D. Nualart. *Renormalized self-intersection local time for fractional Brownian motion*. Ann. Probab., 33(3):948–983, 2005.

-
- [23] S. Iacus and N. Yoshida. *Simulation and Inference for Stochastic Processes with YUIMA: A Comprehensive R Framework for SDEs and Other Stochastic Processes*. Springer, 2018.
- [24] L. V. Kantorovich and V. I. Krylov. *Approximate methods of higher analysis*. Translated from the 3rd Russian edition by C. D. Benster. Interscience Publishers, Inc., New York; P. Noordhoff Ltd., Groningen, 1958.
- [25] T. Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.
- [26] M.L. Kleptsyna and A. Le Breton. *Optimal linear filtering of general multidimensional Gaussian processes and its application to Laplace transforms of quadratic functionals*. J. Appl. Math. Stochastic Anal., 14(3):215-226, 2001.
- [27] A.N. Kolmogorov. *The Wiener spiral and some other interesting curves in Hilbert space*. Dokl. Akad. Nauk SSSR 26 (1940), 115–118.
- [28] J. Lebovits, M. Podolskij. *Estimation of the global regularity of a multifractional Brownian motion*. Electronic Journal of Statistics, Vol. 11, No. 1, 2017, p. 78-98.
- [29] E. L. Lehmann and J. P. Romano. *Testing statistical hypotheses*. Springer Texts in Statistics. Springer, New York, third edition, 2005.
- [30] N. N. Leonenko, M. D. Ruiz-Medina, M. S. Taqqu. *Rosenblatt distribution subordinated to Gaussian random fields with long-range dependence*, Stoch. Anal. Appl. 35 (1) (2017) 144–177.
- [31] W. V. Li, Q.-M. Shao. *Gaussian processes: inequalities, small ball probabilities and applications*. Stochastic processes: theory and methods, Vol. 19 of Handbook of Statist., North-Holland, Amsterdam, 2001, pp. 533–597.
- [32] R. S. Liptser and A. N. Shiryaev. *Statistics of random processes. II*, volume 6 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, expanded edition, 2001. Applications, Translated from the 1974 Russian original by A. B. Aries, Stochastic Modelling and Applied Probability.
- [33] H. Luschgy and G. Pagès. *Sharp asymptotics of the functional quantization problem for Gaussian processes*. Ann. Probab., 32(2):1574–1599, 2004.

- [34] A. MacKay, A. Melnikov and Y. Mishura. *Optimization of small deviation for mixed fractional brownian motion with trend*. Stochastics, 2018, forthcoming.
- [35] B. B. Mandelbrot, J. W. Van Ness. *Fractional Brownian motions, fractional noises and applications*. SIAM Rev. 10 (1968) 422–437.
- [36] B. B. Mandelbrot. *On an eigenfunction expansion and on fractional Brownian motions*. Lett. Nuovo Cimento (2), 33(17):549–550, 1982.
- [37] Y. S. Mishura. *Stochastic calculus for fractional Brownian motion and related processes*. Vol. 1929 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2008.
- [38] N. I. Muskhelishvili. *Singular integral equations*. Dover Publications, Inc., New York, 1992. Boundary problems of function theory and their application to mathematical physics, Translated from the second (1946) Russian edition and with a preface by J. R. M. Radok, Corrected reprint of the 1953 English translation.
- [39] A. I. Nazarov and Ya. Yu. Nikitin. *Exact L_2 -small ball behavior of integrated Gaussian processes and spectral asymptotics of boundary value problems*. Probab. Theory Related Fields, 129(4):469–494, 2004.
- [40] A. I. Nazarov and Ya. Yu. Nikitin. *Logarithmic asymptotics of small deviations in the L_2 -norm for some fractional Gaussian processes*. Teor. Veroyatn. Primen., 49(4):695–711, 2004.
- [41] A.I. Nazarov and Y.Y. Nikitin. *On small deviation asymptotics in L^2 of some mixed Gaussian processes*. Mathematics, 6(4):55, 2018.
- [42] A. I. Nazarov. *Exact L_2 -small ball asymptotics of Gaussian processes and the spectrum of boundary-value problems*. J. Theoret. Probab., 22(3):640–665, 2009.
- [43] A. I. Nazarov. *On a family of transformations of Gaussian random functions*. Teor. Veroyatn. Primen., 54(2):209–225, 2009.
- [44] E. Neuman , M. Rosenbaum *Fractional Brownian motion with zero Hurst parameter: a rough volatility viewpoint*. Electronic Communications in Probability, Volume 23, paper no. 61, 12 pp., 2018.

-
- [45] D. Nualart. *Fractional Brownian motion: stochastic calculus and applications*. International Congress of Mathematicians. Vol. III, Eur. Math. Soc., Zürich, 2006, pp. 1541–1562.
- [46] V. Pipiras and M.S. Taqqu. *Long-Range Dependence and Self-Similarity*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2017.
- [47] H.J. Reinhardt. *Analysis of approximation methods for differential and integral equations*, volume 57 of Applied Mathematical Sciences. Springer-Verlag, New York, 1985.
- [48] L. A. Shepp. *Radon-Nikodým derivatives of Gaussian measures*. Ann. Math. Statist. 37 (1966) 321–354.
- [49] T. Sottinen and A. Yazigi. *Generalized Gaussian bridges*. Stochastic Process. Appl., 124(9):3084–3105, 2014.
- [50] S. Sukhatme. *Fredholm determinant of a positive definite kernel of a special type and its application*. Ann. Math. Statist., 43:1914–1926, 1972.
- [51] G. N. Sytaja. *Certain asymptotic representations for a Gaussian measure in Hilbert space*. Theory of random processes, No. 2 (Russian), pages 93–104, 140. Izdat. “Naukova Dumka”, Kiev, 1974.
- [52] M. S. Taqqu. *Fractional Brownian motion and long-range dependence*. Theory and applications of long-range dependence, Birkhäuser Boston, Boston, MA, 2003, pp. 5–38.
- [53] A. B. Tsybakov. *Introduction to nonparametric estimation*. Springer Series in Statistics. Springer, New York, 2009. Revised and extended from the 2004 French original, Translated by Vladimir Zaiats.
- [54] S. Ukai. *Asymptotic distribution of eigenvalues of the kernel in the Kirkwood-Riseman integral equation*. J. Mathematical Phys., 12:83–92, 1971.
- [55] M. S. Veillette, M. S. Taqqu. *Properties and numerical evaluation of the Rosenblatt distribution*. Bernoulli 19 (3) (2013) 982–1005.
- [56] M. Vidyasagar. *Nonlinear systems analysis, 2ed*. SIAM, 2002.

BIBLIOGRAPHY

- [57] V. Li. Wenbo *Comparison results for the lower tail of Gaussian seminorms.* J. Theoret. Probab., 5(1):1–31, 1992.
- [58] Harry van Zanten. *When is a linear combination of independent fBm's equivalent to a single fBm?* Stochastic Process. Appl., 117(1):57–70, 2007.

Title: Asymptotic study of covariance operator of fractional processes: analytic approach with applications

Key words: fractional Gaussian processes, covariance operator, eigenproblem, estimation of parameters, optimal linear filtering.

Abstract: Eigenproblems frequently arise in theory and applications of stochastic processes, but only a few have explicit solutions. Those which do are usually solved by reduction to the generalized Sturm-Liouville theory for differential operators. This includes the Brownian motion and a whole class of processes, which derive from it by means of linear transformations. The more general eigenproblems are not solvable in closed form and the subject of this thesis is the asymptotic spectral analysis of the fractional Gaussian processes and its applications.

In the first part, we develop methodology for the spectral analysis of the fractional type covariance operators, corresponding to an important family of processes that includes the fractional Ornstein-Uhlenbeck process, the integrated fractional Brownian motion and the mixed fractional Brownian motion. We obtain accurate second order asymptotic approximations for both the eigenvalues and the eigenfunctions.

In Chapter 2 we consider the covariance eigenproblem for Gaussian bridges. Given a base process, its bridge is obtained by conditioning the trajectories to start and terminate at the given points. We show how the spectral asymptotics of a bridge can be derived from that of its base process, considering, as an example, the case of the fractional Brownian bridge.

In the final part we consider three representative applications of the developed theory. We consider the filtering problem of fractional Gaussian signals in white noise and find the high signal-to-noise and large time asymptotics of the estimation error. We investigate large deviation properties of the maximum likelihood drift parameter estimator for the Ornstein-Uhlenbeck process driven by mixed fractional Brownian motion. We obtain exact asymptotics of small ball probabilities for the fractional Gaussian processes, which exhibits an interesting stratification of scales with respect to the Hurst parameter of the fractional component.

Titre: Études asymptotiques de l'opérateur de covariance pour les processus fractionnaires: approche analytique avec applications

Mots-clés : processus gaussiens fractionnaires, opérateur de covariance, problème aux valeurs et fonctions propres, estimation de paramètres, filtrage linéaire optimal.

Résumé : Les problèmes aux valeurs et fonctions propres surviennent fréquemment dans la théorie et dans les applications des processus stochastiques. Cependant quelques-uns seulement admettent une solution explicite; la résolution est alors généralement obtenue par la théorie généralisée de Sturm-Liouville pour les opérateurs différentiels. Cela inclut le mouvement brownien et tous les processus qui en dérivent au moyen de transformations linéaires. Les problèmes plus généraux ne peuvent pas être résolus sous une forme fermée et le sujet de cette thèse est l'analyse spectrale asymptotique des processus gaussiens fractionnaires et ses applications.

Dans la première partie, nous développons une méthodologie pour l'analyse spectrale des opérateurs de covariance de type fractionnaire, correspondant à une famille importante de processus, incluant le processus fractionnaire d'Ornstein-Uhlenbeck, le mouvement brownien fractionnaire intégré et le mouvement brownien fractionnaire mixte. Nous obtenons des approximations asymptotiques du second ordre pour les valeurs propres et les fonctions propres.

Au chapitre 2, nous considérons le problème aux

valeurs et fonctions propres pour l'opérateur de covariance des ponts gaussiens. Étant donné un processus de base et deux points, son pont est obtenu en conditionnant les trajectoires à démarrer et se terminer aux points fixés. Nous montrons comment l'asymptotique spectrale d'un pont peut être dérivée de celle de son processus de base, en prenant comme exemple le cas du pont brownien fractionnaire.

Dans la dernière partie, nous considérons trois applications représentatives de la théorie développée. Pour le problème de filtrage des signaux gaussiens fractionnaires dans le bruit blanc, nous trouvons l'asymptotique de l'erreur d'estimation. Nous étudions ensuite les propriétés de grande déviation de l'estimateur du maximum de vraisemblance du paramètre de dérive dans le modèle de processus d'Ornstein-Uhlenbeck gouverné par un mouvement brownien fractionnaire mixte. Nous obtenons enfin une asymptotique exacte des probabilités des petites boules pour les processus gaussiens fractionnaires, qui présente une stratification intéressante des échelles en fonction du paramètre de Hurst de la composante fractionnaire.