Geometric characteristics of smooth anisotropic random fields.

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Caractéristiques géométriques de champs aléatoires anisotropes réguliers

dirigée par

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soutenue le 10 septembre 2018

devant le jury composé de

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Contents

Introduction

Random fields are random functions defined on a multiparameter space. They are used to modelize structures or phenomena that arise or take place in a spatial or spatio-temporal setting. The mathematical theory of random fields is of interest in many fields such as image analysis, medical imaging, optics, material science, environmental science, physical oceanography and cosmology.

This thesis is about smooth real-valued random fields defined on $\mathbb{R}^d$, for some positive integer d, with a focus on dimension two in some parts of our study. Random fields are the generalization in higher dimensions of random processes on the line (where the real variable represents either time or space). In higher dimensions, the parameter space is considered as spatial with sometimes an additional dimension corresponding to time. The multidimensionality of the parameter space entails more complexity as well as the increased possibility to exploit geometrical aspects.

The axis of our study is definitely geometric: it is about geometric random objects linked to smooth real-valued random fields. We focus on the discrete set of critical points, the level sets and the excursion sets of a random field restricted to a finite regular domain. As it will be detailed in the first chapter, we study topological characteristics or the measures of these sets, which are real random variables.

More precisely, we are interested in the number of critical points, the length of the level sets and the Euler characteristic of the excursion sets. We focus on the first and the second moments of these random variables, which provide information about their distribution, the latter being out of reach. Hence, they allow us to learn about distributional properties of the random field itself. Moreover, for two-dimensional and three-dimensional random fields, these geometric characteristics could be measured on realizations of a random field and then used to compare a chosen model with reality.

We will make great use of Rice formulas that precisely provide expressions for the moments of some geometric characteristics, in particular under the assumption that the considered random field is Gaussian. Therefore, most of this thesis (yet not all of it) is written in the Gaussian setting. This allows us to use the rich theory mainly developped in this framework and to derive more explicit formulas under fairly simple assumptions.

When the modelized structure is homogeneous or invariant under rotation, it may be justified to add the assumption that a random field is stationary or isotropic. These invariance assumptions have strong consequences on the covariance structure of the random field, therefore they simplify the study. However, it is sometimes too much of a simplification to assume them as they may not reflect reality. Besides, a big demand for anisotropic models is nowadays observed, in particular by practitioners in

geostatistics, offshore engineering, heterogeneous material or medical imaging (see for instance [RB10], [Kla16], [ASP16]), but also for more theoretical studies dedicated to image synthesis and analysis, cosmology or arithmetic ([ORS14], [BMR15], [PAA$^+$16], [KW17]). In this work, we have chosen to focus on anisotropic random fields, which are even not stationary in one part of our study.

Chapter I is an introductive chapter, where we introduce all the notions and tools that we need in order to formulate properly our contributions to the geometric study of smooth random fields. The chapters that come next are based on research papers that are either already published, accepted for publication or submitted to a journal. Their content is thoroughly described at the end of the first chapter, but we sketch it here.

First, we focus on the number of critical points of a stationary and Gaussian random field defined on $\mathbb{R}^d$. In Chapter II, derived from the published article [EF16] coauthored with Anne Estrade, we present an extension of Geman condition, a condition for the finiteness of the variance of a stationary and Gaussian process, to the multidimensional and anisotropic case.

The purpose of Chapter III is a specific model of anisotropic and non-stationary random fields in the planar framework. It is based on [Fou18], an article accepted for publication. The so-called deformed random fields are built from a stationary and isotropic random field and a bijective and deterministic deformation of the plane. For this model, the cases of isotropy are proved to match a certain invariance property of the expected Euler characteristic of the excursion sets. The mean Euler characteristic of excursion sets also allows to determine the deformation of the model, when the latter is unknown. In the deformed random fields model, anisotropy stems from the parameter space.

However, anisotropy can also stem from the spectral domain. This is evidenced in Chapter IV derived from [EF18], where we study random wave models. Some distributional properties of a random wave are related to the ones of its random wavevector. We focus on the expected measure of the planar and Gaussian random wave's level curves, which is proved to decrease as the anisotropy of its random wavevector increases. The mean length of a random wave's crest line in a fixed direction is also at stake in this chapter. Our results are applied to several models such as anisotropic generalizations of Berry's planar random wave model and a Gaussian spatio-temporal sea wave model.

Here are the precise references of the research articles which form the basis of this thesis :

- [EF16] Estrade, A. and Fournier, J., Number of critical points of a gaussian random field: Condition for a finite variance, *Statistic & Probability Letters*, Vol. 118, 94–99, 2016;
- [Fou18] Fournier, J., Identification and isotropy characterization of deformed random fields through excursion sets, accepted for publication in the *Applied Probability Journals*, hal-01495157, 2017;
- [EF18] Estrade, A. and Fournier, J., Anisotropic random wave models, hal-01745706, 2018.

Chapter I

Framework

This introductory chapter has been written with the help of two major references: [AT07] and [AW09]. Its purpose is to gather all the notions and results which are used in the following chapters. Each of them, based on a published or a submitted article, is presented at the end of this chapter.

General notions and results about random fields are introduced in Section I.1. From Section I.2, we adopt a geometric viewpoint: we present some random geometric characteristics linked to a random field and we explain how their first moments can be computed thanks to Rice formulas.

Notations and conventions

Let d be a positive integer.

- $\#A$: cardinality of a finite set A.
- $\mathbb{N}_0$: set of the the non-negative integers; $\mathbb{N}$: set of the positive integers.
- $\overline{\cdot}$: complex conjugation.
- $\mathring{A}$, $\overline{A}$, ∂A: respectively interior, closure and border of the set A.
- $\mathscr{B}(\mathbb{R}^d)$: Borel σ-algebra on $\mathbb{R}^d$.
- $\mathscr{H}_k$: k-dimensional Hausdorff measure if $k \in \mathbb{N}$ and counting measure if $k = 0$; $|A|_k$ also denotes the k-dimensional Hausdorff measure of the set A.
- In $\mathbb{R}^d$, the canonical Euclidian scalar product between two vectors x and y is written $x \cdot y$, the associated Euclidian norm is written $\|\cdot\|$. We occasionaly use the same notation for $t \in \mathbb{R}^d$ and for the column vector containing its coordinates.
- A subset T of $\mathbb{R}^d$ is a rectangle if there exist $(s_i, t_i)_{1 \leq i \leq d} \in (\mathbb{R}^2)^d$ satisfying for any $i \in \{1, \cdots, d\}$, $s_i \leq t_i$ and an orthonormal basis of $\mathbb{R}^d$ where we can write

$$T = \prod_{i=1}^{d} [s_i, t_i].$$

The rectangle T is called a segment if there exists $k \in \{1, \cdots, d\}$ such that $s_k < t_k$ and $\forall i \neq k,\ s_i = t_i$. Note that by definition, all rectangles are bounded.

- $\mathbb{S}^{d-1}$: $(d-1)$-dimensional sphere in $\mathbb{R}^d$.
- I_d: identity matrix of size $d \times d$.
- A^T: transpose of the matrix A.
- $SO(d)$: group of the rotations in $\mathbb{R}^d$ (that is, orthogonal transforms with determinant equal to one).
- index (S), where S is a symmetric real matrix: number of negative eigenvalues of S.
- $\mathfrak{S}_d$: set of the symmetric real matrices of size $d \times d$; $\mathfrak{S}_d^j$: set of the matrices in $\mathfrak{S}_d$ with exactly j negative eigenvalues.
- Let $f\colon \mathbb{R}^d \to \mathbb{R}$. We fix a basis in $\mathbb{R}^d$. For some positive integer k, we say that f is of class $\mathscr{C}^k$ on $\mathbb{R}^d$ if all the k^{th}-order partial derivatives of f in a fixed basis of $\mathbb{R}^d$ exist and are continuous on $\mathbb{R}^d$. If f is of class $\mathscr{C}^4$, we write f'_i its first-order partial derivative in the i^{th} direction of the basis and $f''_{i,j}(t)$ its second-order partial derivative in the i^{th} and j^{th} directions. The third-order and fourth-order partial derivatives are denoted by $f^{(3)}_{i,j,k}$ and $f^{(4)}_{i,j,k,l}$, respectively. We refer to the gradient $(f'_i(t))_{1\leq i\leq d}$ of f at t as $f'(t)$ and to the Hessian matrix $(f''_{ij}(t))_{1\leq i,j\leq d}$ of f at t as $f''(t)$. For the higher-order partial derivatives (but also occasionally for the first-order and second-order ones), we use the notation with multi-indices introduced in the following point.
- For $\mathbf{j} = (j_1, \cdots, j_d) \in {\mathbb{N}_0}^d$, we write $|\mathbf{j}| = \sum_{\ell=1}^{d} j_\ell$. Moreover, if $\lambda \in \mathbb{R}^d$ and if f is a map from $\mathbb{R}^d$ to $\mathbb{R}$ with adequate regularity, we write
$$\lambda^{\mathbf{j}} = \prod_{\ell=1}^{d} \lambda_\ell^{j_\ell} \quad \text{and} \quad \frac{\partial^{\mathbf{j}} f}{\partial t^{\mathbf{j}}} = \frac{\partial^{|\mathbf{j}|} f}{\partial t_1^{j_1} \cdots \partial t_d^{j_d}}.$$
- The letters "a.s." are short for "almost surely" or "almost sure".
- If X and Y are two random variables on some probability space, we write $X \overset{\mathscr{L}}{\sim} Y$ if X and Y have the same distribution.
- If a random variable $X \in \mathbb{R}^d$ admits a probability density function with respect to the Lebesgue measure on $\mathbb{R}^d$, we write it p_X.
- $\mathscr{N}(m, \Sigma)$, with $m \in \mathbb{R}^d$ and Σ a positive and symmetric matrix of size $d \times d$: normal distribution with expectation m and covariance matrix Σ. We say that a Gaussian random vector is degenerate if its covariance matrix is not invertible.
- Ψ: tail probability of a Gaussian standard variable.
- $(H_i)_{i\in\mathbb{N}_0}$: the Hermite polynomials. We also set $H_{-1}\colon x \mapsto \sqrt{2\pi}\Psi(x)\exp(x^2/2)$.

Random fields

The random fields that we consider are defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Their parameter space is the Euclidian space $\mathbb{R}^d$, where d is a positive integer, or a restriction of it. They take real values. In the following, most of the time, we will not note the dependence on ω of a random field $X = \{X(t,\omega),\, t \in \mathbb{R}^d,\, \omega \in \Omega\}$, writing simply $\{X(t),\, t \in \mathbb{R}^d\}$. In the whole section, we denote by X a real-valued random field defined on $\mathbb{R}^d$ that admits a finite second moment at each point.

Let m be its expectation function $\mathbb{R}^d \to \mathbb{R},\ t \mapsto \mathbb{E}[X(t)]$. Let C be its covariance function $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R},\ (s,t) \mapsto \mathbb{E}[(X(s) - m(s))(X(t) - m(t))]$; C is a symmetric positive semidefinite function. Note that in the complex case (that we shall discuss in Section I.1.4), the covariance function is defined by $C\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C},\ (s,t) \mapsto \mathbb{E}[(X(s) - m(s))(\overline{X(t) - m(t)})]$ and it is a symmetric positive semidefinite function with Hermitian symmetry.

Stationarity and isotropy

Some invariance properties, such as stationarity and isotropy, can be added on the law of X.

The random field X is said to be stationary (or homogeneous) if its law is invariant under any translation of the parameter space:

$$\forall a \in \mathbb{R}^d, \quad X(\cdot + a) \overset{\mathscr{L}}{\sim} X.$$

That means that X has stochastically the same behaviour around any point of $\mathbb{R}^d$. It implies that the expectation function m as well as the variance of X are constant and that

$$\forall (s,t) \in (\mathbb{R}^d)^2, \quad C(s,t) = C(s-t, 0). \tag{1}$$

In this case, we introduce $r\colon \mathbb{R}^d \to \mathbb{R},\ t \mapsto C(t,0)$, which exactly describes the covariance structure of X, since for any $(s,t) \in (\mathbb{R}^d)^2$, $\mathrm{Cov}(X(s), X(t)) = r(s-t)$. By abusing the notation, in the stationary case, we call r the covariance function of X. It is also positive, semidefinite and even. Reciprocally, if the covariance function C of a centred random field X satisfies (1) then X is said to be weakly stationary.

Figure I.1 provides simulations of a stationary random field (Figure I.1b) and of a non-stationary and anisotropic random field (Figure I.1a).

Since we deal with distributional properties (mainly the expectation and the variance of random variables linked to a random field), we may equivalently study X or any random field with the same law as X. Thus if X is stationary, we may assume it centred, since X has the same law as $X - m$.

The random field X is said to be isotropic if its law is invariant under any rotation ρ of the parameter space:

$$\forall \rho \in SO(d), \quad X \circ \rho \overset{\mathscr{L}}{\sim} X.$$

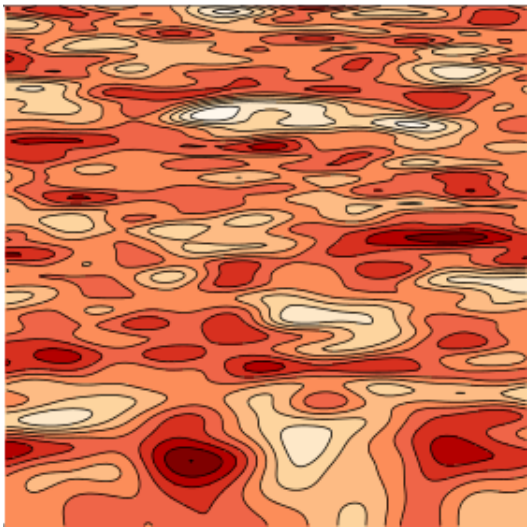
That means that X has stochastically the same behaviour on any direction of $\mathbb{R}^d$. Equivalently, isotropy corresponds to the lack of preferred directions for X. If X is

isotropic, then

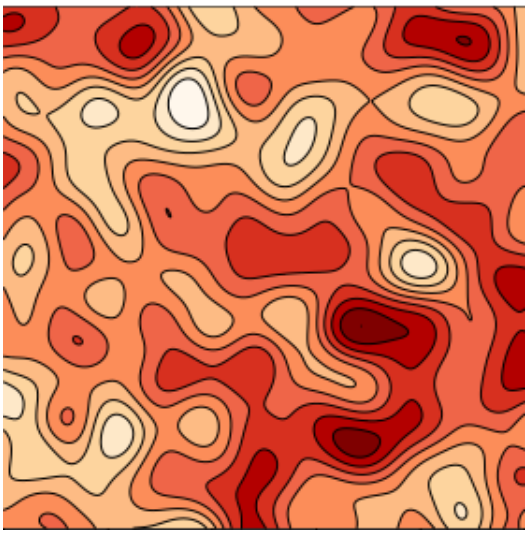
$$\forall \rho \in SO(d),\ \forall (s,t) \in (\mathbb{R}^d)^2, \quad C(\rho(s), \rho(t)) = C(s,t). \tag{2}$$

Reciprocally, if the covariance function C of a centred random field X satisfies condition (2), then X is said weakly isotropic. If X is stationary as well as isotropic, then its covariance function $r\colon \mathbb{R}^d \to \mathbb{R}$ is radial: for any $t \in \mathbb{R}^d$, $r(t)$ only depends on t through $\|t\|$. Figure I.1b shows a simulation of a stationary and isotropic random field while at the end of Chapter III, Figures III.1 present simulations of isotropic but non-stationary random fields.

The assumptions of stationarity and/or isotropy are sometimes justified by modelization purpose. For instance, in image analysis and synthesis, the homogeneous nature of texture entails that stationary random fields are commonly used to model them. However, these invariance assumptions are also sometimes added with a view to simplification, due to the constraints they add on the covariance structure of the random field. We will mention other implications of these assumptions in Sections I.1.3 and I.1.4.



(a) A non-stationary and anisotropic random field, actually a deformed random field X_θ (see Chapter III) with X Gaussian with Gaussian covariance and θ a tensorial deformation $(s,t) \mapsto (s, t^{1,55})$.



(b) A stationary and isotropic random field.

Figure I.1 – Simulation of the level sets of two random fields.

Gaussian random fields

In this thesis, the assumption that the considered random fields are Gaussian will often be in force. According to Kolmogorov theorem (see for instance [AW09] Theorem 1.1), the law of a random field X is determined by its finite-dimensional distributions, that is to say by the set of the laws of the random vectors

$$(X(t_1), \cdots, X(t_n)) \ :\ n \in \mathbb{N},\ (t_1, \cdots, t_n) \in (\mathbb{R}^d)^n.$$

The random field X is said to be Gaussian if all these random vectors are Gaussian vectors. In this case, the law of one of them $(X(t_1), \cdots, X(t_n))$ is determined by its covariance matrix $(C(t_i, t_j))_{1\leq i\leq d,\, 1\leq j\leq d}$ and by its expectation vector $(\mathbb{E}[X(t_i)])_{i\in\{1,\cdots,d\}}$. Therefore, a consequence of Kolmogorov's theorem is that for any function $m\colon \mathbb{R}^d \to \mathbb{R}$ and for any positive semidefinite function $C\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, there exists a Gaussian random field with expectation function m and with covariance function C, and its distribution is uniquely determined.

The fact that the distribution of a Gaussian random field is determined by its expectation and covariance functions simplifies a lot their study. For instance, if a centred Gaussian random field is weakly stationary then it is stationary. Similarly, condition (2) is not only a necessary condition for isotropy but also a sufficient condition in the case of a centred Gaussian random field.

To end with, we present two results that will come into play in Chapter II, when we study joint Gaussian variables: Wick formula and a regression result.

Wick formula expresses the expectation of the product of an even number, say $2m$, of centred Gaussian variables as a homogeneous polynomial function of degree m evaluated at the covariances of these Gaussian variables. Its proof can be found in [AT07] (see Lemma 11.6.1).

Proposition I.1 (Wick formula) *Let $X_1, X_2 \cdots, X_n$ be real random variables with a joint Gaussian distribution and zero means. Then, for any positive integer m such that $2m+1 \leq n$,*

- $\mathbb{E}[X_1 \cdots X_{2m+1}] = 0$,
- $\mathbb{E}[X_1 \cdots X_{2m}] = \sum \mathbb{E}[X_{i_1} X_{i_2}] \cdots \mathbb{E}[X_{i_{2m-1}} X_{i_{2m}}]$,

where the sum is taken over the $\dfrac{(2m)!}{m!2^m}$ different ways of grouping $X_1, \cdots, X_{2m}$ into m pairs.

The following regression formula may be proved through a simple conditional Gaussian density computation.

Proposition I.2 (Gaussian regression) *Let $n < d$ and let $X = (X_1, X_2)$ be a centred Gaussian vector in $\mathbb{R}^d$, such that $X_1 \in \mathbb{R}^n$ and $X_2 \in \mathbb{R}^{d-n}$. We assume that X_2 is not degenerate. We write C_{11} and C_{22} the covariance matrices of X_1 and X_2, respectively, and C_{12} the $n \times (d-n)$ matrix of the covariances between the coordinates of X_1 and X_2, such that the covariance matrix of X is*

$$\begin{pmatrix} C_{11} & C_{12}^T \\ C_{12} & C_{22} \end{pmatrix}.$$

Then the law of $X_1|X_2$, that is the conditional law of X_1 given X_2, is

$$\mathcal{N}(C_{12}C_{22}^{-1}X_2,\, C_{11} - C_{12}C_{22}^{-1}C_{12}^T).$$

In other words, there exists a random variable $\varepsilon \overset{\mathcal{L}}{\sim} \mathcal{N}(0,\, C_{11} - C_{12}C_{22}^{-1}C_{12}^T)$ independent of X_2 such that

$$X_1 = C_{12}C_{22}^{-1}X_2 + \varepsilon$$

Regularity

Our work is about smooth random fields, for which the realizations of the set of critical points, of the level sets and of the excursion sets are non-degenerate geometric objects. Therefore, the kind of regularity that we need is almost sure regularity. In fact, nearly all the random fields considered in this thesis will be at least almost surely of class $\mathscr{C}^2$. However, it is also useful to discuss the regularity in quadratic mean, also called L^2-regularity or mean square regularity, which is linked to the regularity of the covariance function. Let us first recall the definitions of these different types of regularity. The topology on $\mathbb{R}^d$ is the one given by the Euclidian metric. For the proofs of the results stated in this section, we refer to [CL67, AT07] and [AW09].

Definition I.3 (almost sure regularity) *1. We say that X is a.s. continuous on $\mathbb{R}^d$ if, for almost any $\omega \in \Omega$, $X(\cdot, \omega)$ is continuous on $\mathbb{R}^d$.*

2. Let $k \in \mathbb{N}$. We say that X is a.s. of class $\mathscr{C}^k$ if, for almost any $\omega \in \Omega$, $X(\cdot, \omega)$ is of class $\mathscr{C}^k$ on $\mathbb{R}^d$.

If X is a.s. of class $\mathscr{C}^1$, we introduce the vector-valued random field $X' = (X'_j)_{1 \leq j \leq d} \colon \mathbb{R}^d \to \mathbb{R}^d$ corresponding to the gradient of X in the canonical basis of $\mathbb{R}^d$. If X is a.s. of class $\mathscr{C}^2$, the vector-valued random field $X'' = (X''_{ij})_{1 \leq i,j \leq d} \colon \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ corresponds to the Hessian matrix of X in the canonical basis of $\mathbb{R}^d$. Note that if X is a Gaussian random field, its almost sure partial derivatives are also Gaussian random fields.

Definition I.4 ($\mathbf{L^2}$-regularity) *1. The random field X is said to be continuous in quadratic mean at point $t \in \mathbb{R}^d$ if $X(s) \xrightarrow[s \to t]{L^2} X(t)$, that is if $\mathbb{E}[(X(s) - X(t))^2] \underset{s \to t}{\longrightarrow} 0$.*

2. Let $u \in \mathbb{S}^{d-1}$. The random field X is said to be differentiable in quadratic mean at point t along vector u if $\dfrac{X(t+hu) - X(t)}{h}$ admits a limit with respect to the topology of the L^2-norm, as $h \in \mathbb{R}$ tends to zero.

If X is a Gaussian random field, its mean square partial derivatives are also Gaussian.

The upcoming results (Propositions I.5 and I.6) formulate the links between the almost sure regularity, the L^2-regularity and the regularity of the covariance function, partly in the Gaussian framework. Let us make a first observation: if X is almost surely differentiable at $t \in \mathbb{R}^d$ along direction $u \in \mathbb{S}^{d-1}$ and if it admits a L^2-derivative at t along direction u, then the two derivatives coincide almost surely. Indeed, convergence in quadratic mean implies the almost sure convergence of a subsequence.

Proposition I.5 *Let $k \in \mathbb{N}$. Let us assume that X is a centred Gaussian random field with realizations almost surely of class $\mathscr{C}^k$ on $\mathbb{R}^d$. Then its covariance function C is of class $\mathscr{C}^{2k}$ on $\mathbb{R}^d$.*

The above proposition results from the fact that almost sure convergence entails the convergence in distribution, which itself can be related to a convergence result of the characteristic function, according to Lévy theorem. This allows us to conclude on the regularity of the covariance function when the considered random field is Gaussian.

The following property, which is not specific to the Gaussian framework, relates the existence of L^2-derivatives with the existence of derivatives of the covariance function. Remember that the conventions about multi-indices have been introduced in the notations at the beginning of the current chapter.

Proposition I.6 *Let us assume that the random field X is centred.*

1. *For any $t_0 \in \mathbb{R}^d$, X is continuous in quadratic mean at t_0 if and only if its covariance function C is continuous at (t_0, t_0). Moreover, if C is continuous on $\{(t,t), t \in \mathbb{R}^d\}$ then C is continuous on $\mathbb{R}^d \times \mathbb{R}^d$.*
2. *Let $t_0 \in \mathbb{R}^d$ and let $\mathbf{j} \in \mathbb{N}_0^d$. If the covariance function $C\colon (s,t) \mapsto C(s,t)$ admits a partial derivative $\dfrac{\partial^{2|\mathbf{j}|} C}{\partial s^{\mathbf{j}}\, \partial t^{\mathbf{j}}}$ at (t_0, t_0), then the random field X admits a mean square derivative $\dfrac{\partial^{|\mathbf{j}|} X}{\partial t^{\mathbf{j}}}$ at t_0. Moreover, if $\dfrac{\partial^{2|\mathbf{j}|} C}{\partial s^{\mathbf{j}}\, \partial t^{\mathbf{j}}}$ exists on $\{(t,t), t \in \mathbb{R}^d\}$, then it exists on $\mathbb{R}^d \times \mathbb{R}^d$.*

The above proposition together with Proposition I.5 entail that in the Gaussian case, if X is almost surely differentiable on $\mathbb{R}^d$, then it is mean square differentiable on $\mathbb{R}^d$. We have decided to use the same notations for mean square derivatives as for almost sure derivatives, having in mind that they coincide in the Gaussian and almost surely regular framework that will interest us.

In the stationary case, Proposition I.6 may be stated in a simpler way in terms of the regularity of the covariance function r at zero. Indeed, in the stationary case, for any $t_0 \in \mathbb{R}^d$, the continuity of C and the existence of $\dfrac{\partial^2 C}{\partial s_j \partial t_j}$ at point (t_0, t_0) boil down, respectively, to the continuity of r at zero and to the existence of $\dfrac{\partial^2 r}{\partial s_j \partial t_j}(0,0)$.

Thanks to computations similar to the ones leading to Proposition I.6, it is possible to relate partial derivatives of the covariance function C to the covariances between partial derivatives of X. We will especially use them in the case of a stationary random field in Chapter II and in Chapter IV, therefore we state them here in the stationary case. Let $(\mathbf{j}, \mathbf{k}) \in \mathbb{N}_0^{2d}$, let us assume that X is centred, weakly stationary, and that its covariance function r admits a partial derivative $\dfrac{\partial^{|\mathbf{j}+\mathbf{k}|} r}{\partial t^{\mathbf{j}+\mathbf{k}}}$ on $\mathbb{R}^d$. Then it occurs that X admits mean square partial derivatives $\dfrac{\partial^{|\mathbf{j}|} X}{\partial t^{\mathbf{j}}}$ and $\dfrac{\partial^{|\mathbf{k}|} X}{\partial t^{\mathbf{k}}}$ on $\mathbb{R}^d$ and that for any $(s,t) \in (\mathbb{R}^d)^2$,

$$\mathbb{E}\left[\frac{\partial^{|\mathbf{j}|} X}{\partial t^{\mathbf{j}}}(s)\, \frac{\partial^{|\mathbf{k}|} X}{\partial t^{\mathbf{k}}}(t)\right] = (-1)^{|\mathbf{k}|}\, \frac{\partial^{|\mathbf{j}+\mathbf{k}|} r}{\partial t^{\mathbf{j}+\mathbf{k}}}(s-t). \tag{3}$$

The fact that r is an even function explains the apparent asymmetry of the right-hand term of this equality. This formula has simple interesting consequences. First, for

$(i,j) \in \{1,\cdots,d\}^2$, the random field X_i' admits the covariance function $-r_{ii}''$, and $X_{i,j}''$ admits the covariance function $r_{iijj}^{(4)}$. In the next section, we will also see that for $s=t$, the above expectation gives a spectral moment of the random field X, up to a factor ± 1.

Moreover, Formula (3) leads to simplifications in the covariance structure of a weakly stationary random field. Indeed, if X is weakly stationary, its covariance function r is an even function, whence (assuming that $r_{i,j,m}^{(3)}(0)$ exists) $r_i'(0) = r_{i,j,m}^{(3)}(0) = 0$, which entails

$$\forall t \in \mathbb{R}^d, \quad \mathbb{E}[X_i'(t)\, X(t)] = \mathbb{E}[X_{i,j}''(t)\, X_m'(t)] = 0. \tag{4}$$

Thus, if X is centred, stationary and Gaussian, at any $t \in \mathbb{R}^d$, $X(t)$ is independent of any of its first-order partial derivatives at point t, and the same holds for a first-order partial derivative at point t with any second-order partial derivative at point t. This will make simpler some of our computations in Chapter II and in Chapter IV.

In the weakly stationary and isotropic case, other simplifications occur because r is a radial function: for any $(i,j) \in \{1,\cdots,d\}^2$,

$$-\mathbb{E}[X_{ij}''(t)\, X(t)] = \mathbb{E}[X_i'(t)\, X_j'(t)] = -r_{i,j}''(0) = \lambda_2\, \delta_i^j, \tag{5}$$

where δ is here the Kronecker delta and $\lambda_2 \geq 0$ will be defined in the next section as the second-order spectral moment. Consequently, if X is Gaussian and centred, the first-order partial derivatives at a fixed point are independent of one another.

To end with, there is no converse implication to the one of Proposition I.5, that is mean square regularity does not imply almost sure regularity. However, it is possible to deduce from the L^2-regularity of X the existence of a version of X that is almost surely regular, under certain conditions. In particular, the following proposition concerns $\mathscr{C}^\infty$-regularity and it will be used in Chapter IV Section IV.1.3.

Proposition I.7 *Let $C\colon \mathbb{R}^d \to \mathbb{R}$ be a covariance function of class $\mathscr{C}^\infty$. Then there exists a Gaussian centred random field with covariance function C that is almost surely of class $\mathscr{C}^\infty$.*

Spectral representation of stationary random fields

A theorem due to Bochner and dating back from 1933 (see [Boc33]) provides an integral expression of the covariance function of a weakly stationary random field. Originally designed in the setting of Fourier analysis, it applies to any complex-valued positive semidefinite function on $\mathbb{R}^d$, hence exactly to covariance functions of complex-valued, weakly stationary random fields. Thus this section concerns complex-valued weakly stationary random fields on $\mathbb{R}^d$, although we will only be interested in the real-valued random fields in the following chapters. Let us recall that a function $r\colon \mathbb{R}^d \to \mathbb{C}$ is said to be positive semidefinite if

$$\forall n \in \mathbb{N},\ \forall (t_i)_{i\in\{1,\cdots,n\}} \in \mathbb{R}^{dn},\ \forall (z_i)_{i\in\{1\cdots,n\}} \in \mathbb{C}^n, \quad \sum_{i=1}^{n}\sum_{j=1}^{n} z_i\, r(t_i - t_j)\, \bar{z_j} \geq 0.$$

Theorem I.8 (Bochner theorem.) *A continuous function $r\colon \mathbb{R}^d \to \mathbb{R}$ is positive semidefinite if and only if there exists a finite measure F on the Borel σ-algebra $\mathscr{B}(\mathbb{R}^d)$ such that*

$$\forall t \in \mathbb{R}^d, \quad r(t) = \int_{\mathbb{R}^d} e^{it\cdot\lambda}\, dF(\lambda). \tag{6}$$

Moreover, if F exists, it is unique.

If function r is the covariance function of a weakly stationary random field X then F is called the spectral measure of X. Note that $r(0) = F(\mathbb{R}^d)$. Moreover, if F admits a density $f\colon \mathbb{R}^d \to \mathbb{R}$ with respect to the Lebesgue measure in $\mathbb{R}^d$ then f is called the spectral density of X.

In the real case (which we are going to stick to), r is even. Hence, the uniqueness of the spectral measure yields that it is a symmetric measure (that is, for any $B \in \mathscr{B}(\mathbb{R}^d)$, $F(-B) = F(B)$). Therefore, if the spectral density exists, it is an even function. Note also that (6) may be transformed into

$$\forall t \in \mathbb{R}^d,\ r(t) = \frac{1}{2}(r(t) + r(-t)) = \int_{\mathbb{R}^d} \cos(t\cdot\lambda)\, dF(\lambda).$$

If X is weakly stationary and weakly isotropic, r is radial, which entails that F is invariant under rotations in $\mathbb{R}^d$ and that f is radial if it exists. Reciprocally, if F is invariant under rotations then the weakly stationary random field X is weakly isotropic.

The spectral moments are parameters of the random field arising with the spectral representation. Let F be the spectral measure of the weakly stationary random field $X\colon \mathbb{R}^d \to \mathbb{R}$. Let $\mathbf{j} \in \mathbb{N}_0^d$. If $\lambda \mapsto \lambda^{\mathbf{j}}$ is integrable with respect to measure F on $\mathbb{R}^d$ then

$$\int_{\mathbb{R}^d} \lambda^{\mathbf{j}}\, dF(\lambda)$$

defines a spectral moment of X denoted by $\lambda_{\mathbf{j}}$. Its order is $|\mathbf{j}|$. Because of the symmetry of F, if an odd-order spectral moment exists, it is necessarily zero.

The spectral moments may be expressed as some partial derivatives at zero of the covariance function r and thus their existence is linked to the L^2-regularity of X, according to Proposition I.6. More precisely, let $\mathbf{j} \in \mathbb{N}_0^d$; assuming that $\dfrac{\partial^{|\mathbf{j}|} r}{\partial t^{\mathbf{j}}}$ exists, Formula (6) yields

$$\forall \mathbf{j} \in \mathbb{N}_0^d, \quad \frac{\partial^{|\mathbf{j}|} r}{\partial t^{\mathbf{j}}}(0) = i^{|\mathbf{j}|}\lambda_{\mathbf{j}},$$

the two terms being equal to zero if $|\mathbf{j}|$ is odd.

According to Formula (3), these quantities also correspond, up to a factor ± 1, to the covariances between mean square partial derivatives of the random field at a fixed point, if it is centred.

Therefore, it occurs that in the weakly stationary and isotropic case, some of the spectral moments are zero, because of the cancellation of some derivatives of r at zero, already explained in the previous section (see for instance Formulas (4) and (5)).

If X is a weakly isotropic random field, F is invariant under rotations, thus we may define the second-order spectral moment of X as

$$\lambda_2 = \int_{\mathbb{R}^d} \lambda_j^2\, dF(\lambda), \quad \forall j \in \{1, \cdots, d\},$$

the moments $\int_{\mathbb{R}^d} \lambda_i \lambda_j \, dF(\lambda)$ being equal to zero if $i \neq j$. The second-order spectral moment λ_2 is also the variance of any first-order partial derivative of X, thus it describes the variability of the velocity of change of X in the neighborhood of any point in $\mathbb{R}^d$. In this case, the matrix of the second-order moments of X

$$\left(\int_{\mathbb{R}^d} \lambda_i \lambda_j \, dF(\lambda) \right)_{1 \leq i,j \leq d}$$

is simply $\lambda_2 \, I_d$.

Now, let us turn to the spectral reprentation of a weakly stationary random field that can be obtained thanks to Bochner theorem. To state it, we need to define a complex noise based on a measure on $\mathbb{R}^d$. We restrict ourselves to the Gaussian case, for we will only need the spectral representation of Gaussian stationary random fields in Section IV.1.2.

Let F be a σ-finite measure on $\mathbb{R}^d$. A complex Gaussian F-noise on $\mathbb{R}^d$ is a $\mathbb{C}$-valued process W_F defined on $\mathscr{B}(\mathbb{R}^d)$ such that

- *a.s.* W_F is a complex-valued measure on $\mathscr{B}(\mathbb{R}^d)$,
- $\forall A \in \mathscr{B}(\mathbb{R}^d)$, $W_F(A)$ is a complex-valued Gaussian variable with $\mathbb{E}[W_F(A)] = 0$ and $\mathbb{E}[W_F(A)\overline{W_F(A)}] = F(A)$,
- for any sequence $(A_n)_n$ of pairwise disjoint Borel sets, $(W_F(A_n))_n$ are independent random variables.

Then it is possible to define an integration with respect to the F-noise W_F, for all deterministic complex functions that are square integrable with respect to measure F. In particular, if f and g are two complex square integrable functions with respect to measure F, then

$$\mathbb{E}\left[\int_{\mathbb{R}^d} f(\lambda) \, dW_F(\lambda) \right] = 0$$

$$\mathbb{E}\left[\int_{\mathbb{R}^d} f(\lambda) \, dW_F(\lambda) \overline{\int_{\mathbb{R}^d} g(\lambda) \, dW_F(\lambda)} \right] = \int_{\mathbb{R}^d} f(t) \, \overline{g(t)} \, dF(t)$$

Now we are able to state the representation theorem.

Theorem I.9 (Spectral representation theorem.) *Let F be a finite measure on the Borel σ-field $\mathscr{B}(\mathbb{R}^d)$ of $\mathbb{R}^d$ and let W_F be a Gaussian F-noise. Then the complex-valued random field*

$$\left(\int_{\mathbb{R}^d} e^{it \cdot \lambda} \, dW_F(\lambda) \right)_{t \in \mathbb{R}^d}. \tag{7}$$

is centred, stationary, Gaussian, and it admits the covariance function defined by (6)*.*

Reciprocally, let X be a complex-valued, mean square continuous, centred, stationary and Gaussian random field with covariance function r defined by (6)*. Then there exists a complex F-noise W_F such that X admits the representation* (7)*.*

For the representation of real-valued random fields, the above theorem is still correct as soon as we adapt the definition of the complex F-noise W_F at stake, so that it satisfies

$$\forall A \in \mathscr{B}(\mathbb{R}^d), \quad \overline{W_F(A)} = W_F(-A).$$

Spectral method for simulation of Gaussian random fields

This section is about one of the methods used to produce approximate simulate realizations of Gaussian random fields, called the spectral method and designed by Shinozuka and Jan (see [Shi71] and [SJ72]). It was used in this thesis to produce the simulations of realizations of Gaussian random fields which illustrate Chapters I and III.

Let $X\colon \mathbb{R}^d \to \mathbb{R}$ be a Gaussian stationary centred random field with covariance function r and with variance equal to one. (If $m \neq 0$ and $r(0) \neq 1$, the following method allows us to simulate $r(0)^{-1/2}(X - m)$.) Let F be the spectral measure of X. Since $r(0) = 1$, F is a probability measure on $\mathbb{R}^d$. Let $\mathbf{k}$ be a random variable in $\mathbb{R}^d$ with probability measure F. Let η be a uniform random variable on $[0, 2\pi]$, such that η and $\mathbf{k}$ are independent. The random field Z defined by

$$\forall t \in \mathbb{R}^d,\ Z(t) = \sqrt{2}\cos(\mathbf{k} \cdot t + \eta)$$

is centred, stationary and it admits the same covariance function as X. Note that the random field Z corresponds to a random wave model with random wavevector $\mathbf{k}$ and that this model is the purpose of Chapter IV.

Now, we consider independent and identically distributed versions of η and of $\mathbf{k}$, denoted respectively by $(\eta_j)_{j\in\mathbb{N}}$ and by $(\mathbf{k}_j)_{j\in\mathbb{N}}$. According to the central limit theorem applied to finite-dimensional distributions, the distribution of

$$\left(\sqrt{\frac{2}{N}}\sum_{j=1}^{N}\cos(\mathbf{k}_j \cdot t + \eta_j)\right)_{t\in\mathbb{R}^d}$$

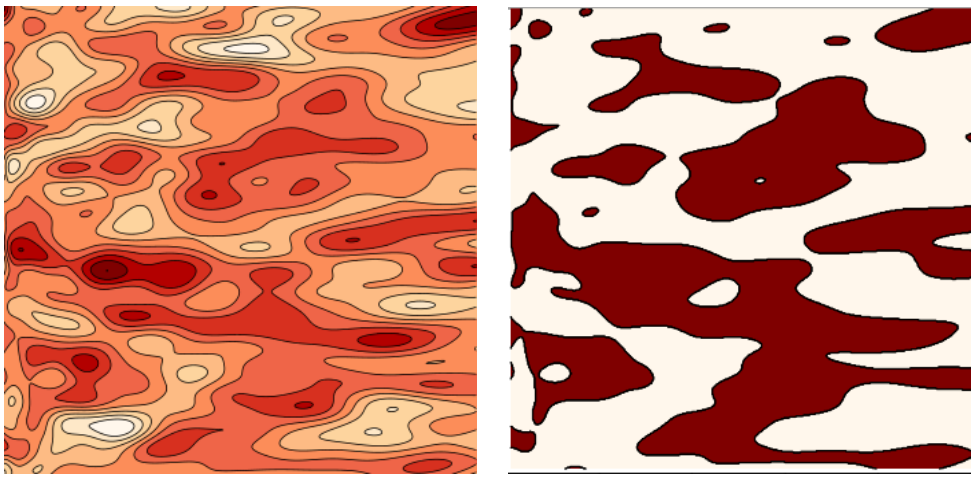
converges as N tends to ∞ towards the distribution of a centred and stationary Gaussian random field with covariance function r, that is exactly the law of X.

Through the choice of a large enough N, this method allows to produce approximate simulations of realizations of X. The only restriction of this method is that it requires to be able to simulate a random variable distributed according to law F.

Gaussian deformed random fields, studied in Chapter III, can be simulated thanks to the spectral method. This kind of random field is of the form $X \circ \theta$, where $X\colon \mathbb{R}^2 \to \mathbb{R}$ is a centred, stationary and isotropic Gaussian random field and where $\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a bijective mapping. Generally speaking, $X \circ \theta$ is non stationary and anisotropic. Our simulation of a realization of $X \circ \theta$ on a fixed window requires to fix a grid on it. Sequences of realizations of random variables denoted above by $(\mathbf{k}_j)_{1\leq j\leq N}$ and $(\eta_j)_{1\leq j\leq N}$ are simulated. Then, at each vertex t of the grid, the spectral method allows to compute an approximate realization of the random variable $X(\theta(t))$.

Some geometric characteristics

This section is dedicated to three characteristics that can be used to study a real random field with a geometric perspective: the number of critical points, the Hausdorff measure of level sets and the Euler characteristic of excursion sets. In this thesis, these characteristics are random variables but originally, they were introduced and studied in a deterministic framework. Consequently, some of the content of this section is deterministic. In the random fields framework, the exact distribution of these random variables is out of reach. However, under certain assumptions on the considered random field, moments of these geometric characteristics are given by rather tractable formulas. We will particularly focus on their expectation, sometimes supplemented with their variance, which may be expressed thanks to the Rice formulas.



(a) Level sets. (b) Excursion set.

Figure I.2 – A realization of a non stationary and anisotropic random field: level sets and excursion set.

Rice formulas

Let $d \geq d'$ be two integers. Let $Z\colon \mathbb{R}^d \to \mathbb{R}^{d'}$ be an a.s. of class $\mathscr{C}^1$ random field and let $v \in \mathbb{R}^{d'}$. For a fixed $v \in \mathbb{R}^{d'}$, $Z^{-1}(\{v\})$ is the level set of Z at level v. Provided that a.s. for any $t \in Z^{-1}(\{v\})$, the Jacobian matrix of Z at t is of rank d', $Z^{-1}(\{v\})$ is a $\mathscr{C}^1$-submanifold of dimension $d-d'$ in $\mathbb{R}^d$. Recall that $\mathscr{H}_{d-d'}$ is a notation for the $(d-d')$-Hausdorff measure. If T is a compact set in $\mathbb{R}^d$, we consider its intersection with $Z^{-1}(\{v\})$. Under certain assumptions on Z, Rice formulas provide expressions for the moments of random variables that are integrals of random fields over the level set $Z^{-1}(\{v\}) \cap T$. More precisely, if $G\colon \mathbb{R}^d \to \mathbb{R}$ is a random field that may depend on Z, the expectation of

$$\int_{Z^{-1}(\{v\})\cap T} G(t)\, d\mathscr{H}_{d-d'}(t), \tag{8}$$

can be expressed as an integral on T, the integrated function being an expectation conditioned on the event $Z(t) = v$, multiplied by the density of the random variable $Z(t)$ at point v. Similar formulas exist for the higher moments of this random variable. The simplest and yet very useful case consists in G being a.s. constant equal to one, so that (8) is the $(d - d')$-Hausdorff measure of the level set $Z^{-1}(\{v\}) \cap T$.

These kind of formulas were developed by Rice and Kac independently in the forties, in dimension one ([Kac43] and [Ric44] were their first contributions). The former considered the number of crossings of a stationary Gaussian process and applied its results to telecommunications. The latter was interested in the number of roots of random polynomials with coefficients independently distributed according to the standard Gaussian law. Because of their dual origins, these formulas are often called Kac-Rice formulas. The formula was then generalized to different kinds of Gaussian processes. In the fifties, Longuett-Higgins brought the first contributions to the multiparameter case in the domain of physical oceanography (see [LH57] for instance). A general presentation of Rice formulas in the random fields setting was done by Adler in [Adl81]. The formulas were also extended to some non-Gaussian models. To prove them, one can start in a deterministic framework, using the area formula (for the case $d = d'$) and the coarea formula (for the case $d > d'$), which are change of variable formulas for multiple integrals (see [BLL17] for a thorough presentation).

We give two different versions of Rice formulas. The first one concerns the case $d = d'$, and contains both an expectation and a second factorial moment formula. The second one allows to handle the case $d \geq d'$. We state versions of the Rice formulas that will meet our future needs. Their proofs can be found in [AW05] (Theorems 6.3, 6.4, and 6.8).

Theorem I.10 (Rice formula, case $\mathbf{d = d'}$.) *Let $(d, k) \in \mathbb{N}^2$ and let $v \in \mathbb{R}^d$. Let U an open subset of $\mathbb{R}^d$, T a compact subset of U and $B \in \mathscr{B}(\mathbb{R}^k)$. Let $Z \colon \mathbb{R}^d \to \mathbb{R}^d$ and $Y \colon \mathbb{R}^d \to \mathbb{R}^k$ be random fields such that*

- *Z is a.s. of class $\mathscr{C}^1$ on $\mathbb{R}^d$ and Y is a.s. continuous on $\mathbb{R}^d$,*
- *the vector-valued random field $(Z, Y) \colon \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^k$ is Gaussian,*
- *$\mathbb{P}(\exists t \in U \colon Z(t) = v,\, \det(Z'(t)) = 0) = 0$.*

1. *If for any $t \in U$, the Gaussian vector $Z(t)$ is non degenerate, then*

$$\mathbb{E}[\sum_{t \in Z^{-1}(\{v\}) \cap T} \mathbb{1}_B(Y(t))] = \int_T \mathbb{E}[\mathbb{1}_B(Y(t)) \,|\, \det(Z'(t))| \,|\, Z(t) = v]\, p_{Z(t)}(v)\, dt, \tag{9}$$

   and in the case where for any $t \in T$, $\mathbb{1}_B(Y(t)) = 1$, this integral is finite.

2. *If for any $(s, t) \in U^2$ such that $s \neq t$, the Gaussian vector $(Z(s), Z(t))$ is non degenerate, then*

$$\mathbb{E}\left[\sum_{t \in Z^{-1}(\{v\}) \cap T} \mathbb{1}_B(Y(t)) \left(\left(\sum_{t \in Z^{-1}(\{v\}) \cap T} \mathbb{1}_B(Y(t))\right) - 1\right)\right] =$$

$$\int_{T \times T} \mathbb{E}[\mathbb{1}_B(Y(s))\, \mathbb{1}_B(Y(t)) \,|\, \det(Z'(s))\, \det(Z'(t))| \,|Z(s) = Z(t) = v]\, p_{(Z(s), Z(t))}(v, v)\, ds\, dt,$$

and in the case where for any $t \in T$, $\mathbb{1}_B(Y(t)) = 1$, both sides may be infinite.

Remark I.11 *In the above theorem, if Z is furthermore stationary and if $Z'(0)$ is a non-degenerate Gaussian vector, then $\mathbb{P}(\exists t \in U : Z(t) = v, \det(Z'(t)) = 0) = 0$. This is a simple consequence of Proposition 6.5 in [AW05].*

In Chapter II, we consider the number of points in a compact set T in $\mathbb{R}^d$ where the gradient of a Gaussian and stationary random field $X : \mathbb{R}^d \to \mathbb{R}$ takes a fixed value $v \in \mathbb{R}^d$. Under the condition that a.s. for any $t \in \mathbb{R}^d$, the Hessian matrix $X''(t)$ is of rank d, $(X')^{-1}(\{v\})$ is a manifold of dimension 0 in $\mathbb{R}^d$. Then $(X')^{-1}(\{v\}) \cap T$ is a.s. a discrete set and it is finite if T is bounded. If $v = 0$, it is the set of critical points of X in T. Assuming that the hypothesis of Theorem I.10 are satisfied by the gradient field X' and applying it with trivial choices for Y and B (for instance Y a.s. a real constant and $B = \mathbb{R}$), we obtain expressions of the expectation and the variance of the cardinal of the random set $(X')^{-1}(\{v\}) \cap T$. The above theorem also comes into play in a less trivial setting in the proof of a formula for the expectation of the Euler characteristic of an excursion set, as we will explain in Section I.2.2.c.

Now we turn to a second Rice formula, which will allow us to deal with the case where the dimension d' of the space where the considered random field takes value is stricty less than the dimension d of the parameter space. We introduce

$$H : \mathbb{R}^d \times \mathbb{R}^{d'} \to [0, +\infty) \quad A \mapsto (\det(AA^T))^{1/2}.$$

Thus if $d = d'$, $H = |\det(\cdot)|$ and if $d' = 1$, H is the Euclidian norm in $\mathbb{R}^d$.

Theorem I.12 (Rice formula, case $\mathbf{d \geq d'}$) *Let $d \geq d'$. Let $Z : \mathbb{R}^d \to \mathbb{R}^{d'}$ be a Gaussian random field. We assume that Z is almost surely of class $\mathscr{C}^2$ and that for any $t \in \mathbb{R}^d$, the Gaussian vector $(Z(t), Z'(t))$ is not degenerate. For any $v \in \mathbb{R}^{d'}$ and for $T \in \mathscr{B}(\mathbb{R}^d)$, we write $l(v, Z, T)$ the $(d - d')$-Hausdorff measure of $Z^{-1}(\{v\}) \cap T$. Then*

$$\mathbb{E}[l(v, Z, T)] = \int_T \mathbb{E}[H(Z'(t)) | Z(t) = v] \, p_{Z(t)}(v) \, dt,$$

where both sides are finite if T is a compact set.

Note that the above assumptions are more restrictive than in the case $d = d'$. The proof of this theorem can be found in [AW05] (see Theorem 6.8 and Proposition 6.12). For instance, let us consider an a.s. of class $\mathscr{C}^2$, Gaussian random field $X : \mathbb{R}^d \to \mathbb{R}$, a compact set T in $\mathbb{R}^d$ and a real v. If a.s. for any $t \in X^{-1}(\{v\})$, $X'(t) \neq 0$, the level set $X^{-1}(\{v\})$ is a.s. a $\mathscr{C}^2$-manifold of dimension $d - 1$ in $\mathbb{R}^d$. If $v = 0$, it is called the nodal set of X. Provided that X satisfies the assumptions of Theorem I.12, we obtain the following expectation formula:

$$\mathbb{E}\left[l(v, X, T)\right] = \int_T \mathbb{E}[\|X'(t)\| \mid X(t) = v] \, p_{X(t)}(v) \, dt.$$

We apply it in Chapter IV to specific models of anisotropic random waves.

Euler characteristic of excursion sets

The last geometric characteristic that we present here is the Euler characteristic of excursion sets. Let $X\colon \mathbb{R}^d \to \mathbb{R}$ an a.s. continous random field, let T a subset of $\mathbb{R}^d$ and let u a fixed real level. We write

$$A(u, X, T) = \{t \in T \ / \ X(t) \geq u\},$$

the excursion set of X restricted to T above level u, that is $X^{-1}([u, +\infty)) \cap T$. If T is compact, then so is $A(u, X, T)$. Figure I.2a shows a simulation of the realization of a random field's level sets, while a corresponding excursion set appears on Figure I.2b.

Although the distribution of the excursion sets of a fixed random field is mathematically out of reach, it may be possible to study the distribution of some of their characteristics such as their measure, the measure of their frontier, or their Euler characteristic. Concerning their frontier, if T is an open bounded subset of $\mathbb{R}^d$ and if for instance X admits no local minimum in $X^{-1}(\{u\})$, then the frontier of $A(u, X, T)$ is the level curve $X^{-1}(\{u\}) \cap T$, whose expectation of length can be expressed, under appropriate assumptions, thanks to Theorem I.12. We are going to focus on the Euler characteristic of excursion sets, which describes their topology.

In many applicative fields, it is important to know the topology of excursion sets. For instance, at a microscopic scale, in material science, the excursion sets of a metallic surface impacts its physical properties. At a macroscopic scale, the surface of the ocean may also be studied by considering its excursion sets. A naïve explanation in dimension two, taken form the introduction of |Adler1981], is that a surface can be described through the repartition and the size of its hills and valleys. To some extent, the Euler characteristic of excursion sets considered at different levels provides these information.

This section begins in a deterministic setting. We provide a general definition of the Euler characteristic in Section I.2.2.a. In Section I.2.2.b, we start focusing on excursion sets and we explain how Morse theory comes into play to express their Euler characteristic, when the function satisfies some regularity properties. In Section I.2.2.c, returning to the random setting, we give explicit expressions of the expectation and the variance of excursion sets of random fields.

Definition and properties

The Euler characteristic, denoted by χ, and also called the Euler-Poincaré characteristic, is an integer-valued topological functional, used to describe the structure of a set. Here we wish to introduce the Euler characteristic in a general setting, before applying it exclusively to excursion sets. However, since we will not need its general definition in the upcoming chapters, we shall not go into details. Several definitions exist; we have chosen one from combinatorial geometry, based on triangulation.

Let us first recall that a simplex K in $\mathbb{R}^d$ is the convex hull of a finite number of affinely independent points $(A_i)_{0\leq i\leq n}$ in $\mathbb{R}^d$; its dimension is defined as n, necessarily less or equal to d. Simplexes of dimensions zero to four in $\mathbb{R}^d$ ($d \geq 3$) are respectively called vertices, edges, triangles and tetrahedrons. A face of the n-dimensional simplex K is a simplex generated by a subfamily of the points $(A_i)_{0\leq i\leq n}$.

Simplicial complexes are formed by joining together simplexes in an appropriate way. More precisely, a simplicial complex $\mathscr{S}$ is a finite collection of simplexes such that if K belongs to $\mathscr{S}$ then a face of K also belongs to $\mathscr{S}$ and if K and K' are in $\mathscr{S}$ then $K \cap K'$ is either empty or a face of both. Abusing the notation, we also call simplicial complex the geometric realization of $\mathscr{S}$ as a subset of $\mathbb{R}^d$.

The Euler characteristic of a simplicial complex $\mathscr{S}$ of dimension n is given by:

$$\chi(\mathscr{S}) = \sum_{i=0}^{n} (-1)^{n-i} \alpha_i(\mathscr{S}), \tag{10}$$

where $\alpha_i(\mathscr{S})$ gives the number of i-dimensional simplexes in $\mathscr{S}$. Thus the Euler characteristic is integer-valued. Also note that $\chi(\mathscr{S})$ is independent of the dimension of the ambiant space $\mathbb{R}^d$ in which $\mathscr{S}$ sits.

This definition may be extended to any triangulable subset M of $\mathbb{R}^d$. A subset M of $\mathbb{R}^d$ is triangulable if there exists a simplicial complex $\mathscr{S}$ and a homeomorphism between M and $\mathscr{S}$. Let $\mathscr{S}_M$ be a simplicial complex associated with a triangulable subset $M \in \mathbb{R}^d$ through a triangulation. The Euler characteristic of M is defined by $\chi(M) = \chi(\mathscr{S}_M)$, which is given by (10) and is proved not to depend on the choice of the triangulation.

Let us write this formula in small dimensions. Let M be a triangulable subset of $\mathbb{R}^d$ of dimension $N \in \{1, 2, 3\}$. The Euler characteristic is obtained by counting the number of vertices, edges and faces in any triangulation of M:

$$\begin{aligned} &\text{if } \dim(M) = 1, \quad \chi(M) = \#\text{edges} - \#\text{vertices}; \\ &\text{if } \dim(M) = 2, \quad \chi(M) = \#\text{faces} - \#\text{edges} + \#\text{vertices}. \end{aligned} \tag{11}$$

Descartes (around 1640) and Euler (in 1752) were the first to notice the existence of such a characteristic, proving that for convex polyhedra, the alternating sum (11) was equal to 2, but that it was not necessarily the case for non-convex polyhedra.

The Euler characteristic satisfies two crucial properties.

Proposition I.13 *1. The Euler characteristic is an additive functional: if A and B are two triangulable subsets of $\mathbb{R}^d$ then*

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

2. The Euler characteristic is homotopy invariant.

For more details about the second property, we refer to [AT07] Definitions 6.1.3 and 6.1.4, and to [Lee00] Theorem 13.36. In the following section, we are going to exploit one of its consequences: if A and B are two triangulable subsets in $\mathbb{R}^d$ such that B is image of A through a bijective and bicontinuous mapping, then $\chi(A) = \chi(B)$. A non-rigorous explanation of the homotopy property is that the Euler characteristic characterizes the general shape of a set.

Through simple triangulations, we obtain that the Euler characteristic of the circle $\mathbb{S}^1$ is zero, the one of a disk is one, the one of the sphere $\mathbb{S}^2$ is 2 and the one of a three-dimensional ball is one. The following heuristic formulas use non-mathematical

vocabulary with intuitive meaning. It may be understood thanks to the above simple examples and the additivity property. If M is a triangulable set,

$$\begin{aligned}
&\text{if } \dim(M) = 1, \quad \chi(M) = \#\text{connected components in } M\\
&\text{if } \dim(M) = 2, \quad \chi(M) = \#\text{connected components} - \#\text{holes in } M\\
&\text{if } \dim(M) = 3, \quad \chi(M) = \#\text{connected components} - \#\text{holes} + \#\text{handles in } M
\end{aligned}$$

The Euler characteristic is one in a finite sequence of functionals that allow to describe the geometry of a set. They are called the intrinsic volumes and are first defined for convex sets. The higher-order intrinsic volumes do not satisfy the homotopy property. The Minkowski functionals form another sequence of functionals that are very close to the intrinsic volumes, since they are equal up to a choice of constants and a renumbering. The intrinsic volumes can be generalized to manifolds and in this setting they are sometimes renamed Lipschitz-Killing curvatures. For a n-dimensional rectangle, which can be written, up to a proper choice of basis, $T = \prod_{i=1}^{n}[0, T_i]$, its intrinsic volumes are defined by

$$\forall j \in \{0, \cdots, n\}, \quad \mathscr{L}_j(T) = \sum T_{i_1} \cdots T_{i_j}, \tag{12}$$

where the sum is take over the $\binom{n}{j}$ choices of the indices $\{i_1, \cdots, i_j\}$. With this definition, $\mathscr{L}_0(T) = 1 = \chi(T)$. Moreover, in small dimensions, we obtain

$$\begin{cases}
\mathscr{L}_1(T) = |T|_1 & \text{if } T \text{ is a segment,}\\
\mathscr{L}_2(T) = |T|_2,\ \mathscr{L}_1(T) = \dfrac{1}{2}|\partial T|_1 & \text{if } T \text{ is a rectangle,}
\end{cases}$$

where the notations used have been introduced at the beginning of the chapter. This can be extended to sets of the form $\theta(T)$, where T is a segment or a rectangle in $\mathbb{R}^2$ and $\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a $\mathscr{C}^2$-diffeomorphism, which will be at stake in Theorem I.17.

Now we return to the random setting until the end of the paragraph, in order to explain one of the reasons why the Euler characteristic started to be applied to the excursion sets of random fields. This tradition takes its roots in the so-called expected Euler characteristic heuristic in extremal theory. This heuristic provides a way to approximate the excursion probability at a large level (see for instance [TTA05]).

Informally, if X is a smooth random field on a planar compact domain T, for large u, it is likely that the excursion set $A(u, X, T)$ has only one component (if $\sup_{t\in T} X(t) \geq u$) or zero component (if $\sup_{t\in T} X(t) < u$). These cases correspond respectively to $\chi(A(u, X, T)) = 1$ and to $\chi(A(u, X, T)) = 0$. Consequently, for large u, $\mathbb{1}(\sup_{t\in T} X(t) \geq u) \approx \chi(A(u, X, T))$. Taking the expectation of both sides of this equality yields

$$\mathbb{P}(\sup_{t\in T} X(t) \geq u) \approx \mathbb{E}[\chi(A(u, X, T)].$$

For some models of random fields (mainly Gaussian random fields with enough regularity), this heuristic is a proved fact, and the error resulting from such an estimate can be evaluated (see [AT07] Section 14.3).

Euler characteristics of excursion sets

In the deterministic framework, an expression of the Euler characteristic of an excursion set is provided by Morse formula. Morse theory, also known as the critical point theory was developed during the twentieth century. It allows to express some geometrical characteristics of a manifold by considering the different types of critical points of a function defined on it (see for instance the original treatise [MC69]).

Let us introduce some notations to state Morse formula for the Euler characteristic of the excursion of a function above a d-dimensional rectangle T in $\mathbb{R}^d$. There exists an orthonormal basis of $\mathbb{R}^d$ where T can be written

$$T = \prod_{i=1}^{d} [s_i, t_i], \quad \text{where for any } 1 \leq i \leq d,\ -\infty < s_i < t_i < +\infty.$$

For $\ell \in \{1, \cdots, d\}$, a ℓ-dimensional face J of T is defined by a subset $\sigma(J)$ of size ℓ and by a sequence of $d - \ell$ zeros and ones, denoted by $\varepsilon(J) = (\varepsilon_i)_{i \notin \sigma(J)}$, such that

$$J = \{v \in T \colon \forall i \in \sigma(J),\ s_i < v_i < t_i,\ \forall i \notin \sigma(J),\ v_i = (1 - \varepsilon_i)s_i + \varepsilon_i t_i\}.$$

For $\ell \in \{1, \cdots, d\}$, the notation $\partial_\ell T$ means the collection of all the ℓ-dimensional faces of T. For instance, the unique d-dimensional face of T is its interior $\mathring{T}$, for which $\sigma(\mathring{T}) = \{1, \cdots, d\}$. Its one-dimensional faces are its open edges. A zero-dimensional face J is formed of a vertex of T, for which $\sigma(J) = \emptyset$. The frontier of T is given by the disjoint union

$$\partial T = \bigcup_{\ell=0}^{d-1} \bigcup_{J \in \partial_\ell T} J.$$

We need to consider functions with enough regularity, in order that the Euler characteristic of their excursions sets is well-defined. Note that, even though the compact set above which an excursion is considered is smooth, with smooth boundaries, the excursion set may not be as regular. Let $f \colon \mathbb{R}^d \to \mathbb{R}$, let T a rectangle in $\mathbb{R}^d$ and let $u \in \mathbb{R}$. Following [AT07] Definition 6.2.1, we write Condition $(\mathbf{R}_{u,T})$ the collection of the three following conditions.

1. f is of class $\mathscr{C}^2$ on $\mathbb{R}^d$;
2. for each face J of T such that $d \in \sigma(J)$, $f_{|J}$ has no critical point in $f^{-1}(\{u\})$;
3. for each face J of T such that $d \in \sigma(J)$, the matrix $(f''_{i,j})_{(i,j) \in (\sigma(J) \backslash \{d\})^2}$ denoted by Δ_J is such that

$$\#\{t \in J \colon \forall j \in \sigma(J) \backslash \{d\},\ 0 = f(t) - u = \det(\Delta_J) = f'_j(t)\} = 0.$$

If S is a symmetric matrix, recall that we denote by $\mathrm{index}(S)$ the number of its negative eigenvalues.

Theorem I.14 (Morse formula) *Let T be a d-dimensional rectangle in $\mathbb{R}^d$, let $u \in \mathbb{R}$ and let $f \colon \mathbb{R}^d \to \mathbb{R}$ satisfying condition $(\mathbf{R}_{u,T})$.* The Euler characteristic of $A(u, f, T)$ is given by

$$\chi(A(u, f, T)) = \sum_{0 \leq \ell \leq d} \sum_{J \in \partial_\ell T} \psi(J) \quad \text{with} \quad \psi(J) = \sum_{k=0}^{\ell} (-1)^k \mu_k(J), \tag{13}$$

where for any ℓ-dimensional face J of T, for $k \in \{1, \cdots, \ell\}$,

$$\begin{aligned}\mu_k(J) = \#\{v \in J :\ & f(v) \geq u,\ \forall i \in \sigma(J),\ f_i'(v) = 0,\\ & \forall i \notin \sigma(J),\ (2\varepsilon_i - 1) f_i'(v) > 0,\ \operatorname{index}((f_{i,j}''(v))_{i,j\in\sigma(J)}) = \ell - k\}.\end{aligned} \tag{14}$$

The proof of this formula, inspired by Morse's theorem, may be found in [AT07] Section 9.4. The term $\psi(\mathring{T})$ corresponding to $\ell = d$ is simply the alternate sum of different types of critical points of f restricted to $\mathring{T}$, classified according to the signs of the eigenvalues of the Hessian matrix of f at these points. If J is a face of T of dimension ℓ strictly less than d, it occurs that the $(\mu_k(J))_{1\leq k\leq \ell}$ also count critical points: those of the restriction of f to the face J. If T is a rectangle of dimension ℓ strictly less than d, we notice that $\chi(A(u, f, T)) = \chi(A(u, f_{|T}, T))$, thus we may apply Theorem I.14 to function $f_{|T}$ in dimension ℓ, in order to get a Morse formula for $\chi(A(u, f, T))$.

In Chapter III, we will need a little more general version of Theorem I.14. A generalized version of Morse formula exists, for excursion sets above stratified manifolds satisfying regularity assumptions (see [AT07] Theorem 9.3.2). However, it would have been too much of an investment to introduce all the framework for the general setting, given that we would finally only need to apply it to very simple manifolds. Indeed, the manifolds of the plane that we consider in Chapter III, are of the form $\theta(T)$, where $\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a deterministic $\mathscr{C}^2$-diffeomorphism and T is a rectangle or a segment in $\mathbb{R}^2$. Let us explain how to generalize Theorem I.14 to this setting with simple arguments.

Since θ is an homeomorphism (that is, a bijective and bicontinous mapping), if T is a rectangle in $\mathbb{R}^2$, then $A(u, f, \theta(T)) = \theta(A(u, f\circ\theta, T))$, therefore the sets $A(u, f, \theta(T))$ and $A(u, f \circ \theta, T)$ are homotopic. Due to the homotopy invariance of χ stated in Proposition I.13, the above relation leads to

$$\chi(A(u, f \circ \theta, T)) = \chi(A(u, f, \theta(T))), \tag{15}$$

and Theorem I.14 may be applied to function $f \circ \theta$ on T to obtain a formula for $\chi(A(u, f, \theta(T)))$. The same holds if T is a segment in $\mathbb{R}^2$.

Expectation formula

Let us come back to the random framework, by considering the excursion of a random field $X\colon \mathbb{R}^d \to \mathbb{R}$ restricted to a rectangle T in $\mathbb{R}^d$, above level u. We wish to get a formula for the expectation of $\chi(A(u, X, T))$. This is possible thanks to Rice formula, which can be applied to express the expectation of each of the terms involved in Morse formula (13), under certain assumptions on X including Gaussianity. In this section and the following, we also add the assumption of stationarity.

Let us explain in a few words how Theorem I.10 applies. We introduce the following regularity condition on a stationary Gaussian random field X with mean zero, satisfying almost sure $\mathscr{C}^2$-regularity. Let $r\colon \mathbb{R}^d \to \mathbb{R},\ t \mapsto \mathbb{E}[X(0)\,X(t)]$ be its covariance function.

$$(\mathbf{D})\colon \begin{cases} \exists \varepsilon > 0,\ \exists \alpha > 0,\ \exists K > 0,\ \forall t \in \mathbb{R}^d \\ \|t\| \leq \varepsilon \ \Rightarrow \max\limits_{1\leq i,j\leq d} \left| r^{(4)}_{i,i,j,j}(t) - r^{(4)}_{i,i,j,j}(0) \right| \leq K |\log(\|t\|)|^{-(1+\alpha)}. \end{cases}$$

We first need to check that the assumptions of Theorem I.14 are a.s. satisfied by X and thus that $\chi(A(u,X,T))$ may be expressed through Morse formula. Secondly, we wish to explain informally how Rice formula given by Theorem I.10 provides an expression for the expectation of each alternate term in the sum (13). For the sake of simplicity, we will focus on the terms related to the interior of T in Formula (13). The following lemma allows us to achieve both points. It is a consequence of a multidimensional version of Bulinskaya lemma (originally stated in [Bul61]) and its proof can be found in [AT07] Lemmas 11.2.10 to 11.2.12. We use notations introduced in the statement of Theorem I.14.

Lemma I.15 *Let T be a d-dimensional rectangle in $\mathbb{R}^d$ and let $u \in \mathbb{R}$. We assume that $X\colon \mathbb{R}^d \to \mathbb{R}$ is a Gaussian and stationary random field satisfying a.s. $\mathscr{C}^2$-regularity and Condition* (**D**)*. Then*

1. $\mathbb{P}(\exists t \in \partial T\colon X'(t) = 0) = 0$;
2. $\mathbb{P}(\exists t \in T\colon X'(t) = 0,\ \det(X''(t)) = 0) = 0$;
3. *almost surely, X satifies Condition* $(\mathbf{R}_{u,T})$.

In the following paragraph, we assume that X satisfies the hypothesis of Lemma I.15. Then the first item entails that for any $k \in \{1,\cdots,d\}$, a.s.

$$\mu_k(\mathring{T}) = \sum_{t \in (X')^{-1}(\{0\}) \cap T} \mathbb{1}_{[u,+\infty) \cap \mathfrak{S}_d^{d-k}}(X(t), X''(t)),$$

where we recall that $\mathfrak{S}_d^{d-k}$ is the set of symmetric matrices of size $d \times d$ with $d-k$ negative eigenvalues. Moreover, the second point allows us to apply to X' the expectation formula in Theorem I.10, if we furthermore assume that the Gaussian vector $(X_i'(0))_{1 \le i \le d}$ is not degenerate. More precisely, we set $v = 0$ in $\mathbb{R}^d$, $Z = X'$, $Y = (X, X'')$ and $B = [u,+\infty) \cap \mathfrak{S}_d^{d-k}$ in Formula (9).

$$\mathbb{E}[\mu_k(\mathring{T})] = \int_T \mathbb{E}[|\det(X''(t))|\, \mathbb{1}_{[u,+\infty)}(X(t))\, \mathbb{1}_{\mathfrak{S}_d^{d-k}}(X''(t)) \,|\, X'(t) = 0]\, p_{X'(t)}(0)\, dt, \tag{16}$$

where $p_{X'(t)}$ is the density of X' with respect to the Lebesgue measure on $\mathbb{R}^d$.

The above explanations give an idea of how to derive a formula for the expectation of $\chi(A(u,X,T))$. The complete proof of this result, which was completed by Adler and Hasofer in [AH76] and [Adl81], can be found in [AT07] (proof of Theorem 11.7.2). Their proof also involves to rescale the random field and to exploit the Gaussian assumption.

Before stating the expectation formula, we need to introduce some more notations. Let T be a d-dimensional rectangle in $\mathbb{R}^d$. Before stating the expectation formula in the stationary case, we may work with the image of T through a translation, hence we may assume that one vertex of T is the origin. For any $k \in \{1,\cdots,d\}$, we write $\mathscr{O}_k = \{J \in \partial_k T\colon 0 \in \overline{J}\}$, which is of cardinal $\binom{d}{k}$. If Λ is a matrix of size $d \times d$, for any face J of T, we write Λ_J the matrix $(\Lambda_{i,j})_{(i,j) \in \sigma(J)^2}$.

We also write $(H_i)_{i \in \mathbb{N}}$ the Hermite polynomials and, for any real x, $H_{-1}(x) = \sqrt{2\pi}\Psi(x)\exp(x^2/2)$, where we recall that Ψ is the tail probability of a standard Gaussian variable.

Theorem I.16 *Let $X\colon \mathbb{R}^d \to \mathbb{R}$ be an a.s. of class $\mathscr{C}^2$, centred, stationary and Gaussian random field. We write r its covariance function, $\sigma^2 = r(0)$ its variance and $\Lambda = -r''(0)$ the matrix of its second-order spectral moments. We assume that X is a.s. of class $\mathscr{C}^2$, that it satisfies condition* (**D**) *and that the Gaussian vector $(X'_i(0), X''_{i,j}(0))_{i\leq j,\,(i,j)\in\{1,2\}^2}$ is not degenerate.*

Then, for any level $u \in \mathbb{R}$, for any d-dimensional rectangle T,

$$\mathbb{E}[\chi(A(u,X,T))] = \sum_{0\leq \ell\leq d} \mathscr{L}_\ell^X(T)\, \rho_\ell(u/\sigma), \tag{17}$$

$$\textit{where} \quad \forall\, 0 \leq \ell \leq d,\ \rho_\ell\colon y \mapsto (2\pi)^{-(\ell+1)/2} H_{\ell-1}(y) e^{-y^2/2} \tag{18}$$

$$\textit{and} \quad \mathscr{L}_\ell^X(T) = \sum_{J\in\mathscr{O}_\ell} \frac{|J|_\ell\ \det(\Lambda_\ell)^{1/2}}{\sigma^\ell}. \tag{19}$$

This formula could surprise, since it shows that $\mathbb{E}[\chi(A(u,X,T))]$ uniquely depends on the covariance structure of X through its variance and its second-order moment matrix. Yet, the expectation of each $\mu_k(J)$ (given for instance by (16) for $J = \mathring{T}$) involves fourth-order spectral moments of X, due to the presence of the second-order partial derivatives of X in their definition (14). The reason why fourth-order spectral moments do not appear in the final result is that simplifications occur when for a fixed face J, the corresponding $(\mathbb{E}[\mu_k(J)])_{1\leq k\leq \dim(J)}$ are summed with alternate signs. Note that if the random field X is furthermore isotropic, then $\Lambda = \lambda_2\, I_d$, where λ_2 is the second-order spectral moment, whence for $\ell \in \{1\cdots, d\}$, $\det(\Lambda_\ell)^{1/2} = \lambda_2^{\ell/2}$.

Let us justify a choice of notation for $(\mathscr{L}_\ell^X(T))_{1\leq\ell\leq d}$ in (17) that is very close to the one used to denote the intrinsic volumes of T, the $(\mathscr{L}_\ell(T))_{1\leq\ell\leq d}$ given by (12). The $(\mathscr{L}_\ell^X(T))_{1\leq\ell\leq d}$ are actually called Lipschitz-Killing curvatures of T; their definition extends the one of the intrinsic volumes, in the sense that they are not computed with respect to the Euclidian metric on $\mathbb{R}^d$, but with respect to the metric induced by the random field X on T (for precise definitions, see [AT07] Section 12.2). In Theorem I.16, since the random field X is stationary, the metric it induces on T only depends on its variance and its second-order spectral moments.

Now, we state a result similar to Theorem I.16, corresponding to our needs in Chapter III. We focus on a stationary and Gaussian random field defined on the plane, adding the assumption of isotropy but generalizing a bit the setting of excursions above rectangles: we now want to deal with excursions above manifolds of the form $\theta(T)$, where T is a rectangle or a segment in $\mathbb{R}^2$, and $\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a $\mathscr{C}^2$-diffeomorphism. If M is a regular manifold, an expectation formula exists for $\chi(A(u,X,M))$ under certain regularity assumptions on X and M, but it is in general more complicated than the one of the rectangle case. It is stated in [AT07] Theorem 12.4.2 and it involves complex expressions for the intrinsic volumes $\mathscr{L}_\ell^X(M)$ relative to the Riemannian metric induced by X on the manifold M. However, if X is stationary and isotropic, these intrinsic volumes have very simple forms, since the Riemannian metric induced by X is the Euclidian metric, up to a multiplicative factor λ_2/σ^2 (see [AT07] Section 12.5: "Isotropic Fields over Smooth Domains"). However, this dependency is not visible in the following theorem since the variance as well as the second-order spectral moment

are assumed to be equal to one. Thus the Lipschitz-Killing curvatures involved are exactly the intrinsic volumes.

Theorem I.17 *Let $X\colon \mathbb{R}^2 \to \mathbb{R}$ be a centred, stationary and isotropic Gaussian random field, such that $r(0) = 1$, $(X_i'(0), X_{i,j}''(0))_{i\leq j,\,(i,j)\in\{1,2\}^2}$ is not degenerate and $r''(0) = -I_2$. We assume that X is a.s. of class $\mathscr{C}^2$ and that it satisfies condition* (**D**).

Then, for any level $u \in \mathbb{R}$, for any $\mathscr{C}^2$-diffeomorphism $\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$, for any segment or rectangle T in $\mathbb{R}^2$,

$$\mathbb{E}[\chi(A(u, X, \theta(T)))] = \sum_{0\leq i\leq \dim(T)} \mathscr{L}_i(\theta(T))\, \rho_i(u),$$

where functions $(\rho_i)_{1\leq i\leq \dim(T)}$ are given by (18) *and where*

$$\begin{cases} \mathscr{L}_1(\theta(T)) = |\theta(T)|_1,\ \mathscr{L}_0(\theta(T)) = 1 \quad \text{if } \dim(T) = 1, \\ \mathscr{L}_2(\theta(T)) = |\theta(T)|_2,\ \mathscr{L}_1(\theta(T)) = \dfrac{1}{2}|\partial\theta(T)|_1,\ \mathscr{L}_0(\theta(T)) = 1 \quad \text{if } \dim(T) = 2. \end{cases} \tag{20}$$

Modified Euler characteristic of excursion sets

While Morse formula stated in Theorem I.14 provides an expression of the Euler characteristic as the sum of different terms, it is sometimes easier to focus on the term of highest index, which only involves the behaviour of the function on the interior of the set above which the excursion is considered. That is what motivates the introduction of the modified Euler characteristic of excursion sets, denoted by ϕ, and only defined for excursion sets. It was introduced in [EL16] (where [AT07] Lemma 11.7.1 is acknowledged as a source of inspiration); it was then used in [DBEL17].

Definition I.18 *Let T a d-dimensional rectangle in $\mathbb{R}^d$, let $u \in \mathbb{R}$ and let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a function of class $\mathscr{C}^2$. The modified Euler characteristic of $A(u, f, T)$ is defined as*

$$\phi(A(u, f, T)) = \sum_{k=0}^{d} (-1)^k \widetilde{\mu}_k(u, f, T),$$

where for any $k \in \{1, \cdots, d\}$,

$$\widetilde{\mu}_k(u, f, T) = \#\{t \in T :\ f(t) \geq u,\ f'(t) = 0,\ \textit{index}\ (f''(t)) = d - k\}.$$

We extend this definition in order to be able to apply it to excursions of $f\colon \mathbb{R}^2 \to \mathbb{R}$ of class $\mathscr{C}^2$ above sets having the form $\theta(T)$, where $\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a deterministic $\mathscr{C}^2$-diffeomorphism and T is a rectangle or a segment in $\mathbb{R}^2$. Thus we define

$$\phi(A(u, f, \theta(T))) = \phi(A(u, f \circ \theta, T)). \tag{21}$$

Let us remark that the only difference between the $(\widetilde{\mu}_k(u, f, T))_{1\leq k\leq d}$ above and the $(\mu_k(\mathring{T}))_{1\leq k\leq d}$ involved in the term of highest index d in the first sum involved in Morse formula (13) is that the former count points situated in T while the latter count points situated in $\mathring{T}$. However, in our setting, $X\colon \mathbb{R}^d \to \mathbb{R}$ is a stationary

and Gaussian random field, a.s. of class $\mathscr{C}^2$ and it satisfies Condition (**D**). Hence the first point of Lemma I.15 applies and entails that almost surely, $\widetilde{\mu}_k(u,f,T) = \mu_k(\mathring{T})$. Hence, referring to a notation introduced in Theorem I.14, we obtain a.s. $\psi(\mathring{T}) = \phi(A(u,f,T))$. Thus, we can state the following corollary of Theorem I.14 and Definition I.18.

Corollary I.19 *Let T be a d-dimensional rectangle in $\mathbb{R}^d$ and let $u \in \mathbb{R}$. We assume that $X\colon \mathbb{R}^d \to \mathbb{R}$ is a stationary and Gaussian random field satisfying a.s. $\mathscr{C}^2$-regularity and Condition* (**D**)*. Then*

$$\chi(A(u,X,T)) = \sum_{0\le \ell < d-1} \sum_{J\in\partial_\ell T} \psi(J) \,+\, \phi(A(u,X,T)).$$

We end this section with expectation and variance formulas for the expected modified Euler characteristic of excursions sets. Under the same assumptions as the ones of Theorem I.16, as it appears in the proof of [AT07] Theorem 12.4.2, the term of highest index in the sum (17) in Theorem I.16 corresponds to the expectation of $\phi(A_u(X,T))$:

Corollary I.20 *Let T be a d-dimensional rectangle in $\mathbb{R}^d$ and let $u \in \mathbb{R}$. We assume that $X\colon \mathbb{R}^d \to \mathbb{R}$ is an a.s. of class $\mathscr{C}^2$, Gaussian and stationary random field, such that $(X_i'(0), X_{i,j}''(0))_{i\le j,\,(i,j)\in\{1,2\}^2}$ is not degenerate and Condition* (**D**) *is satisfied. We write r its covariance function, $\sigma^2 = r(0)$ its variance and $\Lambda = -r''(0)$ the matrix of its second-order spectral moments. Then*

$$\mathbb{E}[\phi(A(u,X,T))] = \mathscr{L}_d^X(T)\,\rho_d(u/\sigma),$$

where $\mathscr{L}_d^X(T)$ and ρ_d are given by (19) *and* (18) *respectively.*

Thanks to the extension (21) of Definition I.18 and to Formula (15) (due to the homotopy invariance of the Euler characteristic), Corollary I.19 extends to the case of the excursion of a random field $X\colon \mathbb{R}^2 \to \mathbb{R}$ above images of rectangles or segments under a diffeomorphism of the plane. Thus, following the proof of Theorem 12.4.2 in [AT07], it is clear that the rectangle case of Corollary I.20 admits the following generalization in dimension two.

Corollary I.21 *Let $X\colon \mathbb{R}^2 \to \mathbb{R}$ be a centred, stationary and isotropic Gaussian random field, such that $r(0) = 1$, $(X_i'(0), X_{i,j}''(0))_{i\le j,\,(i,j)\in\{1,2\}^2}$ is not degenerate and $r''(0) = -I_2$. We assume that X is a.s. of class $\mathscr{C}^2$ and that it satisfies condition* (**D**)*.*

Then, for any level $u \in \mathbb{R}$, for any $\mathscr{C}^2$-diffeomorphism θ, for any segment or rectangle T in $\mathbb{R}^2$,

$$\mathbb{E}[\phi(A(u,X,\theta(T)))] = \mathscr{L}_{\dim(T)}(\theta(T))\,\rho_{\dim(T)}(u),$$

where $\mathscr{L}_{\dim(T)}(\theta(T))$ is given by (20)*.*

To end with, coming back to the d-dimensional framework, we present a result stated in [DBEL17] Proposition 1. Furthermore assuming that the random field X is isotropic, that it satisfies a.s. $\mathscr{C}^3$-regularity and non-degeneracy conditions, the modified Euler characteristic of an excursion set of X over a d-dimensional square admits a finite second moment, which can be expressed through an integral formula.

Theorem I.22 *Let $X\colon \mathbb{R}^d \to \mathbb{R}$ be a centred, stationary and isotropic Gaussian random field such that for any $t \in \mathbb{R}^d$, the Gaussian vector $(X(0), X(t))$ is not degenerate. We also assume that X is unit-variance, that it satisfies a.s. $\mathscr{C}^3$-regularity and that its second-order spectral moment λ_2 is non zero.*

Let T be a d-dimensional square in $\mathbb{R}^d$ and let $u \in \mathbb{R}$. Then $\phi(A(u, X, T))$ admits a finite second moment, given by

$$\mathbb{E}[\phi(A(u,X,T))^2] = \int_{\mathbb{R}^d} |T \cap (T-t)|_d \, E(u,t) D(t)^{-1/2} \, dt \; + \; |T|_d \, (2\pi\lambda_2)^{-d/2} g(u),$$

where

$$\begin{aligned}
&E(u,t) = \mathbb{E}[\mathbf{1}_{[u,+\infty)}(X(0)) \mathbf{1}_{[u,+\infty)}(X(t)) |\det(X''(0)) \det(X''(t))| | X'(0) = X'(t) = 0], \\
&D(t) = (2\pi)^{2d} \det(\lambda_2^2 I_d - r''(t)^2), \\
&g(u) = \mathbb{E}[\mathbf{1}_{[u,+\infty)}(X(0)) |\det(X''(0))|)].
\end{aligned}$$

This theorem can be easily generalized by replacing T by a set of the form $\theta(T)$, where $\theta\colon \mathbb{R}^d \to \mathbb{R}^d$ is a deterministic $\mathscr{C}^2$-diffeomorphism and T is a d-dimensional rectangle. We will apply this generalized version in Section III.5.2 in dimension two. In the same chapter, Section III.3.2 contains other results about the modified Euler characteristic of excursion sets.

Description of our contributions

Coming to the end of this introductive chapter, we are now able to present the contributions of this thesis, which are divided into three parts.

Our first object of study is the number of points where the gradient of a stationary and Gaussian random field restricted to a compact set takes a fixed value. In particular, if this value is zero, what is at stake is the number of critical points of the random field. We investigate the existence of a second moment for this random variable. Let us write $X\colon \mathbb{R}^d \to \mathbb{R}$ a stationary Gaussian random field with a.s. $\mathscr{C}^2$-regularity, let T be a compact domain in $\mathbb{R}^d$ and let $v \in \mathbb{R}^d$. We write $N^{X'}(T, v) = \#(X')^{-1}(\{v\}) \cap T$. For $d = 1$, the so-called "Geman condition" has been proved to be a necessary and sufficient condition for $N^{X'}(T, v)$ to admit a finite second moment. This condition on the fourth derivative $r^{(4)}$ of the covariance function of X does not depend on v and requires $t \mapsto t^{-1}\left(r^{(4)}(0) - r^{(4)}(t)\right)$ to be integrable in a neighbourhood of zero. We prove that for $d \geq 1$, a generalization of the Geman condition remains a sufficient condition for $N^{X'}(T, v)$ to admit a second moment. More precisely, this generalized condition is the integrability of

$$t \mapsto \frac{\|r^{(4)}(0) - r^{(4)}(t)\|}{\|t\|^d}$$

in a neighbourhood of zero in $\mathbb{R}^d$. No assumption of isotropy is required. This result and its proof, derived with Anne Estrade, is the purpose of Chapter II, based on the published paper [EF16]. It will mainly use notions about the regularity of Gaussian random fields and Rice formulas.

The other parts of this thesis are dedicated to the study of two specific models of anisotropic random fields, conducted with a geometric perspective.

Chapter III deals with the deformed random field model in the planar setting. It is built thanks to a deterministic bijective mapping that deforms regularly the parameter space of a stationary and isotropic random field. We write $\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ a bijective mapping and $X\colon \mathbb{R}^2 \to \mathbb{R}$ a stationary and isotropic random field. The deformed field $X \circ \theta$ is in general not isotropic (and not even stationary). However we give an explicit characterization of the deformations θ that preserve the isotropy. Further assuming that X is Gaussian, we introduce a weak form of isotropy of the field $X \circ \theta$, defined by an invariance property of the mean Euler characteristic of some of its excursion sets:

$$\forall T \text{ rectangle or segment in } \mathbb{R}^2,\ \forall u \in \mathbb{R},\ \forall \rho \in SO(2),$$
$$\mathbb{E}[\chi(A(u, X \circ \theta, \rho(T)))] = \mathbb{E}[\chi(A(u, X \circ \theta, T))].$$

Deformed fields satisfying this property are actually proved to be strictly isotropic. Besides, assuming that the mean Euler characteristic of excursions sets of $X \circ \theta$ over some basic domains is known, we are able to identify θ. In this chapter, based on the article [Fou18] accepted for publication, we mainly refer to expectation formulas of the (modified) Euler characteristic of excursion sets presented in the current chapter in Sections I.2.2.c and I.2.2.d.

The thesis ends with the study of the anisotropic random wave model in Chapter IV. If $\mathbf{k}$ is a d-dimensional random vector, we call random wave model with random wavevector $\mathbf{k}$ any centred and stationary random field defined on $\mathbb{R}^d$ with covariance function $t \mapsto \mathbb{E}[\cos(\mathbf{k}.t)]$. Therefore, in the Gaussian case, the distribution of the considered random wave is unique. The purpose of the chapter is to link distributional properties of the random wave, mainly geometric ones, to the ones of the random wavevector and in particular to its anisotropy. In the Gaussian setting, we focus on the expected measure of a level set. For instance, when $\mathbf{k}$ almost surely belongs to the unit sphere in $\mathbb{R}^2$ and the random wave model is nothing but the anisotropic version of Berry's planar waves, we prove that the expected length of the nodal lines is decreasing as the anisotropy of the random wavevector is increasing. Also, when $\mathbf{k}$ almost surely belongs to the Airy surface in $\mathbb{R}^3$ (characterized by the equation $x^2 + y^2 = z^4$) and the associated Gaussian random wave serves as a model for sea waves, we compare the direction that maximises the expected length of the static crests and the mode of the random wavevector's direction, which are orthogonal in some models but not in others. Chapter IV is based on the preprint [EF18] coauthored with Anne Estrade. It mainly refers to the spectral representation stated in Section I.1.4 and to Rice formulas of Section I.2.1.

Let us remark that the anisotropies at stake in Chapters III and IV are different in essence. The anisotropy of the deformed random field model is due to a deformation of the parameter space. This is not the case for the random wave model, for which the anisotropy is linked to assymetries in the spectral domain. The two models meet when $\mathbf{k}$ is equal to $A\mathbf{u}$ with A a matrix and $\mathbf{u}$ a random vector in $\mathbb{R}^d$ whose distribution is

invariant under rotations, since the associated random wave has the same distribution as an isotropic random wave deformed by the linear transformation A^T. In that case, the study of anisotropy, either in the spectral domain, or in the parameter domain, is equivalent, but in the general case where no linear deformation is involved, these are two different approaches.

Let us end on a practical note. For the sake of consistency, some references to this introductive chapter are made in the following ones. Besides, at the beginning of each of them, notations are introduced, which are either the same as the ones used in this chapter, or very close to them. The following chapters can almost entirely be read independently from each other, thus each of them ends with its own conclusion. Note that the numbering of formulas is reset to zero at the beginning of each chapter.

Chapter II

Finite variance of the number of stationary points of a Gaussian random field

In this chapter, we investigate the issue of the finiteness of the second moment of the number of critical points of a stationary random field, without the assumption of isotropy. It corresponds to a paper [EF16] published in *Statistics and Probability Letters*.

Let d be a positive integer and let $X : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a stationary Gaussian random field. We assume that almost every realization of X is of class $\mathscr{C}^2$ on $\mathbb{R}^d$. For any compact subset T in $\mathbb{R}^d$, for any $v \in \mathbb{R}^d$, we consider the number $N^{X'}(T, v)$ of points in T where the gradient of X, denoted by X', reaches the value v:

$$N^{X'}(T, v) = \#\{t \in T : X'(t) = v\}.$$

For $v = 0$, it is nothing but the number of stationary points of X in T. In this chapter, we establish a sufficient condition on the covariance function r of the random field X in order that $N^{X'}(T, v)$ admits a finite variance.

The existence of the second moment of $N^{X'}(T, v)$ has been studied since the late 60s, first in dimension one and for a level equal to the mean, *i.e.* $v = 0$. Cramér and Leadbetter were the first to propose in [CL67] a sufficient condition on the covariance function r in order that $N^{X'}(T, 0)$ belongs to $L^2(\Omega)$. If X satisfies some non-degeneracy assumptions, this simple condition requires that the fourth derivative $r^{(4)}$ satisfies

$$\exists \delta > 0, \quad \int_0^\delta \frac{r^{(4)}(0) - r^{(4)}(t)}{t}\, dt \ < \ +\infty.$$

It is known as the Geman condition for Geman proved some years after in [Gem72] that it was not only sufficient but also necessary. The issue of the finiteness of the higher moments of $N^{X'}(T, 0)$ has also been discussed in many papers (see [Bel66, Cuz75, Mal85] for instance and references therein). Kratz and León generalized Geman's result in [KL06] to the number of crossings of any level $v \in \mathbb{R}$ and also to the number of a curve crossings.

Concerning the problem in higher dimension, it has been an open question for a long time. Elizarov gave in [Eli85] a sufficient condition for $N^{X'}(T,0)$ to be in $L^2(\Omega)$. Even though his condition is weaker than ours, his proof is short and elliptical and it only concerns the number of stationary points. Under the additional hypothesis that X is isotropic and of class $\mathscr{C}^3$, Estrade and León proved in [EL16] that for any $v \in \mathbb{R}^d$, $N^{X'}(T,v)$ admits a finite second moment.

The chapter is organized as follows. In Section 1, we introduce our notations and assumptions. Our proof begins with the use of Rice formulas in Section II.2.1 to give an expression of $N^{X'}(T,v)$ in an integral form. It allows us to restrict the problem to the one of the integrability of function

$$t \mapsto \mathbb{E}[\,(\det X''(0))^2 \,/\, X'(0) = X'(t) = v\,]\,\|t\|^{-d}$$

in a neighbourhood of zero in $\mathbb{R}^d$. We are able to bound this function and, thanks to a regression method implemented in Section II.2.2, to study the asymptotic properties of the bound around zero. Section 3 is devoted to the main result of this chapter, namely Theorem II.3. It gives an extension of Geman condition in dimension $d > 1$ that is sufficient to establish that $N^{X'}(T,v)$ is square integrable for any v.

Notations and derivatives

We deal with a centred stationary Gaussian random field $X = \{X(t),\, t \in \mathbb{R}^d\}$ and we denote by r its covariance function $t \mapsto \mathrm{Cov}(X(0), X(t))$. We assume that almost every realization of X is of class $\mathscr{C}^2$ on $\mathbb{R}^d$. That implies that r is of class $\mathscr{C}^4$ on $\mathbb{R}^d$, according to Proposition I.5.

We fix an othonormal basis of $\mathbb{R}^d$, according to the canonical scalar product that we denote here by $\langle \cdot\,, \cdot \rangle$. We consider the partial derivatives of X and r computed on this basis. As in Chapter I, we write $(X'_i)_{1\le i\le d}$ and $(X''_{i,j})_{1\le i,j\le d}$ the partial derivatives of X of first and second order, respectively, and r'_i, $r''_{i,j}$, $r^{(3)}_{i,j,m}$ and $r^{(4)}_{i,j,m,n}$ the partial derivatives of r, from order one to four, respectively. We refer to the gradient of X at t as $X'(t)$ and to the Hessian matrix of X at t as $X''(t)$. Similarly, we write $r''(t)$ the Hessian of r at t. We will sometimes denote by $r^{(3)}_{i,j}(t)$ the vector $(r^{(3)}_{i,j,m}(t))_{1\le m\le d}$ and by $r^{(4)}_{i,j}(t)$ the matrix $(r^{(4)}_{i,j,m,n}(t))_{1\le m,n\le d}$. We also use the same notation for $t \in \mathbb{R}^d$ and the column vector containing its coordinates.

In every space $\mathbb{R}^m$ (m is any positive integer), we denote by $\|\cdot\|$ the norm associated to the canonical scalar product. We use the standard notations $o(\cdot)$ and $O(\cdot)$ to describe the behaviour of some functions in a neighbourhood of zero.

In this chapter, we will extensively exploit the relationships between the partial derivatives of r and the covariances between the partial derivatives of X, for which a general formula is given by (3) in Section I.1.3. Here are precisely the formulas that we are going to use. For $(s,t) \in (\mathbb{R}^d)^2$ and for $1 \le i,j,m,n \le d$,

$$\begin{array}{ll}
\mathrm{Cov}(X(s), X(t)) = r(s-t) & \mathrm{Cov}(X'_i(s), X(t)) = r'_i(s-t) \\
\mathrm{Cov}(X'_i(s), X'_j(t)) = -r''_{i,j}(s-t) & \mathrm{Cov}(X''_{i,j}(s), X(t)) = r''_{i,j}(s-t) \\
\mathrm{Cov}(X''_{i,j}(s), X'_m(t)) = -r^{(3)}_{i,j,m}(s-t) & \mathrm{Cov}(X''_{i,j}(s), X''_{m,n}(t)) = r^{(4)}_{i,j,m,n}(s-t).
\end{array}$$

We will need the assumption that for any $t \in \mathbb{R}^d \backslash \{0\}$, the Gaussian vectors

$$(X'(0), X'(t)) \quad \text{and} \quad (X''_{i,j}(0))_{1 \le i \le j \le d}$$

are not degenerate. As a consequence, $\mathrm{Var}[X(t)] = r(0) \neq 0$ and so we may assume that $r(0) = 1$. As another consequence, the covariance matrix of $X'(0)$ is not degenerate, which allows us to assume that $-r''(0) = I_d$ or, equivalently, that the first-order derivatives of X are uncorrelated and of unit variance. This assumption is taken from the proof of Lemma 11.7.1 in [AT07]. We explain it here in a few words. The covariance matrix of $X'(0)$ is $-r''(0)$. A square root Q of $(-r''(0))^{-1}$ will satisfy $-Q\, r''(0)\, Q = I_d$. We now define a new random field X^Q on $\mathbb{R}^d$ by $X^Q(t) = X(Q\, t)$. It is not hard to see that X^Q is still stationary, with unit variance, and that the covariance matrix of $\left(X^Q\right)'(0)$ is I_d. Note that this does not imply that X^Q is isotropic. From now on, we will abandon the notation X^Q, although we will still assume that $-r''(0) = I_d$.

We gather all the assumptions made on X in one, referred to as **(A)**:

$$\textbf{(A)} \begin{cases} X \text{ is Gaussian,} \\ \text{almost every realization of } X \text{ is of class } \mathscr{C}^2, \\ \forall t \neq 0,\ (X'(0), X'(t)) \text{ and } (X''_{i,j}(0))_{1 \le i \le j \le d} \text{ are not degenerate,} \\ X \text{ is centred, } \ r(0) = 1 \ \text{ and } \ -r''(0) = I_d. \end{cases}$$

Note that the major assumptions in Condition **(A)** are the first three ones. The last assumption has been added to make the intermediate proofs and computations easier, but our main result Theorem II.3 remains true if we remove it.

With these assumptions in mind, we are able to write the next Taylor formulas around 0 for the covariance function r and its derivatives:

$$\begin{cases} r(t) = 1 - \dfrac{1}{2} \displaystyle\sum_{1 \le i \le d} t_i^2 + \dfrac{1}{4!} \sum_{i,j,m,n} r^{(4)}_{i,j,m,n}(0)\, t_i\, t_j\, t_m\, t_n + o(\|t\|^4) \\ r''(t) = -I_d + \dfrac{1}{2}\Theta(t) + o(\|t\|^2) \\ r^{(3)}_{i,j}(t) = r^{(4)}_{i,j}(0)t + o(\|t\|), \quad \text{for all } \ 1 \le i, j \le d \\ r^{(4)}_{i,j}(t) = r^{(4)}_{i,j}(0) + o(1), \ \text{ for all } \ 1 \le i, j \le d\,, \end{cases}$$

where the $d \times d$ matrix $\Theta(t)$ is defined by $\Theta(t)_{m,n} = \langle r^{(4)}_{m,n}(0)t\,, t\rangle = \displaystyle\sum_{1 \le i,j \le d} r^{(4)}_{i,j,m,n}(0) t_i t_j$.
We note that, for any $t \neq 0$, $\Theta(t)$ is inversible. Indeed, since $\Theta(t)$ is the covariance matrix of vector $X''(0)\, t$, if it was not inversible, $X''(0)\, t$ would be a degenerate Gaussian vector and so there would exist a linear dependence between the coordinates of $X''(0)$. That would be inconsistent with assumption **(A)**. Hence, in what follows, we denote by $\Delta(t)$ the inverse matrix of $\Theta(t)$ for $t \neq 0$. Besides, we also remark that $t \mapsto \Theta(t)$ and $t \mapsto \Delta(t)$ are homogeneous functions of respective degrees 2 and -2.

We fix a compact set T in $\mathbb{R}^d$.

Preliminary results

Rice formula

For any $v \in \mathbb{R}^d$, $N^{X'}(T,v)$ is the number of roots in T of the vectorial random field $X' - v$. In our setting, we can apply the expectation formula stated in Theorem I.10 supplemented by Remark I.11. Rice formula not only gives a closed formula for the expectation of $N^{X'}(T,v)$ but also states that it is finite in our context. So the variance of $N^{X'}(T,v)$ is finite if and only if its second-order factorial moment is finite. Under Condition **(A)** on X, another Rice formula, also stated in Theorem I.10, gives the second factorial moment of $N^{X'}(T,v)$:

$$\begin{aligned}&\mathbb{E}[N^{X'}(T,v)(N^{X'}(T,v)-1)]\\&= \int_{T\times T} \mathbb{E}[|\det X''(s)\ \det X''(t)| \,/\, X'(s) = X'(t) = v]\, p_{s,t}(v,v)\, ds\, dt\,,\end{aligned}$$

where $p_{s,t}$ denotes the probability density function of the Gaussian vector $(X'(s), X'(t))$. This formula holds whether both sides are finite or not. We introduce

$$F(v,t) = \mathbb{E}[|\det X''(0)\ \det X''(t)| \,/\, X'(0) = X'(t) = v]\ ;\ v,t \in \mathbb{R}^d\,,$$

and we use the stationarity of X to transform the double integral in the Rice formula into a simple integral:

$$\mathbb{E}[N^{X'}(T,v)(N^{X'}(T,v)-1)] = \int_{T_0} |T \cap (T-t)|\, F(v,t)\, p_{0,t}(v,v)\, dt\,,$$

where $|T\cap(T-t)|$ is the Lebesgue measure of $T\cap(T-t)$ and $T_0 = \left\{t - t',\ (t,t') \in T^2\right\}$. This formula allows us to give a simple criteria for $N^{X'}(T,v)$ to be square integrable.

Notation. Let $u : \mathbb{R}^d \to \mathbb{R}^m$. We write $u \in L^1(\mathscr{V}_0, \|t\|^{-d}\, dt)$ if there exists a positive constant δ such that $\displaystyle\int_{B(0,\delta)} \frac{\|u(t)\|}{\|t\|^d}\, dt < +\infty$.

Lemma II.1 *Assume that X fulfills Condition **(A)**. For any $v \in \mathbb{R}^d$, we introduce*

$$G(v,\cdot) : t \in \mathbb{R}^d \longmapsto G(v,t) = \mathbb{E}[(\det X''(0))^2 \,/\, X'(0) = X'(t) = v]\,.$$

Then

$$G(v,\cdot) \in L^1(\mathscr{V}_0, \|t\|^{-d} dt) \Rightarrow N^{X'}(T,v) \in L^2(\Omega).$$

Proof. Note that the function $t \mapsto |T \cap (T-t)|\, F(v,t)\, p_{0,t}(v,v)$ is continuous on $\mathbb{R}^d\backslash\{0\}$, because the random field X is Gaussian. So it is integrable in every bounded domain that does not include a neighbourhood of zero.
We are now concerned with its behaviour in a neighbourhood of zero. We first remark that, as t tends to 0, the term $|T \cap (T-t)|$ is equivalent to $|T|$. Next, we use Cauchy-Schwarz inequality and stationarity to write

$$F(v,t) \le (G(v,t)\, G(v,-t))^{1/2}\,.$$

Let us now study $t \mapsto p_{0,t}(v,v)$ as t tends to 0. We know that

$$p_{0,t}(v,v) \leq p_{0,t}(0,0) = (2\pi)^{-d/2} \left(\det \Gamma(t)\right)^{-1/2},$$

where $\Gamma(t)$ is the covariance matrix of the $2d$-dimensional Gaussian vector $(X'(0), X'(t))$. It is given blockwise by $\Gamma(t) = \begin{pmatrix} I_d & -r''(t) \\ -r''(t) & I_d \end{pmatrix}$ and so

$$\begin{aligned} \det \Gamma(t) &= \det(I_d - r''(t)^2) = \det\Big(\Theta(t) + o(\|t\|^2)\Big) \\ &= \det \Theta(t)\, \det\Big(I_d + o(\|t\|^2)\, \Delta(t)\Big) \\ &= \|t\|^{2d} \det \Theta(\frac{t}{\|t\|})\, \det\Big(I_d + o(1)\, \Delta(\frac{t}{\|t\|})\Big), \end{aligned}$$

where we have used the homogeneity properties of Θ and Δ. Since $\min_{u \in \mathbb{S}^{d-1}} \det \Theta(u)$ is strictly positive and $t \mapsto \Delta(\frac{t}{\|t\|})$ is bounded, there exists $c > 0$ such that $\det \Gamma(t) \geq c\, \|t\|^{2d}$ for t in a neighbourhood of zero. Hence, for some positive constant C, $p_{0,t}(v,v) \leq C\, \|t\|^{-d}$.

Consequently, if $G(v,\cdot) \in L^1(\mathscr{V}_0, \|t\|^{-d}\, dt)$ then $t \mapsto |T \cap (T-t)|\, F(v,t)\, p_{0,t}(v,v)$ is bounded by a function that is integrable in a neighbourhood of 0, thanks to Cauchy-Schwarz inequality. That concludes the proof of the lemma. □

Our aim is now to study the behaviour of $G(v,t)$ as $t \to 0$, for a fixed $v \in \mathbb{R}^d$. Precisely, we will provide a sufficient condition for $G(v,\cdot)$ to belong to $L^1(\mathscr{V}_0, \|t\|^{-d} dt)$.

Regression

In order to get an estimate for $G(v,t)$, we compute the conditional law of $X''(0)$ with respect to the event $\{X'(0) = X'(t) = v\}$. Let $K = d(d+1)/2$. We consider the symmetric matrix $X''(0)$ as a K-dimensional Gaussian column vector by putting the coefficients of its upper triangular part in a vector that we write $\nabla^2 X(0)$. So the indices $1 \leq \mathbf{k} \leq K$ of this vector have to be seen as double indices ($\mathbf{k} = (i,j)$ with $1 \leq i \leq j \leq d$). For $t \neq 0$, we write the following K-dimensional regression system:

$$\nabla^2 X(0) = A(t)\, X'(0) + B(t)\, X'(t) + Z(t), \tag{1}$$

where $A(t)$ and $B(t)$ are matrices of size $K \times d$ and $Z(t)$ is a K-dimensional centred Gaussian vector, independent from $X'(0)$ and $X'(t)$. Hence, conditioned on $\{X'(0) = X'(t) = v\}$, $\nabla^2 X(0)$ is a Gaussian vector with mean $(A(t) + B(t))v$ and covariance matrix $\Gamma^Z(t)$. Next proposition is an application of the Gaussian regression result stated in Proposition I.2.

Proposition II.2 *If X fulfills Condition **(A)**, then the regression coefficients of System* (1) *are given by*

$$A(t) = r^{(3)}(t)\, N_2(t) \quad \textit{and} \quad B(t) = r^{(3)}(t)\, N_1(t), \tag{2}$$

where $r^{(3)}(t)$ *stands for the* $K \times d$ *matrix* $(r^{(3)}_{\mathbf{k},i}(t))_{1\leq \mathbf{k}\leq K,\, 1\leq i\leq d}$ *and* $N_1(t)$ *and* $N_2(t)$ *are two* $d \times d$ *matrices defined on* $\mathbb{R}^d\backslash\{0\}$ *by*

$$N_1(t) = (I_d - (r''(t))^2)^{-1} \quad \text{and} \quad N_2(t) = r''(t)\,(I_d - (r''(t))^2)^{-1}. \tag{3}$$

Besides, the covariance matrix $\Gamma^Z(t)$ *of the* K*-dimensional Gaussian vector* $Z(t)$ *is such that for any* $1 \leq \mathbf{k}, \mathbf{l} \leq K$ *and for* $t \in \mathbb{R}^d\backslash\{0\}$*,*

$$\Gamma^Z(t)_{\mathbf{k},\mathbf{l}} = \mathrm{Cov}(Z(t)_{\mathbf{k}}, Z(t)_{\mathbf{l}}) = r^{(4)}_{\mathbf{k},\mathbf{l}}(0) - \langle r^{(3)}_{\mathbf{k}}(t)\,, N_1(t)\, r^{(3)}_{\mathbf{l}}(t)\rangle\,. \tag{4}$$

Proof. We denote by $\mathbf{X_1}$ the vector $\nabla^2 X(0)$ of size K and by $\mathbf{X_2}$ the vector $(X'(0), X'(t))$ of size $2d$. We write C_1 the $K \times K$ covariance matrix of $\mathbf{X_1}$, C_2 the $2d \times 2d$ covariance matrix of $\mathbf{X_2}$ and C_{12} the $K \times 2d$ matrix of the covariances between the coordinates of $\mathbf{X_1}$ and those of $\mathbf{X_2}$. Then, let us recall that the conditional distribution of $\mathbf{X}_1$ with respect to $\mathbf{X_2}$ (that are both centred) is Gaussian, with mean vector $C_{12}C_2^{-1}\mathbf{X_2}$ and covariance matrix $C_1 - C_{12}C_2^{-1}C_{12}^T$.

Thanks to the relations recalled in Section II.1, we have

$$C_1 = \left(r^{(4)}(0)\right),\ C_2 = \begin{pmatrix} I_d & -r''(t) \\ -r''(t) & I_d \end{pmatrix},\ C_{12} = \begin{pmatrix} O_{K,d} & r^{(3)}(t) \end{pmatrix},$$

where $r^{(4)}(0)$ stands for the $K \times K$ matrix $(r^{(4)}_{\mathbf{k},\mathbf{l}}(0))_{1\leq \mathbf{k},\mathbf{l}\leq K}$ and $O_{K,d}$ for the $K \times d$ zero matrix.

Let us note that C_2, which is the covariance matrix of $(X'(0), X'(t))$, is not degenerate for $t \neq 0$ because of hypothesis **(A)**. We note $N(t)$ its inverse. It is not hard to find that $N(t) = \begin{pmatrix} N_1(t) & N_2(t) \\ N_2(t) & N_1(t) \end{pmatrix}$ where $N_1(t)$ and $N_2(t)$ are two square matrices of dimensions $d \times d$. To show (3), we just have to solve the system

$$\begin{cases} N_1(t) - r''(t)\, N_2(t) = I_d \\ -r''(t)\, N_1(t) + N_2(t) = O_{dd}. \end{cases}$$

Computing the conditional mean of $\mathbf{X_1}$ with respect to $\mathbf{X_2}$, we get

$$C_{12}C_2^{-1}\mathbf{X_2} = r^{(3)}(t)\, N_2(t)\, X'(0) + r^{(3)}(t)\, N_1(t)\, X'(t),$$

and thus we deduce the regression coefficients as announced in (2). Moreover, the covariance matrix of the conditional distribution of $\mathbf{X}_1$ with respect to $\mathbf{X_2}$ is given by

$$C_1 - C_{12}C_2^{-1}C_{12}^T = r^{(4)}(0) - r^{(3)}(t)\, N_1(t)\, r^{(3)}(t)^T\,.$$

Its coefficients are exactly those written in Formula (4). That concludes the proof. □

Sufficient Geman condition

We now state our main result. Assumption **(A)** is still in force and we introduce a new condition:

$$\textbf{(G)}: \quad \text{there exists } \delta > 0 \text{ such that } \int_{B(0,\delta)} \frac{\|r^{(4)}(0) - r^{(4)}(t)\|}{\|t\|^d}\, dt < +\infty\,.$$

Condition **(G)** is weaker than X almost surely of class $\mathscr{C}^3$, since in that case, $r^{(4)}(0) - r^{(4)}(t) = o(\|t\|)$ as t tends to zero. It is a generalization of Geman condition known in dimension $d = 1$. In this particular case, it has been proved to be a sufficient and necessary condition to have $N^{X'}(T, v) \in L^2(\Omega)$ for any $v \in \mathbb{R}$ (see [KL06]).

It turns out that our Condition **(G)** remains a sufficient condition in dimension $d > 1$ for $N^{X'}(T, v)$ to be in $L^2(\Omega)$.

Theorem II.3 *If X fulfills Conditions **(A)** and **(G)**, then*

$$\text{for any } v \in \mathbb{R}^d,\ N^{X'}(T, v) \in L^2(\Omega)\,.$$

Proof of Theorem II.3. We will proceed in several steps.

First step: study of function G. Recall that G has been introduced in Lemma II.1.

Lemma II.4 *Suppose that X fulfills Condition **(A)** and let $\mathscr{V} \subset \mathbb{R}^d$ be a compact set. Then*

(i) for any $v \in \mathscr{V}$, $G(v, t) = G(0, t) + o(\|t\|)$,

(ii) there exists a homogeneous polynomial $Q_{(d)}$ of degree d, which does not depend on X, such that $G(0, t) = Q_{(d)}(\Gamma^Z(t))$, where $Q_{(d)}(\Gamma^Z(t))$ is the evaluation of polynomial $Q_{(d)}$ at the coefficients of matrix $\Gamma^Z(t)$.

Proof.
(i) We use the natural identification between symmetric $d \times d$ matrices and vectors in $\mathbb{R}^K$, where $K = d(d+1)/2$, to define $\tilde{\det}(y)$ as the determinant of the $d \times d$ symmetric matrix whose upper triangular part contains the coordinates of $y \in \mathbb{R}^K$. It is a degree d homogeneous polynomial function of K variables. With this notation, and using the regression system (1), we get for $v \in \mathbb{R}^d$, $t \in \mathbb{R}^d \backslash \{0\}$,

$$G(v, t) \;=\; \mathbb{E}\left[\tilde{\det}(S(t)v + Z(t))^2\right],$$

where $S(t)$ stands for $A(t) + B(t)$. Thanks to Formula (3),

$$S(t) = r^{(3)}(t)\,(N_2(t) + N_1(t)) = r^{(3)}(t)\,(I_d - r''(t))^{-1}\,,$$

and since $I_d - r''(t) \to 2I_d$ and $r^{(3)}(t) = O(\|t\|)$ as $t \to 0$, we get the following asymptotics:

$$S(t) = A(t) + B(t) = O(\|t\|) \text{ as } t \to 0. \tag{5}$$

Let us come back to the computation of G. By developing the square of the determinant and bringing together the terms according to the powers of the coordinates $(S(t)v)_{\mathbf{k}}$ of the K-dimensional vector $S(t)v$, we get

$$\begin{aligned}\tilde{\det}\left(S(t)v+Z(t)\right)^2 &= \tilde{\det}(Z(t))^2+\sum_{1\le\mathbf{k}\le K}(S(t)v)_{\mathbf{k}}\,Q^{\mathbf{k}}_{(2d-1)}(Z(t))\\ &+\sum_{1\le\mathbf{k},\mathbf{l}\le K}(S(t)v)_{\mathbf{k}}\,(S(t)v)_{\mathbf{l}}\,Q^{\mathbf{kl}}_{(2d-2)}(S(t)v+Z(t))\,,\end{aligned}$$

where the $Q^{\mathbf{k}}_{(2d-1)}$'s and the $Q^{\mathbf{kl}}_{(2d-2)}$'s are multivariate polynomial functions of respective degrees $2d-1$ and $2d-2$. Note that $\mathbb{E}\left[Q^{\mathbf{k}}_{(2d-1)}(Z(t))\right]=0$ since $Z(t)$ is a centred Gaussian vector and $Q^{\mathbf{k}}_{(2d-1)}$ has an odd degree. Also recall the asymptotics (5) and note that $\Gamma^Z(t)$ is bounded for t in any compact set; this can be justified properly thanks to the upcoming Lemma II.7. Therefore, by taking the expectation of the above equality, we obtain that uniformly with respect to $v\in\mathscr{V}$,

$$G(v,t)=\mathbb{E}[\tilde{\det}(Z(t))^2]+o(\|t\|)\text{ as }t\to 0. \tag{6}$$

Recall that $G(0,t)=\mathbb{E}[\tilde{\det}(Z(t))^2]$, hence point (i) is proved.

(ii) We now compute $\mathbb{E}[\tilde{\det}(Z(t))^2]$ by applying Wick formula. Actually, let us consider for a while a K-dimensional centred Gaussian vector Y and let us compute $\mathbb{E}[\tilde{\det}(Y)^2]$. This quantity is equal to an alternate sum of terms with the following shape: $\mathbb{E}[Y_{i_1}\cdots Y_{i_{2d}}]$ where $i_1,\dots,i_{2d}$ belong to $\{1,\dots,K\}$. According to Wick formula stated in Proposition I.1, there exists a degree d homogeneous polynomial function $Q_{(d)}$ such that

$$\mathbb{E}[\tilde{\det}(Y)^2]=Q_{(d)}(\Gamma^Y)\,, \tag{7}$$

where Γ^Y is the covariance matrix of Y. Taking $Y=Z(t)$, we deduce from (6) that $G(v,t)=Q_{(d)}(\Gamma^Z(t))+o(\|t\|)$. Lemma II.4 is then proved. □

Second step: an auxiliary function. This step is dedicated to the properties of a function that will turn out to be, to some extent, close to $\Gamma^Z(t)$, as t tends to zero. Let us recall that the expression of $\Gamma^Z(t)$ is given by Formula (4).

We introduce $\gamma(t)=(\gamma(t)_{\mathbf{k},\mathbf{l}})_{1\le\mathbf{k},\mathbf{l}\le K}$ defined for $t\neq 0$ by

$$\begin{aligned}\gamma(t)_{\mathbf{k},\mathbf{l}} &= r^{(4)}_{\mathbf{k},\mathbf{l}}(0)-\sum_{1\le i,j,m,n\le d}r^{(4)}_{\mathbf{k},i,m}(0)r^{(4)}_{\mathbf{l},j,n}(0)\Delta(t)_{m,n}t_it_j\\ &= r^{(4)}_{\mathbf{k},\mathbf{l}}(0)-\langle r^{(4)}_{\mathbf{k}}(0)\,t\,,\Delta(t)r^{(4)}_{\mathbf{l}}(0)\,t\rangle,\end{aligned} \tag{8}$$

matrix $\Delta(t)$ being the inverse matrix of $\Theta(t)$ introduced in Section II.1. Function γ only depends on r through its fourth-order derivatives at zero. Clearly, it is homogeneous of degree zero: for any t in $\mathbb{R}^d\backslash\{0\}$, $\gamma(t)=\gamma\left(\frac{t}{\|t\|}\right)$.

Remark II.5 *For any $t\neq 0$, $\gamma(t)$ is the covariance matrix of $\nabla^2X(0)$ conditioned on $\{X'(0)=X''(0)t=0\}$.*

Proof of Remark II.5. The conditional covariance matrix can be computed thanks to the formula recalled in the proof of Proposition II.2. The covariance matrix of vector $\nabla^2 X(0)$ is the $K \times K$ matrix $C_1 = (r^{(4)}(0))$. The covariance matrix of vector $(X'(0), X''(0)t)$ is the $2d \times 2d$ matrix $C_2 = \begin{pmatrix} I_d & 0 \\ 0 & \Theta(t) \end{pmatrix}$ and the matrix of the covariances between the coordinates of vector $\nabla^2 X(0)$ and those of $(X'(0), X''(0)t)$ is the $K \times 2d$ matrix $C_{12} = \left(O_{K,d} \quad (r^{(4)}_{\mathbf{k},i}(0)t)_{\substack{1 \le \mathbf{k} \le K \\ 1 \le i \le d}} \right)$, where $r^{(4)}_{\mathbf{k},i}(0)$ stands for the d-dimensional line vector $(r^{(4)}_{\mathbf{k},i,j}(0))_{1 \le j \le d}$ (i^{th} line of matrix $r^{(4)}_{\mathbf{k}}(0)$). Hence, the covariance matrix of $\nabla^2 X(0) \,/ X'(0) = X''(0)t = 0$ is the $K \times K$ matrix $C_1 - C_{12} C_2^{-1} C_{12}^T$. Its $(\mathbf{k}, \mathbf{l})$-coefficient is exactly $\gamma(t)_{\mathbf{k},\mathbf{l}}$. □

We now state a property of the auxiliary function γ that is interesting for its own.

Proposition II.6 *If X satisfies Condition **(A)**, then $\forall t \in \mathbb{R}^d \backslash \{0\}$, $Q_{(d)}(\gamma(t)) = 0$.*

Proof of Proposition II.6. We first check the result in the particular case of dimension one. For $d = 1$, $K = 1$ and $Q_{(1)}$ is a one variable polynomial such that, if Y is a Gaussian centred random variable, $Q_{(1)}(\Gamma^Y) = \mathbb{E}\left[\tilde{\det}(Y)^2\right] = \mathbb{E}[Y^2] = \mathrm{Var}[Y] = \Gamma^Y$. Hence, for any $x \in \mathbb{R}$, $Q_{(1)}(x) = x$. Moreover, according to the definition of γ (see (8)), for $t \neq 0$, $\gamma(t) = r^{(4)}(0) - \dfrac{(r^{(4)}(0)t)^2}{r^{(4)}(0)t^2} = 0$.

By computing explicitely the polynomial $Q_{(2)}$ and the function $\gamma(t)$, we give in the Appendix an alternative proof of Proposition II.6 in the case $d = 2$.

We now give a general proof. According to Remark II.5 and to the definition of $Q_{(d)}$ prescribed in (7), we have

$$Q_{(d)}(\gamma(t)) = \mathbb{E}\left[\det(X''(0)^2) \,/\, X'(0) = X''(0)t = 0\right].$$

Besides, one can check the following result that we read in [AW05]. *Let M be a $d \times d$ symmetric positive matrix and let $(v_i)_{1 \le i \le d}$ be an orthonormal basis of $\mathbb{R}^d$. Then, denoting by $\widetilde{M}$ the $(d-1) \times (d-1)$ matrix $(\langle M v_i \,, v_j \rangle)_{2 \le i,j \le d}$, the following inequality holds:*

$$\det(M) \le \langle M v_1 \,, v_1 \rangle \, \det(\widetilde{M}) \,.$$

We apply this result with $M = X''(0)^2$, $v_1 = \dfrac{t}{\|t\|}$, taking for $(v_i)_{2 \le i \le d}$ any vectors satisfying the required hypothesis. As a result,

$$\det\left(X''(0)^2\right) \le \left\langle X''(0)^2 \frac{t}{\|t\|} \,, \frac{t}{\|t\|} \right\rangle \det(\widetilde{M}) = \|t\|^{-2} \left\langle X''(0)t \,, X''(0)t \right\rangle \det(\widetilde{M}).$$

So, applying the conditional expectation with respect to the event $\{X'(0) = X''(0)t = 0\}$, we get $Q_{(d)}(\gamma(t)) \le 0$. That concludes the proof since $Q_{(d)}(\gamma(t))$ only takes non negative values. □

Third step: a comparison between $\Gamma^Z(t)$ and $\gamma(t)$. We introduce the following functions defined on $\mathbb{R}^d$,

$$\varepsilon \ : \ t \mapsto r^{(4)}(0) - r^{(4)}(t), \quad \bar{\varepsilon} \ : \ t \mapsto \int_0^1 \varepsilon(ut)\, du, \quad \hat{\varepsilon} \ : \ t \mapsto \int_0^1 \varepsilon(ut)(1-u)\, du.$$

They all take values in $\mathbb{R}^{d^4}$ and are symmetric functions with respect to the indices $(i,j,m,n) \in [\![1,d]\!]^4$. Since r is $\mathscr{C}^4$, ε is continuous and $\varepsilon(t) = o(1)$ as t tends to 0. The same holds for $\bar{\varepsilon}(t)$ and $\hat{\varepsilon}(t)$.

Lemma II.7 *If X satisfies Condition **(A)**, then there exist a neighbourhood $\mathscr{W}$ of zero in $\mathbb{R}^d$ and a positive constant c such that, for any $t \in \mathscr{W}\backslash\{0\}$,*

$$\left\| \Gamma^Z(t) - \gamma(t) \right\| \leq c \left(\|\bar{\varepsilon}(t)\| + \|\hat{\varepsilon}(t)\| + \|t\|^2 \right).$$

Proof of Lemma II.7. Formulas (4) and (8) allow us to write:

$$\begin{aligned} \Gamma^Z(t)_{\mathbf{k},\mathbf{l}} - \gamma(t)_{\mathbf{k},\mathbf{l}} &= \left\langle r^{(4)}_{\mathbf{k}}(0)\, t\, , \Delta(t) r^{(4)}_{\mathbf{l}}(0)\, t \right\rangle - \left\langle r^{(3)}_{\mathbf{k}}(t)\, , N_1(t) r^{(3)}_{\mathbf{l}}(t) \right\rangle \\ &= \left\langle r^{(4)}_{\mathbf{k}}(0)\, t - r^{(3)}_{\mathbf{k}}(t)\, , \Delta(t) r^{(4)}_{\mathbf{l}}(0)\, t \right\rangle \\ &+ \left\langle r^{(3)}_{\mathbf{k}}(t)\, , \Delta(t) \left(r^{(4)}_{\mathbf{l}}(0)\, t - r^{(3)}_{\mathbf{l}}(t) \right) \right\rangle \\ &+ \left\langle r^{(3)}_{\mathbf{k}}(t)\, , (\Delta(t) - N_1(t))\, r^{(3)}_{\mathbf{l}}(t) \right\rangle. \end{aligned} \tag{9}$$

We now use Taylor expansions to get precise upperbounds. For any $t \in \mathbb{R}^d$, for any $1 \leq \mathbf{k} \leq K$ and for any $1 \leq i,j \leq d$, let us consider the functions $u \in \mathbb{R} \mapsto r^{(3)}_{\mathbf{k}}(ut)$ and $u \in \mathbb{R} \mapsto r''_{i,j}(ut)$. We can write the following Taylor expansions with integral remainders between $u = 0$ and $u = 1$, up to order zero and to order one, respectively. That yields:

$$\begin{aligned} r^{(3)}_{\mathbf{k}}(t) &= \int_0^1 r^{(4)}_{\mathbf{k}}(ut)\, t\, du \\ r''_{i,j}(t) &= -\delta_{i,j} + \int_0^1 \left\langle r^{(4)}_{i,j}(ut)\, t, t \right\rangle (1-u)\, du. \end{aligned}$$

Hence, using functions $\bar{\varepsilon}$ and $\hat{\varepsilon}$, and the fact that $\Theta(t)_{i,j} = \left\langle r^{(4)}_{i,j}(0)\, t\, , t \right\rangle$, we get

$$\begin{aligned} r^{(4)}_{\mathbf{k}}(0)\, t - r^{(3)}_{\mathbf{k}}(t) &= \bar{\varepsilon}(t)_{\mathbf{k}}\, t \\ r''_{i,j}(t) &= -\delta_{i,j} + \frac{1}{2}\Theta(t)_{i,j} - \langle \hat{\varepsilon}_{i,j}(t)\, t\, , t \rangle. \end{aligned} \tag{10}$$

We denote by $\widehat{E}(t)$ the $d \times d$ matrix such that $\widehat{E}(t)_{i,j} = 2\langle \hat{\varepsilon}_{i,j}(t)\, t\, , t \rangle$, which allows us to rewrite the last equality $r''(t) = -I_d + \frac{1}{2}\Theta(t) - \frac{1}{2}\widehat{E}(t)$. That yields

$$r''(t)^2 = I_d - \Theta(t) + \widehat{E}(t) + O(\|t\|^4), \tag{11}$$

because $\Theta(t) = O(\|t\|^2)$ and $\widehat{E}(t) = o(\|t\|^2)$. Thanks to (10), we rewrite (9) in the following way:

$$\begin{aligned}\Gamma^Z(t)_{\mathbf{k},\mathbf{l}} - \gamma(t)_{\mathbf{k},\mathbf{l}} = &\left\langle \bar{\varepsilon}(t)_{\mathbf{k}}\, t\ , \Delta(t) r^{(4)}_{\mathbf{l}}(0)\, t \right\rangle \\ &+ \left\langle r^{(4)}_{\mathbf{k}}(0)\, t - \bar{\varepsilon}(t)_{\mathbf{k}}\, t\, , \Delta(t)\bar{\varepsilon}(t)_{\mathbf{l}}\, t \right\rangle \\ &+ \left\langle r^{(4)}_{\mathbf{k}}(0)\, t - \bar{\varepsilon}(t)_{\mathbf{k}}\, t\, , (\Delta(t) - N_1(t))\, (r^{(4)}_{\mathbf{l}}(0)\, t - \bar{\varepsilon}(t)_{\mathbf{l}}\, t) \right\rangle .\end{aligned}$$

We write S_1, S_2 and S_3 the first, the second and the third terms of the sum, respectively. Let $\rho := \|r^{(4)}(0)\|$. For the following computations, we recall that Θ is continuous and homogeneous of degree 2 on $\mathbb{R}^d$ and that for $t \in \mathbb{R}^d \backslash \{0\}$, $\Delta(t) = \Theta(t)^{-1}$. We introduce $\delta := \max_{v \in \mathbb{S}^{d-1}} \|\Delta(v)\|$. Thanks to Cauchy-Schwarz inequality, we may bound the first term S_1 and the second one S_2 as follows:

$$\begin{aligned}&|S_1| \leq \delta\rho\|\bar{\varepsilon}(t)\|, \\ &|S_2| \leq \delta\|r^{(4)}(0) - \bar{\varepsilon}(t)\|\|\bar{\varepsilon}(t)\| \leq \delta\rho\|\bar{\varepsilon}(t)\| + \delta\|\bar{\varepsilon}(t)\|^2.\end{aligned}$$

We now focus on the third term S_3. In order to bound it, we write a precise expansion of $N_1(t) - \Delta(t)$ around zero, based on Formula (11). We have

$$\begin{aligned}N_1(t) &= \left(I_d - r''(t)^2\right)^{-1} = \left(\Theta(t) - \widehat{E}(t) + O(\|t\|^4)\right)^{-1} \\ &= \Delta(t)\, \left(I_d - \widehat{E}(t)\Delta(t) + O(\|t\|^4)\Delta(t)\right)^{-1}\end{aligned}$$

where $\widehat{E}(t)\Delta(t) + O(\|t\|^4)\Delta(t) = \left(\dfrac{\widehat{E}(t)}{\|t\|^2} + O(\|t\|^2)\right)\, \Delta(\dfrac{t}{\|t\|})$, which tends to 0.
If A is a $d \times d$ matrix, $(I_d - A)^{-1} = I_d + A + o(A)$ as $\|A\|$ tends to zero, so we get

$$N_1(t) = \Delta(t) + \Delta(t)\, \widehat{E}(t)\, \Delta(t) + \Delta(t)\, o\Big(\widehat{E}(t)\Delta(t)\Big) + O(1),$$

and hence

$$\|t\|^2\, (N_1(t) - \Delta(t)) = \Delta(\frac{t}{\|t\|})\, \frac{\widehat{E}(t)}{\|t\|^2}\, \Delta(\frac{t}{\|t\|}) + \Delta(\frac{t}{\|t\|})\, o\left(\frac{\widehat{E}(t)}{\|t\|^2}\right)\, \Delta(\frac{t}{\|t\|}) + O(\|t\|^2).$$

Since $\|\widehat{E}(t)\| \leq 2\, \|\hat{\varepsilon}(t)\|\, \|t\|^2$ and $t \mapsto \Delta(\dfrac{t}{\|t\|})$ is bounded, there exists a neighbourhood $\mathscr{V}$ of zero in $\mathbb{R}^d$ and constants $c, d > 0$ such that, for any $t \in \mathscr{V}\backslash\{0\}$,

$$\|t\|^2\, \|N_1(t) - \Delta(t)\| \leq c\, \delta^2\, \|\hat{\varepsilon}(t)\| + d\|t\|^2.$$

Consequently, since $\bar{\varepsilon}(t)$ tends to zero as t tends to zero, there exists a neighbourhood $\mathscr{V}'$ of zero in $\mathbb{R}^d$ and $c' > 0$ such that, for any $t \in \mathscr{V}'\backslash\{0\}$,

$$|S_3| \leq (\rho + \|\bar{\varepsilon}(t)\|)^2(c\delta^2\|\hat{\varepsilon}(t)\| + d\|t\|^2) \leq c'(\|\hat{\varepsilon}(t)\| + \|t\|^2).$$

So, for any $1 \leq \mathbf{k}, \mathbf{l} \leq K$, $\left|\Gamma^Z(t)_{\mathbf{k},\mathbf{l}} - \gamma(t)_{\mathbf{k},\mathbf{l}}\right|$ is bounded by a term proportional to $\|\bar{\varepsilon}(t)\| + \|\hat{\varepsilon}(t)\| + \|t\|^2$ in a neighbourhood of zero. This concludes the proof of Lemma

II.7. □

Let us now combine all our intermediate results to complete the proof of the theorem. Our aim is to prove that $G(v,\cdot) \in L^1(\mathscr{V}_0, \|t\|^{-d}dt)$ in order to conclude thanks to Lemma II.1. We recall that Lemma II.4 allows us to write that $G(v,t) = Q_{(d)}(\Gamma^Z(t)) + o(\|t\|)$ as t tends to zero. Using Proposition II.6, we get for $t \neq 0$

$$\begin{aligned} G(v,t) &= Q_{(d)}(\Gamma^Z(t)) - Q_{(d)}(\gamma(t)) + o(\|t\|) \\ &= \left\langle \left(Q_{(d)}\right)'(\gamma(t)), \Gamma^Z(t) - \gamma(t) \right\rangle + o\left(\|\Gamma^Z(t) - \gamma(t)\|\right) + o(\|t\|). \end{aligned}$$

Since $\left\| \left(Q_{(d)}\right)'(\gamma(t)) \right\|$ is bounded if t belongs to any compact set of $\mathbb{R}^d$ that does not contain 0, we deduce from Lemma II.7 that there exist a neighbourhood of zero $\mathscr{W}$ in $\mathbb{R}^d$ and a positive constant c such that, for any $t \in \mathscr{W}\backslash\{0\}$,

$$G(v,t) \leq c\left(\|\bar{\varepsilon}(t)\| + \|\hat{\varepsilon}(t)\| + \|t\|\right).$$

A change of variable easily shows that Condition **(G)** implies that $\bar{\varepsilon} \in L^1(\mathscr{V}_0, \|t\|^{-d}dt)$ and the same holds for $\hat{\varepsilon}$. Obviously, we also have $t \mapsto \|t\| \in L^1(\mathscr{V}_0, \|t\|^{-d}dt)$. Consequently, under Condition **(G)**, $G(v,\cdot) \in L^1(\mathscr{V}_0, \|t\|^{-d}dt)$. The proof of Theorem II.3 is complete. □

Conclusion and perspectives

In brief, in this chapter, we have addressed the issue of the finiteness of the variance of $N^{X'}(T,v)$ in dimension $d > 1$, with no assumption of isotropy on X. We do not restrict ourselves to the number of stationary points $N^{X'}(T,0)$. In fact, a sufficient condition is established in Theorem II.3. It is named **(G)** and appears as a generalization to higher dimensions of Geman condition. As in dimension one, it does not depend on the considered level $v \in \mathbb{R}^d$.

To end with, let us mention a notable consequence of our result. Recall that the stationary points of X are involved in the computation of another random variable linked to the geometrical properties of X: the Euler characteristic of any of its excursion sets. The question of the finiteness of its second moment has been raised forty years ago in [AH76] and is still subject to investigation, see [TSJJ09] for instance. It turns out that Condition **(G)** is also a necessary condition for the modified Euler characteristic of an excursion set to admit a finite second moment.

Indeed, assume that X is a random field satisfying the assumptions introduced in Section II.1, that T is a d-dimensional compact rectangle and that u is a fixed real level. The modified Euler characteristic of the excursion set of X restricted to T above level u,

$$A(u,X,T) := \{t \in T \ : \ X(t) \geq u\},$$

is given by Definition I.18:

$$\phi(A(u,X,T)) = \sum_{i=0}^{d} (-1)^i \widetilde{\mu}_i(u,X,T),$$

where $\widetilde{\mu}_i(u, X, T) = \#\{t \in T : X(t) \geq u,\, X'(t) = 0,\, \mathrm{index}(X''(t)) = d - i\}$ and the "index" stands for the number of negative eigenvalues. Since each $\widetilde{\mu}_i(u, X, T)$ is bounded by $N^{X'}(T, 0)$, it is clear that $\phi(A(u, X, T))$ is square integrable as soon as it is the case for $N^{X'}(T, 0)$. So Condition **(G)** appears as a sufficient condition for the existence of a finite second moment for the modified Euler characteristic of any excursion set of X above a finite bounded rectangle. The existence of the same sufficient condition is not obvious when the modified Euler characteristic is replaced by the Euler characteristic χ itself. Indeed, Morse formula stated in Theorem I.14 not only involves critical points of X situated on $\mathring{T}$ (through the term $\psi(\mathring{T}) \leq N^{X'}(T, 0)$), but also critical points of restrictions of X to lower-dimensional faces of T.

An open question is whether, in dimension $d > 1$, **(G)** remains a necessary condition for $N^{X'}(T, v)$ to admit a second moment. Another natural question concerns the finiteness of the moments of $N^{X'}(T, v)$ of order higher than two. In particular, sufficient conditions on the covariance function of X should be investigated. Note that in [Mal85], the author deals with the higher moments of $N^{Y}(T, v)$, where $Y \colon \mathbb{R}^d \to \mathbb{R}^d$ is a multivariate random field, and an answer is given through a condition on the spectral density. The latter problem is not the same, but close to ours.

As it is done in dimension one in [KL06], our work could be extended to the study of the finiteness of the variance of $N^{X'}(T, g) := \{t \in T \colon X'(t) = g(t)\}$, where $g \colon \mathbb{R}^d \to \mathbb{R}^d$ is a function of class $\mathscr{C}^1$. Does our condition remain sufficient with some assumptions on g?

Appendix: computations in dimension $\mathrm{d} = 2$

Here is an alternative proof of Proposition II.6 in the case $d = 2$, based on the explicit computations of the polynomial $Q_{(2)}$ introduced in Lemma II.4 and of function $\gamma(t)$ defined by (8). We conclude with a remark concerning the meaning of Geman condition on a separable covariance in dimension two.

We introduce the following notations:

$$\mu_1 = r^{(4)}_{1111}(0)\ ;\ \nu_1 = r^{(4)}_{1112}(0)\ ;\ \nu = r^{(4)}_{1122}(0)\ ;\ \nu_2 = r^{(4)}_{1222}(0)\ ;\ \mu_2 = r^{(4)}_{2222}(0)\,.$$

Note that in the case $d = 2$, we have $K = 3$ and from now on, we use the lexicographic order to denote the "double" indices $\mathbf{k}$, i.e. $\mathbf{1} = (1, 1)\,,\ \mathbf{2} = (1, 2)\,,\ \mathbf{3} = (2, 2)$.

Matrix $\gamma(t)$. The coefficients of the 3×3 symmetric matrix $\gamma(t)$ are defined by (8). For any $t = (t_1, t_2) \neq (0, 0)$, we have

$$\begin{array}{lll} \gamma_{\mathbf{11}}(t) = \dfrac{\alpha}{D(t)}\, t_2^4\,; & \gamma_{\mathbf{12}}(t) = -\dfrac{\alpha}{D(t)}\, t_1 t_2^3\,; & \gamma_{\mathbf{13}}(t) = \dfrac{\alpha}{D(t)}\, t_1^2 t_2^2\,; \\ \gamma_{\mathbf{22}}(t) = \dfrac{\alpha}{D(t)}\, t_1^2 t_2^2\,; & \gamma_{\mathbf{23}}(t) = -\dfrac{\alpha}{D(t)}\, t_1^3 t_2\,; & \gamma_{\mathbf{33}}(t) = \dfrac{\alpha}{D(t)}\, t_1^4\,. \end{array}$$

where $\alpha = \mu_1\mu_2\nu - \mu_1\nu_2^2 - \mu_2\nu_1^2 - \nu^3 + 2\nu\nu_1\nu_2$
and $D(t) = \det \begin{pmatrix} \mu_1 t_1^2 + \nu t_2^2 + 2\nu_1 t_1 t_2 & \nu_1 t_1^2 + \nu_2 t_2^2 + 2\nu t_1 t_2 \\ \nu_1 t_1^2 + \nu_2 t_2^2 + 2\nu t_1 t_2 & \nu t_1^2 + \mu_2 t_2^2 + 2\nu_2 t_1 t_2 \end{pmatrix}$.

Polynomial function $Q_{(2)}$. Let $Y = (Y_1, Y_2, Y_3)$ be a centred Gaussian vector. Then

$$\mathbb{E}[\tilde{\det}(Y)^2] = \mathbb{E}[(Y_1Y_3 - (Y_2)^2)^2] = \mathbb{E}[(Y_1Y_3)^2] - 2\mathbb{E}[Y_1Y_3(Y_2)^2] + \mathbb{E}[(Y_2)^4]$$

$$= 2\mathbb{E}[Y_1Y_3]^2 + \mathbb{E}[(Y_1)^2]\mathbb{E}[(Y_3)^2] - 2\mathbb{E}[Y_1Y_3]\mathbb{E}[(Y_2)^2] - 4\mathbb{E}[Y_1Y_2]\mathbb{E}[Y_2Y_3] + 3\mathbb{E}[(Y_2)^2]^2$$

where we have used Wick formula to get the last line. Formula (7) defining the polynomial function $Q_{(2)}$ entails that

$$Q_{(2)}(\gamma) \quad = \quad 2\gamma_{\mathbf{13}}{}^2 + \gamma_{\mathbf{11}}\gamma_{\mathbf{33}} - 2\gamma_{\mathbf{13}}\gamma_{\mathbf{22}} - 4\gamma_{\mathbf{12}}\gamma_{\mathbf{23}} + 3\gamma_{\mathbf{22}}{}^2.$$

Using the expression of $\gamma(t)$, we recover that $Q_{(2)}(\gamma(t)) = 0\,,\ \forall t \neq 0$.

Case of a separable covariance. Let us focus on the special case where $r(t_1, t_2) = R_1(t_1)R_2(t_2)$, R_1 and R_2 being two one-dimensional covariance functions, each of them of class $\mathscr{C}^4$. Then the fourth derivatives of r are such that $r^{(4)}_{\mathbf{kl}}(0) - r^{(4)}_{\mathbf{kl}}(t) = o(\|t\|)$ for any $(\mathbf{k}, \mathbf{l}) \neq (1,1), (3,3)$. Indeed, for instance $r^{(4)}_{\mathbf{12}}(t) = R_1^{(3)}(t_1)R_2'(t_2)$, and all the $R_i^{(j)}$'s for $i = 1, 2$ and $j = 0, 1, 2, 3$ are at least of class $\mathscr{C}^1$. Hence, r satisfies our Geman condition **(G)** if and only if R_1 and R_2 both satisfy the usual one-dimensional Geman condition.

Chapter III

Characterization of deformed random fields through their excursion sets

This chapter, based on the paper [Fou18] accepted for publication in the *Applied Probability Journals*, deals with a class of non-stationary and non-isotropic fields called deformed random fields. They are obtained by deforming a fixed stationary and isotropic random field thanks to a deterministic function that transforms bijectively the index set. Deformed fields respond to the need to model spatial and physical phenomena that are in numerous cases not stationary nor isotropic. For example, they are currently widely used in cosmology to model the cosmic microwave background (CMB) deformed anisotropically by the gravitational lensing effect, with mass reconstruction as an objective [HO02].

Our framework is two-dimensional: we set $X\colon \mathbb{R}^2 \to \mathbb{R}$ the underlying stationary and isotropic field, $\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ a bijective deterministic function and $X_\theta = X \circ \theta$ the deformed field. In fact, most studies on the deformed field model deal with dimension two. The reason for this is that it is the simplest case of multi-dimensionality, the results can be illustrated easily thanks to simulations and it still covers a lot of possible applications, particularly in image analysis. For instance, deformed fields are involved in the "shape from texture" issue, that is, the problem of recovering a 3-dimensional textured surface thanks to a 2-dimensional projection [CM02].

The model of deformed fields was introduced in 1992 in a spatial statistics framework by Sampson and Guttorp in [SG92], with only a stationarity assumption on X. It is also studied through the covariance function in [PM99] and in [PS00]. In [ASP16], the authors investigate the case of a linear deformation with a matrix representation as the product of a diagonal and a rotation matrix, which produces what they call "geometric anisotropy". In [CM03], the deformed field model is studied as a particular case of a model of deterministic deformation operator applied to a stationary field X. A lot of papers also propose methods to estimate θ, as we will see a little further on in this introduction, when we come to our own contribution to the estimation matter.

Unless otherwise specified, the kind of stationarity and isotropy that we consider consists in an invariance of the field's law under translations or, respectively, rotations. Even though the underlying field X is stationary and isotropic, a lot of deformations transform the index space $\mathbb{R}^2$ in such a way that the stationarity and/or the isotropy

are lost when it comes to the deformed field. The deformations preserving stationarity are the linear deformations. Concerning isotropy, a natural question arises: which are the deformations θ that preserve isotropy, for any underlying field X? It is solved in Section III.2. We give an explicit form for this kind of deformations and we call them spiral deformations. Let us point out here that the question of preserving the isotropy for one fixed underlying field X is different, and it is solved in Section III.4.

For the rest of the paper, we have in mind the following practical problem: the covariance function of the underlying field X and the deformation θ are unknown. We try to study and even to identify θ through observations of some excursion sets of X_θ above fixed levels. For this, we add some assumptions on X (Gaussianity, a little more than $\mathscr{C}^2$-regularity, a non-degeneracy assumption) and on θ ($\mathscr{C}^2$-regularity), which are precisely described and justified in Section III.1, and we focus on the mean Euler characteristic of the excursion sets. This additive topological functional has been introduced in Section I.2.2. Heuristically, the Euler characteristic of a set is determined by its topology: for a two-dimensional compact set, it is the number of connected components minus the number of holes in this set; for a one-dimensional set, it is simply the number of closed intervals that compose the set. Note that a modified version of the Euler characteristic of excursion sets will be more adequate to address our problem. The formulas of the expectation of the (modified) Euler characteristic of an excursion set of X_θ can be found in Section III.3.

More precisely, let T be a bounded rectangle domain or segment in $\mathbb{R}^2$. We are interested in the Euler characteristic χ of the excursion set of X_θ restricted to T above a level $u \in \mathbb{R}$, $A_u(X_\theta, T) = \{t \in T,\ X(\theta(t)) \geq u\}$. However, we may study equivalently the stationary and isotropic field X on the transformed set $\theta(T)$ instead of the deformed field X_θ on the set T, since

$$\chi(A_u(X_\theta, T)) = \chi(A_u(X, \theta(T))).$$

In Section III.4, we introduce the notion of χ-isotropic deformation: it applies to a deformation θ such that, for any level u and for any rectangle T, $\mathbb{E}[\chi(A_u(X_\theta, T))]$ does not vary under any rotation of T. This is in particular true if the deformed field is isotropic, hence this property can be viewed as a weak notion of isotropy. However, it turns out that this weak notion implies the strong one (isotropy in law), that is, the χ-isotropic deformations are exactly the spiral deformations.

In Section III.5, we tackle the problem of identifying θ, assuming that we only have at our disposal the mean Euler characteristic of some excursion sets of the deformed field. The problem of the estimation of a deformation θ thanks to the observation of the deformed random field X_θ is originally a spatial statistics problem and it has been studied under different angles since it was introduced. At first, [SG92] used several observations on a sparse grid to estimate θ. Another approach is to use only one observation of the deformed field on a dense grid; it is adopted in [GP00], [CM03], [AS08], [AC09], and [FDR15] with an underlying field that is stationary and/or isotropic. These different papers involve convergence results on quadratic variations and quasi-conformal theory. The study in [AG16] applies in particular to deformed fields of the form $\{X(x + \nabla\eta(x)),\ x \in \mathbb{R}^2\}$ (where $\eta\colon \mathbb{R}^2 \to \mathbb{R}$ is a deterministic function), which modelize the CMB [HO02]. The authors propose a method to estimate function η that corresponds to the gravitational lensing of the CMB.

Our approach differs from the previous ones, since our observations are limited to realizations of X_θ over a fixed level, and not to the whole realizations. Our method is closer to the one in [Cab87], where the inference of the deformation is based on the size and shape of the deformed field's level curves; however, the author restricts the deformations to linear ones given by symmetric, positive and definite matrices. An analogous approach is adopted in [Ber18] thanks to more general functionals of the level sets. With our sparse observations, we manage as well as in [AS08] to compute the complex dilatation of θ up to a conformal map, at every point of the domain. The complex dilatation provides a characterization of the deformation.

In this paper, we prove four main results. Theorem III.5 states that the deformations preserving isotropy are exactly the spiral deformations. In Theorem III.13, the class of deformations satisfying the invariance condition of the mean Euler characteristic of excursion sets is identified with the spiral deformations. A consequence of this theorem is Corollary III.17. Roughly speaking, it states that three notions of preservation of isotropy coincide and correspond to the set of spiral deformations. In Section III.5, we show how to almost entirely identify θ through the mean Euler characteristic of its excursion sets over basic domains. The general case is described by Method III.19. To end with, in Section III.5.2, limiting ourselves to spiral deformations, we finally propose an estimation method based on one single observation of the deformed field.

Notations and assumptions

For any compact set A in $\mathbb{R}^2$, we write $\dim(A)$ its Hausdorff dimension; if $\dim(A) = 1$, we write $|A|_1$ its one-dimensional Hausdorff measure; if $\dim(A) = 2$, we write $|A|_2$ its two-dimensional Hausdorff measure; we also write ∂A the frontier of A and $\mathring{A}$ its interior.

We work in a fixed orthonormal basis in $\mathbb{R}^2$ and we use the same notation for a linear application defined on $\mathbb{R}^2$ and taking values in $\mathbb{R}^2$ and for its matrix in this basis of $\mathbb{R}^2$. We denote by $O(2)$ the group of orthogonal transformations in $\mathbb{R}^2$ and by $SO(2)$ the group of rotations in $\mathbb{R}^2$. For any $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$, ρ_α stands for the rotation of angle α and u_α denotes the unit vector $(\cos\alpha, \sin\alpha)$. The canonical Euclidian scalar product in $\mathbb{R}^2$ and the associated norm are written $\langle \cdot\,, \cdot \rangle$ and $\| \cdot \|$, respectively.

For any real s, we write $[0, s] = \{x \in \mathbb{R},\, 0 \leq x \leq s\}$ if $s \geq 0$ and $[0, s] = \{x \in \mathbb{R},\, s \leq x \leq 0\}$ if $s < 0$. We say a set T in $\mathbb{R}^2$ is a segment if there exists $(a, b) \in (\mathbb{R}^2)^2$ with $a \neq b$ such that $T = \{a + t(b - a),\, t \in [0, 1]\}$. For any $(s, t) \in \mathbb{R}^2$, we write $T(s, t) = [0, s] \times [0, t]$ and we say a set T in $\mathbb{R}^2$ is a rectangle if there exist $(s, t) \in (\mathbb{R}\backslash\{0\})^2$, $\rho \in SO(2)$ and a translation τ such that $T = \rho \circ \tau(T(s, t))$.

If $f = (f_1, f_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$, with $f_i\colon \mathbb{R}^2 \to \mathbb{R}$ for $i \in \{1, 2\}$, is a differentiable function, for any $x = (s, t) \in \mathbb{R}^2$, we use the notations $J^1_f(x)$ for the vector $\partial_s f(x) = (\partial_s f_1(x), \partial_s f_2(x))$, $J^2_f(x)$ for the vector $\partial_t f(x)$ and $J_f(x)$ for the Jacobian matrix of f at point x. More generally, if M is a 2×2 matrix, for $i \in \{1, 2\}$, we write M^i the i^{th} column of M.

Let X be a Gaussian, stationary and isotropic random field, defined on $\mathbb{R}^2$ and taking real values; we write $C\colon \mathbb{R}^2 \to \mathbb{R}$ its covariance function. Since X is station-

ary, we may assume it is centred. We shall also assume that $C(0) = 1$ since if not, we consider the field $\frac{1}{\sqrt{C(0)}}X$ instead of X. As for the regularity of X, we make the assumption that almost every realization of X is of class $\mathscr{C}^2$ on $\mathbb{R}^2$. As a consequence, according to Proposition I.5 C is of class $\mathscr{C}^4$. We denote by $X'(t)$ and by $X''(t)$ the gradient vector and the Hessian matrix of X at point t, respectively, and by $C''(t)$ the Hessian matrix of C at point t. In order to be able to apply the mean Euler characteristic of excursion sets formula, we make some assumptions on the covariance function of X. In particular, we assume that the joint distribution of $(X'_i(0), X''_{i,j}(0))_{(i,j)\in\{1,2\}^2, i\leq j}$ is not degenerate. Due to stationnarity, this exactly means that for any $t \in \mathbb{R}^2$, $(X'_i(t), X''_{i,j}(t))_{(i,j)\in\{1,2\}^2, i\leq j}$ is not degenerate, and it is equivalent to the non-degeneracy of both $(X'_i(0))_{i\in\{1,2\}}$ and $(X''_{i,j}(0))_{(i,j)\in\{1,2\}^2, i\leq j}$. Consequently, the covariance matrix of $X'(0)$ is not degenerate; since X is isotropic, there exists $\lambda > 0$ such that $\mathrm{Cov}(X'(0)) = \lambda I_2$. If $\lambda \neq 1$, X_θ is nevertheless equal to $\tilde{X}_{\tilde{\theta}}$, with $\tilde{\theta} = \sqrt{\lambda}\theta$ and with $\tilde{X}(\cdot) = X(\sqrt{\lambda}^{-1}\cdot)$ satisfying $\mathrm{Cov}(\tilde{X}'(0)) = I_2$. Consequently, without loss of generality, we shall assume that $C''(0) = -I_2$.

We gather all the assumptions on X that will be in force in Sections III.3, III.4 and III.5 under the name **(H)**:

$$(\mathbf{H})\begin{cases} X \text{ is Gaussian,} \\ X \text{ is stationary and isotropic,} \\ X \text{ is almost surely of class } \mathscr{C}^2, \\ \exists \varepsilon > 0,\ \exists \alpha > 0,\ \exists K > 0,\ \forall t \in \mathbb{R}^2, \\ \qquad \|t\| \leq \varepsilon \ \Rightarrow \left| \dfrac{\partial^4 C}{\partial t_i^2 \partial t_j^2}(t) - \dfrac{\partial^4 C}{\partial t_i^2 \partial t_j^2}(0) \right| \leq K |\log(\|t\|)|^{-(1+\alpha)}, \\ \text{the joint distribution of } (X'_i(0), X''_{i,j}(0))_{\substack{(i,j)\in\{1,2\}^2 \\ i\leq j}} \text{ is not degenerate,} \\ X \text{ is centred, } C(0) = 1 \text{ and } C''(0) = -I_2. \end{cases}$$

Our ambition in Section III.5 is to identify the deformation θ assuming that we only have at our disposal the expectation of $\chi(A_u(X_\theta, T))$ for different sets T and for a fixed level u. However, it is not possible to distinguish between θ and another deformation $\tilde{\theta}$ such that the random fields X_θ and $X_{\tilde{\theta}}$ have the same law. Because of the stationarity and the isotropy of X, it occurs if $\tilde{\theta} = \rho \circ \theta + u$, where $\rho \in O(2)$ and $u \in \mathbb{R}^2$. Therefore, we can only hope to determine a deformation θ up to left-composition with a rotation and with a translation.

Consequently, without loss of generality, we can make the assumption that $\theta(0) = 0$. If θ is differentiable, we shall also assume that for any $x \in \mathbb{R}^2$, $\det(J_\theta(x))$ is positive or, in other words, that θ preserves orientation. Indeed, function $x \mapsto \det(J_\theta(x))$ is continuous on $\mathbb{R}^2$ and does not take zero value, hence it takes either only positive or only negative values. If for all $x \in \mathbb{R}^2$, $\det(J_\theta(x)) < 0$, we can replace θ by $\sigma \circ \theta$, where $\sigma \in O(2)$ is the symmetry with respect to the axis of abscissa; then for any $x \in \mathbb{R}^2$, $J_{\sigma\circ\theta}(x) = \sigma \circ J_\theta(x)$ and so the Jacobian determinant of $\sigma \circ \theta$ is positive on $\mathbb{R}^2$. Those two transformations on θ (translation along vector $-\theta(0)$ and left composition with σ) do not change the law of X_θ. Note that the class of linear as well as tensorial deformations

considered as examples in Section III.5 are stable under those transformations made in order to simplify our study.

We define $\mathscr{D}^0(\mathbb{R}^2)$ the set of continuous and bijective functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ with a continuous inverse, taking value 0 at 0. For $i \in \{1,2\}$, we define $\mathscr{D}^i(\mathbb{R}^2)$ the set of $\mathscr{C}^i$-diffeomorphisms from $\mathbb{R}^2$ to $\mathbb{R}^2$ taking value 0 at 0. We call such functions (in $\mathscr{D}^0(\mathbb{R}^2)$ or in $\mathscr{D}^2(\mathbb{R}^2)$, according to the section of this paper) deformations.

Note that the assumptions on X and on θ that we have just listed are not all in force in Section III.2, where we soften the regularity assumptions on X and θ and we replace the Gaussian hypothesis on X by the assumption of the existence of a second moment.

For which θ is X_θ isotropic?

In this section, Assumption **(H)** on X is **not** in force. We only assume that X is stationary, isotropic and that it admits a second moment. Considering θ in $\mathscr{D}^1(\mathbb{R}^2)$, we denote by C_θ the covariance function of the deformed field X_θ. Because the field X is stationary, for any $(x,y) \in (\mathbb{R}^2)^2$,

$$C_\theta(x,y) = \mathrm{Cov}(X_\theta(x), X_\theta(y)) = C(\theta(x) - \theta(y)). \tag{1}$$

In the following, we exhibit the deformations θ in $\mathscr{D}^1(\mathbb{R}^2)$ that leave the field X_θ isotropic, for any stationary and isotropic field X. Note that the underlying field X is not fixed. Our approach is analogous to the one in [PS00], where the objective is, starting with a random field Y with a known covariance function, to find a deformation θ such that $Y = X \circ \theta$, with $X\colon \mathbb{R}^2 \to \mathbb{R}$ a stationary, or stationary and isotropic random field.

We begin with a short introduction of notations relative to polar representation. We denote by S the transformation of polar coordinates to cartesian coordinates in the plane deprived of the origin:

$$S\colon (0,+\infty) \times \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^2\backslash\{0\} \quad (r,\varphi) \mapsto (r\cos\varphi, r\sin\varphi).$$

We define $\mathscr{D}^0\left((0,+\infty) \times \mathbb{R}/2\pi\mathbb{Z}\right)$ the set of continuous and bijective functions $\hat{\theta}\colon (0,+\infty)\times \mathbb{R}/2\pi\mathbb{Z} \to (0,+\infty) \times \mathbb{R}/2\pi\mathbb{Z}$ with continuous inverses. For any deformation $\theta \in \mathscr{D}^0\left(\mathbb{R}^2\right)$, we write $\theta_0 = \theta_{|\mathbb{R}^2\backslash\{0\}}$, we define the deformation $\hat{\theta} \in \mathscr{D}^0\left((0,+\infty) \times \mathbb{R}/2\pi\mathbb{Z}\right)$ by $\hat{\theta} = S^{-1} \circ \theta_0 \circ S$ and we denote by $\hat{\theta}^1$ and $\hat{\theta}^2$ its coordinate functions.

Proposition III.1 *The mapping $\mathscr{D}^0\left(\mathbb{R}^2\right) \to \mathscr{D}^0\left((0,+\infty) \times \mathbb{R}/2\pi\mathbb{Z}\right),\ \theta \mapsto \hat{\theta}$ is injective and it is a group morphism, that is to say if η and θ belong to $\mathscr{D}^0\left(\mathbb{R}^2\right)$ then $\widehat{\eta\circ\theta} = \hat{\eta}\circ\hat{\theta}$. Moreover, the coordinate functions of the composition $\widehat{\eta\circ\theta}$ are*

$$\widehat{\eta\circ\theta}^1 = \hat{\eta}^1 \circ \hat{\theta} \quad \textit{and} \quad \widehat{\eta\circ\theta}^2 = \hat{\eta}^2 \circ \hat{\theta}.$$

Proof. The above application is obviously injective and if η and θ belong to $\mathscr{D}^0\left(\mathbb{R}^2\right)$, then

$$(\eta\circ\theta)_0 = \eta_0 \circ \theta_0 = (S\circ\hat{\eta}\circ S^{-1}) \circ (S\circ\hat{\theta}\circ S^{-1}) = S\circ\hat{\eta}\circ\hat{\theta}\circ S^{-1},$$

hence we get the homomorphism property. Consequently, for $i \in \{1,2\}$, the coordinate function $\widehat{\eta \circ \theta}^i$ satisfies

$$(\widehat{\eta \circ \theta}^1, \widehat{\eta \circ \theta}^2) = \widehat{\eta \circ \theta} = \hat{\eta} \circ \hat{\theta} = (\hat{\eta}^1 \circ \hat{\theta}, \hat{\eta}^2 \circ \hat{\theta}).$$

■

Definition III.2 *A deformation $\theta \in \mathscr{D}^1(\mathbb{R}^2)$ is a spiral deformation if there exist $f\colon (0,+\infty) \to (0,+\infty)$ strictly increasing and surjective, $g\colon (0,+\infty) \to \mathbb{R}/2\pi\mathbb{Z}$ and $\varepsilon \in \{\pm 1\}$ such that θ satisfies*

$$\forall (r,\varphi) \in (0,+\infty) \times \mathbb{R}/2\pi\mathbb{Z}, \quad \hat{\theta}(r,\varphi) = (f(r),\, g(r) + \varepsilon\varphi). \tag{2}$$

Remark III.3 *Note that the set of spiral deformations forms a group for the operation of composition. The choice of f strictly increasing is due to the conditions of continuity and inversibility on θ and to the fact that $\theta(0) = 0$. The 2π-periodicity of $\hat{\theta^2}$ entails that the coefficient ε in the angular part of (2) is an integer and the inversibility of θ implies that ε belongs to $\{\pm 1\}$. If we only consider deformations with positive Jacobian determinants, in accordance with our explanations in Section III.1, then we can set $\varepsilon = 1$. Indeed, the positivity of the Jacobian determinant of θ is equivalent to the positivity of the one of $\hat{\theta}$ (see Formula (17) in the following by way of justification).*

Example III.4 *Linear spiral deformations. A linear spiral deformation is a deformation with polar representation either $(r,\varphi) \mapsto (\lambda r, \varphi + \alpha)$ or $(r,\varphi) \mapsto (\lambda r, -\varphi + \alpha)$, with $\lambda > 0$ and $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$, that is to say it is of the form $\lambda\rho$, with $\lambda > 0$ and $\rho \in O(2)$.*

In [Cab87] and in [Ber18], the deformations are restricted to the ones given by symmetric, positive and definite matrices. In that case, the field X_θ is isotropic if and only if the two positive eigenvalues of θ are equal. In the following theorem, we also determine the deformations preserving isotropy but in the general case.

Theorem III.5 *The deformations in $\mathscr{D}^1(\mathbb{R}^2)$ such that for any stationary and isotropic field X, X_θ is isotropic are the spiral deformations.*

Proof. To prove the direct implication, let us assume that a deformation θ is a spiral deformation with polar representation (2) and let $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$. We recall that ρ_α stands for the rotation of angle α in $\mathbb{R}^2$.

$$\begin{aligned}\forall (r,\varphi) \in (0,+\infty) \times \mathbb{R}/2\pi\mathbb{Z}, \quad \hat{\theta} \circ \hat{\rho_\alpha}(r,\varphi) &= (f(r), g(r) + \varepsilon(\varphi + \alpha)) \\ &= \hat{\rho_{\varepsilon\alpha}} \circ \hat{\theta}(r,\varphi).\end{aligned}$$

Therefore, θ satisfies the following property:

$$\forall \rho \in SO(2), \quad \exists \rho' \in SO(2) \,/\, \theta \circ \rho = \rho' \circ \theta.$$

This entails that $X_\theta \circ \rho = X \circ \rho' \circ \theta$. The isotropy of X implies that $X \circ \rho'$ has the same law as X. Consequently, $X_\theta \circ \rho$ has the same law as X_θ. Thus the isotropy of X_θ is proved.

We now turn to the converse implication. Let us assume that for any stationary and isotropic field X, the field X_θ is isotropic. Hence its covariance function, given by (1) is invariant under the action of any rotation:

$$\forall \rho \in SO(2),\ \forall (x,y) \in (\mathbb{R}^2)^2, \quad C_\theta(\rho(x),\rho(y)) = C_\theta(x,y).$$

In particular, if we use the Gaussian covariance function $C(x) = \exp(-\|x\|^2)$, we obtain

$$\forall \rho \in SO(2),\ \forall (x,y) \in (\mathbb{R}^2)^2, \quad \|\theta(\rho(x)) - \theta(\rho(y))\| = \|\theta(x) - \theta(y)\|. \tag{3}$$

Taking $y = 0$, we deduce from (3) that $\hat{\theta}^1$ is radial. We set for any $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$ and for any $r > 0$, $\hat{\theta}^1(r,\varphi) = f(r)$. Since θ is bijective, continuous and $\theta(0) = 0$, f is necessarily strictly increasing with $\lim_{r\to 0} f(r) = 0$ and $\lim_{r\to+\infty} f(r) = +\infty$.

To infer the form of $\hat{\theta}^2$, we fix $r > 0$ and, for any $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$, we use the complex representation to write Formula (3) for $x = re^{i\varphi}$, $y = r$ and for any angle α of the rotation ρ. Dividing the equality by $f(r)$, we get

$$|e^{i\hat{\theta}^2(r,\varphi+\alpha)} - e^{i\hat{\theta}^2(r,\alpha)}| = |e^{i\hat{\theta}^2(r,\varphi)} - e^{i\hat{\theta}^2(r,0)}|,$$

hence

$$|1 - e^{i(\hat{\theta}^2(r,\varphi+\alpha) - \hat{\theta}^2(r,\alpha))}| = |1 - e^{i(\hat{\theta}^2(r,\varphi) - \hat{\theta}^2(r,0))}|.$$

Since 1 as well as each exponential term belongs to $\{z \in \mathbb{C} \,/\, |z| = 1\}$, a geometric interpretation of the above equality entails that for any $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$, there exists $\epsilon(r,\varphi,\alpha) \in \{\pm 1\}$ such that

$$\hat{\theta}^2(r,\varphi+\alpha) - \hat{\theta}^2(r,\alpha) = \epsilon(r,\varphi,\alpha)\,(\hat{\theta}^2(r,\varphi) - \hat{\theta}^2(r,0)). \tag{4}$$

Assuming that there exists $\varphi \neq 0$ such that $\hat{\theta}^2(r,\varphi) - \hat{\theta}^2(r,0) = 0$, we deduce from (4) that $\hat{\theta}^2(r,\cdot)$ is constant on $\mathbb{R}/2\pi\mathbb{Z}$, which contradicts the bijectivity of θ. Consequently, for any $\varphi \neq 0$,

$$\epsilon(r,\varphi,\alpha) = \frac{\hat{\theta}^2(r,\varphi+\alpha) - \hat{\theta}^2(r,\alpha)}{\hat{\theta}^2(r,\varphi) - \hat{\theta}^2(r,0)}.$$

This implies that ϵ is continuous from $(0,+\infty) \times \mathbb{R}/2\pi\mathbb{Z}\backslash\{0\} \times \mathbb{R}/2\pi\mathbb{Z}$ onto $\{\pm 1\}$. A connexity argument applies and implies that ϵ is constant. We write $\epsilon(r,\varphi,\alpha) = \epsilon \in \{\pm 1\}$.

We fix $r > 0$. For any $(\varphi,\alpha) \in (\mathbb{R}/2\pi\mathbb{Z})^2$, we can rewrite (4)

$$\hat{\theta}^2(r,\varphi+\alpha) = \hat{\theta}^2(r,\alpha) + \epsilon(\hat{\theta}^2(r,\varphi) - \hat{\theta}^2(r,0)).$$

By differentiating the above equality with respect to α, for a fixed $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$, we deduce that $\partial_\varphi \hat{\theta}^2(r,\cdot)$ is constant on $\mathbb{R}/2\pi\mathbb{Z}$. Therefore, there exists $k(r) \in \{\pm 1\}$ and $g(r) \in \mathbb{R}/2\pi\mathbb{Z}$ such that

$$\forall r > 0,\ \forall \varphi \in \mathbb{R}/2\pi\mathbb{Z}, \quad \hat{\theta}^2(r,\varphi) = k(r)\varphi + g(r).$$

Note that the reason why $k(r)$ must belong to $\{\pm 1\}$ has already been explained in Remarks III.3. Finally, since $\hat{\theta}^2$ is continuous, $k(r)$ is necessarily constant, which concludes the proof of Theorem III.5. ■

Remark III.6 *Considering the proof of Theorem III.5, we could state an equivalent version of it, requiring only one fixed stationary and isotropic random field X such that its covariance function is injective: the deformations in $\mathscr{D}^1(\mathbb{R}^2)$ such that X_θ is isotropic are the spiral deformations.*

Expectation formulas

Euler characteristic of an excursion set

From now on, X is a random field assumed to satisfy **(H)** and θ is a deformation in $\mathscr{D}^2(\mathbb{R}^2)$. Consequently, even though X_θ is in general not stationary nor isotropic, it is Gaussian and its realizations are almost surely of class $\mathscr{C}^2$. In this section, we recall the expectation result of Section I.2.2.c and we apply it to our framework.

We start by introducing the general formula for the expectation of the Euler characteristic of an excursion set of X_θ, above a d-dimensional rectangle T and then we show how it adapts to dimensions $d = 1$ and $d = 2$. Let us first explain why we may study equivalently the stationary and isotropic field X on the transformed set $\theta(T)$ or the non-stationary and anisotropic field X_θ on the set T. The deformation θ is an homeomorphism and it satisfies $A_u(X_\theta, T) = \theta^{-1}(A_u(X, \theta(T)))$, therefore the sets $A_u(X_\theta, T)$ and $A_u(X, \theta(T))$ are homotopic. Since the Euler characteristic is homotopy invariant (see Proposition I.13), the above relation leads to

$$\chi(A_u(X_\theta, T)) = \chi(A_u(X, \theta(T))).$$

Consequently, we can focus on $\mathbb{E}[\chi(A_u(X, \theta(T)))]$. It can be computed thanks to Theorem I.17. Assumption **(H)** on X allows to apply it here. We write $(H_i)_{i\in\mathbb{N}}$ the Hermite polynomials and, for any real x, $H_{-1}(x) = \sqrt{2\pi}\Psi(x)\exp(x^2/2)$, where Ψ is the tail probability of a standard Gaussian variable. For T rectangle or segment in $\mathbb{R}^2$,

$$\mathbb{E}[\chi(A_u(X_\theta, T))] = \mathbb{E}[\chi(A_u(X, \theta(T)))] = \sum_{0\le i\le \dim(T)} \mathscr{L}_i(\theta(T))\,\rho_i(u), \tag{5}$$

$$\text{with} \quad \forall\, 0 \le i \le d,\ \rho_i(u) = (2\pi)^{-(i+1)/2} H_{i-1}(u) e^{-u^2/2},$$

and with $\mathscr{L}_i(\theta(T))$ the i^{th} Lipschitz-Killing curvature of $\theta(T)$. Thanks to the isotropy assumption on X and to the hypothesis $C''(0) = -I_2$, the Lipschitz-Killing curvatures have a very simple expression:

$$\text{if } d = 1,\ \mathscr{L}_1(\theta(T)) = |\theta(T)|_1,\ \mathscr{L}_0(\theta(T)) = \chi(\theta(T)) = 1,$$
$$\text{if } d = 2,\ \mathscr{L}_2(\theta(T)) = |\theta(T)|_2,\ \mathscr{L}_1(\theta(T)) = \frac{1}{2}|\partial\theta(T)|_1,\ \mathscr{L}_0(\theta(T)) = \chi(\theta(T)) = 1.$$

Thus, if T is a segment in $\mathbb{R}^2$ then Formula (5) setting $d = 1$ yields:

$$\mathbb{E}[\chi(A_u(X, \theta(T)))] = e^{-u^2/2}\frac{|\theta(T)|_1}{2\pi} + \Psi(u). \tag{6}$$

If T is a two-dimensional rectangle in $\mathbb{R}^2$, we get

$$\mathbb{E}[\chi(A_u(X, \theta(T)))] = e^{-u^2/2}\left(u\frac{|\theta(T)|_2}{(2\pi)^{3/2}} + \frac{|\partial\theta(T)|_1}{4\pi}\right) + \Psi(u). \tag{7}$$

We now state a continuity result on the mean Euler characteristic of excursion sets. The proposition hereafter shows that if T is a segment in $\mathbb{R}^2$, the mean Euler characteristic of the excursion set of X_θ above T may be seen as the limit of the mean Euler characteristic of excursion sets of X_θ over a sequence of two-dimensional sets, decreasing in the sense of set inclusion and approaching T.

Proposition III.7 *Let T be a segment in $\mathbb{R}^2$. Let v be a unit vector orthogonal to T and, for any $\rho > 0$, let T_ρ be the rectangle $\{t + \delta v,\, t \in T,\, -\rho \leq \delta \leq \rho\}$. Then, for any random field X satisfying Assumption* (**H**)*, as ρ decreases towards 0,*

$$\mathbb{E}[\chi(A_u(X_\theta, T_\rho))] \underset{\rho \to 0}{\longrightarrow} \mathbb{E}[\chi(A_u(X_\theta, T))]$$

Proof. We write $T = \{a + \lambda(b-a),\, \lambda \in [0,1]\}$, where a and b belong to $\mathbb{R}^2$. The set $\theta(T)$ is one-dimensional while for any $\rho > 0$, $\theta(T_\rho)$ is two-dimensional. Therefore, according to Formulas (6) and (7),

$$\mathbb{E}[\chi(A_u(X, \theta(T))] = e^{-u^2/2} \frac{|\theta(T)|_1}{2\pi} + \Psi(u),$$
$$\forall \rho > 0, \quad \mathbb{E}[\chi(A_u(X, \theta(T_\rho)))] = e^{-u^2/2} \left(u \frac{|\theta(T_\rho)|_2}{(2\pi)^{3/2}} + \frac{|\partial\theta(T_\rho)|_1}{4\pi} \right) + \Psi(u).$$

For any sequence $(\rho_n)_{n\in\mathbb{N}}$ of positive terms decreasing towards 0, the sequence of sets $(\theta(T_{\rho_n}))_{n\in\mathbb{N}}$, decreases to $\cap_{n\in\mathbb{N}}\theta(T_{\rho_n}) = \theta(T)$ thus the limit of $|\theta(T_{\rho_n})|_2$ as n tends to infinity is zero.

For any $\rho > 0$, the frontier of $\theta(T_\rho)$ is

$$\partial\theta(T_\rho) = \theta(\partial T_\rho) = \{\theta(t + \rho v),\, t \in T\} \cup \{\theta(t - \rho v),\, t \in T\}$$
$$\cup \{\theta(a + \delta v),\, \delta \in (-\rho, \rho)\} \cup \{\theta(b + \delta v),\, \delta \in (-\rho, \rho)\}.$$

As ρ tends to 0, the one-dimensional measure of each of the first two sets of this disjoint union tends to $|\theta(T)|_1$, while the one of the last two tends to zero; therefore, $|\partial\theta(T_\rho)|_1$ tends to $2|\theta(T)|_1$. This concludes the proof. ■

Remark III.8 *Proposition III.7 could be adapted in various ways. First, we could generalize it to a one-dimensional compact set T satisfying certain regularity assumptions. Besides, the sequence of sets $\{T_\rho,\, \rho > 0\}$ approaching T could be defined differently, for instance as the sequence of ρ-tubes around T, that is,*

$$\forall \rho > 0,\, T_\rho = \{z \in \mathbb{R}^2 \,/\, dist(T, z) \leq \rho\},\ \text{where } dist(T, z) = \min_{x \in T}\{\|x - z\|\}.$$

Modified Euler characteristic of an excursion set

For our approach in Section III.5, where we want to identify θ by considering some well-chosen excursion sets of X_θ, it will be easier to consider the modified Euler characteristic instead of the Euler characteristic of the excursion sets. The modified Euler characteristic has been introduced in Section I.2.2.d and we still denote it by ϕ. It will allow us to limit ourselves to the term of highest index in (5). According to Definition I.18 and its extension through Formula (21), for any segment or rectangle T in $\mathbb{R}^2$, writing $d = \dim(T)$,

$$\phi(A_u(X_\theta, T)) = \phi(A_u(X, \theta(T))) = \sum_{0 \leq k \leq d} (-1)^k \widetilde{\mu}_k(u, X, \theta(T)).$$

$$\text{where} \quad \widetilde{\mu}_k(u, X, \theta(T)) = \#\{t \in \theta(T) :\, X(t) \geq u,\, X'(t) = 0,\, \text{index } (X''(t)) = d - k\}$$

(we recall that # stands for the cardinality and index$(X''(t))$ is the number of negative eigenvalues of the Hessian matrix of X at point t).

If $d = 1$, the above formula becomes

$$\begin{aligned}\phi(A_u(X, \theta(T)) = \#\{&\text{local maxima of } X \text{ above } u \text{ in } \theta(\mathring{T})\} \\ &- \#\{\text{local minima of } X \text{ above } u \text{ in } \theta(\mathring{T})\}.\end{aligned}$$

If $d = 2$, it yields

$$\begin{aligned}\phi(A_u(X, \theta(T))) = 2\#\{&\text{local extrema of } X \text{ above } u \text{ in } \theta(\mathring{T})\} \\ &- \#\{\text{stationary points of } X \text{ above } u \text{ in } \theta(\mathring{T})\}.\end{aligned} \tag{8}$$

Remark III.9 (Additivity property) *The modified Euler characteristic of excursion sets, like the Euler characteristic, satisfies an additivity property. Let T and T' be two rectangles in $\mathbb{R}^2$ such that $T \cap T' = \emptyset$ then*

$$\phi(A_u(X_\theta, T \cup T')) = \phi(A_u(X_\theta, T)) + \phi(A_u(X_\theta, T')).$$

This property is a consequence of (8)*. Even if the rectangles T and T' have a non-empty but one-dimensional intersection, the additivity property is still satisfied. Indeed, in this case, according to Bulinskaya lemma (first point of Lemma I.15), almost surely, X_θ admits no stationary points in $T \cap T'$; consequently, the modified Euler characteristic of the excursion set of X_θ over $T \cap T'$ is almost surely 0.*

Under our assumptions on X, Corollary I.21 applies to T segment or rectangle in $\mathbb{R}^2$. Hence writing $d = \dim(T)$,

$$\mathbb{E}[\phi(A_u(X_\theta, T))] = \mathbb{E}[\phi(A_u(X, \theta(T)))] = \mathscr{L}_d(\theta(T))\, \rho_d(u),$$

It follows that

$$\text{if } \dim(T) = 1, \qquad \mathbb{E}[\phi(A_u(X_\theta, T))] = \mathbb{E}[\phi(A_u(X, \theta(T)))] = e^{-u^2/2} \frac{|\theta(T)|_1}{2\pi}; \tag{9}$$

$$\text{if } \dim(T) = 2, \qquad \mathbb{E}[\phi(A_u(X_\theta, T))] = \mathbb{E}[\phi(A_u(X, \theta(T)))] = e^{-u^2/2} u \frac{|\theta(T)|_2}{(2\pi)^{3/2}}. \tag{10}$$

The expectation of the modified Euler characteristic of excursion sets does not satisfy the same continuity result as the one stated in Proposition III.7 about the Euler characteristic. Indeed, consider for instance $(T_N)_{N \in \mathbb{N} \setminus \{0\}} = ([a, b] \times [-N^{-1}, N^{-1}])_{N \in \mathbb{N} \setminus \{0\}}$ converging toward $T = [a, b] \times \{0\}$, then Formula (10) yields

$$\mathbb{E}[\phi(A_u(X, \theta(T_N)))] \underset{N \to +\infty}{\longrightarrow} 0,$$

whereas, according to Formula (9),

$$\mathbb{E}[\phi(A_u(X, \theta(T)))] = \exp(-u^2/2) \frac{|\theta([a, b] \times \{0\})|_1}{2\pi} \neq 0.$$

In Section III.5.2, using the modified Euler characteristic of excursion sets, we construct estimators of the determinant and the norms of the columns of the Jacobian

matrix of θ. There we need a second moment formula, stated in Theorem I.22, that we recall here, adapted to our setting. Let T be a rectangle in $\mathbb{R}^2$. Even though the random field X does not satisfy a.s. $\mathscr{C}^3$-regularity, the result of Theorem I.22 is still valid here. Indeed, reading the proof given after Proposition 1 in [DBEL17], one can realize that the $\mathscr{C}^3$-regularity assumption only allows to ensure the finiteness of the second moment, but it does not have any relevance for the derivation of the formula. Therefore, we only need to justify the finiteness of the second moment of $\phi(A_u(X_\theta, T)$ under Condition **(H)**. Geman condition denoted by **(G)** in Section II.3 is a sufficient condition for the existence of a second moment for $\phi(A_u(X_\theta, T))$; this is justified in Section II.4. It is also clear that Condition **(D)** in Section I.2.2.c, which is the fourth item in Condition **(H)**, entails the Geman condition. Consequently, if we add the assumption that for any $t \in \mathbb{R}^2$, the Gaussian vector $(X(0), X(t))$ is not degenerate, to Condition **(H)**, Theorem I.22 applies: $\phi(A_u(X_\theta, T))$ admits a finite variance given by

$$\begin{aligned}\mathrm{Var}[\phi(A_u(X_\theta, T))] &= \mathrm{Var}[\phi(A_u(X, \theta(T)))] \\ &= \int_{\mathbb{R}^2} |\theta(T) \cap (\theta(T) - t)|_2 (E(u,t)D(t)^{-1/2} - h(u)^2)\, dt \\ &\qquad + |\theta(T)|_2 (2\pi)^{-1} g(u),\end{aligned} \tag{11}$$

where

$$\begin{aligned}&E(u,t) = \mathbb{E}[\mathbf{1}_{[u,+\infty)}(X(0))\mathbf{1}_{[u,+\infty)}(X(t))|\det(X''(0))\det(X''(t))|\,|\,X'(0) = X'(t) = 0], \\ &D(t) = (2\pi)^4 \det(I_2 - C''(t)^2), \\ &g(u) = \mathbb{E}[\mathbf{1}_{[u,+\infty)}(X(0))|\det(X''(0))|)], \\ &h(u) = (2\pi)^{-3/2}\, u\, e^{-u^2/2}.\end{aligned}$$

Notion of χ-isotropic deformation

In this section, the underlying field X is fixed and it satisfies Assumption **(H)**. We define χ-isotropic deformations, characterized by an invariance condition of the mean Euler characteristic of some excursion sets of the associated deformed field. We show that the only deformations that satisfy this invariance property are the spiral deformations, that is to say the ones that were proved to preserve isotropy in Section III.2.

Definition III.10 (χ-isotropic deformation) *A deformation $\theta \in \mathscr{D}^2(\mathbb{R}^2)$ is χ-isotropic if for any rectangle T in $\mathbb{R}^2$, for any $u \in \mathbb{R}$ and for any $\rho \in SO(2)$,*

$$\mathbb{E}[\chi(A_u(X_\theta, \rho(T))] = \mathbb{E}[\chi(A_u(X_\theta, T)]. \tag{12}$$

Remarks III.11 *Note that the notion of χ-isotropy seems to be dependent on the underlying random field X involved in* (12)*. However, after the statement and the proof of Theorem III.13, it will be clear that it is in fact not the case. It will also be clear that an equivalent definition of χ-isotropic deformations could be given by replacing "for any $u \in \mathbb{R}$" by "for a fixed $u \neq 0$". Besides, an equivalent version of Definition III.10, using the modified Euler characteristic instead of the Euler characteristic of*

excursion sets, will appear to hold true: a deformation $\theta \in \mathscr{D}^2(\mathbb{R}^2)$ is χ-isotropic if for any rectangle or segment T in $\mathbb{R}^2$, for any $u \in \mathbb{R}$ and for any $\rho \in SO(2)$,

$$\mathbb{E}[\phi(A_u(X_\theta, \rho(T))] = \mathbb{E}[\phi(A_u(X_\theta, T)].$$

Example III.12 *Spiral deformations defined in Section III.2 are χ-isotropic deformations. Indeed, if a deformation θ is such that X_θ is isotropic then it satisfies the above definition, because for any $\rho \in SO(2)$, $X_{\theta\circ\rho}$ has the same law as X_θ. But according to Theorem III.5, the deformations that preserve isotropy are exactly the spiral deformations.*

Here comes the main result of Section III.4.

Theorem III.13 *The χ-isotropic deformations are exactly the spiral deformations in $\mathscr{D}^2(\mathbb{R}^2)$.*

Proof. Spiral deformations are χ-isotropic deformations according to Example III.12; we prove that they are the only χ-isotropic deformations thanks to two lemmas and one result from [BF17].

The first lemma gives a characterization of χ-isotropic deformations involving invariance properties of the Jacobian matrix under rotations. To formulate it, we need to introduce an equivalence relation, denoted by $\overset{SO(2)}{\sim}$, on the space of invertible matrices of size 2×2: if M and N are two square matrices of size 2×2, $M \overset{SO(2)}{\sim} N$ if there exists $\rho \in SO(2)$ such that $M = \rho N$. Note that $M \overset{SO(2)}{\sim} N$ is equivalent to the conditions for all $(i,j) \in \{1,2\}^2$, $\langle M^i, M^j\rangle = \langle N^i, N^j\rangle$ and $\det(M) = \det(N)$.

Lemma III.14 *A deformation $\theta \in \mathscr{D}^2(\mathbb{R}^2)$ is χ-isotropic if and only if*

$$\forall \rho \in SO(2),\ \forall x \in \mathbb{R}^2,\ J_{\theta\circ\rho}(x) \overset{SO(2)}{\sim} J_\theta(x). \tag{13}$$

Proof. Let $\theta \in \mathscr{D}^2(\mathbb{R}^2)$. As explained above, Condition (13) is equivalent to the following condition:

$$\forall \rho \in SO(2),\ \forall x \in \mathbb{R}^2, \begin{cases} (i)' & \forall i \in \{1,2\},\ \|J^i_{\theta\circ\rho}(x)\| = \|J^i_\theta(x)\|, \\ (i)'' & \langle J^1_{\theta\circ\rho}(x), J^2_{\theta\circ\rho}(x)\rangle = \langle J^1_\theta(x), J^2_\theta(x)\rangle, \\ (ii) & \det(J_{\theta\circ\rho}(x)) = \det(J_\theta(x)). \end{cases} \tag{14}$$

First, we assume that θ is a χ-isotropic deformation. We fix $\rho \in SO(2)$, $(s,t) \in (\mathbb{R}\backslash\{0\})^2$ and $u \in \mathbb{R}\backslash\{0\}$. Identity (12) is satisfied for rectangle $T = T(s,t)$, thus Formula (7) applied at two different levels u and u' implies that $|\theta\circ\rho(T(s,t))|_2 = |\theta(T(s,t))|_2$, whence

$$\int_{[0,s]}\int_{[0,t]} |\det(J_{\theta\circ\rho}(x,y))|\,dx\,dy = \int_{[0,s]}\int_{[0,t]} |\det(J_\theta(x,y)|\,dx\,dy.$$

Differentiating twice the above equality with respect to s and to t yields, for any $(s,t) \in (\mathbb{R}\backslash\{0\})^2$, $|\det(J_{\theta\circ\rho}(s,t))| = |\det(J_\theta(s,t))|$, but $|\det(J_{\theta\circ\rho}(s,t))| = |\det(J_\theta(\rho(s,t)))|$ and the Jacobian determinant of θ has a fixed sign on $\mathbb{R}^2$, hence (14) Condition *(ii)* is

satisfied. Now let us prove Condition *(i)'*, for $i = 1$ for instance. For any $n \in \mathbb{N}\backslash\{0\}$, according to the definition of χ-isotropy,

$$\mathbb{E}[\chi(A_u(X_\theta, [0,s] \times [t-n^{-1}, t+n^{-1}]))] = \mathbb{E}[\chi(A_u(X_\theta, \rho([0,s] \times [t-n^{-1}, t+n^{-1}])))].$$

Then we apply Proposition III.7 to the set $[0,s] \times \{t\}$, intersection of the sets $\{[0,s] \times [t-n^{-1}, t+n^{-1}],\ n \in \mathbb{N}\backslash\{0\}\}$ (respectively to the set $\rho([0,s] \times \{t\})$, intersection of the sets $\{\rho([0,s] \times [t-n^{-1}, t+n^{-1}]),\ n \in \mathbb{N}\backslash\{0\}\}$). This yields

$$\mathbb{E}[\chi(A_u(X_\theta, [0,s] \times \{t\}))] = \mathbb{E}[\chi(A_u(X_\theta, \rho([0,s] \times \{t\})))],$$

and, thanks to Formula (6),

$$|\theta \circ \rho([0,s] \times \{t\})|_1 = |\theta([0,s] \times \{t\})|_1,$$

which can be written

$$\int_{[0,s]} \|J^1_{\theta\circ\rho}(x,t)\|\, dx = \int_{[0,s]} \|J^1_\theta(x,t)\|\, dx.$$

Differentiating this integral equality with respect to s, we obtain $\|J^1_{\theta\circ\rho}(s,t)\| = \|J^1_\theta(s,t)\|$. Similarly, we get $\|J^2_{\theta\circ\rho}(s,t)\| = \|J^2_\theta(s,t)\|$. We are left to show Condition *(i)"*. Let $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$, $t \in \mathbb{R}\backslash\{0\}$ and let $T = \{\lambda u_\alpha,\ \lambda \in [0,t]\}$ a segment. Using the same arguments as above, the χ-isotropy of θ entails that $|\theta \circ \rho(T)|_1 = |\theta(T)|_1$, that is

$$\int_{[0,t]} \|J_{\theta\circ\rho}(\lambda u_\alpha)\, u_\alpha\|\, d\lambda = \int_{[0,t]} \|J_\theta(\lambda u_\alpha)\, u_\alpha\|\, d\lambda. \tag{15}$$

Writing $\eta \in \{\theta \circ \rho, \theta\}$, for any $\lambda \in [0,t]$, the norm in the integrand of each integral can be developed into

$$\left(\|J^1_\eta(\lambda u_\alpha)\|^2 \cos^2\alpha + \|J^2_\eta(\lambda u_\alpha)\|^2 \sin^2\alpha + 2\,\langle J^1_\eta(\lambda u_\alpha), J^2_\eta(\lambda u_\alpha)\rangle\, \cos\alpha \sin\alpha\right)^{1/2}.$$

Differentiating Equality (15) with respect to t and using the proved Condition *(i)'*, for any $\alpha \notin \{0, \pi/2\}$ modulus π, we get

$$\langle J^1_{\theta\circ\rho}(tu_\alpha), J^2_{\theta\circ\rho}(tu_\alpha)\rangle = \langle J^1_\theta(tu_\alpha), J^2_\theta(tu_\alpha)\rangle.$$

By continuity of J^i_η, for $i \in \{1,2\}$, we conclude that for any $x \in \mathbb{R}^2$, Condition *(i)"* is satisfied.

Hence we have proved the direct implication of Lemma III.14 and we turn to the converse implication. We assume that θ satisfies Condition (14). The proof will actually only exploit Conditions *(i)'* and *(ii)*.

Let T be a rectangle in $\mathbb{R}^2$. In the first place, there exist $(s,t) \in (\mathbb{R}\backslash\{0\})^2$, $\rho_0 \in SO(2)$ and a translation by vector $(a,b) \in \mathbb{R}^2$, denoted by $\tau_{a,b}$, such that $T = \rho_0 \circ \tau_{a,b}(T(s,t))$. Let $\theta \in \mathscr{D}^2(\mathbb{R}^2)$ satisfying (14) for any $\rho \in SO(2)$ and for any $x \in \mathbb{R}^2$. Therefore

$$\begin{aligned}
|\theta \circ \rho(T)|_2 &= |\theta \circ \rho \circ \rho_0(\tau_{a,b}(T(s,t)))|_2 \\
&= \int_{[0,s]} \int_{[0,t]} |\det(J_{\theta\circ\rho\circ\rho_0}(a+x, b+y)|\, dx\, dy \\
&= \int_{[0,s]} \int_{[0,t]} |\det(J_{\theta\circ\rho_0}(a+x, b+y)|\, dx\, dy \\
&= |\theta(T)|_2.
\end{aligned}$$

The third equality results from (14) Condition *(ii)*. Now, we express the perimeter length of $\theta \circ \rho(T)$.

$$\begin{aligned}
|\partial\theta \circ \rho(T)|_1 =& |\partial\theta \circ \rho \circ \rho_0(\tau_{a,b}(T(s,t)))|_1 \\
=& \int_{[0,s]} \|J^1_{\theta\circ\rho\circ\rho_0}(a+x,b)\|\,dx + \int_{[0,s]} \|J^1_{\theta\circ\rho\circ\rho_0}(a+x,b+t)\|\,dx \\
&+ \int_{[0,t]} \|J^2_{\theta\circ\rho\circ\rho_0}(a,b+y)\|\,dy + \int_{[0,t]} \|J^2_{\theta\circ\rho\circ\rho_0}(a+s,b+y)\|\,dy \\
=& \int_{[0,s]} \|J^1_{\theta\circ\rho_0}(a+x,b)\|\,dx + \int_{[0,s]} \|J^1_{\theta\circ\rho_0}(a+x,b+t)\|\,dx \\
&+ \int_{[0,t]} \|J^2_{\theta\circ\rho_0}(a,b+y)\|\,dy + \int_{[0,t]} \|J^2_{\theta\circ\rho_0}(a+s,b+y)\|\,dy \\
=& |\partial\theta(T)|_1.
\end{aligned}$$

The third equality results from (14) Condition *(i)'*. Thanks to Formula (7), this proves that $\mathbb{E}[\chi(A_u(X, \theta \circ \rho(T)))] = \mathbb{E}[\chi(A_u(X, \theta(T)))]$. Hence θ is a χ-isotropic deformation and the proof of Lemma III.14 is completed. ■

Our second lemma states a property, involving the polar representation, satisfied by χ-isotropic deformations.

Lemma III.15 *If a deformation $\theta \in \mathscr{D}^2(\mathbb{R}^2)$ is a χ-isotropic deformation then functions*

$$\begin{cases}
(r,\varphi) \mapsto (\partial_r \hat\theta^1(r,\varphi))^2 + (\hat\theta^1(r,\varphi)\,\partial_r \hat\theta^2(r,\varphi))^2 \\
(r,\varphi) \mapsto (\partial_\varphi \hat\theta^1(r,\varphi))^2 + (\hat\theta^1(r,\varphi)\,\partial_\varphi \hat\theta^2(r,\varphi))^2 \\
(r,\varphi) \mapsto \hat\theta^1(r,\varphi)\ \det(J_{\hat\theta}(r,\varphi))
\end{cases} \tag{16}$$

are radial, i.e. if they do not depend on φ.

Proof. We use the notations introduced at the beginning of Section III.2. The Jacobian matrix of S at point $(r,\varphi) \in (0,+\infty) \times \mathbb{R}/2\pi\mathbb{Z}$ is

$$J_S(r,\varphi) = \rho_\varphi \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}.$$

Consequently,

$$J_{S^{-1}}(S(r,\varphi)) = (J_S(r,\varphi))^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-1} \end{pmatrix} \rho_{-\varphi}.$$

Now for any rotation $\rho \in SO(2)$ and for any $(r,\varphi) \in (0,+\infty) \times \mathbb{R}/2\pi\mathbb{Z}$, writing $\theta_0 = S \circ \hat\theta \circ S^{-1}$, we get

$$J_{\theta_0}(S(r,\varphi)) = \rho_{\hat\theta^2(r,\varphi)} \begin{pmatrix} 1 & 0 \\ 0 & \hat\theta^1(r,\varphi) \end{pmatrix} J_{\hat\theta}(r,\varphi) \begin{pmatrix} 1 & 0 \\ 0 & r^{-1} \end{pmatrix} \rho_{-\varphi}. \tag{17}$$

We use the characterization of χ-isotropy given by Lemma III.14. A deformation $\theta \in \mathscr{D}^2(\mathbb{R}^2)$ is a χ-isotropic deformation if and only if for any $(r,\varphi,\alpha) \in (0,+\infty) \times$

$(\mathbb{R}/2\pi\mathbb{Z})^2$, $J_{\theta_0\circ\rho_\alpha}(S(r,\varphi)) = J_{\theta_0}(S(r,\varphi+\alpha))\,\rho_\alpha$ is equivalent to $J_{\theta_0}(S(r,\varphi))$. Equivalently, for any $(r,\varphi,\alpha) \in (0,+\infty)\times(\mathbb{R}/2\pi\mathbb{Z})^2$, the equivalence relation

$$\begin{pmatrix} 1 & 0 \\ 0 & \hat{\theta}^1(r,\varphi+\alpha) \end{pmatrix} J_{\hat{\theta}}(r,\varphi+\alpha) \overset{SO(2)}{\sim} \begin{pmatrix} 1 & 0 \\ 0 & \hat{\theta}^1(r,\varphi) \end{pmatrix} J_{\hat{\theta}}(r,\varphi)$$

holds. This implies that the above matrices have the same determinant and the same norm of columns, which means that the functions defined by (16) do not depend on their second variable. ■

To conclude the proof of Theorem III.13, we refer to [BF17] that solves the differential system (16) and proves that the solutions are the spiral deformations. Consequently, this proves that χ-isotropic deformations are spiral deformations and the converse inclusion has already been pointed out in Example III.12.

Remark III.16 *The first step in the resolution derived in [BF17] is to transform the quadratic differential system* (16) *into a non quadratic equivalent one, which is done by considering that the two first functions in* (16) *are the square moduli of complex numbers functions.*

■

Let us write $\mathscr{S}$ the set of spiral deformations in $\mathscr{D}^2(\mathbb{R}^2)$, $\mathscr{X}$ the set of χ-isotropic deformations, $\mathscr{I}$ the set of deformations θ in $\mathscr{D}^2(\mathbb{R}^2)$ such that for any isotropic and stationary field X satisfying (**H**), X_θ is isotropic and, finally, for a fixed stationary and isotropic field X satisfying (**H**), $\mathscr{I}(X)$ the set of deformations θ in $\mathscr{D}^2(\mathbb{R}^2)$ such that X_θ is isotropic. These sets satisfy the following chain of inclusions or equalities:

$$\mathscr{S} = \mathscr{I} \subset \mathscr{I}(X) \subset \mathscr{X} = \mathscr{S}.$$

The first and the last equalities come respectively from Theorem III.5 and Theorem III.13; the first inclusion is obvious and the second one is a consequence of Example III.12. As a result, the following corollary holds.

Corollary III.17 *Let X be a stationary and isotropic random field satisfying Assumption* (**H**)*. Then $\mathscr{S} = \mathscr{I}(X) = \mathscr{I} = \mathscr{X}$.*

To conclude, it occurs that the different notions that we have introduced so far to describe the isotropic behaviour of a deterministic deformation are in fact one and correspond to the spiral case.

Identification of θ through excursion sets

As explained in the introduction of this paper, we consider the case of an unknown deformation θ, which we want to identify using sparse data: the observations of excursion sets of X_θ over well-chosen domains. More precisely, we assume that the mean modified Euler characteristic of some excursion sets of X_θ has been computed and we explain how we can almost uniquely characterize θ. The modified Euler characteristic is more adapted to our method than the Euler characteristic itself. This is due to the dependence of the mean Euler characteristic of an excursion set over a two-dimensional

domain on both the perimeter length and the area of the domain, whereas its mean modified version only depends on the area (compare Formulas (7) and (10)). In the second place, we limit ourselves to spiral deformations and we show that in this case, we can easily estimate θ thanks to only one realization of the deformed field X_θ.

The underlying field X is unknown but it is still assumed to satisfy assumption **(H)**. The unknown deformation θ belongs to $\mathscr{D}^2(\mathbb{R}^2)$ and at each point in $\mathbb{R}^2$, its Jacobian determinant is positive.

Identification of θ

Case of a linear deformation.

Here comes the simple case of a linear deformation that we use as a first step towards the general case. Let us assume that θ is a linear function and let us write it matricially in a fixed orthonormal basis of $\mathbb{R}^2$: $\theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$. In this case, we only have to consider the excursion sets over one horizontal segment, one vertical segment and one rectangle (product of two segments): we fix $(s,t) \in (\mathbb{R}\backslash\{0\})^2$, $u \neq 0$ and we assume that we know $\mathbb{E}[\phi(A_u(X_\theta, [0,s]\times\{0\}))]$, $\mathbb{E}[\phi(A_u(X_\theta, \{0\}\times[0,t]))]$ and $\mathbb{E}[\phi(A_u(X_\theta, T(s,t))]$. The three real numbers

$$a = \sqrt{\theta_{11}^2 + \theta_{21}^2}, \quad b = \sqrt{\theta_{12}^2 + \theta_{22}^2} \quad \text{and} \quad c = \theta_{11}\theta_{22} - \theta_{21}\theta_{12} \tag{18}$$

satisfy

$$|\theta([0,s]\times\{0\})|_1 = |s|a, \quad |\theta(\{0\}\times[0,t])|_1 = |t|b, \quad \text{and} \quad |\theta(T(s,t))|_2 = |st|c.$$

Therefore, they are solutions of equations given by Formulas (9) and (10) and they can be used to write another expression of matrix θ: there exists $(\alpha,\beta) \in (\mathbb{R}/2\pi\mathbb{Z})^2$ such that $\theta = \begin{pmatrix} a\cos(\alpha) & b\cos(\beta) \\ a\sin(\alpha) & b\sin(\beta) \end{pmatrix}$. Let $\delta = \beta - \alpha$ be the angle between the two column vectors. It satisfies $c = ab\sin(\delta)$, whence

$$\delta \in \{\delta_0, \delta_1\}, \text{ where } \begin{cases} \delta_0 = \arcsin\left(\dfrac{c}{ab}\right) \in (0, \pi/2] \\ \delta_1 = \pi - \arcsin\left(\dfrac{c}{ab}\right) \in [\pi/2, \pi). \end{cases}$$

Consequently, we are able to determine matrix θ up to an unknown rotation, with two possibilities concerning the angle between its two column vectors: θ belongs to the set $\mathscr{M}(a,b,c)$ defined by

$$\mathscr{M}(a,b,c) = \left\{ \rho_\alpha \begin{pmatrix} a & \sqrt{b^2 - (ca^{-1})^2} \\ 0 & ca^{-1} \end{pmatrix},\ \rho_\alpha \begin{pmatrix} a & -\sqrt{b^2 - (ca^{-1})^2} \\ 0 & ca^{-1} \end{pmatrix},\ \alpha \in \mathbb{R}/2\pi\mathbb{Z} \right\} \tag{19}$$

If the determinant of θ was not assumed to be positive, there would be two other possibilities, up to a rotation, because δ could take four possible values. Note that according to Example III.4, X_θ is isotropic in the case where $a = b = \sqrt{c}$, which implies $\delta = \pi/2$.

Of course, because of the isotropy of X, we obtain θ up to post-composition with an unknown rotation. Our method is based on the mean Euler characteristic of excursion sets of X_θ over some sets, which only depends on θ through the perimeter and area of the set's image by θ. Consequently, we can not differentiate between two deformations that transform any set into sets with the same perimeter and the same area.

We summarize our approach in the following method.

Method III.18 *Let* $\theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$ *be an unknown linear deformation with positive determinant. For a fixed* $(s,t) \in (\mathbb{R}\backslash\{0\})^2$*, for a fixed* $u \in \mathbb{R}\backslash\{0\}$*, we assume that* $\mathbb{E}[\phi(A_u(X_\theta, T))]$ *is known for* T *of the form* $[0,s]\times\{0\}$*,* $\{0\}\times[0,t]$ *and* $[0,s]\times[0,t]$*. Then* a*,* b *and* c *given by* (18) *are computable thanks to Formulas* (9) *and* (10) *and* θ *belongs to the set* $\mathscr{M}(a,b,c)$ *defined by* (19)*.*

General method.

We refer to the appendix of [AS08] for a precise definition of the complex dilatation and for the statement of the mapping theorem that formulates a characterization of a deformation up to a conformal mapping through its complex dilatation. To be able to apply it, we add an hypothesis on θ: from now on, we assume that θ has uniformly bounded distortion. Writing θ as a mapping from the complex plane $\mathbb{C}$ to itself and $|\cdot|$ the complex modulus, this means that the ratio of

$$\limsup_{x\to x_0} \frac{|\theta(x)-\theta(x_0)|}{|x-x_0|} \quad \text{and of} \quad \liminf_{x\to x_0} \frac{|\theta(x)-\theta(x_0)|}{|x-x_0|}$$

is uniformly bounded for $x_0 \in \mathbb{C}$.

We fix $u \neq 0$, $S > 0$ and we assume that $\mathbb{E}[\phi(A_u(X_\theta, [0,s]\times\{t\}))]$, $\mathbb{E}[\phi(A_u(X_\theta, \{s\}\times[0,t]))]$ and $\mathbb{E}[\phi(A_u(X_\theta, T(s,t))]$ are known for any $(s,t) \in [-S,S]^2$. Then for any $(s,t) \in [-S,S]^2$, we can deduce $|\theta([0,s]\times\{t\})|_1$ and $|\theta(\{s\}\times[0,t])|_1$ from Formula (9) by simply solving a linear system. Besides

$$|\theta([0,s]\times\{t\})|_1 = \int_{[0,s]} \|J^1_\theta(x,t)\| dx = \int_{[0,s]} \sqrt{\partial_x\theta_1(x,t)^2 + \partial_x\theta_2(x,t)^2}\, dx,$$
$$|\theta(\{s\}\times[0,t])|_1 = \int_{[0,t]} \|J^2_\theta(s,y)\| dy = \int_{[0,t]} \sqrt{\partial_y\theta_1(s,y)^2 + \partial_y\theta_2(s,y)^2}\, dy.$$

The first-order partial derivatives of θ are continuous. By differentiating the functions $s \mapsto |\theta([0,s]\times\{t\})|_1$ and $t \mapsto |\theta(\{s\}\times[0,t])|_1$, we obtain functions $s \mapsto \|J^1_\theta(s,t)\|$ and $t \mapsto \|J^2_\theta(s,t)\|$ on segment $[-S,S]$.

Now considering the rectangle domains $\{T(s,t),\ (s,t) \in ([-S,S]\backslash\{0\})^2\}$, we assume that $\mathbb{E}[\phi(A_u(X_\theta, T(s,t)))]$ is known. Since $u \neq 0$, we can compute $|T(s,t)|_2$ thanks to Formula (10). Then, by differentiating twice the function

$$(s,t) \mapsto |\theta(T(s,t))|_2 = \int_{[0,s]}\int_{[0,t]} |\det(J_\theta(x,y))|\, dx\, dy,$$

with respect to s and to t on the square $[-S,S]^2$, we obtain function $(s,t) \mapsto |\det(J_\theta(s,t))|$ on the same square.

Now, we fix $x \in ([-S,S]\backslash\{0\})^2$, we write $J_\theta(x) = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$ and we use the same notations a, b and c defined by (18) as in the linear case, although they now depend on x. The explanations given in Section III.5.1.a apply here and consequently, $J_\theta(x)$ belongs to $\mathscr{M}(a,b,c)$. Moreover, let us express the complex dilatation μ, given by

$$\mu = \frac{\partial_{\bar{z}}\theta}{\partial_z\theta},$$

$$\text{where } \begin{cases} \partial_z\theta = \dfrac{1}{2}(\partial_s\theta_1 + \partial_t\theta_2) + \dfrac{i}{2}(\partial_s\theta_2 - \partial_t\theta_1) \\ \partial_{\bar{z}}\theta = \dfrac{1}{2}(\partial_s\theta_1 - \partial_t\theta_2) + \dfrac{i}{2}(\partial_s\theta_2 + \partial_t\theta_1). \end{cases}$$

At point x, a short computation shows that $\mu(x)$ takes two possible values in the set $\mathscr{C}(a,b,c)$ defined by

$$\mathscr{C}(a,b,c) = \left\{ \frac{1}{a^2+b^2+2c}(a^2 - b^2 \pm 2i\sqrt{a^2b^2 - c^2}) \right\}. \tag{20}$$

The general method is summarized below.

Method III.19 *Let $\theta \in \mathscr{D}^2(\mathbb{R}^2)$ be a deformation with a positive Jacobian on $\mathbb{R}^2$. Let $S > 0$ and let $u \in \mathbb{R}\backslash\{0\}$ be fixed. Assuming that for any $x = (s,t) \in [-S,S]^2$, for any $T \in \{[0,s]\times\{t\},\ \{s\}\times[0,t],\ [0,s]\times[0,t]\}$, we know $\mathbb{E}[\phi(A_u(X_\theta,T))]$, we may compute $a = \|J^1_\theta(x)\|$, $b = \|J^2_\theta(x)\|$ and $c = \det(J_\theta(x))$. Consequently, for each $x \in [-S,S]^2$, the Jacobian matrix at point x, $J_\theta(x)$ belongs to $\mathscr{M}(a,b,c)$ defined by* (19) *and the complex dilatation at point x, $\mu(x)$ belongs to $\mathscr{C}(a,b,c)$ defined by* (20).

Remark III.20 (Numerical approach) *In practise, we can only have at our disposal a finite amount of data. Let σ be a partition of $[-S,S]$. If we know $\left\{\mathbb{E}[\phi(A_u(X_\theta,T))],\ T \in \{[0,s]\times\{t\},\ \{s\}\times[0,t],\ [0,s]\times[0,t]\},\ (s,t) \in \sigma^2\right\}$, numerical approaches such as Runge-Kutta methods allow to compute approximate values of $\|J^1_\theta(s,t)\|$, $\|J^2_\theta(s,t)\|$ and $\det(J_\theta(s,t))$ for any $(s,t) \in \sigma^2$ and the approximate values for $J_\theta(s,t)$ and $\mu(s,t)$.*

Case of a tensorial deformation

We now study the particular case of tensorial deformations, where we can completely identify θ if we make an assumption of monotonicity on its coordinate functions. Let $\theta(s,t) = (\theta_1(s),\theta_2(t))$. Our hypotheses on θ mean that for $i \in \{1,2\}$, $\theta_i\colon \mathbb{R} \to \mathbb{R}$ satisfies $\theta_i(0) = 0$, θ_i is a bijective function of class $\mathscr{C}^2$ and therefore it is monotonous. Note that θ transforms a rectangle $[s,v]\times[t,w]$ into another rectangle $\theta_1([s,v]) \times \theta_2([t,w])$.

Let $s \in \mathbb{R}\backslash\{0\}$. We deduce from Formula (9) that

$$\begin{cases} \mathbb{E}[\phi(A_u(X_\theta,[0,s]\times\{0\}))] = \dfrac{e^{-u^2/2}}{2\pi} \displaystyle\int_0^s |\theta_1'(x)|\,dx \\ \mathbb{E}[\phi(A_u(X_\theta,\{0\}\times[0,s]))] = \dfrac{e^{-u^2/2}}{2\pi} \displaystyle\int_0^s |\theta_2'(x)|\,dx \end{cases}$$

and consequently, we can state the following method.

Method III.21 *Let $(s,t) \mapsto \theta(s,t) = (\theta_1(s), \theta_2(t)) \in \mathscr{D}^2(\mathbb{R}^2)$ be a tensorial deformation. We fix $S > 0$ and $u \in \mathbb{R}$. We assume that for any real number $s \in [-S,S]\backslash\{0\}$ and for $T \in \{[0,s] \times \{0\}, \{0\} \times [0,s]\}$, we know $\mathbb{E}[\phi(A_u(X_\theta, T))]$. Then we determine functions $s \mapsto |\theta_1'(s)|$ and $s \mapsto |\theta_2'(s)|$ on $[-S,S]$ thanks to Formula (9). If the sign of each coordinate function is known then θ is completely determined on $[-S,S]^2$.*

Example III.22 *Let $(\alpha,\beta) \in (\mathbb{R}\backslash\{0\})^2$, let θ be defined on $[0,1]^2$ by $\theta(s,t) = (s^\alpha, t^\beta)$ and let σ be a partition of $(0,1]$. To identify θ_1, we follow the above method adapted to a numerical approach; thus we obtain approximate values for $\{|\theta_1'(s)|,\ s \in \sigma\}$. Constant values correspond to the case of $\alpha = 1$. Otherwise, we have $|\theta_1'(s)| = |\alpha|\, s^{\alpha-1}$, therefore coefficient α can be computed through a regression method: $\alpha - 1$ is the slope of the line representing $\log(|\theta_1'(s)|) = \log(|\alpha|) + (\alpha-1)\log(s)$ as a function of $\log(s)$ on $(0,1]$. The same method holds to get coefficient β.*

Remark III.23 *The three methods III.18, III.19 and III.21 can be easily adapted if the modified Euler characteristic ϕ is replaced by the Euler characteristic χ itself.*

Estimation in the spiral case

We have assumed all along the first part of this section that $\mathbb{E}[\phi(A_u(X_\theta, T))]$ was known for some basic domains T, but we have not yet discussed estimation matters. Without any hypothesis on θ, this expectation seems uneasy to estimate from one single realization of X_θ, since the deformed field is non-stationary, except in the linear case. Yet it is possible in the spiral case thanks to the isotropy of the deformed field.

In this section, in order to derive results about the variance of our estimators, we furthermore assume that for any $t \in \mathbb{R}^2$, the vector $(X(0), X(t))$ is not degenerate and that function C and its derivatives satisfy the following condition at infinity:

$$\nu(t) \underset{\|t\| \to +\infty}{\longrightarrow} 0 \quad \text{and} \quad \nu \in L^1(\mathbb{R}^2), \quad \text{where} \quad \nu(t) = \max\{|\frac{\partial^{\mathbf{k}} C}{\partial t^{\mathbf{k}}}(t)|,\ \mathbf{k} \in \mathbb{N}_0^2,\ |\mathbf{k}| \leq 4\}. \tag{21}$$

Let $\theta \in \mathscr{D}^2(\mathbb{R}^2)$ be a spiral deformation; we show in the following how to estimate $\|J_\theta^1(x)\|$, $\|J_\theta^2(x)\|$ and $\det(J_\theta(x))$ at each point x in a chosen domain. Then Method III.19 applies to identify θ.

Let $x \in \mathbb{R}^2\backslash\{0\}$, let (r_0, φ_0) be its polar coordinates and for $N \in \mathbb{N}\backslash\{0\}$, let $T_N^0 = S\left(\{(r,\varphi) \in (0,+\infty) \times \mathbb{R}/2\pi\mathbb{Z} \,/\, r_0 \leq r \leq r_0 + N^{-1},\ \varphi_0 \leq \varphi \leq \varphi_0 + 2\pi N^{-1}\}\right)$.

For any $k \in \{0, \cdots, N-1\}$, we write $T_N^k = \rho_{2k\pi/N}(T_N^0)$. We fix $u \neq 0$ and we define

$$Z_N = N^{-1} \sum_{k=0}^{N-1} \phi(A_u(X_\theta, T_N^k)) = N^{-1} \sum_{k=0}^{N-1} \phi(A_u(X, \theta(T_N^k))),$$

where ϕ is the modified Euler characteristic. Remember that according to Remark III.9, ϕ satisfies an additivity property that applies here because the two-dimensional sets (T_N^k) have a one-dimensional intersection. Thus,

$$\begin{aligned} Z_N &= N^{-1}\phi\left(A_u\left(X, \overset{N-1}{\underset{k=0}{\cup}} \theta(T_N^k)\right)\right) \\ &= N^{-1}\phi(A_u(X, \theta(U_N))), \quad \text{where} \quad U_N = \overset{N-1}{\underset{k=0}{\cup}} T_N^k. \end{aligned}$$

We derive the asymptotic behaviour of the expectation and the variance of Z_N from the χ-isotropy property satisfied by θ. If $(u_N)_{N\in\mathbb{N}}$ and $(v_N)_{N\in\mathbb{N}}$ are real sequences, we write $u_N \underset{N\to+\infty}{\sim} v_N$ if there exist $N_0 \in \mathbb{N}$ and a real sequence $(\varepsilon_N)_{N\geq N_0}$ converging to one such that for $N \geq N_0$, $u_N = \varepsilon_N v_N$.

Proposition III.24 *There exist constants $a \neq 0$ and $c > 0$ (depending only on u) and $n \in \mathbb{N}\backslash\{0\}$ such that*

$$\mathbb{E}[Z_N] \underset{N\to+\infty}{\sim} a\,|\det(J_\theta(x))||T_N^0|_2$$

and for $N \geq n$,

$$\mathrm{Var}[Z_N] \leq c\,\frac{|\det(J_\theta(x))|\,|T_N^0|_2}{N}.$$

Proof. Let $N \in \mathbb{N}\backslash\{0\}$. According to Theorem III.13, θ is χ-isotropic. Using Definition III.10 and the last point of Remarks III.11, this implies that for any $k \in \{0,\cdots,N-1\}$, $\mathbb{E}[\phi(A_u(X,\theta(T_N^k)))] = \mathbb{E}[\phi(A_u(X,\theta(T_N^0)))]$. Therefore, the expectation of Z_N is

$$\mathbb{E}[Z_N] = \mathbb{E}[\phi(A_u(X,\theta(T_N^0)))] = \frac{u\,e^{-u^2/2}}{(2\pi)^{3/2}}|\theta(T_N^0)|_2.$$

We study the asymptotic behaviour of this sequence.

$$\begin{aligned}
&\Big||\theta(T_N^0)|_2 - |\det(J_\theta(x))||T_N^0|_2\Big| \\
&\leq \int_{r_0}^{r_0+N^{-1}}\int_{\varphi_0}^{\varphi_0+2\pi N^{-1}} \big|\,|\det(J_\theta(S(r,\varphi)))| - |\det(J_\theta(S(r_0,\varphi_0)))|\,\big|\,r\,dr\,d\varphi \\
&\leq \sup_{\substack{r_0\leq r\leq r_0+N^{-1}\\ \varphi_0\leq\varphi\leq\varphi_0+2\pi N^{-1}}} \big|\,|\det(J_\theta(S(r,\varphi)))| - |\det(J_\theta(S(r_0,\varphi_0)))|\,\big|\,|T_N^0|_2,
\end{aligned}$$

with $\sup_{\substack{r_0\leq r\leq r_0+N^{-1}\\ \varphi_0\leq\varphi\leq\varphi_0+2\pi N^{-1}}} \big|\,|\det(J_\theta(S(r,\varphi)))| - |\det(J_\theta(S(r_0,\varphi_0)))|\,\big| \underset{N\to+\infty}{\longrightarrow} 0$. Consequently,

$$|\theta(T_N^0)|_2 \underset{N\to+\infty}{\sim} |\det(J_\theta(x))||T_N^0|_2 \tag{22}$$

and the result about the asymptotic expectation holds.

Now we use Formula (11) (with its notations) to get an integral expression of the variance of $Z_N = N^{-1}\phi(A_u(X,\theta(U_N)))$. An asymptotic upper-bound is obtained under Condition (21), which entails that the map $t \mapsto E(u,t)D(t)^{-1/2} - h(u)^2$ has a finite integral on $\mathbb{R}^2$, according to Lemma 3 in [DBEL17].

$$\begin{aligned}
\mathrm{Var}[\phi(A_u(X,\theta(U_N)))] &= \int_{\mathbb{R}^2} |\theta(U_N)\cap(\theta(U_N)-t)|_2(E(u,t)D(t)^{-1/2} - h(u)^2)\,dt \\
&\qquad + |\theta(U_N)|_2(2\pi)^{-1}g(u) \\
&\leq |\theta(U_N)|_2\left(\int_{\mathbb{R}^2}(E(u,t)D(t)^{-1/2} - h(u)^2)\,dt + (2\pi)^{-1}g(u)\right) \\
&\leq c\,|\theta(U_N)|_2 = c\,N|\theta(T_N^0)|_2,
\end{aligned}$$

where $c > 0$. Note that the integration domain is in fact the compact subset $\{t - t',\ (t,t') \in \theta(U_N)^2\}$. The last equality is a consequence of the third point of Condition (14) satisfied by θ according to Lemma III.14. Consequently, using (22), we get

$$\mathrm{Var}[Z_N] \leq c\, N^{-1}\, |\theta(T_N^0)|_2 \underset{N\to+\infty}{\sim} c\, N^{-1}\, |\det(J_\theta(x))|\, |T_N^0|_2.$$

This concludes the proof. ■

Proposition III.24 shows that, asymptotically, the variance of Z_N is negligible with respect to its expectation. Practically, we could obtain $|\det(J_\theta(x))|$ through a regression method since, up to a constant, it is the coefficient of the linear relation linking asymptotically $|\theta(T_N^0)|_2$ and $|T_N^0|_2$. Constant a is totally explicit and constant c may be numerically computed.

We can adopt the same approach to get an estimation of $\|J_\theta^i(x)\|$, for $i \in \{1,2\}$. We will only state the asymptotic result (for $i = 1$) because the proof is very similar to the one of Proposition III.24. Let $x = (x_1, x_2) \in \mathbb{R}^2$ and $S_N^0 = [x_1, x_1 + N^{-1}] \times \{x_2\}$. For any $N \in \mathbb{N}\backslash\{0\}$, for any $k \in \{0, \cdots, N-1\}$, we write $S_N^k = \rho_{2k\pi/N}(S_N^0)$ and we define

$$Y_N = N^{-1} \sum_{k=0}^{N-1} \phi(A_u(X_\theta, S_N^k)).$$

Proposition III.25 *There exist constants $d \neq 0$ and $k > 0$ (depending only on u) and $n \in \mathbb{N}\backslash\{0\}$ such that*

$$\mathbb{E}[Y_N] \underset{N\to+\infty}{\sim} d\, \|J_\theta^1(x)\| |S_N^0|_1$$

and for $N \geq n$,

$$\mathrm{Var}[Y_N] \leq k\, \frac{\|J_\theta^1(x)\|\, |S_N^0|_1}{N}.$$

Estimates of $|\det(J_\theta)|$, $\|J_\theta^1(x)\|$ and $\|J_\theta^2(x)\|$ bring a nearly complete characterization of the Jacobian matrix of θ, as explained in Section III.5.1.

Conclusion and perspectives

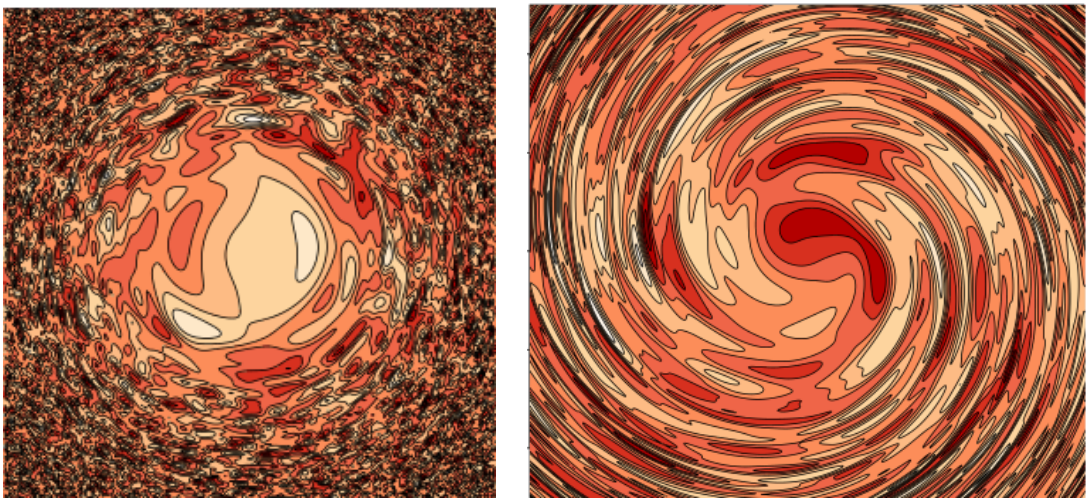
In our study of deformed random fields, a generally speaking anisotropic and non-stationary model, the first question we addressed was the one of the deformations preserving isotropy. The answer, stated in Theorem III.5, is the set of spiral deformations defined by Definition III.2. They provide a model of isotropic and non-stationary random fields.

Then our guideline was to study deformed random fields through the mean Euler characteristic of their excursion sets and to understand how much it could characterize the deformation. First, we introduced in Definition III.10 an invariance condition of the mean Euler characteristic of excursion sets above rectangles, under the action of rotations. However, it occured that this weak isotropy condition is equivalent to the isotropy of the distribution of the deformed random field and, thus, that the deformation of the model is in that case a spiral deformation.

Identification matters were addressed in Section III.5, where we assumed that the deformation was unknown and that observations were made above a fixed level. Thus, our method does not require whole realizations of a random field, but only its excursion sets above a fixed level. We have shown that the mean Euler characteristic of excursion sets above rectangles and segment characterizes the deformation of the model. In the isotropic case, that is, in the case of a spiral deformation, we could provide a method to determine the deformation through estimation. An interesting question would be whether the implementation of our method provides interesting results for the practical estimation of the deformation.

The assets of the Euler characteristic χ of excursion sets is the homotopy invariance satisfied by χ, which theoretically allows to study excursion sets of a deformed random field through the ones of the underlying field. However, we could extend this study to other geometric characteristics. They would possibly allow us to infer properties of the distribution of a deformed random field as well as to characterize its deformation.

Having identified the isotropic deformed random fields, we could now develop an isotropy test for the deformed random field model, based on the mean Euler characteristic of excursion sets. It would follow the geometric approach of isotropy testing in [Ber18], where functionals of level sets provide statistics to test the isotropy of a deformed random field with linear deformation.



(a) X_θ with $\theta : x \mapsto \|x\| x$ and X Gaussian with Gaussian covariance.

(b) X_θ with a deformation with polar representation $\hat{\theta} : (r, \varphi) \mapsto (\sqrt{r}, r + \varphi)$ and X Gaussian with Gaussian covariance.

Figure III.1 – Simulations of the level sets of isotropic deformed random fields. Figure I.1a provides a simulation of a deformed random field constructed with a tensorial deformation.

Chapter IV
Anisotropic random wave models

In this last chapter, based on the preprint [EF18], we study anisotropic random waves. For many centuries, physicists have been using wave models defined on a multidimensional space in various domains as different as acoustics, electronics, geophysics, oceanography or seismology. In order to take into account variability or uncertainty, it is useful to consider random wave models. It is the exact purpose of a pioneer exhaustive study by Longuet and Higgins [LH57] that was concerned by sea waves modelized as a random moving surface. Another mathematical pioneer study was raised by Berry in several papers, [Ber77] or [BD00] for instance. These seminal works opened a wide area of research in the last decades, either for statistical purposes ([ALO05], [Lin06], [ARRL08], [ALW11], [BCW17], [NPR17]), or more recently for topological purposes in link with number theory ([RW08], [KKW13], [MPRW16]). Ten years ago, the interest for nodal sets or level sets met the theory of crossings developed by Rice for one-dimensional stochastic processes fifty years before, presented in Section I.2.1. The present chapter is clearly inspired by all the above references but to the best of our knowledge it is the first time that the different models are gathered in the same work and are studied under the same focus, the influence of anisotropy. Thus it meets the big demand for anisotropic models nowadays observed among practitioners, as mentioned earlier in this thesis.

We explore the anisotropy of random waves that are defined on a d-dimensional space with $d \geq 1$. Our first model is a single random wave given by $t \in \mathbb{R}^d \mapsto a\cos(\mathbf{k} \cdot t + \eta)$, where the directional structure is given by a d-dimensional random wavevector $\mathbf{k}$, the random phase η is uniformly distributed on $[0, 2\pi]$ and independent of $\mathbf{k}$, and the amplitude a is kept constant. Since our focus is on anisotropy, the latter assumption will remain unchanged all along the chapter. We also study the stationary Gaussian counterpart, *i.e.* a stationary Gaussian random field on $\mathbb{R}^d$ with the same covariance function $t \in \mathbb{R}^d \mapsto a\,\mathbb{E}[\cos(\mathbf{k} \cdot t)]$. Our purpose is to link the geometric and anisotropic behaviour properties of the random wave with the distribution of its random wavevector, in particular its moments of finite order and its directional statistics. In particular, considering Berry's anisotropic planar waves, we prove that the expected length of the nodal lines is a decreasing function of the (properly quantified) anisotropy of the random wavevector. In our study of random sea waves, we prove that the direction that maximises the expected length of the static crests is not necessarily orthogonal to the mode of the directional distribution of the wavevector.

As already mentioned, when $\mathbf{k}$ is equal to $A\mathbf{u}$ with A a matrix and $\mathbf{u}$ a random vector in $\mathbb{R}^d$ whose distribution is invariant under rotations, the associated random wave has the same distribution as an isotropic random wave deformed by the linear transformation A^T. In that case, the random wave coincides with an isotropic and stationary deformed random field, a specific case of the model studied in Chapter III. In that case, the study of anisotropy, either in the spectral domain, or in the parameter domain, is equivalent. However, in the general case, the deformation is not linear, and studying anisotropy in those two domains are two different approaches. The latter point of view is adopted in [ASP16] for instance, whereas our approach definitively belongs to the former type as did [BE03] or [RB10].

The chapter is organised as follows. General facts are presented in Section IV.1, in particular the key point of spectral representation. We also investigate how partial differential equations can be solved by some random waves according to the support of their random wavevector's distribution. Section IV.2 deals with the study of planar waves through specific tools that are used in directional statistics studies in dimension two, that is the most probable, the favorite and the principal directions. In Section IV.3, we introduce anisotropic versions of Berry's random waves, which are anisotropic solutions of Helmholtz equation. We focus on the Hausdorff measure and the directional statistics of their nodal sets. Section IV.4 is devoted to a a space-time model for sea waves. We study the mean length of static crests from a directional point of view. All over the chapter, two specific distributions of the random wavevector in dimension two are examined. One is called "elementary model". It is described by a main direction and a bandwidth that quantifies the anisotropy. The other one is called "toy model". It is given by a positive probability density function only depending on a single parameter that carries out the whole quantified information on anisotropy. The technical computations are detailed in the Appendix Section.

Notations.
Let d be a positive integer. We fix an orthonormal basis of $\mathbb{R}^2$ and we use the same notation for a vector z in $\mathbb{R}^d$ and the vector of its coordinates in this basis. For any z and z' in $\mathbb{R}^d$, we write $z \cdot z'$ the canonical Euclidian scalar product of z and z', $\|\cdot\|$ the associated norm and zz^T the $d \times d$ matrix $(z_i z_j)_{1\leq i,j\leq d}$.

For $\varphi \in [0, 2\pi]$, u_φ denotes the vector $(\cos\varphi, \sin\varphi)$ in $\mathbb{R}^2$.

We write $\mathbb{N}_0$ for the set $\{0, 1, 2, \cdots\}$ of the non-negative integers and $\mathbb{N}$ the set of postive integers.

For $\mathbf{j} = (j_1, \cdots, j_d) \in {\mathbb{N}_0}^d$, we write $|\mathbf{j}| = \sum_{l=1}^{d} j_l$. Moreover, if $\lambda \in \mathbb{R}^d$ and if F is a smooth map from $\mathbb{R}^d$ to $\mathbb{R}$, we write

$$\lambda^{\mathbf{j}} = \prod_{l=1}^{d} \lambda_l^{j_l} \quad \text{and} \quad \partial^{\mathbf{j}} F = \frac{\partial^{|\mathbf{j}|} F}{\partial^{j_1}\lambda_1 \cdots \partial^{j_d}\lambda_d}.$$

We also denote by $F'(t)$ and by $F''(t)$ the gradient vector and the Hessian matrix of F at point t, respectively.

For any positive integer s, $\mathscr{H}_s$ denotes the Hausdorff measure of dimension s.

Denoting by $\mathbf{k}$ a random vector in $\mathbb{R}^d$, we respectively write $\mathbb{E}[\mathbf{k}]$ and $\mathbb{V}[\mathbf{k}]$ the expectation (d-dimensional vector) of $\mathbf{k}$ and the variance ($d \times d$ matrix) of $\mathbf{k}$.

General setting

Anisotropic single random wave

Let d be a positive integer. We consider a random multidimensional model of single wave defined by,

$$\forall t \in \mathbb{R}^d, \quad X_{\mathbf{k}}(t) = \sqrt{2}\cos(\mathbf{k}\cdot t + \eta), \tag{1}$$

where $\mathbf{k}$ is a d-dimensional random vector called the random wavevector and where the random phase η is uniformly distributed on $[0, 2\pi]$ and independent of $\mathbf{k}$.

The random field $X_{\mathbf{k}}$ is clearly not isotropic and the kind of anisotropy depends on the law of $\mathbf{k}$. As it will be stated in Proposition IV.1, isotropy occurs if and only if $\mathbf{k}$ is isotropically distributed. If $||\mathbf{k}||$ is almost surely constant, we define $\kappa = ||\mathbf{k}||$ the wavenumber of $X_{\mathbf{k}}$.

We will be particularly interested in examples where the random wavevector $\mathbf{k}$ is supported by $\{\lambda \in \mathbb{R}^d : P(\lambda) = 0\}$, the zero set of a multivariate polynomial P.

Example 1 (Toy model) A particular model in the planar case ($d = 2$) is studied in [Kla16]. The random wavevector is prescribed by $\mathbf{k} = (\cos\Theta, \sin\Theta)$ with Θ a random variable with support in $\mathbb{R}/2\pi\mathbb{Z}$ such that, for a fixed $\alpha \geq 0$, the density of Θ with respect to Lebesgue measure on $[0, 2\pi]$ is given by

$$\theta \mapsto C_\alpha \,|\cos\theta|^\alpha, \text{ with } C_\alpha = \frac{\Gamma(1+\alpha/2)}{2\sqrt{\pi}\Gamma(1/2+\alpha/2)}, \tag{2}$$

where Γ is the usual Gamma function. Parameter α is considered as an anisotropy parameter. Indeed, taking $\alpha = 0$, one gets the isotropic version of model (1), whereas, at the opposite, the case $\alpha \to +\infty$ corresponds with a totally anisotropic version of the model where $\mathbf{k}$ is $a.s.$ along the x-axis.

Example 2 (Elementary model) Another particular case with $d = 2$ is studied in [BMR15] and [PCPC17]. The random wavevector is prescribed by $\mathbf{k} = (\cos\Theta, \sin\Theta)$ with Θ a random variable uniformly distributed on $[\alpha_0 - \delta, \alpha_0 + \delta]$ with $0 \leq \delta \leq \pi$. Parameter α_0 indicates the main direction whereas parameter δ quantifies anisotropy. Actually, the model is totally anisotropic if $\delta = 0$, the case $\delta \approx 0$ corresponds with a narrow spectrum model, and $\delta = \pi$ makes the random field isotropic. To get a symmetric model, one can also consider Θ uniformly distributed on $[\alpha_0 - \delta, \alpha_0 + \delta] \cup [\alpha_0 + \pi - \delta, \alpha_0 + \pi + \delta]$ with $0 \leq \delta \leq \pi/2$.

Example 3 (Berry's random wave) We assume that the wavevector $\mathbf{k} \in \mathbb{R}^d$ satisfies $||\mathbf{k}|| = \kappa$, $a.s.$ for some constant $\kappa > 0$ and that it is not necessarily isotropically distributed. The associated single wave is an anisotropic generalization of Berry's random wave model, an isotropic model that has originally been presented in [Ber77] and intensively studied in the last years. This model is the purpose of Section IV.3, and we study it study it more specifically in the Gaussian and planar case.

Example 4 (Sea waves) We will also examine the case where the random wavevector

$\mathbf{k}$ is supported by the Airy surface in $\mathbb{R}^3$ ($d = 3$), namely $\{(x, y, z) \in \mathbb{R}^3 \,;\, x^2+y^2-z^4 = 0\}$. The associated single wave is related to the space-time model used for the modelization of sea waves, assuming that the depth of the sea is infinite (see [LH57] for the original idea and [ALO05] or [AW09] for more recent developments). Section IV.4 is devoted to the study of this model.

Some basic properties of the covariance function of $X_{\mathbf{k}}$ are stated in the following proposition.

Proposition IV.1 *1. The random field $X_{\mathbf{k}}$ is centred and second-order stationary with covariance function*

$$r(t) := \mathrm{Cov}[X_{\mathbf{k}}(0), X_{\mathbf{k}}(t)] = \mathbb{E}[\cos(\mathbf{k} \cdot t)], \quad t \in \mathbb{R}^d. \tag{3}$$

In particular, $\mathrm{Var}(X_{\mathbf{k}}(0)) = 1$.

2. Let $\mathbf{k}^s$ be the symmetrized random variable associated to $\mathbf{k}$ and let F be its probability measure[1]*. Then*

$$r(t) = \mathbb{E}[\exp(i\mathbf{k}^s \cdot t)] = \int_{\mathbb{R}^d} \exp(iu \cdot t)\, dF(u), \tag{4}$$

which means that r is the characteristic function of the random variable $\mathbf{k}^s$ and that F is the spectral measure of $X_{\mathbf{k}}$.
Moreover, $X_{\mathbf{k}}$ is second-order isotropic if and only if the law of $\mathbf{k}^s$ is invariant under rotations.

3. The covariance function r admits derivatives up to order m ($m \in \mathbb{N}$) if and only if $\mathbf{k}$ admits moments of order m. In this case, for any $\mathbf{j} \in \mathbb{N}^d$ such that $0 < |\mathbf{j}| \leq m$, we have

$$\partial^{\mathbf{j}} r(0) = 0 \quad \text{if } |\mathbf{j}| \text{ is odd} \;\; ; \;\; \partial^{\mathbf{j}} r(0) = (-1)^{|\mathbf{j}|/2}\, \mathbb{E}[\mathbf{k}^{\mathbf{j}}] \quad \text{if } |\mathbf{j}| \text{ is even.}$$

In particular, $r''(0) = -\,\mathbb{E}[\mathbf{k}\mathbf{k}^T]$.

Anisotropic Gaussian wave model

We are still given a random vector $\mathbf{k}$ in $\mathbb{R}^d$ and we now consider a Gaussian, stationary and centred random field with the same covariance function as the single random wave $X_{\mathbf{k}}$ introduced in the previous section. Such a field exists, consequently to Kolmogorov's extension theorem (see [AW09] Sections 1.1 and 1.2 for instance), and its distribution is unique. Consequently, we call it the Gaussian random wave associated with the wavevector $\mathbf{k}$, and we name it $G_{\mathbf{k}}$.

Note that such a Gaussian field can be obtained as a limit by considering independent and identically distributed versions of η and of $\mathbf{k}$, denoted respectively by $(\eta_j)_{j\in\mathbb{N}}$

[1] If $F_{\mathbf{k}}$ and $F_{-\mathbf{k}}$ are respectively the probability measures of $\mathbf{k}$ and $-\mathbf{k}$, then the symmetrized random variable associated with $\mathbf{k}$ is defined as the random variable with probability measure $F = \frac{1}{2}(F_{\mathbf{k}}+F_{-\mathbf{k}})$.

and by $(\mathbf{k}_j)_{j\in\mathbb{N}}$. According to the central limit theorem applied to finite-dimensional distributions, the distribution of

$$\left(\sqrt{\frac{2}{N}}\sum_{j=1}^{N}\cos(\mathbf{k}_j\cdot t+\eta_j)\right)_{t\in\mathbb{R}^d}$$

converges as N tends to ∞ towards a Gaussian random field with the adequate covariance function.

The covariance function of $G_{\mathbf{k}}$ is given by (4) in Proposition IV.1: $\forall t\in\mathbb{R}^d$, $r(t)=\int_{\mathbb{R}^d}\exp(iu\cdot t)\,dF(u)$, where F is the distribution of $\mathbf{k}^s$. From this, we deduce a spectral representation of the field $G_{\mathbf{k}}$. Let W_F be a complex Gaussian F-noise on $\mathbb{R}^d$ that satisfies the condition

$$\forall A\in\mathscr{B}(\mathbb{R}^d),\quad \overline{W_F(A)}=W_F(-A).$$

As it is stated in Theorem I.9, the stationary random field prescribed by

$$\left(\int_{\mathbb{R}^d}e^{it\cdot u}\,dW_F(u)\right)_{t\in\mathbb{R}^d} \tag{5}$$

is real-valued, centred and Gaussian and its covariance function is given by (4).

Reciprocally, if $Y:\mathbb{R}^d\to\mathbb{R}$ is a centred and stationary Gaussian random field with unit variance, according to Bochner theorem (Theorem I.8), there exists a symmetric probability measure on $\mathbb{R}^d$, denoted by F, such that the covariance function r of Y is given by (4). It follows that we can associate with Y a symmetric random variable in $\mathbb{R}^d$ of probability measure F, denoted by $\mathbf{k}_Y$ and referred to in the following as "the random wavevector of Y".

Link with partial differential equation

We will show that both $X_{\mathbf{k}}$ and $G_{\mathbf{k}}$ satisfy a specific partial differential equation if and only if the random wavevector $\mathbf{k}$ is supported by a specific hypersurface of $\mathbb{R}^d$.

Let P be an even d-multivariate polynomial. Then there exists a sequence of real numbers $(\alpha_{\mathbf{j}})_{\mathbf{j}\in{\mathbb{N}_0}^d}$ with only finitely many non-zero terms, such that

$$\forall\lambda\in\mathbb{R}^d,\qquad P(\lambda)=\sum_{\mathbf{j}\in{\mathbb{N}_0}^d;\,|\mathbf{j}|\,\text{even}}\alpha_{\mathbf{j}}\,\lambda^{\mathbf{j}}. \tag{6}$$

We associate with P the following differential operator:

$$\mathscr{L}_P(X)=\sum_{\mathbf{j}\in{\mathbb{N}_0}^d;\,|\mathbf{j}|\,\text{even}}(-1)^{|\mathbf{j}|/2}\alpha_{\mathbf{j}}\,\partial^{\mathbf{j}}X,$$

Let us remark that the random field $X_{\mathbf{k}}$ defined by (1) is obviously almost surely of class $\mathscr{C}^\infty$.

Proposition IV.2 *Let P be an even multivariate polynomial given by* (6)*. Then $X_{\mathbf{k}}$ almost surely satisfies the partial differential equation*

$$\forall t \in \mathbb{R}^d, \qquad \mathscr{L}_P(X)(t) = 0 \tag{7}$$

if and only if $P(\mathbf{k}) = 0$ a.s.

Proof. For any $\mathbf{j} \in {\mathbb{N}_0}^d$ such that $|\mathbf{j}|$ is even, we have $\partial^{\mathbf{j}} X_{\mathbf{k}}(t) = (-1)^{|\mathbf{j}|/2}\mathbf{k}^{\mathbf{j}} \cos(\mathbf{k}\cdot t+\eta)$. Hence, we get $\mathscr{L}_P(X_{\mathbf{k}})(t) = P(\mathbf{k})\, X_{\mathbf{k}}(t)$ and the proof follows immediately. ■

Applying Proposition IV.2 to the examples given at the beginning of the section provides random anisotropic solutions of some famous partial differential equations. We recall that the Laplacian operator Δ on $\mathbb{R}^d$ is defined by $\Delta = \sum_{1\leq j\leq d} \frac{\partial^2}{\partial t_j^2}$. Then, the single random wave associated with Example 1 (case $d = 2$ and $\kappa = 1$) and Example 3 (any d and any κ) is an almost sure solution of Helmholtz equation $\Delta X + \kappa^2 X = 0$. In the same vein, the single random wave associated with Example 4 is an almost sure solution of the partial differential equation $\frac{\partial^2 X}{\partial x^2} + \frac{\partial^2 X}{\partial y^2} + \frac{\partial^4 X}{\partial z^4} = 0$.

Let us now be concerned with $G_{\mathbf{k}}$. We assume that the random wavevector $\mathbf{k}$ admits moments of any order. Hence, the covariance function r of $G_{\mathbf{k}}$ is of class $\mathscr{C}^\infty$ and consequently to Proposition I.7 there exists a version of $G_{\mathbf{k}}$ with almost every realization is of class $\mathscr{C}^\infty$; it is given by representation (5) for instance. First, let us point out that $G_{\mathbf{k}}$ satisfies Proposition IV.2 as well as $X_{\mathbf{k}}$. Indeed, $G_{\mathbf{k}}$ is centred and admits the same covariance function as $X_{\mathbf{k}}$; therefore for any multivariate polynomial P given by (6), for any $t \in \mathbb{R}^d$, $\mathrm{Var}\left(\mathscr{L}_P(G_{\mathbf{k}})(t)\right) = \mathrm{Var}\left(\mathscr{L}_P(X_{\mathbf{k}})(t)\right)$. However, the following theorem is a more general result: it provides a sufficient and necessary condition for any stationary Gaussian random field to satisfy Equation (7).

Theorem IV.3 *Let P be an even multivariate polynomial defined by* (6) *and let Y be Gaussian random field defined on $\mathbb{R}^d$ that is centred, stationary, with unit variance and almost surely of class $\mathscr{C}^\infty$. The following propositions are equivalent.*

1. *The random field Y almost surely satisfies the partial differential equation*

$$\forall t \in \mathbb{R}^d, \qquad \mathscr{L}_P(Y)(t) = 0.$$

2. *The random field Y admits a spectral representation given by* (5)*, where F is a probability measure supported by $\{\lambda \in \mathbb{R}^d : P(\lambda) = 0\}$ and W_F is a complex Gaussian F-noise on $\mathbb{R}^d$.*

3. *The random wavevector $\mathbf{k}_Y$ associated with Y almost surely satisfies $P(\mathbf{k}_Y) = 0$.*

We insist on the fact that the above theorem provides all the Gaussian *a.*s. solutions, isotropic or not, of the partial differential equation $\mathscr{L}_P(Y) = 0$ (for instance Helmholtz equation in the case of Example 1). Moreover, the equation gives information on the localization of the random variable $\mathbf{k}$.

Proof. Items 2 and 3 in Theorem IV.3 are clearly equivalent as F is the distribution of $\mathbf{k}_Y$. Since Y is centred, so are all its derivatives and the stationary random field $\mathscr{L}_P(Y)$. Therefore, $\mathscr{L}_P(Y)$ is almost surely identically zero if and only if its variance at each point is zero. But $\mathrm{Var}(\mathscr{L}_P(Y)(t))$ can be expressed as a linear combination of derivatives of the covariance function r_Y of Y. Hence Y is an *a.*s. solution of the partial differential equation $\mathscr{L}_P(Y) = 0$ if and only if its covariance function r_Y satisfies

$$\sum_{\mathbf{j},\mathbf{k}\in{\mathbb{N}_0}^d;\, |\mathbf{j}|,|\mathbf{k}| \text{ even}} (-1)^{(|\mathbf{j}|+|\mathbf{k}|)/2}\, \alpha_{\mathbf{j}}\, \alpha_{\mathbf{k}}\, \partial^{(\mathbf{j}+\mathbf{k})} r_Y(0) = 0. \tag{8}$$

On the other hand, as it is the covariance function of a stationary centred field, r_Y satisfies Bochner Theorem: there exists a Radon finite measure F on $\mathbb{R}^d$ such that $r_Y(t) = \hat{F}(t)$, where $\hat{F}$ denotes the Fourier transform of F, *i.*e. $\hat{F}(t) = \displaystyle\int_{\mathbb{R}^d} e^{it\cdot\lambda}\, dF(\lambda)$. Then r_Y satisfies (8) if and only if

$$0 = \int_{\mathbb{R}^d} \Big(\sum_{\mathbf{j},\mathbf{k}\in{\mathbb{N}_0}^d;\, |\mathbf{j}|,|\mathbf{k}| \text{ even}} (-1)^{|\mathbf{j}|+|\mathbf{k}|}\, \alpha_{\mathbf{j}}\, \alpha_{\mathbf{k}}\, \lambda^{\mathbf{j}}\, \lambda^{\mathbf{k}} \Big)\, dF(\lambda) = \int_{\mathbb{R}^d} P(\lambda)^2\, dF(\lambda).$$

The above integral vanishes if and only if the measure F is supported by $\{\lambda \in \mathbb{R}^d : P(\lambda) = 0\}$. ■

Remarkable directions in the planar case

We introduce some definitions related to planar models. To get more details in the domain of directional statistics, one can read [MJ09] or [LV17].

Most probable and favorite directions

When Z is a two-dimensional random vector, one can write it out either using Euclidean coordinates $Z = (Z_1, Z_2)$ or, if $Z \neq 0$, *a.s.*, using polar coordinates $Z = Ru_\Theta$, where R is a positive real random variable and Θ is a random variable in $\mathbb{R}/2\pi\mathbb{Z}$. (We recall that u_Θ denotes the vector $(\cos\Theta, \sin\Theta)$). Hence, we introduce two remarkable directions.

Definition IV.4 *Let Z be a random vector in $\mathbb{R}^2$ such that a.s. $Z \neq 0$.*
If the mode of the random variable Θ exists and is unique, we call it the most probable direction of Z. If there exists a mode that is not unique, we define the set of most probable directions of Z in $\mathbb{R}/2\pi\mathbb{Z}$ as the set of all modes of Θ.

If Θ is a discrete random variable then (at least) one most probable direction exists. If Θ is a continuous random variable with a probability density function (p.d.f.) h admitting a maximum on $\mathbb{R}/2\pi\mathbb{Z}$ (which is ensured if h is continuous), the set of the most probable directions can be expressed as the direction(s) in the set

$$\operatorname*{Argmax}_{\theta\in\mathbb{R}/2\pi\mathbb{Z}} h(\theta).$$

Note that if the distribution of the random vector (R, Θ) admits a probability density function $(r, \theta) \mapsto \tilde{f}(r, \theta)$ with respect to Lebesgue measure on $(0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z}$, then for any $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, $h(\theta) = \int_{(0,+\infty)} \tilde{f}(r, \theta)\, dr$.

Definition IV.5 *We assume that the matrix* $\mathbb{E}[ZZ^T]$ *does not belong to the set* $\{\alpha I_2,\ \alpha \geq 0\}$. *Then the favorite direction of* Z *is defined as the only element in*

$$\underset{\varphi \in \mathbb{R}/\pi\mathbb{Z}}{Argmax} \left(\mathbb{E}[(Z \cdot u_\varphi)^2] \right) = \underset{\varphi \in \mathbb{R}/\pi\mathbb{Z}}{Argmax} \left(u_\varphi \cdot \mathbb{E}[ZZ^T] u_\varphi \right).$$

Consequently, the favorite direction is nothing but the direction in $\mathbb{R}/\pi\mathbb{Z}$ *of the eigensubspace of* $\mathbb{R}^2$ *associated with the largest eigenvalue of the symmetric positive matrix* $\mathbb{E}[ZZ^T]$.

If $\mathbb{E}[ZZ^T] = \alpha I_2$ *with* $\alpha \geq 0$, *then* $\underset{\varphi \in \mathbb{R}/\pi\mathbb{Z}}{Argmax} \left(\mathbb{E}[(Z \cdot u_\varphi)^2] \right) = \mathbb{R}/\pi\mathbb{Z}$.

In some cases, such as in the following Examples 1, 3, 5 and 6, the most probable direction(s) modulo π coincides with the favorite direction(s). Nevertheless, in the general case, they don't.

Examples Let $Z = Ru_\Theta$ be a two-dimensional random vector such that $R \in (0, +\infty)$, *a.s.* and $\Theta \in [0, 2\pi)$, *a.s.*

1. If Θ almost surely takes a fixed value $\theta_0 \in [0, 2\pi)$, that is $Z = Ru_{\theta_0}$, then the most probable direction of Z is θ_0. On the other hand, for any $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$, $Z \cdot u_\varphi = R\cos(\theta_0 - \varphi)$ and hence the favorite direction of Z is θ_0 modulo π.

2. If (R, Θ) is distributed as $F_R \otimes \frac{1}{2}(\delta_0 + \delta_{\pi/2})$ on $(0, +\infty) \times [0, 2\pi)$, where δ stands for the Dirac distribution, then the most probable directions are 0 and $\pi/2$ modulo 2π whereas there is no favorite direction. In the same vein, with the distribution $F_R \otimes \frac{1}{2}(\delta_0 + \delta_{\pi/4})$, the most probable directions are 0 and $\pi/4$ modulo 2π whereas the favorite direction is $\pi/8$ modulo π.

3. If Θ and R are independent and if Θ is uniformly distributed on $[0, 2\pi]$, then Z admits $\mathbb{R}/2\pi\mathbb{Z}$ as its set of most probable directions. Moreover, Z is centred and $\mathbb{E}[ZZ^T] = \frac{1}{2}\mathbb{E}[R^2] I_2$, thus the set of favorite directions of Z is $\mathbb{R}/\pi\mathbb{Z}$.

4. If Θ and R are independent and if Θ is uniformly distributed on $[\alpha_0 - \delta, \alpha_0 + \delta]$ (see Example 2 in Section IV.1) , then the set of most probable directions is the whole interval $[\alpha_0 - \delta, \alpha_0 + \delta]$, whereas the favorite direction is reduced to the value α_0 modulo π.

5. If Θ admits a p.d.f. given by (2) (see Example 1 in Section IV.1), for a given $\alpha > 0$, and if Θ and R are independent, then the most probable direction of Z is clearly 0. On the other hand, Z is centred and $\mathbb{V}[Z] = \frac{\mathbb{E}[R^2]}{\alpha + 2} \begin{pmatrix} \alpha + 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence, the favorite direction of Z is 0 as well. We refer to Lemma IV.14 in Appendix section for the detailed computation of the moments.

6. Let Z be a 2-dimensional centred Gaussian vector with a variance matrix $\mathbb{V}[Z]$ that does not belong to $\{\alpha I_2,\, \alpha \in \mathbb{R}\}$. Then, the most probable direction of Z is $\mathrm{Argmin}_\varphi \left(u_\varphi \cdot \mathbb{V}[Z]^{-1} u_\varphi\right) = \mathrm{Argmax}_\varphi \left(u_\varphi \cdot \mathbb{V}[Z] u_\varphi\right)$, thus it is equal modulo π to the favorite direction of Z.

Principal direction

We now introduce a remarkable direction for real-valued and planar stationary random fields. Let $X : \mathbb{R}^2 \to \mathbb{R}$ be a stationary random field that is *a.s.* differentiable and satisfies $\mathbb{E}[||X'(0)||^2] < +\infty$. For φ a direction in $\mathbb{R}/\pi\mathbb{Z}$, we denote by $X^\varphi = (X(xu_\varphi))_{x\in\mathbb{R}}$ the one-dimensional stationary process obtained by restricting the X to the line $\mathbb{R}u_\varphi$.

Definition IV.6 *The principal direction of X is defined as*

$$\underset{\varphi\in\mathbb{R}/\pi\mathbb{Z}}{Argmax}\left(m_2(\varphi)\right), \quad \textit{where} \quad m_2(\varphi) = \mathbb{E}[(X^\varphi)'(0)^2],$$

understood as a certain value if the maximum is unique and as a set of values if it is not.

The latter notion has been introduced by Longuet-Higgins in [LH57] in his study of a planar random wave model for sea waves. Note that $m_2(\varphi)$ is nothing but the second spectral moment of X^φ and that restricting X to a certain line of the plane or to any parallel line does not change the law of the obtained process because X is stationary. We have also,

$$m_2(\varphi) = \mathbb{E}[(X^\varphi)'(0)^2] = \mathbb{E}[(X'(0)\cdot u_\varphi)^2] = \mathbb{E}[(X'(t)\cdot u_\varphi)^2],$$

for any $t \in \mathbb{R}^2$ by stationarity. It yields the following remark.

Remark IV.7 *For any $t \in \mathbb{R}^2$, the principal direction of X coincides with the favorite direction of $X'(t)$.*

Random planar waves

Let $\mathbf{k}$ be a random vector in $\mathbb{R}^2$ and let us consider the associated planar single random wave $X_\mathbf{k}$ and its Gaussian counterpart $G_\mathbf{k}$ as defined in Section IV.1. We now study these random fields from a directional point of view.

Proposition IV.8 *Let $\mathbf{k}$ be a random vector in $\mathbb{R}^2$ and let $Y : \mathbb{R}^2 \to \mathbb{R}$ be a stationary and centred random field with covariance function given by* (3)*. We assume that $\mathbb{E}[\mathbf{k}\mathbf{k}^T] \notin \{\alpha I_2, \alpha \in \mathbb{R}^+\}$.*
Then, the next three remarkable directions in $\mathbb{R}/\pi\mathbb{Z}$ coincide

- *the favorite direction of* $\mathbf{k}$
- *the principal direction of Y*
- *the favorite direction of $Y'(t)$ for any $t \in \mathbb{R}^2$.*

They are given by the direction of the eigensubspace of $\mathbb{R}^2$ *associated with the largest eigenvalue of matrix* $\mathbb{E}[\mathbf{k}\mathbf{k}^T]$.

Proof. It is enough to prove that the principal direction of Y is the favorite direction of $\mathbf{k}$ and then to apply Remark IV.7.
Let φ be fixed, the covariance function of the univariate process Y_φ along a line of direction φ is given for any $x \in \mathbb{R}$ by $r_\varphi(x) = r(xu_\varphi) = \mathbb{E}[\cos(x\,\mathbf{k}.u_\varphi)]$. Hence $m_2(\varphi) = -r''_\varphi(0) = \mathbb{E}[(\mathbf{k}.u_\varphi)^2]$, which clearly yields the equality between the principal direction of Y and the favorite direction of $\mathbf{k}$. ■

We now turn to the directional study of the level sets of the Gaussian planar random waves $G_\mathbf{k}$. We assume that $\mathbb{E}[\mathbf{k}\mathbf{k}^T] \notin \{\alpha I_2, \alpha \in \mathbb{R}^+\}$ and we fix $a \in \mathbb{R}$. The level set

$$G_\mathbf{k}^{-1}(a) = \{t \in \mathbb{R}^2 :\ G_\mathbf{k}(t) = a\}$$

is a finite union of curves whose direction at point $t \in G_\mathbf{k}^{-1}(a)$ is orthogonal to the vector $G'_\mathbf{k}(t)$. Applying Proposition IV.8 yields the next statement, that sounds physically intuitive.

Proposition IV.9 *Let* $a \in \mathbb{R}$. *Let* τ_a *be a two-dimensional vector field defined on the level set* $G_\mathbf{k}^{-1}(a)$ *such that, at any point* t, $\tau_a(t)$ *is tangent to* $G_\mathbf{k}^{-1}(a)$ *at* t. *Then, for any* $t \in G_\mathbf{k}^{-1}(a)$, *the favorite direction of* $\tau_a(t)$ *is orthogonal to the favorite direction of* $\mathbf{k}$.

Let us mention that the above proposition still holds in dimension $d > 2$ once the favorite direction of a d-dimensional random vector is defined as the direction of the eigenspace that is associated with the largest eigenvalue of $\mathbb{V}[Z]$.

Berry's anisotropic random waves

In this section, we focus on Example 3 of Section IV.1, *i.e.* on the case where the random wavevector $\mathbf{k}$ is such that, for some $\kappa > 0$,

$$\kappa^{-1}\mathbf{k} \in \mathbb{S}^{d-1}\ a.s.$$

As previously, we consider the (unique in distribution) associated stationary centred Gaussian random field $G_\mathbf{k}$ on $\mathbb{R}^d$ whose covariance function r is given by (3). Since $||\mathbf{k}||$ is $a.s.$ bounded, it is clear that $G_\mathbf{k}$ is $a.s.$ smooth then, rephrasing Theorem IV.3, we get that $G_\mathbf{k}$ is the generic Gaussian solution of Helmholtz equation

$$\Delta Y + \kappa^2 Y = 0.$$

Equivalently, $G_\mathbf{k}$ is an eigenfunction of the operator $-\Delta$, for the eigenvalue κ^2. Therefore, extending the definition introduced by Berry in [Ber77], we refer to $G_\mathbf{k}$ as a Berry's anisotropic wave with random wavenumber κ.

Applying the appropriate change of variables $t \mapsto \kappa t$ yields the scaling property that $(G_\mathbf{k}(t))_{t\in\mathbb{R}^d}$ and $(G_{\kappa^{-1}\mathbf{k}}(\kappa t))_{t\in\mathbb{R}^d}$ have the same distribution, where we recall that the random vector $\kappa^{-1}\mathbf{k}$ takes its values in $\mathbb{S}^{d-1}$. We also remark that, if the distribution

of $\kappa^{-1}\mathbf{k}^s$ admits a density $\tilde{f}$ with respect to the surface measure σ on $\mathbb{S}^{d-1}$, we can deduce from (4) that the covariance function of $G_{\mathbf{k}}$ is given by

$$r(t) = \int_{\mathbb{S}^{d-1}} e^{i\kappa u\cdot t}\tilde{f}(u)\, d\sigma(u).$$

Expected measure of level sets

We are now interested in the random level sets: for any $a \in \mathbb{R}$,

$$G_{\mathbf{k}}^{-1}(a) = \{t \in \mathbb{R}^d \,/\, G_{\mathbf{k}}(t) = a\},$$

which has Hausdorff dimension $d-1$ *a.s.*. If $a = 0$, this is exactly the nodal set of $G_{\mathbf{k}}$ and more precisely in the case $d = 2$, it is the nodal line of a Berry's anisotropic planar wave.

Let Q be a compact set in $\mathbb{R}^d$ with non empty interior and let $a \in \mathbb{R}$. We focus on the $(d-1)$-dimensional Hausdorff measure of the a-level set of $G_{\mathbf{k}}$ restricted to Q, namely

$$\ell(a, \mathbf{k}, Q) = \mathscr{H}_{d-1}\left(G_{\mathbf{k}}^{-1}(a) \cap Q\right) = \mathscr{H}_{d-1}\left(\{t \in Q \,/\, G_{\mathbf{k}}(t) = a\}\right).$$

Proposition IV.10 *Let $\kappa > 0$ and assume that $\mathbf{k}$ is a random vector in $\mathbb{R}^d$ such that $\widetilde{\mathbf{k}} := \kappa^{-1}\mathbf{k} \in \mathbb{S}^{d-1}$ a.s.*
Let Φ_d stand for the standard Gaussian probability density function on $\mathbb{R}^d$. Then,

$$\mathbb{E}[\ell(a, \mathbf{k}, Q)] = \mathscr{H}_d(Q)\, \frac{e^{-a^2/2}}{\sqrt{2\pi}}\, \kappa \int_{\mathbb{R}^d} (\mathbb{E}[\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T]x \cdot x)^{1/2} \Phi_d(x)\, dx. \tag{9}$$

Proof. This result is an application of Rice formula stated in Theorem I.12 to $G_{\mathbf{k}}$, which satisfies the required hypothesis. It yields

$$\mathbb{E}[\ell(a, \mathbf{k}, Q)] = \int_Q \mathbb{E}[\parallel G'_{\mathbf{k}}(t) \parallel \mid G_{\mathbf{k}}(t) = a]\, p_{G_{\mathbf{k}}(t)}(a)\, dt,$$

where $p_{G_{\mathbf{k}}(t)}$, the probability density function of $G_{\mathbf{k}}(t)$, is actually given by the standard Gaussian distribution. Using the stationarity of $G_{\mathbf{k}}$ and the fact that for a fixed point t, $G_{\mathbf{k}}(t)$ and $G'_{\mathbf{k}}(t)$ are independent random variables, we have

$$\mathbb{E}[\ell(a, \mathbf{k}, Q)] = \mathscr{H}_d(Q)\, \frac{e^{-a^2/2}}{\sqrt{2\pi}}\, \mathbb{E}[\parallel G'_{\mathbf{k}}(0) \parallel].$$

In order to conclude, it only remains to state that $\parallel G'_{\mathbf{k}}(0) \parallel$ is the Euclidean norm of a d-dimensional centred Gaussian vector with variance matrix $-r''(0) = \mathbb{E}[\mathbf{k}\mathbf{k}^T] = \kappa^2\mathbb{E}[\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T]$. ■

We remark that the same proof (except last equality) can be applied to any random wavevector $\mathbf{k}$ with finite moments, even if $||\mathbf{k}||$ is not constant. It yields the

following identity that is valid when dropping the condition $\widetilde{\mathbf{k}} := \kappa^{-1}\mathbf{k} \in \mathbb{S}^{d-1}$ *a.s.* in Proposition IV.10,

$$\mathbb{E}[\ell(a,\mathbf{k},Q)] = \mathscr{H}_d(Q)\,\frac{e^{-a^2/2}}{\sqrt{2\pi}}\,\int_{\mathbb{R}^d}(\mathbb{E}[\mathbf{k}\mathbf{k}^T]x\cdot x)^{1/2}\Phi_d(x)\,dx.$$

Let us come back to Berry's random waves. In the isotropic case, that is to say when $\widetilde{\mathbf{k}}$ is uniformly distributed on $\mathbb{S}^{d-1}$, $\mathbb{E}[\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T] = \mathbb{V}[\widetilde{\mathbf{k}}] = (1/d)\,I_d$. Hence,

$$\mathbb{E}[\ell(a,\mathbf{k},Q)] = \mathscr{H}_d(Q)\,\frac{e^{-a^2/2}}{\sqrt{2\pi d}}\,\kappa\,\int_{\mathbb{R}^d}\parallel x\parallel\;\Phi_d(x)\,dx,$$

where the above integral is the mean of a χ-distributed random variable with d degrees of freedom and is known to be equal to $\sqrt{2}\,\dfrac{\Gamma((d+1)/2)}{\Gamma(d/2)}$.

Expected length of level curves

In the planar case, *i.e.* $d = 2$, the level sets $G_{\mathbf{k}}^{-1}(a)$ are one-dimensional and Formula (9) can be made much more precise. In particular, the following proposition states that the level curves mean length is decreasing as anisotropy is increasing.

Proposition IV.11 *Let* $\mathbf{k}$ *be a random vector in* $\mathbb{R}^2$ *such that* $\mathbf{k} = \kappa\,\widetilde{\mathbf{k}}$ *with* κ *a positive constant and* $\widetilde{\mathbf{k}} \in \mathbb{S}^1$ *a.s. Let us denote by* $c(\widetilde{\mathbf{k}})$ *the difference between the eigenvalues of* $\mathbb{E}[\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T]$ *(*$0 \le c(\widetilde{\mathbf{k}}) \le 1$*). Let* $\mathscr{E}$ *be the elliptic integral given by* $\mathscr{E}(x) = \displaystyle\int_0^{\pi/2}(1 - x^2\,\sin^2\theta)^{1/2}d\theta$*, for* $x \in [0,1]$*. Then,*

$$\mathbb{E}[\ell(a,\mathbf{k},Q)] = \mathscr{H}_2(Q)\,\frac{e^{-a^2/2}}{\pi\sqrt{2}}\,\kappa\,\mathscr{F}\big(c(\widetilde{\mathbf{k}})\big),$$

where the map $\mathscr{F} : c \in [0,1] \mapsto (1+c)^{1/2}\,\mathscr{E}\left(\left(\dfrac{2c}{1+c}\right)^{1/2}\right)$ *is strictly decreasing.*

Remark IV.12 *In the isotropic case,* $c(\widetilde{\mathbf{k}}) = 0$ *and hence we recover the following result concerning the nodal line of the isotropic Berry's planar wave (see [Ber77]):* $\mathbb{E}[\ell(0,\mathbf{k},Q)] = \mathscr{H}_2(Q)\,\dfrac{\kappa}{\pi\sqrt{2}}\mathscr{E}(0) = \mathscr{H}_2(Q)\,\dfrac{\kappa}{2\sqrt{2}}$.

Remark IV.13 *In directional statistics, it is usual to introduce a parameter termed coherency index and defined as the ratio between the difference of eigenvalues and the sum of eigenvalues of a certain positive symmetric matrix* M*, see [MJ09]. This index is performed in [PCPC17] (see also [Kla16]) with* M *given by the so-named structure tensor in order to quantify the anisotropy of an anisotropic Gaussian planar field. In our context, we like to remark that the trace of matrix* $M = \mathbb{E}[\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T]$ *is equal to one, since* $||\widetilde{\mathbf{k}}|| = 1$*, a.s.. Parameter* $c(\widetilde{\mathbf{k}})$ *actually coincides with the coherency index of our model and hence quantifies its anisotropy.*

Proof. We use Proposition IV.10 in the case $d = 2$. For computing the integral in the right-hand side of (9), we use the following well known fact, that can be proved with simple algebra.
If M is a symmetric positive semidefinite matrix with eigenvalues γ_- and γ_+ such that $0 \leq \gamma_- \leq \gamma_+$ and $\gamma_+ > 0$, then

$$\int_{\mathbb{R}^2} (Mx \cdot x)^{1/2} \Phi_2(x)\, dx = \left(\frac{2\gamma_+}{\pi}\right)^{1/2} \mathscr{E}\left((1 - \gamma_-/\gamma_+)^{1/2}\right). \tag{10}$$

In our case, $M = \mathbb{E}[\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T]$ and $\gamma_- + \gamma_+ = 1$. Hence, $2\gamma_+ = 1 + c$ and $1 - \gamma_-/\gamma_+ = \dfrac{2c}{1+c}$. The proof of the decreasing of mapping $\mathscr{F}$ is postponed to the Appendix section, see Lemma IV.15. ∎

We end the section applying Proposition IV.11 to our two favorite examples.

Example 1 (Toy model) Take $\widetilde{\mathbf{k}}$ distributed on $\mathbb{S}^1$ with probability density function given by (2) for some positive α (see Example 1 Section IV.1). The moments of $\widetilde{\mathbf{k}}$ are computed in the Appendix section, Lemma IV.14. In particular, it holds $\mathbb{E}[\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T] = \dfrac{1}{\alpha+2}\begin{pmatrix} \alpha+1 & 0 \\ 0 & 1 \end{pmatrix}$. Consequently, $c(\widetilde{\mathbf{k}}) = \dfrac{\alpha}{\alpha+2}$, which is an increasing function of parameter α. Thus, the more anisotropic the model is, the smaller the expected length of level sets is.

Example 2 (Elementary model) We choose the random wavevector $\mathbf{k} = \kappa\,\widetilde{\mathbf{k}}$ with $\widetilde{\mathbf{k}}$ uniformly distributed on $[\alpha_0 - \delta, \alpha_0 + \delta] \cup [\alpha_0 + \pi - \delta, \alpha_0 + \pi + \delta]$ for some $0 < \delta \leq \pi/2$, see Example 2 of Section IV.1. In order to simplify the computation, let us assume that $\alpha_0 = 0$. In that case, $\mathbb{E}[\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T] = \dfrac{1}{2}\begin{pmatrix} 1 + \dfrac{\sin(2\delta)}{2\delta} & 0 \\ 0 & 1 - \dfrac{\sin(2\delta)}{2\delta} \end{pmatrix}$ and hence $c(\widetilde{\mathbf{k}}) = \dfrac{\sin(2\delta)}{2\delta}$, which is decreasing on $[0, \pi/2]$. Again, the mean length of level sets is decreasing as the anisotropy of $\mathbf{k}$ is increasing, *i.e.* as δ is decreasing.

Gaussian sea waves

In this section, we now concentrate on Example 4 in Section IV.1 that considers the case where the random wavevector is 3-dimensional and *a.s.* belongs to Airy surface, *i.e.*

$$\mathbf{k} \in \Lambda = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3\,;\, (\lambda_1)^2 + (\lambda_2)^2 = (\lambda_3)^4\} \ a.s..$$

We study the Gaussian random wave $G_{\mathbf{k}}$ associated with $\mathbf{k}$, as defined in Section IV.1.2. Its covariance function is

$$r(t) = \int_\Lambda \cos(t \cdot \lambda)\, dF(\lambda)\ ,\ t \in \mathbb{R}^3,$$

where F is the probability distribution of $\mathbf{k}$.

The field $G_{\mathbf{k}}$ coincides with the spatio-temporal Gaussian random fields that are used for the modelization of sea waves [LH57, ALO05, AW09]. Indeed, for $(x, y, s) \in \mathbb{R}^2 \times \mathbb{R}$, $G_{\mathbf{k}}(x, y, s)$ can be seen as the algebraic height of a wave at point (x, y) and time s.

We use the following parametrization of Λ,

$$(\theta, z) \in [0, 2\pi) \times \mathbb{R} \mapsto (z^2 \cos\theta, z^2 \sin\theta, z),$$

which provides a bijection φ from $[0, 2\pi) \times \mathbb{R} \setminus \{0\}$ onto $\Lambda \setminus \{(0, 0, 0)\}$. Performing the appropriate change of variables yields

$$r(x, y, s) = \int_{(0,2\pi)\times\mathbb{R}} \cos(xz^2 \cos\theta + yz^2 \sin\theta + sz)\, d\widetilde{F}(\theta, z),$$

where $\widetilde{F}$ is the image of measure F by the map φ^{-1}. When $\mathbf{k}$ admits f as probability density function with respect to the surface measure on Λ, consequently to coarea formula (see for instance [AT07] Section 7.4), we get

$$r(x, y, s) = \int_{(0,2\pi)\times\mathbb{R}} \cos(xz^2 \cos\theta + yz^2 \sin\theta + sz)\widetilde{f}(\theta, z)\, d\theta dz,$$

where the map $\widetilde{f}$ is given by

$$\widetilde{f}(\theta, z) = f(z^2 \cos\theta, z^2 \sin\theta, z)\, z^2(1 + 4z^2)^{1/2}.$$

Following the literature, $\widetilde{f}$ is called *directional power spectrum* of $G_{\mathbf{k}}$ (see [ALO05] and [AW09] Chapter 11). Experimental directional power spectra are exhibited in [ALO05], derived from sea data provided by Ifremer.

In order to avoid heavy notations, from now on we assume that the random wavevector $\mathbf{k}$ is symmetrically distributed. Hence, until the end of the present section we deal with the following covariance function

$$r(x, y, s) = \int_{\Lambda} e^{i((x,y,s)\cdot\lambda)}\, dF(\lambda),$$

where F is a probability measure on Λ satisfying $F(-A) = F(A)$ for any Borelian set $A \subset \Lambda$. In other words,

$$r(x, y, s) = \int_{(0,2\pi)\times\mathbb{R}} e^{i(xz^2 \cos\theta + yz^2 \sin\theta + sz)}\, d\widetilde{F}(\theta, z)$$

with $\widetilde{F}$ a probability measure on $\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R} \setminus \{0\}$ that is invariant under the mapping $(\theta, z) \mapsto (\theta + \pi, -z)$. If $\mathbf{k}$ is not symmetrically distributed, the key to get the above expressions is to use the symmetrized probability measure of $\mathbf{k}$ instead of its probability measure.

Let us fix time $s = s_0$ and look at the random field defined on $\mathbb{R}^2$,

$$Z_{\mathbf{k}}(x, y) = G_{\mathbf{k}}(x, y, s_0) \qquad (x, y) \in \mathbb{R}^2,$$

as a picture of the sea height at time s_0. It is a two-dimensional stationary centred Gaussian random field, whose covariance function is given by

$$\Gamma(x,y) = r(x,y,0) = \int_{(0,2\pi)\times\mathbb{R}} e^{i(xz^2\cos\theta + yz^2\sin\theta)}\, d\widetilde{F}(\theta,z).$$

Actually, the random wavevector associated with $Z_{\mathbf{k}}$ is nothing but the projection of the Λ-valued random wavevector $\mathbf{k}$ onto the first two coordinates. We call it $\pi(\mathbf{k})$ in what follows. We will also need the spectral moments of $Z_{\mathbf{k}}$, namely for any integers j and k in $\mathbb{N}_0$

$$\begin{aligned} m_{j,k} &:= (-i)^{j+k}\, \partial^{(j,k)}\Gamma(0,0) \\ &= \int_{(0,2\pi)\times\mathbb{R}} (z^2\cos\theta)^j (z^2\sin\theta)^k\, d\widetilde{F}(\theta,z). \end{aligned} \tag{11}$$

Mean length of static crests

We are now interested in the (static) crest in direction $\varphi \in \mathbb{R}/\pi\mathbb{Z}$. More precisely, we introduce the random set

$$\{(x,y)\in\mathbb{R}^2\,;\, Z'_{\mathbf{k}}(x,y)\cdot u_\varphi = 0\},$$

which contains all points (x,y) in $\mathbb{R}^2$ such that the gradient of $Z_{\mathbf{k}}$ at point (x,y) is orthogonal to direction φ. One can also say that the derivative of $Z_{\mathbf{k}}$ in direction φ at those points is zero. Hence, the crest in direction φ is the nodal set of the derivative of $Z_{\mathbf{k}}$ in direction φ. Note that it is a special case of a *specular points* set as defined in [LH57]. Its Hausdorff dimension is clearly equal to one and one can compute its length within a compact domain $Q \subset \mathbb{R}^2$ such that $\mathscr{H}_1(Q) > 0$,

$$l(\mathbf{k}, Q, \varphi) := \mathscr{H}_1\left(\{(x,y)\in Q\,;\, Z'_{\mathbf{k}}(x,y)\cdot u_\varphi = 0\}\right).$$

Using the same arguments as for the proof of Proposition IV.10 and Formula (10), based on Rice formula, we get the following result that is also stated in [AW09] (Proposition 11.4) or in [ALO05] (Assertion 3).

$$\mathbb{E}[l(\mathbf{k},Q,\varphi)] = \mathscr{H}_2(Q)\,\frac{1}{\pi}\left(\frac{\gamma_+(\varphi)}{v(\varphi)}\right)^{1/2} \mathscr{E}\big((1-\gamma_-(\varphi)/\gamma_+(\varphi))^{1/2}\big), \tag{12}$$

where $v(\varphi) = \mathrm{Var}(Z'_{\mathbf{k}}(0)\cdot u_\varphi)$ and $\gamma_-(\varphi) \le \gamma_+(\varphi)$ are the eigenvalues of the covariance matrix of the gradient of $Z'_{\mathbf{k}}\cdot u_\varphi$, which we can write $\mathbb{V}[Z''_{\mathbf{k}}(0)\,u_\varphi]$. We could also have considered the gradient of $Z'_{\mathbf{k}}\cdot u_\varphi$ computed by derivating in the directions of the vectors of the orthonormal basis $(u_\varphi, u_{\varphi+\pi/2})$. At a fixed $t\in\mathbb{R}^2$, it is equal to $Z''_{\mathbf{k}}(t)\,u_\varphi$ up to a change of basis and its covariance matrix is similar to $\mathbb{V}[Z''_{\mathbf{k}}(0)\,u_\varphi]$, thus it has the same eigenvalues. Therefore, the formula given in [AW09] Proposition 11.4 or [ALO05] Assertion 3 seems different from ours but it is in fact the same as (12). In Lemma IV.16 in the Appendix section, the coefficients of both covariance matrices are given, expressed in terms of the spectral moments $m_{j,k}$ of $Z_{\mathbf{k}}$. We also give the simple expressions for their eigenvalues.

The end of this section is dedicated at showing that the direction that maximises the expected crest length may be orthogonal to the most probable direction of the wavevector $\pi(\mathbf{k})$, introduced in Definition IV.4, but not necessarily (if such directions exist). It is a clear consequence of Formula (12), which shows a dependence on both the second-order and the fourth-order moments of $\mathbf{k}$, while the most probable direction depends on the mode of $\mathbf{k}$. Nevertheless, a rule of thumb is suggested in [LH57] and in [ALO05], claiming that *the direction [that maximises the expected length of crests] is orthogonal to the direction for the maximum integral of the spectrum, i.e. is the most probable direction for the waves.* In this statement, the "most probable direction for the waves" has to be understood as the most probable direction of the random wavevector $\pi(\mathbf{k})$.

Example 4.1 (Elementary wave) We consider a random wavevector $\mathbf{k}$ with values in Λ *a.s.* and with a deterministic orientation. Precisely, the distribution of $\mathbf{k}$ is prescribed on $[0, 2\pi) \times \mathbb{R}$ by $\widetilde{F} = \frac{1}{2}(\delta_{\alpha_0} + \delta_{\alpha_0+\pi}) \otimes h$, where δ_{α_0} stands for the Dirac measure at $\alpha_0 \in [0, \pi)$ and h is any symmetric probability measure on $\mathbb{R} \setminus \{0\}$.
On the one hand, the most probable direction of $\pi(\mathbf{k})$ is clearly α_0 modulo π.
On the other hand, the spectral moments of $Z_{\mathbf{k}}$ are easy to compute from (11). In the simplest case where $\alpha_0 = 0$, we get that $m_{2,0} = M_2$ and $m_{4,0} = M_4$ with $M_k := \int_R z^{2k}\, dh(z)$, and that all the other moments up to order 4 are vanishing. Hence, following Lemma IV.16, $v(\varphi) = M_2 \cos^2\varphi$ and

$$\mathbb{V}[Z''_{\mathbf{k}}(0)\, u_\varphi] = M_4 \cos^2\varphi \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence $\gamma_-(\varphi) = 0$ and $\gamma_+(\varphi) = M_4 \cos^2\varphi$, and Formula (12) allows us to state that the expected length of the crests of $Z_{\mathbf{k}}$ in direction φ does not depend on φ. Therefore, for this model, no link can be established between the direction that maximises the expected length of crests and the most probable direction of $\pi(\mathbf{k})$.

Example 4.2 (Counter-example) We now consider a random vector $\mathbf{k}$ whose distribution is prescribed on $[0, 2\pi) \times \mathbb{R}$ by $\widetilde{F} = (\frac{1}{4}\sum_{j=0}^{3} \delta_{j\pi/2}) \otimes h$, with h any symmetric probability measure on $\mathbb{R} \setminus \{0\}$. Then, the set of most probable directions of $\mathbf{k}$ is $\{j\pi/2 : j = 0, 1, 2, 3\}$.
Computing the spectral moments of $Z_{\mathbf{k}}$ from (11), we get $m_{1,1} = m_{2,2} = m_{3,1} = m_{1,3} = 0$ whereas $m_{2,0} = m_{0,2} = \dfrac{M_2}{2}$ and $m_{4,0} = m_{0,4} = \dfrac{M_4}{2}$, where we have introduced $M_k := \int_{\mathbb{R}} z^{2k}\, dh(z)$. Consequently, using Lemma IV.16, for any $\varphi \in [0, 2\pi]$, $v(\varphi) = \mathrm{Var}(Z'_{\mathbf{k}}(0) \cdot u_\varphi) = \dfrac{M_2}{2}$ and

$$\mathbb{V}[Z''_{\mathbf{k}}(0)\, u_\varphi] = \frac{M_4}{2} \begin{pmatrix} \cos^2\varphi & 0 \\ 0 & \sin^2\varphi \end{pmatrix}.$$

The eigenvalues of the later matrix being $\frac{M_4}{2}\cos^2\varphi$ and $\frac{M_4}{2}\sin^2\varphi$, we apply (12) to get the expected length of the crests of $Z_{\mathbf{k}}$ in direction φ. Up to a positive multiplicative constant that does not depend on φ, $\mathbb{E}[l(\mathbf{k}, Q, \varphi)]$ is equal to

$$(1+|\cos(2\varphi)|)^{1/2}\,\epsilon\Big(\Big(\frac{2\,|\cos(2\varphi)|}{1+|\cos(2\varphi)|}\Big)^{1/2}\Big) = \mathscr{F}(|\cos(2\varphi)|),$$

where the function $\mathscr{F}$ is defined in Proposition IV.11. Since $\mathscr{F}$ is strictly decreasing on $[0,1]$ (see Lemma IV.15), the mean length of crests is maximal when $\cos(2\varphi) = 0$, *i.e.* for $\varphi = \pi/4$ or $3\pi/4$ modulo π. These directions are not orthogonal to the most probable directions of $\pi(\mathbf{k})$.

Mean length of static crests with the toy model

We consider the Gaussian wave with directional power spectrum $\widetilde{f}$ given by

$$\widetilde{f}(\theta, z) = C_\alpha\,|\cos\theta|^\alpha\,h(z), \tag{13}$$

where α is a positive real number (see Equation (2)) and h is an even probability density function on $\mathbb{R}$. As already mentioned, the most probable direction of $\pi(\mathbf{k})$ is 0 in that case.

The spectral moments (see (11)) of this particular Gaussian wave are given, for any j, k in $\mathbb{N}_0$, by

$$\begin{aligned} m_{j,k} &= \Big(\int_{\mathbb{R}} z^{2j+2k}\,h(z)\,dz\Big)\,\left(C_\alpha\int_{(0,2\pi)}(\cos\theta)^j\,(\sin\theta)^k\,|\cos\theta|^\alpha\,d\theta\right) \\ &:= M_{j+k}\,\mu_{j,k}. \end{aligned} \tag{14}$$

The first integral M_{j+k} equals the moment of order $2j+2k$ of h. Note that it does not contain any information on the anisotropy of the model. The second integral, named as $\mu_{j,k}$, is computed in Lemma IV.14 in Appendix section.

Hence, the expected length of crests in a given direction φ can be evaluated through Formula (12) applied to this specific model. An asymptotic expansion of $\varphi \mapsto \mathbb{E}[\ell(\mathbf{k}, Q, \varphi)]$ near $\varphi = \pi/2$ is performed in Lemma IV.17. It shows that the expected length of crests admits a local maximum at $\varphi = \pi/2$, which is precisely orthogonal to the most probable direction of $\pi(\mathbf{k})$ and to its favorite direction as well (see the fifth example in Section IV.2.1).

Conclusion and perspectives

A general definition of an anisotropic random wave model is the entry point of this chapter. This stationary model has an almost sure non Gaussian definition (the so-called single wave model) but also has a Gaussian counterpart. Two notable existing models fall within ours: Berry's isotropic random wave model and the see wave model

first investigated by Longuet-Higgins. Note that any stationary, normalized and centred Gaussian field can actually be seen as a random wave, its wavevector being a random variable distributed according to its spectral measure.

We propose to study the distributional properties of a random wave through the ones of its random wavevector, with a geometric perspective. In Proposition IV.2 and Theorem IV.3, we show that the geometric support of the wavevector may provides a partial differential equation satisfied by the random wave. In the planar case, thanks to directional tools for random vectors and random fields, we exhibit links between notable directions of a random wave and its wavevector. Deriving a formula for the mean length of level sets in Berry's anisotropic wave model yields the interesting conclusion that the expected length of level sets decreases as the random wavevector's anisotropy increases (see Proposition IV.11). In the Gaussian sea waves model, we investigate the link between the direction(s) that maximize(s) the expected length of static crests and the mode of the random wavevector's directional distribution. We come to the conclusion that these directions are not necessarily orthogonal.

To investigate further the links between the geometric distributional properties of a random wave and its wavevector's distribution, the next step would be to study the variance of the geometric characteristics at stake. For a random wave model with a wavevector following a parametric law, this would allow to derive estimators and possibly to develop a test of isotropy.

Appendix

Moments of a random wavevector given by the toy model

We perform some computations related to our toy model given by Example 1 in Section IV.1. We fix $\alpha \geq 0$ and we consider a two-dimensional random wavevector $\mathbf{k} = (\cos\Theta, \sin\Theta)$, with Θ that takes value in $[0, 2\pi]$ with a probability density function given by

$$\theta \mapsto C_\alpha |\cos\theta|^\alpha \quad \textit{with} \quad C_\alpha = \frac{\Gamma(1+\alpha/2)}{2\sqrt{\pi}\Gamma(1/2+\alpha/2)}.$$

Lemma IV.14 *For any non negative integers j and k, let $\mu_{j,k}$ be the (j,k)-moment of $\mathbf{k}$, i.e.*

$$\mu_{j,k} = \mathbb{E}[(\cos\Theta)^j\,(\sin\Theta)^k] = C_\alpha \int_{[0,2\pi]} (\cos\theta)^j (\sin\theta)^k\,|\cos\theta|^\alpha\,d\theta.$$

Then

- $\mu_{0,0} = 1$
- $\mu_{j,k} = 0$ *whenever j or k is odd*
- $\mu_{j,0} = \dfrac{C_\alpha}{C_{\alpha+j}} = \dfrac{(\alpha+1)(\alpha+3)\cdots(\alpha+j-1)}{(\alpha+2)(\alpha+4)\cdots(\alpha+j)}$ *for j even ≥ 2*
- *for any even integers j and k,* $\mu_{j,k} = \displaystyle\sum_{i=0}^{k/2} (-1)^i \binom{k/2}{i} \mu_{j+2i,0}$.

In particular, it yields the non-zero second-order and fourth-order moments of $\mathbf{k}$:

$$\mu_{2,0} = \frac{\alpha+1}{\alpha+2};\ \mu_{0,2} = \frac{1}{\alpha+2} \quad \text{and hence} \quad \mathbb{E}[\mathbf{k}\mathbf{k}^T] = \frac{1}{\alpha+2}\begin{pmatrix} \alpha+1 & 0 \\ 0 & 1 \end{pmatrix};$$
$$\mu_{4,0} = \frac{(\alpha+1)(\alpha+3)}{(\alpha+2)(\alpha+4)};\ \mu_{0,4} = \frac{3}{(\alpha+2)(\alpha+4)};\ \mu_{2,2} = \frac{\alpha+1}{(\alpha+2)(\alpha+4)}.$$

Proof. It is clear that $\mu_{0,0} = 1$, $\mu_{j,k} = 0$ whenever j or k is odd and that $\mu_{j,0} = C_\alpha / C_{\alpha+j}$ for any even integer j. Using the explicit value of C_α yields the value of $\mu_{j,0}$. Finally, for any even integers j and k, writing $\sin^2\theta = 1 - \cos^2\theta$ yields the formula for $\mu_{j,k}$. ■

Variations of map $\mathscr{F}$

Lemma IV.15 *The map* $\mathscr{F} : c \mapsto (1+c)^{1/2}\,\mathscr{E}\left(\left(\frac{2c}{1+c}\right)^{1/2}\right)$ *is strictly decreasing on* $[0,1]$.

Proof. Recall that $\mathscr{E}(k) = \displaystyle\int_0^{\pi/2} (1 - k^2\sin^2\theta)^{1/2}\,d\theta$ for $k \in [0,1]$. Then, for any $k \in [0,1)$, $\mathscr{E}'(k) = -k\displaystyle\int_0^{\pi/2} \frac{\sin^2\theta}{(1-k^2\sin^2\theta)^{1/2}}\,d\theta$. Therefore, for any $c \in [0,1)$,

$$\begin{aligned}
\mathscr{F}'(c) &= \frac{1}{2}(1+c)^{-1/2}\mathscr{E}\left(\left(\frac{2c}{1+c}\right)^{1/2}\right) + (1+c)^{1/2}\frac{(2c)^{-1/2}}{(1+c)^{3/2}}\,\mathscr{E}'\left(\left(\frac{2c}{1+c}\right)^{1/2}\right) \\
&= \frac{1}{2}(1+c)^{-1/2}\int_0^{\pi/2}\left[(1-\frac{2c}{1+c}\sin^2\theta)^{1/2} - \frac{\frac{2}{1+c}\sin^2\theta}{(1-\frac{2c}{1+c}\sin^2\theta)^{1/2}}\right]d\theta \\
&= \frac{1}{2}(1+c)^{-1/2}\int_0^{\pi/2}\frac{\cos(2\theta)}{(1-\frac{2c}{1+c}\sin^2\theta)^{1/2}}\,d\theta.
\end{aligned}$$

It remains to show that the above integral, which we call $J(k)$ with $k = \left(\frac{2c}{1+c}\right)^{1/2}$, is negative. Splitting the integral $J(k) := \displaystyle\int_0^{\pi/2}\frac{\cos(2\theta)}{(1-k^2\sin^2\theta)^{1/2}}\,d\theta$ into two parts, on $[0,\pi/4]$ and on $[\pi/4,\pi/2]$, and performing the change of variables $\theta' = \pi/2 - \theta$ within the second part, we get

$$J(k) = \int_0^{\pi/4}\cos(2\theta)\left[\frac{1}{(1-k^2\sin^2\theta)^{1/2}} - \frac{1}{(1-k^2\cos^2\theta)^{1/2}}\right]d\theta, \tag{15}$$

which is negative since $\cos\theta > \sin\theta$ for $\theta \in (0,\pi/4)$. ■

Second moments of the directional derivatives

Let Z be a two-dimensional stationary Gaussian field that is centred and that admits a spectral density f on $\mathbb{R}^2$. We assume that Z admits spectral moments of all orders

and we denote them by $(m_{j,k})_{(j,k)\in\mathbb{N}_0^2}$, *i.e.*

$$m_{j,k} = \int_{\mathbb{R}^2} (\lambda_1)^j \, (\lambda_2)^k \, f(\lambda) \, d\lambda.$$

The following lemma gives the second moment of the derivative of Z in direction u_φ that we write $Z' \cdot u_\varphi$, as well as the covariance matrix of the gradient of this derivative. Actually, we consider two gradient vectors. One is obtained by derivating $Z' \cdot u_\varphi$ in the directions of the vectors of the canonical basis of $\mathbb{R}^2$ and it can be written $Z''(t)u_\varphi$. Another is obtained by derivating $Z' \cdot u_\varphi$ in the directions of the vector of the orthonormal basis $(u_\varphi, u_{\varphi+\pi/2})$. For any $t \in \mathbb{R}^2$, we write it $\nabla_\varphi(Z'(t) \cdot u_\varphi)$ and it satisfies $\nabla_\varphi(Z'(t) \cdot u_\varphi) = R_{-\varphi} Z''(t)u_\varphi$, where $R_{-\varphi}$ is the matrix of the rotation of angle $-\varphi$. The covariance matrix of $\nabla_\varphi(Z' \cdot u_\varphi)$ is borrowed from [ALO05] page 412. Note that it simply satisfies $\mathbb{V}[\nabla_\varphi(Z'(0) \cdot u_\varphi)] = R_{-\varphi}\mathbb{V}[Z''(0)u_\varphi]R_\varphi$.

Lemma IV.16 *For any $\varphi \in [0, 2\pi]$,*

- $v(\varphi) = \mathrm{Var}(Z'(0) \cdot u_\varphi) = m_{2,0}\cos^2\varphi + 2m_{1,1}\cos\varphi\sin\varphi + m_{0,2}\sin^2\varphi.$
- $\mathbb{V}[Z''(0)u_\varphi] = \begin{pmatrix} \alpha_{22}(\varphi) & \alpha_{23}(\varphi) \\ \alpha_{23}(\varphi) & \alpha_{33}(\varphi) \end{pmatrix}$, *where*

$$\begin{aligned} \alpha_{22}(\varphi) &= m_{4,0}\cos^2\varphi + 2m_{3,1}\cos\varphi\sin\varphi + m_{2,2}\sin^2\varphi, \\ \alpha_{33}(\varphi) &= m_{2,2}\cos^2\varphi + 2m_{1,3}\cos\varphi\sin\varphi + m_{0,4}\sin^2\varphi, \\ \alpha_{23}(\varphi) &= m_{3,1}\cos^2\varphi + 2m_{2,2}\cos\varphi\sin\varphi + m_{1,3}\sin^2\varphi. \end{aligned}$$

- $\mathbb{V}[\nabla_\varphi(Z'(0) \cdot u_\varphi)] = \begin{pmatrix} a_{22}(\varphi) & a_{23}(\varphi) \\ a_{23}(\varphi) & a_{33}(\varphi) \end{pmatrix}$, *where*

$$\begin{aligned} a_{22}(\varphi) &= m_{4,0}\cos^4\varphi + m_{0,4}\sin^4\varphi + 6m_{2,2}\cos^2\varphi\sin^2\varphi \\ &\quad + 4m_{3,1}\cos^3\varphi\sin\varphi + 4m_{1,3}\cos\varphi\sin^3\varphi, \\ a_{33}(\varphi) &= (m_{4,0} + m_{0,4})\cos^2\varphi\sin^2\varphi + m_{2,2}\left((\cos^2\varphi - \sin^2\varphi)^2 - 2\cos^2\varphi\sin^2\varphi\right) \\ &\quad + 2(m_{1,3} - m_{3,1})\cos\varphi\sin\varphi(\cos^2\varphi - \sin^2\varphi), \\ a_{23}(\varphi) &= -m_{4,0}\cos^3\varphi\sin\varphi + m_{3,1}\cos^2\varphi(\cos^2\varphi - 3\sin^2\varphi) \\ &\quad + 3m_{2,2}\cos\varphi\sin\varphi(\cos^2\varphi - \sin^2\varphi) \\ &\quad + m_{1,3}\sin^2\varphi(3\cos^2\varphi - \sin^2\varphi) + m_{0,4}\cos\varphi\sin^3\varphi. \end{aligned}$$

- *Moreover the eigenvalues $\gamma_+(\varphi)$ and $\gamma_-(\varphi)$ of matrices $\mathbb{V}[Z''(0)u_\varphi]$ and $\mathbb{V}[\nabla_\varphi(Z'(0) \cdot u_\varphi)]$ are equal to*

$$\gamma_\pm(\varphi) = \frac{1}{2}(T(\varphi) \pm \sqrt{\Delta(\varphi)}), \tag{16}$$

where $T(\varphi)$ is the trace of either of both matrices and $\Delta(\varphi) = (\alpha_{22}(\varphi) - \alpha_{33}(\varphi))^2 + 4\alpha_{23}(\varphi)^2 = (a_{22}(\varphi) - a_{33}(\varphi))^2 + 4a_{23}(\varphi)^2$.

Length of crests with the toy model

Considering Formula (12) prescribing the expected length of crests in a given direction and a given domain, we focus on the case where the direction of $\mathbf{k}$ is given by the toy model (see Example 1 in Section IV.1). As φ tends to 0, we write $g(\varphi) = \mathscr{O}(h(\varphi))$ if there exists $\varphi_0 \in (0, 2\pi)$ and $c > 0$ such that

$$\forall \varphi \in [0, 2\pi],\ |\varphi| < |\varphi_0| \ \Rightarrow\ |g(\varphi)| \leq c|h(\varphi)|.$$

Lemma IV.17 *Let Q be a compact set in $\mathbb{R}^2$ and let $\mathbf{k}$ be a random wavevector in $\mathbb{R}^2$ prescribed by its directional spectral density $\tilde{f}(\theta, z)$ given by (13) for a fixed $\alpha > 0$. Let $f(\varphi) = \mathbb{E}[\ell(\mathbf{k}, Q, \pi/2 + \varphi)]$ where $\mathbb{E}[\ell(\mathbf{k}, Q, \varphi)]$ is given by (12). Then, as φ tends to 0,*

$$f(\varphi) = f(0) - K\varphi^2 + \mathscr{O}(\varphi^4), \qquad \text{with}\ \ K > 0.$$

Proof. From (12), we get

$$f(\varphi) = \mathscr{H}_2(Q)\,\frac{1}{\pi}\left(\frac{\gamma_+(\pi/2+\varphi)}{v(\pi/2+\varphi)}\right)^{1/2} \mathscr{E}\left(\left(1 - \frac{\gamma_-(\pi/2+\varphi)}{\gamma_+(\pi/2+\varphi)}\right)^{1/2}\right),$$

where $\gamma_-(\varphi)$, $\gamma_+(\varphi)$ and $v(\varphi)$ are given in Lemma IV.16. Moreover, the spectral moments $m_{j,k}$ are prescribed by (14) with $\mu_{j,k}$ given by Lemma IV.14.

Since $\cos(\pi/2 + \varphi) = -\varphi + \dfrac{\varphi^3}{6} + \mathscr{O}(\varphi^4)$ and $\sin(\pi/2 + \varphi) = 1 - \dfrac{\varphi^2}{2} + \mathscr{O}(\varphi^4)$, we use the first and the third points of Lemma IV.16 to get

$$\begin{aligned}
v(\pi/2+\varphi) &= M_2(\mu_{20}\sin^2(\varphi) + \mu_{02}\cos^2(\varphi)) = M_2(\mu_{02} + (\mu_{20} - \mu_{02})\varphi^2) + \mathscr{O}(\varphi^4)\\
&= \frac{M_2}{\alpha+2}(1 + \alpha\varphi^2) + \mathscr{O}(\varphi^4)\\
a_{22}(\pi/2+\varphi) &= M_4\left(\mu_{40}\sin^4(\varphi) + \mu_{04}\cos^4(\varphi) + 6\mu_{22}\sin^2(\varphi)\cos^2(\varphi)\right)\\
&= M_4\left(\mu_{04} + 2(3\mu_{22} - \mu_{04})\varphi^2\right) + \mathscr{O}(\varphi^4)\\
&= \frac{3M_4}{(\alpha+2)(\alpha+4)}(1 + 2\alpha\varphi^2) + \mathscr{O}(\varphi^4)\\
a_{33}(\pi/2+\varphi) &= M_4\left((\mu_{40}+\mu_{04})\sin^2(\varphi)\cos^2(\varphi) + \mu_{22}((\sin^2(\varphi) - \cos^2(\varphi))^2 - 2\sin^2(\varphi)\cos^2(\varphi))\right)\\
&= M_4\left(\mu_{22} + (\mu_{40} + \mu_{04} - 6\mu_{22})\varphi^2\right) + \mathscr{O}(\varphi^4)\\
&= \frac{M_4}{(\alpha+2)(\alpha+4)}(\alpha + 1 + \alpha(\alpha-2)\varphi^2) + \mathscr{O}(\varphi^4)\\
a_{23}(\pi/2+\varphi) &= M_4\left(\mu_{40}\sin^3(\varphi)\cos(\varphi) - 3\mu_{22}\sin(\varphi)\cos(\varphi)(\sin^2(\varphi) - \cos^2(\varphi)) - \mu_{04}\sin(\varphi)\cos^3(\varphi)\right)\\
&= M_4\left((3\mu_{22} - \mu_{04})\varphi + (\mu_{40} + \frac{5}{3}\mu_{04} - 8\mu_{22})\varphi^3\right) + \mathscr{O}(\varphi^4)\\
&= \frac{M_4}{(\alpha+2)(\alpha+4)}[3\alpha\varphi + \alpha(\alpha-4)\varphi^3] + \mathscr{O}(\varphi^4).
\end{aligned}$$

The eigenvalues $\gamma_\pm(\pi/2+\varphi)$ are given by (16):

$$\gamma_\pm(\pi/2+\varphi) = \frac{1}{2}\left(a_{22}(\pi/2+\varphi) + a_{33}(\pi/2+\varphi) \pm \sqrt{\Delta(\pi/2+\varphi)}\right),$$

with discriminant

$$\begin{aligned}\Delta(\pi/2+\varphi) &= (a_{22}(\pi/2+\varphi) - a_{33}(\pi/2+\varphi))^2 + 4\,a_{23}(\pi/2+\varphi)^2 \\ &= \frac{M_4^2}{(\alpha+2)^2(\alpha+4)^2}[(2-\alpha)^2 + 2\alpha(\alpha+4)^2\varphi^2] + \mathscr{O}(\varphi^4).\end{aligned}$$

For $\alpha \neq 2$, it yields the following expansions

$$\begin{aligned}\sqrt{\Delta(\pi/2+\varphi)} &= M_4\frac{|2-\alpha|}{(\alpha+2)(\alpha+4)}\left(1 + \frac{\alpha(\alpha+4)^2}{(2-\alpha)^2}\varphi^2\right) + \mathscr{O}(\varphi^4), \\ \gamma_-(\pi/2+\varphi) &= M_4\frac{\min(3,\alpha+1)}{(\alpha+2)(\alpha+4)}\left(1 - \frac{\alpha(\alpha+4)}{|\alpha-2|}\varphi^2\right) + \mathscr{O}(\varphi^4), \\ \gamma_+(\pi/2+\varphi) &= M_4\frac{\max(3,\alpha+1)}{(\alpha+2)(\alpha+4)}\left(1 + \frac{\alpha(\alpha+4)}{|\alpha-2|}\varphi^2\right) + \mathscr{O}(\varphi^4).\end{aligned}$$

The quantity $k(\pi/2+\varphi) := \left(1 - \dfrac{\gamma_-(\pi/2+\varphi)}{\gamma_+(\pi/2+\varphi)}\right)^{1/2} \in [0,1]$ admits the following expansion

$$k(\pi/2+\varphi) = k_0\left[1 + \frac{m\alpha(\alpha+4)}{(\alpha-2)^2}\varphi^2\right] + \mathscr{O}(\varphi^4),$$

where $m = \min(3,\alpha+1)$, $M = \max(3,\alpha+1)$ and $k_0 = \left(1 - \dfrac{m}{M}\right)^{1/2} = \left(\dfrac{|\alpha-2|}{M}\right)^{1/2}$.

We expand the map $\mathscr{E} : k \mapsto \displaystyle\int_0^{\pi/2}(1-k^2\sin^2\theta)^{1/2}\,d\theta$.

$$\mathscr{E}(k(\pi/2+\varphi)) = \mathscr{E}(k_0) + \frac{m\alpha(\alpha+4)}{(\alpha-2)^2}k_0\,\mathscr{E}'(k_0)\,\varphi^2 + \mathscr{O}(\varphi^4),$$

where $\mathscr{E}'(k) = -k\displaystyle\int_0^{\pi/2}\frac{\sin^2\theta}{(1-k^2\sin^2\theta)^{1/2}}\,d\theta$ and $\mathscr{E}'(0) = 0$.

It remains to expand $\left(\dfrac{\gamma_+(\pi/2+\varphi)}{v(\pi/2+\varphi)}\right)^{1/2}$:

$$\left(\frac{\gamma_+(\pi/2+\varphi)}{v(\pi/2+\varphi)}\right)^{1/2} = \left(\frac{M_4}{M_2}\frac{M}{\alpha+4}\right)^{1/2}\left[1 + \frac{m\alpha}{|\alpha-2|}\varphi^2\right] + \mathscr{O}(\varphi^4).$$

Finally, as claimed, we get the asymptotic expansion of function f as φ tends to 0:

$$f(\varphi) = f(0) - K\varphi^2 + \mathscr{O}(\varphi^4),$$

with $f(0) = \mathscr{H}_2(Q)\,\dfrac{1}{\pi}\left(\dfrac{M_4}{M_2}\dfrac{M}{\alpha+4}\right)^{1/2}\mathscr{E}((1-m/M)^{1/2})$ and

$$K = -\,\mathscr{H}_2(Q)\,\frac{1}{\pi}\left(\frac{M_4}{M_2}\frac{M}{\alpha+4}\right)^{1/2}\frac{m\alpha}{|\alpha-2|}\,J(k_0),$$

where $J(k)$ is introduced within the proof of Lemma IV.15. As $J(k)$ is proved to be negative for any k, see (15), we obtain that $K > 0$ and the lemma is established in the case $\alpha \neq 2$.

For $\alpha = 2$, we get $\sqrt{\Delta(\pi/2+\varphi)} = \dfrac{M_4}{2}\,|\varphi|(1+\mathscr{O}(\varphi^2))$, $\gamma_\pm(\pi/2+\varphi) = \dfrac{M_4}{8}\,(1 \pm 2|\varphi| + 2\varphi^2) + \mathscr{O}(\varphi^3)$ and hence $k(\pi/2+\varphi) = 2\sqrt{|\varphi|}(1 - |\varphi| + \mathscr{O}(\varphi^2))$. Then, performing a Taylor expansion at order 4 of function $\mathscr{E}$ and using $\mathscr{E}''(0) = -\dfrac{\pi}{4}$, $\mathscr{E}^{(3)}(0) = 0$, $\mathscr{E}^{(4)}(0) = -\dfrac{9\pi}{16}$, we obtain

$$f(\varphi) = f(0)\left(1 - \frac{1}{4}\varphi^2\right) + \mathscr{O}(\varphi^3),$$

with $f(0) = \dfrac{\mathscr{H}_2(Q)}{2\sqrt{2}}\left(\dfrac{M_4}{M_2}\right)^{1/2}$. Lemma IV.17 is proved. ■

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Sujet : Caractéristiques géométriques de champs aléatoires anisotropes réguliers

Résumé : Cette thèse a pour sujet l'étude géométrique de champs aléatoires anisotropes réguliers, définis sur l'espace euclidien, en partie dans le cadre gaussien. Nous nous intéressons à trois caractéristiques géométriques: le nombre de points critiques, la mesure des ensembles de niveaux et la caractéristique d'Euler des ensembles d'excursion. Des formules de Rice permettent d'exprimer leur espérance ou leur variance. Nous proposons d'abord une condition suffisante sous laquelle le nombre de points critiques d'un champ stationnaire gaussien est de variance finie. Cette condition s'avère être une généralisation de la condition de Geman, connue en dimension un, au cadre multidimensionnel et anisotrope. Nous étudions ensuite deux modèles de champs aléatoires anisotropes. Pour les champs déformés, en dimension deux, l'anisotropie est liée à une déformation de l'espace des paramètres par une bijection du plan déterministe. Nous exhibons les déformations qui préservent l'isotropie, et nous prouvons que les champs déformés correspondants sont caractérisés par une propriété d'invariance de la caractéristique d'Euler moyenne de leurs ensembles d'excursion. Cette même caractéristique permet d'identifier les déformations en jeu, lorsque celles-ci sont inconnues. L'anisotropie des ondes aléatoires, quant à elles, est liée au domaine spectral. Notre modèle d'onde aléatoire anisotrope permet de généraliser plusieurs modèles existant, tels que les ondes planaires de Berry et un modèle spatio-temporel pour l'étude des vagues. On met en évidence la dépendance entre des caractéristiques géométriques d'une onde (en particulier l'espérance de la mesure de ses ensembles de niveau) et la loi de son vecteur d'onde (notamment ses moments et ses statistiques directionnelles). La longueur moyenne des lignes nodales du modèle planaire anisotrope de Berry s'avère décroître à mesure que l'anisotropie du vecteur d'onde augmente.

Mots clés : champ aléatoire ; champ gaussien ; anisotropie ; ensemble d'excursion ; caractéristique d'Euler ; points stationnaires ; ensemble de niveau ; théorie des franchissements ; onde aléatoire ; statistiques nodales.

Subject : Geometric characteristics of regular anisotropic random fields

Abstract: This thesis deals with anisotropic regular random fields, defined on the Euclidian space and studied from a geometric perspective. Some of our framework is Gaussian. We focus on three geometric characteristics: the number of critical points, the level sets measure and the Euler characteristic of excursion sets. Our main tools are Rice formulas for the expectation and the variance. We first address the question of the finiteness of the variance of the number of critical points of a stationary and Gaussian random field. The so-called Geman condition, which is known as a sufficient condition in dimension one, is extended to higher dimensions and to an anisotropic setting. Then two different anisotropic models are studied. On the one hand, the anisotropy of the deformed random field model (studied in dimension two) is due to a deterministic deformation of the parameter space. We give an explicit characterization of the deformations that preserve the isotropy of deformed random field. The cases of isotropy are proved to match a certain invariance property of the expected Euler characteristic of some excursion sets. This geometric characteristic also allows to identify the deformation of the model, when the latter is unknown. On the other hand, the anisotropy of the random wave model stems from the spectral domain. Our anisotropic random wave model allows to generalize existing models, for instance Berry's planar waves and a spatiotemporal sea wave model. Our purpose is to link geometric characteristics of a random wave, such as the expected measure of its level sets, with the distribution of its random wavevector, in particular its moments of finite order and its directional statistics. Considering Berry's anisotropic planar waves, we prove that the expected length of its nodal lines is a decreasing function of the anisotropy of the random wavevector.

Keywords : random field; Gaussian field; anisotropy; excursion set, Euler characteristic; stationary points; level set; crossing theory; random wave; nodal statistics.