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THÈSE DE DOCTORAT

de

L'UNIVERSITÉ PARIS-SACLAY

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Construction de dynamiques à fortes interactions d'EDP
non linéaires dispersives

Thèse présentée et soutenue à Palaiseau, le 26 Juin 2019

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Chapitre 0

Introduction (version française)

0.1 Contexte général

Le but de cette thèse est d'étudier la dynamique asymptotique des ondes multi-solitaires (aussi appelées solitons) d'équations non-linéaires dispersives.

On parlera d'équations dispersives pour désigner des équations qui combinent une équation linéaire avec un comportement dispersif (toute solution se désintègre uniformément en temps) et une non-linéarité appropriée, typiquement, de la forme suivante :

$$u_t = Lu + F(u, Du, \dots)$$

où $u(t, x)$ est une fonction à valeurs réelles ou complexes et $t \in \mathbb{R}, x \in \mathbb{R}^d$. L'opérateur linéaire L est supposé anti-adjoint, c'est-à-dire

$$\mathcal{F}(Lu)(\xi) = ip(\xi)\mathcal{F}u(\xi), \quad p(\xi) \in \mathbb{R},$$

ici \mathcal{F} est la transformée de Fourier et $D_\xi^2 p(\xi) \neq 0$ pour tout $\xi \neq 0$.

Des exemples d'équations non-linéaires dispersives sont l'équation de Schrödinger non-linéaire, l'équation de Korteweg-de Vries, l'équation de Benjamin-Ono, l'équation de BBM, les équations de KP-I, KP-II.

Avant de commenter les principaux résultats de la thèse, présentons brièvement quelques aspects de la théorie des ondes solitaires de type solitons pour les équations non-linéaires dispersives.

0.1.1 Préliminaires

Considérons l'une des plus typiques équations non-linéaires dispersives, l'équation de Schrödinger non-linéaire dispersive (NLS) dans \mathbb{R}^d :

$$i\partial_t u + \Delta u + |u|^{p-1}u = 0 \tag{NLS}$$

où Δ est le laplacien en d variables spatiales, la non-linéarité $p > 1$ et avec la donnée initiale

$$u(0, x) = u_0, \quad u_0 \in H^1 : \mathbb{R}^d \rightarrow \mathbb{C}.$$

L'existence locale en temps de la solution de (NLS) est bien étudiée. Dans cette thèse, nous utilisons principalement le résultat suivant d'existence locale et d'unicité :

Proposition 0.1.1 (Localement bien posé dans H^1 , Ginibre et Velo [16], voir aussi [3]). Soit

$$2^* = \begin{cases} +\infty & \text{si } d = 1, 2 \\ \frac{2d}{d-2} & \text{si } d \geq 3. \end{cases}$$

Pour $1 < p < 2^* - 1$ et $u_0 \in H^1(\mathbb{R}^d)$, il existe un temps maximal d'existence $T = T(\|u_0\|_{H^1}) > 0$ et une solution $u(t, x) \in \mathcal{C}([0, T]; H^1)$ of (NLS).

De plus, nous avons le critère d'explosion suivant : soit $T = +\infty$, la solution $u(t)$ est dite **globale en temps** ; soit $T < +\infty$ et alors la solution **explose en temps fini**

$$\lim_{t \rightarrow T} \|u(t)\|_{H^1} = +\infty.$$

D'après des calculs directes, l'équation (NLS) admet les groupes de symétrie suivants :

- Phase : pour $\gamma \in \mathbb{R}$, si $u(t, x)$ est une solution de (NLS), alors $w(t, x) = u(t, x)e^{i\gamma}$ l'est aussi.
- Scaling : pour $\lambda > 0$, si $u(t, x)$ est une solution de (NLS), alors $w(t, x) = \lambda^{\frac{2}{p-1}}u(\lambda^2 t, \lambda x)$ l'est aussi.
- Translation : pour $x_0 \in \mathbb{R}^d$, $t_0 \in \mathbb{R}$, si $u(t, x)$ est une solution de (NLS), alors $w(t, x) = u(t - t_0, x - x_0)$ l'est aussi.
- Galiléen : pour $\beta \in \mathbb{R}^d$, si $u(t, x)$ est une solution de (NLS), alors $w(t, x) = u(t, x - \beta t)e^{i\frac{\beta}{2}(x - \frac{\beta}{2}t)}$ l'est aussi.

Une autre propriété de (NLS) est que les symétries induisent des structures hamiltoniennes naturelles : on a **les lois de conservation** suivantes

- Masse :

$$M(u(t)) = \int_{\mathbb{R}^d} |u(t, x)|^2 dx = M(u_0) \quad (0.1.2)$$

- Énergie :

$$E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(t, x)|^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^d} |u(t, x)|^{p+1} dx = E(u_0) \quad (0.1.3)$$

- Moment :

$$J(u(t)) = \text{Im} \int_{\mathbb{R}^d} \nabla u(t, x) \bar{u}(t, x) dx = J(u_0) \quad (0.1.4)$$

On note $u_\lambda(t, x) = \lambda^{\frac{2}{p-1}}u(\lambda^2 t, \lambda x)$ la symétrie par scaling du flot. *L'espace critique* est un concept fondamental dans l'analyse et défini par le nombre de dérivées qui est invariant par la symétrie de scaling :

$$\|u_\lambda(t, x)\|_{\dot{H}^{s_c}} = \|u(t, x)\|_{\dot{H}^{s_c}} \text{ pour } s_c = \frac{d}{2} - \frac{2}{p-1}.$$

Observons que $s_c < 1$ dans Proposition 0.1.1.

- Si $s_c < 0$ (ou $1 < p < 1 + \frac{4}{d}$), l'équation (NLS) est dite L^2 sous-critique.
- Si $s_c = 0$ (ou $p = 1 + \frac{4}{d}$), l'équation (NLS) est dite L^2 critique.
- Si $s_c > 0$ (ou $p > 1 + \frac{4}{d}$), l'équation (NLS) est dite L^2 sur-critique.

Une conséquence directe de la théorie de Cauchy, les lois de conservations et l'inégalité de Gagliardo-Nirenberg suivante

$$\|f\|_{L^{p+1}} \leq C_p \|f\|_{L^2}^{1-\sigma} \|\nabla f\|_{L^2}^\sigma \quad \text{avec} \quad \sigma = \frac{d(p-1)}{p+1} \quad (\text{Gagliardo-Nirenberg})$$

est le résultat d'existence globale suivant, voir [40] :

Proposition 0.1.5 (Existence globale dans le cas sous-critique). Soit $d \geq 1$ et $1 < p < 1 + \frac{4}{d}$, alors toute solution de (NLS) est globale et bornée dans H^1 .

Par une autre quantité appelée le viriel $\int_{\mathbb{R}^d} |x|^2 |u(t, x)|^2 dx$ et l'identité suivante concernant sa dérivée seconde

$$\frac{d^2}{dt^2} \int_{\mathbb{R}^d} |x|^2 |u(t, x)|^2 dx = 16E(u_0) - \frac{4[d(p-1) - 4]}{p+1} |u(t, x)|^{p+1} dx,$$

on peut prouver l'existence de solutions qui explosent en temps fini dans le cas critique et sur-critique, voir [3, 40].

Proposition 0.1.6 (Explosion en temps fini pour $E(u_0) < 0$). Soit $1 + \frac{4}{d} \leq p < 2^* - 1$ et $u_0 \in \Sigma = H^1 \cap L^2(|x|^2 dx)$ avec $E(u_0) < 0$. Alors la solution associée de (NLS) explose en temps fini $0 < T < +\infty$.

Dans le cas $d = 1$ and $p = 3$, on obtient l'équation (NLS) cubique 1D

$$i\partial_t u + u_{xx} + |u|^2 u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}.$$

L'équation (NLS) cubique 1D est complètement intégrable, c'est-à-dire qu'il existe une paire de Lax (voir [9, 14, 23, 49]). Le cas susmentionné a été beaucoup étudié en utilisant la méthode de scattering inverse. En particulier, l'équation (NLS) cubique 1D a pour propriétés : *une infinité de lois de conservation, la collision purement élastique de solitons et la décomposition en solitons*. Les autres cas ne sont pas connus pour être intégrables.

Un autre exemple d'équation non-linéaire dispersive que nous étudions dans cette thèse est l'équation Korteweg-de Vries (gKdV) généralisée :

$$\partial_t u + \partial_x(\partial_x^2 u + u^p) = 0, \quad t, x \in \mathbb{R}. \quad (\text{gKdV})$$

où $p > 1$ est un entier. L'équation est complètement intégrable dans le cas quadratique $p = 2$ et le cas cubique $p = 3$.

Nous étudions également le système couplé d'équations de Schrödinger :

$$\begin{cases} i\partial_t u + \Delta u + (|u|^2 + \omega|v|^2)u = 0 \\ i\partial_t v + \Delta v + (|v|^2 + \omega|u|^2)v = 0 \end{cases} \quad t \in \mathbb{R}, x \in \mathbb{R} \quad (\text{coupled NLS})$$

pour un paramètre $0 < \omega < 1$. Remarque que dans le cas $\omega = 1$, le système (coupled NLS) est dit le système intégrable Manakov (voir [25, 47])

$$\begin{cases} i\partial_t u + \Delta u + (|u|^2 + |v|^2)u = 0 \\ i\partial_t v + \Delta v + (|v|^2 + |u|^2)v = 0. \end{cases}$$

0.1.2 Onde solitaire

Une caractéristique fondamentale de (NLS) est l'existence d'une *solution périodique en temps et localisée en espace*, appelée **onde solitaire** ou **soliton**. Plus précisément,

$$u(t, x) = \phi(x)e^{it}$$

est une solution dans H^1 de (NLS) si et seulement si ϕ résout l'équation non-linéaire

$$\Delta\phi - \phi + |\phi|^{p-1}\phi = 0, \quad \phi \in H^1(\mathbb{R}^d). \quad (\text{soliton eq.})$$

Il y a plusieurs façons de construire une solution de (soliton eq.), la plus simple étant de chercher une solution radiale par la méthode de tir [1]. Rappelons que la théorie des équations linéaires dispersives prédit que toute solution se disperse et désintègre uniformément en temps [43]. Par conséquent, l'existence d'une telle solution est un équilibre subtil entre *la dispersion* induit par le Laplacien et *la concentration* créée par la partie non-linéaire de l'équation (NLS). Une propriété importante de rigidité de (soliton eq.) est *l'unicité de la solution positive*.

Proposition 0.1.7 (Unicité de l'état fondamental, Kwong [22]). Toute solution de

$$\Delta\phi - \phi + |\phi|^{p-1}\phi = 0, \quad \phi \in H^1(\mathbb{R}^d), \quad \phi(x) > 0$$

est une translation d'un profil radial décroissant exponentiellement $Q(x) = q(|x|)$ qui est l'unique solution radiale positive de (soliton eq.). Q est appelé *l'état fondamental*.

L'état fondamental Q est donné par une forme explicite en dimension 1

$$Q(x) = \left(\frac{p+1}{2 \cosh^2\left(\frac{p-1}{2}x\right)} \right)$$

et décroît exponentiellement en toute dimension : il existe une constant $c_Q > 0$

$$\left| q(r) - c_Q r^{-\frac{d-1}{2}} e^{-r} \right| + \left| q'(r) + c_Q r^{-\frac{d-1}{2}} e^{-r} \right| \lesssim r^{-\frac{d-1}{2}-1} e^{-r}.$$

Selon [44], l'état fondamental Q est liée à la meilleure constante de l'inégalité de Gagliardo-Nirenberg. Plus précisément, Q est l'unique minimiseur de l'inégalité et la constante optimale est donnée par

$$C_p = J(Q) = \min_{f \in H^1, f \neq 0} J(f)$$

où pour tout $f \in H^1, f \neq 0$

$$J(f) = \frac{\|f\|_{L^2}^{1-\sigma} \|\nabla f\|_{L^2}^\sigma}{\|f\|_{L^{p+1}}} \quad \text{avec} \quad \sigma = \frac{d(p-1)}{p+1}.$$

Notons $Q_\lambda(x) = \lambda^{\frac{2}{p-1}} Q(\lambda x)$, à partir des symétries de (NLS), il y a une famille à $(2d+2)$ paramètres de solitons :

$$u(t, x) = Q_{\lambda_0}(x - \sigma_0 - \beta_0 t) e^{i\gamma_0 + i\lambda_0^2 t + i\frac{\beta_0}{2}(x - \frac{\beta_0}{2}t)} \quad (0.1.8)$$

avec le scaling $\lambda_0 \in \mathbb{R}_+$, la position $\sigma_0 \in \mathbb{R}^d$, la phase $\gamma_0 \in \mathbb{R}$ et la vitesse $\beta_0 \in \mathbb{R}^d$.

Une fois acquise l'existence d'ondes solitaires, nous voudrions étudier leur stabilité par perturbation de données initiales dans l'espace d'énergie. On observe que les ondes solitaires ont deux instabilités triviales : l'instabilité de scaling ¹ et l'instabilité galiléenne ². Une question naturelle est de s'interroger si ces instabilités triviales sont les seules ou non.

Definition 0.1.9 (La stabilité orbitale). On dit que l'onde solitaire (0.1.8) est orbitalement stable si pour tout $\epsilon > 0$, il existe $\delta > 0$ tel que si

$$\left\| u(0) - Q_{\lambda_0}(x - \sigma_0) e^{i(\gamma_0 + \frac{\beta_0}{2}x)} \right\|_{H^1} \leq \delta,$$

alors pour tout $t \in \mathbb{R}$, il existe $\sigma(t), \gamma(t)$ tels que la solution $u(t)$ de (NLS) satisfait

$$\left\| u(t, \cdot) - Q_{\lambda_0}(\cdot - \sigma(t)) e^{i(\gamma(t) + \frac{\beta_0}{2}x)} \right\|_{H^1} \leq \epsilon.$$

¹ $\forall \lambda > 0$, la solution correspondante de (NLS) avec donnée initiale $u_0(x) = \lambda^{\frac{2}{p-1}} Q(\lambda x)$ est $u(t, x) = \lambda^{\frac{2}{p-1}} Q(\lambda x) e^{i\lambda^2 t}$.

² $\forall \beta > 0$, la solution correspondante de (NLS) avec donnée initiale $u_0(x) = Q(x) e^{i\beta x}$ est $u(t, x) = Q(x - \beta t) e^{it + \frac{\beta}{2}(x - \frac{\beta}{2}t)}$.

En utilisant la méthode de concentration-compacité, Cazenave et Lions [4] a montré que les ondes solitaires sont stables pour tout $1 < p < 1 + \frac{4}{d}$, i.e. quand la non-linearité est L^2 sous-critique. Par une approche différente basée sur le développement des lois de conservation autour d'une onde solitaire, Weinstein [45] a montré la stabilité orbitale sous la condition de positivité

$$\frac{d}{d\lambda} \int Q_\lambda^2(x) dx \Big|_{\lambda=\lambda_0} > 0$$

qui est vérifiée dans le cas sous-critique. Inversement, par le travail de Grillakis, Shatah et Strauss [17, 18], si

$$\frac{d}{d\lambda} \int Q_\lambda^2(x) dx \Big|_{\lambda=\lambda_0} < 0$$

qui est vérifiée dans le cas sur-critique ($1 + \frac{4}{d} < p < 2^* - 1$), alors l'onde solitaire est instable. Notons que le phénomène d'instabilité est très différent dans le cas critique (l'instabilité linéaire et la direction instable est liée au paramètre de scaling) et dans le cas sur-critique (l'instabilité est exponentielle).

Pour le cas critique ($p = 1 + \frac{4}{d}$), la transformation pseudo-conforme

$$v(t, x) = \frac{1}{|t|^{\frac{d}{2}}} u\left(\frac{-1}{t}, \frac{x}{t}\right) e^{i\frac{|x|^2}{4t}} \quad (\text{conformal transform})$$

appliquée à l'onde solitaire $u(t, x) = Q(x)e^{it}$ engendre *une solution explosive* :

$$S(t, x) = \frac{1}{|t|^{\frac{d}{2}}} Q\left(\frac{x}{|t|}\right) e^{-i\frac{|x|^2}{4t} + \frac{i}{t}}, \quad \|S(t)\|_{L^2} = \|Q\|_{L^2} \quad (0.1.10)$$

avec le taux d'explosion (appelé le taux d'explosion conforme)

$$\|\nabla S(t)\|_{L^2} \sim \frac{1}{|t|} \quad \text{quand } t \rightarrow 0^-.$$

$S(t, x)$ est aussi l'unique solution explosive (aux invariances près) avec la masse $\|u_0\|_{L^2} = \|Q\|_{L^2}$, voir [33]. La solution est appelée *l'explosion à masse minimale* car en-dessous de cette masse, $\|u_0\|_{L^2} < \|Q\|_{L^2}$, toute solution est globale et même disperse, i.e. il existe $v^\pm(t, x)$ solutions de l'équation linéaire telles que

$$\|u(t) - v^\pm(t)\|_{H^1} \rightarrow 0$$

quand $t \rightarrow \pm\infty$, voir [10, 21].

En plus de l'explosion avec la masse minimale, il existe également d'autres solutions explosives avec le taux d'explosion conforme (voir [2]). Pour le cas d'une masse légèrement au-dessus du seuil (et avec une énergie négative), i.e.

$$\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha_0, \quad 0 < \alpha_0 \ll 1, \quad (0.1.11)$$

Merle et Raphaël [34, 35, 41] ont prouvé l'existence et la stabilité d'une dynamique explosive avec le taux d'explosion log-log :

$$\|\nabla u(t)\|_{L^2} \sim c \sqrt{\frac{\log|\log(T-t)|}{T-t}} \quad \text{quand } t \rightarrow T. \quad (0.1.12)$$

Ensuite, [41] a également montré que sous la condition (0.1.11), il y a une dichotomie sur le taux d'explosion : soit la solution explose avec le régime log-log (0.1.12), soit elle explosive avec un taux d'explosion plus rapide que le conforme

$$\|\nabla u(t)\|_{L^2} \gtrsim \frac{1}{T-t}. \quad (0.1.13)$$

De même que (NLS), (gKdV) admet une famille d'ondes solitaires à deux paramètres, de la forme

$$R_{v_0, x_0} = Q_{v_0}(x - x_0 - v_0 t)$$

où $Q_v = v^{\frac{1}{p-1}} Q(\sqrt{v}x)$ et Q est l'état fondamental de (soliton eq.) en dimension 1.

Nous remarquons aussi que quand $v = 0$ (ou $u = 0$), l'équation (coupled NLS) se simplifie en (NLS) cubique 1D. Nous en déduisons des solutions spéciales de (coupled NLS) définies par :

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} e^{i\gamma_u + i\lambda_u^2 t + i\frac{\beta_u}{2}(x - \frac{\beta_u}{2}t)} Q_{\lambda_u}(x - \sigma_u - \beta_u t) \\ 0 \end{pmatrix}$$

et

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ e^{i\gamma_v + i\lambda_v^2 t + i\frac{\beta_v}{2}(x - \frac{\beta_v}{2}t)} Q_{\lambda_v}(x - \sigma_v - \beta_v t) \end{pmatrix}$$

pour tout $\lambda_j > 0, \sigma_j, \beta_j, \gamma_j \in \mathbb{R}$ ($j = u, v$).

0.1.3 Onde multi-solitaire

Une **onde multi-solitaire**, aussi appelée **multi-soliton**, est une solution qui se comporte en temps long comme plusieurs solitons. Par exemple, considérons $K \geq 2$ et pour $k \in \{1, \dots, K\}$, soient $\lambda_k > 0, \beta_k \in \mathbb{R}^d, \sigma_k \in \mathbb{R}^d$ et $\gamma_k \in \mathbb{R}$. Définissons le k -ième soliton

$$R_k(t, x) = Q_{\lambda_k}(x - \sigma_k(t)) e^{i\Gamma_k(t, x)}$$

avec

$$\sigma_k(t) = \sigma_k + \beta_k t, \quad \Gamma_k(t, x) = \frac{1}{2}(\beta_k \cdot x) - \frac{1}{4}|\beta_k|^2 t + \lambda_k t + \gamma_k.$$

Remarquons qu'en raison de la non-linéarité, la somme

$$R(t) = \sum_{k=1}^K R_k(t)$$

n'est pas une solution de (NLS). Néanmoins, on peut espérer l'existence d'une solution de (NLS) qui se comporte asymptotiquement comme

$$\|u(t) - R(t)\|_{H^1} \rightarrow 0 \quad \text{quand} \quad t \rightarrow +\infty.$$

Des multi-solitons ont été construites pour la première fois pour le cas intégrable (NLS) cubique 1D ($d = 1$ et $p = 3$), [49] : dans ce cas, par la méthode de *scattering inverse*, il y a une formule de multi-solitons et l'on observe directement que ces solutions se découpent en une somme de solitons à la fois quand $t \rightarrow +\infty$ et quand $t \rightarrow -\infty$ (la collision est élastique). Récemment, la stabilité asymptotique d'un tel objet pour le cas intégrable a été montré dans un espace à poids [8]. D'ailleurs, on peut observer qu'en plus de **l'interaction faible** (c'est-à-dire que les trajectoires des ondes solitaires ne sont pas affectées asymptotiquement par l'interaction de solitons), existe aussi le cas de l'interaction forte où la distance entre solitons est de l'ordre de $\log t$ ou la distance est finie avec un comportement périodique (la forme de $\sigma_k(t)$ est affectée par l'interaction entre solitons).

Le multi-soliton est fondamental en vue de la conjecture de **résolution en solitons**. Cette conjecture (*assez vaguement définie*)³ prévoit qu'**une solution générique devrait se décomposer en temps long comme une somme d'objets rigides non-linéaires (typiquement solitons) et un terme purement dispersif**. Par la méthode de scattering inverse,

³La conjecture est aussi soutenue par un nombre important de travaux numériques [9, 15, 24, 48].

la décomposition en solitons a été obtenue dans le cas intégrable ((NLS) cubique 1D, (KdV) quadratique et cubique). Dans le cas non-intégrable, récemment, dans une suite d'articles remarquables [11, 12, 13] concernant l'équation des ondes $\dot{H}^1 \times L^2$ critique, et en particulier le cas 3D avec données radiales, la conjecture a été résolue. Néanmoins, la conjecture reste encore largement inaccessible en général.

Retournons à notre équation (NLS), pour la configuration non-intégrable, nous rappelons le résultat suivant sur l'existence de multi-solitons avec la condition découplage (and restreint à une direction de temps).

Theorem 0.1.14 (Existence d'onde multi-solitaire pour (NLS), [7, 27, 33]). Soit $1 < p < \frac{d+2}{d-2}$. Considérons $K \geq 1$, pour $k \in \{1, \dots, K\}$, soient $\lambda_k > 0, \beta_k \in \mathbb{R}^d, \sigma_k \in \mathbb{R}^d$ et $\gamma_k \in \mathbb{R}$. Supposons que les vitesses soient découplées, pour $k \neq k', \beta_k \neq \beta_{k'}$. Notons

$$R_k(t, x) = Q_{\lambda_k}(x - \beta_k t - \sigma_k) e^{i\Gamma_k(t, x)} \quad \text{et} \quad \Gamma_k(t, x) = \frac{1}{2}(\beta_k \cdot x) - \frac{1}{4}|\beta_k|^2 t + \lambda_k t + \gamma_k.$$

Alors, il existe $T_0 > 0, \kappa > 0$ et une H^1 solution $u(t)$ de (NLS) telle que pour tout $t \geq T_0$,

$$\left\| u(t, x) - \sum_{k=1}^K R_k(t, x) \right\|_{H^1} \lesssim e^{-\kappa|t|}.$$

En raison de la condition de découplage, les solitons ont des vitesses deux-à-deux différentes et l'interaction entre eux décroît exponentiellement en temps. Dans le cas critique $p = 1 + \frac{4}{d}$, pour la première fois où, tel multi-soliton a été construit par Merle [33]. En plus, par la transformation pseudo-conforme (conformal transform), le résultat a aussi construit des solutions explosives en K points distincts. Le cas sous-critique $1 < p < 1 + \frac{4}{d}$ a été traité par Martel et Merle [27], suivant le résultat de Martel [26] sur (gKdV). Le cas sur-critique a été fait par Côte, Martel et Merle [7], puis Combet [6] a construit une famille à K -paramètres de K -solitons dans ce cas.

Notons que tous les théorèmes ci-dessus dans le cas non-intégrable ne décrivent pas sur le comportement de la solution quand $t \rightarrow -\infty$. Au contraire du cas intégrable, pour le cas non-intégrable, les multi-solitons pour $t \rightarrow +\infty$ pourraient avoir des comportements différents pour $t \rightarrow -\infty$ (qui ne sont plus les multi-solitons purs) car l'interaction entre ondes solitaires devrait être, en général, inélastique (voir [28, 39]). La stabilité des multi-solitons a été étudiée dans le cas sous-critique dans [30]. Les résultats similaires pour d'autres modèles non-linéaires avec onde solitaire : (gKdV) [5, 7, 29], (coupled NLS) [19], par exemple.

Concernant l'interaction forte (c'est-à-dire, des cas où l'interaction change effectivement la trajectoire du soliton), l'un des tout premiers travaux dans l'étude de multi-solitons à fortes interactions pour (NLS) est l'article de Martel et Raphaël [32] où un premier exemple de solutions globales qui explosent en temps infini est construit :

Theorem 0.1.15 (Explosion à fortes interactions en temps infini [32]). Pour l'équation (NLS) cubique 2D, il existe une solution globale $u(t)$ sur $[0, +\infty)$ qui se décompose en une somme d'ondes solitaires : quand $t \rightarrow +\infty$

$$\left\| u(t) - e^{i\gamma(t)} \sum_{k=1}^K \frac{1}{\lambda(t)} Q \left(\frac{\cdot - x_k(t)}{\lambda(t)} \right) \right\|_{H^1} \rightarrow 0, \quad \lambda(t) = \frac{1 + o(1)}{\log t},$$

où $x_k(t)$ converge vers les sommets d'un K -polygone régulier et la solution explose en temps infini avec le taux

$$\|\nabla u(t)\|_{L^2} \sim |\log t|, \quad \text{quand } t \rightarrow +\infty.$$

En utilisant la transformation (conformal transform), l'on obtient l'explosion en temps fini suivante :

Theorem 0.1.16 (Collision et explosion en temps fini [32]). Soient $u(t)$ donnée dans Theorème 0.1.15 et $v \in \mathcal{C}((-\infty, 0), \Sigma)$ la transformation pseudo-conforme de $u(t)$. Alors $v(t)$ explose en $T = 0$ avec

$$\|\nabla v(t)\|_{L^2} \sim K^{\frac{1}{2}} \|\nabla Q\|_{L^2} \left| \frac{\log |t|}{t} \right|, \quad |v(t)|^2 \rightarrow K \|Q\|_{L^2}^2 \delta_{x=0} \text{ quand } t \rightarrow 0.$$

Cela est également le premier exemple d'une explosion dont *le taux d'explosion est plus rapide que le taux d'explosion conforme*.

0.2 Principaux résultats

À partir de la section précédente, nous rappelons les observations suivantes :

◆ Premièrement, on voit que dans le cas intégrable $d = 1$ et $p = 3$, en utilisant la théorie de scattering inverse, il y a **trois types de 2-solitons** pour ((NLS) cubique 1D) :

- (a) Deux solitons avec vitesses différentes : quand $t \rightarrow +\infty$, la distance entre les solitons est de l'ordre t , [49].
- (b) La solution dipôle : les deux solitons sont symétriques (en particulier, ils ont la même d'amplitude) et leur distance est logarithmique en t , [38, 49].
- (c) 2-solitons périodique : les deux solitons ont une amplitude différente et leur distance est une fonction périodique du temps, [47, 49].

Notons que le régime (a) correspond à un comportement Galiléen libre, tandis les régimes (b) et (c) correspondent à un comportement Galiléen non-libre, c'est-à-dire que le comportement spécifique du soliton est dû aux interactions non-linéaires, et n'existe pas pour un soliton seul.

◆ Deuxièmement, pour le cas non-intégrable, la plupart des résultats connus sont dans le cas d'interactions faibles, perturbatives où il y a un découplage fort entre les solitons comme dans Theorème 0.1.14.

Nous posons la question de l'existence de comportements possibles de multi-solitons. En particulier, on se demande si les dynamiques non-génériques susmentionnées (b), (c) du cas intégrable persistent encore pour les modèles non-intégrables. Les travaux dans cette thèse s'orientent vers l'identification des cas où les interactions ont des effets remarquables sur la dynamique des objets non linéaires (**multi-soliton à interaction forte**). Les résultats peuvent être divisés en trois parties :

La première partie montre que la dynamique **distance logarithmique avec symétrie** (le comportement (b)) est un *comportement universel à la fois dans le cas sous-critique et sur-critique* (pour toute dimension $d \geq 1$) pour (NLS). Le régime correspond à l'interaction forte attractive. Dans le cas intégrable ($d = 1$ et $p = 3$), l'existence d'une telle solution est connue par la méthode dite d'inverse scattering, [38, 49]. Le cas d'une masse critique $p = 1 + \frac{4}{d}$ introduit un comportement spécifique lié à l'explosion, étudié dans [32].

Dans la deuxième partie, on étend le résultat de multi-soliton logarithmique à l'équation (gKdV). Pour l'équation (gKdV), il y a des difficultés techniques supplémentaires : la solution approchée (ansatz) pour le 2-soliton n'est pas dans l'espace d'énergie et il nous faut donc la modifier. Pour le cas sous-critique $1 < p < 5$, on sait que les ondes solitaires de signe opposés sont attractives. Pour le cas sur-critique $p > 5$, on déduit de notre calcul que les ondes solitaires de même signe sont attractives.

Le but de la troisième partie est, d'abord, de montrer la persistance du comportement (b) pour (coupled NLS) non-intégrable en présence de symétrie, similaire aux articles [32, 36] pour (NLS). Ensuite, et de façon plus importante, on examine la question de la (non-)persistance

du comportement (c). En fait, on décrit un nouveau régime logarithmique correspondant à 2-solitons non-symétrique avec distance logarithmique qui *remplace* le comportement (c) du cas non-intégrable. De manière formelle, le système d'équations pour les paramètres du 2-solitons n'est plus intégrable et les solutions périodiques disparaissent. Un nouveau régime logarithmique (voir (ii) du Théorème 0.2.9) prend place, qui n'existe pas dans le cas intégrable $\omega = 0$ ou $\omega = 1$.

Les résultats mentionnés ci-dessus sont basés sur les travaux suivants :

- T. V. Nguyen. Existence of multi-solitary waves with logarithmic relative distances for the NLS equations. *C. R. Acad. Sci. Paris, Ser. I* 357 (2019) 13–58.
- T. V. Nguyen. Strongly interacting multi-solitons with logarithmic relative distance for the gKdV equation. *Nonlinearity*, 30(12) :4614, 2017.
- Y. Martel and T. V. Nguyen. Construction of 2-solitons with logarithmic distance for the one-dimensional cubic Schrödinger system. *Preprint*

0.2.1 Existence d'ondes multi-solitaires avec distance relative logarithmique de l'équation de Schrödinger non linéaire.

Dans cette partie, on s'intéresse à l'extension de la dynamique logarithmique (b) au cas non-intégrable (NLS). Nous prouvons le résultat général suivant :

Theorem 0.2.1 (Onde multi-solitaire avec distance logarithmique, [36]). Pour $d \geq 1$, soit

$$1 < p < \frac{d+2}{d-2} \quad (p > 1 \text{ pour } d = 1, 2) \quad \text{et} \quad p \neq 1 + \frac{4}{d}.$$

Il existe une solution H^1 de (NLS) telle que $|z_1(t) - z_2(t)| \sim 2 \log t$ quand $t \rightarrow +\infty$ et

$$\left\| u(t) - e^{i\gamma(t)} \sum_{k=1}^2 Q(\cdot - z_k(t)) \right\|_{H^1(\mathbb{R}^d)} \rightarrow 0.$$

Il est essentiel que les deux solitons aient le même signe, le même scaling et la même phase, en effet, la solution est symétrique. Remarque que la situation reste similaire pour les multi-solitons construits dans [38, 49] pour le cas intégrable.

La distance logarithmique est due aux interactions fortes entre ondes solitaires car les interactions apparaissent à l'ordre principal des équations de modulation. En effet, la dynamique du Théorème 0.2.1 est liée à l'équation différentielle $\ddot{z} = -e^{-z}$, où $z(t)$ est la distance entre les solitons, et pour laquelle $2 \log t$ est une solution particulière.

Le résultat s'applique à toute dimension et toute non-linéarité \dot{H}^1 sous-critique, à l'exception de la puissance masse critique $p = 1 + \frac{4}{d}$. Techniquement, le cas $1 < p \leq 2$ est plus difficile puisque la norme L^∞ de l'interaction est pire ($\sim s^{-p}$ donc nous pouvons intégrer une seule fois), ce qui requiert la construction d'un ansatz raffiné.

Pour le cas critique ($p = 1 + \frac{4}{d}$), on n'espère pas l'existence de multi-solitons avec distance logarithmique. Plus précisément, pour une telle solution, on aurait

$$\int_{\mathbb{R}^d} |x|^2 |u(t, x)|^2 dx \sim \log^2(t),$$

qui serait en contradiction avec l'identité du viriel

$$\frac{d^2}{dt^2} \int_{\mathbb{R}^d} |x|^2 |u|^2 = 16E(u_0).$$

Dans ce cas, la *direction instable liée au scaling* est affectée par les interactions non-linéaires en premier lieu, ce qui conduit à la concentration en temps infini décrite dans Théorème 0.1.15 ([32]) tandis que dans le cas sous-critique et sur-critique, les interactions affectent d'abord le paramètre vitesse.

La preuve est basée sur la construction d'une solution approchée de (NLS) composée de deux solitons modulés symétriques, qui se déplacent dans des directions opposées, plus un petit terme correctif :

$$u(t, x) = \frac{e^{i\gamma(s)}}{\lambda^{\frac{2}{p-1}}(s)} \left[e^{iv_1(s)(\cdot - z_1)} Q(\cdot - z_1(s)) + e^{iv_2(s)(\cdot - z_2)} Q(\cdot - z_2(s)) + \epsilon(s) \right] \left(\frac{x}{\lambda(s)} \right)$$

avec

$$s = \int_0^t \frac{1}{\lambda^2(\tau)} d\tau, \quad v_1(s) = -v_2(s) = \frac{1}{2}v(s), \quad z_1(s) = -z_2(s) = \frac{1}{2}z(s).$$

On suit l'argument de modulation standard pour la solution, qui a été développé dans [27, 35, 42] pour l'équation (NLS). En utilisant les conditions d'orthogonalité liées au noyau généralisé de l'opérateur linéarisé, on en déduit un système d'équations modulation avec un terme de forçage non-linéaire. On cherche une solution approchée qui est pertinente pour notre régime. Formellement, nous avons les estimations suivantes :

$$\left| \frac{\dot{\lambda}}{\lambda} \right| + \left| \dot{z} - 2v + \frac{\dot{\lambda}}{\lambda} z \right| \lesssim |z|^{-\frac{d-1}{2}} e^{-|z|}. \quad (0.2.2)$$

Notamment, l'équation pour \dot{v} contient le terme de forçage due aux interactions non-linéaires entre les solitons :

$$\left| \dot{v} + c \frac{z}{|z|} |z|^{-\frac{d-1}{2}} e^{-|z|} \right| \lesssim |z|^{-\frac{d-1}{2}-1} e^{-|z|} \quad (0.2.3)$$

pour une constante $c > 0$. On peut vérifier que proche des fonctions $z_{mod}(s)$, $\lambda_{mod}(s)$, $v_{mod}(s)$ telles que

$$\lambda_{mod}^{-1}(s) = 1, \quad v_{mod}(s) = s^{-1}, \quad z_{mod}^{-\frac{d-1}{2}} e^{-z_{mod}} = \frac{s^{-2}}{c} \quad (0.2.4)$$

il existe une solution particulière de (0.2.2)-(0.2.3). Nous avons leur asymptotique quand $s \rightarrow +\infty$

$$\begin{aligned} z_{mod}(s) &\sim 2 \log(s), & \dot{v}_{mod}(s) &= -c z_{mod}^{-\frac{d-1}{2}}(s) e^{-z_{mod}(s)}, \\ |\dot{z}_{mod}(s) - 2v_{mod}(s)| &\lesssim s^{-1} \log^{-1}(s), & |\dot{v}_{mod}(s)| &\lesssim s^{-2}. \end{aligned} \quad (0.2.5)$$

Cela suggère que (2.2.39) est proche au premier ordre pour une solution de (NLS) et convient au régime du Théorème 0.2.1.

0.2.2 Multi-solitons à interactions fortes avec distance logarithmique pour l'équation (gKdV).

Nous prouvons également l'existence des *ondes multi-solitaires à interaction forte avec distance relative logarithmique* pour

$$\partial_t u + \partial_x(\partial_x^2 u + |u|^{p-1}u) = 0, \quad t, x \in \mathbb{R}. \quad (0.2.6)$$

Theorem 0.2.7 (Onde multi-solitaire avec distance logarithmique, [37]). Soit p un entier, $p \neq 5$ et $p > 2$. Il existe $t_0 > 0$ et une H^1 solution sur $[t_0, +\infty)$ de (0.2.6) qui se décompose asymptotiquement comme deux ondes solitaires

$$\left\| u(t) - \left(Q(\cdot - t - \log(ct)) + \sigma Q(\cdot - t + \log(ct)) \right) \right\|_{H^1} \rightarrow 0 \quad \text{quand } t \rightarrow +\infty, \quad (0.2.8)$$

où $c = c(p) > 0$ une constante, $\sigma = -1$ dans le cas sous-critique ($2 < p < 5$) et $\sigma = 1$ dans le cas sur-critique ($p > 5$).

Le signe négatif dans le cas sous-critique ($p < 5$) veut dire que dans ce cas l'interaction entre solitons est répulsive tandis que le signe positif dans le cas sur-critique ($p > 5$) veut dire que dans ce cas l'interaction est attractive. Remarquons que la distance relative de deux ondes solitaires est $2 \log(ct)$ avec c donné par

$$c = c(p) = \sqrt{\frac{8(p-1)}{|5-p|}} (2p+2)^{\frac{1}{p-1}} \|Q\|_{L^2}^{-1}.$$

Dans [20], il a été montré par la suite que c'est l'unique régime de distance logarithmique de ce scaling dans la configuration instable ($p > 5$).

En raison de la structure du noyau de l'opérateur linéarisé, l'équation pour le paramètre de position $z_1(t), z_2(t)$ n'est plus quadratique en fonction du terme d'erreur $\epsilon(t)$. Il est nécessaire d'ajouter une partie raffinée de la solution approchée correspondant à l'interaction non-linéaire pour fermer le bootstrap

$$e^{-z(t)} (A_1(\cdot - z_1(t)) + A_2(\cdot - z_2(t))) \varphi$$

où φ est une fonction régularisante. En fait, sans ce terme, on obtiendrait un terme d'erreur d'ordre t^{-2} , ce qui n'est pas suffisamment fin pour exploiter les équations de modulations. Comme les termes ajoutés A_1, A_2 pourraient ne pas être dans H^1 (ils pourraient avoir une limite non-nulle à $-\infty$), il nous faut introduire une fonction régularisante adaptée φ .

0.2.3 Construction de 2-solitons avec distance relative logarithmique pour le système de Schrödinger cubique 1D.

Considérons le système couplé des équations NLS cubiques 1D écrit sous la forme

$$\begin{cases} i\partial_t u + \partial_x^2 u + (|u|^2 + \omega|v|^2)u = 0 \\ i\partial_t v + \partial_x^2 v + (|v|^2 + \omega|u|^2)v = 0, & t \in \mathbb{R}, x \in \mathbb{R} \\ u(0, x) = u_0, \quad v(0, x) = v_0. \end{cases} \quad (\text{coupled NLS})$$

Notons que quand $v = 0$ (ou $u = 0$), le système (coupled NLS) est simplifié comme (NLS)

$$i\partial_t u + \Delta u + |u|^2 u = 0.$$

Par conséquent, (coupled NLS) admet des solutions spéciales de la forme :

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} e^{i\gamma_u + i\lambda_u^2 t + i\frac{\beta_u}{2}(x - \frac{\beta_u}{2}t)} Q_{\lambda_u}(x - \sigma_u - \beta_u t) \\ 0 \end{pmatrix}$$

et

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ e^{i\gamma_v + i\lambda_v^2 t + i\frac{\beta_v}{2}(x - \frac{\beta_v}{2}t)} Q_{\lambda_v}(x - \sigma_v - \beta_v t) \end{pmatrix}$$

pour $\lambda_j > 0, \sigma_j, \beta_j, \gamma_j \in \mathbb{R}, (j = u, v)$. Nous construisons **un 2-soliton logarithmique dans le cas symétrique**, i.e., $u(t, x) = v(t, -x)$ et puis **une nouvelle dynamique logarithmique de 2-solitons dans le cas où les solitons sont non-symétriques** ⁴

Theorem 0.2.9 (Deux familles d'ondes multi-solitaires logarithmiques, [31]). Soit

$$0 < \omega < 1$$

(i) Il existe une solution globale H^1 de (coupled NLS) qui se décompose en deux ondes solitaires

$$\left\| \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} - e^{\gamma(t)} \begin{pmatrix} Q(\cdot - \frac{1}{2} \log(\Omega t) - \frac{1}{4} \log(\log t)) \\ Q(\cdot + \frac{1}{2} \log(\Omega t) + \frac{1}{4} \log(\log t)) \end{pmatrix} \right\|_{H^1 \times H^1} \rightarrow 0 \quad \text{quand } t \rightarrow +\infty. \quad (0.2.10)$$

La distance entre les deux ondes solitaires est $\log t + \frac{1}{2} \log(\log t) + \log(\Omega) + o(1)$.

(ii) On suppose :

$$\omega < \frac{c(c+1)}{2} < 1.$$

Il existe une solution globale H^1 de (coupled NLS) qui se décompose en deux ondes solitaires d'amplitude c et 1 respectivement

$$\left\| \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} - \begin{pmatrix} e^{ic^2t} Q_c(\cdot - \frac{1}{(1+c)c} \log(\Omega_c t)) \\ e^{it} Q(\cdot + \frac{1}{1+c} \log(\Omega_c t)) \end{pmatrix} \right\|_{H^1 \times H^1} \rightarrow 0 \quad \text{quand } t \rightarrow +\infty. \quad (0.2.11)$$

La distance entre les deux ondes solitaires est $\frac{1}{c} \log t + \frac{\log(\Omega_c)}{c} + o(1)$.

À notre connaissance, la deuxième partie donne le premier exemple d'une telle dynamique pour des ondes multi-solitaires (distance relative logarithmique sans symétrie). Le nouveau comportement logarithmique pour le cas non-symétrique *provient de la propriété de non-intégrabilité pour les interactions entre ondes solitaires*. Il y a une résonance non-linéaire qui détruit le régime libre des solitons et les force à se séparer et à créer la dynamique(0.2.11). Cette situation diffère des équations intégrables où les solutions périodiques existent (le comportement (c)). Les termes de forçage apparaissent à *l'ordre quadratique* des interactions non-linéaires. Remarquons que les deux régimes logarithmiques (dans le cas symétrique et le cas non-symétrique) sont différents.

La question du *cas non-symétrique pour l'équation (NLS) scalaire non-intégrable est encore un problème ouvert*. En fait, notre construction de la solution approchée fonctionne aussi bien pour (NLS) que pour (coupled NLS). Nous obtenons aussi une résonance non-linéaire et formellement, les deux solitons se séparent pour former **un régime "logarithmique avec oscillations"**. Néanmoins, les lois de conservation sont moins favorables et il nous manque des arguments pour terminer la méthode d'énergie.

⁴voir le résumé des comportements possibles pour 2-solitons de l'équation (NLS) intégrable dans la section précédente pour une comparaison.

Chapter 1

Introduction (English versions)

1.1 General context

The aim of this thesis is to study the asymptotic dynamics for **multi-solitary wave solutions** (also called **solitons**) of some **non-linear dispersive equations**.

By nonlinear dispersive equations, we mean equations which combine a linear equation with dispersive behavior (any solution decays uniformly in time) and a suitable non-linearity, typically, of the following form:

$$u_t = Lu + F(u, Du, \dots)$$

where $u(t, x)$ is real or complex valued, and $t \in \mathbb{R}, x \in \mathbb{R}^d$. The linear operator L is assumed anti-adjoint, that is

$$\mathcal{F}(Lu)(\xi) = ip(\xi)\mathcal{F}u(\xi), \quad p(\xi) \in \mathbb{R}$$

here \mathcal{F} is the Fourier transform and $D_\xi^2 p(\xi) \neq 0$ for all $\xi \neq 0$.

Example of nonlinear dispersive equations are nonlinear Schrödinger equation, Korteweg-de Vries equation, Benjamin-Ono equation, BBM equation, KP-I, KP-II equations.

Before stating the main results of the thesis, let us briefly present some aspect of the theory of solitons for nonlinear dispersive equations.

1.1.1 Preliminary

Let us consider one of the most typical dispersive equations, the nonlinear Schrödinger equation (NLS) in \mathbb{R}^d :

$$i\partial_t u + \Delta u + |u|^{p-1}u = 0 \tag{NLS}$$

where Δ is the Laplacian in the d space variables, the non-linearity $p > 1$ and with initial data

$$u(0, x) = u_0, \quad u_0 \in H^1 : \mathbb{R}^d \rightarrow \mathbb{C}.$$

Local existence in time for solution of (NLS) is well-studied. In this thesis, we mainly use the following local existence and uniqueness results:

Proposition 1.1.1 (Local wellposedness in H^1 , Ginibre and Velo [16], see also [3]). Let

$$2^* = \begin{cases} +\infty & \text{if } d = 1, 2 \\ \frac{2d}{d-2} & \text{if } d \geq 3. \end{cases}$$

For $1 < p < 2^* - 1$ and $u_0 \in H^1(\mathbb{R}^d)$, there exist a maximal lifespan $T = T(\|u_0\|_{H^1}) > 0$ and a unique solution $u(t) \in \mathcal{C}([0, T]; H^1)$ of (NLS).

Moreover, we have the following blow-up criterion: either $T = +\infty$, solution $u(t)$ is **global in time**; or $T < +\infty$ and then the solution **blows up in finite time**

$$\lim_{t \rightarrow T} \|u(t)\|_{H^1} = +\infty.$$

From direct computations, the equation (NLS) admits the following group of **symmetry**:

- Phase: for $\gamma \in \mathbb{R}$, if $u(t, x)$ is a solution of (NLS), then $w(t, x) = u(t, x)e^{i\gamma}$ is also a solution.
- Scaling: for $\lambda > 0$, if $u(t, x)$ is a solution of (NLS), then $w(t, x) = \lambda^{\frac{2}{p-1}}u(\lambda^2t, \lambda x)$ is also a solution.
- Translation: for $x_0 \in \mathbb{R}^d$, $t_0 \in \mathbb{R}$, if $u(t, x)$ is a solution of (NLS), then $w(t, x) = u(t - t_0, x - x_0)$ is also a solution.
- Galilean: for $\beta \in \mathbb{R}^d$, if $u(t, x)$ is a solution of (NLS), then $w(t, x) = u(t, x - \beta t)e^{i\frac{\beta}{2}(x - \frac{\beta}{2}t)}$ is also a solution.

Another property of the equation (NLS) is that these symmetries induce some natural Hamiltonian structures: we have the following **conservation laws**

- Mass :

$$M(u(t)) = \int_{\mathbb{R}^d} |u(t, x)|^2 dx = M(u_0) \quad (1.1.2)$$

- Energy :

$$E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(t, x)|^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^d} |u(t, x)|^{p+1} dx = E(u_0) \quad (1.1.3)$$

- Momentum:

$$J(u(t)) = \text{Im} \int_{\mathbb{R}^d} \nabla u(t, x) \bar{u}(t, x) dx = J(u_0) \quad (1.1.4)$$

Denote $u_\lambda(t, x) = \lambda^{\frac{2}{p-1}}u(\lambda^2t, \lambda x)$ the scaling symmetry of the flow. The *critical space* is a fundamental phenomenological number for the analysis and is defined as the number of derivatives in L^2 which are left invariant by the scaling symmetry:

$$\|u_\lambda(t, x)\|_{\dot{H}^{s_c}} = \|u(t, x)\|_{\dot{H}^{s_c}} \text{ for } s_c = \frac{d}{2} - \frac{2}{p-1}.$$

Observe that $s_c < 1$ in Proposition 1.1.1.

- If $s_c < 0$ (or equivalently $1 < p < 1 + \frac{4}{d}$), the equation (NLS) is called L^2 subcritical.
- If $s_c = 0$ (or $p = 1 + \frac{4}{d}$), the equation (NLS) is called L^2 critical.
- If $s_c > 0$ (or $p > 1 + \frac{4}{d}$), the equation (NLS) is called L^2 supercritical.

A direct consequence of the Cauchy theory, the conservation laws and the following Gagliardo-Nirenberg inequality

$$\|f\|_{L^{p+1}} \leq C_p \|f\|_{L^2}^{1-\sigma} \|\nabla f\|_{L^2}^\sigma \quad \text{with} \quad \sigma = \frac{d(p-1)}{p+1} \quad (\text{Gagliardo-Nirenberg})$$

is the following global existence result, see [40].

Proposition 1.1.5 (Global wellposedness in the subcritical case). Let $d \geq 1$ and $1 < p < 1 + \frac{4}{d}$, then all solutions to (NLS) are global and bounded in H^1 .

By another quantity called the virial $\int_{\mathbb{R}^d} |x|^2 |u(t, x)|^2 dx$ and the following identity involving its second derivative

$$\frac{d^2}{dt^2} \int_{\mathbb{R}^d} |x|^2 |u(t, x)|^2 dx = 16E(u_0) - \frac{4[d(p-1) - 4]}{p+1} |u(t, x)|^{p+1} dx,$$

one proves the existence of finite blow-up solutions in the critical and supercritical case, see [3, 40].

Proposition 1.1.6 (Virial blow up for $E(u_0) < 0$). Let $1 + \frac{4}{d} \leq p < 2^* - 1$ and $u_0 \in \Sigma = H^1 \cap L^2(|x|^2 dx)$ with $E(u_0) < 0$. Then the corresponding solution to (NLS) blows up in finite time $0 < T < +\infty$.

In the case $d = 1$ and $p = 3$, one obtains 1D cubic (NLS)

$$i\partial_t u + u_{xx} + |u|^2 u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}.$$

The equation 1D cubic (NLS) is completely integrable, which means that there exists a pair of Lax (see [9, 14, 23, 49]). This case has been studied extensively using the method of inverse scattering. In particular, 1D cubic (NLS) equation enjoys *infinite many conservation laws*, the *purely elastic collision* of solitons and the *decomposition into solitons*. In other case, the equation (NLS) is not known to be integrable.

Another example of nonlinear dispersive equations that we study in this thesis is the generalized Korteweg-de Vries equation (gKdV):

$$\partial_t u + \partial_x(\partial_x^2 u + u^p) = 0, \quad t, x \in \mathbb{R}. \quad (\text{gKdV})$$

where $p > 1$ is an integer. The equation is completely integrable in the quadratic case $p = 2$ and cubic case $p = 3$.

We also study the coupled system of cubic Schrödinger equations:

$$\begin{cases} i\partial_t u + \Delta u + (|u|^2 + \omega|v|^2)u = 0 \\ i\partial_t v + \Delta v + (|v|^2 + \omega|u|^2)v = 0 \end{cases} \quad t \in \mathbb{R}, x \in \mathbb{R} \quad (\text{coupled NLS})$$

for a parameter $0 < \omega < 1$. Remark that in the case $\omega = 1$, the (coupled NLS) system is called the integrable Manakov system (see [25, 47])

$$\begin{cases} i\partial_t u + \Delta u + (|u|^2 + |v|^2)u = 0 \\ i\partial_t v + \Delta v + (|v|^2 + |u|^2)v = 0. \end{cases}$$

1.1.2 The solitary wave

A fundamental feature of (NLS) is the existence of *space localized time periodic solutions*, called **solitary waves** or **solitons**. Indeed,

$$u(t, x) = \phi(x)e^{it}$$

is an H^1 solution to (NLS) if and only if ϕ solves the nonlinear elliptic equation:

$$\Delta\phi - \phi + |\phi|^{p-1}\phi = 0, \quad \phi \in H^1(\mathbb{R}^d). \quad (\text{soliton eq.})$$

There are various ways to construct solutions to (soliton eq.), the simplest one being to look for radial solutions via a shooting method [1]. Remark that the theory of linear dispersive predicts that waves of linear Schrödinger equation should spread out and disperse over time [43]. Therefore, the existence of such solution is a fine balance between *the dispersion* induced by the Laplacian part and *the concentration* created by the nonlinear part of the equation (NLS). An important rigidity property of (soliton eq.) is *the uniqueness of the nonnegative solution*.

Proposition 1.1.7 (Uniqueness of the ground state, Kwong [22]). All solutions to

$$\Delta\phi - \phi + |\phi|^{p-1}\phi = 0, \quad \phi \in H^1(\mathbb{R}^d), \quad \phi(x) > 0$$

are a translation of an exponentially decay radial profile $Q(x) = q(|x|)$ which is the unique nonnegative radial solution of (soliton eq.). Q is the so called *ground state* solution.

The ground state Q has an explicit form in one dimension

$$Q(x) = \left(\frac{p+1}{2 \cosh^2\left(\frac{p-1}{2}x\right)} \right)$$

and an exponential decay in all dimensions: for some constant $c_Q > 0$

$$\left| q(r) - c_Q r^{-\frac{d-1}{2}} e^{-r} \right| + \left| q'(r) + c_Q r^{-\frac{d-1}{2}} e^{-r} \right| \lesssim r^{-\frac{d-1}{2}-1} e^{-r}.$$

From [44], the ground state Q is related to the best constant of the (Gagliardo-Nirenberg) inequality. More precisely, Q is the unique minimizer of the inequality and the optimal constant is given by

$$C_p = J(Q) = \min_{f \in H^1, f \neq 0} J(f)$$

where for all $f \in H^1, f \neq 0$

$$J(f) = \frac{\|f\|_{L^2}^{1-\sigma} \|\nabla f\|_{L^2}^\sigma}{\|f\|_{L^{p+1}}} \quad \text{with} \quad \sigma = \frac{d(p-1)}{p+1}.$$

Denote $Q_\lambda(x) = \lambda^{\frac{2}{p-1}} Q(\lambda x)$, from the symmetries of (NLS), there is a $(2d+2)$ parameters family of solitons:

$$u(t, x) = Q_{\lambda_0}(x - \sigma_0 - \beta_0 t) e^{i\gamma_0 + i\lambda_0^2 t + i\frac{\beta_0}{2}(x - \frac{\beta_0}{2}t)} \quad (1.1.8)$$

with scaling $\lambda_0 \in \mathbb{R}_+$, position $\sigma_0 \in \mathbb{R}^d$, phase $\gamma_0 \in \mathbb{R}$ and speed $\beta_0 \in \mathbb{R}^d$.

Once the existence of solitary wave solutions is established, we would like to study their stability by perturbation of the initial data in the energy space. We observe that the solitary wave has two trivial instabilities: scaling instability ¹ and Galilean instability ². A natural question is that whether these trivial instabilities are the only ones or not.

Definition 1.1.9 (Orbital stability). We say that a solitary wave solution (1.1.8) is orbitally stable if for all $\epsilon > 0$, there exists $\delta > 0$ such that if

$$\left\| u(0) - Q_{\lambda_0}(x - \sigma_0) e^{i(\gamma_0 + \frac{\beta_0}{2}x)} \right\|_{H^1} \leq \delta,$$

then for all $t \in \mathbb{R}$, there exist $\sigma(t), \gamma(t)$ such that the solution $u(t)$ of (NLS) satisfies

$$\left\| u(t, \cdot) - Q_{\lambda_0}(\cdot - \sigma(t)) e^{i(\gamma(t) + \frac{\beta_0}{2}x)} \right\|_{H^1} \leq \epsilon.$$

¹ $\forall \lambda > 0$, the solution to (NLS) with initial data $u_0(x) = \lambda^{\frac{2}{p-1}} Q(\lambda x)$ is $u(t, x) = \lambda^{\frac{2}{p-1}} Q(\lambda x) e^{i\lambda^2 t}$.

² $\forall \beta > 0$, the solution to (NLS) with initial data $u_0(x) = Q(x) e^{i\beta x}$ is $u(t, x) = Q(x - \beta t) e^{i\beta x + i\frac{\beta}{2}(x - \frac{\beta}{2}t)}$.

Using the concentration-compactness method, Cazenave and Lions [4] proved that these solitary waves are stable when $1 < p < 1 + \frac{4}{d}$, i.e. when the nonlinearity is L^2 subcritical. By a different approach based on the expansion of the conservation laws around the solitary wave, Weinstein [45] proved the orbital stability of a solitary wave solution under the nondegeneracy condition

$$\left. \frac{d}{d\lambda} \int Q_\lambda^2(x) dx \right|_{\lambda=\lambda_0} > 0$$

which holds in the subcritical case. Conversely, it is known from the work of Grillakis, Shatah and Strauss [17, 18] that if

$$\left. \frac{d}{d\lambda} \int Q_\lambda^2(x) dx \right|_{\lambda=\lambda_0} < 0$$

which holds in the supercritical case ($1 + \frac{4}{d} < p < 2^* - 1$), then the solitary wave solution is unstable. Note that the instability phenomenon is quite different in the critical case (linear instability and the unstable direction is related to the scaling parameter) and in the supercritical case (exponential instability).

For the critical case ($p = 1 + \frac{4}{d}$), the pseudo conformal transform

$$v(t, x) = \frac{1}{|t|^{\frac{d}{2}}} u\left(\frac{-1}{t}, \frac{x}{t}\right) e^{i\frac{|x|^2}{4t}} \quad (\text{conformal transform})$$

applied to the solitary wave solution $u(t, x) = Q(x)e^{it}$ gives an *explicit blow-up element*:

$$S(t, x) = \frac{1}{|t|^{\frac{d}{2}}} Q\left(\frac{x}{|t|}\right) e^{-i\frac{|x|^2}{4t} + \frac{i}{t}}, \quad \|S(t)\|_{L^2} = \|Q\|_{L^2} \quad (1.1.10)$$

with blow-up rate (called conformal blow-up rate)

$$\|\nabla S(t)\|_{L^2} \sim \frac{1}{|t|} \quad \text{as } t \rightarrow 0^-.$$

$S(t, x)$ is also the unique (up to invariances) blow-up solution with the mass $\|u_0\|_{L^2} = \|Q\|_{L^2}$, see [33]. It is called the *minimal mass blow-up* because under this mass, $\|u_0\|_{L^2} < \|Q\|_{L^2}$, all solutions are global and even scatters, i.e. there exist $v^\pm(t, x)$ solutions of the linear equation such that

$$\|u(t) - v^\pm(t)\|_{H^1} \rightarrow 0$$

as $t \rightarrow \pm\infty$, see [10, 21].

Besides the minimal mass blow-up solution, there also exist other blow-up solutions with conformal blow-up rate (see [2]). For the case of mass slightly above the threshold (and with negative energy), i.e.

$$\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha_0, \quad 0 < \alpha_0 \ll 1, \quad (1.1.11)$$

Merle and Raphaël [34, 35, 41] proved the existence and stability of a blow-up dynamics with log-log blow-up rate:

$$\|\nabla u(t)\|_{L^2} \sim c \sqrt{\frac{|\log|\log(T-t)||}{T-t}} \quad \text{as } t \rightarrow T. \quad (1.1.12)$$

Then, [41] also showed that under (1.1.11), there is a gap on the blow-up rate: either the solution blows up in the log-log regime (1.1.12), or it blows up faster than the conformal rate

$$\|\nabla u(t)\|_{L^2} \gtrsim \frac{1}{T-t}. \quad (1.1.13)$$

Similarly to (NLS), (gKdV) also admits a family of solitary wave solutions with two parameters, of the form

$$R_{v_0, x_0} = Q_{v_0}(x - x_0 - v_0 t)$$

where $Q_v = v^{\frac{1}{p-1}} Q(\sqrt{v}x)$ and Q is the ground state of (soliton eq.) in dimension 1.

We also note that when $v = 0$ (or $u = 0$), the (coupled NLS) is simplified as 1D cubic (NLS). Thus, we deduce some special solutions of (coupled NLS) defined as follows:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} e^{i\gamma_u + i\lambda_u^2 t + i\frac{\beta_u}{2}(x - \frac{\beta_u}{2}t)} Q_{\lambda_u}(x - \sigma_u - \beta_u t) \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ e^{i\gamma_v + i\lambda_v^2 t + i\frac{\beta_v}{2}(x - \frac{\beta_v}{2}t)} Q_{\lambda_v}(x - \sigma_v - \beta_v t) \end{pmatrix}$$

for any $\lambda_j > 0, \sigma_j, \beta_j, \gamma_j \in \mathbb{R}$ ($j = u, v$).

1.1.3 Multi-solitary wave

A **multi-solitary wave**, also called **multi-soliton**, is a solution behaving in large time as several solitons. For example, consider $K \geq 2$ and for $k \in \{1, \dots, K\}$, let $\lambda_k > 0, \beta_k \in \mathbb{R}^d, \sigma_k \in \mathbb{R}^d$ and $\gamma_k \in \mathbb{R}$. Define the k -th soliton

$$R_k(t, x) = Q_{\lambda_k}(x - \sigma_k(t)) e^{i\Gamma_k(t, x)}$$

with

$$\sigma_k(t) = \sigma_k + \beta_k t, \quad \Gamma_k(t, x) = \frac{1}{2}(\beta_k \cdot x) - \frac{1}{4}|\beta_k|^2 t + \lambda_k t + \gamma_k.$$

Note that because of the non-linearity, the sum

$$R(t) = \sum_{k=1}^K R_k(t)$$

is not a solution of (NLS). However, we may expect existence of a solution of (NLS) which behaves asymptotically as

$$\|u(t) - R(t)\|_{H^1} \rightarrow 0 \quad \text{as} \quad t \rightarrow +\infty.$$

Multi-solitons were first constructed for the integrable 1D cubic (NLS) ($d = 1$ and $p = 3$), [49]: in that case, by the *scattering inverse method*, there is an explicit formula of multi-solitons, and one observes by inspection that these solutions decouple into a sum of solitons both as $t \rightarrow +\infty$ and as $t \rightarrow -\infty$ (collisions are elastic). Recently, the asymptotic stability of such objects for the integrable case was shown in some weighted space [8]. Moreover, one can observe that besides **weak interactions** (roughly speaking, means that the trajectories of the solitary waves are not affected asymptotically by the interaction of solitons, i.e. $\sigma_k(t)$ has the same form as for a single soliton), there are cases of strong interaction where the distance between solitons is of order $\log t$ or a finite distance with periodic behavior, i.e. the form of $\sigma_k(t)$ is affected by the interaction between solitons.

The multi-soliton behavior is fundamental in view of the **soliton resolution conjecture**. This (*rather vaguely defined*) conjecture³ says that **generic solution should decompose for large time into a sum of rigid nonlinear objects (typically solitons) and a purely**

³The conjecture is also supported by extensive numerical works [9, 15, 24, 48].

dispersive term. By the inverse scattering method, the soliton decomposition was obtained in the setting of integrable equation (1D cubic (NLS), quadratic and cubic (KdV)). In the non-integrable setting, recently, in the notable series of papers [11, 12, 13], results of decomposition into solitons were proved for energy critical wave equation, in particular the 3D case with radial data. However, the conjecture still remains widely open in general.

Returning to (NLS), for non-integrable setting, we recall the following existence result of multi-solitons with decoupling conditions (and limited to one direction of time).

Theorem 1.1.14 (Existence of (NLS) multi-solitary waves, [7, 27, 33]). Let $1 < p < \frac{d+2}{d-2}$. Consider $K \geq 1$, for $k \in \{1, \dots, K\}$, let $\lambda_k > 0, \beta_k \in \mathbb{R}^d, \sigma_k \in \mathbb{R}^d$ and $\gamma_k \in \mathbb{R}$. Assume that the velocities are decoupled, for any $k \neq k', \beta_k \neq \beta_{k'}$. Denote

$$R_k(t, x) = Q_{\lambda_k}(x - \beta_k t - \sigma_k) e^{i\Gamma_k(t, x)} \quad \text{and} \quad \Gamma_k(t, x) = \frac{1}{2}(\beta_k \cdot x) - \frac{1}{4}|\beta_k|^2 t + \lambda_k t + \gamma_k.$$

Then, there exists $T_0 > 0, \kappa > 0$ and an H^1 solution $u(t)$ of (NLS) such that for all $t \geq T_0$,

$$\left\| u(t, x) - \sum_{k=1}^K R_k(t, x) \right\|_{H^1} \lesssim e^{-\kappa|t|}.$$

Because of the decoupling conditions, the solitons have two-by-two different velocities so the interaction between them decays exponentially in time. The critical case $p = 1 + \frac{4}{d}$ was in fact the first setting of non-integrable case where such multi-solitons were constructed by Merle [33]. Moreover, by the pseudo-conformal transform (conformal transform), this result also constructed solutions where blow-up simultaneously occurs at K distinct points. The subcritical case $1 < p < 1 + \frac{4}{d}$ was treated by Martel and Merle [27], following the result by Martel [26] on (gKdV). The supercritical case was done by Cote, Martel and Merle [7], then Combet [6] constructed an K -parameter family of K -solitons in this case.

Note that all above theorems in the non-integrable case say nothing about the behavior of the solution as $t \rightarrow -\infty$. In contrast with the integrable case, for non-integrable case, multi-solitons for $t \rightarrow +\infty$ are expected to have different behavior for $t \rightarrow -\infty$ (not again pure multi-soliton) as the interaction between solitary waves would be, in general, inelastic (see [28, 39]). The stability of the multi-soliton structure was studied in the subcritical case in [30]. Similar results for other nonlinear models with solitary wave: (gKdV) [5, 7, 29], coupled (NLS) [19], for example.

Concerning **strong interactions** (that means, the interactions indeed affect the trajectory of solitons), one of the first intrusion into the study of strongly interacting multi-solitons for (NLS) is the work of Martel and Raphaël [32] where they constructed a first example of global solutions blowing up in infinite time:

Theorem 1.1.15 (Infinite time strongly interacting blow-up [32]). For the 2D cubic (NLS), there exists a global solution $u(t)$ on $[0, +\infty)$ which decomposes into a sum of solitary waves: as $t \rightarrow +\infty$

$$\left\| u(t) - e^{i\gamma(t)} \sum_{k=1}^K \frac{1}{\lambda(t)} Q \left(\frac{\cdot - x_k(t)}{\lambda(t)} \right) \right\|_{H^1} \rightarrow 0, \quad \lambda(t) = \frac{1 + o(1)}{\log t},$$

where $x_k(t)$ converges to the vertices of a K -sided regular polygon and the solution blows up in infinite time with the rate

$$\|\nabla u(t)\|_{L^2} \sim |\log t|, \quad \text{as } t \rightarrow +\infty.$$

Using the (conformal transform), we obtain the following finite time blow-up:

Theorem 1.1.16 (Finite time blow-up collision [32]). Let $u(t)$ be given in Theorem 1.1.15 and let $v \in \mathcal{C}((-\infty, 0), \Sigma)$ be the pseudo conformal transform of $u(t)$. Then $v(t)$ blows up at $T = 0$ with

$$\|\nabla v(t)\|_{L^2} \sim K^{\frac{1}{2}} \|\nabla Q\|_{L^2} \left| \frac{\log |t|}{t} \right|, \quad |v(t)|^2 \rightharpoonup K \|Q\|_{L^2}^2 \delta_{x=0} \text{ as } t \rightarrow 0.$$

This is also the first example of a blow-up solution *whose blow-up rate is faster than the conformal blow-up rate*.

1.2 Main results

From the previous section, we have the following observations:

◆ First, we see that in the integrable case $d = 1$ and $p = 3$, by using the inverse scattering theory, there are **three types of 2-solitons** for (1D cubic (NLS)):

- (a) Two solitons with different velocities: as $t \rightarrow +\infty$, the distance between the solitons is of order t , [49].
- (b) Double pole solutions: the two solitons are symmetric (in particular, they have same amplitude) and their distance is logarithmic in t , [38, 49].
- (c) Periodic 2-solitons: the two solitons have different amplitudes and their distance is a periodic function of time, [47, 49].

Note that (a) corresponds to a free Galilean motion, while (b) and (c) correspond to a non free Galilean motion, which means that the specific behaviors of the solitons are due to nonlinear interactions, and do not exist for a single soliton.

◆ Second, for non-integrable case, the study of multi-solitons is mostly limited to situations where there is a strong decoupling between the solitons (weak interactions) as in Theorem 1.1.14.

We raise the question of other possible behaviors of multi-solitons. In particular, we ask whether the above non generic dynamics (b), (c) of the integrable case persist for non-integrable models. The work in this thesis is oriented to the identification of some new cases where the interactions between traveling waves indeed affect their trajectories (**multi-soliton with strong interaction**). The results are split into three parts:

The first part shows that the dynamics **logarithmic distance with symmetric** (the behavior (b)) is a *universal behavior in both subcritical and supercritical case* (in all dimension $d \geq 1$) for (NLS). This regime corresponds to strong attractive interactions. In the integrable case ($d = 1$ and $p = 3$), the existence of such behaviors is known by inverse scattering, [38, 49]. The mass critical case $p = 1 + \frac{4}{d}$ exhibits a specific behavior related to blow-up, studied in [32].

In the second part, we extend the result of logarithmic multi-solitons to the (gKdV) equation. For the (gKdV) equation, there are several technical additional difficulties: the natural ansatz for the 2-soliton is not in the energy space and thus it has to be modified. For sub-critical $1 < p < 5$, it was known that opposite sign traveling waves are attractive. For super-critical $p > 5$, we derive from our computations that same sign traveling waves are attractive.

The goal of the third part is, first, to show the persistence of the behavior (b) for the non-integrable (coupled NLS) in presence of symmetry, similarly as in the articles [32, 36] for (NLS). Second, and more importantly, we investigate the question of the (non-)persistence of the behavior (c). Actually, we exhibit a new logarithmic regime corresponding to non-symmetric 2-solitons with logarithmic distance which *replaces* the behavior (c) of the nonintegrable case. At the formal level, the system of parameters of the 2-solitons is not anymore integrable and

periodic solutions disappear. A new logarithmic regime (see (ii) of Theorem 1.2.9) then takes place, which does not exist in the integrable case $\omega = 0$ or $\omega = 1$.

The results mentioned above are based on the following works:

- T. V. Nguyen. Existence of multi-solitary waves with logarithmic relative distances for the NLS equations. *C. R. Acad. Sci. Paris, Ser. I* 357 (2019) 13–58.
- T. V. Nguyen. Strongly interacting multi-solitons with logarithmic relative distance for the gKdV equation. *Nonlinearity*, 30(12):4614, 2017.
- Y. Martel and T. V. Nguyen. Construction of 2-solitons with logarithmic distance for the one-dimensional cubic Schrödinger system. *Preprint*

1.2.1 Existence of multi-solitary waves with logarithmic relative distances for the NLS equations.

In this part, we address the extension of the logarithmic dynamics (b) to the non-integrable (NLS). We prove the following general existence result:

Theorem 1.2.1 (Multi-solitary waves with logarithmic distance, [36]). Let $d \geq 1$. Let

$$1 < p < \frac{d+2}{d-2} \quad (p > 1 \text{ for } d = 1, 2) \quad \text{and} \quad p \neq 1 + \frac{4}{d}.$$

There exists an H^1 solution of (NLS) such that $|z_1(t) - z_2(t)| \sim 2 \log t$ as $t \rightarrow +\infty$ and

$$\left\| u(t) - e^{i\gamma(t)} \sum_{k=1}^2 Q(\cdot - z_k(t)) \right\|_{H^1(\mathbb{R}^d)} \rightarrow 0.$$

It is important that the two solitons have the same sign, the same scaling and the same phase, in fact, the solution is symmetric. Remark that this situation is similar to multi-solitons constructed in [38, 49] for the integrable case.

The logarithmic distance is due to strong interactions between solitary waves since the interactions appear at the main order of the modulation equations. Indeed, the dynamics in Theorem 1.2.1 is related in some sense to the simple differential equation $\ddot{z} = -e^{-z(t)}$, where $z(t)$ is the distance between the solitons, and for which $2 \log t$ is a special solution.

The result holds for any space dimension and any \dot{H}^1 sub-critical nonlinearity, except the mass critical power $p = 1 + \frac{4}{d}$. Technically, the case $1 < p \leq 2$ is more involved because the L^∞ norm of the interaction is worse ($\sim s^{-p}$ so we can only integrate once), which requires the construction of a refined ansatz.

For the critical case ($p = 1 + \frac{4}{d}$), the existence of multi-solitons with logarithmic distances is not expected. Indeed, for such solutions, one would have

$$\int_{\mathbb{R}^d} |x|^2 |u(t, x)|^2 dx \sim \log^2(t)$$

which is in contradiction with the virial identity

$$\frac{d^2}{dt^2} \int_{\mathbb{R}^d} |x|^2 |u|^2 = 16E(u_0).$$

In this case, the *scaling instability directions are excited by the nonlinear interactions first which leads to the infinite time concentration* described in Theorem 1.1.15 ([32]) while on subcritical and supercritical case, the interactions primarily affect the velocity parameter.

The proof is based on the construction of an approximate solution of (NLS) composed of two symmetric modulated solitons moving in opposite direction plus a small correction term:

$$u(t, x) = \frac{e^{i\gamma(s)}}{\lambda^{\frac{2}{p-1}}(s)} \left[e^{iv_1(s)(\cdot - z_1)} Q(\cdot - z_1(s)) + e^{iv_2(s)(\cdot - z_2)} Q(\cdot - z_2(s)) + \epsilon(s) \right] \left(\frac{x}{\lambda(s)} \right)$$

with

$$s = \int_0^t \frac{1}{\lambda^2(\tau)} d\tau, \quad v_1(s) = -v_2(s) = \frac{1}{2}v(s), \quad z_1(s) = -z_2(s) = \frac{1}{2}z(s).$$

We follow the standard modulation argument for solution which was developed in [27, 35, 42] for (NLS) equation. Using the orthogonality conditions related to the generalized null space of the linearized equation, we derive a simplified modulation system with forcing term and we determine one of its approximate solutions that is relevant for the regime. Formally, we have the following estimates

$$\left| \frac{\dot{\lambda}}{\lambda} \right| + \left| \dot{z} - 2v + \frac{\dot{\lambda}}{\lambda} z \right| \lesssim |z|^{-\frac{d-1}{2}} e^{-|z|}. \quad (1.2.2)$$

Especially, the equation for \dot{v} contains forcing terms due to the nonlinear interaction of the solitons:

$$\left| \dot{v} + c \frac{z}{|z|} |z|^{-\frac{d-1}{2}} e^{-|z|} \right| \lesssim |z|^{-\frac{d-1}{2}-1} e^{-|z|} \quad (1.2.3)$$

for some constant $c > 0$. It can be checked that closed to real functions $z_{mod}(s)$, $\lambda_{mod}(s)$, $v_{mod}(s)$ such that

$$\lambda_{mod}^{-1}(s) = 1, \quad v_{mod}(s) = s^{-1}, \quad z_{mod}^{-\frac{d-1}{2}} e^{-z_{mod}} = \frac{s^{-2}}{c} \quad (1.2.4)$$

there exists a particular solution of (1.2.2)-(1.2.3). We have the asymptotics as $s \rightarrow +\infty$

$$\begin{aligned} z_{mod}(s) &\sim 2 \log(s), & \dot{v}_{mod}(s) &= -c z_{mod}^{-\frac{d-1}{2}}(s) e^{-z_{mod}(s)}, \\ | \dot{z}_{mod}(s) - 2v_{mod}(s) | &\lesssim s^{-1} \log^{-1}(s), & | \dot{v}_{mod}(s) | &\lesssim s^{-2}. \end{aligned} \quad (1.2.5)$$

This suggests that (1.2.4) is close to the first order asymptotics as $s \rightarrow +\infty$ for some solutions of (NLS) and matches the regime of Theorem 1.2.1.

1.2.2 Strongly interacting multi-solitons with logarithmic relative distance for the gKdV equation.

We also proved the existence of *strongly interacting multi-solitons with logarithmic relative distance for*

$$\partial_t u + \partial_x (\partial_x^2 u + |u|^{p-1} u) = 0, \quad t, x \in \mathbb{R}. \quad (1.2.6)$$

Theorem 1.2.7 (Multi-solitary waves with logarithmic distance, [37]). Let p integer, $p \neq 5$ and $p > 2$. There exist $t_0 > 0$ and an H^1 solution on $[t_0, +\infty)$ of (1.2.6) which decomposes asymptotically into two solitary waves

$$\left\| u(t) - \left(Q(\cdot - t - \log(ct)) + \sigma Q(\cdot - t + \log(ct)) \right) \right\|_{H^1} \rightarrow 0 \quad \text{as } t \rightarrow +\infty, \quad (1.2.8)$$

where $c = c(p) > 0$ a constant, $\sigma = -1$ in sub-critical cases ($2 < p < 5$) and $\sigma = 1$ in super-critical cases ($p > 5$).

The negative sign in the subcritical case ($p < 5$) means that in this case the interaction of solitons is repulsive while the positive sign in the supercritical case ($p > 5$) means that in this case the interaction is attractive. Remark that the relative distance of the two traveling waves is $2 \log(ct)$ with c given by

$$c = c(p) = \sqrt{\frac{8(p-1)}{|5-p|}} (2p+2)^{\frac{1}{p-1}} \|Q\|_{L^2}^{-1}.$$

In [20], it was later proved that this is the unique regime of logarithmic relative distance for this scaling in the unstable context ($p > 5$).

Because of the structure of the null space of the linearized equation, the equation for the position parameter $z_1(t), z_2(t)$ is not quadratic in the error term $\epsilon(t)$. Therefore, a refined part of the approximate solution corresponding to the nonlinear interaction is used to close the bootstrap

$$e^{-z(t)} (A_1(\cdot - z_1(t)) + A_2(\cdot - z_2(t))) \varphi$$

where φ is some cut-off function. Indeed without this term, one would obtain the remainder term of size t^{-2} and ϵ of size t^{-1} , which is not sharp enough to exploit the modulation equations. Since the extra term A_1, A_2 may not be in H^1 (it may have nonzero limits at $-\infty$), we have to introduce a suitable cut-off function φ .

1.2.3 Construction of 2-solitons with logarithmic distance for the one-dimensional cubic Schrödinger system.

Consider the coupled system of 1D cubic NLS equations written in the form

$$\begin{aligned} i\partial_t u + \partial_x^2 u + (|u|^2 + \omega|v|^2) u &= 0 \\ i\partial_t v + \partial_x^2 v + (|v|^2 + \omega|u|^2) v &= 0, \quad t \in \mathbb{R}, x \in \mathbb{R} \\ u(0, x) = u_0, \quad v(0, x) &= v_0. \end{aligned} \tag{coupled NLS}$$

Note that when $v = 0$ (or $u = 0$), the (coupled NLS) simplifies as (NLS)

$$i\partial_t u + \Delta u + |u|^2 u = 0.$$

Therefore, (coupled NLS) admits some special solutions of the form:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} e^{i\gamma_u + i\lambda_u^2 t + i\frac{\beta_u}{2}(x - \frac{\beta_u}{2}t)} Q_{\lambda_u}(x - \sigma_u - \beta_u t) \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ e^{i\gamma_v + i\lambda_v^2 t + i\frac{\beta_v}{2}(x - \frac{\beta_v}{2}t)} Q_{\lambda_v}(x - \sigma_v - \beta_v t) \end{pmatrix}$$

for any $\lambda_j > 0, \sigma_j, \beta_j, \gamma_j \in \mathbb{R}$, ($j = u, v$). We construct a **logarithmic 2-solitary waves in the symmetric case**, i.e., $u(t, x) = v(t, -x)$ and then a **new logarithmic dynamics of 2-solitary waves in the case where the solitons are non-symmetric** ⁴

Theorem 1.2.9 (Two families of logarithmic multi-solitary waves, [31]). Let

$$0 < \omega < 1$$

⁴see the summary on possible 2-soliton behaviors of integrable (NLS) equation in the previous section for a comparison.

(i) There exists a symmetric global H^1 solution of (coupled NLS) which decomposes into two solitary waves

$$\left\| \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} - e^{\gamma(t)} \begin{pmatrix} Q(\cdot - \frac{1}{2} \log(\Omega t) - \frac{1}{4} \log(\log t)) \\ Q(\cdot + \frac{1}{2} \log(\Omega t) + \frac{1}{4} \log(\log t)) \end{pmatrix} \right\|_{H^1 \times H^1} \rightarrow 0 \quad \text{as } t \rightarrow +\infty. \quad (1.2.10)$$

The distance between two solitary waves is $\log t + \frac{1}{2} \log(\log t) + \log(\Omega) + o(1)$.

(ii) Let

$$\omega < \frac{c(c+1)}{2} < 1,$$

there exists a global H^1 solution of (coupled NLS) which decomposes into two solitary waves of scaling c and 1 respectively

$$\left\| \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} - \begin{pmatrix} e^{ic^2t} Q_c(\cdot - \frac{1}{(1+c)c} \log(\Omega_c t)) \\ e^{it} Q(\cdot + \frac{1}{1+c} \log(\Omega_c t)) \end{pmatrix} \right\|_{H^1 \times H^1} \rightarrow 0 \quad \text{as } t \rightarrow +\infty. \quad (1.2.11)$$

The distance between two solitary waves is $\frac{1}{c} \log t + \frac{\log(\Omega_c)}{c} + o(1)$.

To our knowledge, the second part gives the first example of such dynamics for multi-solitary waves (logarithmic relative distance without symmetry). This new logarithmic behavior for non-symmetric case exists due to *the non-integrability property of the interactions between solitary waves*. There is a nonlinear resonance which breaks the free regime of the solitons and forces them to separate and to create the dynamics (1.2.11). This is in contrast with the integrable equations where periodic solutions exist (the behavior (c)). The forcing terms appear at *the quadratic order* of the nonlinear interactions. Remark that the two logarithmic regimes (in the symmetric case and non-symmetric case) are slightly different.

The question of *the non-symmetric case for the non-integrable scalar (NLS) is still an open problem*. In fact, our construction of the approximate solution works equally well for (NLS) as for (coupled NLS). We also obtain the nonlinear resonance and formally, the two solitons separate to form **a regime "logarithmic with an oscillation"**. However, conservation laws are less favorable and we miss some arguments to close the energy method.

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Chapter 2

Existence of multi-solitary waves with logarithmic relative distances for the NLS equation

Abstract

We construct 2-solitary wave solutions with logarithmic distance of the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u + |u|^{p-1}u = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^d,$$

in mass sub-critical cases $1 < p < 1 + \frac{4}{d}$ and mass super-critical cases $1 + \frac{4}{d} < p < \frac{d+2}{d-2}$, i.e. solutions $u(t)$ satisfying

$$\left\| u(t) - e^{i\gamma(t)} \sum_{k=1}^2 Q(\cdot - x_k(t)) \right\|_{H^1} \rightarrow 0$$

and

$$|x_1(t) - x_2(t)| \sim 2 \log t, \quad \text{as } t \rightarrow +\infty,$$

where Q is the ground state. The logarithmic distance is related to strong interactions between solitary waves.

In the integrable case ($d = 1$ and $p = 3$), the existence of such solutions is known by inverse scattering, [19, 33]. The mass critical case $p = 1 + \frac{4}{d}$ exhibits a specific behavior related to blow-up, previously studied in [24].

2.1 Introduction

We consider the nonlinear Schrödinger equation in \mathbb{R}^d , for any $d \geq 1$:

$$\begin{cases} i\partial_t u = -\Delta u - |u|^{p-1}u, & (t, x) \in [0, T) \times \mathbb{R}^d \\ u(0, x) = u_0, & u_0 \in H^1 : \mathbb{R}^d \rightarrow \mathbb{C}. \end{cases} \quad (\text{NLS})$$

It is well-known (see e.g. [2], [10]) that the (NLS) equation is locally well-posed in $H^1(\mathbb{R}^d)$ for $1 < p < \frac{d+2}{d-2}$ ($p > 1$ if $d = 1, 2$): for any $u_0 \in H^1(\mathbb{R}^d)$, there exist $T^* > 0$ and a unique maximal solution $u \in \mathcal{C}([0, T^*), H^1(\mathbb{R}^d))$ of (NLS). Moreover, the following blow up criterion holds

$$T^* < +\infty \text{ implies } \lim_{t \uparrow T^*} \|\nabla u(t)\|_{L^2} = +\infty. \quad (2.1.1)$$

Recall that the solution u satisfies the following three conservation laws:

- Mass :

$$\int_{\mathbb{R}^d} |u(t, x)|^2 dx = \int_{\mathbb{R}^d} |u_0(x)|^2 dx \quad (2.1.2)$$

- Energy :

$$E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(t, x)|^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^d} |u(t, x)|^{p+1} dx = E(u_0) \quad (2.1.3)$$

- Momentum:

$$M(u(t)) = \text{Im} \int_{\mathbb{R}^d} \nabla u(t, x) \bar{u}(t, x) dx = M(u_0) \quad (2.1.4)$$

for all $t \in [0, T^*)$. Recall also that (NLS) admits the following symmetries: the transformation of initial data implies the corresponding transformation of solution:

- Scaling: $\lambda > 0, \lambda^{\frac{2}{p-1}} u_0(\lambda x) \mapsto \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$;
- Space translation: $x_0 \in \mathbb{R}^d, u_0(x + x_0) \mapsto u(t, x + x_0)$;
- Time translation: $t_0 \in \mathbb{R}, u_{t_0}(x) \mapsto u(t + t_0, x)$;
- Space rotation: $A \in SO(d), u_0(A \cdot x_0) \mapsto u(t, A \cdot x_0)$;
- Phase: $\gamma \in \mathbb{R}, u_0(x) e^{i\gamma} \mapsto u(t, x) e^{i\gamma}$;
- Galilean: $\beta \in \mathbb{R}^d, u_0(x) e^{i\beta x} \mapsto u(t, x - \beta t) e^{i\beta(x - \frac{\beta}{2}t)}$.

As a consequence of (2.1.2), (2.1.3) and the Gagliardo-Nirenberg inequality, all solutions of (NLS) are global in the L^2 sub-critical case ($1 < p < 1 + \frac{4}{d}$). In contrast, blow-up solutions exist in the L^2 critical case ($p = 1 + \frac{4}{d}$) and the L^2 super-critical case ($1 + \frac{4}{d} < p < \frac{d+2}{d-2}$). See e.g. [2].

This article is concerned with the construction of special solutions of (NLS) involving solitary wave solutions (or solitons). We recall the expression of the (standing) solitary waves

$$u(t, x) = e^{i\lambda_0^2 t} Q_{\lambda_0}(x) \quad \text{with} \quad Q_{\lambda_0}(x) = \lambda_0^{\frac{2}{p-1}} Q(\lambda_0 x)$$

where $\lambda_0 > 0$ and Q is the ground state, *i.e.* the unique radial positive solution of

$$\Delta Q - Q + Q^p = 0, \quad Q > 0, \quad Q \in H^1(\mathbb{R}^d). \quad (2.1.5)$$

The whole family of ground state solitary waves is obtained using the above symmetries. For more properties of Q , see for example [2] and [28]. Recall that in the L^2 sub-critical case, the solitary waves are stable ([2, 31]) and in the L^2 critical and L^2 super-critical cases, the solitary waves are unstable ([2, 12]).

2.1.1 Motivation

So far, the problem of multi-solitary wave solutions for (NLS) has been studied intensively in the integrable case, i.e. for $d = 1$ and $p = 3$, as well as for some nearly integrable models; see [9, 11, 16, 19, 32, 33]. In particular, it is known from the inverse scattering theory that there are three possible 2-soliton behaviors in the integrable case:

- (a) relative distance between solitons of order t , [33];
- (b) logarithmic relative distance with symmetric solitons (*double-pole* solutions), [19, 33];
- (c) finite relative distance periodic in time, [32, 33].

Note that (a) corresponds to a free Galilean motion, while (b) and (c) correspond to a non free Galilean motion. Remarkably, these solutions admit a pure 2-soliton behavior both for $t \rightarrow +\infty$ and $t \rightarrow -\infty$. They describe perfectly elastic interactions between solitary waves in the integrable case.

In the non-integrable cases, the problem is much less comprehended except for multi-solitary waves with free Galilean motion (a), in one direction of time; see Remark 2.1.8 below for a precise statement. In the present paper, we raise the question of other possible behaviors of multi-solitons. In other words, we ask whether the above non generic dynamics (b), (c) of the integrable case persist for non-integrable models. Previous works, see *e.g.* [8, 11, 16], study formally the dynamics of interacting pulses for several integrable or non-integrable models, and predict the persistence of the logarithmic regime. Indeed, the 2-soliton dynamics is related in some sense to the simple differential equation $\ddot{z}(t) = -e^{-2z(t)}$, where $z(t)$ is half of the distance between the solitons, and for which $\log t$ is a special solution. The main point of the present work is to justify that 2-solitons with logarithmic relative distance is a universal behavior in both sub-critical and super-critical (NLS) in presence of symmetry, thus proving rigorously the persistence of the behavior (b) in the non-integrable case.

2.1.2 Main result

In this article, we prove the following general existence result.

[(Multi-solitary waves with logarithmic distance). Let $d \geq 1$. Let

$$1 < p < \frac{d+2}{d-2} \quad (p > 1 \text{ for } d = 1, 2) \quad \text{and} \quad p \neq 1 + \frac{4}{d}.$$

There exists an H^1 solution $u(t)$ of (NLS) on $[0, +\infty)$ which decomposes asymptotically into two solitary waves, for all $t > 0$,

$$\left\| u(t) - e^{i\gamma(t)} \sum_{k=1}^2 Q(\cdot - x_k(t)) \right\|_{H^1(\mathbb{R}^d)} \lesssim \frac{1}{t} \quad (2.1.6)$$

where $x_1(t) = -x_2(t)$ and

$$|x_1(t) - x_2(t)| = 2(1 + o(1)) \log t, \quad \text{as } t \rightarrow +\infty. \quad (2.1.7)$$

The Main Theorem holds for any space dimension and any \dot{H}^1 sub-critical nonlinearity, except the mass critical power $p = 1 + \frac{4}{d}$. Indeed, the critical nonlinearity exhibits a different phenomenon of strong interactions due to blow-up, previously studied in [24]; see Remark 2.1.10.

Note that the result should hold with a similar proof for any number $K \geq 2$ of solitons located on a regular polygon of size $\log t$. By scaling invariance, we can replace Q in (2.1.6) by Q_{λ_0} for any $\lambda_0 > 0$. We observe that in the result, solitons need to have the same sign, the same scaling and the same phase; in fact, the solution is symmetric by $\tau : x \mapsto -x$. Moreover, the solution is also symmetric by the reflection across the axis passing by the center of the two solitons. Remark that the situation is the same with the multi-solitons constructed in [19, 33] for the integrable case.

Remark 2.1.8. For the (NLS) equation, multiple solitary wave solutions with weak interactions, *i.e.* relative distance between solitons of order t , have been constructed in various settings, both in stable and unstable contexts, see in particular [4, 20, 25]. A typical result of weakly interacting dynamics is the existence of multi-solitary wave solutions of (NLS) satisfying as $t \rightarrow +\infty$,

$$\left\| u(t) - \sum_{k=1}^K e^{-i\Gamma_k(t,x)} Q_{\lambda_k}(\cdot - \nu_k t) \right\|_{H^1(\mathbb{R}^d)} \lesssim e^{-ct}, \quad c > 0, \quad (2.1.9)$$

for any given set of parameters $\{\nu_k, \lambda_k\}_k \in \mathbb{R}^d \times (0, \infty)$ provided that the following decoupling condition holds: $\nu_k \neq \nu_{k'}$ if $k \neq k'$.

Remark 2.1.10. For the L^2 critical case ($p = 1 + \frac{4}{d}$), the existence of bounded multi-solitary wave solutions with logarithmic distances as (2.1.6)–(2.1.7) is ruled out. Indeed, for such solutions, one would have

$$\int_{\mathbb{R}^d} |x|^2 |u(t, x)|^2 dx \sim \log^2(t) \quad (2.1.11)$$

which is in contradiction with the virial identity

$$\frac{d^2}{dt^2} \int_{\mathbb{R}^d} |x|^2 |u|^2 = 16E(u_0).$$

In fact, in the mass critical case, the scaling instability directions are excited by the nonlinear interactions which leads to the infinite time concentration as shown by Theorem 1 in [24]: for the L^2 critical two dimensional case, there exists a global (for $t \geq 0$) solution $u(t)$ that decomposes asymptotically into a sum of solitary waves

$$\left\| u(t) - e^{i\gamma(t)} \sum_{k=1}^K \frac{1}{\lambda(t)} Q\left(\frac{\cdot - x_k(t)}{\lambda(t)}\right) \right\|_{H^1(\mathbb{R}^d)} \rightarrow 0, \quad \lambda(t) = \frac{1 + o(1)}{\log t} \quad \text{as } t \rightarrow +\infty, \quad (2.1.12)$$

where the translation parameters $x_k(t)$ converge to the vertices of a K -sided regular polygon and the solution blows up in infinite time with the rate

$$\|\nabla u(t)\|_{L^2} \sim |\log t| \quad \text{as } t \rightarrow +\infty.$$

The regime justified in the present paper is thus different from the one in [24] since for the critical case, the interactions primarily affect the scaling parameter to lead to blow-up. This notable difference with the sub and super-critical cases shows that a formal approach may not be sufficient to correctly address such subtle regimes.

We also refer to [14, 15, 17, 18, 22] for previous works on other nonlinear equations where a refined analysis of interactions between solitons is a key point.

Remark 2.1.13. We expect solutions in Main Theorem to be unstable, even in L^2 sub-critical cases, since generic perturbation can give collision or on the contrary weak interaction. Recall that the appearance of the log regime is closely related to the equation

$$\ddot{z}(t) = -e^{-2z(t)}$$

where $\log t$ is a solution with initial conditions $z(1) = 0$, $\dot{z}(1) = 1$. From the theory of perturbation, for $z(t) = \log t + \epsilon v_1 + \dots$ with initial conditions $z(1) = \epsilon$, $\dot{z}(1) = 1$, one has at the linear level

$$\ddot{v}_1 = \frac{2v_1}{t^2}, \quad v_1(1) = 1, \quad \dot{v}_1(1) = 0,$$

whose solution is $\frac{1}{3}t^2 + \frac{2}{3}\frac{1}{t}$ so we see that the $\log t$ solution is an unstable state as $t \rightarrow +\infty$.

Remark 2.1.14. We believe that our approach is general. In particular, the strategy of this article can be applied to construct multi-solitary waves with logarithmic relative distance for more general nonlinearity $f(s)$

$$i\partial_t u + \Delta u + f(|u|^2)u = 0$$

where $f(s)$ satisfies standard conditions for the existence of solitary waves (see [23]). Moreover, combining the construction in this paper and the construction of multi-soliton solutions with weak interactions in [4], [20], we prove the existence of multi-solitons, with both solitons distant as Ct and solitons distant as $C \log t$.

Remark 2.1.15. One can give a more precise asymptotic description of the distance (2.1.7) between solitons

$$|x_1(t) - x_2(t)| = 2 \log t - \frac{d-1}{2} \log(\log t) - C + O(\log^{-\frac{1}{2}}(t)) \quad \text{as } t \rightarrow +\infty$$

where $C > 0$ a constant depending only on d and p (see (2.3.25)).

The article is organized as follows. Sections 2.2, 2.3 and 2.4 concern the proof of Main Theorem in L^2 sub-critical cases with $p > 2$. In Section 2.2, we consider an approximate solution (an ansatz solution) to (NLS) made of two symmetric bubbles and extract the formal evolution system of the geometrical parameters of the bubbles (scaling, position, phase). The key observation is that this system contains forcing terms due to the nonlinear interactions of the waves, and has a special solution corresponding at the main order to the regime of Main Theorem. Here, in contrast with free Galilean motion, the construction of a non free Galilean motion as (2.1.7) requires a refined control of strong interactions between the solitary waves to bound the error terms. In Section 2.3, we consider, using modulation, particular backwards solutions of (NLS) related to the special regime of Main Theorem and prove backward uniform estimates by energy method. In Section 2.4, we use compactness arguments on a suitable sequence of such backwards solutions to finish the proof. Section 2.5 deals with the case $1 < p \leq 2$; in this case, there are some extra technical difficulties, even if the strategy of the proof is similar: the interaction becomes stronger, we have to add extra terms in the approximate solution and due to lost of regularity, we have to use some truncations. Finally, the algebraic computations in the proof for L^2 sub-critical cases are still valid in L^2 super-critical cases, Section 2.6 presents additional arguments and modifications needed for L^2 super-critical cases.

2.1.3 Notation

The L^2 scalar product of two complex valued functions $f, g \in L^2(\mathbb{R}^d)$ is denoted by

$$\langle f, g \rangle = \operatorname{Re} \left(\int_{\mathbb{R}^d} f(x) \bar{g}(x) dx \right).$$

We denote by $Q(x) := q(|x|)$ the unique radial positive ground state of (NLS):

$$q'' + \frac{d-1}{r}q' - q + q^p = 0, \quad q'(0) = 0, \quad \lim_{r \rightarrow +\infty} q(r) = 0. \quad (2.1.16)$$

It is well-known and easily checked by ODE arguments that for some constant $c_Q > 0$,

$$\text{for all } r > 1, \quad \left| q(r) - c_Q r^{-\frac{d-1}{2}} e^{-r} \right| + \left| q'(r) + c_Q r^{-\frac{d-1}{2}} e^{-r} \right| \lesssim r^{-\frac{d-1}{2}-1} e^{-r}. \quad (2.1.17)$$

We set

$$I_Q = \int Q^p(x) e^{-x_1} dx, \quad x = (x_1, \dots, x_d).$$

We denote by \mathcal{Y} the set of smooth functions f such that

$$\text{for all } p \in \mathbb{N}, \text{ there exists } q \in \mathbb{N}, \text{ s.t. for all } x \in \mathbb{R}^d, \quad |f^{(p)}(x)| \lesssim |x|^q e^{-|x|}. \quad (2.1.18)$$

Let Λ be the generator of L^2 -scaling corresponding to (NLS):

$$\Lambda f = \frac{2}{p-1} f + x \cdot \nabla f.$$

The linearization of (NLS) around Q involves the following Schrödinger operators:

$$L_+ := -\Delta + 1 - pQ^{p-1}, \quad L_- := -\Delta + 1 - Q^{p-1}.$$

From [30], recall the generalized null space relations in sub-critical and super-critical cases:

$$\begin{aligned} L_- Q &= 0, & L_+(\Lambda Q) &= -2Q, \\ L_+(\nabla Q) &= 0, & L_-(xQ) &= -2\nabla Q. \end{aligned} \quad (2.1.19)$$

We recall the coercivity property in L^2 sub-critical (see [20], [26], [30], [31]): there exists $\mu > 0$ such that for all $\eta \in H^1$,

$$\langle L_+ \operatorname{Re} \eta, \operatorname{Re} \eta \rangle + \langle L_- \operatorname{Im} \eta, \operatorname{Im} \eta \rangle \geq \mu \|\eta\|_{H^1}^2 - \frac{1}{\mu} (\langle \eta, Q \rangle^2 + |\langle \eta, xQ \rangle|^2 + \langle \eta, i\Lambda Q \rangle^2). \quad (2.1.20)$$

In L^2 super-critical (but H^1 sub-critical), we do not have the same situation since the negative direction can not be controlled by the scaling parameter. We consider the operator

$$\mathcal{L}v = iL_+v_1 - L_-v_2 \quad \text{with } v = v_1 + iv_2.$$

The spectrum $\sigma(\mathcal{L})$ of \mathcal{L} satisfies

$$\sigma(\mathcal{L}) \cap \mathbb{R} = \{-e_0, 0, e_0\}.$$

It is easy to see that $iQ, \nabla Q$ are independent and belong to the kernel of \mathcal{L} . In [4], [6], [7], [13], it is proved that there exist two eigenfunctions Y^\pm (normalized by $\|Y^\pm\|_{L^2} = 1$) associated to eigenvalues $\pm e_0$

$$\mathcal{L}(Y^\pm) = \pm e_0 Y^\pm \quad (2.1.21)$$

and $Y^+ = \overline{Y^-}$ belong to \mathcal{Y} , in other words, $\operatorname{Re} Y^+, \operatorname{Im} Y^+ \in \mathcal{Y}$. Moreover, there holds a property of positivity based on Y^\pm : there exists $\mu > 0$ such that for all $\eta \in H^1$,

$$\begin{aligned} \langle L_+ \operatorname{Re} \eta, \operatorname{Re} \eta \rangle + \langle L_- \operatorname{Im} \eta, \operatorname{Im} \eta \rangle &\geq \mu \|\eta\|_{H^1}^2 \\ &- \frac{1}{\mu} (\langle \eta, iY^+ \rangle^2 + \langle \eta, iY^- \rangle^2 + |\langle \eta, xQ \rangle|^2 + \langle \eta, i\Lambda Q \rangle^2). \end{aligned} \quad (2.1.22)$$

2.2 Approximate solution for $p > 2$

2.2.1 System of modulation equations

Let $p > 2$. Consider a time dependent \mathcal{C}^1 function of parameters \vec{q} of the form

$$\vec{q} = (\lambda, z, \gamma, v) \in (0, +\infty) \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d,$$

with $|v| \ll 1$ and $|z| \gg 1$. We renormalize the flow by considering

$$u(t, x) = \frac{e^{i\gamma(s)}}{\lambda^{\frac{2}{p-1}}(s)} w(s, y), \quad dt = \lambda^2(s) ds, \quad y = \frac{x}{\lambda(s)}, \quad (2.2.1)$$

so that

$$i\partial_t u + \Delta u + |u|^{p-1}u = \frac{e^{i\gamma}}{\lambda^{2+\frac{2}{p-1}}} \left[i\dot{w} + \Delta w - w + |w|^{p-1}w - i\frac{\dot{\lambda}}{\lambda}\Lambda w + (1 - \dot{\gamma})w \right] \quad (2.2.2)$$

(\dot{w} denotes derivation with respect to s). We introduce the following \vec{q} -modulated ground state solitary waves, for $k \in \{1, 2\}$,

$$P_k(s, y) = e^{i\Gamma_k(s, y - z_k(s))} Q(y - z_k(s)) = e^{iv_k(s)(y - z_k(s))} Q(y - z_k(s)), \quad (2.2.3)$$

where we set

$$v_1(s) = -v_2(s) = \frac{1}{2}v(s), \quad z_1(s) = -z_2(s) = \frac{1}{2}z(s), \quad \Gamma_k(s, y) = v_k(s) \cdot y, \quad (2.2.4)$$

Let

$$\mathbf{P}(s, y) = \mathbf{P}(y; (z(s), v(s))) = \sum_{k=1}^2 P_k(s, y). \quad (2.2.5)$$

Then, \mathbf{P} is an approximate solution of the rescaled equation in the following sense.

Lemma 2.2.6 (Leading order approximate flow). Let the vectors of modulation equations be

$$\vec{m}_k = \begin{pmatrix} \frac{\dot{\lambda}}{\lambda} \\ \dot{z}_k - 2v_k + \frac{\dot{\lambda}}{\lambda}z_k \\ \dot{\gamma} - 1 + |v_k|^2 - \frac{\dot{\lambda}}{\lambda}(v_k \cdot z_k) - (v_k \cdot \dot{z}_k) \\ \dot{v}_k - \frac{\dot{\lambda}}{\lambda}v_k \end{pmatrix}, \quad \vec{M}V = \begin{pmatrix} -i\Lambda V \\ -i\nabla V \\ -V \\ -yV \end{pmatrix}. \quad (2.2.7)$$

Then the error $\mathcal{E}_{\mathbf{P}}$ to the re-normalized flow (2.2.2) at \mathbf{P} ,

$$\mathcal{E}_{\mathbf{P}} = i\dot{\mathbf{P}} + \Delta \mathbf{P} - \mathbf{P} + |\mathbf{P}|^{p-1}\mathbf{P} - i\frac{\dot{\lambda}}{\lambda}\Lambda \mathbf{P} + (1 - \dot{\gamma})\mathbf{P} \quad (2.2.8)$$

decomposes as

$$\mathcal{E}_{\mathbf{P}} = [e^{i\Gamma_1} \vec{m}_1 \cdot \vec{M}Q](y - z_1(s)) + [e^{i\Gamma_2} \vec{m}_2 \cdot \vec{M}Q](y - z_2(s)) + G \quad (2.2.9)$$

where the interaction term $G = |\mathbf{P}|^{p-1}\mathbf{P} - |P_1|^{p-1}P_1 - |P_2|^{p-1}P_2$ satisfies

$$\|G\|_{L^\infty} \lesssim |z|^{-\frac{d-1}{2}} e^{-|z|}, \quad \|\nabla G\|_{L^\infty} \lesssim |z|^{-\frac{d-1}{2}} e^{-|z|}. \quad (2.2.10)$$

Proof of Lemma 2.2.6. Firstly, we compute $\mathcal{E}_{P_k} = i\dot{P}_k + \Delta P_k - P_k + |P_k|^{p-1}P_k - i\frac{\dot{\lambda}}{\lambda}\Lambda P_k + (1-\dot{\gamma})P_k$. Let $y_{z_k} = y - z_k$, by computations

$$\begin{aligned} i\dot{P}_k &= \left[-(\dot{v}_k \cdot y_{z_k})Q(y_{z_k}) + (v_k \cdot \dot{z}_k)Q(y_{z_k}) - i\dot{z}_k \cdot \nabla Q(y_{z_k}) \right] e^{iv_k \cdot y_{z_k}} \\ \nabla P_k &= \left[\nabla Q(y_{z_k}) + iv_k Q(y_{z_k}) \right] e^{iv_k \cdot y_{z_k}} \\ \Delta P_k &= \left[\Delta Q(y_{z_k}) + 2iv_k \cdot \nabla Q(y_{z_k}) - v_k^2 Q(y_{z_k}) \right] e^{iv_k \cdot y_{z_k}} \\ \Lambda P_k &= \left[\frac{2}{p-1}Q(y_{z_k}) + y \cdot [\nabla Q(y_{z_k}) + iv_k Q(y_{z_k})] \right] e^{iv_k \cdot y_{z_k}} \\ &= \left[\Lambda Q(y_{z_k}) + iv_k \cdot y_{z_k} Q(y_{z_k}) + iv_k \cdot z_k Q(y_{z_k}) + z_k \cdot \nabla Q(y_{z_k}) \right] e^{iv_k \cdot y_{z_k}}. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \mathcal{E}_{P_k} &= \left[-i\frac{\dot{\lambda}}{\lambda}\Lambda Q(y_{z_k}) - i(\dot{z}_k - 2v_k + z_k \frac{\dot{\lambda}}{\lambda}) \cdot \nabla Q(y_{z_k}) \right. \\ &\quad - (\dot{\gamma} - 1 - v_k \cdot \dot{z}_k + |v_k|^2 - v_k \cdot z_k \frac{\dot{\lambda}}{\lambda})Q(y_{z_k}) \\ &\quad \left. - (\dot{v}_k - v_k \frac{\dot{\lambda}}{\lambda}) \cdot y_{z_k} Q(y_{z_k}) + \Delta Q(y_{z_k}) - Q(y_{z_k}) + |Q(y_{z_k})|^{p-1}Q(y_{z_k}) \right] e^{i\Gamma_k(s, y-z_k)}. \end{aligned}$$

Since $\Delta Q - Q + |Q|^{p-1}Q = 0$, we have

$$\mathcal{E}_{P_k} = [e^{i\Gamma_k} \vec{m}_k \cdot \vec{M}Q](y - z_k(s)). \quad (2.2.11)$$

Returning to the error of renormalized flow, we obtain

$$\mathcal{E}_{\mathbf{P}} = \mathcal{E}_{P_1} + \mathcal{E}_{P_2} + |\mathbf{P}|^{p-1}\mathbf{P} - \sum_{k=1}^2 |P_k|^{p-1}P_k. \quad (2.2.12)$$

Next, we estimate the interaction term $G = |\mathbf{P}|^{p-1}\mathbf{P} - |P_1|^{p-1}P_1 - |P_2|^{p-1}P_2$. Clearly,

$$|G| \lesssim |P_1|^{p-1}|P_2| + |P_2|^{p-1}|P_1|.$$

We observe that for $z = z_1 - z_2$, by (2.1.17),

$$Q(y)Q(y-z) \lesssim (1+|y|)^{-\frac{d-1}{2}}(1+|y-z|)^{-\frac{d-1}{2}}e^{-|y|}e^{-|z|+|y|} \lesssim |z|^{-\frac{d-1}{2}}e^{-|z|} \quad (2.2.13)$$

which yields

$$|P_1|^{p-1}|P_2| \lesssim |P_1||P_2||P_1|^{p-2} \lesssim |z|^{-\frac{d-1}{2}}e^{-|z|}|P_1|^{p-2}.$$

Thus,

$$|G(s, y)| \lesssim |z|^{-\frac{d-1}{2}}e^{-|z|} \sum_{k=1}^2 Q^{p-2}(y - z_k(s)) \quad (2.2.14)$$

and since $p > 2$, we get

$$\|G\|_{L^\infty} \lesssim |z|^{-\frac{d-1}{2}}e^{-|z|}. \quad (2.2.15)$$

Similarly, by (2.1.17) and as $|v| \ll 1$,

$$\|\nabla G\|_{L^\infty} \lesssim |z|^{-\frac{d-1}{2}}e^{-|z|}.$$

□

2.2.2 Nonlinear forcing

For the next parts of the article, we will need the first-order and the second-order approximations of $F(u) = |u|^{p-1}u$ where $u = a + ib$. We consider the expansion for $|u| \ll 1$

$$F(1+u) = 1 + pa + ib + \frac{p(p-1)}{2}a^2 + \frac{p-1}{2}b^2 + (p-1)iab + O(|u|^k) \quad (2.2.16)$$

for any $2 < k \leq 3$. From which, we can deduce formally

$$F'(\mathbf{P}).\epsilon = \frac{p+1}{2}|\mathbf{P}|^{p-1}\epsilon + \frac{p-1}{2}|\mathbf{P}|^{p-3}\mathbf{P}^2\bar{\epsilon} \quad (2.2.17)$$

and

$$\frac{\bar{\epsilon}.F''(\mathbf{P}).\epsilon}{2} = \frac{p-1}{2}\epsilon^2\bar{\mathbf{P}}|\mathbf{P}|^{p-3} + (p-1)|\epsilon|^2\mathbf{P}|\mathbf{P}|^{p-3} + (p-1)\left(\frac{p}{2} - \frac{3}{2}\right)(\operatorname{Re}(\epsilon\bar{\mathbf{P}}))^2\mathbf{P}|\mathbf{P}|^{p-5}.$$

In the case $p > 2$, set

$$2^+ = \min\left(3, \frac{p+2}{2}\right).$$

Remark that $2^+ < 2^*$ when $p > 2$ (where $2^* = \frac{2d}{d-2}$ is the critical exponent of the Sobolev injection). Then, from (2.2.16), we have

$$F(\mathbf{P} + \epsilon) = F(\mathbf{P}) + F'(\mathbf{P}).\epsilon + O(|\epsilon|^p) + O\left(\left|\frac{\epsilon}{\mathbf{P}}\right|^{2^+}|\mathbf{P}|^p\right) \quad (2.2.18)$$

and

$$F(\mathbf{P} + \epsilon) = F(\mathbf{P}) + F'(\mathbf{P}).\epsilon + \frac{\bar{\epsilon}.F''(\mathbf{P}).\epsilon}{2} + O(|\epsilon|^p) + O\left(\left|\frac{\epsilon}{\mathbf{P}}\right|^{2^+}|\mathbf{P}|^p\right) \quad (2.2.19)$$

(note that for $\left|\frac{\epsilon}{\mathbf{P}}\right| \gg 1$ we have $F(\mathbf{P} + \epsilon) \sim F(\epsilon)$).

Lemma 2.2.20 (Nonlinear interaction estimates). For $|z| \gg 1, |v| \ll 1$, let

$$H(z) = p \left[\int_{y \cdot \frac{z}{|z|} > -\frac{|z|}{2}} Q^{p-1}(y) \nabla Q(y) Q(y+z) dy + \int_{y \cdot \frac{z}{|z|} < -\frac{|z|}{2}} Q^{p-1}(y+z) \nabla Q(y) Q(y) dy \right]. \quad (2.2.21)$$

Then the following estimates hold:

$$\left| \langle G, e^{i\Gamma_1(y-z_1(s))} \nabla Q(y - z_1(s)) \rangle - H(z) \right| \lesssim (|v|^2|z|^2 + |v|^2)|z|^{-\frac{d-1}{2}} e^{-|z|} + |z|^{-\frac{3(d-1)}{4}} e^{-\frac{3}{2}|z|} \quad (2.2.22)$$

and

$$\left| H(z) - C_p \frac{z}{|z|} |z|^{-\frac{d-1}{2}} e^{-|z|} \right| \lesssim |z|^{-\frac{d-1}{2}-1} e^{-|z|} \quad (2.2.23)$$

where $C_p > 0$.

Remark 2.2.24. The estimate (2.2.23) on the leading order of the core part $H(z)$ of the projection $\langle G, [e^{i\Gamma_1} \nabla Q](y - z_1(s)) \rangle$ is valid not only in the case $p > 2$ but also in the case $1 < p \leq 2$.

Proof of Lemma 2.2.20. step 1 Nonlinear interaction estimates. We prove the estimate (2.2.23) and in this step we will have $p > 1$. Consider

$$H(z) = p \int_{y \cdot \frac{z}{|z|} < -\frac{|z|}{2}} Q^{p-1}(y+z) \nabla Q(y) Q(y) dy + p \int_{y \cdot \frac{z}{|z|} > -\frac{|z|}{2}} Q^{p-1}(y) \nabla Q(y) Q(y+z) dy.$$

Recall that

$$\begin{aligned} Q(y)Q(y+z) &\lesssim |z|^{-\frac{d-1}{2}} e^{-|z|} \\ Q(y)|\nabla Q(y+z)| &\lesssim |z|^{-\frac{d-1}{2}} e^{-|z|} \end{aligned}$$

then with $p > 2$, we have

$$\left| \int_{y \cdot \frac{z}{|z|} < -\frac{|z|}{2}} Q^{p-1}(y+z) \nabla Q(y) Q(y) dy \right| \lesssim e^{-\min(p-1, \frac{3}{2})|z|}$$

and with $1 < p \leq 2$, from the decay property of Q , we have for $\delta = \frac{p-1}{2}$

$$\begin{aligned} \left| \int_{y \cdot \frac{z}{|z|} < -\frac{|z|}{2}} Q^{p-1}(y+z) \nabla Q(y) Q(y) dy \right| &\lesssim e^{-(p-1)|z|} \left| Q\left(\frac{|z|}{2}\right) \right|^{3-p-\delta} \int Q^\delta(y) dy \\ &\lesssim e^{-\frac{p+3}{4}|z|}. \end{aligned}$$

We claim that

$$\begin{aligned} \left| \int_{y \cdot \frac{z}{|z|} > -\frac{|z|}{2}} Q^{p-1}(y) \nabla Q(y) Q(y+z) dy - c_Q |z|^{-\frac{d-1}{2}} e^{-|z|} \int Q^{p-1}(y) \nabla Q(y) e^{-y \cdot \frac{z}{|z|}} dy \right| \\ \lesssim |z|^{-1-\frac{d-1}{2}} e^{-|z|}. \end{aligned} \quad (2.2.25)$$

Indeed, let $0 < \theta < 1$ such that $p\theta > 1$. For $|y| \geq \theta|z|$, we have:

$$\begin{aligned} \left| \int_{\substack{|y| \geq \theta|z| \\ y \cdot \frac{z}{|z|} > -\frac{|z|}{2}}} Q^{p-1}(y) \nabla Q(y) Q(y+z) dy \right| &\lesssim e^{-p\theta|z|} \left| \int Q(y+z) dy \right| \\ &\lesssim e^{-p\theta|z|}. \end{aligned}$$

For $|y| < \theta|z|$, as $Q(x) = q(|x|)$ and $|q(r) - c_Q r^{-\frac{d-1}{2}} e^{-r}| \lesssim r^{-\frac{d-1}{2}-1} e^{-r}$, we have:

$$\begin{aligned} \left| Q(y+z) - c_Q |y+z|^{-\frac{d-1}{2}} e^{-|y+z|} \right| &\lesssim |y+z|^{-1-\frac{d-1}{2}} e^{-|y+z|} \\ &\leq |1-\theta| |z|^{-1-\frac{d-1}{2}} e^{-|z|} e^{|y|}. \end{aligned}$$

Thus we get:

$$\begin{aligned} \left| \int_{\substack{|y| < \theta|z| \\ y \cdot \frac{z}{|z|} > -\frac{|z|}{2}}} Q^{p-1}(y) \nabla Q(y) \nabla Q(y+z) dy - c_Q \int_{\substack{|y| < \theta|z| \\ y \cdot \frac{z}{|z|} > -\frac{|z|}{2}}} Q^{p-1}(y) \nabla Q(y) |y+z|^{-\frac{d-1}{2}} e^{-|y+z|} dy \right| \\ \lesssim |z|^{-1-\frac{d-1}{2}} e^{-|z|} \end{aligned}$$

since $\int Q^{p-1}(y) |\nabla Q(y)| e^{|y|} dy < +\infty$. On the other hand, $|y| < \theta|z|$ implies

$$\left| |y+z|^{-k} - |z|^{-k} \right| \lesssim |z|^{-1-k} |y|$$

for any $k > 0$ and

$$\left| \frac{|y+z|}{|y+z|} - \frac{|z|}{|z|} \right| \lesssim |z|^{-1} |y|.$$

Moreover

$$\left| |y+z| - |z| - y \cdot \frac{z}{|z|} \right| \lesssim |z|^{-1} |y|^2$$

then

$$\left| e^{-|y+z|} - e^{-|z| - y \cdot \frac{z}{|z|}} \right| \lesssim |z|^{-1} |y|^2 e^{-|z|} e^{|y|}.$$

Thus we obtain that

$$\left| |y+z|^{-\frac{d-1}{2}} e^{-|y+z|} - |z|^{-\frac{d-1}{2}} e^{-|z| - y \cdot \frac{z}{|z|}} \right| \lesssim (1+|y|^2) |z|^{-1-\frac{d-1}{2}} e^{-|z|} e^{|y|}.$$

Therefore we have

$$\begin{aligned} & \left| \int_{\substack{|y| < \theta|z| \\ y \cdot \frac{z}{|z|} > -\frac{|z|}{2}}} Q^{p-1}(y) \nabla Q(y) |y+z|^{-\frac{d-1}{2}} e^{-|y+z|} dy \right. \\ & \quad \left. - c_Q |z|^{-\frac{d-1}{2}} e^{-|z|} \int_{\substack{|y| < \theta|z| \\ y \cdot \frac{z}{|z|} > -\frac{|z|}{2}}} Q^{p-1}(y) \nabla Q(y) e^{-y \cdot \frac{z}{|z|}} dy \right| \lesssim |z|^{-1-\frac{d-1}{2}} e^{-|z|}. \end{aligned}$$

Next we observe that

$$|z|^{-\frac{d-1}{2}} e^{-|z|} \int_{\substack{|y| \geq \theta|z| \\ y \cdot \frac{z}{|z|} > -\frac{|z|}{2}}} Q^{p-1}(y) \nabla Q(y) e^{-y \cdot \frac{z}{|z|}} dy \lesssim e^{-p\theta|z|}$$

and by (2.1.17)

$$\left| \int_{y \cdot \frac{z}{|z|} < -\frac{|z|}{2}} Q^{p-1}(y) \nabla Q(y) e^{-y \cdot \frac{z}{|z|}} dy \right| \lesssim e^{-\frac{p-1}{4}|z|}$$

which finish the proof of (2.2.25). Finally, in order to obtain (2.2.23) with $C_p = c_Q I_Q$, we use integration by parts

$$p \int Q^{p-1}(y) \nabla Q(y) e^{-y \cdot \frac{z}{|z|}} dy = \frac{z}{|z|} \int Q^p(y) e^{-y \cdot \frac{z}{|z|}} dy$$

and remark from the parity of the integral that

$$\int Q^p(y) e^{-y \cdot \frac{z}{|z|}} dy = \int Q^p(y) e^{-y_1} dy = I_Q.$$

step 2 Error bound. Recall the interaction term

$$G = |\mathbf{P}|^{p-1} \mathbf{P} - |P_1|^{p-1} P_1 - |P_2|^{p-1} P_2.$$

From (2.2.16), we have the following estimates : if $y \cdot \frac{z}{|z|} > 0$ then $|P_1| > |P_2|$

$$\left| G - \frac{p+1}{2} |P_1|^{p-1} P_2 - \frac{p-1}{2} |P_1|^{p-3} P_1^2 \overline{P_2} \right| \lesssim |P_2|^2 |P_1|^{p-2} \quad (2.2.26)$$

and if $y \cdot \frac{z}{|z|} < 0$ then $|P_2| > |P_1|$

$$\left| G - \frac{p+1}{2} |P_2|^{p-1} P_1 - \frac{p-1}{2} |P_2|^{p-3} P_2^2 \overline{P_1} \right| \lesssim |P_1|^2 |P_2|^{p-2}. \quad (2.2.27)$$

We combine (2.2.26)–(2.2.27) to obtain, for all y ,

$$\begin{aligned} & \left| G - \left[\frac{p+1}{2} |P_1|^{p-1} P_2 + \frac{p-1}{2} |P_1|^{p-3} P_1^2 \overline{P_2} \right] \cdot \mathbf{1}_{y \cdot \frac{z}{|z|} > 0} - \left[\frac{p+1}{2} |P_2|^{p-1} P_1 \right. \right. \\ & \quad \left. \left. + \frac{p-1}{2} |P_2|^{p-3} P_2^2 \overline{P_1} \right] \cdot \mathbf{1}_{y \cdot \frac{z}{|z|} < 0} \right| \lesssim \min(|P_1|^2, |P_2|^2) \max(|P_1|^{p-2}, |P_2|^{p-2}). \end{aligned} \quad (2.2.28)$$

step 3 Projection estimates. Since $\min(|P_1|^2, |P_2|^2) \leq |P_2|^{\frac{3}{2}} |P_1|^{\frac{1}{2}}$ and $\max(|P_1|^{p-2}, |P_2|^{p-2}) \leq |P_1|^{p-2} + |P_2|^{p-2}$, we have

$$\begin{aligned} & \int Q^{\frac{3}{2}}(y-z) |\nabla Q(y)| Q^{\frac{1}{2}}(y) (Q^{p-2}(y) + Q^{p-2}(y+z)) dy \\ & \lesssim |z|^{-\frac{3(d-1)}{4}} e^{-\frac{3}{2}|z|} \int (Q^{p-2}(y) + Q^{p-2}(y+z)) dy \lesssim |z|^{-\frac{3(d-1)}{4}} e^{-\frac{3}{2}|z|} \end{aligned}$$

so we deduce from the error bound (2.2.28)

$$\begin{aligned} & \left| \langle G, [e^{i\Gamma_1} \nabla Q](y - z_1(s)) \rangle - \left\langle \left[\frac{p+1}{2} |P_1|^{p-1} P_2 + \frac{p-1}{2} |P_1|^{p-3} P_1^2 \overline{P_2} \right] \cdot \mathbf{1}_{y \cdot \frac{z}{|z|} > 0} \right. \right. \\ & \quad \left. \left. + \left[\frac{p+1}{2} |P_2|^{p-1} P_1 + \frac{p-1}{2} |P_2|^{p-3} P_2^2 \overline{P_1} \right] \cdot \mathbf{1}_{y \cdot \frac{z}{|z|} < 0}, [e^{i\Gamma_1} \nabla Q](y - z_1(s)) \right\rangle \right| \lesssim |z|^{-\frac{3(d-1)}{4}} e^{-\frac{3}{2}|z|}. \end{aligned} \quad (2.2.29)$$

Using a change of variables, we have

$$\begin{aligned} & \langle |P_1|^{p-1} P_2 \mathbf{1}_{y \cdot \frac{z}{|z|} > 0}, [e^{i\Gamma_1} \nabla Q](y - z_1(s)) \rangle \\ & = \operatorname{Re} \int_{y \cdot \frac{z}{|z|} > -\frac{|z|}{2}} Q^{p-1}(y) \nabla Q(y) Q(y - z_2 + z_1) e^{iv_2 \cdot (y - z_2 + z_1) - iv_1 \cdot y} dy \\ & = \int_{y \cdot \frac{z}{|z|} > -\frac{|z|}{2}} Q^{p-1}(y) \nabla Q(y) Q(y+z) \cos(v_2 \cdot (y+z) - v_1 \cdot y) dy \end{aligned}$$

with $z(s) = z_1(s) - z_2(s)$. Note that

$$|\cos(v_2 \cdot (y+z) - v_1 \cdot y) - 1| \lesssim |v|^2 |z|^2 + |v|^2 |y|^2$$

as the same method to prove (2.2.25), we get

$$\begin{aligned} & \left| \langle |P_1|^{p-1} P_2 \cdot \mathbf{1}_{y \cdot \frac{z}{|z|} > 0}, [e^{i\Gamma_1} \nabla Q](y - z_1(s)) \rangle - \int_{y \cdot \frac{z}{|z|} > -\frac{|z|}{2}} Q^{p-1}(y) \nabla Q(y) Q(y+z) dy \right| \\ & \lesssim (|v|^2 |z|^2 + |v|^2) |z|^{-\frac{d-1}{2}} e^{-|z|}. \end{aligned} \quad (2.2.30)$$

Similarly, for the other projections, we have

$$\begin{aligned} & \left| \langle |P_1|^{p-3} P_1^2 \overline{P_2} \cdot \mathbf{1}_{y \cdot \frac{z}{|z|} > 0}, [e^{i\Gamma_1} \nabla Q](y - z_1(s)) \rangle - \int_{y \cdot \frac{z}{|z|} > -\frac{|z|}{2}} Q^{p-1}(y) \nabla Q(y) Q(y+z) dy \right| \\ & \lesssim (|v|^2 |z|^2 + |v|^2) |z|^{-\frac{d-1}{2}} e^{-|z|} \end{aligned} \quad (2.2.31)$$

$$\begin{aligned} & \left| \langle |P_1|^{p-3} P_1^2 \overline{P_2} \cdot \mathbf{1}_{y \cdot \frac{z}{|z|} < 0}, [e^{i\Gamma_1} \nabla Q](y - z_1(s)) \rangle - \int_{y \cdot \frac{z}{|z|} < -\frac{|z|}{2}} Q^{p-1}(y+z) \nabla Q(y) Q(y) dy \right| \\ & \lesssim (|v|^2 |z|^2 + |v|^2) |z|^{-\frac{d-1}{2}} e^{-|z|} \end{aligned} \quad (2.2.32)$$

and finally

$$\begin{aligned}
& \langle |P_2|^{p-1} P_1 \cdot \mathbb{1}_{y \cdot \frac{z}{|z|} < 0}, [e^{i\Gamma_1} \nabla Q](y - z_1(s)) \rangle \\
&= \operatorname{Re} \int_{y \cdot \frac{z}{|z|} < 0} Q^{p-1}(y - z_2(s)) Q(y - z_1(s)) \nabla Q(y - z_1(s)) dy \\
&= \int_{y \cdot \frac{z}{|z|} < -\frac{|z|}{2}} Q^{p-1}(y + z) \nabla Q(y) Q(y) dy. \quad (2.2.33)
\end{aligned}$$

From (2.2.29)–(2.2.33), we obtain the desired result (2.2.22). \square

2.2.3 Formal resolution and estimates of leading order

From Lemma 2.2.6, we derive a simplified modulation system with forcing term and we determine one of its approximate solution that is relevant for the regime of Main Theorem. Formally, we have the following bounds (making this rigorous will be the goal of the bootstrap estimates in Sect. 2.3.2)

$$|\vec{m}_1| \lesssim |z|^{-\frac{d-1}{2}} e^{-|z|}, \quad (2.2.34)$$

from which we derive a simplified system (\vec{m}_k is defined in (2.2.7)):

$$\left| \frac{\dot{\lambda}}{\lambda} \right| + |\dot{z} - 2v + \frac{\dot{\lambda}}{\lambda} z| \lesssim |z|^{-\frac{d-1}{2}} e^{-|z|}. \quad (2.2.35)$$

Furthermore, since we expect the interaction to be strong enough such that it will affect the main order of the modulation equations so by projecting $\mathcal{E}_{\mathbf{P}}$ onto the direction $e^{i\Gamma_1} \nabla Q(y - z_1(s))$, we obtain formally that

$$c_2 \dot{v}_1 \approx -\langle G, e^{i\Gamma_1} \nabla Q(y - z_1(s)) \rangle \approx -H(z)$$

with $c_2 = \langle -yQ, \nabla Q \rangle > 0$. This remark suggests us to fix

$$\dot{v} = -\frac{2p}{c_2} \left[\int Q^{p-1}(y) \nabla Q(y) Q(y + z) dy + \int Q^{p-1}(y + z) \nabla Q(y) Q(y) dy \right] = -\frac{2}{c_2} H(z) \quad (2.2.36)$$

so $v(s)$ is completely determined by $z(s)$ and initial data v^{in} . In consequence, there are only three free parameters left (λ, z, γ) corresponding to the scaling, translation and phase parameters which we will modulate to obtain orthogonality conditions (as shown below in Lemma 2.3.2). We use (2.2.23) to estimate the main order of \dot{v}

$$\left| \dot{v} + c \frac{z}{|z|} |z|^{-\frac{d-1}{2}} e^{-|z|} \right| \lesssim |z|^{-\frac{d-1}{2}-1} e^{-|z|} \quad (2.2.37)$$

with

$$c = \frac{2C_p}{c_2} = \frac{2c_Q I_Q}{c_2} > 0. \quad (2.2.38)$$

It can be checked that for some real functions $z_{mod}(s), \lambda_{mod}(s), v_{mod}(s)$ such that

$$\lambda_{mod}^{-1}(s) = 1, \quad v_{mod}(s) = s^{-1}, \quad z_{mod}^{-\frac{d-1}{2}} e^{-z_{mod}} = \frac{s^{-2}}{c} \quad (2.2.39)$$

then we have the asymptotics as $s \rightarrow +\infty$

$$\begin{aligned}
z_{mod}(s) &\sim 2 \log(s), & \dot{v}_{mod}(s) &= -c z_{mod}^{-\frac{d-1}{2}}(s) e^{-z_{mod}(s)}, \\
|\dot{z}_{mod}(s) - 2v_{mod}(s)| &\lesssim s^{-1} \log^{-1}(s), & |\dot{v}_{mod}(s)| &\lesssim s^{-2}.
\end{aligned} \quad (2.2.40)$$

Indeed, obviously $\dot{v}_{mod}(s) = -s^{-2} = -cz_{mod}^{-\frac{d-1}{2}}(s)e^{-z_{mod}(s)}$ and by differentiating the equation of z_{mod} , we get

$$-\dot{z}_{mod}z_{mod}^{-\frac{d-1}{2}}e^{-z_{mod}} - \frac{d-1}{2}\dot{z}_{mod}z_{mod}^{-\frac{d-1}{2}-1}e^{-z_{mod}} = -2\frac{s^{-3}}{c}$$

(in the case $d-1=0$, $-\dot{z}_{mod}e^{-z_{mod}} = -2\frac{s^{-3}}{c}$) so $|\dot{z}_{mod} - 2s^{-1}| \lesssim s^{-1}\log^{-1}(s)$ thus we can deduce $|\dot{z}_{mod}(s) - 2v_{mod}(s)| \lesssim s^{-1}\log^{-1}(s)$. The above estimates suggest that (2.2.39) is close to the first order asymptotics as $s \rightarrow +\infty$ for some particular solutions of (2.2.35) and matches the regime in Main Theorem.

2.3 Modulation and backward uniform estimates

Let $(\lambda^{in}, z^{in}, v^{in}) \in (0, +\infty) \times (0, +\infty) \times \mathbb{R}$ to be chosen with $|z^{in}| \gg 1$, $|v^{in}| \ll 1$, $T_{mod} > 0$ and $(\vec{e}_1, \dots, \vec{e}_d)$ standard basis of \mathbb{R}^d . Recall that in this section $p > 2$. Let $u(t, x)$ be the backward solution of (NLS) with initial data

$$u(T_{mod}, x) = \frac{1}{(\lambda^{in})^{\frac{2}{p-1}}} \mathbf{P}^{in} \left(\frac{x}{\lambda^{in}} \right) \quad \text{where} \quad \mathbf{P}^{in}(y) = \mathbf{P}(y; (z^{in}\vec{e}_1, v^{in})) \quad (2.3.1)$$

on some time interval including T_{mod} . Note that the (NLS) equation is invariant by rotation and reflection. In particular, if a solution of (NLS) is invariant by the symmetries $\tau : x \mapsto -x$ and $v : (x_1, x_2, \dots, x_d) \mapsto (x_1, -x_2, \dots, -x_d)$ at some time, then it is invariant by the symmetry at any time.

2.3.1 Decomposition of $u(t)$

We will state a standard modulation result with the same idea as in Lemma 3 of [20] or Lemma 2 of [27]. The choice of the special orthogonality conditions (2.3.6) is related to the generalized null space of the linearized equation around Q in (2.1.19) and to the coercivity property (2.1.20) in sub-critical cases. See the proof of Lemma 2.3.28 for a technical justification of these choices. For $s^{in} \gg 1$ fixed.

Lemma 2.3.2 (Modulation of the approximate solution). Let $u(t, x)$ a solution invariant by τ and v on an interval $[T, T_{mod}]$ satisfying $u(T_{mod}, x) \in H^2(\mathbb{R}^d)$ and

$$\left\| e^{-i\gamma^{in}} (\lambda^{in})^{\frac{2}{p-1}} u(T_{mod}, \lambda^{in}y) - \mathbf{P}(y; (z^{in}\vec{e}_1, v^{in})) \right\|_{H^1} \ll 1$$

for $\mathbf{P}(s, y) = \mathbf{P}(y; (z(s), v(s)))$ as defined in (2.2.5). Then there exist a unique \mathcal{C}^1 function on an open interval $I \ni s^{in}$

$$\vec{q}(s) = (\lambda, z, \gamma, v) : I \rightarrow (0, +\infty) \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d,$$

with $\vec{q}(s^{in}) = (\lambda^{in}, z^{in}\vec{e}_1, \gamma^{in}, v^{in})$ and a rescaling time function

$$t(s) = T_{mod} - \int_s^{s^{in}} \lambda^2(\tau) d\tau \quad (2.3.3)$$

such that $u(t, x)$ decomposes as follows

$$u(t(s), x) = \frac{e^{i\gamma(s)}}{\lambda^{\frac{2}{p-1}}(s)} (\mathbf{P} + \epsilon)(s, y), \quad y = \frac{x}{\lambda(s)} \quad (2.3.4)$$

where by setting

$$\epsilon(s, y) = [e^{i\Gamma_1}\eta_1](s, y - z_1), \quad \Gamma_k(s, y) = v_k(s) \cdot y, \quad (2.3.5)$$

if initially $\langle \eta_1(s^{in}), Q \rangle = \langle \eta_1(s^{in}), yQ \rangle = \langle \eta_1(s^{in}), i\Lambda Q \rangle = 0$, the decomposition satisfies orthogonality conditions

$$\langle \eta_1(s), Q \rangle = \langle \eta_1(s), yQ \rangle = \langle \eta_1(s), i\Lambda Q \rangle = 0 \quad (2.3.6)$$

and the extra relation

$$\dot{v}(s) = -\frac{2}{c_2}H(z(s)). \quad (2.3.7)$$

Moreover, ϵ is also invariant by the symmetry τ and v .

Proof of Lemma 2.3.2. step 1 Orthogonality conditions. We show that the orthogonality conditions (2.3.6) and the extra relation (2.3.7) are equivalent to solve a system of ODEs. Remark that we can go easily from the rescaled time s to t and conversely

$$s = s(t) = s^{in} - \int_t^{T_{mod}} \frac{d\tau}{\lambda^2(\tau)} \quad (2.3.8)$$

with $T_{mod} = t(s^{in})$. Denote

$$\begin{aligned} \mathbf{P}(s, y) &= [e^{i\Gamma_1}\mathbf{P}_1](s, y - z_1), & \mathcal{E}_{\mathbf{P}}(s, y) &= [e^{i\Gamma_1}\mathcal{E}_{\mathbf{P}_1}](s, y - z_1) \\ G(s, y) &= [e^{i\Gamma_1}G_1](s, y - z_1) \end{aligned}$$

where $G = |\mathbf{P}|^{p-1}\mathbf{P} - |P_1|^{p-1}P_1 - |P_2|^{p-1}P_2$. Let $w = \mathbf{P} + \epsilon$ as in (2.2.1). It follows from the equation of w (2.2.2) and the equation of \mathbf{P} (2.2.8) that

$$i\dot{\epsilon} + \Delta\epsilon - \epsilon + (|\mathbf{P} + \epsilon|^{p-1}(\mathbf{P} + \epsilon) - |\mathbf{P}|^{p-1}\mathbf{P}) - i\frac{\dot{\lambda}}{\lambda}\Lambda\epsilon + (1 - \dot{\gamma})\epsilon + \mathcal{E}_{\mathbf{P}} = 0. \quad (2.3.9)$$

We rewrite the equation of ϵ into the following equation for η_1 (see also the proof of Lemma 2.2.6)

$$i\dot{\eta}_1 + \Delta\eta_1 - \eta_1 + (|\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1) + \vec{m}_1 \cdot \vec{M}\eta_1 + \mathcal{E}_{\mathbf{P}_1} = 0. \quad (2.3.10)$$

Thus, for $A(y), B(y) \in \mathcal{Y}$, we get

$$\begin{aligned} \frac{d}{ds}\langle \eta_1, A + iB \rangle &= -\langle \Delta\eta_1 - \eta_1 + (|\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1) \\ &\quad + \vec{m}_1 \cdot \vec{M}\eta_1 + \mathcal{E}_{\mathbf{P}_1}, iA - B \rangle. \end{aligned}$$

Choose $A = Q, B = 0$ and $A = yQ, B = 0$ and $A = 0, B = \Lambda Q$ then the conditions

$$\frac{d}{ds}\langle \eta_1(s), Q \rangle = \frac{d}{ds}\langle \eta_1(s), yQ \rangle = \frac{d}{ds}\langle \eta_1(s), i\Lambda Q \rangle = 0$$

are equivalent to

$$\begin{cases} \left\langle \Delta\eta_1 - \eta_1 + (|\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1) + \vec{m}_1 \cdot \vec{M}\eta_1 + \mathcal{E}_{\mathbf{P}_1}, iQ \right\rangle = 0 \\ \left\langle \Delta\eta_1 - \eta_1 + (|\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1) + \vec{m}_1 \cdot \vec{M}\eta_1 + \mathcal{E}_{\mathbf{P}_1}, iyQ \right\rangle = 0 \\ \left\langle \Delta\eta_1 - \eta_1 + (|\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1) + \vec{m}_1 \cdot \vec{M}\eta_1 + \mathcal{E}_{\mathbf{P}_1}, -\Lambda Q \right\rangle = 0. \end{cases}$$

We claim that the above system is equivalent to an autonomous system of ordinary differential equations on $(\theta(s), z(s), \gamma(s), v(s), t(s))$ where $\theta(s) = \ln(\lambda(s))$. Indeed, remark that

$$\epsilon(s, y) = e^{\frac{2}{p-1}\theta(s)}u(t(s), e^{\theta(s)}y) - \mathbf{P}(y; (z(s), v(s))) \quad (2.3.11)$$

and the expression of $\mathcal{E}_{\mathbf{P}_1}$ (from (2.2.8)–(2.2.9))

$$\mathcal{E}_{\mathbf{P}_1} = [\vec{m}_1 \cdot \vec{M}Q](y) + [e^{i(\Gamma_2(y+z)-\Gamma_1(y))}\vec{m}_2 \cdot \vec{M}Q](y+z) + G_1$$

then we get

$$\begin{cases} \langle \vec{m}_1 \cdot \vec{M}Q, iQ \rangle + \langle e^{i(\Gamma_2(y+z)-\Gamma_1(y))}\vec{m}_2 \cdot \vec{M}Q(y+z), iQ \rangle + \langle \vec{m}_1 \cdot \vec{M}\eta_1, iQ \rangle = \mathcal{F}_1(\theta, z, \gamma, v, t) \\ \langle \vec{m}_1 \cdot \vec{M}Q, iyQ \rangle + \langle e^{i(\Gamma_2(y+z)-\Gamma_1(y))}\vec{m}_2 \cdot \vec{M}Q(y+z), iyQ \rangle + \langle \vec{m}_1 \cdot \vec{M}\eta_1, iyQ \rangle \\ \hspace{15em} = \mathcal{F}_2(\theta, z, \gamma, v, t) \\ \langle \vec{m}_1 \cdot \vec{M}Q, -\Lambda Q \rangle + \langle e^{i(\Gamma_2(y+z)-\Gamma_1(y))}\vec{m}_2 \cdot \vec{M}Q(y+z), -\Lambda Q \rangle + \langle \vec{m}_1 \cdot \vec{M}\eta_1, -\Lambda Q \rangle \\ \hspace{15em} = \mathcal{F}_3(\theta, z, \gamma, v, t) \end{cases} \quad (2.3.12)$$

with

$$\begin{aligned} \mathcal{F}_1(\theta, z, \gamma, v, t) &= -\langle \Delta\eta_1 - \eta_1 + (|\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1) + G_1, iQ \rangle \\ \mathcal{F}_2(\theta, z, \gamma, v, t) &= -\langle \Delta\eta_1 - \eta_1 + (|\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1) + G_1, iyQ \rangle \\ \mathcal{F}_3(\theta, z, \gamma, v, t) &= -\langle \Delta\eta_1 - \eta_1 + (|\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1) + G_1, -\Lambda Q \rangle. \end{aligned}$$

Note that $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ are \mathcal{C}^1 functions. Indeed, if we replace η_1 by the expression (2.3.11) and its definition, it is clear that any term not containing u is continuously differentiable. For terms concerning $u(t, x)$, by integration by parts and chain rule, we show how to prove that typical terms, integrals of the form

$$\frac{d}{dt} \operatorname{Re} \left(\int u(t, x) A(x) dx \right), \quad \frac{d}{dt} \operatorname{Re} \left(\int |u(t, x)|^{p-1} u(t, x) A(x) dx \right)$$

for $A(x)$ some complex functions such that $\operatorname{Re} A(x), \operatorname{Im} A(x) \in \mathcal{Y}$, are continuous. We have

$$\frac{d}{dt} \operatorname{Re} \left(\int u(t, x) A(x) dx \right) = -\operatorname{Im} \left(\int u(t, x) \Delta A(x) dx \right) - \operatorname{Im} \left(\int |u(t, x)|^{p-1} u(t, x) A(x) dx \right) \quad (2.3.13)$$

and

$$\begin{aligned} \frac{d}{dt} \operatorname{Re} \left(\int |u(t, x)|^{p-1} u(t, x) A(x) dx \right) &= p \operatorname{Re} \left(\int \partial_t u(t, x) |u(t, x)|^{p-1} A(x) dx \right) = \\ &= -p \operatorname{Im} \left(\int \Delta u(t, x) |u(t, x)|^{p-1} A(x) dx \right) - p \operatorname{Im} \left(\int |u(t, x)|^{2p-2} u(t, x) A(x) dx \right). \end{aligned} \quad (2.3.14)$$

Recall the persistence of H^2 regularity for (NLS) equation (see Theorem 5.3.1 in [2]), since $u(T_{mod}, x) \in H^2(\mathbb{R}^d)$ then $u \in \mathcal{C}^1([0, T_{mod}], L^2(\mathbb{R}^d)) \cap \mathcal{C}([0, T_{mod}], H^2(\mathbb{R}^d))$. By Sobolev's injection ($\frac{d+6}{d-2} < \frac{2d}{d-4}$), we have $u \in \mathcal{C}([0, T_{mod}], L^{2p-1}(\mathbb{R}^d))$ and thus the right-hand sides of (2.3.13), (2.3.14) are well-defined and continuous. Therefore, in particular, since initially

$$\langle \eta_1(s^{in}), Q \rangle = \langle \eta_1(s^{in}), yQ \rangle = \langle \eta_1(s^{in}), i\Lambda Q \rangle = 0,$$

the decomposition (\vec{q}, ϵ) will satisfy (2.3.6) if (2.3.12) holds.

step 2 System of ODEs. We show the existence of the decomposition (\vec{q}, ϵ) for $u(t)$ and a

rescaling time $t(s)$ by solving the following system on $(\theta, z, \gamma, v, t)$

$$\begin{cases} \langle \vec{m}_1 \cdot \vec{M}Q, iQ \rangle + \langle e^{i(\Gamma_2(y+z)-\Gamma_1(y))} \vec{m}_2 \cdot \vec{M}Q(y+z), iQ \rangle + \langle \vec{m}_1 \cdot \vec{M}\eta_1, iQ \rangle = \mathcal{F}_1(\theta, z, \gamma, v, t) \\ \langle \vec{m}_1 \cdot \vec{M}Q, iyQ \rangle + \langle e^{i(\Gamma_2(y+z)-\Gamma_1(y))} \vec{m}_2 \cdot \vec{M}Q(y+z), iyQ \rangle + \langle \vec{m}_1 \cdot \vec{M}\eta_1, iyQ \rangle \\ \hspace{20em} = \mathcal{F}_2(\theta, z, \gamma, v, t) \\ \langle \vec{m}_1 \cdot \vec{M}Q, -\Lambda Q \rangle + \langle e^{i(\Gamma_2(y+z)-\Gamma_1(y))} \vec{m}_2 \cdot \vec{M}Q(y+z), -\Lambda Q \rangle + \langle \vec{m}_1 \cdot \vec{M}\eta_1, -\Lambda Q \rangle \\ \hspace{20em} = \mathcal{F}_3(\theta, z, \gamma, v, t) \\ \dot{v} = -\frac{2}{c_2}H(z) \\ \dot{t}(s) = \lambda^2(s). \end{cases} \quad (2.3.15)$$

On the one hand, we calculate

$$\begin{aligned} \langle \vec{m}_1 \cdot \vec{M}Q, iQ \rangle &= \left(\frac{\dot{\lambda}}{\lambda}\right) \langle -i\Lambda Q, iQ \rangle = -c_1 \left(\frac{\dot{\lambda}}{\lambda}\right) \\ \langle \vec{m}_1 \cdot \vec{M}Q, iyQ \rangle &= (\dot{z} - 2v + \frac{\dot{\lambda}}{\lambda}z) \langle -i\nabla Q, iyQ \rangle = c_2(\dot{z} - 2v + \frac{\dot{\lambda}}{\lambda}z) \\ \langle \vec{m}_1 \cdot \vec{M}Q, -\Lambda Q \rangle &= c_1(\dot{\gamma} - 1 + |v|^2 - \frac{\dot{\lambda}}{\lambda}(v \cdot z) - (v \cdot \dot{z})) \end{aligned}$$

with $c_1 = \langle \Lambda Q, Q \rangle, c_2 = \langle -\nabla Q, yQ \rangle$ non-zero. On the other hand, there exist a matrix $\mathcal{M}(\theta, z, \gamma, v, t) = (m_{ij})_{5 \times 5}$ and $\vec{\mathcal{G}}(\theta, z, \gamma, v, t)$ such that

$$\begin{pmatrix} \langle e^{i(\Gamma_2(y+z)-\Gamma_1(y))} \vec{m}_2 \cdot \vec{M}Q(y+z), iQ \rangle + \langle \vec{m}_1 \cdot \vec{M}\eta_1, iQ \rangle \\ \langle e^{i(\Gamma_2(y+z)-\Gamma_1(y))} \vec{m}_2 \cdot \vec{M}Q(y+z), iyQ \rangle + \langle \vec{m}_1 \cdot \vec{M}\eta_1, iyQ \rangle \\ \langle e^{i(\Gamma_2(y+z)-\Gamma_1(y))} \vec{m}_2 \cdot \vec{M}Q(y+z), -\Lambda Q \rangle + \langle \vec{m}_1 \cdot \vec{M}\eta_1, -\Lambda Q \rangle \\ 0 \\ 0 \end{pmatrix} = (\dot{\theta}, \dot{z}, \dot{\gamma}, \dot{v}, \dot{t})\mathcal{M}(\theta, z, \gamma, v, t) + \vec{\mathcal{G}}(\theta, z, \gamma, v, t) \quad (2.3.16)$$

where all entries of $\mathcal{M}(\theta, z, \gamma, v, t)$ are small $|m_{ij}| \ll 1$ as $z^{in} \gg 1$ and $\|\epsilon(s^{in})\|_{H^1} \ll 1$ (from hypothesis). Then the system (2.3.15) can be rewritten as an autonomous system

$$(\dot{\theta}, \dot{z}, \dot{\gamma}, \dot{v}, \dot{t})\mathcal{A}(\theta, z, \gamma, v, t) + (\dot{\theta}, \dot{z}, \dot{\gamma}, \dot{v}, \dot{t})\mathcal{M}(\theta, z, \gamma, v, t) = \vec{\mathcal{H}}(\theta, z, \gamma, v, t) \quad (2.3.17)$$

where

$$\vec{\mathcal{H}}(\theta, z, \gamma, v, t) = \begin{pmatrix} \mathcal{F}_1(\theta, z, \gamma, v, t) \\ \mathcal{F}_2(\theta, z, \gamma, v, t) + 2c_2v \\ \mathcal{F}_3(\theta, z, \gamma, v, t) + c_1 - c_1|v|^2 \\ -\frac{2}{c_2}H(z) \\ e^{2\theta} \end{pmatrix} - \vec{\mathcal{G}}(\theta, z, \gamma, v, t)$$

and the matrix \mathcal{A} is given by

$$\mathcal{A} = \begin{pmatrix} -c_1 & c_2z & c_1(v \cdot z) & 0 & 0 \\ 0 & c_2 & c_1v & 0 & 0 \\ 0 & 0 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore the perturbed matrix $(\mathcal{A} + \mathcal{M})(\theta, z, \gamma, v, t)$ is invertible ($\det \mathcal{A} = -c_1^2 c_2 < 0$). As same as the way to deal with \mathcal{F} , one can check that $\mathcal{M}, \vec{\mathcal{G}}$ are continuously differentiable thus so are entries of $(\mathcal{A} + \mathcal{M})^{-1}$ and $\vec{\mathcal{H}}$. Therefore,

$$\mathcal{R}(\theta, z, \gamma, v, t) = [(\mathcal{A} + \mathcal{M})^{-1} \cdot \vec{\mathcal{H}}](\theta, z, \gamma, v, t)$$

satisfies the hypothesis of Cauchy-Lipschitz theorem and the system of ODEs

$$(\dot{\theta}, \dot{z}, \dot{\gamma}, \dot{v}, \dot{t}) = \mathcal{R}(\theta, z, \gamma, v, t) \quad (2.3.18)$$

admits a unique solution $(\theta(s), z(s), \gamma(s), v(s), t(s))$ to the initial value problem. We obtain the decomposition $(\lambda(s), z(s), \gamma(s), v(s))$ of $u(t)$ and the renormalization of time $t(s)$. \square

Observe from (2.3.1) that the initial data

$$\begin{aligned} w(s^{in}) &= \mathbf{P}^{in}(y; (z^{in} \vec{e}_1, v^{in})), & \lambda(s^{in}) &= \lambda^{in}, & \gamma(s^{in}) &= 0, \\ z(s^{in}) &= z^{in} \vec{e}_1, & v(s^{in}) &= v^{in}, & \epsilon(s^{in}) &\equiv 0 \end{aligned} \quad (2.3.19)$$

and $u(T_{mod}, x)$ satisfy the hypothesis of Lemma 2.3.2.

Proposition 2.3.20 (Uniform backwards estimates for $p > 2$). There exists $s_0 > 10$ satisfying the following condition: for all $s^{in} > s_0$, there is a choice of initial parameters $(\lambda^{in}, z^{in}, v^{in})$ with

$$\begin{aligned} \left| c^{-\frac{1}{2}} (z^{in})^{\frac{d-1}{4}} e^{\frac{1}{2} z^{in}} - s^{in} \right| &< s^{in} \log^{-\frac{1}{2}}(s^{in}), & z^{in} &> 0, \\ \lambda^{in} &= 1, & v^{in} &= c^{\frac{1}{2}} (z^{in})^{-\frac{d-1}{4}} e^{-\frac{1}{2} z^{in}} \cdot \vec{e}_1, \end{aligned} \quad (2.3.21)$$

such that the solution u of (NLS) corresponding to (2.3.1) exists. Moreover, the decomposition of u given by Lemma 2.3.2 on the rescaled interval of time $[s_0, s^{in}]$

$$u(s, x) = \frac{e^{i\gamma(s)}}{\lambda^{\frac{2}{p-1}}(s)} (\mathbf{P} + \epsilon)(s, y), \quad y = \frac{x}{\lambda(s)}, \quad dt = \lambda^2(s) ds$$

verifies the uniform estimates for all $s \in [s_0, s^{in}]$

$$\begin{aligned} ||z(s)| - 2 \log(s)| &\lesssim \log(\log(s)), & |\lambda^{-1}(s) - 1| &\lesssim s^{-1}, \\ |v(s)| &\lesssim s^{-1}, & \|\epsilon(s)\|_{H^1} &\lesssim s^{-1}, & \left| |z(s)|^{\frac{d-1}{2}} e^{|z(s)|} - cs^2 \right| &\lesssim s^2 \log^{-\frac{1}{2}}(s). \end{aligned} \quad (2.3.22)$$

Remark 2.3.23. The key point in Proposition 2.3.20 is that s_0 and the constants in (2.3.22) are independent of s^{in} as $s^{in} \rightarrow +\infty$. Observe that the estimates (2.3.22) match the discussion in Sect. 2.2.3. The decomposition in Lemma 2.3.2 is only local but the estimates in (2.3.22) guarantee the global existence of the decomposition. The choice of v^{in} is direct while the choice of z^{in} is based on a contradiction argument and a topological constraint.

The next subsection is devoted to the proof of Proposition 2.3.20 containing several technical steps. The proof relies on a bootstrap argument, integration of the differential system of geometrical parameters and energy estimates. Pick a smooth function $\tilde{\chi} : [0, +\infty) \rightarrow [0, \infty)$, non increasing, with $\tilde{\chi} \equiv 1$ on $[0, \frac{1}{10}]$, $\tilde{\chi} \equiv 0$ on $[\frac{1}{8}, +\infty)$. We define the localized momentum:

$$\mathcal{M}_k(s, \epsilon) = \text{Im} \int (\nabla \epsilon \bar{\epsilon}) \chi_k = \text{Im} \int (\nabla \eta_k \bar{\eta}_k) \chi \quad (2.3.24)$$

for $\chi_k(s, y) = \tilde{\chi}(\log^{-1}(s)|y - z_k(s)|)$ and $\chi = \tilde{\chi}(|\log^{-1}(s)y|)$

2.3.2 Proof of Proposition 2.3.20

2.3.2.1 Bootstrap bounds

We shall consider the following bootstrap estimates

$$\begin{aligned} \left| c^{-\frac{1}{2}} |z|^{\frac{d-1}{4}} e^{\frac{1}{2}|z|} - s \right| &\leq s \log^{-\frac{1}{2}}(s), \\ \|\epsilon(s)\|_{H^1} &\leq C^* s^{-1} \end{aligned} \quad (2.3.25)$$

with $C^* > 1$ a constant to be chosen large enough. Note that the estimate on z and the estimate (2.2.37) of \dot{v} imply that, for s large

$$\left| |z| - 2 \log(s) \right| \lesssim \log(\log(s)), \quad \left| |\dot{v}| - s^{-2} \right| \lesssim s^{-2} \log^{-\frac{1}{2}}(s), \quad \left| |v| - s^{-1} \right| \lesssim s^{-1} \log^{-\frac{1}{2}}(s) \quad (2.3.26)$$

where the last inequality is obtained by integrating the second one with the choice of initial data v^{in} in (2.3.21). Next, we define

$$s^* = \inf\{\tau \in [s_0, s^{in}]; (2.3.25) \text{ holds on } [\tau, s^{in}]\}. \quad (2.3.27)$$

2.3.2.2 Control of the modulation equations

Denote \vec{m}_k^* the system \vec{m}_k without equation $\dot{z}_k - 2v_k + \frac{\dot{\lambda}}{\lambda} z_k$ and M^* the vector M without the direction $-i\nabla V$.

Lemma 2.3.28 (Pointwise control of the modulation equations and the error). The following estimates hold on $[s^*, s^{in}]$.

$$|\vec{m}_k^*(s)| \lesssim (C^*)^2 s^{-2}. \quad (2.3.29)$$

$$|\langle \eta_1(s), i\nabla Q \rangle| \lesssim (C^*)^2 s^{-1} \log^{-1}(s), \quad (2.3.30)$$

$$|\dot{z} - 2v| \lesssim s^{-1} \log^{-\frac{3}{4}}(s). \quad (2.3.31)$$

Moreover, for all $s \in [s^*, s^{in}]$, for all $y \in \mathbb{R}^2$,

$$|\mathcal{E}_{\mathbf{P}}(s, y)| \lesssim s^{-1} \log^{-\frac{3}{4}}(s) \sum_{k=1}^2 Q(y - z_k(s)) + |G(s, y)|. \quad (2.3.32)$$

Proof of Lemma 2.3.28. Since $\epsilon(s^{in}) \equiv 0$, we may define

$$s^{**} = \inf\{s \in [s^*, s^{in}]; |\langle \eta_1(\tau), i\nabla Q \rangle| \leq C^{**} \tau^{-1} \log^{-1}(\tau) \text{ holds on } [s, s^{in}]\},$$

for some constant $C^{**} > 0$ to be chosen large enough. We work on the interval $[s^{**}, s^{in}]$. Recall equation for η_1 (2.3.10) as below

$$i\dot{\eta}_1 + \Delta \eta_1 - \eta_1 + (|\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1) + \vec{m}_1 \cdot \vec{M} \eta_1 + \mathcal{E}_{\mathbf{P}_1} = 0.$$

Let $A(y)$ and $B(y)$ be two real-valued functions in \mathcal{Y} . We claim the following estimate on $[s^{**}, s^{in}]$

$$\left| \frac{d}{ds} \langle \eta_1, A + iB \rangle - \left[\langle \eta_1, iL_- A - L_+ B \rangle - \langle \vec{m}_1 \cdot \vec{M} Q, iA - B \rangle \right] \right| \lesssim (C^*)^2 s^{-2} + s^{-1} |\vec{m}_1|. \quad (2.3.33)$$

We compute from (2.3.10),

$$\begin{aligned}
\frac{d}{ds}\langle \eta_1, A + iB \rangle &= \langle \dot{\eta}_1, A + iB \rangle = \langle i\dot{\eta}_1, iA - B \rangle \\
&= \langle -\Delta\eta_1 + \eta_1 - \left(\frac{p+1}{2}Q^{p-1}\eta_1 + \frac{p-1}{2}Q^{p-1}\bar{\eta}_1\right), iA - B \rangle \\
&\quad - \langle |\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1 - \frac{p+1}{2}Q^{p-1}\eta_1 - \frac{p-1}{2}Q^{p-1}\bar{\eta}_1, iA - B \rangle \\
&\quad - \langle \vec{m}_1 \cdot \vec{M}\eta_1, iA - B \rangle - \langle \mathcal{E}_{\mathbf{P}_1}, iA - B \rangle.
\end{aligned}$$

First, since A and B are real-valued, we have

$$\langle -\Delta\eta_1 + \eta_1 - \left(\frac{p+1}{2}Q^{p-1}\eta_1 + \frac{p-1}{2}Q^{p-1}\bar{\eta}_1\right), iA - B \rangle = \langle \eta_1, iL_-A - L_+B \rangle.$$

Second, recall the expression of \mathbf{P}_1

$$\mathbf{P}_1 = Q(y) + e^{i(\Gamma_2(y-(z_2-z_1))-\Gamma_1(y))}Q(y - (z_2 - z_1)).$$

By the expansion in (2.2.18), we can deduce the first order and the error of

$$\begin{aligned}
&|\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1 - \frac{p+1}{2}Q^{p-1}\eta_1 - \frac{p-1}{2}Q^{p-1}\bar{\eta}_1 \\
&= \frac{p+1}{2}(|\mathbf{P}_1|^{p-1} - Q^{p-1})\eta_1 + \frac{p-1}{2}(|\mathbf{P}_1|^{p-3}\mathbf{P}_1^2 - Q^{p-1})\bar{\eta}_1 + O\left(\left|\frac{\eta_1}{\mathbf{P}_1}\right|^2 |\mathbf{P}_1|^p\right) + O(|\eta_1|^p).
\end{aligned}$$

By (2.3.25)–(2.3.26) for some $q > 0$,

$$\begin{aligned}
&| \langle (|\mathbf{P}_1|^{p-1} - Q^{p-1})\eta_1, (iA - B) \rangle | + | \langle (|\mathbf{P}_1|^{p-3}\mathbf{P}_1^2 - Q^{p-1})\bar{\eta}_1, (iA - B) \rangle | \\
&\lesssim |z|^q e^{-|z|} \|\eta_1\|_{L^2} \lesssim C^* s^{-3} \log^q(s).
\end{aligned}$$

Using Cauchy-Schwarz and Gagliardo-Nirenberg inequalities (as $p > 2$)

$$\begin{aligned}
\left\langle \left|\frac{\eta_1}{\mathbf{P}_1}\right|^2 |\mathbf{P}_1|^p, (iA - B) \right\rangle &\lesssim \|\epsilon\|_{L^2}^2 \lesssim (C^*)^2 s^{-2}, \\
\langle |\eta_1|^p, (iA - B) \rangle &\lesssim \|\epsilon\|_{H^1}^p \lesssim (C^*)^2 s^{-2}.
\end{aligned}$$

Therefore

$$\left| \langle |\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1 - \frac{p+1}{2}Q^{p-1}\eta_1 - \frac{p-1}{2}Q^{p-1}\bar{\eta}_1, iA - B \rangle \right| \lesssim (C^*)^2 s^{-2}. \quad (2.3.34)$$

Next, using (2.3.25)–(2.3.26), we obtain

$$| \langle \vec{m}_1 \cdot \vec{M}\eta_1, iA - B \rangle | \lesssim C^* s^{-1} |\vec{m}_1(s)|.$$

Finally, we need to prove following estimate

$$\left| \langle \mathcal{E}_{\mathbf{P}_1}, iA - B \rangle - \langle \vec{m}_1 \cdot \vec{M}Q, iA - B \rangle \right| \lesssim s^{-2} + s^{-1} |\vec{m}_1|. \quad (2.3.35)$$

Indeed, recall that we have

$$\mathcal{E}_{\mathbf{P}_1} = [\vec{m}_1 \cdot \vec{M}Q](y) + [e^{i(\Gamma_2(y-(z_2-z_1))-\Gamma_1(y))} \vec{m}_2 \cdot \vec{M}Q](y - (z_2 - z_1)) + G_1.$$

From (2.2.15) and (2.3.25)–(2.3.26),

$$|\langle G_1, iA - B \rangle| \lesssim \|G\|_{L^\infty} \lesssim |z|^{-\frac{d-1}{2}} e^{-|z|} \lesssim s^{-2}.$$

Since $A, B \in \mathcal{Y}$, we have

$$|\langle e^{i(\Gamma_2(y-(z_2-z_1))-\Gamma_1(y))}(\vec{m}_2 \cdot \vec{M}Q(\cdot - (z_2 - z_1))), iA - B \rangle| \lesssim s^{-1} |\vec{m}_1|,$$

so the proof of (2.3.35) is complete.

We now use (2.3.33) to control the modulation vector \vec{m}_1 . Note that η_1 satisfies the orthogonality conditions (2.3.6).

$\langle \eta_1, Q \rangle = 0$. Let $A = Q$ and $B = 0$. Since $L_-Q = 0$ and $\langle \vec{m}_1 \cdot \vec{M}Q, iQ \rangle = -c_1(\frac{\dot{\lambda}}{\lambda})$, we obtain

$$\left| \frac{\dot{\lambda}}{\lambda} \right| \lesssim (C^*)^2 s^{-2} + s^{-1} |\vec{m}_1|. \quad (2.3.36)$$

$\langle \eta_1, i\Lambda Q \rangle = 0$. Let $A = 0$ and $B = \Lambda Q$. Since $L_+(\Lambda Q) = -2Q$, $\langle \eta_1, Q \rangle = 0$ and $\langle \vec{m}_1 \cdot \vec{M}Q, -\Lambda Q \rangle = c_1(\dot{\gamma} - 1 + |v|^2 - \frac{\dot{\lambda}}{\lambda}(v \cdot z) - (v \cdot \dot{z}))$, we obtain

$$\left| \dot{\gamma} - 1 + |v|^2 - \frac{\dot{\lambda}}{\lambda}(v \cdot z) - (v \cdot \dot{z}) \right| \lesssim (C^*)^2 s^{-2} + s^{-1} |\vec{m}_1|. \quad (2.3.37)$$

$\langle \eta_1, yQ \rangle = 0$. Let $A = yQ$ and $B = 0$. Since $L_-(yQ) = -2\nabla Q$, $|\langle \eta_1, i\nabla Q \rangle| \lesssim C^{**} s^{-1} \log^{-1}(s)$ and $\langle \vec{m}_1 \cdot \vec{M}Q, iyQ \rangle = c_2(\dot{z} - 2v + \frac{\dot{\lambda}}{\lambda}z)$, we obtain

$$\left| \dot{z} - 2v + \frac{\dot{\lambda}}{\lambda}z \right| \lesssim C^{**} s^{-1} \log^{-1}(s) + (C^*)^2 s^{-2} + s^{-1} |\vec{m}_1|. \quad (2.3.38)$$

By (2.3.26) and (2.3.36),

$$\left| \dot{v} - \frac{\dot{\lambda}}{\lambda}v \right| \lesssim |\dot{v}| + \left| \frac{\dot{\lambda}}{\lambda} \right| |v| \lesssim s^{-2}. \quad (2.3.39)$$

Combining (2.3.36)–(2.3.39), we have proved, for all $s \in [s^{**}, s^{in}]$,

$$|\vec{m}_1^*(s)| \lesssim (C^*)^2 s^{-2} \quad (2.3.40)$$

and

$$|\dot{z} - 2v| \lesssim s^{-1} \log^{-\frac{3}{4}}(s). \quad (2.3.41)$$

Now we turn to the study of localized momentum \mathcal{M}_k :

$$\frac{d}{ds} \mathcal{M}_1 = \text{Im} \int (\nabla \eta_1 \bar{\eta}_1) \dot{\chi} + \langle i\dot{\eta}_1, 2\chi \nabla \eta_1 + \eta_1 \nabla \chi \rangle.$$

We claim that

$$\frac{1}{2} \frac{d}{ds} \mathcal{M}_1 = \left\langle \frac{\bar{\eta}_1 \cdot F''(\mathbf{P}_1) \cdot \eta_1}{2}, \nabla Q \right\rangle + \left(\dot{z}_1 - 2v_1 + \frac{\dot{\lambda}}{\lambda} z_1 \right) \langle i\nabla Q, \nabla \eta_1 \rangle + O(\log^{-1}(s) \|\eta_1\|_{H^1}^2). \quad (2.3.42)$$

Note that by direct computations

$$|\dot{\chi}| \lesssim |s^{-1} \log^{-2}(s) y \tilde{\chi}'(\log^{-1}(s) y)| \lesssim s^{-1} \log^{-1}(s)$$

and so, by (2.3.25)–(2.3.26),

$$\left| \operatorname{Im} \int (\nabla \eta_1 \bar{\eta}_1) \dot{\chi} \right| \lesssim s^{-1} \log^{-1}(s) \|\eta_1\|_{H^1}^2 \lesssim s^{-3} \log^{-\frac{1}{2}}(s).$$

Now, we use the equation (2.3.10) of η_1

$$\begin{aligned} i\dot{\eta}_1 + \Delta \eta_1 - \eta_1 + (|\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1) + \vec{m}_1^* \cdot \vec{M}^* \eta_1 - (z_1 - 2v_1 + \frac{\dot{\lambda}}{\lambda} z_1) \cdot i \nabla \eta_1 \\ + \vec{m}_1^* \cdot \vec{M}^* Q - (z_1 - 2v_1 + \frac{\dot{\lambda}}{\lambda} z_1) \cdot i \nabla Q + [e^{i(\Gamma_2(y-z) - \Gamma_1(y))} \vec{m}_2 \cdot \vec{M} Q](y-z) + G_1 = 0 \end{aligned}$$

to estimate $\langle i\dot{\eta}_1, 2\chi \nabla \eta_1 + \eta_1 \nabla \chi \rangle$. By integration by parts, we check the following

$$\langle \Delta \eta_1, 2\chi \nabla \eta_1 + \eta_1 \nabla \chi \rangle = -2 \langle \nabla \eta_1 \cdot \nabla \chi, \nabla \eta_1 \rangle + \frac{1}{2} \int |\eta_1|^2 \nabla(\Delta \chi).$$

We have

$$|\langle \nabla \eta_1 \cdot \nabla \chi, \nabla \eta_1 \rangle| \lesssim \log^{-1}(s) \|\eta_1\|_{H^1}^2$$

and as $|\nabla(\Delta \chi)| \lesssim \log^{-3}(s)$ we obtain

$$\left| \int |\eta_1|^2 \nabla(\Delta \chi) \right| \lesssim \log^{-3}(s) \|\eta_1\|_{H^1}^2.$$

In conclusion for term $\Delta \eta_1$ in the equation of η_1 , we get

$$|\langle \Delta \eta_1, 2\chi \nabla \eta_1 + \eta_1 \nabla \chi \rangle| \lesssim \log^{-1}(s) \|\eta_1\|_{H^1}^2.$$

For the term η_1 , we simply verify by integration by parts that

$$\langle \eta_1, 2\chi \nabla \eta_1 + \eta_1 \nabla \chi \rangle = 0.$$

From (2.3.40) and (2.3.41), we also have that

$$\left| \langle \vec{m}_1 \cdot \vec{M} \eta_1, 2\chi \nabla \eta_1 + \eta_1 \nabla \chi \rangle \right| \lesssim s^{-1} \log^{-\frac{1}{2}}(s) \|\eta_1\|_{H^1}^2,$$

$$\left| \langle \vec{m}_1^* \cdot \vec{M}^* Q, 2\chi \nabla \eta_1 + \eta_1 \nabla \chi \rangle \right| \lesssim (C^*)^2 s^{-2} \|\eta_1\|_{H^1} \lesssim s^{-\frac{5}{2}},$$

$$\left| \langle [\vec{m}_2 \cdot \vec{M} Q](\cdot - z), 2\chi \nabla \eta_1 + \eta_1 \nabla \chi \rangle \right| \lesssim s^{-1} \log^{-\frac{1}{2}}(s) e^{-\frac{7}{8}z} \|\eta_1\|_{H^1} \lesssim s^{-3},$$

and

$$\left| \langle G_1, 2\chi \nabla \eta_1 + \eta_1 \nabla \chi \rangle \right| \lesssim \|G_1\|_{L^\infty} \log^{\frac{d}{2}}(s) \|\epsilon\|_{H^1} \lesssim s^{-\frac{3}{2}} \|\eta_1\|_{H^1}.$$

where we use Cauchy-Schwarz inequality and the fact that the support of χ is contained in $\{|y| \leq \frac{1}{8} \log(s)\}$. Now we will deal with the term $\langle |\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1, 2\chi \nabla \eta_1 + \eta_1 \nabla \chi \rangle$. By (2.2.18), we consider

$$|\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1 = F'(\mathbf{P}_1) \cdot \eta_1 + O(|\eta_1|^p) + O\left(\left|\frac{\eta_1}{\mathbf{P}_1}\right|^2 |\mathbf{P}_1|^p\right)$$

and using Gagliardo-Nirenberg inequality (note that if $p > 2$ then $3 < 2^*$)

$$\begin{aligned} \left| \left\langle \left| \frac{\eta_1}{\mathbf{P}_1} \right|^2 |\mathbf{P}_1|^p, 2\chi \nabla \eta_1 + \eta_1 \nabla \chi \right\rangle \right| &\lesssim \|\eta_1\|_{H^1}^3 \lesssim s^{-\frac{5}{2}}, \\ |\langle \eta_1|^p, 2\chi \nabla \eta_1 + \eta_1 \nabla \chi \rangle| &\lesssim \|\eta_1\|_{H^1}^{p+1} \lesssim s^{-2} \log^{-2}(s). \end{aligned}$$

Then, we have

$$|\langle F'(\mathbf{P}_1) \cdot \eta_1, \eta_1 \nabla \chi \rangle| \lesssim |\nabla \chi| \|\epsilon\|_{H^1}^2 \lesssim \log^{-1}(s) \|\eta_1\|_{H^1}^2.$$

Finally by integration by parts, we get

$$\langle F'(\mathbf{P}_1) \cdot \eta_1, 2\chi \nabla \eta_1 \rangle = -2 \left\langle \nabla \mathbf{P}_1 \chi, \frac{\bar{\eta}_1 \cdot F''(\mathbf{P}_1) \cdot \eta_1}{2} \right\rangle - \langle F'(\mathbf{P}_1) \cdot \eta_1, \eta_1 \nabla \chi \rangle,$$

therefore the collection of above bounds gives

$$\frac{d}{ds} \mathcal{M}_1 = 2 \left\langle \nabla \mathbf{P}_1 \chi, \frac{\bar{\eta}_1 \cdot F''(\mathbf{P}_1) \cdot \eta_1}{2} \right\rangle + 2 \left(\dot{z}_1 - 2v_1 + \frac{\dot{\lambda}}{\lambda} z_1 \right) \langle i \nabla Q, \chi \nabla \eta_1 \rangle + O(\log^{-1}(s) \|\eta_1(s)\|_{H^1}^2). \quad (2.3.43)$$

We finish the proof of (2.3.42) by showing the following estimate

$$\begin{aligned} |\langle \nabla \mathbf{P}_1 \chi, \bar{\eta}_1 \cdot F''(\mathbf{P}_1) \cdot \eta_1 \rangle - \langle \nabla Q, \bar{\eta}_1 \cdot F''(\mathbf{P}_1) \cdot \eta_1 \rangle| &\lesssim \left| \int_{|y| < \frac{1}{8} \log s} (\bar{\eta}_1 \cdot F''(\mathbf{P}_1) \cdot \eta_1) \nabla Q(\cdot + z) \right| \\ &+ \left| \int_{|y| > \frac{1}{10} \log s} (\bar{\eta}_1 \cdot F''(\mathbf{P}_1) \cdot \eta_1) \nabla Q \right| \lesssim s^{-\frac{1}{20}} \|\epsilon\|_{H^1}^2, \end{aligned}$$

and

$$\begin{aligned} \left(\dot{z}_1 - 2v_1 + \frac{\dot{\lambda}}{\lambda} z_1 \right) |\langle i \nabla Q, \chi \nabla \eta_1 \rangle - \langle i \nabla Q, \nabla \eta_1 \rangle| &\lesssim s^{-1} \log^{-\frac{1}{2}}(s) \left| \int_{|y| > \frac{1}{10} \log s} \nabla Q \nabla \bar{\eta}_1 \right| \\ &\lesssim s^{-1 - \frac{1}{20}} \log^{-\frac{3}{4}}(s) \|\eta_1\|_{H^1} \end{aligned}$$

here we use (2.1.17). On the other hand, from (2.3.33), refining up to order s^{-2} , using $L_+(\nabla Q) = 0$ and (2.2.19), we have that

$$\begin{aligned} \frac{d}{ds} \langle \eta_1, i \nabla Q \rangle &= \left\langle \frac{\bar{\eta}_1 \cdot F''(\mathbf{P}_1) \cdot \eta_1}{2}, \nabla Q \right\rangle - c_2 \left(\dot{v}_1 - \frac{\dot{\lambda}}{\lambda} v_1 \right) \\ &\quad - \langle G_1, \nabla Q \rangle + \langle \vec{m}_1 \cdot \vec{M} \eta_1, \nabla Q \rangle + O(s^{-2+}). \end{aligned}$$

From (2.2.22) and the choice of v in (2.3.7), we get

$$\left| c_2 \dot{v}_1 + \langle G_1, \nabla Q \rangle - c_2 \frac{\dot{\lambda}}{\lambda} v_1 \right| \lesssim (|v|^2 |z|^2 + |v|^2) |z|^{-\frac{1}{2}} e^{-|z|} + |z|^{-\frac{3(d-1)}{4}} e^{-\frac{3}{2}|z|} + |v| \left| \frac{\dot{\lambda}}{\lambda} \right| \lesssim s^{-3},$$

then from (2.3.40), we obtain

$$|\langle \vec{m}_1^* \cdot \vec{M}^* \eta_1, \nabla Q \rangle| \lesssim s^{-2} \|\eta_1\|_{H^1}.$$

Thus, we deduce that

$$\frac{d}{ds} \langle \eta_1, i \nabla Q \rangle = \left\langle \frac{\bar{\eta}_1 \cdot F''(\mathbf{P}_1) \cdot \eta_1}{2}, \nabla Q \right\rangle - \left(\dot{z}_1 - 2v_1 + \frac{\dot{\lambda}}{\lambda} z_1 \right) \langle i \nabla \eta_1, \nabla Q \rangle + O(s^{-2+}).$$

Note that $\langle i\nabla\eta_1, \nabla Q \rangle = -\langle i\nabla Q, \nabla\eta_1 \rangle$, we obtain

$$\frac{d}{ds} \langle \eta_1, i\nabla Q \rangle = \frac{1}{2} \frac{d}{ds} \operatorname{Im} \int (\nabla\eta_1 \bar{\eta}_1) \chi + O(\log^{-1}(s) \|\eta_1\|_{H^1}^2).$$

This information combining with $\langle \eta_1(s^{in}), i\nabla Q \rangle = 0$ and $\mathcal{M}_1(t^{in}) = 0$ implies that

$$\left| \langle \eta_1, i\nabla Q \rangle - \frac{1}{2} \mathcal{M}_1 \right| \lesssim (C^*)^2 s^{-1} \log^{-1}(s).$$

From the bootstrap (2.3.25), we deduce that $|\langle \eta_1, i\nabla Q \rangle| \lesssim (C^*)^2 s^{-1} \log^{-1}$ so if we take C^{**} big enough such that $\frac{C^{**}}{2} \gtrsim (C^*)^2$ then $s^{**} = s^*$. Those estimates (2.3.31) and (2.3.32) are direct consequences of (2.3.26), (2.3.29) and (2.3.36). \square

2.3.2.3 Energy functional

Consider the nonlinear energy functional for ϵ

$$\mathbf{K}(s, \epsilon) = \frac{1}{2} \int \left(|\nabla\epsilon|^2 + |\epsilon|^2 - \frac{2}{p+1} (|\mathbf{P} + \epsilon|^{p+1} - |\mathbf{P}|^{p+1} - (p+1)|\mathbf{P}|^{p-1} \operatorname{Re}(\epsilon \bar{\mathbf{P}})) \right).$$

and

$$\mathbf{J} = \sum_k J_k, \quad J_k(s, \epsilon) = v_k \cdot \mathcal{M}_k(s, \epsilon).$$

where $\mathcal{M}_k(s, \epsilon)$ the localized moment defined in (2.3.24). Finally, we set

$$\mathbf{W}(s, \epsilon) = \mathbf{K}(s, \epsilon) - \mathbf{J}(s, \epsilon).$$

The functional \mathbf{W} is coercive in ϵ at the main order and it is an almost conserved quantity for the problem (see [29] for a similar functional).

Proposition 2.3.44 (Coercivity and time control of the energy functional). For all $s \in [s^*, s^{in}]$,

$$\mathbf{W}(s, \epsilon(s)) \gtrsim \|\epsilon(s)\|_{H^1}^2, \quad (2.3.45)$$

and

$$\left| \frac{d}{ds} [\mathbf{W}(s, \epsilon(s))] \right| \lesssim s^{-2} \|\epsilon(s)\|_{H^1}. \quad (2.3.46)$$

Proof of Proposition 2.3.44. step 1 Coercivity. The proof of the coercivity (2.3.45) is a standard consequence of the coercivity property (2.1.20) around one solitary wave with the orthogonality properties (2.3.6), (2.3.30), and an elementary localization argument. We refer to the proof of Lemma 4.1 in Appendix B of [23] for a similar proof.

step 2 Variation of the energy. We estimate the time variation of the functional \mathbf{K} and claim that for all $s \in [s^*, s^{in}]$,

$$\left| \frac{d}{ds} [\mathbf{K}(s, \epsilon(s))] - \sum_{k=1}^2 \dot{z}_k \cdot \left\langle \nabla P_k, \frac{\bar{\epsilon} \cdot F''(\mathbf{P}) \cdot \epsilon}{2} \right\rangle \right| \lesssim s^{-2} \|\epsilon(s)\|_{H^1} + s^{-1} \log^{-\frac{3}{4}}(s) \|\epsilon\|_{H^1}^2. \quad (2.3.47)$$

The time derivative of $s \mapsto H(s, \epsilon(s))$ splits into two parts

$$\frac{d}{ds} [\mathbf{K}(s, \epsilon(s))] = D_s \mathbf{K}(s, \epsilon(s)) + \langle D_\epsilon \mathbf{K}(s, \epsilon(s)), \dot{\epsilon}_s \rangle,$$

where D_s denotes differentiation of \mathbf{K} with respect to s and D_ϵ denotes differentiation of \mathbf{K} with respect to ϵ . Firstly we compute:

$$\begin{aligned}
D_s \mathbf{K} &= - \operatorname{Re} \int [\dot{\mathbf{P}}(|\mathbf{P} + \epsilon|^{p-1} \overline{(\mathbf{P} + \epsilon)} - |\mathbf{P}|^{p-1} \overline{\mathbf{P}}) \\
&\quad - \frac{p-1}{2} |\mathbf{P}|^{p-3} (\dot{\mathbf{P}} \overline{\mathbf{P}} + \dot{\overline{\mathbf{P}}} \mathbf{P}) \operatorname{Re}(\overline{\epsilon} \mathbf{P}) - |\mathbf{P}|^{p-1} \overline{\epsilon} \dot{\mathbf{P}}](y) dy \\
&= - \operatorname{Re} \int [\dot{\mathbf{P}}(|\mathbf{P} + \epsilon|^{p-1} \overline{(\mathbf{P} + \epsilon)} - |\mathbf{P}|^{p-1} \overline{\mathbf{P}}) \\
&\quad - \frac{p-1}{2} |\mathbf{P}|^{p-3} \frac{\epsilon \overline{\mathbf{P}}^2 \dot{\mathbf{P}} + \overline{\epsilon} |\mathbf{P}|^2 \dot{\mathbf{P}} + \epsilon |\mathbf{P}|^2 \dot{\overline{\mathbf{P}}} + \overline{\epsilon} \mathbf{P}^2 \dot{\overline{\mathbf{P}}}}{2} - |\mathbf{P}|^{p-1} \overline{\epsilon} \dot{\mathbf{P}}](y) dy \\
&= - \langle \dot{\mathbf{P}}, |\mathbf{P} + \epsilon|^{p-1} (\mathbf{P} + \epsilon) - |\mathbf{P}|^{p-1} \mathbf{P} - \frac{p+1}{2} \epsilon |\mathbf{P}|^{p-1} - \frac{p-1}{2} \overline{\epsilon} \mathbf{P}^2 |\mathbf{P}|^{p-3} \rangle.
\end{aligned}$$

We observe that $\dot{P}_k = -\dot{z}_k \cdot \nabla P_k + i\dot{v}_k \cdot (y - z_k) P_k$. Denote

$$K = |\mathbf{P} + \epsilon|^{p-1} (\mathbf{P} + \epsilon) - |\mathbf{P}|^{p-1} \mathbf{P} - \frac{p+1}{2} \epsilon |\mathbf{P}|^{p-1} - \frac{p-1}{2} \overline{\epsilon} \mathbf{P}^2 |\mathbf{P}|^{p-3}$$

then by (2.2.17), $K = |\mathbf{P} + \epsilon|^{p-1} (\mathbf{P} + \epsilon) - |\mathbf{P}|^{p-1} \mathbf{P} - F'(\mathbf{P}) \cdot \epsilon$, we deduce from (2.2.18) that

$$|K| \lesssim |\epsilon|^2 |\mathbf{P}|^{p-2} + |\epsilon|^p$$

so we obtain

$$|\langle i\dot{v}_k \cdot (y - z_k) P_k, K \rangle| \lesssim (\|\epsilon\|_{H^1}^2 + \|\epsilon\|_{H^1}^p) |\dot{v}| \lesssim s^{-2} \|\epsilon\|_{H^1}^2.$$

Next we look more precisely at K

$$K = \frac{\overline{\epsilon} \cdot F''(\mathbf{P}) \cdot \epsilon}{2} + O\left(\left|\frac{\epsilon}{\mathbf{P}}\right|^{2+} |\mathbf{P}|^p\right) + O(|\epsilon|^p)$$

as $|\dot{z}_k| \lesssim s^{-1}$ and $p - 2^+ > 0$, we have

$$\left| \left\langle -\dot{z}_k \cdot \nabla P_k, \left| \frac{\epsilon}{\mathbf{P}} \right|^{2+} |\mathbf{P}|^p \right\rangle \right| \lesssim s^{-1} \|\epsilon\|_{H^1}^{2+}$$

and

$$|\langle -\dot{z}_k \cdot \nabla P_k, |\epsilon|^p \rangle| \lesssim s^{-1} \|\epsilon\|_{H^1}^p.$$

Combining these computations, we get

$$D_s \mathbf{K}(s, \epsilon) = \sum_{k=1}^2 \langle \dot{z}_k \cdot \nabla P_k, \frac{\overline{\epsilon} \cdot F''(\mathbf{P}) \cdot \epsilon}{2} \rangle + O(s^{-1} \|\epsilon\|_{H^1}^{2+}) + O(s^{-2} \|\epsilon\|_{H^1}^2) + O(s^{-1} \|\epsilon\|_{H^1}^p). \quad (2.3.48)$$

Secondly we consider

$$D_\epsilon \mathbf{K}(s, \epsilon) = -\Delta \epsilon + \epsilon - (|\mathbf{P} + \epsilon|^{p-1} (\mathbf{P} + \epsilon) - |\mathbf{P}|^{p-1} \mathbf{P})$$

and note that the equation (2.3.9) of ϵ can be rewritten as

$$i\dot{\epsilon} - D_\epsilon \mathbf{K}(s, \epsilon) - i \frac{\dot{\lambda}}{\lambda} \Lambda \epsilon + (1 - \dot{\gamma}) \epsilon + \mathcal{E}_{\mathbf{P}} = 0$$

so that

$$\begin{aligned}\langle D_\epsilon \mathbf{K}(s, \epsilon), \dot{\epsilon} \rangle &= \langle i D_\epsilon \mathbf{K}(s, \epsilon), i \dot{\epsilon} \rangle \\ &= \frac{\dot{\lambda}}{\lambda} \langle D_\epsilon \mathbf{K}(s, \epsilon), \Lambda \epsilon \rangle - (1 - \dot{\gamma}) \langle i D_\epsilon \mathbf{K}(s, \epsilon), \epsilon \rangle - \langle i D_\epsilon \mathbf{K}(s, \epsilon), \mathcal{E}_P \rangle.\end{aligned}$$

On the other hand, from (2.3.29) and (2.3.25)–(2.3.26), we have

$$\begin{aligned}\left| \frac{\dot{\lambda}}{\lambda} \langle D_\epsilon \mathbf{K}(s, \epsilon), \Lambda \epsilon \rangle \right| &\lesssim \left| \frac{\dot{\lambda}}{\lambda} \right| \left(\|\epsilon\|_{H^1}^2 + \|\epsilon\|_{H^1}^{p+1} \right) \lesssim (C^*)^2 s^{-2} \|\epsilon\|_{H^1}^2, \\ |(1 - \dot{\gamma}) \langle i D_\epsilon \mathbf{K}(s, \epsilon), \epsilon \rangle| &\lesssim |1 - \dot{\gamma}| (\|\epsilon\|_{H^1}^2 + \|\epsilon\|_{H^1}^{p+1}) \lesssim (C^*)^2 s^{-2} \|\epsilon\|_{H^1}^2.\end{aligned}$$

For the last term, we rewrite

$$\begin{aligned}\langle i D_\epsilon \mathbf{K}(s, \epsilon), \mathcal{E}_P \rangle &= \langle -i \Delta \epsilon + i \epsilon - i (|\mathbf{P} + \epsilon|^{p-1} (\mathbf{P} + \epsilon) - |\mathbf{P}|^{p-1} \mathbf{P}), \\ &\quad [e^{i\Gamma_1} \vec{m}_1 \cdot \vec{M}Q](y - z_1(s)) + [e^{i\Gamma_2} \vec{m}_2 \cdot \vec{M}Q](y - z_2(s)) + G \rangle.\end{aligned}$$

Recall that with $\eta_1 = \eta_1^1 + i\eta_1^2$ for η_1^1, η_1^2 real, from the expression of operators L_+ and L_-

$$\begin{aligned}I_1 &= \langle -i \Delta \epsilon + i \epsilon - i (|\mathbf{P} + \epsilon|^{p-1} (\mathbf{P} + \epsilon) - |\mathbf{P}|^{p-1} \mathbf{P}), [e^{i\Gamma_1} \vec{m}_1 \cdot \vec{M}Q](y - z_1(s)) \rangle \\ &= \langle -i \Delta \eta_1 + i \eta_1 - i (|\mathbf{P}_1 + \eta_1|^{p-1} (\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1} \mathbf{P}_1), \vec{m}_1 \cdot \vec{M}Q \rangle \\ &= \langle i L_+ \eta_1^1 - L_- \eta_1^2, \vec{m}_1 \cdot \vec{M}Q \rangle \\ &\quad - \left\langle i \left(|\mathbf{P}_1 + \eta_1|^{p-1} (\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1} \mathbf{P}_1 - \frac{p+1}{2} Q^{p-1} \eta_1 - \frac{p-1}{2} Q^{p-1} \bar{\eta}_1 \right), \vec{m}_1 \cdot \vec{M}Q \right\rangle \\ &= -\frac{\dot{\lambda}}{\lambda} \langle \eta_1, -2Q \rangle + \left(\dot{v} - \frac{\dot{\lambda}}{\lambda} v \right) \langle \eta_1, -2i \nabla Q \rangle \\ &\quad - \left\langle i \left(|\mathbf{P}_1 + \eta_1|^{p-1} (\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1} \mathbf{P}_1 - \frac{p+1}{2} Q^{p-1} \eta_1 - \frac{p-1}{2} Q^{p-1} \bar{\eta}_1 \right), \vec{m}_1 \cdot \vec{M}Q \right\rangle.\end{aligned}$$

By orthogonality of η_1 (2.3.6), (2.3.30) and the estimate (2.3.29), (2.3.34), we get

$$|I_1| = O((C^*)^2 s^{-3} \log^{-\frac{3}{4}}(s)).$$

By symmetry, we have the same estimate for I_2 . Finally, from (2.2.14) and (2.3.25), we have $\|G\|_{H^1} \lesssim s^{-2}$ so using integration by parts and Cauchy-Schwarz inequality

$$|\langle -i \Delta \epsilon + i \epsilon - i (|\mathbf{P} + \epsilon|^{p-1} (\mathbf{P} + \epsilon) - |\mathbf{P}|^{p-1} \mathbf{P}), G \rangle| \lesssim s^{-2} \|\epsilon\|_{H^1}. \quad (2.3.49)$$

The collection of above estimates finishes the proof of (2.3.47).

step 3 Variation of the localized momentum. We now claim: for all $s \in [s^*, s^{in}]$,

$$\left| \frac{d}{ds} [\mathbf{J}(s, \epsilon(s))] - \sum_{k=1}^2 2v_k \cdot \left\langle \nabla P_k, \frac{\bar{\epsilon} \cdot F''(\mathbf{P}) \cdot \epsilon}{2} \right\rangle \right| \lesssim s^{-2} \log^{-\frac{3}{4}}(s) \|\epsilon(s)\|_{H^1}. \quad (2.3.50)$$

Indeed, we compute, for any k ,

$$\frac{d}{ds} [J_k(s, \epsilon(s))] = \dot{v}_k \cdot \text{Im} \int (\nabla \epsilon \bar{\epsilon}) \chi_k + v_k \cdot \frac{d}{ds} \text{Im} \int (\nabla \epsilon \bar{\epsilon}) \chi_k.$$

By (2.3.25) and (2.3.26), we have

$$\left| \dot{v}_k \cdot \text{Im} \int (\nabla \epsilon \bar{\epsilon}) \chi_k \right| \lesssim s^{-2} \|\epsilon\|_{H^1}^2.$$

Recall from (2.3.42) that

$$\begin{aligned} |v_k| \left| \frac{d}{ds} [\mathcal{M}_k(s, \epsilon(s))] - 2 \cdot \langle \nabla P_k, \frac{\bar{\epsilon} \cdot F''(\mathbf{P}) \cdot \epsilon}{2} \rangle - 2 \left(\dot{z}_k - 2v_k + \frac{\dot{\lambda}}{\lambda} z_k \right) \langle i \nabla Q, \nabla \eta_k \rangle \right| \\ \lesssim s^{-1} \log^{-1}(s) \|\epsilon(s)\|_{H^1}^2. \end{aligned}$$

From (2.3.31),

$$|v_k| \left| \left(\dot{z}_k - 2v_k + \frac{\dot{\lambda}}{\lambda} z_k \right) \langle i \nabla Q, \nabla \eta_k \rangle \right| \lesssim s^{-2} \log^{-\frac{3}{4}}(s) \|\epsilon\|_{H^1}$$

so we get (2.3.50).

step 4 Conclusion. Recall that, by (2.3.31), $|\dot{z}_k - 2v_k| \lesssim s^{-1} \log^{-\frac{3}{4}}(s)$ so

$$\left| (\dot{z}_k - 2v_k) \cdot \langle \nabla P_k, \frac{\bar{\epsilon} \cdot F''(\mathbf{P}) \cdot \epsilon}{2} \rangle \right| \lesssim s^{-1} \log^{-\frac{3}{4}}(s) \|\epsilon\|_{H^1}^2,$$

and (2.3.46) now follows from (2.3.47), (2.3.50). This concludes the proof of Proposition 2.3.44. \square

2.3.2.4 End of the bootstrap argument

We close the bootstrap estimates (2.3.25).

step 1 Closing the estimate in ϵ . By (2.3.46) in Proposition 2.3.44 and then (2.3.25)–(2.3.26), we have

$$\left| \frac{d}{ds} [\mathbf{W}(s, \epsilon(s))] \right| \lesssim s^{-2} \|\epsilon\|_{H^1} \lesssim C^* s^{-3}.$$

Thus, by integration on $[s, s^{in}]$ for any $s \in [s^*, s^{in}]$, using $\epsilon(s^{in}) = 0$ (see (2.3.19)), we obtain

$$|\mathbf{W}(s, \epsilon(s))| \lesssim C^* s^{-2}.$$

By (2.3.45) in Proposition 2.3.44, we get

$$\|\epsilon(s)\|_{H^1}^2 \leq C_0 C^* s^{-2}.$$

Therefore, for C^* large enough such that $C_0 C^* \leq \frac{(C^*)^2}{4}$, we have $\|\epsilon\|_{H^1} \leq \frac{C^*}{2} s^{-1}$ which strictly improves the estimate on $\|\epsilon\|_{H^1}$ in (2.3.25).

step 2 Closing the parameter z . Now, we need to finish the bootstrap argument for $z(s)$. Note that

$$\begin{aligned} \left| \dot{v} + c \frac{z}{|z|} |z|^{-\frac{d-1}{2}} e^{-|z|} \right| &\lesssim s^{-2} \log^{-1}(s) \\ |\dot{z} - 2v| &\lesssim s^{-1} \log^{-\frac{3}{4}}(s) \end{aligned}$$

thus we deduce

$$\begin{aligned} \left| \dot{v} \cdot \frac{z}{|z|} + c |z|^{-\frac{d-1}{2}} e^{-|z|} \right| &\lesssim s^{-2} \log^{-1}(s) \\ \left| \dot{z} \cdot \frac{z}{|z|} - 2v \cdot \frac{z}{|z|} \right| &\lesssim s^{-1} \log^{-\frac{3}{4}}(s). \end{aligned}$$

We get

$$\left| 2 \left(v \cdot \frac{z}{|z|} \right) \left(\dot{v} \cdot \frac{z}{|z|} \right) + c \dot{z} \cdot \frac{z}{|z|} |z|^{-\frac{d-1}{2}} e^{-|z|} \right| \lesssim s^{-3} \log^{-\frac{3}{4}}(s)$$

since $|v| \lesssim s^{-1}$, $|\dot{v}| \lesssim s^{-2}$. Therefore, by the explicit choice of initial data

$$v(s^{in}) = \sqrt{c}(z^{in})^{-\frac{d-1}{4}} e^{-\frac{1}{2}z^{in}} \vec{e}_1, \quad z(s^{in}) = z^{in} \vec{e}_1,$$

we integrate on $[s, s^{in}]$ for any $s \in [s^*, s^{in})$, if $d-1 > 0$

$$\left| \left(v \cdot \frac{z}{|z|} \right)^2 - c|z|^{-\frac{d-1}{2}} e^{-|z|} \right| \lesssim s^{-2} \log^{-\frac{3}{4}}(s) + \int_s^{s^{in}} |\dot{z}| |z|^{-\frac{d-1}{2}-1} e^{-|z|} \lesssim s^{-2} \log^{-\frac{3}{4}}(s),$$

if $d-1 = 0$, $\left| 2(v \cdot \frac{z}{|z|})(\dot{v} \cdot \frac{z}{|z|}) + c \dot{z} \cdot \frac{z}{|z|} e^{-|z|} \right| \lesssim s^{-3} \log^{-\frac{3}{4}}(s)(s)$ implies also $\left| (v \cdot \frac{z}{|z|})^2 - c e^{-|z|} \right| \lesssim s^{-2} \log^{-\frac{3}{4}}(s)$. In both cases, combining with (2.3.31), we get

$$\left| \left(v \cdot \frac{z}{|z|} \right) - \sqrt{c}|z|^{-\frac{d-1}{4}} e^{-\frac{1}{2}|z|} \right| + \left| \left(\dot{z} \cdot \frac{z}{|z|} \right) - 2\left(v \cdot \frac{z}{|z|} \right) \right| \lesssim s^{-1} \log^{-\frac{3}{4}}(s)$$

so $\left| \left(\dot{z} \cdot \frac{z}{|z|} \right) - 2\sqrt{c}|z|^{-\frac{d-1}{4}} e^{-\frac{1}{2}|z|} \right| \lesssim s^{-1} \log^{-\frac{3}{4}}(s)$. Next, note that if $d-1 > 0$

$$\frac{d}{ds} (|z|^{\frac{d-1}{4}} e^{\frac{1}{2}|z|}) = \frac{1}{2} \dot{z} \cdot \frac{z}{|z|} |z|^{\frac{d-1}{4}} e^{\frac{1}{2}|z|} + \frac{d-1}{4} \dot{z} \cdot \frac{z}{|z|} |z|^{\frac{d-1}{4}-1} e^{\frac{1}{2}|z|}$$

and if $d-1 = 0$

$$\frac{d}{ds} (e^{\frac{1}{2}|z|}) = \frac{1}{2} \dot{z} \cdot \frac{z}{|z|} e^{\frac{1}{2}|z|}$$

thus

$$\left| \frac{d}{ds} (|z|^{\frac{d-1}{4}} e^{\frac{1}{2}|z|}) - c^{\frac{1}{2}} \right| \lesssim \log^{-\frac{3}{4}}(s) + \frac{d-1}{4} |\dot{z}| |z|^{\frac{d-1}{4}-1} e^{\frac{1}{2}|z|} \lesssim \log^{-\frac{3}{4}}(s) \quad (2.3.51)$$

here we use $|z| \lesssim \log^{-1}(s)$ and $|\dot{z}| \lesssim s^{-1}$. Next, we need to adjust the initial choice of z^{in} through a topological argument (see [4] for a similar argument). We define ζ and ξ the following two functions on $[s^*, s^{in}]$

$$\zeta(s) = c^{-\frac{1}{2}} |z|^{\frac{d-1}{4}} e^{\frac{1}{2}|z|}, \quad \xi(s) = (\zeta(s) - s)^2 s^{-2} \log(s). \quad (2.3.52)$$

Then, (2.3.51) writes

$$|\dot{\zeta}(s) - 1| \lesssim \log^{-\frac{3}{4}}(s). \quad (2.3.53)$$

According to (2.3.25), our objective is to prove that there exists a suitable choice of

$$\zeta(s^{in}) = \zeta^{in} \in [s^{in} - s^{in} \log^{-\frac{1}{2}}(s^{in}), s^{in} + s^{in} \log^{-\frac{1}{2}}(s^{in})],$$

so that $s^* = s_0$. Assume for the sake of contradiction that for all $\zeta^\sharp \in [-1, 1]$, the choice

$$\zeta^{in} = s^{in} + \zeta^\sharp s^{in} \log^{-\frac{1}{2}}(s^{in})$$

leads to $s^* = s^*(\zeta^\sharp) \in (s_0, s^{in})$. Since all estimates in (2.3.25) except the one on $z(s)$ have been strictly improved on $[s^*, s^{in}]$, it follows from $s^*(\zeta^\sharp) \in (s_0, s^{in}]$ and continuity that

$$|\zeta(s^*(\zeta^\sharp)) - s^*| = s^* \log^{-\frac{1}{2}} s^* \quad \text{i.e.} \quad \zeta(s^*(\zeta^\sharp)) = s^* \pm s^* \log^{-\frac{1}{2}} s^*.$$

We need a transversality condition to reach a contradiction. We compute:

$$\dot{\xi}(s) = 2(\zeta(s) - s)(\dot{\zeta}(s) - 1)s^{-2} \log(s) - (\zeta(s) - s)^2 (2s^{-3} \log(s) - s^{-3}). \quad (2.3.54)$$

At $s = s^*$, this gives

$$|\dot{\xi}(s^*) + 2(s^*)^{-1}| \lesssim (s^*)^{-1} \log^{-\frac{1}{4}}(s^*).$$

Thus, for s_0 large enough,

$$\dot{\xi}(s^*) < -(s^*)^{-1}. \quad (2.3.55)$$

A consequence of the transversality property (2.3.55) is the continuity of the function $\zeta^\sharp \in [-1, 1] \mapsto s^*(\zeta^\sharp)$. Indeed, let $\epsilon > 0$ then there exists $\delta > 0$ such that $\xi(s^*(\zeta^\sharp) - \epsilon) > 1 + \delta$ and $\xi(s^*(\zeta^\sharp) + \epsilon) < 1 - \delta$. Moreover, by definition of $s^*(\zeta^\sharp)$ (choosing δ small enough) for all $s \in [s^*(\zeta^\sharp) + \epsilon, s^{in}]$ we have $\xi(s) < 1 - \delta$. But from the continuity of the flow, there exists $\iota > 0$ such that for all $|\zeta^\sharp - \tilde{\zeta}^\sharp| < \iota$

$$\forall s \in [s^*(\tilde{\zeta}^\sharp) - \epsilon, s^{in}], \quad |\tilde{\xi}(s) - \xi(s)| \leq \delta/2$$

so we obtain that $s^*(\zeta^\sharp) - \epsilon \leq s^*(\tilde{\zeta}^\sharp) \leq s^*(\zeta^\sharp) + \epsilon$ and the continuity of $s^*(\zeta^\sharp)$ as expected. Thus we deduce the continuity of the function Φ defined by

$$\Phi : \zeta^\sharp \in [-1, 1] \mapsto (\zeta(s^*) - s^*)(s^*)^{-1} \log^{\frac{1}{2}}(s^*) \in \{-1, 1\}.$$

Moreover, for $\zeta^\sharp = -1$ and $\zeta^\sharp = 1$, in these two cases $\xi(s^{in}) = 1$, from (2.3.54) we have that $\dot{\xi}(s^{in}) < 0$ thus $s^* = s^{in}$. Therefore, $\Phi(-1) = -1$ and $\Phi(1) = 1$, but this is a contradiction with the continuity.

In conclusion, there exists at least a choice of

$$\zeta(s^{in}) = \zeta^{in} \in (s^{in} - s^{in} \log^{-\frac{1}{2}}(s^{in}), s^{in} + s^{in} \log^{-\frac{1}{2}}(s^{in}))$$

such that $s^* = s_0$. This concludes our bootstrap argument.

step 3 Estimate on the parameter λ . From (2.3.29), we obtain

$$\left| \frac{\dot{\lambda}}{\lambda} \right| \lesssim s^{-2}.$$

By integration on $[s, s^{in}]$, for any $s \in [s_0, s^{in}]$, using the value $\lambda(s^{in}) = \lambda^{in} = 1$ (see (2.3.21)), we have

$$|\log(\lambda(s))| \lesssim s^{-1},$$

and thus

$$|\lambda(s) - 1| \lesssim s^{-1}$$

or in other words

$$|\lambda^{-1}(s) - 1| \lesssim s^{-1}. \quad (2.3.56)$$

□

2.4 Compactness arguments

2.4.1 Construction of a sequence of backwards solutions

Lemma 2.4.1. There exist $t_0 > 1$ and a sequence of solutions $u_n \in \mathcal{C}([t_0, T_n], H^1)$ of (NLS), where

$$T_n \rightarrow +\infty \quad \text{as } n \rightarrow +\infty, \quad (2.4.2)$$

satisfying the following estimates, for all $t \in [t_0, T_n]$,

$$\begin{aligned} & ||z_n(t)| - 2 \log t| \lesssim \log(\log t), \quad |\lambda_n^{-1}(t) - 1| \lesssim t^{-1}, \\ & |v_n(t)| \lesssim t^{-1}, \quad \|\epsilon_n(t)\|_{H^1} \lesssim t^{-1}, \quad \left| |z_n(t)|^{\frac{d-1}{2}} e^{|z_n(t)|} - ct^2 \right| \lesssim t^2 \log^{-\frac{1}{2}}(t), \end{aligned} \quad (2.4.3)$$

where $(\lambda_n, z_n, \gamma_n, v_n)$ are the parameters of the decomposition of u_n , i.e.

$$u_n(t, x) = \frac{e^{i\gamma_n(t)}}{\lambda_n^{\frac{2}{p-1}}(t)} \left(\sum_{k=1}^2 [e^{i\Gamma_{k,n}} Q] \left(\frac{x}{\lambda_n(t)} + \frac{(-1)^k}{2} z_n(t) \right) + \epsilon_n \left(t, \frac{x}{\lambda_n(t)} \right) \right), \quad (2.4.4)$$

with $\Gamma_{k,n}(t, x) = \frac{(-1)^{k+1}}{2} v_n(t) \cdot \frac{x}{\lambda_n(t)}$.

Proof of Lemma 2.4.1. Applying Proposition 2.3.20 with $s^{in} = n$ for any large n , there exists a solution $u_n(t)$ of (NLS) defined on the time interval $[0, T_n]$ where

$$T_n = \int_{s_0}^n \lambda_n^2(s) ds.$$

and whose decomposition satisfies the uniform estimates (2.3.22). First, we see that $T_n \rightarrow +\infty$ as $n \rightarrow +\infty$ which follows directly from the estimate on $\lambda_n(s)$. From the definition of the rescaled time s (see (2.3.3)), for any $s \in [s_0, n]$, we have

$$t(s) = \int_{s_0}^s \lambda_n^2(s') ds' \quad \text{where} \quad |\lambda_n^2(s) - 1| \lesssim s^{-1}.$$

Fix $t_0 = \bar{s}_0$ with $\bar{s}_0 > s_0$ large enough independent of n such that for all s with $n \geq s > \bar{s}_0$

$$\frac{1}{2}s \leq \int_{s_0}^s \lambda_n^2(s') ds' = s(1 + O(s^{-1})) \leq \frac{3}{2}s$$

then, for all $t \in [t_0, T_n]$

$$t(s) = s(1 + O(s^{-1})) \geq \frac{1}{2}s$$

and

$$s = t(1 + O(t^{-1})).$$

Thus, we get from (2.3.22)

$$\begin{aligned} & ||z_n(s)| - 2 \log(s)| \lesssim \log(\log(s)) \Leftrightarrow ||z_n(s(t))| - 2 \log(t)| \lesssim \log(\log(t)) \\ & |\lambda_n^{-1}(s) - 1| \lesssim s^{-1} \Leftrightarrow |\lambda_n^{-1}(s(t)) - 1| \lesssim t^{-1} \\ & \|\epsilon_n(s)\|_{H^1} \lesssim s^{-1} \Leftrightarrow \|\epsilon_n(s(t))\|_{H^1} \lesssim t^{-1} \\ & |v_n(s)| \lesssim s^{-1} \Leftrightarrow |v_n(s(t))| \lesssim t^{-1}. \end{aligned}$$

□

2.4.2 Compactness argument

Next, we claim a strong compactness result in $L^2(\mathbb{R}^d)$.

Lemma 2.4.5. There exist $u_0 \in H^1(\mathbb{R}^d)$ and a sub-sequence, still denoted u_n , such that

$$\begin{aligned} u_n(t_0) &\rightharpoonup u_0 \text{ weakly in } H^1(\mathbb{R}^d) \\ u_n(t_0) &\rightarrow u_0 \text{ in } H^\sigma(\mathbb{R}^d), \text{ for } 0 \leq \sigma < 1 \end{aligned}$$

as $n \rightarrow \infty$.

Proof of Lemma 2.4.5. By interpolation, it is enough to prove that the sub-sequence $u_n(t_0) \xrightarrow{L^2} u_0$ as $n \rightarrow \infty$. First, we claim the following: $\forall \delta_1 > 0, \delta_1 \ll 1, \exists n_0 \gg 1, \exists K_1 = K_1(\delta_1) > 0$ such that $\forall n \geq n_0$

$$\int_{|x| > K_1} |u_n(t_0, x)|^2 dx < \delta_1. \quad (2.4.6)$$

Indeed, denote $x_n(t) = z_n(t)\lambda_n(t)$ and

$$\begin{aligned} \tilde{R}_n(t, x) &= e^{i\gamma_n(t)} \sum_{k=1}^2 [e^{i\Gamma_{k,n}} Q_{\lambda_n^{-1}(t)}] \left(x + \frac{(-1)^k}{2} x_n(t) \right) \\ R_n(t, x) &= e^{i\gamma_n(t)} \sum_{k=1}^2 Q \left(x + \frac{(-1)^k}{2} x_n(t) \right) \end{aligned}$$

then we have

$$\begin{aligned} \|u_n(t) - R_n(t)\|_{H^1} &\leq \|\epsilon_n(t)\|_{H^1} + 2\|\tilde{R}_n(t) - R_n(t)\|_{H^1} \\ &\lesssim \|\epsilon_n(t)\|_{H^1} + |\lambda_n^{-1}(t) - 1| + |v_n(t)| \lesssim t^{-1}. \end{aligned} \quad (2.4.7)$$

We get a direct consequence of the above estimate

$$\|u_n(t)\|_{H^1} < C \quad (2.4.8)$$

for all $t \in [t_0, T_n]$ since $\|R_n(t)\|_{H^1} \leq 2\|Q\|_{H^1}$. Furthermore, for fixed δ_1 , there exists $t_1 > t_0$ such that

$$\|u_n(t_1) - R_n(t_1)\|_{H^1} \lesssim (t_1)^{-1} < \sqrt{\delta_1}$$

for n large enough that $T_n > t_1$; in others words, we have

$$\int |u_n(t_1, x) - R_n(t_1, x)|^2 dx < \delta_1.$$

Besides, $|x_n(t_1) - 2 \log(t_1)| \lesssim \log(\log t_1)$ then for $K_2 \gg 1$ large enough we have

$$\int_{|x| > K_2} |R_n(t_1, x)|^2 dx < \delta_1.$$

Consider now a \mathcal{C}^1 cut-off function $g : \mathbb{R} \rightarrow [0, 1]$ such that : $g \equiv 0$ on $(-\infty, 1]$, $0 < g' < 2$ on $(1, 2)$ and $g \equiv 1$ on $[2, +\infty)$. Since $\|u_n(t)\|_{H^1} < C$ bounded in H^1 independently of n and $t \in [t_0, T_n]$, we can choose $\gamma_1 > 0$ independent of n such that

$$\gamma_1 \geq \frac{2}{\delta_1} (t_1 - t_0) C^2.$$

We have by direct calculations, for $t \in [t_0, T_n]$

$$\begin{aligned} \left| \frac{d}{dt} \int |u_n(t, x)|^2 g \left(\frac{|x| - K_2}{\gamma_1} \right) dx \right| &= \left| \frac{1}{\gamma_1} \operatorname{Im} \int u \left(\nabla \bar{u} \cdot \frac{x}{|x|} \right) g' \left(\frac{|x| - K_2}{\gamma_1} \right) dx \right| \\ &\leq \frac{2}{\gamma_1} \sup_{T_n \geq t \geq t_0} \|u_n(t)\|_{H^1}^2 \leq \frac{\delta_1}{t_1 - t_0}. \end{aligned}$$

By integration from t_0 to t_1

$$\begin{aligned} \int |u_n(t_0, x)|^2 g \left(\frac{|x| - K_2}{\gamma_1} \right) dx - \int |u_n(t_1, x)|^2 g \left(\frac{|x| - K_2}{\gamma_1} \right) dx \\ \leq \int_{t_0}^{t_1} \left| \frac{d}{dt} \int |u_n(t, x)|^2 g \left(\frac{|x| - K_2}{\gamma_1} \right) dx \right| \leq \delta_1. \end{aligned}$$

From the properties of g we conclude:

$$\begin{aligned} \int_{|x| > 2\gamma_1 + K_2} |u_n(t_0, x)|^2 dx &\leq \int |u_n(t_0, x)|^2 g \left(\frac{|x| - K_2}{\gamma_1} \right) dx \\ &\leq \int |u_n(t_1, x)|^2 g \left(\frac{|x| - K_2}{\gamma_1} \right) dx + \delta_1 \leq \int_{|x| > K_2} |u_n(t_1, x)|^2 dx + \delta_1 \leq 5\delta_1. \end{aligned}$$

Thus (2.4.6) is proved. As $\|u_n(t_0)\|_{H^1} < C$, there exists a subsequence of (u_n) (still denoted by (u_n)) and $u_0 \in H^1$ such that

$$\begin{aligned} u_n(t_0) &\rightharpoonup u_0 \quad \text{weakly in } H^1(\mathbb{R}^2), \\ u_n(t_0) &\rightarrow u_0 \quad \text{in } L_{loc}^2(\mathbb{R}^d), \text{ as } n \rightarrow +\infty \end{aligned}$$

and by (2.4.6), we conclude that $u_n(t_0) \xrightarrow{L^2} u_0$ as required. \square

Let us finish the proof of the Main Theorem in sub-critical cases with $p > 2$. We consider u the solution of (NLS) corresponding to $u(t_0) = u_0$. By continuous dependence of the solution upon the initial data (see [2] and [3]), for all $0 \leq \sigma < 1$, for all $t \in [t_0, +\infty)$,

$$u_n(t) \rightarrow u(t) \quad \text{in } H^\sigma(\mathbb{R}^d).$$

Moreover, the decomposition (\vec{q}, ϵ) of u satisfies, for all $t \geq t_0$,

$$\vec{q}_n(t) \rightarrow \vec{q}(t), \quad \epsilon_n(t) \rightarrow \epsilon(t) \text{ in } H^\sigma, \quad \epsilon_n(t) \rightharpoonup \epsilon(t) \text{ in } H^1 \quad (2.4.9)$$

(see e.g. [26], Claim p.598). In particular, for all $t \in [t_0, +\infty)$, $u(t)$ decomposes as

$$u(t, x) = \frac{e^{i\gamma(t)}}{\lambda^{\frac{p-1}{2}}(t)} \left(\sum_{k=1}^2 [e^{i\Gamma_k} Q] \left(\frac{x + \frac{(-1)^k \lambda(t) z(t)}{2}}{\lambda(t)} \right) + \epsilon \left(t, \frac{x}{\lambda(t)} \right) \right), \quad (2.4.10)$$

where $\Gamma_k(t, y) = \frac{(-1)^{k+1}}{2} v(t) \cdot y$ and it follows from the uniform estimates (2.4.3) that

$$\begin{aligned} ||z(t)| - 2 \log t| &\lesssim \log(\log t), \quad |\lambda^{-1}(t) - 1| \lesssim t^{-1}, \\ |v(t)| &\lesssim t^{-1}, \quad \|\epsilon(t)\|_{H^1} \lesssim t^{-1}, \quad \left| |z(t)|^{\frac{d-1}{2}} e^{|z(t)|} - ct^2 \right| \lesssim t^2 \log^{-\frac{1}{2}}(t). \end{aligned} \quad (2.4.11)$$

We obtain $|x_1(t) - x_2(t)| = \lambda(t)|z(t)| \rightarrow 2(1 + o(1)) \log t$, more precisely

$$||x_1(t) - x_2(t)| - 2 \log(t)| \lesssim \log(\log(t))$$

and the following estimate

$$\left\| u(t) - e^{i\gamma(t)} \sum_{k=1}^2 Q(x - x_k(t)) \right\|_{H^1} \lesssim \|\epsilon(t)\|_{H^1} + |\lambda^{-1}(t) - 1| + |v(t)| \lesssim t^{-1}. \quad (2.4.12)$$

2.5 Sub-critical cases with $1 < p \leq 2$

In this section, we show the difficulties occurring and sketch the proof of Main Theorem in the case $1 < p \leq 2$. In this case, let

$$2^+ = \min(2^*, \frac{p+3}{2}).$$

Note that $p - 2^+ > -1$. From (2.2.16), we deduce the following Taylor expansions:

$$F(\mathbf{P} + \epsilon) = F(\mathbf{P}) + F'(\mathbf{P}) \cdot \epsilon + O(|\epsilon|^p) \quad (2.5.1)$$

$$F(\mathbf{P} + \epsilon) = F(\mathbf{P}) + F'(\mathbf{P}) \cdot \epsilon + O\left(\left|\frac{\epsilon}{\mathbf{P}}\right|^2 |\mathbf{P}|^p\right) \quad (2.5.2)$$

(since $|\epsilon| > \frac{|\mathbf{P}|}{2}$ then $|\epsilon|^p \lesssim |\frac{\epsilon}{\mathbf{P}}|^2 |\mathbf{P}|^p$ and $|\epsilon| \leq \frac{|\mathbf{P}|}{2}$ then $|\frac{\epsilon}{\mathbf{P}}|^2 |\mathbf{P}|^p \lesssim |\epsilon|^p$) and

$$F(\mathbf{P} + \epsilon) = F(\mathbf{P}) + F'(\mathbf{P}) \cdot \epsilon + \frac{\bar{\epsilon} \cdot F''(\mathbf{P}) \cdot \epsilon}{2} + O\left(\left|\frac{\epsilon}{\mathbf{P}}\right|^{2^+} |\mathbf{P}|^p\right). \quad (2.5.3)$$

In the following remark, we identify new problems compared with the case $p > 2$.

Remark 2.5.4. Let us try to control the nonlinear interaction term

$$G(y; (z(s), v(s))) = |\mathbf{P}|^{p-1} \mathbf{P} - |P_1|^{p-1} P_1 - |P_2|^{p-1} P_2.$$

Since $|P_1| > |P_2|$ for $y \cdot \frac{z}{|z|} > 0$ and $|P_2| > |P_1|$ for $y \cdot \frac{z}{|z|} < 0$, one has by (2.2.16)

$$\begin{aligned} |G(y; (z(s), v(s)))| &= \left| |P_1 + P_2|^{p-1} (P_1 + P_2) - |P_1|^{p-1} P_1 - |P_2|^{p-1} P_2 \right| \\ &\lesssim |P_1|^{p-1} |P_2| \cdot \mathbf{1}_{y \cdot \frac{z}{|z|} > 0} + |P_2|^{p-1} |P_1| \cdot \mathbf{1}_{y \cdot \frac{z}{|z|} < 0}. \end{aligned} \quad (2.5.5)$$

Using the asymptotic behavior of Q , on the half space $\{y \cdot \frac{z}{|z|} > 0\}$

$$\begin{aligned} |P_1|^{p-1} |P_2| \cdot \mathbf{1}_{y \cdot \frac{z}{|z|} > 0} &\lesssim |P_1 P_2|^{p-1} |P_2|^{2-p} \cdot \mathbf{1}_{y \cdot \frac{z}{|z|} > 0} \lesssim |z|^{-\frac{(p-1)(d-1)}{2}} e^{-(p-1)|z|} |P_2|^{2-p} \cdot \mathbf{1}_{y \cdot \frac{z}{|z|} > 0} \\ &\lesssim |z|^{-\frac{(p-1)(d-1)}{2}} e^{-(p-1)|z|} \left|\frac{z}{2}\right|^{-\frac{(2-p)(d-1)}{2}} e^{-\frac{2-p}{2}|z|} \lesssim |z|^{-\frac{d-1}{2}} e^{-\frac{p}{2}|z|}. \end{aligned} \quad (2.5.6)$$

By symmetry, we have the same estimate on the other half space $\{y \cdot \frac{z}{|z|} < 0\}$ and thus

$$\|G\|_{L^\infty} \lesssim |z|^{-\frac{d-1}{2}} e^{-\frac{p}{2}|z|} \sim s^{-p} \quad (2.5.7)$$

(to be compared with (2.2.10)). Now for the projection of interaction, we recall that its core part (as identified in the proof of Lemma 2.2.20 and in step 4 of Proposition 2.5.10) is given by

$$H(z) = p \int_{y \cdot \frac{z}{|z|} > -\frac{|z|}{2}} Q^{p-1}(y) \nabla Q(y) Q(y+z) dy + p \int_{y \cdot \frac{z}{|z|} < -\frac{|z|}{2}} Q^{p-1}(y+z) \nabla Q(y) Q(y) dy$$

and the following estimate of $H(z)$ is still valid for $1 < p \leq 2$ (see Lemma 2.2.20)

$$\left| H(z) - c_Q I_Q \frac{z}{|z|} |z|^{-\frac{d-1}{2}} e^{-|z|} \right| \lesssim |z|^{-1-\frac{d-1}{2}} e^{-|z|}. \quad (2.5.8)$$

In summary, the projection $\langle G, e^{i\Gamma_1} \nabla Q(y - z_1(s)) \rangle$ and thus \dot{v} are still of order s^{-2} , however the interaction G is of order $s^{-p} \gg s^{-2}$ in L^∞ norm. Therefore, there still exist some terms in the interaction that perturb our regime and prevent us to close the bootstrap arguments (for example (2.3.49)).

In view of the above remark, we look for a refined approximate solution \mathbf{P} of the form

$$\begin{aligned}\mathbf{P}(s, y) &= \mathbf{P}(y; (z(s), v(s))) = \sum_{k=1}^2 e^{iv_k(s)(y-z_k(s))} Q(y - z_k(s)) + W(y; (z(s), v(s))) \\ &= \sum_{k=1}^2 P_k(s, y) + W(y; (z(s), v(s))),\end{aligned}\tag{2.5.9}$$

where $W(y; (z(s), v(s)))$ to be determined.

Proposition 2.5.10 (Expansion of the refined approximate solution). There exists a series of $(J + 1)$ functions $R_j(y; (z(s), v(s)))$ which are invariant by τ and v such that by setting

$$W(y; (z(s), v(s))) = \sum_{j=0}^J R_j(y; (z(s), v(s))),$$

the error $\mathcal{E}_{\mathbf{P}}$ defined as in (2.2.8) admits the decomposition

$$\mathcal{E}_{\mathbf{P}} = [e^{i\Gamma_1} \vec{m}_1 \cdot \vec{M}Q](y - z_1(s)) + [e^{i\Gamma_2} \vec{m}_2 \cdot \vec{M}Q](y - z_2(s)) + G_0,\tag{2.5.11}$$

where under the bootstrap assumptions (2.3.25) and the pointwise control of the modulation equation (2.3.29)–(2.3.31)

$$|z| \lesssim \log(s), \quad |\dot{z}| \lesssim s^{-1}, \quad |v| \lesssim s^{-1}, \quad |\dot{v}| \lesssim s^{-2}, \quad \left| \frac{\dot{\lambda}}{\lambda} \right| \lesssim (C^*)^2 s^{-2}, \quad |\dot{\gamma} - 1| \lesssim (C^*)^2 s^{-2},$$

the corrected interaction term G_0 satisfies

$$\|G_0\|_{L^2} \lesssim s^{-2}, \quad \|\nabla G_0\|_{L^2} \lesssim s^{-2}.\tag{2.5.12}$$

Moreover, G_0 is symmetric and

$$\left| \langle G_0, e^{i\Gamma_1(y-z_1(s))} \nabla Q(y - z_1(s)) \rangle - C_p \frac{z}{|z|} |z|^{-\frac{d-1}{2}} e^{-|z|} \right| \lesssim s^{-2} \log^{-1}(s)\tag{2.5.13}$$

with $C_p > 0$.

Remark 2.5.14. In fact, before the pointwise control of the modulation equations in Lemma 2.3.28, we bound $\|G_0\|_{L^2}, \|\nabla G_0\|_{L^2}$ by z, v and $s^{-p} |\vec{m}_1|$ then once we have the control on \vec{m}_1 , we will obtain (2.5.12).

Proof of Proposition 2.5.10. step 1 Properties of the Helmholtz operators. We recall well-known properties of $(-\Delta + 1)u_s(y) = f_s(y)$ in \mathbb{R}^d . The operator $(-\Delta + 1)^{-1}$ is continuous from L^2 to H^1 , in particular

$$\|u\|_{H^1} \leq \|f\|_{L^2}.$$

It is self-adjoint

$$\langle u, (-\Delta + 1)g \rangle = \langle (-\Delta + 1)u, g \rangle = \langle f, g \rangle,\tag{2.5.15}$$

invariant by τ, v and $(-\Delta + 1)\dot{u}_s(y) = \dot{f}_s(y)$ (\dot{f} denotes the derivative with respect to time s). Moreover, by theory of elliptic equation (see e.g [1]), we have an explicit kernel representation E_d for $(-\Delta + 1)^{-1}$ as follows

$$E_d(x) = -(2\pi)^{-\frac{d}{2}} \left(\frac{1}{|x|} \right)^{\frac{d}{2}-1} \mathcal{K}_{\frac{d}{2}-1}(|x|)$$

$$u(x) = \int_{\mathbb{R}^d} E_d(x-y)f(y)dy \quad (2.5.16)$$

where \mathcal{K}_α is modified Bessel functions of second kind which is decreasing exponentially when $|x| \rightarrow +\infty$. This is a convolution of type $L^1 \star L^\infty$ so we deduce that

$$\|u\|_{L^\infty} \lesssim \|f\|_{L^\infty}. \quad (2.5.17)$$

Next, we claim the exponential decay property: assume that a regular function f is exponentially decreasing in the direction e_j , $e^{\delta|y_j|}|f(y)| \leq C$ with $0 < \delta < 1$, then so is the solution u of $(-\Delta + 1)^{-1}$.

Indeed, we consider

$$\begin{aligned} e^{\delta|x_j|}|u(x)| &= e^{\delta|x_j|} \left| \int_{\mathbb{R}^d} E_d(x-y)f(y)dy \right| \\ &\lesssim C \left| \int_{\mathbb{R}^d} \left(\frac{1}{|x-y|} \right)^{\frac{d}{2}-1} e^{-|x-y|} e^{\delta(|x_j|-|y_j|)} dy \right| \lesssim C \left\| \left(\frac{1}{|x|} \right)^{\frac{d}{2}-1} e^{-(1-\delta)|x|} \right\|_{L^1} \lesssim C. \end{aligned}$$

step 2 Iteration of R_j . We introduce a suitable smooth cut-off function that localizes the points whose distances to center of two solitons are smaller than $|z|$. Denote $\psi_0 : \mathbb{R} \rightarrow [0, 1]$ such that

$$0 \leq \psi'_0 \leq C, \quad \psi_0 \equiv 0 \text{ on } (-\infty, -1], \quad \psi_0 \equiv 1 \text{ on } [0, +\infty)$$

and

$$\psi(y; z(s)) = \psi_0 \left(|z(s)| - \left| y + \frac{z(s)}{2} \right| \right) \psi_0 \left(|z(s)| - \left| y - \frac{z(s)}{2} \right| \right).$$

Recall the definition of G

$$G(y; (z(s), v(s))) = |P_1 + P_2|^{p-1}(P_1 + P_2) - |P_1|^{p-1}P_1 - |P_2|^{p-1}P_2$$

and denote \mathbf{pr}_i the projection on the direction ∇Q around each soliton

$$\mathbf{pr}_i(f) = \frac{\langle f(\cdot), \nabla Q(\cdot + \frac{(-1)^i}{2}z(s)) \rangle}{\|\nabla Q(\cdot + \frac{(-1)^i}{2}z(s))\|_{L^2}^2} \nabla Q(\cdot + \frac{(-1)^i}{2}z(s)).$$

Setting

$$\begin{aligned} A_0(y; (z(s), v(s))) &= G(y; (z(s), v(s)))\psi(y; z(s)), \\ \tilde{A}_0 &= A_0 - \mathbf{pr}_1(A_0) - pr_2(A_0), \\ A_1 &= |P_1 + P_2 + R_0|^{p-1}(P_1 + P_2 + R_0) - |P_1 + P_2|^{p-1}(P_1 + P_2), \\ \tilde{A}_1 &= A_1 - \mathbf{pr}_1(A_1) - pr_2(A_1) \end{aligned}$$

and for $j \geq 2$

$$\begin{aligned} A_j &= |P_1 + P_2 + \sum_{k=0}^{j-1} R_k|^{p-1}(P_1 + P_2 + \sum_{k=0}^{j-1} R_k) - |P_1 + P_2 + \sum_{k=0}^{j-2} R_k|^{p-1}(P_1 + P_2 + \sum_{k=0}^{j-2} R_k), \\ \tilde{A}_j &= A_j - \mathbf{pr}_1(A_j) - pr_2(A_j). \end{aligned}$$

Observe that

$$\sum_{j=1}^J A_j = |P_1 + P_2 + \sum_{k=0}^{j-1} R_k|^{p-1}(P_1 + P_2 + \sum_{k=0}^{j-1} R_k) - |P_1 + P_2|^{p-1}(P_1 + P_2). \quad (2.5.18)$$

Then let

$$R_j(y; (z(s), v(s))) = (-\Delta + 1)^{-1}\tilde{A}_j.$$

We will show by induction on j the following properties

- R_j is almost orthogonal to $\nabla(Q^p)(\cdot \pm \frac{1}{2}z)$, i.e,

$$\langle R_j(\cdot), \nabla(Q^p)(\cdot \pm \frac{1}{2}z) \rangle \lesssim s^{-3}. \quad (2.5.19)$$

- The L^∞, H^1 norm of R_j satisfy

$$\|R_{j+1}\|_{L^\infty} \lesssim s^{-(p-1)} \|R_j\|_{L^\infty} \lesssim s^{-p},$$

$$\|R_{j+1}\|_{H^1} \lesssim s^{-(p-1-\kappa)} \|R_j\|_{H^1} \lesssim s^{-p} \log^{dp}(s)$$

with $0 < \kappa \ll 1$ to be determined (see (2.5.36), (2.5.37)).

- After a finite number $(J+1)$ of steps, the function R_J satisfies the two following estimates: there is $\epsilon > 0$

$$|Q^{p-1}(y)R_J(y + \frac{z}{2})| + |Q^{p-1}(y)R_J(y - \frac{z}{2})| \lesssim e^{-\epsilon|y|} s^{-2} \quad (2.5.20)$$

$$\|R_J\|_{H^1}^p + s^{p(p-1)} \|R_J\|_{H^1} \ll s^{-2} \quad (2.5.21)$$

independently of z, v ((2.5.21) means that there exists $\delta > 0$ such that $\|R_J\|_{H^1}^p + s^{-p(p-1)} \|R_J\|_{H^1} \lesssim s^{-2-\delta}$).

Note that a direct consequence of the above estimates is

$$\begin{aligned} & \|A_{J+1}\|_{L^2} \\ &= \left\| \left| P_1 + P_2 + \sum_{j=0}^J R_j \right|^{p-1} (P_1 + P_2 + \sum_{j=0}^J R_j) - \left| P_1 + P_2 + \sum_{j=0}^{J-1} R_j \right|^{p-1} (P_1 + P_2 + \sum_{j=0}^{J-1} R_j) \right\|_{L^2} \\ &\lesssim \left\| \left| P_1 + P_2 + \sum_{j=0}^{J-1} R_j \right|^{p-1} |R_J| + |R_J|^p \right\|_{L^2} \\ &\lesssim \|Q^{p-1}(\cdot)R_J(\cdot + \frac{z}{2})\|_{L^2} + \|R_J\|_{L^2}^p + s^{p(p-1)} \|R_J\|_{L^2} \lesssim s^{-2} \end{aligned} \quad (2.5.22)$$

since $\|R_j\|_{L^\infty} \lesssim \|R_0\|_{L^\infty} \lesssim s^{-p}, \forall j = \overline{1, J}$.

Let us begin with R_0 . We have that

$$\begin{aligned} |G(y; (z(s), v(s)))| &= \left| |P_1 + P_2|^{p-1} (P_1 + P_2) - |P_1|^{p-1} P_1 - |P_2|^{p-1} P_2 \right| \\ &\lesssim |P_1|^{p-1} |P_2| \cdot \mathbb{1}_{y \cdot \frac{z}{|z|} > 0} + |P_2|^{p-1} |P_1| \cdot \mathbb{1}_{y \cdot \frac{z}{|z|} < 0}. \end{aligned}$$

Consider

$$\begin{aligned} |P_1|^{p-1} |P_2| \cdot \mathbb{1}_{y \cdot \frac{z}{|z|} > 0} &\lesssim e^{-(p-1)(z_1 - y \cdot \frac{z}{|z|})} \left| \frac{z}{2} \right|^{-\frac{d-1}{2}} e^{-(y \cdot \frac{z}{|z|} - z_2)} \\ &\lesssim |z|^{-\frac{d-1}{2}} e^{-\frac{p}{2}|z|} e^{-(2-p)y \cdot \frac{z}{|z|}} \lesssim s^{-p} e^{-(2-p)y \cdot \frac{z}{|z|}} |z|^{-\frac{(2-p)(d-1)}{2}}, \end{aligned} \quad (2.5.23)$$

by symmetry, we also have the same estimate on $\{y \cdot \frac{z}{|z|} < 0\}$. Thus, from definition of ψ , we get

$$\|e^{(2-p)y \cdot \frac{z}{|z|}} A_0(y; (z(s), v(s)))\|_{L^\infty} \lesssim s^{-p} |z|^{-\frac{(2-p)(d-1)}{2}} \lesssim s^{-p} \quad (2.5.24)$$

and

$$\|A_0(y; (z(s), v(s)))\|_{L^2} \lesssim s^{-p} \log^d(s). \quad (2.5.25)$$

The estimate (2.5.24) yields

$$\begin{aligned} |A_0(y + \frac{z}{2})| &\lesssim e^{-(2-p)|y \cdot \frac{z}{|z|} + \frac{z}{2}|} s^{-p} |z|^{-\frac{(2-p)(d-1)}{2}} \\ &\lesssim e^{(2-p)|y \cdot \frac{z}{|z|}|} e^{-(2-p)\frac{|z|}{2}} |z|^{-\frac{(2-p)(d-1)}{2}} s^{-p} \lesssim e^{(2-p)|y \cdot \frac{z}{|z|}|} s^{-2} \end{aligned}$$

so it gives a control on projections of A_0

$$\left| \int_{\mathbb{R}^d} A_0(y + \frac{z}{2}) \nabla Q(y) dy \right| \lesssim s^{-2}. \quad (2.5.26)$$

Therefore, from definition of \tilde{A}_0

$$\|e^{(2-p)|y \cdot \frac{z}{|z|}|} \tilde{A}_0\|_{L^\infty} \lesssim s^{-p} |z|^{-\frac{(2-p)(d-1)}{2}}, \quad \|\tilde{A}_0\|_{L^2} \lesssim s^{-p} \log^d(s).$$

From step 1, we can transfer these properties to $R_0(y; (z(s), v(s)))$

$$\|e^{(2-p)|y \cdot \frac{z}{|z|}|} R_0(y; (z(s), v(s)))\|_{L^\infty} \lesssim s^{-p} |z|^{-\frac{(2-p)(d-1)}{2}}, \quad (2.5.27)$$

$$\|R_0(y; (z(s), v(s)))\|_{H^1} \lesssim s^{-p} \log^d(s). \quad (2.5.28)$$

To show the almost orthogonality condition, we note that $(-\Delta + 1)\nabla Q = \nabla(Q^p)$ so from self-adjoint property (2.5.15) of $(-\Delta + 1)$, we have

$$\begin{aligned} \left| \langle R_0, \nabla(Q^p)(\cdot + \frac{z}{2}) \rangle \right| &= \left| \langle A_0 - \mathbf{pr}_1(A_0) - \mathbf{pr}_2(A_0), \nabla Q(\cdot + \frac{z}{2}) \rangle \right| \\ &= \left| \langle \mathbf{pr}_1(A_0), \nabla Q(\cdot + \frac{z}{2}) \rangle \right| \lesssim s^{-2} \langle \nabla Q(\cdot - \frac{z}{2}), \nabla Q(\cdot + \frac{z}{2}) \rangle \lesssim s^{-3}. \end{aligned}$$

If $\frac{3}{2} < p \leq 2$, we see that R_0 satisfies already the conditions (2.5.20), (2.5.21) as

$$\begin{aligned} \|R_0\|_{H^1}^p &\lesssim s^{-p^2} \log^{dp}(s) \leq s^{-\frac{9}{4}} \log^{dp}(s) \ll s^{-2} \\ s^{p(p-1)} \|R_0\|_{H^1} &\lesssim s^{-\frac{3}{4}} s^{-\frac{3}{2}} \ll s^{-2} \end{aligned}$$

and $|R_0(y + \frac{z}{2})| \lesssim e^{-(2-p)|y \cdot \frac{z}{|z|} + \frac{z}{2}|} s^{-p} |z|^{-\frac{(2-p)(d-1)}{2}} \lesssim e^{(2-p)|y \cdot \frac{z}{|z|}|} s^{-2}$ so for $\epsilon = 2p - 3 > 0$

$$|Q^{p-1}(y) R_0(y + \frac{z}{2})| \lesssim e^{(2-p)|y \cdot \frac{z}{|z|}|} Q^{(2-p)}(y) s^{-2} Q^{(2p-3)}(y) \lesssim e^{-\epsilon|y|} s^{-2}.$$

Thus $J = 0$ and $W = R_0(y; (z(s), v(s)))$ in this case.

If $\frac{4}{3} < p \leq \frac{3}{2}$, we consider $A_1(y; (z(s), v(s)))$, by (2.2.16), we obtain

$$\begin{aligned} &\left| |P_1 + P_2 + R_0|^{p-1} (P_1 + P_2 + R_0) - |P_1 + P_2|^{p-1} (P_1 + P_2) \right. \\ &\quad \left. - \frac{p+1}{2} |P_1 + P_2|^{p-1} R_0 - \frac{p-1}{2} |P_1 + P_2|^{p-3} (P_1 + P_2)^2 \overline{R_0} \right| \lesssim |R_0|^p. \quad (2.5.29) \end{aligned}$$

Next remark that for $1 < p \leq 2$, $||P_1 + P_2|^{p-1} - |P_1|^{p-1} - |P_2|^{p-1}| \lesssim \min(|P_1|^{p-1}, |P_2|^{p-1})$ so the main part of $A_1 = |P_1 + P_2 + R_0|^{p-1} (P_1 + P_2 + R_0) - |P_1 + P_2|^{p-1} (P_1 + P_2)$ can be computed by

$$\begin{aligned} &\left\| \frac{p+1}{2} |P_1 + P_2|^{p-1} R_0 + \frac{p-1}{2} |P_1 + P_2|^{p-3} (P_1 + P_2)^2 \overline{R_0} - p |P_1 + P_2|^{p-1} R_0 \right\|_{L^2} \\ &\quad \lesssim \left\| (|v|^2 |y|^2 + |v|^2 |z|^2) |R_0| (|P_1|^{p-1} + |P_2|^{p-1}) \right\|_{L^2} \ll s^{-2} \end{aligned} \quad (2.5.30)$$

$$\left| |P_1 + P_2|^{p-1} R_0 - (|P_1|^{p-1} + |P_2|^{p-1}) R_0 \right| \lesssim \min(|P_1|^{p-1}, |P_2|^{p-1}) |R_0| \quad (2.5.31)$$

here in (2.5.30) we use the bootstrap assumptions and the control of modulation equations. Let estimate $R_0(y)Q^{p-1}(y + \frac{z}{2})$, from the decreasing properties of R_0 (2.5.27), we have

$$\begin{aligned} |R_0(y)Q^{p-1}(y + \frac{z}{2})| &\lesssim e^{-(2-p)|y \cdot \frac{z}{|z|}|} s^{-p} |z|^{-\frac{(2-p)(d-1)}{2}} e^{(p-1)|y \cdot \frac{z}{|z|}|} e^{-(p-1)\frac{|z|}{2}} \\ &\lesssim e^{-(3-2p)|y \cdot \frac{z}{|z|}|} s^{-(2p-1)} |z|^{-\frac{(3-2p)(d-1)}{2}} \end{aligned} \quad (2.5.32)$$

so for $\kappa \ll 1$ determined later in (2.5.36)

$$\|R_0(y)Q^{p-1}(y + \frac{z}{2})\|_{L^2} \lesssim s^{-(2p-1-\kappa)}. \quad (2.5.33)$$

The collection of above estimates gives a bound on norm L^2 and on the decay property of A_1

$$\begin{aligned} \|A_1\|_{L^2} &\lesssim \|R_0\|_{L^2}^p + \|R_0|P_2|^{p-1}\|_{L^2} + \|R_0|P_2|^{p-1}\|_{L^2} \\ &\lesssim s^{-p^2} \log^{dp}(s) + s^{-(2p-1-\kappa)} \leq s^{-(2p-1-\kappa)}, \end{aligned}$$

$$\begin{aligned} &\|e^{(3-2p)|y \cdot \frac{z}{|z|}|} A_1\|_{L^\infty} \\ &\lesssim \|e^{(3-2p)|y \cdot \frac{z}{|z|}|} |R_0|^p\|_{L^\infty} + \|e^{(3-2p)|y \cdot \frac{z}{|z|}|} R_0 |P_2|^{p-1}\|_{L^\infty} + \|e^{(3-2p)|y \cdot \frac{z}{|z|}|} R_0 |P_2|^{p-1}\|_{L^\infty} \\ &\lesssim s^{-p^2} + s^{-(2p-1)} |z|^{-\frac{(3-2p)(d-1)}{2}} \leq s^{-(2p-1)} |z|^{-\frac{(3-2p)(d-1)}{2}} \end{aligned}$$

as the decay $e^{-(2-p)|y \cdot \frac{z}{|z|}|}$ of R_0 is faster than the one of $e^{-(3-2p)|y \cdot \frac{z}{|z|}|}$. Finally, we consider

$$\begin{aligned} &\left| \langle A_1, \nabla Q(y + \frac{z}{2}) \rangle - p \left\langle Q^{p-1}(y - \frac{z}{2}) R_0 + Q^{p-1}(y + \frac{z}{2}) R_0, \nabla Q(y + \frac{z}{2}) \right\rangle \right| \\ &\lesssim \left\langle |R_0|^p, \nabla Q(y + \frac{z}{2}) \right\rangle + \left\langle \min(|P_1|^{p-1}, |P_2|^{p-1}) |R_0|, \nabla Q(y + \frac{z}{2}) \right\rangle \\ &\lesssim \left\langle e^{-(2-p)p|y \cdot \frac{z}{|z|}|} s^{-p^2} |z|^{-\frac{(2-p)p(d-1)}{2}} e^{(2-p)p|y \cdot \frac{z}{|z|}|} e^{-(2-p)p\frac{|z|}{2}}, Q^{1-(2-p)p}(y + \frac{z}{2}) \right\rangle \\ &\quad + \left\langle s^{-(p-1)} e^{-(2-p)|y \cdot \frac{z}{|z|}|} s^{-p} |z|^{-\frac{(2-p)(d-1)}{2}} e^{(2-p)|y \cdot \frac{z}{|z|}|} e^{-(2-p)\frac{|z|}{2}}, Q^{1-(2-p)}(y + \frac{z}{2}) \right\rangle \\ &\lesssim s^{-2p} + s^{-(p+1)} \ll s^{-2}. \end{aligned}$$

We can deduce from the almost orthogonality (2.5.19) that

$$\langle A_1, \nabla Q(y \pm \frac{z}{2}) \rangle \ll s^{-2}, \quad (2.5.34)$$

in other words, we have

$$\|\mathbf{pr}_i(A_1)\|_{L^2} \ll s^{-2}, \quad i = 1, 2. \quad (2.5.35)$$

Therefore, we have the following estimates for $\tilde{A}_1 = A_1 - \mathbf{pr}_1(A_1) - \mathbf{pr}_2(A_2)$

$$\|\tilde{A}_1\|_{L^2} \lesssim s^{-(2p-1-\kappa)}, \quad \|e^{(3-2p)|y \cdot \frac{z}{|z|}|} \tilde{A}_1\|_{L^\infty} \lesssim s^{-(2p-1)} |z|^{-\frac{(3-2p)(d-1)}{2}}$$

and the analogue for R_1

$$\|R_1\|_{H^1} \lesssim s^{-(2p-1-\kappa)}, \quad \|e^{(3-2p)|y \cdot \frac{z}{|z|}|} R_1\|_{L^\infty} \lesssim s^{-(2p-1)} |z|^{-\frac{(3-2p)(d-1)}{2}}.$$

There exists $0 < \kappa \ll 1$ such that for all $p > \frac{4}{3}$

$$-(2p-1-\kappa)p < -2, \quad -(2p-1-\kappa) - p(p-1) < -2 \quad (2.5.36)$$

so $\|R_J\|_{H^1}^p + s^{p(p-1)}\|R_J\|_{H^1} \lesssim s^{-(2p-1-\kappa)p} + s^{-(2p-1-\kappa)-p(p-1)} \ll s^{-2}$ and for $\epsilon = 3p - 4 > 0$

$$\begin{aligned} |Q^{p-1}(y)R_1(y + \frac{\tilde{z}}{2})| &\lesssim e^{-(3-2p)|y \cdot \frac{\tilde{z}}{|\tilde{z}|} + \frac{\tilde{z}}{2}|} s^{-(2p-1)} |z|^{-\frac{(3-2p)(d-1)}{2}} Q^{p-1}(y) \\ &\leq e^{(3-2p)|y \cdot \frac{\tilde{z}}{|\tilde{z}|}|} Q^{(3-2p)}(y) s^{-2} Q^{(3p-4)}(y) \lesssim e^{-\epsilon|y|} s^{-2}. \end{aligned}$$

The almost orthogonal property of V_1 is a direct consequence of $\langle \tilde{A}_1(\cdot \pm \frac{\tilde{z}}{2}), \nabla Q \rangle \lesssim s^{-3}$. Thus $J = 1$ and $W = R_0(y; (z(s), v(s))) + R_1(y; (z(s), v(s)))$ in this case.

If $\frac{J+3}{J+2} < p \leq \frac{J+2}{J+1}$, we proceed the same way and after $(J+1)$ steps, our process will finish with

$$W = \sum_{j=0}^J R_j(y; (z(s), v(s))),$$

$\epsilon = (J+2)p - (J+3) > 0$ and $0 < \kappa \ll 1$ such that for all $\frac{J+2}{J+1} < p \leq \frac{J+1}{J}$

$$-((J+1)p - J - \kappa)p < -2, \quad -((J+1)p - J - \kappa) - p(p-1) < -2. \quad (2.5.37)$$

step 3 Estimate of G_0 . Let $\mathbf{P} = P_1 + P_2 + W$ and put into the definition $\mathcal{E}_{\mathbf{P}}$, it follows from the computations in Lemma 2.2.6 that

$$\begin{aligned} \mathcal{E}_{\mathbf{P}} &= [e^{i\Gamma_1} \vec{m}_1 \cdot \vec{M}Q](y - z_1(s)) + [e^{i\Gamma_2} \vec{m}_2 \cdot \vec{M}Q](y - z_2(s)) + |\mathbf{P}|^{p-1} \mathbf{P} - |P_1|^{p-1} P_1 - |P_2|^{p-1} P_2 \\ &\quad + \sum_{j=0}^J (\Delta - 1) R_j + \sum_{j=0}^J [i\dot{R}_j - i\frac{\dot{\lambda}}{\lambda} \Lambda R_j + (1 - \dot{\gamma}) R_j]. \quad (2.5.38) \end{aligned}$$

Note that

$$\sum_{j=1}^J (\Delta - 1) R_j = - \sum_{j=1}^J \tilde{A}_j = - \sum_{j=1}^J A_j + \sum_{j=1}^J [\mathfrak{pr}_1(A_j) + \mathfrak{pr}_2(A_j)]$$

thus following (2.5.11) and (2.5.18), we have the explicit expression of G_0

$$\begin{aligned} G_0 &= |P_1 + P_2 + \sum_{j=0}^J R_j|^{p-1} (P_1 + P_2 + \sum_{j=0}^J R_j) - |P_1 + P_2 + \sum_{j=0}^{J-1} R_j|^{p-1} (P_1 + P_2 + \sum_{j=0}^{J-1} R_j) \\ &\quad + \sum_{j=1}^J [A_j + (\Delta - 1) R_j] + |P_1 + P_2|^{p-1} (P_1 + P_2) - |P_1|^{p-1} P_1 - |P_2|^{p-1} P_2 + (\Delta - 1) R_0 \\ &\quad + \sum_{j=0}^J [i\dot{R}_j - i\frac{\dot{\lambda}}{\lambda} \Lambda R_j + (1 - \dot{\gamma}) R_j] \\ &= |P_1 + P_2 + \sum_{j=0}^J R_j|^{p-1} (P_1 + P_2 + \sum_{j=0}^J R_j) - |P_1 + P_2 + \sum_{j=0}^{J-1} R_j|^{p-1} (P_1 + P_2 + \sum_{j=0}^{J-1} R_j) \\ &\quad + \sum_{j=1}^J [\mathfrak{pr}_1(A_j) + \mathfrak{pr}_2(A_j)] + G + (\Delta - 1) R_0 + \sum_{j=0}^J [i\dot{R}_j - i\frac{\dot{\lambda}}{\lambda} \Lambda R_j + (1 - \dot{\gamma}) R_j] \\ &= A_{J+1} + \sum_{j=1}^J [\mathfrak{pr}_1(A_j) + \mathfrak{pr}_2(A_j)] + \mathfrak{pr}_1(G\psi) + \mathfrak{pr}_2(G\psi) + G(1 - \psi) + \sum_{j=0}^J [i\dot{R}_j - i\frac{\dot{\lambda}}{\lambda} \Lambda R_j + (1 - \dot{\gamma}) R_j]. \end{aligned}$$

We bound the first term by (2.5.22)

$$\|A_{J+1}\|_{L^2} \lesssim s^{-2}.$$

Next, from pointwise control of the modulation equations, we have $\left|\frac{\dot{\lambda}}{\lambda}\right|, |1 - \dot{\gamma}| \lesssim (C^*)^2 s^{-2}$ and $\|R_j\|_{H^1} < \|R_0\|_{H^1} \lesssim s^{-p} \log^d(s)$, therefore

$$\left\| \sum_{j=0}^J i \frac{\dot{\lambda}}{\lambda} \Lambda R_j - (1 - \dot{\gamma}) R_j \right\|_{L^2} \ll s^{-2}. \quad (2.5.39)$$

We recall (2.5.26) that

$$\|\mathbf{pr}_1(G\psi)\|_{L^2} + \|\mathbf{pr}_2(G\psi)\|_{L^2} \lesssim s^{-2}$$

and similarly to (2.5.35), we have

$$\|\mathbf{pr}_1(A_j)\|_{L^2} + \|\mathbf{pr}_2(A_j)\|_{L^2} \ll s^{-2}, \quad \forall j \geq 1.$$

The term

$$\|G(1 - \psi)\|_{L^2} \lesssim |z|^{-\frac{d-1}{2}} e^{-|z|} (\|P_1\|_{L^2}^{p-1} + \|P_2\|_{L^2}^{p-1}) \lesssim s^{-2},$$

this is a consequence of the choice of localized cut-off function ψ and the decay property of Q .

For the last term, we have $\dot{R}_j = (-\Delta + 1)^{-1} \dot{A}_j$, so

$$\|\dot{R}_j\|_{H^1} \leq \|\dot{A}_j\|_{L^2}.$$

We consider R_0 and A_0 , proceeding as the way we control G in (2.5.5), we have that \dot{G} decays more rapidly because of extra terms \dot{z} and \dot{v} . In fact, we have

$$\begin{aligned} |\dot{G}| &\leq \left| (\dot{P}_1 + \dot{P}_2) |P_1 + P_2|^{p-1} - \dot{P}_1 |P_1|^{p-1} - \dot{P}_2 |P_2|^{p-1} \right| \\ &+ \left| (\dot{P}_1 + \dot{P}_2) |P_1 + P_2|^{p-2} (P_1 + P_2) - \dot{P}_1 |P_1|^{p-2} P_1 - \dot{P}_2 |P_2|^{p-2} P_2 \right| \end{aligned}$$

and

$$\dot{P}_k = \dot{z}_k \nabla P_k + i \dot{v}_k (y - z_k) P_k.$$

Then for $|P_1| > |P_2|$, we deduce from the asymptotic behavior of $Q, \nabla Q$ at infinity that

$$\begin{aligned} &\left| (\nabla P_1 - \nabla P_2) |P_1 + P_2|^{p-1} - \nabla P_1 |P_1|^{p-1} + \nabla P_2 |P_2|^{p-1} \right| \\ &= \left| \nabla P_1 |P_1|^{p-1} \left[\left(1 - \frac{\nabla P_2}{\nabla P_1}\right) \left|1 + \frac{P_2}{P_1}\right|^{p-1} - 1 + \frac{\nabla P_2}{\nabla P_1} \left|\frac{P_2}{P_1}\right|^{p-1} \right] \right| \lesssim |P_1|^{p-1} |P_2| \cdot \mathbf{1}_{y \cdot \frac{z}{|z|} > 0} \end{aligned}$$

and

$$\begin{aligned} &\left| (\nabla P_1 - \nabla P_2) |P_1 + P_2|^{p-2} (P_1 + P_2) - \nabla P_1 |P_1|^{p-2} P_1 + \nabla P_2 |P_2|^{p-2} P_2 \right| \\ &= \left| \nabla P_1 |P_1|^{p-2} P_1 \left[\left(1 - \frac{\nabla P_2}{\nabla P_1}\right) \left|1 + \frac{P_2}{P_1}\right|^{p-2} \left(1 + \frac{P_2}{P_1}\right) - 1 + \frac{\nabla P_2}{\nabla P_1} \left|\frac{P_2}{P_1}\right|^{p-2} \frac{P_2}{P_1} \right] \right| \\ &\lesssim |P_1|^{p-1} |P_2| \cdot \mathbf{1}_{y \cdot \frac{z}{|z|} > 0}. \end{aligned}$$

We do the same way in case $|P_2| > |P_1|$ and for function $(y - z_k)P_k$ thus we obtain from (2.5.23) that

$$|\dot{G}| \lesssim |\dot{z}|s^{-p} + |\dot{v}|s^{-p} \lesssim s^{-(p+1)}$$

so $\|\dot{A}_0\|_{L^2} \lesssim \|\dot{G}\psi\|_{L^2} + |\dot{z}|\|G\nabla\psi\|_{L^2} \ll s^{-2}$. Next remark that for a function f

$$\left| \frac{d}{ds} \mathbf{pr}_i(f) \right| \lesssim \left| \mathbf{pr}_i(\dot{f}) \right| + |\dot{z}| |\mathbf{pr}_i(f)|, \quad i = 1, 2 \quad (2.5.40)$$

thus $\|\dot{A}_0\|_{L^2} \ll s^{-2}$, by properties of $(-\Delta + 1)^{-1}$, this implies $\|\dot{R}_0\|_{L^2} \ll s^{-2}$. We will prove by induction that

$$\|R_j\|_{L^2}, \forall j \geq 1.$$

For A_j ($j \geq 1$), we have

$$|\dot{A}_j| \lesssim \left| (\dot{P}_1 + \dot{P}_2 + \sum_{k=0}^{j-1} \dot{R}_k) |P_1 + P_2 + \sum_{k=0}^{j-1} R_k|^{p-1} - (\dot{P}_1 + \dot{P}_2 + \sum_{k=0}^{j-2} \dot{R}_k) |P_1 + P_2 + \sum_{k=0}^{j-2} R_k|^{p-1} \right|.$$

As $\|\dot{R}_k\|_{L^2} \ll s^{-2}$ for $0 \leq k < j$, it is sufficient to prove that

$$\left| (\dot{P}_1 + \dot{P}_2) |P_1 + P_2 + \sum_{k=0}^{j-1} R_k|^{p-1} - (\dot{P}_1 + \dot{P}_2) |P_1 + P_2 + \sum_{k=0}^{j-2} R_k|^{p-1} \right| \ll s^{-2}. \quad (2.5.41)$$

Let estimate

$$B_j = \left| (\nabla P_1 - \nabla P_2) |P_1 + P_2 + \sum_{k=0}^{j-1} R_k|^{p-1} - (\nabla P_1 - \nabla P_2) |P_1 + P_2 + \sum_{k=0}^{j-2} R_k|^{p-1} \right|.$$

We have three cases to consider.

At a given point x , if it holds $\max(|P_1|, |P_2|, |V_0|, \dots, |V_{j-1}|) > \max(|P_1|, |P_2|)$, then

$$B_j \lesssim \sum_{k=0}^{j-1} |V_k|^p \lesssim s^{-p};$$

otherwise if $\max(|P_1|, |P_2|, |V_0|, \dots, |V_{j-1}|) = |P_1|$ then, by the first-order Taylor expansion

$$\begin{aligned} B_j &= \left| \nabla P_1 |P_1|^{p-1} \left[\frac{1 - \nabla P_2 / \nabla P_1}{1 + P_2 / P_1 + \sum_{k=0}^{j-1} R_k / P_1} \left(1 + \frac{P_2}{P_1} \sum_{k=0}^{j-1} \frac{R_k}{P_1} \right) \right] - \left(1 - \frac{\nabla P_2}{\nabla P_1} \right) \left[1 + \frac{P_2}{P_1} + \sum_{k=0}^{j-2} \frac{R_k}{P_1} \right]^{p-1} \right| \\ &\lesssim |P_1|^{p-1} |P_2| \cdot \mathbb{1}_{y - \frac{z}{|z|} > 0} + \sum_{k=0}^{j-1} |P_1|^{p-1} |R_k| \\ &\lesssim s^{-p}, \end{aligned}$$

and similarly for the case $\max(|P_1|, |P_2|, |V_0|, \dots, |V_{j-1}|) = |P_2|$. Thus, $B_j \lesssim s^{-p}$, from which we deduce (2.5.41). Recall the estimate for the derivative of a projection (2.5.40) so we get $\|\dot{A}_j\|_{L^2} \ll s^{-2}$. In conclusion, we have $\|G_0\|_{L^2} \lesssim s^{-2}$. Similarly, the same estimate holds for ∇G_0 , which finishes the proof of (2.5.12).

step 4 Estimate of projection. From step 3, the terms whose norm L^2 is of order s^{-2} are

$A_{J+1}, \mathbf{pr}_1(G\psi), \mathbf{pr}_2(G\psi), G(1-\psi)$. As $|\langle \mathbf{pr}_2(G\psi), e^{i\Gamma_1(y-z_1(s))} \nabla Q(y-z_1(s)) \rangle| \ll s^{-2}$ and similarly to (2.5.35), we can show $|\mathbf{pr}_1(A_{J+1})| \ll s^{-2}$ thus

$$\langle G_0, e^{i\Gamma_1(y-z_1(s))} \nabla Q(y-z_1(s)) \rangle = \langle G, e^{i\Gamma_1(y-z_1(s))} \nabla Q(y-z_1(s)) \rangle + o(s^{-2}).$$

For $1 < p \leq 2$, we also have the analogous estimates of (2.2.26), (2.2.27)

$$\begin{aligned} & \left| |\mathbf{P}|^{p-1} \mathbf{P} - |P_1|^{p-1} P_1 - |P_2|^{p-1} P_2 - \left[\frac{p+1}{2} |P_1|^{p-1} P_2 + \frac{p-1}{2} |P_1|^{p-3} P_1^2 \overline{P_2} \right] \cdot \mathbf{1}_{y \cdot \frac{z}{|z|} > 0} \right. \\ & \quad \left. - \left[\frac{p+1}{2} |P_2|^{p-1} P_1 + \frac{p-1}{2} |P_2|^{p-3} P_2^2 \overline{P_1} \right] \cdot \mathbf{1}_{y \cdot \frac{z}{|z|} < 0} \right| \lesssim \min(|P_1|^p, |P_2|^p). \end{aligned} \quad (2.5.42)$$

We note that for $\delta = \frac{p-1}{2} > 0$

$$\begin{aligned} \left| \int_{y \cdot \frac{z}{|z|} > -\frac{|z|}{2}} Q^p(y+z) \nabla Q(y) dy \right| & \lesssim |z|^{-\frac{d-1}{2}} e^{-|z|} Q^{(p-1)-\delta} \left(\frac{|z|}{2} \right) \int Q^\delta(y) dy \\ & \lesssim s^{-(p+1-\delta)} \ll s^{-2} \log^{-1}(s), \end{aligned}$$

$$\left| \int_{y \cdot \frac{z}{|z|} < -\frac{|z|}{2}} Q^p(y) \nabla Q(y) dy \right| \lesssim Q^{(p+1)-\delta} \left(\frac{|z|}{2} \right) \int Q^\delta(y) dy \lesssim s^{-(p+1-\delta)} \ll s^{-2} \log^{-1}(s).$$

We repeat the approach in step 3 of Lemma 2.2.20 and combine it with (2.5.8) to conclude that

$$\left| \langle G_0, e^{i\Gamma_1(y-z_1(s))} \nabla Q(y-z_1(s)) \rangle - C_p \frac{z}{|z|} |z|^{-\frac{d-1}{2}} e^{-|z|} \right| \lesssim |z|^{-1-\frac{d-1}{2}} e^{-|z|} \lesssim s^{-2} \log^{-1}(s)$$

as required. \square

The modulation part remains the same as for $p > 2$ (see Lemma 2.3.2) except the extra relation will be

$$\dot{v} = -\frac{2}{c_2} H_0(v, z) \quad (2.5.43)$$

where

$$\begin{aligned} H_0(v, z) & = \left\langle G_0(y; (v(s), z(s))), e^{i\frac{v(s)}{2}(y-\frac{z(s)}{2}(s))} \nabla Q \left(y - \frac{z(s)}{2} \right) \right\rangle \\ & = \langle G_0, e^{i\Gamma_1(y-z_1(s))} \nabla Q(y-z_1(s)) \rangle. \end{aligned} \quad (2.5.44)$$

Remark that by (2.5.13), the main order of \dot{v} still remains

$$\left| \dot{v} + c \frac{z}{|z|} |z|^{-\frac{d-1}{2}} e^{-|z|} \right| \lesssim |z|^{-\frac{d-1}{2}-1} e^{-|z|}.$$

We claim the following analogue of Proposition 2.3.20 in the context $1 < p \leq 2$ for L^2 sub-critical.

Proposition 2.5.45 (Uniform backwards estimates for $1 < p \leq 2$). There exists $s_0 \gg 1$ satisfying the following condition: for all $s^{in} > s_0$, there is a choice of initial parameters $(\lambda^{in}, z^{in}, v^{in})$ such that the solution u of (NLS) corresponding to (2.3.1) exists. Moreover, the decomposition of u with extra relation (2.5.43) on the rescaled interval of time $[s_0, s^{in}]$

$$u(s, x) = \frac{e^{i\gamma(s)}}{\lambda^{\frac{2}{p-1}}(s)} (\mathbf{P} + \epsilon)(s, y), \quad y = \frac{x}{\lambda(s)}, \quad dt = \lambda^2(s) ds$$

verifies the uniform estimates for all $s \in [s_0, s^{in}]$

$$\begin{aligned} & ||z(s)| - 2 \log(s)| \lesssim \log(\log(s)), \quad |\lambda^{-1}(s) - 1| \lesssim s^{-1}, \\ & |v(s)| \lesssim s^{-1}, \quad \|\epsilon(s)\|_{H^1} \lesssim s^{-1}, \quad \left| |z(s)|^{\frac{d-1}{2}} e^{|z(s)|} - cs^2 \right| \lesssim s^2 \log^{-\frac{1}{2}}(s). \end{aligned} \quad (2.5.46)$$

Proof of Proposition 2.5.45. We only sketch the proof since it is very similar to Section 2.3.2, the main difference is the localization to avoid singularities due to the small power p in Taylor expansions (2.5.1)–(2.5.3).

step 1 Modulation equations. Consider

$$\begin{aligned} \frac{d}{ds} \langle \eta_1, A + iB \rangle &= \langle \eta_1, iL_- A - L_+ B \rangle - \langle \vec{m}_1 \cdot \vec{M} \eta_1, iA - B \rangle - \langle \mathcal{E}_{\mathbf{P}_1}, iA - B \rangle \\ &- \langle |\mathbf{P}_1 + \eta_1|^{p-1} (\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1} \mathbf{P}_1 - \frac{p+1}{2} Q^{p-1} \eta_1 - \frac{p-1}{2} Q^{p-1} \bar{\eta}_1, iA - B \rangle \end{aligned} \quad (2.5.47)$$

where the expression of \mathbf{P}_1 is given by

$$\mathbf{P}_1 = Q(y) + e^{i(\Gamma_2(y - (z_2 - z_1)) - \Gamma_1(y))} Q(y - (z_2 - z_1)) + \sum_{j=0}^J e^{-i\Gamma_1(y)} R_j(y + z_1).$$

Let C the set such that $\max(|R_0(y + z_1)|, \dots, |R_J(y + z_1)|) \geq \frac{1}{J+2} Q(y)$ then for $y \in C$

$$|Q(y)| \lesssim \|R_i\|_{L^\infty} \leq s^{-p}, \quad \text{for some } i \in \{0, \dots, J\}.$$

Since $|A|, |B| \lesssim |x|^q e^{-|x|}$, from the asymptotic behavior (2.1.17) of Q , over the set C , we have

$$|A| + |B| \lesssim s^{-p} \log^q s. \quad (2.5.48)$$

Next, denote

$$\Gamma(s, y) = \Gamma_2(y - (z_2 - z_1)) - \Gamma_1(y) = -\frac{1}{2} i v \cdot (y + z) - \frac{1}{2} i v \cdot y, \quad (2.5.49)$$

from the estimates $||z| - 2 \log(s)| \lesssim \log(\log(s))$ and $||v| - s^{-1}| \lesssim s^{-1} \log^{-1}(s)$, there exists a constant c_0 (independent of s^{in}) such that if $|y| \leq c_0 s$ then $|\Gamma(s, y)| \leq \frac{\pi}{2}$. Let $D = \{y \in \mathbb{R}^d, |y| > c_0 s\}$, we have for $y \in C^c \cap D^c$

$$\frac{1}{J+2} Q(y) \leq |\mathbf{P}_1(y)| \lesssim 1 \quad (2.5.50)$$

since $|R_0(y + z_1)|, \dots, |R_J(y + z_1)| < \frac{1}{J+2} Q(y)$ and $\text{Re}[e^{i\Gamma} Q(y + z)] > 0$. And we have for $y \in C \cup D$, using $A, B \in \mathcal{Y}$ and (2.5.48),

$$|A(y)| + |B(y)| \lesssim \min(e^{-\frac{c_0}{2} s}, s^{-p} \log^q(s)) \lesssim s^{-1^+} \quad (2.5.51)$$

with $1^+ = \frac{p+1}{2}$. We denote

$$\varphi(s, y) = \mathbf{1}_{D^c} \mathbf{1}_{C^c}. \quad (2.5.52)$$

A consequence of (2.5.50) and (2.5.51) is that

$$|\mathbf{P}_1(y)|^{-m} Q(y)^n \varphi(s, y) \lesssim 1 \quad \text{for } n \geq m > 0 \quad (2.5.53)$$

and

$$(|A(y)| + |B(y)|)(1 - \varphi(s, y)) \lesssim s^{-1^+}. \quad (2.5.54)$$

By Cauchy-Schwarz and Gagliardo-Nirenberg inequalities

$$\begin{aligned} & | \langle (|\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1 - \frac{p+1}{2}Q^{p-1}\eta_1 - \frac{p-1}{2}Q^{p-1}\bar{\eta}_1, (iA - B)(1 - \varphi(s, y))) \rangle | \\ & \lesssim \langle |\eta_1| + |\eta_1|^p, (iA + B)(1 - \varphi(s, \cdot)) \rangle \lesssim s^{-1+} (\|\eta_1\|_{H^1} + \|\eta_1\|_{H^1}^p) \lesssim C^* s^{-(1+1^+)}. \end{aligned}$$

From the expansion in (2.5.1), we get

$$\begin{aligned} & \left[|\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1 - \frac{p+1}{2}Q^{p-1}\eta_1 - \frac{p-1}{2}Q^{p-1}\bar{\eta}_1 \right] \varphi(s, y) \\ & = \left[\frac{p+1}{2}(|\mathbf{P}_1|^{p-1} - Q^{p-1})\eta_1 + \frac{p-1}{2}(|\mathbf{P}_1|^{p-3}\mathbf{P}_1^2 - Q^{p-1})\bar{\eta}_1 + O\left(\left|\frac{\eta_1}{\mathbf{P}_1}\right|^2 |\mathbf{P}_1|^p\right) \right] \varphi(s, y). \end{aligned}$$

We control the first two terms as before in the case $p > 2$

$$\begin{aligned} & | \langle (|\mathbf{P}_1|^{p-1} - Q^{p-1})\eta_1, (iA - B)\varphi(s, \cdot) \rangle | + | \langle (|\mathbf{P}_1|^{p-3}\mathbf{P}_1^2 - Q^{p-1})\bar{\eta}_1, (iA - B)\varphi(s, \cdot) \rangle | \\ & \lesssim C^* s^{-(p+1)} \log^q(s) \end{aligned}$$

and for the last term, we use (2.5.53) to remark that $|\mathbf{P}_1|^{p-2}|iA - B|\varphi(s, \cdot) \lesssim 1$ then deduce the inequality

$$\left\langle \left| \frac{\eta_1}{\mathbf{P}_1} \right|^2 |\mathbf{P}_1|^p, (iA - B)\varphi(s, \cdot) \right\rangle \lesssim \|\epsilon\|_{L^2}^2 \lesssim (C^*)^2 s^{-2}.$$

To summarize, we have shown that

$$\left\langle |\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1 - \frac{p+1}{2}Q^{p-1}\eta_1 - \frac{p-1}{2}Q^{p-1}\bar{\eta}_1, iA - B \right\rangle \lesssim s^{-2}. \quad (2.5.55)$$

Next, it is obvious that we still have as before

$$| \langle \vec{m}_1 \cdot \vec{M}\eta_1, iA - B \rangle | \lesssim C^* s^{-1} |\vec{m}_1(s)|.$$

To prove the estimate

$$\left| \langle \mathcal{E}_{\mathbf{P}_1}, iA - B \rangle - \langle \vec{m}_1 \cdot \vec{M}Q, iA - B \rangle \right| \lesssim s^{-2} + s^{-1} |\vec{m}_1|, \quad (2.5.56)$$

we recall $\mathcal{E}_{\mathbf{P}_1} = [\vec{m}_1 \cdot \vec{M}Q](y) + [e^{i(\Gamma_2(y-(z_2-z_1))-\Gamma_1(y))}\vec{m}_2 \cdot \vec{M}Q](y - (z_2 - z_1)) + e^{-i\Gamma_1(y)}G_0(y + z_1)$. From (2.5.12)

$$| \langle e^{-i\Gamma_1(y)}G_0(y + z_1), iA - B \rangle | \lesssim \|G_0\|_{L^2} \lesssim s^{-2}$$

and finally since $A, B \in \mathcal{Y}$, we have

$$| \langle e^{i(\Gamma_2(y-(z_2-z_1))-\Gamma_1(y))}(\vec{m}_2 \cdot \vec{M}Q(\cdot - (z_2 - z_1))), iA - B \rangle | \lesssim s^{-1} |\vec{m}_1|,$$

which yields the estimate (2.3.33) in the case $1 < p \leq 2$. We project η_1 onto three null spaces of the linearized equation around Q and obtain the almost orthogonality for the fourth null space by the localized momentum thanks to the special choice of \dot{v} in (2.5.44) (as in Section 2.3.2.2). Indeed, proceeding the same way as (2.5.47), taking into account the terms of order s^{-2} , we have that

$$\frac{d}{ds} \langle \eta_k, i\nabla Q \rangle = \left\langle \frac{\bar{\eta}_k \cdot F''(\mathbf{P}_k) \cdot \eta_k}{2}, \nabla Q \right\rangle + \left(\dot{z}_k - 2v_k + \frac{\dot{\lambda}}{\lambda} z_k \right) \langle i\nabla Q, \nabla \eta_k \rangle + O(C^* s^{-(1+1^+)}).$$

For the estimate of localized momentum \mathcal{M}_k : for all $s \in [s^*, s^{in}]$,

$$\frac{1}{2} \frac{d}{ds} \mathcal{M}_k = \left\langle \frac{\bar{\eta}_k \cdot F''(\mathbf{P}_k) \cdot \eta_k}{2}, \nabla Q \right\rangle + \left(\dot{z}_k - 2v_k + \frac{\dot{\lambda}}{\lambda} z_k \right) \langle i \nabla Q, \nabla \eta_k \rangle + O(\log^{-1}(s) \|\eta_k\|_{H^1}^2). \quad (2.5.57)$$

Recall that from the equation of $i\dot{\eta}_k$ (2.3.10), we have

$$\begin{aligned} \frac{d}{ds} \mathcal{M}_k &= \text{Im} \int (\nabla \eta_k \bar{\eta}_k) \dot{\chi} - \langle \Delta \eta_k - \eta_k + (|\mathbf{P}_k + \eta_k|^{p-1} (\mathbf{P}_k + \eta_k) - |\mathbf{P}_k|^{p-1} \mathbf{P}_k) \\ &\quad + \vec{m}_k^* \cdot \vec{M}^* \eta_k - (\dot{z}_k - 2v_k + \frac{\dot{\lambda}}{\lambda} z_k) \cdot i \nabla \eta_k + \vec{m}_k^* \cdot \vec{M}^* Q - (\dot{z}_k - 2v_k + \frac{\dot{\lambda}}{\lambda} z_k) \cdot i \nabla Q \\ &\quad + [e^{i(\Gamma_j(y-z) - \Gamma_k(y))} \vec{m}_j \cdot \vec{M} Q](y \pm z) + G_k, 2\chi \nabla \eta_k + \eta_k \nabla \chi \rangle. \end{aligned}$$

We proceed the same way as in Lemma 2.3.28 for L^2 sub-critical cases with $p > 2$, except for the term

$$\langle |\mathbf{P}_k + \eta_k|^{p-1} (\mathbf{P}_k + \eta_k) - |\mathbf{P}_k|^{p-1} \mathbf{P}_k, 2\chi \nabla \eta_k + \eta_k \nabla \chi \rangle.$$

First, by (2.5.1)

$$|\mathbf{P}_k + \eta_k|^{p-1} (\mathbf{P}_k + \eta_k) - |\mathbf{P}_k|^{p-1} \mathbf{P}_k = F'(\mathbf{P}_k) \cdot \epsilon + O(\|\eta_k\|^p)$$

and then we have

$$|\langle |\eta_p|^p, 2\chi \nabla \eta_k + \eta_k \nabla \chi \rangle| \lesssim \|\epsilon\|_{H^1}^{p+1} \lesssim s^{-2} \log^{-2}(s).$$

Second, we consider

$$|\langle F'(\mathbf{P}_k) \cdot \eta_k, \eta_k \nabla \chi \rangle| \lesssim |\nabla \chi| \|\eta_k\|_{H^1}^2 \lesssim \log^{-1}(s) \|\epsilon\|_{H^1}^2.$$

Finally by integration by parts, we obtain

$$\langle F'(\mathbf{P}_k) \cdot \eta_k, \chi \nabla \eta_k \rangle = -\frac{1}{2} \langle \nabla \mathbf{P}_k \chi, \bar{\eta}_k \cdot F''(\mathbf{P}_k) \cdot \eta_k \rangle - \frac{1}{2} \langle F'(\mathbf{P}_k) \cdot \eta_k, \eta_k \nabla \chi \rangle.$$

These estimates yield (2.5.57) since in the support of χ , we have $|P_k| \gtrsim s^{-\frac{1}{8}} \geq \|V_j\|_{L^\infty}, \forall j = \overline{0, J}$ so $\varphi_k \equiv 1$ then

$$\begin{aligned} &|\langle \nabla \mathbf{P}_k \chi, \bar{\eta}_k \cdot F''(\mathbf{P}_k) \cdot \eta_k \rangle - \langle \varphi_k \nabla P_k, \bar{\eta}_k \cdot F''(\mathbf{P}_k) \cdot \eta_k \rangle| \\ &\lesssim \left[\sum_{j \neq k} \left| \int_{|y| < \frac{1}{8} \log s} (\bar{\eta}_k \cdot F''(\mathbf{P}_k) \cdot \eta_k) \nabla Q \right| + \left| \int_{|y| > \frac{1}{10} \log s} \varphi_k (\bar{\eta}_k \cdot F''(\mathbf{P}_k) \cdot \eta_k) \nabla Q(y \pm z) \right| \right] \\ &\lesssim s^{-\frac{p-1}{20}} \|\epsilon\|_{H^1}^2 \end{aligned}$$

and

$$|\langle i \nabla Q, \chi \nabla \eta_k \rangle - \langle i \nabla Q, \nabla \eta_k \rangle| \lesssim \left| \int_{|y| > \frac{1}{10} \log s} \nabla Q \nabla \bar{\eta}_k \right| \lesssim s^{-\frac{1}{20}} \|\eta_k\|_{H^1}$$

here we use the property (2.5.53) of φ_k that $\varphi_k \neq 0$ implies $\left| \frac{\nabla P_j}{\mathbf{P}} \right| \lesssim 1$ and $\left| \frac{\nabla P_k}{\mathbf{P}} \right| \lesssim 1$.

step 2 Control the energy functional. We still consider the energy functional

$$\begin{aligned} \mathbf{W}(s, \epsilon) &= \mathbf{K}(s, \epsilon) - \mathbf{J}(s, \epsilon) \\ &= \frac{1}{2} \int \left(|\nabla \epsilon|^2 + |\epsilon|^2 - \frac{2}{p+1} (|\mathbf{P} + \epsilon|^{p+1} - |\mathbf{P}|^{p+1} - (p+1)|\mathbf{P}|^{p-1} \text{Re}(\epsilon \bar{\mathbf{P}})) \right) \\ &\quad - \sum_{k=1}^2 v_k \cdot \text{Im} \int (\nabla \epsilon \bar{\epsilon}) \chi_k \end{aligned}$$

and remark that we still have the coercivity property

$$\mathbf{W}(s, \epsilon(s)) \gtrsim \|\epsilon(s)\|_{H^1}^2$$

(see for example [17], [22]). Define

$$\varphi_1(s, y) = \varphi(s, y - z_1(s)) \quad (2.5.58)$$

a function localized to the first soliton \mathbf{P}_1 . Similarly, we can define an analogous function $\varphi_2(s, y)$ localized to the second soliton \mathbf{P}_2 .

We claim an estimate on the derivative of \mathbf{K} by $\dot{z}_k \cdot \langle \nabla P_k, \frac{\bar{\epsilon} \cdot F''(\mathbf{P}) \cdot \epsilon}{2} \rangle$ but now localized by φ_k

$$\left| \frac{d}{ds} [\mathbf{K}(s, \epsilon(s))] - \sum_{k=1}^2 \dot{z}_k \cdot \langle \varphi_k \nabla P_k, \frac{\bar{\epsilon} \cdot F''(\mathbf{P}) \cdot \epsilon}{2} \rangle \right| \lesssim s^{-2} \|\epsilon(s)\|_{H^1} + s^{-2} \|\epsilon\|_{H^1}^2. \quad (2.5.59)$$

Recall that we have

$$\frac{d}{ds} [\mathbf{K}(s, \epsilon(s))] = D_s \mathbf{K}(s, \epsilon(s)) + \langle D_\epsilon \mathbf{K}(s, \epsilon(s)), \dot{\epsilon}_s \rangle,$$

and

$$D_s \mathbf{K} = \langle \dot{\mathbf{P}}, K \rangle, \quad \langle D_\epsilon \mathbf{K}(s, \epsilon), \dot{\epsilon} \rangle = \frac{\dot{\lambda}}{\lambda} \langle D_\epsilon \mathbf{K}(s, \epsilon), \Lambda \epsilon \rangle - (1 - \dot{\gamma}) \langle i D_\epsilon \mathbf{K}(s, \epsilon), \epsilon \rangle - \langle i D_\epsilon \mathbf{K}(s, \epsilon), \mathcal{E}_{\mathbf{P}} \rangle$$

with $K = |\mathbf{P} + \epsilon|^{p-1}(\mathbf{P} + \epsilon) - |\mathbf{P}|^{p-1}\mathbf{P} - \frac{p+1}{2}\epsilon|\mathbf{P}|^{p-1} - \frac{p-1}{2}\bar{\epsilon}\mathbf{P}^2|\mathbf{P}|^{p-3}$. We observe from (2.5.51) that for $\dot{P}_k = -\dot{z}_k \cdot \nabla P_k + i\dot{v}_k \cdot (y - z_k)P_k$, over the set $C \cup D$, $|\dot{P}_k| \lesssim s^{-(1+1^+)}$ then

$$|\langle \dot{P}_k, K(1 - \varphi_k) \rangle| \lesssim s^{-(1+1^+)} \|\epsilon\|_{H^1}.$$

From (2.5.1), $|K| \lesssim |\epsilon|^2 |\mathbf{P}|^{p-2}$ so we obtain

$$|\langle i\dot{v}_k \cdot (y - z_k)P_k, K\varphi_k \rangle| \lesssim |\dot{v}| \|\epsilon\|_{H^1}^2 \lesssim s^{-2} \|\epsilon\|_{H^1}^2$$

since $\frac{Q(y-z_k)}{|\mathbf{P}|^{\varphi_k}} \lesssim 1$ by (2.5.53). Next we look more precisely at K

$$K = \frac{\bar{\epsilon} \cdot F''(\mathbf{P}) \cdot \epsilon}{2} + O\left(\left|\frac{\epsilon}{\mathbf{P}}\right|^{2^+} |\mathbf{P}|^p\right)$$

since $|\dot{z}_k| \lesssim s^{-1}$ and $p - 2^+ > -1$, we also have

$$\left| \left\langle -\dot{z}_k \cdot \nabla P_k, \left|\frac{\epsilon}{\mathbf{P}}\right|^{2^+} |\mathbf{P}|^p \varphi_k \right\rangle \right| \lesssim s^{-1} \|\epsilon\|_{H^1}^{2^+}.$$

We deal the first two terms of $\langle D_\epsilon \mathbf{K}(s, \epsilon), \dot{\epsilon} \rangle$ as in the case $p > 2$

$$\left| \frac{\dot{\lambda}}{\lambda} \langle D_\epsilon \mathbf{K}(s, \epsilon), \Lambda \epsilon \rangle \right| \lesssim \left| \frac{\dot{\lambda}}{\lambda} \right| \left(\|\epsilon\|_{H^1}^2 + \|\epsilon\|_{H^1}^{p+1} \right) \lesssim (C^*)^2 s^{-2} \|\epsilon\|_{H^1}^2,$$

$$|(1 - \dot{\gamma}) \langle i D_\epsilon \mathbf{K}(s, \epsilon), \epsilon \rangle| \lesssim |1 - \dot{\gamma}| (\|\epsilon\|_{H^1}^2 + \|\epsilon\|_{H^1}^{p+1}) \lesssim (C^*)^2 s^{-2} \|\epsilon\|_{H^1}^2.$$

Recall that for the last term we have

$$\begin{aligned} \langle i D_\epsilon \mathbf{K}(s, \epsilon), \mathcal{E}_{\mathbf{P}} \rangle &= \langle -i\Delta \epsilon + i\epsilon - i(|\mathbf{P} + \epsilon|^{p-1}(\mathbf{P} + \epsilon) - |\mathbf{P}|^{p-1}\mathbf{P}), \\ &\quad [e^{i\Gamma_1} \bar{m}_1 \cdot \bar{M}Q](y - z_1(s)) + [e^{i\Gamma_2} \bar{m}_2 \cdot \bar{M}Q](y - z_2(s)) + G_0 \rangle \end{aligned}$$

so from the properties of operators L_+ and L_-

$$\begin{aligned} I_1 &= \langle -i\Delta\epsilon + i\epsilon - i(|\mathbf{P} + \epsilon|^{p-1}(\mathbf{P} + \epsilon) - |\mathbf{P}|^{p-1}\mathbf{P}), [e^{i\Gamma_1}\vec{m}_1 \cdot \vec{M}Q](y - z_1(s)) \rangle \\ &= -\frac{\dot{\lambda}}{\lambda} \langle \eta_1, -2Q \rangle + (\dot{v} - \frac{\dot{\lambda}}{\lambda}v) \langle \eta_1, -2i\nabla Q \rangle \\ &\quad - \left\langle i \left(|\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1 - \frac{p+1}{2}Q^{p-1}\eta_1 - \frac{p-1}{2}Q^{p-1}\bar{\eta}_1 \right), \vec{m}_1 \cdot \vec{M}Q \right\rangle. \end{aligned}$$

By the same way to prove (2.5.55), combining with the orthogonality of η_1 (2.3.6), (2.3.30) and the estimate of modulation equation (2.3.29), we get

$$|I_1| = O((C^*)^4 s^{-4}) + O((C^*)^2 s^{-4}).$$

Finally, using integration by parts and Cauchy-Schwarz inequality, from the bound for H^1 norm of G_0 (2.5.12), we obtain

$$|\langle -i\Delta\epsilon + i\epsilon - i(|\mathbf{P} + \epsilon|^{p-1}(\mathbf{P} + \epsilon) - |\mathbf{P}|^{p-1}\mathbf{P}), G_0 \rangle| \lesssim s^{-2} \|\epsilon\|_{H^1}.$$

Combining these computations, the proof of (2.5.59) is finished. We still have the same estimate for localized momentum J_k : for all $s \in [s^*, s^{in}]$,

$$\left| \frac{d}{ds} [\mathbf{J}(s, \epsilon(s))] - \sum_{k=1}^2 2v_k \cdot \left\langle \varphi_k \nabla P_k, \frac{\bar{\epsilon} \cdot F''(\mathbf{P}) \cdot \epsilon}{2} \right\rangle \right| \lesssim s^{-2} \log^{-\frac{3}{4}}(s) \|\epsilon(s)\|_{H^1}. \quad (2.5.60)$$

(by using (2.5.57)). Then we can deduce from the modulation equation $|\dot{z}_k - 2v_k| \lesssim s^{-1} \log^{-\frac{3}{4}}(s)$ that

$$\left| \frac{d}{ds} [\mathbf{W}(s, \epsilon(s))] \right| \lesssim s^{-2} \|\epsilon(s)\|_{H^1}.$$

The rest of the proof stays unchanged in comparison to the case $p > 2$ in Section 2.3.2.4. \square

From the uniform backwards estimates in Proposition 2.5.45, since $\|R_j\|_{H^1} \ll s^{-1}$ for $j = \bar{0}, \bar{J}$, we have that

$$\begin{aligned} \left\| u(t(s), x) - \frac{e^{i\gamma(s)}}{\lambda^{\frac{2}{p-1}}(s)} \sum_{k=1}^2 [e^{i\Gamma_k} Q] \left(\frac{x}{\lambda(s)} + \frac{(-1)^k}{2} z(s) \right) \right\|_{H^1} &\lesssim \|\epsilon(s)\|_{H^1} + \sum_{j=0}^J \|R_j(s)\|_{H^1} \\ &\lesssim s^{-1} \end{aligned}$$

then we proceed like in Section 2.4 to obtain the existence of a solution $u(t)$ satisfying the regime (2.1.6) in sub-critical cases with $1 < p \leq 2$

$$\left\| u(t) - e^{i\gamma(t)} \sum_{k=1}^2 Q(\cdot - x_k(t)) \right\|_{H^1} \lesssim \frac{1}{t}.$$

2.6 Super-critical cases

In this section, we will present the necessary modifications to prove the result in the L^2 super-critical cases ($1 + \frac{4}{d} < p < \frac{d+2}{d-2}$) (see [4]). For $k \in \{1, 2\}$, $z_1(s) = -z_2(s) = \frac{1}{2}z(s)$, $v_1(s) = -v_2(s) = \frac{1}{2}v(s)$, denote

$$\begin{aligned} Y_k^\pm(s, y) &= e^{i\Gamma_k(s, y - z_k(s))} Y^\pm(y - z_k(s)) \\ Z_k(s, y) &= e^{i\Gamma_k(s, y - z_k(s))} i\Lambda Q(y - z_k(s)) \end{aligned} \quad (2.6.1)$$

$$V_k(s, y) = e^{i\Gamma_k(s, y - z_k(s))} i \nabla Q(y - z_k(s))$$

$$W_k(s, y) = e^{i\Gamma_k(s, y - z_k(s))} (y - z_k(s)) Q(y - z_k(s)).$$

Let

$$\mathbf{Y}^\pm(s, y) = \mathbf{Y}^\pm(y; (z(s), v(s))) = \sum_{k=1}^2 Y_k^\pm(s, y), \quad \mathbf{Z}(s, y) = \mathbf{Z}(y; (z(s), v(s))) = \sum_{k=1}^2 Z_k(s, y),$$

$$\mathbf{V}(s, y) = \mathbf{V}(y; (z(s), v(s))) = V_1(s, y) - V_2(s, y),$$

$$\mathbf{W}(s, y) = \mathbf{W}(y; (z(s), v(s))) = W_1(s, y) - W_2(s, y).$$

We need some extra parameters to control the instability created by Y^\pm . Consider a solution of (NLS) with symmetric initial data like below: for $\mathbf{b} = (\mathbf{b}^+, \mathbf{b}^-, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) \in \mathbb{R}^5$, $\|\mathbf{b}\| \leq C(s^{in})^{-\frac{3}{2}}$ (the constant C independent of s^{in} and given in Lemma 2.6.4)

$$u(T_{mod}, x) = \frac{1}{(\lambda^{in})^{\frac{2}{p-1}}} w(s^{in}, y), \quad y = \frac{x}{\lambda^{in}} \quad (2.6.2)$$

with

$$w(s^{in}) = \mathbf{P}^{in}(y; (z^{in} \vec{e}_1, v^{in})) + \mathbf{b}^+ i \mathbf{Y}^+(y; (z^{in} \vec{e}_1, v^{in})) + \mathbf{b}^- i \mathbf{Y}^-(y; (z^{in} \vec{e}_1, v^{in}))$$

$$+ \mathbf{b}_1 \mathbf{Z}(y; (z^{in} \vec{e}_1, v^{in})) + \mathbf{b}_2 \mathbf{V}(y; (z^{in} \vec{e}_1, v^{in})) + \mathbf{b}_3 \mathbf{W}(y; (z^{in} \vec{e}_1, v^{in})). \quad (2.6.3)$$

Then we get

$$\epsilon(s^{in}) = \mathbf{b}^+ i \mathbf{Y}^+(y; (z^{in} \vec{e}_1, v^{in})) + \mathbf{b}^- i \mathbf{Y}^-(y; (z^{in} \vec{e}_1, v^{in}))$$

$$+ \mathbf{b}_1 \mathbf{Z}(y; (z^{in} \vec{e}_1, v^{in})) + \mathbf{b}_2 \mathbf{V}(y; (z^{in} \vec{e}_1, v^{in})) + \mathbf{b}_3 \mathbf{W}(y; (z^{in} \vec{e}_1, v^{in})).$$

Lemma 2.6.4 (Modulated data in direction Y^\pm). There exists $C > 0$ such that for all $s^{in} \geq s_0$ and for all $a^{in} \in [-(s^{in})^{-\frac{3}{2}}, (s^{in})^{-\frac{3}{2}}]$, there is a unique \mathbf{b} so that $\|\mathbf{b}\| \leq C|a^{in}|$ (C independent of s^{in}) and the initial data satisfies

$$\langle \eta_1(s^{in}), iY^- \rangle = a^{in}, \quad \langle \eta_1(s^{in}), iY^+ \rangle = \langle \eta_1(s^{in}), i\Lambda Q \rangle = \langle \eta_1(s^{in}), yQ \rangle = \langle \eta_1(s^{in}), i\nabla Q \rangle = 0 \quad (2.6.5)$$

with η_1 defined as in (2.3.5).

Proof of Lemma 2.6.4. Let

$$\mathbf{c} = (\langle \eta_1(s^{in}), iY^+ \rangle, \langle \eta_1(s^{in}), iY^- \rangle, \langle \eta_1(s^{in}), i\Lambda Q \rangle, \langle \eta_1(s^{in}), i\nabla Q \rangle, \langle \eta_1(s^{in}), yQ \rangle).$$

We consider the linear maps

$$\Psi : \mathbb{R}^5 \rightarrow H^1(\mathbb{R}^d) \quad \Phi : H^1(\mathbb{R}^d) \rightarrow \mathbb{R}^5$$

$$\mathbf{b} \mapsto \epsilon(s^{in}) \quad \epsilon(s^{in}) \rightarrow \mathbf{c}$$

and $\Omega = \Phi \circ \Psi : \mathbb{R}^5 \rightarrow \mathbb{R}^5$. We compute

$$\Psi(h) = (i\mathbf{Y}^+(y; (z^{in} \vec{e}_1, v^{in})), i\mathbf{Y}^-(y; (z^{in} \vec{e}_1, v^{in})),$$

$$\mathbf{Z}(y; (z^{in} \vec{e}_1, v^{in})), \mathbf{V}(y; (z^{in} \vec{e}_1, v^{in})), \mathbf{W}(y; (z^{in} \vec{e}_1, v^{in}))) \cdot h$$

and

$$\Phi(v) = \begin{pmatrix} \int v(y)[e^{-i\Gamma_1 \overline{iY^+}}](y - \frac{1}{2}z^{in}\vec{e}_1)dy \\ \int v(y)[e^{-i\Gamma_1 \overline{iY^-}}](y - \frac{1}{2}z^{in}\vec{e}_1)dy \\ \int v(y)[e^{-i\Gamma_1 \overline{i\Lambda Q}}](y - \frac{1}{2}z^{in}\vec{e}_1)dy \\ \int v(y)[e^{-i\Gamma_1 \overline{i\nabla Q}}](y - \frac{1}{2}z^{in}\vec{e}_1)dy \\ \int v(y)[e^{-i\Gamma_1 \overline{yQ}}](y - \frac{1}{2}z^{in}\vec{e}_1)dy \end{pmatrix}$$

then we can deduce that for some complex functions $A(y), B(y) \in \mathbf{Y}$

$$\Omega = \Phi \circ \Psi = N + O(|\langle A(y + z^{in}\vec{e}_1), B(y) \rangle|) = N + O(e^{-|z^{in}|})$$

where

$$N = \begin{pmatrix} \langle iY^+, iY^+ \rangle & \langle iY^-, iY^+ \rangle & \langle i\Lambda Q, iY^+ \rangle & \langle i\nabla Q, iY^+ \rangle & \langle yQ, iY^+ \rangle \\ \langle iY^+, iY^- \rangle & \langle iY^-, iY^- \rangle & \langle i\Lambda Q, iY^- \rangle & \langle i\nabla Q, iY^- \rangle & \langle yQ, iY^- \rangle \\ \langle iY^+, i\Lambda Q \rangle & \langle iY^-, i\Lambda Q \rangle & \langle i\Lambda Q, i\Lambda Q \rangle & \langle i\nabla Q, i\Lambda Q \rangle & \langle yQ, i\Lambda Q \rangle \\ \langle iY^+, i\nabla Q \rangle & \langle iY^-, i\nabla Q \rangle & \langle i\Lambda Q, i\nabla Q \rangle & \langle i\nabla Q, i\nabla Q \rangle & \langle yQ, i\nabla Q \rangle \\ \langle iY^+, yQ \rangle & \langle iY^-, yQ \rangle & \langle i\Lambda Q, yQ \rangle & \langle i\nabla Q, yQ \rangle & \langle yQ, yQ \rangle \end{pmatrix}$$

and $\Omega(0) = 0$. Remark that N is the Gramian matrix of $iY^+, iY^-, i\Lambda Q, i\nabla Q, yQ$ which are linearly independent since if for some $m, n, p, q, r \in \mathbb{R}$ (not all zeros)

$$m iY^+ + n iY^- + p i\Lambda Q + q yQ + r i\nabla Q = 0$$

then $mY^+ + nY^- + p\Lambda Q - qiyQ + r\nabla Q = 0$. We apply \mathcal{L} to both sides of the equality ($L_+(\Lambda Q) = -2Q, L_-(xQ) = -2\nabla Q, L_+(\nabla Q) = 0$) and get

$$me_0Y^- - ne_0Y^- - 2piQ - 2q\nabla Q = 0$$

so $m = n = p = q = 0$ as $Y^+, Y^-, iQ, \nabla Q$ are linearly independent thus $r = 0$, a contradiction. Therefore, $\det N \neq 0$ and with $|z^{in}| \gg 1$, we have that Ω is invertible around 0 and

$$\|\Omega^{-1}\| \leq \|\text{Gram}(iY^+, iY^-, i\Lambda Q, i\nabla Q, yQ)\| + 2$$

Therefore, for any $a^{in} \in [-(s^{in})^{-\frac{3}{2}}, (s^{in})^{-\frac{3}{2}}]$, we can choose

$$\mathbf{b} = \Omega^{-1}((0, a^{in}, 0, 0, 0)), \quad \|\mathbf{b}\| \leq \|\Omega^{-1}\| |a^{in}|$$

to conclude the lemma. □

In fact, the coefficients $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ can be determined explicitly from $\mathbf{b}^+, \mathbf{b}^-$ as follows

$$\mathbf{b}_1 = \frac{1}{\|\Lambda Q\|_{L^2}^2 + \langle e^{i\Gamma_0(\cdot)} i\Lambda Q(\cdot + z^{in}\vec{e}_1), i\Lambda Q \rangle} (\mathbf{b}^+ \langle iY^+, i\Lambda Q \rangle + \mathbf{b}^+ \langle e^{i\Gamma_0(\cdot)} iY^+(\cdot + z^{in}\vec{e}_1), i\Lambda Q \rangle + \mathbf{b}^- \langle iY^-, i\Lambda Q \rangle + \mathbf{b}^- \langle e^{i\Gamma_0(\cdot)} iY^-(\cdot + z^{in}\vec{e}_1), i\Lambda Q \rangle)$$

$$\mathbf{b}_2 = \frac{1}{\|\nabla Q\|_{L^2}^2 - \langle e^{i\Gamma_0(\cdot)} [i\nabla Q](\cdot + z^{in}\vec{e}_1), i\nabla Q \rangle} (\mathbf{b}^+ \langle iY^+, i\nabla Q \rangle + \mathbf{b}^+ \langle e^{i\Gamma_0(\cdot)} iY^+(\cdot + z^{in}\vec{e}_1), i\nabla Q \rangle + \mathbf{b}^- \langle iY^-, i\nabla Q \rangle + \mathbf{b}^- \langle e^{i\Gamma_0(\cdot)} iY^-(\cdot + z^{in}\vec{e}_1), i\nabla Q \rangle)$$

$$\mathbf{b}_3 = \frac{1}{\|yQ\|_{L^2}^2 - \langle e^{i\Gamma_0(\cdot)}[yQ](\cdot + z^{in}\vec{e}_1), yQ \rangle} (\mathbf{b}^+ \langle iY^+, yQ \rangle + \mathbf{b}^+ \langle e^{i\Gamma_0(\cdot)} iY^+(\cdot + z^{in}\vec{e}_1), yQ \rangle + \mathbf{b}^- \langle iY^-, yQ \rangle + \mathbf{b}^- \langle e^{i\Gamma_0(\cdot)} iY^-(\cdot + z^{in}\vec{e}_1), yQ \rangle)$$

where $\Gamma_0(y) = -\frac{1}{2}iv^{in} \cdot (y + z^{in}\vec{e}_1) - \frac{1}{2}iv^{in} \cdot y$. This specific choice is made in order that initially, we have the following orthogonality conditions

$$\langle \eta_1(s^{in}), i\Lambda \rangle = \langle \eta_1(s^{in}), yQ \rangle = 0 \quad (2.6.6)$$

and $\langle \eta_1(s^{in}), i\nabla Q \rangle = 0$. We recall the decomposition of $u(t)$: there exists a \mathcal{C}^1 function

$$\vec{q}(t) = (\lambda, z, \gamma, v) : [s_0, s^{in}] \rightarrow (0, +\infty) \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$$

such that we can modulate $u(t)$ on $[s_0, s^{in}]$ as

$$u(t(s), x) = \frac{e^{i\gamma(s)}}{\lambda(s)} (\mathbf{P} + \epsilon)(s, y)$$

and $\langle \eta_1(s), i\Lambda \rangle = \langle \eta_1(s), yQ \rangle = 0$. In here we obtain only two orthogonality conditions as the initial data satisfies only two (2.6.6). The proof of uniform estimates will remain the same except for some modifications that we will clarify immediately. Denote

$$a^\pm(s) = \langle \eta_1(s), iY^\pm \rangle, \quad (2.6.7)$$

Lemma 2.6.4 allows us to establish a one-to-one mapping between the choice of $(\mathbf{b}^+, \mathbf{b}^-)$ and the constraints $a^+(s^{in}) = 0, a^-(s^{in}) = a^{in}$ for any choice of a^{in} . We now define the maximal time interval $[S(a^{in}), s^{in}]$ on which (2.3.25) holds and

$$|a^\pm(s)| \leq s^{-\frac{3}{2}} \quad (2.6.8)$$

for all $s \in [S(a^{in}), s^{in}]$. We will prove that there exists a choice of

$$a^{in} \in [-(s^{in})^{-\frac{3}{2}}, (s^{in})^{-\frac{3}{2}}]$$

and z^{in} such that $S(a^{in}) = s_0$. The first thing changed is that $\epsilon(s^{in})$ may not be zero, but we still have $\epsilon(s^{in}) \lesssim \|\mathbf{b}\| \lesssim (s^{in})^{-\frac{3}{2}}$. This is enough to conclude that $|\mathbf{W}(s, \epsilon(s))| \lesssim C^* s^{-2}$ from the fact $|\frac{d}{ds} \mathbf{W}(s, \epsilon(s))| \lesssim C^* s^{-3}$. Next, from $\langle \eta_1(s^{in}), i\nabla Q \rangle = 0$, we still deduce that $|\langle \eta_1, i\nabla Q \rangle| \lesssim (C^*)^2 s^{-1} \log^{-1}(s)$ by considering the localized momentum \mathcal{M}_k . The second thing which need to be modified is the coercivity of \mathbf{W} . By (2.1.22)

$$\mathbf{W}(s, \epsilon(s)) \gtrsim \|\epsilon(s)\|_{H^1}^2 + O(s^{-3})$$

the process in Section 2.3 is still valid as long as we have (2.6.8). We claim the following preliminary estimates on the parameters $a^\pm(s)$.

Lemma 2.6.9. For all $s \in [S(a^{in}), s^{in}]$,

$$\left| \frac{da^\pm}{ds}(s) \mp e_0 a^\pm(s) \right| \lesssim \|\epsilon\|_{H^1}^2 \quad (2.6.10)$$

Proof of Lemma 2.6.9. Applying the inequality (2.3.33) with $A = -\text{Im} Y^+, B = \text{Re} Y^+$ and using the equation of Y^\pm (2.1.21)

$$\left| \frac{d}{ds} \langle \eta_1, i \text{Re} Y^+ - \text{Im} Y^+ \rangle - [\langle \eta_1, -iL_-(\text{Im} Y^+) - L_+(\text{Re} Y^+) \rangle - \langle \vec{m}_1 \cdot \vec{M}Q, -i \text{Im} Y^+ - \text{Re} Y^+ \rangle] \right| \lesssim (C^*)^2 s^{-2} + s^{-1} |\vec{m}_1| \quad (2.6.11)$$

so we get

$$\left| \frac{d}{ds} \langle \eta_1, iY^+ \rangle - \langle \eta_1, i\mathcal{L}(Y^+) \rangle \right| \lesssim (C^*)^2 s^{-2} + s^{-1} |\vec{m}_1| + |\langle \vec{m}_1 \cdot \vec{M}Q, Y^+ \rangle|.$$

This implies $\left| \frac{da^+}{ds}(s) - e_0 a^+(s) \right| \lesssim \|\epsilon\|_{H^1}^2$. In the same way, we also obtain

$$\left| \frac{da^-}{ds}(s) + e_0 a^-(s) \right| \lesssim \|\epsilon\|_{H^1}^2$$

as desired. \square

By the same arguments in Section 2.3, we improve all estimates in the bootstrap bounds except those of $a^\pm(s)$ and $z(s)$. It seems to us that the reasoning to close the bootstrap bound of $z(s)$ still works, in fact, it is, however we will control $a^\pm(s)$ through a suitable value of a^{in} also by a topological argument so we have to choose (z^{in}, a^{in}) in the same time.

Lemma 2.6.12 (Control of $a^+(s)$). For all $a^{in} \in [-(s^{in})^{-\frac{3}{2}}, (s^{in})^{-\frac{3}{2}}]$, the following inequality holds for all $s \in [S(a^{in}), s^{in}]$

$$|a^+(s)| \leq \frac{1}{2} s^{-\frac{3}{2}}. \quad (2.6.13)$$

Proof of Lemma 2.6.12. It follows (2.3.25), (2.6.10) and $a^+(s^{in}) = 0$ that for all $s \in [S(a^{in}), s^{in}]$

$$\begin{aligned} |a^+(s)| &\lesssim (C^*)^2 e^{e_0 s} \int_s^{s^{in}} e^{-e_0 \tau} \tau^{-2} d\tau \\ &= \frac{(C^*)^2}{e_0} e^{e_0 s} [e^{-e_0 s} s^{-2} - e^{-e_0 s^{in}} (s^{in})^{-2}] - 2 \frac{(C^*)^2}{e_0} e^{e_0 s} \int_s^{s^{in}} e^{-e_0 \tau} \tau^{-3} d\tau \\ &\leq \frac{(C^*)^2}{e_0} s^{-2} \leq \frac{1}{2} s^{-\frac{3}{2}} \end{aligned}$$

for s_0 to be large enough. \square

Lemma 2.6.14 (Control of $a^-(s)$ and closing the parameter z). There exist z^{in} and $a^{in} \in [-(s^{in})^{-\frac{3}{2}}, (s^{in})^{-\frac{3}{2}}]$ such that $S(a^{in}) = s_0$.

Proof of Lemma 2.6.14. We argue by contradiction. Consider $\zeta(s), \xi(s)$ as defined in (2.3.52) and

$$\mathcal{N}(s) = s^3 (a^-(s))^2.$$

Suppose for all $(\zeta^\sharp, a^\sharp) \in \mathbb{D} = [-1, 1] \times [-1, 1]$, the choice of

$$\zeta^{in} = s^{in} + \zeta^\sharp s^{in} \log^{-\frac{1}{2}}(s^{in}), \quad a^{in} = a^\sharp (s^{in})^{-\frac{3}{2}}$$

gives us $S(a^{in}) = S(\zeta^\sharp, a^\sharp) \in (s_0, s^{in})$. Recall that

$$\dot{\xi}(s) = 2(\zeta(s) - s)(\dot{\zeta}(s) - 1)s^{-2} \log(s) - (\zeta(s) - s)^2 (2s^{-3} \log(s) - s^{-3}). \quad (2.6.15)$$

On the other hand, for $s \in (S(\zeta^\sharp, a^\sharp), s^{in}]$, then by (2.3.25) and (2.6.10), we have

$$\begin{aligned} \dot{\mathcal{N}}(s) &= s^3 (3s^{-1} a^-(s) + 2 \frac{da^-}{ds}(s)) a^-(s) \\ &= s^3 (3s^{-1} - 2e_0) (a^-(s))^2 + O(\|\epsilon\|_{H^1}^2 s^3 |a^-(s)|). \end{aligned}$$

Due to the bound on $\|\epsilon\|_{H^1}^2$, we obtain

$$\dot{\mathcal{N}}(s) \leq s^3(3s^{-1} - 2e_0)(a^-(s))^2 + C(C^*)^2 s^{-\frac{1}{2}} \sqrt{\mathcal{N}(s)}$$

then for s_0 large enough ($\frac{3}{s_0} < \frac{1}{2}e_0$ and $C(C^*)^2 s_0^{-\frac{1}{2}} < \frac{1}{2}e_0$), the estimate becomes

$$\dot{\mathcal{N}}(s) \leq -\frac{3}{2}e_0 \mathcal{N}(s) + C(C^*)^2 s^{-\frac{1}{2}} \sqrt{\mathcal{N}(s)}. \quad (2.6.16)$$

Denote

$$\begin{aligned} \Psi_1(s) &= (\zeta(s) - s)(s)^{-1} \log^{\frac{1}{2}}(s), \\ \Psi_2(s) &= a^-(s)(s)^{\frac{3}{2}}. \end{aligned}$$

From the definition of $S(a^{in})$ and the continuity of flow, at the limit $S(\zeta^\#, a^\#)$, we have one of the following situation

$$\Psi_1(S(\zeta^\#, a^\#)) = \pm 1, \quad \Psi_2 \in [-1, 1] \quad (2.6.17)$$

or

$$\Psi_2(S(\zeta^\#, a^\#)) = \pm 1, \quad \Psi_1 \in [-1, 1]. \quad (2.6.18)$$

Remark that in the first case, we have

$$\dot{\xi}(S(\zeta^\#, a^\#)) < -(S(\zeta^\#, a^\#))^{-1} < 0$$

and in the second case we have $\mathcal{N}(S(\zeta^\#, a^\#)) = 1$

$$\dot{\mathcal{N}}(S(\zeta^\#, a^\#)) \leq -e_0 < 0.$$

A consequence of the above transversality property is the continuity of the map $(\zeta^\#, a^\#) \mapsto S((\zeta^\#, a^\#))$ thus the following map

$$\begin{aligned} \Psi : \quad \mathbb{D} &\rightarrow \partial\mathbb{D} \\ (\zeta^\#, a^\#) &\mapsto (\Psi_1(S(\zeta^\#, a^\#)), \Psi_2(S(\zeta^\#, a^\#))) \end{aligned}$$

is also continuous where $\partial\mathbb{D}$ is the boundary of \mathbb{D} . Note that if $a^\# = \pm 1$, then from (2.6.16), $\dot{\mathcal{N}}(s^{in}) < 0$, we have $S(\zeta^\#, a^\#) = s^{in}$ and if $\zeta^\# = \pm 1$, then from (2.6.15), $\dot{\xi}(s^{in}) < 0$, we also have $S(\zeta^\#, a^\#) = s^{in}$. Thus $\Psi(\zeta^\#, a^\#) = (\zeta^\#, a^\#)$ for all $(\zeta^\#, a^\#) \in \partial\mathbb{D}$, which means that the restriction of Ψ to the boundary of \mathbb{D} is the identity. But the existence of such a map contradicts the Brouwer fixed point theorem. In conclusion, there exists a final data (z^{in}, a^{in}) such that $S(a^{in}) = s_0$. \square

Finally, we still have the strong compactness result as in Lemma 2.4.5

$$u_n(t_0) \rightharpoonup u_0 \text{ weakly in } H^1(\mathbb{R}^d)$$

$$u_n(t_0) \rightarrow u_0 \text{ in } H^\sigma(\mathbb{R}^d), \text{ for } 0 \leq \sigma < 1$$

then we also consider u the solution of (NLS) corresponding to u_0 , by local well-posedness and continuous dependence (in [3]) for L^2 super-critical of (NLS), we have for all $t \in [t_0, +\infty)$,

$$u_n(t) \rightarrow u(t) \text{ in } H^\sigma(\mathbb{R}^d), \quad s_c \leq \sigma < 1$$

where s_c is the critical exponent $s_c = \frac{d}{2} - \frac{2}{p-1} < 1$. Thus we can pass to the limit the decomposition (\vec{q}, ϵ) and get

$$\left\| u(t) - e^{i\gamma(t)} \sum_{k=1}^2 Q(x - x_k(t)) \right\|_{H^1} \lesssim t^{-1}. \quad (2.6.19)$$

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Chapter 3

Strongly interacting multi-solitons with logarithmic relative distance for gKdV equation

Abstract

We consider the following class of equations of (gKdV) type

$$\partial_t u + \partial_x(\partial_x^2 u + |u|^{p-1}u) = 0, \quad p \text{ integer}, \quad t, x \in \mathbb{R}$$

with mass sub-critical ($2 < p < 5$) and mass super-critical nonlinearities ($p > 5$). We prove the existence of 2-solitary wave solutions with logarithmic relative distance, i.e. solutions $u(t)$ satisfying

$$\left\| u(t) - \left(Q(\cdot - t - \log(ct)) + \sigma Q(\cdot - t + \log(ct)) \right) \right\|_{H^1} \rightarrow 0 \quad \text{as } t \rightarrow +\infty,$$

where $c = c(p) > 0$ is a fixed constant, $\sigma = -1$ in sub-critical cases and $\sigma = 1$ in super-critical cases. This regime corresponds to strong attractive interactions. For sub-critical p , it was known that opposite sign traveling waves are attractive. For super-critical p , we derive from our computations that same sign traveling waves are attractive.

3.1 Introduction

We consider the following class of equations of (gKdV) type

$$\begin{cases} \partial_t u + \partial_x(\partial_x^2 u + |u|^{p-1}u) = 0, & t, x \in \mathbb{R} \\ u(0, x) = u_0(x), & u_0 \in H^1 : \mathbb{R} \rightarrow \mathbb{R} \end{cases} \quad (3.1.1)$$

where $p > 2$ is an integer. When p is odd ($p = 3, 5, 7, \dots$), (3.1.1) is the usual generalized Korteweg-de Vries equations (gKdV). In particular, the case $p = 3$ corresponds to the modified Korteweg-de Vries (mKdV) equation, which is a completely integrable model [24].

From [7], the Cauchy problem for (3.1.1) is well-posed: for all $u_0 \in H^1(\mathbb{R})$, there exists $T^* > 0$ and a solution $u \in \mathcal{C}([0, T^*], H^1(\mathbb{R}))$ to (3.1.1), unique in some class $Y_{T^*} \subset \mathcal{C}([0, T^*], H^1(\mathbb{R}))$. Moreover, the following blow up criterion holds:

$$T^* < +\infty \text{ implies } \lim_{t \rightarrow T^*} \|u(t)\|_{H^1} = +\infty.$$

Recall that such H^1 solutions $u(t)$ satisfy the conservation of mass and energy:

$$\begin{aligned} M(u(t)) &= \int u^2(t, x) dx = M_0, \\ E(u(t)) &= \frac{1}{2} \int (\partial_x u)^2(t, x) dx - \frac{1}{p+1} \int |u|^{p+1}(t, x) dx = E_0. \end{aligned}$$

For $p < 5$, all H^1 solutions are global in time, as a consequence of the conservation laws and the Gagliardo-Nirenberg inequality

$$\|u\|_{L^{p+1}} \lesssim \|u\|_{L^2}^{1-\sigma} \|\partial_x u\|_{L^2}^\sigma \quad \text{with} \quad \sigma = \frac{p-1}{2(p+1)}.$$

For $p = 5$, called the critical case, blow-up in finite time is possible (see e.g. [14]).

A very important feature of these equations is the existence of traveling wave solutions, usually called solitons, of the form

$$R_{v_0, x_0} = Q_{v_0}(x - x_0 - v_0 t)$$

where $Q_v = v^{\frac{1}{p-1}} Q(\sqrt{v}x)$ and Q is the ground state solitary wave, i.e. the unique positive even solution of the equation

$$Q'' + Q^p = Q, \quad Q(x) = \left(\frac{p+1}{2 \cosh^2\left(\frac{p-1}{2}x\right)} \right)^{\frac{1}{p-1}}.$$

Recall that for sub-critical cases ($p < 5$), the solitons are stable and asymptotically stable in H^1 in some sense (see [1], [11], [21], [27]) while for critical and super-critical cases ($p \geq 5$), the solitons are unstable (see [3], [6], [14], [22]).

3.1.1 Main results

In this article, we construct a 2-solitary wave solution with logarithmic relative distance.

[(Multi-solitons with logarithmic distance). Let p integer, $p \neq 5$ and $p > 2$. There exist $t_0 > 0$ and an H^1 solution on $[t_0, +\infty)$ of (3.1.1) which decomposes asymptotically into two solitary waves

$$\left\| u(t) - \left(Q(\cdot - t - \log(ct)) + \sigma Q(\cdot - t + \log(ct)) \right) \right\|_{H^1} \rightarrow 0 \text{ as } t \rightarrow +\infty, \quad (3.1.2)$$

where

$$c = c(p) = \sqrt{\frac{8(p-1)}{|5-p|}} (2p+2)^{\frac{1}{p-1}} \|Q\|_{L^2}^{-1} > 0, \quad (3.1.3)$$

$\sigma = -1$ in sub-critical cases ($2 < p < 5$) and $\sigma = 1$ in super-critical cases ($p > 5$).

In sub-critical cases, [18] proved that the interaction of two solitons of same sign is repulsive. The regime displayed in Main Theorem corresponds to attractive interaction between solitons and thus $\sigma = -1$ for $p < 5$. For the integrable case ($p = 3$), the existence of "double pole solutions", two solitons with alternative sign corresponding to the regime in Main Theorem, was reported in [25] by using inverse scattering transform (see also Remark 3.1.6).

In super-critical cases, we derive from our computations that the interaction between two solitons with the same sign is attractive which explains that $\sigma = 1$ for $p > 5$. In particular, we can apply the strategy of this article to construct multi-solitary waves with logarithmic relative distance for super-critical (gKdV) $\partial_t u + \partial_x(\partial_x^2 u + u^p) = 0$ with p even ($p = 6, 8, \dots$).

We observe that the relative distance of the two traveling waves is $2 \log(ct)$ with c given in (3.1.3). We expect that this is the unique regime of logarithmic relative distance for this scaling.

We point out similarity with the result proved by the author in [19] for nonlinear Schrödinger equations $i\partial_t u + \Delta u + |u|^{p-1}u = 0$: for any dimension $d \geq 1$ and any H^1 sub-critical nonlinearity p , except the L^2 critical one $p = 1 + \frac{4}{d}$, there exists an H^1 solution $u(t)$ which decomposes asymptotically

$$\left\| u(t) - e^{i\gamma(t)} \sum_{k=1}^2 Q(\cdot - x_k(t)) \right\|_{H^1(\mathbb{R}^d)} \lesssim \frac{1}{t}$$

where $x_1(t) = -x_2(t)$ and $|x_1(t) - x_2(t)| = 2(1 + o(1)) \log t$ as $t \rightarrow +\infty$.

Remark 3.1.4. Our result holds in both mass sub-critical ($p < 5$) and mass super-critical cases ($p > 5$). For the mass critical case $p = 5$, we conjecture that solution such as in Main Theorem does not exist. Indeed, as for nonlinear Schrödinger equations, the instability directions related to scaling should be excited by the nonlinear interactions (see [16], [19]).

By scaling and translation, for any $v > 0$ and $x_0 \in \mathbb{R}$, there exists an H^1 solution such that as $t \rightarrow +\infty$

$$\left\| u(t) - \left(Q_v(\cdot - x_0 - vt - \frac{1}{\sqrt{v}} \log(cv^{\frac{3}{2}}t)) + \sigma Q_v(\cdot - x_0 - vt + \frac{1}{\sqrt{v}} \log(cv^{\frac{3}{2}}t)) \right) \right\|_{H^1} \rightarrow 0.$$

It is mandatory that the scaling v is the same for both solitons since otherwise they would have different speeds, see next remark.

Remark 3.1.5. In our main result, the logarithmic distance is due to strong attractive interaction between the two solitary waves. This is in contrast with most previous works on multi-solitary waves of (gKdV) where weak interactions do not change the behavior of solitons, see in particular [2], [4], [10], [15]. A typical example to illustrate weakly interacting dynamics is the existence of multi-soliton solutions $u(t)$ of (gKdV) with any different speeds $0 < v_1 < \dots < v_K$ and any $x_1, \dots, x_K \in \mathbb{R}$,

$$\left\| u(t) - \sum_{k=1}^K Q_{v_k}(\cdot - v_k t - x_k) \right\|_{H^1} \rightarrow 0, \quad \text{as } t \rightarrow +\infty.$$

Combining the construction of this paper and the construction of multi-soliton solutions with weak interactions in [2], [10], we prove the existence of $u(t)$ such that as $t \rightarrow +\infty$

$$\left\| u(t) - \left[\sum_{k=1}^K \left(Q_{v_k}(\cdot - v_k t - \frac{1}{\sqrt{v_k}} \log(cv_k^{\frac{3}{2}} t)) + \sigma Q_{v_k}(\cdot - v_k t + \frac{1}{\sqrt{v_k}} \log(cv_k^{\frac{3}{2}} t)) \right) + \sum_{k'=1}^{K'} Q_{w_{k'}}(\cdot - w_{k'} t) \right] \right\|_{H^1} \rightarrow 0$$

for any $v_k, w_{k'} \neq 1$, $0 < v_1 < \dots < v_K$, $0 < w_1 < \dots < w_{K'}$ and $v_k \neq w_{k'}$. Since configurations of 2-soliton with logarithmic distance like in (3.1.2) are determined by its strong interaction, which will not be affected by weak interactions.

Remark 3.1.6. It is informative to observe the asymptotic form of the double pole solution in [25] as $t \rightarrow \pm\infty$ to remark the perfect interaction of the solitons in integrable case. For $p = 3$, the ground state solitary wave is $Q(x) = \sqrt{2} \cosh^{-1}(x)$ and the behavior of double pole solution at $t \rightarrow \pm\infty$ writes

$$\begin{aligned} u(x, t) &\sim Q(x - t + \log(4t)) - Q(x - t - \log(4t)), & t \rightarrow +\infty \\ &\sim -Q(x - t + \log(-4t)) + Q(x - t - \log(-4t)), & t \rightarrow -\infty \end{aligned}$$

(see the formula 3.13 in [25] with $\eta = 1$ after suitable scaling, note that $c = 4$ for $p = 3$ matches (3.1.3)) so soliton and antisoliton approach very slowly, interact nonlinearly and separate again very slowly. The distance between soliton and antisoliton is asymptotically proportional to $\log|t|$ both at $t \rightarrow +\infty$ and $t \rightarrow -\infty$. An interesting question is to understand the behavior of solutions in Main Theorem for $t \leq 0$ in non-integrable case ($p = 4$). We conjecture that the behavior as $t \rightarrow -\infty$ for $p = 4$ is not the same, the relative distance being of order $|t|$. A hint for this observation comes from computations in [13]: when the dispersion is nontrivial, the faster soliton becomes bigger and the slower becomes smaller and then they should split linearly in time for $t \rightarrow -\infty$, in contrast with the integrable case.

We summarize the organization of the paper. In Section 3.2, we construct an approximate solution (an ansatz solution) and find the main order of all terms in the formal evolution system of the geometrical parameters (scaling, position). In Section 3.3, we prove backward uniform estimates, note that the proof of these estimates is slightly different in super-critical cases due to unstable directions (see [2]). In Section 3.4, we finish the proof of Main Theorem by compactness arguments on a suitable sequence of backward solutions.

3.1.2 Notation and identities on solitons

The L^2 scalar product of two real valued functions $f, g \in L^2(\mathbb{R})$ is denoted by

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)g(x)dx.$$

Recall the equation of Q_v

$$Q_v = v^{\frac{1}{p-1}} Q(\sqrt{v}x), \quad Q_v'' + Q_v^p = vQ_v \quad (3.1.7)$$

for $v > 0$ where

$$Q(x) = \left(\frac{p+1}{2 \cosh^2\left(\frac{p-1}{2}x\right)} \right)^{\frac{1}{p-1}}$$

solves

$$Q'' + Q^p = Q \quad \text{and} \quad (Q')^2 + \frac{2}{p+1}Q^{p+1} = Q^2. \quad (3.1.8)$$

It is easily checked that for $p > 2$, as $x \rightarrow +\infty$

$$Q(x) = c_Q e^{-x} + O(e^{-2x}), \quad Q'(x) = -c_Q e^{-x} + O(e^{-2x}). \quad (3.1.9)$$

with $c_Q = (2p+2)^{\frac{1}{p-1}}$. Let

$$P = \frac{Q'}{Q} + 1 - \frac{2}{c_Q} e^{-x} Q,$$

then $|P(x)| \lesssim e^{-2|x|}$.

We denote by \mathcal{Y} the set of smooth functions f satisfying

$$\text{for all } p \in \mathbb{N}, \text{ there exists } q \in \mathbb{N}, \text{ such that } \forall x \in \mathbb{R} \quad |f^{(p)}| \lesssim |x|^q e^{-x}.$$

Let Λ be the generator of L^2 -scaling

$$\Lambda Q_v = \left(\frac{\partial}{\partial v'} Q_{v'} \right)_{|v'=v} = \frac{1}{v} \left(\frac{1}{p-1} Q_v + \frac{1}{2} x Q'_v \right), \quad (3.1.10)$$

$$\Lambda^2 Q_v = \left(\frac{\partial^2}{\partial v'^2} Q_{v'} \right)_{|v'=v}. \quad (3.1.11)$$

The linearization of (3.1.1) involves the following self-adjoint operator

$$L f = -f'' + f - p Q^{p-1} f.$$

We recall the coercivity property of L (see [15], [27]) in sub-critical cases: there exists $\mu > 0$ such that for $f \in H^1(\mathbb{R})$,

$$\langle L f, f \rangle \geq \mu \|f\|_{H^1}^2 - \frac{1}{\mu} (\langle f, Q \rangle^2 + \langle f, Q' \rangle^2). \quad (3.1.12)$$

The situation is different in super-critical cases since the direction related to the eigenvector $Q^{\frac{p+1}{2}}$ cannot be controlled by the scaling parameter. This is due to the unstable nature of the soliton and to the existence of eigenfunctions Z^\pm with real nonzero eigenvalues of the operator $L \partial_x$:

$$L(\partial_x Z^\pm) = \pm e_0 Z^\pm, \quad e_0 > 0 \quad (3.1.13)$$

constructed in [9], [21]. The functions Z^\pm are normalized so that $\|Z^\pm\|_{L^2} = 1$. We recall from [21] that $Z^\pm \in \mathcal{Y}$ and from [2] (see also [5]) that there holds a property of positivity based on Z^\pm : there exists $\mu > 0$ such that for $f \in H^1(\mathbb{R})$,

$$\langle L f, f \rangle \geq \mu \|f\|_{H^1}^2 - \frac{1}{\mu} (\langle f, Z^+ \rangle^2 + \langle f, Z^- \rangle^2 + \langle f, Q' \rangle^2). \quad (3.1.14)$$

Now, we give here some explicit antecedents and integral identities for L :

$$L Q = -(p-1)Q^p, \quad L Q' = 0, \quad \left(\frac{Q'}{Q} \right)' = -\frac{p-1}{p+1} Q^{p-1}, \quad (3.1.15)$$

$$L \left(\frac{Q'}{Q} \right) = -\frac{3p-1}{p+1} Q^{p-2} Q' + \frac{Q'}{Q}, \quad \left(L \left(\frac{Q'}{Q} \right) \right)' = -\frac{3p(p-1)}{p+1} Q^{p-1} + \frac{3(3p-1)(p-1)}{(p+1)^2} Q^{2p-2}, \quad (3.1.16)$$

$$\int Q \Lambda Q = \left(\frac{1}{p-1} - \frac{1}{4} \right) \int Q^2, \quad \int Q^{r+p-1} = \frac{r(p+1)}{2r+p-1} \int Q^r \text{ for } r \geq 1. \quad (3.1.17)$$

We introduce here some notation of order

- $f(t, x) = O(g(t, x))$ if $\exists C > 0$ such that

$$|f(t, x)| \leq Cg(t, x) \quad \text{and} \quad |\partial_x f(t, x)| \leq Cg(t, x)$$

- $f(t, x) = \mathcal{O}(g(t, x))$ if $\exists C > 0$ such that

$$\left(1 + e^{\frac{1}{2}(x-z_1(t))}\right) |f(t, x)| \leq Cg(t, x) \quad \text{and} \quad \left(1 + e^{\frac{1}{2}(x-z_1(t))}\right) |\partial_x f(t, x)| \leq Cg(t, x) \quad (3.1.18)$$

- $f(t, x) = O_{H^1}(g(t))$ if $\exists C > 0$ such that

$$f(t, x) = O(g(t)) \quad \text{and} \quad \|f(t, x)\|_{H^1} \leq Cg(t).$$

3.2 Approximate solution

In this section, we first construct an almost symmetric 2-bubble approximate solution to renormalized equations of (3.1.1) and then extract the evolution system of the geometrical parameters of the bubbles. The approximate solution contains special terms due to the nonlinear interactions of the waves (see Lemma 3.2.23) which appear at the main order of the evolution system (see (3.2.11)). This tail of order e^{-z} is indeed relevant in the description of the exact solution, see Remark 3.4.5. We also state a standard modulation lemma around the approximate solution.

3.2.1 System of modulation equations

We renormalize the flow by considering

$$u(t, x) = w(t, y), \quad x = y + t \quad (3.2.1)$$

so that $w(t, y)$ verifies the equation

$$\partial_t w + \partial_y (\partial_y^2 w - w + |w|^{p-1} w) = 0. \quad (3.2.2)$$

Consider a time dependent \mathcal{C}^1 function of parameters $\Gamma(t)$ of the form

$$\Gamma(t) = (\mu_1(t), \mu_2(t), z_1(t), z_2(t)) \in \mathbb{R}^+ \times \mathbb{R}^- \times \mathbb{R} \times \mathbb{R}$$

with $|\mu_1|, |\mu_2| \ll 1$, $z(t) \gg 1$ and $|z_1(t) + z_2(t)| \ll 1$ where we denote

$$z(t) = z_1(t) - z_2(t).$$

We look for an approximate solution to the problem. By expanding the first order of the interaction of the two solitons which is of order e^{-z} , we guess an ansatz $r(t, y)$ for this order and deduce from the computations the evolution system of the geometrical parameters $\Gamma(t)$. Since the extra term $r(t, y)$ due to the interactions may not be in H^1 (it may have nonzero limits at $-\infty$), we have to introduce an L^2 approximation of these terms, using suitable cut-off functions. Note that in the integrable case ($p = 3$), one should have $r(t, y)$ in L^2 (see Remark 3.2.28), thus the phenomenon is related to nonintegrability (see [13] for a similar phenomenon).

Let $\psi : \mathbb{R} \rightarrow [0, 1]$ be a \mathcal{C}^∞ function such that

$$\psi \equiv 0 \text{ on } (-\infty, 0], \quad \psi \equiv 1 \text{ on } \left[\frac{1}{2}, +\infty\right), \quad \psi' \geq 0$$

and

$$\varphi(t, y) = \psi \left(e^{-\frac{1}{2}z(t)}y + 1 \right), \quad \tilde{\varphi}(t, y) = \psi' \left(e^{-\frac{1}{2}z(t)}y + 1 \right).$$

Then remark that

$$\|\varphi(t, y)/(1 + e^{\frac{1}{2}(y-z_1(t))})\|_{L^2} + \|\tilde{\varphi}(t, y)/(1 + e^{\frac{1}{2}(y-z_1(t))})\|_{L^2} \lesssim e^{\frac{1}{4}z(t)}, \quad (3.2.3)$$

$$\|\partial_y^k \varphi(t, y)\|_{L^2} \lesssim e^{\frac{1}{4}z(t) - \frac{k}{2}z(t)}, \quad \text{for } k \in \mathbb{N} \quad (3.2.4)$$

and

$$\left| \frac{\partial \varphi(t, y)}{\partial z_k} \right| = \left| \frac{1}{2} e^{-\frac{1}{2}z(t)} y \psi' \left(e^{-\frac{1}{2}z(t)}y + 1 \right) \right| \lesssim |\tilde{\varphi}(t, y)|, \quad \text{for } k = 1, 2 \quad (3.2.5)$$

as $\psi' \left(e^{-\frac{1}{2}z(t)}y + 1 \right) \equiv 0$ for $|y| \geq e^{\frac{1}{2}z(t)}$. Next, we set

$$\begin{aligned} \tilde{R}_k(t, y) &= Q_{1+\mu_k(t)}(y - z_k(t)), & R_k(t, y) &= Q(y - z_k(t)) \\ \Lambda \tilde{R}_k(t, y) &= \Lambda Q_{1+\mu_k(t)}(y - z_k(t)), & \Lambda R_k(t, y) &= \Lambda Q(y - z_k(t)) \end{aligned}$$

and similarly for $\Lambda^2 R_k$, where ΛQ_v and $\Lambda^2 Q_v$ are defined in (3.1.10), (3.1.11). Denote $\tilde{R} = \tilde{R}_1 + \sigma \tilde{R}_2$, let consider the approximate solution of the form

$$V = \tilde{R}_1 + \sigma \tilde{R}_2 + \tilde{r} = \tilde{R} + \tilde{r}, \quad \text{where } \tilde{r}(t, y) = r(t, y)\varphi(t, y), \quad (3.2.6)$$

with $r(t, y)$ to be determined.

Proposition 3.2.7 (Approximate solution and leading order flow). Let I be some interval and a function of parameters $\Gamma(t)$ on I such that

$$|\mu_1(t) + \mu_2(t)| \leq e^{-\frac{9}{16}z(t)}, \quad |z_1(t) + z_2(t)| \leq e^{-\frac{1}{32}z(t)}, \quad z_1(t) - z_2(t) \geq 0. \quad (3.2.8)$$

Then there exist unique real-valued functions $A_1(y), A_2(y)$ and some constants $\alpha > 0, \theta, a_1, a_2$ satisfying:

1. $V(t, y)$ defined as in (3.2.6) with $r(t, y) = e^{-z(t)}[A_1(y - z_1(t)) + A_2(y - z_2(t))]$

$$\begin{aligned} V(t, y) &= V(y; \Gamma(t)) \\ &= Q_{1+\mu_1(t)}(y - z_1(t)) + \sigma Q_{1+\mu_2(t)}(y - z_2(t)) \\ &\quad + e^{-z(t)} (A_1(y - z_1(t)) + A_2(y - z_2(t))) \varphi(t, y) \end{aligned} \quad (3.2.9)$$

is an approximate solution of equation (3.2.2) in the following sense: the error to the flow at V

$$\mathcal{E}_V = \partial_t V + \partial_y (\partial_y^2 V - V + |V|^{p-1}V) \quad (3.2.10)$$

decomposes as

$$\mathcal{E}_V = \vec{m}_1 \cdot \vec{M}_1 + \vec{m}_2 \cdot \vec{M}_2 + E(t, y) \quad (3.2.11)$$

where

$$\vec{m}_1 = \begin{pmatrix} \dot{\mu}_1 + \alpha e^{-z} \\ \dot{z}_1 - \mu_1 - a_1 e^{-z} \end{pmatrix}, \quad \vec{m}_2 = \begin{pmatrix} \dot{\mu}_2 - \alpha e^{-z} \\ \dot{z}_2 - \mu_2 + a_2 e^{-z} \end{pmatrix}, \quad (3.2.12)$$

$$\vec{M}_1 = \begin{pmatrix} \Lambda \tilde{R}_1 \\ -\partial_y \tilde{R}_1 \end{pmatrix}, \quad \vec{M}_2 = \sigma \begin{pmatrix} \Lambda \tilde{R}_2 \\ -\partial_y \tilde{R}_2 \end{pmatrix} \quad (3.2.13)$$

and

$$\|E(t, y)\|_{H^1} \lesssim |\mu_1| z e^{-\frac{3}{4}z} + |\mu_2| z e^{-\frac{3}{4}z} + e^{-\frac{5}{4}z} + \sum_{j=1}^2 |\vec{m}_j| \left(e^{-z} + |\mu_1| e^{-\frac{3}{4}z} + |\mu_2| e^{-\frac{3}{4}z} \right). \quad (3.2.14)$$

2. Closeness to the sum of two solitons. For some $q > 0$,

$$\|V(t) - \{Q_{1+\mu_1(t)}(\cdot - z_1(t)) + \sigma Q_{1+\mu_2(t)}(\cdot - z_2(t))\}\|_{H^1} \lesssim |z(t)|^q e^{-\frac{3}{4}z(t)}, \quad t \in I. \quad (3.2.15)$$

3. $A_j \in L^\infty(\mathbb{R})$, $j = 1, 2$ are the unique solutions of:

$$\begin{aligned} (-LA_1)' - \alpha\Lambda Q - a_1Q' &= -\sigma pc_Q \partial_x(e^{-x}Q^{p-1}), \\ (-LA_2)' + 2p\theta(Q^{p-1})' + \alpha\sigma\Lambda Q + a_2\sigma Q' &= -pc_Q \partial_x(e^xQ^{p-1}) \end{aligned} \quad (3.2.16)$$

satisfying

$$\lim_{y \rightarrow +\infty} A_1 = \lim_{y \rightarrow +\infty} A_2 = 0, \quad \lim_{y \rightarrow -\infty} A_1 = \lim_{y \rightarrow -\infty} A_2 = 2\theta \quad (3.2.17)$$

and

$$\int A_1 Q' = \int A_1 Q = 0, \quad \int A_2 Q' = \int (A_2 + 2\theta)Q = 0.$$

3.2.2 Proof of Proposition 3.2.7

We compute \mathcal{E}_V .

Using $\frac{\partial \tilde{R}_j}{\partial \mu_j} = \Lambda \tilde{R}_j$, $\frac{\partial \tilde{R}_j}{\partial y_j} = -\partial_y \tilde{R}_j$ and $\partial_y^2 \tilde{R}_k - (1 + \mu_k) \tilde{R}_k + |\tilde{R}_k|^{p-1} \tilde{R}_k = 0$, we have

$$\begin{aligned} \mathcal{E}_{\tilde{R}} &= \dot{\mu}_1 \Lambda \tilde{R}_1 - (\dot{z}_1 - \mu_1) \partial_y \tilde{R}_1 + \dot{\mu}_2 \sigma \Lambda \tilde{R}_2 - (\dot{z}_2 - \mu_2) \sigma \partial_y \tilde{R}_2 \\ &\quad + \partial_y [|\tilde{R}_1 + \sigma \tilde{R}_2|^{p-1} (\tilde{R}_1 + \sigma \tilde{R}_2) - \tilde{R}_1^p - \sigma \tilde{R}_2^p]. \end{aligned} \quad (3.2.18)$$

Next, let

$$\begin{aligned} I(r) &= \partial_y \left[\partial_y^2 r - r + p(\tilde{R}_1^{p-1} + \tilde{R}_2^{p-1})r \right], \\ K_1(r, \varphi) &= \mu_1 \frac{\partial r}{\partial z_1} \varphi + \mu_2 \frac{\partial r}{\partial z_2} \varphi + \mu_1 r \frac{\partial \varphi}{\partial z_1} + \mu_2 r \frac{\partial \varphi}{\partial z_2}, \\ K_2(r, \varphi) &= p(\tilde{R}_1^{p-1} + \tilde{R}_2^{p-1})r \partial_y \varphi - r \partial_y \varphi + r \partial_y^3 \varphi + 3\partial_y r \partial_y^2 \varphi + 3\partial_y^2 r \partial_y \varphi \end{aligned}$$

(note that $I(r), K_1(r, \varphi), K_2(r, \varphi)$ are linear in r) and

$$J_1(\tilde{r}) = \dot{\mu}_1 \frac{\partial \tilde{r}}{\partial \mu_1} + (\dot{z}_1 - \mu_1) \frac{\partial \tilde{r}}{\partial z_1} + \dot{\mu}_2 \frac{\partial \tilde{r}}{\partial \mu_2} + (\dot{z}_2 - \mu_2) \frac{\partial \tilde{r}}{\partial z_2},$$

$$J_2(\tilde{r}) = \partial_y [|\tilde{R}_1 + \sigma \tilde{R}_2 + \tilde{r}|^{p-1} (\tilde{R}_1 + \sigma \tilde{R}_2 + \tilde{r}) - |\tilde{R}_1 + \sigma \tilde{R}_2|^{p-1} (\tilde{R}_1 + \sigma \tilde{R}_2) - p(\tilde{R}_1^{p-1} + \tilde{R}_2^{p-1})\tilde{r}].$$

Then by computation, we see that

$$\mathcal{E}_V = \mathcal{E}_{\tilde{R}} + I(r)\varphi + K_1(r, \varphi) + J_1(\tilde{r}) + J_2(\tilde{r}) + K_2(r, \varphi). \quad (3.2.19)$$

Lemma 3.2.20 (Expansion of nonlinear interaction). The nonlinear interaction term

$$G = \partial_y [|\tilde{R}|^{p-1} \tilde{R} - \tilde{R}_1^p - \sigma \tilde{R}_2^p]$$

can be decomposed asymptotically as

$$G = pc_Q \left(\sigma e^{-z} \partial_y [e^{-(y-z_1)} R_1^{p-1}] + e^{-z} \partial_y [e^{(y-z_2)} R_2^{p-1}] \right) + \sum_{k=1}^2 O_{H^1}(|\mu_k| z e^{-z}) + O_{H^1}(e^{-\frac{3}{2}z}). \quad (3.2.21)$$

Proof of Lemma 3.2.20. We observe that for $z = z_1 - z_2$, by (3.1.9),

$$\begin{aligned} Q(y)Q(y-z) &\lesssim e^{-z} & \text{and} & & \|Q(y)Q(y-z)\|_{L^2} &\lesssim ze^{-z} \\ |Q'(y)|Q(y-z) &\lesssim e^{-z} & \text{and} & & \|Q'(y)Q(y-z)\|_{L^2} &\lesssim ze^{-z} \end{aligned}$$

so that (recall $p-1 \geq 2$)

$$\begin{aligned} \partial_y[|\tilde{R}_1 + \sigma\tilde{R}_2|^{p-1}(\tilde{R}_1 + \sigma\tilde{R}_2) - \tilde{R}_1^p - \sigma\tilde{R}_2^p] &= p\partial_y[\sigma\tilde{R}_1^{p-1}\tilde{R}_2 + \tilde{R}_1\tilde{R}_2^{p-1}] + O(e^{-z}R_1R_2) \\ &= p\partial_y[\sigma\tilde{R}_1^{p-1}\tilde{R}_2 + \tilde{R}_1\tilde{R}_2^{p-1}] + O_{H^1}(ze^{-2z}). \end{aligned}$$

Next, also from the asymptotic behavior of Q and Q' , we deduce the Taylor formula

$$\begin{aligned} \tilde{R}_k(t, y) &= Q(y-z_k) + \mu_k\Lambda Q(y-z_k) + \mu_k^2 \int_0^1 (1-s)\Lambda^2 Q_{1+s\mu_k}(y-z_k)ds \\ &= Q(y-z_k) + \mu_k\Lambda Q(y-z_k) + O(\mu_k^2(1+|y-z_k|^2)e^{-|y-z_k|}) \quad (3.2.22) \end{aligned}$$

thus we find

$$\sigma\partial_y[\tilde{R}_1^{p-1}\tilde{R}_2] = \sigma\partial_y(R_1^{p-1}R_2) + O_{H^1}(|\mu_1|ze^{-z}) + O_{H^1}(|\mu_2|ze^{-z}).$$

On the other hand, we claim that

$$\partial_y(R_1^{p-1}R_2) = c_Q e^{-z}\partial_y[e^{-(y-z_1)}R_1^{p-1}] + O_{H^1}(e^{-\frac{3}{2}z}).$$

Indeed, consider

$$R_1^{p-1}\partial_y R_2 + c_Q e^{-z}e^{-(y-z_1)}R_1^{p-1} = R_1^{p-1}\partial_y R_2 + c_Q e^{-(y-z_2)}R_1^{p-1}.$$

For $y < z_2$, by the exponential decay of R_1 , we have $e^{-(y-z_2)}R_1 = e^{-z}$ and $|R_1(y)|^p \lesssim e^{-pz}$ so as $p-1 \geq 2$,

$$|R_1^{p-1}\partial_y R_2 + c_Q e^{-(y-z_2)}R_1^{p-1}| \lesssim e^{-\frac{3}{2}z}|R_1|^{\frac{1}{2}}.$$

For $y > z_2$, as $\partial_y R_2 = -c_Q e^{-(y-z_2)} + O(e^{-2(y-z_2)})$, we also have:

$$|R_1^{p-1}\partial_y R_2 + c_Q e^{-(y-z_2)}R_1^{p-1}| \lesssim e^{-\frac{3}{2}z}|R_1|^{\frac{1}{2}}.$$

By the same way, we consider $R_1^{p-2}\partial_y R_1 R_2 - c_Q e^{-z}e^{-(y-z_1)}R_1^{p-2}\partial_y R_1$ and have

$$|R_1^{p-2}\partial_y R_1 R_2 - c_Q e^{-z}e^{-(y-z_1)}R_1^{p-2}\partial_y R_1| \lesssim e^{-\frac{3}{2}z}|R_1|^{\frac{1}{2}}.$$

Therefore,

$$\sigma\partial_y[\tilde{R}_1^{p-1}\tilde{R}_2] = \sigma c_Q e^{-z}\partial_y[e^{-(y-z_1)}R_1^{p-1}] + O_{H^1}(|\mu_1|ze^{-z}) + O_{H^1}(|\mu_2|ze^{-z}) + O_{H^1}(e^{-\frac{3}{2}z}).$$

Similarly

$$\partial_y[\tilde{R}_2^{p-1}\tilde{R}_1] = c_Q e^{-z}\partial_y[e^{(y-z_2)}R_2^{p-1}] + O_{H^1}(|\mu_1|ze^{-z}) + O_{H^1}(|\mu_2|ze^{-z}) + O_{H^1}(e^{-\frac{3}{2}z}).$$

□

Now we construct the refined term $r(y; \Gamma(t))$ to match the order $e^{-z(t)}$ of the nonlinear interaction.

Lemma 3.2.23 (Definition and equation of $r(t, y)$). There exist $\alpha > 0$, θ , a_1, a_2 , $\hat{A}_1 \in \mathcal{Y}$, $\hat{A}_2 \in \mathcal{Y}$ such that the two functions

$$A_1 = \hat{A}_1 + \theta \left(1 + \frac{Q'}{Q}\right), \quad A_2 = \hat{A}_2 - \sigma\theta \left(1 + \frac{Q'}{Q}\right)$$

solves

$$(-LA_1)' - \alpha\Lambda Q - a_1Q' = -\sigma pc_Q \partial_x(e^{-x}Q^{p-1}), \quad \int A_1Q' = \int A_1Q = 0, \quad (3.2.24)$$

$$(-LA_2)' + 2p\theta(Q^{p-1})' + \alpha\sigma\Lambda Q + a_2\sigma Q' = -pc_Q \partial_x(e^xQ^{p-1}), \quad \int A_2Q' = \int (A_2 + 2\theta)Q = 0. \quad (3.2.25)$$

Moreover, by setting

$$r(y; (\mu(t), z(t))) = e^{-z(t)}[A_1(y - z_1(t)) + A_2(y - z_2(t))] \quad (3.2.26)$$

then

$$\mathcal{E}_{\tilde{R}} + I(r)\varphi = \vec{m}_1 \cdot \vec{M}_1 + \vec{m}_2 \cdot \vec{M}_2 + O_{H^1}(|\mu_1|ze^{-z}) + O_{H^1}(|\mu_2|ze^{-z}) + O_{H^1}(e^{-\frac{5}{4}z}). \quad (3.2.27)$$

Remark 3.2.28. Note that $\lim_{y \rightarrow +\infty} r(y) = 0$ while the limit at $-\infty$ of $r(s, y)$ may be non-zero, in others words, the function $r(t, y)$ may have a tail on the left of two solitons which corresponds to a dispersion of size $e^{-z(t)}$ (in the integrable case $p = 3$, we have $\theta = 0$ (see (3.2.34)) so $r(y)$ has no tail, which is compatible to the property of integrable model).

Proof of Lemma 3.2.23. First, assume A_1 solves

$$(-LA_1)' - \alpha_1\Lambda Q - a_1Q' = -\sigma pc_Q \partial_x(e^{-x}Q^{p-1}) \quad (3.2.29)$$

and A_2 solves

$$(-LA_2)' + 6\theta_2(Q^{p-1})' + \alpha_2\sigma\Lambda Q + a_2\sigma Q' = -pc_Q \partial_x(e^xQ^{p-1}). \quad (3.2.30)$$

To show $\alpha_1 = \alpha_2 = \alpha$, we multiply both sides of (3.2.29), (3.2.30) with Q , integrate and use $LQ' = 0$ and parity properties to obtain

$$-\alpha_1 \int Q\Lambda Q = -\sigma pc_Q \int (e^{-x}Q^{p-1})'Q = \sigma c_Q \int e^{-x}Q^p$$

and so

$$\alpha_1 = -\frac{\sigma c_Q \int e^{-x}Q^p}{\int Q\Lambda Q}.$$

Similarly, we also have

$$\alpha_2\sigma \int Q\Lambda Q = -pc_Q \int (e^xQ^{p-1})'Q = -c_Q \int e^xQ^p = -c_Q \int e^{-x}Q^p$$

and so we deduce the unique possible value for α_1 and α_2

$$\alpha_1 = \alpha_2 = \alpha = -\frac{\sigma c_Q \int e^{-x}Q^p}{\int Q\Lambda Q}.$$

Remark from (3.1.17) that in sub-critical cases $\int Q\Lambda Q > 0$ and in super-critical cases $\int Q\Lambda Q < 0$ thus by choice of sign of σ : in sub-critical cases $\sigma = -1$ and in super-critical cases $\sigma = 1$, we have $\alpha > 0$ in both cases as required. By the parity of Q and integration by parts, we obtain

$$\begin{aligned} \int e^{-x} Q^p dx &= \int \frac{e^{-x} + e^x}{2} Q^p dx = \int_0^\infty (e^{-x} + e^x) Q^p dx \\ &= \lim_{R \rightarrow +\infty} \int_0^R (e^{-x} + e^x) Q^p dx = \lim_{R \rightarrow +\infty} \int_0^R (e^{-x} + e^x) (Q - Q'') dx \\ &= \lim_{R \rightarrow +\infty} \left(\int_0^R (e^{-x} + e^x) Q dx + \int_0^R (-e^{-x} + e^x) Q' dx - [(e^{-x} + e^x) Q'] \Big|_0^R \right) \\ &= \lim_{R \rightarrow +\infty} \left([(-e^{-x} + e^x) Q] \Big|_0^R - [(e^{-x} + e^x) Q'] \Big|_0^R \right) = 2c_Q = 2(2p+2)^{\frac{1}{p-1}} \end{aligned}$$

using the asymptotic behavior (3.1.9). Combine with (3.1.17) to get

$$\alpha = \frac{8(p-1)}{|5-p|} (2p+2)^{\frac{2}{p-1}} \|Q\|_{L^2}^{-2}. \quad (3.2.31)$$

Second, let us look for a solution A_1 under the form $A_1 = \hat{A}_1 + \theta_1 \left(1 + \frac{Q'}{Q}\right)$ then by (3.1.15), (3.1.16), we deduce the equation of \hat{A}_1

$$\begin{aligned} (-L\hat{A}_1)' - \theta_1 \left(-\frac{3p(p-1)}{p+1} Q^{p-1} + \frac{3(3p-1)(p-1)}{(p+1)^2} Q^{2p-2} \right) \\ - p\theta_1 (Q^{p-1})' - \alpha\Lambda Q - a_1 Q' = -\sigma p c_Q \partial_x (e^{-x} Q^{p-1}). \end{aligned}$$

To find $\hat{A}_1 \in \mathcal{Y}$, which implies $L\hat{A}_1 \in \mathcal{Y}$, we need to impose

$$\theta_1 \int \left(-\frac{3p(p-1)}{p+1} Q^{p-1} + \frac{3(3p-1)(p-1)}{(p+1)^2} Q^{2p-2} \right) + \alpha \int \Lambda Q = 0 \quad (3.2.32)$$

so from (3.1.17), we get

$$\theta_1 = \frac{(p+1)\alpha \int \Lambda Q}{(p-1) \int Q^{p-1}}.$$

Similarly, we consider the equation of \hat{A}_2 and obtain the same equation for θ_2

$$\sigma\theta_2 \int \left(-\frac{3p(p-1)}{p+1} Q^{p-1} + \frac{3(3p-1)(p-1)}{(p+1)^2} Q^{2p-2} \right) + \sigma\alpha \int \Lambda Q = 0 \quad (3.2.33)$$

thus

$$\theta_1 = \theta_2 = \theta = \frac{(p+1)\alpha \int \Lambda Q}{(p-1) \int Q^{p-1}} = \frac{(p+1)(3-p)\alpha \int Q}{(p-1)^2 \int Q^{p-1}}. \quad (3.2.34)$$

Next, let $Z \in \mathcal{Y}$, $\int ZQ' = 0$ be such that

$$Z' = \theta \left(-\frac{3p(p-1)}{p+1} Q^{p-1} + \frac{3(3p-1)(p-1)}{(p+1)^2} Q^{2p-2} \right) + \alpha\Lambda Q - \sigma p c_Q \partial_x (e^{-x} Q^{p-1}).$$

Then it suffices to solve $-L(\hat{A}_1 + a_1\Lambda Q) = Z$. Indeed, from properties of the linearized operator L , there exists unique $A \in \mathcal{Y}$, $\int AQ' = 0$ such that $-LA = Z$. Therefore, we set $\hat{A}_1 = A - a_1\Lambda Q$ solves the equation. We uniquely fix a_1 so that $\int \hat{A}_1 Q = 0$ as $\int Q\Lambda Q \neq 0$. It is straightforward

to check that $A_1 = \hat{A}_1 + \theta \left(1 + \frac{Q'}{Q}\right)$ satisfies desired conditions. We do similarly for A_2 . Note from the definition of A_1, A_2 that $A'_1, A'_2 \in \mathcal{Y}$ as $\hat{A}_1, \hat{A}_2 \in \mathcal{Y}$ and

$$\left(1 + \frac{Q'}{Q}\right)' = \frac{(p-1)Q^{p-1}}{p+1} \in \mathcal{Y}.$$

Third, set

$$r(y; (\mu(t), z(t))) = e^{-z(t)}[A_1(y - z_1(t)) + A_2(y - z_2(t))].$$

By (3.2.22), we have

$$I(r) = \partial_y[\partial_y^2 r - r + p(R_1^{p-1} + R_2^{p-1})r] + O_{H^1}(|\mu_1|e^{-z}) + O_{H^1}(|\mu_1|e^{-z}).$$

Consider

$$\begin{aligned} \partial_y[\partial_y^2 r - r + p(R_1^{p-1} + R_2^{p-1})r] &= e^{-z}(-LA_1)'(x - z_1) + e^{-z}\partial_y(pR_1^{p-1}A_2(x - z_2)) \\ &\quad + e^{-z}(-LA_2 + 2p\theta Q^{p-1})'(x - z_2) + e^{-z}\partial_y(pR_2^{p-1}(A_1(x - z_1) - 2\theta)). \end{aligned} \quad (3.2.35)$$

Using the estimates

$$|A_2(x - z_2)| \lesssim (1 + |x - z_2|^q)e^{-(x-z_2)}, \quad \text{for } x > z_2$$

and

$$|A_1(x - z_1) - 2\theta| \lesssim (1 + |x - z_1|^q)e^{-(x-z_1)}, \quad \text{for } x < z_1$$

with $A'_1, A'_2 \in \mathcal{Y}$, we have that

$$\begin{aligned} I(r) &= e^{-z}(-LA_1)'(x - z_1) + e^{-z}(-LA_2 + 2p\theta Q^{p-1})'(x - z_2) \\ &\quad + O_{H^1}(|\mu_1|e^{-z}) + O_{H^1}(|\mu_1|e^{-z}) + \mathcal{O}(e^{-\frac{3}{2}z}). \end{aligned} \quad (3.2.36)$$

Then, by (3.2.3),

$$\|\varphi(t, y)/(1 + e^{\frac{1}{2}(y-z_1(t))})\|_{L^2} \lesssim e^{\frac{1}{4}z(t)}$$

and we obtain

$$\begin{aligned} I(r)\varphi &= e^{-z}(-LA_1)'(x - z_1)\psi(e^{-\frac{1}{2}z}y + 1) + e^{-z}(-LA_2 + 2p\theta Q^{p-1})'(x - z_2)\psi(e^{-\frac{1}{2}z}y + 1) \\ &\quad + O_{H^1}(|\mu_1|e^{-z}) + O_{H^1}(|\mu_1|e^{-z}) + O_{H^1}(e^{-\frac{5}{4}z}). \end{aligned} \quad (3.2.37)$$

Forth, we deduce from (3.2.18), the expansion of nonlinear interaction (3.2.21), the equation of A_1 (3.2.24) and A_2 (3.2.25) that

$$\begin{aligned} &\mathcal{E}_{\tilde{R}} + I(r)\varphi \\ &= e^{-z}[\alpha\Lambda Q + a_1Q' - \sigma pc_Q\partial_y(e^{-y}Q^{p-1})](y - z_1)\psi(e^{-\frac{1}{2}z}y + 1) + \dot{\mu}_1\Lambda\tilde{R}_1 - (\dot{z}_1 - \mu_1)\partial_y\tilde{R}_1 \\ &\quad + e^{-z}[-\alpha\sigma\Lambda Q - a_2\sigma Q' - pc_Q\partial_y(e^yQ^{p-1})](y - z_2)\psi(e^{-\frac{1}{2}z}y + 1) + \dot{\mu}_2\sigma\Lambda\tilde{R}_2 - (\dot{z}_2 - \mu_2)\sigma\partial_y\tilde{R}_2 \\ &\quad + \sigma pc_Qe^{-z}\partial_y(e^{-(y-z_1)}Q^{p-1}(y - z_1)) + pc_Qe^{-z}\partial_y(e^{(y-z_2)}Q^{p-1}(y - z_2)) \\ &\quad + O_{H^1}(|\mu_1|ze^{-z}) + O_{H^1}(|\mu_2|ze^{-z}) + O_{H^1}(e^{-\frac{5}{4}z}) \\ &= (\dot{\mu}_1 + \alpha e^{-z})\Lambda\tilde{R}_1 - (\dot{z}_1 - \mu_1 - a_1e^{-z})\partial_y\tilde{R}_1 \\ &\quad + (\dot{\mu}_2 - \alpha e^{-z})\sigma\Lambda\tilde{R}_2 - (\dot{z}_2 - \mu_2 + a_2e^{-z})\sigma\partial_y\tilde{R}_2 \\ &\quad + O_{H^1}(|\mu_1|ze^{-z}) + O_{H^1}(|\mu_2|ze^{-z}) + O_{H^1}(e^{-\frac{5}{4}z}) \\ &= \vec{m}_1 \cdot \vec{M}_1 + \vec{m}_2 \cdot \vec{M}_2 + O_{H^1}(|\mu_1|ze^{-z}) + O_{H^1}(|\mu_2|ze^{-z}) + O_{H^1}(e^{-\frac{5}{4}z}) \end{aligned}$$

here we use (3.2.22) and $\|Q(y - z_i)(1 - \varphi(y))\|_{H^1} \lesssim \exp(-\frac{1}{4}e^{-\frac{1}{2}z}) \ll e^{-\frac{5}{4}z}, i = 1, 2$. Therefore, we obtain the estimation (3.2.27) as required. \square

Finally, we will control other terms in (3.2.19). By the definition of $r(t, y)$, we have

$$|r| + |\partial_y r| + |\partial_y^2 r| \lesssim e^{-z}.$$

Let consider $\frac{\partial r}{\partial z_k} = (-1)^k r - e^{-z} A'_k(y - z_k)$, as $A'_k \in \mathcal{Y}$ so we can control $\mu_k \frac{\partial r}{\partial z_k} \varphi$ as $\mathcal{O}(|\mu_1| e^{-z}) \varphi$. Moreover, recall (3.2.5), $\left| \frac{\partial \varphi(t, y)}{\partial z_k} \right| \lesssim |\tilde{\varphi}(t, y)|$, we deduce that

$$K_1(r, \varphi) = \mathcal{O}(|\mu_1| e^{-z}) (\varphi + \tilde{\varphi}) + \mathcal{O}(|\mu_2| e^{-z}) (\varphi + \tilde{\varphi}) = \sum_{j=1}^2 \mathcal{O}_{H^1}(|\mu_j| e^{-\frac{3}{4}z})$$

since $\|\varphi(y)/(1 + e^{\frac{1}{2}(y-z_1(t))})\|_{L^2} + \|\tilde{\varphi}(t, y)/(1 + e^{\frac{1}{2}(y-z_1(t))})\|_{L^2} \lesssim e^{\frac{1}{4}z(t)}$. For $J_1(\tilde{r})$, note that \tilde{r} does not depend on μ_1, μ_2 and by the product rule, the same way as we control $K_1(r, \varphi)$, we have $(\dot{z}_k - \mu_k) \frac{\partial \tilde{z}}{\partial z_k} = \mathcal{O}(|\vec{m}_k| e^{-z}) \varphi$ thus

$$J_1(\tilde{r}) = \sum_{j=1}^2 \mathcal{O}(|\vec{m}_j| e^{-z}) (\varphi + \tilde{\varphi}) = \sum_{j=1}^2 \mathcal{O}_{H^1}(|\vec{m}_j| e^{-\frac{3}{4}z}).$$

The term $J_2(\tilde{r})$ is quadratic in \tilde{r} so $J_2(\tilde{r}) = \mathcal{O}_{H^1}(e^{-2z})$. Recall from (3.2.4) that

$$\|\partial_y^k \varphi(t, y)\|_{L^2} \lesssim e^{\frac{1}{4}z(t) - \frac{k}{2}z(t)}, \quad \text{for } k \in \mathbb{N}$$

so all terms in $K_2(r, \varphi)$ can be controlled in H^1 as $\mathcal{O}_{H^1}(e^{-\frac{5}{4}z})$. Therefore,

$$\|E\|_{H^1} \lesssim |\mu_1| z e^{-\frac{3}{4}z} + |\mu_2| z e^{-\frac{3}{4}z} + e^{-\frac{5}{4}z} + \sum_{j=1}^2 |\vec{m}_j| \left(e^{-z} + |\mu_1| e^{-\frac{3}{4}z} + |\mu_2| e^{-\frac{3}{4}z} \right). \quad (3.2.38)$$

The estimate (3.2.15) is a direct consequence of the definition of $r(s, y)$ (see Lemma 3.2.23) and the choice (3.2.3) of $\varphi(y)$. \square

3.2.3 Modulation of the approximate solution

We state a standard modulation result around V based on the Implicit Function Theorem (see e.g. Lemma 3.1 in [13]) and we omit its proof.

Lemma 3.2.39 (Modulation around V). For $p \neq 5$, there exist $\bar{\omega}_0 > 0, \bar{z}_0 > 0, C > 0$ such that if $w(t)$ is a solution of (3.2.2) on some interval I satisfying for some $0 < \omega_0 < \bar{\omega}_0, z_0 > \bar{z}_0$

$$\forall t \in I, \quad \inf_{z_1 - z_2 > z_0} \|w(t) - V(\cdot; (0, 0, z_1, z_2))\|_{H^1} \leq \omega_0. \quad (3.2.40)$$

Then there exists a unique \mathcal{C}^1 function $\Gamma(t) = (\mu_1(t), \mu_2(t), z_1(t), z_2(t))$ such that $w(t, y)$ decomposes on I as

$$w(t, y) = V(y; \Gamma(t)) + \epsilon(t, y) \quad (3.2.41)$$

which satisfies the orthogonality conditions

$$\int \epsilon(t) \tilde{R}_1(t) = \int \epsilon(t) \partial_y \tilde{R}_1(t) = \int \epsilon(t) \tilde{R}_2(t) = \int \epsilon(t) \partial_y \tilde{R}_2(t) = 0 \quad (3.2.42)$$

and for all $t \in I$

$$z(t) = z_1(t) - z_2(t) > z_0 - C\omega_0, \quad \|\epsilon(t)\|_{H^1} + |\mu_1(t)| + |\mu_2(t)| \leq C\omega_0. \quad (3.2.43)$$

Moreover, the equation of the rest term $\epsilon(t, y)$ writes

$$\partial_t \epsilon + \partial_y (\partial_y^2 \epsilon - \epsilon + |V + \epsilon|^{p-1} (V + \epsilon) - |V|^{p-1} V) + \vec{m}_1 \cdot \vec{M}_1 + \vec{m}_2 \cdot \vec{M}_2 + E = 0 \quad (3.2.44)$$

where \vec{m}_k, \vec{M}_k and E defined in Proposition 3.2.7.

Note that the choice of the special orthogonality conditions (3.2.42) is related to the coercivity property (3.1.12).

3.3 Backward uniform estimates

Let $(\mu^{in}, z^{in}) \in \mathbb{R} \times (0, +\infty)$ to be chosen with $0 < \mu^{in} \ll 1, z^{in} \gg 1$. Let $u(t, x)$ be solution of (3.1.1) with initial data

$$\begin{aligned} u(t^{in}, x) &= V(x - t^{in}; (\mu^{in}, -\mu^{in}, z_{in}, -z^{in})) + \epsilon^{in}(x - t^{in}) \\ &= [Q_{1+\mu^{in}}(\cdot - z^{in}) + \sigma Q_{1-\mu^{in}}(\cdot + z^{in})] (x - t^{in}) \\ &\quad + e^{-2z^{in}} \left[(A_1(\cdot - z^{in}) + A_2(\cdot + z^{in})) \psi(e^{-z^{in}} \cdot + 1) \right] (x - t^{in}) + \epsilon^{in}(x - t^{in}) \end{aligned}$$

where $\sigma = -1, \epsilon^{in} \equiv 0$ for sub-critical cases while $\sigma = 1, \epsilon^{in}$ chosen in an appropriate way with $\|\epsilon^{in}\|_{H^1} \leq C(t^{in})^{-\frac{3}{2}}$ for super-critical cases (see Section 3.3.2). By the renormalization (3.2.1), we consider $w(t, y) = u(t, y+t)$ solution of (3.2.2) on some open interval containing t^{in} , observe that

$$\begin{aligned} w(t^{in}) &= V(y; (\mu^{in}, -\mu^{in}, z_{in}, -z^{in})), \quad \epsilon(t^{in}) = \epsilon^{in}(y), \\ \mu_1(t^{in}) &= -\mu_2(t^{in}) = \mu^{in}, \quad z_1(t^{in}) = -z_2(t^{in}) = z^{in}. \end{aligned}$$

Denote $\bar{z} = z_1 + z_2, \bar{\mu} = \mu_1 + \mu_2$, we claim the following uniform estimates:

Proposition 3.3.1 (Uniform backward estimates). There exists $t_0 \gg 1$ such that for all $t^{in} > t_0$, there is a choice of parameters (μ^{in}, z^{in}) with

$$\mu_1(t^{in}) = -\mu_2(t^{in}) = \mu^{in} = \sqrt{\alpha} e^{-z^{in}}, \quad z_1(t^{in}) = -z_2(t^{in}) = z^{in} \gg 1 \quad (3.3.2)$$

such that the solution u of (3.1.1) exists and satisfies the hypothesis (3.2.40) of Lemma 3.2.39 on the rescaled frame (t, y) . Moreover, the decomposition given in Lemma 3.2.39 of u satisfies the following uniform estimates, for all $t \in [t_0, t^{in}]$

$$\begin{aligned} |z(t) - 2 \log(\sqrt{\alpha} t)| &\lesssim t^{-\frac{1}{16}}, \quad |\mu(t)| \lesssim t^{-1} \\ |\bar{\mu}(t)| &\lesssim t^{-\frac{9}{8}}, \quad |\bar{z}(t)| \lesssim t^{-\frac{1}{16}}, \quad \|\epsilon(t)\|_{H^1} \lesssim t^{-\frac{9}{8}} \end{aligned} \quad (3.3.3)$$

where

$$\alpha = -\frac{\sigma c_Q \int e^{-x} Q^p}{\int Q \Lambda Q} = \frac{8(p-1)}{|5-p|} (2p+2)^{\frac{2}{p-1}} \|Q\|_{L^2}^{-2} > 0. \quad (3.3.4)$$

Notice in Proposition 3.3.1 that all estimates are independent of t^{in} , thus the distance between $u(t)$ and the approximate solution $V(t)$ depends only on t and not on the time t^{in} where $u(t)$ was taken equal to $V(t) + \epsilon^{in}$.

3.3.1 Proof of the uniform estimates in sub-critical cases

3.3.1.1 Bootstrap bounds

The proof of Proposition 3.3.1 follows from bootstrapping the following estimates

$$|\mu_1 + \mu_2| \leq t^{-\frac{9}{8}} \quad (3.3.5)$$

$$|z_1 + z_2| \leq t^{-\frac{1}{16}} \quad (3.3.6)$$

$$\|\epsilon\|_{H^1} \leq t^{-\frac{9}{8}} \quad (3.3.7)$$

$$\left| \frac{e^{\frac{1}{2}z}}{\sqrt{\alpha}} - t \right| \leq t^{\frac{15}{16}} \quad (3.3.8)$$

$$\frac{1}{2}t^{-1} \leq \mu \leq 2t^{-1}. \quad (3.3.9)$$

The bootstrap regime implies immediately that

$$z(t) = 2 \log t + \log \alpha + O(t^{-\frac{1}{16}}) \quad (3.3.10)$$

and

$$|\mu_1| + |\mu_2| \lesssim t^{-1}, \quad |z_1| + |z_2| \lesssim \log t. \quad (3.3.11)$$

For t_0 to be chosen large enough (independent of t^{in}), and all $t^{in} > t_0$, we define in view of Lemma 3.2.39:

$$t^* = \inf\{\tau \in [t_0, t^{in}]; (3.3.5) - (3.3.9) \text{ hold on } [\tau, t^{in}]\}.$$

3.3.1.2 Control of modulation equation

Lemma 3.3.12 (Pointwise control of the modulation equations and the error). The following estimates hold on $[t_0, t^{in}]$

$$|\dot{\mu}_1(t) + \alpha e^{-z(t)}| + |\dot{\mu}_2(t) - \alpha e^{-z(t)}| \lesssim t^{-\frac{9}{4}}, \quad (3.3.13)$$

$$|\mu_j(t) - \dot{z}_j(t)| \lesssim t^{-\frac{9}{8}}. \quad (3.3.14)$$

and

$$\|E(t, \cdot)\|_{H^1} \lesssim t^{-\frac{9}{4}}. \quad (3.3.15)$$

Proof of Lemma 3.3.12. We claim the following estimates for the modulation equations

$$|\dot{\mu}_1(t) + \alpha e^{-z(t)}| + |\dot{\mu}_2(t) - \alpha e^{-z(t)}| \lesssim \|\epsilon(t)\|_{L^2}^2 + z(t)e^{-z(t)}\|\epsilon\|_{L^2} + \|E\|_{L^2} \quad (3.3.16)$$

$$|\mu_j(t) - \dot{z}_j(t)| \lesssim \|\epsilon\|_{L^2} + \|E\|_{L^2} + e^{-z(t)}. \quad (3.3.17)$$

From (3.2.14), (3.3.7), (3.3.11), we have

$$\|E\|_{H^1} \lesssim t^{-\frac{5}{2}} \log(t) + t^{-\frac{5}{2}} + \sum_{j=1}^2 |\vec{m}_j| t^{-2} \lesssim t^{-\frac{9}{4}} + \sum_{j=1}^2 |\vec{m}_j| t^{-2}. \quad (3.3.18)$$

Therefore $|\dot{\mu}_1(t) + \alpha e^{-z(t)}| + |\dot{\mu}_2(t) - \alpha e^{-z(t)}| \lesssim t^{-\frac{9}{4}}$, $|\mu_j - \dot{z}_j| \lesssim t^{-\frac{9}{8}}$ follow from the bootstrap bounds on $\|\epsilon\|_{H^1}$ and $e^{-z(t)}$. In order to prove (3.3.16), (3.3.17), recall the equation of ϵ , we have

$$\partial_t \epsilon + \partial_y (\partial_y^2 \epsilon - \epsilon + |V + \epsilon|^{p-1}(V + \epsilon) - |V|^{p-1}V) + \sum_{j=1}^2 \vec{m}_j \cdot \vec{M}_j + E = 0$$

with

$$\begin{aligned} E = & \partial_y [|\tilde{R}_1 + \sigma \tilde{R}_2|^{p-1}(\tilde{R}_1 + \sigma \tilde{R}_2) - \tilde{R}_1^p - \sigma \tilde{R}_2^p] + I(r)\varphi + K(r, \varphi) + H(\tilde{r}) \\ & - e^{-\frac{1}{2}z(t)} r \varphi' + e^{-\frac{3}{2}z(t)} r \varphi''' + 3e^{-z(t)} \partial_y r \varphi'' + 3e^{-\frac{1}{2}z(t)} \partial_y^2 r \varphi'. \end{aligned}$$

From the orthogonality condition $\int \epsilon \tilde{R}_k = 0$, we expand $\frac{\partial}{\partial t} \int \epsilon \tilde{R}_1$ and using the equation of $\epsilon(t)$ to obtain

$$\begin{aligned} 0 = & \frac{\partial}{\partial t} \int \epsilon \tilde{R}_1 \\ = & \int \epsilon \partial_t \tilde{R}_1 + \int (\partial_y^2 \epsilon - \epsilon + p|V|^{p-1}\epsilon) \partial_y \tilde{R}_1 \\ & + \int (|V + \epsilon|^{p-1}(V + \epsilon) - |V|^{p-1}V - p|V|^{p-1}\epsilon) \partial_y \tilde{R}_1 \\ & - (\dot{\mu}_1 + \alpha e^{-z}) \int \Lambda \tilde{R}_1 \tilde{R}_1 + \sigma(\mu_2 - \alpha e^{-z}) \int \Lambda \tilde{R}_2 \tilde{R}_1 - \int E \tilde{R}_1. \end{aligned}$$

Using (3.2.15), the equation of Q_v and $\frac{\partial \tilde{R}_j}{\partial \mu_j} = \Lambda \tilde{R}_j$, $\frac{\partial \tilde{R}_j}{\partial y_j} = -\partial_y \tilde{R}_j$, we get

$$0 = \frac{\partial}{\partial t} \int \epsilon \tilde{R}_1 = \dot{\mu}_1 \int \epsilon \Lambda \tilde{R}_1 - (\dot{z}_1 - \mu_1) \int \epsilon \partial_y \tilde{R}_1 + \|\epsilon\|_{L^2} O(ze^{-z}) + O(\|\epsilon\|_{L^2}^2) \\ - (\dot{\mu}_1 + \alpha e^{-z}) \int \Lambda \tilde{R}_1 \tilde{R}_1 + \sigma(\dot{\mu}_2 - \alpha e^{-z}) O(ze^{-z}) - \int E \tilde{R}_1$$

so for $z \gg 1$ and $\|\epsilon\|_{H^1} \ll 1$

$$|\dot{\mu}_1 + \alpha e^{-z}| \leq C (\|\epsilon\|_{L^2}^2 + ze^{-z} \|\epsilon\|_{L^2}^2 + \|E\|_{L^2}) \\ + |\dot{z}_1 - \mu_1| O(\|\epsilon\|_{L^2}) + |\dot{\mu}_2 - \alpha e^{-z}| O(ze^{-z}). \quad (3.3.19)$$

Next, we consider $\int \epsilon \partial_y \tilde{R}_1 = 0$ so

$$0 = \frac{\partial}{\partial t} \int \epsilon \partial_y \tilde{R}_1 \\ = \int \partial_y \epsilon \partial_t \tilde{R}_1 + \int (\partial_y^2 \epsilon - \epsilon + p|V|^{p-1} \epsilon) \partial_y^2 \tilde{R}_1 \\ + \int (|V + \epsilon|^{p-1} (V + \epsilon) - |V|^{p-1} V - p|V|^{p-1} \epsilon) \partial_y^2 \tilde{R}_1 \\ + (\dot{z}_1 - \mu_1 - a_1 e^{-z}) \int \partial_y \tilde{R}_1 \partial_y \tilde{R}_1 + \sigma(\dot{z}_2 - \mu_2 + a_2 e^{-z}) \int \partial_y \tilde{R}_2 \partial_y \tilde{R}_1 - \int E \partial_y \tilde{R}_1 \\ = \dot{\mu}_1 \int \partial_y \epsilon \Lambda \tilde{R}_1 - (\dot{z}_1 - \mu_1) \int \partial_y \epsilon \partial_y \tilde{R}_1 + \|\epsilon\|_{H^1} O(|\mu_1|) + O(\|\epsilon\|_{L^2}) \\ + (\dot{z}_1 - \mu_1 - a_1 e^{-z}) \int \partial_y \tilde{R}_1 \partial_y \tilde{R}_1 + \sigma(\dot{z}_2 - \mu_2 + a_2 e^{-z}) \int \partial_y \tilde{R}_2 \partial_y \tilde{R}_1 - \int E \partial_y \tilde{R}_1$$

so we deduce that

$$|\dot{z}_1 - \mu_1| \leq |a_1| e^{-z} + C \|\epsilon\|_{H^1} (1 + |\mu_1| + e^{-z}) + C \|E\|_{L^2} \\ + |\dot{\mu}_1 + \alpha e^{-z}| O(\|\epsilon\|_{L^2}) + |\dot{z}_2 - \mu_2| O(ze^{-z}) + O(ze^{-2z}). \quad (3.3.20)$$

Combining two estimates (3.3.19), (3.3.20) with their analogues for $|\dot{\mu}_2 + \alpha e^{-z}|$ and $|\dot{z}_2 - \mu_2|$, the estimates (3.3.16), (3.3.17) are proved. Finally, the estimate (3.3.15) is a direct consequence of (3.3.13), (3.3.14) and (3.3.18). \square

3.3.1.3 Energy functional

We introduce a nonlinear energy functional for $\epsilon(t)$: choose $\rho = \frac{1}{32}$ and set

$$\phi(y) = \frac{2}{\pi} \arctan(\exp(8\rho y))$$

so that $\lim_{y \rightarrow -\infty} \phi = 0$ and $\lim_{y \rightarrow +\infty} \phi = 1$. We see that $\forall y \in \mathbb{R}$,

$$\phi(-y) = 1 - \phi(y), \quad \phi'(y) = \frac{8\rho}{\pi \cosh(8\rho y)}, \\ |\phi''(y)| \leq 8\rho |\phi'(y)|, \quad |\phi'''(y)| \leq (8\rho)^2 |\phi'(y)|.$$

Let

$$\Phi_1(t, y) = \frac{\phi(y)}{(1 + \mu_1(t))^2} + \frac{1 - \phi(y)}{(1 + \mu_2(t))^2}, \quad \Phi_2(t, y) = \frac{\mu_1(t) \phi(y)}{(1 + \mu_1(t))^2} + \frac{\mu_2(t) (1 - \phi(y))}{(1 + \mu_2(t))^2}$$

and consider

$$\mathbf{W}(t) = \int \left[\left((\partial_y \epsilon)^2 + \epsilon^2 - \frac{2}{p+1} (|\epsilon + V|^{p+1} - |V|^{p+1} - (p+1)|V|^{p-1}V\epsilon) \right) \Phi_1(t, y) + \epsilon^2 \Phi_2(t, y) \right] dy. \quad (3.3.21)$$

The functional \mathbf{W} is coercive in ϵ at the main order and it is an almost conserved quantity for the problem (see [13], [23] for a similar functional).

Proposition 3.3.22 (Coercivity and time control of energy functional). For all $t \in [t^*, t^{in}]$,

$$\|\epsilon(t)\|_{H^1}^2 \lesssim \mathbf{W}(t) \quad (3.3.23)$$

and

$$\frac{\partial}{\partial t} \mathbf{W}(t) \geq -C_0 \left(t^{-\frac{9}{8}} \|\epsilon\|_{H^1}^2 + t^{-\frac{9}{4}} \|\epsilon\|_{H^1} \right) \quad (3.3.24)$$

where $C_0 > 0$ a constant independent of t^{in} .

Proof of Proposition 3.3.22.

(a) The proof of the coercivity property (3.3.23) is a standard consequence of (3.1.12) and the orthogonality properties (3.2.42) by an elementary localization argument. We refer to the proof of Lemma 4 in [15]. We observe that locally around each soliton \tilde{R}_j , the functional behaves essentially as

$$\int (\partial_y \epsilon)^2 + (1 + \mu_j) \epsilon^2 - p \tilde{R}_j^{p-1} \epsilon^2,$$

which is a rescaled version of $\langle L\epsilon, \epsilon \rangle$.

(b) Now we compute $\frac{\partial}{\partial t} \mathbf{W}(t)$

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} \mathbf{W}(t) &= \int \partial_t \epsilon \left(-\partial_y^2 \epsilon + \epsilon - (|V + \epsilon|^{p-1}(V + \epsilon) - |V|^{p-1}V) \right) \Phi_1 \\ &\quad - \int \partial_t \epsilon \partial_y \epsilon \partial_y \Phi_1 + \int \partial_t \epsilon \epsilon \Phi_2 - \int \partial_t V \left(|V + \epsilon|^{p-1}(V + \epsilon) - |V|^{p-1}V - p|V|^{p-1}\epsilon \right) \Phi_1 \\ &\quad + \frac{1}{2} \int \left((\partial_y \epsilon)^2 + \epsilon^2 - \frac{2}{p+1} (|\epsilon + V|^{p+1} - |V|^{p+1} - (p+1)|V|^{p-1}V\epsilon) \right) \partial_t \Phi_1 + \frac{1}{2} \int \epsilon^2 \partial_t \Phi_2. \end{aligned}$$

First, we consider

$$W_1(t) = \int \partial_t \epsilon \left(-\partial_y^2 \epsilon + \epsilon - (|V + \epsilon|^{p-1}(V + \epsilon) - |V|^{p-1}V) \right) \Phi_1.$$

Using the equation (3.2.44) of ϵ

$$\partial_t \epsilon = -\partial_y \left(\partial_y^2 \epsilon - \epsilon + (|V + \epsilon|^{p-1}(V + \epsilon) - |V|^{p-1}V) \right) - \sum_{j=1}^2 \vec{m}_j \cdot \vec{M}_j - E,$$

we get

$$\begin{aligned} W_1 &= -\frac{1}{2} \int \left(-\partial_y^2 \epsilon + \epsilon - (|V + \epsilon|^{p-1}(V + \epsilon) - |V|^{p-1}V) \right)^2 \partial_y \Phi_1 \\ &\quad + \int E \left(\partial_y^2 \epsilon - \epsilon + |V + \epsilon|^{p-1}(V + \epsilon) - |V|^{p-1}V \right) \Phi_1 + \sum_{j=1}^2 \int \vec{m}_j \cdot \vec{M}_j \left(\partial_y^2 \epsilon - \epsilon + p \tilde{R}_j^{p-1} \epsilon \right) \Phi_1 \\ &\quad + \sum_{j=1}^2 \int \vec{m}_j \cdot \vec{M}_j \left(|V + \epsilon|^{p-1}(V + \epsilon) - |V|^{p-1}V - p|V|^{p-1}\epsilon \right) \Phi_1 + \sum_{j=1}^2 \int \vec{m}_j \cdot \vec{M}_j \left(p|V|^{p-1}\epsilon - p \tilde{R}_j^{p-1} \epsilon \right) \Phi_1. \end{aligned}$$

On the one hand, we have, by (3.3.15)

$$\left| \int E (\partial_y^2 \epsilon - \epsilon + |V + \epsilon|^{p-1}(V + \epsilon) - |V|^{p-1}V) \Phi_1 \right| \lesssim \|\epsilon\|_{H^1} \|E\|_{H^1} \lesssim t^{-\frac{9}{4}} \|\epsilon\|_{H^1}$$

and by (3.3.13), (3.3.14)

$$\left| \int \vec{m}_j \cdot \vec{M}_j (|V + \epsilon|^{p-1}(V + \epsilon) - |V|^{p-1}V - p|V|^{p-1}\epsilon) \Phi_1 \right| \lesssim t^{-\frac{9}{8}} \|\epsilon\|_{H^1}^2.$$

Using (3.3.10) and the asymptotic behavior of Q (3.1.9), we obtain $\vec{M}_j(|V|^{p-1} - \tilde{R}_j^{p-1}) \lesssim |z|^q e^{-z} \lesssim t^{-\frac{3}{2}}$ so by pointwise control modulation equations (3.3.16), (3.3.17)

$$\left| \int \vec{m}_j \cdot \vec{M}_j (p|V|^{p-1}\epsilon - p\tilde{R}_j^{p-1}\epsilon) \Phi_1 \right| \lesssim t^{-\frac{3}{2}} \|\epsilon\|^2.$$

On the other hand, by (3.3.13),

$$\left| \int (\dot{\mu}_1 + \alpha e^{-z}) \Lambda \tilde{R}_1 (\partial_y^2 \epsilon - \epsilon + p\tilde{R}_j^{p-1}\epsilon) \Phi_1 \right| \lesssim t^{-\frac{9}{4}} \|\epsilon\|_{H^1}.$$

Denote $L_j = -f'' + f - p\tilde{R}_j^{p-1}f$ then

$$L_j(\partial_y \tilde{R}_j) = \mu_j \partial_y \tilde{R}_j.$$

Thus by (3.2.42), (3.3.14) and remark that by the decay property of Q , ϕ' , ϕ''

$$\begin{aligned} \|e^{-\frac{1}{2}|y-z_j|} \partial_y \Phi_1\|_{L^\infty} &\lesssim (|\mu_1| + |\mu_2|) e^{-2\rho z} \lesssim t^{-\frac{9}{8}} \\ \|(\mu_j \Phi_1 - \Phi_2) e^{-\frac{1}{2}|y-z_j|}\|_{L^\infty} &\lesssim (|\mu_1| + |\mu_2|) e^{-2\rho z} \lesssim t^{-\frac{9}{8}} \end{aligned} \quad (3.3.25)$$

for $j = 1, 2$, here note that $\rho = \frac{1}{32}$, so we obtain

$$\begin{aligned} &\int (\dot{z}_1 - \mu_1 - a_1 e^{-z}) \partial_y \tilde{R}_1 (\partial_y^2 \epsilon - \epsilon + p\tilde{R}_j^{p-1}\epsilon) \Phi_1 \\ &= \int (\dot{z}_1 - \mu_1 - a_1 e^{-z}) L_1 (\partial_y \tilde{R}_1) \epsilon \Phi_1 + O(\|\epsilon\|_{H^1}^2 (|\mu_1| + |\mu_2|) e^{-2\rho z}) \\ &= (\dot{z}_1 - \mu_1 - a_1 e^{-z}) \int \mu_1 \partial_y \tilde{R}_1 \epsilon \Phi_1 + O(t^{-\frac{9}{8}} \|\epsilon\|_{H^1}^2) \\ &= (\dot{z}_1 - \mu_1 - a_1 e^{-z}) \int \partial_y \tilde{R}_1 \epsilon \Phi_2 + O(t^{-\frac{9}{8}} \|\epsilon\|_{H^1}^2). \end{aligned}$$

The same estimates hold for \tilde{R}_2 hence the first term W_1 of $\frac{\partial}{\partial t} \mathbf{W}$ verifies

$$\begin{aligned} W_1(t) &= -\frac{1}{2} \int (-\partial_y^2 \epsilon + \epsilon - (|V + \epsilon|^{p-1}(V + \epsilon) - |V|^{p-1}V))^2 \partial_y \Phi_1 \\ &+ \sum_{j=1}^2 (\dot{z}_j - \mu_j - a_j e^{-z}) \int \partial_y \tilde{R}_j \epsilon \Phi_2 + O(t^{-\frac{9}{8}} \|\epsilon\|_{H^1}^2) + O(t^{-\frac{9}{4}} \|\epsilon\|_{H^1}). \end{aligned}$$

For the first term, using integration by parts, $\|e^{-\frac{1}{2}|y-z_j|} \partial_y \Phi_1\|_{L^\infty} \lesssim (|\mu_1| + |\mu_2|) e^{-2\rho z} \lesssim t^{-\frac{9}{8}}$, $|\phi'''| \leq (8\rho)^2 |\phi'| \leq \frac{1}{16} |\phi'|$ and the fact that since $\mu_1(t) \geq \mu_2(t)$ so $\frac{1}{(1+\mu_1(t))^2} \leq \frac{1}{(1+\mu_2(t))^2}$, $\partial_y \Phi_1 \leq 0$,

we get

$$\begin{aligned}
W_1(t) &= -\frac{1}{2} \int (-\partial_y^2 \epsilon)^2 \partial_y \Phi_1 - \frac{1}{2} \int \epsilon^2 \partial_y \Phi_1 + \int \partial_y^2 \epsilon \epsilon \partial_y \Phi_1 + O(t^{-\frac{9}{8}} \|\epsilon\|_{H^1}^2) \\
&\quad + \sum_{j=1}^2 (\dot{z}_j - \mu_j - a_j e^{-z}) \int \partial_y \tilde{R}_j \epsilon \Phi_2 + O(t^{-\frac{9}{8}} \|\epsilon\|_{H^1}^2) + O(t^{-\frac{9}{4}} \|\epsilon\|_{H^1}) \\
&\geq -\frac{3}{4} \int (\partial_y \epsilon)^2 \partial_y \Phi_1 - \frac{3}{8} \int \epsilon^2 \partial_y \Phi_1 + \sum_{j=1}^2 (\dot{z}_j - \mu_j - a_j e^{-z}) \int \partial_y \tilde{R}_j \epsilon \Phi_2 \\
&\quad + O(t^{-\frac{9}{8}} \|\epsilon\|_{H^1}^2) + O(t^{-\frac{9}{4}} \|\epsilon\|_{H^1}).
\end{aligned} \tag{3.3.26}$$

Second, consider

$$W_2 = - \int \partial_t \epsilon \partial_y \epsilon \partial_y \Phi_1 + \int \partial_t \epsilon \epsilon \Phi_2.$$

From the equation of $\partial_t \epsilon$ (3.2.44)

$$\begin{aligned}
\int \partial_t \epsilon \epsilon \Phi_2 &= \int (\partial_y^2 \epsilon - \epsilon + |V + \epsilon|^{p-1}(V + \epsilon) - |V|^{p-1}V) \partial_y (\epsilon \Phi_2) - \int E \epsilon \Phi_2 \\
&\quad - \sum_{j=1}^2 \int \vec{m}_j \cdot \vec{M}_j \epsilon \Phi_2.
\end{aligned} \tag{3.3.27}$$

We have

$$\left| \int E(\epsilon \Phi_2) \right| \lesssim \|\epsilon\|_{H^1} \|E\|_{H^1} \lesssim t^{-\frac{9}{4}} \|\epsilon\|_{H^1}$$

and from (3.3.13) $|\dot{\mu}_1 + \alpha e^{-z}| \lesssim t^{-\frac{9}{4}}$ so

$$\left| \int (\dot{\mu}_1 + \alpha e^{-z}) \Lambda \tilde{R}_1(\epsilon \Phi_2) \right| \lesssim t^{-\frac{9}{4}} \|\epsilon\|_{H^1} |\mu_1| \lesssim t^{-3} \|\epsilon\|_{H^1}.$$

And for the first term, using integration by parts and the fact $\partial_y \Phi_2 \geq 0$, we get

$$\begin{aligned}
&\int (\partial_y^2 \epsilon - \epsilon + |V + \epsilon|^{p-1}(V + \epsilon) - |V|^{p-1}V) \partial_y (\epsilon \Phi_2) \\
&= -\frac{3}{2} \int (\partial_y \epsilon)^2 \partial_y \Phi_2 - \frac{1}{2} \int \epsilon^2 \partial_y \Phi_2 + \frac{1}{2} \int \epsilon^2 \partial_y^3 \Phi_2 + \int (|V + \epsilon|^{p-1}(V + \epsilon) - |V|^{p-1}V) \partial_y (\epsilon \Phi_2) \\
&\geq -\frac{3}{2} \int (\partial_y \epsilon)^2 \partial_y \Phi_2 - \frac{3}{4} \int \epsilon^2 \partial_y \Phi_2 + \int (|V + \epsilon|^{p-1}(V + \epsilon) - |V|^{p-1}V) \partial_y (\epsilon \Phi_2).
\end{aligned}$$

As $(|V + \epsilon|^{p-1}(V + \epsilon) - |V|^{p-1}V) - p|V|^{p-1}\epsilon = O(\epsilon^2)$, let consider

$$\begin{aligned}
&\int (|V + \epsilon|^{p-1}(V + \epsilon) - |V|^{p-1}V) \partial_y (\epsilon \Phi_2) = \int p|V|^{p-1}\epsilon \partial_y (\epsilon \Phi_2) + \sum_{j=1}^2 |\mu_j| \|\epsilon\|_{H^1}^3 \\
&= - \int \frac{p(p-1)}{2} |V|^{p-3} V (\partial_y V \Phi_2) \epsilon^2 - \left(\frac{p}{2} - 1\right) \int p|V|^{p-1} \epsilon^2 \partial_y \Phi_2 + O(s^{-2} \|\epsilon\|_{H^1}^2).
\end{aligned}$$

However, by the decay property of V and Φ , we have

$$\begin{aligned}
\|V \partial_y \Phi_2\|_{L^\infty} &\lesssim (|\mu_1| + |\mu_2|) e^{-2\rho z}, \\
\left\| \Phi_2 \partial_y V - \sum_{j=1}^2 \mu_j \partial_y \tilde{R}_j \right\|_{L^\infty} &\lesssim (|\mu_1| + |\mu_2|) e^{-2\rho z} + e^{-z},
\end{aligned}$$

so gathering these computations

$$\begin{aligned}
\int \partial_t \epsilon \in \Phi_2 &\geq -\frac{p(p-1)}{2} \sum_{j=1}^2 \mu_j \int \partial_y \tilde{R}_j |V|^{p-3} V \epsilon^2 - \sum_{j=1}^2 (\dot{z}_j - \mu_j - a_j e^{-z}) \int \partial_y \tilde{R}_j \epsilon \Phi_2 \\
&\quad - \frac{3}{2} \int (\partial_y \epsilon)^2 \partial_y \Phi_2 - \frac{3}{4} \int \epsilon^2 \partial_y \Phi_2 + O(s^{-2} \|\epsilon\|_{H^1}^2) + O(\|\epsilon\|_{H^1}^2 (|\mu_1| + |\mu_2|) e^{-2\rho z}) \\
&= -\frac{p(p-1)}{2} \sum_{j=1}^2 \mu_j \int \partial_y \tilde{R}_j |V|^{p-3} V \epsilon^2 - \sum_{j=1}^2 (\dot{z}_j - \mu_j - a_j e^{-z}) \int \partial_y \tilde{R}_j \epsilon \Phi_2 \\
&\quad - \frac{3}{2} \int (\partial_y \epsilon)^2 \partial_y \Phi_2 - \frac{3}{4} \int \epsilon^2 \partial_y \Phi_2 + O(t^{-\frac{9}{8}} \|\epsilon\|_{H^1}^2).
\end{aligned} \tag{3.3.28}$$

Moreover, by (3.2.44), (3.3.25), integrating by parts and arguing as in (3.3.27), we have

$$\begin{aligned}
-\int \partial_t \epsilon \partial_y \epsilon \partial_y \Phi_1 &= -\int (\partial_y^2 \epsilon - \epsilon + (|V + \epsilon|^{p-1}(V + \epsilon) - |V|^{p-1}V) \partial_y (\partial_y \epsilon \partial_y \Phi_1) \\
&\quad + \sum_{j=1}^2 \int \vec{m}_j \cdot \vec{M}_j \partial_y \epsilon \partial_y \Phi_1 + \int E \partial_y \epsilon \partial_y \Phi_1 \\
&\geq -\int (\partial_y^2 \epsilon)^2 \partial_y \Phi_1 - \frac{7}{8} \int (\partial_y \epsilon)^2 \partial_y \Phi_1 + O(t^{-\frac{9}{8}} \|\epsilon\|_{H^1}^2) \geq O(t^{-\frac{9}{8}} \|\epsilon\|_{H^1}^2)
\end{aligned}$$

as $\partial_y \Phi_1 \leq 0$ and $\|E\|_{H^1} \lesssim t^{-\frac{9}{4}}$, $|m_j| \lesssim t^{-\frac{9}{8}}$, $\|e^{-\frac{1}{2}|y-z_j|} \partial_y \Phi_1\|_{L^\infty} \lesssim t^{-\frac{9}{8}}$. Therefore, we deduce that

$$\begin{aligned}
W_2 &\geq -\frac{p(p-1)}{2} \sum_{j=1}^2 \mu_j \int \partial_y \tilde{R}_j |V|^{p-3} V \epsilon^2 - \sum_{j=1}^2 (\dot{z}_j - \mu_j - a_j e^{-z}) \int \partial_y \tilde{R}_j \epsilon \Phi_2 \\
&\quad - \frac{3}{2} \int (\partial_y \epsilon)^2 \partial_y \Phi_2 - \frac{3}{4} \int \epsilon^2 \partial_y \Phi_2 + O(t^{-\frac{9}{8}} \|\epsilon\|_{H^1}^2).
\end{aligned} \tag{3.3.29}$$

Next, let

$$W_3 = -\int \partial_t V (|V + \epsilon|^{p-1}(V + \epsilon) - |V|^{p-1}V - p|V|^{p-1}\epsilon) \Phi_1.$$

Remark that from the definition of V and ϕ

$$\begin{aligned}
\left\| \partial_t V \Phi_1 - \left(\dot{\mu}_1 \Lambda \tilde{R}_1 - \dot{z}_1 \partial_y \tilde{R}_1 \right) - \sigma \left(\dot{\mu}_2 \Lambda \tilde{R}_2 - \dot{z}_2 \partial_y \tilde{R}_2 \right) \right\|_{L^\infty} \\
\lesssim |z|^q e^{-z} + (|\mu_1| + |\mu_2|) e^{-2\rho z} \lesssim t^{-\frac{9}{8}}
\end{aligned} \tag{3.3.30}$$

as $|\dot{z}_j| \sim |\mu_j| \lesssim t^{-1}$, $|\dot{\mu}_j| \sim e^{-z} \lesssim t^{-2}$. And from the expansion of $|V + \epsilon|^{p-1}(V + \epsilon)$

$$\left\| (|V + \epsilon|^{p-1}(V + \epsilon) - |V|^{p-1}V - p|V|^{p-1}\epsilon) - \frac{p(p-1)}{2} \epsilon^2 V |V|^{p-3} \right\|_{L^\infty} \lesssim \|\epsilon\|_{L^\infty}^3.$$

Combining with (3.3.5)-(3.3.9), (3.3.13), (3.3.14), we have

$$\begin{aligned}
\left| W_3 - \frac{p(p-1)}{2} \dot{z}_1 \int \partial_y \tilde{R}_1 V |V|^{p-3} \epsilon^2 - \frac{p(p-1)}{2} \dot{z}_2 \int \sigma \partial_y \tilde{R}_2 V |V|^{p-3} \epsilon^2 \right| \\
\lesssim \sum_{j=1}^2 |\dot{z}_j| \|\epsilon\|_{H^1}^3 + \sum_{j=1}^2 |\dot{\mu}_j| \|\epsilon\|_{H^1}^2 + t^{-\frac{9}{8}} \|\epsilon\|_{H^1}^2 \lesssim t^{-\frac{9}{8}} \|\epsilon\|_{H^1}^2,
\end{aligned}$$

in other words,

$$W_3 = \frac{p(p-1)}{2} \dot{z}_1 \int \partial_y \tilde{R}_1 V |V|^{p-3} \epsilon^2 + \frac{p(p-1)}{2} \dot{z}_2 \int \sigma \partial_y \tilde{R}_2 V |V|^{p-3} \epsilon^2 + O(t^{-\frac{9}{8}} \|\epsilon\|_{H^1}^2). \quad (3.3.31)$$

Finally,

$$W_4 = \frac{1}{2} \int \left((\partial_y \epsilon)^2 + \epsilon^2 - \frac{2}{p+1} (|\epsilon + V|^{p+1} - |V|^{p+1} - (p+1)|V|^{p-1} V \epsilon) \right) \partial_t \Phi_1 + \frac{1}{2} \int \epsilon^2 \partial_t \Phi_2.$$

Since

$$\partial_t \Phi_1 = \frac{-2(1 + \mu_1(t)) \dot{\mu}_1(t) \phi(y)}{(1 + \mu_1(t))^3} + \frac{-2(1 + \mu_2(t)) \dot{\mu}_2(t) (1 - \phi(y))}{(1 + \mu_2(t))^3},$$

$$\partial_t \Phi_2(t, y) = \frac{[(1 + \mu_1(t)) \dot{\mu}_1(t) - 2\mu_1(t) \dot{\mu}_1(t)] \phi(y)}{(1 + \mu_1(t))^3} + \frac{[(1 + \mu_2(t)) \dot{\mu}_2(t) - 2\mu_2(t) \dot{\mu}_2(t)] (1 - \phi(y))}{(1 + \mu_2(t))^3},$$

we get that

$$|W_4| \lesssim (|\dot{\mu}_1| + |\dot{\mu}_2|) \|\epsilon\|_{H^1}^2 \lesssim t^{-2} \|\epsilon\|_{H^1}^2$$

as $|\dot{\mu}_1 + \alpha e^{-z}| + |\dot{\mu}_2 - \alpha e^{-z}| \lesssim t^{-\frac{9}{4}}$ and $|e^{-z}| \lesssim t^{-2}$ so

$$W_4 = O(t^{-2} \|\epsilon\|_{H^1}^2). \quad (3.3.32)$$

To conclude, recall that by (3.3.14), $|\mu_j(t) - \dot{z}_j(t)| \lesssim t^{-\frac{9}{8}}$ and remark that by explicit computations

$$|\partial_y \Phi_1 + 2\partial_y \Phi_2| = \left| \frac{\mu_1^2(t) \phi'(y)}{(1 + \mu_1(t))^2} + \frac{-\mu_2^2(t) \phi'(y)}{(1 + \mu_2(t))^2} \right| \lesssim |\mu_1(t)|^2 + |\mu_2(t)|^2 \lesssim t^{-2},$$

we can deduce from (3.3.26), (3.3.29), (3.3.31), (3.3.32) that

$$\frac{d}{ds} \mathbf{W}(t) = W_1(t) + W_2(t) + W_3(t) + W_4(t) \geq -C_0 \left(t^{-\frac{9}{8}} \|\epsilon\|_{H^1}^2 + t^{-\frac{9}{4}} \|\epsilon\|_{H^1} \right)$$

for some $C_0 > 0$ as required. \square

3.3.1.4 End of the proof of Proposition 3.3.22

We close the bootstrap estimates (3.3.5)-(3.3.9).

step 1 Closing the estimates in ϵ (3.3.7). By (3.3.24) in Proposition 3.3.22, we have

$$\frac{\partial}{\partial t} \mathbf{W}(t) \geq -C_0 \left(t^{-\frac{9}{8}} \|\epsilon\|_{H^1}^2 + t^{-\frac{9}{4}} \|\epsilon\|_{H^1} \right) \geq -C t^{-(\frac{9}{4} + \frac{9}{8})}.$$

Thus, by integration on $[t, t^{in}]$ for any $t \in [t^*, t^{in}]$, using $\epsilon(t^{in}) = 0$ in sub-critical cases (in super-critical cases $\|\epsilon(t^{in})\|_{H^1} \leq C(t^{in})^{-\frac{3}{2}}$, see Section 3.3.2 for details), we obtain

$$\mathbf{W}(t) \lesssim t^{-(\frac{9}{4} + \frac{1}{8})}$$

so by (3.3.23)

$$\|\epsilon\|_{H^1}^2 \lesssim t^{-(\frac{9}{4} + \frac{1}{8})}.$$

Therefore, for t_0 large enough, for all $t \in [t^*, t^{in}]$,

$$\|\epsilon\|_{H^1}^2 \leq \frac{1}{2} t^{-\frac{9}{4}}$$

which strictly improves the estimates on $\|\epsilon\|_{H^1}$ in (3.3.7).

step 2 Closing the parameters $\bar{\mu}, \bar{z}$. From the equations (3.3.13), (3.3.14) and the estimates (3.3.5), (3.3.8) in the bootstrap regime, we obtain

$$|\dot{\mu}_1 + \dot{\mu}_2| \lesssim t^{-\frac{9}{4}}, \quad |\dot{z}_1 + \dot{z}_2| \lesssim t^{-\frac{9}{8}}.$$

By integrating on $[t, t^{in}]$,

$$|\mu_1 + \mu_2| \leq t^{-\frac{5}{4}}, \quad |z_1 + z_2| \leq t^{-\frac{1}{8}}$$

as we choose initial data

$$\mu_1(t^{in}) = -\mu_2(t^{in}), \quad z_1(t^{in}) = -z_2(t^{in}).$$

This improves the estimates (3.3.5), (3.3.6).

step 3 Closing the parameters μ, z . Recall that by Lemma 3.3.12

$$|\dot{\mu}_1(t) + \alpha e^{-z(t)}| + |\dot{\mu}_2(t) - \alpha e^{-z(t)}| \lesssim t^{-\frac{9}{4}},$$

$$|\mu_j(t) - \dot{z}_j(t)| \lesssim t^{-\frac{9}{8}}.$$

Thus we deduce for $\mu(t) = \mu_1(t) - \mu_2(t)$ and $z(t) = z_1(t) - z_2(t)$

$$|\dot{\mu} + 2\alpha e^{-z}| \lesssim t^{-\frac{9}{4}},$$

$$|\mu - \dot{z}| \lesssim t^{-\frac{9}{8}}.$$

We get

$$|\dot{\mu}\mu + 2\alpha \dot{z}e^{-z}| \lesssim t^{-(3+\frac{1}{8})}.$$

since $|\mu| \lesssim t^{-1}$, $|\dot{\mu}| \lesssim t^{-2}$. Therefore, by explicit choice of initial data

$$\mu_1(t^{in}) = -\mu_2(t^{in}) = \sqrt{\alpha}e^{-z^{in}}, \quad z_1(t^{in}) = -z_2(t^{in}) = z^{in}$$

then $\mu(t^{in}) = 2\sqrt{\alpha}e^{-\frac{z(t^{in})}{2}}$, we integrate on $[t, t^{in}]$: for any $t \in [t^*, t^{in}]$

$$|\mu^2 - 4\alpha e^{-z}| \lesssim t^{-(2+\frac{1}{8})}.$$

Combining with (3.3.8), $|\mu - 2s^{-1}| \lesssim t^{-(1+\frac{1}{16})}$ which closes (3.3.9). Now, we need to finish the bootstrap argument for z (3.3.8). Let consider

$$\left| \mu - 2\sqrt{\alpha}e^{-\frac{1}{2}z} \right| + |\mu - \dot{z}| \lesssim t^{-\frac{9}{8}}$$

then we get

$$\left| \dot{z} - 2\sqrt{\alpha}e^{-\frac{1}{2}z} \right| \lesssim t^{-\frac{9}{8}}.$$

Note that $\frac{d}{dt} \left(e^{\frac{1}{2}z} \right) = \frac{1}{2}\dot{z}e^{\frac{1}{2}z}$ thus

$$\left| \frac{d}{dt} \left(e^{\frac{1}{2}z} \right) - \sqrt{\alpha} \right| \lesssim t^{-\frac{1}{8}} \tag{3.3.33}$$

here we use $|e^{-z}| \lesssim t^{-2}$.

Next, we need to adjust the initial choice of $z^{in} \gg 1$ through a topological argument (see [2] for a similar argument). We define ζ and ξ the following two functions on $[T^*, T^{in}]$

$$\zeta(t) = \frac{e^{\frac{1}{2}z}}{\sqrt{\alpha}}, \quad \xi(t) = (\zeta(t) - t)^2 t^{-\frac{15}{8}}. \quad (3.3.34)$$

Then, (3.3.33) writes

$$|\dot{\zeta}(t) - 1| \lesssim t^{-\frac{1}{8}}. \quad (3.3.35)$$

According to (3.3.8), our objective is to prove that there exists a suitable choice of

$$\zeta(t^{in}) = \zeta^{in} \in [t^{in} - (t^{in})^{\frac{15}{16}}, t^{in} + (t^{in})^{\frac{15}{16}}],$$

so that $t^* = t_0$. Assume for the sake of contradiction that for all $\zeta^\sharp \in [-1, 1]$, the choice

$$\zeta^{in} = t^{in} + (t^{in})^{-\frac{15}{16}} \zeta^\sharp$$

leads to $t^* = t^*(\zeta^\sharp) \in (t_0, t^{in}]$. Since all estimates in the bootstrap regime except the one on z have been strictly improved on $[s^*, s^{in}]$, it follows from $t^*(\zeta^\sharp) \in (t_0, t^{in}]$ and continuity that

$$|\zeta(t^*(\zeta^\sharp)) - t^*| = (t^*)^{\frac{15}{16}} \quad \text{i.e.} \quad \zeta(t^*(\zeta^\sharp)) = t^* \pm (t^*)^{\frac{15}{16}}.$$

We need a transversality condition to reach a contradiction. We compute:

$$\dot{\xi}(t) = 2(\zeta(t) - t)(\dot{\zeta}(t) - 1)t^{-\frac{15}{8}} - \frac{15}{8}(\zeta(t) - t)^2 t^{-\frac{23}{8}}. \quad (3.3.36)$$

At $t = t^*$, this gives

$$\left| \dot{\xi}(t^*) + \frac{15}{8}(t^*)^{-1} \right| \lesssim (t^*)^{-\frac{17}{16}}.$$

Thus, for t_0 large enough,

$$\dot{\xi}(t^*) < -(t^*)^{-1}. \quad (3.3.37)$$

A consequence of the transversality property (3.3.37) is the continuity of the function $\zeta^\sharp \in [-1, 1] \mapsto t^*(\zeta^\sharp)$. Indeed, let $\epsilon > 0$ then there exists $\delta > 0$ such that $\xi(t^*(\zeta^\sharp) - \epsilon) > 1 + \delta$ and $\xi(t^*(\zeta^\sharp) + \epsilon) < 1 - \delta$. Moreover, by definition of $t^*(\zeta^\sharp)$ (choosing δ small enough) for all $t \in [t^*(\zeta^\sharp) + \epsilon, t^{in}]$ we have $\xi(t) < 1 - \delta$. But from the continuity of the flow, there exists $\iota > 0$ such that for all $|\tilde{\zeta}^\sharp - \zeta^\sharp| < \iota$

$$\forall t \in [t^*(\zeta^\sharp) - \epsilon, t^{in}], \quad |\tilde{\xi}(t) - \xi(t)| \leq \delta/2$$

so we obtain that $t^*(\zeta^\sharp) - \epsilon \leq t^*(\tilde{\zeta}^\sharp) \leq t^*(\xi^\sharp) + \epsilon$ and the continuity of $t^*(\zeta^\sharp)$ as expected. Thus we deduce the continuity of the function Φ defined by

$$\Phi : \zeta^\sharp \in [-1, 1] \mapsto (\zeta(t^*) - t^*)(t^*)^{\frac{15}{16}} \in \{-1, 1\}.$$

Moreover, for $\zeta^\sharp = -1$ and $\zeta^\sharp = 1$, in these two cases $\xi(t^{in}) = 1$, from (3.3.36) we have that $\dot{\xi}(t^{in}) < 0$ thus $t^* = t^{in}$. Therefore, $\Phi(-1) = -1$ and $\Phi(1) = 1$, but this is a contradiction with the continuity.

In conclusion, there exists at least a choice of

$$\zeta(t^{in}) = \zeta^{in} \in \left(t^{in} - (t^{in})^{\frac{15}{16}}, t^{in} + (t^{in})^{\frac{15}{16}} \right)$$

such that $t^* = t_0$. This concludes our bootstrap argument for (3.3.8).

3.3.2 Proof of the uniform estimates in super-critical cases

In this section, we present some modifications to prove the result in super-critical cases. Some extra parameters are needed in order to control the instability created by Z^\pm . Denote

$$\tilde{Z}_k^\pm(t, y) = Z_{1+\mu_k}^\pm(y - z_k(t)).$$

Thus, instead of considering the final data $u(t^{in}) = V(x - t^{in}; (\mu^{in}, -\mu^{in}, z_{in}, -z^{in}))$ as in sub-critical cases, we look at solution $u(t)$ of (3.1.1) with final data

$$u(t^{in}, x) = V(x - t^{in}; (\mu^{in}, -\mu^{in}, z_{in}, -z^{in})) + \epsilon^{in}(x - t^{in})$$

where

$$\begin{aligned} \epsilon^{in}(y) = & b_1^+ \tilde{Z}_1^+(t^{in}, y) + b_1^- \tilde{Z}_1^-(t^{in}, y) + b_2^+ \tilde{Z}_2^+(t^{in}, y) + b_2^- \tilde{Z}_2^-(t^{in}, y) \\ & + b_3 \tilde{R}_1(t^{in}, y) + b_4 \tilde{R}_2(t^{in}, y) + b_5 \partial_y \tilde{R}_1(t^{in}, y) + b_6 \partial_y \tilde{R}_2(t^{in}, y) \end{aligned} \quad (3.3.38)$$

and $b = (b_1^+, b_1^-, b_2^+, b_2^-, b_3, b_4, b_5, b_6)$ belongs to some small neighborhood of 0 in \mathbb{R}^8 .

We consider the decomposition of $w(t, y) = u(t, y + t)$ by Lemma 3.2.39

$$w(t, y) = V(y; \Gamma(t)) + \epsilon(t, y)$$

that satisfies the orthogonality conditions (3.2.42). Define

$$a_k^\pm(t) = \int \epsilon(t, y) \tilde{Z}_k^\pm(t, y) dy, \quad a^\pm(t) = (a_1^\pm(t), a_2^\pm(t)). \quad (3.3.39)$$

The following lemma allows us to establish a one-to-one mapping between the choice of $b = (b_1^+, b_1^-, b_2^+, b_2^-, b_3, b_4, b_5, b_6)$ and the initial constraints $a^+(t^{in}) = 0, a^-(t^{in}) = a^{in}$.

Lemma 3.3.40 (Modulated data in direction Y^\pm). There exists $C > 0$ such that for all $t^{in} \geq t_0$ and for all $a^{in} = (a_1^{in}, a_2^{in}) \in B_{\mathbb{R}^2}(0, (t^{in})^{-\frac{3}{2}})$, there is a unique b so that $\|b\| \leq C \|a^{in}\|$ (C independent of t^{in}) and the initial data satisfies

$$\begin{aligned} \left(\int \epsilon^{in}(y) \tilde{Z}_1^-(t^{in}, y) dy, \int \epsilon^{in}(y) \tilde{Z}_2^-(t^{in}, y) dy \right) &= (a_1^{in}, a_2^{in}), \\ \int \epsilon^{in} \tilde{Z}_1^+(t^{in}) &= \int \epsilon^{in} \tilde{Z}_2^+(t^{in}) = \int \epsilon^{in} \tilde{R}_1(t^{in}) = \int \epsilon^{in} \partial_y \tilde{R}_1(t^{in}) \\ &= \int \epsilon^{in} \tilde{R}_2(t^{in}) = \int \epsilon^{in} \partial_y \tilde{R}_2(t^{in}) = 0. \end{aligned} \quad (3.3.41)$$

Proof of Lemma 3.3.40. Denote

$$c = \left(\int \epsilon^{in} \tilde{Z}_1^-, \int \epsilon^{in} \tilde{Z}_1^+, \int \epsilon^{in} \tilde{R}_1, \int \epsilon^{in} \partial_y \tilde{R}_1, \int \epsilon^{in} \tilde{Z}_2^-, \int \epsilon^{in} \tilde{Z}_2^+, \int \epsilon^{in} \tilde{R}_2, \int \epsilon^{in} \partial_y \tilde{R}_2 \right)$$

and consider the linear maps

$$\begin{array}{ll} \Psi : \mathbb{R}^8 \rightarrow H^1(\mathbb{R}) & \Phi : H^1(\mathbb{R}) \rightarrow \mathbb{R}^8 \\ b \mapsto \epsilon^{in} & \epsilon^{in} \rightarrow c \end{array}$$

and $\Omega = \Phi \circ \Psi : \mathbb{R}^8 \rightarrow \mathbb{R}^8$. We can check that for some functions $A(y), B(y) \in \mathcal{Y}$

$$\begin{aligned} \Omega = \Phi \circ \Psi &= \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix} + O(|\mu^{in}|) + O(|\langle A(y + z^{in}\vec{e}_1), B(y) \rangle|) \\ &= \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix} + O(|\mu^{in}|) + O(e^{-z^{in}}) \end{aligned}$$

where N is the Gramian matrix of $Z^\pm, Q, \partial_y Q$ which are linearly independent. Indeed, Z^+, Z^-, Q are linearly independent and orthogonal since they are eigenfunctions of $L\partial_y$ corresponding to different eigenvalues $e_0, -e_0, 0$. On the other hand, $\partial_y Q$ are orthogonal to Z^+, Z^-, Q (see Lemma 4.9 in [3] for more properties of Z^\pm) so they are linearly independent. Thus $\det N \neq 0$ and with $z^{in} \gg 1, 0 < \mu^{in} \ll 1$, we have that Ω is invertible around 0. Therefore, for any $a^{in} \in B_{\mathbb{R}^2}(0, (t^{in})^{-\frac{3}{2}})$, we can choose

$$b = \Omega^{-1}((a_1^{in}, 0, 0, 0, a_2^{in}, 0, 0, 0)), \quad \|b\| \leq \|\Omega^{-1}\| |a^{in}|$$

to conclude the lemma. \square

We now define the maximal time interval $[T(a^{in}), t^{in}]$ on which the bootstrap bounds (3.3.5)–(3.3.9) hold and

$$\|a^\pm(t)\| \leq t^{-\frac{3}{2}} \quad (3.3.42)$$

for all $t \in [T(a^{in}), t^{in}]$. The uniform backward estimates of Proposition 3.3.1 state that there is a choice of $(\mu^{in}, z^{in}, a^{in})$ with

$$\mu^{in} = \sqrt{\alpha} e^{-z^{in}}, \quad z^{in} \gg 1, \quad a^{in} \in B_{\mathbb{R}^2}(0, (t^{in})^{-\frac{3}{2}}) \quad (3.3.43)$$

such that $T(a^{in}) = t_0$. Indeed, we proceed as for sub-critical cases in Section 3.3.1 and improve all estimates in the bootstrap bounds except those of $a^\pm(t)$ and $z(t)$. Remark that

$$\epsilon(t^{in}) \lesssim \|b\|$$

so if we choose $a^{in} \in B_{\mathbb{R}^2}(0, (t^{in})^{-\frac{3}{2}})$, by Lemma 3.3.40, we have $\|\epsilon(t^{in})\|_{H^1}^2 \lesssim t^{-3}$ which is still enough to conclude

$$\mathbf{W}(t) \lesssim t^{-\left(\frac{9}{4} + \frac{1}{8}\right)}$$

from the fact that $\frac{\partial}{\partial t} \mathbf{W}(t) \geq -C_0 \left(t^{-\frac{9}{8}} \|\epsilon\|_{H^1}^2 + t^{-\frac{9}{4}} \|\epsilon\|_{H^1} \right) \geq -Ct^{-\left(\frac{9}{4} + \frac{9}{8}\right)}$. It seems to us that the reasoning to close the bootstrap bound of $z(t)$ still works, in fact, it is, however we will control $a^\pm(t)$ through a suitable value of a^{in} also by a topological argument so we have to choose (z^{in}, a^{in}) in the same time. Now we claim the following preliminary estimates on the parameters $a^\pm(t)$.

Lemma 3.3.44. For all $t \in [T(a^{in}), t^{in}]$,

$$\left| \frac{da^\pm}{dt}(t) \mp e_0 a^\pm(t) \right| \lesssim \|\epsilon\|_{H^1}^2 + t^{-\frac{9}{4}} \quad (3.3.45)$$

Proof of Lemma 3.3.44. Recall the equation of ϵ :

$$\partial_t \epsilon + \partial_y (\partial_y^2 \epsilon - \epsilon + |V + \epsilon|^{p-1}(V + \epsilon) - |V|^{p-1}V) + \sum_{j=1}^2 \vec{m}_j \cdot \vec{M}_j + E = 0$$

and note that $\int \partial_y \tilde{R}_k \tilde{Z}_k^\pm = 0$ which follows from

$$\int \partial_y Q Z^\pm = \pm \frac{1}{e_0} \int \partial_y Q L(\partial_y Z^\pm) = \pm \frac{1}{e_0} \int \partial_y L(\partial_y Q) Z^\pm = 0.$$

Then, we have

$$\begin{aligned} \frac{da_k^\pm}{dt}(t) &= \int \partial_t \epsilon \tilde{Z}_k^\pm + \int \epsilon \partial_t \tilde{Z}_k^\pm \\ &= \int (\partial_y^2 \epsilon - (1 + \mu_k) \epsilon + p|V|^{p-1} \epsilon) \partial_y \tilde{Z}_k^\pm \\ &\quad + \int (|V + \epsilon|^{p-1} (V + \epsilon) - |V|^{p-1} V - p|V|^{p-1} \epsilon) \partial_y \tilde{Z}_k^\pm \\ &\quad - (\dot{\mu}_1 + \alpha e^{-z}) \int \Lambda \tilde{R}_1 \tilde{Z}_k^\pm + \sigma(\mu_2 - \alpha e^{-z}) \int \Lambda \tilde{R}_2 \tilde{Z}_k^\pm - \int E \tilde{Z}_k^\pm \\ &\quad + \dot{\mu}_k \int \epsilon \Lambda \tilde{Z}_k^\pm - (\dot{z}_k - \mu_k) \int \epsilon \partial_y \tilde{Z}_k^\pm. \end{aligned}$$

Using $Z \in \mathcal{Y}$, for $k \neq j$

$$\int |\tilde{R}_j| (|\tilde{Z}_k^\pm| + |\partial_y \tilde{Z}_k^\pm|) \lesssim z^q e^{-z}$$

hence denote $L_k = -f'' + (1 + \mu_k)f - p\tilde{R}_k^{p-1}f$, we have

$$\int (\partial_y^2 \epsilon - (1 + \mu_k) \epsilon + p|V|^{p-1} \epsilon) \partial_y \tilde{Z}_k^\pm = \int \epsilon L_k (\partial_y \tilde{Z}_k^\pm) + O(\|\epsilon\|_{H^1} e^{-\frac{3}{4}z}).$$

Moreover from (3.3.13), (3.3.14), (3.3.15), we get

$$\begin{aligned} \left| (\dot{\mu}_1 + \alpha e^{-z}) \int \Lambda \tilde{R}_1 \tilde{Z}_k^\pm \right| + \left| \sigma(\mu_2 - \alpha e^{-z}) \int \Lambda \tilde{R}_2 \tilde{Z}_k^\pm \right| &\lesssim \|\epsilon\|_{H^1}^2, \\ \left| \int E \tilde{Z}_k^\pm \right| &\lesssim t^{-\frac{9}{4}}, \\ \left| \dot{\mu}_k \int \epsilon \Lambda \tilde{Z}_k^\pm \right| + \left| (\dot{z}_k - \mu_k) \int \epsilon \partial_y \tilde{Z}_k^\pm \right| &\lesssim \|\epsilon\|_{H^1}^2. \end{aligned}$$

Using the equation of Z^\pm (3.1.13), we obtain

$$\int \epsilon L_k (\partial_y \tilde{Z}_k^\pm) = \pm e_0 (1 + \mu_k)^{\frac{3}{2}} \int \epsilon \tilde{Z}_k^\pm.$$

As $|\mu_k| \lesssim t^{-1}$, $|a^\pm| \lesssim t^{-\frac{3}{2}}$, we get

$$\frac{da_k^\pm}{dt}(t) = \pm e_0 a_k^\pm(t) + O(\|\epsilon\|_{H^1}^2) + O(t^{-\frac{9}{4}})$$

as required. \square

We now control $a^\pm(t)$ through topological arguments by noticing that the direction $a^+(t)$ is already stable. Consider $\zeta(t) = \frac{e^{\frac{1}{2}z}}{\sqrt{\alpha}}$ and $\xi(t)$ as defined in (3.3.34).

Lemma 3.3.46 (Control of $a^\pm(t)$). There exist $\zeta^{in} = \zeta(t^{in}) \in [t^{in} - (t^{in})^{-\frac{15}{16}}, t^{in} + (t^{in})^{-\frac{15}{16}}]$ (in consequence, $z^{in} \gg 1$) and $a^{in} \in B_{\mathbb{R}^2}(0, (t^{in})^{-\frac{3}{2}})$ such that $T(a^{in}) = t_0$.

Proof of Lemma 3.3.46. First of all, we claim that for all $a^{in} \in B_{\mathbb{R}^2}(0, (t^{in})^{-\frac{3}{2}})$, the following inequality holds for all $t \in [T(a^{in}), t^{in}]$

$$|a^+(t)| \leq \frac{1}{2}t^{-\frac{3}{2}}. \quad (3.3.47)$$

Indeed, it follows the bootstrap bounds, (3.3.45) and $a^+(t^{in}) = 0$ that for all $t \in [T(a^{in}), t^{in}]$

$$\begin{aligned} |a^+(t)| &\lesssim e^{e_0 t} \int_t^{t^{in}} e^{-e_0 \tau} \tau^{-\frac{9}{4}} d\tau \\ &= \frac{e^{e_0 t}}{e_0} [e^{-e_0 t} t^{-\frac{9}{4}} - e^{-e_0 t^{in}} (t^{in})^{-\frac{9}{4}}] - \frac{9e^{e_0 t}}{4e_0} \int_t^{t^{in}} e^{-e_0 \tau} \tau^{-\frac{13}{4}} d\tau \\ &\leq \frac{1}{e_0} t^{-\frac{9}{4}} \leq \frac{1}{2} t^{-\frac{3}{2}} \end{aligned}$$

for t_0 to be large enough.

Let $\mathbb{D} = [-1, 1] \times B_{\mathbb{R}^2}(0, 1)$ equipped with the norm $\|(x, y)\| = \max(\|x\|, \|y\|)$. Now we suppose that for all $(\zeta^\sharp, a^\sharp) \in \mathbb{D}$, the choice

$$\zeta^{in} = t^{in} + (t^{in})^{-\frac{15}{16}} \zeta^\sharp, \quad a^{in} = a^\sharp (t^{in})^{-\frac{3}{2}}$$

gives us $T(a^{in}) = T(\zeta^\sharp, a^\sharp) \in (t_0, t^{in}]$. Recall that

$$\dot{\zeta}(t) = 2(\zeta(t) - t)(\dot{\zeta}(t) - 1)t^{-\frac{15}{8}} - \frac{15}{8}(\zeta(t) - t)^2 t^{-\frac{23}{8}}. \quad (3.3.48)$$

On the other hand, consider

$$\mathcal{N}(t) = t^3 \|a^-(t)\|^2$$

then for $t \in (T(\zeta^\sharp, a^\sharp), t^{in}]$, by the bound on $\|\epsilon\|_{H^1}^2$ and (3.3.45), we have

$$\begin{aligned} \dot{\mathcal{N}}(t) &= t^3 \langle 3t^{-1} a^-(t) + 2 \frac{da^-}{dt}(t), a^-(t) \rangle \\ &= t^3 (3t^{-1} - 2e_0) \|a^-(t)\|^2 + O\left(t^{-\frac{3}{4}} t^{\frac{3}{2}} \|a^-(t)\|\right). \end{aligned}$$

Therefore, with t_0 large enough ($\frac{3}{t_0} < \frac{1}{2}e_0$), we deduce that

$$\dot{\mathcal{N}}(t) \leq -\frac{3}{2}e_0 \mathcal{N}(t) + Ct^{-\frac{3}{4}} \sqrt{\mathcal{N}(t)}. \quad (3.3.49)$$

Denote

$$\begin{aligned} \Psi_1(t) &= (\zeta(t) - t)t^{\frac{15}{16}}, \\ \Psi_2(t) &= a^-(t)t^{\frac{3}{2}}. \end{aligned}$$

From the definition of $T(a^{in})$ and the continuity of flow, at the limit $T(\zeta^\sharp, a^\sharp)$, we have one of the following situation

$$\Psi_1(S(\zeta^\sharp, a^\sharp)) = \pm 1, \quad \Psi_2 \in B_{\mathbb{R}^2}(0, 1) \quad (3.3.50)$$

or

$$\|\Psi_2(S(\zeta^\sharp, a^\sharp))\| = 1 \Leftrightarrow \Psi_2 \in \partial B_{\mathbb{R}^2}(0, 1), \quad \Psi_1 \in [-1, 1] \quad (3.3.51)$$

where $\partial B_{\mathbb{R}^2}(0, 1)$ is the boundary of $B_{\mathbb{R}^2}(0, 1)$. Remark that in the first case, we have

$$\dot{\zeta}(T(\zeta^\sharp, a^\sharp)) < -(T(\zeta^\sharp, a^\sharp))^{-1} < 0$$

and in the second case we have $\mathcal{N}(T(\zeta^\sharp, a^\sharp)) = 1$

$$\dot{\mathcal{N}}(T(\zeta^\sharp, a^\sharp)) \leq -e_0 < 0.$$

A consequence of the above transversality property is the continuity of $(\zeta^\sharp, a^\sharp) \mapsto T((\zeta^\sharp, a^\sharp))$ thus the following map

$$\begin{aligned} \Psi : \mathbb{D} &\rightarrow \partial\mathbb{D} \\ (\zeta^\sharp, a^\sharp) &\mapsto (\Psi_1(T(\zeta^\sharp, a^\sharp)), \Psi_2(T(\zeta^\sharp, a^\sharp))) \end{aligned}$$

is also continuous where $\partial\mathbb{D}$ is the boundary of \mathbb{D} . Note that if $a^\sharp \in \partial B_{\mathbb{R}^2}(0, 1)$, then from (3.3.49), $\dot{\mathcal{N}}(t^{in}) < 0$, we have $T(\zeta^\sharp, a^\sharp) = t^{in}$ and if $\zeta^\sharp = \pm 1$, then from (3.3.48), $\dot{\xi}(t^{in}) < 0$, we also have $T(\zeta^\sharp, a^\sharp) = t^{in}$. Thus $\Psi(\zeta^\sharp, a^\sharp) = (\zeta^\sharp, a^\sharp)$ for all $(\zeta^\sharp, a^\sharp) \in \partial\mathbb{D}$, which means that the restriction of Ψ to the boundary of \mathbb{D} is the identity. But the existence of such a map contradicts the Brouwer fixed point theorem. In conclusion, there exists a final data (z^{in}, a^{in}) such that $T(a^{in}) = t_0$, which concludes the proof of Proposition 3.3.1 in super-critical cases. \square

3.4 Construction of solution

Applying Proposition 3.3.1 with $t^{in} = n$ for any n large enough, there exists a solution $u_n(t)$ of (3.1.1) on the interval $[t_0, n]$ whose decomposition

$$(\Gamma^n(t); \epsilon_n(t)) = ((z_1^n(t), z_2^n(t), \mu_1^n(t), \mu_2^n(t)); \epsilon_n(t))$$

satisfies the uniform estimates (3.3.3). Denote

$$\tilde{N}_n(t, x) = Q_{1+\mu_1^n(t)}(x - t - z_1^n(t)) + \sigma Q_{1+\mu_2^n(t)}(x - t - z_2^n(t)).$$

From (3.2.15) $\|V_n(t, x) - \tilde{N}_n(t, x)\|_{H^1} \lesssim t^{-\frac{3}{2}} \log^q(t)$ and (3.3.3) $\|\epsilon_n(t)\|_{H^1} \leq t^{-\frac{9}{8}}$, we have

$$\|u_n(t) - \tilde{N}_n(t, x)\|_{H^1} \lesssim t^{-\frac{9}{8}}. \quad (3.4.1)$$

On the other hand, by setting

$$N(t, x) = Q(x - t - \log(\sqrt{\alpha t})) + \sigma Q(x - t + \log(\sqrt{\alpha t})) \quad (3.4.2)$$

we deduce from (3.3.3) that

$$\begin{aligned} &\|\tilde{N}_n(t, x) - N(t, x)\|_{H^1} \lesssim |\mu_1^n(t)| + |\mu_2^n(t)| + |z_1^n(t) - \log(\sqrt{\alpha t})| + |z_2^n(t) + \log(\sqrt{\alpha t})| \\ &\lesssim \left| \frac{\bar{\mu}(t) + \mu(t)}{2} \right| + \left| \frac{\bar{\mu}(t) + \mu(t)}{2} \right| + \left| \frac{\bar{z}(t) + z(t)}{2} - \log(\sqrt{\alpha t}) \right| + \left| \frac{\bar{z}(t) - z(t)}{2} - \log(\sqrt{\alpha t}) \right| \\ &\lesssim t^{-\frac{1}{16}}. \end{aligned} \quad (3.4.3)$$

Therefore, there exist a sequence of backward solutions $u_n \in \mathcal{C}([t_0, n], H^1)$ of (3.1.1) such that for all $t \in [t_0, n]$,

$$\|u_n(t) - N(t, x)\|_{H^1} \lesssim t^{-\frac{1}{16}}. \quad (3.4.4)$$

Remark 3.4.5. We see that the size of the extra term $\tilde{r}(t, y)$ in the definition (3.2.6) of the approximate solution $V(t, y)$ is much smaller than the estimate on $\epsilon(t, y)$. However, by Lemma 3.3.12, the term

$$\tilde{r}(t, y) = e^{-z(t)} (A_1(y - z_1(t)) + A_2(y - z_2(t))) \varphi(t, y)$$

improves the computation of the error to the flow (3.2.11) to obtain (3.3.15). This refinement is essential to close the bootstrap (3.3.7) on $\epsilon(t, y)$, see Proposition 3.3.22, since without it, one would obtain E of size t^{-2} and ϵ of size t^{-1} , which is not sharp enough to exploit the modulation equations (3.3.13), (3.3.14).

Next, we construct a function $u_0 \in H^1(\mathbb{R})$ as a strong limit of a subsequence of $u_n(t_0)$.

Lemma 3.4.6. There exist $u_0 \in H^1(\mathbb{R})$ and a sub-sequence, still denoted u_n , such that

$$\begin{aligned} u_n(t_0) &\rightharpoonup u_0 \text{ weakly in } H^1(\mathbb{R}) \\ u_n(t_0) &\rightarrow u_0 \text{ in } H^\sigma(\mathbb{R}), \text{ for } 0 \leq \sigma < 1 \end{aligned}$$

as $n \rightarrow \infty$.

Proof of Lemma 3.4.6. By the bounds on u_n and interpolation, it is enough to prove that the sub-sequence $u_n(t_0) \xrightarrow{L^2} u_0$ as $n \rightarrow \infty$. First, we claim the following: $\forall \delta_1 > 0, \delta_1 \ll 1, \exists n_0 \gg 1, \exists K_1 = K_1(\delta_1) > 0$ such that $\forall n \geq n_0$

$$\int_{|x| > K_1} |u_n(t_0, x)|^2 dx < \delta_1. \quad (3.4.7)$$

Indeed, by (3.4.4), we have, for all n

$$\|u_n(t) - N(t, x)\|_{H^1} \lesssim t^{-\frac{1}{16}}.$$

A direct consequence of the above estimate is

$$\|u_n(t)\|_{H^1} < C \quad (3.4.8)$$

for all $t \in [t_0, n]$ since $\|N(t)\|_{H^1} \leq 2\|Q\|_{H^1}$. Furthermore, for fixed δ_1 , there exists $t_1 > t_0$ such that

$$\|u_n(t_1) - N(t_1)\|_{H^1} \lesssim (t_1)^{-\frac{1}{16}} < \sqrt{\delta_1}$$

for n large enough that $n > t_1$; in others words, we have

$$\int |u_n(t_1, x) - N(t_1, x)|^2 dx < \delta_1.$$

Besides, for $K_2 \gg 1$ large enough we have

$$\int_{|x| > K_2} |N(t_1, x)|^2 dx < \delta_1.$$

Consider now a C^1 cut-off function $g : \mathbb{R} \rightarrow [0, 1]$ such that : $g \equiv 0$ on $(-\infty, 1]$, $0 < g' < 2$ on $(1, 2)$ and $g \equiv 1$ on $[2, +\infty)$. Since $\|u_n(t)\|_{H^1} < C$ bounded in H^1 independently of n and $t \in [t_0, n]$, we can choose $\gamma_1 > 0$ independent of n such that

$$\gamma_1 \geq \frac{2}{\delta_1} (t_1 - t_0) C^2.$$

We have by direct calculations, for $t \in [t_0, n]$

$$\begin{aligned} \left| \frac{d}{dt} \int |u_n(t, x)|^2 g \left(\frac{|x| - K_2}{\gamma_1} \right) dx \right| &= \left| \frac{1}{\gamma_1} \operatorname{Im} \int u \left(\nabla \bar{u} \cdot \frac{x}{|x|} \right) g' \left(\frac{|x| - K_2}{\gamma_1} \right) \right| \\ &\leq \frac{2}{\gamma_1} \sup_{n \geq t \geq t_0} \|u_n(t)\|_{H^1}^2 \leq \frac{\delta_1}{t_1 - t_0}. \end{aligned}$$

By integration from t_0 to t_1

$$\begin{aligned} \int |u_n(t_0, x)|^2 g \left(\frac{|x| - K_2}{\gamma_1} \right) dx - \int |u_n(t_1, x)|^2 g \left(\frac{|x| - K_2}{\gamma_1} \right) dx \\ \leq \int_{t_0}^{t_1} \left| \frac{d}{dt} \int |u_n(t, x)|^2 g \left(\frac{|x| - K_2}{\gamma_1} \right) dx \right| \leq \delta_1. \end{aligned}$$

From the properties of g we conclude:

$$\begin{aligned} \int_{|x|>2\gamma_1+K_2} |u_n(t_0, x)|^2 dx &\leq \int |u_n(t_0, x)|^2 g\left(\frac{|x| - K_2}{\gamma_1}\right) dx \\ &\leq \int |u_n(t_1, x)|^2 g\left(\frac{|x| - K_2}{\gamma_1}\right) dx + \delta_1 \leq \int_{|x|>K_2} |u_n(t_1, x)|^2 dx + \delta_1 \leq 5\delta_1. \end{aligned}$$

Thus (3.4.7) is proved. As $\|u_n(t_0)\|_{H^1} < C$, there exists a subsequence of (u_n) (still denoted by (u_n)) and $u_0 \in H^1$ such that

$$\begin{aligned} u_n(t_0) &\rightharpoonup u_0 \quad \text{weakly in } H^1(\mathbb{R}), \\ u_n(t_0) &\rightarrow u_0 \quad \text{in } L^2_{loc}(\mathbb{R}), \text{ as } n \rightarrow +\infty \end{aligned}$$

and by (3.4.7), we obtain that $u_n(t_0) \xrightarrow{H^\sigma} u_0$, for $0 \leq \sigma < 1$. □

To conclude the proof of Main Theorem, we consider $u(t)$ the solution of (3.1.1) corresponding to $u(t_0) = u_0$. From [7], we have the continuous dependence of the solution upon the initial data, so for all $t \in [t_0, +\infty)$,

$$\begin{aligned} u_n(t) &\rightarrow u(t) \quad \text{in } H^\sigma(\mathbb{R}), \quad s_c \leq \sigma < 1 \\ u_n(t) &\rightharpoonup u(t) \quad \text{in } H^1(\mathbb{R}). \end{aligned}$$

where $s_c < 1$ is the critical exponent. Thus, from (3.4.4), $\|u_n(t) - N(t, x)\|_{H^1} \lesssim t^{-\frac{1}{16}}$, passing to the weak limit as $n \rightarrow +\infty$, we have

$$\|u(t) - N(t, x)\|_{H^1} \leq Ct^{-\frac{1}{16}}.$$

Therefore, recall the value of α given in (3.2.31), we have constructed a solution $u(t)$ satisfying the conclusion of Main Theorem.

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Chapter 4

Construction of 2-solitons with logarithmic distance for the one-dimensional cubic Schrödinger system

Abstract

We consider a system of coupled cubic Schrödinger equations in one space dimension

$$\begin{cases} i\partial_t u + \partial_x^2 u + (|u|^2 + \omega|v|^2)u = 0 \\ i\partial_t v + \partial_x^2 v + (|v|^2 + \omega|u|^2)v = 0 \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R},$$

in the non-integrable case $0 < \omega < 1$.

First, we justify the existence of a symmetric 2-solitary wave with logarithmic distance, more precisely a solution of the system satisfying

$$\lim_{t \rightarrow +\infty} \left\| \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} - \begin{pmatrix} e^{it} Q(\cdot - \frac{1}{2} \log(\Omega t) - \frac{1}{4} \log \log t) \\ e^{it} Q(\cdot + \frac{1}{2} \log(\Omega t) + \frac{1}{4} \log \log t) \end{pmatrix} \right\|_{H^1 \times H^1} = 0$$

where $Q = \sqrt{2} \operatorname{sech}$ is the explicit solution of $Q'' - Q + Q^3 = 0$ and $\Omega > 0$ is a constant. This result extends to the non-integrable case the existence of symmetric 2-solitons with logarithmic distance known in the integrable case $\omega = 0$ and $\omega = 1$ ([15, 33]). Such strongly interacting symmetric 2-solitary waves were also previously constructed for the non-integrable scalar nonlinear Schrödinger equation in any space dimension and for any energy-subcritical power nonlinearity ([20, 22]).

Second, under the conditions $0 < c < 1$ and $0 < \omega < \frac{1}{2}c(c+1)$, we construct solutions of the system satisfying

$$\lim_{t \rightarrow +\infty} \left\| \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} - \begin{pmatrix} e^{ic^2 t} Q_c(\cdot - \frac{1}{(c+1)c} \log(\Omega_c t)) \\ e^{it} Q(\cdot + \frac{1}{c+1} \log(\Omega_c t)) \end{pmatrix} \right\|_{H^1 \times H^1} = 0$$

where $Q_c(x) = cQ(cx)$ and $\Omega_c > 0$ is a constant. Such logarithmic regime with non-symmetric solitons does not exist in the integrable cases $\omega = 0$ and $\omega = 1$ and is still unknown in the non-integrable scalar case.

4.1 Introduction

4.1.1 System of cubic Schrödinger equations

We consider the following one dimensional focusing-focusing system of coupled cubic Schrödinger equations

$$\begin{cases} i\partial_t u + \partial_x^2 u + (|u|^2 + \omega|v|^2)u = 0 \\ i\partial_t v + \partial_x^2 v + (|v|^2 + \omega|u|^2)v = 0 \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R} \quad (\text{coupled NLS})$$

for $u(t, x), v(t, x) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ and for any parameter $0 < \omega < 1$. The initial data $u(0, x) = u_0(x)$, $v(0, x) = v_0(x)$ is taken in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$. The Hamiltonian system ([coupled NLS](#)) arises as a model for the propagation of the electrical field in nonlinear optics. Such systems also appear to model the interaction of two Bose-Einstein condensates in different spin states. See [[1](#), [2](#), [32](#)].

For $\omega = 0$, the system ([coupled NLS](#)) simply reduces to two cubic focusing Schrödinger equations without coupling (see [[1](#), [32](#), [33](#)])

$$i\partial_t u + \partial_x^2 u + |u|^2 u = 0 \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \quad (\text{cubic NLS})$$

For $\omega = 1$, the system ([coupled NLS](#)) is called the Manakov system (see [[1](#), [15](#), [32](#)])

$$\begin{cases} i\partial_t u + \partial_x^2 u + (|u|^2 + |v|^2)u = 0 \\ i\partial_t v + \partial_x^2 v + (|v|^2 + |u|^2)v = 0. \end{cases} \quad (\text{MS})$$

Both ([cubic NLS](#)) and ([MS](#)) are completely integrable. For $0 < \omega < 1$, the system is not known to be integrable.

It follows from standard arguments (see *e.g.* [[3](#), [10](#)]) that the system ([coupled NLS](#)) is locally well-posed in $H^1 \times H^1$. In this paper, we work in the framework of such $H^1 \times H^1$ solutions. Moreover, the system is invariant under the following symmetries:

- Phase: $\gamma, \gamma' \in \mathbb{R}$, $\begin{pmatrix} u_0(x)e^{i\gamma} \\ v_0(x)e^{i\gamma'} \end{pmatrix} \mapsto \begin{pmatrix} u(t, x)e^{i\gamma} \\ v(t, x)e^{i\gamma'} \end{pmatrix}$;
- Scaling: $\lambda > 0$, $\lambda \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} (\lambda x) \mapsto \lambda \begin{pmatrix} u \\ v \end{pmatrix} (\lambda^2 t, \lambda x)$;
- Space translation: $\sigma \in \mathbb{R}$, $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} (x + \sigma) \mapsto \begin{pmatrix} u \\ v \end{pmatrix} (t, x + \sigma)$;
- Galilean invariance: $\beta \in \mathbb{R}$, $e^{i\beta x} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} (x) \mapsto e^{i\beta(x-\beta t)} \begin{pmatrix} u \\ v \end{pmatrix} (t, x - 2\beta t)$.

For $H^1 \times H^1$ solutions, the following quantities are constant:

- Masses:

$$M(u(t)) = \int_{\mathbb{R}} |u(t, x)|^2 dx = M(u_0), \quad M(v(t)) = \int_{\mathbb{R}} |v(t, x)|^2 dx = M(v_0);$$

- Energy:

$$\begin{aligned} E(u(t), v(t)) &= \frac{1}{2} \int_{\mathbb{R}} (|\partial_x u|^2 + |\partial_x v|^2) (t, x) dx - \frac{1}{4} \int_{\mathbb{R}} (|u|^4 + |v|^4 + 2\omega|u|^2|v|^2) (t, x) dx \\ &= E(u_0, v_0); \end{aligned}$$

- Momentum:

$$J(u(t), v(t)) = \Im \int_{\mathbb{R}} \partial_x u(t, x) \bar{u}(t, x) dx + \Im \int_{\mathbb{R}} \partial_x v(t, x) \bar{v}(t, x) dx = J(u_0, v_0).$$

By the Gagliardo-Nirenberg inequality $\|u\|_{L^4}^4 \lesssim \|u\|_{L^2}^3 \|\partial_x u\|_{L^2}$ and standard arguments, the system is globally well-posed in $H^1 \times H^1$ (see *e.g.* [3, 28]).

Let Q be the ground state, defined as

$$Q(x) = \frac{\sqrt{2}}{\cosh(x)} \text{ unique (up to translation) } H^1 \text{ solution of } Q'' - Q + Q^3 = 0 \text{ on } \mathbb{R}.$$

Recall that (cubic NLS) admits solitary wave solutions, also called solitons, of the form

$$u(t, x) = e^{i\gamma + i\lambda^2 t + i\beta(x - \beta t)} Q_\lambda(x - \sigma - 2\beta t) \quad \text{with} \quad Q_\lambda(x) = \lambda Q(\lambda x)$$

where $\lambda > 0, \gamma, \sigma, \beta \in \mathbb{R}$. When $v = 0$ (or $u = 0$), the system (coupled NLS) simplifies into (cubic NLS), and thus we deduce soliton solutions of (coupled NLS):

$$\begin{pmatrix} u \\ v \end{pmatrix} (t, x) = \begin{pmatrix} e^{i\Gamma_1(t, x)} Q_{\lambda_1}(x - \sigma_1 - 2\beta_1 t) \\ 0 \end{pmatrix}, \quad \Gamma_1(t, x) = \gamma_1 + \lambda_1^2 t + \beta_1(x - \beta_1 t)$$

and

$$\begin{pmatrix} u \\ v \end{pmatrix} (t, x) = \begin{pmatrix} 0 \\ e^{i\Gamma_2(t, x)} Q_{\lambda_2}(x - \sigma_2 - 2\beta_2 t) \end{pmatrix}, \quad \Gamma_2(t, x) = \gamma_2 + \lambda_2^2 t + \beta_2(x - \beta_2 t)$$

for any $\lambda_j > 0, \gamma_j, \sigma_j, \beta_j \in \mathbb{R}$ ($j = 1, 2$). By definition, a multi-solitary wave (or multi-soliton) is a solution behaving in large time as a sum of such single solitons. In this article, we focus on 2-solitons such that one solitary wave is carried by u and the other one by v .

4.1.2 Previous results and motivation

Multi-solitons have been studied intensively in the integrable case, *i.e.* for (cubic NLS) and (MS), as well as for some nearly integrable models; see [1, 7, 8, 13, 24, 32, 33]. From the inverse scattering theory, there are three types of 2-solitons for (cubic NLS):

- Two solitons with different velocities: as $t \rightarrow +\infty$, the distance between the solitons is of order t ([33]).
- Double pole solutions: the two solitons have the same amplitude and their distance is logarithmic in t ([24, 33]).
- Periodic 2-solitons: the two solitons have different amplitudes and their distance is a periodic function of time ([32, 33]).

More generally, the integrability theory treats the case of K -solitary waves for any $K \geq 2$. Moreover, in the integrable case, multi-solitons have a pure soliton behavior for both $t \rightarrow +\infty$ and $t \rightarrow -\infty$ and describe the elastic interactions between solitons. For (MS), a trichotomy similar to (a)-(b)-(c) is studied formally and numerically in [31].

For non-integrable models, the study of multi-solitons is mostly limited to situations where solitons are decoupled, in particular, asymptotically in large time. Consider first the scalar nonlinear Schrödinger equation

$$i\partial_t u + \Delta u + |u|^{p-1}u = 0, \quad u(0, x) = u_0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \quad (\text{NLS})$$

in any space dimension $d \geq 1$ and for any energy subcritical power nonlinearity (*i.e.* $p > 1$ for $d = 1, 2$ and $1 < p < 1 + \frac{4}{d-2}$ for $d \geq 3$). This equation is known to be completely integrable only for $d = 1$ and $p = 3$, *i.e.* (**cubic NLS**). Define the ground state Q as the unique radial positive H^1 solution (up to symmetries) of $\Delta Q - Q + Q^p = 0$ in \mathbb{R}^d (for more properties of the ground state, see [3, 9, 25, 30]) and $Q_\lambda(x) = \lambda^{\frac{2}{p-1}} Q(\lambda x)$, for any $\lambda > 0$. The existence of K -solitary waves for (**NLS**) corresponding to case (a), *i.e.* solutions $u(t)$ of (**NLS**) such that

$$\lim_{t \rightarrow +\infty} \left\| u(t) - \sum_{k=1}^K e^{-i\Gamma_k(t, \cdot)} Q_{\lambda_k}(\cdot - \sigma_k - 2\beta_k t) \right\|_{H^1(\mathbb{R}^d)} = 0$$

for any $\lambda_k > 0$ and any two-by-two different $\beta_k \in \mathbb{R}^d$, was established in [5, 17, 21].

Recently, the second author proved that the dynamics (b) is also a universal regime for (**NLS**), by constructing two symmetric solitary waves with logarithmic distance, [22]. The L^2 critical case ($p = 1 + \frac{4}{d}$), previously studied in [20], exhibits a specific blow-up behavior also related to symmetric 2-solitons with logarithmic distance in rescaled variables.

Turning back to the system (**coupled NLS**) in the non-integrable case, *i.e.* for $0 < \omega < 1$, the existence of multi-solitary wave solutions corresponding to case (a)

$$\lim_{t \rightarrow +\infty} \left\| \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} - \begin{pmatrix} e^{-i\Gamma_1(t, \cdot)} Q_c(\cdot - \sigma_1 - 2\beta_1 t) \\ e^{-i\Gamma_2(t, \cdot)} Q(\cdot - \sigma_2 - 2\beta_2 t) \end{pmatrix} \right\|_{H^1} = 0,$$

for any $c > 0$ and any different velocities $\beta_1 \neq \beta_2$ was proved in [6] (see also [11]).

A first goal of this paper is to justify the persistence of the regime (b) for the non-integrable (**coupled NLS**) in presence of symmetry, following the articles [20, 22] for the scalar (**NLS**) equation.

Second, and more importantly, we investigate the question of the (non-)persistence of the regime (c). Indeed, we exhibit a new logarithmic regime corresponding to non-symmetric 2-solitons with logarithmic distance which replaces the behavior (c). At the formal level, the system of parameters of the 2-solitons is not anymore integrable and periodic solutions disappear, see Remark 4.1.7. A logarithmic regime (see Theorem 4.1.4 and Remark 4.1.6) then takes place, which does not exist in the integrable cases $\omega = 0$ and $\omega = 1$. To our knowledge, such question is open for the scalar equation (**NLS**) in the non-integrable case (see Section 4.5).

4.1.3 Main results.

First, we present the symmetric logarithmic regime.

Theorem 4.1.1. For any $0 < \omega < 1$, there exists a solution $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{C}(\mathbb{R}, H^1 \times H^1)$ of (**coupled NLS**) such that

$$\lim_{t \rightarrow +\infty} \left\| \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} - \begin{pmatrix} e^{it} Q(\cdot - \frac{1}{2} \log(\Omega t) - \frac{1}{4} \log \log t) \\ e^{it} Q(\cdot + \frac{1}{2} \log(\Omega t) + \frac{1}{4} \log \log t) \end{pmatrix} \right\|_{H^1 \times H^1} = 0$$

where $\Omega > 0$ is a constant depending on ω .

Note that as $t \rightarrow +\infty$, the distance between the two solitary waves is asymptotic to

$$y(t) = \log t + \frac{1}{2} \log \log t + \log \Omega. \quad (4.1.2)$$

Remark 4.1.3. An analogous dynamics was constructed for (**cubic NLS**) in [24, 33] and for (**NLS**) in [20, 22].

Second, we construct for ([coupled NLS](#)) a new logarithmic dynamics of 2-solitary waves with different amplitude.

Theorem 4.1.4. For any $0 < c < 1$ and $0 < \omega < \frac{1}{2}c(c+1) < 1$, there exists a solution $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{C}(\mathbb{R}, H^1 \times H^1)$ of ([coupled NLS](#)) such that

$$\lim_{t \rightarrow +\infty} \left\| \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} - \begin{pmatrix} e^{ic^2t} Q_c(\cdot - \frac{1}{(c+1)c} \log(\Omega_c t)) \\ e^{it} Q(\cdot + \frac{1}{c+1} \log(\Omega_c t)) \end{pmatrix} \right\|_{H^1 \times H^1} = 0$$

where $\Omega_c > 0$ is a constant depending on c and ω .

Note that as $t \rightarrow +\infty$, the distance between the two solitary waves is asymptotic to

$$y_c(t) = \frac{1}{c} \log t + \frac{1}{c} \log \Omega_c. \quad (4.1.5)$$

As mentioned before, such solution does not exist in the integrable cases and the analogous question for the non-integrable scalar equation ([NLS](#)) seems open. See [Section 4.5](#).

Remark 4.1.6. The slight difference between the two regimes ([4.1.2](#)) and ([4.1.5](#)) is due to stronger interactions when solitary waves have equal amplitudes. We refer to [Sections 4.4.2](#) and [4.2.3](#) for formal derivations of the regimes ([4.1.2](#)) and ([4.1.5](#)).

We believe that there is no other logarithmic regime for ([coupled NLS](#)). In support of this conjecture, we refer to the case of the generalized Korteweg-de Vries equation, for which existence of a logarithmic regime was proved in [\[23\]](#) and uniqueness (in the super-critical case) was established in [\[12\]](#).

The case $\frac{1}{2}c(c+1) \leq \omega < 1$ in [Theorem 4.1.4](#) is open (see step 1 of the proof of [Proposition 4.3.16](#)).

Remark 4.1.7. The dynamics of the distance between the two solitary waves is related to nonlinear interactions. A formal study (see notably [\[8, 13\]](#) and Chapter 4 in [\[32\]](#)) shows that the three behaviors (a), (b) and (c) are related to different solutions of

$$\begin{cases} \ddot{\gamma} = c_\gamma e^{-\sigma} \sin \gamma \\ \ddot{\sigma} = -c_\sigma e^{-\sigma} \cos \gamma \end{cases}$$

where γ is the phase difference, σ the relative distance and c_γ, c_σ are constants. For ([cubic NLS](#)), it holds $c_\gamma = c_\sigma > 0$. Denoting $Y = \sigma + i\gamma$, the resulting equation $\ddot{Y} = -c_\gamma e^Y$ is integrable and admits nontrivial solutions for which σ is periodic.

Remark 4.1.8. The proofs of [Theorems 4.1.1](#) and [4.1.4](#) follow the overall strategy of several previous articles on multi-solitons ([\[14, 16, 17, 18, 19, 20, 21, 22, 26\]](#)), particularly of [\[20, 22\]](#) which started the study of multi-solitons with logarithmic distance in a non-integrable setting. We focus on the proof of [Theorem 4.1.4](#), which is more original in the construction of a suitable approximate solution and the determination of the asymptotic regime (see [Remark 4.2.8](#)).

See [Section 4.5](#) for a comment on the introduction of a refined energy method.

4.1.4 Notation and preliminaries

For complex-valued functions $f, g \in L^2(\mathbb{R})$, we denote

$$\langle f, g \rangle = \Re \left(\int f \bar{g} \right).$$

For r a positive function of time, the notation $f(t, x) = O_{H^1}(r(t))$ means that there exists a constant $C > 0$ such that $\|f(t)\|_{H^1} \leq Cr(t)$.

For any $\lambda > 0$ and any function f , let

$$f_\lambda(x) = \lambda f(\lambda x) \quad \text{and} \quad \Lambda f(x) = f(x) + xf'(x) = \partial_\lambda f_\lambda(x)|_{\lambda=1}.$$

Note the following relation which describes the asymptotics of $Q(x)$ as $x \rightarrow -\infty$,

$$Q(x) = \kappa e^x - e^{2x}Q(x) \quad \text{on } \mathbb{R} \quad \text{where } \kappa = 2\sqrt{2}. \quad (4.1.9)$$

Throughout this paper, we consider ω and c such that

$$0 < c \leq 1 \quad \text{and} \quad 0 < \omega < \frac{c(c+1)}{2}. \quad (4.1.10)$$

The linearization of (coupled NLS) around solitons involves the following operators:

$$\mathcal{L}_+ = -\partial_x^2 + 1 - 3Q^2, \quad \mathcal{L}_- = -\partial_x^2 + 1 - Q^2, \quad \mathcal{L}_c = -\partial_x^2 + c^2 - \omega Q^2.$$

Recall the special relations ([29])

$$\mathcal{L}_- Q = 0, \quad \mathcal{L}_+(\Lambda Q) = -2Q, \quad \mathcal{L}_+(Q') = 0, \quad \mathcal{L}_-(xQ) = -2Q'. \quad (4.1.11)$$

We will use the following properties of these operators.

Lemma 4.1.12. Assume (4.1.10).

1. There exists $\mu > 0$ such that, for all $z \in H^1$,

$$\begin{aligned} \langle \mathcal{L}_+ \Re z, \Re z \rangle + \langle \mathcal{L}_- \Im z, \Im z \rangle &\geq \mu \|z\|_{H^1}^2 - \frac{1}{\mu} (\langle z, Q \rangle^2 + \langle z, xQ \rangle^2 + \langle z, i\Lambda Q \rangle^2), \\ \langle \mathcal{L}_c z, z \rangle &\geq \mu \|z\|_{H^1}^2. \end{aligned}$$

2. For any $f \in L^2$, there exists a unique solution $u \in H^2$ of $\mathcal{L}_c u = f$. Moreover,

- If $|f(x)| \lesssim e^{-\lambda|x|}$ for some $\lambda > c$, then $|u(x)| \lesssim e^{-c|x|}$.
- If $|f(x)| \lesssim e^{-c|x|}$ then $|u(x)| \lesssim (1 + |x|)e^{-c|x|}$.

Proof. (i) The coercivity properties of \mathcal{L}_+ and \mathcal{L}_- (here in the L^2 sub-critical case) are well-known facts (see e.g. [17, 29, 30]).

Let $0 < \rho < c$ be such that $\omega = \frac{1}{2}\rho(\rho + 1)$. By [27] or direct computation, we see that the positive function Q^ρ satisfies $\mathcal{L}_c Q^\rho = (c^2 - \rho^2)Q^\rho$. The coercivity property follows.

(ii) Let $c \leq \lambda \leq 1$. If $\mathcal{L}_c u = f$ with $|f(x)| \lesssim e^{-\lambda|x|}$ then $-u'' + c^2 u = g$ where $g = f + \omega Q^2 u$ also satisfies $|g(x)| \lesssim e^{-\lambda|x|}$. The decay properties of u then follows from standard arguments. \square

The following result follows directly from Lemma 4.1.12.

Lemma 4.1.13. 1. Assume $0 < c < 1$. There exists a solution A of

$$\mathcal{L}_c A = -A'' + c^2 A - \omega Q^2 A = c\kappa\omega e^{cx} Q^2 \quad (4.1.14)$$

satisfying

$$|A(x)| + |A'(x)| + |A''(x)| \lesssim Q_c(x) \quad \text{on } \mathbb{R}. \quad (4.1.15)$$

2. There exists a solution B of

$$\mathcal{L}_1 B = -B'' + B - \omega Q^2 B = \kappa\omega e^x Q^2 \quad (4.1.16)$$

satisfying

$$|B(x)| + |B'(x)| + |B''(x)| \lesssim (1 + |x|)Q(x) \quad \text{on } \mathbb{R}. \quad (4.1.17)$$

4.2 Approximate solution in the case $0 < c < 1$

4.2.1 Definition of the approximate solution

Consider \mathcal{C}^1 time-dependent real-valued functions $\sigma_1, \sigma_2, \gamma_1, \gamma_2, \beta_1, \beta_2$, to be fixed later and set

$$\sigma = \sigma_1 - \sigma_2, \quad \beta = \beta_1 - \beta_2, \quad \gamma = \gamma_1 - \gamma_2.$$

Denote

$$\begin{aligned} U &= P + \varphi, \quad P(t, x) = Q_c(x - \sigma_1(t))e^{i\Gamma_1(t, x)}, \quad \varphi(t, x) = e^{-c\sigma(t)}A(x - \sigma_2(t))e^{i\Gamma_1(t, x)}, \\ V &= R, \quad R(t, x) = Q(x - \sigma_2(t))e^{i\Gamma_2(t, x)}, \end{aligned}$$

where

$$\Gamma_1(t, x) = c^2t + \gamma_1(t) + \beta_1(t)x, \quad \Gamma_2(t, x) = t + \gamma_2(t) + \beta_2(t)x.$$

Introduce the notation

$$\begin{aligned} \partial_1 P &= Q'_c(x - \sigma_1)e^{i\Gamma_1}, \quad x_1 P = (x - \sigma_1)P, \quad \Lambda_1 P = \Lambda Q_c(x - \sigma_1)e^{i\Gamma_1}, \\ \partial_1 \varphi &= e^{-c\sigma}A'(x - \sigma_2)e^{i\Gamma_1}, \quad x_2 \varphi = (x - \sigma_2)\varphi, \\ \partial_2 R &= Q'(x - \sigma_2)e^{i\Gamma_2}, \quad x_2 R = (x - \sigma_2)R, \quad \Lambda_2 R = \Lambda Q(x - \sigma_2)e^{i\Gamma_2}. \end{aligned}$$

Define the approximate solution

$$\mathbf{Z} = \begin{pmatrix} U \\ V \end{pmatrix} \quad \text{and set} \quad \mathcal{E}_{\mathbf{Z}} = \begin{pmatrix} \mathcal{E}_U \\ \mathcal{E}_V \end{pmatrix} = \begin{pmatrix} i\partial_t U + \partial_x^2 U + (|U|^2 + \omega|V|^2)U \\ i\partial_t V + \partial_x^2 V + (|V|^2 + \omega|U|^2)V \end{pmatrix}.$$

Lemma 4.2.1. It holds

$$\begin{cases} \mathcal{E}_U = F - \vec{m}_1 \cdot \vec{M}_1 - \vec{m}_\varphi \cdot \vec{M}_\varphi \\ \mathcal{E}_V = G - \vec{m}_2 \cdot \vec{M}_2 \end{cases} \quad (4.2.2)$$

where

$$\begin{cases} F = 3|P|^2\varphi + 3|\varphi|^2P + |\varphi|^2\varphi - \omega e^{2c(x-\sigma_1)}|R|^2P \\ G = \omega|P + \varphi|^2R \end{cases} \quad (4.2.3)$$

and

$$\begin{aligned} \vec{m}_1 &= \begin{pmatrix} \dot{\sigma}_1 - 2\beta_1 \\ \dot{\gamma}_1 + \dot{\beta}_1\sigma_1 + \beta_1^2 \\ \dot{\beta}_1 \end{pmatrix}, \quad \vec{M}_1 = \begin{pmatrix} i\partial_1 P \\ P \\ x_1 P \end{pmatrix} \\ \vec{m}_\varphi &= \begin{pmatrix} \dot{\sigma}_2 - 2\beta_1 \\ \dot{\gamma}_1 + \dot{\beta}_1\sigma_2 + \beta_1^2 + ic\dot{\sigma} \\ \dot{\beta}_1 \end{pmatrix}, \quad \vec{M}_\varphi = \begin{pmatrix} i\partial_1 \varphi \\ \varphi \\ x_2 \varphi \end{pmatrix} \\ \vec{m}_2 &= \begin{pmatrix} \dot{\sigma}_2 - 2\beta_2 \\ \dot{\gamma}_2 + \dot{\beta}_2\sigma_2 + \beta_2^2 \\ \dot{\beta}_2 \end{pmatrix}, \quad \vec{M}_2 = \begin{pmatrix} i\partial_2 R \\ R \\ x_2 R \end{pmatrix}. \end{aligned}$$

Proof. Using $Q_c'' - c^2Q_c = Q_c^3$ and (4.1.14), we compute

$$\begin{aligned} i\partial_t P + \partial_x^2 P + |P|^2P &= -\vec{m}_1 \cdot \vec{M}_1, \\ i\partial_t \varphi + \partial_x^2 \varphi + \omega|R|^2(P + \varphi) &= -\vec{m}_\varphi \cdot \vec{M}_\varphi + \omega|R|^2 [Q_c(x - \sigma_1) - ck e^{c(x-\sigma_1)}] e^{i\Gamma_1}. \end{aligned}$$

Using (4.1.9), we obtain (4.2.2) for \mathcal{E}_U with F defined as in (4.2.3).

Similarly, the equation

$$i\partial_t R + \partial_x^2 R + |R|^2R = -\vec{m}_2 \cdot \vec{M}_2$$

implies (4.2.2) for \mathcal{E}_V with G defined as in (4.2.3). \square

4.2.2 Projection of the error terms

The soliton dynamics is expected to be determined by the following projections

$$a = \frac{1}{2c} \langle F, \partial_1 P \rangle \quad \text{and} \quad b = \frac{1}{2} \langle G, \partial_2 R \rangle.$$

Using $\langle \partial_1 P, x_1 P \rangle = \langle Q'_c, x Q_c \rangle = -\frac{1}{2} \|Q_c\|_{L^2}^2 = -2c$ and $\langle \partial_2 R, x_2 R \rangle = -\frac{1}{2} \|Q\|_{L^2}^2 = -2$, we decompose F and G as follows

$$\begin{cases} F = F^\perp - ax_1 P, & \langle F^\perp, \partial_1 P \rangle = 0 \\ G = G^\perp - bx_2 R, & \langle G^\perp, \partial_2 R \rangle = 0 \end{cases} \quad (4.2.4)$$

so that (4.2.2) rewrites

$$\begin{cases} \mathcal{E}_U = F^\perp - \vec{m}_1^a \cdot \vec{M}_1 - \vec{m}_\varphi \cdot \vec{M}_\varphi \\ \mathcal{E}_V = G^\perp - \vec{m}_2^b \cdot \vec{M}_2 \end{cases} \quad (4.2.5)$$

with

$$\vec{m}_1^a = \begin{pmatrix} \dot{\sigma}_1 - 2\beta_1 \\ \dot{\gamma}_1 + \dot{\beta}_1 \sigma_1 + \beta_1^2 \\ \dot{\beta}_1 + a \end{pmatrix} \quad \text{and} \quad \vec{m}_2^b = \begin{pmatrix} \dot{\sigma}_2 - 2\beta_2 \\ \dot{\gamma}_2 + \dot{\beta}_2 \sigma_2 + \beta_2^2 \\ \dot{\beta}_2 + b \end{pmatrix}.$$

We compute the main order of these projections.

Lemma 4.2.6. Let $1 < \theta < \min\{\frac{1}{c}; 2\}$. It holds

$$a = \alpha_c e^{-2c\sigma} + O(e^{-2c\theta\sigma}), \quad b = -c\alpha_c e^{-2c\sigma} + O(e^{-2c\theta\sigma}) \quad (4.2.7)$$

where

$$\alpha_c = 4c^2 \omega \|e^{cx} Q\|_{L^2}^2 + \frac{1}{2} \langle \mathcal{L}_c A, A \rangle > 0.$$

Remark 4.2.8. The expression of the positive constant α_c , relevant in the dynamics of the 2-soliton (see Section 4.2.3), suggests that even at the formal level, the introduction of the approximate solution $(\begin{smallmatrix} U \\ V \end{smallmatrix})$ including the refined term φ is necessary to determine correctly the non-symmetric logarithmic regime.

Proof. We start by proving the following estimates

$$\int e^{2c(x-\sigma)} Q_c^2(x-\sigma) Q^2(x) dx = O(e^{-2c\theta\sigma}), \quad (4.2.9)$$

$$\int Q_c^2(x-\sigma) Q(x) Q'(x) dx = -c^3 \kappa^2 e^{-2c\sigma} \int e^{2cx} Q^2(x) dx + O(e^{-2c\theta\sigma}). \quad (4.2.10)$$

Proof of (4.2.9). By (4.1.9) and the condition on θ , we have

$$e^{2c(x-\sigma)} Q_c^2(x-\sigma) Q^2(x) \lesssim e^{2c\theta(x-\sigma)} Q^2(x) \lesssim e^{-2c\theta\sigma} e^{-2(1-c\theta)|x|},$$

and (4.2.10) follows.

Proof of (4.2.10). It follows from (4.1.9) that

$$Q_c^2(x) = c^2 \kappa^2 e^{2cx} + O(e^{3cx} Q_c(x)),$$

and so

$$Q_c^2(x-\sigma) = c^2 \kappa^2 e^{-2c\sigma} e^{2cx} + O(e^{-2c\theta\sigma} e^{2c\theta x}).$$

Thus

$$\int Q_c^2(x - \sigma)Q(x)Q'(x)dx = c^2\kappa^2e^{-2c\sigma} \int e^{2cx}Q(x)Q'(x)dx + O(e^{-2c\theta\sigma}).$$

and (4.2.10) follows by integration by parts.

From the expression of F in (4.2.3), we have

$$\begin{aligned} \langle F, \partial_1 P \rangle &= 3e^{-c\sigma} \int Q_c^2(x)Q'_c(x)A(x + \sigma)dx + 3e^{-2c\sigma} \int Q_c(x)Q'_c(x)A^2(x + \sigma)dx \\ &\quad + e^{-3c\sigma} \int Q'_c(x)A^3(x + \sigma)dx - \omega \int e^{2cx}Q_c(x)Q'_c(x)Q^2(x + \sigma)dx. \end{aligned}$$

For the first term, using $-(Q'_c)'' + c^2Q'_c = 3Q_c^2Q'_c$ (obtained by differentiating the equation of Q_c) and the equation A in (4.1.14), we compute

$$\begin{aligned} 3 \int Q_c^2(x)Q'_c(x)A(x + \sigma)dx &= \int Q'_c(x - \sigma)(-A''(x) + c^2A(x))dx \\ &= \omega \int Q'_c(x - \sigma) [Q^2(x)A(x) + c\kappa e^{cx}Q^2(x)] dx. \end{aligned}$$

Similarly as in the proof of (4.2.10), using (4.1.9) we observe

$$\begin{aligned} \int Q'_c(x - \sigma)Q^2(x)A(x)dx &= c^2\kappa e^{-c\sigma} \int e^{cx}Q^2(x)A(x)dx + O(e^{-c\theta\sigma}), \\ \int Q'_c(x - \sigma)e^{cx}Q^2(x)dx &= c^2\kappa e^{-c\sigma} \int e^{2cx}Q^2(x)dx + O(e^{-c\theta\sigma}). \end{aligned}$$

Moreover, it follows from (4.1.14) and the coercivity of the operator \mathcal{L}_c that

$$c\kappa\omega \int e^{cx}Q^2(x)A(x)dx = \langle \mathcal{L}_c A, A \rangle > 0.$$

Last, we check using the decay property of A in (4.1.15) and the condition on θ that

$$\int Q_c(x)Q'_c(x)A^2(x + \sigma)dx = O(e^{-c\theta\sigma}), \quad \int Q'_c(x)A^3(x + \sigma)dx = O(e^{-c\sigma}).$$

Using also (4.2.9) and $\kappa^2 = 8$, we find

$$a = \frac{e^{-2c\sigma}}{2} \left[c^2\kappa^2\omega \int e^{2cx}Q^2(x)dx + \langle \mathcal{L}_c A, A \rangle \right] + O(e^{-2c\theta\sigma}) = \alpha_c e^{-2c\sigma} + O(e^{-2c\theta\sigma}).$$

From the definition of G , we have

$$\begin{aligned} \langle G, \partial_2 R \rangle &= \omega \int Q_c^2(x - \sigma)Q(x)Q'(x)dx \\ &\quad + 2\omega e^{-c\sigma} \int Q_c(x - \sigma)A(x)Q(x)Q'(x)dx + \omega e^{-2c\sigma} \int A^2(x)Q(x)Q'(x)dx. \end{aligned}$$

On the one hand, integrating by parts, it holds

$$\langle \mathcal{L}_c A, A' \rangle = -\omega \int Q^2(x)A(x)A'(x)dx = \omega \int A^2(x)Q(x)Q'(x)dx.$$

On the other hand, using (4.1.14) and then integration by parts, it holds

$$\begin{aligned}\langle \mathcal{L}_c A, A' \rangle &= c\kappa\omega \int e^{cx} Q^2(x) A'(x) dx \\ &= -c^2\kappa\omega \int e^{cx} Q^2(x) A(x) dx - 2c\kappa\omega \int e^{cx} Q(x) Q'(x) A(x) dx \\ &= -c\langle \mathcal{L}_c A, A \rangle - 2c\kappa\omega \int e^{cx} Q(x) Q'(x) A(x) dx.\end{aligned}$$

Thus, also using

$$\int Q_c(x - \sigma) A(x) Q(x) Q'(x) dx = c\kappa e^{-c\sigma} \int e^{cx} Q(x) Q'(x) A(x) Q dx + O(e^{-c\theta\sigma})$$

and (4.2.10), we obtain $b = -c\alpha_c e^{-2c\sigma} + O(e^{-2c\theta\sigma})$. \square

4.2.3 Formal discussion

Formally, the previous computations lead us to the system

$$\dot{\sigma}_1 = 2\beta_1, \quad \dot{\beta}_1 = -\alpha_c e^{-2c\sigma}, \quad \dot{\sigma}_2 = 2\beta_2, \quad \dot{\beta}_2 = c\alpha_c e^{-2c\sigma}.$$

Recalling $\sigma = \sigma_1 - \sigma_2$ and $\beta = \beta_1 - \beta_2$, this gives

$$\ddot{\sigma} = -2(c+1)\alpha_c e^{-2c\sigma}, \quad 2\beta = \dot{\sigma},$$

which admits the following solution

$$\sigma(t) = \frac{1}{c} \log(\Omega_c t), \quad 2\beta(t) = \frac{1}{ct} = \frac{\Omega_c}{c} e^{-c\sigma} \quad \text{where} \quad \Omega_c = [2c(c+1)\alpha_c]^{\frac{1}{2}} > 0.$$

This justifies the existence of the regime (4.1.5) of Theorem 4.1.4. In particular, observe that the positive sign of the constant α_c is responsible for the emergence of the special non-symmetric logarithmic regime. The phase parameters γ_1 and γ_2 are not essential for the dynamics and so we do not discuss them here.

4.2.4 Decomposition around the approximate solution

Let $T_\infty \gg 1$ to be fixed later and consider a solution $\begin{pmatrix} u \\ v \end{pmatrix}$ of (coupled NLS) under the form

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix} (T_\infty) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (4.2.11)$$

Then, using the notation

$$h(u, v) = (|u|^2 + \omega|v|^2) u$$

the function $\begin{pmatrix} \varepsilon \\ \eta \end{pmatrix}$ satisfies the system

$$\begin{cases} i\partial_t \varepsilon + \partial_x^2 \varepsilon + h(U + \varepsilon, V + \eta) - h(U, V) + \mathcal{E}_U = 0 \\ i\partial_t \eta + \partial_x^2 \eta + h(V + \eta, U + \varepsilon) - h(V, U) + \mathcal{E}_V = 0 \end{cases} \quad (4.2.12)$$

The parameters σ_1 , σ_2 , γ_1 , γ_2 , β_1 and β_2 in the definition of $\begin{pmatrix} U \\ V \end{pmatrix}$ are fixed by imposing the following orthogonality conditions

$$\begin{cases} \langle \varepsilon, x_1 P \rangle = \langle \varepsilon, i\Lambda_1 P \rangle = \langle \varepsilon, i\partial_1 P \rangle = 0 \\ \langle \eta, x_2 R \rangle = \langle \eta, i\Lambda_2 R \rangle = \langle \eta, i\partial_2 R \rangle = 0 \end{cases} \quad (4.2.13)$$

and initial conditions

$$\begin{cases} \sigma_1(T_\infty) = \frac{\sigma_\infty}{c+1}, & \sigma_2(T_\infty) = -\frac{c\sigma_\infty}{c+1}, \\ \beta_1(T_\infty) = \frac{\beta_\infty}{c+1}, & \beta_2(T_\infty) = -\frac{c\beta_\infty}{c+1}, \\ \gamma_1(T_\infty) = 0, & \gamma_2(T_\infty) = 0, \end{cases} \quad (4.2.14)$$

where σ_∞ is to be chosen later close to $\frac{1}{c} \log(\Omega_c T_\infty)$ (see below (4.3.2)) and

$$\beta_\infty = \frac{\Omega_c}{2c} e^{-c\sigma_\infty}. \quad (4.2.15)$$

Indeed, by a standard argument and the initial conditions (including $\varepsilon(T_\infty) = \eta(T_\infty) = 0$), the orthogonality conditions are equivalent to a first order differential system in the parameters $(\sigma_1, \sigma_2, \gamma_1, \gamma_2, \beta_1, \beta_2)$, which admits a unique local solution in the regime considered in this paper. See *e.g.* Lemma 2.7 in [4] for a detailed argument in the case of the (gKdV) equation, and Lemma 4.3.10 in the present paper for the corresponding estimates on the time derivatives of the parameters. For technical reasons, one can fix zero initial conditions on γ_1, γ_2 as in (4.2.14), but the initial conditions on $\sigma_1, \sigma_2, \beta_1$ and β_2 have to depend on a parameter σ_∞ to be fixed later by a topological argument.

As in [20, 22, 26], the orthogonality conditions in (4.2.13) are related to (4.1.11). Using the conservation of masses and L^2 sub-criticality, we avoid the modulation of the scaling parameters of the solitons (see [30] and the proof of Lemma 4.3.10).

4.3 Proof of Theorem 4.1.4

4.3.1 Bootstrap bounds

Fix θ_1, θ_2 and θ_3 such that $1 < \theta_3 < \theta_2 < \theta_1 < \min\{\frac{1}{c}; 2\}$. Following Section 4.2.3, we work under the following bootstrap estimates, for $1 \ll t \leq T_\infty$,

$$\begin{cases} \|\varepsilon\|_{H^1} + \|\eta\|_{H^1} \leq t^{-\theta_1}, \\ \left| \beta - \frac{1}{2ct} \right| + \left| \beta_1 - \frac{1}{2c(c+1)t} \right| + \left| \beta_2 + \frac{1}{2(c+1)t} \right| \leq t^{-\theta_3}, \\ \left| \sigma_1 - \frac{\log(\Omega_c t)}{c(c+1)} \right| + \left| \sigma_2 + \frac{\log(\Omega_c t)}{c+1} \right| \leq t^{1-\theta_3}, \\ \left| \frac{e^{c\sigma}}{\Omega_c t} - 1 \right| \leq t^{1-\theta_2}. \end{cases} \quad (4.3.1)$$

For consistency, the free parameter σ_∞ in (4.2.14) will have to be chosen such that

$$\left| \frac{e^{c\sigma_\infty}}{\Omega_c T_\infty} - 1 \right| \leq T_\infty^{1-\theta_2}. \quad (4.3.2)$$

Lemma 4.3.3. Let $0 < c_1 \leq c_2$ and $q \geq 0$. It holds, for $\sigma > 1$,

$$\int (1 + |x - \sigma|)^q e^{-c_1|x-\sigma|} e^{-c_2|x|} dx \lesssim \begin{cases} \sigma^{q+1} e^{-c_1\sigma} & \text{if } c_1 = c_2 \\ \sigma^q e^{-c_1\sigma} & \text{if } c_1 \neq c_2. \end{cases}$$

Proof. We decompose

$$\begin{aligned} \int (1 + |x - \sigma|)^q e^{-c_1|x-\sigma|} e^{-c_2|x|} dx &= e^{-c_1\sigma} \int_{-\infty}^0 (1 + |x - \sigma|)^q e^{(c_1+c_2)x} dx \\ &+ e^{-c_1\sigma} \int_0^\sigma (1 + |x - \sigma|)^q e^{-(c_2-c_1)x} dx + e^{c_1\sigma} \int_\sigma^{+\infty} (1 + |x - \sigma|)^q e^{-(c_1+c_2)x} dx. \end{aligned}$$

The result follows by integration. \square

Lemma 4.3.4. The following hold

$$\begin{aligned}\|\partial_t P - ic^2 P\|_{L^2} &\lesssim \left(|\dot{\gamma}_1| + |\dot{\beta}_1| |\sigma_1| + |\dot{\sigma}_1|\right), \\ \|\partial_t \varphi - ic^2 \varphi\|_{L^2} &\lesssim \left(|\dot{\gamma}_1| + |\dot{\beta}_1| |\sigma_2| + |\dot{\sigma}_2| + |\dot{\sigma}|\right) e^{-c\sigma}, \\ \|\partial_t R - iR\|_{L^2} &\lesssim \left(|\dot{\gamma}_2| + |\dot{\beta}_2| |\sigma_2| + |\dot{\sigma}_2|\right).\end{aligned}\tag{4.3.5}$$

Let $1 < \theta < \min\{\frac{1}{c}; 2\}$. The following hold

$$\|F\|_{L^2} + \|F^\perp\|_{L^2} \lesssim e^{-2c\sigma},\tag{4.3.6}$$

$$\|\partial_t F - ic^2 F\|_{L^2} + \|\partial_t F^\perp - ic^2 F^\perp\|_{L^2} \lesssim \left(|\dot{\gamma}_1| + |\dot{\beta}_1| |\sigma_1| + |\dot{\sigma}_1| + |\dot{\sigma}|\right) e^{-2c\sigma},\tag{4.3.7}$$

$$\|G\|_{L^2} + \|G^\perp\|_{L^2} \lesssim e^{-c\theta\sigma},\tag{4.3.8}$$

$$\|\partial_t G - iG\|_{L^2} + \|\partial_t G^\perp - iG^\perp\|_{L^2} \lesssim \left(|\dot{\gamma}_2| + |\dot{\beta}_2| |\sigma_2| + |\dot{\sigma}_2| + |\dot{\sigma}|\right) e^{-c\theta\sigma}.\tag{4.3.9}$$

Proof. Estimates (4.3.5) are simple consequences of the definitions of P , φ and R .

Proof of (4.3.6). Recall that $F(t, x) = F_1(t, x - \sigma_1(t))e^{i\Gamma_1(t, x)}$, where

$$F_1 = 3e^{-c\sigma} Q_c^2 A(x + \sigma) + 3e^{-2c\sigma} Q_c A^2(x + \sigma) + e^{-3c\sigma} A^3(x + \sigma) - \omega e^{2cx} Q^2(x + \sigma) Q_c.$$

Moreover, from (4.1.15) and Lemma 4.3.3, it holds

$$\|Q_c^2 A(x + \sigma)\|_{L^2} + \|Q_c A^2(x + \sigma)\|_{L^2} \lesssim e^{-c\sigma},$$

and $\|e^{2cx} Q^2(x + \sigma) Q_c\|_{L^2} \lesssim e^{-2c\sigma} \|e^{2cx} Q^2\|_{L^2} \lesssim e^{-2c\sigma}$.

Proof of (4.3.7). Note that

$$\partial_t F - ic^2 F = i(\dot{\gamma}_1 + \dot{\beta}_1 \sigma_1) F + i\dot{\beta}_1 (x - \sigma_1) F - \dot{\sigma}_1 \partial_x F_1(t, x - \sigma_1) e^{i\Gamma_1} + \partial_t F_1(t, x - \sigma_1) e^{i\Gamma_1}.$$

We see from the expression of F_1 and similar estimates that the following hold

$$\begin{aligned}\|(\dot{\gamma}_1 + \dot{\beta}_1 \sigma_1) F_1\|_{L^2} &\lesssim (|\dot{\gamma}_1| + |\dot{\beta}_1| |\sigma_1|) e^{-2c\sigma}, \quad \|x F_1\|_{L^2} \lesssim e^{-2c\sigma}, \\ \|\partial_x F_1\|_{L^2} &\lesssim e^{-2c\sigma}, \quad \|\partial_t F_1\|_{L^2} \lesssim |\dot{\sigma}| e^{-2c\sigma}.\end{aligned}$$

This proves estimate (4.3.7) for F .

Next, note that from the definition of a , we have

$$\dot{a} = \frac{1}{2c} \langle \partial_t F - ic^2 F, \partial_1 P \rangle + \frac{1}{2c} \langle F, \partial_t \partial_1 P - ic^2 \partial_1 P \rangle.$$

Thus, from the analogue of (4.3.5) for $\partial_1 P$ and (4.3.6)-(4.3.7), we deduce

$$|\dot{a}| \lesssim \left(|\dot{\gamma}_1| + |\dot{\beta}_1| |\sigma_1| + |\dot{\sigma}_1| + |\dot{\sigma}|\right) e^{-2c\sigma}.$$

Estimate (4.3.7) for F^\perp then comes from

$$\partial_t F^\perp - ic^2 F^\perp = \partial_t F - ic^2 F + \dot{a} x_1 P + a [\partial_t(x_1 P) - ic^2(x_1 P)]$$

and the analogue of (4.3.5) for $x_1 P$.

Proof of (4.3.8). We rewrite $G(t, x) = G_2(x - \sigma_2(t))e^{i\Gamma_2(t, x)}$, where

$$G_2 = \omega Q_c^2(x - \sigma) Q + 2\omega e^{-c\sigma} Q_c(x - \sigma) A Q + \omega e^{-2c\sigma} A^2 Q.$$

From Lemma 4.3.3 and the definition of θ , we have

$$\|Q_c^2(x - \sigma)Q\|_{L^2} \lesssim e^{-c\theta\sigma}, \quad \|Q_c(x - \sigma)AQ\|_{L^2} \lesssim e^{-c\sigma}.$$

Proof of (4.3.9). We have

$$\partial_t G - iG = i(\dot{\gamma}_2 + \dot{\beta}_2\sigma_2)G + \dot{\beta}_2(x - \sigma_2)G - \dot{\sigma}_2\partial_x G_2(t, x - \sigma_2)e^{i\Gamma_2} + \partial_t G_2(t, x - \sigma_2)e^{i\Gamma_2}.$$

As before, we use the following estimates to prove (4.3.9) for G

$$\begin{aligned} \|(\dot{\gamma}_2 + \dot{\beta}_2\sigma_2)G_2\|_{L^2} &\lesssim (|\dot{\gamma}_2| + |\dot{\beta}_2|\|\sigma_2\|)e^{-c\theta\sigma}, & \|xG_2\|_{L^2} &\lesssim e^{-c\theta\sigma}, \\ \|\partial_x G_2\|_{L^2} &\lesssim e^{-c\theta\sigma}, & \|\partial_t G_2\|_{L^2} &\lesssim |\dot{\sigma}|e^{-c\theta\sigma}. \end{aligned}$$

The proof of (4.3.9) for G^\perp follows from similar arguments and it is omitted. \square

4.3.2 Modulation equations

Lemma 4.3.10. Let $\theta_1 < \theta < \min\{\frac{1}{c}; 2\}$. It holds

$$|\langle \varepsilon, P \rangle| \lesssim t^{-2} \log t, \quad |\langle \eta, R \rangle| \lesssim t^{-2\theta_1}, \quad (4.3.11)$$

$$|\dot{\sigma}_1 - 2\beta_1| + |\dot{\sigma}_2 - 2\beta_2| + |\dot{\gamma}_1| + |\dot{\gamma}_2| \lesssim t^{-\theta}. \quad (4.3.12)$$

$$|\vec{m}_1| + |\vec{m}_2| + |\vec{m}_1^a| + |\vec{m}_2^b| \lesssim t^{-\theta}, \quad |\vec{m}_\varphi| \lesssim t^{-1}, \quad (4.3.13)$$

$$|\dot{\beta}_1 + a| + |\dot{\beta}_2 + b| \lesssim t^{-1-\theta_1}. \quad (4.3.14)$$

Proof. Proof of (4.3.11). First, it follows from Lemma 4.3.3 and (4.3.1) that

$$\|U\|_{L^2}^2 = \|Q_c + e^{-c\sigma}A(\cdot + \sigma)\|_{L^2}^2 = \|Q_c\|_{L^2}^2 + O(t^{-2} \log t).$$

We use the mass conservation for u and $\varepsilon(T_\infty) = 0$,

$$\|U + \varepsilon\|_{L^2}^2 = \|u\|_{L^2}^2 = \|u(T_\infty)\|_{L^2}^2 = \|U(T_\infty)\|_{L^2}^2 = \|Q_c\|_{L^2}^2 + O(T_\infty^{-2} \log T_\infty),$$

and thus by (4.3.1),

$$2\langle \varepsilon, U \rangle = \|U + \varepsilon\|_{L^2}^2 - \|U\|_{L^2}^2 - \|\varepsilon\|_{L^2}^2 = O(t^{-2} \log t).$$

Last, using $|\langle \varepsilon, \varphi \rangle| \leq \|\varepsilon\|_{L^2} \|\varphi\|_{L^2} \lesssim t^{-1} \|\varepsilon\|_{L^2} \lesssim t^{-1-\theta_1}$ and $2\langle \varepsilon, P \rangle = 2\langle \varepsilon, U \rangle - 2\langle \varepsilon, \varphi \rangle$, we obtain $|\langle \varepsilon, P \rangle| \lesssim t^{-2} \log t$. The estimate on $\langle \eta, R \rangle$ follows directly from $\|v\|_{L^2} = \|v(T_\infty)\|_{L^2}$.

Proof of (4.3.12)-(4.3.13)-(4.3.14). We use the special choice of orthogonality conditions (4.2.13) as well as the relations (4.1.11). We refer to the proof of Lemma 7 in [20] for a similar argument. First, differentiating the second orthogonality in (4.2.13) and using (4.2.12),

$$\begin{aligned} 0 &= \frac{d}{dt} \langle \varepsilon, i\Lambda_1 P \rangle = -\langle i\partial_t \varepsilon, \Lambda_1 P \rangle + \langle \varepsilon, i\partial_t \Lambda_1 P \rangle \\ &= -\langle -\partial_x^2 \varepsilon + c^2 \varepsilon + h(U + \varepsilon, V + \eta) - h(U, V), \Lambda_1 P \rangle \\ &\quad + \langle F, \Lambda_1 P \rangle - \langle \vec{m}_1 \cdot \vec{M}_1, \Lambda_1 P \rangle - \langle \vec{m}_\varphi \cdot \vec{M}_\varphi, \Lambda_1 P \rangle - \langle i\varepsilon, \partial_t(\Lambda_1 P) - ic^2 \Lambda_1 P \rangle. \end{aligned}$$

We claim

$$\left| \langle -\partial_x^2 \varepsilon + c^2 \varepsilon + h(U + \varepsilon, V + \eta) - h(U, V), \Lambda_1 P \rangle \right| \lesssim t^{-2} \log t. \quad (4.3.15)$$

Observe that

$$h(U + \varepsilon, V + \eta) - h(U, V) = 2|U|^2 \varepsilon + U^2 \bar{\varepsilon} + \omega|V|^2 \varepsilon + 2\omega U \Re(V \bar{\eta}) + O(|\varepsilon|^2 + |\eta|^2).$$

By Lemma 4.3.3, $\|V^2\Lambda_1P\|_{L^2} \lesssim t^{-1} \log t$, $\|UV\Lambda_1P\|_{L^2} \lesssim t^{-1}$, and thus

$$\begin{aligned} & |\langle -\partial_x^2\varepsilon + c^2\varepsilon + h(U + \varepsilon, V + \eta) - h(U, V), \Lambda_1P \rangle - \langle \varepsilon, -\partial_x^2(\Lambda_1P) + c^2\Lambda_1P + 3|P|^2\Lambda_1P \rangle| \\ & \lesssim t^{-1}(\log t) (\|\varepsilon\|_{L^2} + \|\eta\|_{L^2}) + \|\varepsilon\|_{L^2}^2 + \|\eta\|_{L^2}^2 \lesssim t^{-1-\theta_1} \log t. \end{aligned}$$

Using $\mathcal{L}_+(\Lambda Q) = -2Q$ from (4.1.11) and $\|\partial_x^2(\Lambda_1P) - \partial_1^2(\Lambda_1P)\|_{L^2} \lesssim |\beta_1| \lesssim t^{-1}$ (by analogy with the notation introduced in §4.2.1, we set $\partial_1^2(\Lambda_1P) = (\Lambda Q_c)''(x - \sigma_1)e^{i\Gamma_1}$) we see that

$$\|[-\partial_x^2(\Lambda_1P) + c^2\Lambda_1P + 3|P|^2\Lambda_1P] + 2c^2P\|_{L^2} \lesssim t^{-1}.$$

Thus, by (4.3.1) and (4.3.11), we obtain (4.3.15).

The estimate $|\langle F, \Lambda_1P \rangle| \lesssim e^{-2c\sigma} \lesssim t^{-2}$ is clear from (4.3.6) and then (4.3.1). Next, using $\langle P, \Lambda_1P \rangle = \langle Q_c, \Lambda Q_c \rangle = \frac{1}{2}\|Q_c\|_{L^2}^2 = 2c$ and $\langle iQ_c', \Lambda Q_c \rangle = \langle xQ_c, \Lambda Q_c \rangle = 0$, we obtain

$$-\langle \vec{m}_1 \cdot \vec{M}_1, \Lambda_1P \rangle = -2c(\dot{\gamma}_1 + \dot{\beta}_1\sigma_1 + \beta_1^2).$$

Moreover, using Lemma 4.3.3,

$$\begin{aligned} -\langle \vec{m}_\varphi \cdot \vec{M}_\varphi, \Lambda_1P \rangle &= -(\dot{\gamma}_1 + \dot{\beta}_1\sigma_2 + \beta_1^2)\langle \varphi, \Lambda_1P \rangle + \dot{\beta}_1\langle x_2\varphi, \Lambda_1P \rangle \\ &= (|\dot{\gamma}_1| + |\dot{\beta}_1|\sigma_2 + \beta_1^2)O(\sigma^2e^{-2c\sigma}). \end{aligned}$$

Last, using the analogue of (4.3.5) for Λ_1P , we have

$$|i\langle \varepsilon, \partial_t(\Lambda_1P) - ic^2\Lambda_1P \rangle| \lesssim (|\dot{\gamma}_1| + |\dot{\beta}_1|\sigma_1 + |\dot{\sigma}_1 - 2\beta_1| + |\beta_1|)\|\varepsilon\|_{L^2}.$$

The conclusion of these estimates is

$$|\dot{\gamma}_1| \lesssim t^{-2} \log t + t^{-1}|\dot{\sigma}_1 - 2\beta_1| + |\dot{\beta}_1| \log t.$$

Proceeding similarly with the orthogonality condition $\langle \eta, i\Lambda_2R \rangle = 0$, we check

$$|\dot{\gamma}_2| \lesssim t^{-\theta} + t^{-1}|\dot{\sigma}_2 - 2\beta_2| + |\dot{\beta}_2| \log t.$$

Note that we again use $\mathcal{L}_+(\Lambda Q) = -2Q$ and (4.3.11) for η . The term $t^{-\theta}$ comes from estimate of G in (4.3.8), which is to be compared with (4.3.6) for F .

Next, differentiating the orthogonality conditions $\langle \varepsilon, x_1P \rangle = \langle \eta, x_2R \rangle = 0$, using the relation $\mathcal{L}_-(xQ) = -2Q'$ from (4.1.11) and last $\langle \varepsilon, i\partial_1P \rangle = \langle \eta, i\partial_2R \rangle = 0$, we find

$$\begin{aligned} |\dot{\sigma}_1 - 2\beta_1| &\lesssim t^{-1-\theta_1} \log t + t^{-1}(|\dot{\gamma}_1| + |\dot{\beta}_1|\sigma_1 + |\dot{\sigma}_2 - 2\beta_2|), \\ |\dot{\sigma}_2 - 2\beta_2| &\lesssim t^{-1-\theta_1} \log t + t^{-1}(|\dot{\gamma}_2| + |\dot{\beta}_2|\sigma_2 + |\dot{\sigma}_1 - 2\beta_1|). \end{aligned}$$

Note that for these estimates, we have also used $\langle F, ix_1P \rangle = 0$ and $\langle G, ix_2R \rangle = 0$.

Last, differentiating the orthogonality conditions $\langle \varepsilon, i\partial_1P \rangle = \langle \eta, i\partial_2R \rangle = 0$, using the relation $\mathcal{L}_+(Q') = 0$ from (4.1.11) and $\langle F^\perp, \partial_1P \rangle = \langle G^\perp, \partial_2R \rangle = 0$, we check that

$$|\dot{\beta}_1 + a| + |\dot{\beta}_2 + b| \lesssim t^{-1-\theta_1} + t^{-1}(|\dot{\gamma}_1| + |\dot{\sigma}_1 - 2\beta_1| + |\dot{\gamma}_2| + |\dot{\sigma}_2 - 2\beta_2|).$$

The proof of (4.3.12)-(4.3.13)-(4.3.14) follows from the above estimates and (4.2.7). \square

4.3.3 Energy estimates

Let

$$H(u, v) = \frac{1}{4}|u|^4 + \frac{1}{4}|v|^4 + \frac{\omega}{2}|u|^2|v|^2, \quad h(u, v) = (|u|^2 + \omega|v|^2)u.$$

and remark that

$$\begin{aligned} d_1H(U, V)(\varepsilon) &= \frac{1}{2}(|U|^2 + \omega|V|^2)(U\bar{\varepsilon} + \bar{U}\varepsilon) = \Re(h(U, V)\varepsilon), \\ d_2H(U, V)(\eta) &= \frac{1}{2}(|V|^2 + \omega|U|^2)(V\bar{\eta} + \bar{V}\eta) = \Re(h(V, U)\eta), \\ d_1h(U, V)(\varepsilon) &= 2|U|^2\varepsilon + U^2\bar{\varepsilon} + \omega|V|^2\varepsilon, \quad d_2h(U, V)(\eta) = \omega(V\bar{\eta} + \bar{V}\eta)U, \\ \frac{1}{2}(\varepsilon, \eta)^\top (d^2h)(U, V)(\varepsilon, \eta) &= 2\varepsilon\Re(U\bar{\varepsilon}) + U|\varepsilon|^2 + 2\omega\varepsilon\Re(V\bar{\eta}) + \omega U|\eta|^2. \end{aligned}$$

Consider the energy functional for $(\frac{\varepsilon}{\eta})$

$$\mathbf{K}(t, \varepsilon, \eta) = \frac{1}{2} \int \left\{ |\partial_x \varepsilon|^2 + |\partial_x \eta|^2 - 2[H(U + \varepsilon, V + \eta) - H(U, V) - d_1H(U, V)(\varepsilon) - d_2H(U, V)(\eta)] \right\}$$

and the mass functionals for ε and η

$$\mathbf{M} = M_1 + M_2, \quad M_1(\varepsilon) = \frac{c^2}{2} \int |\varepsilon|^2, \quad M_2(\eta) = \frac{1}{2} \int |\eta|^2.$$

Let $\chi : [0, +\infty) \rightarrow [0, +\infty)$ be a smooth non-increasing function satisfying $\chi \equiv 1$ on $[0, \frac{1}{4}]$ and $\chi \equiv 0$ on $[\frac{1}{2}, +\infty)$. Denote $\mathbf{J} = J_1 + J_2$ where, for $j = 1, 2$,

$$J_j(t, \varepsilon, \eta) = \beta_j \Im \int [(\partial_x \varepsilon)\bar{\varepsilon} + (\partial_x \eta)\bar{\eta}] \chi_j \quad \text{where} \quad \chi_j(t, x) = \chi\left(\frac{|x - \sigma_j(t)|}{\log t}\right).$$

Last, we set

$$\mathbf{S}(t, \varepsilon, \eta) = \langle \varepsilon, F^\perp \rangle + 2\beta \langle \varepsilon, i\phi \rangle + \langle \eta, G^\perp \rangle \quad \text{where} \quad \phi = \partial_1 \varphi - c\varphi.$$

Last, set

$$\mathbf{W}(t, \varepsilon, \eta) = \mathbf{K}(t, \varepsilon, \eta) + \mathbf{M}(t, \varepsilon, \eta) - \mathbf{J}(t, \varepsilon, \eta) - \mathbf{S}(t, \varepsilon, \eta).$$

We refer to [14, 16, 17, 19, 20, 22, 26] for similar energy functionals. However, the introduction of the correcting term \mathbf{S} seems to be a previously unnoticed general improvement of the energy method in this context. See Section 4.5.

Under the bootstrap (4.3.1), we prove the following estimates.

Proposition 4.3.16. Let $\theta_1 < \theta < \min\{\frac{1}{c}; 2\}$. It holds

$$\|\varepsilon\|_{H^1}^2 + \|\eta\|_{H^1}^2 \lesssim \mathbf{W}(t, \varepsilon, \eta) + Ct^{-2\theta}, \quad (4.3.17)$$

and

$$\left| \frac{d}{dt} [\mathbf{W}(t, \varepsilon, \eta)] \right| \lesssim t^{-1-2\theta_1} (\log t)^{-1}. \quad (4.3.18)$$

Proof of Proposition 4.3.16. step 1. The coercivity property (4.3.17) is a consequence of the coercivity property around one solitary wave in Lemma 4.1.12, the orthogonality relations (4.2.13)-(4.3.11)) and the positivity of \mathcal{L}_c . It also involves a localization argument similar to the proof of Lemma 4.1 in [19] for the scalar case.

Note that by (4.3.1),

$$|\mathbf{J}(t, \varepsilon, \eta)| \lesssim t^{-1}(\|\varepsilon\|_{H^1}^2 + \|\eta\|_{H^1}^2)$$

and by (4.3.6) and (4.3.8),

$$|\mathbf{S}(t, \varepsilon, \eta)| \lesssim t^{-\theta}(\|\varepsilon\|_{H^1} + \|\eta\|_{H^1}).$$

Next, we see that the following terms in the functional \mathbf{K} are easily controlled

$$\int (|P\varphi| + |\varphi|^2)|\varepsilon|^2 + \int |U\varepsilon||V\eta| \lesssim t^{-1}(\|\varepsilon\|_{H^1}^2 + \|\eta\|_{H^1}^2).$$

Moreover, cubic and higher order terms in ε or η are of order $t^{-\theta_1}(\|\varepsilon\|_{H^1}^2 + \|\eta\|_{H^1}^2)$.

Therefore, we are reduced to consider the following two decoupled functionals

$$\begin{aligned} \mathbf{W}_1 &= \int \{|\partial_x \varepsilon|^2 + c^2|\varepsilon|^2 - |P|^2|\varepsilon|^2 - 2[\Re(P\bar{\varepsilon})]^2 - \omega|R|^2|\varepsilon|^2\}, \\ \mathbf{W}_2 &= \int \{|\partial_x \eta|^2 + |\eta|^2 - |R|^2|\eta|^2 - 2[\Re(R\bar{\eta})]^2 - \omega|P|^2|\eta|^2\}. \end{aligned}$$

We focus on the coercivity property for \mathbf{W}_1 , the case of \mathbf{W}_2 is similar.

Denote $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ an even function of class \mathcal{C}^2 such that

$$\Phi \equiv 1 \text{ on } [0, 1], \quad \Phi \equiv e^{-x} \text{ on } [2, +\infty), \quad e^{-x} \leq \Phi(x) \leq e^{-3x}, \quad \Phi' \leq 0 \text{ on } \mathbb{R}.$$

Let $B > 1$ and $\Phi_B(x) = \Phi(x/B)$. We claim that for B large enough, there exists $\mu_1 > 0$, such that for any $\hat{\varepsilon}$ satisfying $\langle \hat{\varepsilon}, Q \rangle = \langle \hat{\varepsilon}, xQ \rangle = \langle \hat{\varepsilon}, i\Lambda Q \rangle = 0$, and any $\tilde{\varepsilon}$, it holds

$$\begin{aligned} \mathcal{N}_1(\hat{\varepsilon}) &:= \int \Phi_B \{|\partial_x \hat{\varepsilon}|^2 + |\hat{\varepsilon}|^2 - Q^2|\hat{\varepsilon}|^2 - 2[\Re(Q\bar{\hat{\varepsilon}})]^2\} \geq \mu_1 \int \Phi_B (|\partial_x \hat{\varepsilon}|^2 + |\hat{\varepsilon}|^2), \\ \mathcal{N}_2(\tilde{\varepsilon}) &:= \int \Phi_B \{|\partial_x \tilde{\varepsilon}|^2 + c^2|\tilde{\varepsilon}|^2 - \omega Q^2|\tilde{\varepsilon}|^2\} \geq \mu_1 \int \Phi_B (|\partial_x \tilde{\varepsilon}|^2 + |\tilde{\varepsilon}|^2). \end{aligned}$$

Setting $z = \hat{\varepsilon}\Phi_B^{\frac{1}{2}}$ and following the proof of Claim 8 in [19], the coercivity of \mathcal{N}_1 follows from (i) of Lemma 4.1.12 applied to the function z . A similar localization argument, using the coercivity property of \mathcal{L}_c proves the estimate for $\mathcal{N}_2(\tilde{\varepsilon})$ without any orthogonality condition on $\tilde{\varepsilon}$. This is where our proof needs the condition (4.1.10).

Using these estimates with $\hat{\varepsilon}$ and $\tilde{\varepsilon}$ such that $\varepsilon = c\hat{\varepsilon}(c(x - \sigma_1))e^{i\Gamma_1}$ and $\varepsilon = \tilde{\varepsilon}(x - \sigma_2)e^{i\Gamma_2}$, the orthogonality conditions (4.2.13) and the almost orthogonality relation (4.3.11), we obtain the estimate $\|\varepsilon\|_{H^1}^2 \lesssim \mathbf{W}_1 + t^{-4}(\log t)^2$.

step 2. Time variation of the energy. Denote

$$K_1 = h(U + \varepsilon, V + \eta) - h(U, V) - d_1 h(U, V)(\varepsilon) - d_2 h(U, V)(\eta),$$

$$K_2 = h(V + \eta, U + \varepsilon) - h(V, U) - d_1 h(V, U)(\eta) - d_2 h(V, U)(\varepsilon),$$

so that

$$\begin{aligned} K_1 &= \frac{1}{2}(\varepsilon, \eta)^T (d^2 h)(U, V)(\varepsilon, \eta) + O(|\varepsilon|^3 + |\eta|^3), \\ K_2 &= \frac{1}{2}(\eta, \varepsilon)^T (d^2 h)(V, U)(\eta, \varepsilon) + O(|\varepsilon|^3 + |\eta|^3). \end{aligned}$$

We prove the following estimate

$$\begin{aligned} \frac{d}{dt}[\mathbf{K}(t, \varepsilon, \eta)] &= 2\beta_1 \langle \partial_x U, K_1 \rangle + 2\beta_2 \langle \partial_x V, K_2 \rangle - c^2 \langle iU, K_1 \rangle - \langle iV, K_2 \rangle \\ &\quad - \langle iD_\varepsilon \mathbf{K}, \mathcal{E}_U \rangle - \langle iD_\eta \mathbf{K}, \mathcal{E}_V \rangle + O(t^{-1-2\theta_1} (\log t)^{-1}). \end{aligned} \quad (4.3.19)$$

The time derivative of $t \mapsto \mathbf{K}(t, \varepsilon(t), \eta(t))$ splits into three parts

$$\frac{d}{dt}[\mathbf{K}(t, \varepsilon, \eta)] = D_t \mathbf{K}(t, \varepsilon, \eta) + \langle D_\varepsilon \mathbf{K}(t, \varepsilon, \eta), \partial_t \varepsilon \rangle + \langle D_\eta \mathbf{K}(t, \varepsilon, \eta), \partial_t \eta \rangle,$$

where D_t denotes the differentiation of \mathbf{K} with respect to t , and D_ε, D_η the differentiation of \mathbf{K} with respect to ε and η . In particular, $D_t \mathbf{K} = -\langle \partial_t U, K_1 \rangle - \langle \partial_t V, K_2 \rangle$.

We claim

$$\begin{aligned} \partial_t U &= ic^2 U - 2\beta_1 \partial_x U + O_{H^1}(t^{-\theta}), \\ \partial_t V &= iV - 2\beta_2 \partial_x V + O_{H^1}(t^{-\theta}). \end{aligned} \quad (4.3.20)$$

Indeed, from the definition of U

$$\begin{aligned} \partial_t U &= ic^2 U - 2\beta_1 \partial_1 U - (\dot{\sigma}_1 - 2\beta_1) \partial_1 P + i(\dot{\gamma}_1 + \dot{\beta}_1 \sigma_1) P + i\dot{\beta}_1 x_1 P \\ &\quad - (\dot{\sigma}_2 - 2\beta_1) \partial_1 \varphi + i(\dot{\gamma}_1 + \dot{\beta}_1 \sigma_2 + ic\dot{\sigma}) \varphi + i\dot{\beta}_1 x_2 \varphi. \end{aligned}$$

Thus, using (4.3.12) and (4.3.14), we obtain (4.3.20) for U . The proof for V is similar.

Using (4.3.20) and (4.3.1), we obtain

$$D_t \mathbf{K}(t, \varepsilon, \eta) = 2\beta_1 \langle \partial_x U, K_1 \rangle + 2\beta_2 \langle \partial_x V, K_2 \rangle - c^2 \langle iU, K_1 \rangle - \langle iV, K_2 \rangle + O(t^{-\theta-2\theta_1}).$$

Next, we observe

$$D_\varepsilon \mathbf{K}(t, \varepsilon, \eta) = -\partial_x^2 \varepsilon - h(U + \varepsilon, V + \eta) + h(U, V)$$

so that the equation of ε in (4.2.12) rewrites $i\partial_t \varepsilon - D_\varepsilon \mathbf{K}(t, \varepsilon, \eta) + \mathcal{E}_U = 0$ and thus

$$\langle D_\varepsilon \mathbf{K}(t, \varepsilon, \eta), \partial_t \varepsilon \rangle = -\langle iD_\varepsilon \mathbf{K}(t, \varepsilon, \eta), \mathcal{E}_U \rangle.$$

Similarly,

$$\langle D_\eta \mathbf{K}(t, \varepsilon, \eta), \partial_t \eta \rangle = -\langle iD_\eta \mathbf{K}(t, \varepsilon, \eta), \mathcal{E}_V \rangle.$$

We have proved (4.3.19).

step 3. Time variation of the total mass. We claim

$$\frac{d}{dt}[\mathbf{M}(\varepsilon, \eta)] = c^2 \langle iU, K_1 \rangle + \langle iV, K_2 \rangle - \langle ic^2 \varepsilon, \mathcal{E}_U \rangle - \langle i\eta, \mathcal{E}_V \rangle. \quad (4.3.21)$$

By integration by parts, we have $\langle i\partial_x^2 \varepsilon, \varepsilon \rangle = 0$ so from (4.2.12),

$$\frac{d}{dt}[M_1(\varepsilon)] = c^2 \langle \partial_t \varepsilon, \varepsilon \rangle = -c^2 \langle i\varepsilon, h(U + \varepsilon, V + \eta) - h(U, V) \rangle - c^2 \langle i\varepsilon, \mathcal{E}_U \rangle.$$

We claim the following identity

$$\langle iU, K_1 \rangle + \langle i\varepsilon, h(U + \varepsilon, V + \eta) - h(U, V) \rangle = 0. \quad (4.3.22)$$

Indeed, since $h(u, v)\bar{u}$ is real, for all $\theta \in \mathbb{R}$, it holds

$$\langle i(U + \theta\varepsilon), h(U + \theta\varepsilon, V + \theta\eta) \rangle = 0.$$

Differentiating with respect to θ , and taking $\theta = 0$, we obtain

$$\langle i\varepsilon, h(U, V) \rangle + \langle iU, d_1 h(U, V)(\varepsilon) \rangle + \langle iU, d_2 h(U, V)(\eta) \rangle = 0$$

Moreover, with $\theta = 0$ and $\theta = 1$

$$\langle iU, h(U, V) \rangle = 0, \quad \langle i(U + \varepsilon), h(U + \varepsilon, V + \eta) \rangle = 0.$$

We see that (4.3.22) follows from combining these identities.

This yields $\frac{d}{dt} M_1 = c^2 \langle iU, K_1 \rangle - c^2 \langle i\varepsilon, \mathcal{E}_U \rangle$. Computing also $\frac{d}{dt} M_2$, we obtain (4.3.21).

step 4. Time variation of the localized momentum. We claim

$$\frac{d}{dt} [\mathbf{J}(t, \varepsilon, \eta)] = 2\beta_1 \langle \partial_x U, K_1 \rangle + 2\beta_2 \langle \partial_x V, K_2 \rangle + O(t^{-1-2\theta_1} (\log t)^{-1}). \quad (4.3.23)$$

By direct computation,

$$\begin{aligned} \frac{d}{dt} [J_1(t, \varepsilon, \eta)] &= \dot{\beta}_1 \Im \int [(\partial_x \varepsilon) \bar{\varepsilon} + (\partial_x \eta) \bar{\eta}] \chi_1 + \beta_1 \Im \int [(\partial_x \varepsilon) \bar{\varepsilon} + (\partial_x \eta) \bar{\eta}] \partial_t \chi_1 \\ &\quad + \beta_1 \langle i \partial_t \varepsilon, 2\chi_1 \partial_x \varepsilon + \varepsilon \partial_x \chi_1 \rangle + \beta_1 \langle i \partial_t \eta, 2\chi_1 \partial_x \eta + \eta \partial_x \chi_1 \rangle. \end{aligned}$$

By (4.3.1) and (4.3.14), we have

$$\left| \dot{\beta}_1 \Im \int [(\partial_x \varepsilon) \bar{\varepsilon} + (\partial_x \eta) \bar{\eta}] \chi_1 \right| \lesssim t^{-2} (\|\varepsilon\|_{H^1}^2 + \|\eta\|_{H^1}^2) \lesssim t^{-2-2\theta_1}.$$

By direct computations,

$$\partial_t \chi_j(t, x) = - \left[\frac{\dot{\sigma}_j}{\log t} \frac{x - \sigma_j}{|x - \sigma_j|} + \frac{|x - \sigma_j|}{t(\log t)^2} \right] \chi' \left(\frac{|x - \sigma_j|}{\log t} \right)$$

and so by (4.3.1), (4.3.12) and the properties of χ , $|\partial_t \chi_j| \lesssim t^{-1} (\log t)^{-1}$. It follows that

$$\left| \beta_1 \Im \int [(\partial_x \varepsilon) \bar{\varepsilon} + (\partial_x \eta) \bar{\eta}] \partial_t \chi_1 \right| \lesssim t^{-2} (\log t)^{-1} (\|\varepsilon\|_{H^1}^2 + \|\eta\|_{H^2}^2) \lesssim t^{-2-2\theta_1} (\log t)^{-1}.$$

Next, using the equation (4.2.12)

$$\begin{aligned} \langle i \partial_t \varepsilon, 2\chi_1 \partial_x \varepsilon + \varepsilon \partial_x \chi_1 \rangle &= - \langle \partial_x^2 \varepsilon, 2\chi_1 \partial_x \varepsilon + \varepsilon \partial_x \chi_1 \rangle \\ &\quad - \langle h(U + \varepsilon, V + \eta) - h(U, V), 2\chi_1 \partial_x \varepsilon + \varepsilon \partial_x \chi_1 \rangle \\ &\quad - \langle \mathcal{E}_U, 2\chi_1 \partial_x \varepsilon + \varepsilon \partial_x \chi_1 \rangle. \end{aligned}$$

Integrating by parts, we have

$$- \langle \partial_x^2 \varepsilon, 2\chi_1 \partial_x \varepsilon + \varepsilon \partial_x \chi_1 \rangle = \int |\partial_x \varepsilon|^2 \partial_x \chi_1 - \frac{1}{2} \int |\varepsilon|^2 \partial_x^3 \chi_1.$$

Since $|\partial_x \chi_1| \lesssim (\log t)^{-1}$ and $|\partial_x^3 \chi_1| \lesssim (\log t)^{-3}$, from (4.3.1), we have

$$\left| \beta_1 \langle \partial_x^2 \varepsilon, 2\chi_1 \partial_x \varepsilon + \varepsilon \partial_x \chi_1 \rangle \right| \lesssim t^{-1-2\theta_1} (\log t)^{-1}.$$

For the term containing \mathcal{E}_U , we use (4.2.2), (4.3.1), (4.3.6) and (4.3.13),

$$\left| \beta_1 \langle \mathcal{E}_U, 2\chi_1 \partial_x \varepsilon + \varepsilon \partial_x \chi_1 \rangle \right| \lesssim t^{-1-\theta} \|\varepsilon\|_{H^1} \lesssim t^{-1-2\theta_1} (\log t)^{-1}.$$

Then, we estimate, using $|\partial_x \chi_1| \lesssim (\log t)^{-1}$,

$$|\langle h(U + \varepsilon, V + \eta) - h(U, V), \varepsilon \partial_x \chi_1 \rangle| \lesssim (\log t)^{-1} (\|\varepsilon\|_{H^1}^2 + \|\eta\|_{H^1}^2) \lesssim t^{-2\theta_1} (\log t)^{-1}.$$

Collecting the above estimates, we obtain

$$\begin{aligned} \frac{d}{dt}[J_1(t, \varepsilon, \eta)] &= -2\beta_1 \langle \chi_1 \partial_x \varepsilon, h(U + \varepsilon, V + \eta) - h(U, V) \rangle \\ &\quad - 2\beta_1 \langle \chi_1 \partial_x \eta, h(V + \eta, U + \varepsilon) - h(V, U) \rangle + O(t^{-1-2\theta_1} (\log t)^{-1}). \end{aligned}$$

We complete the proof of (4.3.23) by showing the following

$$\begin{aligned} \langle \partial_x U, K_1 \rangle + \langle \chi_1 \partial_x \varepsilon, h(U + \varepsilon, V + \eta) - h(U, V) \rangle \\ + \langle \chi_1 \partial_x \eta, h(V + \eta, U + \varepsilon) - h(V, U) \rangle = O(t^{-2\theta_1} (\log t)^{-1}). \end{aligned} \quad (4.3.24)$$

First, we prove the identity

$$\begin{aligned} \langle \partial_x U, K_1 \rangle + \langle \partial_x \varepsilon, h(U + \varepsilon, V + \eta) - h(U, V) \rangle \\ + \langle \partial_x V, K_2 \rangle + \langle \partial_x \eta, h(V + \eta, U + \varepsilon) - h(V, U) \rangle = 0. \end{aligned} \quad (4.3.25)$$

Indeed, we have

$$\langle \partial_x u, h(u, v) \rangle + \langle \partial_x v, h(v, u) \rangle = \int \partial_x [H(u, v)] = 0.$$

Applying this to $u = U + \theta \varepsilon$ and $v = V + \theta \eta$, we have that for all $\theta \in \mathbb{R}$

$$\langle \partial_x (U + \theta \varepsilon), h(U + \theta \varepsilon, V + \theta \eta) \rangle + \langle \partial_x (V + \theta \eta), h(V + \theta \eta, U + \theta \varepsilon) \rangle = 0.$$

Taking the derivative with respect to θ at $\theta = 0$, we obtain

$$\begin{aligned} \langle \partial_x \varepsilon, h(U, V) \rangle + \langle \partial_x \eta, h(V, U) \rangle + \langle \partial_x U, d_1 h(U, V)(\varepsilon) \rangle + \langle \partial_x U, d_2 h(U, V)(\eta) \rangle \\ + \langle \partial_x V, d_1 h(V, U)(\eta) \rangle + \langle \partial_x V, d_2 h(V, U)(\varepsilon) \rangle = 0. \end{aligned}$$

Moreover, using the above identity with $\theta = 0$ and $\theta = 1$, we have

$$\begin{aligned} \langle \partial_x U, h(U, V) \rangle + \langle \partial_x V, h(V, U) \rangle = 0, \\ \langle \partial_x (U + \varepsilon), h(U + \varepsilon, V + \eta) \rangle + \langle \partial_x (V + \eta), h(V + \eta, U + \varepsilon) \rangle = 0. \end{aligned}$$

Gathering these identities, we obtain (4.3.25).

We apply identity (4.3.25) to $\chi_1^{\frac{1}{4}} U$, $\chi_1^{\frac{1}{4}} V$, $\chi_1^{\frac{1}{4}} \varepsilon$ and $\chi_1^{\frac{1}{4}} \eta$. Recall that $|\partial_x \chi_1| \lesssim (\log t)^{-1}$ and also note that by the definition of χ , $|\chi_1 V| + (1 - \chi_1) |\partial_x U| \lesssim (\log t)^{-1}$. In particular, this shows that

$$\begin{aligned} |\langle \partial_x (\chi_1^{\frac{1}{4}} U), K_1 \chi_1^{\frac{3}{4}} \rangle - \langle \partial_x U, K_1 \rangle| + |\langle \partial_x (\chi_1^{\frac{1}{4}} V), K_2 \chi_1^{\frac{3}{4}} \rangle| = O(t^{-2\theta_1} (\log t)^{-1}), \\ \langle [\chi_1^{\frac{3}{4}} \partial_x (\chi_1^{\frac{1}{4}} \varepsilon) - \chi_1 \partial_x \varepsilon], h(U + \varepsilon, V + \eta) - h(U, V) \rangle = O(t^{-2\theta_1} (\log t)^{-1}), \end{aligned}$$

and

$$\langle [\chi_1^{\frac{3}{4}} \partial_x (\chi_1^{\frac{1}{4}} \eta) - \chi_1 \partial_x \eta], h(V + \eta, U + \varepsilon) - h(V, U) \rangle = O(t^{-2\theta_1} (\log t)^{-1}).$$

This proves (4.3.24) and then (4.3.23), the computations for J_2 being identical.

step 5. Additional correction terms. We claim

$$\frac{d}{dt}[\mathbf{S}(t, \varepsilon, \eta)] = -\langle i(D_\varepsilon \mathbf{K} + c^2 \varepsilon), F^\perp + 2i\beta \phi \rangle - \langle i(D_\eta \mathbf{K} + \eta), G^\perp \rangle + O(t^{-(1+\theta+\theta_1)}). \quad (4.3.26)$$

We compute, using (4.2.12),

$$\frac{d}{dt}\langle \varepsilon, F^\perp \rangle = -\langle i(D_\varepsilon \mathbf{K} + c^2 \varepsilon), F^\perp \rangle - \langle \mathcal{E}_U, iF^\perp \rangle + \langle \varepsilon, \partial_t F^\perp - ic^2 F^\perp \rangle.$$

From (4.2.4) and $F^\perp e^{-i\Gamma_1} \in \mathbb{R}$, it follows that $\langle \vec{m}_1^a \cdot \vec{M}_1, iF^\perp \rangle = 0$. One also observes that

$$\langle \vec{m}_\varphi \cdot M_\varphi, iF^\perp \rangle = c\dot{\sigma}\langle \varphi, F^\perp \rangle + (\dot{\sigma}_2 - 2\beta_1)\langle \partial_1 \varphi, F^\perp \rangle = O(t^{-5}(\log t)^2) = O(t^{-(1+2\theta)}),$$

where we have used (4.3.12) and (from Lemma 4.3.3 and the definitions of F^\perp and φ)

$$|\langle \varphi, F^\perp \rangle| + |\langle \partial_1 \varphi, F^\perp \rangle| \lesssim t^{-4}(\log t)^2. \quad (4.3.27)$$

Since $\langle F^\perp, iF^\perp \rangle = 0$, it follows that $\langle \mathcal{E}_U, iF^\perp \rangle = O(t^{-(1+2\theta)})$. Last, it follows from (4.3.7), (4.3.12) and (4.3.1) that

$$|\langle \varepsilon, \partial_t F^\perp - ic^2 F^\perp \rangle| \lesssim t^{-3-\theta_1} \lesssim t^{-1-\theta_1-\theta}.$$

Thus, using (4.2.5),

$$\frac{d}{dt}\langle \varepsilon, F^\perp \rangle = -\langle i(D_\varepsilon \mathbf{K} + c^2 \varepsilon), F^\perp \rangle + O(t^{-1-\theta_1-\theta}).$$

From (4.3.9) and similar estimates, we also obtain

$$\frac{d}{dt}\langle \eta, G^\perp \rangle = -\langle i(D_\eta \mathbf{K} + \eta), G^\perp \rangle + O(t^{-1-\theta_1-\theta})$$

Finally, we compute

$$\frac{d}{dt}[2\beta\langle \varepsilon, i\phi \rangle] = 2\dot{\beta}\langle \varepsilon, i\phi \rangle + 2\beta\langle \partial_t \varepsilon, i\phi \rangle + 2\beta\langle \varepsilon, i\partial_t \phi \rangle.$$

The first term is estimated $|\dot{\beta}\langle \varepsilon, i\phi \rangle| \lesssim t^{-3}\|\varepsilon\|_{H^1} \lesssim t^{-3-\theta_1}$ using (4.3.14). Then, using (4.2.12),

$$\langle \partial_t \varepsilon, i\phi \rangle + \langle \varepsilon, i\partial_t \phi \rangle = -\langle i(D_\varepsilon \mathbf{K} + c^2 \varepsilon), i\phi \rangle + \langle \mathcal{E}_U, \phi \rangle - \langle i\varepsilon, \partial_t \phi - ic^2 \phi \rangle.$$

From (4.3.27), $|\beta\langle F^\perp, \phi \rangle| \lesssim t^{-5}(\log t)^2 \lesssim t^{-1-2\theta}$. From (4.3.13), the expression of $\vec{m}_1^a \cdot \vec{M}_1$ and Lemma 4.3.3,

$$|\beta\langle \vec{m}_1^a \cdot \vec{M}_1, \phi \rangle| \lesssim t^{-1}|\vec{m}_1^a| (|\langle P, \phi \rangle| + |\langle x_1 P, \phi \rangle|) \lesssim t^{-3-\theta}(\log t)^2 \lesssim t^{-1-\theta_1-\theta}.$$

Next, from (4.3.12), the expression of $\vec{m}_\varphi \cdot \vec{M}_\varphi$ and Lemma 4.3.3,

$$|\beta\langle \vec{m}_\varphi \cdot \vec{M}_\varphi, \phi \rangle| \lesssim t^{-1} \left(|\dot{\gamma}_1 + \dot{\beta}_1 \sigma_2 + \beta_1^2| |\langle \varphi, \phi \rangle| + |\dot{\beta}_1 + a| |\langle x_2 \varphi, \phi \rangle| \right) \lesssim t^{-3-\theta} \lesssim t^{-1-\theta_1-\theta}.$$

Last, using (4.3.5) and (4.3.12),

$$|\beta\langle i\varepsilon, \partial_t \phi - ic^2 \phi \rangle| \lesssim t^{-3}\|\varepsilon\|_{L^2} \lesssim t^{-3-\theta_1}.$$

Estimate (4.3.26) is now proved.

step 6. Conclusion. Combining the estimates (4.3.19), (4.3.21), (4.3.23), (4.3.26) and using the decompositions of \mathcal{E}_U and \mathcal{E}_V in (4.2.5), we have obtained

$$\begin{aligned} \frac{d}{dt}\mathbf{W}(t, \varepsilon, \eta) &= \langle i(D_\varepsilon \mathbf{K} + c^2 \varepsilon), \vec{m}_1^a \cdot \vec{M}_1 \rangle + \langle i(D_\varepsilon \mathbf{K} + c^2 \varepsilon), 2i\beta\phi + \vec{m}_\varphi \cdot \vec{M}_\varphi \rangle \\ &\quad + \langle i(D_\eta \mathbf{K} + \eta), \vec{m}_2^b \cdot \vec{M}_2 \rangle + O(t^{-1-2\theta_1}(\log t)^{-1}). \end{aligned}$$

We claim

$$|\langle i(D_\varepsilon \mathbf{K} + c^2 \varepsilon), \vec{m}_1^a \cdot \vec{M}_1 \rangle| \lesssim t^{-(1+\theta_1+\theta)}. \quad (4.3.28)$$

Indeed, following the proof of (4.3.15), using Lemma 4.3.3, the relations (4.1.11), (4.3.1) and the third orthogonality condition in (4.2.13), it holds

$$\begin{aligned} |\langle -\partial_x^2 \varepsilon + c^2 \varepsilon - h(U + \varepsilon, V + \eta) + h(U, V), \partial_1 P \rangle| &\lesssim t^{-(1+\theta_1)}, \\ |\langle -\partial_x^2 \varepsilon + c^2 \varepsilon - h(U + \varepsilon, V + \eta) + h(U, V), iP \rangle| &\lesssim t^{-(1+\theta_1)}, \\ |\langle -\partial_x^2 \varepsilon + c^2 \varepsilon - h(U + \varepsilon, V + \eta) + h(U, V), ix_1 P \rangle| &\lesssim t^{-(1+\theta_1)} \log t. \end{aligned}$$

Thus, (4.3.28) follows from (4.3.13) and (4.3.14). Similarly,

$$|\langle i(D_\eta \mathbf{K} + \eta), \vec{m}_2^b \cdot \vec{M}_2 \rangle| \lesssim t^{-(1+\theta_1+\theta)}.$$

Finally, we remark that from the explicit expression of $\vec{m}_\varphi \cdot \vec{M}_\varphi$ and (4.3.12)

$$\|2i\beta\phi + \vec{m}_\varphi \cdot \vec{M}_\varphi\|_{H^1} \lesssim t^{-1-\theta},$$

which implies by integration by parts and then (4.3.1)

$$|\langle i(D_\varepsilon \mathbf{K} + c^2 \varepsilon), 2i\beta\phi + \vec{m}_\varphi \cdot \vec{M}_\varphi \rangle| \lesssim t^{-1-\theta_1-\theta}.$$

The proof of Proposition 4.3.16 is complete. \square

4.3.4 Bootstrap argument

Proposition 4.3.29. There exists $T_0 > 1$ large enough and for any $T_\infty \geq T_0$, there exists σ_∞ satisfying (4.3.2) such that the solution $\begin{pmatrix} u \\ v \end{pmatrix}$ of (coupled NLS) corresponding to initial data $\begin{pmatrix} U \\ V \end{pmatrix}(T_\infty)$ at $t = T_\infty$ with parameters chosen as in (4.2.14)-(4.2.15) admits a decomposition (4.2.11)-(4.2.13) which satisfies (4.3.1) on $[T_0, T_\infty]$. Moreover, $|\gamma_1| + |\gamma_2| \lesssim t^{1-\theta_1}$ on $[T_0, T_\infty]$.

Proof. For T_0 large enough, for any $T_\infty \geq T_0$ and any σ_∞ satisfying (4.3.2), we define

$$T_\star = T_\star(T_\infty, \sigma_\infty) = \inf\{t \in [T_0, T_\infty] \text{ such that (4.3.1) holds on } [t, T_\infty]\} \in [T_0, T_\infty].$$

We prove by contradiction that, provided T_0 is large enough independent of T_∞ , there exists at least a value of σ_∞ satisfying (4.3.2) such that $T_\star = T_0$. We work only on the time interval $[T_\star, T_\infty]$ on which the bootstrap estimates (4.3.1) hold.

First, we strictly improve the estimates of ε and η in (4.3.1). Indeed, integrating (4.3.18) on $[t, T_\infty]$ and using (4.3.17), it holds

$$\|\varepsilon\|_{H^1}^2 + \|\eta\|_{H^1}^2 \lesssim t^{-2\theta_1} (\log t)^{-1},$$

which strictly improves the estimate in (4.3.1) for large t .

Next, we close the estimates on β_1 , β_2 and β in (4.3.1). Using the estimate of σ in (4.3.1), (4.3.14), (4.2.7) and the expression of Ω_c , it holds

$$\left| \dot{\beta}_1 + \frac{1}{2c(c+1)t^2} \right| \lesssim t^{-1-\theta_2}.$$

At T_∞ , we remark that by (4.2.15) and (4.3.2),

$$\left| \beta_\infty - \frac{1}{2cT_\infty} \right| \lesssim T_\infty^{-\theta_2} \quad \text{and so} \quad \left| \beta_1(T_\infty) - \frac{1}{2c(c+1)T_\infty} \right| \lesssim T_\infty^{-\theta_2}.$$

Integrating on $[t, T_\infty]$ and using (4.2.14) for β_1 , we obtain

$$\left| \beta_1 - \frac{1}{2c(c+1)t} \right| \lesssim t^{-\theta_2},$$

which strictly improves (4.3.1) for β_1 provided that t is large enough. Improving the estimate for β_2 (and then β) is similar.

Then, using (4.3.12), we find

$$\left| \dot{\sigma}_1 - \frac{1}{c(c+1)t} \right| \lesssim t^{-\theta_2}.$$

Integrating on $[t, T_\infty]$, using (4.2.14) and (4.3.2) we obtain

$$\left| \sigma_1 - \frac{\log(\Omega_c t)}{c(c+1)} \right| \lesssim t^{1-\theta_2},$$

which strictly improves the estimate in (4.3.1). The estimate on σ_2 is improved similarly.

We only have to improve the estimate on σ to finish the bootstrap argument. This is where we need to argue by contradiction (see [5] for a similar argument). Using (4.3.12), (4.3.14) and (4.2.7), it holds, on the interval $[T_\star, T_\infty]$,

$$|\dot{\sigma} - 2\beta| \lesssim t^{-\theta_1} \quad \text{and} \quad \left| \dot{\beta} + (1+c)\alpha_c e^{-2c\sigma} \right| \lesssim t^{-1-\theta_1}.$$

Set $g = \beta^2 - \frac{(1+c)\alpha_c}{2c} e^{-2c\sigma}$, so that by the above estimates and (4.2.15) it holds

$$\dot{g} = 2\beta\dot{\beta} + (1+c)\alpha_c\dot{\sigma}e^{-2c\sigma} = O(t^{-2-\theta_1}) \quad \text{and} \quad g(T_\infty) = 0.$$

By integration on $[t, T_\infty]$, this yields

$$\left| \beta^2 - \frac{(1+c)\alpha_c}{2c} e^{-2c\sigma} \right| \lesssim t^{-1-\theta_1} \quad \text{and so} \quad \left| 2\beta - \frac{\Omega_c}{c} e^{-c\sigma} \right| \lesssim t^{-\theta_1}.$$

Define

$$\zeta(t) = \frac{e^{c\sigma}}{\Omega_c} \quad \text{and} \quad \xi(t) = \left(\frac{\zeta(t)}{t} - 1 \right)^2.$$

The previous estimates imply

$$|\dot{\zeta}(t) - 1| \lesssim t^{1-\theta_1}. \tag{4.3.30}$$

Assume for the sake of contradiction that for all $\zeta_\# \in [-1, 1]$, the choice

$$\zeta(T_\infty) = T_\infty + \zeta_\# t^{2-\theta_2}$$

leads to $T_\star \in (T_0, T_\infty]$. By a continuity argument, this means that the bootstrap estimates are reached at T_\star . Since all estimates in (4.3.1) except the one on σ , have been strictly improved on $[T_\star, T_\infty]$, this yields

$$\left| \frac{e^{c\sigma(T_\star)}}{\Omega_c T_\star} - 1 \right| = T_\star^{1-\theta_2}. \tag{4.3.31}$$

Following the argument of [5], we remark that for any $t \in [T_\star, T_\infty]$ satisfying (4.3.31), using (4.3.30) and $\theta_2 < \theta_1$, it holds (taking T_0 large enough)

$$\dot{\xi}(t) = 2(\dot{\zeta}(t) - 1)(\zeta(t) - t)t^{-2} - 2(\zeta(t) - t)^2 t^{-3} = -2t^{1-2\theta_2} (1 + O(t^{\theta_2-\theta_1})) < 0.$$

This transversality condition implies that T_\star is a continuous function of σ_∞ and thus

$$\Phi : \zeta_\# \in [-1, 1] \mapsto T_\star^{\theta_2-2} (\zeta(T_\star) - T_\star) \in \{-1, 1\}$$

is also a continuous function whose image is $\{-1, 1\}$, which is contradictory.

To complete the proof of Proposition 4.3.29, we observe that from (4.3.12), $|\dot{\gamma}_1| + |\dot{\gamma}_2| \lesssim t^{-\theta}$ holds on the interval $[T_0, T_\infty]$. Integrating and using (4.2.14), this gives the uniform estimate $|\gamma_1| + |\gamma_2| \lesssim t^{1-\theta}$ on $[T_0, T_\infty]$. \square

4.3.5 End of the proof of Theorem 4.1.4 by compactness

We use Proposition 4.3.29 with $T_\infty = n$, for any $n \geq T_0$, to construct a sequence of solutions $(\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}) \in \mathcal{C}([T_0, n], H^1 \times H^1)$ of (coupled NLS) such that, for some $\delta > 0$, on $[T_0, n]$,

$$\left\| \begin{pmatrix} u_n \\ v_n \end{pmatrix} - \begin{pmatrix} e^{ic^2t} Q_c \left(\cdot - \frac{\log t}{c(c+1)} - \frac{\log \Omega_\varepsilon}{c(c+1)} \right) \\ e^{it} Q \left(\cdot + \frac{\log t}{c+1} - \frac{\log \Omega_\varepsilon}{c+1} \right) \end{pmatrix} \right\|_{H^1 \times H^1} \lesssim t^{-\delta}. \quad (4.3.32)$$

Now, we adapt from [17] (in the scalar case) and from [11] (for the vector case), the following convergence result.

Lemma 4.3.33. There exists $(\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ such that up to a subsequence, as $n \rightarrow \infty$

$$\begin{aligned} \begin{pmatrix} u_n \\ v_n \end{pmatrix}(T_0) &\rightharpoonup \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \text{ weakly in } H^1(\mathbb{R}) \times H^1(\mathbb{R}) \\ \begin{pmatrix} u_n \\ v_n \end{pmatrix}(T_0) &\rightarrow \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \text{ in } H^s(\mathbb{R}) \times H^s(\mathbb{R}) \text{ for any } 0 \leq s < 1. \end{aligned}$$

We consider $(\begin{smallmatrix} u \\ v \end{smallmatrix})$ the solution of (coupled NLS) corresponding to initial data $(\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix})$ at $t = T_0$. By $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ boundedness and local well-posedness of Cauchy problem in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for any $0 \leq s < 1$ (see *e.g.* [3]), we have the continuous dependence of the solution on the initial data, so for all $t \in [T_0, +\infty)$, as $n \rightarrow \infty$,

$$\begin{aligned} \begin{pmatrix} u_n \\ v_n \end{pmatrix}(t) &\rightharpoonup \begin{pmatrix} u \\ v \end{pmatrix}(t) \text{ in } H^1(\mathbb{R}) \times H^1(\mathbb{R}), \\ \begin{pmatrix} u_n \\ v_n \end{pmatrix}(t) &\rightarrow \begin{pmatrix} u \\ v \end{pmatrix}(t) \text{ in } H^s(\mathbb{R}) \times H^s(\mathbb{R}), \quad 0 \leq s < 1. \end{aligned}$$

Passing to the weak limit as $n \rightarrow \infty$ in the uniform estimates (4.3.32), the solution $(\begin{smallmatrix} u \\ v \end{smallmatrix})$ satisfies Theorem 4.1.4.

4.4 Sketch of the proof of Theorem 4.1.1

4.4.1 Approximate solution in the case $c = 1$

In this case, the approximate solution and the solution are symmetric (*i.e.* $u(t, x) = v(t, -x)$) and thus we have $\sigma_1 = -\sigma_2 = \frac{\sigma}{2}$, $\beta_1 = -\beta_2 = \frac{\beta}{2}$ and $\gamma_1 = \gamma_2$. Using the same notation as in Sections 4.2 and 4.3, we define (the function B is introduced in Lemma 4.1.13)

$$\begin{aligned} U &= P + \varphi, & P(t, x) &= Q(x - \sigma_1(t))e^{i\Gamma_1(t, x)}, & \varphi(t, x) &= e^{-\sigma(t)}B(x - \sigma_2(t))e^{i\Gamma_1(t, x)}, \\ V &= R + \psi, & R(t, x) &= Q(x - \sigma_2(t))e^{i\Gamma_2(t, x)}, & \psi(t, x) &= e^{-\sigma(t)}B(x - \sigma_1(t))e^{i\Gamma_2(t, x)}. \end{aligned}$$

Lemma 4.4.1. It holds

$$\mathcal{E}_U = F - \vec{m}_1 \cdot \vec{M}_1 - \vec{m}_\varphi \cdot \vec{M}_\varphi,$$

where

$$F = 3|P|^2\varphi + 3|\varphi|^2P + |\varphi|^2\varphi - \omega e^{2(x-\sigma_1)}|R|^2P + \omega(2|R\psi| + |\psi|^2)P,$$

and

$$\begin{aligned} \vec{m}_1 &= \begin{pmatrix} \dot{\gamma}_1 + \dot{\beta}_1\sigma_1 + \beta_1^2 \\ \dot{\sigma}_1 - 2\beta_1 \\ \dot{\beta}_1 \end{pmatrix}, & \vec{M}_1 &= \begin{pmatrix} i\partial_1 P \\ P \\ x_1 P \end{pmatrix}, \\ \vec{m}_\varphi &= \begin{pmatrix} \dot{\gamma}_1 + \dot{\beta}_1\sigma_2 + \beta_1^2 + i\dot{\sigma} \\ \dot{\sigma}_2 - 2\beta_1 \\ \dot{\beta}_1 \end{pmatrix}, & \vec{M}_\varphi &= \begin{pmatrix} i\partial_1 \varphi \\ \varphi \\ x_2 \varphi \end{pmatrix}. \end{aligned}$$

We set

$$a = \frac{1}{2} \langle F, \partial_1 P \rangle.$$

Lemma 4.4.2. It holds

$$a = \alpha \sigma e^{-2\sigma} + O(e^{-2\sigma}) \quad \text{where} \quad \alpha = 32\omega. \quad (4.4.3)$$

Proof. From the expression of F , one has

$$\begin{aligned} \langle F, \partial_1 P \rangle &= 3e^{-\sigma} \int Q^2(x)Q'(x)B(x+\sigma)dx + 3e^{-2\sigma} \int Q(x)Q'(x)B^2(x+\sigma)dx \\ &\quad + e^{-3\sigma} \int Q'(x)B^3(x+\sigma)dx - \omega \int e^{2x}Q(x)Q'(x)Q^2(x+\sigma)dx. \end{aligned}$$

From (4.1.17) and Lemma 4.3.3, the second and third terms in the right-hand side are bounded by $\sigma^3 e^{-4\sigma}$. The last term is bounded by

$$\int e^{2x}Q^2(x)Q^2(x+\sigma)dx = e^{-2\sigma} \int e^{2x}Q^2(x-\sigma)Q^2(x)dx \lesssim e^{-2\sigma} \int Q^2(x-\sigma)dx \lesssim e^{-2\sigma}.$$

For the first term, using $\mathcal{L}_+Q' = 0$ and then (4.1.16), we compute

$$\begin{aligned} 3 \int Q^2(x)Q'(x)B(x+\sigma)dx &= \int Q'(x-\sigma)(-B''(x) + B(x))dx \\ &= \omega \int Q'(x-\sigma) [Q^2(x)B(x) + \kappa e^x Q^2(x)] dx \end{aligned}$$

By Lemma 4.3.3, we have $\int |Q'(x-\sigma)Q^2(x)B(x)|dx \lesssim e^{-\sigma}$.

We only have to compute $\int Q'(x-\sigma)e^x Q^2(x)dx$. First, we see

$$\begin{aligned} \int_{x<0} Q'(x-\sigma)e^x Q^2(x)dx &\lesssim e^{-\sigma} \int_{x<0} e^{4x}dx \lesssim e^{-\sigma}, \\ \int_{x>\sigma} |Q'(x-\sigma)|e^x Q^2(x)dx &\lesssim e^{\sigma} \int_{x>\sigma} e^{-2x}dx \lesssim e^{-\sigma}. \end{aligned}$$

Second, using (4.1.9)

$$Q'(x-\sigma) = \kappa e^{x-\sigma} - e^{2x-2\sigma}Q(x-\sigma), \quad Q^2(x) = \kappa^2 e^{-2x} + O(e^{-3x}Q(x)),$$

and thus

$$\int_0^{\sigma} Q'(x-\sigma)e^x Q^2(x)dx = \kappa^3 \sigma e^{-\sigma} + O(e^{-\sigma}).$$

In conclusion, $a = \omega \frac{\kappa^4}{2} \sigma e^{-2\sigma} + O(e^{-2\sigma}) = 32\omega \sigma e^{-2\sigma}$. □

4.4.2 Formal discussion for $c = 1$

The previous computations leads us to

$$\ddot{\sigma} = -4\alpha \sigma e^{-2\sigma}, \quad 2\dot{\beta} = \dot{\sigma},$$

for which the following function is an *approximate solution*

$$\sigma_0(t) = \log t + \frac{1}{2} \log \log t + \log \Omega, \quad 2\beta_0(t) = \frac{1}{t} \quad \text{where} \quad \Omega = \sqrt{4\alpha} = 8\sqrt{2\omega}.$$

4.4.3 Bootstrap estimates in the case $c = 1$

Fix θ_1 such that $1 < \theta_1 < 2$. The following bootstrap estimates are used in this case: for $1 \ll t \leq T_\infty$,

$$\begin{cases} \|\varepsilon\|_{H^1} + \|\eta\|_{H^1} \leq t^{-\theta_1}, \\ \left| \beta - \frac{1}{2t} \right| \leq t^{-1}(\log t)^{-\frac{1}{4}}, \\ \left| \frac{e^\sigma}{\Omega\sigma^{\frac{1}{2}}t} - 1 \right| \leq (\log t)^{-\frac{1}{2}}, \end{cases}$$

where σ_∞ is to be chosen satisfying

$$\left| \frac{e^{\sigma_\infty}}{\Omega\sigma_\infty^{\frac{1}{2}}T_\infty} - 1 \right| \leq (\log T_\infty)^{-\frac{1}{2}}.$$

We refer to [20, 22] for similar bootstrap estimates.

The rest of the proof is similar to the one of Theorem 4.1.4 and we omit it.

4.5 Discussion

For (coupled NLS), with any coupling coefficient $0 < \omega < 1$, we have proved the existence of symmetric 2-solitary waves (Theorem 4.1.1) and of non-symmetric 2-solitary waves (Theorem 4.1.4) with logarithmic distance. Symmetric 2-solitons with logarithmic distance were already known in the literature for the integrable cases ($\omega = 0$ and $\omega = 1$) and in the scalar case (NLS). In contrast, the existence of non-symmetric 2-solitary waves with logarithmic distance is new. In particular, it does not hold for the integrable case where instead a periodic regime exists.

An interesting remaining open question is whether non-symmetric logarithmic 2-solitary waves exist for the non-integrable scalar (NLS). We conjecture that it is indeed the case, as long as $p \neq 3$. Indeed, the first step of the strategy used in this paper, *i.e.* the computation of an approximate solution involving the main interaction terms, works equally well for (NLS) as for (coupled NLS). We expect a logarithmic regime with oscillations. However, whereas (coupled NLS) enjoys two L^2 conservation laws, the scalar equation (NLS) enjoys only one, which does not seem sufficient for the energy method to apply in a context of two solitons with logarithmic distance without symmetry.

A more technical original aspect of this article is the introduction of a refinement of the energy method. In previous articles using approximate solutions in the context of error terms of order t^{-k} (*e.g.* in [20, 22, 23]), the energy method induces a loss of decay. Here, the additional correction term \mathbf{S} in Section 4.3.3 allows an estimate of the remainder $\begin{pmatrix} \varepsilon \\ \eta \end{pmatrix}$ directly related to the size of the error term $\begin{pmatrix} \varepsilon_U \\ \varepsilon_V \end{pmatrix}$. We believe that this general observation will be useful elsewhere.

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Titre : Construction de dynamiques à fortes interactions d'EDP non linéaires dispersives .

Mots Clefs : Multi-solitons, fortes interactions, gKdV, NLS, système de Schrödinger, comportement asymptotique.

Résumé : Cette thèse est consacrée à l'étude des propriétés dynamiques des solutions de type soliton d'équations aux dérivées partielles (EDP) dispersives non linéaires. À travers des exemples-type de telles équations, l'équation de Schrödinger non-linéaire (NLS), l'équation de Korteweg-de Vries généralisée (gKdV) et le système de Schrödinger, on traite du comportement des solutions convergeant en temps grand vers des sommes de solitons (multi-solitons). Dans un premier temps, nous montrons que dans une configuration symétrique, avec des interactions fortes, le comportement de séparation des solitons logarithmique en temps est universel à la fois dans le cas sous-critique et sur-critique pour (NLS). Ensuite, en adaptant les techniques précédentes à l'équation (gKdV), nous prouvons un résultat similaire de l'existence de multi-solitons avec distance relative logarithmique; pour (gKdV), les solitons sont répulsifs dans le cas sous-critique et attractifs dans le cas sur-critique. Finalement, nous identifions un nouveau régime de distance logarithmique où les solitons sont non-symétriques pour le système de Schrödinger non-intégrable; une telle solution n'existe pas dans le cas intégrable pour le système et pour (NLS).

Title : Construction of dynamics with strongly interacting for non-linear dispersive PDE .

Keys words : Multi-solitons, strongly interacting, gKdV, NLS, Schrödinger system, asymptotic behavior.

Abstract : This thesis deals with long time dynamics of soliton solutions for nonlinear dispersive partial differential equation (PDE). Through typical examples of such equations, the nonlinear Schrödinger equation (NLS), the generalized Korteweg-de Vries equation (gKdV) and the coupled system of Schrödinger, we study the behavior of solutions, when time goes to infinity, towards sums of solitons (multi-solitons). First, we show that in the symmetric setting, with strong interactions, the behavior of logarithmic separation in time between solitons is universal in both subcritical and supercritical case. Next, adapting previous techniques to (gKdV) equation, we prove a similar result of existence of multi-solitons with logarithmic relative distance; for (gKdV), the solitons are repulsive in the subcritical case and attractive in the supercritical case. Finally, we identify a new logarithmic regime where the solitons are non-symmetric for the non-integrable coupled system of Schrödinger; such solution does not exist in the integrable case for the system and for (NLS).

