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A new look at the interfaces in the percolation and Ising models

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THÈSE DE DOCTORAT

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Un nouveau regard sur les interfaces
dans les modèles de percolation et d'Ising

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Résumé

Titre : Un nouveau regard sur les interfaces dans les modèles de percolation et d'Ising

Mots Clefs : Interface, localisation, percolation, FK-percolation, Ising.

Résumé : Les interfaces dans les modèles de percolation et d'Ising jouent un rôle crucial dans la compréhension de ces modèles et sont au cœur de plusieurs problématiques : la construction de Wulff, le mouvement par courbure moyenne, la théorie du SLE. Dans son célèbre article de 1972, Roland Dobrushin a montré que le modèle d'Ising en dimension $d \geq 3$ admet une mesure de Gibbs qui n'est pas invariante par translation à l'aide d'une étude sur l'interface entre le haut et le bas d'une boîte droite de taille finie. Le cas d'une boîte penchée est très différent et plus difficile à analyser. Nous proposons dans cette thèse une nouvelle définition de l'interface. Cette définition est construite dans le modèle de percolation Bernoulli à l'aide d'un couplage dynamique de deux configurations. Nous montrons que cette interface est localisée autour des arêtes pivot à une distance d'ordre de $\ln^2 n$ dans une boîte de taille n . Notre méthode de preuve utilise les chemins espace-temps, qui permettent de contrôler la vitesse de déplacement de l'interface. Nous montrons aussi que la vitesse des arêtes pivot est au plus de l'ordre de $\ln n$. Nous étendons ces résultats au modèle de FK-percolation, nous montrons la localisation de l'interface à distance d'ordre $\ln^2 n$ autour des arêtes pivot. En utilisant une modification du couplage classique d'Edwards-Sokal, nous obtenons des résultats analogues sur la localisation de l'interface dans le modèle d'Ising.

Abstract

Title : A new look at the interfaces in the percolation and Ising models

Keywords : Interface, localisation, percolation, FK-percolation, Ising model

Abstract : The interfaces in the percolation and Ising models play an important role in the understanding of these models and are at the heart of several problematics : the Wulff construction, the mean curvature motion and the SLE theory. In his famous 1972 paper, Roland Dobrushin showed that the Ising model in dimensions $d \geq 3$ has a Gibbs measure which is not invariant by translation by studying the interface between the top and the bottom of a straight finite box. The case of a tilted box is very different and more difficult to analyse. In this thesis, we propose a new definition of the interface. This definition is constructed in the Bernoulli percolation model with the help of a dynamical coupling between two configurations. We show that this interface is localised around the pivotal edges within a distance of order $\ln^2 n$ inside a box of size n . The proof relies on space-time paths which allow us to control the speed of the interface. We also show that the speed of the pivotal edges is at most of order $\ln n$. We extend these results to the FK-percolation model, we show the localisation of the interface at distance of order $\ln^2 n$ around the pivotal edges. Using a modification of the classical Edwards-Sokal coupling, we obtain analogous results on the localisation of the interface in the Ising model.



Harry Kesten, Rudolf Peierls et Roland Dobrushin, à Oxford, 1993

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Première partie

Présentation des résultats

Chapitre 1

Introduction générale

Cette thèse est consacrée à l'étude des interfaces dans des modèles de physique statistique, en particulier le modèle de percolation et le modèle d'Ising. Nous nous intéressons à la définition des interfaces et aux propriétés géométriques des interfaces. Nous nous concentrons tout d'abord sur le modèle de percolation. Nous proposons une définition des interfaces à l'aide d'un couplage entre deux processus de percolations dynamiques et nous étudions la structure de l'interface. Ensuite, nous adaptons la définition de l'interface dans le modèle de FK-percolation et le modèle d'Ising.

1.1 Les modèles de physique statistique

Nous commençons par présenter les modèles de physique statistique dans lesquels nous avons étudié les interfaces. Nous allons d'abord définir le cadre géométrique dans lequel nous allons travailler.

1.1.1 Les objets géométriques

Le réseau \mathbb{L}^d . Soit x, y deux points de \mathbb{Z}^d , nous disons que x, y sont voisins s'ils sont à distance 1 en norme euclidienne. L'ensemble \mathbb{E}^d est l'ensemble des paires $\langle x, y \rangle$, où x, y sont deux points voisins de \mathbb{Z}^d . Le réseau \mathbb{L}^d est le graphe dont \mathbb{Z}^d est l'ensemble des sommets et \mathbb{E}^d est l'ensemble des arêtes. Soit A un sous-ensemble de \mathbb{R}^d , nous disons que l'arête $e = \langle x, y \rangle$ est incluse dans A si le segment ouvert $]x, y[$ est inclus dans A .

Les bords d'un ensemble. Soit A un sous-ensemble de \mathbb{R}^d , nous appelons le bord extérieur de A , noté par ∂A , l'ensemble de sommets défini comme suit :

$$\partial A = \{ x \in \mathbb{Z}^d : \exists y \in \mathbb{Z}^d \cap A, \langle x, y \rangle \in \mathbb{E}^d \}.$$

Nous définissons aussi le bord intérieur, noté par $\partial_{in}A$, comme l'ensemble

$$\partial_{in}A = \{x \in \mathbb{Z}^d \cap A : \exists y \in \mathbb{Z}^d \setminus A, \langle x, y \rangle \in \mathbb{E}^d\}.$$

Les chemins. Soient x et y deux sommets dans \mathbb{Z}^d , un chemin entre x et y est une suite $x_0, e_0, x_1, e_1, \dots, e_n, x_{n+1}$ de sommets x_i et d'arêtes e_i distincts où $x_0 = x$ et $x_{n+1} = y$ et e_i est l'arête joignant x_i à x_{i+1} . Pour simplifier les notations, nous notons le chemin $x_0, e_0, x_1, e_1, \dots, e_n, x_{n+1}$ uniquement par sa suite d'arêtes

$$(e_0, e_1, \dots, e_n).$$

Les ensembles séparants. Soient A, B deux sous-ensembles de \mathbb{Z}^d . Nous disons qu'un ensemble d'arêtes $S \subset \mathbb{E}^d$ sépare A et B si aucune partie connexe du graphe $(\mathbb{Z}^d, \mathbb{E}^d \setminus S)$ n'intersecte simultanément A et B . Un tel ensemble est appelé un ensemble séparant pour A et B . Nous disons que S est un ensemble séparant minimal pour A, B si aucun sous-ensemble strict de S ne sépare A et B .

1.1.2 Le modèle de percolation Bernoulli

Le modèle mathématique de percolation a été introduit par John Hammersley en 1957 et ce modèle a été l'origine de plusieurs problèmes qui fascinent de nombreux mathématiciens : des problèmes qui peuvent être énoncés avec peu de prérequis mais dont les solutions sont difficiles et demandent des nouvelles idées. Commençons par les définitions de base du modèle de percolation par arête.

Les configurations. L'espace de configurations est $\Omega = \{0, 1\}^{\mathbb{E}^d}$. Une configuration est une fonction $\omega : \mathbb{E}^d \rightarrow \Omega$. Pour une arête $e \in \mathbb{E}^d$, nous disons que e est ouverte si $\omega(e) = 1$ et fermée si $\omega(e) = 0$. Soient A un sous-ensemble de \mathbb{Z}^d et ω une configuration, la configuration ω restreinte à A , notée $\omega|_A$, est la restriction de ω aux arêtes dont les deux extrémités sont incluses dans A . Soient $e \in \mathbb{E}^d$ une arête et $\omega \in \Omega$ une configuration, nous définissons les configurations ω^e, ω_e par :

$$\forall f \in \mathbb{E}^d \quad \omega^e(f) = \begin{cases} \omega(f) & f \neq e \\ 1 & f = e \end{cases}, \quad \omega_e(f) = \begin{cases} \omega(f) & f \neq e \\ 0 & f = e \end{cases}.$$

Les configurations ω^e, ω_e sont obtenues à partir de ω en ouvrant ou fermant l'arête e . Il existe un ordre partiel naturel dans l'ensemble Ω . Pour deux configurations $\omega_1, \omega_2 \in \Omega$, nous disons que ω_1 domine ω_2 , ce que nous notons par $\omega_1 \geq \omega_2$, si

$$\forall e \in \mathbb{E}^d \quad \omega_1(e) \geq \omega_2(e).$$

La probabilité de percolation Bernoulli. Soit un réel $p \in [0, 1]$. Sur l'espace Ω , nous considérons la tribu cylindrique \mathcal{F} . Nous considérons la probabilité produit

$$P_p = (p\delta_1 + (1-p)\delta_0)^{\otimes \mathbb{E}^d}.$$

Intuitivement, nous obtenons une configuration en fermant indépendamment chaque arête de \mathbb{E}^d avec une probabilité $1-p$. Une autre façon de construire la probabilité est de considérer une famille de variables i.i.d. $(X_e)_{e \in \mathbb{E}^d}$, de loi uniforme sur l'intervalle $[0, 1]$. Nous posons

$$\omega(e) = \begin{cases} 1 & \text{si } X_e \leq p \\ 0 & \text{si } X_e > p \end{cases}.$$

Les clusters. Considérons le graphe aléatoire, formé des sommets de \mathbb{Z}^d et des arêtes ouvertes de \mathbb{E}^d . Une composante connexe de ce graphe est appelée un cluster ouvert. Pour un sommet x , nous notons $C(x)$ le cluster ouvert qui contient x . Les sommets de $C(x)$ sont les sommets connectés à x par un chemin ouvert et les arêtes de $C(x)$ sont les arêtes ouvertes joignant deux sommets de $C(x)$. Nous notons $C(x) = \{x\}$ si toutes les arêtes qui ont une extrémité x sont fermées. Dans notre étude, nous considérons $C(x)$ plutôt comme l'ensemble des arêtes ouvertes connectées à x au lieu du sous graphe contenant x .

Pour deux ensembles de sommets A, B de \mathbb{Z}^d . Nous écrivons $A \longleftrightarrow B$ s'il existe un chemin ouvert qui relie un sommet de A à un sommet de B .

1.1.3 Le modèle de FK-percolation

Aussi connu sous le nom de *random cluster model*, le modèle de FK-percolation a été inventé par Cees Fortuin et Piet Kasteleyn vers 1969 dans le but d'unifier les modèles de percolation, d'Ising et de Potts. L'importance du modèle pour les probabilités et la mécanique statistique n'a été réalisée qu'à la fin des années 80 et depuis, plusieurs résultats dans le modèle d'Ising et de Potts ont été démontrés à l'aide de la FK-percolation. Par exemple, l'existence d'une mesure de Gibbs qui n'est pas invariante par translation [Dob72] et la construction de Wulff pour les dimensions deux et supérieures [CP00].

Les mesures de probabilité de FK-percolation. Soit $G = (V, E)$ un graphe fini. L'espace de configuration est $\Omega = \{0, 1\}^E$. Ce modèle est différent du modèle de percolation à cause de la présence de corrélations entre les arêtes. Plus précisément, pour une configuration $\omega \in \Omega$, nous notons $k(\omega)$ le nombre de clusters ouverts dans cette configuration. La probabilité de FK-percolation $\Phi_{p,q}$ sur le graphe G est définie avec deux paramètres

$p \in [0, 1]$ et $q \in]0, \infty[$ comme suit :

$$\Phi_{p,q}[\omega] = \frac{1}{Z_{FK}} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega)}, \quad \omega \in \{0, 1\}^E,$$

où la constante Z_{FK} , que nous appelons la fonction de partition, est égale à

$$\sum_{\omega \in \{0,1\}^E} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega)}.$$

Notons que dans le cas où le paramètre $q = 1$, la probabilité $\Phi_{p,1}$ est exactement celle de la percolation Bernoulli où les états de de chaque arête sont indépendants. Pour $q < 1$, les configurations avec peu de clusters sont favorisées, et pour $q > 1$, les configurations avec beaucoup de clusters sont favorisées. En particulier, les cas avec une valeur de $q \in \{2, 3, \dots\}$ sont les plus intéressants car ils peuvent être reliés au modèle d'Ising et de Potts. Dans notre étude, nous allons nous concentrer sur les cas $q = 1$ et $q = 2$ mais les méthodes utilisées pour étudier le cas $q = 2$ sont valables pour les cas généraux $q > 1$. Notons que nous pouvons définir la probabilité de FK-percolation sur le réseau \mathbb{L}^d tout entier comme la limite faible de la suite des probabilités définies dans les boîtes finies $\Lambda_n = [-n, n]^d$. Comme nous étudions principalement les graphes finis dans la thèse, nous ne présentons pas les problèmes concernant les probabilités de volume infini. Nous faisons référence aux chapitres correspondant de [Gri06] pour les détails.

Les conditions aux bords. Une question importante dans les modèles de mécanique statistique est de comprendre comment une condition à la frontière d'une région influence ce qui se passe à l'intérieur. Pour formaliser cette question, nous introduisons ce que nous appelons les conditions aux bords. Soit $G = (V, E)$ un sous-graphe fini du réseau \mathbb{L}^d . Nous considérons une configuration $\xi \in \{0, 1\}^{\mathbb{E}^d}$ et nous notons Ω_G^ξ l'ensemble (fini) des configurations ω telles que

$$\forall e \in \mathbb{E}^d \setminus E \quad \omega(e) = \xi(e).$$

Pour $\xi \in \{0, 1\}^{\mathbb{E}^d}$, $p \in [0, 1]$ et $q \in]0, \infty[$, nous notons $\Phi_{G,p,q}^\xi$ la probabilité de FK-percolation sur le graphe G avec les conditions aux bords ξ , définie par

$$\Phi_{G,p,q}^\xi[\omega] = \begin{cases} \frac{1}{Z^\xi} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega,G)} & \text{si } \omega \in \Omega_G^\xi, \\ 0 & \text{sinon,} \end{cases}$$

où $k(\omega, G)$ est le nombre de clusters ouverts de ω qui intersectent V et Z^ξ est la constante de normalisation telle que $\Phi_{G,p,q}^\xi[\Omega_G^\xi] = 1$. Les conditions

aux bords influencent la probabilité à travers le nombre de clusters ouverts $k(\omega, G)$. Soit $x, y \in \partial V$ et supposons qu'il existe un chemin d'arêtes dans $\mathbb{E}^d \setminus E$ ouvert dans ξ qui relie x et y . Alors, les clusters ouverts de ω contenant x ou y vont contribuer seulement 1 dans le compte de $k(\omega, G)$.

Nous allons considérer en particulier trois conditions aux bords dans cette thèse.

- La 0-condition correspond à la condition où toutes les arêtes sont fermées dans ξ . Cette condition est aussi appelée la condition *free*.
- La 1-condition correspond à la condition où toutes les arêtes sont ouvertes. Nous pouvons aussi obtenir cette condition en ajoutant un sommet fictif et relier ce sommet avec toutes les sommets de $\partial_{in} V$. Pour cette raison, cette condition est aussi appelée la condition *wired*.
- La *TB*-condition qui correspond à la condition aux bords de Dobrushin pour le modèle d'Ising et qui fut introduite dans [Dob72]. Nous allons détailler cette condition aux bords dans la suite de notre étude.

La propriété de Markov spatiale. Une des propriétés importantes de la FK-percolation concernant les conditions aux bords est la propriété de Markov spatiale. Nous notons \mathcal{F} la tribu engendrée par les configurations des arêtes de E . Pour un sous-graphe fini Λ , nous notons \mathcal{T}_Λ la tribu engendrée par les configurations des arêtes de $\mathbb{E}^d \setminus E$. Nous avons la proposition suivante :

Proposition 1.1.1 (Lemme 4.13,[Gri06]). *Soient $p \in [0, 1]$ et $q \in]0, \infty[$. Pour tout Λ sous-graphe de G , toute configuration $\xi \in \Omega$ et tout événement $A \in \mathcal{F}$, nous avons*

$$\Phi_{G,p,q}^\xi[A \mid \mathcal{T}_\Lambda] = \Phi_{\Lambda,p,q}^\omega[A],$$

où $\omega \in \Omega_\Lambda^\xi$ est la condition aux bords induite par ξ .

1.1.4 Le modèle d'Ising

Le modèle d'Ising fut introduit dans [Isi25] pour étudier la fameuse expérience de Pierre Curie. Considérons un bloc de fer plongé dans un champ magnétique. L'intensité du champ varie de zéro jusqu'à un certain maximum et puis elle redescend à zéro. Si la température est suffisamment basse, le bloc de fer reste magnétisé, mais dans le cas contraire, il ne l'est pas. Pour donner une image simplifiée de cette expérience, nous supposons que les particules sont sur les sommets du réseau \mathbb{L}^d et que chaque particule possède un spin qui peut être soit dirigé vers le haut soit vers le bas. Les spins sont choisis d'une manière aléatoire selon une loi que nous appelons la mesure de Gibbs. Nous présentons cette mesure dans la suite.

La mesure de Gibbs. Soient $G = (V, E)$ un graphe fini dans le réseau \mathbb{L}^d et $\Sigma = \{-1, +1\}^V$ l'espace des configurations. Nous considérons trois paramètres $\beta, J \in [0, \infty[$ et $h \in \mathbb{R}$. La mesure de probabilité $\pi_{\beta, J, h}$ sur Σ est définie par

$$\forall \sigma \in \Sigma, \quad \pi_{\beta, J, h}(\sigma) = \frac{1}{Z_I} e^{-\beta H(\sigma)},$$

où la fonction de partition Z_I et l'hamiltonien $H : \Sigma \rightarrow \mathbb{R}$ sont définis par

$$H(\sigma) = -J \sum_{e=\langle x, y \rangle \in E} \sigma_x \sigma_y - h \sum_{x \in V} \sigma_x,$$

et

$$Z_I = \sum_{\sigma \in \Sigma} e^{-\beta H(\sigma)}.$$

En physique, le paramètre β est interprété comme l'inverse de la température T , le paramètre J modélise la force d'interaction entre les plus proches voisins et le paramètre h l'intensité du champ magnétique extérieur. Dans notre étude, nous considérons uniquement le cas où il n'y a pas de champ magnétique extérieur, i.e., $h = 0$. Chaque arête a la même force d'interaction J dans la précédente définition et comme J n'intervient que dans le produit βJ , nous pouvons supposer que $J = 1$ et nous écrivons $\pi_{\beta, J, h} = \pi_\beta$. Mentionnons une généralisation du modèle d'Ising, le modèle de Potts. Au lieu d'avoir un spin à deux valeurs sur chaque sommet, le spin peut prendre ses valeurs dans l'ensemble $\{1, \dots, q\}$ avec $q \in \mathbb{N}$. Les résultats obtenus dans cette thèse sur le modèle d'Ising s'adaptent au modèle de Potts car notre méthode repose essentiellement sur le couplage d'Edwards et Sokal introduit dans [ES88].

Le couplage FK-Ising. Nous construisons un espace de probabilité qui contient à la fois le modèle d'Ising et de FK-percolation. Soient $G = (V, E)$ un graphe fini, $p \in [0, 1]$ et $q = 2$. Considérons l'espace des configurations $\Sigma \times \Omega$ où $\Sigma = \{-1, +1\}^V$ et $\Omega = \{0, 1\}^E$. Pour une arête $e = \langle x, y \rangle$, nous notons $\delta_e(\sigma) = \delta_{\sigma_x, \sigma_y}$ où δ est le symbole de Kronecker et nous définissons la probabilité

$$\forall (\sigma, \omega) \in \Sigma \times \Omega, \quad \mu(\sigma, \omega) = \frac{1}{Z} \prod_{e \in E} \{(1-p)\delta_{\omega(e), 0} + p\delta_{\omega(e), 1}\delta_e(\sigma)\},$$

où Z est la constante de normalisation telle que

$$\sum_{(\sigma, \omega) \in \Sigma \times \Omega} \mu(\sigma, \omega) = 1.$$

Remarquons que la probabilité μ peut être factorisée sous la forme

$$\mu(\sigma, \omega) \propto \psi(\sigma)\phi_p(\omega)\mathbb{1}_F(\sigma, \omega),$$

où ψ est la probabilité uniforme sur Σ et $\mathbb{1}_F$ est la fonction indicatrice de l'événement

$$F = \{ \forall e \text{ telle que } \omega(e) = 1, \delta_e(\sigma) = 1 \text{ pour tout } e \text{ telle que } \omega(e) = 1 \}.$$

Nous pouvons voir μ comme le produit de ψ et ϕ conditionné par F . Les deux théorèmes suivants décrivent les marginales de μ .

Théorème 1.1.2 (Les marginales de μ , [ES88]). *Soient $p \in [0, 1]$ et $p = 1 - e^{-\beta}$. Nous avons*

- *Marginale sur Σ . Soit $\mu_1(\sigma) = \sum_{\omega \in \Omega} \mu(\sigma, \omega)$. La probabilité μ_1 sur Σ est la mesure de Gibbs*

$$\forall \sigma \in \Sigma, \quad \mu_1(\sigma) = \frac{1}{Z_I} e^{\beta \sum_{e \in E} \delta_e(\sigma)}.$$

- *Marginale sur Ω . Soit $\mu_2(\omega) = \sum_{\sigma \in \Sigma} \mu(\sigma, \omega)$. La probabilité μ_2 sur Ω est la mesure de FK-percolation*

$$\forall \omega \in \Omega, \quad \mu_2(\omega) = \frac{1}{Z_{FK}} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} 2^{k(\omega)}.$$

Théorème 1.1.3 (Les lois conditionnelles de μ , [ES88]). *Soient $p \in [0, 1]$ et $p = 1 - e^{-\beta}$. Nous avons*

- *Pour $\omega \in \Omega$, la loi conditionnelle $\mu(\cdot | \omega)$ sur Σ est obtenue en choisissant les spins aléatoirement sur les clusters de ω . Les spins sont identiques sur chaque cluster et indépendants entre les différents clusters. Chaque spin est distribué selon la loi uniforme sur $\{-1, +1\}$.*
- *Pour $\sigma \in \Sigma$, la loi conditionnelle $\mu(\cdot | \sigma)$ est obtenue en fermant toutes les arêtes $e = \langle x, y \rangle$ telles que $\sigma(x) \neq \sigma(y)$; si $\sigma(x) = \sigma(y)$, $\omega(e)$ est donnée par*

$$\omega(e) = \begin{cases} 1 & \text{avec probabilité } p \\ 0 & \text{sinon} \end{cases}.$$

Les variables $(\omega(e))_{e \in E}$ sont indépendantes.

Nous allons adapter ce couplage pour relier l'interface dans le modèle de FK-percolation et l'interface dans le modèle d'Ising.

1.2 Les dynamiques dans les modèles

Une des stratégies principales de notre étude ressemble à une méthode de Monte-Carlo par chaînes de Markov. La méthode MCMC consiste à utiliser une chaîne de Markov qui permet d'approcher la loi d'équilibre. Nous construisons des chaînes de Markov sur l'espace des configurations dont la loi stationnaire décrit exactement les interfaces et nous obtenons de l'information sur les interfaces en étudiant le comportement des chaînes. Nous introduisons maintenant les dynamiques classiques que nous allons adapter dans la suite.

1.2.1 La percolation dynamique

L'étude de la percolation dynamique a été initiée par Häggström, Peres et Steif dans [HPS97] en 1995. Benjamini a proposé indépendamment ce modèle. Dans ce modèle, le paramètre p est fixé et la configuration d'une arête est déterminée indépendamment des autres par un processus de sauts à deux états. Plus formellement, nous considérons le réseau \mathbb{L}^d et une famille indépendante de processus de Poisson $(N_e(t))_{e \in \mathbb{E}^d}$ d'intensité 1. Nous notons les instants de sauts par

$$T_n^e = \inf\{t \geq 0, N_e(t) \geq n\}.$$

L'état d'une arête à l'instant t est noté par $\omega(e, t)$. A chaque instant T_n^e , la configuration $\omega(e, T_n^e)$ est tirée indépendamment selon une variable de loi Bernoulli de paramètre p , i.e.,

$$\forall n \geq 0, \quad \omega(e, T_n^e) = \begin{cases} 1 & \text{avec probabilité } p \\ 0 & \text{avec probabilité } 1 - p \end{cases}.$$

L'arête e ne change pas d'état en dehors des instants T_n^e : si $T_n^e \leq t < T_{n+1}^e$, alors $\omega(e, t) = \omega(e, T_n^e)$.

La percolation dynamique sur un graphe fini. Dans notre étude, nous nous concentrons sur le cas plus simple où le processus est défini sur un graphe fini $G = (V, E)$ et la percolation dynamique devient une chaîne de Markov à espace d'états fini. Nous pouvons définir le processus à l'aide d'une configuration graphique : il s'agit d'une suite de triplets $(X_t, E_t, B_t)_{t \in \mathbb{N}^*}$, où $(X_t)_{t \in \mathbb{N}^*}$ est un processus à valeurs dans $\{0, 1\}^E$, $(E_t)_{t \in \mathbb{N}^*}$ est une suite d'arêtes dans l'ensemble E et $(B_t)_{t \in \mathbb{N}^*}$ est une suite de variables aléatoires à valeurs dans $\{0, 1\}$. Décrivons maintenant la dynamique du processus. La suite $(E_t)_{t \in \mathbb{N}^*}$ est une suite indépendante d'arêtes aléatoires et chaque E_t suit la loi uniforme sur E . La suite $(B_t)_{t \in \mathbb{N}^*}$ est une suite indépendante de variables de Bernoulli de paramètre p . Le processus $(X_t)_{t \in \mathbb{N}}$ est construit par récurrence sur t , comme suit. Nous partons d'une configuration initiale

X_0 . A l'instant $t \geq 1$, nous changeons l'état de l'arête E_t en B_t , i.e., nous définissons,

$$X_t(e) = \begin{cases} X_{t-1}(e) & \text{si } e \neq E_t \\ B_t & \text{si } e = E_t \end{cases} .$$

La chaîne de Markov $(X_t)_{t \geq 0}$ est irréductible apériodique et d'espace d'états fini donc elle admet une unique probabilité invariante qui est simplement la probabilité de la percolation Bernoulli de paramètre p .

1.2.2 Les dynamiques de FK-percolation

Il existe plusieurs dynamiques pour le modèle de FK-percolation. Nous allons étudier en particulier un type de dynamiques sur les graphes finis, les dynamiques de Glauber. Dans ces dynamiques, seulement une arête change son état à un instant et la loi d'équilibre des dynamiques est la mesure de FK-percolation. Nous étudions une certaine dynamique de Glauber appelée le *Gibbs sampler* qui un processus à temps discret.

Le Gibbs sampler. Soient $p \in [0, 1]$, $q > 0$ et $G = (V, E)$ un graphe fini. Le Gibbs sampler est une chaîne de Markov $(X_t)_{t \in \mathbb{N}}$ sur $\{0, 1\}^E$ définie comme suit. Soient $(E_t)_{t \in \mathbb{N}}$ une suite i.i.d. d'arêtes de loi uniforme sur E et $(U_t)_{t \in \mathbb{N}}$ une suite i.i.d. de variables de loi uniforme sur $[0, 1]$. Les suites $(E_t)_{t \in \mathbb{N}}$ et $(U_t)_{t \in \mathbb{N}}$ sont indépendantes. Nous construisons la suite $(X_t)_{t \in \mathbb{N}}$ par récurrence. Soit $X_0 \in \{0, 1\}^E$ une configuration initiale. Supposons que, à l'instant $t - 1$, nous avons $X_{t-1} = \omega$. Nous posons alors

$$X_t(e) = \begin{cases} \omega(e) & \text{si } E_t \neq e \\ 1 & \text{si } E_t = e \text{ et } U_t \geq \frac{\Phi(\omega_e)}{\Phi(\omega^e) + \Phi(\omega_e)} \\ 0 & \text{si } E_t = e \text{ et } U_t < \frac{\Phi(\omega_e)}{\Phi(\omega^e) + \Phi(\omega_e)} \end{cases} ,$$

où Φ est la probabilité de FK-percolation sur G de paramètre p et q . Cette définition de dynamiques a deux avantages. Le premier c'est que cette construction permet d'étudier le comportement du processus via seulement deux suites de variables très simples. Le deuxième c'est qu'elle permet de coupler des dynamiques avec des paramètres p, q différents ou des conditions aux bords différents. Nous allons exploiter ces deux points dans notre étude sur la FK-percolation.

1.2.3 Les dynamiques du modèle d'Ising

Nous allons présenter une dynamique pour le modèle d'Ising que nous espérons pouvoir mieux comprendre avec notre méthode. Ce processus appelé la dynamique non conservative de Glauber, noté par $(\sigma_t)_{t \in \mathbb{N}}$, est défini pour un

graphe fini $G = (V, E)$. Soient $\Sigma = \{-1, +1\}^V$ et $\beta > 0$. Pour $x \in V$ et $\sigma \in \Sigma$, nous posons

$$S(\sigma, x) = \sum_{y: y \sim x} \sigma(y)$$

la somme des spins des voisins de x . À l'instant t , supposons que σ_t est connue, nous construisons la configuration σ_{t+1} comme suit. Nous choisissons d'abord un site $x \in V$ avec la loi uniforme sur V . Nous calculons

$$\Delta(x) = 2\sigma_t(x)S(\sigma_t(x), x).$$

Nous déterminons le spin $\sigma_{t+1}(x)$ selon le signe de $\Delta(x)$:

- Si $\Delta(x) < 0$, nous posons $\sigma_{t+1}(x) = -\sigma_t(x)$.
- Si $\Delta(x) \geq 0$, nous changeons le spin en x avec probabilité $e^{-\beta\Delta(x)}$.

Les spins sur les sommets différents de x restent identiques à l'instant t et $t + 1$.

Notre résultat sur l'interface dans le modèle d'Ising n'est pas obtenu via la dynamique de Glauber mais à l'aide d'un couplage avec la FK-percolation. Cependant les objets que nous avons étudiés peuvent être définis dans le contexte de la dynamique de Glauber. Comprendre cette dynamique fait partie de nos futurs projets.

1.3 L'interface classique

Dans cette section, nous allons présenter la définition de l'interface au sens classique et quelques résultats connus sur l'interface. Une interface est induite directement par la condition aux bords de Dobrushin. Initialement, Dobrushin a introduit cette définition dans [Dob72] pour montrer l'existence d'une mesure de Gibbs qui n'est pas invariante par translation dans le modèle d'Ising à basse température en dimension $d \geq 3$.

1.3.1 La définition de l'interface

Nous présentons la définition classique de l'interface, aussi appelée l'interface de Dobrushin. Nous considérons une boîte Λ_L de côté $2L$ dans \mathbb{R}^d , i.e., $\Lambda_L = [-L, L]^d$ avec les conditions aux bords de Dobrushin que nous définissons comme suit.

Les conditions aux bords de Dobrushin. Cette condition aux bords a été construite dans [Dob72] dans le contexte du modèle d'Ising mais elle peut aussi être construite dans le modèle de FK-percolation. Nous séparons le bord $\partial\Lambda$ en deux parties :

$$\begin{aligned} \partial^+ \Lambda_L &= \{x \in \partial\Lambda : x_d > 0\}, \\ \partial^- \Lambda_L &= \{x \in \partial\Lambda : x_d \leq 0\}. \end{aligned}$$

Dans le modèle d'Ising, les sommets de $\partial^+ \Lambda_L$ sont munis d'un spin $+$ et les sommets de $\partial^- \Lambda_L$ sont munis d'un spin $-$.

Pour la FK-percolation, nous ajoutons deux sommets fictifs f^+ et f^- à l'extérieur de la boîte Λ_L et nous obtenons la condition aux bords voulue en reliant tous les sommets de $\partial^+ \Lambda_L$ à f^+ et tous les sommets de $\partial^- \Lambda_L$ à f^- et en fermant les arêtes de l'ensemble

$$\mathcal{D}_L = \{ \langle x, y \rangle \in \mathbb{E}^d : x \in \partial^+ \Lambda_L, y \in \partial^- \Lambda_L \}.$$

Les plaquettes. Pour $x \in \mathbb{Z}^d$, nous notons \mathcal{K}_x le cube unité centré en x :

$$\mathcal{K}_x = x + \left[-\frac{1}{2}, \frac{1}{2} \right]^d.$$

Soit $\langle x, y \rangle \in \mathbb{E}^d$, la plaquette entre x et y , notée par $\mathcal{Q}_{x,y}$ est la face commune entre \mathcal{K}_x et \mathcal{K}_y (illustré en figure 1.1), i.e.,

$$\mathcal{Q}_{x,y} = \mathcal{K}_x \cap \mathcal{K}_y.$$

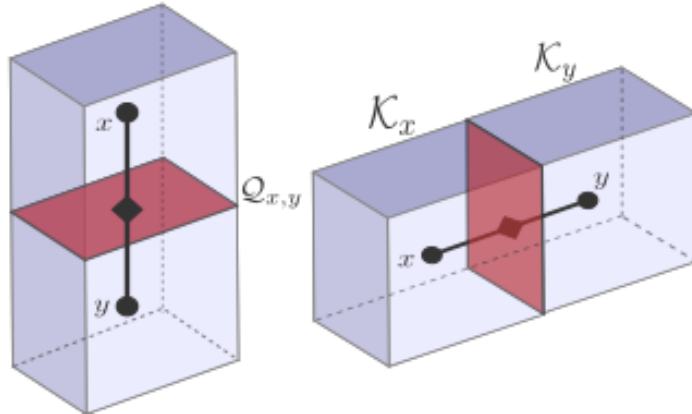


FIGURE 1.1 – La plaquette $\mathcal{Q}_{x,y}$ (en rouge) est l'intersection entre les deux cubes \mathcal{K}_x et \mathcal{K}_y .

Notons que, en dimension $d = 2$, la plaquette d'une arête est exactement l'arête duale. Quelques propriétés géométriques des plaquettes sont données dans [Gri06, Chapitre 7].

L'interface dans le modèle d'Ising. Soit σ une configuration avec la condition aux bords de Dobrushin, nous définissons

$$\mathcal{B}(\sigma) = \bigcup_{\substack{\langle x, y \rangle \in \mathbb{E}^d \cap \Lambda \\ \sigma(x) \neq \sigma(y)}} \mathcal{Q}_{x,y}.$$

Dans l'ensemble $\mathcal{B}(\sigma)$, il existe une unique partie connexe I maximale au sens de l'inclusion qui contient l'ensemble

$$\bigcup_{\substack{\langle x, y \rangle \in \mathbb{E}^d \\ x \in \partial^+ \Lambda, y \in \partial^- \Lambda}} \mathcal{Q}_{x,y}$$

que nous appelons l'interface dans la configuration ω . Remarquons que cette définition de l'interface n'est intéressante que dans les cas où la température est basse. En effet, dans le cas où T est proche de T_c , avec une grande probabilité, cet ensemble va remplir toute la boîte.

L'interface dans la FK-percolation. L'interface dans la FK-percolation est définie pour une boîte rectangulaire, i.e.,

$$\Lambda_{L,M} = [-L, L]^{d-1} \times [-M, M].$$

De la même façon, nous pouvons définir la condition aux bords de Dobrushin pour une telle boîte. Nous considérons uniquement les configurations de l'événement

$$\{ \partial^+ \Lambda_{L,M} \leftrightarrow \partial^- \Lambda_{L,M} \}.$$

Il existe un ensemble d'arêtes fermées $*$ -connecté maximal au sens d'inclusion qui contient l'ensemble $\mathcal{D}_{L,M}$. Nous l'appelons l'interface d'une telle configuration. Notons que cette définition reste valable pour le cas où $q = 1$ qui correspond au modèle de percolation. De plus, nous remarquons que l'interface ainsi définie n'est intéressante que pour le cas p proche de 1. Pour p proche de p_c , l'interface remplit toute la boîte avec une grande probabilité.

1.3.2 La localisation des interfaces en dimensions $d \geq 3$

Les résultats importants en dimensions $d \geq 3$ sur la géométrie des interfaces dans ces modèles concernent la rigidité de l'interface.

La rigidité dans le modèle d'Ising. Dans la boîte $\Lambda(L) = [-L, L]^d$ munie de la condition aux bords de Dobrushin, l'interface coïncide avec le plan $\{x_d = 1/2\}$ à part des différences locales. Le travail original de Dobrushin [Dob72] concerne l'interface dans le modèle d'Ising. Il utilise des techniques de *Clusters Expansion* et il décompose les plaquettes de l'interface en deux ensembles qu'il appelle les plafonds et les murs. Plus formellement, soit Q une plaquette horizontale, i.e.,

$$Q = \mathcal{Q}_{(x_1, \dots, x_d), (x_1, \dots, x_d+1)},$$

sa projection horizontale $p(Q)$ est la plaquette

$$p(Q) = \mathcal{Q}_{(x_1, \dots, 0), (x_1, \dots, 1)}.$$

Une plaquette Q est dans l'ensemble des plafonds si elle est l'unique plaquette qui admet la projection $p(Q)$ et les autres plaquettes sont les murs.

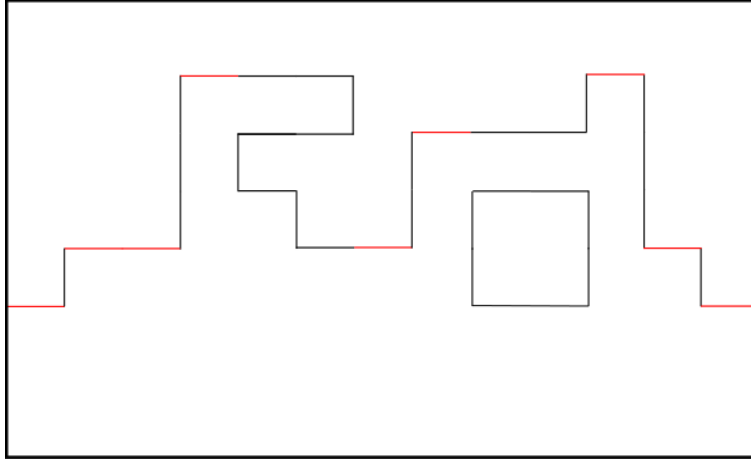


FIGURE 1.2 – Les plaquettes de plafond sont en rouge et les autres plaquettes sont les murs.

Pour un ensemble de murs B , il considère le nombre

$$\pi(B) = \text{card}(B) - \text{card}(p(B)),$$

où $p(B)$ est l'ensemble de la projection des plaquettes de B . Dobrushin a montré que la probabilité d'avoir une interface qui contient un ensemble de murs B décroît exponentiellement vite en fonction du nombre $\pi(B)$. L'énoncé précis peut se trouver dans le lemme 8 de [Dob72]. Pour donner une idée des résultats, nous donnons ici un théorème plus faible mais qui nécessite moins de notations.

Théorème 1.3.1 (Théorème 3.60,[FV17]). *Supposons que $d \geq 3$. Il existe une constante $c'(\beta) > 0$ qui tend vers 0 quand β tend vers l'infini telle que pour tout $n \in \mathbb{N}$ et $i \in \{j \in \Lambda(n) : j_d = 0\}$, nous avons*

$$\pi_{\Lambda(n),\beta}^{Dob}(I \supset \mathcal{Q}_{i,\bar{i}}) \geq 1 - c'(\beta),$$

où $\pi_{\Lambda(n),\beta}^{Dob}$ est la mesure de Gibbs dans la boîte $\Lambda(n)$ avec la condition aux bords de Dobrushin et le sommet \bar{i} est $(i_1, \dots, i_d + 1)$.

La rigidité dans la FK-percolation. Le théorème précédent donne une description locale de l'interface et la preuve n'utilise pas les techniques de clusters expansion. En utilisant les méthodes similaires à celles de Dobrushin, Gielis et Grimmett montrent la rigidité de l'interface dans le modèle de

percolation et de FK-percolation dans [GG02]. Par contre, la probabilité utilisée pour étudier l'interface est conditionnée par l'événement

$$D_{L,M} = \{ \partial^+ \Lambda_{L,M} \leftrightarrow \partial^- \Lambda_{L,M} \}$$

qui arrive avec une probabilité de l'ordre $e^{-cL^{d-1}}$ (voir [CP00] et [DP96]). Dans [GG02], ils considèrent la boîte $\Lambda_{L,M}$ en dimension trois et ils montrent que la probabilité $\bar{\Phi}_L$ définie comme la limite

$$\lim_{M \rightarrow \infty} \Phi_{\Lambda_{L,M}}^{Dob} [\cdot \mid D_{L,M}]$$

existe et que sous cette probabilité, comme dans le théorème 1.3.1, l'interface est localisée au plan $\{x_3 = 1/2\}$, voir [GG02, Théorème 2]. En plus, la hauteur $h(x_1, x_2)$ d'une interface I définie comme

$$h(x_1, x_2) = \sup \left\{ \left| d - \frac{1}{2} \right| : (x_1, x_2, d) \in I \right\}$$

décroit exponentiellement. Plus précisément, nous avons le théorème suivant :

Théorème 1.3.2 (Théorème 3, [GG02]). *Soit $q \geq 1$. Il existe $\hat{p} < 1$ et $\alpha(p) > 0$ tels que pour tout $p \geq \hat{p}$ et $x_1, x_2 \in \{-L, \dots, L\}$, nous avons*

$$\bar{\Phi}_L [h(x_1, x_2) \geq d] \leq e^{-\alpha(p)d}.$$

Ce théorème montre que non seulement l'interface se confond avec le plan $\{x_3 = 1/2\}$, mais de plus, son déplacement vertical est au plus de l'ordre de $\ln L$ dans une boîte de côté L .

La loi des grands nombres pour la hauteur. Très récemment, Gheisari et Lubetzky ont amélioré le résultat de Dobrushin pour le modèle d'Ising en montrant une loi des grands nombres pour la hauteur de l'interface dans le preprint [GL19]. Considérons le cylindre droit de côté n :

$$\Lambda_n = [-n, n]^2 \times]-\infty, \infty[$$

muni de la condition aux bords de Dobrushin. Le comportement asymptotique de la hauteur maximale de l'interface I ,

$$M_n = \max\{h : \exists(x, y) \in \mathbb{Z}^2, (x, y, h) \in I\},$$

est décrit dans le théorème suivant :

Théorème 1.3.3 (Théorème 1, [GL19]). *Il existe β_0 tel que, pour tout $\beta > \beta_0$, la hauteur maximale M_n de l'interface \mathcal{I} dans le modèle d'Ising 3D sous probabilité $\pi_{\Lambda_n, \beta}^{Dob}$ satisfait*

$$\lim_{n \rightarrow \infty} \frac{M_n}{\ln n} = \frac{2}{\alpha_\beta}$$

en probabilité, où la constante $\alpha_\beta > 0$ est donnée par

$$\alpha_\beta = \lim_{h \rightarrow \infty} -\frac{1}{h} \ln \pi_{\mathbb{Z}^3}^{Dob} \left(\begin{array}{l} (0, 0, 1) \text{ est connecté à } \mathbb{Z}^2 \times \{h\} \\ \text{dans la tranche } \mathbb{Z}^2 \times \{1, \dots, h\} \\ \text{par un chemin de spins } + \end{array} \right).$$

De plus, $\alpha_\beta/\beta \rightarrow 4$ quand $\beta \rightarrow \infty$.

Pour montrer ce résultat, ils utilisent les techniques de clusters expansion qui permettent de définir et d'étudier la structure d'un pilier \mathcal{P}_x qui correspond au bord du cluster $+$ au-dessus d'un point x . Ils décomposent un pilier en plusieurs incréments et ils comparent les incréments avec les points de régénération de marches aléatoires.

1.3.3 Les interfaces en dimension deux

Les définitions de l'interface de Dobrushin restent valables pour les modèles en dimension deux et le comportement des interfaces est bien différent par rapport aux dimensions supérieures. De plus, les modèles en dimension deux sont mieux compris qu'en dimensions supérieures. En particulier, nous avons une image relativement complète de l'interface pour $\beta \geq \beta_c$ dans le cadre du modèle d'Ising.

L'interface d'Ising à basse température. Nous avons une description du comportement de l'interface en dimension $d = 2$ pour $\beta > \beta_c$ grâce à la théorie d'Ornstein-Zernike. Considérons à nouveau une configuration σ dans la boîte $\Lambda(n)$, obtenue avec la condition aux bords de Dobrushin. Nous définissons Γ^+ l'interface la plus haute et Γ^- l'interface la plus basse comme suit :

$$\begin{aligned} \Gamma^+(i) &= \max \{ j \in \mathbb{Z} : \sigma(i, j) = -1 \} + 1 \\ \Gamma^-(i) &= \min \{ j \in \mathbb{Z} : \sigma(i, j) = +1 \} - 1 \end{aligned} .$$

Nous pouvons d'abord montrer que Γ^+ et Γ^- sont proches [CIV03], i.e., il existe $K < \infty$ telle que

$$\pi_{\Lambda(n), \beta}^{Dob} \left(\max_{i \in \mathbb{Z}} \Gamma^+(i) - \Gamma^-(i) \leq K \ln n \right) \xrightarrow{n \rightarrow \infty} 1.$$

En plus, nous pouvons définir pour $x \in [-1, 1]$, l'interface renormalisée

$$\hat{\Gamma}^\pm(x) = \frac{1}{\sqrt{n}} \Gamma^\pm(\lfloor nx \rfloor).$$

L'interface de Dobrushin renormalisée est comprise entre $\hat{\Gamma}^+$ et $\hat{\Gamma}^-$ et ces deux dernières interfaces sont quasiment confondues après la renormalisation. La théorie d'Ornstein-Zernike permet de comparer l'interface Γ^+ à une marche aléatoire effective et de montrer que l'interface renormalisée se comporte comme un pont brownien.

Théorème 1.3.4 (Théorème 1.2,[GI05]). *Pour tout $\beta > \beta_c(2)$, il existe $\kappa \in]0, \infty[$ telle que $\hat{\Gamma}^+$ converge faiblement vers un pont brownien sur $[-1, 1]$ avec constante de diffusivité κ .*

Les interfaces à paramètre critique. Nous voudrions aussi mentionner les résultats sur le comportement des interfaces dans les modèles précédents au point critique. Ce sont de très jolis résultats liés aux propriétés d'invariance conforme. Les physiciens ont effectué des calculs exacts pour prédire la forme des interfaces renormalisées. Grâce à la théorie des *Stochastic Löwner Evolutions* (SLE) introduite par Schramm [Sch00], certaines de ces prédictions ont pu être justifiées rigoureusement. Le SLE_κ est un processus stochastique défini comme la solution de l'équation différentielle

$$\frac{d}{dt}g_t(z) = \frac{2}{g_t(z) - B_{\kappa t}}$$

définie sur la fermeture du demi plan

$$\mathbb{U} =] - \infty, \infty[\times]0, \infty[.$$

Pour le modèle de percolation, le résultat principal est obtenu dans le contexte du réseau triangulaire. Smirnov [Smi01] a montré que l'interface définie comme le bord des composantes connexes converge vers les processus SLE_6 . Pour le modèle d'Ising et la FK-percolation avec paramètre $q = 2$, Smirnov [Smi06] a d'abord montré que l'interface converge respectivement vers SLE_3 et $SLE_{16/3}$ au sens des observables. Depuis, Benoist, Chelkak, Hongler, Kemppainen et Smirnov ont amélioré et développé plus de méthodes pour obtenir les convergences aux d'autres sens.

1.4 L'interface dynamique en percolation

Dans ma thèse, je me suis concentré sur les interfaces en dimensions $d \geq 3$ et j'ai étudié l'évolution de l'interface sous les dynamiques présentées dans la section 1.2. Les techniques de clusters expansion sont bien adaptées pour étudier l'interface dans une boîte droite, mais nous rencontrons des obstacles quand nous voulons étudier les interfaces dans une boîte penchée. Une des raisons est que ces techniques permettent de comparer une interface quelconque avec l'interface de référence qui est le plan $\{x_d = 1/2\}$. Mais pour une boîte penchée, l'interface de référence n'est pas simple à trouver. Une façon de contourner cet obstacle est de trouver une définition d'interface qui demande moins d'information géométrique sur la configuration. Nous avons donc proposé une définition de l'interface à partir d'un couplage entre les dynamiques. Nous proposons une nouvelle définition de l'interface dans les modèles et nous montrons des résultats de type localisation dans la thèse.

1.4.1 La définition de l'interface

Dans une boîte Λ centrée à l'origine (mais pas forcément parallèle aux axes), nous identifions deux côtés opposés notés par T et B . Soit $(X_t)_{t \in \mathbb{N}}$ le processus de percolation dynamique dans Λ . Nous construisons un autre processus $(Y_t)_{t \in \mathbb{N}}$ couplé avec $(X_t)_{t \in \mathbb{N}}$ et qui reste dans l'ensemble de configurations $\{T \leftrightarrow B\}$. Nous reprenons les deux suites $(E_t)_{t \in \mathbb{N}}$ et $(B_t)_{t \in \mathbb{N}}$ de la construction graphique et le paramètre $p \in [0, 1]$. Nous commençons par une condition initiale $(X_0, Y_0) \in \Omega \times \{T \leftrightarrow B\}$, et nous définissons $(X_t)_{t \in \mathbb{N}}$ par récurrence :

$$X_t(e) = \begin{cases} X_{t-1}(e) & \text{si } e \neq E_t \\ B_t & \text{si } e = E_t \end{cases} .$$

Par contre pour le processus $(Y_t)_{t \in \mathbb{N}}$, nous interdisons les changements qui réalisent la connexion entre T et B , i.e.,

$$\forall e \subset \Lambda \quad Y_t(e) = \begin{cases} Y_{t-1}(e) & \text{if } e \neq E_t \\ 0 & \text{si } e = E_t \text{ et } B_t = 0 \\ 1 & \text{si } e = E_t, B_t = 1 \text{ et } T \xrightarrow{Y_{t-1}^{E_t}} B \\ 0 & \text{si } e = E_t, B_t = 1 \text{ et } T \xleftarrow{Y_{t-1}^{E_t}} B \end{cases} .$$

Voici un exemple illustré dans la figure 1.3, à l'instant $t + 1$, nous ouvrons l'arête blue. Par contre, à l'instant $t + 2$, lorsque nous essayons d'ouvrir l'arête rouge, l'arête s'ouvre dans la configuration X_{t+2} mais reste fermée dans Y_{t+2} . Le couple $(X_t, Y_t)_{t \in \mathbb{N}}$ a une unique mesure d'équilibre μ_p . Nous

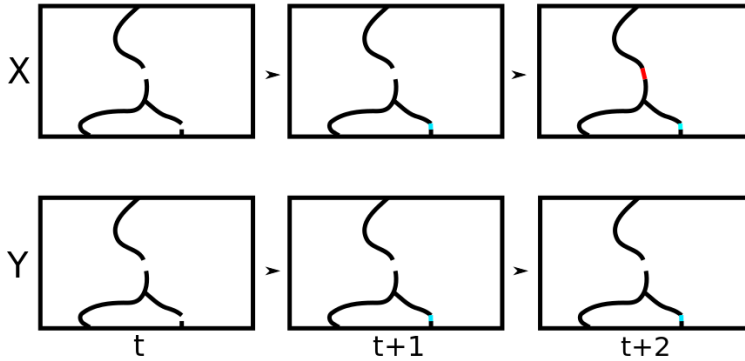


FIGURE 1.3 – Une illustration du couplage $(X_t, Y_t)_{t \in \mathbb{N}}$

définissons l'interface comme suit :

Definition 1.4.1. *L'interface au temps t entre T et B , notée par \mathcal{I}_t , est l'ensemble des arêtes dans Λ qui diffèrent entre les configurations X_t et Y_t , i.e.,*

$$\mathcal{I}_t = \{e \subset \Lambda : X_t(e) \neq Y_t(e)\} .$$

Pour étudier l'interface, nous avons besoin de l'ensemble des arêtes pivot dans la configuration Y_t :

Définition 1.4.2. *L'ensemble des arêtes pivot \mathcal{P}_t dans Y_t est constitué des arêtes dont l'ouverture crée une connexion entre T et B , i.e.,*

$$\mathcal{P}_t = \{ e \subset \Lambda : T \xleftrightarrow{Y_t^e} B \}.$$

Enfin, nous appelons un cut un ensemble d'arêtes fermées qui sépare T et B (qui correspond à la définition classique d'une interface) et nous notons \mathcal{C}_t l'ensemble des cuts dans la configuration Y_t .

1.4.2 La localisation de l'interface

Nous pouvons montrer que l'interface définie dans la section précédente est localisée autour des arêtes pivot sous la mesure μ_p quand p est assez proche de 1. Ce résultat sera présenté en détail dans le chapitre 3.

Théorème 1.4.3 ([CZ18a]). *Il existe $\tilde{p} < 1$ et $\kappa > 0$ tels que, pour tout $p \geq \tilde{p}$, $c \geq 1$ et les boîtes Λ qui satisfont*

$$|\Lambda| \geq \max \{ (cd)^{cd^2}, 3^{6d} \},$$

nous avons

$$\mu_p \left(\exists e \in \mathcal{P} \cup \mathcal{I}, d(e, \Lambda^c \cup \mathcal{P} \setminus \{e\}) \geq \kappa c^2 \ln^2 |\Lambda| \right) \leq \frac{1}{|\Lambda|^c}.$$

Expliquons brièvement les deux étapes de la preuve. Premièrement, nous montrons qu'il n'existe pas d'arêtes pivot isolées. En effet, pour $p \geq \tilde{p}$, nous avons

$$P_p \left(\exists e \in \mathcal{P}, d(e, \Lambda^c \cup \mathcal{P} \setminus \{e\}) \geq \kappa c \ln |\Lambda| \mid T \leftrightarrow B \right) \leq \frac{1}{|\Lambda|^c}. \quad (1.4.1)$$

Nous avons cette inégalité car la probabilité d'obtenir un long chemin fermé décroît exponentiellement avec la longueur de chemin en percolation surcritique. Le deuxième point nécessaire pour obtenir le théorème vient de l'étude de la vitesse de déplacement des cuts et des arêtes pivots. Nous notons P_μ la loi du processus $(X_t, Y_t)_{t \in \mathbb{N}}$ démarré avec une condition initiale aléatoire de loi μ_p . Sous cette probabilité, nous avons la proposition suivante :

Proposition 1.4.4 ([CZ18a]). *Il existe $\tilde{p} < 1$ tel que, pour tout $p \geq \tilde{p}$, $\ell \geq 2$, $t \in \mathbb{N}$, $s \in \{0, \dots, |\Lambda|\}$ et une arête e de distance plus que ℓ à Λ^c , nous avons l'inégalité suivante :*

$$P_\mu \left(\begin{array}{c} e \in \mathcal{P}_{t+s} \\ \forall r \in [t, t+s] \quad \mathcal{P}_r \neq \emptyset \end{array} \mid \exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell \right) \leq \exp(-\ell).$$

En effet, comme au plus une arête est changée à un instant, un grand déplacement d'un cut implique la fermeture de nombreuses arêtes et ces arêtes forment un *chemin espace-temps* qui est l'objet central pour contrôler les déplacements. Avec des arguments de type Peierls, nous montrons qu'un long chemin espace-temps fermé est réalisé avec une faible probabilité. De plus, nous observons aussi qu'une arête de l'interface ne peut pas rester différente dans les deux configurations pendant longtemps. S'il existe une arête de l'interface loin des arêtes pivot, elle est soit créée par une arête pivot isolée, ou bien les arêtes pivot ont effectué un grand déplacement depuis sa création. Les deux scénarios sont contrôlés par les deux arguments présentés précédemment.

1.4.3 La loi de la configuration conditionnée

Nous avons aussi obtenu une description quantitative de la loi d'une arête dans une configuration de l'événement $\{T \leftrightarrow B\}$. Plus précisément, à l'aide du couplage (X_t, Y_t) , nous pouvons comparer l'état d'une arête dans les deux configurations comme suit :

Théorème 1.4.5 ([CZ18a]). *Nous avons l'inégalité suivante :*

$$\begin{aligned} \exists \tilde{p} < 1 \quad \exists \kappa > 0 \quad \forall p \geq \tilde{p} \quad \forall c \geq 2 \quad \forall \Lambda \quad \ln |\Lambda| > 4 + c + 2dc^2 + 12(2\kappa d)^d \\ \forall e \in \Lambda \quad d(e, \Lambda^c) \geq \kappa c^2 \ln^2 |\Lambda| \\ \mu_p \left(e \in \mathcal{I} \mid \exists C \in \mathcal{C}, d(e, C) \geq \kappa c^2 \ln^2 |\Lambda| \right) \leq \frac{1}{|\Lambda|^c}. \end{aligned}$$

Dans une configuration qui satisfait $\{T \leftrightarrow B\}$, nous pouvons observer au moins un cut entre T et B . Le théorème précédent montre qu'une arête dont la distance à un cut est supérieure à $\ln^2 |\Lambda|$ se comporte comme dans une configuration de percolation Bernoulli. La preuve repose sur les deux arguments utilisés pour prouver le théorème 1.4.3, mais nous avons besoin de résultats plus précis sur la vitesse de déplacement des cuts. Ce résultat sera présenté dans le chapitre 3.

1.4.4 Une tentative d'amélioration sur la localisation

Les résultats de localisation pour les interfaces classiques montrent que l'interface est localisée autour du plan $\{x_d = 1/2\}$ et la distance à ce plan est au plus de l'ordre $\ln |\Lambda|$. Nous voudrions aussi avoir un contrôle de l'ordre de $\ln |\Lambda|$ au lieu de $\ln^2 |\Lambda|$ pour les deux théorèmes présentés précédemment. Une raison principale pour laquelle nous obtenons $\ln^2 |\Lambda|$ est que nous ne pouvons contrôler la vitesse des cuts pendant un intervalle de temps de taille $|\Lambda|$. Grâce à une nouvelle construction du chemin espace-temps, nous arrivons à contrôler les déplacements des arêtes pivot pendant un intervalle de temps $|\Lambda| \ln |\Lambda|$. Nous obtenons le théorème suivant :

Théorème 1.4.6. *Il existe $\tilde{p} < 1$ et $\kappa \geq 2d$ tels que, pour $p \geq \tilde{p}$, $c \geq 1$ et les boîtes Λ qui satisfont $|\Lambda| \geq e^{2d^2c}$, nous avons*

$$P_\mu \left(d_H^{2dc \ln |\Lambda|} \left(\bigcup_{r \in [t-c|\Lambda| \ln \Lambda, t]} \mathcal{P}_r, \bigcup_{s \in [t, t+c|\Lambda| \ln \Lambda]} \mathcal{P}_s \right) \geq 2dc \ln |\Lambda| \right) \leq \frac{1}{|\Lambda|^c},$$

où d_H^ℓ est la semi-distance adaptée de la distance de Hausdorff définie par

$$d_H^\ell(A, B) = \inf \left\{ r \geq 0 : \begin{array}{l} A \setminus \mathcal{V}(\Lambda^c, \ell) \subset \mathcal{V}(B, r) \\ B \setminus \mathcal{V}(\Lambda^c, \ell) \subset \mathcal{V}(A, r) \end{array} \right\}.$$

Nous avons une amélioration de l'ordre de l'intervalle de temps par rapport à proposition 1.4.4. Par contre, nous devons considérer l'union des arêtes pivot pendant cet intervalle de temps.

1.5 L'interface dynamique en FK-percolation et d'Ising

Comme le Gibbs sampler joue le rôle de percolation dynamique en FK-percolation, nous pouvons aussi définir l'interface à l'aide d'un couplage de processus dans le contexte de la FK-percolation. De plus, à l'aide d'un couplage inspiré par le couplage d'Edwards et Sokal, nous pouvons définir l'interface dans le modèle d'Ising.

1.5.1 L'interface en FK-percolation

Nous allons construire un couplage de deux processus de Gibbs samplers comme pour la percolation dans une boîte Λ . Considérons les deux suites $(E_t)_{t \in \mathbb{N}}$ et $(U_t)_{t \in \mathbb{N}}$ dans la construction du Gibbs sampler. Rappelons la relation de récurrence pour la définition du processus $(X_t)_{t \in \mathbb{N}}$:

$$X_t(e) = \begin{cases} \omega(e) & \text{si } E_t \neq e \\ 1 & \text{si } E_t = e \text{ et } U_t \geq \frac{\Phi(\omega_e)}{\Phi(\omega^e) + \Phi(\omega_e)} \\ 0 & \text{si } E_t = e \text{ et } U_t < \frac{\Phi(\omega_e)}{\Phi(\omega^e) + \Phi(\omega_e)} \end{cases}.$$

Comme pour la percolation, nous définissons le processus $(Y_t)_{t \in \mathbb{N}}$ qui reste dans l'ensemble $\{T \leftrightarrow B\}$:

$$Y_t(e) = \begin{cases} X_{t-1}(e) & \text{si } E_t \neq e \\ 1 & \text{si } E_t = e, U_t \geq \frac{\Phi(X_{t-1}e)}{\Phi(X_{t-1}^e) + \Phi(X_{t-1}e)} \text{ et } T \xrightarrow{Y_{t-1}^e} B \\ 0 & \text{si } E_t = e, U_t \geq \frac{\Phi(X_{t-1}e)}{\Phi(X_{t-1}^e) + \Phi(X_{t-1}e)} \text{ et } T \xleftarrow{Y_{t-1}^e} B \\ 0 & \text{si } E_t = e, U_t < \frac{\Phi(X_{t-1}e)}{\Phi(X_{t-1}^e) + \Phi(X_{t-1}e)} \end{cases}.$$

Le couple $(X_t, Y_t)_{t \in \mathbb{N}}$ admet une mesure d'équilibre que nous notons par $\mu_{\Lambda, p, q}$. Nous pouvons encore définir l'interface \mathcal{I}_t comme l'ensemble des arêtes différentes dans les deux configurations à l'instant t et \mathcal{P}_t comme l'ensemble des arêtes pivot pour l'événement $\{T \leftrightarrow B\}$ dans la configuration Y_t .

La localisation autour des arêtes pivot. Nous pouvons encore étudier les chemins espace-temps que nous avons utilisés dans [CZ18a] pour analyser les déplacements des arêtes pivot. Nous obtenons la localisation de l'interface \mathcal{I} autour de \mathcal{P} comme dans le cas de percolation Bernoulli.

Théorème 1.5.1 ([Zho18]). *Pour tout $q \geq 1$, il existe $\tilde{p} < 1$ et $\kappa > 0$ tels que, pour $p \geq \tilde{p}$ et $c \geq 1$ et toute boîte Λ , nous avons*

$$\mu_{\Lambda, p, q} \left(\exists e \in \mathcal{P} \cup \mathcal{I}, d(e, \Lambda^c \cup \mathcal{P} \setminus \{e\}) \geq \kappa c^2 \ln^2 |\Lambda| \right) \leq \frac{1}{|\Lambda|^c}.$$

Un cut dans le modèle d'Ising est un ensemble d'arêtes qui ont des spins différents sur les sommets qui sépare T et B . De plus, nous montrons que'un sommet en haut (resp. en bas) et loin d'un cut a le même spin dans la configuration σ^+ (resp. σ^-) et σ^D , i.e.,

Theorem 1.5.2 ([Zho18]). *$0 < \tilde{\beta} < \infty$ and $\kappa \geq 0$, such that for $\beta \geq \tilde{\beta}$, $c > 0$ and any Λ such that $|\Lambda| \geq \max\{3^{6d}, (cd)^{cd^2}\}$, we have*

$$\pi_\beta \left(\begin{array}{l} \exists x \in \Lambda \quad \sigma^+(x) = +1, \quad \sigma^D(x) = -1 \\ \exists C \text{ un cut separant } x \text{ de } B \\ d(x, C) \geq \kappa c^2 \ln^2 |\Lambda| \end{array} \right) \leq \frac{1}{|\Lambda|^c},$$

et

$$\pi_\beta \left(\begin{array}{l} \exists x \in \Lambda \quad \sigma^+(x) = -1, \quad \sigma^D(x) = +1 \\ \exists C \text{ un cut separant } x \text{ de } T \\ d(x, C) \geq \kappa c^2 \ln^2 |\Lambda| \end{array} \right) \leq \frac{1}{|\Lambda|^c}.$$

Notons qu'à cause de la dépendance entre les arêtes dans le modèle de FK-percolation, l'inégalité BK [Gri99] que nous avons utilisée à plusieurs reprises dans [CZ18a] n'est plus valable dans ce modèle. Par contre, la preuve utilise les deux arguments déjà présentés pour la preuve du théorème 1.4.3. Le premier argument qui contrôle la distance entre les arêtes pivot peut être adapté à l'aide d'un processus d'exploration qui génère un sous-graphe de Λ contenant un chemin fermé disjoint d'un cut et d'une inégalité de comparaison entre les mesures de FK-percolation associées à différents paramètres p, q . Le deuxième argument peut être remplacé par une comparaison entre le Gibbs sampler et un processus de Poisson bien choisi.

1.5.2 L'interface dans le modèle d'Ising

Le couplage d'Edwards-Sokal permet de transférer de nombreux résultats de la FK-percolation au modèle d'Ising. Nous allons adapter ce couplage

pour construire un triplet de configurations de spins $(\sigma^+, \sigma^-, \sigma^D)$ à partir des configurations (X_t, Y_t) de FK-percolation. La configuration σ^+ (respectivement σ^-) est construite à partir de X_t , et elle est associée à la condition aux bords $+$ (respectivement $-$) et la configuration σ^D est construite à partir de Y_t , et elle est associée à la condition aux bords de Dobrushin. Nous mettons d'abord les spins dans les clusters qui touchent le bord, en fonction des conditions aux bords. En effet, comme la configuration X_t domine Y_t , un cluster ouvert dans Y_t est inclus dans un cluster ouvert de X_t . Si deux clusters sont identiques dans X_t et Y_t , alors nous mettons le même spin dans les trois configurations σ^+, σ^- et σ^D . Sinon, nous retirons aléatoirement les spins du cluster dans σ^- et σ^+ indépendamment de ceux de σ^D . Les trois configurations σ^+, σ^- et σ^D sont distribuées selon la mesure de Gibbs avec les conditions aux bords correspondantes. Nous notons $\pi_{\Lambda, \beta}$ la loi du triplet $(\sigma^+, \sigma^-, \sigma^D)$ et nous définissons les ensembles suivants :

Définition 1.5.3. *L'ensemble \mathcal{P}_I est l'ensemble des arêtes $\langle x, y \rangle$ telles que*

$$\begin{aligned} \sigma^D(x) &= +1 \text{ et } x \text{ est connecté à } T \text{ par un chemin de spins } +1 \text{ dans } \sigma^D \\ \sigma^D(y) &= -1 \text{ et } y \text{ est connecté à } B \text{ par un chemin de spins } -1 \text{ dans } \sigma^D \end{aligned}$$

et l'interface \mathcal{I}_I est l'ensemble des arêtes $\langle x, y \rangle$ telles que

$$\sigma^+(x) = \sigma^+(y), \quad \sigma^-(x) = \sigma^-(y), \quad \sigma^D(x) \neq \sigma^D(y).$$

L'ensemble \mathcal{I}_I correspond aux différences créées par la condition aux bords de Dobrushin et l'ensemble \mathcal{P}_I est obtenu après avoir enlevé les "bulles" dans l'interface de Dobrushin.

La localisation dans le modèle d'Ising. La localisation de l'interface en FK-percolation induit la localisation de \mathcal{I}_I autour de \mathcal{P}_I comme énoncé dans le théorème suivant :

Théorème 1.5.4. *Il existe $0 < \tilde{\beta} < \infty$ et $\kappa \geq 0$ tels que, pour $\beta \geq \tilde{\beta}$, $c > 0$ et toute boîte Λ , nous avons*

$$\pi_{\Lambda, \beta} \left(\exists e \in \mathcal{I}_I, d(e, \Lambda^c \cup \mathcal{P}_I) \geq \kappa c^2 \ln^2 |\Lambda| \right) \leq \frac{1}{|\Lambda|^c}.$$

Ce théorème donne une localisation de l'interface par rapport à un cut. Pour la preuve, nous remarquons d'abord qu'une arête de \mathcal{I}_I est dans le même cluster qu'une arête de \mathcal{I} dans Y_t . En FK-percolation, nous contrôlons la taille d'un cluster ouvert dans Y_t disjoint d'un cut et nous appliquons le théorème 1.5.1 pour enfin montrer que la distance entre \mathcal{I}_I et \mathcal{P}_I n'est pas trop grande.

1.6 Les chemins espace-temps

Comme mentionné dans la section précédente, pour estimer la vitesse de déplacement des arêtes pivots, nous utilisons un objet que nous appelons chemin espace-temps. C'est une généralisation naturelle d'un chemin dans le contexte de la percolation dynamique. Plus formellement, c'est une suite d'arête-temps $(e_i, t_i)_{i \in \mathbb{N}}$ qui vérifie la relation suivante :

$$\forall i \in \mathbb{N} \quad (e_i = e_{i+1} \text{ et } t_i \neq t_{i+1}) \\ \text{ou } (t_i = t_{i+1} \text{ et } e_i, e_{i+1} \text{ ont une extrémité commune}).$$

Expliquons d'abord pourquoi un chemin espace-temps est relié au déplacement des arêtes pivot. Remarquons d'abord qu'à un instant donné, les arêtes pivots sont reliées entre elles par un chemin fermé. Considérons deux instants s, t avec $s < t$, nous pouvons montrer qu'une arête de \mathcal{P}_t et une autre arête de \mathcal{P}_s sont reliées par un chemin espace-temps fermé dans le processus $(Y_t)_{t \in \mathbb{N}}$. Si la vitesse de déplacement des arêtes pivot est grande, alors nous pouvons trouver deux arêtes pivot éloignées spatialement mais à des instants assez proches. Ainsi, nous obtenons un long chemin espace-temps fermé qui a une durée petite. Nous avons montré que ce type de chemin espace-temps arrive avec une faible probabilité.

Deux difficultés se présentent pour notre étude. Tout d'abord, nous avons besoin d'une propriété de type décroissance exponentielle pour la longueur d'un chemin espace-temps fermé en régime sur-critique. Ce résultat est bien connu pour un chemin fermé, qui correspond à un chemin espace-temps où tous les t_i sont égaux (voir [Gri99, chapitre 5]). Cependant, un chemin espace-temps qui relie plusieurs chemins à différents instants peut parcourir une longue distance à l'aide de nombreux chemins courts qui sont fermés à différents instants. Ainsi, il faut contrôler à la fois la distance parcourue par le chemin et le nombre de morceaux pour qu'un chemin réalise un tel parcours. Nous avons introduit la notion de chemin impatient pour réaliser ce contrôle que nous allons détailler dans le chapitre 2. Ensuite, le chemin espace-temps que nous allons étudier est défini via le processus $(Y_t)_{t \in \mathbb{N}}$, qui est conditionné par l'événement $\{T \leftrightarrow B\}$. Ce conditionnement introduit des corrélations positives entre différentes arêtes aux instants différents. Nous avons réussi à surmonter cet obstacle avec une modification qui rend un chemin espace-temps "simple" (voir chapitre 3).

1.7 Perspectives

La géométrie de l'interface est le sujet central de plusieurs questions ouvertes. Mentionnons les deux questions les plus connues : la conjecture de *roughening transition* et la question sur les mesures de Gibbs qui ne sont pas invariantes par translation.

Problème 1 : Pour $d \geq 3$, existe-t-il une température $\beta_r > \beta_c$, telle que pour $\beta \in]0, \beta_r[$, l'interface de Dobrushin n'est pas localisée autour du plan $\{x_d = 1/2\}$?

Plusieurs questions intéressantes font le lien entre nos résultats et cette fameuse conjecture. La première question naturelle est de demander la relation entre notre nouvelle définition de l'interface et l'interface classique de Dobrushin. En fait, notre interface permet de comparer une configuration "standard" (percolation Bernoulli, FK-percolation ou Ising avec conditions aux bords simple), qui n'admet a priori pas d'interface, avec une configuration avec l'interface. Nos résultats nous indiquent les positions de l'interface de Dobrushin en éliminant les zones qui sont confondues avec les configurations "standards". La deuxième question est de demander si les résultats sont valables jusqu'au point critique, i.e., $p > p_c$ pour la percolation et $\beta > \beta_c$ pour le modèle d'Ising. Par exemple, dans le contexte de la percolation, plusieurs arguments que nous utilisons restent valables pour $p > p_c$. Le seul point qui pose vraiment un problème, c'est que l'estimée sur la décroissance exponentielle d'un chemin espace-temps n'est valable que pour p proche de 1. Ceci est dû au fait que nous utilisons un argument de type Peierls qui ne fonctionne que pour p proche de 1. Pour obtenir la décroissance exponentielle pour $p > p_c$, nous pourrions par exemple utiliser la formule de Russo (voir [Gri99]) ou adapter les techniques développées dans [DCRT19] pour obtenir une inégalité différentielle.

Problème 2 : Existe-t-il d'autres mesures de Gibbs que celle qui correspond à l'interface de Dobrushin pour la boîte droite et qui ne sont pas invariantes par translation ?

Pour apporter quelques éléments de réponse à cette question, nous remarquons d'abord que nos résultats ont l'avantage d'être valables pour un domaine plus général qu'une boîte droite. Le point crucial est que les résultats de localisation sont relatifs à l'ensemble des arêtes pivot. La compréhension plus détaillée de la position des arêtes pivot serait un grand pas vers la réponse. Nous remarquons que les arêtes pivot sont comprises entre deux cuts. Ce type de structure est déjà présent dans la théorie d'Orstein-Zernike pour le modèle d'Ising en dimension deux où il y a une structure de diamants enchaînés. Mais pour l'instant cette théorie n'est développée que pour les objets de dimension un et une interface en dimension $d \geq 3$ correspond à une surface de dimension plus grande que deux. Remarquons que des résultats sur les positions des arêtes pivot pourraient peut-être donner un élément de réponse à la question précédente, en combinant la localisation de l'interface autour des arêtes pivot jusqu'au point critique.

Quelques questions moins ambitieuses. Sans mentionner de fameuses conjectures, notre couplage présente plusieurs points intéressants. Notons que ce couplage apparaît déjà dans la preuve classique de l'inégalité d'Holley [Gri06]. Il est loin d'être bien compris. Par exemple, nous pouvons nous demander si ce couplage minimise le nombre d'arêtes différentes entre une configuration standard et une configuration conditionnée.

Nous pouvons aussi nous demander si ce couplage permet d'analyser directement le mouvement de l'interface dans le modèle d'Ising sous la dynamique de Glauber. Pour l'instant, seulement des résultats partiels sont obtenus dans [CL07, CSS95, KS95, DMOPT94, Sow99, Spo93]. Récemment, Lacoïn, Simenhaus et Toninelli [LST, LST12] ont montré que l'interface suit un mouvement par courbure moyenne dans le cas du modèle d'Ising 2D à température zéro. De plus, un chemin espace-temps construit directement dans le modèle d'Ising pourrait nous donner une description plus intrinsèque de l'interface d'Ising au lieu de récupérer de l'information à partir de la FK-percolation, comme ce que nous avons fait jusqu'à maintenant.

Comme cette construction de l'interface demande seulement d'avoir un processus dont la mesure d'équilibre décrit l'interface, nous pourrions définir de même l'interface dans d'autres modèles avec cette méthode. Par exemple, dans le contexte de la percolation de premier passage, nous pouvons conditionner la configuration dans une boîte par l'existence d'au plus m connexions entre le haut et le bas et étudier la géométrie du cut minimal. Avec les techniques déjà développées dans notre étude de percolation, nous pouvons obtenir un résultat similaire pour

$$m = O\left(\frac{\ln n}{\ln \ln n}\right)$$

dans une boîte cubique de côté n , mais pour les déviations d'ordre $m = \epsilon n^{d-1}$, nous avons besoin d'arguments plus complexes pour relier le comportement global d'un cut et les changements locaux près d'une arête.

1.8 L'organisation de la thèse

Nous présentons les travaux dans les quatre chapitres qui suivent. Nous commençons par un préliminaire sur les chemins espace-temps, qui est un objet essentiel pour nos études sur les couplages. Nous étudions l'interface dans le modèle de percolation dans les deux chapitres qui suivent. Nous finissons avec un chapitre sur l'interface en FK-percolation et d'Ising.

Deuxième partie

L'interface en percolation

Chapitre 2

Un premier résultat sur les chemins espace-temps

Plutôt que de rentrer directement dans les études de l'interface, nous présentons dans ce chapitre les chemins espace-temps qui seront l'outil central pour notre étude. Nous discutons certaines propriétés des chemins espace-temps, en particulier, nous montrons un premier résultat de type décroissance exponentielle des chemins espace-temps dans le contexte de percolation dynamique classique.

2.1 Les définitions et l'énoncé du théorème

Nous considérons les sommets et les arêtes qui sont inclus dans une boîte $\Lambda_L = [-L, L]^d$. Nous étudions les chemins espace-temps fermés dans la boîte pour la percolation dynamique de paramètre p .

Les arêtes-temps. Une arête-temps est un couple (e, t) où e est une arête de \mathbb{E}^d et t un entier naturel.

La relation de connexion. Sur l'espace $\mathbb{E}^d \times \mathbb{N}$, nous définissons la relation de connexion \sim de la manière suivante. Soient (e, t) et (f, s) deux arêtes-temps, nous disons qu'elles sont connectées, ce que nous notons $(e, t) \sim (f, s)$, si

$$(e = f \text{ et } s \neq t) \text{ ou } (s = t \text{ et } e, f \text{ ont une extrémité commune}).$$

Les chemins espace-temps. Un chemin espace-temps est une suite alternée finie et de sommets x_i et d'arêtes-temps (e_i, t_i) ,

$$x_1, (e_1, t_1), y_1, x_2, (e_2, t_2), y_2, \dots, x_n, (e_n, t_n), y_n$$

telle que, pour $1 \leq i \leq n$, e_i est l'arête qui relie x_i à y_i et pour $1 \leq i \leq n-1$, (e_i, t_i) et (e_{i+1}, t_{i+1}) sont connectées de la manière suivante :

$$(e_i = e_{i+1} \text{ et } t_i \neq t_{i+1}) \text{ ou } (t_i = t_{i+1} \text{ et } y_i = x_{i+1}).$$

Nous définissons la longueur d'un chemin espace-temps comme le nombre de ses arêtes-temps. Pour simplifier les notations, lorsqu'il n'y a pas d'ambiguïté, nous notons un chemin espace-temps seulement par la suite de ses arêtes-temps $(e_1, t_1), \dots, (e_n, t_n)$. Soient x, y deux sommets dans Λ_L , nous disons qu'un chemin espace-temps $(e_1, t_1), \dots, (e_n, t_n)$ relie x à y si x est une extrémité de e_1 et y une extrémité de e_n .

Les changements de temps. Soit $(e_i, t_i)_{1 \leq i \leq n}$ un chemin espace-temps, nous disons que (e_i, t_i) est un changement de temps si $e_{i+1} = e_i$ et $t_{i+1} \neq t_i$ et dans ce cas nous disons que l'intervalle $[\min(t_i, t_{i+1}), \max(t_i, t_{i+1})]$ est un intervalle de changement de temps.

Quelques propriétés basiques. Un chemin espace-temps $(e_i, t_i)_{1 \leq i \leq n}$ est dit *croissant* (respectivement *décroissant*) si

$$t_1 \leq \dots \leq t_n \text{ (resp. } t_1 \geq \dots \geq t_n),$$

et il est dit *fermé* si, pour tout $1 \leq i \leq n$, l'arête e_i est fermé à l'instant t_i . La projection spatiale d'un chemin espace-temps γ est la suite d'arêtes obtenues en enlevant une arête dans chaque changement de temps de γ , i.e., pour un chemin espace temps $(e_1, t_1), \dots, (e_n, t_n)$ qui a m changements de temps, nous définissons la fonction $\phi : 1, \dots, n-m \rightarrow \mathbb{N}$ avec $\phi(1) = 1$ et

$$\forall i \in \{1, \dots, n-m\} \quad \phi(i+1) = \begin{cases} \phi(i) + 1 & \text{si } e_{\phi(i)} \neq e_{\phi(i)+1} \\ \phi(i) + 2 & \text{si } e_{\phi(i)} = e_{\phi(i)+1} \end{cases}.$$

La suite $(e_{\phi(i)})_{1 \leq i \leq n-m}$ est la projection spatiale de γ , notée par $\text{Space}(\gamma)$. La longueur d'un chemin espace-temps γ est définie comme la longueur de la suite $\text{Space}(\gamma)$ et le support de γ est défini comme

$$\text{support}(\gamma) = \{e \in \Lambda : \exists i \in \{1, \dots, k\} \quad e_i = e\},$$

qui est aussi le support de la suite $\text{Space}(\gamma)$.

Nous énonçons maintenant notre résultat de décroissance exponentielle.

Théorème 2.1.1. *Il existe $\tilde{p} < 1$ tel que, pour $p > \tilde{p}$,*

$$\forall s, t \in \mathbb{N}, \quad s < t, \quad \forall n \in \mathbb{N},$$

$$P \left(\begin{array}{l} \text{il existe un chemin espace-temps décroissant} \\ \text{de longueur } n \text{ partant de } O \text{ fermé entre } s \text{ et } t \\ \text{et fermé pendant les changements de temps} \end{array} \right) \leq \exp \left(\frac{n(t-s)}{|\Lambda|} + \frac{n}{2} \ln(3-3p) + n \ln \alpha(d) \right),$$

où $\alpha(d)$ est le nombre de voisins d'une arête en dimension d .

2.2 Les chemins espace-temps simples

Les chemins espace-temps simples. Nous considérons le processus de percolation dynamique à temps discret et ω une trajectoire. Nous disons que ce chemin espace-temps fermé est simple de longueur n avec m changements de temps s'il existe m indices $1 \leq k(1) < k(2) < \dots < k(m) \leq n$ tels que :

- Les changements de temps arrivent aux instants $t_{k(1)}, \dots, t_{k(m)}$, i.e.,

$$\forall i \in \{1, \dots, m-1\} \quad e_{k(i)} = e_{k(i)+1}, \\ t_{k(i)} \neq t_{k(i)+1}, \quad t_{k(i)+1} = t_{k(i)+2} = \dots = t_{k(i+1)}.$$

- Les arêtes visitées à un instant donné sont 2 à 2 distinctes, i.e.,

$$\forall i, j \in \{1, \dots, n\} \quad (i \neq j, t_i = t_j) \Rightarrow e_i \neq e_j.$$

- Les fermetures d'arêtes arrivent disjointement, i.e., pour tout i, j tels que $1 \leq i < j \leq n$ et $e_i = e_j$, l'une des 3 conditions suivantes est vérifiée :

- ◊ $j = i + 1$ et $i \in \{k(1), \dots, k(m)\}$;
- ◊ $t_i < t_j$ et il existe un instant $s \in]t_i, t_j[$ tel que e_j est ouverte à l'instant s dans ω ;
- ◊ $t_j < t_i$ et il existe un instant $s \in]t_j, t_i[$ tel que e_j est ouverte à l'instant s dans ω .

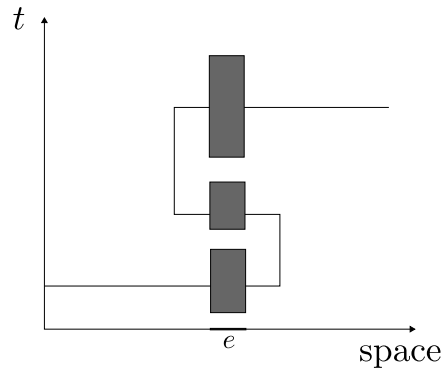


FIGURE 2.1 – Un chemin espace-temps simple, les intervalles où l'arête e est fermée sont en gris

Désormais, nous considérons les chemins espace-temps qui n'admettent pas deux changements de temps consécutifs. En fait, tout chemin espace-temps peut être modifié en un chemin espace-temps qui n'admet pas de changements de temps consécutifs. Nous exhibons un algorithme de modification.

Algorithme 2.2.1. Soit $(e_1, t_1), \dots, (e_n, t_n)$ un chemin espace-temps. Nous allons remplacer récursivement les arêtes de changements de temps consécutifs par un seul changement de temps. Nous commençons avec (e_1, t_1) et trois cas se présentent :

- Si (e_1, t_1) n'est pas une arête de changement de temps, nous ne modifions pas l'arête et nous continuons l'algorithme avec le sous chemin qui débute à partir de l'arête-temps (e_2, t_2) .
- Si (e_1, t_1) est une arête de changement de temps mais $e_3 \neq e_1$, alors (e_2, t_2) n'est pas suivie par une arête de changement de temps. Nous ne modifions pas le chemin et nous recommençons l'algorithme avec le chemin $(e_3, t_3), \dots, (e_n, t_n)$.
- Si (e_1, t_1) est une arête de changement de temps et $e_3 = e_1$, (e_2, t_2) est suivie par un changement de temps, nous considérons l'indice I défini par

$$I = \max \{1 < i \leq n : \forall j \leq i \quad e_j = e_1\}.$$

Si $t_1 \neq t_I$, nous remplaçons $(e_1, t_1), \dots, (e_I, t_I)$ par $(e_1, t_1), (e_I, t_I)$ et si $t_1 = t_I$, nous remplaçons $(e_1, t_1), \dots, (e_I, t_I)$ par (e_1, t_1) . Enfin, nous recommençons l'algorithme avec le chemin $(e_{i+1}, t_{i+1}), \dots, (e_n, t_n)$.

Nous remarquons d'abord que la longueur de chemin qui reste à modifier diminue après chaque itération, donc l'algorithme se termine. Nous remarquons aussi qu'un chemin simple n'admet pas de changements de temps consécutifs. Dans la suite, nous appliquons systématiquement l'algorithme précédent à tout chemin que nous considérons pour obtenir un chemin qui n'a pas de changements de temps consécutifs. Nous montrons dans la suite que, de tout chemin espace-temps, nous pouvons extraire un chemin simple.

Proposition 2.2.2. Soit $(e_i, t_i)_{1 \leq i \leq N}$ un chemin espace-temps fermé qui relie x à y . Il existe une fonction $\phi : \{1, \dots, n\} \rightarrow \{1, \dots, N\}$ strictement croissante telle que $(e_{\phi(1)}, t_{\phi(1)}), \dots, (e_{\phi(n)}, t_{\phi(n)})$ est un chemin espace-temps fermé simple qui relie x à y .

Démonstration. Nous allons obtenir un chemin espace-temps simple par une modification itérative à partir du chemin espace-temps $(e_i, t_i)_{1 \leq i \leq N}$. Commençons par (e_1, t_1) et examinons les arêtes-temps qui restent dans l'ordre. Supposons que les arêtes-temps $(e_1, t_1), \dots, (e_{i-1}, t_{i-1})$ ont été modifiés et nous examinons l'arête-temps (e_i, t_i) . Les trois cas suivants se présentent :

- Pour tout $j \in \{i+1, \dots, N\}$, nous avons $e_j \neq e_i$. Dans ce cas, nous ne modifions pas (e_i, t_i) et nous continuons avec (e_{i+1}, t_{i+1}) .
- Il existe un indice $j \in \{i+1, \dots, N\}$ tel que $e_j = e_i$, mais pour tout indice j telle que $e_j = e_i$, il existe un instant θ_j dans l'intervalle

$$] \min(t_i, t_j), \max(t_i, t_j)[$$

où l'arête e_i est ouverte à l'instant θ_j . Nous ne modifions rien et nous continuons avec (e_{i+1}, t_{i+1}) .

- Il existe un indice $j \in \{i + 1, \dots, N\}$ tel que $e_j = e_i$ et que e_i reste fermée sur l'intervalle $[\min(t_i, t_j), \max(t_i, t_j)]$. Dans ce cas, soit k le dernier indice qui satisfait cette condition, nous supprimons tous les arêtes-temps entre les indices i et k et les remplaçons par un changement de temps $(e_i, t_i), (e_k, t_k)$. Nous continuons avec l'arête temps (e_{k+1}, t_{k+1}) .

Le chemin espace-temps qui reste à modifier devient strictement plus court après chaque étape et cette modification se termine après un nombre fini d'étape. Le chemin espace-temps que nous obtenons à la fin est extrait du $(e_i, t_i)_{1 \leq i \leq N}$ et il est simple. \square

2.3 Chemins espace-temps impatients

Les chemins espace-temps impatients. Un chemin espace-temps fermé décroissant que nous notons par $(e_i, t_i)_{1 \leq i \leq n}$ est dit impatient si toute arête de changement de temps e_k est suivie par une arête e_{k+2} qui se ferme à l'instant t_{k+2} . Nous allons montrer que tout chemin espace-temps admet une modification temporelle qui est impatiente. Pour cela, nous introduisons l'algorithme de modification récursive suivant :

Algorithme 2.3.1. *Soit $(e_1, t_1), \dots, (e_n, t_n)$ un chemin espace-temps fermé décroissant. Nous allons modifier la première arête e_1 du chemin. Nous considérons les cas suivants :*

- Si $e_2 \neq e_1$, alors nécessairement $t_1 = t_2$, et nous ne modifions pas (e_1, t_1) . Nous recommençons l'algorithme avec le chemin $(e_2, t_2), \dots, (e_n, t_n)$;
- Si $e_2 = e_1$ et $t_1 > t_2$, soit $\tau_3 + 1$ le dernier instant avant t_1 où e_3 s'ouvre. Si $t_1 \leq \tau_3$, nous remplaçons $(e_1, t_1), (e_2, t_2)$ par $(e_1, t_1), (e_3, t_1)$. Nous recommençons l'algorithme avec le chemin $(e_3, t_1), (e_3, t_3), \dots, (e_n, t_n)$. Si $t_1 > \tau_3$, nous remplaçons $(e_1, t_1), (e_2, t_2)$ par $(e_1, t_1), (e_2, \tau_3), (e_3, \tau_3)$. Nous recommençons l'algorithme avec le chemin $(e_3, \tau_3), (e_3, t_3), \dots, (e_n, t_n)$.

Nous remarquons que la longueur du chemin espace-temps à modifier diminue après chaque itération, donc l'algorithme se termine. Le chemin espace-temps obtenu à la fin de l'algorithme 2.3.1 est impatient, nous avons donc le résultat suivant :

Proposition 2.3.2. *Soit γ un chemin espace-temps fermé décroissant qui relie x à y . Sa modification Γ obtenue selon l'algorithme 2.3.1 est un chemin fermé décroissant impatient qui relie x à y . De plus, les intervalles de changement de temps de Γ sont inclus dans les intervalles de changement de temps de γ .*

Nous montrons maintenant qu'un chemin simple est toujours simple après la modification selon l'algorithme.

Proposition 2.3.3. *Soit γ un chemin espace-temps fermé décroissant et simple. Soit Γ le chemin obtenu en modifiant γ selon l'algorithme 2.3.1. Le chemin Γ est fermé décroissant simple et impatient.*

Démonstration. Nous vérifions que la condition du chemin simple est satisfaite à chaque étape de l'algorithme. Soit $(e_i, t_i), (e_{i+1}, t_{i+1})$ le changement de temps qui est modifié lors d'une itération, et supposons que le chemin visite e_i ou e_{i+2} plus d'une fois. Nous examinons les deux résultats possibles de la modification. Si nous obtenons $(e_i, t_i), (e_{i+2}, t_i)$ après la modification, nous devons vérifier qu'il existe un instant entre chaque visite de e_{i+2} et t_i tel que e_{i+2} est ouverte à cet instant. Or (e_{i+2}, t_{i+2}) est dans γ qui est un chemin simple, donc l'arête e_{i+2} s'ouvre entre t_{i+2} et les autres instants de visites de e_{i+2} . Vu que l'arête e_{i+2} est fermée entre t_i et t_{i+2} , cette dernière propriété est encore vraie pour t_i . Si nous obtenons $(e_i, t_i), (e_{i+1}, \tau_{i+2}), (e_{i+2}, \tau_{i+2})$ après la modification, nous vérifions la condition pour e_i et e_{i+2} . Nous rappelons que $e_{i+1} = e_i$ et que τ_{i+2} est le dernier instant avant t_{i+1} où e_{i+2} se ferme. Or l'arête e_i est fermée entre t_i et τ_{i+2} , donc e_i s'ouvre entre τ_{i+2} et les autres instants de visites de e_i . De même, l'arête e_{i+2} s'ouvre entre τ_{i+2} et les autres instants de visites de e_{i+1} car e_{i+2} est fermée entre τ_{i+2} et t_{i+2} . \square

2.4 La décroissance exponentielle

Nous démontrons ici que, pour p proche de 1, la probabilité d'avoir un chemin espace-temps fermé décroissant qui relie deux points décroît exponentiellement vite avec la distance entre les deux points. Nous commençons par énoncer un lemme combinatoire.

Lemme 2.4.1. *Soit $S(n, m)$ l'ensemble des m -uplets d'entiers défini par :*

$$S(n, m) = \{ (u_1, \dots, u_m) \in \{1, \dots, n\}^m : u_{i+1} > u_i + 1, \quad 1 \leq i \leq m-1 \}.$$

Alors

$$|S(n, m)| = \binom{n-m+1}{m}.$$

Démonstration. Nous considérons l'application

$$\Phi : (u_1, \dots, u_m) \rightarrow (u_1, \dots, u_i - i + 1, \dots, u_m - m + 1).$$

L'application Φ est une bijection de $S(n, m)$ sur l'ensemble des m -uplets d'entiers strictement croissants entre 1 et $n - m + 1$, i.e.,

$$\{ (u_1, \dots, u_m) \in \{1, \dots, n - m + 1\}^m : u_{i+1} > u_i, \quad 1 \leq i \leq m-1 \}.$$

Ce dernier ensemble est de cardinal $\binom{n-m+1}{m}$. \square

Nous montrons le théorème 2.1.1.

Démonstration. Par les propositions 2.2.2 et 2.3.3, nous pouvons supposer que le chemin espace-temps est simple et impatient et nous notons

$$\gamma = (e_1, t_1), \dots, (e_N, t_N)$$

ce chemin espace-temps. Fixons d'abord la projection spatiale de γ et notons $\Gamma = \text{space}(\gamma)$ et $\mathcal{E}(\Gamma)$ l'événement

$$\left\{ \begin{array}{l} \text{il existe un chemin espace-temps décroissant de projection spatiale } \Gamma \\ \text{de longueur } n \text{ partant de } O \text{ et fermé dans } X \text{ entre } s \text{ et } t \\ \text{et reste fermé pendant les changements de temps} \end{array} \right\}.$$

Nous avons

$$P \left(\begin{array}{l} \text{il existe un chemin espace-temps décroissant} \\ \text{de longueur } n \text{ partant de } O \text{ fermé entre } s \text{ et } t \\ \text{et reste fermé pendant les changements de temps} \end{array} \right) \leq \sum_{\Gamma} P(\mathcal{E}(\Gamma)).$$

Notons

$$k(1), \dots, k(m)$$

les indices où les changements de temps ont lieu dans γ . Par définition, nous avons $t_{k(i)+1} = t_{k(i+1)}$ pour $1 \leq i \leq m$. Par convention, nous posons $k(0) = 1$ et $k(m+1) = N$. Quitte à arrêter γ à l'instant où il visite y pour la première fois, nous pouvons supposer que γ ne se termine pas par un changement de temps, de sorte que $k(m) < n - 1$. Pour $0 \leq i \leq m$ et pour une arête $e \in \text{support}(\gamma)$, nous notons $J(e)$ l'ensemble des indices défini par

$$J(e) = \{ i \in \{1, \dots, N\} : e_i = e \}$$

et $\mathcal{E}(e)$ l'événement :

$$\mathcal{E}(e) = \left\{ \begin{array}{l} \forall i \in J(e) \quad e \text{ est fermée à } t_i \\ \forall i \in \{k(1), \dots, k(m)\} \cap J(e) \quad e_{i+1} \text{ se ferme à } t_i \\ \text{et } e \text{ reste fermée entre } t_{i-1} \text{ et } t_i \end{array} \right\}.$$

Comme les différentes arêtes sont indépendantes dans le processus X , nous pouvons factoriser la probabilité $P(\mathcal{E}(\Gamma))$ selon les arêtes et écrire

$$P(\mathcal{E}(\Gamma)) = \sum_{0 \leq m \leq n/2} \sum_{1 \leq k(1) < \dots < k(m) \leq n} \sum_{t_{k(1)}, \dots, t_{k(m)}} \prod_{e \in \text{support}(\Gamma)} P(\mathcal{E}(e)).$$

Nous étudions maintenant chaque terme $P(\mathcal{E}(e))$. Comme l'état de l'arête e est donné par une chaîne de Markov à deux états et comme γ est simple, entre deux visites consécutives de l'arête e autres que les changements de temps, il existe un instant r où l'arête e est ouverte, i.e.,

$$\forall i, j \in J(e) \quad (j > i, i - j \neq 1) \Rightarrow \exists r \in]t_j, t_i[, X_r(e) = 1.$$

La probabilité d'avoir $|J(e) \setminus \{k(1), \dots, k(m)\}|$ visites où e est fermée à chaque visite est majorée par

$$(1-p)^{|J(e) \setminus \{k(1), \dots, k(m)\}|}.$$

Soit maintenant

$$k(i) \in J(e) \cap \{k(1), \dots, k(m)\}.$$

A chaque instant t , nous choisissons une arête uniformément parmi toutes les arêtes de $\Lambda(l)$ et nous déterminons le nouvel état de cette arête selon une loi de Bernoulli de paramètre p , la probabilité que e reste fermée entre $t_{k(i)}$ et $t_{k(i-1)}$ (remarquons ici que $t_{k(i-1)} = t_{k(i)-1}$) est donc

$$\left(1 - \frac{p}{|\Lambda|}\right)^{t_{k(i-1)} - t_{k(i)}}.$$

Enfin, la probabilité que $e_{k(i)+1}$ change son état à l'instant $t_{k(i)}$ est majorée par $1/|\Lambda|$. Nous obtenons

$$P(\mathcal{E}(e)) \leq (1-p)^{|J(e) \setminus \{k(1), \dots, k(m)\}|} \prod_{i \in J(e) \cap \{k(1), \dots, k(m)\}} \left(1 - \frac{p}{|\Lambda|}\right)^{t_{k(i-1)} - t_{k(i)}} \frac{1}{|\Lambda|}.$$

Nous injectons cette majoration dans le produit précédent et nous obtenons

$$P(\mathcal{E}(\Gamma)) \leq \sum_{0 \leq m \leq n/2} \sum_{1 \leq k(1) < \dots < k(m) \leq n} \sum_{t_{k(1)}, \dots, t_{k(m)}} (1-p)^{n-m} \left(1 - \frac{p}{|\Lambda|}\right)^{\sum_{i=1}^m t_{k(i-1)} - t_{k(i)}} \frac{1}{|\Lambda|^m}.$$

Calculons d'abord la somme sur les instants $t_{k(1)}, \dots, t_{k(m)}$. Posons

$$\forall i \in \{0, \dots, m\} \quad \Delta_i = t_{k(i-1)} - t_{k(i)}.$$

Si m et les indices $k(1), \dots, k(m)$ sont fixés, la suite $t_{k(1)}, \dots, t_{k(m)}$ est déterminée par $t_{k(1)}$ et les valeurs $\Delta_0, \dots, \Delta_{m-1}$, d'où

$$\begin{aligned} \sum_{t_{k(1)}, \dots, t_{k(m)}} \left(1 - \frac{p}{|\Lambda|}\right)^{\sum_{i=1}^m t_{k(i-1)} - t_{k(i)}} \\ \leq (t-s) \sum_{1 \leq \Delta_1, \dots, \Delta_{m-1} \leq t-s} \left(1 - \frac{p}{|\Lambda|}\right)^{\Delta_1 + \dots + \Delta_{m-1}}. \end{aligned}$$

Nous échangeons la somme et le produit et nous obtenons

$$\begin{aligned} \sum_{1 \leq \Delta_1, \dots, \Delta_{m-1} \leq t-s} \left(1 - \frac{p}{|\Lambda|}\right)^{\Delta_1 + \dots + \Delta_{m-1}} &= \prod_{i=1}^{m-1} \left(\sum_{\Delta_i=1}^{t-s} \left(1 - \frac{p}{|\Lambda|}\right)^{\Delta_i} \right) \\ &= \prod_{i=1}^{m-1} \left(1 - \frac{p}{|\Lambda|}\right)^{1 - \left(1 - \frac{p}{|\Lambda|}\right)^{t-s}} \leq \prod_{i=1}^{m-1} \frac{1 - \left(1 - \frac{p}{|\Lambda|}\right)^{t-s}}{\frac{p}{|\Lambda|}}. \end{aligned}$$

Comme $(1-x)^\alpha \geq 1 - \alpha x$ pour $0 < x < 1$ et $\alpha \geq 1$, nous avons

$$\prod_{i=1}^{m-1} \frac{1 - \left(1 - \frac{p}{|\Lambda|}\right)^{t-s}}{\frac{p}{|\Lambda|}} \leq (t-s)^{m-1}.$$

Nous avons donc

$$P(\mathcal{E}(\Gamma)) \leq \sum_{0 \leq m \leq n/2} \sum_{1 \leq k(1) < \dots < k(m) < n} (1-p)^{n-m} (t-s)^m \frac{1}{|\Lambda|^m}.$$

Or, d'après le lemme 1, le nombre de termes dans la seconde somme est $\binom{n-m+1}{m}$ et nous avons

$$\begin{aligned} P(\mathcal{E}(\Gamma)) &\leq \sum_{0 \leq m \leq \frac{n}{2}} \binom{n-m+1}{m} (1-p)^{n-m} (t-s)^m \frac{1}{|\Lambda|^m} \\ &\leq (1-p)^{n/2} \sum_{0 \leq m \leq \frac{n}{2}} \frac{n^m}{m! |\Lambda|^m} (t-s)^m \\ &\leq \exp\left(\frac{n(t-s)}{|\Lambda|} + \frac{n}{2} \ln(1-p)\right). \end{aligned}$$

Enfin, sommons selon les choix de Γ . Comme le nombre d'un chemin de longueur n est majoré par $\alpha(d)^n$ où $\alpha(d)$ est le nombre de voisins d'une arête en dimension d , nous avons ainsi l'inégalité voulue. \square

Remarquons que cet estimé reste valable si nous considérons un chemin espace-temps fermé et qui n'est pas forcément fermé pendant les changements de temps. Dans la suite de notre étude, le chemin espace-temps sera un objet essentiel pour analyser les mouvements des arêtes pivot. En fait, nous pouvons relier les arêtes pivots à différents instants par un chemin espace-temps. Par contre, les chemins espace-temps n'auront pas les mêmes propriétés que celui présenté dans ce chapitre et seront plus difficiles à étudier. Par exemple, dans le chapitre trois, le chemin espace-temps utilisé

est construit à partir d'un processus conditionné où les arêtes ne sont pas indépendantes. Dans le chapitre quatre, le chemin espace-temps contient des arêtes ouvertes à certain instant. Dans ces cas plus complexes, nous pouvons encore montrer des résultats de type décroissance exponentielle.

Chapitre 3

Un nouveau regard sur l'interface

Ce chapitre est adapté de l'article [CZ18a], réalisé en collaboration avec Raphaël Cerf. La version préprint est disponible sur: <https://arxiv.org/abs/1806.08576>

We propose a new definition of the interface in the context of the Bernoulli percolation model. We construct a coupling between two percolation configurations, one which is a standard percolation configuration, and one which is a percolation configuration conditioned on a disconnection event. We define the interface as the random set of the edges where these two configurations differ. We prove that, inside a cubic box Λ , the interface between the top and the bottom of the box is typically localised within a distance of order $(\ln |\Lambda|)^2$ of the set of the pivotal edges.

3.1 Introduction

At the macroscopic level, the interface between two pure phases seems to be deterministic. In fact, such an interface obeys a minimal action principle: it minimizes the surface tension between the two phases and it is close to the solution of a variational problem. This can be seen as an empirical law, derived from the observation at the macroscopic level. This law has been justified from a microscopic point of view in the context of the Ising model [DKS92]. One starts with a simple model of particles located on a discrete lattice. There are two types of particles, which have a slight tendency to repel each other. In the limit where the number of particles tends to ∞ , at low temperatures, the system presents a phenomenon of phase segregation, with the formation of interfaces between two pure phases. On a suitable scale, these interfaces converge towards deterministic shapes, a prominent example being the Wulff crystal of the Ising model, which is the typical shape of the Ising droplets. Although the limit is deterministic on the

macroscopic level, the interfaces are intrinsically random objects and their structure is extremely complex. In two dimensions, the fluctuations of the Ising interfaces were precisely analysed in the DKS theory, with the help of cluster expansions [DH97, DKS92]. In higher dimensions, there is essentially one result on the fluctuations of the interfaces, due to Dobrushin [Dob72], which says that horizontal interfaces stay localised at low temperatures. When dealing with interfaces in the Ising model, the first difficulty is to get a proper definition of the interface itself. The usual way is to start with the Dobrushin type boundary conditions, that is a box with pluses on its upper half boundary and minuses on its lower half boundary. This automatically creates an Ising configuration in the box with a microscopic interface between the pluses and the minuses which separates the upper half and the lower half of the box. Yet it is still not obvious how one should define the interface in this case, because several such microscopic interfaces exist, and a lot of different choices are possible. Dobrushin, Kotecky and Shlosman [DKS92] introduced a splitting rule between contours, which leads to pick up one particular microscopic interface. The potential problem with this approach is that the outcome is likely to include microscopic interfaces which are not necessarily relevant, for instance interfaces between opposite signs which would have been present anyway, and which are not induced by the Dobrushin boundary conditions.

Our goal here is to propose a new way to look at the random interfaces, in any dimension $d \geq 2$. We start our investigation in the framework of the Bernoulli percolation model, for several reasons. First, the probabilistic structure of the percolation model is simpler than the one of the Ising model. Another reason is that the Wulff theorem in dimensions three was first derived for the percolation model [Cer00] and then extended to the Ising model [Bod99, CP00]. A key fact was that the definition of the surface tension is much simpler for the percolation model than for the Ising model. This leads naturally to hope that the probabilistic structure of the interfaces should be easier to apprehend as well in the percolation model. Finally, in the context of percolation, one sees directly which edges are essential or not in an interface: these are the pivotal edges. There is no corresponding notion in the Ising model. For all these reasons, it seems wise to try to develop a probabilistic description of random interfaces in the framework of Bernoulli percolation.

In this paper, we consider the Bernoulli bond percolation model with a parameter p close to 1. Interfaces in a cubic box Λ are naturally created when the configuration is conditioned on the event that the top T and the bottom B of the box are disconnected. From now onwards, this event is denoted by

$$\{ T \leftrightarrow B \}.$$

Our goal is to gain some understanding on the typical configurations realiz-

ing such a disconnection event. To do so, we build a coupling between two percolation configurations X, Y in the box Λ such that:

- The edges in X are i.i.d., open with probability p and closed with probability $1 - p$.
- The distribution of Y is the distribution of the Bernoulli percolation conditioned on $\{T \leftrightarrow B\}$.
- Every edge open in Y is also open in X .

We define then the random interface between the top T and the bottom B of the box Λ as the random set \mathcal{I} of the edges where X and Y differ:

$$\mathcal{I} = \{e \subset \Lambda : X(e) \text{ is open, } Y(e) \text{ is closed}\}.$$

Among these edges, some are essential for the disconnection between T and B to occur. These edges are called pivotal and they are denoted by \mathcal{P} :

$$\mathcal{P} = \left\{ e \in \mathcal{I} : \begin{array}{l} \text{the opening of } e \text{ in } Y \text{ would create} \\ \text{an open connection between } T \text{ and } B \end{array} \right\}.$$

When conditioning on the disconnection between T and B , a lot of pivotal edges are created. Yet another collection of edges which are not essential for the disconnection event turn out to be closed as well. Therefore it becomes extremely difficult to understand the effect of the conditioning on the distribution by looking at the conditioned probability measure alone. This is why we build a coupling and we define the interface as the set of the edges where the two percolation configurations differ. The set \mathcal{P} of the pivotal edges can be detected by a direct inspection of the conditioned configuration, but not the interface \mathcal{I} . Our main result provides a quantitative control on the interface \mathcal{I} with respect to the set \mathcal{P} of the pivotal edges. We denote by μ_p the coupling probability measure between the configurations X and Y . The precise construction of μ_p is done in section 3.2. We denote by d the usual Euclidean distance on \mathbb{R}^d , by Λ a cubic box with sides parallel to the axis of \mathbb{Z}^d , and by $|\Lambda|$ the cardinality of $\Lambda \cap \mathbb{Z}^d$.

Theorem 3.1.1. *There exists $\tilde{p} < 1$ and $\kappa > 0$, such that, for $p \geq \tilde{p}$, any $c \geq 1$ and any box Λ satisfying $|\Lambda| \geq \max\{(cd)^{cd^2}, 3^{6d}\}$,*

$$\mu_p\left(\exists e \in \mathcal{P} \cup \mathcal{I}, d(e, \Lambda^c \cup \mathcal{P} \setminus \{e\}) \geq \kappa c^2 \ln^2 |\Lambda|\right) \leq \frac{1}{|\Lambda|^c}.$$

The typical picture which emerges from theorem 3.1.1 is the following. In the configuration conditioned on the event $\{T \leftrightarrow B\}$, there is a set \mathcal{P} of pivotal edges. These are the edges having one extremity connected by an open path to the top T and the other extremity connected by an open path to the bottom B . Because of the conditioning, compared to the i.i.d. configuration, some additional edges are closed, but they are typically within a distance of order $(\ln |\Lambda|)^2$ of the set \mathcal{P} of the pivotal edges. The edges

which are further away from \mathcal{P} behave as in the ordinary unconditioned percolation. Therefore the interface \mathcal{I} is strongly localised around the set \mathcal{P} of the pivotal edges. The interface is a dust of closed edges pinned around the pivotal edges.

Our next endeavour was to obtain a conditional version of theorem 3.1.1. More precisely, we would like to estimate the conditional probability

$$\mu_p \left(e \in \mathcal{P} \cup \mathcal{I} \mid d(e, \mathcal{P} \setminus \{e\}) \geq \kappa (\ln |\Lambda|)^2 \right).$$

We did not really succeed so far, however we are able to control the interface conditionally on the distance to a cut. Before stating our result, let us recall the definition of a cut.

Definition 3.1.2. *A set S of edges separates the top T and the bottom B in Λ if every deterministic path of edges from T to B in Λ intersects S . A cut C between T and B in Λ is a set of edges which separates T and B in Λ and which is minimal for the inclusion.*

A cut C is closed in the configuration Y if all the edges of C are closed in Y . We denote by \mathcal{C} the collection of the closed cuts present in Y . Since Y realizes the event $\{T \leftrightarrow B\}$, the collection \mathcal{C} is not empty.

Theorem 3.1.3. *We have the following inequality:*

$$\begin{aligned} \exists \tilde{p} < 1 \quad \exists \kappa > 0 \quad \forall p \geq \tilde{p} \quad \forall c \geq 2 \quad \forall \Lambda \quad \ln |\Lambda| > 4 + c + 2dc^2 + 12(2\kappa d)^d \\ \forall e \in \Lambda \quad d(e, \Lambda^c) \geq \kappa c^2 \ln^2 |\Lambda| \\ \mu_p \left(e \in \mathcal{I} \mid \exists C \in \mathcal{C}, d(e, C) \geq \kappa c^2 \ln^2 |\Lambda| \right) \leq \frac{1}{|\Lambda|^c}. \end{aligned}$$

Let us explain briefly how we build the coupling probability measure μ_p , as well as the strategy for proving theorem 3.1.1. Conditioning on the event $\{T \leftrightarrow B\}$ creates non trivial correlations between the edges, and there is no simple tractable formula giving for instance the conditional distribution of a finite set of edges. Yet a standard application of the FKG inequality yields that, for any increasing event A , we have

$$P_p(A \mid T \leftrightarrow B) \leq P_p(A).$$

Thus the product measure P_p stochastically dominates the conditional measure $P_p(\cdot \mid T \leftrightarrow B)$. Strassen's theorem tells us that there exists a monotone coupling between these two probability measures. In order to derive quantitative estimates on the differences between the coupled configurations, we build our coupling measure as the invariant measure of a dynamical process. This method of coupling is standard, for instance it is used in the proof of Holley inequality (see chapter 2 of [Gri95]). Our contribution is to

study some specific properties of this coupling in the context of percolation, and to relate it to the geometry of the interfaces. To do so, we consider the classical dynamical percolation process in the box Λ , see [Ste09]. Since we always work in a finite box, we use the following discrete time version. We start with an initial configuration X_0 . At each step, we choose one edge uniformly at random, and we update its state with a coin of parameter p . Of course all the random choices are independent. The resulting process is denoted by $(X_t)_{t \in \mathbb{N}}$. Obviously the invariant probability measure of $(X_t)_{t \in \mathbb{N}}$ is the product measure P_p and the process $(X_t)_{t \in \mathbb{N}}$ is reversible with respect to P_p . Next, we duplicate the initial configuration X_0 , thereby getting a second configuration Y_0 . We use the same random variables as before to update this second configuration, with one essential difference. In the second configuration, we prohibit the opening of an edge if this opening creates a connection between the top T and the bottom B . This mechanism ensures that X_t is always above Y_t . Moreover, a classical result on reversible Markov chains ensures that the invariant probability measure of the process $(Y_t)_{t \in \mathbb{N}}$ is the conditional probability measure $P_p(\cdot \mid T \leftrightarrow B)$. Our coupling probability measure μ_p is defined as the invariant probability measure of the process $(X_t, Y_t)_{t \in \mathbb{N}}$. In the case of the Ising model, where one has access to an explicit formula for the equilibrium measure, one usually derives results on the dynamics (for instance the Glauber dynamics) from results on the Ising Gibbs measure. We go here in the reverse direction: we use our dynamical construction to derive results on the equilibrium measure μ_p . For the proof, we consider the stationary process $(X_t, Y_t)_{t \in \mathbb{N}}$ starting from its equilibrium distribution μ_p . We fix a time t and we estimate the probability that the configuration (X_t, Y_t) realizes the event appearing in the statement of theorem 3.1.1. We distinguish the case of edges in the interface which are pivotal or not. For pivotal edges, we shall prove the following slightly stronger result.

Proposition 3.1.4. *There exists $\tilde{p} < 1$ and $\kappa > 1$ such that, for $p \geq \tilde{p}$, and for any $c \geq 1$ and any box Λ satisfying $|\Lambda| > 3^{6d}$, we have*

$$P_p\left(\exists e \in \mathcal{P}, d(e, \Lambda^c \cup \mathcal{P} \setminus \{e\}) \geq \kappa c \ln |\Lambda| \mid T \leftrightarrow B\right) \leq \frac{1}{|\Lambda|^c}.$$

The proof of this proposition relies on the BK inequality. We consider next the case of an edge e in the interface which is not pivotal. Such an edge e can be opened at any time in the configuration Y . Therefore, unless it becomes pivotal again, it cannot stay for a long time in the interface. In addition, before becoming part of the interface \mathcal{I} , the edge e must have been pivotal. Indeed, non-pivotal edges in the process $(Y_t)_{t \in \mathbb{N}}$ evolve exactly as in the process $(X_t)_{t \in \mathbb{N}}$. We look backwards in the past at the last time when the edge e was still pivotal. As said before, this time must be quite close from t . However, at time t , it turns out that the set of the pivotal edges is quite far

from e . We conclude that the set of the pivotal edges must have moved away from e very fast. To estimate the probability of a fast movement of the set \mathcal{P} , we derive an estimate on the speed of the set of the pivotal edges, which is stated in proposition 3.4.1. This estimate is at the heart of the argument. It relies on the construction of specific space–time paths, which describe how the influence of the conditioning propagates in the box. If a space–time path travels over a long distance in a short time, then this implies that a certain sequence of closing events has occurred, and we estimate the corresponding probability. This estimate is delicate, because the closed space–time path can take advantage of the pivotal edges which remain closed thanks to the conditioning. The computation relies again on the BK inequality, this time applied to the space–time paths.

The statement of theorem 3.1.1 naturally prompts several questions. First, the results presented here hold only for values of p sufficiently close to 1, because the proofs rely on Peierls arguments.

Question 1. Is it possible to prove an analogous result throughout the supercritical regime $p > p_c$?

Proposition 3.1.4 shows that, typically, each pivotal edge is within a distance of order $\ln |\Lambda|$ of another pivotal edge. Of course, we would like to understand better the random set \mathcal{P} .

Question 2. What else can be said about the structure of the set \mathcal{P} ?

Since there is no square in the logarithm appearing in proposition 3.1.4, we suspect that it should also be the case in the statement of theorem 3.1.1.

Question 3. Is it possible to replace $(\ln |\Lambda|)^2$ by $\ln |\Lambda|$ in the statement of theorem 3.1.1?

Ultimately, we would like to gain some understanding on the Ising interfaces. The natural road to transfer percolation results towards the Ising model is to use the FK percolation model. However, there are several difficulties to overcome in order to adapt the proof to FK percolation. First, we use the BK inequality twice in the proof, and this inequality is not available in the FK model. Second, the dynamics for the FK model is more complicated.

Question 4. Does theorem 3.1.1 extend to the FK percolation model?

Suppose that the answer to question 4 is positive. It is not obvious to transcribe theorem 3.1.1 in the Ising context. For instance, the pivotal edges, which can be detected by visual inspection of a percolation configuration, are hidden inside the associated Ising configuration.

Question 5. What is the counterpart of theorem 3.1.1 for Ising interfaces?

We hope to attack successfully these questions in future works. A partial step in the direction of question 3 is done in [CZ18b]. The question 4 and 5 are tackled in [Zho18].

The paper is organized as follows. In section 3.2, we define precisely the model and the notations. Beyond the classical percolation definitions, this section contains the definition of the space–time paths and the graphical construction of the coupling. Section 3.3 is devoted to the proof of proposition 3.1.4. In section 3.4, we prove the central result on the control of the speed of the set of the pivotal edges. Then, the theorem 3.1.1 is proved in section 3.5. In section 3.6, we improve the results obtained in section 3.4 and finally we prove the theorem 3.1.3 in section 3.7.

3.2 The model and notations

3.2.1 Geometric definitions

We give standard geometric definitions.

The edges \mathbb{E}^d . The set of edges \mathbb{E}^d is the set of the pairs $\{x, y\}$ of points of \mathbb{Z}^d which are at Euclidean distance 1.

The box Λ . We will mostly work in a closed box Λ centred at the origin. We denote by T the top side of Λ and by B its bottom side.

The separating sets. Let A, B be two subsets of Λ . We say that a set of edges $S \subset \Lambda$ separates A and B if no connected subset of $\Lambda \cap \mathbb{E}^d \setminus S$ intersects both A and B . Such a set S is called a separating set for A and B . We say that a separating set is minimal if there does not exist a strict subset of S which separates A and B .

The cuts. We say that S is a cut if S separates T and B , and S is minimal for the inclusion.

The usual paths. We say that two edges e and f are neighbours if they have one endpoint in common. A usual path is a sequence of edges (e_1, \dots, e_n) such that for $1 \leq i < n$, the edge e_i and e_{i+1} are neighbours.

The *-paths. In order to study the cuts in any dimension $d \geq 2$, we use *-connectedness on the edges as in [DP96]. We consider the supremum norm on \mathbb{R}^d :

$$\forall x = (x_1, \dots, x_d) \in \mathbb{R}^d \quad \|x\|_\infty = \max_{i=1, \dots, d} |x_i|.$$

For e an edge in \mathbb{E}^d , we denote by m_e the center of the unit segment associated to e . We say that two edges e and f of \mathbb{E}^d are *-neighbours if $\|m_e - m_f\|_\infty \leq 1$. A *-path is a sequence of edges (e_1, \dots, e_n) such that, for $1 \leq i < n$, the edge e_i and e_{i+1} are *-neighbours.

3.2.2 The dynamical percolation.

We define the dynamical percolation and the space-time paths.

Percolation configurations. A percolation configuration in Λ is a map from the set of the edges included in Λ to $\{0, 1\}$. An edge $e \subset \Lambda$ is said to be open in a configuration ω if $\omega(e) = 1$ and closed if $\omega(e) = 0$. For two subsets A, B of Λ and a configuration $\omega \in \Omega$, we denote by $A \xleftrightarrow{\omega} B$ the event that there is an open path between a vertex of A and a vertex of B in the configuration ω .

Probability measures. We denote by P_p the law of the Bernoulli bond percolation in the box Λ with parameter p . The probability P_p is the probability measure on the set of bond configurations which is the product of the Bernoulli distribution $(1-p)\delta_0 + p\delta_1$ over the edges included in Λ . We define $P_{\mathcal{D}}$ as the probability measure P_p conditioned on the event $\{T \leftrightarrow B\}$, i.e.,

$$P_{\mathcal{D}}(\cdot) = P_p(\cdot \mid T \leftrightarrow B).$$

Probability space. Throughout the paper, we assume that all the random variables used in the proofs are defined on the same probability space Ω . For instance, this space contains the random variables used in the graphical construction presented below, as well as the random variables generating the initial configurations of the Markov chains. We denote simply by P the probability measure on Ω .

Graphical construction. We now present a graphical construction of the dynamical percolation in the box Λ . We build a sequence of triplets $(X_t, E_t, B_t)_{t \in \mathbb{N}}$, where X_t is the percolation configuration in Λ at time t , E_t is a random edge in the box Λ and B_t is a Bernoulli random variable. The sequence $(E_t)_{t \in \mathbb{N}}$ is an i.i.d. sequence of edges, with uniform distribution over the edges included in Λ . The sequence $(B_t)_{t \in \mathbb{N}}$ is an i.i.d. sequence of Bernoulli random variables with parameter p . The sequence $(E_t)_{t \in \mathbb{N}}$ and $(B_t)_{t \in \mathbb{N}}$ are independent. The process $(X_t)_{t \in \mathbb{N}}$ is built iteratively as follows. At time 0, we start from the configuration X_0 , which might be random. At time t , we change the state of E_t to B_t and we set

$$\forall t \geq 1 \quad X_t(e) = \begin{cases} X_{t-1}(e) & \text{if } E_t \neq e \\ B_t & \text{if } E_t = e \end{cases}.$$

The process $(X_t)_{t \in \mathbb{N}}$ is the dynamical percolation process in the box Λ .

The space-time paths. We introduce the space-time paths which generalise both the usual paths and the $*$ -paths to the dynamical percolation. A

space-time path is a sequence of pairs, called time-edges, $(e_i, t_i)_{1 \leq i \leq n}$, such that, for $1 \leq i \leq n-1$, we have either $e_i = e_{i+1}$, or (e_i, e_{i+1}) are neighbours and $t_i = t_{i+1}$. We say that a space-time path $(e_i, t_i)_{1 \leq i \leq n}$ is during a time interval $[s, t]$ if for all $1 \leq i \leq n$, we have $t_i \in [s, t]$. We define also space-time $*$ -paths, by using edges which are $*$ -neighbours in the above definition. For s, t two integers, we define

$$s \wedge t = \min(s, t), \quad s \vee t = \max(s, t).$$

A space-time path $(e_i, t_i)_{1 \leq i \leq n}$ is open in the dynamical percolation process $(X_t)_{t \in \mathbb{N}}$ if

$$\forall i \in \{1, \dots, n\} \quad X_{t_i}(e_i) = 1$$

and

$$\forall i \in \{1, \dots, n-1\} \quad e_i = e_{i+1} \implies \forall t \in [t_i \wedge t_{i+1}, t_i \vee t_{i+1}] \quad X_t(e_i) = 1.$$

In the same way, we can define a closed space-time path by changing 1 to 0 in the previous definition. In the remaining of the article, we use the abbreviation STP to design a space-time path. Moreover, unless otherwise specified, the closed paths (and the closed STPs) are defined with the relation $*$ and the open paths (and the open STPs) are defined with the usual relation. This is because the closed paths come from the cuts, while the open paths come from existing connexions.

3.2.3 The interfaces by coupling.

We propose a new way of defining the interfaces by coupling two processes of dynamical percolation. We start with the graphical construction $(X_t, E_t, B_t)_{t \in \mathbb{N}}$ of the dynamical percolation. We define a further process $(Y_t)_{t \in \mathbb{N}}$ as follows: at time 0, we set $X_0 = Y_0$, and for all $t \geq 1$, we set

$$\forall e \in \Lambda \quad Y_t(e) = \begin{cases} Y_{t-1}(e) & \text{if } e \neq E_t \\ 0 & \text{if } e = E_t \text{ and } B_t = 0 \\ 1 & \text{if } e = E_t, B_t = 1 \text{ and } T \xrightarrow{Y_{t-1}^{E_t}} B \\ 0 & \text{if } e = E_t, B_t = 1 \text{ and } T \xrightarrow{Y_t^{E_t}} B \end{cases},$$

where, for a configuration ω and an edge e , the notation ω^e means the configuration obtained by opening e in ω . Typically, we start with a configuration Y_0 realizing the event $\{T \leftrightarrow B\}$, but this is not mandatory in the above definition. An illustration of this dynamics is given in the figure 3.1. We denote by $P_{\mathcal{D}}$ the equilibrium distribution of the process $(Y_t)_{t \in \mathbb{N}}$. Before opening a closed edge e at time t , we verify whether this will create a connexion between T and B . If it is the case, the edge e stays closed in the

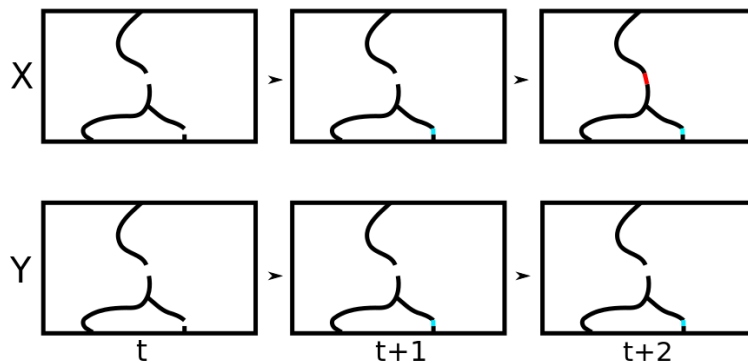


Figure 3.1: A coupling of the process $(X_t, Y_t)_{t \in \mathbb{N}}$. At time $t + 1$ we try to open the blue edge and at time $t + 2$, we try to open the red edge.

process $(Y_t)_{t \in \mathbb{N}}$ but can be opened in the process $(X_t)_{t \in \mathbb{N}}$, otherwise the edge e is opened in both processes $(X_t)_{t \in \mathbb{N}}$ and $(Y_t)_{t \in \mathbb{N}}$. On the contrary, the two processes behave similarly for the edge closing events since we cannot create a new connexion by closing an edge. The set of the configurations satisfying $\{T \leftrightarrow B\}$ is irreducible and the process $(X_t)_{t \in \mathbb{N}}$ is reversible. Therefore, the process $(Y_t)_{t \in \mathbb{N}}$ is the dynamical percolation conditioned to satisfy the event $\{T \leftrightarrow B\}$. According to the lemma 1.9 of [Kel11], the invariant probability measure of $(Y_t)_{t \in \mathbb{N}}$ is $P_{\mathcal{D}}$, the probability P_p conditioned by the event $\{T \leftrightarrow B\}$, i.e.,

$$P_{\mathcal{D}}(\cdot) = P_p(\cdot | T \leftrightarrow B).$$

Suppose that we start from a configuration (X_0, Y_0) belonging to the set

$$\mathcal{E} = \{(\omega_1, \omega_2) \in \{0, 1\}^{\mathbb{E}^d \cap \Lambda} \times \{T \leftrightarrow B\} : \forall e \subset \Lambda \quad \omega_1(e) \geq \omega_2(e)\}.$$

The set \mathcal{E} is irreducible and aperiodic. In fact, each configuration of \mathcal{E} communicates with the configuration where all edges are closed. The state space \mathcal{E} is finite, therefore the Markov chain $(X_t, Y_t)_{t \in \mathbb{N}}$ admits a unique equilibrium distribution μ_p . We denote by P_μ the law of the process $(X_t, Y_t)_{t \in \mathbb{N}}$ starting from a random initial configuration (X_0, Y_0) with distribution μ_p . We now present a definition of the interface between T and B based on the previous coupling.

Definition 3.2.1. *The interface at time t between T and B , denoted by \mathcal{I}_t , is the set of the edges in Λ that differ in the configurations X_t and Y_t , i.e.,*

$$\mathcal{I}_t = \{e \subset \Lambda : X_t(e) \neq Y_t(e)\}.$$

The edges of \mathcal{I}_t are open in X_t but closed in Y_t and the configuration X_t is above the configuration Y_t . We define next the set \mathcal{P}_t of the pivotal edges for the event $\{T \leftrightarrow B\}$ in the configuration Y_t .

Definition 3.2.2. *The set \mathcal{P}_t of the pivotal edges in Y_t is the collection of the edges in Λ whose opening would create a connection between T and B , i.e.,*

$$\mathcal{P}_t = \{ e \in \Lambda : T \xleftrightarrow{Y_t^e} B \}.$$

We define finally the set \mathcal{C}_t of the cuts in Y_t .

Definition 3.2.3. *The set \mathcal{C}_t of the cuts in Y_t is the collection of the cuts in Λ at time t .*

3.3 The isolated pivotal edges

In this section, we will show the proposition 3.1.4. We first investigate the structure of the set of the cuts. In a configuration ω realizing the event $\{T \leftrightarrow B\}$, we will identify two separating sets S^+ and S^- . We construct S^+ by considering the open cluster

$$O(T) = \{ x \in \mathbb{Z}^d \cap \Lambda : x \xleftrightarrow{\omega} T \}.$$

We consider the set $O(T)^c = \mathbb{Z}^d \setminus O(T)$. As $\mathbb{Z}^d \setminus \Lambda$ is $*$ -connected, there exists only finitely many $*$ -connected components of $O(T)^c$ and exactly one of them is of infinite size. We denote these components by G, H_1, \dots, H_k where G is the unique infinite component. We set

$$O'(T) = O(T) \cup H_1 \cup \dots \cup H_k.$$

The set $O'(T)$ is $*$ -connected and has no holes. For a $*$ -connected set $A \subset \mathbb{Z}^d$, we define the external boundary of A , denoted by $\partial^{ext} A$, as

$$\partial^{ext} A = \{ \{x, y\} \in \mathbb{E}^d : x \in A, y \notin A \}.$$

We then define S^+ as the subset of $\partial^{ext} O'(T)$ consisting of the edges of $\partial^{ext} O'(T)$ which are included in Λ . In a similar way, we define S^- by replacing T by B in the previous construction. Each of the two sets contains a cut. An illustration of these two separating sets can be found in the figure 3.2.

Lemma 3.3.1. *The sets $\partial^{ext} O'(T)$ and $\partial^{ext} O'(B)$ are $*$ -connected.*

This result is a direct consequence of the first point in lemma 2.1 in [DP96]. We also mention the lemma 2.23 in [Kes86] for a similar result on the set of vertices and a shorter argument presented in [Tim07]. We explain next the relation between the sets S^+ , S^- and \mathcal{P} .

Lemma 3.3.2. *The set \mathcal{P} of the pivotal edges is the intersection between S^+ and S^- .*

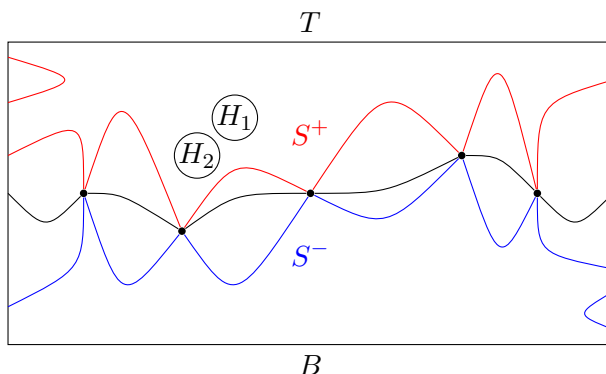


Figure 3.2: The sets S^+ (red) and the set S^- (blue).

Proof. We have the inclusion $\mathcal{P} \subset S^+ \cap S^-$ since all the pivotal edges are in all the cuts. Both S^+ and S^- contain a cut. We consider next an edge e in $S^+ \cap S^-$. Since S^+ consists of the boundary edges of $O(T)$, there is an open path between T and e . The same result holds for S^- . Therefore, there is a path between T and B whose edges other than e are open. By opening e , we realise the event $\{T \longleftrightarrow B\}$. In other words, the edge e is included in \mathcal{P} . We conclude that $S^+ \cap S^- \subset \mathcal{P}$. \square

We also need a combinatoric result on the $*$ -connectedness in dimension d .

Lemma 3.3.3. *In the d -dimensional lattice, the number of $*$ -neighbours of an edge e is*

$$\alpha(d) = 3^d + 4(d-1)3^{d-2} - 1.$$

Proof. An $*$ -neighbour edge f of e is either parallel to e or belongs to the $(d-1)$ -cube centred at a vertex of e and of side-length 2 perpendicular to e . For the edges that are parallel to e , the distance between their centres is 1 and there are $3^d - 1$ such edges. A $(d-1)$ -cube of side-length 2 has $2(d-1)3^{d-2}$ edges. Hence there are $3^d + 4(d-1)3^{d-2} - 1$ $*$ -neighbours of e . \square

We now prove the proposition 3.1.4. The main idea of the proof is to observe a long closed path outside of a cut whenever a pivotal edge is isolated. We use then the BK inequality and we conclude with the help of classical arguments of exponential decay.

Proof of proposition 3.1.4. Since there is a pivotal edge which is at distance more than 1 from the others, there is a cut which contains at least one non pivotal edge. By lemma 3.3.2, this cut is not included in $S^- \cap S^+$, thus there are at least two distinct cuts in the configuration. Let e be an edge of \mathcal{P} which is at distance at least $\kappa c \ln |\Lambda|$ from $\Lambda^c \cup \mathcal{P} \setminus \{e\}$. Let e' be the pivotal edge which is nearest to e or one of them if there are several. By

lemma 3.3.1, there is a closed $*$ -path included in $\partial^{ext}O(T)$ between e and e' . This path might exit from the box Λ , since $\partial^{ext}O(T)$ is defined as the external boundary of $O'(T)$, where $O'(T)$ is seen as a subset of \mathbb{Z}^d , not of Λ . However, since e is at distance at least $\kappa c \ln |\Lambda|$ from $\mathcal{P} \setminus \{e\}$ and from Λ^c , the initial portion of the closed $*$ -path from its origin until it has travelled a distance $\kappa c \ln |\Lambda|$ is inside the box Λ , it consists of closed edges which are not pivotal, and therefore, by lemma 3.3.2, it is also disjoint from the set S^- . Let us denote by $\mathcal{E}(e)$ the event:

$$\mathcal{E}(e) = \left\{ \begin{array}{l} \text{there exists a closed } * \text{-path starting at one } * \text{-neighbour of } e \\ \text{which travels a distance at least } \kappa c \ln |\Lambda| - 2d \end{array} \right\}.$$

From the previous discussion, we conclude that

$$\{e \in \mathcal{P}, d(e, \Lambda^c \cup \mathcal{P} \setminus \{e\}) \geq \kappa c \ln |\Lambda|\} \cap \{T \leftrightarrow B\} \subset \mathcal{E}(e) \circ \{T \leftrightarrow B\},$$

where \circ means the disjoint occurrence. Therefore, we have the following inequality:

$$P_{\mathcal{D}}(e \in \mathcal{P}, d(e, \Lambda^c \cup \mathcal{P} \setminus \{e\}) \geq \kappa c \ln |\Lambda|) \leq P_{\mathcal{D}}(\mathcal{E}(e) \circ \{T \leftrightarrow B\}).$$

By the definition of $P_{\mathcal{D}}$, we have

$$P_{\mathcal{D}}(\mathcal{E}(e) \circ \{T \leftrightarrow B\}) = \frac{P_p(\mathcal{E}(e) \circ \{T \leftrightarrow B\})}{P_p(T \leftrightarrow B)}.$$

Note that the event $\mathcal{E}(e)$ and $\{T \leftrightarrow B\}$ are both decreasing. Applying the BK inequality (see [Gri99]), we get

$$P_{\mathcal{D}}(e \in \mathcal{P}, d(e, \Lambda^c \cup \mathcal{P} \setminus \{e\}) \geq \kappa c \ln |\Lambda|) \leq P_p(\mathcal{E}(e)).$$

The closed $*$ -path in the event $\mathcal{E}(e)$ starts at a neighbour of e and travels a distance at least $\kappa c \ln |\Lambda| - 2d$. By this, we mean that there is an Euclidean distance at least $\kappa c \ln |\Lambda| - 2d$ from one endpoint of the first edge of the path to one endpoint of the last edge of the path. The distance between the centres of two $*$ -neighbouring edges is at most d , therefore the number of edges in such a path is at least

$$\frac{1}{d}(\kappa c \ln |\Lambda| - 2d - 1).$$

We assume that $|\Lambda| \geq 3^{6d}$ and we choose $\kappa > 1$, whence, for $c \geq 1$,

$$\kappa c \ln |\Lambda| - 2d - 1 \geq \frac{\kappa c}{2} \ln |\Lambda|.$$

Hence

$$P_p(\mathcal{E}(e)) \leq (1-p) \frac{\kappa c}{2d} \ln |\Lambda| \alpha(d) \frac{\kappa c}{2d} \ln |\Lambda|.$$

We then sum the probability over all the edges e in Λ . We obtain

$$P_{\mathcal{D}}\left(\exists e \in \mathcal{P}, d(e, \Lambda^c \cup \mathcal{P} \setminus \{e\}) \geq \kappa c \ln |\Lambda|\right) \leq d|\Lambda|^{1 + \frac{\kappa c}{2d} \ln\left((1-p)\alpha(d)\right)}.$$

We choose $\tilde{p} < 1$ such that $(1-\tilde{p})\alpha(d) < 1$. There exists a $\kappa > 0$ such that, for any $p \geq \tilde{p}$ and any $c > 0$, we have

$$d|\Lambda|^{1 + \frac{\kappa c}{2d} \ln\left((1-p)\alpha(d)\right)} \leq \frac{1}{|\Lambda|^c},$$

and we obtain the desired inequality. \square

We state now a corollary of the proposition 3.1.4 which controls the distance between any cut present in the configuration and the set \mathcal{P} .

Corollary 3.3.4. *There exists $\tilde{p} < 1$ and $\kappa > 1$ such that, for $p \geq \tilde{p}$, for any constant $c \geq 1$ and any box Λ satisfying $|\Lambda| > 3^{6d}$, the following inequality holds:*

$$P_{\mathcal{D}}\left(\exists C \in \mathcal{C}, \exists e \in C, d(e, \mathcal{P} \cup \Lambda^c) \geq \kappa c \ln |\Lambda|\right) \leq \frac{1}{|\Lambda|^c}.$$

Proof. Let C be a cut and let e be an edge of C such that $d(e, \mathcal{P} \cup \Lambda^c) \geq \kappa c \ln |\Lambda|$. There exists a closed $*$ -path included in C which connects e to a pivotal edge f . Within a distance less than $\kappa c \ln |\Lambda|$ from e , there is no pivotal edge. By stopping the path at the first pivotal edge that it encounters or at the first edge intersecting the boundary of Λ , we obtain a path (e_1, \dots, e_n) without pivotal edge. Suppose that this path encounters the set S^+ or the set S^- . Let e_j be the first edge of the path which is in $S^+ \cup S^-$. By lemma 3.3.2, the edge e_j doesn't belong to $S^+ \cap S^-$. Without loss of generality, we can suppose that $e_j \in S^+ \setminus S^-$. We concatenate (e_1, \dots, e_j) and a closed path in $\partial^{ext} O'(T)$ from e_j to a pivotal edge or to an edge on the boundary of Λ . We obtain a closed path disjoint from S^- . We reuse the same techniques as in the proof of 3.1.4 and we obtain the desired result. \square

We shall also study the case where there is no pivotal edge in a configuration.

Proposition 3.3.5. *There exists a constant $\tilde{p} < 1$, such that,*

$$\forall p \geq \tilde{p} \quad \forall \Lambda \quad P_{\mathcal{D}}(\mathcal{P} = \emptyset) \leq d|\Lambda| \exp(-D),$$

where D is the diameter of T (or B).

Proof. Suppose that \mathcal{P} is empty. By lemma 3.3.2, the set S^+ and the set S^- are then disjoint. Each of them contains a cut. Therefore, there are two disjoint closed $*$ -paths travelling a distance at least $|\Lambda|^{1/d}$. By the same reasoning as in the proof of proposition 3.1.4, the $P_{\mathcal{D}}$ probability of this event can be bounded by

$$P_p(\exists \gamma \text{ closed path } \subset \Lambda, \gamma \text{ travels a distance at least } D).$$

Since there are at least D/d edges in such a path γ , this probability is less than

$$d|\Lambda|(\alpha(d)(1-p))^{\frac{D}{d}}.$$

There exists $\tilde{p} < 1$ such that, for all Λ , we have

$$\forall p \geq \tilde{p} \quad d|\Lambda|(\alpha(d)(1-p))^{\frac{D}{d}} \leq d|\Lambda| \exp(-D).$$

This yields the desired inequality. \square

3.4 Speed of the cuts

We state now the crucial proposition which gives a control on the speed of the cuts.

Proposition 3.4.1. *There exists $\tilde{p} < 1$, such that for $p \geq \tilde{p}$, for any $\ell \geq 2$, $t \in \mathbb{N}$, $s \in \{0, \dots, |\Lambda|\}$ and any edge $e \subset \Lambda$ at distance more than ℓ from Λ^c ,*

$$P_\mu \left(\begin{array}{c} e \in \mathcal{P}_{t+s} \\ \forall r \in [t, t+s] \quad \mathcal{P}_r \neq \emptyset \end{array} \mid \exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell \right) \leq \exp(-\ell).$$

To prove this result, we will construct a STP associated to the movement of the pivotal edges and then show that the probability to have such a long STP decreases exponentially fast as the length of the path grows.

3.4.1 Construction of the STP

We start by defining some properties of a STP. In the rest of the paper, unless otherwise specified, all the closed paths (and the closed STPs) are defined with the relation $*$ and the open paths (and the open STPs) are defined with the usual relation.

Definition 3.4.2. *A STP $(e_1, t_1), \dots, (e_n, t_n)$ is increasing (respectively decreasing) if*

$$t_1 \leq \dots \leq t_n \text{ (resp. } t_1 \geq \dots \geq t_n).$$

If a STP is increasing or decreasing, we say that it is monotone.

We consider the function $\theta(t)$ giving the time when the oldest edge of \mathcal{P}_t appeared, i.e.,

$$\theta(t) = \min_{\varepsilon \in \mathcal{P}_t} \min \{ r \leq t : \varepsilon \in \mathcal{P}_r, \varepsilon \notin \mathcal{P}_{r-1}, \forall \alpha \in [r, t], \varepsilon \in \mathcal{P}_\alpha \}.$$

We denote by e_1 one of the edges realizing the minimum $\theta(t)$. We claim that $\theta(t) < t$. Indeed, suppose first that an edge closes at time t . Then a pivotal edge cannot be created at time t and all the pivotal edges present at time t were also pivotal at time $t-1$. Therefore $\theta(t) \leq t-1 < t$. Suppose next that an edge opens at time t . Let us consider an edge ε of \mathcal{P}_{t-1} , which is assumed to be not empty. At time $t-1$, there is one open path which connects ε to T and another one which connects ε to B . Since one edge opens at time t , these two paths remain open at time t . Therefore ε is still pivotal at time t . We have thus $\mathcal{P}_{t-1} \subset \mathcal{P}_t$ and it follows that $\theta(t) \leq t-1 < t$. We have proved that $\theta(t) < t$.

If $\theta(t) \leq s$, we consider the STP obtained by connecting the path between (e, t) , (e_1, t) and the path between (e_1, s) , (f, s) with a time change from t to s on the edge e_1 . If this STP does not encounter $\partial\Lambda$ then it answers the question. If it encounters $\partial\Lambda$, then we stop the STP at the first edge intersecting $\partial\Lambda$, we obtain a STP satisfying the second condition of the proposition. Suppose now that $\theta(t) > s$. We consider the edge e_1 at time $\theta(t)$. By construction, the edge e_1 belongs to $\mathcal{P}_{\theta(t)}$. Moreover, using lemma 3.3.1, $(e_1, \theta(t))$ is connected to (e, t) by a STP consisting of a closed path at time t and a time change from t to $\theta(t)$ on the edge e_1 . We take $(e_1, \theta(t))$ as the new starting point. We repeat the procedure above and we obtain a sequence of times $(\theta(t), \theta(\theta(t)), \dots, \theta^{(i)}(t), \dots)$ by defining iteratively

$$\theta^{i+1}(t) = \min_{\varepsilon \in \mathcal{P}_{\theta^i(t)}} \{ r \leq \theta^i(t) : \varepsilon \in \mathcal{P}_r, \varepsilon \notin \mathcal{P}_{r-1}, \forall \alpha \in [r, \theta^i(t)], \varepsilon \in \mathcal{P}_\alpha \}.$$

For each index i , we choose an edge $e_i \in \mathcal{P}_{\theta^{i-1}(t)}$ which becomes pivotal at time $\theta^{(i)}(t)$. From the argument above, we obtain a strictly decreasing sequence

$$t > \theta(t) > \dots > \theta^i(t).$$

Therefore, there exists an index k such that

$$\theta^{k+1}(t) \leq s < \theta^k(t).$$

For $i \in \{0, \dots, k-1\}$, the edge-time $(e_i, \theta^i(t))$ is connected to $(e_{i+1}, \theta^{i+1}(t))$ by a decreasing STP γ_i which is closed in Y . By concatenating these STPs, we obtain a decreasing STP between (e, t) and $(e_k, \theta^k(t))$. At time $\theta^k(t)$, there exists also a closed path ρ between e_k and e_{k+1} . We stop the time change at s on the edge e_{k+1} in order to arrive at an edge of \mathcal{P}_s . By lemma 3.3.1, there is a closed path ρ between e_{k+1} and f at time s . Therefore, the STP

$$(e, t), \gamma_0, (e_1, \theta(t)), \gamma_1, \dots, \gamma_{k-1}, (e_k, \theta^k(t)), \rho, (e_{k+1}, \theta^k(t)), (e_{k+1}, s), \rho, (f, s)$$

is decreasing, closed in Y and it connects (e, t) and (f, s) . If this STP exits the box Λ , then the initial portion starting from e until the first edge intersecting $\partial\Lambda$ satisfies the second condition of the proposition.

In order to obtain a STP which is simple in Y , we consider the following iterative procedure to modify a path. Let us denote by $(e_i, t_i)_{0 \leq i \leq N}$ the STP obtained previously. Starting with the edge e_0 , we examine the rest of the edges one by one. Let $i \in \{0, \dots, N\}$. Suppose that the edges e_0, \dots, e_{i-1} have been examined and let us focus on e_i . We encounter three cases:

- For every index $j \in \{i + 1, \dots, N\}$, we have $e_j \neq e_i$. Then, we don't modify anything and we start examining the edge e_{i+1} .
- There is an index $j \in \{i + 1, \dots, N\}$ such that $e_i = e_j$, but for the first index $k > i + 1$ such that $e_i = e_k$, there is a time $\alpha \in]t_k, t_i[$ when $Y_\alpha(e_i) = 1$. Then we don't modify anything and we start examining the next edge e_{i+1} .
- There is an index $j \in \{i + 1, \dots, N\}$ such that $e_i = e_j$ and for the first index $k > i + 1$ such that $e_i = e_k$, we have $Y_\alpha(e_i) = 0$ for all $\alpha \in]t_k, t_i[$. In this case, we remove all the time-edges whose indices are strictly between i and k . We then have a simple time change between t_i and t_k on the edge e_i . We continue the procedure from the index e_k .

The STP becomes strictly shorter after every modification, and the procedure will end after a finite number of modifications. We obtain in the end a simple path in Y . Since the procedure doesn't change the order of the times t_i , we still have a decreasing path. \square

3.4.2 The BK inequality applied to a STP

Before embarking in technicalities, let us discuss the differences between the processes $(X_t)_{t \in \mathbb{N}}$ and $(Y_t)_{t \in \mathbb{N}}$. Let (e, t) be a closed time-edge in Y . Since there is no constraint in the process X , the edge e can be open in the configuration X_t . If the edge e is open in X_t , then it belongs to \mathcal{I}_t . Now let us consider a time t for which $E_t = e$ and $B_t = 0$. Closing an edge doesn't create an open path between T and B , thus the edge e will be closed in both X_t and Y_t . On the contrary, for a time t such that $B_t = 1$ and $E_t = e$ is pivotal at time $t - 1$, the edge e can be opened in the process $(X_t)_{t \in \mathbb{N}}$ but it remains closed in the process $(Y_t)_{t \in \mathbb{N}}$. Now let us consider the STP constructed in proposition 3.4.4. Since the STP is closed in Y , each edge e visited by the path is either closed at time s or there is a time $r \in]s, t]$ when $E_r = e$ and $B_r = 0$. In fact, since the STP is simple, then each edge is reopened and closed between two successive visits of the STP. Our first goal is to introduce the necessary notation in order to keep track of all the closing events implied by the STP.

We shall define the space projection of a STP. Given $k \in \mathbb{N}^*$ and a sequence $\Gamma = (e_i)_{1 \leq i \leq k}$ of edges, we say that it has length k , which we denote by $\text{length}(\Gamma) = k$, and we define its support

$$\text{support}(\Gamma) = \{ e \subset \Lambda : \exists i \in \{1, \dots, k\} \quad e_i = e \}.$$

Let $\gamma = (e_i, t_i)_{1 \leq i \leq n}$ be a simple STP, the space projection of γ is obtained by removing one edge in every time change in the sequence $(e_i)_{1 \leq i \leq n}$. More precisely, let m be the number of time changes in γ . We define the function $\phi : \{1, \dots, n - m\} \rightarrow \mathbb{N}$ by setting $\phi(1) = 1$ and

$$\forall i \in \{1, \dots, n - m\} \quad \phi(i + 1) = \begin{cases} \phi(i) + 1 & \text{if } e_{\phi(i)} \neq e_{\phi(i)+1} \\ \phi(i) + 2 & \text{if } e_{\phi(i)} = e_{\phi(i)+1} \end{cases}.$$

The sequence $(e_{\phi(i)})_{1 \leq i \leq n - m}$ is called the space projection of γ , denoted by $\text{Space}(\gamma)$. We say that $\text{length}(\text{Space}(\gamma))$ is the length of the STP γ , denoted also by $\text{length}(\gamma)$. We shall distinguish $\text{Space}(\gamma)$ from the support of γ , denoted by $\text{support}(\gamma)$, which we define as:

$$\text{support}(\gamma) = \text{support}(\text{Space}(\gamma)).$$

We say that a sequence of edges $\Gamma = (e_i)_{1 \leq i \leq k}$ is visitable if there exists a STP γ such that $\text{Space}(\gamma) = \Gamma$.

We prove next a key inequality to control the number of closing events along a simple STP.

Proposition 3.4.5. *Let Γ be a visitable sequence of edges and $[s, t]$ a time interval. For any $k \in \{0, \dots, |\text{support}(\Gamma)|\}$ and any percolation configuration y such that there are exactly k closed edges in $\text{support}(\Gamma)$, we have the following inequality:*

$$P \left(\begin{array}{l} \exists \gamma \text{ decreasing closed} \\ \text{simple STP in } Y \text{ during } [s, t] \\ \text{such that } \text{Space}(\gamma) = \Gamma \end{array} \middle| Y_s = y \right) \leq \left(\frac{(t - s)(1 - p)}{|\Lambda|} \right)^{\text{length}(\Gamma) - k}.$$

Proof. We denote by n the length of Γ and (e_1, \dots, e_n) the sequence Γ . We consider a STP γ such that $\text{Space}(\gamma) = \Gamma$. Since γ is closed, all the edges of Γ are closed at time s or become closed after s . For an edge $e \in \text{support}(\Gamma)$, we denote by $v(e)$ the number of times that Γ visits e :

$$v(e) = |\{j \in \{1, \dots, n\} : e_j = e\}|.$$

Since γ is simple, between two consecutive visits, there exists a time when the edge e is open, as illustrated in the figure 3.5.

For each edge e visited by γ , we distinguish two cases according to the configuration Y_s . If $Y_s(e) = 1$, there is a time between s and the first visit when e becomes closed and the edge e closes at least $v(e)$ times during the

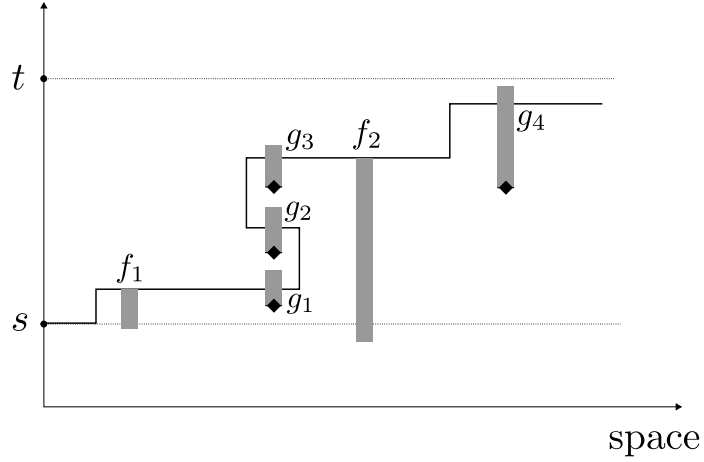


Figure 3.5: The edges f_1, f_2 are closed at time s . The edges g_1, g_2, g_3 and g_4 closes after s . We see that $g_1 = g_2 = g_3$ and the simplicity of the path implies that the edge opens and closes between two consecutive visits.

time interval $]s, t]$. If $Y_s(e) = 0$, then, between the time s and the first visit of e , it can remain closed and the edge e becomes closed at least $v(e) - 1$ times during $]s, t]$. Notice that the numbers $v(e)$ depend on the sequence Γ . The probability in the proposition is therefore less than or equal to

$$P \left(\left\{ \begin{array}{l} \forall e \in \text{support}(\Gamma) \quad y(e) = 1 \\ e \text{ closes at least } v(e) \text{ times during }]s, t] \end{array} \right\} \cap \left\{ \begin{array}{l} \forall f \in \text{support}(\Gamma) \quad y(f) = 0 \\ f \text{ closes at least } v(f) - 1 \text{ times during }]s, t] \end{array} \right\} \middle| Y_s = y \right). \quad (3.4.1)$$

Notice that for any edge e such that $y(e) = 1$ (respectively $y(e) = 0$), the event

$$\{ e \text{ closes at least } v(e) \text{ (resp. } v(e) - 1 \text{) times during }]s, t] \}$$

depends on the collection of random variables

$$F(e) = \{ (E_r, B_r) : s < r \leq t, E_r = e, B_r = 0 \}.$$

Therefore these events are independent of the event $\{ Y_s = y \}$ which depends on (X_0, Y_0) and $\{ (E_r, B_r) : r \leq s \}$. The probability in (3.4.1) is thus equal to

$$P \left(\left\{ \begin{array}{l} \forall e \in \text{support}(\Gamma) \quad y(e) = 1 \\ e \text{ closes at least } v(e) \text{ times during }]s, t] \end{array} \right\} \cap \left\{ \begin{array}{l} \forall f \in \text{support}(\Gamma) \quad y(f) = 0 \\ f \text{ closes at least } v(f) - 1 \text{ times during }]s, t] \end{array} \right\} \right).$$

For any edge $e \in \text{support}(\Gamma)$, we define $J(e)$ as the set of indices

$$J(e) = \{s < j \leq t : E_j = e, B_j = 0\}.$$

We notice that the sets $(J(e), e \in \text{support}(\Gamma))$ are pairwise disjoint subsets of \mathbb{N}^* . By the BK inequality applied to the random variables $(E_t, B_t)_{t \in \mathbb{N}}$, the probability in (3.4.1) is less than

$$\begin{aligned} & \prod_{e \in \text{support}(\Gamma), y(e)=1} P(e \text{ closes at least } v(e) \text{ times during }]s, t]) \\ & \times \prod_{f \in \text{support}(\Gamma), y(f)=0} P(f \text{ closes at least } v(f) - 1 \text{ times during }]s, t]). \end{aligned}$$

We obtain therefore

$$\begin{aligned} & P \left(\begin{array}{l} \exists \gamma \text{ decreasing closed simple STP} \\ \text{in } Y \text{ during }]s, t] \quad \text{Space}(\gamma) = \Gamma \end{array} \middle| Y_s = y \right) \\ & \leq \prod_{e \in \text{support}(\Gamma), y(e)=1} P(e \text{ closes } v(e) \text{ times during }]s, t]) \\ & \times \prod_{f \in \text{support}(\Gamma), y(f)=0} P(f \text{ closes } v(f) - 1 \text{ times during }]s, t]). \quad (3.4.2) \end{aligned}$$

For any edge $e \in \text{support}(\Gamma)$ and any $m \in \mathbb{N}$, we have

$$\begin{aligned} & P(e \text{ closes at least } m \text{ times during }]s, t]) \\ & \leq P \left(\begin{array}{l} \exists J \subset \{s+1, \dots, t\} \quad |J| = m \\ \forall j \in J \quad E_j = e \quad B_j = 0 \end{array} \right) \\ & \leq \left(\frac{(t-s)(1-p)}{|\Lambda|} \right)^m. \end{aligned}$$

We use this inequality in (3.4.2) and we obtain

$$\begin{aligned} & P \left(\begin{array}{l} \exists \gamma \text{ decreasing closed simple STP} \\ \text{in } Y \text{ during }]s, t] \quad \text{Space}(\gamma) = \Gamma \end{array} \middle| Y_s = y \right) \\ & \leq \left(\frac{(t-s)(1-p)}{|\Lambda|} \right)^{\sum_{e \in \text{support}(\Gamma), y(e)=1} v(e) + \sum_{f \in \text{support}(\Gamma), y(f)=0} v(f) - 1} \\ & = \left(\frac{(t-s)(1-p)}{|\Lambda|} \right)^{n-k}. \end{aligned}$$

This is the desired result. \square

3.4.3 Proof of proposition 3.4.1

Our goal is to study the speed of a cut during a time interval of size $|\Lambda|$. We start by using the results in the previous section to control the length of a STP far from a cut.

Proposition 3.4.6. *Let ℓ be a positive constant, Γ be a visitable sequence of edges starting from an edge e such that Γ travels a distance less than ℓ and t be a time. For $s \in \mathbb{N}$, we have the following inequality:*

$$P_\mu \left(\begin{array}{l} \exists \gamma \text{ decreasing closed} \\ \text{simple STP in } Y \text{ during } [t, t+s] \\ \text{such that } \text{Space}(\gamma) = \Gamma \end{array} \middle| \begin{array}{l} \exists c_t \in \mathcal{C}_t \\ d(e, c_t) \geq \ell \end{array} \right) \leq \left(\left(1 + \frac{s}{|\Lambda|} \right) (1-p) \right)^{\text{length}(\Gamma)}.$$

Proof. We start by rewriting the conditional probability in the proposition as

$$P_\mu \left(\begin{array}{l} \exists \gamma \text{ decreasing closed} \\ \text{simple STP in } Y \text{ during } [t, t+s] \\ \text{such that } \text{Space}(\gamma) = \Gamma \end{array} \middle| \exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell \right) = \frac{P_\mu \left(\left\{ \begin{array}{l} \exists \gamma \text{ decreasing closed} \\ \text{simple STP in } Y \text{ during } [t, t+s] \\ \text{such that } \text{Space}(\gamma) = \Gamma \end{array} \right\} \cap \left\{ \begin{array}{l} \exists c_t \in \mathcal{C}_t \\ d(e, c_t) \geq \ell \end{array} \right\} \right)}{P_\mu \left(\begin{array}{l} \exists c_t \in \mathcal{C}_t \\ d(e, c_t) \geq \ell \end{array} \right)}.$$

We denote by $(e_i, t_i)_{1 \leq i \leq N}$ the time-edges of γ . Let n be the length of γ . We consider the case where there are exactly k edges of $\text{support}(\gamma)$ that are closed at time t . We shall estimate the following probability:

$$P_\mu \left(\left(\left\{ \begin{array}{l} \exists \gamma \text{ simple decreasing STP} \\ \text{closed in } Y \text{ of length } n \\ \gamma \text{ starts at } (e, t+s) \text{ and ends after } t \\ \text{Space}(\gamma) = \Gamma \end{array} \right\} \cap \left\{ \begin{array}{l} \exists F \subset \text{support}(\gamma) \quad |F| = k \\ \forall f \in F \quad Y_t(f) = 0 \\ \forall f \in \text{support}(\gamma) \setminus F \quad Y_t(f) = 1 \end{array} \right\} \cap \left\{ \begin{array}{l} \exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell \end{array} \right\} \right) \right). \quad (3.4.3)$$

We consider the set $M(k)$ of the configurations defined as

$$M(k) = \left\{ \omega : \begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad \omega(f) = 0 \\ \forall f \in \text{support}(\Gamma) \setminus F \quad \omega(f) = 1 \\ \exists C \in \mathcal{C} \quad d(e, C) \geq \ell \end{array} \right\}.$$

The probability in (3.4.3) is bounded from above by

$$\sum_{y \in M(k)} P_\mu \left(\begin{array}{l} \exists \gamma \text{ decreasing simple closed STP,} \\ \text{length}(\gamma) = n, \quad \text{Space}(\gamma) = \Gamma, \\ \gamma \text{ starts at } (e, t+s) \text{ and ends after } t \end{array} \middle| Y_t = y \right) P_\mu(Y_t = y).$$

By proposition 3.4.5, for any $y \in M(k)$, we have

$$P_\mu \left(\begin{array}{l} \exists \gamma \text{ decreasing simple closed STP,} \\ \text{length}(\gamma) = n, \quad \text{Space}(\gamma) = \Gamma, \\ \gamma \text{ starts at } (e, t+s) \text{ and ends after } t \end{array} \middle| Y_t = y \right) \leq \left(\frac{s}{|\Lambda|} (1-p) \right)^{n-k}. \quad (3.4.4)$$

We compute now the probability $P_\mu(Y_t \in M(k))$. Notice that

$$P_\mu(Y_t \in M(k)) \leq P_\mu \left(\begin{array}{l} \exists F \subset \text{support}(\Gamma) \quad |F| = k \\ \forall f \in F \quad Y_t(f) = 0 \\ \exists c_t \in \mathcal{C}_t \quad d(e, c_t) \geq \ell \end{array} \right).$$

The event in the last probability depends only on the configuration Y_t . Since the initial configuration (X_0, Y_0) is distributed according to μ_p , so is the couple (X_t, Y_t) . The configuration Y_t is distributed according to the second marginal distribution $P_{\mathcal{D}}$. We have therefore

$$P_\mu \left(\begin{array}{l} \exists F \subset \text{support}(\Gamma) \quad |F| = k \\ \forall f \in F \quad Y_t(f) = 0 \\ \exists c_t \in \mathcal{C}_t \quad d(e, c_t) \geq \ell \end{array} \right) = P_{\mathcal{D}} \left(\begin{array}{l} \exists F \subset \text{support}(\Gamma) \quad |F| = k \\ \forall f \in F \quad f \text{ closed} \\ \exists C \in \mathcal{C} \quad d(e, C) \geq \ell \end{array} \right).$$

By the definition of $P_{\mathcal{D}}$, we have

$$\begin{aligned} & P_{\mathcal{D}} \left(\begin{array}{l} \exists F \subset \text{support}(\Gamma) \quad |F| = k \\ \forall f \in F \quad f \text{ closed} \\ \exists C \in \mathcal{C} \quad d(e, C) \geq \ell \end{array} \right) \\ &= P_p \left(\begin{array}{l} \exists F \subset \text{support}(\Gamma) \quad |F| = k \\ \forall f \in F \quad f \text{ closed} \\ \exists C \in \mathcal{C} \quad d(e, C) \geq \ell \end{array} \middle| T \leftrightarrow B \right) \\ &= \frac{P_p \left(\left\{ \begin{array}{l} \exists F \subset \text{support}(\Gamma) \quad |F| = k \\ \forall f \in F \quad f \text{ closed} \end{array} \right\} \cap \left\{ \begin{array}{l} \exists C \in \mathcal{C} \\ d(e, C) \geq \ell \end{array} \right\} \cap \{T \leftrightarrow B\} \right)}{P_p(T \leftrightarrow B)}. \end{aligned} \quad (3.4.5)$$

The existence of a cut implies the event $\{T \leftrightarrow B\}$, thus we can rewrite the numerator as

$$P_p \left(\left\{ \begin{array}{l} \exists F \subset \text{support}(\Gamma) \quad |F| = k \\ \forall f \in F \quad f \text{ closed} \end{array} \right\} \cap \left\{ \exists C \in \mathcal{C}, d(e, C) \geq \ell \right\} \right).$$

The edges of $\text{support}(\Gamma)$ are at distance less than ℓ from the edge e and the event $\{\exists C \in \mathcal{C}, d(e, C) \geq \ell\}$ depends on the edges at distance more than ℓ from e . It follows that the two events in the previous probability are independent and we have

$$\begin{aligned} P_p \left(\left\{ \begin{array}{l} \exists F \subset \text{support}(\Gamma) \quad |F| = k \\ \forall f \in F \quad f \text{ closed} \end{array} \right\} \cap \{\exists C \in \mathcal{C}, d(e, C) \geq \ell\} \right) \\ = P_p \left(\begin{array}{l} \exists F \subset \text{support}(\Gamma) \quad |F| = k \\ \forall f \in F \quad f \text{ closed} \end{array} \right) P_p(\exists C \in \mathcal{C}, d(e, C) \geq \ell). \end{aligned}$$

Replacing the numerator in (3.4.5) by this product, we obtain

$$\begin{aligned} P_p \left(\left\{ \begin{array}{l} \exists F \subset \text{support}(\Gamma) \quad |F| = k \\ \forall f \in F \quad f \text{ closed} \end{array} \right\} \cap \{\exists C \in \mathcal{C}, d(e, C) \geq \ell\} \middle| T \leftrightarrow B \right) \\ = \frac{P_p \left(\begin{array}{l} \exists F \subset \text{support}(\Gamma) \quad |F| = k \\ \forall f \in F \quad f \text{ closed} \end{array} \right) P_p(\exists C \in \mathcal{C}, d(e, C) \geq \ell)}{P_p(T \leftrightarrow B)} \\ = P_p \left(\begin{array}{l} \exists F \subset \text{support}(\Gamma) \quad |F| = k \\ \forall f \in F \quad f \text{ closed} \end{array} \right) P_{\mathcal{D}}(\exists C \in \mathcal{C}, d(e, C) \geq \ell). \end{aligned}$$

Since the edges of F are distinct, we have

$$P_p \left(\begin{array}{l} \exists F \subset \text{support}(\Gamma) \quad |F| = k \\ \forall f \in F \quad f \text{ closed} \end{array} \right) \leq \binom{|\text{support}(\Gamma)|}{k} (1-p)^k.$$

Combined with (3.4.4), we obtain that, for Γ and k fixed, the probability in (3.4.3) is bounded from above by

$$\binom{|\text{support}(\Gamma)|}{k} \left(\frac{s}{|\Lambda|} \right)^{n-k} (1-p)^n P_{\mathcal{D}}(\exists C \in \mathcal{C}, d(e, C) \geq \ell).$$

We sum on the number k from 0 to $|\text{support}(\Gamma)|$, and we recall that

$$|\text{support}(\Gamma)| \leq n.$$

We have therefore

$$\begin{aligned} P_{\mu} \left(\left\{ \begin{array}{l} \exists \gamma \text{ simple closed decreasing STP,} \\ \text{length}(\gamma) = n, \quad \text{Space}(\gamma) = \Gamma, \\ \gamma \text{ starts at } (e, t+s) \text{ and ends after } t \end{array} \right\} \cap \left\{ \begin{array}{l} \exists c_t \in \mathcal{C}_t \\ d(e, c_t) \geq \ell \end{array} \right\} \right) \\ \leq \sum_{0 \leq k \leq \ell/2d} \binom{n}{k} \left(\frac{s}{|\Lambda|} \right)^{n-k} (1-p)^n P_{\mathcal{D}}(\exists C \in \mathcal{C}, d(e, C) \geq \ell) \\ = \left(\left(1 + \frac{s}{|\Lambda|} \right) (1-p) \right)^n P_{\mathcal{D}}(\exists C \in \mathcal{C}, d(e, C) \geq \ell). \end{aligned}$$

Since the second marginal of P_μ is $P_{\mathcal{D}}$, we have

$$P_\mu \left(\begin{array}{c} \exists c_t \in \mathcal{C}_t \\ d(e, c_t) \geq \ell \end{array} \right) = P_{\mathcal{D}} \left(\exists C \in \mathcal{C}, d(e, C) \geq \ell \right).$$

This yields the inequality in the proposition. \square

We can now estimate the probability that the set of the pivotal edges moves fast. To do so, we study the STP constructed in proposition 3.4.4 and we use the previous results.

Proof of proposition 3.4.1. We rewrite the conditional probability appearing in the proposition as

$$\frac{P_\mu \left(\{e \in \mathcal{P}_{t+s}\} \cap \{\forall r \in [t, t+s] \quad \mathcal{P}_r \neq \emptyset\} \cap \{\exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell\} \right)}{P_\mu \left(\exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell \right)}.$$

Let us estimate the probability in the numerator. By proposition 3.4.4, there exists a closed decreasing simple STP γ inside of Λ which connects $(e, t+s)$ to either an edge of \mathcal{P}_t at time t or to an edge intersecting the boundary of Λ after time t . In both cases, this STP travels a distance at least ℓ because all the edges of \mathcal{P}_t are included in the cuts and e is at distance more than ℓ from Λ^c . Since the STP is a $*$ -STP, the distance between two consecutive edges is at most d , and the length of the STP is at least $(\ell-1)/d$. We denote by $(e_i, t_i)_{1 \leq i \leq N}$ the time-edges of γ . Let n be the first index such that the STP

$$(e_1, t_1), \dots, (e_n, t_n)$$

is longer than $\ell/2d$, i.e.,

$$n = \inf \left\{ k \geq 1 : \text{length} \left((e_1, t_1), \dots, (e_k, t_k) \right) \geq \frac{\ell}{2d} \right\}.$$

We set $\Gamma = \text{Space}((e_i, t_i)_{1 \leq i \leq n})$ and we denote $\Gamma = (f_i)_{i \in I}$. We have the following inequality:

$$\begin{aligned} & P_\mu \left(\begin{array}{c} e \in \mathcal{P}_{t+s} \\ \forall r \in [t, t+s] \quad \mathcal{P}_r \neq \emptyset \end{array} \mid \begin{array}{c} \exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell \end{array} \right) \\ & \leq \sum_{\Gamma} P_\mu \left(\begin{array}{c} \exists \gamma \text{ simple closed decreasing STP,} \\ \text{length}(\gamma) = \ell/2d, \quad \text{Space}(\gamma) = \Gamma, \\ \gamma \text{ starts at } (e, t+s) \text{ and ends after } t \end{array} \mid \begin{array}{c} \exists c_t \in \mathcal{C}_t, \\ d(e, c_t) \geq \ell \end{array} \right). \end{aligned}$$

By proposition 3.4.6, each term in the sum is less than

$$\left(\left(1 + \frac{s}{|\Lambda|} \right) (1-p) \right)^{\ell/2d}.$$

We sum next on all the possible choices of Γ . By lemma 3.3.3, we have

$$P_\mu \left(\begin{array}{c} e \in \mathcal{P}_{t+s} \\ \forall r \in [t, t+s] \quad \mathcal{P}_r \neq \emptyset \end{array} \middle| \exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell \right) \leq \left(\alpha(d) \left(1 + \frac{s}{|\Lambda|} \right) (1-p) \right)^{\ell/2d}.$$

There is a constant $\tilde{p} < 1$ such that, for all $p \geq \tilde{p}$, $s \leq |\Lambda|$ and $\ell \geq 2$,

$$\left(\alpha(d) \left(1 + \frac{s}{|\Lambda|} \right) (1-p) \right)^{\ell/2d} \leq e^{-\ell}.$$

We have obtained the result stated in the proposition 3.4.1. \square

3.5 The localisation around pivotal edges

We start by stating a corollary of proposition 3.4.1. Recall at first that the Hausdorff distance between two subsets A and B of \mathbb{R}^d , denoted by $d_H(A, B)$, is

$$d_H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}.$$

For A a subset of \mathbb{R}^d and $r > 0$, we define the neighbourhood

$$\mathcal{V}(A, r) = \{ x \in \mathbb{R}^d : d(x, A) < r \}.$$

The Hausdorff distance is also equal to

$$\inf \{ r \geq 0 : A \subset \mathcal{V}(B, r), \quad B \subset \mathcal{V}(A, r) \}.$$

For $\ell \geq 0$, we consider two subsets A, B of Λ and we define a semi-distance between two such subsets, denoted by $d_H^\ell(A, B)$, adapted to our study, by

$$d_H^\ell(A, B) = \inf \left\{ r \geq 0 : \begin{array}{l} A \setminus \mathcal{V}(\Lambda^c, \ell) \subset \mathcal{V}(B, r) \\ B \setminus \mathcal{V}(\Lambda^c, \ell) \subset \mathcal{V}(A, r) \end{array} \right\}.$$

Notice that d_H^ℓ is a semi-distance, in fact the triangle inequality is not satisfied. However, the following lemma allow us to compare d_H^ℓ with the Hausdorff distance and provides us an alternative to the triangle inequality.

Lemma 3.5.1. *For two subsets A, B of Λ and for all $\ell \geq 0$, we have*

$$d_H^\ell(A, B) \vee \ell \geq d_H(A \cup \Lambda^c, B \cup \Lambda^c).$$

Proof. Let A, B be two subsets of Λ , and let us set $d_1 = d_H^\ell(A, B)$. We claim that

$$A \cup \Lambda^c \subset \mathcal{V}(B \cup \Lambda^c, d_1 \vee \ell).$$

Let $x \in A \cup \Lambda^c$, we will show that x belongs to $\mathcal{V}(B \cup \Lambda^c, d_1 \vee \ell)$. We distinguish two cases. If $x \in \mathcal{V}(\Lambda^c, \ell)$, then we have

$$x \in \mathcal{V}(B \cup \Lambda^c, \ell) \subset \mathcal{V}(B \cup \Lambda^c, d_1 \vee \ell).$$

In the other case, if $x \in A \setminus \mathcal{V}(\Lambda^c, \ell)$ and by the definition of d_H^ℓ , we have

$$x \in \mathcal{V}(B, d_1) \subset \mathcal{V}(B \cup \Lambda^c, d_1) \subset \mathcal{V}(B \cup \Lambda^c, d_1 \vee \ell).$$

By exchanging A and B , we have

$$B \cup \Lambda^c \subset \mathcal{V}(A \cup \Lambda^c, d_1 \vee \ell).$$

By the definition of d_H , we obtain the desired claim, which in turn proves the lemma. \square

Proposition 3.5.2. *We have the following result:*

$$\begin{aligned} \exists \tilde{p} < 1 \quad \exists \kappa > 1 \quad \forall p \geq \tilde{p} \quad \forall c \geq 1 \quad \forall \Lambda \quad |\Lambda| \geq (cd)^{cd^2} \quad \forall t \geq 0 \\ P_\mu \left(\exists s \leq |\Lambda| \quad d_H^{\kappa c \ln |\Lambda|}(\mathcal{P}_t, \mathcal{P}_{t+s}) \geq \kappa c \ln |\Lambda| \right) \leq \frac{10d}{|\Lambda|^c}. \end{aligned}$$

Proof. We fix $s \in \{1, \dots, |\Lambda|\}$. By the definition of the distance d_H^ℓ , we have, for any $\kappa > 1$,

$$\begin{aligned} P_\mu \left(d_H^{\kappa c \ln |\Lambda|}(\mathcal{P}_t, \mathcal{P}_{t+s}) \geq \kappa c \ln |\Lambda| \right) \leq \\ P_\mu \left(\mathcal{P}_{t+s} \setminus \mathcal{V}(\Lambda^c, \kappa c \ln |\Lambda|) \not\subseteq \mathcal{V}(\mathcal{P}_t, \kappa c \ln |\Lambda|) \right) \\ + P_\mu \left(\mathcal{P}_t \setminus \mathcal{V}(\Lambda^c, \kappa c \ln |\Lambda|) \not\subseteq \mathcal{V}(\mathcal{P}_{t+s}, \kappa c \ln |\Lambda|) \right). \quad (3.5.1) \end{aligned}$$

Since the two probabilities in the sum depend only on the process Y , which is reversible, they are in fact equal to each other. We shall estimate the first probability. We discuss first the case where there is a time $r \in \{t, \dots, t+s\}$ when $\mathcal{P}_r = \emptyset$. By proposition 3.3.5, there is a $\tilde{p} < 1$ such that, for $p \geq \tilde{p}$ and all Λ ,

$$\forall r \in \mathbb{N} \quad P_{\mathcal{D}}(\mathcal{P}_r = \emptyset) \leq d|\Lambda| \exp(-D),$$

where D is the diameter of T . By summing over the time r , we have

$$P_{\mathcal{D}}(\exists r \in [t, t+s] \quad \mathcal{P}_r = \emptyset) \leq d|\Lambda|^2 \exp(-D). \quad (3.5.2)$$

We now consider the case where there exists always at least one pivotal edge during the time interval $[t, t+s]$. We can then apply proposition 3.4.1 with

an ℓ which will be determined later. There exists $\tilde{p} < 1$ such that for $p \geq \tilde{p}$, for $t \geq 0$, and for any $s \leq |\Lambda|$ and e an edge such that $d(e, \Lambda^c) \geq \ell$,

$$P_\mu \left(\begin{array}{c} e \in \mathcal{P}_{t+s} \\ \forall r \in [t, t+s] \quad \mathcal{P}_r \neq \emptyset \end{array} \mid \exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell \right) \leq e^{-\ell}.$$

Let us fix $t \geq 0$, $s \leq |\Lambda|$ and e an edge such that $d(e, \Lambda^c) \geq \ell$. The previous inequality implies that

$$P_\mu \left(\begin{array}{c} e \in \mathcal{P}_{t+s} \\ \forall r \in [t, t+s], \mathcal{P}_r \neq \emptyset \\ \exists c_t \in \mathcal{C}_t \quad d(e, c_t) \geq \ell \end{array} \right) \leq e^{-\ell}.$$

In order to replace c_t by \mathcal{P}_t in the last probability, we use the corollary 3.3.4. At the time t , the configuration Y_t follows the distribution $P_{\mathcal{D}}$. Therefore, there exists $\tilde{p} < 1$ and a $\kappa' > 1$ such that for $p \geq \tilde{p}$, for all $c \geq 1$ and all Λ such that $|\Lambda| \geq 3^{6d}$, we have

$$P_\mu \left(\begin{array}{c} \exists C \in \mathcal{C}_t \quad \exists f \in C \\ d(f, \Lambda^c \cup \mathcal{P}_t \setminus \{f\}) \geq \kappa' c \ln |\Lambda| \\ \forall r \in [t, t+s] \quad \mathcal{P}_r \neq \emptyset \end{array} \right) \leq \frac{1}{|\Lambda|^c}.$$

From now onwards, we suppose that p is larger than the three previous \tilde{p} . Let $c > 0$ be fixed and let κ' be associated to c as above. We distinguish two cases to control the following probability:

$$\begin{aligned} & P_\mu(e \in \mathcal{P}_{t+s}, d(e, \mathcal{P}_t) \geq \kappa c \ln |\Lambda|, \forall r \in [t, t+s] \quad \mathcal{P}_r \neq \emptyset) \leq \\ & P_\mu \left(\begin{array}{c} e \in \mathcal{P}_{t+s}, \quad d(e, \mathcal{P}_t) \geq \kappa c \ln |\Lambda|, \\ \forall C \in \mathcal{C}_t \quad \forall f \in C \setminus \mathcal{V}(\Lambda^c, \kappa' c \ln |\Lambda|) \\ \quad \quad \quad d(f, \mathcal{P}_t) < \kappa' c \ln |\Lambda|, \\ \forall r \in [t, t+s] \quad \mathcal{P}_r \neq \emptyset \end{array} \right) \\ & + P_\mu(\exists C \in \mathcal{C}_t, \exists f \in C, d(f, \Lambda^c \cup \mathcal{P}_t \setminus \{f\}) \geq \kappa' c \ln |\Lambda|). \end{aligned}$$

The second probability is less than $1/|\Lambda|^c$. Let us study the first probability. Since all the edges of a cut at time t are either at distance less than $\kappa' c \ln |\Lambda|$ from Λ^c or at distance less than $\kappa' c \ln |\Lambda|$ from \mathcal{P}_t and the distance between e and $\mathcal{P}_t \cup \Lambda^c$ is larger than $\kappa c \ln |\Lambda|$, then all the cuts at time t are at distance

more than $(\kappa - \kappa')c \ln |\Lambda|$ from e . Hence, for $\kappa > \kappa'$,

$$\begin{aligned} P_\mu \left(\begin{array}{l} e \in \mathcal{P}_{t+s} \quad d(e, \Lambda^c \cup \mathcal{P}_t) \geq \kappa c \ln |\Lambda| \\ \forall C \in \mathcal{C}_t \quad \forall f \in C \setminus \mathcal{V}(\Lambda^c, \kappa' c \ln |\Lambda|) \\ \quad d(f, \mathcal{P}_t) < \kappa' c \ln |\Lambda| \\ \forall r \in [t, t+s] \quad \mathcal{P}_r \neq \emptyset \end{array} \right) &\leq \\ P_\mu \left(\begin{array}{l} e \in \mathcal{P}_{t+s} \quad d(e, \Lambda^c) \geq \kappa c \ln |\Lambda| \\ \forall C \in \mathcal{C}_t \quad \forall f \in C \setminus \mathcal{V}(\Lambda^c, \kappa' c \ln |\Lambda|) \\ \quad d(f, e) > (\kappa - \kappa')c \ln |\Lambda| \\ \forall r \in [t, t+s] \quad \mathcal{P}_r \neq \emptyset \end{array} \right) &\leq \\ P_\mu \left(\begin{array}{l} e \in \mathcal{P}_{t+s} \\ \exists c_t \in \mathcal{C}_t \quad d(e, c_t) \geq (\kappa - \kappa')c \ln |\Lambda| \\ \forall r \in [t, t+s] \quad \mathcal{P}_r \neq \emptyset \end{array} \right) &\leq \frac{1}{|\Lambda|^{(\kappa - \kappa')c}}. \end{aligned}$$

We choose now $\kappa = \kappa' + 1$, and we get

$$P_\mu(e \in \mathcal{P}_{t+s}, d(e, \mathcal{P}_t) \geq \kappa c \ln |\Lambda|, \forall r \in [t, t+s] \mathcal{P}_r \neq \emptyset) \leq \frac{2}{|\Lambda|^c}.$$

We sum over e in Λ and $s \in \{1, \dots, |\Lambda|\}$ to get

$$P_\mu \left(\begin{array}{l} \exists s \leq |\Lambda|, \exists e \in \mathcal{P}_{t+s} \\ d(e, \Lambda^c \cup \mathcal{P}_t) \geq \kappa c \ln |\Lambda| \\ \forall r \in [t, t+s] \quad \mathcal{P}_r \neq \emptyset \end{array} \right) \leq \frac{4d}{|\Lambda|^{c-2}}.$$

We add the probability in (3.5.2) and we obtain

$$P_\mu \left(\begin{array}{l} \exists s \leq |\Lambda|, \exists e \in \mathcal{P}_{t+s} \\ d(e, \Lambda^c \cup \mathcal{P}_t) \geq \kappa c \ln |\Lambda| \end{array} \right) \leq \frac{4d}{|\Lambda|^{c-2}} + d|\Lambda|^2 \exp(-D).$$

This is the first probability in (3.5.1) and we conclude that

$$P_\mu \left(\begin{array}{l} \exists s \leq |\Lambda| \\ d_H^\Lambda(\mathcal{P}_t, \mathcal{P}_{t+s}) \geq \kappa c \ln |\Lambda| \end{array} \right) \leq \frac{8d}{|\Lambda|^{c-2}} + 2d|\Lambda|^2 \exp(-D).$$

For all box Λ such that $|\Lambda| \geq (cd)^{cd^2}$, we have

$$\exp(-D) \leq \frac{1}{|\Lambda|^c}.$$

Therefore, for all Λ such that $|\Lambda| \geq (cd)^{cd^2}$, we have

$$\frac{8d}{|\Lambda|^{c-2}} + 2d|\Lambda|^2 \exp(-D) \leq \frac{10d}{|\Lambda|^{c-2}}.$$

In order to obtain $1/|\Lambda|^c$, we replace c by $c+2$, since $(c+2)/c \leq 3$ for $c \geq 1$, we have, for $|\Lambda| \geq \max \left\{ (cd)^{cd^2}, 3^{6d} \right\}$,

$$\begin{aligned} P_\mu \left(\begin{array}{c} \exists s \leq |\Lambda| \\ d_H^\Lambda(\mathcal{P}_t, \mathcal{P}_{t+s}) \geq 3\kappa c \ln |\Lambda| \end{array} \right) \\ \leq P_\mu \left(\begin{array}{c} \exists s \leq |\Lambda| \\ d_H^\Lambda(\mathcal{P}_t, \mathcal{P}_{t+s}) \geq \kappa(c+2) \ln |\Lambda| \end{array} \right) \leq \frac{10d}{|\Lambda|^c}. \end{aligned}$$

This yields the desired inequality. \square

We now complete the proof of the theorem 3.1.1.

Proof of theorem 3.1.1. Let us fix an edge e in Λ and a time t . We distinguish the cases where $e \in \mathcal{I}_t \setminus \mathcal{P}_t$ and $e \in \mathcal{P}_t$. If $e \in \mathcal{P}_t$, then we use the proposition 3.1.4. We consider now the case where $e \in \mathcal{I}_t \setminus \mathcal{P}_t$. We consider the last time τ when e was pivotal,

$$\tau = \max \left\{ 0 \leq s < t : e \in \mathcal{P}_s, e \notin \mathcal{P}_{s+1} \right\}.$$

The edge e has not been modified between τ and t . Let $c \geq 1$. We have

$$P_\mu(t - \tau \geq c|\Lambda| \ln |\Lambda|) \leq P_\mu \left(\begin{array}{c} \forall r \in [t - c|\Lambda| \ln |\Lambda|, t] \\ E_r \neq e \end{array} \right) \leq \frac{1}{|\Lambda|^c}.$$

We consider now the case where $t - \tau < c|\Lambda| \ln |\Lambda|$. We split the interval $[\tau, t]$ into subintervals of length $|\Lambda|$ and we set

$$t_i = \tau + i|\Lambda|, \quad 0 \leq i < \frac{t - \tau}{|\Lambda|} \quad \text{and} \quad t_{\lfloor (t-\tau)/|\Lambda| \rfloor + 1} = t.$$

According to proposition 3.5.2, there exists $\tilde{p} < 1$ and $\kappa' > 1$, such that

$$\begin{aligned} \forall p \geq \tilde{p} \quad \forall c \geq 1 \quad \forall |\Lambda| \geq (cd)^{cd^2} \quad \forall j \geq 0 \\ P_\mu \left(d_H^{\kappa' c \ln |\Lambda|}(\mathcal{P}_{t_j}, \mathcal{P}_{t_{j+1}}) \geq \kappa' c \ln |\Lambda| \right) \leq \frac{10d}{|\Lambda|^c}. \end{aligned}$$

Let $c \geq 1$. We suppose that

$$d(e, \mathcal{P}_t \cup \Lambda^c) \geq 2\kappa' c^2 (\ln |\Lambda|)^2 > (c \ln |\Lambda| + 1) \kappa' c \ln |\Lambda|.$$

We have by lemma 3.5.1, as illustrated in the figure 3.6,

$$\begin{aligned} \sum_{0 \leq i < (t-\tau)/|\Lambda|} d_H^{\kappa' c \ln |\Lambda|}(\mathcal{P}_{t_i}, \mathcal{P}_{t_{i+1}}) \vee \kappa' c \ln |\Lambda| \\ \geq \sum_{0 \leq i < (t-\tau)/|\Lambda|} d_H(\mathcal{P}_{t_i} \cup \Lambda^c, \mathcal{P}_{t_{i+1}} \cup \Lambda^c) \\ \geq d_H(\mathcal{P}_\tau \cup \Lambda^c, \mathcal{P}_t \cup \Lambda^c) \\ \geq d(e, \mathcal{P}_t \cup \Lambda^c) \geq 2\kappa' c^2 (\ln |\Lambda|)^2. \end{aligned}$$

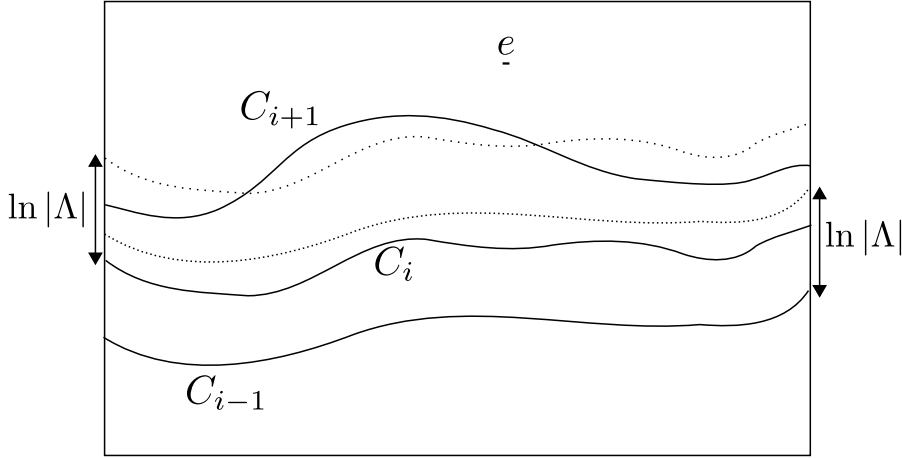


Figure 3.6: The cut C_i at time t_i is at distance less than $\ln |\Lambda|$ from C_{i-1} and the cut C_{i+1} is at distance more than $\ln |\Lambda|$ from C_i .

Necessarily, there is an index $0 \leq j < c \ln |\Lambda|$ such that

$$d_H^{\kappa' c \ln |\Lambda|}(\mathcal{P}_{t_j}, \mathcal{P}_{t_{j+1}}) \vee \kappa' c \ln |\Lambda| \geq 2\kappa' c \ln |\Lambda|.$$

Therefore, we have

$$d_H^{\kappa' c \ln |\Lambda|}(\mathcal{P}_{t_j}, \mathcal{P}_{t_{j+1}}) \geq \kappa' c \ln |\Lambda|.$$

By summing over j from 0 to $\lfloor c \ln |\Lambda| \rfloor$, we have

$$P_\mu \left(e \in \mathcal{I}_t, d(e, \mathcal{P}_t) \geq 2\kappa' c^2 (\ln |\Lambda|)^2, t - \tau < c |\Lambda| \ln |\Lambda| \right) \leq \frac{10d(c \ln |\Lambda| + 1)}{|\Lambda|^c}.$$

We set $\kappa = 2\kappa'$ and we obtain

$$\begin{aligned} & P_\mu \left(e \in \mathcal{P}_t \cup \mathcal{I}_t, d(e, \mathcal{P}_t \setminus \{e\}) \geq \kappa c^2 (\ln |\Lambda|)^2 \right) \\ & \leq P_\mu \left(e \in \mathcal{P}_t, d(e, \mathcal{P}_t \setminus \{e\}) \geq \kappa (c \ln |\Lambda|)^2 \right) \\ & + P_\mu \left(t - \tau \geq c |\Lambda| \ln |\Lambda| \right) + P_\mu \left(e \in \mathcal{I}_t, d(e, \mathcal{P}_t) \geq \kappa (c \ln |\Lambda|)^2, t - \tau \leq c \ln |\Lambda| \right) \\ & \leq \frac{2}{|\Lambda|^c} + \frac{10d(c \ln |\Lambda| + 1)}{|\Lambda|^c}. \end{aligned}$$

We sum over the edge e . For Λ such that $|\Lambda| \geq (cd)^{cd^2}$, we have

$$\begin{aligned} & P_\mu \left(\exists e \in \mathcal{P}_t \cup \mathcal{I}_t, d(e, \mathcal{P}_t \setminus \{e\}) \geq \kappa (c \ln |\Lambda|)^2 \right) \leq \\ & \frac{4d + 20d^2(c \ln |\Lambda| + 1)}{|\Lambda|^{c-1}} \leq \frac{1}{|\Lambda|^{c-2}}. \end{aligned}$$

We apply this result with $c+2$ of c , since for $c \geq 1$, $(c+2)^2/c^2 \leq 9$, we have

$$9\kappa c^2 \geq \kappa(c+2)^2.$$

Therefore, we have

$$P_\mu\left(\exists e \in \mathcal{P}_t \cup \mathcal{I}_t, d(e, \mathcal{P}_t \setminus \{e\}) \geq 9\kappa(c \ln |\Lambda|)^2\right) \leq \frac{1}{|\Lambda|^c}.$$

This proves the statement of theorem 3.1.1. \square

3.6 Speed estimations conditioned by the past

We derive further estimates on the speed of the pivotal edges which will be used in the proof of the theorem 3.1.3. First, we give a corollary of the proposition 3.4.1, which provides a control on the cuts, rather than the pivotal edges.

Corollary 3.6.1. *We have the following inequality:*

$$\begin{aligned} \exists \tilde{p} < 1 \quad \forall p \geq \tilde{p} \quad \forall \Lambda \\ \forall \ell \geq 1 \quad \forall e \in \Lambda \quad d(e, \Lambda^c) \geq \ell \quad \forall t > 0 \quad \forall s \in \{0, \dots, |\Lambda|\} \\ P_\mu\left(\exists C \in \mathcal{C}_{t+s}, e \in C \mid \exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell\right) \leq \exp(-\ell). \end{aligned}$$

Proof. We adapt the construction of the STP done in the proposition 3.4.4. We cannot use directly the STP constructed in proposition 3.4.4 because between the times t and $t+s$, the set of the pivotal edges can be empty. Therefore, we consider τ the last time before $t+s$ when \mathcal{P} is empty, i.e.,

$$\tau = \sup \{ r \leq t+s : \mathcal{P}_r = \emptyset \}.$$

If $\tau \leq t$, the conditions of proposition 3.4.4 are satisfied and there exists a closed decreasing simple STP starting from $(e, t+s)$ and ending after t which travels a distance at least ℓ . If $\tau = t+s$, since the edge e is in a cut, there exists a closed $*$ -path in Y_{t+s} which connects e to an edge intersecting the boundary of Λ . This path travels a distance at least ℓ . If $t < \tau < t+s$, then we have $\mathcal{P}_r \neq \emptyset$ for $\tau < r \leq t+s$. According to proposition 3.4.4, there exists a STP from $(e, t+s)$ to an edge of $\mathcal{P}_{\tau+1}$ at time $\tau+1$ or an edge intersecting the boundary of Λ after time $\tau+1$. If the STP ends at an edge intersecting the boundary, then it travels a distance at least ℓ . If it ends at an edge of $\mathcal{P}_{\tau+1}$ at time $\tau+1$, then, at time $\tau+1$, there must be an edge which becomes open and creates the pivotal edges of $\mathcal{P}_{\tau+1}$ which are on a cut C at time $\tau+1$. Notice that the cut C existed already at time τ because all the edges of C are closed. Therefore, there exists a decreasing closed STP which connects $(e, t+s)$ to an edge intersecting the boundary of Λ at

time τ . We reapply the algorithm of modification described in the proof of proposition 3.4.4 to obtain a simple STP. In all the cases above, we obtain a decreasing closed simple STP starting at $(e, t + s)$ which travels a distance at least ℓ . We apply the same arguments as in the proof of proposition 3.4.1 in order to obtain the desired estimate. \square

We wish to control the movement of the set of the cuts over a time interval. To achieve this goal, we will derive estimates for the appearance of a pivotal edge conditionally on the presence of a cut far away during a whole interval. In proposition 3.4.1, the conditioning gave information on one instant, not a whole interval. In the next lemma, we deal with a time interval of length $|\Lambda|$.

Lemma 3.6.2. *There exist $\tilde{p} < 1$ and $\kappa > 0$ such that for $p \geq \tilde{p}$, any $c \geq 1$, any integer $m \geq 1$, any Λ such that $|\Lambda| \geq 2d$, any edge e at distance more than $\kappa c \ln |\Lambda|$ from Λ^c and for $0 < s \leq |\Lambda| \leq t$, we have*

$$P_\mu \left(\begin{array}{c} \exists C \in \mathcal{C}_{t+s} \\ d(e, C) \leq (m-1)\kappa c \ln |\Lambda| \end{array} \middle| \begin{array}{c} \forall r \in]t - |\Lambda|, t] \\ \exists C_r \in \mathcal{C}_r \quad d(e, C_r) \geq m\kappa c \ln |\Lambda| \\ \exists C' \in \mathcal{C}_{t-|\Lambda|} \quad d(e, C') \geq (m+1)\kappa c \ln |\Lambda| \end{array} \right) \leq \frac{1}{|\Lambda|^c}.$$

Proof. Let κ be a positive constant which will be chosen at the end of the proof. We reuse the construction of the STP in corollary 3.6.1: there exists a decreasing closed simple STP which connects $(e, t + s)$ to a pivotal edge at time t or to an edge intersecting the boundary of Λ at a time after t . Since the edge e is at distance at least $\kappa c \ln |\Lambda|$ from $\mathcal{P}_t \cup \Lambda^c$, in both cases, there exists a decreasing closed simple STP γ of length $(\kappa c \ln |\Lambda|)/2d$ starting from the time-edge $(e, t + s)$ and ending after t which is strictly included in the box Λ . Let Γ be the space projection of γ , i.e.,

$$\Gamma = \text{Space}(\gamma) = (e_1, \dots, e_m).$$

We introduce the following events:

$$D_1 = \{ \forall r \in]t - |\Lambda|, t], \exists C_r \in \mathcal{C}_r, d(e, C_r) \geq m\kappa c \ln |\Lambda| \},$$

$$\bar{D}_1 = \{ \exists C \in \mathcal{C}_{t-|\Lambda|}, d(e, C) \geq (m+1)\kappa c \ln |\Lambda| \},$$

and

$$\mathcal{E}(t, s, \Gamma) = \left\{ \begin{array}{l} \exists \gamma \text{ simple closed decreasing STP} \\ \text{length}(\gamma) = (\kappa c \ln |\Lambda|)/2d, \text{Space}(\gamma) = \Gamma \\ \gamma \text{ starts at an edge } (e', t + s) \\ d(e', e) \leq (m-1)\kappa c \ln |\Lambda| \\ \text{and ends after } t \end{array} \right\}.$$

As in the proof of proposition 3.4.1, the probability appearing in the proposition is less than

$$\sum_{\Gamma} P_{\mu} (\mathcal{E}(t, s, \Gamma) \mid D_1, \bar{D}_1), \quad (3.6.1)$$

where the sum is over the possible choices for Γ . We fix a path Γ and we condition each probability in the sum by the configuration at time t . Let A be a subset of $\text{support}(\Gamma)$, we denote by $M(A)$ the following set of configurations:

$$M(A) = \left\{ \omega : \begin{array}{l} \forall f \in A \quad \omega(f) = 0 \\ \forall f \in \text{support}(\Gamma) \setminus A \quad \omega(f) = 1 \end{array} \right\}.$$

Let y be a configuration in $M(A)$ and let us start by estimating the probability

$$P_{\mu} (\mathcal{E}(t, s, \Gamma) \mid Y_t = y, D_1, \bar{D}_1).$$

By the Markov property, this probability is equal to

$$P_{\mu} (\mathcal{E}(t, s, \Gamma) \mid Y_t = y),$$

and by proposition 3.4.5, it is less than

$$\left(\frac{s}{|\Lambda|} (1-p) \right)^{(\kappa c \ln |\Lambda|)/2d - |A|}. \quad (3.6.2)$$

Each term of the sum in (3.6.1) can be written as

$$\sum_{0 \leq k \leq |\text{support}(\Gamma)|} \sum_{\substack{A \subset \text{support}(\Gamma) \\ |A| = k}} \sum_{y \in M(A)} P_{\mu} (\mathcal{E}(t, s, \Gamma) \mid Y_t = y, D_1, \bar{D}_1) \times P_{\mu} (Y_t = y \mid D_1, \bar{D}_1).$$

Using (3.6.2), we see that each term in (3.6.1) is less than

$$\sum_{0 \leq k \leq |\text{support}(\Gamma)|} \left(\frac{s}{|\Lambda|} (1-p) \right)^{(\kappa c \ln |\Lambda|)/2d - k} \sum_{\substack{A \subset \text{support}(\Gamma) \\ |A| = k}} P_{\mu} (Y_t \in M(A) \mid D_1, \bar{D}_1). \quad (3.6.3)$$

In the rest of the proof, we will calculate an upper bound of

$$\sum_{\substack{A \subset \text{support}(\Gamma) \\ |A| = k}} P_{\mu} (Y_t \in M(A) \mid D_1, \bar{D}_1). \quad (3.6.4)$$

Notice that, for an edge $f \in \Gamma$, if there is a time $r \in [t - |\Lambda|, t]$ such that $E_r = f$, then, under the probability P_p , conditioned on D_1, \bar{D}_1 , the state of f at time t is independent of the other edges of Γ and it follows a Bernoulli variable of parameter p . On the contrary, if $E_r \neq f$ for all $r \in [t - |\Lambda|, t]$,

then the state of f at time t is the same as at time $t - |\Lambda|$. For A a subset of $\text{support}(\Gamma)$ and B a subset of A , we define the event $\text{reset}(B, A)$ as

$$\text{reset}(B, A) = \left\{ \begin{array}{l} \forall e \in B, \exists r \in [t - |\Lambda|, t], E_r = e \\ \forall r \in [t - |\Lambda|, t], E_r \notin A \setminus B \end{array} \right\}.$$

For each subset A , we partition the probability in (3.6.4) according to the subset B of A for which the event $\text{reset}(B, A)$ occurs, and we get

$$\begin{aligned} & \sum_{\substack{A \subset \text{support}(\Gamma) \\ |A| = k, B \subset A}} P_\mu(Y_t \in M(A), \text{reset}(B, A) \mid D_1, \bar{D}_1) \\ &= \sum_{\substack{A \subset \text{support}(\Gamma) \\ |A| = k, B \subset A}} P_\mu \left(\begin{array}{l} \forall f \in B, Y_t(f) = 0 \\ \forall f \in A \setminus B, Y_{t-|\Lambda|}(f) = 0 \\ \text{reset}(B, A) \end{array} \mid D_1, \bar{D}_1 \right). \end{aligned} \quad (3.6.5)$$

We write

$$P_\mu(\cdot) = \sum_{x_0, y_0} P_{x_0, y_0}(\cdot) \mu((x_0, y_0)),$$

where P_{x_0, y_0} is the law of the process $(X_t, Y_t)_{t \in \mathbb{N}}$ starting from the initial configuration (x_0, y_0) . For each term we rewrite the conditioned probability as follows:

$$\begin{aligned} & P_{x_0, y_0} \left(\begin{array}{l} \forall f \in B, Y_t(f) = 0 \\ \forall f \in A \setminus B, Y_{t-|\Lambda|}(f) = 0 \\ \text{reset}(B, A) \end{array} \mid D_1, \bar{D}_1 \right) \\ &= \frac{P_{x_0, y_0} \left(\left\{ \begin{array}{l} \forall f \in B, Y_t(f) = 0 \\ \forall f \in A \setminus B, Y_{t-|\Lambda|}(f) = 0 \\ \text{reset}(B, A) \end{array} \right\} \cap D_1 \cap \bar{D}_1 \right)}{P_{x_0, y_0}(D_1, \bar{D}_1)}. \end{aligned} \quad (3.6.6)$$

Starting from an initial configuration (x_0, y_0) , the process $(Y_t)_{t \in \mathbb{N}}$ is obtained by conditioning to stay in the configurations with disconnexion. We can replace P_μ by P_p in the previous fraction and the numerator can be written as

$$\begin{aligned} & P_p \left(\left\{ \begin{array}{l} \forall f \in B, X_t(f) = 0 \\ \forall f \in A \setminus B, X_{t-|\Lambda|}(f) = 0 \\ \text{reset}(B, A) \end{array} \right\} \cap D_1 \cap \bar{D}_1 \right) \\ &= P_p \left(\left\{ \begin{array}{l} \forall f \in B, B_{\tau(f)} = 0 \\ \forall f \in A \setminus B, X_{t-|\Lambda|}(f) = 0 \\ \text{reset}(B, A) \end{array} \right\} \cap D_1 \cap \bar{D}_1 \right), \end{aligned}$$

where the time $\tau(f)$ is the last time before t when the edge f is chosen, i.e.,

$$\tau(f) = \sup \{ s \leq t : E_s = f \}.$$

Let us fix a sequence of edges $\mathbf{e} = (e_1, \dots, e_t)$ and let us condition this last probability by the event

$$(E_1, \dots, E_t) = \mathbf{e}.$$

We have

$$\begin{aligned} & P_p \left(\left\{ \begin{array}{l} \forall f \in B, B_{\tau(f)} = 0 \\ \forall f \in A \setminus B, X_{t-|\Lambda|}(f) = 0 \\ \text{reset}(B, A) \end{array} \right\} \cap D_1 \cap \bar{D}_1 \right) \\ &= \sum_{\mathbf{e} \in \text{reset}(B, A)} P_p \left(\left\{ \begin{array}{l} \forall f \in B, B_{\tau(f)} = 0 \\ \forall f \in A \setminus B, X_{t-|\Lambda|}(f) = 0 \end{array} \right\} \cap D_1 \cap \bar{D}_1 \mid (E_1, \dots, E_t) = \mathbf{e} \right) \\ & \quad \times P_p((E_1, \dots, E_t) = \mathbf{e}). \quad (3.6.7) \end{aligned}$$

Notice that on the event $\text{reset}(B, A)$, for an edge $f \in B$, we have necessarily $\tau(f) > t - |\Lambda|$. Therefore the event $\{ \forall f \in B, B_{\tau(f)} = 0 \}$ depends on the set of variables $\{ B_s : e_s \in B, s > t - |\Lambda| \}$. The events

$$\{ \forall f \in A \setminus B, X_{t-|\Lambda|}(f) = 0 \}$$

and \bar{D}_1 depend on the variables $\{ B_s : s \leq t - |\Lambda| \}$ and the event D_1 does not depend on the variables $\{ B_s : E_s \in B \}$. All the events above are decreasing, by the BK inequality applied to the random variables $(B_s)_{s \in \mathbb{N}}$, we have

$$\begin{aligned} & P_p \left(\left\{ \begin{array}{l} \forall f \in B, B_{\tau(f)} = 0 \\ \forall f \in A \setminus B, X_{t-|\Lambda|}(f) = 0 \end{array} \right\} \cap D_1 \cap \bar{D}_1 \mid (E_1, \dots, E_t) = \mathbf{e} \right) \\ & \leq P_p(\forall f \in B, B_{\tau(f)} = 0 \mid (E_1, \dots, E_t) = \mathbf{e}) \\ & \quad \times P_p \left(\begin{array}{l} \forall f \in A \setminus B, X_{t-|\Lambda|}(f) = 0 \\ D_1 \cap \bar{D}_1 \end{array} \mid (E_1, \dots, E_t) = \mathbf{e} \right) \\ & \leq (1-p)^{|B|} P_p \left(\begin{array}{l} \forall f \in A \setminus B, X_{t-|\Lambda|}(f) = 0 \\ D_1 \cap \bar{D}_1 \end{array} \mid (E_1, \dots, E_t) = \mathbf{e} \right). \end{aligned}$$

We use this inequality in (3.6.7) and we obtain

$$\begin{aligned} & P_p \left(\left\{ \begin{array}{l} \forall f \in B, B_{\tau(f)} = 0 \\ \forall f \in A \setminus B, X_{t-|\Lambda|}(f) = 0 \\ \text{reset}(B, A) \end{array} \right\} \cap D_1 \cap \bar{D}_1 \right) \\ & \leq (1-p)^{|B|} P_p \left(\left\{ \begin{array}{l} \forall f \in A \setminus B, X_{t-|\Lambda|}(f) = 0 \\ \text{reset}(B, A) \end{array} \right\} \cap D_1 \cap \bar{D}_1 \right). \end{aligned}$$

We replace the numerator in (3.6.6) and we sum over the initial configurations, we have

$$P_\mu \left(\begin{array}{c} \forall f \in B, Y_t(f) = 0 \\ \forall f \in A \setminus B, Y_{t-|\Lambda|}(f) = 0 \\ \text{reset}(B, A) \end{array} \middle| D_1, \bar{D}_1 \right) \\ \leq (1-p)^{|B|} P_\mu \left(\begin{array}{c} \forall f \in A \setminus B, X_{t-|\Lambda|}(f) = 0 \\ \text{reset}(B, A) \end{array} \middle| D_1, \bar{D}_1 \right).$$

This last probability is less than

$$\frac{P_\mu \left(\begin{array}{c} \forall f \in A \setminus B \quad Y_{t-|\Lambda|}(f) = 0 \\ \text{reset}(B, A) \end{array} \middle| \bar{D}_1 \right)}{P_\mu(D_1 | \bar{D}_1)}. \quad (3.6.8)$$

Let us estimate separately the numerator and the denominator. In order to calculate the numerator, we use the notation $\bar{M}(A)$ defined as follows:

$$\bar{M}(A) = \{ \omega : \forall f \in A \quad \omega(f) = 0 \}.$$

We obtain

$$P_\mu \left(\begin{array}{c} \forall f \in A \setminus B \quad Y_{t-|\Lambda|}(f) = 0 \\ \text{reset}(B, A) \end{array} \middle| \bar{D}_1 \right) \\ = \sum_{y \in \bar{M}(A \setminus B)} P_\mu \left(\text{reset}(B, A), Y_{t-|\Lambda|} = y \middle| \bar{D}_1 \right).$$

As in the proof of proposition 3.4.1, we write

$$P_\mu \left(\text{reset}(B, A), Y_{t-|\Lambda|} = y \middle| \bar{D}_1 \right) = \\ P_\mu \left(\text{reset}(B, A) \middle| Y_{t-|\Lambda|} = y, \bar{D}_1 \right) P_\mu \left(Y_{t-|\Lambda|} = y \middle| \bar{D}_1 \right).$$

Since the event $\text{reset}(B, A)$ depends only on the variables

$$\{ (E_r, B_r) : t - |\Lambda| < r \leq t \},$$

it is independent from $Y_{t-|\Lambda|}$ (and also from the event \bar{D}_1 , as \bar{D}_1 is entirely determined by $Y_{t-|\Lambda|}$). We obtain

$$\sum_{y \in \bar{M}(A \setminus B)} P_\mu \left(\text{reset}(B, A), Y_{t-|\Lambda|} = y \middle| \bar{D}_1 \right) = \\ P_\mu \left(\text{reset}(B, A) \right) P_\mu \left(Y_{t-|\Lambda|} \in \bar{M}(A \setminus B) \middle| \bar{D}_1 \right).$$

Let us estimate the last probability. Since the second marginal of P_μ is $P_{\mathcal{D}}$ and $P_{\mathcal{D}}(\cdot) = P_p(\cdot | T \leftrightarrow B)$, we have

$$\begin{aligned} P_\mu\left(Y_{t-|\Lambda|} \in \overline{M}(A \setminus B) \mid \overline{D}_1\right) \\ = \frac{P_p\left(\begin{array}{l} \forall f \in A \setminus B \quad f \text{ closed} \\ \exists C \in \mathcal{C}, d(e, C) \geq (m+1)\kappa c \ln |\Lambda| \end{array}\right)}{P_{\mathcal{D}}\left(\exists C \in \mathcal{C}, d(e, C) \geq (m+1)\kappa c \ln |\Lambda|\right) P_p\left(T \leftrightarrow B\right)}. \end{aligned}$$

The event

$$\{\forall f \in A \setminus B \quad f \text{ closed}\}$$

depends only on the edges at distance less than $(m-1/2)\kappa c \ln |\Lambda|$ from the edge e , while the event

$$\{\exists C \in \mathcal{C}, d(e, C) \geq (m+1)\kappa c \ln |\Lambda|\}$$

depends on the edges at distance larger than $(m+1)\kappa c \ln |\Lambda|$ from e . By independence, we have

$$\begin{aligned} P_p\left(\begin{array}{l} \forall f \in A \setminus B \quad f \text{ closed} \\ \exists C \in \mathcal{C}, d(e, C) \geq (m+1)\kappa c \ln |\Lambda| \end{array}\right) = \\ P_p(\forall f \in A \setminus B \quad f \text{ closed}) P_p(\exists C \in \mathcal{C}, d(e, C) \geq (m+1)\kappa c \ln |\Lambda|). \end{aligned}$$

We obtain therefore

$$P_\mu\left(Y_{t-|\Lambda|} \in \overline{M}(A \setminus B) \mid \overline{D}_1\right) \leq P_p(\forall f \in A \setminus B \quad f \text{ closed}) = (1-p)^{|A \setminus B|}.$$

We conclude that the numerator of (3.6.8) is less than

$$(1-p)^{|A \setminus B|} P_\mu(\text{reset}(B, A)).$$

Now, we estimate the denominator in (3.6.8). In fact, this probability is equal to

$$1 - P_\mu\left(\exists s \in]t-|\Lambda|, t], \forall C \in \mathcal{C}_s, d(e, C) < m\kappa c \ln |\Lambda| \mid \overline{D}_1\right).$$

By corollary 3.6.1, there exists a $\tilde{p} < 1$ such that for $p \geq \tilde{p}$, for any $c, \kappa_1 \geq 1$ and for any edge e at distance more than $\kappa_1 c \ln |\Lambda|$ from Λ^c , we have

$$P_\mu\left(\exists C \in \mathcal{C}_{t+s}, e \in C \mid \exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \kappa_1 c \ln |\Lambda|\right) \leq \frac{1}{|\Lambda|^{\kappa_1 c}}.$$

Since $(c+3)/c \leq 4$ for $c \geq 1$, there exists a $\kappa_1 > 1$, such that for any $c \geq 1$,

$$\frac{1}{|\Lambda|^{\kappa_1 c}} \leq \frac{1}{|\Lambda|^{c+3}}.$$

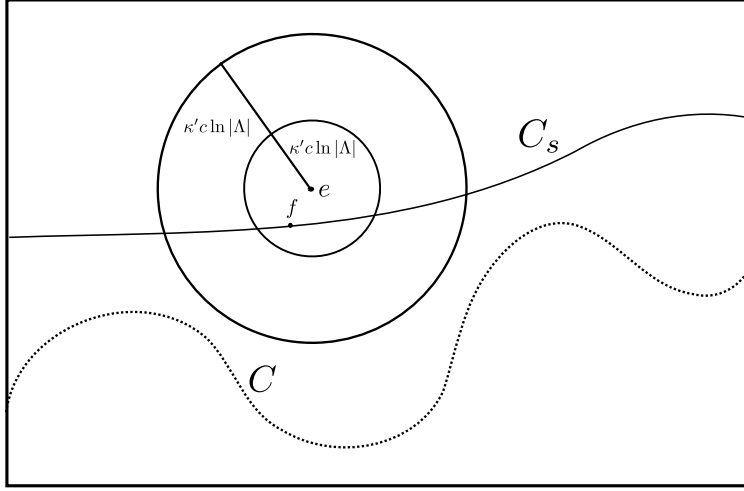


Figure 3.7: The edge $f \in \mathcal{P}_s$ is at distance less than $\kappa'c \ln |\Lambda|$ from e and the cut $C \in \mathcal{C}_{t-|\Lambda|}$ is at distance larger than $2\kappa'c \ln |\Lambda|$ from e .

We have therefore, as illustrated in the figure 3.7, for $|\Lambda| \geq 2d$:

$$\begin{aligned}
 P_\mu \left(\begin{array}{c} \exists s \in]t - |\Lambda|, t], \forall C' \in \mathcal{C}_s \\ d(e, C') < \kappa_1 c \ln |\Lambda| \end{array} \middle| \bar{D}_1 \right) &\leq \\
 \sum_{s \in]t - |\Lambda|, t]} \sum_{f: d(e, f) < \kappa_1 c \ln |\Lambda|} P_\mu \left(\exists C_s \in \mathcal{C}_s, f \in C_s \middle| \bar{D}_1 \right) &\leq \\
 \sum_{s \in]t - |\Lambda|, t]} \sum_{f: d(e, f) < \kappa_1 c \ln |\Lambda|} \frac{1}{|\Lambda|^{c+3}} &\leq \frac{1}{|\Lambda|^c} \leq \frac{1}{2}.
 \end{aligned}$$

Let $\kappa \geq \kappa_1$, the probability (3.6.8) is less than

$$2(1-p)^{|A \setminus B|} P_\mu \left(\text{reset}(B, A) \right).$$

We bound from above each term of (3.6.5) and we obtain an upper bound for (3.6.4):

$$\sum_{\substack{A \subset \text{support}(\Gamma) \\ |A| = k}} P_\mu \left(Y_t \in M(A) \middle| D_1, \bar{D}_1 \right) \leq \sum_{\substack{A \subset \text{support}(\Gamma) \\ |A| = k, B \subset A}} 2(1-p)^k P_\mu \left(\text{reset}(B, A) \right).$$

For each set A fixed, we have

$$\sum_{B \subset A} P_\mu \left(\text{reset}(B, A) \right) = 1.$$

Therefore, we obtain

$$P_\mu \left(Y_t \in M(A) \middle| D_1, \bar{D}_1 \right) \leq 2(1-p)^{|A|} \quad (3.6.9)$$

$$\sum_{\substack{A \subset \text{support}(\Gamma) \\ |A| = k}} P_\mu (Y_t \in M(A) \mid D_1, \bar{D}_1) \leq 2 \binom{|\text{support}(\Gamma)|}{k} (1-p)^k.$$

Finally, combined with (3.6.3), we obtain an upper bound for (3.6.1) which is

$$\begin{aligned} 2(1-p)^{(\kappa c \ln |\Lambda|)/2d} \sum_{0 \leq k \leq (\kappa c \ln |\Lambda|)/2d} \binom{|\text{support}(\Gamma)|}{k} \left(\frac{s}{|\Lambda|} \right)^{(\kappa c \ln |\Lambda|)/2d - k} \\ \leq 2 \left((1-p) \left(1 + \frac{s}{|\Lambda|} \right) \right)^{(\kappa c \ln |\Lambda|)/2d}. \end{aligned}$$

We sum over the possible choices for the path Γ , by the lemma 3.3.3, the sum in (3.6.1) is less than

$$2|\Lambda| (\alpha(d)(1-p) (1 + s/|\Lambda|))^{(\kappa c \ln |\Lambda|)/2d}.$$

There is a $\kappa > 0$, such that for $p \geq \tilde{p}$, such that this term is less than

$$\frac{1}{|\Lambda|^c}.$$

We obtain the result stated in the lemma. \square

We next show a generalisation of proposition 3.4.1 and corollary 3.6.1 which is an essential ingredient for the proof of theorem 3.1.3.

Proposition 3.6.3. *We have the following estimate:*

$$\begin{aligned} \exists \tilde{p} < 1 \quad \exists \kappa > 1 \quad \forall p \geq \tilde{p} \quad \forall c \geq 2 \quad \forall \Lambda \quad |\Lambda| \geq 12(2\kappa d)^d \\ \forall e \subset \Lambda \quad d(e, \Lambda^c) \geq \kappa c^2 \ln^2 |\Lambda| \\ \forall n \geq 1 \quad n \leq c \ln |\Lambda| \quad \forall m \geq 0 \quad n + m \leq c \ln |\Lambda| \\ \forall s \in \{1, \dots, |\Lambda|\} \quad \forall t \geq n|\Lambda| \\ P_\mu \left(\begin{array}{l} \exists C \in \mathcal{C}_{t+s} \\ d(e, C) \leq m\kappa c \ln |\Lambda| \end{array} \middle| \begin{array}{l} \forall k \in \{1, \dots, n\} \\ \forall r \in]t - k|\Lambda|, t - (k-1)|\Lambda| \\ \exists C_r \in \mathcal{C}_r \quad d(e, C_r) \geq (k+m)\kappa c \ln |\Lambda| \\ \exists C' \in \mathcal{C}_{t-n|\Lambda|} \quad d(e, C') \geq (n+m+1)\kappa c \ln |\Lambda| \end{array} \right) \\ \leq \frac{1}{|\Lambda|^c}. \end{aligned}$$

Proof. Notice that for the case $n = 1$, this proposition corresponds to the lemma 3.6.2. Let κ be a constant which will be determined at the end of the proof. We start by introducing some notations. For $m \in \mathbb{N}$ and $k \geq 1$, we define $D_{k,m}$ to be the event

$$D_{k,m} = \left\{ \begin{array}{l} \forall r \in]t - k|\Lambda|, t - (k-1)|\Lambda| \\ \exists C_r \in \mathcal{C}_r \quad d(e, C_r) \geq (k+m)\kappa c \ln |\Lambda| \end{array} \right\}$$

and $\bar{D}_{k,m}$ the event

$$\bar{D}_{k,m} = \left\{ \exists C \in \mathcal{C}_{t-k|\Lambda} \quad d(e, C) \geq (k+m+1)\kappa c \ln |\Lambda| \right\}.$$

For $\kappa \geq 1$, $c \geq 2$, $k \in \mathbb{N}^*$, $e \subset \Lambda$ and $t, s \in \mathbb{N}^*$, we denote by $(H_{k,m})$ the following inequality:

$$(H_{k,m}) : P_\mu \left(\begin{array}{c} \exists C \in \mathcal{C}_{t+s} \\ d(e, C) \leq m\kappa c \ln |\Lambda| \end{array} \middle| D_{1,m}, \dots, D_{k,m}, \bar{D}_{k,m} \right) \leq \frac{1}{|\Lambda|^c}.$$

Our goal is to show that there exist $\tilde{p} < 1$ and $\kappa > 1$ such that for $p \geq \tilde{p}$, $c \geq 2$, $e \subset \Lambda$ at distance larger than $\kappa c^2 \ln^2 |\Lambda|$ from Λ^c , $s \in \{1, \dots, |\Lambda|\}$, the inequality $(H_{k,m})$ holds for any $1 \leq k+m \leq c \ln |\Lambda|$ and $t \geq (k+m)|\Lambda|$. In particular, the inequality stated in the proposition corresponds to the case $(k, m) = (n, 0)$. In order to show this proposition by induction on k , we introduce an auxiliary inequality $(G_{k,m})$ for $A \subset \Lambda$, $d(e, A) \leq (\kappa c \ln |\Lambda|)/2$:

$$(G_{k,m}) : P_\mu \left(\forall f \in A \quad Y_t(f) = 0 \mid D_{1,m}, \dots, D_{k+m}, \bar{D}_{k+m} \right) \leq 2^k (1-p)^{|A|}.$$

By lemma 3.6.2, there exist $\tilde{p} < 1$ and $\kappa > 1$ such that for $p \geq \tilde{p}$, $c \geq 2$, $e \subset \Lambda$ at distance larger than $\kappa c^2 \ln^2 |\Lambda|$ from Λ^c and $t \geq c|\Lambda| \ln |\Lambda|$, $s \in \{1, \dots, |\Lambda|\}$, the inequalities $(H_{1,m})$ hold for all $m \leq c \ln |\Lambda| - 1$, meanwhile, the inequalities $(G_{1,m})$ was also proved in (3.6.9). For this \tilde{p} , there exists a $\kappa > 0$ such that, for any $c \geq 2$, we have

$$\left(\alpha(d) 2^{1+2d/\kappa} (1-\tilde{p}) \right)^{(\kappa c \ln |\Lambda|)/2d} \leq \frac{1}{|\Lambda|^c}.$$

Notice that for this κ , the inequality in lemma 3.6.2 is also satisfied. Let us fix $c \geq 2$ and let us show the inequalities by induction on the integer k . Let $k < c \ln |\Lambda|$, we suppose that the inequalities $(H_{k,m})$ and $(G_{k,m})$ hold for all $m \in \mathbb{N}$ such that $k+m \leq c \ln |\Lambda|$. Let us prove first the inequality $(G_{k+1,m})$ for a $m \in \{0, \dots, \lfloor c \ln |\Lambda| \rfloor - k - 1\}$. We reuse the notations $\text{reset}(I, A, B)$ and $\bar{M}(A)$ defined for a subset B of A and a time interval I :

$$\begin{aligned} \text{reset}(I, A, B) &= \left\{ \begin{array}{l} \forall e \in B, \exists r \in I, E_r = e \\ \forall r \in I, E_r \notin A \setminus B \end{array} \right\}, \\ \bar{M}(A) &= \left\{ \omega : \forall f \in A \quad \omega(f) = 0 \right\}. \end{aligned}$$

We denote by I_1 the interval $]t-|\Lambda|, t]$. We rewrite the probability $(G_{k+1,m})$ as in the proof of lemma 3.6.2:

$$\begin{aligned} &P_\mu \left(\forall f \in A \quad Y_t(f) = 0 \mid D_{1,m}, \dots, D_{k+1,m}, \bar{D}_{k+1,m} \right) \\ &= \sum_{B \subset A} P_\mu \left(\forall f \in A \quad Y_t(f) = 0, \text{reset}(I_1, A, B) \mid D_{1,m}, \dots, D_{k+1,m}, \bar{D}_{k+1,m} \right). \end{aligned}$$

For each $B \subset A$, we have

$$\begin{aligned} & P_\mu \left(\forall f \in A \quad Y_t(f) = 0, \text{reset}(I_1, A, B) \mid D_{1,m}, \dots, D_{k+1,m}, \bar{D}_{k+1,m} \right) \\ &= P_\mu \left(\begin{array}{c} \forall f \in B, Y_t(f) = 0 \\ \forall f \in A \setminus B, Y_{t-|\Lambda|}(f) = 0 \\ \text{reset}(I_1, A, B) \end{array} \mid D_{1,m}, \dots, D_{k+1,m}, \bar{D}_{k+1,m} \right). \end{aligned}$$

We use the same arguments as in the inequality (3.6.6) of the lemma 3.6.2 to obtain a factor $1 - p$ for each edge where the event reset is realised. We have

$$\begin{aligned} & P_\mu \left(\begin{array}{c} \forall f \in B, Y_t(f) = 0 \\ \forall f \in A \setminus B, Y_{t-|\Lambda|}(f) = 0 \\ \text{reset}(I_1, A, B) \end{array} \mid D_{1,m}, \dots, D_{k+1,m}, \bar{D}_{k+1,m} \right) \\ &\leq (1-p)^{|B|} P_\mu \left(Y_{t-|\Lambda|} \in \bar{M}(A \setminus B), \text{reset}(I_1, A, B) \mid D_{1,m}, \dots, D_{k+1,m}, \bar{D}_{k+1,m} \right). \end{aligned}$$

The event $\text{reset}(I_1, A, B)$ is independent of what happens before and until $t - |\Lambda|$ and of $D_{2,m}, \dots, D_{k+1,m}, \bar{D}_{k+1,m}$. Therefore, this last probability is less than or equal to

$$\begin{aligned} & \frac{P_\mu \left(Y_{t-|\Lambda|} \in \bar{M}(A \setminus B), \text{reset}(I_1, A, B) \mid D_{2,m}, \dots, D_{k+1,m}, \bar{D}_{k+1,m} \right)}{P_\mu \left(D_{1,m} \mid D_{2,m}, \dots, D_{k+1,m}, \bar{D}_{k+1,m} \right)} = \\ & \frac{P_\mu \left(\text{reset}(I_1, A, B) \right)}{P_\mu \left(D_{1,m} \mid D_{2,m}, \dots, D_{k+1,m}, \bar{D}_{k+1,m} \right)} \\ & \times P_\mu \left(Y_{t-|\Lambda|} \in \bar{M}(A \setminus B) \mid D_{2,m}, \dots, D_{k+1,m}, \bar{D}_{k+1,m} \right). \end{aligned}$$

We apply the inequality $(G_{k,m+1})$, at time $t - |\Lambda|$. The last probability is less or equal than

$$2^k (1-p)^{|A \setminus B|}.$$

For the denominator, we apply $(H_{k,m+1})$ at time $t - 1$ and we obtain

$$\begin{aligned} & P_\mu \left(D_{1,m} \mid D_{2,m}, \dots, D_{k+1,m}, \bar{D}_{k+1,m} \right) \\ & \geq 1 - P_\mu \left(\begin{array}{c} \exists r \in]t - |\Lambda|, t] \quad \exists C_r \in \mathcal{C}_r \\ d(e, C_r) \leq m\kappa c \ln |\Lambda| \end{array} \mid D_{2,m}, \dots, D_{k+1,m}, \bar{D}_{k+1,m} \right) \\ & \geq 1 - \frac{|\Lambda|}{|\Lambda|^c}. \end{aligned}$$

Therefore, for $|\Lambda| \geq 2$, we have

$$\frac{1}{|\Lambda|^{c-1}} \leq \frac{1}{2}.$$

Therefore, we have for the denominator

$$P_\mu\left(D_{1,m} \mid D_{2,m}, \dots, D_{k+1,m}, \bar{D}_{k+1,m}\right) \geq \frac{1}{2}.$$

We obtain $(G_{k+1,m})$ by summing over the choices of B :

$$\begin{aligned} P_\mu\left(\forall f \in A \quad Y_t(f) = 0 \mid D_{1,m}, \dots, D_{k+1,m}, \bar{D}_{k+1,m}\right) \\ \leq \frac{2^k(1-p)^{|A|}}{1/2} \sum_{B \subset A} P_\mu\left(\text{reset}(I_1, A, B)\right) = 2^{k+1}(1-p)^{|A|}. \end{aligned}$$

In order to obtain $(H_{k+1,m})$, we will study the STP obtained as in the corollary 3.6.1. We recall that this STP is of length at least $(\kappa c \ln |\Lambda|)/2d$. We fix first the space projection of the STP, which we denote by Γ . As in the proof of lemma 3.6.2 and proposition 3.4.1, we study separately the edges that close after the time t and the edges which are closed at time t by conditioning the probability by the configuration Y_t . For the edges which become closed after t , we apply proposition 3.4.5 and we obtain that the probability for obtaining a simple closed decreasing STP γ between t and $t+s$ satisfying $\text{Space}(\gamma) = \Gamma$ is less than

$$\begin{aligned} \sum_{0 \leq j \leq \text{support}(\Gamma)} \sum_{A \subset \text{support}(\Gamma): |A|=j} \left(\frac{s}{|\Lambda|}(1-p)\right)^{(\kappa c \ln |\Lambda|)/2d-j} \times \\ P\left(Y_t \in \bar{M}(A) \mid D_{1,m}, \dots, D_{k+1,m}, \bar{D}_{k+1,m}\right). \end{aligned} \quad (3.6.10)$$

We apply the inequality $(G_{k+1,m})$ for the last probability and we have

$$P\left(Y_t \in \bar{M}(A) \mid D_{1,m}, \dots, D_{k+1,m}, \bar{D}_{k+1,m}\right) \leq 2^{k+1}(1-p)^j.$$

Therefore, the sum in (3.6.10) is less than

$$\begin{aligned} \sum_{0 \leq j \leq \text{support}(\Gamma)} \binom{\text{support}(\Gamma)}{j} \left(\frac{s}{|\Lambda|}\right)^j 2^{k+1}(1-p)^{(\kappa c \ln |\Lambda|)/2d} \leq \\ 2^{k+1} \left((1+s/|\Lambda|)(1-p)\right)^{(\kappa c \ln |\Lambda|)/2d}. \end{aligned}$$

For $|\Lambda| \geq 2d$, $k+1 \leq c \ln |\Lambda|$ and $s \leq |\Lambda|$, we have

$$2^{k+1} \left((1+s/|\Lambda|)(1-p)\right)^{(\kappa c \ln |\Lambda|)/2d} \leq \left(2^{1+2d/\kappa}(1-p)\right)^{(\kappa c \ln |\Lambda|)/2d}.$$

We sum over the choices for Γ by using the lemma 3.3.3, and we have

$$\begin{aligned} P_\mu\left(\begin{array}{c} \exists C \in \mathcal{C}_{t+s} \\ d(e, C) \leq m\kappa c \ln |\Lambda| \end{array} \mid D_{1,m}, \dots, D_{k+1,m}, \bar{D}_{k+1,m}\right) \\ \leq |\Lambda| \left(\alpha(d)2^{1+2d/\kappa}(1-p)\right)^{(\kappa c \ln |\Lambda|)/2d}. \end{aligned}$$

For $p \geq \tilde{p}$ and the κ chosen at the beginning of the proof, for any $c \geq 2$, we have

$$|\Lambda| \left(\alpha(d) 2^{1+2d/\kappa} (1-p) \right)^{(\kappa c \ln |\Lambda|)/2d} \leq \frac{1}{|\Lambda|^c}.$$

Notice that the constant κ doesn't depend on k . Therefore, the inequalities $\{(H_{k,m}), (G_{k,m}) : 1 \leq k+m \leq c \ln |\Lambda|\}$ are all satisfied for $p \geq \tilde{p}$ and this κ . This concludes the induction. \square

3.7 The law of an edge far from a cut

We now show the theorem 3.1.3 with the help of propositions 3.6.3.

Proof of theorem 3.1.3. Since μ is the stationary distribution of the process $(X_t, Y_t)_{t \in \mathbb{N}}$, we can choose a time t and show the result for the configuration (X_t, Y_t) . For a time $r \in \mathbb{N}$ and a distance $\ell > 0$, we introduce the events

$$\overline{D}(r, \ell) = \{ \exists C \in \mathcal{C}_r, d(e, C) \geq \ell \}$$

and

$$D(r, \ell) = \{ \forall \theta \in]r, r + |\Lambda|], \exists C_\theta \in \mathcal{C}_\theta, d(e, C_\theta) \geq \ell \}.$$

We have to estimate the probability

$$P_\mu \left(e \in \mathcal{I}_t \mid \overline{D}(t, \kappa' c^2 \ln^2 |\Lambda|) \right), \quad (3.7.1)$$

where κ' is a constant which will be determined later. For the moment, we can simply consider a large κ' . We notice first that, on the event $\overline{D}(t, \kappa' c^2 \ln^2 |\Lambda|)$, there is a cut which is disjoint from e , so the edge e cannot be pivotal, thus

$$P_\mu \left(e \in \mathcal{I}_t \mid \overline{D}(t, \kappa' c^2 \ln^2 |\Lambda|) \right) = P_\mu \left(e \in \mathcal{I}_t \setminus \mathcal{P}_t \mid \overline{D}(t, \kappa' c^2 \ln^2 |\Lambda|) \right).$$

We consider the last time when e is pivotal, i.e., the time $t - s$ defined by

$$s = \inf \{ r \geq 0 : e \in \mathcal{P}_{t-r} \}.$$

On the interval $]t - s, t]$, the edge e is not pivotal and it remains in the interface. Therefore, this edge is not modified during this interval, so we have

$$P_\mu \left(e \in \mathcal{I}_t \mid \overline{D}(t, \kappa' c^2 \ln^2 |\Lambda|) \right) \leq P_\mu \left(\begin{array}{l} \exists s \geq 0, e \in \mathcal{P}_{t-s} \\ \forall r \in]t - s, t] \\ e \notin \mathcal{P}_r, E_r \neq e \end{array} \mid \overline{D}(t, \kappa' c^2 \ln^2 |\Lambda|) \right).$$

The events appearing in this probability concern only the process $(E_t)_{t \in \mathbb{N}}$ and the process $(Y_t)_{t \in \mathbb{N}}$. These processes are both reversible. By reversing the time, we obtain that

$$P_\mu \left(\begin{array}{l} \exists s \geq 0, e \in \mathcal{P}_{t-s} \\ \forall r \in]t-s, t] \\ e \notin \mathcal{P}_r, E_r \neq e \end{array} \middle| \overline{D}(t, \kappa' c^2 \ln^2 |\Lambda|) \right) = P_\mu \left(\begin{array}{l} \exists s \geq 0, e \in \mathcal{P}_{t+s} \\ \forall r \in]t, t+s] \\ e \notin \mathcal{P}_r, E_r \neq e \end{array} \middle| \overline{D}(t, \kappa' c^2 \ln^2 |\Lambda|) \right).$$

Notice that the sequence $(E_r)_{t < r \leq t+s}$ is independent of the configuration Y_t . We estimate first the probability that the interval $]t, t+s]$ is too large. More precisely, we will show that s is at most of order $|\Lambda| \ln |\Lambda|$. Let $c \geq 1$ be a constant. We have

$$P_\mu \left(\begin{array}{l} \exists s \geq c|\Lambda| \ln |\Lambda|, e \in \mathcal{P}_{t+s} \\ \forall r \in]t, t+s] \\ e \notin \mathcal{P}_r, E_r \neq e \end{array} \middle| \overline{D}(t, \kappa' c^2 \ln^2 |\Lambda|) \right) \leq P_\mu \left(\begin{array}{l} \forall r \in]t, t+c|\Lambda| \ln |\Lambda|] \\ E_r \neq e \end{array} \right) \leq \left(1 - \frac{1}{|\Lambda|} \right)^{c|\Lambda| \ln |\Lambda|} \leq \frac{1}{|\Lambda|^c}. \quad (3.7.2)$$

We now consider the case where $s < c|\Lambda| \ln |\Lambda|$. We split the interval $]t, t+s]$ into subintervals of length $|\Lambda|$. We set, for $0 \leq i < s/|\Lambda|$,

$$t_i = t + i|\Lambda|.$$

Let us distinguish two cases according to the positions of the cuts during the time interval $]t, t_1]$. We consider a constant $\kappa > 0$ which will be chosen later. If the event $D(t, \kappa' c^2 \ln^2 |\Lambda| - \kappa c \ln |\Lambda|)$ doesn't occur, then there exists a time $\tau \in]t, t_1]$ and a cut of \mathcal{C}_τ which visits at least an edge f at distance less than $\kappa' c^2 \ln^2 |\Lambda| - \kappa c \ln |\Lambda|$ from e . Therefore, for a $s < c|\Lambda| \ln |\Lambda|$ fixed, we have

$$P_\mu \left(e \in \mathcal{P}_{t+s} \mid \overline{D}(t, \kappa' c^2 \ln^2 |\Lambda|) \right) \leq P_\mu \left(\begin{array}{l} \exists \tau \in]t, t_1], \exists C_\tau \in \mathcal{C}_\tau, \exists f \in C_\tau \\ d(e, f) \leq \kappa' c^2 \ln^2 |\Lambda| - \kappa c \ln |\Lambda| \end{array} \middle| \overline{D}(t, \kappa' c^2 \ln^2 |\Lambda|) \right) + P_\mu \left(\begin{array}{l} e \in \mathcal{P}_{t+s} \\ D(t, \kappa' c^2 \ln^2 |\Lambda| - \kappa c \ln |\Lambda|) \end{array} \middle| \overline{D}(t, \kappa' c^2 \ln^2 |\Lambda|) \right).$$

We estimate the first probability with the help of corollary 3.6.1. This case is illustrated in figure 3.7 but this time with the radius of the circles taken to be $\kappa' c^2 \ln^2 |\Lambda|$ and $\kappa' c^2 \ln^2 |\Lambda| - \kappa c \ln |\Lambda|$. There is a $\tilde{p} < 1$, such that, for

$p \geq \tilde{p}$ and $\kappa > 0$, for any $c \geq 2$, $0 < \tau \leq |\Lambda|$ and an edge f at distance less than $\kappa' c^2 \ln^2 |\Lambda| - \kappa c \ln |\Lambda|$ from e , we have

$$P_\mu (\exists C_\tau \in \mathcal{C}_\tau, f \in C_\tau \mid \overline{D}(t, \kappa' c^2 \ln^2 |\Lambda|)) \leq \frac{1}{|\Lambda|^{c+2}}.$$

Therefore, the following inequality holds:

$$\begin{aligned} & P_\mu \left(\begin{array}{c} \exists \tau \in]t, t_1], \exists C_\tau \in \mathcal{C}_\tau, \exists f \in C_\tau \\ d(e, f) \leq \kappa' c^2 \ln^2 |\Lambda| - \kappa c \ln |\Lambda| \end{array} \mid \overline{D}(t, \kappa' c^2 \ln^2 |\Lambda|) \right) \leq \\ & \sum_{\tau \in]t, t_1]} P_\mu (\exists C_\tau \in \mathcal{C}_\tau, d(e, C_\tau) \leq \kappa' c^2 \ln^2 |\Lambda| - \kappa c \ln |\Lambda| \mid \overline{D}(t, \kappa' c^2 \ln^2 |\Lambda|)) \leq \frac{2d}{|\Lambda|^c}. \end{aligned}$$

We then obtain

$$\begin{aligned} P_\mu (e \in \mathcal{P}_{t+s} \mid \overline{D}(t, \kappa' \ln^2 |\Lambda|)) \leq \\ \frac{2d}{|\Lambda|^c} + P_\mu \left(e \in \mathcal{P}_{t+s} \mid \begin{array}{c} D(t, \kappa' \ln^2 |\Lambda| - \kappa \ln |\Lambda|) \\ \overline{D}(t, \kappa' \ln^2 |\Lambda|) \end{array} \right). \end{aligned}$$

Starting from this inequality, we apply proposition 3.6.3 and repeat the previous argument at the times t_i , $0 \leq i < s/|\Lambda|$. By iteration, we obtain that, for any $n < s/|\Lambda|$ and $|\Lambda| \geq 12(2\kappa d)^d$,

$$\begin{aligned} P_\mu (e \in \mathcal{P}_{t+s} \mid \overline{D}(t, \kappa' c^2 \ln^2 |\Lambda|)) \leq \\ \frac{2dn}{|\Lambda|^c} + P_\mu \left(e \in \mathcal{P}_{t+s} \mid \begin{array}{c} \bigcap_{1 \leq i \leq n} D(t_i, \kappa' c^2 \ln^2 |\Lambda| - i\kappa c \ln |\Lambda|) \\ \overline{D}(t, \kappa' c^2 \ln^2 |\Lambda|) \end{array} \right). \quad (3.7.3) \end{aligned}$$

We consider this inequality with $n = \lfloor s/|\Lambda| \rfloor < c \ln |\Lambda|$:

$$\begin{aligned} P_\mu (e \in \mathcal{P}_{t+s} \mid \overline{D}(t, \kappa' c^2 \ln^2 |\Lambda|)) \leq \\ \frac{2dc \ln |\Lambda|}{|\Lambda|^c} + P_\mu \left(e \in \mathcal{P}_{t+s} \mid \begin{array}{c} \bigcap_{1 \leq i < s/|\Lambda|} D(t_i, \kappa' c^2 \ln^2 |\Lambda| - i\kappa c \ln |\Lambda|) \\ \overline{D}(t, \kappa' c^2 \ln^2 |\Lambda|) \end{array} \right). \end{aligned}$$

We notice that $s - |\Lambda| \lfloor s/|\Lambda| \rfloor < |\Lambda|$ and there exists a $\kappa' > 0$ such that

$$\kappa' c^2 \ln^2 |\Lambda| - \kappa c \ln |\Lambda| \lfloor s/|\Lambda| \rfloor \geq \kappa c \ln |\Lambda|.$$

We can apply again proposition 3.6.3 at time t_n and we get

$$P_\mu \left(e \in \mathcal{P}_{t+s} \mid \begin{array}{c} \bigcap_{1 \leq i < c \ln |\Lambda|} D(t_i, \kappa' c^2 \ln^2 |\Lambda| - i\kappa c \ln |\Lambda|) \\ \overline{D}(t, \kappa' c^2 \ln^2 |\Lambda|) \end{array} \right) \leq \frac{1}{|\Lambda|^c}.$$

Finally, we obtain the following upper bound for (3.7.3):

$$P_\mu \left(e \in \mathcal{P}_{t+s} \mid \overline{D}(t, \kappa' c^2 \ln^2 |\Lambda|) \right) \leq \frac{2dc \ln |\Lambda| + 1}{|\Lambda|^c}.$$

We sum over the choices of $s < c|\Lambda| \ln |\Lambda|$ and we combine with (3.7.2). We obtain

$$P_\mu \left(\begin{array}{l} \exists s \geq 0, e \in \mathcal{P}_{t+s} \\ \forall r \in]t, t+s] \\ e \notin \mathcal{P}_r, E_r \neq e \end{array} \mid \overline{D}(t, \kappa' c^2 \ln^2 |\Lambda|) \right) \leq \frac{1 + c|\Lambda| \ln |\Lambda| + 2d|\Lambda|c^2 \ln^2 |\Lambda|}{|\Lambda|^c}.$$

For $|\Lambda| > 4 + c + 2dc^2 + 12(2\kappa d)^d$, we have $\ln |\Lambda| \leq |\Lambda|$ and thus

$$\frac{1 + c|\Lambda| \ln |\Lambda| + 2|\Lambda|dc^2 \ln^2 |\Lambda|}{|\Lambda|^c} \leq \frac{1 + c + 2dc^2}{|\Lambda|^{c-3}} \leq \frac{1}{|\Lambda|^{c-4}}.$$

Therefore, there exists a $\tilde{p} < 1$ and a $\kappa' > 0$ such that for $p \geq \tilde{p}$, for any $c \geq 2$, we have

$$P_\mu \left(e \in \mathcal{I}_t \setminus \mathcal{P}_t \mid \overline{D}(t, \kappa' c^2 \ln^2 |\Lambda|) \right) \leq \frac{1}{|\Lambda|^{c-4}}.$$

Since $(c+4)^2/c^2 \leq 25$ for $c \geq 1$, by replacing κ' by $25\kappa'$ in the probability, we can replace $1/|\Lambda|^{c-4}$ by $1/|\Lambda|^c$. Hence the desired result. \square

Chapitre 4

Une tentative d'améliorer le contrôle de la vitesse

Ce chapitre est adapté de l'article [CZ18b], réalisé en collaboration avec Raphaël Cerf. La version préprint est disponible sur: <https://arxiv.org/abs/1811.12368>

We consider the Bernoulli bond percolation model in a box Λ (not necessarily parallel to the directions of the lattice) in the regime where the percolation parameter is close to 1. We condition the configuration on the event that two opposite faces of the box are disconnected. We couple this configuration with an unconstrained percolation configuration. The interface edges are the edges which differ in the two configurations. We prove that the union of the pivotal edges during a time interval of length $|\Lambda| \ln |\Lambda|$ moves at a speed of order $\ln |\Lambda|$.

4.1 Introduction

We pursue here the study of the structure of large interfaces in the percolation model. We consider the Bernoulli bond percolation model in a box Λ (not necessarily parallel to the directions of the lattice) in the regime where the percolation parameter is close to 1. We condition the configuration on the event $\{T \leftrightarrow B\}$ that two opposite faces T and B of the box are disconnected and we wish to gain some insight into the resulting configuration. Since p is close to 1, there will be a lot of pivotal edges, that is closed edges whose opening would create a connection between the faces T and B . However, the effect of the conditioning is complex and is not limited to the presence of the pivotal edges. In [CZ18a], we constructed a coupling between two percolation configurations which allows to keep track of the effect of the conditioning. Let us sum up briefly the strategy of this

construction. We consider the classical dynamical percolation process in the box Λ . We start with an initial configuration X_0 . At each step, we choose one edge uniformly at random, and we update its state with a coin of parameter p . This process is denoted by $(X_t)_{t \in \mathbb{N}}$. Of course all the random choices are independent. Next, we duplicate the initial configuration X_0 , thereby getting a second configuration Y_0 . We use the same random variables as before to update this second configuration, with one essential difference. In the second configuration, we prohibit the opening of an edge if this opening creates a connection between the top T and the bottom B . We denote by μ_p the invariant probability of the process $(X_t, Y_t)_{t \in \mathbb{N}}$. Let now (X, Y) be a pair of percolation configurations distributed according to μ_p . We define the set \mathcal{P} of the pivotal edges

$$\mathcal{P} = \{ e \subset \Lambda : e \text{ is pivotal in } Y \text{ for } T \leftrightarrow B \}$$

and the set \mathcal{I} of the interface edges

$$\mathcal{I} = \{ e \subset \Lambda : X(e) \neq Y(e) \}.$$

The effect of the conditioning is precisely encoded in the set of the interface edges \mathcal{I} . A standard Peierls estimate and the BK inequality yield that, typically, each pivotal edge is within distance of order $\ln |\Lambda|$ of another pivotal edge (see Proposition 1.4 of [CZ18a]). Our main result gives us a control of the displacement of the pivotal edges during a time interval of order $|\Lambda| \ln |\Lambda|$. The box Λ in the next theorem is centred at the origin but its sides are not necessarily parallel to the axis of \mathbb{Z}^d . We introduce a semi-distance d_H^ℓ derived from the Hausdorff distance. For $\ell \geq 0$ and two subsets A, B of Λ , we define

$$d_H^\ell(A, B) = \inf \left\{ r \geq 0 : \begin{array}{l} A \setminus \mathcal{V}(\Lambda^c, \ell) \subset \mathcal{V}(B, r) \\ B \setminus \mathcal{V}(\Lambda^c, \ell) \subset \mathcal{V}(A, r) \end{array} \right\}.$$

Here is our main result.

Theorem 4.1.1. *There exists $\tilde{p} < 1$ such that for $p \geq \tilde{p}$, $c \geq 1$ and any box Λ satisfying $|\Lambda| \geq e^{2d^2c}$, we have*

$$P_\mu \left(d_H^{2dc \ln |\Lambda|} \left(\bigcup_{r \in [t-c|\Lambda| \ln \Lambda, t]} \mathcal{P}_r, \bigcup_{s \in [t, t+c|\Lambda| \ln \Lambda]} \mathcal{P}_s \right) \geq 2dc \ln |\Lambda| \right) \leq \frac{1}{|\Lambda|^{c-3}}.$$

At first sight, theorem 4.1.1 looks like a minor improvement of proposition 1.4 of [CZ18a]. However, this result is a lot different. To control the distance between pivotal edges at two different times, we need to control the speed of the pivotal edges. In [CZ18a], we control the speed of these edges during a time interval of order $|\Lambda|$. When we study the pivotal edges on a

time interval of length $2dc|\Lambda| \ln |\Lambda|$, the previous results give us a control of the distance of order $\ln^2 |\Lambda|$ instead of $\ln |\Lambda|$. This new result requires a new ingredient compared to the previous argument. Here we obtain a speed estimate on a time interval of order $|\Lambda| \ln |\Lambda|$ by studying a new type of space-time path which connects a pivotal edge at time t to an edge of $\mathcal{P}_s \cup \mathcal{I}_s$ at a time $s < t$. The length of this new type of space-time paths has an exponential decay property during a time interval of order $|\Lambda| \ln |\Lambda|$. As a drawback, we have to replace \mathcal{P} by $\mathcal{P} \cup \mathcal{I}$ due to the construction of this new space-time path. As a consequence, we can only study the distance between a pivotal edge and the union of the pivotal edges on a time interval in the past.

Apart from this crucial ingredient just described, the strategy of our proof follows the general ideas in [CZ18a]. We construct a space-time path which represents the movement of the set $\mathcal{P} \cup \mathcal{I}$. The study of the space-time path gives us a control on the speed of the movement. Combined with the fact that an edge of the interface has a limited lifetime, we obtain a control on the distance between an edge of the interface and the set $\mathcal{P} \cup \mathcal{I}$.

Ideally, we would like to prove that the distance between an edge of the interface and the pivotal edges is of order $\ln |\Lambda|$. But due to the lack of reversibility of the process $(X_t, Y_t)_{t \in \mathbb{N}}$, we would need a further argument to conclude.

The paper is organised as follows. In section 2, we define the model and the notations which will be used in the rest of our study. In section 3, we construct the new space-time path which will be used in the proof. In section 4, we control the distance between $\mathcal{P}_t \cup \mathcal{I}_t$ and \mathcal{P}_{t+s} for $s \leq |\Lambda| \ln |\Lambda|$ with the help of this space-time path. Finally, the proof of theorem 4.1.1 is presented in the section 5.

4.2 The model and notations

We will reuse most of the notations in [CZ18a], which we recall briefly.

4.2.1 Geometric definitions

We give some standard geometric definitions.

The edges \mathbb{E}^d . The set \mathbb{E}^d is the set of pairs $\{x, y\}$ of points in \mathbb{Z}^d which are at Euclidean distance 1.

The usual paths. We say that two edges e and f are neighbours if they have one endpoint in common. A usual path is a sequence of edges $(e_i)_{1 \leq i \leq n}$ such that for $1 \leq i < n$, e_i and e_{i+1} are neighbours.

The box Λ . We will mostly work in a closed box Λ centred at the origin. The top side of the box is denoted by T and the bottom side is denoted by B . The box might be tilted, i.e., its sides are not necessarily parallel to the axis of \mathbb{Z}^d .

The cuts. We say that S is a cut if there is no usual path included in $\Lambda \cap \mathbb{E}^d \setminus S$ which connects T and B . Notice that the box might be tilted with respect to the lattice \mathbb{Z}^d .

The *-paths. In order to study the cuts in any dimension $d \geq 2$, we use *-connectedness on the edges as in [DP96]. We consider the supremum norm on \mathbb{R}^d :

$$\forall x = (x_1, \dots, x_d) \in \mathbb{R}^d \quad \|x\|_\infty = \max_{i=1, \dots, d} |x_i|.$$

For e an edge in \mathbb{E}^d , we denote by m_e the center of the unit segment associated to e . We say that two edges e and f of \mathbb{E}^d are *-neighbours if $\|m_e - m_f\|_\infty \leq 1$. A *-path is a sequence of edges (e_1, \dots, e_n) such that, for $1 \leq i < n$, the edge e_i and e_{i+1} are *-neighbours.

4.2.2 The dynamical percolation

We define the dynamical percolation and the space-time paths.

Percolation configurations. A configuration in Λ is a map from the set of edges in Λ to $\{0, 1\}$. In a configuration ω , an edge e is said to be open if $\omega(e) = 1$ and closed if $\omega(e) = 0$.

Probability measures. We denote by P_p the law of the Bernoulli bond percolation in Λ with parameter p . We also define $P_{\mathcal{D}}$ as the probability measure P_p conditioned by the event $\{T \leftrightarrow B\}$, i.e.,

$$P_{\mathcal{D}}(\cdot) = P_p(\cdot | T \leftrightarrow B).$$

Probability space. Throughout the paper, we assume that all the random variables used in the proofs are defined on the same probability space Ω . For instance, this space contains the random variables used in the graphical construction presented below, as well as the random variables generating the initial configurations of the Markov chains. We denote simply by P the probability measure on Ω .

Graphical construction. We construct the dynamical percolation in Λ as a discrete time Markov chain $(X_t)_{t \in \mathbb{N}}$. We will need an i.i.d. sequence of random edges in Λ , denoted by $(E_t)_{t \in \mathbb{N}}$, with uniform distribution over the edges of Λ . We also need an i.i.d. sequence of uniform variables in

the interval $[0, 1]$, denoted by $(U_t)_{t \in \mathbb{N}}$. The sequences $(E)_{t \in \mathbb{N}}$, $(U_t)_{t \in \mathbb{N}}$ are independent. We build the process $(X_t)_{t \in \mathbb{N}}$ iteratively. At time 0, we start from a configuration X_0 and at time t , we set

$$\forall t \geq 1 \quad X_t(e) = \begin{cases} X_{t-1}(e) & \text{if } E_t \neq e \\ \mathbb{1}_{\{U_t \leq p\}} & \text{if } E_t = e \end{cases} .$$

The space-time paths. We introduce the space-time paths which generalise both the usual paths and the $*$ -paths to the dynamical percolation. A space-time path is a sequence of pairs, called time-edges, $(e_i, t_i)_{1 \leq i \leq n}$, such that, for $1 \leq i \leq n-1$, we have either $e_i = e_{i+1}$, or (e_i, e_{i+1}) are neighbours and $t_i = t_{i+1}$. We define also space-time $*$ -paths, by using edges which are $*$ -neighbours in the above definition. For s, t two integers, we define

$$s \wedge t = \min(s, t), \quad s \vee t = \max(s, t).$$

A space-time path $(e_i, t_i)_{1 \leq i \leq n}$ is open in the dynamical percolation process $(X_t)_{t \in \mathbb{N}}$ if

$$\forall i \in \{1, \dots, n\} \quad X_{t_i}(e_i) = 1$$

and

$$\forall i \in \{1, \dots, n-1\} \quad e_i = e_{i+1} \implies \forall t \in [t_i \wedge t_{i+1}, t_i \vee t_{i+1}] \quad X_t(e_i) = 1.$$

In the same way, we can define a closed space-time path by changing 1 to 0 in the previous definition. In the remaining of the article, we use the abbreviation STP to design a space-time path. Moreover, unless otherwise specified, the closed paths (and the closed STPs) are defined with the relation $*$ and the open paths (and the open STPs) are defined with the usual relation. This is because the closed paths come from the cuts, while the open paths come from existing connexions.

We shall define the space projection of a STP. Given $k \in \mathbb{N}^*$ and a sequence $\Gamma = (e_i)_{1 \leq i \leq k}$ of edges, we say that it has length k , which we denote by $\text{length}(\Gamma) = k$, and we define its support

$$\text{support}(\Gamma) = \{ e \subset \Lambda : \exists i \in \{1, \dots, k\} \quad e_i = e \}.$$

Let $\gamma = (e_i, t_i)_{1 \leq i \leq n}$ be a STP. The space projection of γ is obtained by removing one edge in every time change in the sequence $(e_i)_{1 \leq i \leq n}$. More precisely, let m be the number of time changes in γ . We define the function $\phi : \{1, \dots, n-m\} \rightarrow \mathbb{N}$ by setting $\phi(1) = 1$ and

$$\forall i \in \{1, \dots, n-m\} \quad \phi(i+1) = \begin{cases} \phi(i) + 1 & \text{if } e_{\phi(i)} \neq e_{\phi(i)+1} \\ \phi(i) + 2 & \text{if } e_{\phi(i)} = e_{\phi(i)+1} \end{cases} .$$

The sequence $(e_{\phi(i)})_{1 \leq i \leq n-m}$ is called the space projection of γ , denoted by $\text{Space}(\gamma)$. We say that $\text{length}(\text{Space}(\gamma))$ is the length of the STP γ , denoted

also by $\text{length}(\gamma)$. We shall distinguish $\text{Space}(\gamma)$ from the support of γ , denoted by $\text{support}(\gamma)$, which we define as:

$$\text{support}(\gamma) = \text{support}(\text{Space}(\gamma)).$$

4.2.3 The interfaces by coupling.

As in [CZ18a], we define the interface with the help of a coupling between two processes of dynamical percolation. We start with the graphical construction $(X_t, E_t, U_t)_{t \in \mathbb{N}}$ of the dynamical percolation. We define a further process $(Y_t)_{t \in \mathbb{N}}$ as follows: at time 0, we start from an initial condition (X_0, Y_0) belonging to the set

$$\mathcal{E}_0 = \{ (\omega_1, \omega_2) \in \{0, 1\}^{\mathbb{E}^d \cap \Lambda} \times \{T \leftrightarrow B\} : \forall e \subset \Lambda \quad \omega_1(e) \geq \omega_2(e) \}$$

and for all $t \geq 1$, we set

$$\forall e \subset \Lambda \quad Y_t(e) = \begin{cases} Y_{t-1}(e) & \text{if } e \neq E_t \\ 0 & \text{if } e = E_t \text{ and } U_t > p \\ 1 & \text{if } e = E_t, U_t \leq p \text{ and } T \xleftrightarrow{Y_{t-1}^{E_t}} B \\ 0 & \text{if } e = E_t, U_t \leq p \text{ and } T \xleftrightarrow{Y_{t-1}^{E_t}} B \end{cases},$$

where, for a configuration ω and an edge e , the notation ω^e means the configuration obtained by opening e in ω . The set of the configurations satisfying $\{T \leftrightarrow B\}$ is irreducible and the process $(X_t)_{t \in \mathbb{N}}$ is reversible. Therefore, the process $(Y_t)_{t \in \mathbb{N}}$ is the dynamical percolation conditioned to satisfy the event $\{T \leftrightarrow B\}$. According to corollary 1.10 of [Kel11], the invariant probability measure of $(Y_t)_{t \in \mathbb{N}}$ is P_D , the probability P_p conditioned by the event $\{T \leftrightarrow B\}$. The set \mathcal{E}_0 is irreducible and aperiodic. In fact, each configuration of \mathcal{E}_0 communicates with the configuration where all edges are closed. The state space \mathcal{E}_0 is finite, therefore the Markov chain $(X_t, Y_t)_{t \in \mathbb{N}}$ admits a unique equilibrium distribution μ_p . We now present a definition of the interface between T and B for a coupled process $(X_t, Y_t)_{t \in \mathbb{N}}$.

Definition 4.2.1. *The interface at time t between T and B , denoted by \mathcal{I}_t , is the set of the edges in Λ that differ in the configurations X_t and Y_t , i.e.,*

$$\mathcal{I}_t = \{ e \subset \Lambda : X_t(e) \neq Y_t(e) \}.$$

We define next the set \mathcal{P}_t of the pivotal edges for the event $\{T \leftrightarrow B\}$ in the configuration Y_t .

Definition 4.2.2. *The set \mathcal{P}_t of the pivotal edges in Y_t is the collection of the edges in Λ whose opening would create a connection between T and B , i.e.,*

$$\mathcal{P}_t = \{ e \subset \Lambda : T \xleftrightarrow{Y_t^e} B \}.$$

4.3 The construction of the STP

We will construct a STP which connects an edge $e \in \mathcal{P}_t$ at time t and the set $\mathcal{P}_s \cup \mathcal{I}_s$ at time $s < t$. Before starting the construction, we define first some relevant properties of a STP, which will be enjoyed by our construction.

Definition 4.3.1. A STP $(e_1, t_1), \dots, (e_n, t_n)$ is increasing (respectively decreasing) if

$$t_1 \leq \dots \leq t_n \text{ (resp. } t_1 \geq \dots \geq t_n \text{)}.$$

If a STP is increasing or decreasing, we say that it is monotone.

Definition 4.3.2. A STP $(e_1, t_1), \dots, (e_n, t_n)$ in X (respectively Y) is called simple if each edge is visited only once or its status changes at least once between any two consecutive visits, i.e., for any i, j in $\{1, \dots, n\}$ such that $|i - j| \neq 1$,

$$(e_i = e_j \quad t_i < t_j) \implies \exists s \in]t_i, t_j[\quad X_s(e_i) \neq X_{t_i}(e_i) \text{ (resp. } Y_s(e_i) \neq Y_{t_i}(e_i)).$$

Definition 4.3.3. In a STP $(e_1, t_1), \dots, (e_n, t_n)$, for $1 \leq i < n$, we say that the edge e_i is a time-change edge if $e_i = e_{i+1}$.

We define next two properties of a STP related to the time-change edges.

Definition 4.3.4. A STP $(e_1, t_1), \dots, (e_n, t_n)$ is impatient if every time-change is ended by an edge which is updated, i.e.,

$$\forall i \in \{1, \dots, n-2\} \quad e_i = e_{i+1} \implies E_{t_{i+1}+1} = e_{i+1}.$$

Definition 4.3.5. A STP $(e_1, t_1), \dots, (e_n, t_n)$ is called X -closed-moving (resp. Y -closed-moving) if all the edges which are not time-change edges are closed in X (resp. in Y), i.e.,

$$\forall i \in \{1, \dots, n-1\} \quad e_i \neq e_{i+1} \implies X_{t_i}(e_i) = 0 \quad \text{(resp. } Y_{t_i}(e_i) = 0 \text{)}.$$

We now construct a specific STP satisfying some of these properties.

Proposition 4.3.6. Let $s < t$ be two times and $e \in \mathcal{P}_t$. There exists a decreasing simple impatient STP which connects the time-edge (e, t) to an edge of the set $\mathcal{P}_s \cup \mathcal{I}_s \setminus \{e\}$ at time s or an edge f intersecting the boundary of Λ after time s . Moreover this STP is X -closed-moving except on the edge e .

Proof. The proof of this proposition is done in two steps. The first step is to construct a STP which connects certain edges. In the second step, we modify the STP obtained in the first step to get a simple and impatient STP.

Step 1. At time t , the edge e belongs to a cut. Therefore, there exists a path γ_1 which connects e to the boundary of Λ . We start at the edge e and we follow the path γ_1 . If the path γ_1 does not encounter an edge $f \in \mathcal{I}_t \cup \mathcal{P}_{t-1} \setminus \{e\}$ then the STP

$$(e, t), (\gamma_1, t)$$

connects e to the boundary of Λ , where the notation (ρ, t) , for a path $\rho = (e_i)_{1 \leq i \leq n}$ and a time t , means the sequence of time-edges $(e_i, t)_{1 \leq i \leq n}$. Suppose next that there exists an edge of $\mathcal{I}_t \cup \mathcal{P}_{t-1} \setminus \{e\}$ in γ_1 . We enumerate the edges of γ_1 in the order they are visited when starting from e and we consider the first edge e_1 in γ_1 which belongs to the set $\mathcal{I}_t \cup \mathcal{P}_{t-1}$. We denote by ρ_1 the sub-path of γ_1 visited between e and e_1 . We then consider the time $\eta(t)$ defined as follows:

$$\eta(t) = \max \{ r < t : X_r(e_1) = Y_r(e_1) \}.$$

Since $e_1 \in \mathcal{I}_t$, the time $\eta(t)$ when it becomes an edge of the interface is strictly less than t and if $e_1 \in \mathcal{P}_{t-1} \setminus \mathcal{I}_t$, we have $\eta(t) \leq t - 1$. In both cases, we have $\eta(t) < t$ and the edge e_1 is closed in $X_{\eta(t)}$. If the time $\eta(t)$ is before the time s then, at time s , the edge e_1 belongs to the set $\mathcal{I}_s \cup \mathcal{P}_s \setminus \{e\}$. Therefore the STP

$$(e, t), (\rho, t), (e_1, t), (e_1, s)$$

satisfies the conditions in the proposition. If we have $\eta(t) > s$, then we repeat the above argument starting from the edge e_1 at time $\eta(t)$. We obtain either a path γ_2 which connects e_1 to the boundary of Λ at time $\eta(t)$ or a path ρ_2 which connects e_1 to an edge $e_2 \in \mathcal{I}_{\eta(t)} \cup \mathcal{P}_{\eta(t)-1} \setminus \{e\}$ and a time $\eta^2(t) < \eta(t)$. We proceed in this way until we reach a time edge $(e_k, \eta^k(t))$ with $\eta^k(t) \leq s$. Since $\eta(t) < t$, the sequence of times

$$\eta(t), \eta^2(t), \dots, \eta^k(t)$$

decreases strictly through this procedure and this procedure terminates after a finite number of iterations. The concatenation of the paths obtained at the end of the procedure,

$$(e, t), (\rho_1, t), (e_1, \eta(t)), \dots, (\rho^k, \eta^{k-1}(t)), (f_k, s),$$

connects e to an edge of $\mathcal{P}_s \cup \mathcal{I}_s \setminus \{e\}$. Since the sequence $(\eta^i(t))_{1 \leq i \leq k}$ is decreasing, this is a decreasing STP. Each time when the STP meets an edge of \mathcal{I} which is different from e , there is a time change to the time before it opened in X , therefore each movement in space except on the edge e is done through a closed edge in X and the STP is X -closed-moving.

Step 2. We use two iterative procedures to transform the STP in the step 1 into a simple and impatient STP. To get a simple STP, we use the same procedure as in the proof of proposition 4.4 in [CZ18a]. Let us denote by $(e_i, t_i)_{0 \leq i \leq N}$ the STP obtained previously. Starting with the edge e_0 , we examine the rest of the edges one by one. Let $i \in \{0, \dots, N\}$. Suppose that the edges e_0, \dots, e_{i-1} have been examined and let us focus on e_i . We encounter three cases:

- For every index $j \in \{i+1, \dots, N\}$, we have $e_j \neq e_i$. Then, we don't modify anything and we start examining the edge e_{i+1} .
- There is an index $j \in \{i+1, \dots, N\}$ such that $e_i = e_j$, but for the first index $k > i+1$ such that $e_i = e_k$, there is a time $\alpha \in]t_k, t_i[$ when $X_\alpha(e_i) = 1$. Then we don't modify anything and we start examining the next edge e_{i+1} .
- There is an index $j \in \{i+1, \dots, N\}$ such that $e_i = e_j$ and for the first index $k > i+1$ such that $e_i = e_k$, we have $X_\alpha(e_i) = 0$ for all $\alpha \in]t_k, t_i[$. In this case, we remove all the time-edges whose indices are strictly between i and k . We then have a simple time change between t_i and t_k on the edge e_i . We continue the procedure from the index k .

The STP becomes strictly shorter after every modification (we remove systematically the consecutive time changes if there is any), and the procedure will end after a finite number of modifications. We obtain in the end a simple path in X . Since the procedure doesn't change the order of the times t_i , we still have a decreasing path. In order to obtain an impatient STP, we modify the simple decreasing STP obtained above and we use another iterative procedure as follows. We denote again by $(e_i, t_i)_{0 \leq i \leq n}$ the simple STP obtained above. We start by examining the time-edge (e_0, t_0) and then the rest of the time edges of the STP one by one as illustrated in the figure 4.1. Suppose that we have examined the indices $i < k$ and that we are checking the index k . If the edge e_{k+1} is different from e_k , we don't modify the STP at this stage and we continue the procedure from (e_{k+1}, t_{k+1}) . If the edge e_{k+1} is equal to e_k , then the time-edge (e_k, t_k) belongs to a time change. Since the STP is X -closed-moving, then the edge e_{k+1} is closed at time t_{k+1} . Let $[\alpha, \beta]$ be the biggest interval containing t_{k+1} during which the edge e_{k+1} is closed in X . If $\beta \geq t_k$ and $e_{k+2} \neq e_{k+3}$, we replace the sub-sequence

$$(e_k, t_k), (e_{k+1}, t_{k+1}), (e_{k+2}, t_{k+2})$$

by

$$(e_k, t_k), (e_{k+2}, t_k), (e_{k+2}, t_{k+2}),$$

and we continue the modification from the time-edge (e_{k+2}, t_k) . If $\beta \geq t_k$ and $e_{k+2} = e_{k+3}$ we replace the sequence

$$(e_k, t_k), (e_{k+1}, t_{k+1}), (e_{k+2}, t_{k+2}), (e_{k+3}, t_{k+3})$$

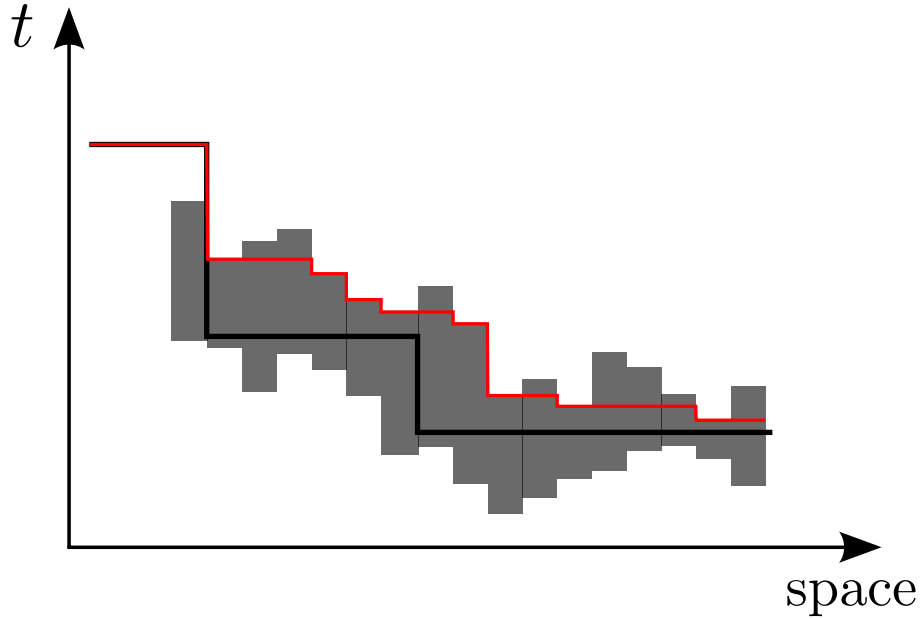


Figure 4.1: An impatient modification (in red) of a STP (in black) according to the intervals when each edge is closed (in gray)

by

$$(e_k, t_k), (e_{k+2}, t_k), (e_{k+2}, t_{k+3}),$$

and we continue the modification from the time-edge (e_{k+2}, t_k) . If $\beta < t_k$ and $e_{k+2} \neq e_{k+3}$, we replace $(e_k, t_k), (e_{k+1}, t_{k+1})$ by

$$(e_k, t_k), (e_k, \beta), (e_{k+2}, \beta).$$

If $\beta < t_k$ and $e_{k+2} = e_{k+3}$, we replace $(e_k, t_k), (e_{k+1}, t_{k+1}), (e_{k+2}, t_{k+2})$ by

$$(e_k, t_k), (e_k, \beta), (e_{k+2}, \beta),$$

and we continue the STP at the time-edge (e_{k+2}, β) . The STP obtained after the modification procedure is decreasing, X -closed-moving and impatient. Moreover, between two consecutive visits of an edge f of the STP, there exists a time when the edge f is open. Therefore, this STP is also simple. \square

4.4 Speed estimates

We show here that the set $\mathcal{P} \cup \mathcal{I}$ cannot move too fast. Typically, during an interval of size $|\Lambda| \ln |\Lambda|$, the set $\mathcal{P} \cup \mathcal{I}$ can at most move a distance of order $\ln |\Lambda|$. This result relies on an estimate for the STP constructed in proposition 4.3.6 which we state in the following proposition.

Lemma 4.4.1. *Let e be an edge in Λ and $\ell \in \mathbb{N}^*$. Let $(\varepsilon_1, \dots, \varepsilon_n)$ be a sequence of edges such that $|\text{support}(\varepsilon_1, \dots, \varepsilon_n)| = \ell$. We have the following inequality:*

$$\begin{aligned} & \exists \tilde{p} < 1 \quad \forall p \geq \tilde{p} \quad \forall s, t \quad 0 < t - s \leq \ell |\Lambda| \\ & P_\mu \left(\begin{array}{l} \exists \gamma \text{ decreasing simple impatient} \\ X\text{-closed-moving STP except on } e, \\ \gamma \text{ starts from } (e, t) \text{ and ends after } s, \\ \text{space}(\gamma) = (\varepsilon_1, \dots, \varepsilon_n) \end{array} \right) \leq \left(1 + \frac{1}{|\Lambda|} \right)^{\ell |\Lambda|} (4 - 4p)^n. \end{aligned}$$

Proof. Let us fix a STP γ satisfying the conditions stated in the probability. We denote by $(e_i, t_i)_{i \in I}$ the sequence of the time-edges of γ . We denote by k the number of the time changes in γ and by T the set of the indices of the time changes, i.e.,

$$T = \{ i \in I : e_i = e_{i+1}, \quad t_i \neq t_{i+1} \}.$$

We shall obtain an upper bound of the probability

$$P \left(\begin{array}{l} (e_i, t_i)_{i \in I} \text{ is a decreasing simple impatient} \\ X\text{-closed-moving STP except on } e \end{array} \right), \quad (4.4.1)$$

which depends only upon the integer n and the number of time changes k . In order to bound the probability appearing in the lemma, we shall sum over the choices of the set of the k times, denoted by K , in the interval $\{s, \dots, t\}$, over the choices of set of the k edges, denoted by A , where the time changes occur and the number k from 1 to n . The probability appearing in the lemma is less or equal than

$$\sum_{1 \leq k \leq n} \sum_{\substack{A \subset \{1, \dots, n\} \\ |A| = k}} \sum_{\substack{K \subset \{s, \dots, t\} \\ |K| = k}} P \left(\begin{array}{l} (e_i, t_i)_{i \in I} \text{ is a decreasing simple impatient} \\ X\text{-closed-moving STP except on } e \end{array} \right). \quad (4.4.2)$$

Let us obtain an upper bound for this probability. The STP is impatient and X -closed-moving, therefore for any $i \in T$, the edge e_{i+1} becomes open at time $t_{i+1} + 1$. Moreover, the STP is simple, thus for any pair of indices $(p, q) \in I \setminus T$, if $e_p = e_q$ and $t_p > t_q$, there exists a time $r \in]t_q, t_p[$, such that the edge e_p is open at time r . We can rewrite the probability inside the sum as

$$P_\mu \left(\begin{array}{l} \forall i \in T \quad E_{t_{i+1}+1} = e_{i+1} \\ \forall i \in I \setminus T \quad X_{t_i}(e_i) = 0 \\ \forall p, q \in I \setminus T \text{ s.t. } e_p = e_q, t_p > t_q \\ \exists r \in]t_q, t_p[\quad X_r(e_p) = 1 \end{array} \right). \quad (4.4.3)$$

Since the times t_i are fixed, this probability can be factorised as a product over the edges. In fact, the event in the probability depends only on the

process $(X_t)_{t \in \mathbb{N}}$. We introduce, for an edge $f \subset \Lambda$, the subset $J(f)$ of I :

$$J(f) = \{i \in I : e_i = f\}.$$

Let us denote by S the set $\text{support}(\gamma)$. The previous probability is less or equal than

$$\prod_{f \in S \setminus \{e\}} P_\mu \left(\begin{array}{l} \forall i \in J(f) \cap (T+1) \quad E_{t_{i+1}} = f \\ \forall i \in J(f) \setminus T \quad X_{t_i}(f) = 0 \\ \forall p, q \in J(f) \setminus T \text{ s.t. } p < q \\ \exists r \in]t_q, t_p[\quad X_r(f) = 1 \end{array} \right). \quad (4.4.4)$$

Let us consider one term of the product. For a fixed edge f , we can order the set $\{t_i : i \in J(f) \setminus T\}$ in an increasing sequence $(\tau_i)_{1 \leq i \leq m_f}$, where $m_f = |J(f) \setminus T|$. Let us denote by $T(f)$ the set of the indices among $\{1, \dots, m_f\}$ which correspond to the end of a time change, i.e., the set corresponding to $J(f) \cap (T+1)$ before the reordering. Since the STP is simple, between two consecutive visits at times τ_i and τ_{i+1} of f , there is a time θ_i when f is open. Moreover the STP is impatient, so for each index $i \in T(f)$, the edge f becomes open at time $\tau_i + 1$. Therefore, each term of the product (4.4.4) is less or equal than

$$P_\mu \left(\begin{array}{l} \forall i \in \{1, \dots, m_f\} \quad X_{\tau_i}(f) = 0 \\ \forall i \in T(f) \quad X_{\tau_i+1}(f) = 1 \\ \forall i \in \{1, \dots, m_f - 1\} \quad \exists \theta_i \in]\tau_i, \tau_{i+1}[\quad X_{\theta_i}(f) = 1 \end{array} \right). \quad (4.4.5)$$

In order to simplify the notations, we define, for a time r , the event

$$\mathcal{E}(r) = \left\{ \begin{array}{l} \forall i \in \{1, \dots, m_f\} \text{ such that } \tau_i \leq r \quad X_{\tau_i}(f) = 0 \\ \forall i \in T(f) \text{ such that } \tau_i + 1 \leq r \quad X_{\tau_i+1}(f) = 1 \\ \forall i \in \{1, \dots, m_f - 1\} \text{ such that } \tau_i \leq r \\ \exists \theta_i \in]\tau_i, \tau_{i+1}[\quad X_{\theta_i}(f) = 1 \end{array} \right\}.$$

The status of the edge f in the process $(X_t)_{t \in \mathbb{N}}$ evolves according to a Markov chain on $\{0, 1\}$. The sequence $(\tau_i)_{1 \leq i \leq m_f}$ being fixed, if $m_f \in T(f)$, we condition 4.4.5 by the events before time τ_{m_f} , we have

$$\begin{aligned} P_\mu \left(\begin{array}{l} \forall i \in \{1, \dots, m_f\} \quad X_{\tau_i}(f) = 0 \\ \forall i \in T(f) \quad X_{\tau_i+1}(f) = 1, \quad m_f \in T(f) \\ \forall i \in \{1, \dots, m_f - 1\} \quad \exists \theta_i \in]\tau_i, \tau_{i+1}[\quad X_{\theta_i}(f) = 1 \end{array} \right) = \\ P_\mu \left(X_{\tau_{m_f}+1}(f) = 1 \mid \mathcal{E}(\tau_{m_f}) \right) P_\mu(\mathcal{E}(\tau_{m_f})) \leq \frac{P_\mu(\mathcal{E}(\tau_{m_f}))}{|\Lambda|}. \end{aligned}$$

If $m_f \notin T(f)$, the probability

$$P_\mu \left(\begin{array}{l} \forall i \in \{1, \dots, m_f\} \quad X_{\tau_i}(f) = 0 \\ \forall i \in T(f) \quad X_{\tau_i+1}(f) = 1, \quad m_f \notin T(f) \\ \forall i \in \{1, \dots, m_f - 1\} \quad \exists \theta_i \in]\tau_i, \tau_{i+1}[\quad X_{\theta_i}(f) = 1 \end{array} \right)$$

is equal to $P_\mu(\mathcal{E}(\tau_{m_f}))$. We then condition $P_\mu(\mathcal{E}(\tau_{m_f}))$ by the events before time τ_{m_f-1} . We shall distinguish two cases according to whether $m_f - 1$ belongs to $T(f)$ or not. If $m_f - 1 \in T(f)$, we have

$$P_\mu(\mathcal{E}(\tau_{m_f})) = P_\mu \left(\begin{array}{c} X_{\tau_{m_f}}(f) = 0 \\ X_{\tau_{m_f-1}+1}(f) = 1 \end{array} \middle| \mathcal{E}(\tau_{m_f-1}) \right) P_\mu(\mathcal{E}(\tau_{m_f-1})),$$

and if $m_f - 1 \notin T(f)$, we have

$$P_\mu(\mathcal{E}(\tau_{m_f})) = P_\mu \left(\begin{array}{c} X_{\tau_{m_f}}(f) = 0 \\ \exists \theta_{m_f} \in]\tau_{m_f-1}, \tau_{m_f}[\\ X_{\theta_{m_f}}(f) = 1 \end{array} \middle| \mathcal{E}(\tau_{m_f-1}) \right) P_\mu(\mathcal{E}(\tau_{m_f-1})).$$

We condition successively the event $P_\mu(\mathcal{E}(\tau_i))$ by $\mathcal{E}(\tau_{i-1})$, we obtain

$$\begin{aligned} P_\mu(\mathcal{E}(\tau_{m_f})) &= P_\mu(\mathcal{E}(\tau_1)) \prod_{1 \leq i < m_f, i \in T(f)} P_\mu \left(\begin{array}{c} X_{\tau_{i+1}}(f) = 0 \\ X_{\tau_{i+1}}(f) = 1 \end{array} \middle| \mathcal{E}(\tau_i) \right) \\ &\times \prod_{1 \leq i < m_f, i \notin T(f)} P_\mu \left(\begin{array}{c} X_{\tau_{i+1}}(f) = 0 \\ \exists \theta_{i+1} \in]\tau_i, \tau_{i+1}[\\ X_{\theta_{i+1}}(f) = 1 \end{array} \middle| \mathcal{E}(\tau_i) \right). \end{aligned} \quad (4.4.6)$$

By the Markov property, each term in the second product is equal to

$$P_\mu \left(\begin{array}{c} X_{\tau_{i+1}}(f) = 0 \\ \exists \theta_i \in]\tau_i, \tau_{i+1}[\\ X_{\theta_{i+1}}(f) = 1 \end{array} \middle| X_{\tau_i}(f) = 0 \right).$$

Since this probability is invariant by translation in time, it is equal to

$$P_0 \left(\begin{array}{c} X_{\tau'}(f) = 0 \\ \exists \theta \in]0, \tau'[\\ X_\theta(f) = 1 \end{array} \right),$$

where we have set $\tau' = \tau_{i+1} - \tau_i$ and P_0 is the law of the Markov chain $(X_t(f))_{t \in \mathbb{N}}$ starting from a closed edge. By considering the stopping time θ' defined as the first time after 0 when f is open, we have by the strong Markov property

$$P_0 \left(\begin{array}{c} X_{\tau'}(f) = 0 \\ \exists \theta \in]0, \tau'[\\ X_\theta(f) = 1 \end{array} \right) \leq P_0(X_{\tau'}(f) = 0 \mid X_{\theta'}(f) = 1) = P_1(X_{\tau'-\theta'}(f) = 0).$$

Notice that for $r \geq 1$, we have

$$P_1(X_r(f) = 0) \leq P_\mu(X_r(f) = 0) = 1 - p.$$

Therefore we have

$$P_0 \left(\begin{array}{l} X_{\tau'}(f) = 0 \\ \exists \theta \in]0, \tau'[\\ X_{\theta}(f) = 1 \end{array} \right) \leq 1 - p.$$

As for the probabilities in the first product of (4.4.6), we can also replace $\mathcal{E}(\tau_i)$ by $\{X_{\tau_i}(f) = 0\}$ in the conditioning. The difference between the previous case is that we have directly $\theta' = 1$, since $X_{\tau_i+1}(f) = 1$. We have

$$P_{\mu} \left(\begin{array}{l} X_{\tau_{i+1}}(f) = 0 \\ X_{\tau_{i+1}}(f) = 1 \end{array} \middle| \mathcal{E}(\tau_i) \right) \leq P_1(X_{\tau'-1}(f) = 0)P_0(X_1(f) = 1) \leq \frac{1-p}{|\Lambda|}.$$

Combining the upper bounds for each term of the product above, we have the following upper bound for $P_{\mu}(\mathcal{E}(\tau_{m_f}))$:

$$P_{\mu}(\mathcal{E}(\tau_{m_f})) \leq \frac{(1-p)^{m_f}}{|\Lambda|^{|T(f) \cap \{1, \dots, m_f-1\}|}},$$

where

$$|T(f) \cap \{1, \dots, m_f - 1\}| = \begin{cases} |J(f) \cap (T+1)| & \text{if } m_f \notin T(f) \\ |J(f) \cap (T+1)| - 1 & \text{if } m_f \in T(f) \end{cases}.$$

In both cases, we have the following upper bound for (4.4.5):

$$P_{\mu} \left(\begin{array}{l} \forall i \in \{1, \dots, m_f\} \quad X_{\tau_i}(f) = 0 \\ \forall i \in T(f) \quad X_{\tau_i+1}(f) = 1 \\ \forall i \in \{1, \dots, m_f - 1\} \quad \exists \theta_i \in]\tau_i, \tau_{i+1}[\quad X_{\theta_i}(f) = 1 \end{array} \right) \leq \frac{2(1-p)^{m_f}}{|\Lambda|^{|J(f) \cap (T+1)|}}.$$

We obtain an upper bound for (4.4.3) by multiplying this inequality over the edges f in $\text{support}(\gamma)$:

$$P_{\mu} \left(\begin{array}{l} \forall i \in T \quad E_{t_{i+1}} = e_{i+1} \\ \forall i \in I \setminus T \quad X_{t_i}(e_i) = 0 \\ \forall p, q \in I \setminus T \text{ s.t. } e_p = e_q, t_p > t_q \\ \exists r \in]t_q, t_p[\quad X_r(e_p) = 1 \end{array} \right) \leq \frac{2^{|S|}(1-p)^{\sum_{f \in S} m_f}}{|\Lambda|^{\sum_f |J(f) \cap (T+1)|}} \leq \frac{2^{|S|}(1-p)^{|I|-k}}{|\Lambda|^k}. \quad (4.4.7)$$

Since $|I| - k \geq n$, and $|S| \leq n$, for k fixed and $(t_i)_{i \in I}$ fixed, we have the following upper bound for (4.4.1),

$$P \left(\begin{array}{l} (e_i, t_i)_{i \in I} \text{ is a decreasing simple impatient} \\ X\text{-closed-moving STP except on } e \end{array} \right) \leq \frac{(2-2p)^n}{|\Lambda|^k}.$$

Finally, we use this upper bound in (4.4.2) and we have

$$\begin{aligned}
P_\mu & \left(\begin{array}{l} \exists \gamma \text{ decreasing simple impatient} \\ X\text{-closed-moving STP except on } e, \\ \gamma \text{ starts from } (e, t) \text{ and ends after } s, \\ \text{space}(\gamma) = (\varepsilon_1, \dots, \varepsilon_n) \end{array} \right) \\
& \leq \sum_{1 \leq k \leq n} \sum_{A \subset \{1, \dots, n\}, |A|=k} \sum_{K \subset \{s, \dots, t\}, |K|=k} \frac{(2-2p)^n}{|\Lambda|^k} \\
& \leq \sum_{1 \leq k \leq n} \binom{n}{k} \binom{\ell|\Lambda|}{k} \frac{(2-2p)^n}{|\Lambda|^k} \leq \sum_{1 \leq k \leq n} \binom{\ell|\Lambda|}{k} \frac{(4-4p)^n}{|\Lambda|^k} \\
& \leq \left(1 + \frac{1}{|\Lambda|}\right)^{\ell|\Lambda|} (4-4p)^n.
\end{aligned}$$

This yields the desired result. \square

We use next proposition 4.3.6 and lemma 4.4.1 to show that the pivotal edges cannot move too fast.

Proposition 4.4.2. *There exists $\tilde{p} < 1$, such that for $p \geq \tilde{p}$, for $\ell \geq 1$, $t \in \mathbb{N}$, $s \in \mathbb{N}$, $s \leq \ell|\Lambda|$ and any edge e at distance at least ℓ from the boundary of Λ ,*

$$P_\mu(e \in \mathcal{P}_{t+s}, d(e, \mathcal{P}_t \cup \mathcal{I}_t \setminus \{e\}) \geq \ell) \leq \exp(-\ell).$$

Proof. By proposition 4.3.6, there exists a STP which is decreasing simple impatient and X -closed-moving except on e which starts from the edge e at time $t+s$ and ends at an edge of $\mathcal{P}_t \cup \mathcal{I}_t \setminus \{e\}$ or an edge intersecting the boundary of Λ after the time t . In both cases, this STP has a length at least ℓ . Therefore, we have the inequality

$$\begin{aligned}
P_\mu & (e \in \mathcal{P}_{t+s}, d(e, \mathcal{P}_t \cup \mathcal{I}_t \setminus \{e\}) \geq \ell) \\
& \leq P_\mu \left(\begin{array}{l} \exists \gamma \text{ decreasing simple impatient} \\ X\text{-closed-moving STP except on } e \\ \gamma \text{ starts from } (e, t+s) \text{ and ends after } t \\ |\text{length}(\gamma)| \geq \ell \end{array} \right).
\end{aligned}$$

Let us fix a path (e_1, \dots, e_n) with $n = \ell$ starting from e . By lemma 4.4.1, for $\ell \geq 1$, we have

$$P_\mu \left(\begin{array}{l} \exists \gamma \text{ decreasing simple impatient} \\ X\text{-closed-moving STP except on } e, \\ \gamma \text{ starts from } (e, t+s) \text{ and ends after } t, \\ \text{space}(\gamma) = (e_1, \dots, e_\ell) \end{array} \right) \leq \left(1 + \frac{1}{|\Lambda|}\right)^{\ell|\Lambda|} (4-4p)^\ell.$$

We sum over the number of the choices for the path (e_1, \dots, e_ℓ) and we obtain

$$P_\mu \left(\begin{array}{l} \exists \gamma \text{ decreasing simple impatient} \\ X\text{-closed-moving STP except on } e \\ \gamma \text{ starts from } (e, t+s) \text{ and ends after } t \\ |\text{length}(\gamma)| \geq \ell \end{array} \right) \leq \left(1 + \frac{1}{|\Lambda|}\right)^{\ell|\Lambda|} \alpha(d)^\ell (4-4p)^\ell,$$

where $\alpha(d)$ is the number of the $*$ -neighbours of an edge in dimension d . There exists a $\tilde{p} < 1$ such that for $p \geq \tilde{p}$, we have

$$\forall \ell \geq 1 \quad \left(1 + \frac{1}{|\Lambda|}\right)^{\ell|\Lambda|} \alpha(d)^\ell (4-4p)^\ell \leq e^{-\ell}.$$

This gives the desired upper bound. \square

4.5 The proof of the main theorem

We now prove theorem 4.1.1 with the help of proposition 4.4.2 and the observation that an edge of the interface cannot survive a time more than $O(|\Lambda| \ln |\Lambda|)$.

Proof of theorem 4.1.1. Let c be a constant bigger than 1. We define two sets \mathfrak{P}_t^- and \mathfrak{P}_t^+ as

$$\begin{aligned} \mathfrak{P}_t^- &= \bigcup_{r \in [t-2dc|\Lambda| \ln |\Lambda|, t]} \mathcal{P}_r \\ \mathfrak{P}_t^+ &= \bigcup_{s \in [t, t+2dc|\Lambda| \ln |\Lambda|]} \mathcal{P}_s. \end{aligned}$$

By the definition of d_H^ℓ , we have

$$\begin{aligned} P_\mu \left(d_H^{2dc \ln |\Lambda|}(\mathfrak{P}_t^+, \mathfrak{P}_t^-) \geq 2dc \ln |\Lambda| \right) \\ = P_\mu \left(\begin{array}{l} \exists s \in [0, 2dc|\Lambda| \ln |\Lambda|] \quad \exists e \in \mathcal{P}_{t+s} \\ d(e, \Lambda^c \cup \mathfrak{P}_t^-) \geq 2dc \ln |\Lambda| \end{array} \right) \\ + P_\mu \left(\begin{array}{l} \exists s \in [0, 2dc|\Lambda| \ln |\Lambda|] \quad \exists e \in \mathcal{P}_{t-s} \\ d(e, \Lambda^c \cup \mathfrak{P}_t^+) \geq 2dc \ln |\Lambda| \end{array} \right). \end{aligned}$$

Since the probability concerns only the process $(Y_t)_{t \in \mathbb{N}}$, which is reversible, we have

$$\begin{aligned} P_\mu \left(\begin{array}{l} \exists s \in [0, 2dc|\Lambda| \ln |\Lambda|] \quad \exists e \in \mathcal{P}_{t+s} \\ d(e, \Lambda^c \cup \mathfrak{P}_t^-) \geq 2dc \ln |\Lambda| \end{array} \right) \\ = P_\mu \left(\begin{array}{l} \exists s \in [0, 2dc|\Lambda| \ln |\Lambda|] \quad \exists e \in \mathcal{P}_{t-s} \\ d(e, \Lambda^c \cup \mathfrak{P}_t^+) \geq 2dc \ln |\Lambda| \end{array} \right). \end{aligned}$$

Therefore, we can concentrate on the following probability

$$P_\mu \left(\begin{array}{l} \exists s \in [0, 2dc|\Lambda| \ln |\Lambda|] \quad \exists e \in \mathcal{P}_s \\ d(e, \Lambda^c \cup \mathfrak{P}_t^-) \geq 2dc \ln |\Lambda| \end{array} \right).$$

Let us fix an edge e in Λ at distance at least $2dc \ln |\Lambda|$ from \mathfrak{P}_t^- and a $s \in [0, 2dc|\Lambda| \ln |\Lambda|]$. We distinguish two cases. If the set $\mathcal{P}_t \cup \mathcal{I}_t$ is at distance more than $2dc \ln |\Lambda|$ from the edge e , then by proposition 4.4.2, there exists a $p_2 < 1$ such that, for $p \geq p_2$ and $c \geq 1$, we have

$$P_\mu(e \in \mathcal{P}_{t+s}, d(e, \mathcal{P}_t \cup \mathcal{I}_t) \geq 2dc \ln |\Lambda|) \leq \exp(-2dc \ln |\Lambda|). \quad (4.5.1)$$

If there exists an edge $f \in \mathcal{P}_t \cup \mathcal{I}_t$, which is at distance less than $2dc \ln |\Lambda|$ from e , we consider the last time when f was pivotal before t and we define the random integer τ such that

$$\tau = \inf \{ r \geq 0 : f \in \mathcal{P}_{t-r} \}.$$

Since $f \notin \mathfrak{P}_t^-$, we must have $\tau \geq 2dc|\Lambda| \ln |\Lambda|$. The edge f is not pivotal during the time interval $[t-\tau+1, t]$ and it belongs to the interface. Moreover, it cannot be chosen to be modified during this interval since it must remain different in the two processes. Therefore, for any $r \in [t-2dc|\Lambda| \ln |\Lambda| + 1, t]$, we have $E_r \neq f$. However, this event is unlikely because the sequence $(E_t)_{t \in \mathbb{N}}$ is a sequence of i.i.d. random edges chosen uniformly in Λ . More precisely, we have the following inequality:

$$\begin{aligned} P_\mu(\tau \geq 2dc|\Lambda| \ln |\Lambda|) &\leq P(\forall r \in [t-2dc|\Lambda| \ln |\Lambda| + 1, t], E_r \neq f) \\ &\leq \left(1 - \frac{1}{2d|\Lambda|}\right)^{2dc|\Lambda| \ln |\Lambda|} \leq \frac{1}{|\Lambda|^c}. \end{aligned}$$

We obtain the following inequality:

$$P_\mu \left(\begin{array}{l} e \in \mathcal{P}_{t+s} \quad \exists f \in \mathcal{P}_t \cup \mathcal{I}_t \\ d(e, f) < 2dc \ln |\Lambda| \\ d(e, \Lambda \cup \mathfrak{P}_t^-) \geq 2dc \ln |\Lambda| \end{array} \right) \leq \frac{\lambda(d)(2dc \ln |\Lambda|)^d}{|\Lambda|^c}, \quad (4.5.2)$$

where $\lambda(d)$ is a constant depending only on the dimension. We combine the two cases (4.5.1) and (4.5.2), we obtain

$$P_\mu(e \in \mathcal{P}_{t+s}, d(e, \Lambda \cup \mathfrak{P}_t^-) \geq 2dc \ln |\Lambda|) \leq \frac{\lambda(d)(2dc \ln |\Lambda|)^d}{|\Lambda|^c} + \frac{1}{|\Lambda|^{2dc}}.$$

We then sum over the number of the choices for the edge e and of the number s from 1 to $2dc|\Lambda| \ln |\Lambda|$. We obtain

$$P_\mu \left(\begin{array}{l} \exists s \in [t, t + 2dc|\Lambda| \ln |\Lambda|] \quad \exists e \in \mathcal{P}_{t+s} \\ d(e, \Lambda^c \cup \mathfrak{P}_t^-) \geq 2dc \ln |\Lambda| \end{array} \right) \leq \frac{\lambda(d)(2dc \ln |\Lambda|)^{d+1}}{|\Lambda|^{c-2}} + \frac{2dc \ln |\Lambda|}{|\Lambda|^{2dc-2}}. \quad (4.5.3)$$

In other words, we have

$$P_\mu (\mathfrak{P}_t^+ \not\subseteq \mathcal{V}(\Lambda^c \cup \mathfrak{P}_t^-, 2dc \ln |\Lambda|)) \leq \frac{\lambda(d)(2dc \ln |\Lambda|)^{d+1}}{|\Lambda|^{c-2}} + \frac{2dc \ln |\Lambda|}{|\Lambda|^{2dc-2}}.$$

By the reversibility of the process $(Y_t)_{t \in \mathbb{N}}$, we also have

$$P_\mu (\mathfrak{P}_t^- \not\subseteq \mathcal{V}(\Lambda^c \cup \mathfrak{P}_t^+, 2dc \ln |\Lambda|)) \leq \frac{\lambda(d)(2dc \ln |\Lambda|)^{d+1}}{|\Lambda|^{c-2}} + \frac{2dc \ln |\Lambda|}{|\Lambda|^{2dc-2}}.$$

Combining the two previous inequalities, we have

$$P_\mu \left(d_H^{2dc \ln |\Lambda|}(\mathfrak{P}_t^-, \mathfrak{P}_t^+) \geq 2dc \ln |\Lambda| \right) \leq \frac{2\lambda(d)(2dc \ln |\Lambda|)^{d+1}}{|\Lambda|^{c-2}} + \frac{4dc \ln |\Lambda|}{|\Lambda|^{2dc-2}}.$$

For $|\Lambda| \geq e^{2d^2c}$, we have

$$\frac{2\lambda(d)(2dc \ln |\Lambda|)^{d+1}}{|\Lambda|^{c-2}} + \frac{4dc \ln |\Lambda|}{|\Lambda|^{2dc-2}} \leq \frac{1}{|\Lambda|^{c-3}}.$$

This yields the desired result. □

Troisième partie

L'interface de la
FK-percolation et d'Ising

Chapitre 5

La localisation de l'interface d'Ising à basse température

Ce chapitre est adapté de l'article [Zho18]. La version préprint est disponible sur: <https://arxiv.org/abs/1901.05787>

We study the Ising model in a box Λ in \mathbb{Z}^d (not necessarily parallel to the directions of the lattice) with Dobrushin boundary conditions at low temperature. We couple the spin configuration with the configurations under $+$ and $-$ boundary conditions and we define the interface as the edges whose endpoints have the same spins in the $+$ and $-$ configurations but different spins with the Dobrushin boundary conditions. We prove that, inside the box Λ , the interface is localised within a distance of order $\ln^2 |\Lambda|$ of the set of the edges which are connected to the top by a $+$ path and connected to the bottom by a $-$ path.

5.1 Introduction

At the macroscopic level, the dynamics of the interface between two pure phases in the Ising model seem to be deterministic. In fact, the interface tends to minimize the surface tension between the two phases. The microscopic justification of this fact in the context of 2D Ising model was achieved in [DKS92]. In the limit where the size of the system grows to infinity, the two types of spins, at low temperature, form two regions separated by the interfaces. After a suitable spatial rescaling, these interfaces converge to deterministic shapes. However, the interfaces remain random and their geometric structure is extremely complex. In two dimensions, the fluctuations of the interfaces are well analysed in [DH97] using the cluster expansions techniques. Recently, Ioffe and Velenik gave a geometric description of the interfaces and their scaling limits with the help of the Ornstein–Zernike

theory in [IV18]. In higher dimensions, the famous result of Dobrushin in [Dob72] says that at low temperature, the interface in a straight box is localised around the middle hyperplane of the box when the temperature is low. One of the difficulties to study the interfaces is to define them properly. The usual way is to consider the Dobrushin boundary conditions. More precisely, the vertices on the upper boundary of the box are pluses and those on the lower boundary are minuses. It is a geometric fact that, with such a boundary condition, the spin configurations present an interface separating a region of plus spins containing the upper boundary and a region of minus spins containing the lower boundary. However, for several reasons, it is still not obvious to define an interface in this setting. For example, there are more than one separating set between the pluses and minuses in a typical configuration with Dobrushin boundary conditions.

Remark. While I was finalizing this paper, Gheissari and Lubetzky completed a very interesting paper [GL19] on the large deviations of the interface in 3D Ising model. They study the height of the interface in a straight box using a decomposition of the pillars and they obtain a localisation result at an order $\ln |\Lambda|$ at low temperature. Our localisation result is clearly weaker in the case of a straight box, however it holds also for a tilted box.

The first goal of this study is to adapt to the Ising model the definition of the interfaces, introduced in [CZ18a] for the percolation model. The second goal is to progress in the geometric description of these interfaces for a box not necessarily straight in dimensions $d \geq 2$ at low temperature. In [CZ18a], we constructed a coupling between the dynamical percolation process and a conditioned process. The interface was defined as the difference between the two processes. We showed a localisation result for the interface around the pivotal edges for the disconnection event. For the Ising model, a coupling can be realised by the Glauber dynamics, yet there is no corresponding notion for the pivotal edges. However, the objects introduced for the percolation model in [CZ18a] are defined naturally for the FK-percolation model. With the help of the Edwards-Sokal coupling, we can define and localise the interfaces using the results obtained in the FK-percolation model. To realise our first goal, we construct three spin configurations $(\sigma^+, \sigma^-, \sigma^D)$, corresponding to the plus, minus and Dobrushin boundary conditions, and a probability measure $\pi_{\Lambda, \beta}$ on this triplet, whose marginals are the Ising measures with the corresponding boundary conditions. We consider a box $\Lambda = (V, E)$ and we define the interface as follows:

Definition 5.1.1. *The set \mathcal{P}_I is the set of the edges $\langle x, y \rangle \in E$ such that*

$$\begin{aligned} \sigma^D(x) &= +1 \text{ and } x \text{ is connected to } T \text{ by a path of vertices with } +1 \\ \sigma^D(y) &= -1 \text{ and } y \text{ is connected to } B \text{ by a path of vertices with } -1. \end{aligned}$$

The set \mathcal{I}_I is the set of the edges $\langle x, y \rangle \in E$ such that

$$\sigma^+(x) = \sigma^+(y), \quad \sigma^-(x) = \sigma^-(y), \quad \sigma^D(x) \neq \sigma^D(y).$$

The interface \mathcal{I}_I is the set of the edges whose endpoints have different spins in σ^D but have the same spins in the other two configurations. The set \mathcal{P}_I corresponds to the edges of \mathcal{I}_I connected to the boundary in σ^D . As for the second goal, we show the following result:

Theorem 5.1.2. *There exist $0 < \tilde{\beta} < \infty$ and $\kappa \geq 0$, such that for $\beta \geq \tilde{\beta}$, $c > 0$ and any Λ such that $|\Lambda| \geq \max\{3^{6d}, (cd)^{cd^2}\}$, we have*

$$\pi_{\Lambda, \beta} \left(\exists e \in \mathcal{I}_I, d(e, \Lambda^c \cup \mathcal{P}_I) \geq \kappa c^2 \ln^2 |\Lambda| \right) \leq \frac{1}{|\Lambda|^c}.$$

We call a cut in a spin configuration a set of edges $e = \langle x, y \rangle$ separating T and B such that

$$\sigma^D(x) \neq \sigma^D(y).$$

Using the same method, we show that, under the probability π , a vertex separated from B by a cut and which is far from this cut has the same spin in σ^+ and σ^D , more precisely, we have:

Theorem 5.1.3. *$0 < \tilde{\beta} < \infty$ and $\kappa \geq 0$, such that for $\beta \geq \tilde{\beta}$, $c > 0$ and any Λ such that $|\Lambda| \geq \max\{3^{6d}, (cd)^{cd^2}\}$, we have*

$$\pi_{\beta} \left(\begin{array}{l} \exists x \in \Lambda \quad \sigma^+(x) = +1, \quad \sigma^D(x) = -1 \\ \exists C \text{ a cut separating } x \text{ from } B \\ d(x, C) \geq \kappa c^2 \ln^2 |\Lambda| \end{array} \right) \leq \frac{1}{|\Lambda|^c},$$

and

$$\pi_{\beta} \left(\begin{array}{l} \exists x \in \Lambda \quad \sigma^+(x) = -1, \quad \sigma^D(x) = +1 \\ \exists C \text{ a cut separating } x \text{ from } T \\ d(x, C) \geq \kappa c^2 \ln^2 |\Lambda| \end{array} \right) \leq \frac{1}{|\Lambda|^c}.$$

The key to obtain these two results is to construct a coupling (X, Y) , where X is a standard FK-percolation configuration and Y is a configuration where the top side T and the bottom side B of the box Λ are disconnected. We denote this event by $\{T \leftrightarrow B\}$. The localisation of the interface in the Ising model is induced by a control of the distance between the interface \mathcal{I} and the set of the pivotal edges \mathcal{P} of the coupling (X, Y) . In this paper, we consider the FK-percolation model with a parameter p close to 1 and q larger than 1. Interfaces in a box Λ are naturally created when the configuration is conditioned to stay in the set $\{T \leftrightarrow B\}$. The interface \mathcal{I} is defined as

$$\mathcal{I} = \{ e \subset \Lambda : X(e) \neq Y(e) \}$$

and we denote by \mathcal{P} the set of the pivotal edges for the event $\{T \leftrightarrow B\}$ in Y . Our main result for the FK model is the following.

Theorem 5.1.4. *For any $q \geq 1$, there exist $\tilde{p} < 1$ and $\kappa > 0$, such that, for $p \geq \tilde{p}$, any $c \geq 1$ and any box Λ such that $|\Lambda| \geq \max\{3^{6d}, (cd)^{cd^2}\}$,*

$$\mu_{\Lambda, p, q} \left(\exists e \in \mathcal{P} \cup \mathcal{I}, d(e, \Lambda^c \cup \mathcal{P} \setminus \{e\}) \geq \kappa c^2 \ln^2 |\Lambda| \right) \leq \frac{1}{|\Lambda|^c}.$$

Let us explain briefly how we build the measures $\mu_{\Lambda,p,q}$ and $\pi_{\Lambda,\beta}$ as well as the strategy for proving theorem 5.1.2. With the help of a Gibbs sampler algorithm (see section 8.4 of [Gri06]), we construct a coupling between two Markov chains $(X_t, Y_t)_{t \in \mathbb{N}}$ on the space of the percolation configurations in a box Λ . The measure $\mu_{\Lambda,p,q}$ is the unique invariant measure of the process $(X_t, Y_t)_{t \in \mathbb{N}}$. Starting from a coupled configuration (ω, ω') under the measure $\mu_{\Lambda,p,q}$, we put spins on the vertices in the box Λ using an adaptation of the Edwards-Sokal coupling. By construction, the configuration ω dominates ω' . We put spins at first on the vertices according to the configuration ω' , under the Dobrushin boundary condition, to obtain a spin configuration σ^D . Then, we put spins according to ω , under the plus (respectively minus) boundary condition to obtain the configuration σ^+ (respectively σ^-) with the restriction that an open cluster in ω also appearing in ω' has the same spin as in σ^D . The measure $\pi_{\Lambda,\beta}$ is the probability distribution of $(\sigma^+, \sigma^-, \sigma^D)$ obtained from $\mu_{\Lambda,p,q}$ and the colouring. Each of its marginals is an Ising measure in the box with the corresponding boundary conditions. As for the proof of theorem 5.1.4, we follow the ideas presented in [CZ18a]. We control the distance between two pivotal edges by identifying a cut and a closed path disjoint from the cut. However, due to the correlations between all the edges in the FK-percolation model, we cannot use the BK inequality which holds for a product space and which is a key ingredient in [CZ18a]. To solve this difficulty, we explore adequately the open clusters and we identify a sub-graph in Λ containing a long closed path and outside of which we can find a cut. The configurations in this sub-graph can be compared to a Bernoulli configuration. To study the case where the distance between an edge of the interface and the pivotal edges is big, we show that the interface edge cannot have been created a long time ago. Moreover, at the time when it is created, it must be a pivotal edge. Therefore, the set of the pivotal edges must move rather fast. We obtain a control over the speed of the pivotal edges. This estimate relies on the study of specific space-time paths, which describe how the cut sets move.

These results answer the question 4 raised in [CZ18a], and also give some information to the subsequent question 5. However, we would like to obtain more information about the structure of the set \mathcal{P}_I .

This paper is organised as follows. In section 2, we give the definitions of the objects and the notations which we will use in this article. In section 3, we show the estimate on the distance between two pivotal edges in the FK-percolation model. In section 4, we control the speed of the pivotal edges. In section 5, we show theorem 5.1.4 and we prove theorem 5.1.2 in section 6.

5.2 The notations

In this section, we present the FK-percolation model which we study and we recall some fundamental tools which we will use in the rest of this paper.

5.2.1 Geometric definitions

We start with some geometric definitions.

The lattice \mathbb{L}^d . For an integer $d \geq 2$, the lattice \mathbb{L}^d is the graph $(\mathbb{Z}^d, \mathbb{E}^d)$, where the set \mathbb{E}^d is the set of pairs $\langle x, y \rangle$ of points in \mathbb{Z}^d which are at Euclidean distance 1.

The usual paths. We say that two edges e and f are neighbours if they have one endpoint in common. A usual path is a sequence of edges $(e_i)_{1 \leq i \leq n}$ such that for $1 \leq i < n$, e_i and e_{i+1} are neighbours.

The *-paths. In order to study the cuts in any dimension $d \geq 2$, we use *-connectedness on the edges as in [DP96]. We consider the supremum norm on \mathbb{R}^d :

$$\forall x = (x_1, \dots, x_d) \in \mathbb{R}^d \quad \|x\|_\infty = \max_{i=1, \dots, d} |x_i|.$$

For e an edge in \mathbb{E}^d , we denote by m_e the center of the unit segment associated to e . We say that two edges e and f of \mathbb{E}^d are *-neighbours if $\|m_e - m_f\|_\infty \leq 1$. A *-path is a sequence of edges (e_1, \dots, e_n) such that, for $1 \leq i < n$, the edge e_i and e_{i+1} are *-neighbours. For a path γ , we denote by $\text{support}(\gamma)$ the set of the edges of γ . We say that a path is simple if the cardinal of its support is equal to its length.

The box Λ . We will mostly work in a box Λ centred at the origin (not necessarily straight) as illustrated in the figure 5.1. More precisely, we will consider a d -cube Λ centred at origin. We can also consider Λ as the graph $\Lambda = (V, E)$ is the sub-graph of \mathbb{L}^d whose vertices are included in the cube. The boundary of Λ , denoted by $\partial\Lambda$, is defined as,

$$\partial\Lambda = \{x \in V : \exists y \notin \Lambda, \langle x, y \rangle \in \mathbb{E}^d\}.$$

We will distinguish two disjoint non-empty subsets of $\partial\Lambda$, denoted by T and B . We consider a $(d-1)$ dimensional plane containing the origin and parallel to a side of Λ . This plane separates Λ into two parts Λ^+ and Λ^- . The set T is the subset of $\partial\Lambda$ included in Λ^+ and B the one included in Λ^- .

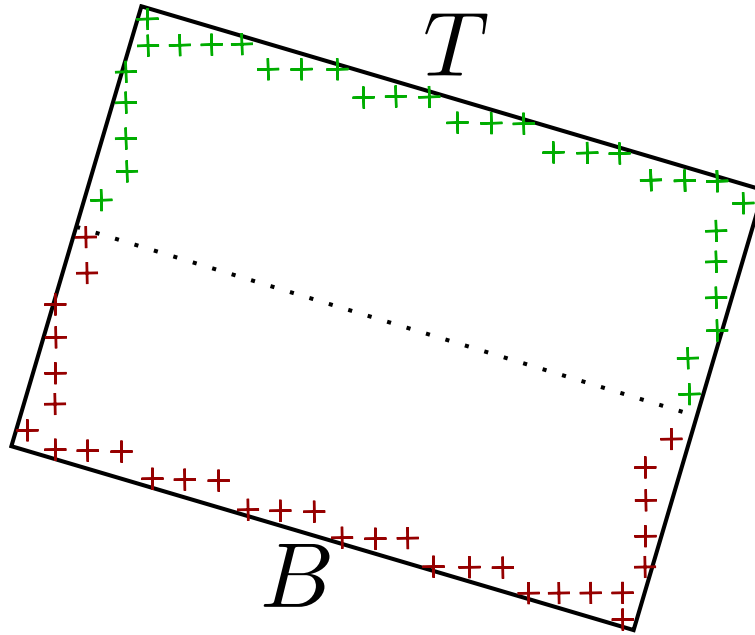


Figure 5.1: The box Λ and its boundary (the crosses). The green crosses form the side T and the red ones form the side B .

The separating sets. Let A, B be two subsets of Λ . We say that a set of edges $S \subset \Lambda$ separates A and B if no connected subset of $\Lambda \cap \mathbb{E}^d \setminus S$ intersects both A and B . Such a set S is called a separating set for A and B . We say that a separating set is minimal if there does not exist a strict subset of S which separates A and B .

The cuts. We say that S is a cut if S separates T and B , and S is minimal for the inclusion.

5.2.2 The Ising model

Let $\Lambda = (V, E)$ be the finite box. We associate to each vertex $x \in V$ a random spin $\sigma(x)$ which can either be $+1$ or -1 . The spin values are chosen according to a certain probability measure λ_β , known as a Gibbs state, which depends on a parameter $\beta \in [0, +\infty[$, and is given by

$$\lambda_\beta(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z_I}, \quad \sigma \in \{+1, -1\}^V,$$

where

$$H(\sigma) = - \sum_{\langle x, y \rangle \in E} \sigma(x)\sigma(y)$$

is the Hamiltonian and Z_I is the normalisation constant called the partition function.

5.2.3 The FK-percolation model

Also known as the random-cluster model, the FK-percolation model is a generalisation of the Bernoulli percolation model, in which we introduce correlations between edges by taking into account the number of open clusters in a configuration. On a finite graph (V, E) , a random cluster-measure is a member of a certain class of probability measures on the space set $\{0, 1\}^E$. Let ω belongs to $\{0, 1\}^E$, we say that an edge e is open if $\omega(e) = 1$ and closed if $\omega(e) = 0$, and we set

$$\eta(\omega) = \{ e \in E : \omega(e) = 1 \}.$$

Let $k(\omega)$ be the number of connected components (or the open clusters) of the graph $(V, \eta(\omega))$, and note that $k(\omega)$ includes the count of the isolated vertices, that is, of vertices incident to no open edge. For two parameters $p \in [0, 1]$ and $q > 0$, the *random-cluster measure* $\Phi_{\Lambda, p, q}$ is defined as

$$\Phi_{\Lambda, p, q}[\omega] = \frac{1}{Z_{RC}} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega)}, \quad \omega \in \{0, 1\}^E,$$

where the partition function Z_{RC} is given by

$$Z_{RC} = \sum_{\omega \in \{0, 1\}^E} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega)}.$$

In our study, we will consider only the case where $q \geq 1$.

Boundary conditions. We will consider different boundary conditions. Let $\xi \in \{0, 1\}^{\mathbb{E}^d}$ and $\Lambda = (V, E)$ be the box. Let Ω^ξ denote the subset of $\{0, 1\}^{\mathbb{E}^d}$ consisting of all the configurations ω satisfying $\omega(e) = \xi(e)$ for $e \in \mathbb{E}^d \setminus E$. We shall write $\Phi_{\Lambda, p, q}^\xi$ for the random-cluster measure on Λ with boundary condition ξ , given by

$$\Phi_{\Lambda, p, q}^\xi[\omega] = \begin{cases} \frac{1}{Z^\xi} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega, \Lambda)} & \text{if } \omega \in \Omega^\xi, \\ 0 & \text{otherwise,} \end{cases}$$

where $k(\omega, \Lambda)$ is the number of components of the graph $(\mathbb{Z}^d, \eta(\omega))$ that intersect Λ , and Z^ξ is the appropriate normalizing constant. In particular, we will consider three special boundary conditions in this paper:

- The 0-boundary condition corresponds to the case where all the edges of ξ are closed. This condition is also called the *free* boundary condition.

- The 1-boundary condition corresponds to the case where all the edges of ξ are open. We can also see this condition as adding one vertex which is connected to all the vertices of $\partial\Lambda$, therefore, this boundary condition is called the *wired* boundary condition.
- The TB -boundary condition corresponds to the Dobrushin boundary condition for the Ising model introduced in [Dob72]. This boundary condition corresponds to a configuration ξ where all the vertices of T are connected and all the vertices of B are connected by open paths of edges outside of Λ , but there is no open path which connects a vertex of T to a vertex of B . We can also see this condition as adding two vertices to the graph Λ , one of which is connected to all the vertices of T and the other one connected to all the vertices of B .

In order to simplify the notations, we omit Λ, p, q, ξ in $\Phi_{\Lambda, p, q}^{\xi}$ if it doesn't create confusions. We will use two fundamental properties of the FK-percolation model called the spatial Markov property (see chapter 4.2 of [Gri06]) and the comparison between different values of p and q stated in chapter 3.4 of [Gri06].

5.2.4 Coupled dynamics of FK-percolation

We will use a special Glauber process called the Gibbs sampler to study the conditioned FK-measure. Consider the finite graph $\Lambda = (V, E)$. To simplify the notations, we omit the Λ, p, q in $\Phi_{\Lambda, p, q}$ in this section. The Gibbs sampler is a Markov chain $(X_t)_{t \in \mathbb{N}}$ on the state space $\Omega = \{0, 1\}^E$. At a time t , we choose an edge e uniformly in E , and we set the status of e according to the current states of the other edges. More precisely, let $(U_t)_{t \in \mathbb{N}}$ be a sequence of uniform variables on $[0, 1]$ and $(E_t)_{t \in \mathbb{N}}$ be a sequence of uniform variables in E . For $\omega \in \Omega$ and $e \in E$, we denote by ω^e the configuration obtained by opening the edge e and by ω_e the one where e is closed. At time t , we suppose $X_{t-1} = \omega$ and we set

$$X_t(e) = \begin{cases} \omega(e) & \text{if } E_t \neq e \\ 1 & \text{if } E_t = e \text{ and } U_t \geq \frac{\Phi(\omega_e)}{\Phi(\omega^e) + \Phi(\omega_e)} \\ 0 & \text{if } E_t = e \text{ and } U_t < \frac{\Phi(\omega_e)}{\Phi(\omega^e) + \Phi(\omega_e)} \end{cases} .$$

We also define a coupled process $(Y_t)_{t \in \mathbb{N}}$ which stays in $\{T \leftrightarrow B\}$. We use the same sequences $(E_t)_{t \in \mathbb{N}}$ and $(U_t)_{t \in \mathbb{N}}$. At time t , we suppose $Y_{t-1}(e) = \omega$

and we change the status of the edge e in Y_t as follows:

$$Y_t(e) = \begin{cases} \omega(e) & \text{if } E_t \neq e \\ 1 & \text{if } E_t = e, U_t \geq \frac{\Phi(\omega_e)}{\Phi(\omega^e) + \Phi(\omega_e)} \text{ and } T \overset{\omega^e}{\longleftrightarrow} B \\ 0 & \text{if } E_t = e, U_t \geq \frac{\Phi(\omega_e)}{\Phi(\omega^e) + \Phi(\omega_e)} \text{ and } T \overset{\omega^e}{\nrightarrow} B \\ 0 & \text{if } E_t = e, U_t < \frac{\Phi(\omega_e)}{\Phi(\omega^e) + \Phi(\omega_e)} \end{cases} .$$

Before opening a closed edge e at time t , we verify whether this will create a connexion between T and B in Y_t . If it is the case, the edge e stays closed in Y_t but can be opened in X_t , otherwise the edge e is opened in both X_t and Y_t . On the contrary, the two processes behave similarly for the edge closing events since we cannot create a new connexion by closing an edge. The set of the configurations satisfying $\{T \leftrightarrow B\}$ is irreducible and the process $(X_t)_{t \in \mathbb{N}}$ is reversible. By the lemma 1.9 of [Kel11], there exists a unique stationary distribution $\Phi^{\mathcal{D}}$ for the process $(Y_t)_{t \in \mathbb{N}}$ and $\Phi^{\mathcal{D}}$ is equal to the probability Φ^{TB} conditioned by the event $\{T \leftrightarrow B\}$, i.e.,

$$\Phi^{\mathcal{D}}(\cdot) = \Phi^{TB}(\cdot | T \leftrightarrow B).$$

Suppose that we start from a configuration (X_0, Y_0) belonging to the set

$$\mathcal{E} = \{(\omega_1, \omega_2) \in \{0, 1\}^{\mathbb{E}^d \cap \Lambda} \times \{T \leftrightarrow B\} : \forall e \in \Lambda \quad \omega_1(e) \geq \omega_2(e)\}.$$

The set \mathcal{E} is irreducible and aperiodic. In fact, each configuration of \mathcal{E} communicates with the configuration where all edges are closed. The state space \mathcal{E} is finite, therefore the Markov chain $(X_t, Y_t)_{t \in \mathbb{N}}$ admits a unique equilibrium distribution μ_p . We denote by P_μ the law of the process $(X_t, Y_t)_{t \in \mathbb{N}}$ starting from a random initial configuration (X_0, Y_0) with distribution $\mu_{\Lambda, p, q}$. We define the following objects using the previous coupling.

Definition 5.2.1. *The interface at time t between T and B , denoted by \mathcal{I}_t , is the set of the edges in Λ that differ in the configurations X_t and Y_t , i.e.,*

$$\mathcal{I}_t = \{e \in \Lambda : X_t(e) \neq Y_t(e)\}.$$

The edges of \mathcal{I}_t are open in X_t but closed in Y_t and the configuration X_t is above the configuration Y_t . We define next the set \mathcal{P}_t of the pivotal edges for the event $\{T \leftrightarrow B\}$ in the configuration Y_t .

Definition 5.2.2. *The set \mathcal{P}_t of the pivotal edges in Y_t is the collection of the edges in Λ whose opening would create a connection between T and B , i.e.,*

$$\mathcal{P}_t = \{e \in \Lambda : T \overset{Y_t^e}{\longleftrightarrow} B\}.$$

We define finally the set \mathcal{C}_t of the cuts in Y_t .

Definition 5.2.3. *The set \mathcal{C}_t of the cuts in Y_t is the collection of the cuts in Λ at time t .*

5.2.5 The classical Edwards-Sokal coupling

We wish to gain insight into the interface in the Ising model with the help of our previous results and the classical coupling of Edwards and Sokal (see chapter 1 of [Gri06] for more details on this coupling). Let $\Lambda = (V, E)$ be the box. We consider the product space $\Sigma \times \Omega$ where $\Sigma = \{-1, 1\}^V$ and $\Omega = \{0, 1\}^E$. We define a probability ν on $\Sigma \times \Omega$ by

$$\nu(\sigma, \omega) \propto \prod_e \left\{ (1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\delta_e(\sigma) \right\}, \quad (\sigma, \omega) \in \Sigma \times \Omega,$$

where $\delta_e(\sigma) = \delta_{\sigma(x),\sigma(y)}$ for $e = \langle x, y \rangle \in E$. The constant of proportionality is the one which ensures the normalization

$$\sum_{(\sigma, \omega) \in \Sigma \times \Omega} \nu(\sigma, \omega) = 1.$$

For $q = 2$, $p = 1 - e^{-\beta}$ and $\omega \in \Omega$, the conditional measure $\nu(\cdot|\omega)$ on Σ is obtained by colouring randomly the clusters of ω . More precisely, conditionally on a percolation configuration ω , the spins are constant on the clusters of ω and they are independent between the clusters. With the help of this coupling, we can transport results in FK-percolation to the Ising model.

5.2.6 The coupling of spin configurations

We construct a coupling of the Ising configurations $(\sigma^+, \sigma^-, \sigma^D)$ with different boundary conditions from a pair of percolation configurations

$$(\omega, \omega') \in \Omega \times \{T \leftrightarrow B\}$$

satisfying $\omega \geq \omega'$. The configuration σ^+ (resp. σ^-) will correspond to the spin configuration with the $+$ boundary condition (resp. $-$ boundary condition) and the configuration σ^D will correspond to the Dobrushin boundary condition. We will put spins on the vertices in Λ as follows. We start by putting spins on the vertices of the clusters of ω' using the Edwards-Sokal coupling with the Dobrushin boundary conditions. This way we obtain a spin configuration, which we denote by σ^D . Notice that the open clusters of ω are unions of the open clusters of ω' . For an open cluster in ω which touches the boundary of Λ , we color its vertices with $+1$ in the configuration σ^+ and with -1 in σ^- . For an open cluster C which does not touch the boundary of Λ , if C is also an open cluster in ω' , we set

$$\forall x \in C \quad \sigma^+(x) = \sigma^-(x) = \sigma^D(x).$$

For an open cluster C of ω which does not touch the boundary of Λ and which is the union of several open clusters of ω' , we set

$$\forall x \in C \quad \sigma^+(x) = \sigma^-(x) = N(C),$$

where $N(C)$ is equal to 1 with probability $1/2$ and -1 with probability $1/2$. Of course, the random variables $N(C)$ associated to the clusters of ω are independent and also independent from the configuration outside C . For $q = 2, p = 1 - e^{-\beta}$ and (ω, ω') distributed under $\mu_{\Lambda, p, 2}$, the configurations σ^+, σ^- and σ^D obtained are distributed according to the Gibbs state at inverse temperature β with boundary conditions $+, -$ and Dobrushin. We denote by $\pi_{\Lambda, \beta}$ the distribution of the triple $(\sigma^+, \sigma^-, \sigma^D)$.

5.3 Localising a cut around the pivotal edges

The following proposition controls the distance between an edge belonging to a cut and the set of the pivotal edges.

Proposition 5.3.1. *For any $q \geq 1$, there exist $\tilde{p} < 1$ and $\kappa > 1$ such that, for $p \geq \tilde{p}$, and for any $c \geq 1$ and any box Λ satisfying $|\Lambda| > 3^{6d}$, we have*

$$\Phi_{\Lambda, p, q}^{TB} [\exists C \in \mathcal{C}, \exists e \in C, d(e, \mathcal{P} \cup \Lambda^c \setminus \{e\}) \geq \kappa c \ln |\Lambda| \mid T \leftrightarrow B] \leq \frac{1}{|\Lambda|^c}.$$

The strategy of the proof follows that of proposition 1.4 of [CZ18a]. We can still observe a closed path which is disjoint from a cut. However, one key ingredient of the proof in [CZ18a] is the BK inequality (see [Gri99]) which doesn't hold for the FK-percolation model. The following lemma gives us an inequality which plays the role of the BK inequality in the proof.

Lemma 5.3.2. *Let $n \geq 2, q \geq 1, p \in [0, 1]$, and let e be an edge in Λ . We define the event*

$$\Gamma(e, n, T) = \left\{ \begin{array}{l} \text{there exists a closed path of length } n \text{ starting from } e \\ \text{and there exists a cut which separates this path from } T \end{array} \right\}.$$

The following inequality holds:

$$\Phi_{\Lambda, p, q}^{TB} [\Gamma(e, n, T)] \leq \alpha^n(d) (1 - f(p, q))^n \Phi_{\Lambda}^{TB} [T \leftrightarrow B],$$

where

$$f(p, q) = \frac{p}{p + q - pq}$$

and $\alpha(d)$ is the number of $*$ -neighbours of an edge in the lattice \mathbb{L}^d .

Notice that this inequality also holds if we consider the symmetric event $\Gamma(e, n, B)$.

Proof. Let us start with the construction of a random graph in Λ in which we can find a path starting from e and the complementary of which contains a cut. The construction is inspired by the proof of theorem 5.3 in [Gri95].

For the topological complications in the construction, we refer to the related passage of [Kes86, Sect. 2]. We define the open cluster of T as

$$O(T) = \{x \in \Lambda : x \longleftrightarrow T\}.$$

We consider a configuration satisfying $\{T \leftrightarrow B\}$. Then we define the set C^+ as the set of the edges f satisfying

1. f has exactly one endpoint in $O(T)$;
2. there exists a geometric path in \mathbb{L}^d which connects B to f and which does not use vertices of $O(T)$.

Notice that the path in point 2 is only geometric, there is no requirement on the status of the edges of this path. We note three facts about the set C^+ :

- a. the edges of C^+ are closed,
- b. the set C^+ contains a cut,
- c. C^+ is measurable with respect to the configuration of the edges which have at least one endpoint in $O(T)$.

Let us now consider a configuration in the event $\Gamma(e, n, T)$. The cut in C^+ separates e from T . We now construct a sub-graph \mathcal{G} of Λ . We define V' as the set of the vertices included in $V \setminus O(T)$ which are connected to B without using an edge of C^+ . We define the sub-graph \mathcal{G} as

$$\mathcal{G} = (V', \{\langle x, y \rangle : x, y \in V'\}).$$

The figure 5.2 illustrates the construction of the graph \mathcal{G} . From the construction, it follows that the cut in C^+ does not contain an edge of the graph \mathcal{G} . For a fixed sub-graph $G_1 = (V_1, E_1)$ of (V, E) , we denote by G' the graph (V', E') , where $E' = E \setminus E_1$ and V' is the set of the endpoints of the edges in E' . We decompose the set of the edge configurations Ω in Λ as

$$\Omega = \Omega_{G_1} \times \Omega_{G'},$$

where Ω_G (respectively $\Omega_{G'}$) is the set of the configurations of the edges in G_1 (respectively G'). We notice that the event $\mathcal{G} = G_1$ is entirely determined by the configurations $\omega_{G'} \in \Omega_{G'}$. We obtain that

$$\begin{aligned} \Phi_{\Lambda, p, q}^{TB}[\Gamma(e, n, T)] &= \sum_{G_1} \Phi_{\Lambda, p, q}^{TB}[\Gamma(e, n, T), \mathcal{G} = G_1] \\ &= \sum_{G_1} \sum_{\eta \in \Omega_{G'}: \mathcal{G} = G_1} \Phi_{\Lambda, p, q}^{TB}[\Gamma(e, n, T), \omega_{G'} = \eta] \\ &= \sum_{G_1} \sum_{\eta \in \Omega_{G'}: \mathcal{G} = G_1} \Phi_{\Lambda, p, q}^{TB}[\Gamma(e, n, T) \mid \omega_{G'} = \eta] \Phi_{\Lambda, p, q}^{TB}[\omega_{G'} = \eta]. \end{aligned} \quad (5.3.1)$$

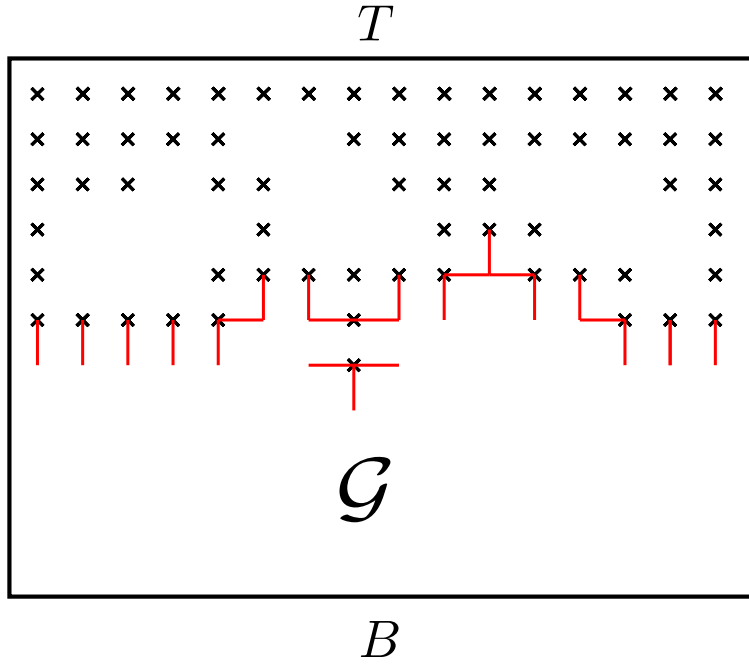


Figure 5.2: The crosses are the vertices of $O(T)$ and the red edges form the set C^+ , the graph \mathcal{G} remains unexplored.

We define the event $\Gamma(e, n)$ as

$$\Gamma(e, n) = \{ \text{there exists a closed } * \text{-path of length } n \text{ starting from } e \}.$$

By the spatial Markov property, for a fixed graph G_1 containing all the edges intersecting B and η such that $\mathcal{G} = G_1$, we have

$$\Phi_{\Lambda, p, q}^{TB}[\Gamma(e, n, T) \mid \omega_{G'} = \eta] = \Phi_{G_1, p, q}^{\xi(\eta)}[\Gamma(e, n)] \leq \Phi_{G_1, p, q}^0[\Gamma(e, n)],$$

where $\xi(\eta)$ is the boundary condition on G_1 induced by η . The last inequality holds because the event $\Gamma(e, n)$ is decreasing. As for the configurations in the graph G_1 , we can compare the measure $\Phi_{G_1, p, q}^0$ with the Bernoulli percolation measure with parameter

$$f(p, q) = \frac{p}{p + q - pq}.$$

Since the event $\Gamma(e, n)$ is decreasing, by the comparison inequalities stated in chapter 3.4 of [Gri06], we have

$$\Phi_{G_1, p, q}^0[\Gamma(e, n)] \leq \Phi_{G_1, f(p, q), 1}^0[\Gamma(e, n)].$$

By a standard Peierls argument, we have then

$$\Phi_{G_1, f(p, q), 1}^0[\Gamma(e, n)] \leq \alpha^n(d)(1 - f(p, q))^n,$$

where $\alpha(d)$ is the number of $*$ -neighbours of an edge in dimension d . We use this upper bound in the sum (5.3.1) and we obtain

$$\begin{aligned} \Phi_{\Lambda}^{TB}[\Gamma(e, n, T)] &\leq \alpha^n(d)(1 - f(p, q))^n \sum_{G_1: B \subset V(G_1)} \sum_{\eta: \mathcal{G}=G_1} \Phi_{\Lambda}^{TB}[\omega_{G'} = \eta] \\ &= \alpha^n(d)(1 - f(p, q))^n \sum_{G_1: B \subset V(G_1)} \Phi_{\Lambda}^{TB}[\mathcal{G} = G_1]. \end{aligned}$$

We now calculate the last sum. We have

$$\begin{aligned} \sum_{G_1: B \subset V(G_1)} \Phi_{\Lambda}^{TB}[\mathcal{G} = G_1] &= \Phi_{\Lambda}^{TB} \left[\begin{array}{l} \exists \mathcal{G} \text{ sub-graph of } \Lambda \\ V(\mathcal{G}) \cap O(T) = \emptyset, \quad B \subset V(\mathcal{G}) \end{array} \right] \\ &= \Phi_{\Lambda}^{TB}[O(T) \cap B = \emptyset]. \end{aligned}$$

The event $\{O(T) \cap B = \emptyset\}$ implies the disconnection between T and B , thus we have

$$\Phi_{\Lambda}^{TB}[\Gamma(e, n, T)] \leq \alpha^n(d)(1 - f(p, q))^n \Phi_{\Lambda}^{TB}[T \leftrightarrow B],$$

which is the desired inequality. \square

We now prove proposition 5.3.1 with the help of the previous lemma.

Proof of proposition 5.3.1. Notice that every edge of a cut is connected to the set of the pivotal edges \mathcal{P} or to the boundary of Λ by a closed path. Let us fix an edge e which belongs to a cut and which is at distance more than $\kappa c \ln |\Lambda|$ from $\mathcal{P} \cup \Lambda^c \setminus \{e\}$. There is a closed path starting from the edge e and which is of length $\kappa c \ln |\Lambda|/4d$. This path is disjoint from a cut since there is no pivotal edge on this path. We refer to the section 3 of [CZ18a] for the detailed geometric justifications. The probability appearing in the proposition is less than

$$\Phi_{\Lambda}^{TB}[A \mid T \leftrightarrow B],$$

where A is the event

$$A = \left\{ \begin{array}{l} \exists \gamma \text{ closed path of length } \frac{\kappa c \ln |\Lambda|}{4d} \text{ starting from } e \\ \exists C \in \mathcal{C} \quad C \text{ disjoint from } \gamma \end{array} \right\}.$$

Since the existence of a cut implies the disconnection between T and B , we can rewrite the conditioned probability as

$$\Phi_{\Lambda}^{TB}[A \mid T \leftrightarrow B] = \frac{\Phi_{\Lambda}^{TB}[A]}{\Phi_{\Lambda}^{TB}[T \leftrightarrow B]}.$$

We distinguish two cases according to the positions of the path γ and the cut C . Since the cut C splits the box Λ into two parts and the path γ starting from e is disjoint from the cut, then it is included in one of the two parts. Therefore, the path γ is either separated from T or from B by C . We obtain

$$\Phi_{\Lambda}^{TB}[A] \leq \Phi_{\Lambda}^{TB}[\Gamma(e, T)] + \Phi_{\Lambda}^{TB}[\Gamma(e, B)],$$

where we reuse the notation Γ introduced in the previous lemma by setting $n = \kappa c \ln |\Lambda| / 4d$ and we omit n in the notation as follows:

$$\Gamma(e, T) = \left\{ \begin{array}{l} \exists \gamma \text{ closed path of length } \frac{\kappa c \ln |\Lambda|}{4d} \text{ starting from } e \\ \exists C \in \mathcal{C}, C \text{ separates } \gamma \text{ from } T \end{array} \right\}$$

and $\Gamma(e, B)$ is defined similarly, with B instead of T . By lemma 5.3.2, we have

$$\Phi_{\Lambda}^{TB}[\Gamma(e, T)] \leq (\alpha(d)(1 - f(p, q))^{\kappa c \ln |\Lambda| / 4d} \Phi_{\Lambda}^{TB}[T \leftrightarrow B])$$

and the same holds for $\Phi_{\Lambda}^{TB}[\Gamma(e, B)]$. Thus, we have

$$\frac{\Phi_{\Lambda}^{TB}[A \circ \{\exists C \in \mathcal{C}\}]}{\Phi_{\Lambda}^{TB}[T \leftrightarrow B]} \leq 2(\alpha(d)(1 - f(p, q))^{\kappa c \ln |\Lambda| / 4d}.$$

For $q \geq 1$, there exist a $\tilde{p} < 1$ and $\kappa \geq 1$, such that, for $p \geq \tilde{p}$ and $c \geq 1$, we have

$$(\alpha(d)(1 - f(p, q))^{\kappa c \ln |\Lambda| / 4d} \leq \frac{1}{2|\Lambda|^c},$$

which yields the desired result. \square

5.4 The speed estimate of the pivotal edges

We now study the difference between a cut at two different times t and $t+s$. We begin with a key lemma which controls the number of closing events which can be realised on an interval of length s .

Lemma 5.4.1. *Let Γ be a simple $*$ -path and t a time and $s \in \mathbb{N}$. For any $k \in \{0, \dots, |\text{support}(\Gamma)|\}$ and any configuration y such that exactly k edges of Γ are closed, we have the following inequality:*

$$P_{\mu}(\Gamma \text{ closed in } Y_{t+s} \mid Y_t = y) \leq \left(\frac{s(1-p)}{|\Lambda|(1+p/q-p)} \right)^{|\text{support}(\Gamma)|-k}.$$

Proof. Let us denote by n the cardinal of $\text{support}(\Gamma)$. If an edge e is visited, then it must be closed at time t or become closed at a time after t . Therefore,

there exist $n - k$ different edges e_1, \dots, e_{n-k} of Γ , and $n - k$ different times t_1, \dots, t_{n-k} strictly bigger than t such that

$$\forall i \in \{1, \dots, n - k\} \quad E_{t_i} = e_i \quad \text{and} \quad U_{t_i} \leq \frac{1 - p}{1 - p + p/q}.$$

These events are independent of the configuration Y_t and we obtain an upper bound on the desired probability as follows:

$$P_\mu(\Gamma \text{ closed in } Y_{t+s} \mid Y_t = y) \leq P \left(\begin{array}{l} \forall i \in \{1, \dots, n - k\} \\ \exists t_i \in]t, t + s] \quad E_{t_i} = e_i \\ U_{t_i} \leq \frac{1 - p}{1 - p + p/q} \end{array} \right). \quad (5.4.1)$$

This last probability depends on the i.i.d. sequence of couples $(E_t, U_t)_{t \in \mathbb{N}}$. Moreover, the sets

$$J(e) = \{s < j \leq t : E_j = e\}, \quad e \in \{e_1, \dots, e_{n-k}\}$$

are pairwise disjoint subsets of \mathbb{N} . For an edge $e \in \{e_1, \dots, e_{n-k}\}$, the event

$$\left\{ \begin{array}{l} \exists r \in]s, t] \quad E_r = e \\ U_r \leq \frac{1 - p}{1 - p + p/q} \end{array} \right\}$$

is entirely determined by $(E_t, U_t)_{t \in J(e)}$. Therefore, by the BK inequality (see [Gri99]) applied with the random variables $(E_r, U_r)_{s < r \leq t}$, the second probability in (5.4.1) is less than

$$\prod_{e \in \{e_1, \dots, e_{n-k}\}} P \left(\begin{array}{l} \exists r \in]t, t + s] \quad E_r = e \\ U_r \leq \frac{1 - p}{1 - p + p/q} \end{array} \right). \quad (5.4.2)$$

For each $e \in \{e_1, \dots, e_{n-k}\}$, we have

$$P \left(\begin{array}{l} \exists r \in]t, t + s] \quad E_r = e \\ U_r \leq \frac{1 - p}{1 - p + p/q} \end{array} \right) \leq \frac{s(1 - p)}{|\Lambda|(1 - p + p/q)}.$$

We conclude that (5.4.2) is less or equal than

$$\left(\frac{s(1 - p)}{|\Lambda|(1 - p + p/q)} \right)^{n-k}.$$

Combined with (5.4.1), we have the inequality stated in the lemma. \square

In order to apply the previous result to control the speed of a cut, we will consider s satisfying $0 < s \leq |\Lambda|$ and we study the probability that there exists a pivotal edge e at time $t + s$ which is at distance ℓ from the pivotal edges at time t . The following proposition gives an upper bound on the speed of a cut.

Proposition 5.4.2. *There exist $\tilde{p} < 1$ and $\kappa \geq 1$, such that for $p \geq \tilde{p}$, $t \in \mathbb{N}$, $s \in \{1, \dots, |\Lambda|\}$, $\ell \geq 1$ and any edge $e \subset \Lambda$ at distance more than ℓ from Λ^c ,*

$$P_\mu\left(e \in \mathcal{P}_{t+s} \mid \exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell\right) \leq e^{-\ell}.$$

The proof follows the ideas in the proof of proposition 4.1 of [CZ18a].

Proof. We start by rewriting the conditioned probability as

$$\frac{P_\mu\left(\{e \in \mathcal{P}_{t+s}\} \cap \{\exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell\}\right)}{P_\mu\left(\exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell\right)}. \quad (5.4.3)$$

Since a pivotal edge is in a cut, there exists a closed simple $*$ -path γ which connects e to an edge intersecting the boundary of Λ at time $t + s$. This $*$ -path travels a distance at least ℓ and the length of γ is at least ℓ/d . Therefore, by taking the first n edges such that the support of this sequence is equal to $\ell/2d$, the numerator is bounded from above by

$$P_\mu\left(\left\{\exists \gamma \text{ simple closed } * \text{-path of length } n \text{ starting at } e \text{ at time } t+s\right\} \cap \left\{\exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell\right\}\right) \leq \sum_{\Gamma} P_\mu\left(\left\{\Gamma \text{ simple closed } * \text{-path of length } n \text{ starting at } e \text{ at time } t+s\right\} \cap \left\{\exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell\right\}\right). \quad (5.4.4)$$

For a $*$ -path Γ of length n starting from e , we consider the set $M(k)$ of the configurations defined as

$$M(k) = \left\{ \omega : \begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad \omega(f) = 0 \\ \forall f \in \text{support}(\Gamma) \setminus F \quad \omega(f) = 1 \\ \exists C \in \mathcal{C} \quad d(e, C) \geq \ell \end{array} \right\}.$$

For k fixed, the probability of having exactly k edges in Γ which are closed at time t is less than

$$\sum_{y \in M(k)} P_\mu\left(\begin{array}{l} \Gamma \text{ simple } * \text{-path} \\ \text{closed at time } t+s \end{array} \mid Y_t = y\right) P_\mu(Y_t = y).$$

By lemma 5.4.1, each term of the sum is less than

$$\left(\frac{s(1-p)}{|\Lambda|(1+p/q-p)}\right)^{n-k} P_\mu(Y_t = y).$$

We obtain an upper bound for (5.4.4) as

$$\sum_{\Gamma} \sum_k \left(\frac{s(1-p)}{|\Lambda|(1+p/q-p)}\right)^{n-k} P_\mu(Y_t \in M(k)).$$

Let us calculate the probability $P_\mu(Y_t \in M(k))$. Notice that this event is determined by the configuration Y_t . This probability is less than

$$\begin{aligned} \Phi_\Lambda^{TB} \left[\begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad f \text{ closed} \\ \exists C \in \mathcal{C} \quad d(e, C) \geq \ell \end{array} \middle| T \leftrightarrow B \right] \\ = \frac{\Phi_\Lambda^{TB} \left[\begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad f \text{ closed} \\ \exists C \in \mathcal{C} \quad d(e, C) \geq \ell \end{array} \right]}{\Phi_\Lambda^{TB}[T \leftrightarrow B]}. \end{aligned} \quad (5.4.5)$$

Denote by Δ the set of the edges at distance less than $\ell - 1$ from e , then the event

$$\left\{ \begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad f \text{ closed} \end{array} \right\}$$

depends on the edges inside Δ , whereas the event

$$D = \{ \exists C \in \mathcal{C} \quad d(e, C) \geq \ell \}$$

depends on the edges in $\Lambda \setminus \Delta$. Denote by $\Omega_{\Lambda \setminus \Delta}$ the set of the configurations of the edges in $\Lambda \setminus \Delta$. By the spatial Markov property, we have

$$\begin{aligned} \Phi_\Lambda^{TB} \left[\begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad f \text{ closed} \\ \exists C \in \mathcal{C} \quad d(e, C) \geq \ell \end{array} \right] \\ = \sum_{\omega \in \Omega_{\Lambda \setminus \Delta} \cap D} \Phi_\Lambda^{TB} \left[\begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad f \text{ closed} \end{array} \middle| \omega \right] \Phi_\Lambda^{TB}[\omega] \\ = \sum_{\omega \in \Omega_{\Lambda \setminus \Delta} \cap D} \Phi_\Delta^{\xi(\omega)} \left[\begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad f \text{ closed} \end{array} \right] \Phi_\Lambda^{TB}[\omega]. \end{aligned} \quad (5.4.6)$$

Since the event

$$\left\{ \begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad f \text{ closed} \end{array} \right\}$$

is decreasing, for any boundary condition $\xi(\omega)$ on Δ induced by ω , we have

$$\Phi_\Delta^{\pi(\omega)} \left[\begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad f \text{ closed} \end{array} \right] \leq \Phi_\Delta^0 \left[\begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad f \text{ closed} \end{array} \right].$$

We compare this last probability to the one under Bernoulli percolation with parameter

$$f(p, q) = \frac{p}{p + q - pq}.$$

By the comparison inequalities between different values of p, q , we have

$$\Phi_{\Delta}^0 \left[\begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad f \text{ closed} \end{array} \right] \leq \binom{\ell/2d}{k} (1 - f(p, q))^k.$$

We use this inequality in (5.4.6) and we obtain

$$\Phi_{\Lambda}^{TB} \left[\begin{array}{l} \exists F \subset \text{support}(\Gamma), |F| = k \\ \forall f \in F \quad f \text{ closed} \\ \exists C \in \mathcal{C} \quad d(e, C) \geq \ell \end{array} \right] \leq \binom{\ell/2d}{k} (1 - f(p, q))^k \Phi_{\Lambda}^{TB}[D].$$

Combined with (5.4.5), we have

$$P_{\mu}(Y_t \in M(k)) \leq \binom{\ell/2d}{k} (1 - f(p, q))^k \Phi_{\Lambda}^{TB}[D \mid T \leftrightarrow B].$$

We replace $P_{\mu}(Y_t \in M(k))$ in the previous upper bound for (5.4.4) and we have another upper bound as follows:

$$\Phi_{\Lambda}^{TB}[D \mid T \leftrightarrow B] \sum_{\Gamma} \sum_k \left(\frac{s(1-p)}{|\Lambda|(1+p/q-p)} \right)^{n-k} \binom{\ell/2d}{k} (1 - f(p, q))^k.$$

We calculate the sum and we get

$$\begin{aligned} & \sum_{\Gamma} \sum_k \left(\frac{s(1-p)}{|\Lambda|(1+p/q-p)} \right)^{n-k} \binom{\ell/2d}{k} (1 - f(p, q))^k \\ & \leq \left(\frac{1-p}{p/q} \right)^n \sum_{\Gamma} \sum_k \binom{\ell/2d}{k} \left(\frac{s}{|\Lambda|} \right)^{n-k} \\ & \leq \sum_{\Gamma} \left(\frac{1-p}{p/q} \right)^n \left(1 + \frac{s}{|\Lambda|} \right)^{\ell/2d} \leq \left(\frac{\alpha(d)(1-p)}{p/q} \right)^n \left(1 + \frac{s}{|\Lambda|} \right)^{\ell/2d}. \end{aligned}$$

Since $s \leq |\Lambda|$, there exists a constant $\tilde{p} < 1$ such that

$$\left(\frac{\alpha(d)(1-p)}{p/q} \right)^n \left(1 + \frac{s}{|\Lambda|} \right)^{\ell/2d} \leq e^{-\ell}.$$

We obtain the following upper bound for (5.4.3):

$$\frac{P_{\mu}(\{e \in \mathcal{P}_{t+s}\} \cap \{\exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell\})}{P_{\mu}(\{\exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell\})} \leq e^{-\ell} \frac{\Phi_{\Lambda}^{TB}[D \mid T \leftrightarrow B]}{P_{\mu}(\{\exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell\})}.$$

Finally, we notice that

$$\Phi_{\Lambda}^{TB}[D \mid T \leftrightarrow B] = P_{\mu}(\{\exists c_t \in \mathcal{C}_t, d(e, c_t) \geq \ell\})$$

and we obtain the desired inequality. \square

5.5 The interface in the FK-percolation model

We now prove the main result stated in theorem 5.1.4. We follow the main steps of the proof of theorem 1.1 in [CZ18a]. We start with the definition of d_H^ℓ , a semi-distance, similar to the Hausdorff distance, between two subsets A, B of Λ ,

$$d_H^\ell(A, B) = \inf \left\{ r \geq 0 : \begin{array}{l} A \setminus \mathcal{V}(\Lambda^c, \ell) \subset \mathcal{V}(B, r) \\ B \setminus \mathcal{V}(\Lambda^c, \ell) \subset \mathcal{V}(A, r) \end{array} \right\}.$$

We show a lemma which controls the speed of the pivotal edges.

Lemma 5.5.1. *We have the following result:*

$$\begin{aligned} \exists \tilde{p} < 1 \quad \exists \kappa > 1 \quad \forall p \geq \tilde{p} \quad \forall c \geq 1 \quad \forall \Lambda \quad |\Lambda| \geq 4 \quad \forall t \geq 0 \\ P_\mu \left(\exists s \leq |\Lambda| \quad d_H^{\kappa c \ln |\Lambda|}(\mathcal{P}_t, \mathcal{P}_{t+s}) \geq \kappa c \ln |\Lambda| \right) \leq \frac{8d}{|\Lambda|^c}. \end{aligned}$$

Proof. We fix $s \in \{1, \dots, |\Lambda|\}$. By the definition of the semi-distance d_H^ℓ , we have, for any $\kappa > 1$,

$$\begin{aligned} P_\mu \left(d_H^{\kappa c \ln |\Lambda|}(\mathcal{P}_t, \mathcal{P}_{t+s}) \geq \kappa c \ln |\Lambda| \right) \leq \\ P_\mu \left(\mathcal{P}_{t+s} \setminus \mathcal{V}(\Lambda^c, \kappa c \ln |\Lambda|) \not\subseteq \mathcal{V}(\mathcal{P}_t, \kappa c \ln |\Lambda|) \right) \\ + P_\mu \left(\mathcal{P}_t \setminus \mathcal{V}(\Lambda^c, \kappa c \ln |\Lambda|) \not\subseteq \mathcal{V}(\mathcal{P}_{t+s}, \kappa c \ln |\Lambda|) \right). \quad (5.5.1) \end{aligned}$$

Since the two probabilities in the sum depend only on the process Y , which is reversible, they are in fact equal to each other. We shall estimate the first probability. By proposition 5.4.2, for any $\ell \geq 1$, we have

$$P_\mu \left(\begin{array}{l} e \in \mathcal{P}_{t+s} \\ \exists c_t \in \mathcal{C}_t \quad d(e, c_t) \geq \ell \end{array} \right) \leq e^{-\ell}.$$

In order to replace c_t by \mathcal{P}_t in the last probability, we use proposition 5.3.1 for the configuration Y_t . We introduce another constant κ' and we have

$$P_\mu \left(\begin{array}{l} \exists C \in \mathcal{C}_t \quad \exists f \in C \\ d(f, \Lambda^c \cup \mathcal{P}_t \setminus \{f\}) \geq \kappa' c \ln |\Lambda| \end{array} \right) \leq \frac{1}{|\Lambda|^c}.$$

Therefore, by distinguishing two cases for the configuration Y_t , we have

$$\begin{aligned} P_\mu(e \in \mathcal{P}_{t+s}, d(e, \mathcal{P}_t) \geq \kappa c \ln |\Lambda|) \leq \\ P_\mu \left(\begin{array}{l} e \in \mathcal{P}_{t+s}, \quad d(e, \mathcal{P}_t) \geq \kappa c \ln |\Lambda|, \\ \forall C \in \mathcal{C}_t \quad \forall f \in C \setminus \mathcal{V}(\Lambda^c, \kappa' c \ln |\Lambda|) \\ \quad d(f, \mathcal{P}_t) < \kappa' c \ln |\Lambda| \end{array} \right) \\ + P_\mu \left(\exists C \in \mathcal{C}_t, \exists f \in C, d(f, \Lambda^c \cup \mathcal{P}_t \setminus \{f\}) \geq \kappa' c \ln |\Lambda| \right). \end{aligned}$$

For $\kappa > \kappa'$, we have

$$P_\mu \left(\begin{array}{l} e \in \mathcal{P}_{t+s} \quad d(e, \Lambda^c \cup \mathcal{P}_t) \geq \kappa c \ln |\Lambda| \\ \forall C \in \mathcal{C}_t \quad \forall f \in C \setminus \mathcal{V}(\Lambda^c, \kappa' c \ln |\Lambda|) \\ d(f, \mathcal{P}_t) < \kappa' c \ln |\Lambda| \end{array} \right) \leq \\ P_\mu \left(\begin{array}{l} e \in \mathcal{P}_{t+s} \\ \exists c_t \in \mathcal{C}_t \quad d(e, c_t) \geq (\kappa - \kappa') c \ln |\Lambda| \end{array} \right) \leq \frac{1}{|\Lambda|^{(\kappa - \kappa')c}}.$$

We choose now $\kappa = \kappa' + 1$, and we sum over e in Λ and $s \in \{1, \dots, |\Lambda|\}$ to get

$$P_\mu \left(\begin{array}{l} \exists s \leq |\Lambda|, \exists e \in \mathcal{P}_{t+s} \\ d(e, \Lambda^c \cup \mathcal{P}_t) \geq \kappa c \ln |\Lambda| \end{array} \right) \leq \frac{4d}{|\Lambda|^{c-2}}.$$

This is the first probability in (5.5.1) and we conclude that

$$P_\mu \left(\begin{array}{l} \exists s \leq |\Lambda| \\ d_H^\Lambda(\mathcal{P}_t, \mathcal{P}_{t+s}) \geq \kappa c \ln |\Lambda| \end{array} \right) \leq \frac{8d}{|\Lambda|^{c-2}}.$$

Finally, we replace c by $c + 2$ and we have

$$P_\mu \left(\begin{array}{l} \exists s \leq |\Lambda| \\ d_H^\Lambda(\mathcal{P}_t, \mathcal{P}_{t+s}) \geq 3\kappa c \ln |\Lambda| \end{array} \right) \leq \frac{8d}{|\Lambda|^c}.$$

This is the desired inequality. \square

The rest of the proof of theorem 5.1.4 is exactly the same as the proof of theorem 1.1 in [CZ18a] which relies essentially on lemma 5.5.1 in [CZ18a] independently from the model. We distinguish the case $e \in \mathcal{P}_t$ and the case $e \in \mathcal{I}_t \setminus \mathcal{P}_t$. For the first case, we apply proposition 5.3.1. For the second case, we notice that this configuration is due to a movement of pivotal edges of distance $\ln^2 |\Lambda|$ during a time interval of order $|\Lambda|$. We then apply lemma 5.5.1 to prove that this event is unlikely.

5.6 The interface in the Ising model

We recall the definition of the two sets \mathcal{P}_I and \mathcal{I}_I :

$$\mathcal{P}_I = \left\{ \langle x, y \rangle \in E : \begin{array}{l} \sigma^D(x) = +1 \quad \sigma^D(y) = -1, \\ x \text{ connected to } T \text{ by a path of spin } +1 \text{ in } \sigma^D, \\ y \text{ connected to } B \text{ by a path of spin } -1 \text{ in } \sigma^D \end{array} \right\}.$$

and

$$\mathcal{I}_I = \left\{ \langle x, y \rangle \in E : \begin{array}{l} \sigma^+(x) = \sigma^+(y) \\ \sigma^-(x) = \sigma^-(y) \\ \sigma^D(x) \neq \sigma^D(y) \end{array} \right\}.$$

We now show our main result on the Ising model.

Proof of theorem 5.1.2. We fix a constant c and we let κ be a constant which will be determined later. Let us consider the FK configurations (ω, ω') associated to $(\sigma^+, \sigma^-, \sigma^D)$. We consider the set of the pivotal edges \mathcal{P} and the set of the interface edges \mathcal{I} in the couple (ω, ω') . We claim that $\mathcal{P} \subset \mathcal{P}_I$. In fact, for an edge $\langle x, y \rangle \in \mathcal{P}$, one of the endpoint x is connected to T and the other B is connected to B in ω' . Therefore, we have $\sigma^D(x) = 1$ and $\sigma^D(y) = -1$. The configuration ω dominates ω' , so both of them are also connected to the boundary of Λ in ω . We conclude that

$$\sigma^+(x) = \sigma^+(y) = 1 \text{ and } \sigma^-(x) = \sigma^-(y) = -1.$$

Let e be an edge in \mathcal{I}_I . Since the endpoints of e have different spins in σ^D , then the edge e must be closed in ω' and its endpoints belong to two distinct open clusters of ω' . We denote by x, y the two endpoints of e and by C_x, C_y the two open clusters of ω' such that $x \in C_x$ and $y \in C_y$. The vertices x and y have the same spin in σ^+ and in σ^- , therefore one of them, as well as its cluster, has different spins in σ^+ and σ^D . Suppose it is C_x . Suppose that e belongs to $\mathcal{I}_I \setminus \mathcal{P}_I$. We distinguish three cases as follows:

- If x is connected to the boundary of Λ in ω' , then it is connected to the boundary in ω and the spin of x is determined by the boundary condition in σ^+ and σ^- . We have $\sigma^+(x) = +$ and $\sigma^-(x) = -$. Since we have $e \in \mathcal{I}_I$, we have also $\sigma^+(y) = +$ and $\sigma^-(y) = -$. Since the difference of the spin between σ^+ and σ^- can only be induced by the boundary, the vertex y is also connected to the boundary in ω . However, since the edge e is not in \mathcal{P}_I , the vertex y can not be connected to the boundary in ω' . Therefore there is an edge $f \in \mathcal{I}$ included in the boundary of the C_y .
- If x is connected to the boundary in ω but not in ω' . Then, x is connected to an edge $f \in \mathcal{I}$. In other words, there exists an edge $f \in \mathcal{I}$ on the boundary of C_x .
- If x is not connected to the boundary in ω , we have then $\sigma^+(x) = \sigma^-(x)$ and x is not connected to the boundary of Λ in ω' . By construction of σ^+ , C_x is contained in an open cluster of ω which is the union of at least two different open clusters of ω' . Therefore, the boundary of C_x contains at least one edge of ω' in \mathcal{I} .

In all cases, at least one endpoint of the edge e is included in an open cluster in ω' which contains an edge $f \in \mathcal{I}$ on the boundary. Let us fix the edge e at distance more than $\kappa c \ln^2 |\Lambda|$ from Λ^c . We distinguish two cases according to the position of the edge $f \in \mathcal{I}$. Let κ' be the constant given by theorem

5.1.4. The probability in the theorem is less than

$$\begin{aligned} & \mu_{p,2} \left(\exists f \in \mathcal{I}, d(f, \Lambda^c \cup \mathcal{P}) \geq \kappa' c^2 \ln |\Lambda| \right) \\ & + \mu_{p,2} \left(\begin{array}{l} \exists f \in \mathcal{I}, d(f, \Lambda^c \cup \mathcal{P}) < \kappa' c^2 \ln |\Lambda| \\ e \in \mathcal{I}_I, d(e, \Lambda^c \cup \mathcal{P}_I) \geq \kappa c^2 \ln^2 |\Lambda| \\ \exists x \text{ endpoint of } e \text{ s.t.} \\ f \text{ is on the boundary of } C_x \\ \text{and } C_x \cap \partial\Lambda = \emptyset \end{array} \right). \end{aligned} \quad (5.6.1)$$

By theorem 5.1.4, there exists a p_1 such that for $p \geq p_1$, we have

$$\mu_{p,2} \left(\exists f \in \mathcal{I}, d(f, \Lambda^c \cup \mathcal{P}) \geq \kappa' c^2 \ln |\Lambda| \right) \leq \frac{1}{|\Lambda|^c}. \quad (5.6.2)$$

We consider the second case where the edge f is at distance less than $\kappa' c^2 \ln^2 |\Lambda|$ from $\Lambda^c \cup \mathcal{P}$. Notice that the cluster C_x is of diameter at least $(\kappa - \kappa') c^2 \ln^2 |\Lambda|$. By taking $\kappa/2 \geq \kappa'$, the diameter of C_x is at least $(\kappa c^2 \ln^2 |\Lambda|)/2$. So the second probability in (5.6.1) is less than

$$\mu_{p,2} \left(\begin{array}{l} \exists x \text{ endpoint of } e \quad \exists f \in \Lambda \\ C_x \text{ is of diameter at least } (\kappa c^2 \ln^2 |\Lambda|)/2 \\ \text{and } f \in \partial_e C_x \quad C_x \cap \partial\Lambda = \emptyset \end{array} \right).$$

We fix an edge $f \in \Lambda$ and we write

$$\begin{aligned} & \mu_{p,2} \left(\begin{array}{l} \exists x \text{ endpoint of } e \quad \exists f \in \Lambda \\ C_x \text{ is of diameter at least } (\kappa c^2 \ln^2 |\Lambda|)/2 \\ \text{and } f \in \partial_e C_x \quad C_x \cap \partial\Lambda = \emptyset \end{array} \right) \\ & \leq \sum_{f \in \Lambda} \frac{\Phi_{\Lambda,p,2}^{TB} \left[\begin{array}{l} \exists C_x \text{ of diameter at least } (\kappa c^2 \ln^2 |\Lambda|)/2 \\ f \in \partial_e C_x \quad C_x \cap \partial\Lambda = \emptyset \quad T \leftrightarrow B \end{array} \right]}{\Phi_{\Lambda,p,2}^{TB} [T \leftrightarrow B]}. \end{aligned} \quad (5.6.3)$$

In order to estimate the numerator, we fix the set $\partial_e C_x$ and we denote by n the cardinal of C_x . We claim that there exists a set of edges A included in $\partial_e C_x$ which is disjoint from a cut and which is of size at least $n/2$. Let us consider the case where every cut C intersects $\partial_e C_x$ (the claim is true in case where there exists a cut which is disjoint from $\partial_e C_x$). Since $\partial_e C_x$ is the outer edge boundary of an open cluster, it does not contain a pivotal edge. We apply again the exploration process described in lemma 5.3.2 starting from T and we reuse the notation \mathcal{G} defined in the proof of the lemma. If the sub-graph \mathcal{G} obtained after exploration contains at least $n/2$ edges of $\partial_e C_x$, then the set $\mathcal{A}_1 = E(\mathcal{G}) \cap \partial_e C_x$ is disjoint from a cut and is of size at least $n/2$. If not, using the same exploration starting from B , we obtain a sub-graph \mathcal{G}' such that

$$(\partial_e C_x \setminus E(\mathcal{G})) \subset E(\mathcal{G}').$$

Actually, an edge of the set $\partial_e C_x \setminus E(\mathcal{G})$ contains at least one endpoint connected to T , since there are no pivotal edges in $\partial_e C_x$, it is not connected to B . By the construction of \mathcal{G}' , this edge is contained in \mathcal{G}' . Therefore, $A_2 = \partial_e C_x \cap \mathcal{E}'$ contains at least $n/2$ and is disjoint from a cut (see figure 5.3).

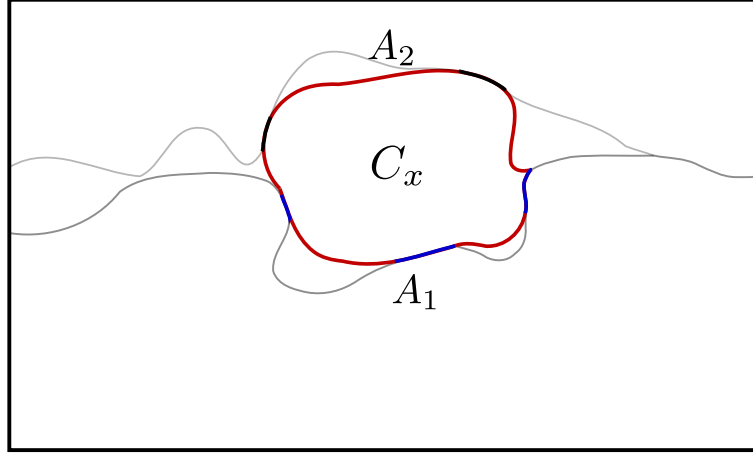


Figure 5.3: The set $\partial_e C_x$ (red, blue and black) and two cuts obtained after the exploration process from T and B (gray). The sets A_1 (red and blue) and A_2 (red and black) are both disjoint from a cut.

For two graphs G_1 containing the edges intersecting B and G_2 containing the edges intersecting T , we set $A_1 = E(G_1) \cap \partial_e C_x$ and $A_2 = E(G_2) \cap \partial_e C_x$. We have

$$\begin{aligned}
& \Phi_{\Lambda,p,2}^{TB} \left[\begin{array}{l} \forall f \in \partial_e C_x \quad f \text{ is closed} \\ |\partial_e C_x| = n \quad T \leftrightarrow B \end{array} \right] \\
& \leq \sum_{G_1} \Phi_{\Lambda,p,2}^{TB} \left[\begin{array}{l} \forall f \in A_1 \quad f \text{ is closed,} \\ |A_1| \geq n/2, \quad \mathcal{G} = G_1 \end{array} \right] \\
& + \sum_{G_2} \Phi_{\Lambda,p,2}^{TB} \left[\begin{array}{l} \forall f \in A_2 \quad f \text{ is closed,} \\ |A_2| \geq n/2, \quad \mathcal{G}' = G_2 \end{array} \right] \\
& + \Phi_{\Lambda,p,2}^{TB} \left[\begin{array}{l} \forall f \in \partial_e C_x \quad f \text{ is closed} \\ |\partial_e C_x| = n \\ \exists C \in \mathcal{C} \quad C \cap \partial_e C_x = \emptyset \end{array} \right]. \quad (5.6.4)
\end{aligned}$$

By the spatial Markov property and the comparison inequality used in the proof of lemma 5.3.2. We have

$$\sum_{G_1} \Phi_{\Lambda,p,2}^{TB} \left(\begin{array}{l} \forall f \in A_1 \quad f \text{ is closed,} \\ |A_1| \geq n/2, \quad \mathcal{G} = G_1 \end{array} \right) \leq (1 - f(p, 2))^{n/2} \sum_{G_1} \Phi_{\Lambda,p,2}^{TB} [\mathcal{G} = G_1].$$

The same goes for the second sum of (5.6.4). The last sum is less than $\Phi_{\Lambda,p,2}^{TB}[T \leftrightarrow B]$. By lemma 5.3.2, last term in (5.6.4) is less than

$$2(1 - f(p, 2))^n \Phi_{\Lambda,p,2}^{TB}[T \leftrightarrow B].$$

We obtain an upper bound for (5.6.4) as follows:

$$\Phi_{\Lambda,p,2}^{TB} \left[\begin{array}{l} \forall f \in \partial_e C_x \quad f \text{ is closed} \\ |\partial_e C_x| = n \quad T \leftrightarrow B \end{array} \right] \leq 4(1 - f(p, 2))^n \Phi_{\Lambda,p,2}^{TB}[T \leftrightarrow B].$$

The set $\partial_e C_x$ is $*$ -connected (see [DP96]). For a fixed edge f , the number of the $*$ -connected sets $\partial_e C_x$ of size n containing the edge f is bounded from above by $C_n \alpha(d)^n$, where

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

is the n th number Catalan number. Using Stirling formula, an upper bound of the number of $\partial_e C_x$ is $4^n \alpha(d)^n$. We would like to mention the arguments of [Kes86][p.82] and [Gri99][theorem 4.20] for an upper bound of lattice animals in \mathbb{Z}^d . We obtain therefore

$$\begin{aligned} \Phi_{\Lambda,p,2}^{TB} \left[\begin{array}{l} \exists C_x \text{ of diameter at least } (\kappa c^2 \ln^2 |\Lambda|)/2 \\ f \in \partial_e C_x \quad \partial_e C_x \cap \partial \Lambda = \emptyset \quad T \leftrightarrow B \end{array} \right] \\ \leq \sum_{n \geq (\kappa c^2 \ln^2 |\Lambda|)/2} 4^{n+1} \alpha(d)^n (1 - f(p, 2))^{n/2} \Phi_{\Lambda,p,2}^{TB}[T \leftrightarrow B] \\ \leq (8\alpha^2(d) - 8\alpha^2(d)f(p, 2))^{\kappa c^2 \ln^2 |\Lambda|/4} \Phi_{\Lambda,p,2}^{TB}[T \leftrightarrow B]. \end{aligned}$$

We sum over the choices of the edge f and we obtain an upper bound for (5.6.3) as follows:

$$\begin{aligned} \mu_{p,2} \left(\begin{array}{l} \exists x \text{ endpoint of } e \quad \exists f \in \Lambda \\ C_x \text{ is of diameter at least } (\kappa c^2 \ln^2 |\Lambda|)/2 \\ \text{and } f \in \partial_e C_x \quad C_x \cap \partial \Lambda = \emptyset \end{array} \right) \\ \leq |\Lambda| (8\alpha^2(d) - 8\alpha^2(d)f(p, q))^{\kappa c^2 \ln^2 |\Lambda|/4} \end{aligned}$$

There exist $p_2 < 1$ and $\kappa \geq 1$ such that for $p \geq p_2$,

$$|\Lambda| (8\alpha^2(d) - 8\alpha^2(d)f(p, q))^{\kappa c^2 \ln^2 |\Lambda|/4} \leq \frac{1}{|\Lambda|^c}.$$

Combined with (5.6.1), (5.6.2), for $p \geq \max(p_1, p_2)$, we have

$$\pi_\beta \left(e \in \mathcal{I}_I, d(e, \Lambda^c \cup \mathcal{P}_I) \geq \kappa c^2 \ln^2 |\Lambda| \right) \leq \frac{2}{|\Lambda|^c}.$$

We then sum over the edges e in Λ and we get

$$\pi_\beta \left(\exists e \in \mathcal{I}_I, d(e, \Lambda^c \cup \mathcal{P}_I) \geq \kappa c^2 \ln^2 |\Lambda| \right) \leq \frac{2}{|\Lambda|^{c-1}}.$$

For $|\Lambda| \geq 4$, we can replace $2/|\Lambda|^{c-1}$ by $1/|\Lambda|^{c-2}$. By taking κ big enough, we can replace $c-2$ by c and we obtain the result announced in the theorem. \square

5.7 Proof of the second result

Proof. By symmetry, it is sufficient to show the first inequality. The proof follows the same arguments for the proof of theorem 5.1.2 and we use the same notations as in the previous proof. We fix a vertex x and we notice that x has different spins only when it is connected to the boundary or it is contained in an open cluster C_1 in ω which is the union of at least two open clusters of ω' . Actually, if x is connected to T in ω , since $\sigma^D(x) = -1$, it is not connected to T in ω' . If x is connected to B in ω , since the cut separates x from B in ω' , it can not be connected to B in ω' . If x is not connected to the boundary of the box, since $\sigma^+(x) \neq \sigma^D(x)$, the open cluster C_x in ω is the union of at least two open clusters of ω' . In all the three cases, the open cluster of C_x in ω' contains an edge $f \in \mathcal{I}$ and C_x does not touch the boundary of Λ . We obtain

$$\begin{aligned} \pi_\beta \left(\begin{array}{l} \sigma^+(x) = +1, \quad \sigma^D(x) = -1 \\ \exists C \text{ a cut separating } x \text{ from } B \\ d(x, C) \geq \kappa c^2 \ln^2 |\Lambda| \end{array} \right) \\ \leq \mu_{p,2} \left(\begin{array}{l} \exists f \in \mathcal{I} \quad f \in \partial_e C_x \\ \exists C \in \mathcal{C} \quad d(x, C) \geq \kappa c^2 \ln^2 |\Lambda| \\ C_x \cap \partial \Lambda = \emptyset \end{array} \right). \end{aligned}$$

The rest of the proof follows exactly the same arguments as in the previous proof. We distinguish two cases, if there is an edge $f \in \mathcal{I}$ far from the cut C , we apply the theorem 5.1.4. If all the edges of \mathcal{I} are close to the cut, the cluster C_x has a diameter at least $\kappa c^2 \ln^2 |\Lambda|/2$, the same reasoning starting from (5.6.3) can be applied to obtain an upper bound in this case. Combining the two cases and we have

$$\pi_\beta \left(\begin{array}{l} \sigma^+(x) = +1, \quad \sigma^D(x) = -1 \\ \exists C \text{ a cut separating } x \text{ from } B \\ d(x, C) \geq \kappa c^2 \ln^2 |\Lambda| \end{array} \right) \leq \frac{1}{|\Lambda|^c}.$$

We sum over the choices of x and we have

$$\pi_\beta \left(\begin{array}{l} \exists x \in \Lambda \quad \sigma^+(x) = +1, \quad \sigma^D(x) = -1 \\ \exists C \text{ a cut separating } x \text{ from } B \\ d(x, C) \geq \kappa c^2 \ln^2 |\Lambda| \end{array} \right) \leq \frac{1}{|\Lambda|^{c-1}}.$$

We can change c by $c+1$ and we obtain the desired result. \square

Liste de notations principales

Graphes et ensembles :

$G = (V, E)$	Grphe avec l'ensemble de sommets V et d'arêtes E , 5
$\langle x, y \rangle$	Arête entre le sommets x et y , 3
\mathbb{R}	L'ensemble des réels, 3
\mathbb{Z}	L'ensemble des entiers, 3
\mathbb{N}	L'ensemble des entiers naturels, 10
\mathbb{L}^d	Le réseau cubique en dimension d , 3
\mathbb{E}^d	L'ensemble des arêtes du réseau \mathbb{L}^d , 3
∂A	Le bord extérieur de A , 3
$\partial_{in} A$	Le bord intérieur de A , 4
$\mathcal{Q}_{x,y}$	La plaquette entre x et y , 13
Λ	Boîte de taille finie dans \mathbb{L}^d , 19
Λ_n	La boîte des sommets inclus dans $[-n, n]^d$, 12
A^c	Le complémentaire de l'ensemble A , 20
$\alpha(d)$	Le nombre de $*$ -voisins d'une arête en dimension d , 52
$\delta_{x,y}$	Le symbole de Kronecker, 8
$ A $	Le cardinal de l'ensemble A , 20
$d(x, y)$	La distance euclidienne entre x et y , 20
$d_H(A, B)$	La distance de Hausdorff entre les ensembles A et B , 66

Notations probabilistes :

p, q	Paramètres de FK-percolation, 5
P_p	Probabilité de percolation Bernoulli avec paramètre p , 5
$\Phi_{G,p,q}$	Probabilité de FK-percolation sur le graphe G , 6
$\Phi_{G,p,q}^\xi$	Probabilité de FK-percolation avec condition aux bords ξ , 6
β	L'inverse de la température T , 8
ω	Configuration formée des arêtes ouvertes et fermées, 4
Ω	Espace de configurations d'arêtes, 4
$\omega _A$	Restriction d'une configuration dans A , 4
ω^e, ω_e	Configuration ω avec l'arête e ouverte/fermée, 4
$\omega_1 \geq \omega_2$	La configuration ω_1 domine ω_2 , 4

σ	Configuration de spins sur les sommets, 8
Σ	Espace de configurations de spins, 8
$\{A \longleftrightarrow B\}$	Événement l'ensemble A est connecté à l'ensemble B , 19
$\{A \nleftrightarrow B\}$	Le complémentaire de $\{A \leftrightarrow B\}$, 19
$\mathbb{1}_A$	La fonction indicatrice de l'événement A , 8
C_x	Cluster de x , 138

Notations dynamiques :

$(X_t)_{t \in \mathbb{N}}$	Gibbs sampler (la percolation dynamique pour $q = 1$), 10
$(Y_t)_{t \in \mathbb{N}}$	Processus conditionné par l'événement $\{T \leftrightarrow B\}$, 10
$(E_t)_{t \in \mathbb{N}}$	Suite d'arêtes aléatoires dans Λ , 10
$(B_t)_{t \in \mathbb{N}}$	Suite de variables de Bernoulli, 10
$(U_t)_{t \in \mathbb{N}}$	Suite de variables uniformes sur $[0, 1]$, 10
\mathcal{I}_t	Interface à l'instant t , 19
\mathcal{P}_t	Ensemble des arêtes pivot à l'instant t , 19
\mathcal{C}_t	Ensemble des cuts à l'instant t , 51
$\mu_{p,q}$	La mesure invariante du processus $(X_t, Y_t)_{t \in \mathbb{N}}$, 20
P_μ	La loi de $(X_t, Y_t)_{t \in \mathbb{N}}$ démarré avec (X_0, Y_0) de loi μ , 20
$(X_t^\varepsilon)_{t \in \mathbb{N}}$	Le processus $(X_t)_{t \in \mathbb{N}}$ perturbé de ε , 93
P^ε	La probabilité P perturbée de ε , 94
$\sigma^+, \sigma^-, \sigma^D$	Configurations de spins obtenues à partir de (X_t, Y_t) , 24
$\pi_{\Lambda, \beta}$	La loi de $(\sigma^+, \sigma^-, \sigma^D)$ avec paramètre β , 24

Chemins espace-temps :

γ	Chemin espace-temps, 32
$\text{Space}(\gamma)$	Projection spatiale de γ , 32
$\text{length}(\gamma)$	Longueur de γ , 32
$\text{support}(\gamma)$	Support de γ , 32

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Titre : Un nouveau regard sur les interfaces dans les modèles de percolation et d'Ising

Mots Clefs : Interface, localisation, percolation, FK-percolation, modèle d'Ising.

Résumé : Les interfaces dans les modèles de percolation et d'Ising jouent un rôle crucial dans la compréhension de ces modèles et sont au cœur de plusieurs problématiques : la construction de Wulff, le mouvement par courbure moyenne, la théorie du SLE. Dans son célèbre article de 1972, Roland Dobrushin a montré que le modèle d'Ising en dimension $d \geq 3$ admet une mesure de Gibbs qui n'est pas invariante par translation à l'aide d'une étude sur l'interface entre le haut et le bas d'une boîte droite de taille finie. Le cas d'une boîte penchée est très différent et plus difficile à analyser. Nous proposons dans cette thèse une nouvelle définition de l'interface. Cette définition est construite dans le modèle de percolation Bernoulli à l'aide d'un couplage dynamique de deux configurations. Nous montrons que cette interface est localisée autour des arêtes pivot à une distance d'ordre de $\ln^2 n$ dans une boîte de taille n . Notre méthode de preuve utilise les chemins espace-temps, qui permettent de contrôler la vitesse de déplacement de l'interface. Nous montrons aussi que la vitesse des arêtes pivot est au plus de l'ordre de $\ln n$. Nous étendons ces résultats au modèle de FK-percolation, nous montrons la localisation de l'interface à distance d'ordre $\ln^2 n$ autour des arêtes pivot. En utilisant une modification du couplage classique d'Edwards-Sokal, nous obtenons des résultats analogues sur la localisation de l'interface dans le modèle d'Ising.

Title : A new look at the interfaces in the percolation and Ising models

Keywords : Interface, localization, percolation, FK-percolation, Ising model

Abstract : The interfaces in the percolation and Ising models play an important role in the understanding of these models and are at the heart of several problematics : the Wulff construction, the mean curvature motion and the SLE theory. In his famous 1972 paper, Roland Dobrushin showed that the Ising model in dimensions $d \geq 3$ has a Gibbs measure which is not invariant by translation by studying the interface between the top and the bottom of a straight finite box. The case of a tilted box is very different and more difficult to analyze. In this thesis, we propose a new definition of the interface. This definition is constructed in the Bernoulli percolation model with the help of a dynamical coupling between two configurations. We show that this interface is localized around the pivotal edges within a distance of order $\ln^2 n$ inside a box of size n . The proof relies on space-time paths which allow us to control the speed of the interface. We also show that the speed of the pivotal edges is at most of order $\ln n$. We extend these results to the FK-percolation model, we show the localization of the interface at distance of order $\ln^2 n$ around the pivotal edges. Using a modification of the classical Edwards-Sokal coupling, we obtain analogous results on the localization of the interface in the Ising model.

